ABSTRACT

This paper illustrates the calibration of a term structure model to market data. The specific application uses Eurodollar futures options prices and the underlying futures contracts to calibrate a one- and two-factor Hull-White term structure model. This paper explains the Hull-White term structure models, Eurodollar futures options contracts, and a numerical methodology (Hull-White trees) that is used to evaluate the American feature of the traded options. When the calibrated models are applied to Eurodollar futures options, the results illustrate that the two-factor model appears to have fewer biases in modeling market prices than a one-factor model.

1. INTRODUCTION

A number of term structure models are available to simulate movements of interest rates. Term structure models play an important role in many types of financial analysis including the valuation of interest rate dependent securities, the measurement of interest rate risk, and even strategic planning exercises. Understanding the characteristics of different term structure models can help the user choose the most appropriate model for his or her application. A vital step to using a term structure model is to choose appropriate parameters. Even if the form of the term structure model incorporates all of the desirable historical movements in yields (such as mean reversion), such models will provide poor results if the parameters are carelessly chosen.

This paper documents a methodology, called calibration, which chooses appropriate parameters for term structure models. The rationale for the approach is that, when a term structure model is used to price interest rate contingent claims, the model should (approximately) replicate market prices. This paper uses Eurodollar (ED) futures options, a specific, but relatively simple type of interest rate option, to calibrate both a one- and two-factor Hull-White term structure model. The methodology of this study is general enough to apply to all models of

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interest rates and any set of securities prices. The motivation for these choices of term structure model and calibration instruments will also be discussed.

This paper is organized as follows. First, an overview of different interest rate models is presented, including a description of the one- and two-factor Hull-White models. Next, a numerical methodology based on trinomial trees is discussed. The following section discusses ED futures contracts and options on ED futures, along with some motivation for choosing these securities as the basis for calibrating the term structure models. The fifth section explains the calibration methodology and the sixth section illustrates the results. Finally, some concluding comments are presented.

2. INTEREST RATE LITERATURE REVIEW

A number of papers discuss general features of modeling term structure movements. Subrahmanyam (1996) provides an overview of the relevant issues in the area of interest rate modeling, from conceptual foundations of alternative models to empirical issues and estimation. Ahlgrim, D'Arcy, and Gorvett (1999) discuss features of several popular term structure models and provide comparative statistics between historical interest rate movements and simulated rates of some models. Chapman and Pearson (2001) provide a summary of term structure research, with a focus on distinguishing areas of agreement and areas for further research.

There are two primary issues to consider when selecting a term structure model to value financial securities. The first decision is based on the theoretical foundation for the interest rate model; there are general equilibrium or arbitrage-free models. General equilibrium models formulate bond yields based on the expectations of investors in the economy. Two popular interest rate models using the general equilibrium approach are the models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) (CIR). One of the major benefits of these models is that interest rate movements and resulting securities' prices frequently have convenient closed-form solutions.

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On the negative side, general equilibrium models are criticized because the yield curves produced by the models do not replicate the term structure built from existing market prices. This makes these models unsatisfactory for pricing derivative securities. Hull (2000) points out that if practitioners cannot rely on the model to accurately portray the existing term structure, they will have little confidence that the model will accurately imitate the dynamics of the yield curve.

The inconsistency between the yield curves from general equilibrium models and asset prices has increased the popularity of arbitrage-free models. Arbitrage-free models take the initial yield curve as given and then generate the future dynamics of the curve. Popular arbitrage-free approaches include the models of Ho and Lee (1986), Black, Derman, and Toy (1990) [BDT], Hull and White (1990), Black and Karasinski (1991), and Heath, Jarrow, and Morton (1992) [HJM]. The difficulty with arbitrage-free models is that, in many cases, one must resort to numerical techniques to value derivative securities. Depending on the model and the particular application, these numerical techniques can become quite cumbersome.

The second consideration in choosing among alternative term structure models is in determining the number of factors that are subject to variation. Early approaches typically identified one stochastic factor – the short-term rate. In continuous time models, this means the instantaneous short-term rate; in discrete time, this factor is the yield on short-term bonds such as one-day, one-week, or one-month. An example of the one-factor approach to term structure modeling is the model of Vasicek (1977):

$$dr_t = \kappa(\theta - r_t)dt + \sigma dB_t \tag{1.1}$$

r_t = current level of the short-term rate
θ = long-run mean level of the short-term rate
κ = coefficient (strength) of mean reversion
σ = instantaneous volatility of the short-term rate
B = a standard Wiener process

As with most term structure models, the Vasicek model is formulated in terms of changes in the instantaneous (short-term) yield. To interpret the Vasicek model, consider the two terms

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separately. The first term represents the expected movement or drift of the short-term rate over the next instant. This term indicates that r_t reverts to its long-term average θ. To see this, consider the case when r_t exceeds θ. In this case, the drift term is negative so that the short rate is expected to decrease when it is above θ. The speed of mean reversion is measured by the parameter κ. To understand how the short rate reverts to its long term average, Vasicek shows that the expected value of the short rate at a future time s ($>t$) is a linear combination of the current short rate (r_t) and its long term average θ:

$$E[r_s|r_t] = e^{-\kappa(s-t)}r_t + (1 - e^{-\kappa(s-t)})\theta \qquad (1.2)$$

The second term in the Vasicek model represents the volatility of changes in the short-term rate. A Wiener process $\{B\}$ is a special type of random process such that $B_s\text{-}B_t$ ($s>t$) has a normal distribution with expected value of 0 and a variance of $s\text{-}t$. Therefore, the volatility of interest rate changes is $\sigma^2 dt$. In the Vasicek model, the coefficient of the Wiener process is constant (σ), implying that the conditional volatility of interest rate changes is also constant.

The parameters of one-factor models specify the expected evolution of the short-term rate (under the risk-neutral probability measure), which consequently determines longer-term bond yields and corresponding prices.

$$P(t,T) = e^{-R(t,T)(T-t)} = E\left[\exp\left(-\int_t^T r_s ds\right)\right] \qquad (1.3)$$

In this equation, $P(t,T)$ is the price of a \$1 face value, zero-coupon bond at time t whose maturity is T and $R(t,T)$ is the yield-to-maturity of the bond. Vasicek (1977) finds that this expectation can be rewritten as[1]

$$P(t,T) = A^V(t,T)e^{-B^V(t,T)r_t} \qquad (1.4)$$

where

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[1] The superscript (V) on the function $A(t,T)$ signifies that this result holds for the Vasicek model. As will be seen later, other term structure models also price bonds using (1.4), but they will have a different functional form for $A(t,T)$. To distinguish the various forms of this function, superscripts will be used.

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$$B^V(t,T) = \frac{1-e^{-a(T-t)}}{a} \qquad (1.5)$$

and

$$\ln A^V(t,T) = \frac{\left(B(t,T) - T + t\right)\left(a^2 b - \frac{\sigma^2}{2}\right)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a} \qquad (1.6)$$

CIR introduces a volatility factor that allows volatility to be related to the level of interest rates:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\,dB_t \qquad (1.7)$$

Because of the inclusion of the square root in the volatility term, CIR is also known as the square root process. The resulting process is more consistent with empirical evidence, which finds that volatility is higher when interest rates are high (see Chapman and Pearson (2001) and Ahlgrim, D'Arcy, and Gorvett (1999)). An additional benefit of relating volatility to interest rates in the CIR model is that negative interest rates are ruled out. Since the process is continuous, the volatility approaches zero as interest rates decline and the mean reversion drift dominates.

To give a sense for the impact that a term structure model can have on security prices, consider how the volatility of interest rates differs in the CIR and Vasicek models. When interest rates are low, CIR implies that volatility also decreases, but the volatility of interest rate changes in the Vasicek model does not change. Thus, if interest rates (or strike levels) are low, Vasicek may provide relatively high option prices compared to when interest rates (or strike levels) are high. Hull and White (1990) provide numerical confirmation of this effect. They use a constant volatility model and the CIR model to price interest rate caps. They price a 5-year, $100 notional principal, interest rate cap with two strike rates. At a 9% strike, they find a value of $5.90 under the constant volatility model and $5.83 under CIR. At an 11% strike, the values are $1.72 and

$1.80. Even with only a 2% difference in the strike level, the relative pricing between the models is different.

Similar to the Vasicek case, bond prices under the CIR model can be directly obtained through (1.4). But in this case,

$$B^{CIR}(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + \kappa)(e^{\gamma(T-t)} - 1) + 2\gamma} \tag{1.8}$$

$$A^{CIR}(t,T) = \left[\frac{2\gamma e^{(\kappa+\gamma)(T-t)/2}}{(\gamma + \kappa)(e^{\gamma(T-t)} - 1) + 2\gamma} \right]^{2\kappa\theta/\sigma^2} \tag{1.9}$$

$$\text{where } \gamma = \sqrt{\kappa^2 + 2\sigma^2}$$

Vasicek and CIR are general equilibrium models. As mentioned above, an important feature of these one-factor, general equilibrium models is that the initial term structure cannot be perfectly replicated, even with careful parameter selection. Hull and White (1990) extend the Vasicek approach to allow the models to fit the initial term structure. The one-factor Hull-White model is:

$$dr_t = (\theta^{HW1}(t) - ar_t)dt + \sigma dB_t \tag{1.10}$$

Because of the similarity to the model proposed by Vasicek, the one-factor Hull-White model is also called the extended-Vasicek model. (A similar formulation corresponding to CIR is known as the extended-CIR model). The important feature of the model is that the drift term has a time dependent factor $\theta^{HW1}(t)$, so that the expected evolution of the short rate yields a term structure that is consistent with the existing market prices of bonds. The $\theta^{HW1}(t)$ term is related to changes in the forward rate as follows

$$\theta^{HW1}(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{1.11}$$

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In this equation, $F(0,t)$ represents the instantaneous forward rate maturing at time t as viewed at time 0 and the subscript represents partial differentiation. Instantaneous forward rates are defined by the relationship between successive bond prices:

$$F(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \qquad (1.12)$$

Again, like Vasicek and CIR, yields of longer maturities in the one-factor Hull-White model use (1.4), where

$$B^{HW1}(t,T) = \frac{1 - e^{-a(T-t)}}{a} \qquad (1.13)$$

(which is the same as the Vasicek case), but

$$\ln A^{HW1}(t,T) = \ln \frac{P(0,T)}{P(0,t)} - B(t,T)\frac{\partial \ln P(0,t)}{\partial t} - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1) \qquad (1.14)$$

An argument against one-factor models is that they place too much rigidity on the yield curve movements. As the short-term rate evolves, other points on the yield curve are deterministic based on the parameters of the model; long-term yields are a one-to-one mapping of the short-term rate. This result can be seen by rearranging (1.3) and (1.4), and solving for the yield $R(t,T)$:

$$R(t,T) = -\frac{\ln A(t,T) - B(t,T)r_t}{T - t} \qquad (1.15)$$

Exhibit 1 provides the historical relationship between 1-year yields and 10-year yields by month from 1953-2000. Note that the relationship between yields is not fixed. When the 1-year yield is 4%, there is a range of observations for the 10-year rate. However, by using a one-factor term structure model, this dispersion is not possible. Exhibit 2 illustrates the relationship between the short-term rate and other yields from a simulation of a one-factor term structure model. For each realization of the short-term rate, there is only one realization of the 1- and 10-year yields. This rigid relationship is at odds with the historical relationships illustrated in Exhibit 1.

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Another way to analyze the relationship between yields of various maturities is to calculate their correlations. Empirical estimation (and intuition) indicates that the correlation decreases as the distance between yields increases. For example, the correlation between the four-year and five-year yields is higher than the correlation between the one- and twenty-year yields. A shortfall of one-factor models is that all resulting yields are perfectly, positively correlated. Although yields of various maturities are positively correlated, the correlation is not 1.0 (see Ahlgrim, D'Arcy, and Gorvett (1999)).

Exhibit 3 illustrates the correlation of yields of various maturities. As expected, the correlations decrease as we look at points on the yield curve that are farther apart. The monthly correlation between 1- and 10-year yield from 1953-2000 was 0.943. This correlation is a little higher than reported in Chapman and Pearson (2001). The difference presented here reflects the addition of almost ten years of earlier data than to the Chapman and Pearson (2001) sample. Exhibit 3 also presents the correlation analysis by decade. The correlations toward the end of the period appear to be quite different from the earlier periods. In fact, the correlation between the 1- and 10-year yields during the 1990s was only 0.670.

The perfectly correlated yield dynamic embedded in one-factor models is clearly inconsistent with the variability exhibited in the market. To enhance the possible interest rate movements, one can introduce additional stochastic factors into a term structure model. Unfortunately, the benefits of incorporating multiple factors may not offset the costs with these models. In particular, they are difficult to implement and estimating parameter values is significantly more challenging than in the one factor case.

The Vasicek and CIR models incorporate only one stochastic factor (the short-term rate), so that bond prices of all maturities are subsequently related to the short-term rate. As a result, bond yields are perfectly correlated. This feature is particularly unsatisfactory when evaluating the securities that are exposed to many types of yield curve shifts. Litterman and Scheinkman (1991) use principal components analysis to rank the important factors that impact returns on

bonds. They find that almost 90% of changes in yields are explained by parallel shifts. The second pervasive factor they found is based on changes in the slope of the yield curve. An additional 9% of the variance of bond returns is explained by changes in steepness. Their findings illustrate that the relationship between the short-term and long-term rates is not as rigid as is implied by one-factor term structure models.

To alleviate the problem of correlated bond prices and capture additional forces that affect movements in bond yields, a term structure model can incorporate two or more stochastic factors. By modeling several factors, the potential range of yield curve dynamics is enhanced. Exhibit 4 presents a simulation of a two-factor term structure model and illustrates the more flexible relationship between yields of various maturities. Note that for a given realization of the short-term rate, longer yields take on more than one possible level. The range of results is more consistent with historical observations.

