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# PROCEEDINGS

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# FOREWORD

The Casualty Actuarial Society was organized in 1914 as the Casualty Actuarial and Statistical Society of America, with 97 charter members of the grade of Fellow; the Society adopted its present name on May 14, 1921.

Actuarial science originated in England in 1792, in the early days of life insurance. Due to the technical nature of the business, the first actuaries were mathematicians; eventually their numerical growth resulted in the formation of the Institute of Actuaries in England in 1848. The Faculty of Actuaries was founded in Scotland in 1856, followed in the United States by the Actuarial Society of America in 1889 and the American Institute of Actuaries in 1909. In 1949 the two American organizations were merged into the Society of Actuaries.

In the beginning of the 20th century in the United States, problems requiring actuarial treatment were emerging in sickness, disability, and casualty insurance—particularly in workers' compensation, which was introduced in 1911. The differences between the new problems and those of traditional life insurance led to the organization of the Society. Dr. I. M. Rubinow, who was responsible for the Society's formation, became its first president. The object of the Society was, and is, the promotion of actuarial and statistical science as applied to insurance other than life insurance. Such promotion is accomplished by communication with those affected by insurance, presentation and discussion of papers, attendance at seminars and workshops, collection of a library, research, and other means.

Since the problems of workers' compensation were the most urgent, many of the Society's original members played a leading part in developing the scientific basis for that line of insurance. From the beginning, however, the Society has grown constantly, not only in membership, but also in range of interest and in scientific and related contributions to all lines of insurance other than life, including automobile, liability other than automobile, fire, homeowners, commercial multiple peril, and others. These contributions are found principally in original papers prepared by members of the Society and published in the annual *Proceedings of the Casualty Actuarial Society*. The presidential addresses, also published in the *Proceedings*, have called attention to the most pressing actuarial problems, some of them still unsolved, that have faced the industry over the years.

The membership of the Society includes actuaries employed by insurance companies, industry advisory organizations, national brokers, accounting firms, educational institutions, state insurance departments, and the federal government; it also includes independent consultants. The Society has two classes of members, Fellows and Associates. Both classes are achieved by successful completion of examinations, which are held in February, May, and November in various cities of the United States, Canada, Bermuda, and selected overseas sites.

The publications of the Society and their respective prices are listed in the *Yearbook* which is published annually. The *Syllabus of Examinations* outlines the course of study recommended for the examinations. Both the *Yearbook*, at a charge of $40, and the *Syllabus of Examinations*, without charge, may be obtained upon request to the Casualty Actuarial Society, Suite 600, 1100 North Glebe Road, Arlington, Virginia 22201.

JANUARY 1, 1992

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NOTICE

Papers submitted to the *Proceedings of the Casualty Actuarial Society* are subject to review by the members of the Committee on Review of Papers and, where appropriate, additional individuals with expertise in the relevant topics. In order to qualify for publication, a paper must be relevant to casualty actuarial science, include original research ideas and/or techniques, or have special educational value, and must not have been previously published or be concurrently considered for publication elsewhere. Specific instructions for preparation and submission of papers are included in the *Yearbook* of the Casualty Actuarial Society.

The Society is not responsible for statements of opinion expressed in the articles, criticisms, and discussions published in these *Proceedings*.

Volume LXXIX, Part 1 No. 150

# PROCEEDINGS

## May 10, 11, 12, 13, 1992

# TESTING FOR SHIFTS IN RESERVE ADEQUACY

RICHARD M. DUVALL

*Abstract*

*This paper develops regression models that can be used to test for the effects of changes in reserving practices. The models include terms for exposure, trend, and loss development. A loss triangle of reported losses at annual valuation dates is used to estimate the parameters of the regression models. Dummy variables are introduced into the loss development factor terms of the models to test for shifts and trends in the loss development factor parameters. The expanded models are estimated, and the parameters associated with the shift and trend variables are tested for significance. If shifts in reserve adequacy are indicated, the models can be used to restate reported incurred losses for the early valuation dates on a basis that is consistent with recent valuation dates. Similar models can be used to test for changes in settlement rates that create changes in the paid loss development pattern. If a change is revealed, the models can be used to estimate the effects of the change.*

## 1. INTRODUCTION

This paper develops regression models that can be used to test for the effects of changes in reserving practices. If shifts in reserve adequacy are indicated, the models can be used to restate reported incurred losses for the early valuation dates in the data sample on a basis that is consistent with recent valuation dates. This topic has been explored by Berquist and Sherman [1] and more recently by Fleming and Mayer [2]. The procedures advocated in both of those approaches rely on subjective estimates for some of the parameters. While actuarial methods rely on the judgment of professionals, the credibility of results is improved when it is possible to obtain objective confirmation of the subjective assessments.

The Berquist and Sherman procedure for testing for shifts in reserve adequacy is to compare, at each valuation, the rate of growth of the per claim reserve for open claims with the rate of growth of the per claim cost for closed claims. They calculate the rate of growth for both averages over the years in the experience period. If reserving practices are consistent, they contend that the rate of growth in average claim reserves should be equal, approximately, to the rate of growth in average closed claims. Unequal rates indicate a change in reserve adequacy over the experience period.

Given a shift in reserving practices, the Berquist-Sherman adjustment for the shift begins by obtaining the rate of inflation in average closed claims. Next, the average reserve at the most recent valuation date is calculated for each year. These average reserves are trended back to earlier valuation dates at the estimated trend rate to obtain the average reserve at each age for each year in the experience period. The computed average reserves are then multiplied by the number of open claims at each age to get the estimated cost of open claims. Cumulative claim payments are then added to get an estimate of incurred losses on a basis that is consistent with current reserving practice.

Fleming and Mayer observed that if there is an increase in the claim closing rate and if claims close at a cost that exceeds the amount reserved, there will be a change in the incurred loss development pattern. They present an addition to the Berquist-Sherman method that adjusts the data for this speed-up in claim settlement rates.

This paper presents a model for estimating reported incurred loss amounts that incorporates a loss development factor (LDF) function. The model is generalized to account for shifts or trends in the LDFs. If the shift or trend parameters are significant, the function can be used to restate incurred losses from prior valuation dates on a basis that is consistent with current levels of reserve adequacy.

## 2. A MODEL FOR REPORTED LOSS

To develop a regression model for estimating reported incurred losses at each valuation date, one begins by assuming the basic relationship that ultimate loss for year $n$, $Y_n^o$, is the product of the number of claims, $F_n$, and the average claim cost, $X_n$,

$$Y_n^o = F_n X_n . \tag{2.1}$$

An estimate of the ultimate cost is the reported amount as of a given valuation date, $Y_{n,k}$, times the to-ultimate loss development factor, $D_k$^, appropriate for the age, $k$, of the year $n$. Alternatively, the reported incurred loss can be expressed as the ultimate cost divided by the LDF,

$$Y_{n,k} = Y_n^o / D_k . \tag{2.2}$$

Substituting Equation 2.1 into Equation 2.2 gives

$$Y_{n,k} = F_n X_n / D_k . \tag{2.3}$$

A model is developed for each of these factors.

Before proceeding with further development of the model, a system for numbering the observations must be explained. The numbering system expresses the observation number, $t$, as a function of $n$ and $k$. Expressing the matrix of loss data as an array is required when using most regression packages. In addition, the model will contain some variables that are functions of the numbering system. Assume there are $N$ years of loss data with annual valuations of each year's losses.

The loss triangle is arranged as follows:

| | | | Age ($k$) | | |
|---|---|---|---|---|---|
| Year ($n$) | 1 ($t$) | 2 ($t$) | 3 ($t$) | 4 ($t$) | 5 ($t$) |
| 19 x 1 | xx 1 | xx 6 | xx 10 | xx 13 | xx 15 |
| 19 x 2 | xx 2 | xx 7 | xx 11 | xx 14 | |
| 19 x 3 | xx 3 | xx 8 | xx 12 | | |
| 19 x 4 | xx 4 | xx 9 | | | |
| 19 x 5 | xx 5 | | | | |

There are $N$ valuations of the earliest year; $N - 1$ valuations of the next earliest. The number of valuations continues to decline until there is one valuation for the most recent year. Assume that the data are arranged such that the first valuations for each of the $N$ years are listed in the first column; the second valuations for each of the $N - 1$ years are listed in the second column; and so on. The observation number is

$$t = n + (k - 1)(2N - k + 2)/2 \tag{2.4}$$

and $Y_{n,k}$ will be referred to as $Y_t$.

Specific forms for each of the factors in Equation 2.3 are now developed. The specification of the model for the number of claims assumes that the number of claims for each year is related to a measure of the exposure for that year, $E_n$. The specific form assumed for the relationship is

$$F_n = a_1 E_n^{B_0}. \tag{2.5}$$

The standard assumption is that $B_0 = 1$, and Equation 2.5 has the form $F_n = a_1 E_n$. Thus, this form is more general than the standard form. The parameters $a_1$ and $B_0$ will be estimated from the company's data.

The model for average claim amount assumes that the average claim size increases exponentially:

$$X_n = a_2\, e^{n B_1}. \tag{2.6}$$

This is the standard form assumed for the trend component of loss costs. Substituting Equations 2.5 and 2.6 into Equation 2.3 gives

$$Y_t = a_1\, E_n{}^{B_0}\, a_2\, e^{n B_1}/D_k\,. \tag{2.7}$$

The specification of the model for the loss development factors consists of two parts. The first part describes the LDF function for early valuations where the LDFs decline fairly rapidly. The second part describes the LDF function for relatively high ages, where the decay toward unity is slight from one valuation to the next. Both branches of the function are assumed to be a trend function with the general form

$$D_k = a_i\, k^{B_j}. \tag{2.8}$$

For the first $m$ valuations, the equation is expressed as

$$D_k = a_3\, k^{B_2},\ \ k = 1, \ldots, m; \tag{2.9}$$

and, similarly, the second part of the function has the equation

$$D_k = a_4\, k^{B_3},\ \ \ k = m + 1, \ldots, N. \tag{2.10}$$

In order to express the LDF function in a more compact form that can be estimated by regression analysis, three additional variables are introduced. First, let $a_4 = a_3\, e^{B_4}$, and $d_1 = 1$ if $k \le m$ or $d_1 = e$ if $k > m$. Also, let $k_1 = k$ if $k \le m$ or $k_1 = 1$ and $k > m$. Similarly, $k_2 = 1$ if $k \le m$ or $k_2 = k$ if $k > m$. Now, the LDF function can be written as

$$D_k = a_3\, d_1{}^{B_4}\, k_1{}^{B_2}\, k_2{}^{B_3}. \tag{2.11}$$

A brief analysis of this model indicates that it is equivalent to Equations 2.9 and 2.10. For the first $m$ observations, $d_1$ and $k_2$ are one, and the expression reduces to Equation 2.9. For the last $N - m$ observations $d_1 = e$, $k_1 = 1$, and $k_2 = k$ , and Equation 2.11 reduces to Equation 2.10.

When estimating this function, a decision has to be made concerning the size of $m$, i.e., at which age the function should be branched. This depends on the exposure that is being studied, but branching the function at an age of three to four years usually gives a good fit for casualty exposures.

The LDF function is central to the objective of this paper. Changes in reserving practices must be manifest in changes in the parameters of this function if they are to be detected. Therefore, it is important that the function be capable of providing an excellent fit to the observed development patterns. On the other hand, it is not important that the function be capable of extrapolation outside of the range of the data since its purpose is to identify and measure shifts within the data sample. The particular form used for the LDF function is flexible enough to fit regular loss development patterns, but it is not appropriate for extrapolation to ages outside the data range. For example, the LDF should approach one as the age of the loss data increases, but the LDF from the function specified above approaches zero if $B_3 < 0$. The assumed form of the LDF function has two positive features: its flexibility and its linearity when expressed in logarithmic form.

Substituting Equation 2.11 into Equation 2.7 and combining the $a_i$ gives the expression

$$Y_t = \frac{a_0 \, E_n{}^{B_0} \, e^{n B_1}}{d_1{}^{B_4} \, k_1{}^{B_2} \, k_2{}^{B_3}}. \tag{2.12}$$

where $a_0 = a_1 \, a_2 / a_3$. This model will be fit to the Berquist-Sherman data and used to test for a shift in reserve adequacy.

## 3. ESTIMATION OF THE MODEL

The model developed above is now applied to the Berquist-Sherman Medical Malpractice data. After estimating the model, it is reestimated in several forms that test for a shift in reserve adequacy. Each of the forms tests for a shift in one of the parameters. If the model indicates that a shift has occurred, the data is adjusted for the indicated shift.

The top section of Table 1 reproduces Exhibit A of Berquist-Sherman. The to-ultimate LDFs derived by Berquist and Sherman are labelled least squares estimates (L.S. Est.). Equation 2.11 is fit to this data with the branching occurring after the fourth valuation (48 months) for each year ($m = 4$). Logarithms of both sides of the equation are taken:

$$\ln(D_k) = \ln(a_3) + B_4 \ln(d_1) + B_2 \ln(k_1) + B_3 \ln(k_2). \qquad (3.1)$$

The bottom section of Table 1 gives the results of the least squares estimation of Equation 3.1. All of the coefficients have the anticipated signs and are significant at the 1% level. The coefficient of determination, $R^2$, is .998, which indicates an excellent fit. The residuals were tested for departures from randomness using the Durbin-Watson test and the von Neumann ratio test. The results of both tests did not indicate a rejection of the null hypothesis of randomness at the 5% level of significance. Autocorrelation in the residuals would be anticipated if the observations were ordered in time (time series data), or if one or more explanatory variables were not included in the model, or if the model being fit to the data had the wrong functional form. The loss development factors are not time dependent observations. Because the error terms exhibit random behavior, the form used to estimate the LDF function has an appropriate shape and includes appropriate explanatory variables.

The actual and estimated loss development factors are compared in Figure 1. The chart demonstrates that the form chosen for the LDF function can give a good fit to the empirical function. An accurate fit is essential if the function is to be used to test for reserve adequacy shifts in the incurred loss data.

The complete model is estimated using the natural logarithms of the reported incurred losses in Table 1. Unfortunately, the Berquist-Sherman paper does not give any exposure data nor the total number of reported claims. To complete the model, the number of claims is estimated from the data and is used as the exposure base for each loss year. Berquist and Sherman report the number of open claims as of each valuation date, and the number of closed claims has been estimated from two of their exhibits. Their Exhibit C gives the average cost of claims closed in the intervals between valuation dates. Their Exhibit E gives the cumulative paid losses

## TABLE 1

### MEDICAL MALPRACTICE INCURRED LOSSES (000s OMITTED)

Months of Development

| Accident Year | 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | Projected Ultimate |
|---|---|---|---|---|---|---|---|---|---|
| 1969 | 2,897 | 5,160 | 10,714 | 15,228 | 16,661 | 20,899 | 22,892 | 23,506 | 23,506 |
| 1970 | 4,828 | 10,707 | 16,907 | 22,840 | 26,211 | 31,970 | 32,316 | | 33,183 |
| 1971 | 5,455 | 11,941 | 20,733 | 30,928 | 42,395 | 48,377 | | | 52,312 |
| 1972 | 8,732 | 18,633 | 32,143 | 57,196 | 61,163 | | | | 79,700 |
| 1973 | 11,228 | 19,967 | 50,143 | 73,733 | | | | | 112,457 |
| 1974 | 8,706 | 33,459 | 63,477 | | | | | | 145,490 |
| 1975 | 12,928 | 48,904 | | | | | | | 215,308 |
| 1976 | 15,791 | | | | | | | | 176,051 |

Age-to-Age Development Factors

| | 12-24 | 24-36 | 36-48 | 48-60 | 60-72 | 72-84 | 84-96 | 96-Ult |
|---|---|---|---|---|---|---|---|---|
| 1969 | 1.7812 | 2.0764 | 1.4213 | 1.0941 | 1.2544 | 1.0954 | 1.0268 | 1.0000 |
| 1970 | 2.2177 | 1.5791 | 1.3509 | 1.1476 | 1.2197 | 1.0108 | | |
| 1971 | 2.1890 | 1.7363 | 1.4917 | 1.3708 | 1.1411 | | | |
| 1972 | 2.1339 | 1.7251 | 1.7794 | 1.0694 | | | | |
| 1973 | 1.7783 | 2.5113 | 1.4705 | | | | | |
| 1974 | 3.8432 | 1.8972 | | | | | | |
| 1975 | 3.7828 | | | | | | | |

Average Incurred Loss Development Factors

| | 12-24 | 24-36 | 36-48 | 48-60 | 60-72 | 72-84 | 84-96 | 96-Ult |
|---|---|---|---|---|---|---|---|---|
| Average | 2.5323 | 1.9209 | 1.5028 | 1.1705 | 1.2051 | 1.0531 | 1.0268 | 1.0000 |
| Cum. | 11.1488 | 4.4027 | 2.2920 | 1.5252 | 1.3031 | 1.0813 | 1.0268 | 1.0000 |
| L.S. Est. | 11.3864 | 4.1892 | 2.3340 | 1.5412 | 1.2578 | 1.1369 | 1.0438 | 1.0000 |

Regression Output:

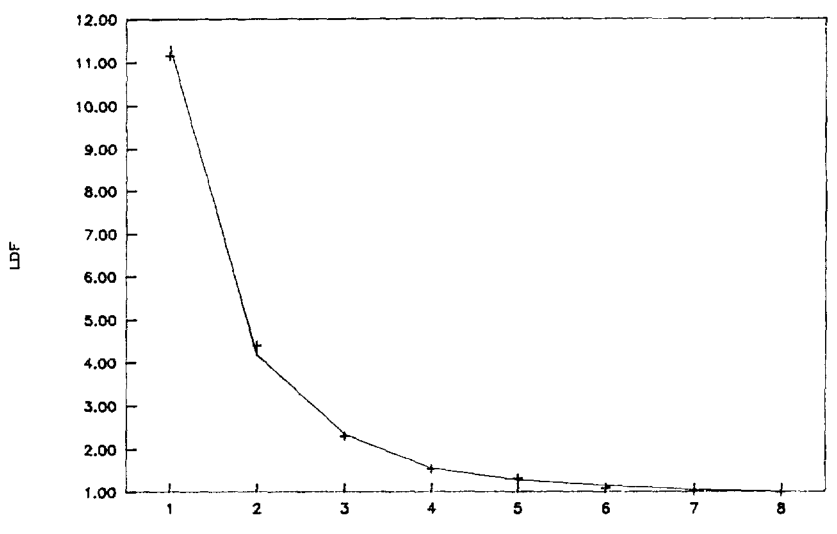
| | $\ln(a_3)$ | $B_2$ | $B_3$ | $B_4$ | $R^2$ |
|---|---|---|---|---|---|
| $X$ Coefficient(s) | 2.432 | -1.443 | -0.554 | -1.311 | 0.998 |
| Standard Error of Coefficient | | 0.044 | 0.130 | 0.247 | |

as of each valuation date. By subtraction, the amount paid between valuation dates is determined. Dividing the average claim payment into the total amount paid is used to approximate the number of claims closed during the period. These closed claim counts are accumulated from period to period. The open claims at each valuation are added to the total number closed to date to give the reported claim counts. The reported claim counts are developed to an estimated ultimate number of claims for each year. The estimated claim counts and their development are presented in Table 2.

