Advanced Exposure Rating

Topics

- History of Casualty Loss Curves
- ISO’s Truncated Pareto
- Casualty Exposure Rating
- Using ILFs for Exposure Rating
- Working with Personal Auto Split Limits
- Property Exposure Rating – Using First Loss Scales
- Property Exposure Rating – Using PSOLD
- Stacking and Participation
- Miscellaneous Topics
Section 1

**History of Casualty Loss Curves**

And Concerns with some of them
ISO Casualty Loss Curves

History

- Truncated Pareto ("5-parameter Pareto") – prior to 1994

  - Single Mixed Pareto approximation
  - Truncated Pareto approximation

- Mixed Exponential (ME) – introduced in 1998/1999
  - Single Mixed Pareto approximation
  - Truncated Pareto approximation
ISO Casualty Loss Curves
Advantages/Disadvantages

- **Pareto Soup**
  - Preferred by many reinsurers because of thicker tail
  - (Not updated, new tables not available)
  - (Many parameters needed)

- **Truncated Pareto Approximation to the Pareto Soup**
  - Same as Pareto Soup
  - Fewer parameters (5)
  - (Difficult to use below truncation point)

- **Mixed Pareto Approximation to the Pareto Soup**
  - Same as Pareto Soup
  - Fewer parameters (5)
ISO Casualty Loss Curves
Advantages/Disadvantages

- Mixed Exponential (ME)
  - Current ISO Methodology
  - Source of Latest Information (by-State Prem/Ops and Auto Groups)
  - Better fit than Mixed Pareto over wide range of loss sizes
  - Simpler, fewer parameters than Pareto Soup, more flexible
  - (Many Reinsurers believe these are too thin in the tail)

- Truncated Pareto Approximation to the Mixed Exponential
  - Same as Mixed Exponential
  - (Difficult to use below truncation point)

- Mixed Pareto Approximation to the Mixed Exponential
  - Same as Mixed Exponential
ISO Casualty Loss Curves
Formulas

Mixed Exponential

$$CDF_{ME}(x; \mu, w) = \sum_{i=1}^{8} w_i \times \left(1 - e^{-\frac{x}{\mu}}\right)$$

Truncated Pareto (w/ Split Uniform)

$$CDF_{TP}(x; b, q, p, s, t) = \frac{p \times (t - s)}{t} + \frac{(x - s) \times p \times s}{t \times (t - s)} \quad t \geq x > s$$

Single Mixed Pareto

$$CDF_{MP}(x; b_1, q_1, p, b_2, q_2) = \left(1 - p\right) \left(1 - \left(1 + \frac{x}{b_1}\right)^{-q_1}\right) + p \left(1 - \left(1 + \frac{x}{b_2}\right)^{-q_2}\right)$$

Full Mixed Pareto (Pareto Soup)

$$CDF_{PS}(x; \overline{b_1}, q_1, \overline{p}, \overline{b_2}, q_2, w) = \sum_{i=1}^{7} w_i CDF_{MP}(x; b_{1,i}, q_1, p_i, b_{2,i}, q_2)$$
Section 2

**ISO’s Truncated Pareto**

Estimating Losses below the Truncation Point T
Truncated Pareto Curve
Estimating below Truncation Point

- Standard Truncated Pareto parameters
  - B, Scale Parameter of the Ballasted Pareto
  - Q, Shape Parameter of the Ballasted Pareto
  - T, Truncation point
  - P, Probability of being less that T
  - S, Mean of the Losses smaller than T

- For losses greater than T, the Curve is a Truncated Pareto
- For losses less than T, the Curve is undefined (other than the mean)
Truncated Pareto Curve
Uniform Option

- One option is to split the curve below $T$ into two Uniform Distributions
  - For $0 < X \leq S$, $f(x) = P \times (T - S) / (S \times T)$
  - For $S < X \leq T$, $f(x) = P \times S / (T \times (T - S))$
  - For $T < X$, $f(x) = \text{Ballasted Pareto with weight } (1-P)$
Another option is to choose another Distribution (e.g. Gamma) to model the losses less than T

Use solver for the Gamma CV and Theta so that the PDFs at T match and so that the mean of the truncated gamma is S.
- You can solve for a CV which will match the two PDFs
- You can solve for Theta which will match the means
- Or solve both at the same time

The Gamma can take many shapes, the CV determines the shape below T
- CV < 1, then you get “Log Normal-ish”
- CV = 1, Exponential Shape
- CV > 1, “Hyper Exponential”
Truncated Pareto Curve
PDF Comparison

Compare PDF's

pdf

0.00E+00
2.00E-04
1.80E-04
1.60E-04
1.40E-04
1.20E-04
1.00E-04
8.00E-05
6.00E-05
4.00E-05
2.00E-05
0.00E+00

0 20,000 40,000 60,000 80,000 100,000

TP(Unif)
TP(Ga cv>1)
TP(Ga cv=1)
TP(Ga cv<1)
MP
ME
Truncated Pareto Curve
CDF Comparison for a representative table

Compare CDF's

0% 10% 20% 30% 40% 50% 60% 70% 80% 90% 100%

0 20,000 40,000 60,000 80,000 100,000

TP(Unif) TP(Ga cv>1) TP(Ga cv=1) TP(Ga cv<1) MP ME
Truncated Pareto Curve
LAS Comparison

Compare LAS's

LAS

0 20,000 40,000 60,000 80,000 100,000

0 1,000 2,000 3,000 4,000 5,000 6,000 7,000 8,000 9,000 10,000

TP(Unif)  
TP(Ga cv>1)  
TP(Ga cv=1)  
TP(Ga cv<1)  
MP  
ME
Section 3

Casualty Exposure Rating
Working with ISO’s Mixed Exponential
The Mixed Exponential
Issues – Data Limitations

Issues

- Many Reinsurers feel that the tail of the Mixed Exponential is too thin
- ISO states that there are limitations to the data and recommended usage of the Mixed Exponential

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<th>Line of Business</th>
<th>Truncation</th>
<th>Tempering</th>
<th>Max Filed</th>
<th>Max Limit</th>
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<tr>
<td>Commercial Auto-Light/Medium/All</td>
<td>2,000,000</td>
<td>Not Tempered</td>
<td>10,000,000</td>
<td>10,000,000</td>
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<td>10,000,000</td>
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<tr>
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<td>200,000</td>
<td>Not Tempered</td>
<td>3,000,000</td>
<td>10,000,000</td>
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</tbody>
</table>

- Unfortunately knowing this doesn’t help us exposure rate high excess layers
The Mixed Exponential
Some Common Solutions

- Use the Mixed Exponential and assume everything is fine
- Use the Truncated Pareto Approximation
- Use the Pareto Soup since many reinsurers feel more comfortable with the tail of the Pareto Soup compared to the Mixed Exponential
The Mixed Exponential
Common Beliefs, Urban Legends and Myths #1

Pareto Soup is thicker tailed than the Mixed Exponential

- Generally True But……

- For the lighter exposures, many times the Pareto Soup is lighter than the Mixed Exponential in the high layers (ie 5M xs 5M)
  - PremOps/Products 1
    - Prem1 – Pareto Soup is much lighter than Mixed Exponential
    - Prod A – Pareto Soup and Mixed Exponential are about the same
  - Commercial Auto
    - Light/Medium – Pareto Soup is lighter than the Mixed Exponential in most state groups
    - All Other – Pareto Soup is lighter than the Mixed Exponential in some state groups
  - Medical
    - Pareto Soup thicker than Mixed Exponential in all cases

