Capital Allocation in the presence of Tail Dependencies

Dr. Christoph Hummel

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Capital allocation – what for?

Theoretical concept in order to answer the question:

- How much does a line of business or even a treaty have to contribute to the expected profit of an insurance company?
  - Retrospectively: Performance measurement
  - Prospectively: Pricing
- Focus : **Pricing** from a reinsurer's perspective

The price of risk

- Let $\mathcal{X}$ be a reinsurance treaty with a profit X for the reinsurer:

$$X = \text{NPV(Premium-Losses-Expenses)}$$

  - Future cash flows are discounted at the risk free rate.
- What is an appropriate amount for EX ?

→ The higher the risk of $\mathcal{X}$ **and** the less $\mathcal{X}$ diversifies the portfolio, the bigger EX should be.

Some elementary premium principles

1. Expected value principle
2. Standard deviation principles
3. Variance and Exponential principle

- **Advantage**: Easy to implement
- **Disadvantage**: They do not take the diversification of $\mathcal{X}$ into account and 1. not even the risk.

Quantile based premium principles

Distorted probabilities: D. Denneberg (1989), S. Wang (1995/6)

- G: $[0,1] \rightarrow [0,1]$ increasing, concave, surjective:

$$E^*(X) := \int_{-\infty}^{+\infty} x \, d(G \circ F_X)(x) \stackrel{!}{=} 0$$

  - G changes the probabilities of possible outcomes.
    - Probabilities of very bad outcomes increase
    - Probabilities of very good outcomes decrease
- Substantial advantage: additivity
- Cf. risk neutral probabilities in Finance

Portfolio Viewpoint

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What is the role of capital?

- The purpose of regulatory solvency requirements is "*to ensure that the insurers have the capacity to meet their obligations to pay the present and future claims to policyholders*."
  International **A**ssociation of **I**nsurance **S**upervisors, On Solvency, Solvency Assessment and Actuarial Issues, An IAIS Issues Paper, March 2000, p. 17, available at www.iaisweg.org under "Publications".

  → Solvency requirements under discussions for insurance industry.

Risk measure: Expected shortfall

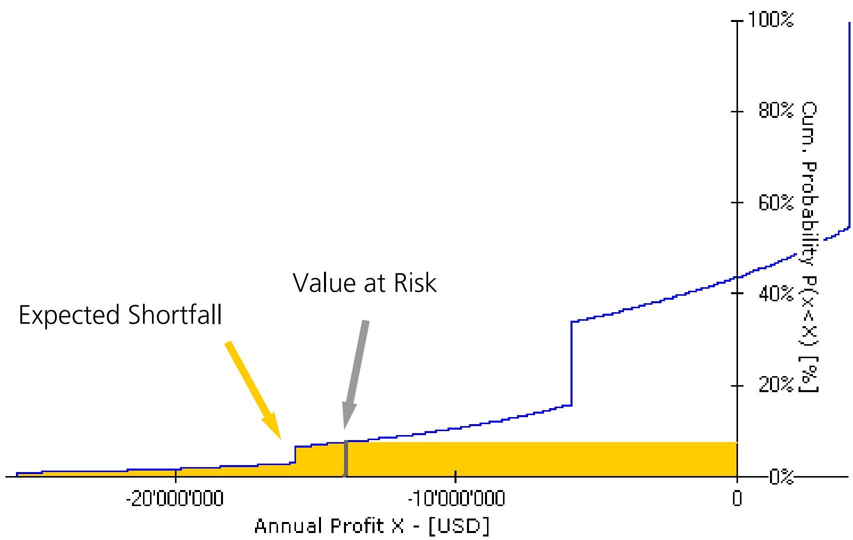
- As a risk measure r to determine the risk based capital K for the portfolio $\mathcal{Z}$, we choose an expected shortfall:

$$K := \rho(Z) := -E(Z \mid Z \leq F_Z^{-1}(\alpha))$$

for some small quantile a>0.