Given the estimated claim count for each year, numbering the years from one to eight, and assigning $d_1$, $k_1$, and $k_2$ their values as defined above, Equation 2.12 is estimated by taking the natural logarithms of both sides and using least squares regression. The results of the estimation are reported on Table 3. The error terms are tested for autocorrelation using

FIGURE 1

LOSS DEVELOPMENT FACTORS—ACTUAL VERSUS ESTIMATED




Age (Years)

— Estimated + Actual

## TABLE 2

### MEDICAL MALPRACTICE
### NUMBER OF REPORTED CLAIMS

Months of Development

| Accident Year | 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | Projected Ultimate |
|---|---|---|---|---|---|---|---|---|---|
| 1969 | 1,060 | 1,672 | 2,182 | 2,566 | 2,555 | 2,579 | 2,608 | 2,625 | 2,625 |
| 1970 | 1,051 | 1,877 | 2,340 | 2,719 | 2,777 | 2,804 | 2,828 | | 2,846 |
| 1971 | 1,296 | 2,511 | 3,138 | 3,743 | 3,859 | 3,909 | | | 3,973 |
| 1972 | 1,354 | 2,725 | 3,515 | 4,210 | 4,459 | | | | 4,581 |
| 1973 | 1,382 | 2,828 | 3,671 | 4,665 | | | | | 4,921 |
| 1974 | 1,365 | 2,765 | 3,623 | | | | | | 4,586 |
| 1975 | 1,544 | 2,785 | | | | | | | 4,524 |
| 1976 | 1,594 | | | | | | | | 4,879 |

Age-to-Age Development Factors

| | 12-24 | 24-36 | 36-48 | 48-60 | 60-72 | 72-84 | 84-96 | 96-Ult |
|---|---|---|---|---|---|---|---|---|
| 1969 | 1.578 | 1.305 | 1.176 | 0.996 | 1.009 | 1.011 | 1.007 | 1.000 |
| 1970 | 1.786 | 1.247 | 1.162 | 1.021 | 1.010 | 1.009 | | |
| 1971 | 1.938 | 1.250 | 1.193 | 1.031 | 1.013 | | | |
| 1972 | 2.013 | 1.290 | 1.198 | 1.059 | | | | |
| 1973 | 2.046 | 1.298 | 1.271 | | | | | |
| 1974 | 2.026 | 1.310 | | | | | | |
| 1975 | 1.804 | | | | | | | |

Average Claim Count Development Factors

| | 12-24 | 24-36 | 36-48 | 48-60 | 60-72 | 72-84 | 84-96 | 96-Ult |
|---|---|---|---|---|---|---|---|---|
| Average | 1.884 | 1.283 | 1.200 | 1.027 | 1.011 | 1.010 | 1.007 | 1.000 |
| Cum. | 3.061 | 1.624 | 1.266 | 1.055 | 1.027 | 1.016 | 1.007 | 1.000 |

## TABLE 3

### BASE MODEL
### ESTIMATED LOSSES (000S OMITTED)

| Accident Year | 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 |
|---|---|---|---|---|---|---|---|---|
| | Months of Development | | | | | | | |
| 1969 | 2,571 | 6,617 | 11,503 | 17,029 | 19,676 | 21,606 | 23,383 | 25,041 |
| 1970 | 3,421 | 8,804 | 15,304 | 22,657 | 26,180 | 28,746 | 31,112 | |
| 1971 | 5,426 | 13,965 | 24,276 | 35,940 | 41,527 | 45,598 | | |
| 1972 | 7,535 | 19,393 | 33,713 | 49,910 | 57,670 | | | |
| 1973 | 9,962 | 25,638 | 44,569 | 65,982 | | | | |
| 1974 | 11,930 | 30,703 | 53,375 | | | | | |
| 1975 | 14,865 | 38,256 | | | | | | |
| 1976 | 19,706 | | | | | | | |

Regression Output:

| | | | | | |
|---|---|---|---|---|---|
| Constant | $\ln(a_0) =$ 2.149 | | $a_0 =$ 8.576 | | |
| Standard Error of Y Est. | 0.163 | | | | |
| $R^2$ | 0.964 | | | | |
| Number of Observations | 36 | | | | |
| Degrees of Freedom | 30 | | | | |
| | $B_0$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
| $X$ Coefficient(s) | 0.695 | 0.229 | -1.364 | -0.513 | -1.210 |
| Standard Error of Coefficient | 0.239 | 0.031 | 0.064 | 0.329 | 0.585 |
| Durbin-Watson | $D = 1.916$ | | | | |
| Trend | $\text{Exp}(B_1) =$ 1.258 | | | | |

the Durbin-Watson statistic, $D = 1.916$. This is very close to the expected value, and the hypothesis of independent error terms is accepted. The data are not a time series in the normal sense. The data are ordered such that the 12-month valuations for all years are grouped, then the 24-month valuations, etc. The presence of independent error terms indicates that the estimates at each age are neither too large nor too small.

The bottom section indicates that the fit is excellent with a coefficient of determination, $R^2$, of .964. All of the individual coefficients are significant at the 5% level, with the exception of $B_3$, which is about 1.56 standard errors above zero. Berquist and Sherman estimated a trend in average claim costs of about 30%, whereas this analysis indicates a trend of 25.8% in the average claim cost. The estimated development of incurred losses using the model is reported in the top portion of Table 3. These may be compared to the actual values which are reported in Table 1. Since the year-to-year development is variable, there are some substantial differences between the individual estimates and the observed values, but, on the whole, the fit is good. Thus, a model that gives good estimates of reported loss amounts has been developed. In the next section, the model will be modified to test changes in loss development patterns. If the revised models give superior results, reserving practices will have changed during the sample period.

## 4. TESTS FOR RESERVING CHANGES

If a shift has occurred in reserving patterns, it would be reflected in a change in the parameters of the LDF function, Equation 2.11. There are several parameters that might change with a shift in reserving. The coefficients $a_3$ and $a_4$ could be affected and/or the exponents $B_2$ and $B_3$ might change. These possibilities are explored beginning with testing for changes in the coefficients $a_3$ and $a_4$.

One procedure for testing for a shift in the parameters is to introduce a variable that has a value of unity for valuations that occurred prior to a certain date, and a value of $e$ for valuations after that date. All of the reported losses on the last diagonal of the loss development triangle have the same valuation date. The diagonal elements of the loss development triangle have values of $t = k(2N - k + 1)/2$. Assume that the $p$ most recent valuations reflect the change in reserving practices, then define $d_2 = 1$ if $t \leq k(2N - k + 1)/2 - p$, and $d_2 = e$ if $t > k(2N - k + 1)/2 - p$. Introducing $d_2$ into Equation 2.12 gives

$$Y_t = \frac{a_0 \, E_n{}^{B_0} \, e^{n B_1}}{d_1{}^{B_4} \, d_2{}^{B_5} \, k_1{}^{B_2} \, k_2{}^{B_3}} \, . \tag{4.1}$$

This equation has been estimated, and the results are given in Table 4.

The coefficient of determination increases slightly from .964 to .972 with the addition of the new variable. The Durbin-Watson statistic is 2.194, indicating a small, insignificant amount of negative autocorrelation in the error terms. The coefficient $B_3$ is substantially less significant than in Model 1; however, the coefficient for the shift variable, $B_5$, is highly significant, and indicates that the more recent reported incurred losses are 27.4% larger, on average, than the estimates at the earlier valuations. Also, the estimated trend has decreased from 25.8% to 18.5%. The trends estimated by Berquist and Sherman dropped from 30% to 15%.

The estimates obtained from this model can be used to restate the reported incurred losses for the earlier valuations on a basis consistent with the reported incurred losses for more recent valuations. The early valuations can be increased by 27.4%, to adjust for the indicated shift in the estimates that has occurred during the past two years. This adjustment has been made for the malpractice data, and the results are displayed in Table 5. The last two diagonals of Table 5 are the same as the corresponding numbers in Table 1. All of the numbers above the last two diagonals have been increased by the indicated 27.4%. The restatement results in lower loss development factors and substantially lower estimates of ultimate incurred losses for the more recent years.

To test for a shift in the exponents $B_2$ and $B_3$, two variables are added to Equation 2.12. The first variable, $d_3$, is assigned a value of unity for valuations before the cutoff date, and a value of $k_1$ after the cutoff date, i.e., for the $p$ most recent valuations for each year. Thus, $d_3 = 1$ if $t \leq k(2N - k + 1)/2 - p$, and $d_3 = k_1$ if $t > k(2N - k + 1)/2 - p$. The second variable, $d_4$, is also assigned a value of unity for valuations before the

## TABLE 4

### MEDICAL MALPRACTICE
### MODEL 2 ESTIMATED LOSSES (000S OMITTED)

| Accident Year | Months of Development 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 |
|---|---|---|---|---|---|---|---|---|
| 1969 | 2,874 | 7,005 | 11,796 | 17,073 | 18,631 | 18,936 | 24,456 | 24,748 |
| 1970 | 3,632 | 8,854 | 14,909 | 21,580 | 23,549 | 30,490 | 30,911 | |
| 1971 | 5,612 | 13,678 | 23,034 | 33,340 | 46,347 | 47,105 | | |
| 1972 | 7,449 | 18,156 | 30,574 | 56,373 | 61,517 | | | |
| 1973 | 9,346 | 22,781 | 48,869 | 70,734 | | | | |
| 1974 | 10,475 | 32,526 | 54,773 | | | | | |
| 1975 | 15,649 | 38,143 | | | | | | |
| 1976 | 19,698 | | | | | | | |

Regression Output:

| | | | | |
|---|---|---|---|---|
| Constant | $\ln(a_0) =$ | 1.543 | $a_0 =$ | 4.677 |
| Standard Error of Y Est. | | 0.147 | | |
| $R^2$ | | 0.972 | | |
| Number of Observations | | 36 | | |
| Degrees of Freedom | | 29 | | |

| | $B_0$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|---|---|---|---|---|---|---|
| $X$ Coefficient(s) | 0.794 | 0.170 | -1.285 | -0.089 | -1.726 | -0.242 |
| Standard Error of Coefficient | 0.218 | 0.035 | 0.064 | 0.332 | 0.559 | 0.086 |

| | | |
|---|---|---|
| Durbin-Watson | $D = 2.194$ | |
| Trend | $\text{Exp}(B_1) =$ | 1.185 |
| Shift | $\text{Exp}(B_5) =$ | 1.274 |

TABLE 5

MEDICAL MALPRACTICE

$B_5$ ADJUSTED INCURRED LOSSES (000S OMITTED)

| Accident Year | 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | Projected Ultimate |
|---|---|---|---|---|---|---|---|---|---|
| | Months of Development | | | | | | | | |
| 1969 | 3,690 | 6,573 | 13,648 | 19,399 | 21,224 | 26,623 | 22,892 | 23,506 | 23,506 |
| 1970 | 6,150 | 13,639 | 21,537 | 29,095 | 33,390 | 31,970 | 32,316 | | 33,183 |
| 1971 | 6,949 | 15,211 | 26,411 | 39,398 | 42,395 | 48,377 | | | 46,463 |
| 1972 | 11,123 | 23,736 | 40,946 | 57,196 | 61,163 | | | | 65,654 |
| 1973 | 14,303 | 25,435 | 50,143 | 73,733 | | | | | 86,807 |
| 1974 | 11,090 | 33,459 | 63,477 | | | | | | 106,587 |
| 1975 | 12,928 | 48,904 | | | | | | | 150,347 |
| 1976 | 15,791 | | | | | | | | 117,204 |

Age-to-Age Development Factors

| | 12-24 | 24-36 | 36-48 | 48-60 | 60-72 | 72-84 | 84-96 | 96-Ult |
|---|---|---|---|---|---|---|---|---|
| 1969 | 1.7812 | 2.0764 | 1.4213 | 1.0941 | 1.2544 | 0.8599 | 1.0268 | 1.0000 |
| 1970 | 2.2177 | 1.5791 | 1.3509 | 1.1476 | 0.9575 | 1.0108 | | |
| 1971 | 2.1890 | 1.7363 | 1.4917 | 1.0761 | 1.1411 | | | |
| 1972 | 2.1339 | 1.7251 | 1.3969 | 1.0694 | | | | |
| 1973 | 1.7783 | 1.9714 | 1.4705 | | | | | |
| 1974 | 3.0169 | 1.8972 | | | | | | |
| 1975 | 3.7828 | | | | | | | |

Average Incurred Loss Development Factors

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Average | 2.4143 | 1.8309 | 1.4263 | 1.0968 | 1.1177 | 0.9353 | 1.0268 | 1.0000 |
| Cum. | 7.4222 | 3.0743 | 1.6791 | 1.1773 | 1.0734 | 0.9604 | 1.0268 | 1.0000 |

cutoff date and a value of $k_2$ for valuations after the cutoff date. With these two variables included, the new equation becomes

$$Y_t = \frac{a_0 \, E_n^{\ B_0} \, e^{n B_1}}{d_1^{\ B_4} \, d_3^{\ B_6} \, d_4^{\ B_7} \, k_1^{\ B_2} \, k_2^{\ B_3}} . \tag{4.2}$$

Equation 4.2 has been fit to the Berquist-Sherman data, and the results are summarized in Table 6. The coefficient of determination is marginally higher than for Equation 4.1, and the Durbin-Watson statistic is 2.3887, indicating an insignificant ($\alpha = .05$) amount of negative autocorrelation in the error terms. As before, all of the coefficients are significant with the exception of $B_3$, and in this case, $B_3$ entered with the wrong sign. This model gives a higher estimate of the trend factor than the previous model by about 4.5 percentage points.

A small table has been inserted to indicate the average ratio of losses valued after the critical date to losses valued before the critical date. The ratios for this model vary with the age of the data at the valuation date. The ratios range from no adjustments for 12-month valuations to a 50.5% adjustment for 48-month valuations. These adjustments have been applied to the loss data, and the results are displayed in Table 7. As for the previous model, the adjusted estimates of ultimate incurred loss are considerably lower than for the unadjusted data.

A combined form of Equations 4.1 and 4.2 that included $d_2$, $d_3$, and $d_4$ was estimated. The variables $d_3$ and $d_4$ entered as significant, but $d_2$ was not significant. This indicates that Equation 4.2 is the appropriate model to describe the shift in the reserving practices for these data.