- For heavier exposures, the Pareto Soup is typically heavier tailed than the Mixed Exponential
Truncated Pareto is thicker tailed than the Mixed Exponential

- Many times the Truncated Pareto is lighter than the Mixed Exponential in the high layers (ie 5M xs 5M)
  - PremOps/Products
    - Prem1,2,3 – Truncated Pareto is lighter than Mixed Exponential
    - ProdA – Truncated Pareto is lighter than Mixed Exponential
  - Commercial Auto
    - Light/Medium – Truncated Pareto is lighter than the Mixed Exponential in most state groups
    - Other Tables – Varied by Table and Group

- For Attachment points less than 100K, Truncated Pareto can be significantly Higher than the Mixed Exponential
  - Especially true when Truncation point, T, is low (Comm Auto)
  - Overstated LAS for low limits, implies an overstated credit for deductible policies
The Mixed Exponential
Advantages of Using the Latest Mixed Exponential

- Mixed Exponential includes the most recent available information
  - State Specific Prem/Ops
  - Updated Class Code definitions associated with the Mixed Exponential
  - New State Groupings for Commercial Auto
  - Variable ALAE factors in MILD were not available prior to the Mixed Exponential

- Latest Pareto Soup parameters are from 1999 and are not getting any younger
“Adjusted” Mixed Exponential
Compromise between Mixed Exponential and Pareto Soup

A proposed method that blends the most recent information from the latest Mixed Exponential methods with a distribution that has a thicker tail than the Mixed Exponential

- Assume there is uncertainty on the mean parameters of the Mixed Exponential
  - Inclusion of uncertainty will thicken the tail of the distribution
  - If you assume the uncertainty is modeled by an Inverse Gamma distribution, the resulting distribution is a Ballasted Pareto
  - Therefore the Mixed Exponential becomes a “Mixed Pareto”
  - This “Mixed Pareto” has the same number of Ballasted Paretos as the Mixed Exponential has Exponentials (not to be confused with the Mixed Pareto which is one of the approximations to the Mixed Exponential published by ISO)
**Adjusted Mixed Exponential**

Using a Mixing Distribution on a Mixed Exponential to get a Mixed Pareto

\[
h(y; \phi, \theta) = \int f(y; \phi, \psi) g(\psi; \theta) d\psi
\]

\[
h(y; \phi, \theta) = f(y; \phi, \psi) \psi \ g(\psi; \theta)
\]

- \( f(y; \phi, \psi) \) - structural loss distribution with independent parameter(s) \( \phi \) and dependent parameter(s) \( \psi \)

- \( g(\psi; \theta) \) - mixing distribution on parameter(s) \( \psi \) with it’s own parameter(s) \( \phi \)

- \( h(y; \phi, \theta) \) - mixed distribution with parameters \( \phi \) and \( \theta \)
Adjusted Mixed Exponential
A Simple Example using Theoretical Mixing

Ballasted Pareto can result from an Exponential distribution mixed with an Inverse Gamma distribution

\[ BP(y; \theta, \alpha) = \text{Exp}(y; \mu) \hat{\mu} \text{InvGA}(\mu; \theta, \alpha) \]

Choosing Parameters for the distributions

The Exponential has an assumed mean, \( \mu \)

• Select parameters for the Inverse Gamma Distribution

• You can assume an Inverse Gamma with a mean, \( \mu \), and an assumed CV
  
  – \( \alpha = 2 + 1/CV^2 \), \( \theta = \mu^*(\alpha-1) \)
  
  – Nice intuitive approach, 2nd moment exists for Ballasted Pareto
  
  – For a thicker tailed Ballasted Pareto, you can select an \( \alpha < 2 \). While this a valid approach, it lacks some of the intuitive appeal.

\[
\begin{align*}
\text{pdf}_{BP}(x; \theta, \alpha) &= \frac{\alpha}{\theta} \frac{1}{(1+(x/\theta))^{(\alpha+1)}} \\
\text{pdf}_{Exp}(x; \mu) &= \frac{1}{\mu} e^{-(x/\mu)} \\
\text{pdf}_{IG}(\mu; \theta, \alpha) &= \frac{1}{\theta} \left( \frac{\theta / \mu}{\Gamma(\alpha)} \right)^{\alpha+1} e^{-\theta/\mu} \\
\text{CDF}_{BP}(x; \theta, \alpha) &= 1 - \frac{1}{(1+(x/\theta))^{\alpha}} \\
\text{CDF}_{Exp}(x; \mu) &= 1 - e^{-(x/\mu)} \\
\end{align*}
\]
Adjusted Mixed Exponential
A Simple Example using Theoretical Mixing

CDF-Ballasted Pareto by 7 Pnt Integration

LAS-Ballasted Pareto by 7 Pnt Integration
Adjusted Mixed Exponential
A “Mixed” Mixed Exponential becomes a “Mixed” Pareto

A single Ballasted Pareto can result from an single Exponential distribution mixed with an Inverse Gamma distribution

\[ BP(y; \theta_i, \alpha) = \text{Exp}(y; \mu_i) \mu_i \text{InvGA}(\mu_i; \theta_i, \alpha) \]

\[ \theta_i = \mu_i \times (\alpha - 1); \alpha = 2 + 1/\text{CV}^2 \]

The same mixing idea can be extended to a weighted average of Exponentials (Mixed Exponential) to created a weighted average of Ballasted Paretos

\[ \sum_i w_i \times BP(y; \theta_i, \alpha) = \sum_i w_i \times \text{Exp}(y; \mu_i) \mu_i \text{InvGA}(\mu_i; \theta_i, \alpha) \]
Section 4

Using ILFs for Exposure Rating
Using ILFs
Considerations

Sometimes the ceding company (or other company) ILFs are the only information available

- If the deal is sessions rated, you may be required to use a company ILF, is the resulting rate fair
- Does ILF include risk and expense loads
- Is ILF sufficiently detailed or is interpolation required
Using Ceding Company ILFs
A Quick Review of ILFs

\[ ILF(X) = \frac{LAS(X)}{LAS(B)} = \frac{\int_{0}^{X} [1 - F(x)]dx}{\int_{0}^{B} [1 - F(x)]dx} = \frac{\int_{0}^{X} [1 - F(x)]dx}{LAS(B)} \]

The ILF is non-decreasing (non-negative 1st derivative)

\[ \frac{d}{dX} \left( \frac{1}{LAS(B)} \int_{0}^{X} [1 - F(x)]dx \right) = \frac{[1 - F(X)]}{LAS(B)} \geq 0 \]

The ILF is non-decreasing at a decreasing rate (non-positive 2nd derivative)

\[ \frac{d^2}{dX^2} \left( \frac{ILF(X)}{LAS(B)} \right) = -\frac{f(X)}{LAS(B)} \leq 0 \]

LAS(B), the limited average severity at the base limit is constant with respect to X
Using Ceding Company ILFs
A few quick tests of ILFs