Even among two-factor models, there are numerous alternatives from which to choose. Heath, Jarrow, and Morton (1992) [HJM] provide the most general approach to multifactor interest rate modeling. Their model simulates the evolution of all *forward* rates.

$$dF(t,T) = \mu(t,T,F(t,T))dt + \sigma(t,T,F(t,T))dB_t \qquad (1.16)$$

where $F(t,T)$ is the instantaneous forward rate (defined above) and μ and σ are the drift and the volatility of the forward rate process. HJM show that the drift of the forward rate can be stated in terms of the volatility structure.

$$\mu(t,T,F(t,T)) = \sigma(t,T,F(t,T))\int_t^T \sigma(t,s,F(t,s))ds \qquad (1.17)$$

Depending on the functional form of the volatility structure, many of the existing term structure models can be expressed as special cases of the HJM approach, including Vasicek and Ho-Lee.

In the two-factor model as described in Brennan and Schwartz (1979, 1982), one factor is used to simulate the short-term rate while a second factor represents the rate on a perpetual bond.

Hull and White (1994b) extend the approach by fitting the two-factor model to the initial yield curve. The two-factor Hull-White model is:

$$dr_t = (\theta^{HW2}(t) + u_t - ar_t)dt + \sigma_1 dB_1$$
$$du_t = -bu_t dt + \sigma_2 dB_2 \tag{1.18}$$
$$\rho \; dt = dB_1 dB_2$$

In the Hull-White two-factor model, the interest rate reverts to a long-term rate that is also stochastic. The mean reversion coefficients (*a* and *b*) are typically chosen so that the short-term rate has high mean reversion, but the long-term rate reverts more slowly. This produces a yield curve where short-term yields are more volatile than long-term yields, which is consistent with historical statistics (see Ahlgrim, D'Arcy, and Gorvett (1999)).

For the two-factor Hull-White model, the time dependent drift has a closed form solution:

$$\theta^{HW2}(t) = F_t(0,t) + aF(0,t) + \phi_t(0,t) + a\phi(0,t) \tag{1.19}$$

$$\phi(t,T) = \frac{1}{2}\sigma_1^2 B^{HW2}(t,T)^2 + \frac{1}{2}\sigma_2^2 C(t,T)^2 + \rho\sigma_1\sigma_2 B^{HW2}(t,T)C(t,T) \tag{1.20}$$

Bond prices are also more complicated than the one-factor case:

$$P(t,T) = A^{HW2}(t,T)\exp\left[-B^{HW2}(t,T)t - C(t,T)u\right] \tag{1.21}$$

where

$$B^{HW2}(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{1.22}$$

$$C(t,T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab} \tag{1.23}$$

$$\ln A^{HW2}(t,T) = \ln\frac{P(0,T)}{P(0,t)} + B^{HW2}(t,T)F(0,t) - \eta \tag{1.24}$$

$$\eta = \frac{\sigma_1^2}{4a}(1-e^{-2at})B^{HW2}(t,T)^2 - \rho\sigma_1\sigma_2\left[B^{HW2}(0,t)C(0,t)B^{HW2}(t,T) + \gamma_4 - \gamma_2\right]$$

$$-\frac{1}{2}\left[\sigma_2^2 C(0,t)^2 B^{HW2}(t,T) + \gamma_6 - \gamma_5\right]$$

(1.25)

and

$$\gamma_1 = \frac{e^{-(a+b)T}\left[e^{(a+b)t}-1\right]}{(a+b)(a-b)} - \frac{e^{-2aT}(e^{2at}-1)}{2a(a-b)}$$

(1.26)

$$\gamma_2 = \frac{1}{ab}\left[\gamma_1 + C(t,T) - C(0,T) + \frac{1}{2}B^{HW2}(t,T)^2 - \frac{1}{2}B^{HW2}(0,T)^2 + \frac{t}{a} - \frac{e^{-a(T-t)}-e^{-aT}}{a^2}\right]$$

(1.27)

$$\gamma_3 = -\frac{e^{-(a+b)t}-1}{(a+b)(a-b)} + \frac{e^{-2at}-1}{2a(a-b)}$$

(1.28)

$$\gamma_4 = \frac{1}{ab}\left[\gamma_3 - C(0,t) - \frac{1}{2}B^{HW2}(0,t)^2 + \frac{t}{a} + \frac{e^{-at}-1}{a^2}\right]$$

(1.29)

$$\gamma_5 = \frac{1}{b}\left[\frac{1}{2}C(t,T)^2 - \frac{1}{2}C(0,T)^2 + \gamma_2\right]$$

(1.30)

$$\gamma_6 = \frac{1}{b}\left[\gamma_4 - \frac{1}{2}C(0,t)^2\right]$$

(1.31)

Choosing a term structure model is usually a tradeoff between accurately describing the historical interest rate process and having a tractable model that can be used to value financial instruments. It is important that a model should be chosen to reflect the needs of the particular application. If the use of the model includes valuing securities with interest rate options, the model should be consistent with the market prices of similar options. In particular, it should fully reflect the sensitivity of those options to several plausible interest rate scenarios.

In some instances, particular term structure models may be more appropriate for a particular application. For example, two papers look at the historical movements in yields to determine more explicitly which models are most representative of actual yield changes.

Balduzzi, Das, and Foresi (1998) show that when both short-term and long-term rates are viewed as stochastic, predicted interest rate movements are more consistent with historical changes than under a one-factor model. Jegadeesh and Pennacchi (1996) conclude that a two-factor model fits the movements in ED futures prices better than a one-factor model.

Jegadeesh (1998) provides empirical support of the superiority of arbitrage-free term structure models. He uses interest rate cap data to compare the Vasicek model against the Hull-White one-factor model. He concludes that no arbitrage models provide more accurate cap pricing than do general equilibrium models.

This paper looks at the importance of the number of stochastic factors in a term structure model in the pricing of Eurodollar futures options. In particular, it applies the one- and the two-factor Hull-White model in order to value Eurodollar futures options. Several recent papers have provided an analysis of these securities. Bhanot (1998) compares several alternative interest rate models to determine which of the models provides the most accurate pricing of Eurodollar futures options. The tests include the Vasicek and CIR model, as well as a two-factor CIR model incorporating stochastic volatility. He finds a reduction in absolute pricing errors for the stochastic volatility model when compared to the alternatives.

Two other papers have looked at Eurodollar futures applications that provide some useful comparison of the results included below. First, Mathis and Bierwag (1999) compare the Ho-Lee and Black-Derman-Toy binomial pricing models. They find that both models price ED options very well. However, they look at options that are very similar in terms of moneyness and maturity. Rather than requiring their fitted models to predict option prices of all options, they focus on a small subset of option prices, which provide a very good fit with market prices. Second, Bali and Karagozoglu (2000) compare the pricing accuracy associated with the use of different yield curve smoothing techniques, including cubic splines and linear interpolation.

This research is most similar to the Bhanot (1998) paper. It provides a comparison between two competing term structure models to see which predicts prices more accurately. Each

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of the models is calibrated to a set of Eurodollar futures options prices and a comparison between the models is presented to illustrate the effects of the introduction of a second stochastic factor.

The next section describes a useful numerical procedure that can be used to price interest rate contingent financial securities under several term structure assumptions. Section 4 then provides details on Eurodollar futures options markets.

3. A NUMERICAL PROCEDURE: HULL-WHITE TREES

One convenient numerical procedure that can be used to evaluate interest rate contingent claims is based on employing interest rate trees. As with the lattices used to represent stock price movements (usually binomial), interest rate trees provide a discrete approximation to the continuous time process represented in term structure models. There are two binomial models that are popular in the term structure literature. Ho and Lee (1986) provide a binomial tree-building procedure that is consistent with the initial term structure, while Black, Derman, and Toy (1990) also incorporate the term structure of volatilities. Hull and White (1993) provide a generic *trinomial* tree approach that can be used for many types of interest rate assumptions. The resulting tree can then be used to value interest rate contingent claims. The Hull and White approach can also be used to accommodate the traditionally binomial approaches of Ho and Lee (1986) and Black, Derman, and Toy (1990).[2]

The main benefit of interest rate trees over the use of Monte Carlo simulation is that American options can be more easily valued. When financial instruments contain American (or Bermudan) style option features, Monte Carlo simulation must take into account the early exercise feature at every potential exercise date. Exercise behavior must consider all potential future economic scenarios at each point in time. Thus, at every exercise date, an additional simulation is required just to determine the optimal exercise strategy. For options with terminal
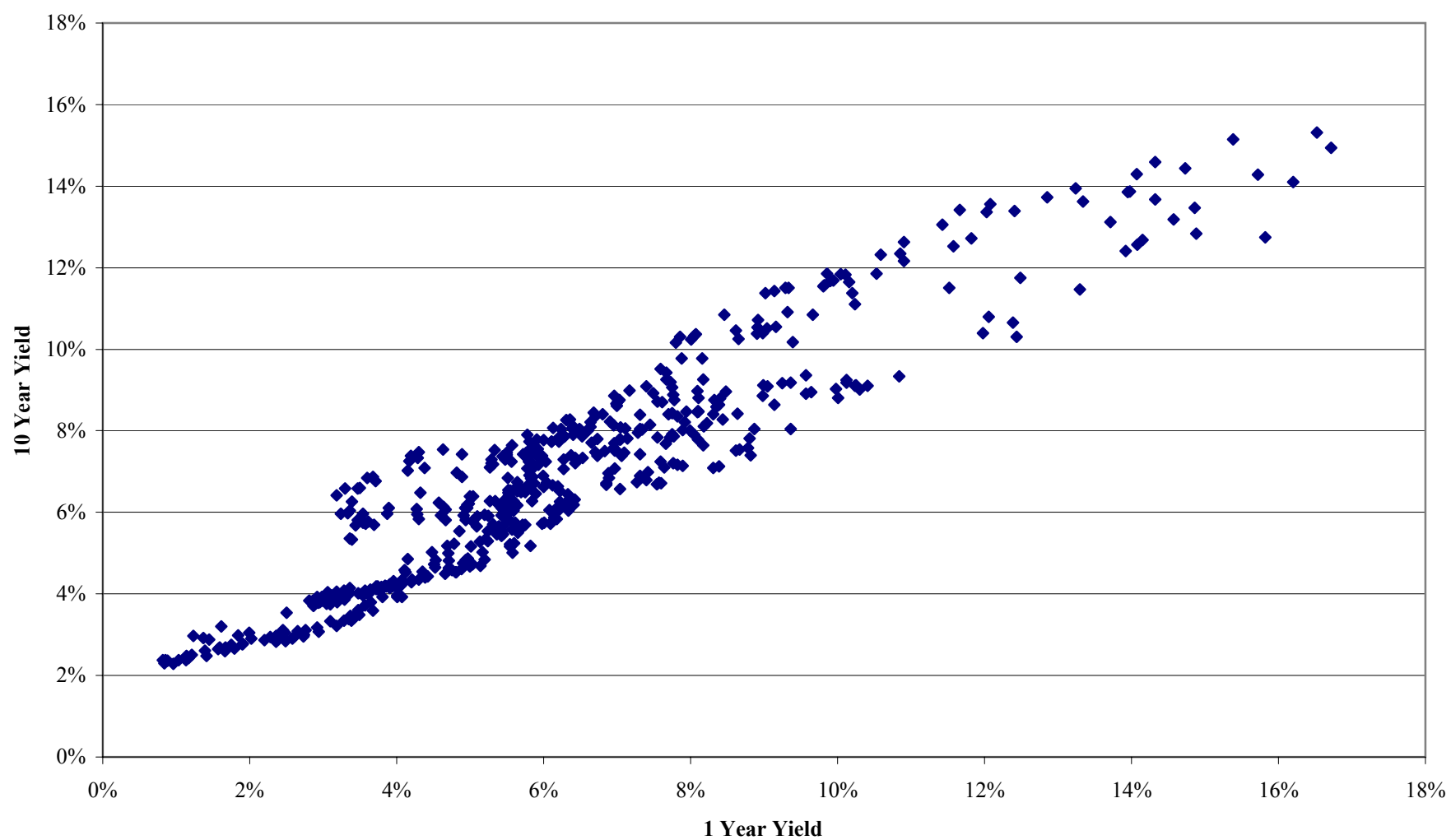
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[2] The continuous time version of the Ho and Lee (1986) model is a special case of the Hull and White (1990) one-factor model. It refers to the case where the mean reversion parameter (*a*) is equal to zero.
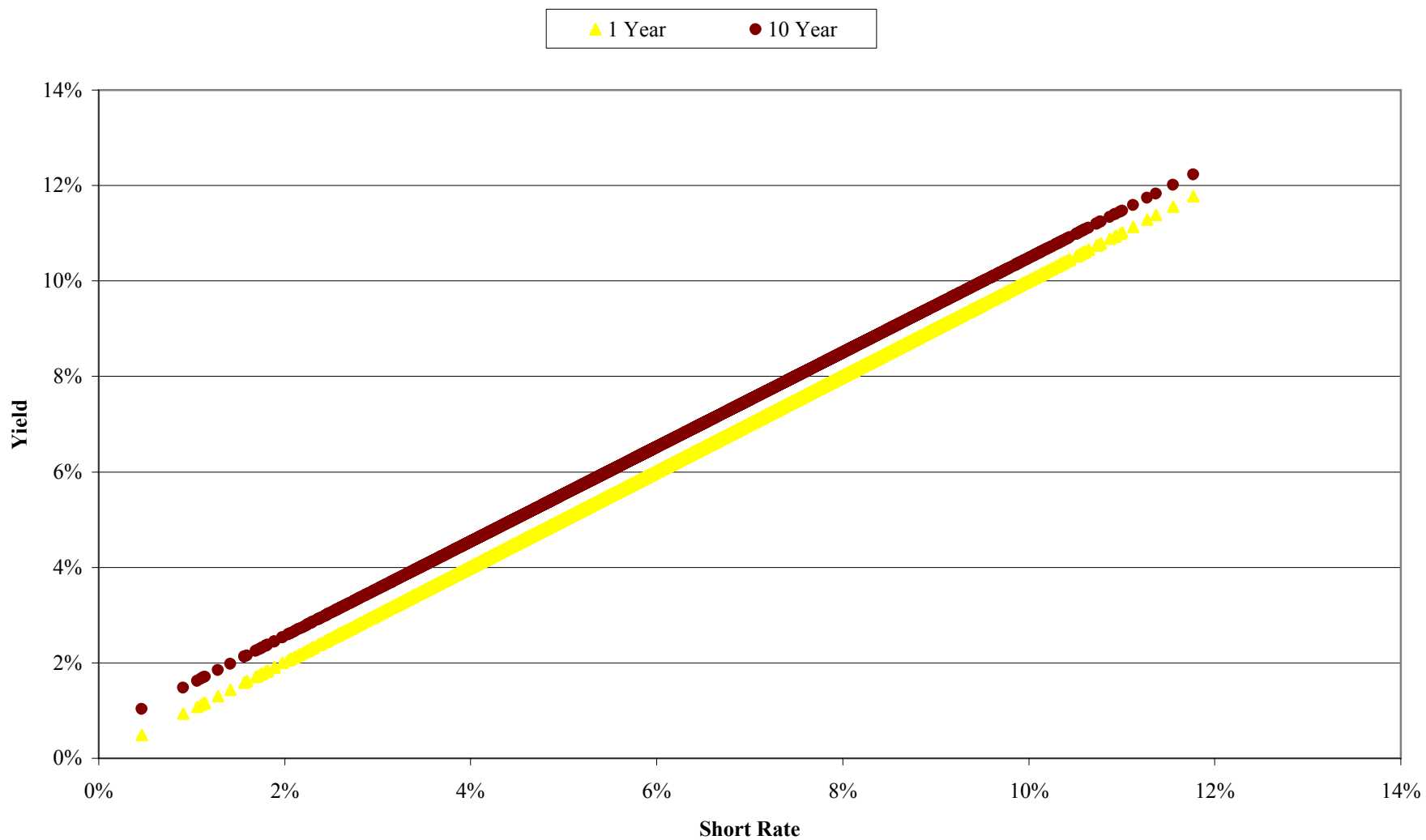
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EXHIBIT 1