- In this way, we take the role of capital into account.
- Expected Shortfall:
  - is coherent
  - in contrast to VaR, it is more stable in simulations
  - focuses on the entire tail of the portfolio
  - can be easily explained to non-actuaries as well

Expected Shortfall vs. VaR



2. Risk contribution

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Portfolio decomposition

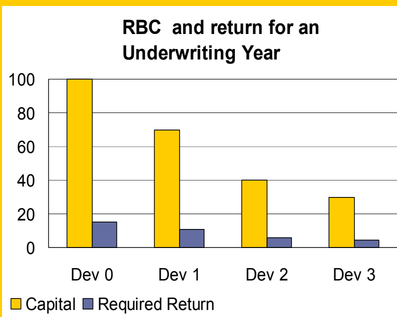
- The portfolio is supported by an RBC K.
- A minimum annual return h has to be generated on K.
- Decomposition of portfolio in risks $X_1, X_2, X_3, \ldots, X_n$
- Allocation of capital $K_1, K_2, K_3, \ldots, K_n$ for $X_1, X_2, X_3, \ldots, X_n$.

Premium for X_i is ***sufficient***, if the expected profit satisfies

$$EX_i \geq h \cdot \tau_i \cdot K_i$$

And the premium is ***technical*** in case of equality. Here τ_i is a time factor for the X_i.

Example: Time factor



NPV of required return:

$$100 \times 15\% \times 1.03^{-1} + 70 \times 15\% \times 1.03^{-2}$$
$$+ 40 \times 15\% \times 1.03^{-3} + 30 \times 15\% \times 1.03^{-4}$$
$$= 33.95 = 100 \times 15\% \times 2.263$$

- Hurdle=15%
- RBC=Initial capital =100
- Risk free rate = 3%

- The annual return of 15% on the initial capital 100 has to be generated 2.263 times.
- The time factor for the risk is 2.263.
- The NPV of the required return is 33.95

Capital allocation: Euler principle

We allocate capital to a subportfolio $\mathcal{S}$ (e.g., treaty, Line of Business) in $\mathcal{Z}$ according to the Euler principle:

$$K_S = \left.\frac{d}{dt}\right|_{t=0} \rho(Z + tS)$$

Theorem (D. Tasche, 1999). Under the above assumptions and some mild differentiability assumptions we have:

$$K_S = -E(S \mid Z \le F_Z^{-1}(\alpha))$$

In other words, the RBC of $\mathcal{S}$ is the expected contribution of $\mathcal{S}$ to the portfolio's shortfall.

Capital via Monte Carlo simulations

The capital allocated for X_i is the expected contribution of $\mathcal{X}$ to the expected shortfall of the portfolio

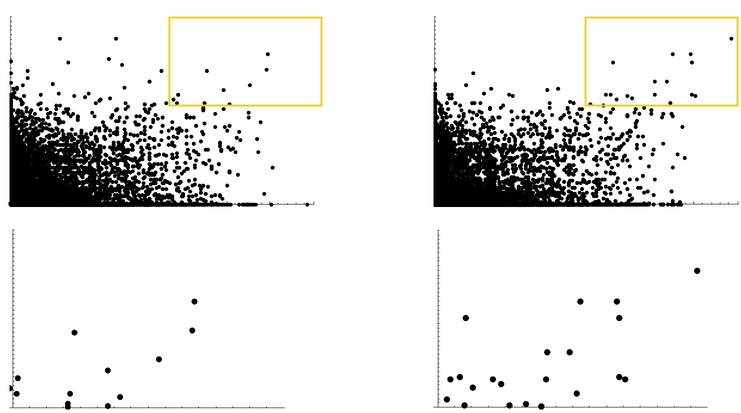
- Consider 1 million iterations of the portfolio's result Z and say that α =1/100:
- The average of the worst 10'000 results is the RBC K for $\mathcal{Z}$.
- The average of X_i on ***these*** 10'000 iterations is its RBC K_i.
- The sum of the K_i is equal to K.

- In order to use this principle in practise, we need a good ***Portfolio-Model***!
- ***Dependencies*** have to be described carefully.

