Modeling Paid and Incurred Losses Together

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Outline

- Review of Linear (or Regression) Models
- GLM? Generalize how?
- Spreadsheet Examples

The Formulation of the Linear Model

$$\begin{bmatrix} \mathbf{y}_{1(t_1 \times 1)} \\ \hdashline \mathbf{y}_{2(t_2 \times 1)} \end{bmatrix} = \begin{bmatrix} \mathrm{X}_{1(t_1 \times k)} \\ \hdashline \mathrm{X}_{2(t_2 \times k)} \end{bmatrix} \beta_{(k \times 1)} + \begin{bmatrix} \mathbf{e}_{1(t_1 \times 1)} \\ \hdashline \mathbf{e}_{2(t_2 \times 1)} \end{bmatrix},$$

$$Var \begin{bmatrix} \mathbf{e}_1 \\ \hdashline \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \Sigma_{11\,(t_1 \times t_1)} & \Sigma_{12\,(t_1 \times t_2)} \\ \hdashline \Sigma_{21\,(t_2 \times t_1)} & \Sigma_{22\,(t_2 \times t_2)} \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} \Phi_{11\,(t_1 \times t_1)} & \Phi_{12\,(t_1 \times t_2)} \\ \hdashline \Phi_{21\,(t_2 \times t_1)} & \Phi_{22\,(t_2 \times t_2)} \end{bmatrix}$$

Trend Example

$$\begin{bmatrix} \mathbf{y}_{1\,(5\times 1)} \\ \hline \mathbf{y}_{2\,(3\times 1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ \hline 1 & 6 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \beta_{(2\times 1)} + \begin{bmatrix} \mathbf{e}_{1\,(5\times 1)} \\ \hline \mathbf{e}_{2\,(3\times 1)} \end{bmatrix},$$

$$Var\begin{bmatrix} \mathbf{e}_1 \\ \hline \mathbf{e}_2 \end{bmatrix} = \sigma^2 \left[\begin{array}{c|c} \mathrm{I}_{(5\times 5)} & 0_{(5\times 3)} \\ \hline 0_{(3\times 5)} & \mathrm{I}_{(3\times 3)} \end{array}\right]$$

The BLUE Solution

$$\hat{\mathbf{y}}_2 = X_2\hat{\boldsymbol{\beta}} + \Phi_{21}\Phi_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)$$

$$\hat{\boldsymbol{\beta}} = \left(X_1'\Phi_{11}^{-1}X_1\right)^{-1}X_1'\Phi_{11}^{-1}\mathbf{y}_1$$

$$Var\left[\mathbf{y}_2 - \hat{\mathbf{y}}_2\right] = \underbrace{\sigma^2\left(\Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12}\right)}_{\text{process variance}}$$

$$+ \underbrace{\left(X_2 - \Phi_{21}\Phi_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(X_2 - \Phi_{21}\Phi_{11}^{-1}X_1\right)'}_{\text{parameter variance}}$$

$$Var\left[\hat{\boldsymbol{\beta}}\right] = \sigma^2\left(X_1'\Phi_{11}^{-1}X_1\right)^{-1}$$

Special Case: $\Phi = \mathbf{I}_t$

$$\hat{\mathbf{y}}_2 = \mathrm{X}_2 \hat{\boldsymbol{\beta}}$$

$$\hat{\boldsymbol{\beta}} = \left(\mathrm{X}_1' \mathrm{X}_1\right)^{-1} \mathrm{X}_1' \mathbf{y}_1$$

$$Var\left[\mathbf{y}_2 - \hat{\mathbf{y}}_2\right] = \sigma^2 \mathrm{I}_{t_2} + \mathrm{X}_2 Var\left[\hat{\boldsymbol{\beta}}\right] \mathrm{X}_2'$$

$$Var\left[\hat{\boldsymbol{\beta}}\right] = \sigma^2 \left(\mathrm{X}_1' \mathrm{X}_1\right)^{-1}$$

Estimator of the Variance Scale

$$\hat{\sigma}^2 = \frac{\left(\mathbf{y}_1 - \mathrm{X}_1\hat{\boldsymbol{\beta}}\right)' \Phi_{11}^{-1} \left(\mathbf{y}_1 - \mathrm{X}_1\hat{\boldsymbol{\beta}}\right)}{t_1 - k}$$

Generalizing the Simple Linear Model

- GLM's

$$\mathbf{y} = g^{-1}(\mathbf{X}\beta) + \mathbf{e}, \text{ diagonal variance}$$

- Judge, *IT&PE*: Covariance, Seemingly unrelated regressions (SUR). Solve:

$$\begin{bmatrix} \mathbf{y}_1 \\ \hline \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \hline \mu_2 \end{bmatrix}(1) + \begin{bmatrix} \mathbf{e}_1 \\ \hline \mathbf{e}_2 \end{bmatrix}, Var\begin{bmatrix} \mathbf{e}_1 \\ \hline \mathbf{e}_2 \end{bmatrix} = \left[\begin{array}{c|c} \sigma_{11} & \sigma_{12} \\ \hline \sigma_{21} & \sigma_{22} \end{array}\right]$$

It's all about Covariance

$$\hat{\mathbf{y}}_2 = \mu_2(1) + \sigma_{21}\sigma_{11}^{-1}(\mathbf{y}_1 - \mu_1(1))$$
$$= \mu_2 + (\rho\sigma_1\sigma_2)(\mathbf{y}_1 - \mu_1)/\sigma_1^2$$

$$\frac{\hat{\mathbf{y}}_2 - \mu_2}{\sigma_2} = \rho\frac{(\mathbf{y}_1 - \mu_1)}{\sigma_1}$$

- GLM, Quarg, et al. attempt to do in the design matrix what should be done with covariance (Halliwell, *PCAS*, 1996)

Vector Means and Variances

$$\mathbf{Y}_{n\times 1} \sim \mu_{n\times 1}, \Sigma_{n\times n}$$

$$\Rightarrow$$

$$\mathbf{A}_{m\times n}\mathbf{Y} \sim \mathbf{A}\mu_{m\times 1}, \mathbf{A}\Sigma\mathbf{A}'_{m\times m}$$

- $A\Sigma A' \geq 0$, i.e., Σ is non-negative (or positive) definite.
- $A\Sigma A' = 0$ for $A \neq 0$ indicates linear dependence within the elements of **Y**.

Modeling Essentials

- Proper design matrix, X (or regressors, or independent variables)
- The only random term on the right side of the equation is **e**, i.e., no stochastic regressors.
- How does each observation covary with the others? Don't assume zero off the diagonal.

Spreadsheet Examples

- Increasing complexity of the variance structures of Models 1-4
- Conjoint model (cf. Halliwell, Summer 1997 Forum, versus Quarg, Variance, Fall 2008