## 5. SUMMARY

A procedure that tests for changes in loss development patterns in an objective manner has been demonstrated. If a change is observed, the models developed can be used to restate the early valuations on a basis that is consistent with the current valuations. These models cannot replace the judgment of the actuary, but they do provide an additional tool with which to analyze this problem.

TABLE 6

MEDICAL MALPRACTICE

MODEL 3 ESTIMATED LOSSES (000S OMITTED)

| Accident Year | Months of Development 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 |
|---|---|---|---|---|---|---|---|---|
| 1969 | 2,988 | 6,964 | 11,426 | 16,234 | 19,286 | 19,263 | 24,344 | 24,718 |
| 1970 | 3,841 | 8,953 | 14,688 | 20,870 | 24,794 | 30,750 | 31,296 | |
| 1971 | 5,670 | 13,217 | 21,684 | 30,810 | 44,460 | 45,396 | | |
| 1972 | 7,539 | 17,574 | 28,832 | 61,647 | 59,116 | | | |
| 1973 | 9,643 | 22,479 | 50,984 | 78,853 | | | | |
| 1974 | 11,411 | 32,632 | 60,334 | | | | | |
| 1975 | 13,932 | 39,840 | | | | | | |
| 1976 | 17,860 | | | | | | | |

LOSS ADJUSTMENT MULTIPLIERS

| $k_1$ | $k_1^{(-B_6)}$ | $k_2$ | $k_2^{(-B_7)}$ |
|---|---|---|---|
| 1 | 1.000 | 5 | 1.215 |
| 2 | 1.227 | 6 | 1.242 |
| 3 | 1.382 | 7 | 1.265 |
| 4 | 1.505 | 8 | 1.286 |

Regression Output:

Constant $\ln(a_0) = 3.489$ $a_0 = 32.744$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Standard Error of Y Est. | 0.133 | | | | | | |
| $R^2$ | 0.978 | | | | | | |
| Number of Observations | 36 | | | | | | |
| Degrees of Freedom | 28 | | | | | | |
| | $B_0$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_6$ | $B_7$ |
| *X* Coefficient(s) | 0.5470 | 0.2070 | -1.2210 | 0.0066 | -1.8756 | -0.2948 | -0.1208 |
| Standard Error of Coefficient | 0.2053 | 0.0267 | 0.0637 | 0.3524 | 0.5695 | 0.0745 | 0.0659 |

Durbin-Watson $D = 2.3887$ Trend $\text{Exp}(B_1) = 1.2300$

# TABLE 7

## MEDICAL MALPRACTICE
## $B_6$ AND $B_7$ ADJUSTED INCURRED LOSSES (000S OMITTED)

Months of Development

| Accident Year | 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | Projected Ultimate |
|---|---|---|---|---|---|---|---|---|---|
| 1969 | 2,897 | 6,330 | 14,812 | 22,916 | 20,238 | 25,951 | 22,892 | 23,506 | 23,506 |
| 1970 | 4,828 | 13,134 | 23,374 | 34,371 | 31,838 | 31,970 | 32,316 | | 33,183 |
| 1971 | 5,455 | 14,648 | 28,663 | 46,542 | 42,395 | 48,377 | | | 47,016 |
| 1972 | 8,732 | 22,857 | 44,437 | 57,196 | 61,163 | | | | 67,913 |
| 1973 | 11,228 | 24,494 | 50,143 | 73,733 | | | | | 77,567 |
| 1974 | 8,706 | 33,459 | 63,477 | | | | | | 98,816 |
| 1975 | 12,928 | 48,904 | | | | | | | 151,813 |
| 1976 | 15,791 | | | | | | | | 140,169 |

Age-to-Age Development Factors

| | 12-24 | 24-36 | 36-48 | 48-60 | 60-72 | 72-84 | 84-96 | 96-Ult |
|---|---|---|---|---|---|---|---|---|
| 1969 | 2.1850 | 2.3400 | 1.5471 | 0.8831 | 1.2823 | 0.8821 | 1.0268 | 1.0000 |
| 1970 | 2.7205 | 1.7796 | 1.4705 | 0.9263 | 1.0041 | 1.0108 | | |
| 1971 | 2.6853 | 1.9567 | 1.6238 | 0.9109 | 1.1411 | | | |
| 1972 | 2.6177 | 1.9441 | 1.2871 | 1.0694 | | | | |
| 1973 | 2.1815 | 2.0472 | 1.4705 | | | | | |
| 1974 | 3.8432 | 1.8972 | | | | | | |
| 1975 | 3.7828 | | | | | | | |

Average Incurred Loss Development Factors

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Average | 2.8594 | 1.9941 | 1.4798 | 0.9474 | 1.1425 | 0.9465 | 1.0268 | 1.0000 |
| Cum. | 8.8765 | 3.1043 | 1.5567 | 1.0520 | 1.1104 | 0.9719 | 1.0268 | 1.0000 |

The models that have been illustrated test for a change in reserving practices as of a specified date. Models that will detect a trend in the loss development factors, rather than an abrupt change in the factors as of the specified date, can also be employed. As above, one can test for a trend in the coefficients, $a_3$ and $a_4$, or in the exponents, $B_2$ and $B_3$. All data on the same diagonal of the loss development triangle have the same valuation date and are given the same time index of $g = n + k - 1$. This index numbers the diagonals beginning with one in the northwest corner of the matrix and increases by one for each diagonal added to the triangle. The LDF model that estimates and tests for a trend in the exponents is

$$D_t = a_3 \, k_1^{\,(B_2 + gB_8)} \, k_2^{\,(B_3 + gB_9)} \, d_1^{\,B_4}. \tag{5.1}$$

Finally, a model for the LDF function that includes a trend factor for the coefficients is

$$D_t = a_3 \, a_5^{\,g} \, k_1^{\,B_2} \, k_2^{\,B_3} \, d_1^{\,B_4}. \tag{5.2}$$

Both Equation 5.1 and Equation 5.2 have been fit to the Berquist-Sherman data, but the results were not as significant as the models that incorporated a jump in the parameters. The results of the estimation are not reported.

Similar models can be employed to test for changes in claim settlement rates that are reflected in changes in paid loss development factors. If paid losses are substituted for reported losses as the dependent variable and the loss development function is interpreted as the paid loss development function, the models can be used to test for parameter changes in the same manner.

This paper has shown how regression models can be used to estimate the effects of changes in reserving practices. Once the effects have been estimated, the appropriate adjustments can be made to past valuations to restate them on a basis consistent with current reserving practices. The models allow one to test for abrupt changes in reserving practices versus changes that emerge progressively. This procedure is flexible and objective.

## REFERENCES

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# PARAMETRIZING THE WORKERS COMPENSATION EXPERIENCE RATING PLAN

WILLIAM R. GILLAM

## *Abstract*

*This paper describes the development of the revised Workers Compensation Experience Rating Plan. The plan is based on sound statistical theory, certain modeling assumptions, and thorough empirical testing. It is heartening that the empirically derived parametrization is consistent with most of the assumptions needed to simplify the algebraic foundation.*

*The paper begins with an heuristic derivation of a general modification formula based on losses split into primary and excess portions. It delineates the assumptions about the components of loss ratio variance leading to the algebraic form of the formulae tested.*

*Iterative testing is used to parametrize those formulae. A simple preliminary test procedure is described to clarify the basic concepts. The operative test procedure is then specified, and results of iterative testing using that procedure are displayed for the selected formulae.*

*The parametrized formula finally approved by the National Council on Compensation Insurance (NCCI) was subject to certain adjustments to maintain continuity during the transition from the old to new plans. Credibilities have been scaled to account for differences in state benefit levels and the effect of inflation.*

## 1. THEORETICAL JUSTIFICATION

### *A. Motivation*

Compensation experience rating is a large-scale, ongoing application of credibility theory. The large volume of data supporting that application provides the raw material for the tests of that theory described below.

Researchers at the NCCI have used testing to create an improved experience rating plan. The power of modern electronic data processing has enabled them to reopen older experience rating files and recalculate experience modifications (mods) as if a hypothetical plan had been in place. The plans tested and the measurement of their performance are described in this paper.

The general strategy was to start with a formula based on sound theory, then use iterative testing to parametrize that formula. Least squares or Bayesian credibility was used to develop an algebraic form for the modification formula. Certain assumptions about loss ratio variance simplified the algebra. The parameters that worked best were consistent with a priori judgments about the components of loss variance.

### *B. Heuristic Derivation of Mod Formula*

This section outlines the theoretical development of the split plan modification formula. Here, a *split plan* is one in which individual losses are split by formula into two components, primary and excess, and separate credibilities are assigned to the totals of the respective loss components.

The formula is based on a Bayesian view of the process of individual risk rating. The reader may refer to papers by Hewitt [1], Meyers [2], Mahler [3], and Venter [4] for more general theoretical background.

The split plan modification formula can be derived with one major simplifying assumption: that unconditional expected primary and excess losses are uncorrelated. This simplification is defensible more on the basis of its usefulness than its veracity. The standard used to select the final plan is how well it works, not how well it satisfies the assumptions. In the course of evaluating plan parameters, NCCI researchers found that a change in the primary/excess split formula improved the performance of the plan. They believe this change places the data used for rating in a form that better fits the assumptions.

The underlying analysis is simplified by assuming that most of the administrative features of the current experience rating plan are fixed.

To begin the derivation, hypothesize a linear approximation to the posterior mean experience, $P_o + X_o$ (split primary + excess), given experience $P_t$ and $X_t$:

$$P_o + X_o = Y + Z_p P_t + Z_x X_t + e, \tag{1.1}$$

where

$P$ = primary loss,

$X$ = excess loss,

$Y$ = constant to be determined,

$t$ = (past) time period,

$o$ = (future) time period, and

$e$ = error.

$Z_p$ and $Z_x$ will be called the respective primary and excess credibilities; they and $Y$ are the coefficients to be evaluated.

Time periods are fixed in the experience rating plan, so that time period $t$ is the three most recently completed one-year policy periods before the prospective single policy period, labeled $o$. For example, the experience of completed policies with inception dates in 1986, 1987, and 1988 will be used to rate a 1990 policy.

Solving this equation for the coefficients that minimize the expected value of $e^2$ (with the assumption mentioned above) yields the following expressions:

$$Z_p = \frac{\text{Var}_S\,[\text{E}\,[P_t \mid S]\,]}{\text{Var}\,[P_t]}, \text{ and}$$

$$Z_x = \frac{\mathrm{Var}_s\,[\mathrm{E}\,[X_t \mid S]\,]}{\mathrm{Var}[X_t]}\,. \tag{1.2}$$

And

$$Y = (1 - Z_p)\,\mathrm{E}\,[P_t] + (1 - Z_x)\,\mathrm{E}\,[X_t]\,. \tag{1.3}$$

where the condition $S$ denotes a particular element of the parameter space (a particular risk) and the subscript $s$ denotes the prior structure (the distribution of risk parameters).

Equation 1.2 has also been written

$$Z_p = \frac{1}{1 + \dfrac{\mathrm{E}_s\,[\mathrm{Var}[P_t \mid S]\,]}{\mathrm{Var}_s\,[\mathrm{E}\,[P_t \mid S]\,]}}\,,$$

using $\mathrm{Var}[P_t] = \mathrm{Var}_s\,[\mathrm{E}[P_t \mid S]\,] + \mathrm{E}_s\,[\mathrm{Var}[P_t \mid S]\,]$. The first term is the *variance of the conditional means* or the *between-variance*. The second term is the *expected value of the conditional variance* or the *within-variance*.

Using these equations, the linear credibility estimate of the posterior mean becomes

$$P_o + X_o = \mathrm{E}[P_t] + \mathrm{E}[X_t] + Z_p\,(P_t - \mathrm{E}[P_t]\,) + Z_x\,(X_t - \mathrm{E}[X_t]\,). \tag{1.4}$$

In practice, the loss functions are ratios to the prior expected total loss, so $\mathrm{E}[P_t] + \mathrm{E}[X_t] = 1$. In this paper, $P$ and $X$ are referred to as *loss ratios*, but the denominator is expected loss, not premium.

The rate modification factor is

$$M = 1 + Z_p\,(P_t - \mathrm{E}[P_t]\,) + Z_x\,(X_t - \mathrm{E}[X_t]\,)\,. \tag{1.5}$$

*C. Variance Assumptions*

More assumptions are needed to derive the form of the components of variance in the formulae for $Z_p$ and $Z_x$.

*First-Level Variance Assumptions*

In Equation 1.4, loss ratio functions $P$ and $X$ were introduced. Those ratios have a variance that decreases as the size of risk increases. The sample ratios $P_t$ and $X_t$ are the emerged primary and excess actual losses of the individual risk divided by the unconditional expected total losses. The denominator is the *exposure*. The simplest assumption is that the large risk is essentially a combination of a large number of independent homogeneous units. That assumption leads to a within-variance of the risk loss ratio inversely proportional to exposure. The increase in exposure from additional time periods can be thought of as adding more independent units of exposure. The process variance decreases proportionately. Also, it is usually assumed that the variance of the conditional means is independent of exposure (i.e., size of risk). With those assumptions,

$$\mathrm{E}_s\left[\mathrm{Var}[P_t \mid S]\right] = a/E ,$$

and

$$\mathrm{Var}_s\left[\mathrm{E}[P_t \mid S]\right] = b ,$$

where $E$ with no subscript represents the total expected losses, or exposure, of the individual risk: $E = \mathrm{E}[P_o + X_o \mid S]$.

Here $b$, the variance of ratios less than one, is small relative to $a$, which is measured in dollars of expected loss.

Using equation 1.2,

$$Z_p = \frac{b}{b + a/E}$$

$$= \frac{1}{1 + a/bE}$$

$$= \frac{E}{E + a/b}$$

This is the familiar expression

$$Z_p = \frac{E}{E + K_p}, \tag{1.6}$$

where $K_p$ is constant. Similarly,

$$Z_x = \frac{E}{E + K_x},$$

where $K_x$ is the excess credibility constant. This *compound fraction* form, with $E$ alone in the numerator and $K$ a ratio of components of variance, helps to simplify the mod formula.

*Second-Level Variance Assumptions*

Several investigators have refuted this simple variance assumption. Meyers [2] and Mahler [3] show that within-variance does not decrease in inverse proportion to exposure. Assuming there is a small, non-diversifiable component of risk loss ratio variance, averaging $c > 0$,

$$\mathrm{E}_s[\mathrm{Var}[P_t \mid S]\,] = c + d/E.$$

Using $b$ again as the between-variance,

$$Z_p = \frac{b}{b + c + d/E}$$

$$= \frac{1}{1 + c/b + d/Be}$$

$$= \frac{E}{E + \frac{cE + d}{b}}, \text{ so}$$

$$Z_p = \frac{E}{E + K_p'}, \text{ where}$$

$$K_p' = \frac{cE + d}{b} \,. \tag{1.7}$$

Now $K_p'$ is a linear function of the exposure. Here, $b$ and $c$ are small relative to $d$. The limiting value of primary credibility for the largest risks is less than unity, or $b/(b + c)$.

This form for $K_p'$ and a similar one for $K_x'$ are among possible formulae tested as described elsewhere in the paper. Because $K_*$ ($*$ either $p$ or $x$) is a linear function of the exposure rather than a constant, it performs better than the constant coefficient $K$, and considerably better than the formula $B$ value of the old plan. However, it is not as good as the third-level formulae described below. The data show that $K$ should not be constant, nor even a linear function of $E$, but rather should be a curve, increasing rapidly at first but then decreasing in slope to a more linear form for large values of $E$.

*Third-Level Variance Assumptions*

The variance assumptions resulting in the formula for $K$ at this level were suggested by Mahler [3]. Mahler, in turn, credits Hewitt [5] with observation of the underlying phenomenon.

For this level, it is assumed that the between-variance is not constant across all risk sizes but has a component inversely proportional to exposure. This would follow if each larger risk was, at least in part, a random combination of non-homogeneous components. The effect is to flatten the variance of the conditional means as risk size increases. In this case,

$$\mathrm{Var}_s\,[\mathrm{E}\,[P_t \mid S]\,] = e + f/E \,.$$

Retaining the second assumption about individual risk variance,

$$Z_p = \frac{e + f/E}{e + f/E + c + d/E} = \frac{1}{1 + \dfrac{c + d/E}{e + f/E}} \,.$$

In compound fraction form, this is

$$Z_p = \cfrac{E}{E + \cfrac{cE + d}{e + f/E}}$$

$$= \frac{E}{E + E\left(\dfrac{cE + d}{eE + f}\right)}, \text{ so}$$

$$Z_p = \frac{E}{E + K_p''},$$

where

$$K_p'' = E\left(\frac{cE + d}{eE + f}\right). \tag{1.8}$$

A similar form follows for $K_x''$. Notice that $d$ and $f$ are quite large compared to $c$ and $e$. Since $c$ is a small component of within loss ratio variance and $e$ is a large component of between loss ratio variance, it is also plausible that $c < e$.