First Order test, ILFs should be strictly non-decreasing
\[ \frac{d \ ILF(X)}{dX} \geq 0 \]
• Fortunately, I think every ILF (excluding typos) I have seen have passed this test

Second Order Test, ILFs should be non-decreasing at an decreasing rate
\[ \frac{d^2 \ ILF(X)}{dX^2} \leq 0 \]
- I have frequently seen this test violated, even when the ILFs were supposedly based on actual data
- Test=0 implies f(x)=0 which means there is zero probability of loss in the range where the second derivative is equal to zero
- Test > 0 implies f(x)<0 which means there is negative probability of loss in the range where the second derivative is positive
Using Ceding Company ILFs

Good ILFs

<table>
<thead>
<tr>
<th>Limit ILF</th>
<th>F.O.</th>
<th>S.O.</th>
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<tbody>
<tr>
<td>1,000,000</td>
<td>1.000</td>
<td>2.3E-07</td>
</tr>
<tr>
<td>2,000,000</td>
<td>1.231</td>
<td>1.6E-07</td>
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<tr>
<td>3,000,000</td>
<td>1.390</td>
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<tr>
<td>4,000,000</td>
<td>1.516</td>
<td>1.0E-07</td>
</tr>
<tr>
<td>5,000,000</td>
<td>1.621</td>
<td>8.4E-08</td>
</tr>
<tr>
<td>7,500,000</td>
<td>1.830</td>
<td>6.6E-08</td>
</tr>
<tr>
<td>10,000,000</td>
<td>1.995</td>
<td></td>
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</table>

Example ILF is of the form

\[ ILF(X) = \left(\frac{X}{1M}\right)^3; \frac{d}{dX} ILF(X) = (0.3)\left(\frac{X}{1M}\right)^{-0.7}; \frac{d^2}{dX^2} ILF(X) = (-0.7)(0.3)\left(\frac{X}{1M}\right)^{-1.7} \]

This is a very common functional form for severe ILFs
### Using Ceding Company ILFs

#### Bad ILFs

<table>
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<th>Limit</th>
<th>ILF</th>
<th>F.O.</th>
<th>S.O.</th>
</tr>
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<tbody>
<tr>
<td>1,000,000</td>
<td>1.000</td>
<td>2.3E-07</td>
<td>0.0E+00</td>
</tr>
<tr>
<td>2,000,000</td>
<td>1.231</td>
<td>2.3E-07</td>
<td>-1.8E-13</td>
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<tr>
<td>10,000,000</td>
<td>1.831</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Probability of exceeding 2M is greater than the probability of exceeding 3M, therefore there is a negative probability of being in the range 3M-4M.
- Probability of exceeding 4M is greater than the probability of exceeding 3M, therefore there is a negative probability of being in the range 3M-4M.

If your ILF fails these tests, then it is an invalid ILF and you likely will not be able to get a good results from the ILF.
Using Ceding Company ILFs

Interpolating between ILFs

Same rules apply to the interpolation routine as apply to the ILFs

• First Order test \( \frac{d}{dX} ILF(X) \geq 0 \)

• Second Order Test \( \frac{d^2}{dX^2} ILF(X) \leq 0 \)

Common Interpolation Routines

• Linear
• Log-Linear
• Log-Log-Linear
• Power
• Fitted CDF (Single Parameter Pareto Example)

Let’s test these routines assuming you have a two point Table of ILFs

\[
\begin{align*}
X_{LO} & \quad \text{ILF}(X_{LO}) \quad \text{where} \quad X_{HI} > X_{LO} \\
X_{HI} & \quad \text{ILF}(X_{HI}) \quad \text{and} \quad \text{ILF}(X_{HI}) \geq \text{ILF}(X_{LO})
\end{align*}
\]
Using Ceding Company ILFs
Testing Linear Interpolation

\[ ILF(x) = ILF(X_{Low}) + \frac{(x - X_{Low})(ILF(X_{Hi}) - ILF(X_{Low}))}{(X_{Hi} - X_{Low})} \]

\[ \frac{d}{dX} ILF(X) = \frac{(ILF(X_{Hi}) - ILF(X_{Low}))}{(X_{Hi} - X_{Low})} \geq 0 \text{ Passes Test 1} \]

\[ \frac{d^2}{dX^2} ILF(X) = 0 \text{ Fails Test 2} \]

• Most people acknowledge that linear interpolation is bad
• Since it is so easy to do, many can’t seem to resist using it
Using Ceding Company ILFs
Testing Log$_y$-Linear Interpolation

\[ ILF(x) = \text{Exp} \left( \ln(\text{ILF}(X_{\text{Low}})) + \frac{(x - X_{\text{Low}})(\ln(\text{ILF}(X_{\text{Hi}})) - \ln(\text{ILF}(X_{\text{Low}})))}{(X_{\text{Hi}} - X_{\text{Low}})} \right) \]

\[ \frac{d \ ILF(X)}{dX} = \text{ILF}(x) \left( \frac{\ln(\text{ILF}(X_{\text{Hi}})) - \ln(\text{ILF}(X_{\text{Low}}))}{X_{\text{Hi}} - X_{\text{Low}}} \right) \geq 0 \quad \text{Passes Test 1} \]

\[ \frac{d \ ILF(X)}{dX} = \text{ILF}(x) \left( \frac{\ln(\text{ILF}(X_{\text{Hi}})) - \ln(\text{ILF}(X_{\text{Low}}))}{X_{\text{Hi}} - X_{\text{Low}}} \right)^2 \geq 0 \quad \text{Fails Test 2} \]

This one is worst than Linear Interpolation which will be shown in an example that follows.
Using Ceding Company ILFs

Testing $\log_x$-Linear Interpolation

$$ILF(x) = ILF(X_{\text{Low}}) + \frac{(\ln(x) - \ln(X_{\text{Low}}))(ILF(X_{\text{Hi}}) - ILF(X_{\text{Low}}))}{(\ln(X_{\text{Hi}}) - \ln(X_{\text{Low}}))}$$

$$\frac{d ILF(X)}{dX} = \frac{1}{x} \times \frac{(ILF(X_{\text{Hi}}) - ILF(X_{\text{Low}}))}{(\ln(X_{\text{Hi}}) - \ln(X_{\text{Low}}))} \geq 0 \quad \text{Passes Test 1}$$

$$\frac{d^2 ILF(X)}{dX^2} = -\frac{1}{x^2} \frac{(ILF(X_{\text{Hi}}) - ILF(X_{\text{Low}}))}{(\ln(X_{\text{Hi}}) - \ln(X_{\text{Low}}))} \geq 0 \quad \text{Passes Test 2}$$