Historical Yield Relationship
Monthly Data 1953-2000

EXHIBIT 2

Relationship Between Yields - One Factor Model

▲ 1 Year   ● 10 Year

EXHIBIT 3

Monthly Yield Correlations

1953-2000

Yield	1 Year	3 Year	5 Year	10 Year
1 Year	1.000			
3 Year	0.984	1.000		
5 Year	0.968	0.996	1.000	
10 Year	0.943	0.984	0.995	1.000

Yield Correlations by Decade

1953-1959

Yield	1 Year	3 Year	5 Year	10 Year
1 Year	1.000			
3 Year	0.989	1.000		
5 Year	0.976	0.997	1.000	
10 Year	0.943	0.977	0.991	1.000

1960-1969

Yield	1 Year	3 Year	5 Year	10 Year
1 Year	1.000			
3 Year	0.994	1.000		
5 Year	0.987	0.997	1.000	
10 Year	0.976	0.987	0.996	1.000

1970-1979

Yield	1 Year	3 Year	5 Year	10 Year
1 Year	1.000			
3 Year	0.971	1.000		
5 Year	0.938	0.992	1.000	
10 Year	0.865	0.943	0.971	1.000

1980-1989

Yield	1 Year	3 Year	5 Year	10 Year
1 Year	1.000			
3 Year	0.976	1.000		
5 Year	0.954	0.996	1.000	
10 Year	0.924	0.982	0.995	1.000

1990-1999

Yield	1 Year	3 Year	5 Year	10 Year
1 Year	1.000			
3 Year	0.945	1.000		
5 Year	0.829	0.964	1.000	
10 Year	0.670	0.866	0.967	1.000

2000

Yield	1 Year	3 Year	5 Year	10 Year
1 Year	1.000			
3 Year	0.888	1.000		
5 Year	0.855	0.991	1.000	
10 Year	0.756	0.926	0.964	1.000

EXHIBIT 4

Relationship Between Yields - Two Factor Model
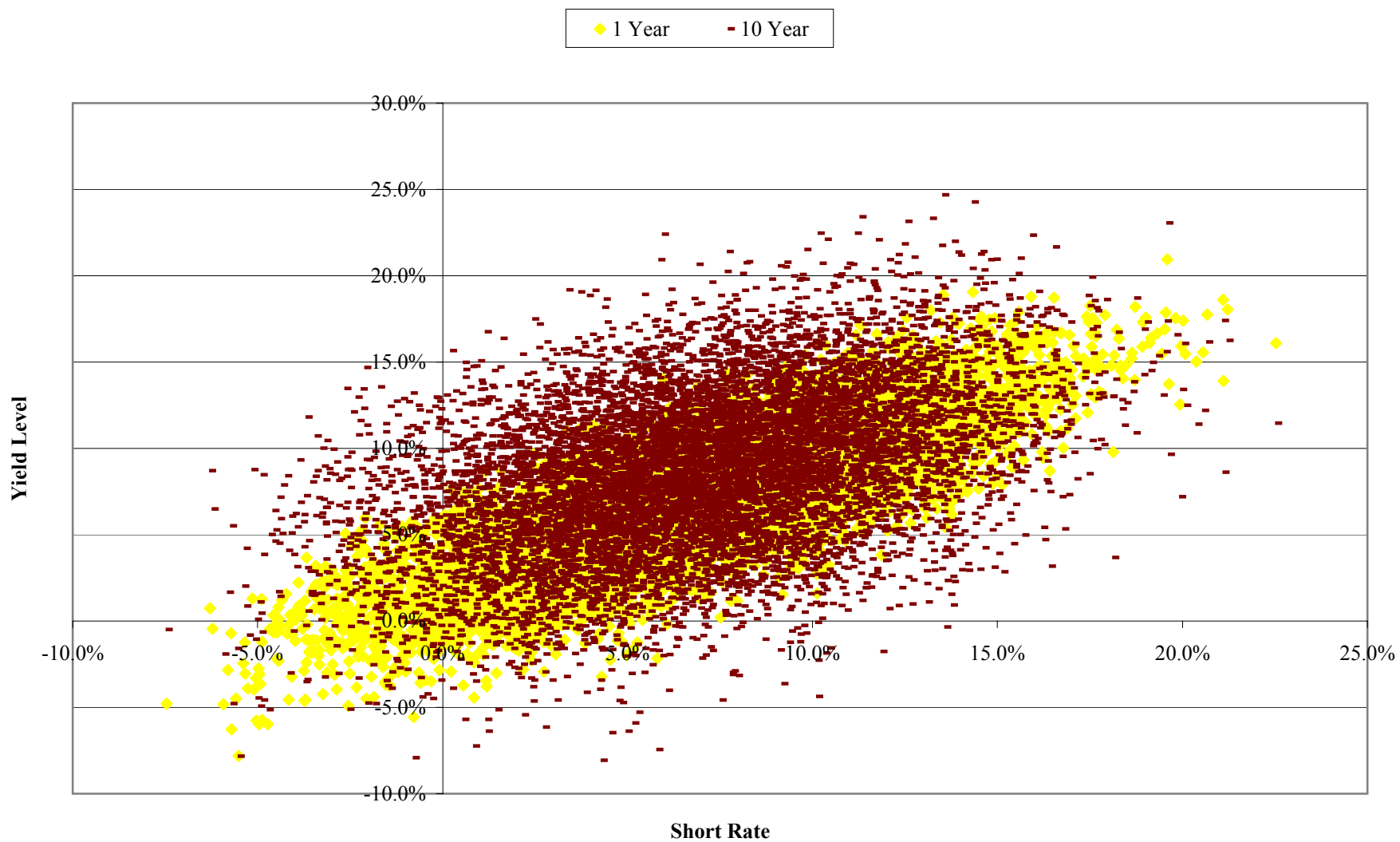
distribution, insurers can set aside appropriate surplus to protect policyholders against adverse circumstances. Insurance regulators may also use the projected distribution of surplus when defining capital requirements. For example, a regulator may require that the insurer set aside a level of capital such that at least 99% of the time, the insurer will remain solvent over the next six months to a year. Using higher probabilities of solvency and/or longer time horizons will lead to higher capital requirements.[11] This approach is similar to the value at risk (VaR) methodology that is often used in the banking industry.[12] Thus, an additional benefit of using DFA is that the methodology is similar to other financial institutions. The comparison between different companies is an important benefit given the increasing merger activity in the financial services industry.

Since both the assets and liabilities of insurers are tied to the level and movement of interest rates, it is an important assumption for any DFA application. This paper investigates whether alternative interest rate assumptions affect the distribution of outcomes for an insurer, which may subsequently affect capital requirements. The next section discusses some of the issues to consider when choosing an interest rate assumption and the following sections apply these concepts to P-L insurers using a DFA model.

4. TERM STRUCTURE ISSUES

4.1 Choosing A Term Structure Model

A term structure model is a tool that is used to simulate paths of interest rates over time. Choosing the specific form of a term structure model is not a trivial decision, especially given the large variety of approaches that are available. When selecting a model, there is typically a

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[11] For an application of this approach applied to a life insurance liability, see Ahlgrim (1999).
[12] For a history of the use of VaR in banking, see Jorion (1998). One should note that it is likely that the time horizon for insurance companies is considerably longer than for banks. Panning (1999) discusses many issues related to using VaR over long horizons.

tradeoff between accurately describing the historical interest rate process and having a tractable model that can be used to value financial instruments.

One important consideration in choosing among alternative term structure models is in determining the number of factors that are subject to variation. Early approaches typically identified one stochastic factor – the short-term rate. In continuous time models, this means the instantaneous short-term rate; in discrete time, this factor is the yield on short-term bonds such as one day, one week, or one month. Yields on longer maturity bonds are determined from the expected evolution of the short-rate over the maturity of the corresponding bond.

One-factor term structure models may place too much rigidity on the yield curve movements. Specifically, the short-term rate uniquely determines all other points on the yield curve; i.e., long-term yields are a one-to-one mapping of the short-term rate. For example, in a one-factor model, when the short-term rate in a one-factor model is 5%, the parameters may specify that the 10-year yield is 7%. With a one-factor model, there is no other possible 10-year yield when the short rate is 5% (see Chapter 1, Exhibit 2). But empirical results illustrate that yield curves can have a variety of shapes when the short rate is 5%: it can be upward sloping, downward sloping, flat, or humped (see Chapter 1, Exhibit 1). The one-factor term structure model implies a rigid yield curve dynamic that is inconsistent with the variability exhibited in the market. Since the entire yield curve dynamics evolve from a single factor, yields of all maturities are perfectly, positively correlated. Although historical statistics do indicate that yields of various maturities are positively correlated, the correlation is not 1.0 (see Chapter 1, Exhibit 3).

Despite their criticisms, a single-factor model does have an important feature – simplicity. Employing one stochastic factor is usually easier to understand conceptually and the numerical methodologies used to value interest rate contingent claims are cleaner when only one process is included. In addition, estimating the parameter values of multifactor models is significantly more challenging than in the one factor case. Empirical evidence also shows that the

parameter estimates are less stable than lower dimensional models (see Chapter 1, Section 6 and

Amin and Morton (1994)).

By introducing additional stochastic factors to a term structure model, interest rate

dynamics can be significantly enhanced. Depending on the particular financial application, the

benefits of incorporating multiple factors may not offset the costs of complexity with these

models. If the increased complexity produces more accurate valuation results, the additional

work may be worthwhile. However, if the additional factor indicates minor deviations from the

one-factor values, then users may be better served to use the simpler approach.

4.2 Term Structure Literature Review

Subrahmanyam (1996) provides an overview of the relevant issues in the area of interest

rate modeling, from conceptual foundations of alternative models to empirical issues and

estimation. Ahlgrim, D'Arcy, and Gorvett (1999) discuss the features of several popular term

structure models, simulate different models, and provide statistics of both historical yield

movements and the simulations.

One of the earliest approaches to term structure modeling is the one-factor model of

Vasicek (1977):

$$dr_t = \kappa(\theta - r_t)dt + \sigma dB_t \qquad (2.1)$$

where r_t = current level of the short-term rate
θ = long-run mean level of the short-term rate
κ = coefficient (strength) of mean reversion
σ = instantaneous volatility of the short-term rate
B = a standard Wiener process

Similar to many interest rate models, the Vasicek model is stated in terms of changes in the

instantaneous spot rate (r_t). As time passes, there are two contributions to interest rate shifts. The

first term is called the drift. It represents the deterministic (or expected) movement in the interest

rate over the next instant dt. The drift is mean-reverting so that if the level of the interest rate at

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time t increases above some level θ, the drift term becomes negative implying that rates are then expected to decline. The second component of interest rate changes is derived from the stochastic movement of a Brownian motion; this second term represents the volatility of interest rates. In the Vasicek model, the random component is scaled by a constant, which indicates that the (time) conditional volatility of interest rate changes is constant. One shortfall of constant volatility in the Vasicek model is that interest rates can be negative.

Cox, Ingersoll, and Ross (1985) [hereafter CIR] introduce a volatility factor that is related to the level of interest rates:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\,dB_t \tag{2.2}$$

Similar to Vasicek, CIR incorporates mean reversion. But in the CIR model as interest rates decline, the volatility component is dampened. Since the process is continuous, this rules out the possibility of negative interest rates.

Vasicek and CIR are general equilibrium models. General equilibrium models formulate bond yields based on the expectations of investors in the economy. One of the major benefits of these models is that interest rate movements and securities' prices frequently have convenient closed-form solutions. On the negative side, general equilibrium models are criticized because the yield curves produced by the models do not replicate the term structure built from existing market prices. This makes these models unsatisfactory for pricing derivative securities. Hull (2000) points out that if practitioners cannot rely on the model to accurately portray the existing term structure, they will have little confidence that the model will accurately imitate the dynamics of the yield curve.

Hull and White (1990) extend the Vasicek approach to allow the model to fit the initial term structure. The Hull-White model one-factor model is:

$$dr_t = (\theta(t) - ar_t)dt + \sigma dB_t \tag{2.3}$$

Because of the similarity to the model proposed by Vasicek (1977), the Hull-White model is also called the extended-Vasicek model. (A similar formulation corresponding to CIR is known as the extended-CIR model). The similarity between the Hull-White one-factor model and the Vasicek model can be seen explicitly by rewriting the Hull-White model as:

$$dr_t = a(\frac{\theta(t)}{a} - r_t)dt + \sigma dB_t \qquad (2.4)$$

Here, a is the mean reversion parameter (κ in the Vasicek model) and σ is the (constant) volatility. The important feature of the model is that the drift term has a time dependent factor $\theta(t)$, so that the yield curve derived from the model is consistent with the existing market prices of bonds.