Dividing through by $e$, we define $C = c/e$, $D = d/e$, and $F = f/e$, so that

$$K_* = E\left(\frac{CE + D}{E + F}\right). \tag{1.9}$$

This form is selected for parametrization, so the superscripts have been dropped.

In all the sample parameters that worked well (as described below), $C$ was consistently between 0 and 1, which is reasonable if $c$ is in fact smaller than $e$. $D$ and $F$ are large positive numbers, as expected.

In a sense, the final parametrization selected is more general than the underlying variance assumptions. This is because performance testing was used to derive parameters for the modification that worked best. This obviates the need to estimate components of variance and reduces reliance on the correctness of assumptions. Thus, the only constraint on plan performance was the algebraic form of Equation 1.9, not the ability to analyze variance. Statisticians use *robust* to refer to models such as these that can fit a variety of processes while not necessarily satisfying the assumptions underlying the model.

With the definition of $K_p$ and a similar one for $K_x$ underlying $Z$ in the form $E/(E + K_*)$, the modification formula becomes,

$$M = 1 + \frac{A_p - E_p}{E + K_p} + \frac{A_x - E_x}{E + K_x}, \tag{1.10}$$

where $A$ and $E$ are the actual and expected losses from the experience period and $p$ denotes primary and $x$ excess. The algebraic form of the modification used for most of the testing was Equation 1.10, with $K_p$ and $K_x$ defined as in Equation 1.9. For each of $K_p$ and $K_x$, there are coefficients to estimate.

## 2. ESTIMATION OF PARAMETERS

### *A. Initial Testing*

The concept of evaluating workers compensation individual risk credibility by looking back at how well it worked was discussed by Dorweiler [6]. He did not use his method to establish credibilities, but to check them for reasonableness.

Bailey and Simon [7] described a variant of the procedure for automobile merit rating wherein they were able to estimate the implicit credibilities of one, two, or three car-years. They were not trying to parametrize a continuous formula for credibility depending on exposure, however, but were trying to estimate only three values for one, two, or three car-years' experience.

Today, we are able to take their ideas a step further, largely because of the power of the computer.

In this study of experience rating, the criterion of "working best" is first measured by the ability of a plan to satisfy Dorweiler's necessary criterion for correct credibilities: that credit risks and debit risks would be made equally desirable insureds in the prospective period. In workers compensation, credibility is a function of risk size, so this property should exist across all size categories. We use this criterion as a *naive test*, which belies its great value to our early investigations. It also serves to simplify the demonstration of the basic idea behind the testing.

An example of this test is included in Exhibit 1. Note that the plan proposed earlier is still a long way from the plan that was eventually selected. The test begins with experience rated risks for policies effective in 1981. Their modifications are computed according to the formula to be tested. The 1981 loss experience that actually emerged may be found in the 1983 rating year files, i.e., the data underlying mods effective 1983. The risks in each size group are stratified by their 1981 modifications, so that risks with mods in the lower $50^{th}$ percentile would be in one stratum and risks with mods in the upper $50^{th}$ percentile would be in the other. (It should be noted that for the smaller size groups, the majority of risks have credit mods, so the upper percentile includes a proportion of risks with small credits.)

A canonical comparison would be of the subsequent loss ratios of the two strata: actual losses to manual premium on one side, and actual losses to modified premium on the other. The first ratios, actual to manual, should follow the predicted quality of the stratum. Risks with credit mods should prove to have favorable loss ratios on average, and those with debits should average poor ratios, showing that the plan was indeed able to "separate the wheat from the chaff." In order to see if the differences in predicted quality were correctly offset by the mod, the ratio of losses to modified premiums for the two groups should equal each other. It would be too much to expect that premium rates be correct in aggregate and that the two subsequent loss-to-modified-premium ratios be equal to the permissible loss ratio.

Effective manual premiums for the three policy periods used for the modification are retained in the experience rating files. Unfortunately, since they are not used for either ratemaking or experience rating, the

# EXHIBIT 1

## 1981
## ACTUAL TO EXPECTED LOSS RATIOS BEFORE AND AFTER EXPERIENCE RATING — 7 STATES TOTAL

### CURRENT EXPERIENCE RATING FORMULA

| Risk Size | Quality Indication | Subsequent Period: Loss Ratios | Subsequent Period: Modified Loss Ratios |
|---|---|---|---|
| 2,500-5,000 | 50% Best | 0.75 | 0.80 |
| | 50% Worst | 1.12 | 1.05 |
| 5,000-10,000 | 50% Best | 0.71 | 0.80 |
| | 50% Worst | 1.11 | 1.01 |
| 10,000-25,000 | 50% Best | 0.79 | 0.92 |
| | 50% Worst | 1.12 | 0.96 |
| 25,000-100,000 | 50% Best | 0.75 | 0.89 |
| | 50% Worst | 1.15 | 0.93 |
| Over 100,000 | 50% Best | 0.71 | 0.93 |
| | 50% Worst | 1.00 | 0.79 |
| Sum of Absolute Differences | | 1.79 | 0.68 |

### EARLY PROPOSED EXPERIENCE RATING FORMULA

| Risk Size | Quality Indication | Subsequent Period: Loss Ratios | Subsequent Period: Modified Loss Ratios |
|---|---|---|---|
| 2,500-5,000 | 50% Best | 0.76 | 0.91 |
| | 50% Worst | 1.10 | 1.00 |
| 5,000-10,000 | 50% Best | 0.71 | 0.89 |
| | 50% Worst | 1.11 | 0.98 |
| 10,000-25,000 | 50% Best | 0.79 | 0.98 |
| | 50% Worst | 1.12 | 0.95 |
| 25,000-100,000 | 50% Best | 0.75 | 0.91 |
| | 50% Worst | 1.15 | 0.94 |
| Over 100,000 | 50% Best | 0.72 | 0.89 |
| | 50% Worst | 0.99 | 0.83 |
| Sum of Absolute Differences | | 1.74 | 0.30 |

numbers are seldom checked and are considered unreliable. The expected loss rates, or ELRs, by class in these files, however, are subject to review by insureds and insurers. These are not the true loss costs underlying the rates, but estimates of emerged loss for three policy years as of a certain evaluation date. The ELRs used to estimate rating year 1981 expected losses are used to compute the expected ratable losses for modifications effective during the 1983 policy year. The ELRs are meant to be correct on average for the losses on three policy years, including, in this case, 1979, 1980, and 1981. The three policy years are at respective third, second, and first reports. ELRs are probably not correct for any single policy year, but should bear some reasonable relation to the rates effective in the latest year. A key assumption is that the ELRs will be uniformly redundant or inadequate over all insureds with the same rate.

The comparison is then between the ratios of actual to manual expected loss for each stratum and ratios of actual to expected losses *adjusted by the modification*, or modified expected loss. Specifically, 1981 actual loss to 1981 expected loss, taken from the 1983 rating year files, should reflect the predicted quality difference. Application of the 1981 modification to 1981 expected losses should make the ratios converge.

Just as it was observed in the case of premiums, it is reasonable to hope that the subsequent ratios to modified expected loss would be close to each other, but it is unreasonable in this case to require values near unity.

The need for credibility in a format not unlike the one finally selected is evidenced by application of the naive test. The smallest ratable risks have non-zero excess as well as primary credibilities. Starting with the smallest risks, credibility increases rapidly with risk size, but then increases at a slower rate, and never reaches full credibility for even the largest risks.

### *B. The Quintiles Test*

As the testing of the plans progresses and more sophisticated actuarial theory is applied to the algebraic form of the credibility constants, it becomes apparent that a more sophisticated test is needed to measure the quality of alternative formulae. Dorweiler's sufficient criterion for correctness of the modification is that any subdivision of risks based on prior experience should produce uniform subsequent loss ratios (to modified premium).

Instead of good versus bad as in the naive test, the risks are grouped into five equal-sized strata according to the value of their modifications. The lowest 20% of the values belong to risks in the first quintile; the next 20% to the second; and so on. This is the prior subdivision. The subsequent aggregate unmodified loss ratios of the strata should reflect the quality difference recognized by the mod. Application of the modifications should cause the ratios to flatten across the strata.

This leads to the ratio of two sums of squared differences: the five squared deviations from the mean of the modified loss ratios, divided by the sum of the squared deviations of the ratios before modification. Lower values indicate greater reduction of loss ratio variances. The statistic would pertain to the experience of each group, so for a particular parametrized mod formula, several values are available for comparison. In most of the NCCI testing, coincidentally, five groups were considered.

The quintiles test was developed without reference to risk theory, but it can be characterized as the ratio of posterior structure variance to prior structure variance. The sum of five squared deviations does not capture the entire structure variance, either prior or posterior; but the ratio is valid. Experience rating should reduce this component of variance. Meyers [2] uses the more theoretically grounded "efficiency" standard: the proportion by which the total variance is reduced. Either statistic is useful; the quintiles test is computationally simpler and has an indisputable best value of zero.

Section 3 outlines the variants of the basic plan for which minimal values of the statistics were sought and it discusses some of the rationale for each. The exhibits shown are the final product of a large number of trial-and-error evaluations.

One sidebar test deserves mention. In this test, primary and excess credibilities were evaluated separately. The mod as a function of either primary or excess losses alone had far less predictive accuracy. Credibilities were lower when losses were used separately as the sole basis for the mod than when they were used together.

This conclusion should be contrasted with that of Meyers [2]—that a best modification formula could be based on primary losses only. His conclusion may be correct in the special case of a uniform, well-behaved severity distribution for all risks, which was the model he tested. The NCCI tests of real-world data support the split formula with a two-part credibility.

The workers compensation severity distribution is composed of many types of losses. An essential component of workers compensation ratemaking and individual risk rating is that the distribution of losses by type varies from class to class and risk to risk.

## 3. PLANS TESTED

### *A. Basic Specifications of the Former Plan*

Potential revised experience rating plans were tested in comparison to the then-current experience rating formula, herein referred to as the "former" plan.

The former formula was derived through practical simplifications that made sense at the time of its development. It was partly these simplifications, however, that moved the plan away from whatever underlying credibility theory it may have had. The former formula is written:

$$M = \frac{A_p + WA_x + (1 - W)E_x + B}{E + B} . \tag{3.1}$$

It is one fraction, with weighting value $W$ and credibility ballast $B$ —both linear functions of total expected losses. $A$ denotes actual and $E$ represents expected loss for the experience period. Subscripts $p$ and $x$ denote primary and excess portions of loss, respectively; and $E$ with no subscript denotes total expected losses.

$$W = \begin{cases} 0 & \text{for } E < 25{,}000 \\ \dfrac{E - 25{,}000}{\text{SRP} - 25{,}000} & \text{for } 25{,}000 \le E \le \text{SRP} \\ 1 & \text{for } E > \text{SRP} \end{cases}$$

$$B = 20{,}000\,(1 - W)$$

Here the SRP is the state *Self-Rating Point*, 25 times the state average serious cost per case. This approach provides a nominal indexing to plan credibilities and ratable loss limits that should vary by state and by year.

In the former multi-split formula, the primary portion of a loss $L$ was $L_p$.

$$L_p = \begin{cases} L & \text{if } L \le 2{,}000 \\ \dfrac{10{,}000L}{8{,}000 + L} & \text{if } L > 2{,}000 \end{cases} \tag{3.2}$$

To calculate the excess portion, losses are limited on a per-claim basis to 10% of the SRP, and on a per-accident basis to 20% of the SRP. Denoting the loss so limited by $L_r$, the excess portion of a loss greater than $2,000 would be

$$L_x = L_r - L_p \tag{3.3}$$

where $L_p$ is calculated as noted above.

*B. Basic Specifications of the Proposed Plan*

Many of the elements of the former plan are retained, including ELRs and D-ratios by class, the primary-excess split formula, and state ratable loss limitations. Payroll (in hundreds) by class is extended by the respective class ELRs to produce the total expected loss. D-ratios, which also vary by class, measure the primary portion of expected loss.

Putting $B = K_p$ and $W = (E + K_p)/(E + K_x)$ into Equation 3.1 results in algebraic equivalence of the new modification formula, Equation 1.10, and the former formula, Equation 3.1. Throughout the testing used to evaluate parameters, the NCCI researchers used Equation 1.10 for the mod, and concentrated on finding best values of $K_p$ and $K_x$. The values of

$K_p$ and $K_x$ that worked best in all the testing lead to values of $W$ and $B$ quite unlike the former plan's values.

It is highly desirable that differences in benefit levels by state be reflected in the credibility constants $K_p$ and $K_x$. The former formula used the SRP to effect a nominal difference in the $W$ and $B$ tables by state, but only really affected the risks whose expected losses were near the SRP. We want to use an adjustment that results in a true scaling by state, which would be valid across all risk sizes. That objective is accomplished by inserting a value $G$, measuring relative benefit levels by state, into the formulae for $K_p$ and $K_x$. Equation 1.9 is modified to make the following expression for $K_p$ by state:

$$K_p = E\left(\frac{CE + GD}{E + GF}\right). \tag{3.4}$$

A similar change was made to the formula for $K_x$.

The $G$-value not only accounts for differences in benefit levels, but also indexes credibility constants for inflation in average claim costs. This property is seen in the following analysis. Assume inflation of $1 + i$ between times $t$ and $s$. For example, let primary credibility at time $t$ be given by

$$Z(t) = \frac{E}{E + K_p}$$

$$= \frac{E}{E + E\left(\frac{CE + GD}{E + GF}\right)}.$$

With inflation but no real growth, both $E$ and $G$ increase by the factor $1 + i$. This factor cancels everywhere in the formula for Z(s) so that

$$Z(s) = Z(t).$$

The formula for $G$ is one of the parameters that can be varied to optimize the test statistic. In most of the initial NCCI testing, $G$ was taken as a linear function of the existing SRP.

The SRP is retained, but only for use in limitation of ratable losses. There can be no self-rating under any analytic plan, so the SRP is renamed the *State Reference Point.*

*C. Tested Plans*

One assumption underlying Equation 1.9 for the credibility ballast values $K_p$ and $K_x$ is that both primary and excess credibilities depend on total expected losses, $E$. The same assumption underlies the former formula, which is Perryman's First Formula. We call the first alternate formula *Perryman I* because it borrows much from the original.

Ultimately, the NCCI researchers tested four alternative plans in addition to the former plan, herein called Current Multi-Split. For each alternative plan, optimal values of the credibility parameters were chosen based on results of the testing. The selection of a final plan from among the four optimized alternatives took into consideration not only the associated values of the test statistic, but also the ease of understanding and implementation.

The tested plans include :

1. Current Multi-Split;
2. Perryman I Multi-Split;
3. Perryman II Multi-Split;
4. Perryman I Single-Split; and
5. Perryman II Single-Split.

Their specifications follow.

*1. Current Multi-Split*

The basic specifications for this plan have been given. They include the formulae for $B$, $W$, the SRP, the primary/excess split of actual losses, and the modification formula itself. They also include calculation of the ELRs and D-ratios by class. The rating values of each insured are included in the experience rating files for each year. In particular, rating years 1981 through 1984 were used in the testing.

*2. Perryman I Multi-Split*

As described in the introduction, this is the first alternative to the former plan. It is Equation 1.10, with Equation 3.2 used to split actual losses into primary and excess components. Values such as ELRs and D-ratios can be carried over directly from the experience rating files, while $K_p$ and $K_x$ can be calculated easily from the elements of the files: namely, total expected losses of the risk, state identification of the risk (which would be used to fetch indexed SRP and $G$ values), and three coefficients for each formula, selected by trial and error.

*3. Perryman II Multi-Split*

This formula results from a different assumption about loss variance than the one used in Perryman I. It is only nominally related to Perryman's Second Formula, as noted below.

In the version tested, it is hypothesized that conditional primary loss variance is a function of expected primary losses and that excess loss variance is a similar function of expected excess losses.

The formula for credibilities takes the following form:

$$Z_p = \frac{E_p}{E_p + \tilde{K}_p} \tag{3.5}$$

where $\tilde{K}_p = CE + GH/(1 + GF/E)$, and $E_p$ is expected primary losses. Notice that $\tilde{K}_p$ ought to be expressed in terms of $E_p$, not $E$. This, however, further complicates the formula. The selection of $C$, $F$, and $H$, as determined by performance, could incorporate average D-ratios, if appropriate, and $\tilde{K}_*$ could be a function of total expected losses. The resulting credibility parameters could be put in tabular form by state according to expected primary or excess losses.

Denoting the average D-ratio by risk as $\delta$ results in the following formulae:

$$Z_p = \frac{E_p}{E_p + \tilde{K}_p}$$

$$= \frac{\delta E}{\delta E + \tilde{K}_p}$$

$$= \frac{E}{E + \tilde{K}_p/\delta} . \tag{3.6}$$

Similarly,

$$Z_x = \frac{E_x}{E_x + \tilde{K}_x} ,$$

which yields

$$Z_x = \frac{E}{E + \tilde{K}_x/(1 - \delta)} . \tag{3.7}$$

Testing of this plan was accomplished using values available from the experience rating files.