3. Modelling dependencies

Linear correlation is not sufficient!

Two joint simulations of losses in two LoBs



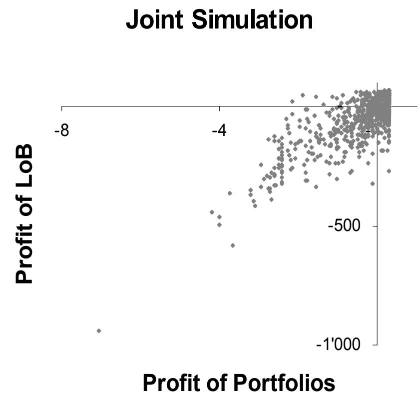
No Tail-Dependence
Correlation = 22.5%

Tail-Dependence
Correlation = 22.5%

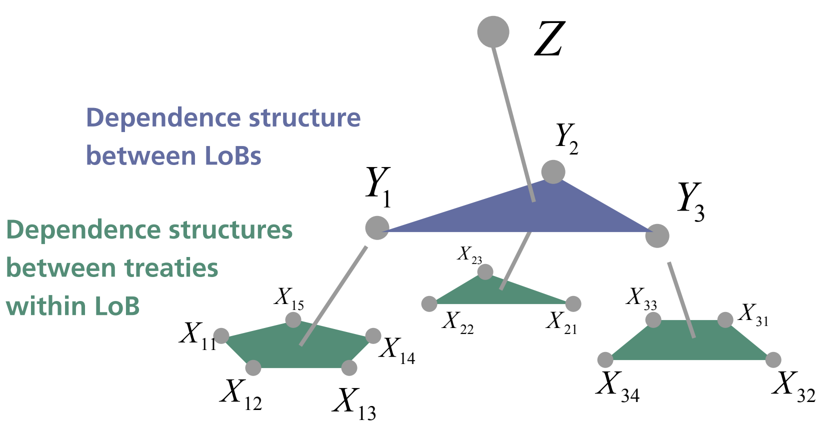
Describing dependencies

- Via joint simulations
- Based on
  - scenarios
  - experience
  - Expert opinion

→ ***Copula***



Hierarchical dependence structure within the portfolio



Accumulation Control for LoB: Scenarios → Copula

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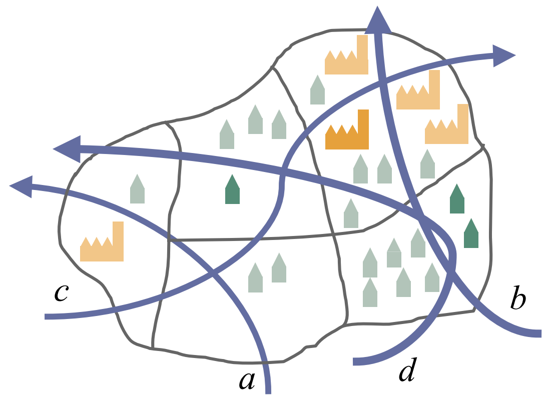
- Determine the loss for $\mathcal{X}$ and $\mathcal{Y}$ for each scenario
- The orders, not the sizes, of the simulated aggregated losses per period losses are relevant for the copula between X and Y.

	$\mathcal{X}$	$\mathcal{Y}$
a	0	3
b	6	29
c	3	15
d	4	12



Dependence between the LoBs

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We estimate / guess a copula between the LoBs.

- In reinsurance, there is not enough data to estimate the copulas.
- But still, copulas can be used to translate an opinion about dependencies in the portfolio into a model:
  - Look at causal relation between LoBs
  - Think about scenarios in the portfolio
  - Try to estimate conditional probabilities by asking questions of the type "What if a particular LoB turned very bad?"

Tail dependence between the LoBs has to be implemented

We focus on copulas with lower tail dependence.