As in most one-factor models, the prices of zero-coupon bonds $P(t,T)$ are determined by the expected evolution of the short term rate over the maturity of the bond. Hull and White (1990) provide closed-form solutions for bonds prices as follows:

$$P(t,T) = A(t,T)e^{-B(t,T)r} \qquad (2.5)$$

where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \qquad (2.6)$$

and

$$\ln A(t,T) = \ln \frac{P(0,T)}{P(0,t)} - B(t,T)\frac{\partial \ln P(0,t)}{\partial t} - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1) \qquad (2.7)$$

Defining the $R(t,T)$ as the continuous compounded yield from time t to maturity T, it follows:

$$P(t,T) = e^{-R(t,T)(T-t)} \qquad (2.8)$$

Therefore, the yields of all maturities can be determined by rearranging (2.5) and (2.8):

$$R(t,T) = -\frac{\ln A(t,T) - B(t,T)r_t}{T - t} \qquad (2.9)$$

The Vasicek, CIR, and one-factor Hull-White models incorporate only one stochastic factor and all bond prices are subsequently related to the level of the short-term rate. As a result, bond yields of all maturities are perfectly correlated, severely constraining implied yield curve movements. This feature is particularly unsatisfactory when evaluating securities that may be sensitive to the shape of yield curve shifts. (For a demonstration of this result, see the simulation in Chapter 3). To alleviate the problem of correlated bond yields, a model can incorporate two or more stochastic factors. By modeling several factors, the potential range of yield curve dynamics is enhanced.

Heath, Jarrow, and Morton (1992) [HJM] provide the most general approach to multifactor interest rate modeling. HJM simulate the evolution of forward rates as follows:

$$df(t,T,f(t,T)) = \mu(t,T,f(t,T))dt + \sigma(t,T,f(t,T))dB_t \tag{2.10}$$

where μ and σ are the drift and volatility parameters of the forward rate process. HJM show that the drift of the forward rate is related to the volatility structure.

$$\mu(t,T,f(t,T)) = \sigma(t,T,f(t,T))\int_t^T \sigma(t,s,f(t,s))ds \tag{2.11}$$

Note that the HJM approach is very general. In particular, the Brownian motion may be a vector process, introducing multiple stochastic factors to drive the evolution of the forward rates. Also the structural form of the volatility parameter provides additional flexibility in setting the yield curve dynamics. Many models are special cases of the HJM approach, including Vasicek and Ho and Lee (1986).

In the two-factor model described in Brennan and Schwartz (1979, 1982), one factor is used to represent the short-term rate while the other factor is the rate on a perpetual bond. Hull and White (1994) extend the approach by fitting the two-factor model to the initial yield curve. The two-factor Hull-White model is:

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$$dr_t = (\theta(t) + u_t - ar_t)dt + \sigma_1 dB_1 \qquad (2.12)$$

$$du_t = -bu_t dt + \sigma_2 dB_2 \qquad (2.13)$$

$$\rho\ dt = dB_1 dB_2 \qquad (2.14)$$

In the Hull-White two-factor model, the instantaneous interest rate reverts to a long-term rate that includes a stochastic component (u_t). The volatility of the short-term rate (r_t) and the mean reversion component (u_t) are σ_1 and σ_2, respectively. There are two mean reversion coefficients: a is the short-term mean reversion speed and b is the long-term mean reversion. When a is high and b is low, the Hull-White two-factor model produces a yield curve where short-term yields are more volatile than long-term yields, which is consistent with historical statistics (see Ahlgrim, D'Arcy, and Gorvett (1999)).

Empirical evidence supports the need for additional dimensions of yield curve shifts as represented by a two-factor model such as Hull-White. Litterman and Scheinkman (1991) use principal components analysis to show that although 90% of changes in yields are explained by a parallel shift, changes in the slope of the yield curve are also significant. Their findings demonstrate that the relationship between the short-term and long-term rates is not as rigid as is implied by one-factor term structure models. Chapman and Pearson (2001) provide subperiod analysis to show that the result of Litterman and Scheinkman (1991) are persistent over time. As discussed in the next section, sensitivity to other factors, including yield curve slope, may have specific financial consequences for insurers.

~~5. MULTIFACTOR TERM STRUCTURE MODELS IN P-L INSURANCE~~

~~The major source of risk for P-L insurers is catastrophic losses, but interest rate risk is also a significant factor. Although prior research in insurance valuation has recognized the need for incorporating a stochastic interest rate assumption in financial analyses, recent applications~~

6.2 Selecting the Parameters for the Term Structure Models

The ability to differentiate among the many term structure models is an important financial exercise. Using an implausible structural model or selecting inappropriate parameters may lead to results that are misleading, causing the end user to make inappropriate financial decisions. For insurers who are using the models to evaluate risk, the use of a model that does not reflect realistic future interest rate dynamics may lead to lower risk measures and inadequate surplus to protect against adverse experience.

There are two approaches that are generally used to evaluate and compare the performance of alternative term structure model specifications. Each of these techniques is analogous to determining the optimal parameters for term structure models. In the first approach, the models are measured against historical interest rate movements. The user selects the best parameter values based on statistical estimation from history. For example, the speed of mean reversion can be measured by looking at the magnitude and direction of interest rate changes. Estimation is straightforward since most of the popular models are formulated in terms of interest rate changes. Papers that use this approach include Chan, Karolyi, Longstaff, and Schwartz (1994), Conley, Hansen, Luttmer, and Scheinkman (1997), Pearson and Sun (1994), Aït-Sahalia (1996), and Stanton (1997). Most studies narrow the candidate models by selecting the term structure model(s) that most accurately fits the historic data – this model is considered "best". Chapman and Pearson (2001) provide a summary of the important findings and performance of various term structure models, using the historical interest rate path as a benchmark of performance. In particular, they focus on the relationship between mean reversion and the level of interest rates, as well as the most appropriate form for the volatility of interest rate changes.

Selecting the appropriate model based on its compatibility with historical yield movements is an approach that works well if the underlying dynamics of interest rate movements are constant over time. If this is the case, then the term structure model will be useful for a variety of purposes because it provides for an accurate representation of future interest rate

scenarios. However, as pointed out in Chapman and Pearson (2001), there can be some historical periods that may have a significant impact on a model's ability to replicate historical interest rate paths. In particular, many models perform poorly because of the period from 1979-1982 when interest rates remained at higher levels than any time prior or since. The possibility of a structural break in the interest rate process (e.g., the period from 1979-1982) introduces a potential hazard for using term structure models that were based on history.

In the second approach for comparing term structure models (and for estimating parameter values), each of the competing models is used to price a fixed set of securities. Modeled prices can then be compared to observed market prices. The model that most closely replicates security prices is deemed the best model. This approach, termed calibration, has been used in Bhanot (1998), Jegadeesh (1998), Mathis and Bierwag (1999), Archer and Ling (1995), and Chen and Yang (1995). Each of these papers was focused on a specific market. For example, Bhanot (1998) and Mathis and Bierwag (1999) used the market for options on Eurodollar (ED) futures contracts to see which term structure model was the most accurate for valuing these options. Jegadeesh (1998) looks at interest rate caps and Archer and Ling (1995) and Chen and Yang (1995) look at mortgage-backed security prices. When the ultimate application of the term structure model corresponds well with the underlying market used to calibrate competing models, the approach is quite plausible. Thus, even though it may appear that market prices of securities do not reflect the historical distributions of interest rate movements, the use of a calibrated model that reflects these prices is internally consistent. The underlying assumption is that there is some interest rate process that is _implied_ by the set of market prices, so that in using the implied interest rate dynamics, one should be able to accurately price securities that are very similar to the instruments used for calibration.

But what if the resulting term structure model is not used to value the same types of securities? For example, is it appropriate to use the dynamics implied by the ED futures options market and apply the result to price mortgage-backed securities (MBS)? This is an important

issue in insurance applications because no liquid secondary market exists for insurance liabilities. Therefore, there is no clear choice for selecting the comparative benchmark market to calibrate and compare the performance of different term structure models. Deciding on the most appropriate model for insurance is complicated by this constraint.

6.3 Comparing Term Structure Models

This research does not attempt to explicitly identify the one, "best" term structure model for evaluating the risk to insurers. Instead, it attempts to understand and illustrate the economic significance of using a multifactor model for projecting capital requirements given the interest rate risk exposure. In order to generate projections of surplus requirements under multiple term structure models and to illustrate the importance of multiple factors in insurance applications, it is important to adequately control the analysis and allow for the most reasonable comparison between competing models. If possible, it would be beneficial to compare models that are reasonably consistent so that the importance of the second stochastic factor can be explicitly evaluated. There are three possible definitions for determining the consistency between models. First, we can insure that competing models are matched to historical data. Second, we can use a specific set of market prices and calibrate each of the models to security data. Given the difficulties associated with these two approaches that were mentioned above, a third measure of consistency can be based on a direct comparison of the interest rate dynamics implied by each of the models.

Of course, if two term structure models have different structural forms (such as one-versus two-factors), the resulting estimates of the parameter values will also be different; correspondingly, the distribution of interest rates at various times in the future will not be identical and the application of the competing models will yield different values for interest rate contingent claims. To illustrate the shortfall of the first two approaches and the suggested remedy, consider Exhibits 4 through 7, which are similar to Exhibits 12 through 15 from

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Chapter 1. In Chapter 1, a one-factor model and a two-factor model were calibrated to market data. Although the models are each internally consistent with the market prices of ED futures options, the resulting distributions of future interest rates are markedly different. A comparison between Exhibits 4 and 5 illustrate that the distribution of the 1-year yield under the two-factor model is much wider than under the one-factor assumption and that the discrepancy holds across time. Exhibits 6 and 7 confirm that the difference in future interest rates carries over to the resulting 10-year yield.

The disparity between future interest rate distributions will have an impact on any application that uses the calibrated models. For the insurance application considered here, it may appear that that two-factor term structure model does in fact indicate a higher level of risk (as hypothesized) simply because the volatility of interest rates is much higher during the projection years. But it is unclear whether the result is attributed to differences in yield curve volatility or due to the addition of a second stochastic factor.

However, if the two-factor model does indicate a higher level of risk, this result in itself would be an important finding even without explicitly comparing the interest rate dynamics under the competing models. It indicates that calibration may not be the best procedure to choose parameters for a term structure model, especially if there is a difference between the market data used for calibration and the application of the model. At a minimum, it suggests that users of calibrated models should be aware of how they were calibrated and what the implied interest rate dynamics look like. For example, it may be that individuals in practice may simply choose to calibrate a two-factor model to market data and incorporate the model in their application, unaware that the interest rate dynamics are completely different than if they used the same set of market prices to calibrate a one-factor model.

One approach that may be used to get around the built-in differences between various term structure models is to explicitly relate the underlying interest rate distributions. Although the structural differences between term structure models will not allow future interest rate

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scenarios to be perfectly consistent across time and at all maturities, we can attempt to minimize the differences by looking at specific distributions of future interest rates under each of the term structure models.

For the insurance application presented here, matching the volatility of future interest rates appears to be a useful approach. It has been argued that the one-factor models may not capture the risks associated with changes in yield curve slope. This is a statement of the relationship of interest rate changes over time. To directly test this issue, we can attempt to (approximately) match the volatilities of future interest rates at a future point in time for the one- and two-factor models. This assures that any difference in risk measures is driven by the variation in yield curve slope and not by differences in the volatility between models.

To generate term structure models that are consistent with historical interest rate data, market prices, or interest rate distributions, one must hold one (or several) dimension(s) constant and fit competing models to these constraints. For example, when estimating parameters from historical data, the unique time series path of interest rates becomes fixed through time. The relevant measure of fit is then based on the relationship between the term structure dynamics and the fixed historical interest rate process. With calibration, the models are individually optimized to fit a fixed set of securities prices.

There are limits to making the models consistent given that they have very different structural forms. Since the one-factor model has two parameters, one approach is to choose those parameters to equate the distributions of future interest rates between the one-factor Hull-White model and the two-factor Hull-White model along two dimensions.

For this research, two separate term structure comparisons are used to evaluate the risk in P-L insurance companies. For the first comparison, both the one- and the two-factor models are directly calibrated using ED futures options prices (this calibration process is described below). Thus, the two competing models are "consistent" in that they are both are based on the same market data. However, as noted above, the interest rate dynamics between the resulting models

are quite different. The top two panels of Exhibit 8 provides statistics to confirm that the standard deviation of the distribution of interest rates is much higher than under the one-factor model. In the second comparison, the one-factor parameters are chosen to replicate the higher volatility of the two-factor Hull-White model. Thus, if there are differences in the results from the first comparison that are driven by the inconsistency between interest rate volatility, the second comparison attempts to control for this distinction by "equating" the interest rate dynamics between the two models.

To create a volatility-matched one-factor Hull-White model, the parameters of the model are chosen to match the volatility (standard deviation) of the one- and ten-year yields of the two-factor model during the fifth projection year; i.e., these are the two dimensions upon which the interest rate dynamics are consistent. Hull and White (1990) provide closed-form solutions for the volatility of bond prices for their one-factor model

$$\frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right) \tag{2.16}$$

The one- and ten-year yields were chosen since they provide information about the slope of the yield curve. The fifth projection year was chosen because this is the final projection year that is used in this analysis. Note that although the distributions of the one- and ten-year yields are (approximately) equal, the correlation between the yields is not the same. By comparing the volatility-matched one-factor model with the results of the two-factor calibrated model, the impact of the slope on insurer risk can be accurately measured.

Exhibits 9 and 10 present the distribution of the one- and ten-year yields for the volatility-matched one-factor Hull-White model. From casual inspection, it can be seen that these distributions are more closely related to the two-factor interest rate dynamics (shown in Exhibits 5 and 7) than the calibrated one-factor model (Exhibits 4 and 6). The bottom panel of Exhibit 8 provides summary statistics to confirm the improved relationship between the volatility-matched one-factor model and the two-factor model.