For the sake of historical accuracy, the true Perryman's Second Formula actually resulted from the unusual expressions for credibilities

$$Z_p = \frac{E}{E_p + K_p} ;$$

$$Z_x = WZ_p .$$

Perryman does not derive these expressions and attempts (somewhat less than successfully) to rationalize their contradiction of credibility principles [8].

*4. Perryman I Single-Split*

One of the key assumptions of the tested formulae is the non-correlation of primary and excess loss components. As long as the primary losses had a severity component, the NCCI researchers were not fully satisfied with a credibility-based plan that uses the former primary-excess split.

It is classically assumed that frequency and severity are independent, hence uncorrelated. This is probably not a valid assumption, but it is reasonable. It is less reasonable to assume that primary and excess losses defined by the multi-split formula are uncorrelated. Thus, the NCCI researchers considered using a modification formula based strictly on frequency and severity. One problem with this idea would be the difficulty of obtaining a valid claim count. (For example, are small medical-only claims recorded on a consistent basis by all carriers for all risks?) Because this change would require the cooperation of so many different interests, it was not pursued.

A compromise is to use a single split (into primary and excess categories) of losses. The portion of a loss below the single threshold value would be primary; and the portion of a loss in excess of that value, if any, would be excess. Using $2,000 as the single-split point is a relatively easy choice: it is the smallest size for which individual claims data is reported, so it is the closest to a frequency/severity dichotomy we can obtain using available data.

To test a single-split plan against actual risk experience, expected losses can be taken directly from experience rating files. New D-ratios corresponding to the new split formula are needed. They are developed by adjusting the multi-split D-ratios in the files to maintain the aggregate adequacy of the D-ratios. For example, if the aggregate emerged actual-primary to ratable-total losses under the former formula had been 0.40, and under the new split it is 0.38, the D-ratios would all be adjusted (downward) by the factor (0.38)/(0.40). With D-ratios so adjusted, the formula is tested with $K_p$ and $K_e$ in the established Equation 1.10, until optimal values for the six coefficients are obtained.

*5. Perryman II Single-Split*

The last plan to test was the one that utilizes two major variations from Plan 2, Perryman I Multi-Split. Plan 5 uses a single primary-excess split of losses, with the credibility formulae from Plan 3. This is the "fully equipped" model as compared to the other "economy" versions. The question is whether there is enough improvement in performance to justify the additional cost and more difficult handling.

*D. Summary*

The NCCI researchers tested experience ratings effective on 1980 and 1981 policies. Best parametrizations for each of the plans tested may be seen in Exhibit 2. Of course, "best" is subjective in that no single set of coefficients in any plan produced a lowest value for all 10 evaluations (five size groups and two years). Still, the pattern that emerged for all evaluations was that the smaller sizes deserved more credibility and the larger sizes deserved much less credibility than under the current plan.

Exhibit 3 shows summary statistics and a sample calculation of the test statistic for Size Group Two in 1980.

Several credibilities are displayed in Figures 1, 2, and 3. The consistent pattern for the four optimized plans can be seen. The plans also bear a fairly logical relation to each other. In particular, credibilities seem to increase substantially in the passage from a multi-split to a single-split formula. This may be due to better satisfaction of the assumption that primary and excess losses are uncorrelated.

By contrast, the use of the Perryman II equation in place of Perryman I does not seem to increase average credibilities much. There is, of course, a slight improvement in the distribution of the credibility assigned to the individual risk, as reflected in the test statistic. As described in Section 4, the evaluation of all the plans included weighing the benefit of increased accuracy against the cost of increased complexity in application.

## EXHIBIT 2

### 1. Perryman I Multi-Split

$$K_p = E\left[\frac{0.067\,E + 17{,}200G}{E + 3{,}100\,G}\right] \qquad K_x = E\left[\frac{0.60\,E + 563{,}000\,G}{E + 5{,}000\,G}\right]$$

$$G = \text{SRP}/570{,}000$$

### 2. Perryman II Multi-Split

$$K_p = E\left[\frac{0.068\,E + 7{,}000\,G}{E + 1{,}600\,G}\right] \qquad K_x = E\left[\frac{0.67\,E + 263{,}700\,G}{E + 5{,}500\,G}\right]$$

$$G = 1$$

### 3. Perryman I Single-Split *

$$K_p = E\left[\frac{0.10\,E + 2{,}570\,G}{E + 700\,G}\right] \qquad K_x = E\left[\frac{0.75\,E + 203{,}825\,G}{E + 5{,}100\,G}\right]$$

$$G = 0.85 + \text{SRP}/2{,}700{,}000$$

### 4. Perryman II Single-Split

$$K_p = 0.04\,E + 850\,G \qquad K_x = E\left[\frac{0.60\,E + 98{,}500\,G}{E + 2{,}500\,G}\right]$$

$$G = \text{SRP}/570{,}000$$

* As developed in the text, this is the form of the credibility constants used in the final plan, except that minimum values were established: min $K_p = 7{,}500$, min $K_x = 150{,}000$.

Also, $G$ was changed so that $G = \frac{\text{SACC}}{1{,}000}$, where SACC is the average cost per case by state. At the same time, the SRP was defined as the State Reference Point, SRP $= 250 \times$ SACC, so $G = \text{SRP}/250{,}000$.

## EXHIBIT 3
## PART 1
### SAMPLE STATISTICS

| Formula | 1980 Size Group One | 1980 Size Group Two | 1980 Size Group Three | 1980 Size Group Four | 1980 Size Group Five | 1981 Size Group One | 1981 Size Group Two | 1981 Size Group Three | 1981 Size Group Four | 1981 Size Group Five |
|---|---|---|---|---|---|---|---|---|---|---|
| Former Plan | 0.3277 | 0.2236 | 0.0918 | 0.0228 | 0.0293 | 0.3230 | 0.2361 | 0.1116 | 0.0453 | 0.2187 |
| Perryman I Multi-Split | 0.1978 | 0.1248 | 0.0994 | 0.0148 | 0.0012 | 0.2664 | 0.1674 | 0.0930 | 0.0380 | 0.0831 |
| Perryman II Multi-Split | 0.1632 | 0.1058 | 0.0976 | 0.0112 | 0.0033 | 0.1809 | 0.1333 | 0.0985 | 0.0414 | 0.0980 |
| Perryman I Single-Split | 0.0852 | 0.0519 | 0.0459 | 0.0169 | 0.0042 | 0.1140 | 0.0838 | 0.0688 | 0.0331 | 0.0782 |
| Perryman II Single-Split | 0.0803 | 0.0366 | 0.0380 | 0.0091 | 0.0075 | 0.0785 | 0.0735 | 0.0583 | 0.0312 | 0.0187 |

Smaller statistics are more desirable. There were many different samples tested. This was one of the series of tests used to select among the choices.

# EXHIBIT 3
## PART 2

## 1980
## ACTUAL TO EXPECTED LOSS RATIOS BEFORE AND AFTER EXPERIENCE RATING

### 15 STATES TOTAL
### Risk Size: \$5,000-\$10,000

### PERRYMAN I SINGLE-SPLIT

| Quintile | Before | Squared Deviation From Mean | After | Squared Deviation From Mean |
|---|---|---|---|---|
| 1 | 0.62 | 972 | 0.83 | 106 |
| 2 | 0.76 | 295 | 0.95 | 2 |
| 3 | 0.86 | 58 | 0.95 | 4 |
| 4 | 1.05 | 148 | 1.00 | 42 |
| 5 | 1.32 | 1,532 | 0.92 | 2 |
| Mean Total: | 0.93 | 3,005 | 0.93 | 156 |

Test Statistic

$$\frac{156}{3{,}005} = 0.0519$$

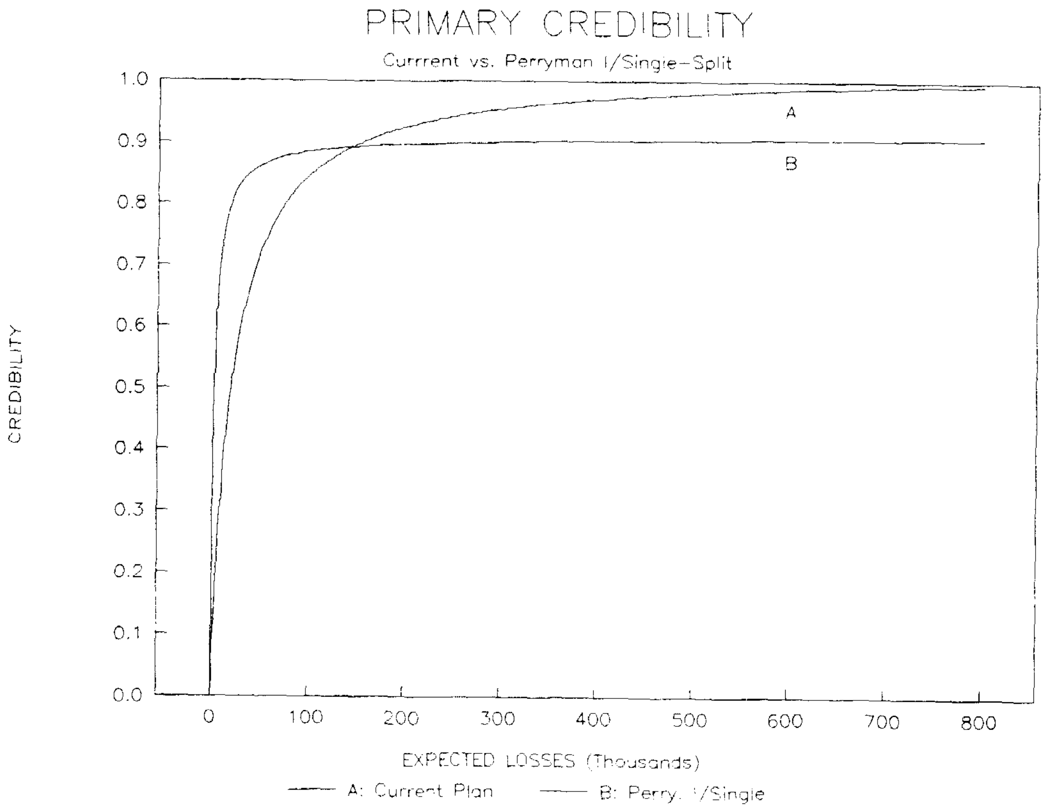
### CURRENT MULTI-SPLIT

| Quintile | Before | Squared Deviation From Mean | After | Squared Deviation From Mean |
|---|---|---|---|---|
| 1 | 0.68 | 623 | 0.79 | 192 |
| 2 | 0.70 | 532 | 0.78 | 214 |
| 3 | 0.87 | 37 | 0.93 | 0 |
| 4 | 1.06 | 173 | 1.04 | 112 |
| 5 | 1.30 | 1,372 | 1.03 | 94 |
| Mean Total: | 0.93 | 2,737 | 0.93 | 612 |

Test Statistic

$$\frac{612}{2{,}737} = 0.2236$$

FIGURE 1



PRIMARY CREDIBILITY
Currrent vs. Perryman I/Single-Split
CREDIBILITY
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
A
B
0
100
200
300
400
500
600
700
800
EXPECTED LOSSES (Thousands)
A: Current Plan
B: Perry. I/Single

**FIGURE 2**

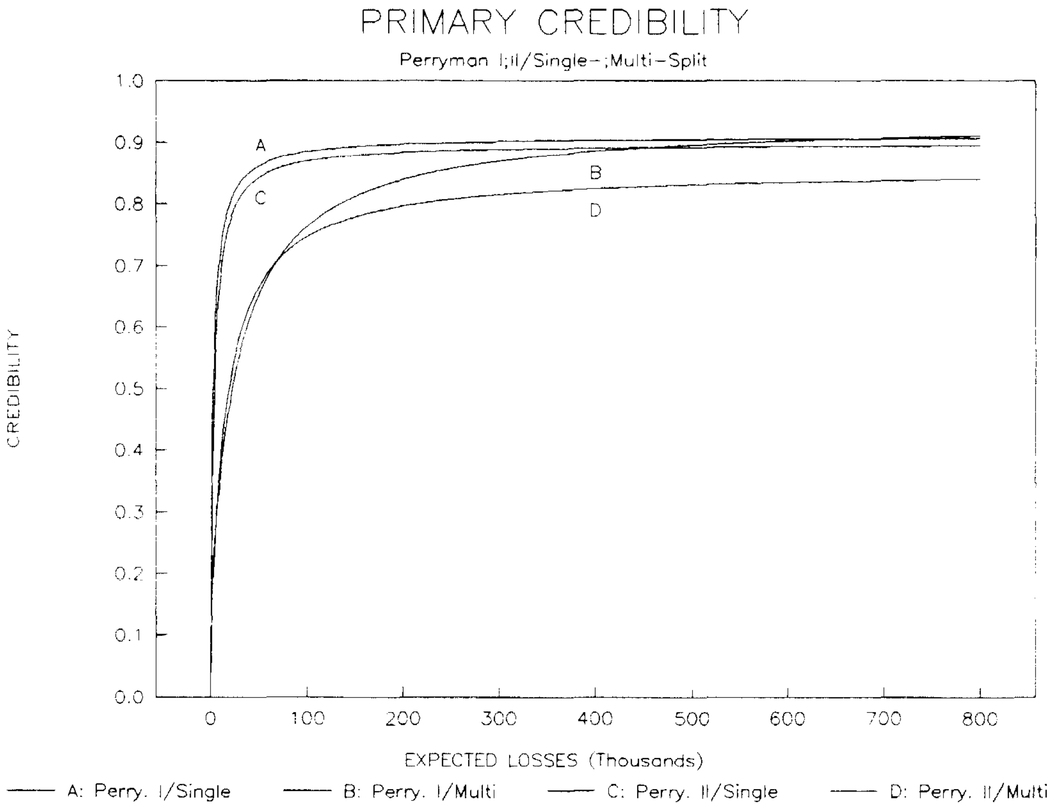
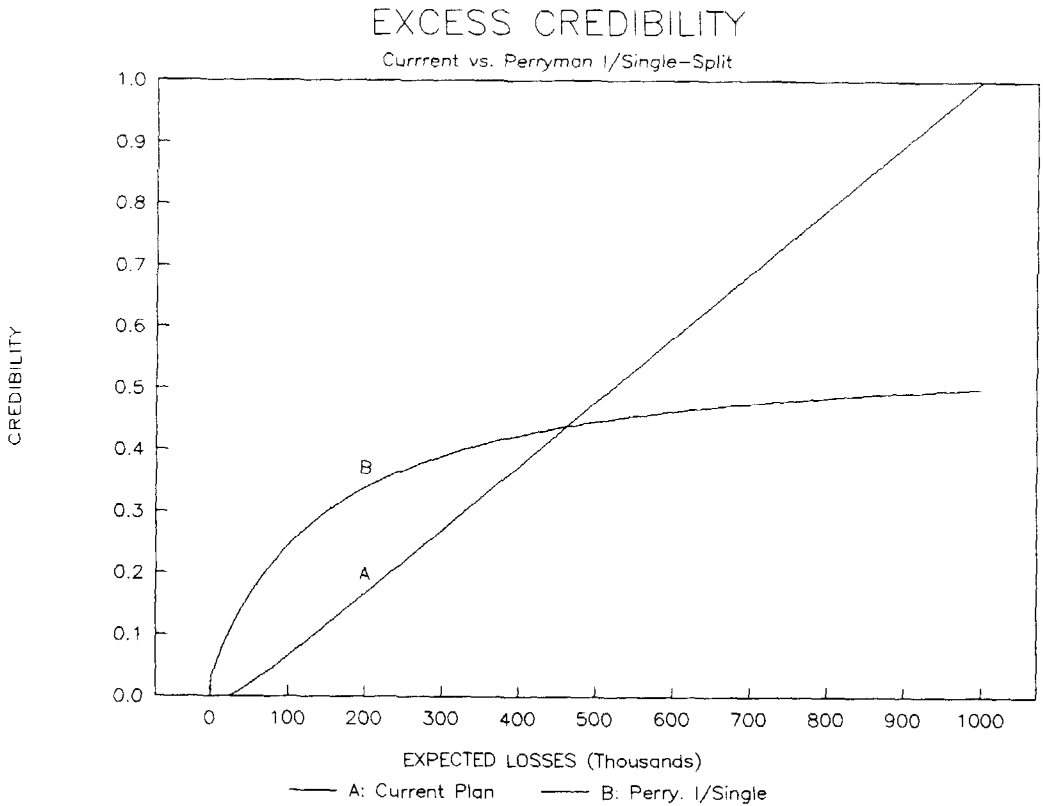

PRIMARY CREDIBILITY
Perryman I;II/Single-;Multi-Split
CREDIBILITY
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
100
200
300
400
500
600
700
800
EXPECTED LOSSES (Thousands)
A
B
C
D
A: Perry. I/Single
B: Perry. I/Multi
C: Perry. II/Single
D: Perry. II/Multi

FIGURE 3


EXCESS CREDIBILITY
Currrent vs. Perryman I/Single-Split
CREDIBILITY
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
B
A
0
100
200
300
400
500
600
700
800
900
1000
EXPECTED LOSSES (Thousands)
A: Current Plan
B: Perry. I/Single

## 4. PARAMETRIZING THE PLAN

### *A. Selection of Perryman I Single-Split*

Selection of an experience rating formula was made primarily on the basis of performance, of course, but also on the basis of practical considerations. Ease of acceptance and implementation were among the considerations. Fortunately, this did not lead to any great compromise of actuarial principles.