• Passes both tests
• Performs poorly when extrapolation below the lowest ILF
• Ln(0) does not exist
Using Ceding Company ILFs
Testing Log-Log-Linear Interpolation

\[ ILF(x) = \exp(\ln(ILF(X_{\text{Low}})) + \frac{(\ln(x) - \ln(X_{\text{Low}}))(\ln(ILF(X_{\text{Hi}})) - \ln(ILF(X_{\text{Low}})))}{(\ln(X_{\text{Hi}}) - \ln(X_{\text{Low}}))}) \]

\[ \frac{d ILF(X)}{dX} = \frac{ILF(x)}{x} \times \frac{\ln(ILF(X_{\text{Hi}})) - \ln(ILF(X_{\text{Low}}))}{\ln(X_{\text{Hi}}) - \ln(X_{\text{Low}})} \geq 0 \] passes Test 1

\[ \frac{d^2 ILF(X)}{dX^2} = -\frac{ILF(x)}{x^2} \times \left[ \frac{(\ln(ILF(X_{\text{Hi}})) - \ln(ILF(X_{\text{Low}})))}{(\ln(X_{\text{Hi}}) - \ln(X_{\text{Low}}))} \right] + \left[ \frac{(\ln(ILF(X_{\text{Hi}})) - \ln(ILF(X_{\text{Low}})))}{(\ln(X_{\text{Hi}}) - \ln(X_{\text{Low}}))} \right]^2 \leq 0 \] passes Test 2
Using Ceding Company ILFs
Power Interpolation

\[ ILF(x) = ILF(X_{Low}) \left( \frac{ILF(X_{Hi})}{ILF(X_{Low})} \right)^{\frac{(x-X_{Low})}{(X_{Hi}-X_{Low})}} \]

\[ ILF(x) = \exp \left( \ln \left( ILF(X_{Low}) \left( \frac{ILF(X_{Hi})}{ILF(X_{Low})} \right) \right)^{\frac{(x-X_{Low})}{(X_{Hi}-X_{Low})}} \right) \]

\[ ILF(x) = \exp(\ln(ILF(X_{Low}))) + \frac{(x-X_{Low})(\ln(ILF(X_{Hi}))-\ln(ILF(X_{Low})))}{(X_{Hi}-X_{Low})} \]

Same as Log\_y-Linear

**Passes Test 1**

**Fails Test 2**
Using Ceding Company ILFs
Single Parameter (Simple) Pareto Interpolation

\[ F(x) = 1 - \left( \frac{c}{x} \right) ^{\alpha} ; x > c \]

\[ LAS(x) = \int_0^x (1 - F(t)) \, dt = \int_0^x \left( \frac{c}{t} \right) ^{\alpha} \, dt = \int_0^x \left( \frac{t}{c} \right) ^{-\alpha} \, dt \]

\[ LAS(x) = \frac{c}{(1 - \alpha)} \left( \frac{x}{c} \right) ^{1-\alpha} = \frac{c}{(1 - \alpha)} \left( \frac{c}{x} \right) ^{\alpha-1} \]

\[ \frac{LAS(X_{Hi})}{LAS(X_{Low})} = \frac{c}{(1 - \alpha)} \left( \frac{c}{X_{Hi}} \right) ^{\alpha-1} = \left( \frac{X_{Hi}}{X_{Low}} \right) ^{1-\alpha} \]

\[ \bar{\alpha} = 1 - \frac{\ln(LAS(X_{Hi})) - \ln(LAS(X_{Low}))}{\ln(X_{Hi}) - \ln(X_{Low})} \]
Using Ceding Company ILFs
Single Parameter (Simple) Pareto Interpolation (cont.)

\[
ILF(x) = \frac{LAS(x)}{LAS(X_{Low})} = \left( \frac{x}{X_{Low}} \right)^{1-\tilde{\alpha}}
\]

\[
\frac{d \ ILF(x)}{dx} = \left( \frac{1-\tilde{\alpha}}{X_{Low}} \right) \left( \frac{x}{X_{Low}} \right)^{-\tilde{\alpha}} > 0 \quad \text{Passes Test 1 If } \alpha < 1
\]

\[
\frac{d^2 \ ILF(x)}{dx^2} = \left( -\tilde{\alpha} \right) (1-\alpha) \left( \frac{x}{c} \right)^{-\tilde{\alpha}-1} < 0 \quad \text{Passes Test 2 If } \alpha > 0
\]

• Any valid distribution function is a candidate for an interpolation routine
• SPP is an easy to use function that many find intuitively appealing
• Questionable when less than the original truncation point (Extrapolation)
• Better at Extrapolating than many other methods
• Essentially the same as the Log-Log method
Using Ceding Company ILFs
Comparing the Interpolation Methods

Comparing Various Interpolation Routines

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Blue -Actual Values, Brown – Interpolated Values

2004 CARe Meeting in Boston - Advanced Exposure Rating
Using Ceding Company ILFs
Comparing the Interpolation Methods

![Graph comparing different interpolation methods]
Using Ceding Company ILFs
Comparing the Interpolation Methods
Using Ceding Company ILFs
Comparing the Extrapolation Methods

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= - The previous range has the same percent of losses exposed, not realistic
+ - The previous range has a lower percent of losses exposed, not possible
Using Ceding Company ILFs
Comparing the Extrapolation Methods

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Using the values at 100K and 200K to extrapolate below 100K
None of the methods are reliable. At least the Log-Log/SPP gives feasible answers

Blue -Actual Values, Red – Extrapolated Values
Using Ceding Company ILFs
Other Ideas

Fit a curve directly to the ILF
• If using CDF, need estimate of either $E(x)$ or $LAS(B)$
• May get improved estimates for values less than $C$ in SPP

Compare ILF to known ILFs
• Calculate SSE comparing ILF to Collections of known ILFs
• Consider inclusion of ALAE and Risk Load
• Possible Method for backing out Risk Load
Section 5

Working with Personal Auto Split Limits
Private Passenger Auto (PPA) Split Limits
Conversion to CSL

- Many PPA Loss Curves are Occurrence-based distributions, for use with Combined Single Limits (CSL) profiles.

- But, most PPA business is written on a Split Limits basis. How do you convert these limits profiles for use with CSL curves?
Private Passenger Auto (PPA) Split Limits
Split Limit to One CSL

Option 1:
- Assume a distribution of number of claimants and convert into each Split Limit into a CSL.
- For example, assume 60% 1 claimant, 30% 2 claimants, 10% 3 claimants.
  - 100k/300k/50k Split Limits would be converted to a 200k CSL, assuming a full 50k PD limit (100k * .6 + 200k * .3 + 300k * .1 + 50k)
- Using this method will not give any exposure greater than 200k, even though there is really a possibility of a 350k loss.
Option 2:

- Assume a distribution of number of claimants and create a new CSL profile.
- For example, assume 60% 1 claimant, 30% 2 claimants, 10% 3 claimants.
  - Split total premium using these percentages
  - Allocate premium to each per person combination
### PPA Split Limit Conversion
#### Option 2 Example

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# PPA Split Limit Conversion

## Option 2 Example

This example assumes a flat $10k PD limit. May want to use actual PD limit or some other assumption.

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Private Passenger Auto (PPA) Split Limits

Questions

- If PPA curves are built using primarily Split Limits data, is there an inherent assumption of a claimant distribution underlying the curve?
  - If so, when we apply a claimant distribution assumption again in converting split limits profiles, are we underestimating the resulting CSL? Or should we simply use the per occurrence limit?

- How are curves loaded for other liability coverages?
Private Passenger Auto (PPA) Split Limits
Loading for other coverages

- Important to understand what your curves include.
  - Some curves may include only BI and PD losses.
  - UM usually mirrors BI limits and does not occur at the same time as BI – probably no adjustment needed
  - Medical Payments are usually very small, so should have minor impact.
  - If PIP is not included, could have significant impact in some states.