- Lots of LoBs seem to be almost independent in “normal” situations, but there are dependencies in the tail:
  - 11. Sept.: Aviation, Worker Compensation, D&O, Property-Cat, Life, Business Interruption
- Parametrised families
  - t-copulas, Gumbel copulas, Clayton copulas (refer e.g. to Embrechts, Lindskog, McNeil.)
  - Calibration by estimating / guessing tail dependencies

The crucial assumption in our portfolio model

- We assume in our portfolio model that

$$P(X \leq x \mid Y = y, Z \leq z) = P(X \leq x \mid Y = y)$$

for each LoB $\mathcal{Y}$ and each treaty $\mathcal{X}$ in $\mathcal{Y}$. In other words:

- Given the outcome of the profit in LoB $\mathcal{Y}$ additional information on the outcome of $\mathcal{Z}$ does not change the profit distribution of treaties $\mathcal{X}$ in $\mathcal{Y}$.

➡ This assumption determines the dependence structure, i.e., the copula, between any treaty and the portfolio.

Calculation of allocated capital

- The capital allocated for a treaty $\mathcal{X}$ in $\mathcal{Z}$ is given by

$$K_X = -E(X \mid Z \le F_Z^{-1}(\alpha)).$$

- Need to change the probability measure to calculate K_S.
- This change can be described by a function H_S.

$$K_X = -E(X \mid Z \le F_Z^{-1}(\alpha)) = -\int_{-\infty}^{+\infty} x\, d(H_X \circ F_X)(x)$$

➡ We call H_X the ***diversification function*** of $\mathcal{X}$ in $\mathcal{Z}$.

- H_X can be calculated from the copula between $\mathcal{X}$ and $\mathcal{Z}$.

Interpretation

Consequently, the treaty $\mathcal{X}$ has the technical premium if

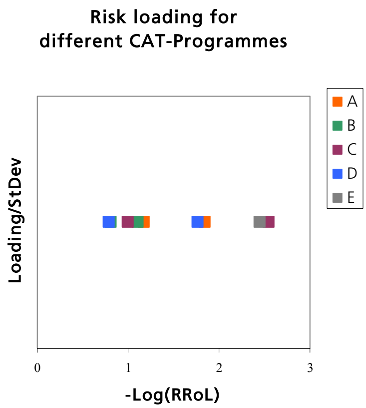
$$EX = \int_{-\infty}^{+\infty} x\, dF_X(x) = -\eta\tau \int_{-\infty}^{+\infty} x\, d(H_X \circ F_X)(x) = \eta\tau K_X$$

This is equivalent to

$$\int_{-\infty}^{+\infty} x\, d(G_X \circ F_X)(x) = 0 \quad \text{with} \quad G_X(p) = \frac{p + \eta\tau H_X(p)}{1 + \eta\tau}$$

Compare this to the quantile based premium principle. In our set up, the distorted probabilities differ from treaty to treaty and are determined from the diversification of the treaty in the portfolio.

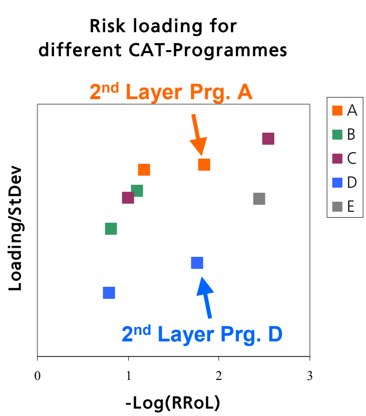
A traditional approach: An example



- Using the standard deviation principle, all treaties are on a horizontal line.
- Risk Rate on Line

$$RoL = \frac{\text{Expected Loss}}{\text{Granted Limit}}$$

Active portfolio management: Same example in portfolio context



- Even treaties with almost the same loss distribution get very different risk loadings:
- Example:
  - The distributions of the second Layers of the programmes **A** and **D** are almost identical.
  - **A** has a high, **D** a low dependency with the portfolio.
- Result: Adequate pricing aids in optimising the portfolio.

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Contact

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Dr. Christoph Hummel
Pricing Actuary
Converium AG

General Guisan Quai 26
CH-8022 Zürich

christoph.hummel@converium.com