The Individual Risk Rating Plans (IRRP) Subcommittee of the NCCI Actuarial Committee approved the Perryman I Single-Split plan as parametrized in Exhibit 2. Its performance was nearly as good as the Perryman II Single-Split plan, but the slight improvement offered by the latter did not appear to outweigh the effort necessary to make the more complex changes. The improvement offered by a single-split over a multi-split plan was significant and the transition would not be difficult.

### *B. Decreasing Swing*

Consider Exhibit 4. It shows the average change in modification, plan to plan, for small risks grouped by value of the former modification. These risks constituted the smallest size group used in testing the original formula, as well as in testing this particular plan. Small risks whose 1985 mod exceeded 1.20 could expect an average increase, or "swing," of 62 points in their mods! Even if this selected plan reflected correctly calibrated credibility, several of the subcommittee representatives thought it would be unacceptable in the market. Some even doubted that it was correct at all, despite the evidence that credibilities were optimal for this size group. Of course, the tests worked on averages and these were extreme cases. Thus, it was possible to believe the tests, yet still believe there was a problem to be fixed.

To address this problem, two changes were made to the plan: the SRP was decreased, as it affected limitations on ratable losses; and minimum values for credibility constants $K_p$ and $K_x$ were established.

It was decided to make the SRP a multiple of average cost per case by state (SACC), rather than a multiple of the average cost per serious case.

# EXHIBIT 4

## 1985

## PERRYMAN I SINGLE-SPLIT

## CHANGES IN AVERAGE MODIFICATIONS FOR RISKS GROUPED BY VALUE OF CURRENT MOD

## \$2,500-\$5,000 EXPECTED LOSSES DURING EXPERIENCE PERIOD

| Range of Current Mods | Number of Risks | Average Current Mod | Average Proposed Mod | Change |
|---|---|---|---|---|
| 0.80-0.84 | 1 | 0.83 | 0.66 | -0.17 |
| 0.85-0.89 | 32 | 0.89 | 0.70 | -0.19 |
| 0.90-0.94 | 9,062 | 0.93 | 0.79 | -0.14 |
| 0.95-0.99 | 9,600 | 0.96 | 0.87 | -0.09 |
| 1.00 | 484 | 1.00 | 1.02 | 0.02 |
| 1.01-1.05 | 1,637 | 1.03 | 1.13 | 0.10 |
| 1.06-1.10 | 1,238 | 1.08 | 1.31 | 0.23 |
| 1.11-1.15 | 944 | 1.13 | 1.42 | 0.29 |
| 1.16-1.20 | 742 | 1.18 | 1.48 | 0.30 |
| Over 1.20 | 2,578 | 1.35 | 1.97 | 0.62 |
| Totals | 26,318 | 1.01 | 1.03 | 0.02 |

Since the average cost per case was between \$750 and \$3,500 for most states in 1981, a multiple of $250 \times$ SACC generally led to smaller SRPs than the typical \$1 million SRPs effective at the time.

In order to test this plan, it was necessary to adjust the ELRs to compensate for the new limits on ratable losses. This was accomplished just like the adjustment of the D-ratios as described in Section 3(C.4).

Several minimum values of $K_p$ and $K_x$ were tested also, but the analysis quickly led to min $K_x$ = 150,000 and min $K_p$ = 7,500, which worked well in conjunction with the new loss limitations in the range above.

Exhibit 5 shows comparisons of the swing in mods for groups of risks by size with a 1986 mod greater than 1.2. (Computed for SRP = 250 or $300 \times$ SACC, and with min $K_p$ = 7,500 and min $K_x$ = 150,000.) In all cases, the swing was less than 25 points.

Changing the SRP formula also led to a re-examination of the calculation of the state scale factor $G$. The older $G$ formula may be seen in Exhibit 2 as $G = 0.85 + \text{SRP}/2{,}700{,}000$, where the SRP was the value from the former plan. The new formula, resulting from some trial and error, was $G = \text{SACC}/1{,}000$, which worked well with the modified plan.

*C. Caps on Modifications*

Independent tests of the new plan still showed the potential for large swings in the values of the mod for risks in the smaller size categories. After considerable discussion, the IRRP subcommittee recommended one more change to the rating plan. Rather than tamper with credibility constants, loss limitations, or split points, the subcommittee decided to put absolute caps on the mods of smaller size risks. In this way a debit under the current formula could increase only a limited amount with the change to the new formula.

# EXHIBIT 5

## 1986

## PERRYMAN I SINGLE-SPLIT, AS MODIFIED*

## CHANGE IN CURRENT MOD TO PROPOSED MOD

## RISK HAVING CURRENT MOD GREATER THAN 1.2

| SRP | Overall | Size Group One | Size Group Two | Size Group Three | Size Group Four | Size Group Five |
|---|---|---|---|---|---|---|
| 250 × SACC | 0.06 | 0.22 | 0.21 | 0.16 | 0.09 | -0.05 |
| 300 × SACC | 0.06 | 0.22 | 0.21 | 0.16 | 0.09 | -0.05 |

* SRP = 250 or 300 × SACC
Minimum $K_p = 7{,}500$ and $K_x = 150{,}000$

# EXHIBIT 6

## 1986

## CHANGES IN AVERAGE MODIFICATIONS FOR RISKS GROUPED BY VALUE OF CURRENT MOD

### \$2,500-\$5,000 EXPECTED LOSSES DURING EXPERIENCE PERIOD

| Range of Current Mod | Number of Risks | Average Current Mod | Average Proposed Mod | Change |
|---|---|---|---|---|
| 0.80-0.84 | 18 | 0.83 | 0.79 | -0.04 |
| 0.85-0.89 | 620 | 0.88 | 0.84 | -0.04 |
| 0.90-0.94 | 10,951 | 0.93 | 0.88 | -0.05 |
| 0.95-0.99 | 13,522 | 0.96 | 0.93 | -0.03 |
| 1.00 | 682 | 1.00 | 1.02 | 0.02 |
| 1.01-1.05 | 2,307 | 1.03 | 1.08 | 0.05 |
| 1.06-1.10 | 1,675 | 1.08 | 1.20 | 0.12 |
| 1.11-1.15 | 1,278 | 1.13 | 1.26 | 0.13 |
| 1.16-1.20 | 1,066 | 1.18 | 1.31 | 0.13 |
| Over 1.20 | 3,576 | 1.35 | 1.49 | 0.14 |
| Totals | 35,695 | 1.01 | 1.02 | 0.01 |

With minimum $K_p = 7{,}500$ and minimum $K_x = 150{,}000$

SRP = 250 × SACC

$G$ = SACC ÷ 1,000

Mods Limited

The following table lists the limits by size.

| Expected Loss Size | Maximum Modification |
|---|---|
| $0 < E < 5{,}000$ | 1.6 |
| $5{,}000 \le E < 10{,}000$ | 1.8 |
| $10{,}000 \le E < 15{,}000$ | 2.0 |

Exhibit 6 is similar to Exhibit 4 and shows that the swing problem is greatly reduced by this action.

No transition program was designed to phase the maximums out, except the impact of inflation. As experience rating eligibility increased, fewer risks would enjoy a potential 1.60 cap.

Although this action was largely pragmatic, it was not without some actuarial justification. Mods higher than the stated limits are probably not deserved, statistical arguments notwithstanding. The test results only showed that mods were correct for the worst 20% of risks *on average*. Other testing (not shown), using higher percentiles than the $80^{\text{th}}$, showed the new formula could result in unreasonably high mods, at least for the smallest risks. In addition, risks just below eligibility have a maximum modification of unity. There should be some continuity at the point of eligibility.

Exhibit 7 shows the quintiles test statistics for the finalized plan. These statistics compare reasonably well with the statistics shown in Exhibit 3—Part I.

*D. Trending the Split Point*

In the matter of a split point, the subcommittee also recommended that appropriate trending be applied to the single-split point used in the testing so that it would have the same relativity to the loss size distribution when actually applied. Trending was based on several years of change in the average cost per case, which led to the single-split point of $5,000 used in the filing. This is a reasonable value, given that it will be well after 1990 before the revised plan is widely accepted, and a few more years after that before any study can be done to revise the point.

# EXHIBIT 7

## 1981

### SUMMARY—QUINTILES TEST STATISTICS

| SRP | Size Group One | Size Group Two | Size Group Three | Size Group Four | Size Group Five |
|---|---|---|---|---|---|
| $250 \times$ SACC | 0.1800 | 0.1183 | 0.0546 | 0.0127 | 0.1588 |
| $300 \times$ SACC | 0.1704 | 0.1150 | 0.0645 | 0.0524 | 0.3138 |

SRP = 250 or $300 \times$ SACC

Minimum $K_p = 7{,}500$ and $K_x = 150{,}000$

Still, several researchers thought $5,000 might be too high. After all, the single-split worked well because of its resemblance to a frequency/severity model, and $5,000 may be too high to resemble just claim frequency.

The rationale for such a high selected value was twofold. First, there would be no automatic adjustment, and the $5,000 would be retained until a study could be made to determine the optimal value. Second, the resultant D-ratios for the single-split point would not have to decrease dramatically from the ones in the former plan. Loss size trend would make primary losses increasingly resemble claim count.

It should be noted that a conscious decision was made not to index the split point. As a consequence, it can be expected that average D-ratios will decrease over time. Primary credibilities should be monitored. An initiative to study indexing the split point can be expected in the future.

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# THE COMPUTATION OF AGGREGATE LOSS DISTRIBUTIONS

JOHN P. ROBERTSON


*Abstract*

*Paul R. Halmos recently hailed the fast Fourier transform as one of the 22 most significant developments in mathematics in the last 75 years. This paper provides an application of this tool to the computation of aggregate loss distributions from arbitrary frequency and severity distributions. All necessary mathematics is developed in the paper, complete algorithms are given, and examples are provided. Sufficient details are given to allow implementation in any computer language, and sample APL computer language routines are given. The final section includes a discussion of excess loss distributions where computation is not limited to the fast Fourier transform based algorithm.*


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## 1. INTRODUCTION

According to Halmos [1], the fast Fourier transform is one of this century's most significant mathematical developments. This paper presents an algorithm for computing aggregate loss distributions using this device. The algorithm assumes that one knows the claim count distribution, $T$ (the probability distribution of the number of claims that will occur), and the severity distribution of a single claim, $S = S_1 = S_2 = \ldots$ (the distribution of the amount of a single claim). The algorithm computes the aggregate loss distribution,

$$AGG = S_1 + S_2 + \ldots + S_T$$

(the distribution of the total amount of claims). The algorithm applies to arbitrary frequency and severity distributions.

As an example, claim counts might be expected to follow a Poisson distribution with mean 10, and severity might be expected to follow some distribution with mean \$10,000. This implies that the mean of the aggregate distribution is \$100,000 (10 times \$10,000). In any given year, the total amount of claims might vary from \$100,000 because the actual number of claims might differ from 10, and because individual claims will vary from the \$10,000 mean. The aggregate distribution expresses the probabilities of the possible total amounts of claims in the same way the severity distribution expresses the probabilities of the amounts of a single claim.

The algorithm given here is less of a "black box" than some other algorithms presented in these *Proceedings* in the following way. The algorithm, as a matter of course, computes the distribution for the sum of $n$ claims, where $n$ is any number of claims with nonzero probability in the claim count distribution. While the computer routines presented herein do not save these distributions, only trivial programming changes are needed to capture and save these distributions for later use. Capturing these distributions can be useful when the resulting aggregate distribution has unexpected properties and one wants to check that it is being correctly computed, or for other reasons.

The method presented here should be considered to be approximate. Technically, it is an exact method, but it is generally necessary to use an approximation of the severity distribution as input, and this makes the output approximate. The running time for the algorithm is roughly proportional to the number of claims expected. For small numbers of claims, this method seems to be faster than other methods (e.g., Heckman and Meyers [2]), but this advantage disappears as the number of claims grows. The algorithm presented here explicitly computes the entire aggregate distribution up to a specified limit, making it easy to derive any statistics of the aggregate distribution.

A quick overview of the algorithm is as follows. Everything in this summary will be described fully below, as it is not possible to give brief rigorous definitions of all the concepts used. The severity distribution will be given a discrete representation; that is, the severity distribution will be represented by a vector. The $n$-fold (discrete) *convolutions* of this vector with itself are computed. The result of these convolutions is very nearly the vector representation of the $n$-fold sum of the severity distribution with itself. The precise representation of an $n$-fold sum will be obtained by computing the convolution of this result with another vector (described later) to "spread out" the result a bit more. This representation of the density function for the $n$-fold sum of claims is multiplied by the probability of there being exactly $n$ claims, and these products are added to get the vector representation of the aggregate distribution.

The *discrete Fourier transform* is used to compute the convolutions, and the *fast Fourier transform* is used for rapid computation of the discrete Fourier transform. Convolutions, discrete Fourier transforms, and fast Fourier transforms are defined and discussed in Section 2. The purpose of this discussion is to introduce these items and to give examples so the main structure of the algorithm will be clear. The technical details are in the appendices. Additionally, Section 2 discusses the vector used to "spread out" the $n$-fold convolutions of the vector representing the severity distribution. Two tactics used to speed the overall computations are also covered.

Section 3 of the paper walks through the full algorithm. Section 4 gives examples and discusses use of the algorithm. Sufficient details are given throughout the paper that it should be possible to implement the algorithm in any computer language. As an example, various appendices show routines implementing the algorithm in the APL computer language.

## 2. CONVOLUTIONS AND THE FAST FOURIER TRANSFORM

### *Convolutions*

The distribution of the sum of two random variables is given by the *convolution* of their respective distributions. Heckman and Meyers [2, p. 32] discuss convolutions for the case of continuous random variables. In this case, if $X_1$ and $X_2$ are independent continuous random variables with density functions $f$ and $g$, then the density of the sum of these two variables, i.e., the density of the random variable $X_1 + X_2$, is given by the convolution of $f$ and $g$, $f*g$, defined as:

$$(f*g)(x) = \int_0^x f(t)\, g(x-t)\, dt\,.$$

For the algorithm presented here, the probability distributions for the severity of a single claim and for the sum of $n$ claims will be given certain discrete representations; that is, they will be represented by certain vectors. It will be necessary to compute the convolutions of these vectors. The definition of the convolution of vectors is similar to the above definition of the convolution of continuous functions. Let $\boldsymbol{U} = (u_0, u_1, \ldots, u_{n-1})$ and $\boldsymbol{V} = (v_0, v_1, \ldots, v_{n-1})$ be two vectors of the same length, $n$. Their discrete convolution, $\boldsymbol{W} = \boldsymbol{U}*\boldsymbol{V}$, is a vector of length $n$ defined by:

$$w_i = \sum_{j=0}^{n-1} u_j\, v_{i-j}\,,$$

where $0 \le i \le n-1$ and the indices of the terms $v_{i-j}$ are taken modulo $n$. For example, if $\boldsymbol{U} = (1, 2, 3)$ and $\boldsymbol{V} = (4, 5, 6)$ , then

$$\begin{aligned}\boldsymbol{U}*\boldsymbol{V} &= (1\times4 + 2\times6 + 3\times5,\ 1\times5 + 2\times4 + 3\times6,\ 1\times6 + 2\times5 + 3\times4)\\ &= (31, 31, 28).\end{aligned}$$

This definition of convolution is not exactly what is needed here. The *no-wrap convolution* of $\boldsymbol{U}$ with $\boldsymbol{V}$ is defined to have the following components:

$$w_i = \sum_{j=0}^{i} u_j\, v_{i-j}\,.$$

That is:

$$w_0 = u_0\, v_0,$$

$$w_1 = u_0\, v_1 + u_1\, v_0\,,$$

$$w_2 = u_0 v_2 + u_1 v_1 + u_2 v_0 ,$$

$$\vdots$$

$$w_{n-1} = u_0 v_{n-1} + u_1 v_{n-2} + \ldots + u_{n-1} v_0 .$$

The no-wrap convolution of $(1, 2, 3)$ with $(4, 5, 6)$ is $(1{\times}4,\ 1{\times}5 + 2{\times}4,\ 1{\times}6 + 2{\times}5 + 3{\times}4)$ or $(4, 13, 28)$.