- Subject Premium Base should include premium from all liability coverages
Section 6

Property Exposure Rating
Working with First Loss Scales
Property Loss Curves
Example

- Curve generally are expressed as percentage of policy limit or TIV.
  - When curves are expressed as a percentage of policy limits or TIV, impact of trend is not as significant – may not need to update as often.

Example of Percentage curve:

Assume: Coverage A limit = 100k, Layer = 100k xs 50k

Given the curve below, what is the % of losses to the layer?

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<th>% of Cvg A Limit</th>
<th>% of Loss Below</th>
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<th>( F(\frac{X}{\text{cvg A}}) )</th>
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<td>0.00%</td>
<td>( P( L &lt; X ) ) = 0 %</td>
<td>( F(\frac{X}{\text{cvg A}}) ) = 0 %</td>
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<tr>
<td>50%</td>
<td>61.70%</td>
<td>( P( L &lt; \text{ret } ) = )</td>
<td>( F(\frac{50k}{100k}) = F(50%) = 61.70% )</td>
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<tr>
<td>100%</td>
<td>84.20%</td>
<td>( P( L &lt; \text{limit + ret } ) = )</td>
<td>( F(\frac{150k}{100k}) = F(150%) = 96.50% )</td>
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<td>150%</td>
<td>96.50%</td>
<td>% of Loss to Layer =</td>
<td>( 96.50% - 61.70% = 34.80% )</td>
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Property Loss Curves

History

- Lloyds
- Salzmann (1960 INA Homeowners data)
- Reinsurer Curves (Swiss Re, Munich Re, etc)
- ISO’s PSOLD (Recent Commercial data)
Property Loss Curves
Advantages/Disadvantages

- Lloyds Curves
  - (Very old data)
  - (Does not vary by amount of insurance or occupancy class)
  - (Underlying data is largely unknown (marine losses?))

- Salzmann (Personal Property)
  - Based on actual Homeowners data
  - Varies by Construction/Protection Class
  - (Very old data – from 1960)
  - (Does not vary by amount of insurance)
  - (Building losses only and Fire losses only)

- Swiss Re Curves
  - Documented study on personal & commercial reinsurance business
  - (Old data)
  - (Does not vary by amount of insurance or occupancy class)
Property Loss Curves
Advantages/Disadvantages

- Ludwig Curves (Personal and Commercial)
  - Based on actual Homeowners and Commercial data, (but uses Hartford small commercial property book – may not be good for large national accts)
  - Varies by Construction/Protection Class for HO and Occupancy Class for Commercial
  - Includes all property coverages and perils
  - (Old data: 1984 - 1988)

- ISO’s PSOLD
  - Recent Data – updated every 2 years
  - Varies by amount of insurance, occupancy class, state, coverage, and peril
  - (Based on ISO data only)
First Loss Scales
A Quick Review of FLSs

\[ FLS(X) = \frac{LAS(X)}{E(X)} = \frac{\int_0^X [1 - F(x)] \, dx}{\int_0^\infty [1 - F(x)] \, dx} = \frac{\int_0^X [1 - F(x)] \, dx}{E(X)} \]

The FLS is also non-decreasing (non-negative 1\textsuperscript{st} derivative), similar to ILF

\[ \frac{d\ ILF(X)}{dX} = \frac{\frac{1}{E(X)} \int_0^X [1 - F(x)] \, dx}{dX} = \frac{[1 - F(X)]}{E(X)} \geq 0 \]

The FLS is also non-decreasing at a decreasing rate (non-positive 2\textsuperscript{nd} derivative)

\[ \frac{d^2\ ILF(X)}{dX^2} = \frac{-f(X)}{E(X)} \leq 0 \]

E(X), the unlimited average severity of X
First Loss Scales
Consistency Tests and Interpolation routines

Consistency tests are exactly the same for Increased Limits Factors and for First Loss Scales

Issues regarding valid interpolation routines are also the same for Increased Limits Factors and First Loss Scales
Section 7

**Property Exposure Rating**
Working with PSOLD
Many users of PSOLD use the model as a source for the underlying parameters for the mixed exponentials, but then use their own model to do the calculations based on the PSOLD curves.

If you are going to make your own model, you should consider making improvements to the methodology:
- Limited Average Severities over a Range of Value within a Single AOI Group
- Weighting between AOI groups
- Weighting between Occupancy Classes
- Stacking and Participation
- Exposure Above Policy Limit and Stacking
- Exposure Above Policy Limit and Margin Clauses
PSOLD Methodology

Notation

\[ LAS_{Exp}(x) = \mu \times \left( 1 - e^{-\frac{x}{\mu}} \right) \]

\[ LAS_{ME}(x) = \sum_{i=1}^{\#Lags} w_i \mu_i \times \left( 1 - e^{-\frac{x}{\mu_i}} \right) \]

\[ \mu_{i, psold} = 10^{.5 \times (i+1)} \]

\( w_i \) varies by: Coverage (B, C, B+C, B+C+I)
Peril (BG1, BG2, Special Causes, All)
Net of Deductible vs Ground Up
Occupancy Class
State
PSOLD Methodology
Interpreting a single policy LAS in an AOI Ranges in PSOLD

- Is the movement from one AOI range to the next a step Function or a smooth progression?
- Consider three policies, two within a single AOI range and the third in the next highest AOI range but close in value to the second policy
- Should two different policy limits within a single AOI range have the same LAS or should the difference in policy limits be reflected?
- PSOLD currently calculates the LAS at a single point, the minimum of the loss limit and 1.5x(upper bound of the AOI range) for all policies in the range.
PSOLD Methodology
Evaluating LAS functions over a range

PSOLD evaluates the function at a fixed point, consider the LAS

\[
LAS(x) = \sum_{i=1}^{\#lags} w_{i,AOI} \mu_i \times \left(1 - e^{-\frac{x}{\mu_i}}\right) = LAS_{ME}(x)
\]

Consider evaluating this over a range of values

\[
LAS(x \mid X_U \geq x > X_L) = \frac{\int_{X_L}^{X_U} g(x)LAS(x)dx}{\int_{X_L}^{X_U} g(x)dx}
\]

Consider two forms for g(x)
1. g(x) is Uniform on the range \((X_U \geq x > X_L)\)
2. g(x) follows same distribution as losses conditional on \((X_U \geq x > X_L)\)
**PSOLD Methodology**  
Evaluating LAS functions over a range – Uniform Distribution

Will use a single exponential in the example for simplicity. This is easily generalized to a mixed exponential

\[
g(x) = \frac{1}{X_U - X_L} \Rightarrow LAS(x \mid X_U \geq x > X_L) = \int_{X_L}^{X_U} \frac{1}{X_U - X_L} LAS(x)dx
\]

\[
LAS(x \mid X_U \geq x > X_L) = \frac{\mu}{X_U - X_L} \int_{X_L}^{X_U} \left(1 - e^{-\frac{x}{\mu}}\right)dx
\]

\[
LAS(x \mid X_U \geq x > X_L) = \frac{\mu(X_U - X_L)}{(X_U - X_L)} - \frac{\mu}{(X_U - X_L)} \int_{X_L}^{X_U} e^{-\frac{x}{\mu}} dx
\]

\[
LAS(x \mid X_U \geq x > X_L) = \mu - \frac{\mu (LAS_{Exp}(X_U) - LAS_{Exp}(X_L))}{(X_U - X_L)}
\]

