Optimal Reinsurance under VaR and CVaR Risk Measures: A Simplified Approach

Ken Seng Tan*

Department of Statistics and Actuarial Science
University of Waterloo, Canada

China Institute for Actuarial Science
Central University of Finance and Economics, Beijing, China

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Outline

- Introduction and motivation
- Risk measure based optimal reinsurance models
  - Cai and Tan (2007)
  - Chi and Tan (2011)
- Conclusion
Without Reinsurance

Policyholders (Insureds) → Premium → Insurance Company (Insurer or Cedent) → Claims

Policyholders

Premium

Claims

Insurance Company

(Insurer or Cedent)
With Reinsurance

Policyholders (Insureds) → Insurance Company (Insurer or Cedent)

Premium

Reinsurance Company (Reinsurer)

Reinsurance Premium → Ceded Claims

Premium

Claims
Reinsurance as a Risk Management Tool

- Reinsurance can be an effective risk management tool for insurers

- Some reasons for Reinsurance:
  - limitation of exposure to risk
  - avoidance of large single losses
  - increasing capacity to accept risk
  - availability of expertise

- The primary goal of reinsurance is to maintain, at an acceptable level, the random fluctuations of the business operation of the insurers:
  - earning volatilities
  - variance of the underlying risk
Reinsurance Contracts

- Let $X$ denote the loss initially assumed by an insurer
  - $X$ is a non-negative r.v. with
    - c.d.f. $F_X(x) = \Pr(X \leq x)$,
    - survival function $S_X(x) = \Pr(X > x)$, and
    - $\mathbb{E}[X] < \infty$

- In the presence of reinsurance, the insurer cedes part of its loss, say $f(X)$, to a reinsurer
  - $f(x)$ is known as a ceded loss function
  - $R_f(x) = x - f(x)$ is the retained loss function
  - $0 \leq f(x) \leq x$ and $0 \leq R_f(x) \leq x$
Samples Ceded Loss Functions

Quota-share reinsurance

\[ f(x) = ax \]

Limited stop-loss reinsurance

\[ f(x) = \min(x - d, m) \]

Stop-loss reinsurance

\[ f(x) = (x - d)_+ \]

Truncated stop-loss reinsurance

\[ f(x) = (x - d)_+ \mathbb{1}(x \leq m) \]
Reinsurance Premium

- By ceding part of its risk to a reinsurer, the insurer incurs an additional cost in the form of reinsurance premium which is payable to a reinsurer

- Let $\Pi_f(X)$ denote the reinsurance premium which corresponds to a ceded loss function $f(x)$

- Under expected premium principle:

$$\Pi_f(X) = (1 + \rho)E[f(X)]$$

where $\rho > 0$ is the relative safety loading
Optimal Reinsurance

- There’s a tradeoff between the amount of loss retained and the reinsurance premium payable to reinsurer
  - optimal reinsurance design?

- Some plausible optimal reinsurance models:
  - Minimizes insurer’s ruin probability
  - Classical result (Borch 1960):
    \[
    \min_{f} \text{Variance}(R_f(X))
    \]
    subject to \( \text{Premium} = (1 + \rho)E[f(X)] \)
    \( \Rightarrow \) Stop-loss reinsurance is optimal
  - By maximizing expected utility of insurer’s terminal wealth, Arrow (1963) shows that stop-loss reinsurance is optimal

- Our approach exploits the risk measure based optimal reinsurance model of Cai and Tan (2007)
Risk Measure based Optimal Reinsurance Model

- Define total "risk exposure" of the insurer in the presence of stop-loss reinsurance as

\[ T_f(X) = R_f(X) + \Pi_f(X) = X - f(X) + \Pi_f(X) \]

⇒ implications?

- Risk measure based optimal reinsurance model:

\[ \min_{f \in C} \psi(T_f(X)) \]
- \( \psi(\cdot) \) is a risk measure
- \( C \) is the set of admissible ceded loss functions
- Complexity of this model?
Cai and Tan (2007): Assumptions

- $\mathcal{C}$ is the stop-loss reinsurance with retention $d > 0$;
  \[ f(x) = (x - d)_+ \]
- $\Pi$ is the expected premium principle
- $\psi$ is either
  - Value at Risk (VaR) or
  - Conditional VaR (CVaR)/Conditional Tail Expectation (CTE)
VaR vs CVaR at Confidence Level $1 - \alpha$