The no-wrap convolution can be visualized, as below, by taking one vector, reversing it, and placing it so that its first element is directly below the first element of the other vector. Then successively shift the vectors together, multiply elements in the same column, and add the products. Repeat this until the vectors are completely aligned.

<u>No-wrap Convolution</u>

$$\begin{array}{ccccc} & & 1 & 2 & 3 \\ 6 & 5 & 4 & & \\ & & 1\times 4 & & \end{array} \qquad \begin{array}{cccc} & 1 & 2 & 3 \\ 6 & 5 & 4 & \\ & 1\times 5+2\times 4 & & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 6 & 5 & 4 \\ & 1\times 6+2\times 5+3\times 4 & \end{array}$$

In contrast, for regular convolutions, the bottom vector is wrapped around as shown below:

<u>Regular Convolution</u>

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 6 & 5 \\ & 1\times 4+2\times 6+3\times 5 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 5 & 4 & 6 \\ & 1\times 5+2\times 4+3\times 6 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 6 & 5 & 4 \\ & 1\times 6+2\times 5+3\times 4 & \end{array}$$

The analogy of the definition of no-wrap convolution for discrete vectors to the definition of convolution in the continuous case should be clear.

One can obtain the no-wrap convolution of two vectors from a routine that computes (regular) convolutions by padding each of the two vectors to the right with enough zeroes to double the length of each vector, performing the regular convolution with these longer vectors, and then taking just the left half of the result. For example, the first three components of the convolution of $U = (u_0, u_1, u_2, 0, 0, 0)$ with $V = (v_0, v_1, v_2, 0, 0, 0)$ are:

$$
\begin{aligned}
w_0 &= u_0\,v_0 \;+\; u_1 \times 0 \;+\; u_2 \times 0 \;+\; 0 \times 0 \;+\; 0 \times v_2 \;+\; 0 \times v_1 \\
&= u_0\,v_0\,; \\
w_1 &= u_0\,v_1 \;+\; u_1\,v_0 \;+\; u_2 \times 0 \;+\; 0 \times 0 \;+\; 0 \times 0 \;+\; 0 \times v_2 \\
&= u_0\,v_1 + u_1\,v_0\,; \\
w_2 &= u_0\,v_2 \;+\; u_1\,v_1 \;+\; u_2\,v_0 \;+\; 0 \times 0 \;+\; 0 \times 0 \;+\; 0 \times 0 \\
&= u_0\,v_2 \;+\; u_1\,v_1 \;+\; u_2\,v_0\,.
\end{aligned}
$$

The first definition of convolution will always be used (unless otherwise noted), but generally zeroes will be added to the vectors being convolved so as to achieve a no-wrap convolution.

Observe that the definition of convolution is valid when the vector elements are complex numbers.

A note on notation is needed. The vector $U$ has components $u_0$, $u_1$, $u_2$, etc., sometimes denoted $U[0]$, $U[1]$, $U[2]$, etc. In particular, the indices of vector elements start at zero.

*The Discrete Fourier Transform*

Complex numbers and complex roots of unity are used extensively in what follows. The *primitive* $n^{th}$ *roots of unity* are

$$\cos(2\pi a/n) + i\,\sin(2\pi a/n),$$

where $a$ and $n$ are relatively prime and $i$ is $\sqrt{-1}$. The properties of complex numbers needed here are reviewed in Appendix A, and are also given in Baase [3, p. 279] and Aho, Hopcraft, and Ullman [4, p. 252].

Given a complex (or real) vector $U$, the discrete Fourier transform of $U$ is a complex vector of the same length. As in Baase [3, p. 269], for $n \geq 1$, let $\omega$ be a primitive $n^{\text{th}}$ root of unity, and let $F$ be the $n \times n$ matrix with entries $f_{i,j} = \omega^{ij}$ where $0 \leq i, j \leq n-1$. The discrete Fourier transform

(DFT) of the $n$-vector $\boldsymbol{U} = (u_0, u_1, \ldots, u_{n-1})$ is the product $\boldsymbol{FU}$ (with $\boldsymbol{U}$ treated as a column vector). This is a vector of length $n$ with components:

$$\begin{array}{l}
\omega^0 u_0 + \omega^0 u_1 + \ldots + \omega^0 u_{n-2} + \omega^0 u_{n-1}\,, \\
\omega^0 u_0 + \omega^1 u_1 + \ldots + \omega^{n-2} u_{n-2} + \omega^{n-1} u_{n-1}\,, \\
\vdots \\
\omega^0 u_0 + \omega^i u_1 + \ldots + \omega^{i(n-2)} u_{n-2} + \omega^{i(n-1)} u_{n-1}\,, \\
\vdots \\
\omega^0 u_0 + \omega^{n-1} u_1 + \ldots + \omega^{(n-1)(n-2)} u_{n-2} + \omega^{(n-1)(n-1)} u_{n-1}\,.
\end{array}$$

(Note that the DFT of $\boldsymbol{U}$ depends on the $\omega$ chosen.)

Let $\boldsymbol{FU}$ be the DFT of $\boldsymbol{U}$. Given $\boldsymbol{FU}$, $\boldsymbol{U}$ is recovered (i.e., the inverse DFT is applied to $\boldsymbol{FU}$) as easily as $\boldsymbol{FU}$ is computed from $\boldsymbol{U}$. To obtain $\boldsymbol{U}$ from $\boldsymbol{FU}$, compute the DFT of $\boldsymbol{FU}$, divide each resulting term by $n$, and reverse the order of the last $n-1$ elements of this result.

*Using the DFT to Compute Convolutions*

The DFT helps compute convolutions because

$$\mathrm{DFT}(\boldsymbol{U}*\boldsymbol{V}) = \mathrm{DFT}(\boldsymbol{U}) \times \mathrm{DFT}(\boldsymbol{V})\text{ , or}$$

$$\boldsymbol{U}*\boldsymbol{V} = \mathrm{INVDFT}(\mathrm{DFT}(\boldsymbol{U}) \times \mathrm{DFT}(\boldsymbol{V})),$$

where INVDFT is the inverse DFT.

Thus, to compute the convolution of two vectors, one can compute the DFT of each vector, multiply the DFTs together pointwise, and compute the inverse DFT. This is known as the convolution theorem, proofs of which are given in Baase [3, p. 278] and Aho, Hopcraft, and Ullman [4, p. 255].

For example, let $\boldsymbol{U}$ be (1, 2, 3), let $\boldsymbol{V}$ be (4, 5, 6), and let $\omega = -0.5 + 0.866i$, a primitive third root of unity. Then $\boldsymbol{FU}$ is $(6, -1.5 - 0.866i, -1.5 + 0.866i)$, $\boldsymbol{FV}$ is $(15, -1.5 - 0.866i, -1.5 + 0.866i)$, and the pointwise product, $\boldsymbol{FU} \times \boldsymbol{FV}$, is $(90, 1.5 + 2.598i, 1.5 - 2.598i)$.

To compute the inverse DFT of this last vector, first apply the (forward) DFT to obtain (93, 84, 93). Then divide each element by 3 and reverse the order of the last two elements, giving (31, 31, 28). This matches the previous computation.

A more thorough example showing the convolution of two vectors representing severity distributions is given in Appendix B. This appendix also discusses the inverse DFT.

*The Fast Fourier Transform*

The *fast Fourier transform* (FFT) is a particularly fast method for computing the DFT (and the inverse DFT) for long vectors. Appendix C gives Baase's [3, p. 273] version of the complete fast Fourier transform and inverse FFT. Appendix D gives an APL implementation of these algorithms.

Why use such a complicated method to compute convolutions (i.e., using the convolution theorem and FFTs)? This method is used because, for long vectors, it's much faster than more straightforward methods (such as computing directly from the definition of convolution). Knuth [5, Vol. 2, p. 651] discusses the number of calculations needed to compute the fast Fourier transform. Using the FFT to convolve vectors of length 1,024 ($2^{10}$) gives a gain in speed by a factor of about 60 compared to the "naive method" of convolution. The time needed by the naive method for computing convolutions of vectors of length $n$ is proportional to $n^2$. More precisely,[1] it is $O(n^2)$. The time needed by the method using the convolution theorem and the fast Fourier transform is proportional to $n\ln(n)(O(n\ln(n)))$, where $n$ is the length of the vectors convolved. For large $n$, $n\ln(n)$ increases more slowly than $n^2$.

In the literature there are several different definitions of the DFT and FFT that accomplish the same goals but differ in certain details. Be careful before trying to use the algorithms presented here in conjunction with algorithms that appear in other sources or are available in libraries of

[1] That the number of computations is $O(f(n))$ means that there is a constant $c$ such that the number of computations is less than $c \times f(n)$ for all $n$ greater than some $n_0$. See Knuth [5, Vol. 1, p. 104].

computer routines. For example, using the forward FFT from another source with the inverse FFT given here could produce erroneous results. Other routines for the DFT and FFT can certainly be used, as long as all the routines used are consistent among themselves.

In addition to Baase [3, p. 268], the fast Fourier transform is discussed from several different viewpoints in Knuth [5, Vol. 2], Press, Flannery, Teukolsky, and Vetterling [6, p. 390], Preparata [7, p. 207], and Aho, Hopcraft, and Ullman [4, p. 257]. Each of these discusses the theory behind the FFT, thereby explaining what the FFT is doing and showing why the FFT is so efficient. Chiu [8] gives an introduction to the fast Fourier transform motivated by the problem of exact multiplication of large integers. A detailed implementation of the fast Fourier transform suitable for Fortran and similar languages is given in Monro [9, p. 153]. Convolution is discussed in Hogg and Klugman [10, p. 42], Feller [11, p. 6], and many other statistics books.

Other methods that use the fast Fourier transform to compute aggregate loss distributions are given by I. J. Good in Borch [12, p. 298] and by Bertram [13, p. 175]. The method presented in the latter is summarized in Bühlmann [14, p. 116].

## 3. THE ALGORITHM

### *The Basic Algorithm*

The *collective risk model* will be used to model the claims process. That is, the aggregate loss distribution is the distribution:

$$AGG = S_1 + S_2 + \ldots + S_T ,$$

where $T$ is a random variable for the number of claims and each $S_i$ is a random variable for severity. It is assumed that the $S_i$ are identically distributed and are pairwise independent and that the $S_i$ are independent of $T$. This definition of aggregate loss distribution is the same as that given by Algorithm 3.2 in Heckman and Meyers [2, p. 30]. This model is discussed in Bühlmann [15, p. 54], Beard, Pentikäinen, and Pesonen [16, p. 50], Patrik and John [17, p. 412], and Mayerson, Jones, and Bowers [18, p. 177].

There are three inputs to the algorithm. The first, denoted $M$, is the smallest number of claims that has nonzero probability in the claim count distribution. The second, $\boldsymbol{P}$, is a vector giving the probability density function of the claim count distribution. $\boldsymbol{P}(i)$ is the probability that there are exactly $i + M$ claims for $i \geq 0$.

The third input, $\boldsymbol{S}$, is a vector representing the severity distribution as a piecewise uniform distribution. Due to technical considerations involving the fast Fourier transform, the length, $n$, of $\boldsymbol{S}$ will be an integral power of two; i.e., $n = 2^k$ for some positive integer $k$. Let $L$ be the maximum size of claim considered. Then each element $s_i$ of $\boldsymbol{S}$ represents the probability that a given claim is at least $\frac{i}{n}L$ but less than $\frac{(i+1)}{n}L$. The probability distribution is uniform across each such interval. In other words, the probability density function, $f(x)$, of the claim size distribution is:

$$f(x) = \begin{cases} s_i \div \left(\dfrac{L}{n}\right), & \text{for } \dfrac{i}{n} L \leq x < \dfrac{i+1}{n} L,\ 0 \leq i \leq n-1; \\ 0, & \text{for } x < 0 \text{ or } x \geq L. \end{cases}$$

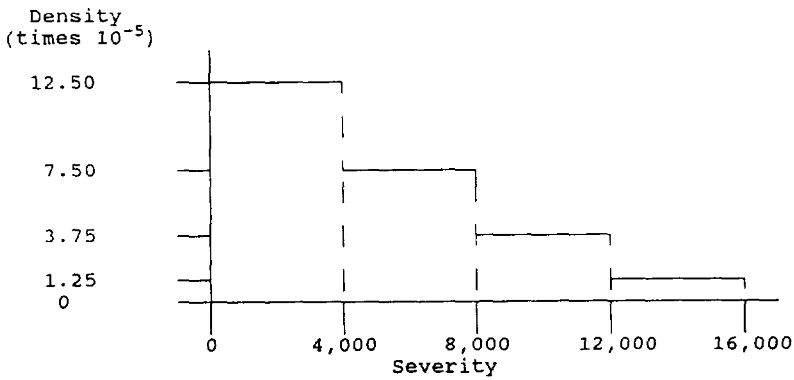
As an example, if $L$ is \$16,000 and $\boldsymbol{S}$ is (0.50, 0.30, 0.15, 0.05), then there is a 50% probability that any given claim is between \$0 and \$4,000, a 30% probability that the claim is between \$4,000 and \$8,000, etc. The density function is uniform over the interval \$0 to \$4,000 at .000125 (= 0.50 ÷ 4,000). Similarly, the density function is uniformly .000075; i.e., (0.30 ÷ 4,000) over the interval \$4,000 to \$8,000, and so on. This is graphed in Figure 1. Note that this specification of the severity distribution is the same as that in Heckman and Meyers [2] but restricted to uniform intervals.

Define $\boldsymbol{S}^i$ inductively as follows. Let $\boldsymbol{S}^1$ be the vector of length $2n$ obtained by catenating $n$ zeroes onto $\boldsymbol{S}$. That is,

$$s_i^1 = \begin{cases} s_i, & \text{if } 0 \leq i \leq n-1; \\ 0, & \text{if } n \leq i \leq 2n-1. \end{cases}$$

FIGURE 1

DENSITY FUNCTION REPRESENTED BY $S$




Let $S^{*i} = S^1 * S^{i-1}$ for $i \geq 2$.

Let $S^i$ be the same as $S^{*i}$ for the first $n$ elements and be 0 for subsequent elements. Then the first $n$ elements of $S^i$ are the first $n$ elements of the no-wrap convolution of $S$ with itself $i$ times.

Everything will be defined in greater detail below, but the algorithm is simply summarized as computing:

$$\sum_{i=M}^{N} P(i-M) \times A^i * S^i,$$

where

$M$ is the smallest number of claims with nonzero probability; $N$ is the largest number,

$P(i - M)$ is the probability of exactly $i$ claims,

$A^i$ is a vector of "spreads" to be defined later, but, for example, $A^3 = (1/6, 2/3, 1/6, 0, 0, 0, \ldots, 0)$,

$*$ is the discrete convolution operator, and

$S^i$ is the vector that is the no-wrap convolution of the severity distribution, $S$, with itself $i$ times.

Very roughly speaking, $S^i$ is the density function of the sum of the original severity distribution, $S$, with itself $i$ times. However, it needs to be "spread out" (in a way that will be made precise below). Certain vectors of coefficients, $A^i$, to be defined shortly, will be used to "spread out" the $S^i$. The distribution of exactly $i$ claims is given by $A^i * S^i$. More precisely, $A^i * S^i[j]$ for $0 \leq j \leq n - 1$ is

$$F^i\left((j+1)\frac{L}{n}\right) - F^i\left(j\frac{L}{n}\right),$$

where $F^i(x)$ is the distribution function for the sum of the severity distribution with itself $i$ times. For $n \leq j \leq 2n - 1$ the values of $A^i * S^i[j]$ are not meaningful.

To see why the $A^i$ are needed, let $S$ be a uniform distribution on the interval 0 to 1. Let $n$ and $L$ be 4. Then $S = (1, 0, 0, 0)$. Doing the convolutions gives $S^i = S^1 = (1, 0, 0, 0, 0, 0, 0, 0)$ for all $i$. But the distribution function of the sum of the uniform distribution on $[0, 1]$ with itself should have nonzero probability not only between 0 and 1, but also between 1 and 2. In fact, this distribution should be $(1/2, 1/2, 0, 0, 0, 0, 0, 0)$. Similarly, the sum of this distribution with itself three times should be $(1/6, 2/3, 1/6, 0, 0, 0, 0, 0)$, which has three nonzero terms. $A^2 * S^2$ will be $(1/2, 1/2, 0, 0, 0, 0, 0, 0)$, and $A^3 * S^3$ will be $(1/6, 2/3, 1/6, 0, 0, 0, 0, 0)$.