Not difficult to calculate
Will use a single exponential in the example for simplicity. This exponential has the same \( \mu \) as the loss distribution but it is conditional on being within the range \((X_L, X_H)\)

\[
g(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} \Rightarrow LAS(x \mid X_U \geq x > X_L) = \left( e^{-\frac{X_L}{\mu}} - e^{-\frac{X_U}{\mu}} \right)^{-1} \int_{X_L}^{X_U} \frac{1}{e^{-\frac{x}{\mu}}} \times \mu \left( 1 - e^{-\frac{x}{\mu}} \right) dx
\]

\[
LAS(x \mid X_U \geq x > X_L) = \left( e^{-\frac{X_L}{\mu}} - e^{-\frac{X_U}{\mu}} \right)^{-1} \int_{X_L}^{X_U} \left( e^{-\frac{x}{\mu}} - e^{-\frac{x}{\mu}} \times e^{-\frac{x}{\mu}} \right) dx
\]

\[
LAS(x \mid X_U \geq x > X_L) = \left( e^{-\frac{X_L}{\mu}} - e^{-\frac{X_U}{\mu}} \right)^{-1} \int_{X_L}^{X_U} \left( e^{-\frac{2x}{\mu}} \right) dx
\]

\[
LAS(x \mid X_U \geq x > X_L) = \left( e^{-\frac{X_L}{\mu}} - e^{-\frac{X_U}{\mu}} \right)^{-1} \int_{X_L}^{X_U} \left( e^{-\frac{x}{\mu}} - e^{-\frac{x}{0.5 \mu}} \right) dx
\]
**PSOLD Methodology**

Evaluating LAS functions over a range – Exponential Distribution

\[
LAS(x | X_U \geq x > X_L) = \frac{\int_{X_L}^{X_U} e^{-\mu x} \, dx - \int_{X_L}^{X_U} e^{-0.5 \times \mu x} \, dx}{\int_{X_L}^{X_U} e^{-\mu x} \, dx - \int_{X_L}^{X_U} e^{-\mu x} \, dx}
\]

\[
LAS(x | X_U \geq x > X_L) = \left[ \mu \left( e^{-\mu X_L} - e^{-\mu X_U} \right) - 0.5 \times \mu \left( e^{-0.5 \times \mu X_L} - e^{-0.5 \times \mu X_U} \right) \right]
\]

\[
LAS(x | X_U \geq x > X_L) \approx \mu \left( 1 - \frac{e^{-\mu X_U} + e^{-\mu X_L}}{2} \right)
\]

This relationship is by observation and not rigorous proof. Intuitively it is around what I would like the result to be.

The average of the LAS(X_H) and LAS(X_LO)
**PSOLD Methodology**

**PSOLD LAS Calculations over Single AOI Range**

LAS for an Mixed Exponential

\[
LAS_{ME}(x) = \sum_{i=1}^{#Lags} w_i \mu_i \times \left(1 - e^{-\frac{x}{\mu_i}}\right)
\]

For Coverages B, C and B+C PSOLD constrains the LAS Calculation

PSOLD has two Ranges of Interest

\[
x_{Limit} < AOI_{Upper}^* : LAS_{PSOLD}(x_{Limit}) = LAS_{ME}(x_{Limit})
\]

\[
x_{Limit} \geq AOI_{Upper}^* : LAS_{PSOLD}(x_{Limit}) = LAS_{ME}(AOI_{Upper}^*)
\]

where \(AOI_{Upper}^* = AOI_{Upper} \times (1 + \text{Additional Exposure}%)\)
Calculating the LAS over a continuous range adds one more degree of complexity

\[
x_{\text{Limit}} < AOI^{*}_{\text{Lower}} : LAS_{\text{ALT}}(x_{\text{Limit}}) = LAS_{\text{ME}}(x_{\text{Limit}})
\]

\[
x_{\text{Limit}} \geq AOI^{*}_{\text{Upper}} : LAS_{\text{ALT}}(x_{\text{Limit}}) = \frac{LAS_{\text{ME}}(AOI^{*}_{\text{Upper}}) + LAS_{\text{ME}}(AOI^{*}_{\text{Lower}})}{2}
\]

\[
AOI^{*}_{\text{Upper}} \geq x_{\text{Limit}} \geq AOI^{*}_{\text{Lower}} :
\]

\[
LAS_{\text{ALT}}(x_{\text{Limit}}) = \left(\frac{LAS_{\text{ME}}(AOI^{*}_{\text{Lower}}) + LAS_{\text{ME}}(x_{\text{Limit}})}{2}\right) + \frac{x_{\text{Limit}} - AOI^{*}_{\text{Lower}}}{AOI^{*}_{\text{Upper}} - AOI^{*}_{\text{Lower}}}
\]

Which simplifies to

\[
LAS_{\text{ALT}}(x_{\text{Limit}}) = LAS_{\text{ME}}(AOI^{*}_{\text{Lower}}) \left(1 + \frac{x_{\text{Limit}} - AOI^{*}_{\text{Lower}}}{2(AOI^{*}_{\text{Upper}} - AOI^{*}_{\text{Lower}})}\right)
\]

where \( AOI^{*}_{\text{Upper}} = AOI_{\text{Upper}} \times (1 + \text{Additional Exposure}\%) \)

where \( AOI^{*}_{\text{Lower}} = AOI_{\text{Lower}} \times (1 + \text{Additional Exposure}\%) \)
PSOLD Methodology
Advantages of the Alternate LAS Calculations

- Different policy limits within the same AOI range will get different LAS
- Smoother transition as you move from one AOI range to the next
- Since this impacts the unlimited average severity for the policy, it will change the allocation of losses to the layer for any exposed policy

- An additional enhancement would be to adjust the $w_i$’s as you move within an AOI range
PSOLD Methodology
Weighting between AOI Ranges

If a range of Insured values spans more than one AOI Range. You need to combine the results of the Individual AOI ranges

- In PSOLD any AOI group included within the range will be given full weight
- An improvement would be to only Include an AOI range in proportion To the percentage that the range is covered
In PSOLD, when using more than one Occupancy class on a single policy group, the relative weight assigned to each occupancy class is based on the occupancy counts in the underlying industry data base.

An improvement would be to allow the user to define the weights between the occupancy classes so that you can more accurately reflect the individual ceding companies exposure.
PSOLD Methodology

Additional Exposure Percentage

PSOLD uses the following additional exposure percentage:

- Building Only – 50%
- Contents Only – 50%
- Building+Contents Only – 50%
- Building+Contents+Business Interruption – Unlimited

You may want to select a different percentage due to any of the following:

- Stacking of Excess Policies – you do not want the policies to overlap
- Margin Clause – contractually limits exposure greater than the limit
- Company Experience
- Judgement
Additional consideration when dealing with stacking and participation

- The selected AOI group should be based on a full value on the insured risk (same AOI group as if the risk was fully covered by a single policy)

- All stacked policies should have the same AOI group

- When stacking, assume additional coverage % is zero or the policies will overlap
Section 8

Stacking and Participation
Participation allows you to correctly model the situation where a contract only covers a proportional share of the underlying loss.