EXHIBIT 4

Distribution of 1-Year Yield
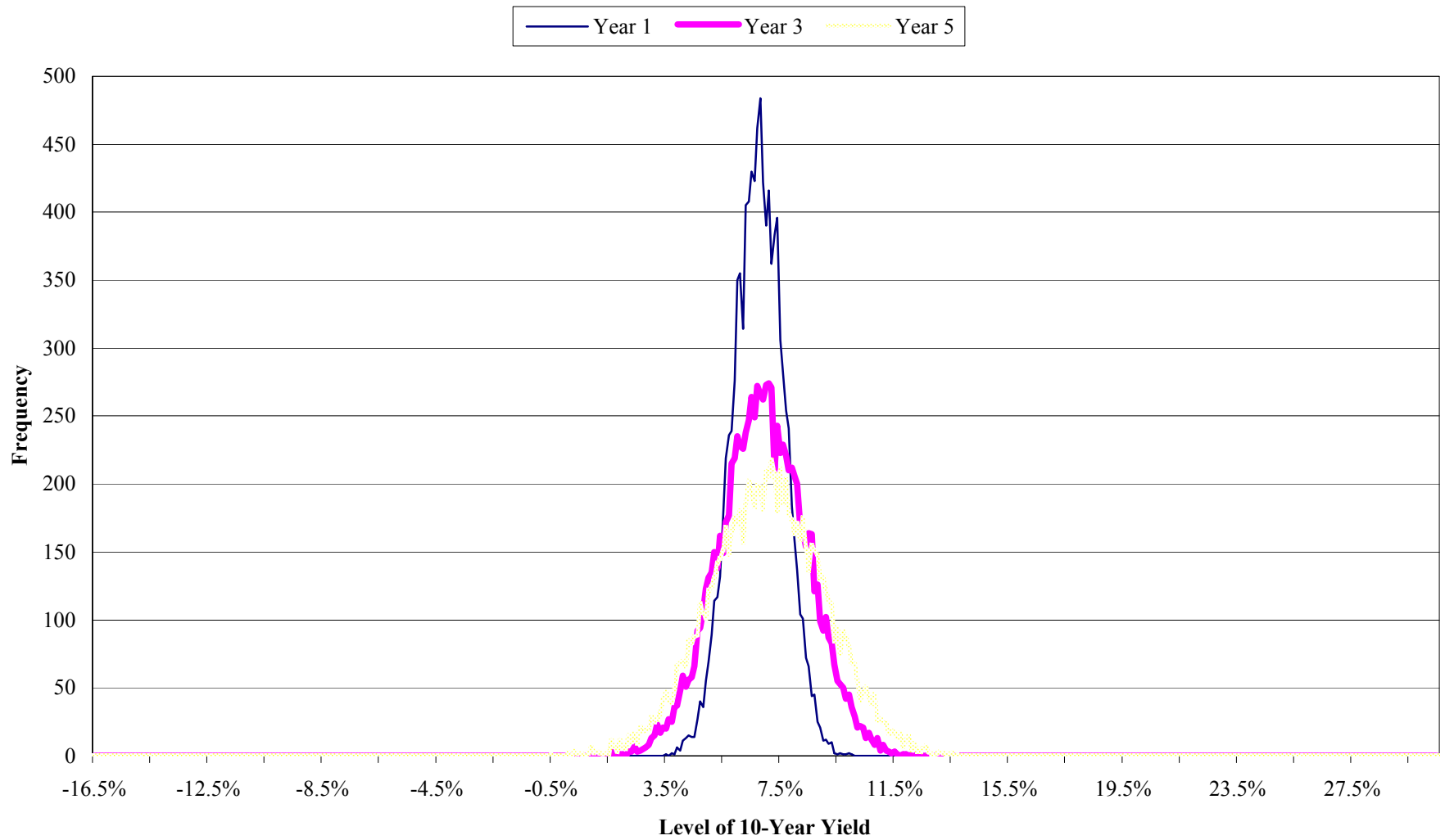*Calibrated One-Factor Hull-White*
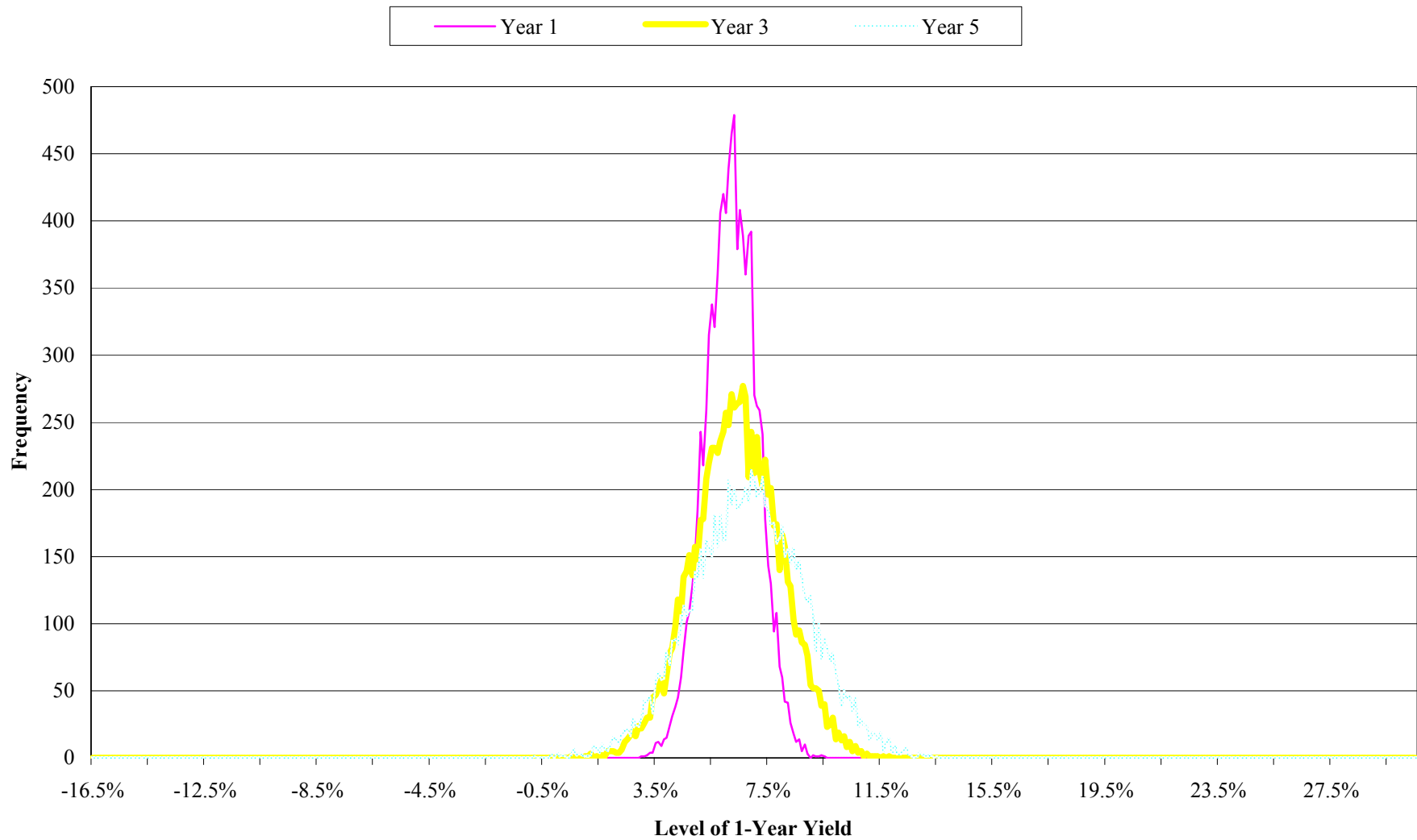
EXHIBIT 5
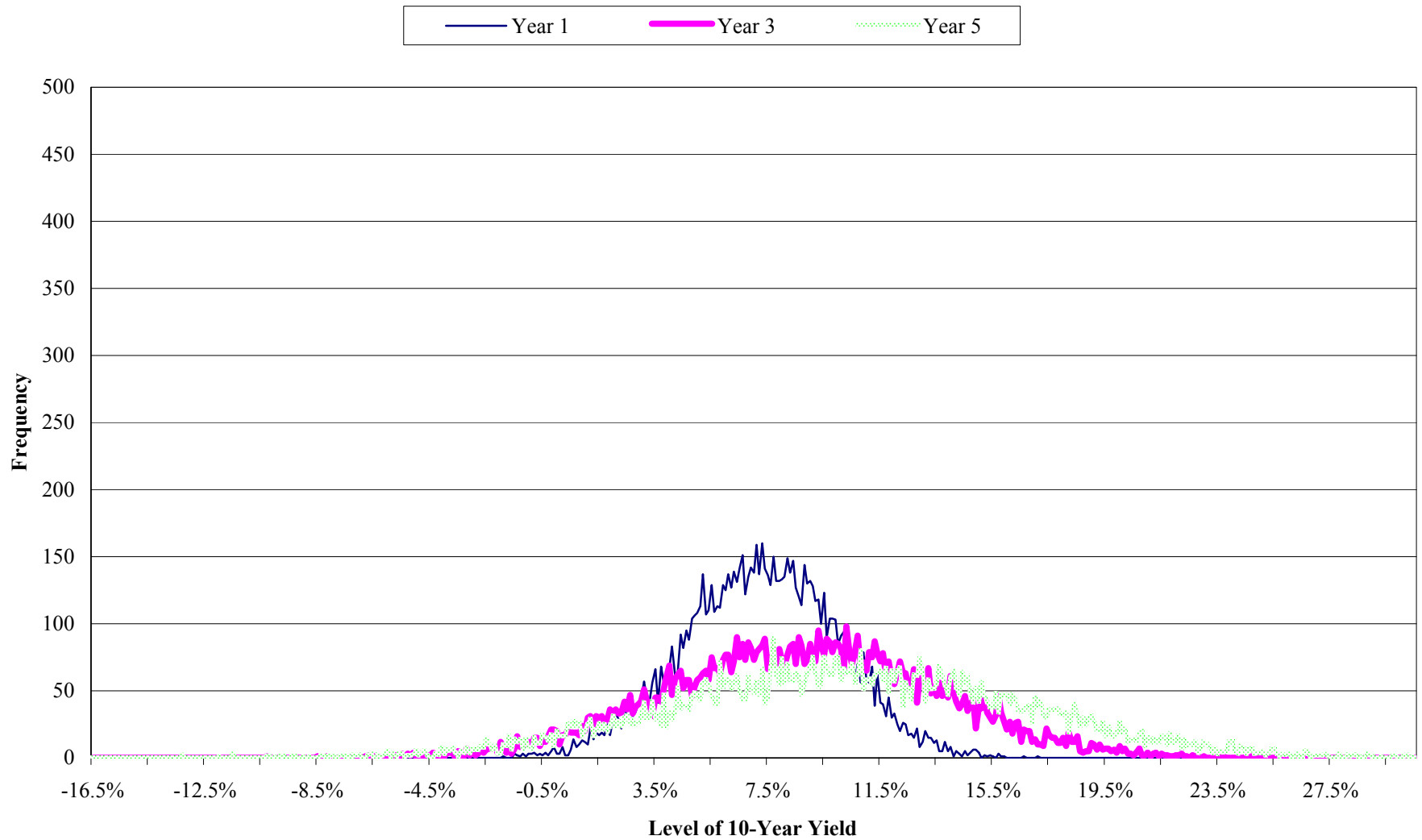
Distribution of 1-Year Yield
Two-Factor Hull-White
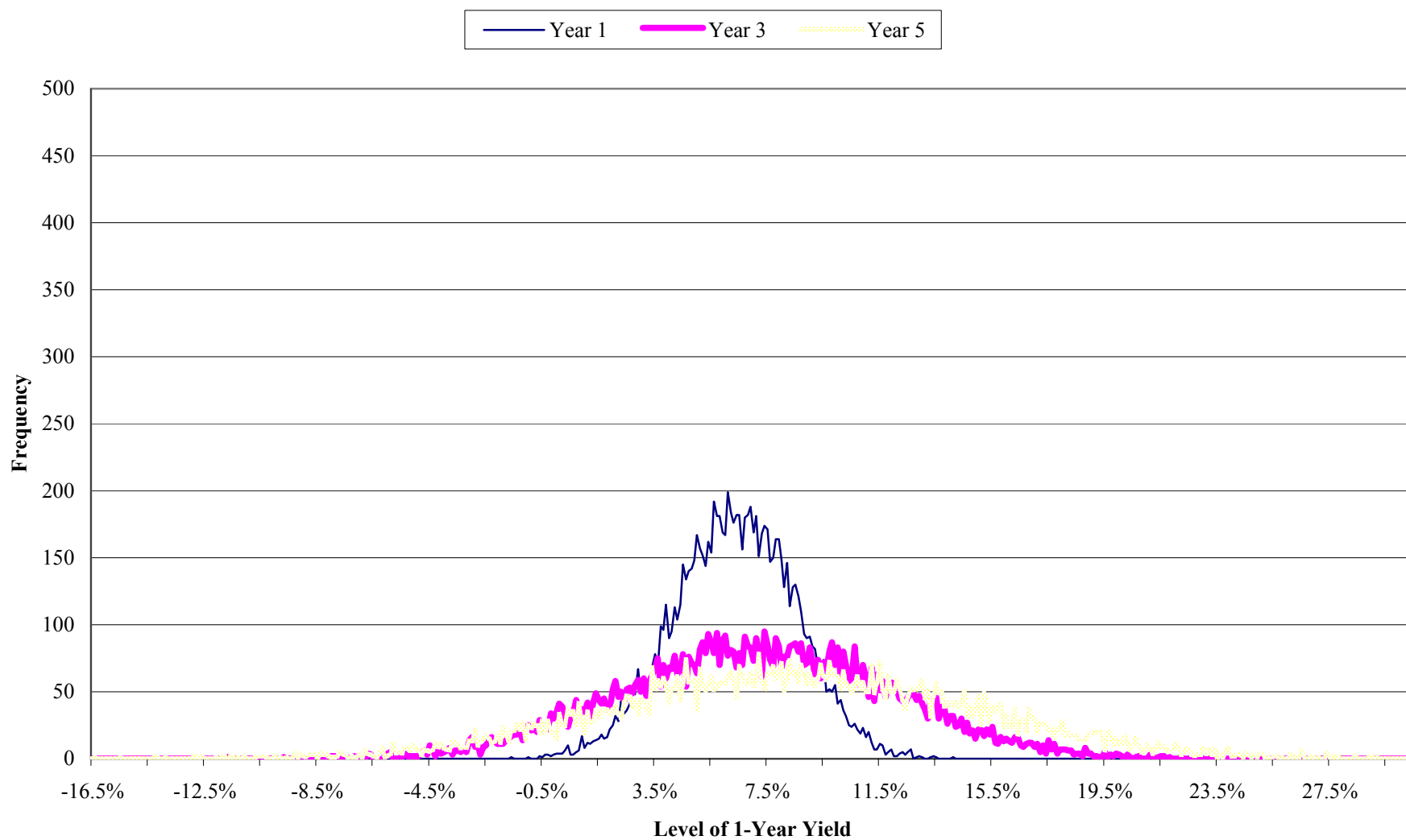
EXHIBIT 6