$\text{CVaR}_\alpha(X) = E[X | X \geq \text{VaR}_\alpha(X)]$

$\text{VaR}_\alpha(X) \equiv S_X^{-1}(\alpha) = F_X^{-1}(1 - \alpha)$

$\Pr\{X > \text{VaR}_\alpha(X)\} = \alpha \iff \Pr\{X \leq \text{VaR}_\alpha(X)\} = 1 - \alpha$
Cai and Tan (2007): Results

- **VaR-optimization:**
  \[ d^* \rightarrow \min_{d > 0} \{ \text{VaR}_\alpha(T_f(X); d) \} \]

- **CVaR-optimization:**
  \[ \tilde{d} \rightarrow \min_{d > 0} \{ \text{CVaR}_\alpha(T_f(X); d) \} \].
An Alternate Justification

- Let $p_X$ be the premium payable by the insured to the insurer.
- Let $r_X$ be the minimum capital set aside by the insurer so that the insurer’s probability of insolvency is at most $\alpha$; i.e.
  \[
  \Pr\{T_f > r_X + p_X\} \leq \alpha.
  \]
- From the definition of VaR:
  \[
  r_X = \text{VaR}_{\alpha}(T_f) - p_X.
  \]
  \[\Rightarrow \min_f \text{VaR}_{\alpha}(T_f(X)) \Leftrightarrow \min_f r_X\]
Cai and Tan (2007): VaR-Optimization

- The optimal retention $d^* > 0$ that minimizes $\text{VaR}_\alpha(T_f(X))$ exists if and only if both

$$\alpha < \rho^* < S_X(0)$$

and

$$S_X^{-1}(\alpha) \geq S_X^{-1}(\rho^*) + \Pi\left(S_X^{-1}(\rho^*)\right)$$

hold, where $\rho^* = \frac{1}{1 + \rho}$.

- When the optimal retention $d^*$ exists, then $d^*$ is given by

$$d^* = S_X^{-1}(\rho^*)$$

and the minimum VaR of $T$ is given by

$$\text{VaR}_\alpha(T_f(X), d^*) = d^* + \Pi(d^*)$$.
Examples

$X \sim$ Exponential Distribution

- $S_X(x) = e^{-0.001x}$
- $E[X] = 1,000$, $\alpha = 0.1$, $\rho = 0.2$
- optimal retention $d^*$ exists and equals to
  
  $$d^* = S_X^{-1}(\rho^*) = 1,000 \log(1 + \rho) = 182.32.$$  

$X \sim$ Pareto Distribution

- $S_X(x) = \left( \frac{2,000}{x + 2,000} \right)^3$, $x \geq 0$.
- $E[X] = 1,000$, $\alpha = 0.1$, $\rho = 0.2$
- optimal retention $d^*$ exists and equals to
  
  $$d^* = S_X^{-1}(\rho^*) = 125.32.$$
The optimal retention $\tilde{d} > 0$ that minimizes $\text{CVaR}_\alpha (T_f(X); d)$ exists if and only if

$$0 < \alpha \leq \rho^* < S_X(0).$$

When the optimal retention $\tilde{d} > 0$ exists, $\tilde{d}$ is given by

$$\tilde{d} = S_X^{-1} (\rho^*) \quad \text{if} \quad \alpha < \rho^*,$$

and

$$\tilde{d} \geq S_X^{-1} (\rho^*) \quad \text{if} \quad \alpha = \rho^*.$$
Cai and Tan (2007): Summary

- The optimal reinsurance model is simple and intuitive
- It exploits two prevalent risk measures
- The optimal retention has a very simple analytic form
- If optimal solutions exist, then both VaR- and CVaR-based optimization criteria yield the same optimal retentions, except when $\alpha = \rho^*$
  \[ d^* = \tilde{d} = S_X^{-1} \left( \frac{1}{1 + \rho} \right) \]
- The optimal retention depends only on the assumed loss distribution and the reinsurer’s safety loading factor

- Limitations?
Chi and Tan (2011)

• Generalize Cai and Tan (2007) by considering more general admission sets of ceded loss functions:

\[ C^1 \triangleq \{ 0 \leq f(x) \leq x : f(x) \text{ is an increasing convex function} \} \]

\[ C^2 \triangleq \{ 0 \leq f(x) \leq x : \text{both } R_f(x) \text{ and } f(x) \text{ are increasing functions} \} \]

\[ C^3 \triangleq \{ 0 \leq f(x) \leq x : R_f(x) \text{ is an increasing and l.c. function} \} . \]

Properties:

• \( C^1 \subsetneq C^2 \subsetneq C^3 \)