It is most common in a subscription type market like Lloyds, but it is also useful for modeling some facultative business.

Example

Assume the following:
- Write 25% participation on a $1M Contract.
- You reinsure a 200K xs 200K layer

In order to get a loss that will expose the Reinsurance Cover
- You must have a loss to the primary contract greater than 800K (200K / 25%)
- The largest loss you can have exposing the layer is 250K (25% of 1M) or 50K to the layer
- Actually, you would take 25% of losses ceded to an 800K xs 800K reinsurance layer. But since the primary policy is $1M, it is effectively 25% of 200k xs 800k.
Stacking and Participation

Stacking

Stacking is where an insurer issues multiple excess contracts covering the same underlying risk

- Assume someone writes a series of policies covering the same risk, 100K x 100K (Yellow), 300K x 200K (Blue), 500K x 500K (Red) and 1M x 1M (Green)

- If all are written at the same level of participation then effectively it is the same as a single 1.9M xs 100K (Purple) policy with the given participation

- In practice, not all contracts are at the same participation and not all contract are written (can be thought of as participation=0%, this is sometimes called ventilation)
Stacking and Participation

Stacking

Now Assume there is a 500K x 500K reinsurance contract covering these contracts

- If the contracts are assumed to be independent, then you would only cover the 500K x 500K layer on the 1M x 1M policy. No other policy would expose.

- If the contracts are assumed to be stacked, then you would cover the 500K x 500K layer on the 1.9M x 100K policy.

- There can be significantly greater exposure to the Reinsurance Contract under the stacked assumption.
Stacking and Participation

Stacking

Stacking is Generally thought of as an International Issue, but…

- Stacking can be used in the Facultative Markets
- Stacking can be used to model Umbrella written over a company’s own underlying policies
- Stacking is commonly used in combination with participation in a subscription market like Lloyds
Umbrella Comparison - Assume: Layer 500k xs 500k, Umbrella Limit = 1M, Underlying Limit = 500k
Stacking and Participation
Partial Participation without Stacking

Layer: 300k xs 200k - no stacking

<table>
<thead>
<tr>
<th>Policy Limit</th>
<th>SIR/Retention</th>
<th>Participation</th>
<th>Rescaled Treaty Limit (Capped)</th>
<th>Rescaled Treaty Retention</th>
</tr>
</thead>
<tbody>
<tr>
<td>100,000</td>
<td>100,000</td>
<td>100.0%</td>
<td>0</td>
<td>200,000</td>
</tr>
<tr>
<td>300,000</td>
<td>200,000</td>
<td>100.0%</td>
<td>100,000</td>
<td>200,000</td>
</tr>
<tr>
<td>500,000</td>
<td>500,000</td>
<td>50.0%</td>
<td>100,000</td>
<td>400,000</td>
</tr>
<tr>
<td>1,000,000</td>
<td>1,000,000</td>
<td>25.0%</td>
<td>200,000</td>
<td>800,000</td>
</tr>
</tbody>
</table>

"Our share" of the layer would be Participation x Capped Treaty Limit
Stacking and Participation
Partial Participation with Stacking

- Assume someone writes a series of policies covering the same risk, 100K x 100K (Yellow), 300K x 200K (Blue), 500K x 500K (Red) and 1M x 1M (Green).
  - Your participation on each is: 100K x 100K (100%), 300K x 200K (100%), 500K x 500K (50%), 1M x 1M (25%)
  - These policies are stacked
  - You reinsure a 500K x 500K layer

- In order to get a loss that will expose the Reinsurance Cover
  - You must have a loss to the excess contracts greater than 600K (100K / 100% + 300K / 100% + 100K / 50%)
  - The largest loss you can have exposing the layer is 900K (100K * 100% + 300K * 100% + 500K * 50% + 1M * 25%) or 400K to the layer
Section 9

Miscellaneous Topics
Miscellaneous Topics

- SIR/Deductibles
- ALAE options
- Policy Count vs Premium
- Deductible/SIR Retains Policy Limit
  - Limit floats on top of the Deductible/SIR

- Deductible /SIR Reduces Policy Limit
  - Limit stays fixed relative to ground-up and the Deductible /SIR effectively reduces or erodes the policy limit
Miscellaneous Topics
Treatment of Self-Insured Retentions (SIRs) and Deductibles
Assume: Layer 100k xs 100k, Policy Limit = 250k, SIR/Ded = 50k

% Exposed = (LAS(250k) – LAS(150k) / (LAS(300k) – LAS(50k))

Treaty Layer

Company Retention

w/o eroding SIR/Ded

with eroding SIR/Ded
Miscellaneous Topics
ALAE Options

- ALAE Excluded
- ALAE Prorata in Addition to Loss
- ALAE Included outside Policy Limit
- ALAE Included within Policy Limit
**Miscellaneous Topics**

**Burning Cost Formulas**

**ALAE Excluded**

\[ BC = LR_{PureLoss} \times \frac{LAS(\text{Min}(\text{LayLmt}, \text{PolLmt}) + \text{PolDed}) - LAS(\text{Min}(\text{Lay Ret}, \text{PolLmt}) + \text{PolDed})}{LAS(\text{PolLmt} + \text{PolDed}) - LAS(\text{PolDed})} \]

\[ BC_{SIR\_Erode} = LR_{PureLoss} \times \frac{LAS(\text{Min}(\text{LayLmt} + \text{PolDed}, \text{PolLmt})) - LAS(\text{Min}(\text{Lay Ret} + \text{PolDed}, \text{PolLmt}))}{LAS(\text{PolLmt}) - LAS(\text{PolDed})} \]

**ALAE Prorata**

\[ BC = LR_{Loss+ALAE} \times \frac{LAS(\text{Min}(\text{LayLmt}, \text{PolLmt}) + \text{PolDed}) - LAS(\text{Min}(\text{Lay Ret}, \text{PolLmt}) + \text{PolDed})}{LAS(\text{PolLmt} + \text{PolDed}) - LAS(\text{PolDed})} \]

\[ BC_{SIR\_Erode} = LR_{Loss+ALAE} \times \frac{LAS(\text{Min}(\text{LayLmt} + \text{PolDed}, \text{PolLmt})) - LAS(\text{Min}(\text{Lay Ret} + \text{PolDed}, \text{PolLmt}))}{LAS(\text{PolLmt}) - LAS(\text{PolDed})} \]

The only difference between Excluded and Prorata is the Loss Ratio assumption.
ALAE Included Within Policy Limit

\[ BC = LR_{Loss + ALAE} \times \frac{LAS\left(\min\left(LayLmt,PolLmt\right) + PolDed/A\right) - LAS\left(\min\left(LayRe,PolLmt\right) + PolDed/A\right)}{LAS\left(PolLmt + PolDed/A\right) - LAS\left(PolDed/A\right)} \]

\[ BC_{SIR\_Erode} = LR_{Loss + ALAE} \times \frac{LAS\left(\min\left(LayLmt + PolDed,PolLmt\right)/A\right) - LAS\left(\min\left(Lay Re + PolDed,PolLmt\right)/A\right)}{LAS\left(PolLmt/A\right) - LAS\left(PolDed/A\right)} \]

ALAE Included Outside Limit

\[ BC = LR_{Loss + ALAE} \times \frac{LAS\left(\min\left(LayLmt/A,PolLmt\right) + PolDed/A\right) - LAS\left(\min\left(Lay Re/A,PolLmt\right) + PolDed/A\right)}{LAS\left(PolLmt + PolDed/A\right) - LAS\left(PolDed/A\right)} \]

\[ BC_{SIR\_Erode} = LR_{Loss + ALAE} \times \frac{LAS\left(\min\left((LayLmt + PolDed)/A,PolLmt\right)\right) - LAS\left(\min\left((Lay Re + PolDed)/A,PolLmt\right)\right)}{LAS\left(PolLmt\right) - LAS\left(PolDed/A\right)} \]

ALAE Included vs Prorata is based on the Reinsurance Contract.
ALAE Included Inside the Limit versus Outside the Limit depends on underlying policy.
Miscellaneous Topics
Premium vs Policy Count

- When exposure rating, it is best to use a policy limits profile by premium.