Distribution of 10-Year Yield
Calibrated One-Factor Hull-White

EXHIBIT 7

Distribution of 10-Year Yield
Two-Factor Hull-White

EXHIBIT 8

Yield Statistics of Term Structure Models
10,000 Simulations

Calibrated One-Factor

	1-Year Yield				10-Year Yield		
	Year 1	*Year 3*	*Year 5*		*Year 1*	*Year 3*	*Year 5*
Mean	6.14%	6.34%	6.75%		6.68%	6.84%	7.00%
Median	6.16%	6.34%	6.76%		6.69%	6.83%	7.01%
Std Dev	0.90%	1.54%	2.01%		0.89%	1.53%	1.99%
Coeff of Var	0.1466	0.2436	0.2972		0.1337	0.2238	0.2843
Skewness	-0.0561	-0.0255	-0.0343		-0.0561	-0.0255	-0.0343
Kurtosis	-0.0278	-0.0570	-0.0428		-0.0278	-0.0570	-0.0428
Min	2.90%	0.48%	-0.84%		3.46%	1.04%	-0.53%
Max	9.45%	11.78%	13.42%		9.96%	12.23%	13.61%

Two-Factor

	1-Year Yield				10-Year Yield		
	Year 1	*Year 3*	*Year 5*		*Year 1*	*Year 3*	*Year 5*
Mean	6.28%	7.03%	8.37%		7.20%	8.61%	10.33%
Median	6.31%	7.04%	8.33%		7.24%	8.65%	10.28%
Std Dev	2.15%	4.79%	6.37%		2.74%	4.79%	6.25%
Coeff of Var	0.3427	0.6813	0.7606		0.3799	0.5566	0.6048
Skewness	-0.0771	-0.0412	-0.0259		-0.0779	-0.0351	-0.0209
Kurtosis	-0.0584	-0.0780	-0.0145		-0.0646	-0.0712	-0.0068
Min	-1.63%	-11.73%	-16.41%		-3.53%	-10.30%	-13.39%
Max	14.02%	24.64%	29.12%		17.08%	25.62%	30.39%

Volatility-Matched One-Factor

	1-Year Yield				10-Year Yield		
	Year 1	*Year 3*	*Year 5*		*Year 1*	*Year 3*	*Year 5*
Mean	6.26%	6.86%	8.00%		7.10%	8.24%	9.69%
Median	6.29%	6.85%	8.04%		7.13%	8.23%	9.72%
Std Dev	2.80%	4.78%	6.20%		2.74%	4.68%	6.06%
Coeff of Var	0.4472	0.6975	0.7741		0.3855	0.5676	0.6253
Skewness	-0.0561	-0.0254	-0.0341		-0.0561	-0.0254	-0.0341
Kurtosis	-0.0278	-0.0568	-0.0430		-0.0278	-0.0568	-0.0430
Min	-3.82%	-11.27%	-15.47%		-2.76%	-9.49%	-13.26%
Max	16.53%	23.69%	28.61%		17.15%	24.70%	29.84%

EXHIBIT 9

Distribution of 1-Year Yield
Volatility-Matched One-Factor Hull-White
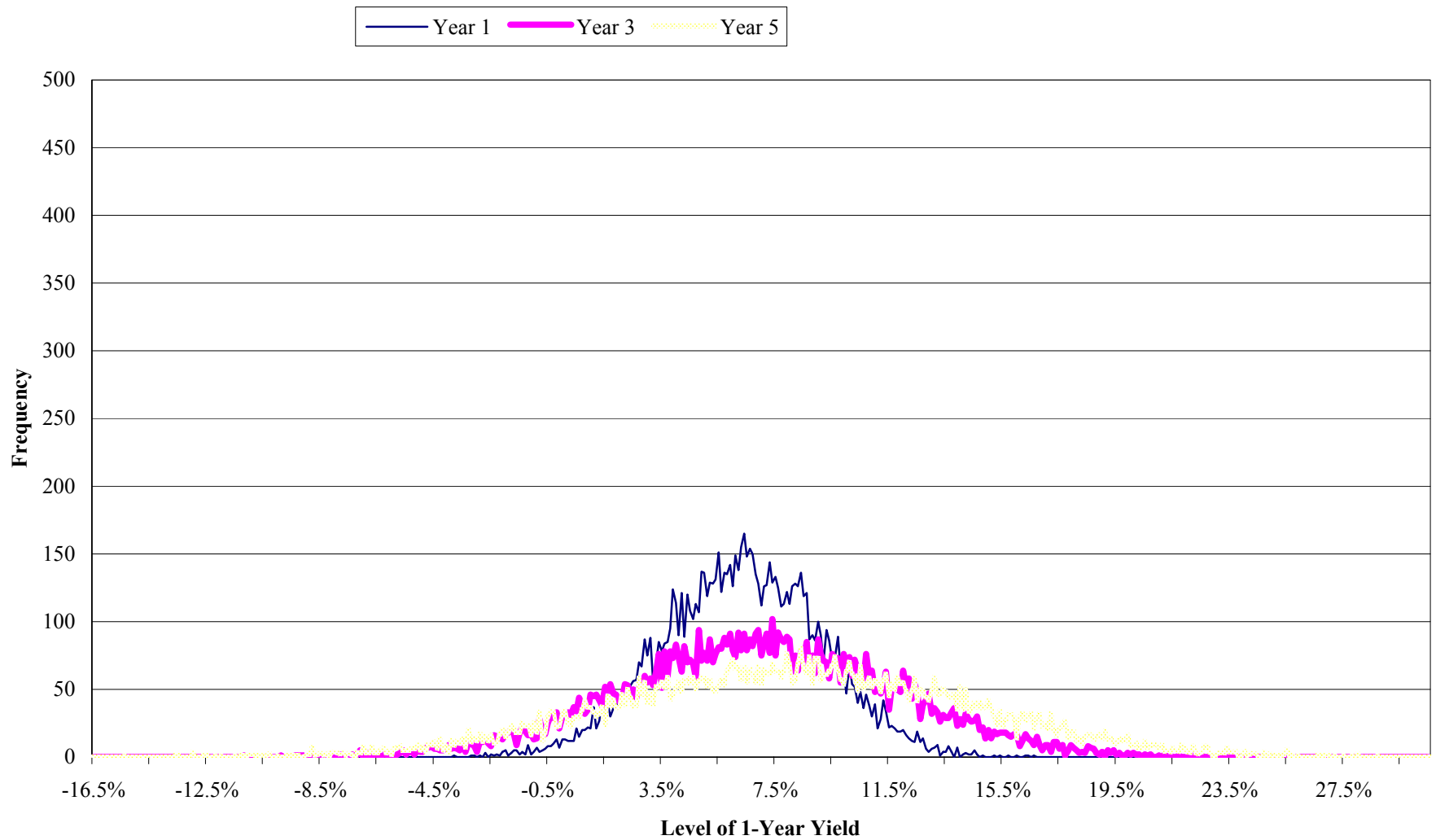
EXHIBIT 10

Distribution of 10-Year Yield
Volatility-Matched One-Factor Hull-White
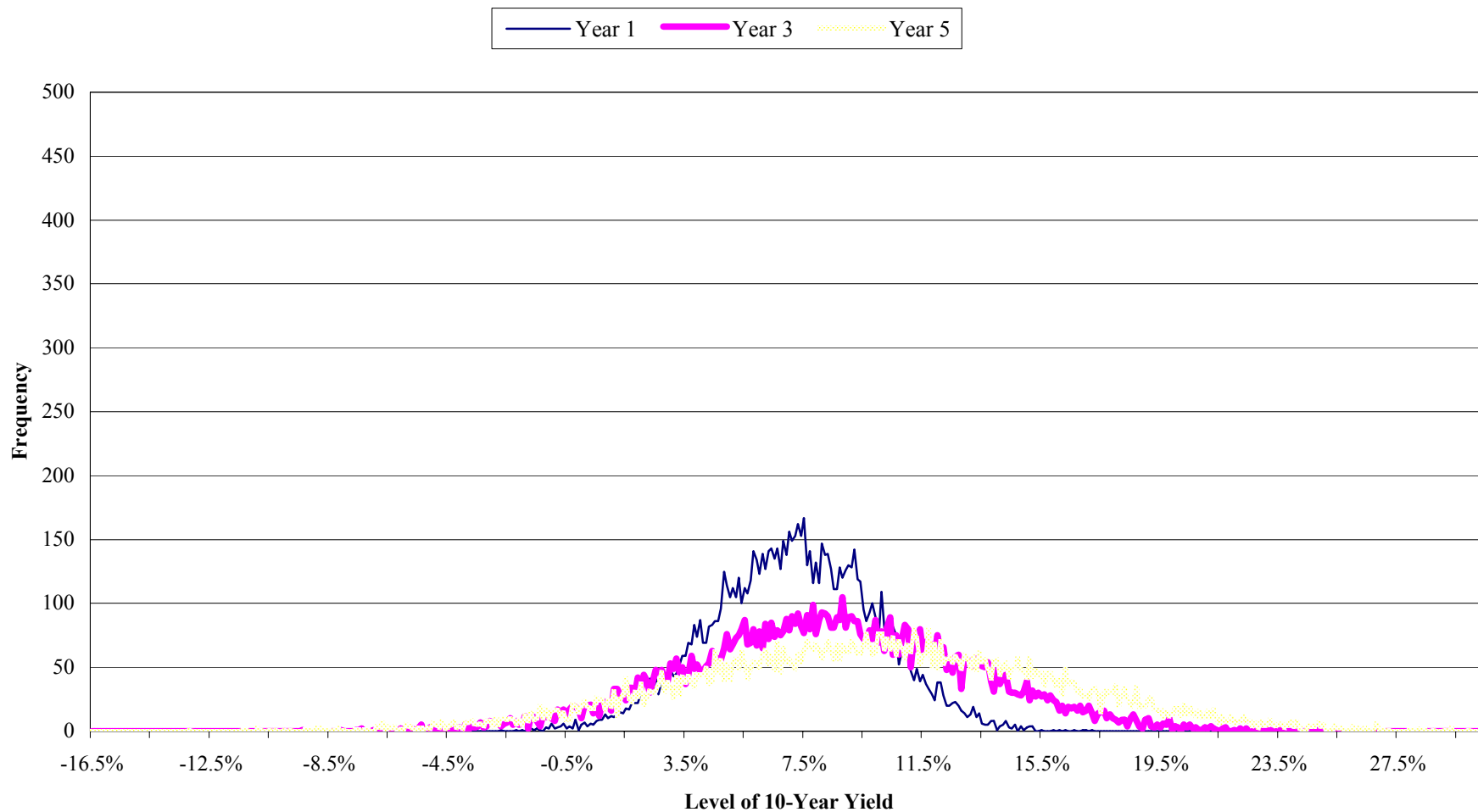
regarding premium payments and dividend declarations to obscure the results. This is especially important since there are no empirical studies available on which to base these assumptions.

2.4 Interest Rate Risk and Capital Requirements of Life Insurance Companies

The products offered by life insurance companies over the last 25 years have been transformed from a rigid product design with static guarantees and minimal interest rate sensitivity to a more dynamic design that has a number of non-guaranteed elements. Policyholders have become more demanding consumers and are increasingly responsive to the rates of return offered by financial products. These characteristics of the changing environment of the life insurance industry have ultimately affected the volatility of life insurers.

Given the disintermediation of the 1980s, life insurers are not completely ignorant of the interest rate risk in their business. Several approaches have been developed to get a better understanding of a life insurer's interest rate risk. However, the analysis that is performed in the industry today is frequently inadequate to capture the true risk exposure. As discussed in Chapter 2, Section 2, a reasonable approach for determining the amount the capital that is needed to support the obligations of an insurer is to stochastically simulate the experience of an insurer. Based on the distribution of the projected surplus, insurers can determine the appropriate amount of capital to set aside. This approach is called dynamic financial analysis (DFA) in property-liability insurance; it has several names in the life insurance industry including dynamic solvency testing (DST), dynamic financial condition analysis, asset adequacy testing, or simply cash flow testing. Although the details of each of these approaches vary a little based on the specific information desired, the general approach remains the same.

Cummins (2000) provides a summary of alternative approaches that have been proposed for establishing risk capital in the insurance industry, including Value at Risk (VaR) and expected

policyholder deficit (EPD).[24] Butsic (1994) suggests that an appropriate way to establish risk-based capital for insurers is to calculate the expected policyholder deficit (EPD). By projecting the experience under a policy, we can determine the potential amount of gains or losses that insurers may face. An alternative approach is by comparing key percentiles in the resulting distribution, as suggested in Panning (1999). The latter approach is called Value at Risk (VaR) and has been an accepted methodology for establishing capital in the banking industry for years. VaR focuses more attention on the tail of the distribution than the average and can be considered an alternative measure of inherent risk. Since many policyholders are more concerned about only one-half of the distribution, namely the left-hand tail, the insurer should be required to hold higher amounts of capital to protect against possible future losses if there are a fair number of scenarios in the analysis indicating that insurer will be insolvent.

3. RESEARCH ISSUES

3.1 Interest Rate Dynamics

Both the assets and liabilities of life insurance companies are directly related to the level of interest rates. Life insurers may have had some rationale for ignoring interest rate risk prior to the 1970s due to the stable interest rate environment. However, as Table 3.1 shows, insurers quickly became aware of their risk exposure as interest volatility increased.