• What is the significance of imposing increasing condition on both retained and ceded loss functions?
VaR-Optimization under $C^1$

Optimal Reinsurance Model:

$$\min_{f \in C^1} \text{VaR}_\alpha(T_f(X))$$

Optimal Solution:

- $f^*_1(x) = \begin{cases} (x - d^*)^+ , & \text{VaR}_\alpha(X) > \beta; \\ c(x - d^*)^+ , & \forall c \in [0, 1], \quad \text{VaR}_\alpha(X) = \beta; \\ 0, & \text{otherwise}, \end{cases}$

where $d^* = \text{VaR}_{\rho^*}(X)$,

$$\beta = d^* + (1 + \rho)E[(X - d^*)^+] .$$

- $\text{VaR}_\alpha(T_{f^*_1}(X)) = \min_{f \in C^1} \text{VaR}_\alpha(T_f(X)) = \min(\beta, \text{VaR}_\alpha(X))$\n
$\Rightarrow$ **Stop-loss reinsurance** is optimal under $C^1$
A Special Case

- If $\rho^* \geq S_X(0)$, then $d^* = 0$.

- The optimal ceded loss function $f^{*1}$ simplifies to

$$f^{*1}(x) \triangleq \begin{cases} x, & \text{VaR}_\alpha(X) > (1 + \rho)\mathbb{E}[X]; \\ cx, & \forall c \in [0, 1], \text{VaR}_\alpha(X) = (1 + \rho)\mathbb{E}[X]; \\ 0, & \text{otherwise.} \end{cases}$$

$\Rightarrow$ Quota-share ceded loss function
VaR-Optimization under $C^2$

Optimal Reinsurance Model:

$$\min_{f \in C^2} \text{VaR}_\alpha(T_f(X))$$

Optimal Solution:

- $f^{*2}(x) = \begin{cases} 
\min \{(x - d^*)_+, \text{VaR}_\alpha(X) - d^*\}, & d^* < \text{VaR}_\alpha(X); \\
0, & \text{otherwise}.
\end{cases}$

- $\text{VaR}_\alpha(T_{f^{*2}}(X)) = \min[d^*, \text{VaR}_\alpha(X)] + (1 + \rho)E[\min \{(X - d^*)_+, (\text{VaR}_\alpha(X) - d^*)_+\}]$.

$\Rightarrow$ Limited stop-loss reinsurance is optimal under $C^2$
VaR-Optimization under $C^3$

Optimal Reinsurance Model:

$$\min_{f \in C^3} \text{VaR}_\alpha(T_f(X))$$

Optimal Solution:

- Let $\gamma = \alpha + \rho^*$, then
  $$f^*3(x) = (x - \gamma) + \mathbb{I}(x \leq \text{VaR}_\alpha(X)),$$
- $\text{VaR}_\alpha(T_{f^*3}(X)) = \gamma + (1 + \rho)E[(X - \gamma) + \mathbb{I}(X \leq \text{VaR}_\alpha(X))].$

$\Rightarrow$ Truncated stop-loss reinsurance is optimal under $C^3$
CVaR-Optimization under $C^j, j = 1, 2, 3$

Optimal Reinsurance Model:

$$\min_{f \in C^j} \text{VaR}_\alpha(T_f(X))$$

Optimal Solution:

- $f^*(x) = \begin{cases} (x - d^*)_+, & \alpha < \rho^*; \\ 0, & \text{otherwise,} \end{cases}$

- $CVaR_\alpha(T_{f^*}(X)) = \min_{f \in C^j} CVaR_\alpha(T_f(X)) = \begin{cases} \beta, & \alpha < \rho^*; \\ CVaR_\alpha(X), & \text{otherwise} \end{cases}$

$\Rightarrow$ **Stop-loss reinsurance** is optimal under $C^j, j = 1, 2, 3$
Summary/Conclusion

• We extended the reinsurance model of Cai and Tan (2007) by analyzing the solutions to the VaR- and CVaR-based optimal reinsurance models over different classes of ceded loss functions with increasing generality.

• The impact of the optimal reinsurance design on the assumed feasible set of ceded loss functions is highlighted in the case of VaR criterion.
  • This suggests a difference in risk management strategy depending on the adopted optimal reinsurance model.
  • The different optimal reinsurance policies also suggest the differences in insurer’s style toward risk management and its attitude towards risk.

• The CVaR-based optimal reinsurance model is quite robust in the sense that the stop-loss reinsurance is always the optimal solution.
Thank You For Your Attention