- If you only have a policy count profile, you can estimate a premium distribution
  - Property – multiply policy count by policy limit to estimate TIV
    - Multiply policy count by the Limited Average Severity for the policy when using size of loss distributions (PSOLD) rather than first loss scales
  - Casualty – multiply policy count by ILF at corresponding policy limit
Miscellaneous Topics
Premium vs Policy Count – Property Example

Multiply policy count by policy limit to estimate TIV

<table>
<thead>
<tr>
<th>Policy Limit</th>
<th>Policy Count</th>
<th>Policy Count Distribution</th>
<th>Estimated Premium (TIV)</th>
<th>Premium Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>75,000</td>
<td>2,755</td>
<td>60.31%</td>
<td>206,625,000</td>
<td>38.31%</td>
</tr>
<tr>
<td>150,000</td>
<td>1,439</td>
<td>31.50%</td>
<td>215,850,000</td>
<td>40.02%</td>
</tr>
<tr>
<td>250,000</td>
<td>268</td>
<td>5.87%</td>
<td>67,000,000</td>
<td>12.42%</td>
</tr>
<tr>
<td>350,000</td>
<td>52</td>
<td>1.14%</td>
<td>18,200,000</td>
<td>3.37%</td>
</tr>
<tr>
<td>450,000</td>
<td>28</td>
<td>0.61%</td>
<td>12,600,000</td>
<td>2.34%</td>
</tr>
<tr>
<td>550,000</td>
<td>12</td>
<td>0.26%</td>
<td>6,600,000</td>
<td>1.22%</td>
</tr>
<tr>
<td>650,000</td>
<td>3</td>
<td>0.07%</td>
<td>1,950,000</td>
<td>0.36%</td>
</tr>
<tr>
<td>750,000</td>
<td>3</td>
<td>0.07%</td>
<td>2,250,000</td>
<td>0.42%</td>
</tr>
<tr>
<td>850,000</td>
<td>4</td>
<td>0.09%</td>
<td>3,400,000</td>
<td>0.63%</td>
</tr>
<tr>
<td>950,000</td>
<td>2</td>
<td>0.04%</td>
<td>1,900,000</td>
<td>0.35%</td>
</tr>
<tr>
<td>1,250,000</td>
<td>1</td>
<td>0.02%</td>
<td>1,250,000</td>
<td>0.23%</td>
</tr>
<tr>
<td>1,750,000</td>
<td>1</td>
<td>0.02%</td>
<td>1,750,000</td>
<td>0.32%</td>
</tr>
</tbody>
</table>

Total 4,568 539,375,000
## Miscellaneous Topics
### Premium vs Policy Count – Property LAS Example

Multiply policy count by LAS

<table>
<thead>
<tr>
<th>Average Policy Limit</th>
<th>Policy Count Distribution</th>
<th>LAS</th>
<th>LAS * Count</th>
<th>Premium Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>75,000</td>
<td>60.31%</td>
<td>8,000</td>
<td>22,040,000</td>
<td>48.41%</td>
</tr>
<tr>
<td>150,000</td>
<td>31.50%</td>
<td>12,000</td>
<td>17,268,000</td>
<td>37.93%</td>
</tr>
<tr>
<td>250,000</td>
<td>5.87%</td>
<td>15,000</td>
<td>4,020,000</td>
<td>8.83%</td>
</tr>
<tr>
<td>350,000</td>
<td>1.14%</td>
<td>17,000</td>
<td>884,000</td>
<td>1.94%</td>
</tr>
<tr>
<td>450,000</td>
<td>0.61%</td>
<td>21,000</td>
<td>588,000</td>
<td>1.29%</td>
</tr>
<tr>
<td>550,000</td>
<td>0.26%</td>
<td>22,000</td>
<td>264,000</td>
<td>0.58%</td>
</tr>
<tr>
<td>650,000</td>
<td>0.07%</td>
<td>25,000</td>
<td>75,000</td>
<td>0.16%</td>
</tr>
<tr>
<td>750,000</td>
<td>0.07%</td>
<td>30,000</td>
<td>90,000</td>
<td>0.20%</td>
</tr>
<tr>
<td>850,000</td>
<td>0.09%</td>
<td>35,000</td>
<td>140,000</td>
<td>0.31%</td>
</tr>
<tr>
<td>950,000</td>
<td>0.04%</td>
<td>37,000</td>
<td>74,000</td>
<td>0.16%</td>
</tr>
<tr>
<td>1,250,000</td>
<td>0.02%</td>
<td>39,000</td>
<td>39,000</td>
<td>0.09%</td>
</tr>
<tr>
<td>1,750,000</td>
<td>0.02%</td>
<td>45,000</td>
<td>45,000</td>
<td>0.10%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>4,568</strong></td>
<td><strong>45,527,000</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Miscellaneous Topics
### Premium vs Policy Count – Casualty Example

Multiply policy count by ILF at corresponding policy limit

<table>
<thead>
<tr>
<th>Policy Limit</th>
<th>Policy Count</th>
<th>Policy Count Distribution</th>
<th>ILF</th>
<th>Estimated &quot;Premium&quot;</th>
<th>Premium Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>25,000</td>
<td>14</td>
<td>0.24%</td>
<td>1.000</td>
<td>14.00</td>
<td>0.13%</td>
</tr>
<tr>
<td>50,000</td>
<td>50</td>
<td>0.87%</td>
<td>1.200</td>
<td>60.00</td>
<td>0.54%</td>
</tr>
<tr>
<td>100,000</td>
<td>636</td>
<td>11.10%</td>
<td>1.500</td>
<td>954.00</td>
<td>8.57%</td>
</tr>
<tr>
<td>300,000</td>
<td>1,817</td>
<td>31.70%</td>
<td>1.800</td>
<td>3,270.60</td>
<td>29.39%</td>
</tr>
<tr>
<td>500,000</td>
<td>1,221</td>
<td>21.30%</td>
<td>2.000</td>
<td>2,442.00</td>
<td>21.95%</td>
</tr>
<tr>
<td>1,000,000</td>
<td>1,994</td>
<td>34.79%</td>
<td>2.200</td>
<td>4,386.80</td>
<td>39.42%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>5,732</strong></td>
<td></td>
<td></td>
<td><strong>11,127.40</strong></td>
<td></td>
</tr>
</tbody>
</table>