Table 3.1

	Standard Deviation of Monthly Rate Changes		
Decade	*One-Year Yield*	*Ten-Year Yield*	*Slope (Difference)*
1953-1959	0.24%	0.13%	0.17%
1960-1969	0.21%	0.13%	0.14%
1970-1979	0.45%	0.23%	0.30%
1980-1989	0.82%	0.50%	0.47%
1990-1999	0.23%	0.22%	0.16%

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[24] See Chapter 2, Section 2 for a discussion of regulatory framework in insurance with particular emphasis on the capital requirements.

Approximately 65% of a life insurer's assets are invested in fixed income securities including corporate/government bonds and mortgage-related investments (Black and Skipper (2000)).[25] The majority (84%) of their liabilities come in the form of policy reserves. Policy reserves represent an amount that, together with future premiums and investment earnings, will fund all future policy benefits. If the insurer cannot earn the rate of return assumed in the reserve calculation, then this is a potential source of financial risk.

Aside from the variability in present value, insurance liabilities are also affected by interest rates because many products contain a number of embedded interest rate options. In life insurance policies, the credited rate on the cash value of a life insurance contract typically has an option component, where the insurer will provide a minimum return regardless of the current interest rate environment. The cost of providing these guarantees is related to movements in the interest rate over the policy's lifetime. Policyholders also have a right to lapse the policy at their discretion (lapses are covered in Section 5). The relationship between the crediting rate of the insurer and the rate offered by competitors will be one of the factors that determines whether an insured will cancel his policy.

Based on the dependency of assets and liabilities to the level of interest rates, it is imperative that the life insurer has an understanding of the sensitivity of its operations to changes in yields. One way to begin this analysis is to use a term structure model. By generating many future interest rate scenarios, the insurer can project financial results to determine if there is an undesirable exposure to interest rate changes. Using the analysis, the insurer can shed any unwanted risk by unloading some of their risky operations or determine appropriate hedging vehicles, if available. For example, if an insurer discovers substantial financial losses in high interest rate scenarios, they may consider purchasing interest rate caps to reduce this exposure.

Chapters 1 and 2 of this research provide an overview of the several term structure models that are available for various financial applications. One of the important issues presented

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[25] Interestingly, six years earlier, the amount allocated to fixed income securities was 75%.

is that when using a one-factor term structure model, longer-term yields are a one-to-one mapping of the stochastic, short-term rate. This result is inconsistent with historical yield movements (see Chapter 1, Exhibits 1 through 3). Given the rigidity of the yield curve fluctuations, the use of a one-factor term structure model may be masking an additional source of risk to the life insurer. If the changes in the slope are not accurately represented in a one-factor model, perhaps a two-factor model may reveal additional insight into a risk analysis.

This study considers the one- and two-factor Hull-White models of interest rate movements. The Hull and White (1990) one-factor model is formulated in terms of changes in the instantaneous short rate (r_t). All yields are directly related to movements in the short-term rate, which implies that the yields of all maturities are perfectly correlated. The one-factor Hull-White model is:

$$dr_t = (\theta(t) - ar_t)dt + \sigma dB_t \tag{3.6}$$

where $\theta(t)$ is the time-dependent drift
a is the parameter controlling mean reversion speed
σ is the volatility of interest rate changes
dB_t is a Brownian motion

Instead of generating the yield dynamics from only one point on the curve, a multiple factor model produces interest rate changes that are driven by two or more factors. In the two-factor Hull-White model, the level of mean reversion is also stochastic. Since each end of the yield curve is allowed to fluctuate, the two-factor Hull-White model reduces the correlation between bond yields and allows for the opportunity for changes in the slope of the yield curve. The two-factor model of Hull and White (1994) is:

$$dr_t = (\theta(t) + u_t - ar_t)dt + \sigma_1 dB_1 \qquad (3.7)$$

$$du_t = -bu_t dt + \sigma_2 dB_2 \qquad (3.8)$$

$$\rho \; dt = dB_1 dB_2 \qquad (3.9)$$

3.2 Importance of Multiple Factors in Life Insurance Valuation

Allowing some variability in the relationship between yields of various maturities is important if the security being valued depends on the yields at different points. In fact, a whole life insurance contract will depend on the relationship between yields since there are many possible cash flow dates. A cohort of insureds will have many future dates of death and their present value will depend on long-term rates of interest. Also, the timing and the amount of the cash flow depends on various points on the yield curve. For example, the CSV (a liability of the insurer) and policyholder lapse behavior is driven by changes in a reference interest rate (the 5-year rate in this study – see the discussion of lapse behavior in Section 5). But the return on an insurer's assets will depend on the maturity distribution of bonds held in the asset portfolio.

This study considers the effects of the choice of the term structure model on the risk profile of life insurers. Since the one-factor Hull-White model has some restrictions on yield curve movements, it may be that the two-factor Hull-White model identifies a portion of interest rate risk that was masked by the one-factor model.

~~4. A SIMPLE ILLUSTRATION~~

~~Given the variety of structural forms for a term structure model and the choice of parameters, no two models will provide identical interest rate dynamics across time. Trying to decide which of the competing models is most appropriate for a particular application depends on the benchmark used. Even if the user attempts to artificially develop a link between any two term~~

interest rate sensitivity of insurance liabilities. Persson and Aase (1997) also use the Vasicek model to price the embedded interest rate guarantee of life insurance products. Noris and Epstein (1988) use stochastic interest rates to determine the most appropriate hedging asset for a life annuity. Asay, Bouyoucos, and Marciano (1993) use the Black, Derman, and Toy (1990) model to look at the costs of embedded options of SPDAs. All of these studies use a one-factor term structure model to generate results. This paper looks at extending the literature by providing a comparison between the results of a one-factor model and a two-factor model.

The importance of a second factor in evaluating insurer risk was pointed out in Santomero and Babbel (1997). In this paper, the authors review the financial risk management practices of the insurance industry. They find that although insurers have a thorough understanding of asset risk, their knowledge of liability risk is not as rigorous. In particular, Santomero and Babbel (1997) point out that the interest rate sensitivity of assets may be measured using two-factor models, but liability values are usually not.

5. METHODOLOGY

5.1 Modeling Issues

When evaluating life insurance products, the actuary must determine which elements of the policy are fixed and which are subject to variation. The simplest ordinary life policy has fixed premiums and guarantees all elements of the contract, such as mortality and expense charges, interest rate credits, and the CSV schedule. Many insurers find that the cost to provide guarantees (such as a fixed interest credit) over long periods of time are extremely prohibitive and they may not be willing to take such long-term pricing risks. On the demand side, policyholders may not feel comfortable being locked into such rigid guarantees. They may feel that if economic conditions become more favorable, they would like to receive part of the windfall. For example, if the rate of return on invested assets exceeds the assumptions used in pricing, policyholders

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generally feel entitled to some portion of the additional investment earnings. In light of this resistance to guaranteed contracts, very little business is currently written on a completely static basis and most policies have some form of non-guaranteed element.

Dividends are the most common type of non-guaranteed element in ordinary life policies. The terminology "dividend" stemmed from the early dominance of mutual life insurers in the life insurance industry. The policyholders of the mutual life insurers own the company and share in its profitability. Instead of receiving dividends on the common shares, mutual policyholders *participate* in the distributable surplus in the company through dividend payments. If the company's mortality, expense, or investment experience is better than expected under the original pricing assumptions, a dividend may be declared and returned to the participating policyholders. Just like corporate dividends, the timing and amount of a life insurance dividend is at the discretion of the insurer. Policyholders can use dividends in a variety of ways including (1) receiving cash, (2) reducing premiums, (3) increasing the amount of insurance, or (4) adding the amount to the CSV.

Most *stock* insurance companies, which are owned by shareholders, do not issue participating policies, but rather issue policies with other non-guaranteed elements such as variable mortality/expense charges and interest rate credits. The existence of non-guaranteed elements complicates any financial analysis of life insurance products. The modeler must incorporate assumptions about the source of favorable experience, the effects on the cost of providing coverage, the dividend payout strategy of the insurer, and the impact on these elements on policyholder persistence. Policyholder behavior for dividend distributions must also be captured.

When studies are performed in attempt to measure non-guaranteed elements, they are usually quite broad and do not capture behavior in a dynamic setting. For example, the LIMRA 1993-94 study provides information about lapse rates for several types of whole life insurance. Although this study provides real data where none existed, it does not provide an adequate

explanation for interest sensitive lapses. From the description of the policies in Section 2, it is evident that each form provides aspects that may complicate the financial analysis, given the lack of reliable data to back the required assumptions.

To simplify the challenges associated with modeling cash values, this research chooses a product that represents a hybrid of traditional ordinary life insurance policies and the newer UL policies. The exact product is a fixed premium, non-participating, interest-sensitive whole life (ISWL) insurance policy. This product reflects the important features of whole life insurance products, but avoids modeling behavior for which no empirical support is available. Because premiums are fixed and no dividends are paid, it is not necessary to model policyholder behavior regarding premium fluctuations. The only required assumption relates to the interest sensitive surrender decision. The CSV of this policy is interest sensitive so that as interest rates increase, so does the return on the cash value. Each year, the crediting rate is subject to a minimum guarantee of the insurer – similar to an interest rate floor.

5.2 Cash Surrender Values

There are regulations that describe the methodologies to determine minimum statutory reserve requirements. The process of establishing reserves is called valuation. There are two methodologies that can be used to calculate reserves, the retrospective and prospective methods. For traditional policies such as term life and ordinary life insurance, the retrospective view is rarely used. For recent policy forms that use varying interest credits on cash value, the retrospective approach is more useful.

The prospective view of reserve determination looks at the shortfall of future premiums to fund future benefits. Reserves are equal to the APV of future benefits less the APV of future premiums.

$$V_n = \sum_{t=0}^{\infty} DB_t \cdot v^t \cdot {}_t p_{x+n} q_{x+n+t} - \sum_{t=0}^{\infty} \pi_t \cdot v^t \cdot {}_t p_{x+n} \qquad (3.10)$$

Note that the reserve amounts are based on a specific rate of interest. The appropriate interest rate established in the regulations is based on a dynamic index of seasoned corporate bonds. Thus, if yields change, the values of reserves are also affected.

In 1990, the NAIC issued the Standard Nonforfeiture Law that describes the minimum allowable CSVs on whole life policies. No specific values are given in the law. Instead, the law gives the methodology to be followed by insurers as a guide to be used in computing their surrender values. The approach is similar to reserve calculations, but allows insurers some flexibility in product design by allowing them to lower (or even eliminate) CSVs in early policy years. To calculate CSV for ISWL products, the retrospective approach is usually employed because the CSV is dependent on the path of interest rates. The retrospective view of CSV is based on the accumulated value of premiums less the mortality charge for the insurance coverage provided. The CSV is defined iteratively:

$$(CSV_t + \pi)(1 + r_c) - DB \cdot q = p \cdot CSV_{t+1} \tag{3.11}$$

For this study, there are no issuing expenses for the ISWL policy (see assumptions below). Therefore, the CSV of early policy years is not reduced and is as illustrated in (3.11). Note that if there is an interest rate guarantee, this formula is altered slightly to account for the minimum return r_{min}.

$$\left(CSV_t + \pi\right)\left(1 + Max(r_c, r_{min})\right) - DB \cdot q = p \cdot CSV_{t+1} \tag{3.12}$$

5.3 Life Insurance Lapses

The decision to lapse is sensitive to the rate that is credited to policyholders' cash values. As long as insurers' can offer rates that are competitive with alternative products, there is little economic reason for canceling a policy. Although cancellations may appear irrational when the crediting rate exceeds that available on other contracts, there may be some secondary purpose for canceling a policy such as (1) an emergency need for cash, (2) a change in demand for life

insurance protection (especially if the original policy does not provide for changes in benefit amounts), or (3) the realization that the policy was not appropriate.

Few studies are available that measure the lapse rates of life insurers by product line. Although insurers must provide information in their statutory annual statements about total lapses from their operations, the lack of detail provided by these numbers does not allow insurers to determine lapses by year of issue. In addition, only imprecise statements can be made to tie this behavior to changes in interest rates. LIMRA International (1998) gathered some lapse data from 25 life insurers over the policy years 1993-94. LIMRA's report provides estimates of lapse behavior by policy year for several lines of business including interest sensitive whole life. However, their study is static in nature. It does not reveal the dynamic quality of lapse behavior because no attempt is made to relate lapses with changes in interest rates.

Despite the lack of rigorous studies relating the movement of interest rates to policyholder behavior, industry actuaries have developed plausible functional forms that provide some assistance when predicting lapses. Essentially, they argue that lapses are a function of the interest rate differential between the crediting rate on their policy (r_c) and a market interest rate (i_m). One popular assumption is based on the difference between the rate credited by the incumbent insurer and the rate offered by a competitor for a similar product.

$$Lapse = a(t) + b(t) \cdot \arctan\left[c \cdot (i_m - r_c - d(t))\right]$$

(3.13)

The arctangent function is popular because of the shape of the lapse assumption as the interest rate differential increases. Thus, as the differential increases between the rates credited by the insurer and the rate provided by similar products, the lapse rate increases. At first, the increase in lapses is slow. As the differential increases, the lapse rate accelerates. Once the differential reaches extraordinary levels, it is argued that if policyholders haven't lapsed, they never will. Thus, there is a maximum lapse rate. The arctangent function provides a convenient approach to capture these lapse dynamics.

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The parameters a, b, c, and d of (3.13) are chosen to reflect the insurer's experience (if available) or, more often, simply reflect actuarial judgment. These parameters may also be a function of time. The parameter a is a baseline lapse that is not interest rate sensitive, b measures the magnitude of the interest sensitive component of lapse behavior, and c determines the sensitivity of lapses to the crediting rate differential. Finally, the parameter d reflects the inertia of policyholders, which attempts to measure a phenomenon that is similar to the burnout of mortgage prepayments. The arctangent formula is used in Asay, Bouyoucos, and Marciano (1993) in their analysis of single premium deferred annuity contracts.

A similar functional form uses the distribution function for a normal distribution.

$$Lapse = a(t) + b(t) \cdot \Phi[c(t) \cdot (i_m - r_c - d(t))]$$
(3.14)

This study uses the latter function to predict interest sensitive lapses. An illustration of the lapse assumptions are provided in Exhibits 7 through 9. Exhibit 7 shows only the interest sensitive portion of lapses by policy year. Note that during early policy years, the policyholder is more sensitive to the difference in the rates. In Exhibit 8, the total lapse is illustrated which is the sum of the interest sensitive component (the second term in the lapse function) and the baseline lapse $a(t)$. Consistent with the LIMRA 1993-94 lapse study of ISWL policies, the baseline lapse assumption $a(t)$ is illustrated in Table 3.3:

Table 3.3

Year(s)	Lapse Rate
1	15%
2	13%
3-5	9%
6-10	7%
11+	5%

A sampling of the interest sensitive lapse assumption at various points is provided in Exhibit 9. In addition to looking at these assumptions, this study will also test the sensitivity of the analysis to the parameters of the lapse function, since the estimates are not based on any empirical data.

5.4 Life Insurer Crediting Strategy

When evaluating lapse-sensitive life insurance products, one important modeling decision is based on changes in the insurer's crediting rate. Since the policyholder's decision to lapse is linked to the rates offered by alternative policies, the insurer must keep pace with increases in interest rates. As interest rates increase, the insurer will not be able to aggressively compete with new policies written by incoming companies. The gap between the reference rate and the crediting rate of the incumbent insurer will lead to policyholder lapses. As the new policy is written, the new insurer is able to invest all new premium inflows into the higher yielding assets that are available. This is exactly what occurred during the 1980s. As interest rates skyrocketed, policyholders wanted the benefit of new money rates on their cash balances. Ordinary life policyholders lapsed their policies to capture the higher returns on new universal life (UL) policies.

Therefore, in order for the insurer to remain competitive, the credited interest rate to policyholder CSV should be representative of the rate available in the market. Crediting a lower rate may increase the lapse experience on life insurance policies, which may impact the insurer's profit and the ability recapture the first year issuing expense. Providing policyholders with a high rate of interest may attract additional business, but this reduces the profit margin on these policies. The optimal crediting rate is the point where the additional savings from a decrease in the rate earned from non-lapsing policyholders is offset by the lost profits from those who do lapse (Asay, Bouyoucos, and Marciano (1993)).

This study chooses the insurer's crediting strategy based on competitive and portfolio constraints. The reference rate of the market is the rate that is available on similar products with comparable credit quality. Given that the investor can buy term and invest the difference, the determination of the reference rate may not be entirely clear. Since life insurance is a long-term product, many companies provide a crediting rate that is related to longer-term yields. Thus,

policyholders compare the return on a life insurance product to another point on the yield curve (such as the one-, five-, or ten-year yield). This study assumes that policyholders consider the 5-year yield.

The incumbent insurer would like to offer competitive rates to their policyholders, but is constrained by current portfolio holdings. When the insurer targets the 5-year yield, only a portion of the portfolio matures each year and is available for reinvestment. Therefore, any rate that is credited to policyholders will adjust slowly to changes in market yields. One possible crediting strategy is based on a moving average yield over n years:

$$CR_t = \frac{\sum_{i=0}^{n-1} R_{t-i}}{n} - Spread \qquad (3.15)$$

where CR_t is the rate credited to policyholder cash values during year t, R_i is the reference interest rate that policyholders are monitoring, and n is the length of time which the average is determined. In practice, insurers may also reduce the crediting rate by some spread. In this study, the insurer credits policyholders the average of the 5-year yield from the last five years less 50 basis points.

5.5 Modeling Approach

Asset shares are calculated for the block of policies under investigation. Asset share calculations are used to determine profitability and uncertainty of a line of business by determining the available assets that have been accumulated on behalf of surviving policyholders.[26] There are two major differences between CSV and asset share calculations. First, whereas CSV was determined using a fixed premium and expected mortality, asset share calculations are based on the actual premiums received from remaining policyholders and the actual lapse and mortality experience for a given year. Second, the return on assets is based on

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[26] For an introduction to asset share calculations, see Atkinson (1993) or Bowers, et.al. (1986).

the actual return of the insurers portfolio instead of the crediting rate. Aside from these differences, asset shares are also determined retrospectively:
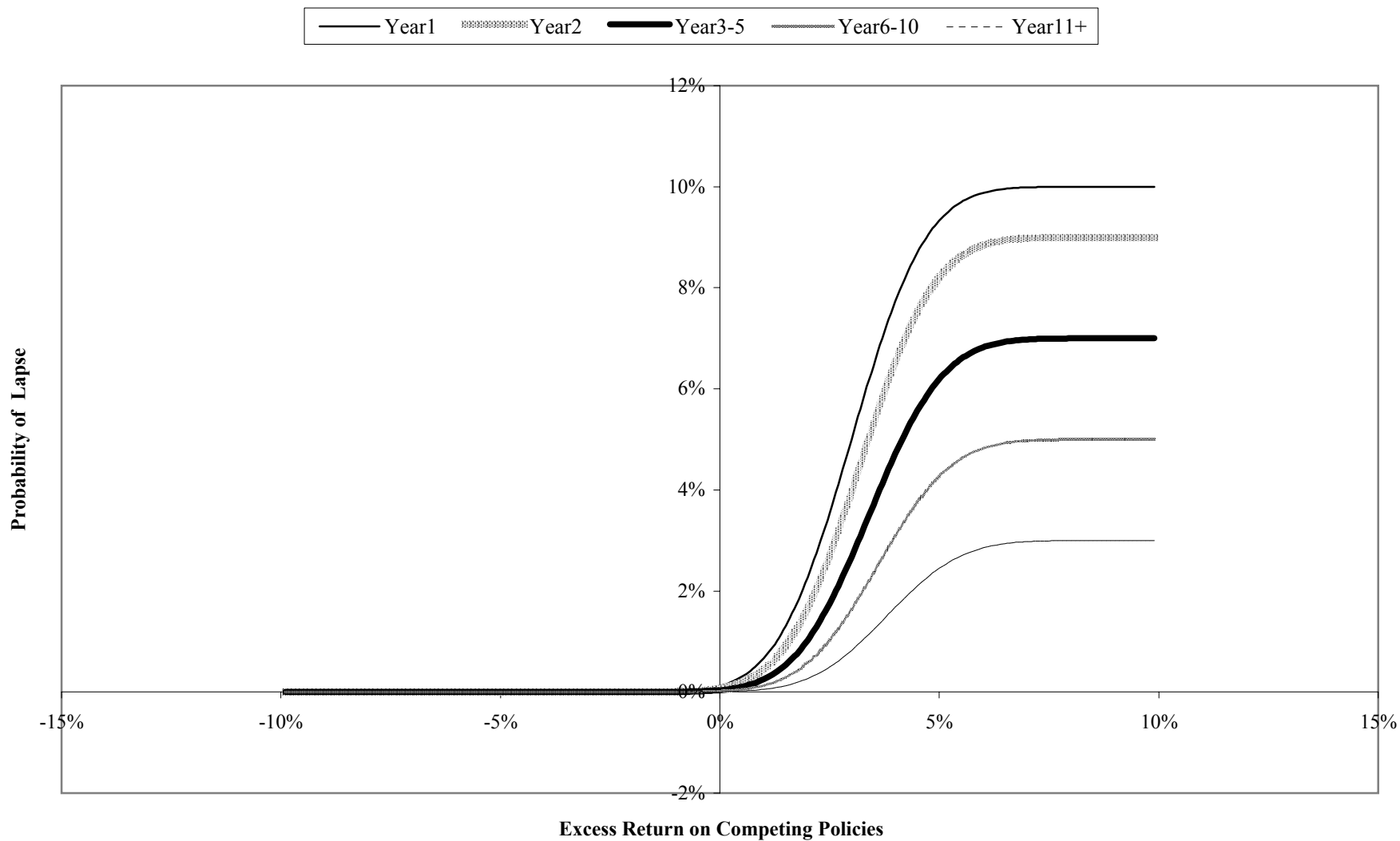
$$AS_{t+1} = \frac{(AS_t + Income)(1 + r_p) - PdBenefits}{p} \qquad (3.16)$$

In equation (3.16), AS_t is the asset share at time t, *Income* is the premium income received at the beginning of the year from the remaining policyholders, r_p is the return on the insurer's portfolio, *PdBenefits* is the value of all benefits that are paid at the end of the year. The surplus of the insurer can then be determined by the difference between the assets (or asset share) and liabilities (CSV).

A model is developed to track a group of policyholders from policy issue through the termination of the policy, whether driven by death or withdrawal. Each year, the policyholder must pay the fixed, required premium to maintain the policy. The use of a fixed premium significantly simplifies the modeling approach. The life of the contract is tracked until the insured either (1) dies and receives the death benefit, (2) lapses to receive the accumulated cash surrender value, or (3) reaches age 100 (mortality table assumes death at age 100).
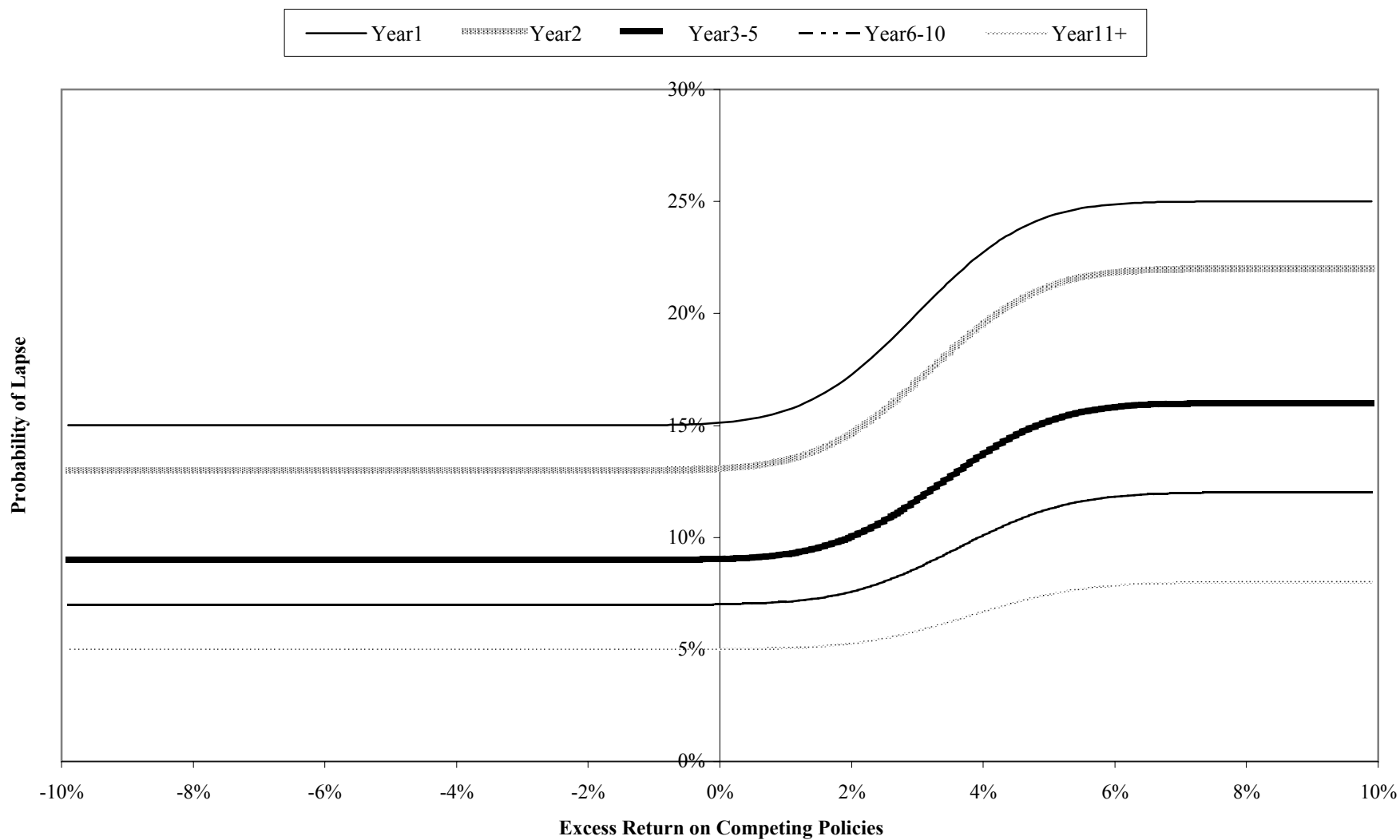
The simulation is repeated 5,000 times to get a distribution of projected surplus levels. For each simulation, a path of interest rates is generated and death experience is sampled. From the path of interest rates, one can determine the crediting rate of the insurer over the life of the policy and compute the proportion of policyholders that lapse based on the interest rate differential between the calculated crediting rate and the market rate of interest. Performing the CSV and asset share calculations allow the insurer to project the surplus per unit inforce at future years. (Per unit inforce is simply the surplus per remaining policyholder at the end of the year). The distribution of surplus projections by year over all of the simulations will provide information on the risk of providing the insurance coverage.

EXHIBIT 7
Interest Sensitive Lapse

Legend: Year1   Year2   Year3-5   Year6-10   - - - - Year11+

Y-axis: **Probability of Lapse** (−2%, 0%, 2%, 4%, 6%, 8%, 10%, 12%)

X-axis: **Excess Return on Competing Policies** (−15%, −10%, −5%, 0%, 5%, 10%, 15%)

EXHIBIT 8
Lapse Rates by Policy Duration

Legend: —— Year1　░░░ Year2　▬▬ Year3-5　– · – Year6-10　······ Year11+

Y-axis: **Probability of Lapse** (0% to 30%)

X-axis: **Excess Return on Competing Policies** (-10% to 10%)

EXHIBIT 9
Sample Lapse Rates by Policy Duration

Policy Duration

Crediting Rate Differential*	Year 1	Year 2	Years 3 – 5	Years 6 – 10	Years 11+
-2%	15.0%	13.0%	9.0%	7.0%	5.0%
0%	15.1%	13.1%	9.0%	7.0%	5.0%
+2%	17.3%	14.7%	10.0%	7.6%	5.3%
+4%	22.7%	19.5%	13.7%	10.1%	6.7%
+6%	24.9%	21.8%	15.8%	11.8%	7.8%

* The crediting rate differential is the difference between the (1) current rate on the policy and (2) rate offered by competitors. Differential = Available Competitor Rate – Current Crediting Rate

For example, + 4% means that competitors are crediting 4% MORE than the insurer. Note that when the differential increases, so does the percentage of policyholders who lapse.