If $S$ were a discrete distribution, instead of the piecewise uniform distribution used here, the $A^i$ would not be necessary. (That $S = (s_0, s_1, \ldots)$ is discrete means that if $x$ is $\frac{i}{n}L$, for some $i$ from 0 to $n - 1$, then the probability that a claim equals $x$ is $s_i$; for other $x$, the probability is 0. An example is given in Appendix B.)

Use of a piecewise uniform severity distribution roughly doubles the running time of the algorithm compared to the time required for a similar algorithm using a discrete severity distribution. The use of a piecewise uniform distribution is suggested because the author believes that frequently a piecewise uniform approximation with $n$ vector elements gives a better approximation to the severity distribution than does a discrete approximation with $2n$ vector elements. If the severity distribution is more accurately approximated, then the resulting aggregate loss distribution is more accurately approximated. Also, due to memory limitations in many computers, it is often possible to compute the aggregate distribution using the piecewise uniform approximation with $n$ vector elements, but it is not possible (easily) to compute the aggregate distribution using a discrete approximation with $2n$ vector elements.

In the next subsection the coefficients $A^i$ are defined so they will provide the needed spread. Then, the following two subsections cover two special tactics that substantially speed the running of the algorithm. Then, the full algorithm is discussed.

*The Coefficients* $A^i$

Define $a_j^i$ by:

$$a_0^i = 1/i! \qquad \text{for } i \geq 1,$$

$$a_j^1 = 0 \qquad \text{for } j \geq 1,$$

$$a_j^i = \left(\frac{1}{i}\right)\left[\,(i-j)\,a_{j-1}^{i-1} + (j+1)\,a_j^{i-1}\,\right] \quad \text{for } i \geq 2, j \geq 1.$$

Table 1 gives the first few values of $a_j^i$. For example, $a_4^6 = \frac{1}{6}\left(2\,a_3^5 + 5a_4^5\right) = \frac{1}{6}\left(2 \cdot \frac{26}{120} + 5 \cdot \frac{1}{120}\right) = \frac{57}{720}$.

$A^i$ is the vector of length $2n$ whose first $n$ elements are given by the $a_j^i$, and whose last $n$ elements are zero. For example, $A^4 = (1/24, 11/24, 11/24, 1/24, 0, 0, \ldots, 0)$.

## TABLE 1

VALUES OF $a_j^i$

| | | | $j$ | | | |
|---|---|---|---|---|---|---|
| $i$ | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1/2 | 1/2 | 0 | 0 | 0 | 0 |
| 3 | 1/6 | 4/6 | 1/6 | 0 | 0 | 0 |
| 4 | 1/24 | 11/24 | 11/24 | 1/24 | 0 | 0 |
| 5 | 1/120 | 26/120 | 66/120 | 26/120 | 1/120 | 0 |
| 6 | 1/720 | 57/720 | 302/720 | 302/720 | 57/720 | 1/720 |

The probability that the sum of $i$ unit rectangular distributions is between $j$ and $j+1$ for $i \geq 1$ and $j \geq 0$ is $a_j^i$. This is the reason these coefficients provide the needed spread. Appendix E shows this and gives a more detailed explanation of the reasons these coefficients are needed.

The numerators of the $a_j^i$, i.e., $i!a_j^i$, are known as *Eulerian numbers* and are discussed in Graham, Knuth, and Patashnik [19, pp. 253-258]. Feller [11, pp. 26-29] gives formulae useful in working with Eulerian numbers, although he does not mention them by name.

### *Packing and Unpacking*

Two special tactics are applied to make the algorithm run faster. One is to "pack" the severity distribution into a vector, so that the computation of the discrete Fourier transform of a given real vector of length $2n$ is accomplished, instead, by the computation of the discrete Fourier transform of a related complex vector of length $n$. This tactic roughly doubles the speed of the algorithm (with no effect on accuracy because the DFT of the original vector of length $2n$ is still what is finally computed), and is discussed in this sub-section. The second tactic is to compute the $M$-fold convolution of the severity distribution with itself using a method which, for $M$ greater than 3, is faster than the naive method that computes $M-1$ convolutions. This tactic is discussed in the next subsection.

The fast Fourier transform operates on vectors of complex numbers, but here it is used only to transform vectors of real numbers. As such, half of the place values are not really being used, because the imaginary parts of the elements of the input vector are all zero. Clever use of certain symmetry properties of discrete Fourier transforms of purely real vectors and purely imaginary vectors, as discussed in Press, et al. [6, p. 398], allows the following.

To transform the length $2n$ vector $\boldsymbol{V} = (v_0, v_1, \ldots, v_{2n-1})$, where each $v_i$ is a real number, rewrite $\boldsymbol{V}$ as $\boldsymbol{PV} = (v_0 + iv_1, v_2 + iv_3, \ldots, v_{2n-2} + iv_{2n-1})$, where $i$ is $\sqrt{-1}$. This is now a complex vector of length $n$. $\boldsymbol{PV}$ is referred to as the *packed untransformed* vector (it is packed because it is written in a more compact form; it is untransformed because the discrete Fourier transformation has not yet been applied). Compute the discrete Fourier transform of $\boldsymbol{PV}$, and call it $\boldsymbol{FPV}$. $\boldsymbol{FPV}$ is the *packed transformed* vector. Some simple computations on $\boldsymbol{FPV}$, called *unpacking*, yield $\boldsymbol{FV}$, the (unpacked) discrete Fourier transform of $\boldsymbol{V}$. While $\boldsymbol{FV}$ is a vector of length $2n$, if the first $n + 1$ elements of $\boldsymbol{FV}$ are known, then the remaining $n - 1$ elements can be deduced using a formula from Appendix F. Similarly, one can pack the transformed vector in such a way that when the inverse discrete Fourier transform is applied, the untransformed vector appears in the form of $\boldsymbol{PV}$. Note, in particular, that to apply the convolution theorem to real vectors of length $2n$, one can instead work with complex vectors whose lengths never exceed $n + 1$.

Packing and unpacking the untransformed vectors is trivial. Depending on how one represents real and complex vectors this might be a simple rearrangement, or just a redefinition of the meaning of each element of an array. Usually no calculations are needed. Packing and unpacking the transformed vectors does involve some calculation, but not a great amount. Details of the algorithms to pack and unpack are given in Appendix F, and an APL implementation is given in the functions PACK and UNPACK in Appendix G.

Table 2 shows the steps involved in computing the convolution of two real vectors when one packs the vectors.

TABLE 2

CONVOLUTION USING PACKED VECTORS

| Vector(s) | Step | Real or Complex | Length | Transf or Untransf | Packed or Unpacked |
|---|---|---|---|---|---|
| $\boldsymbol{V}, \boldsymbol{W}$ ⇓ | Start | Real | $2n$ | Untransf | Unpacked |
| $\boldsymbol{PV}, \boldsymbol{PW}$ ⇓ | Pack | Complex | $n$ | Untransf | Packed |
| $\boldsymbol{FPV}, \boldsymbol{FPW}$ ⇓ | Apply DFT | Complex | $n$ | Transf | Packed |
| $\boldsymbol{FV}, \boldsymbol{FW}$ ⇓ | Unpack | Complex | $n+1$ | Transf | Unpacked |
| $\boldsymbol{FU} = \boldsymbol{FV} \times \boldsymbol{FW}$ ⇓ | Multiply | Complex | $n+1$ | Transf | Unpacked |
| $\boldsymbol{FPU}$ ⇓ | Pack | Complex | $n$ | Transf | Packed |
| $\boldsymbol{PU}$ ⇓ | Inv DFT | Complex | $n$ | Untransf | Packed |
| $\boldsymbol{U}$ | Unpack | Real | $2n$ | Untransf | Unpacked |

While packing and unpacking add four steps to the above, the time saved by transforming vectors of length $n$ instead of $2n$ is more than offsetting. In practice, the untransformed vectors are usually kept packed, thus further reducing the number of steps.

*Binary Exponentiation*

Before discussing the second special tactic directly, consider an analogous question: how many multiplications are needed to compute $2^{100}$? One way to compute $2^{100}$ is to start with 2, and then repeatedly multiply by 2, doing 99 multiplications. Another way is to use the following formula:

$$2^{100} = (((((2^2 \times 2)^2)^2)^2 \times 2)^2)^2 .$$

Each operation of squaring is one multiplication and twice an intermediate result is multiplied by 2, so this computes $2^{100}$ with only eight multiplications.

In general, to compute $a^n$ for $n \geq 1$, one can apply the following algorithm. Express $n$ as a binary number, $b$, and drop the left-most digit (which is always 1). Set $z$ equal to $a$. Loop: if there are no digits left in $b$, then stop; $z$ is $a^n$. If there is at least one digit remaining in $b$, square $z$. If the current left-most digit of $b$ is 1, then multiply $z$ by $a$. Drop the left-most digit from $b$. Go back to the step labeled "Loop."

The binary representation of 100 is 1100100. Dropping the first digit gives 100100. Following the steps above, set $z$ to 2, then square, multiply by 2, square, square, square, multiply by 2, square, and square.

This is called a *left-to-right binary method for exponentiation* and is discussed in Knuth [5, Vol. 2, p. 441] (along with even faster methods).

This is used as follows. For some applications of the overall algorithm, the smallest number of claims with nonzero probability, $M$, will be greater than one. In these cases, this method is used to compute $\boldsymbol{S}^M = \boldsymbol{S} * \boldsymbol{S} * \ldots * \boldsymbol{S}$ (with $M$ factors of $\boldsymbol{S}$, here the $*$ is the no-wrap convolution). That is, $M$ is written as a binary number, and the left-to-right binary method is applied, with no-wrap convolution at each step instead of multiplication. Since convolution is associative, $\boldsymbol{S}^M$ is well defined, and this is a correct way to compute $\boldsymbol{S}^M$.

*The Full Algorithm*

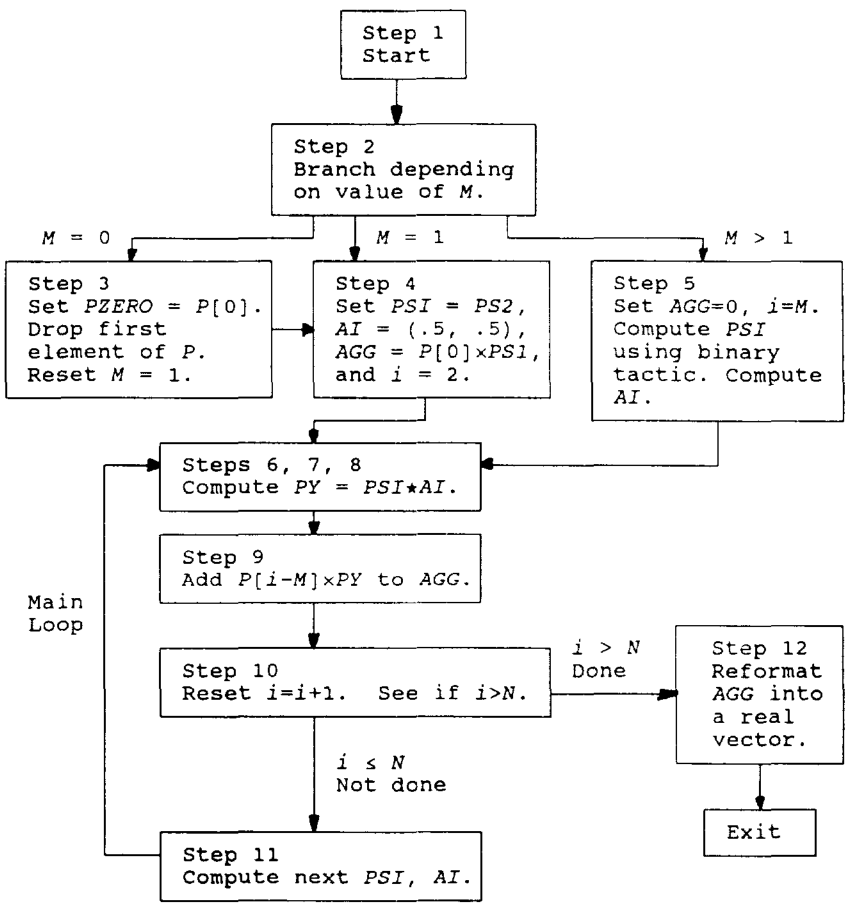
Now the full algorithm can be described. Figure 2 outlines the algorithm using a flowchart. The notation used is described in the presentation of the complete algorithm, below. A summary of the meaning of each variable is given in Table 3.

The complete algorithm is as follows.

*Algorithm for Aggregate Loss Distribution*: Let $M$ be the smallest number of claims with nonzero probability, let $N$ be the largest number of such claims ($N \geq 1$ and $N \geq M$), let $\boldsymbol{P}$ be the vector of probabilities of $M, M+1, M+2, \ldots, N$ claims, and let $\boldsymbol{S}$ be the vector representing the density function of the claim severity distribution. The length of $\boldsymbol{S}$ is $n = 2^k$ for some positive integer $k$. All vectors are indexed starting at 0, so

## FIGURE 2

FLOWCHART FOR COMPUTATION OF
AGGREGATE LOSS DISTRIBUTION



# TABLE 3

## VARIABLES USED IN THE MAIN ALGORITHM

### INPUT VARIABLES:

| | |
|---|---|
| $n$ | Length of vectors which will be subjects of the FFT or inverse FFT. $n = 2^k$ for some positive integer $k$. |
| $M$ | Smallest number of claims with nonzero probability. |
| $N$ | Largest number of claims with nonzero probability. |
| $P$ | Probabilities of $M$ through $N$ claims ($p_0 =$ probability of $M$ claims, $p_1 =$ probability of $M + 1$ claims, ...). |
| $S$ | Severity distribution. $S$ is a real vector of length $n$. $S = (s_0, s_1, \ldots, s_{n-1})$ |

### MAJOR VARIABLES:

| | |
|---|---|
| $i$ | Index giving the current number of claims. |
| ***PS1*** | Packed severity distribution, $(s_0 + is_1, s_2 + is_3, \ldots s_{n-2} + is_{n-1}, 0, 0, \ldots, 0)$. (Here $i$ is $\sqrt{-1}$.) |
| ***FS1*** | Transformed (unpacked) severity distribution. |
| ***PSI*** | Packed severity distribution convolved with self $i$ times. |
| ***SI*** | Severity distribution convolved with self $i$ times. (Unpacked ***PSI***.) |
| ***FSI*** | Transformed severity distribution convolved with self i times. |
| ***AI*** | Vector of spread coefficients, $(a_0^i, a_1^i, \ldots) = \boldsymbol{A}^i$. |
| ***AGG*** | Aggregate distribution. |

### MINOR VARIABLES:

| | |
|---|---|
| *PZERO* | Probability of zero claims. |
| *BIN* | Initialized to the binary representation of M, and used in the binary exponentiation tactic. |
| *FSIFLAG* | Flag to determine whether ***FSI*** has been computed for the current $i$. |
| ***X*** | Unpacked ***PSI***. |
| $j$ | Index used in Step 7. |
| ***Y*** | Becomes ***AI*X***. |
| ***PY*** | Packed ***Y***. |
| ***FAI*** | Transformed ***AI***. |
| ***FY*** | Transformed ***Y*** = ***FSI*** × ***FAI***. |

the indices for $\boldsymbol{S}$ run from 0 to $n-1$. The result will be $\boldsymbol{AGG}$, a vector representing the aggregate distribution.

1. [Some initializations.] Set $\boldsymbol{AGG}$ to be a complex vector of length $n$ all of whose elements are zero. Pack $\boldsymbol{S}$ into the complex vector $\boldsymbol{PS1}$ of length $n$ so:

$$\boldsymbol{PS1} = \left( s_0 + is_1, s_2 + is_3, \ldots, s_{n-2} + is_{n-1}, 0, 0, \ldots, 0 \right).$$

   Here $i$ is $\sqrt{-1}$. Set $PZERO$, the probability of exactly 0 claims, to 0.

2. [Branch depending on the value of $M$.] If $M$ is 0, go to Step 3. If $M$ is 1, go to Step 4. If $M$ is greater than 1, go to Step 5.

3. [Initialize if $M = 0$.] Let $PZERO$ be $\boldsymbol{P}[0]$. Drop the first element from $\boldsymbol{P}$. Let $M$ be 1.

4. [Initialize if $M = 0$ or 1.] Let $\boldsymbol{AGG}$ be $\boldsymbol{P}[0]$ times $\boldsymbol{PS1}$. If $N$ is 1, go to Step 10. Let $\boldsymbol{FS1}$ be the unpacked DFT of $\boldsymbol{PS1}$. Let $\boldsymbol{PSI}$ be the inverse DFT of the packed pointwise product of $\boldsymbol{FS1}$ with itself. Set the last $n/2$ elements of $\boldsymbol{PSI}$ to 0. Let $\boldsymbol{AI}$ be the two-element vector (0.5, 0.5). Let $i$ be 2. Go to Step 6.

5. [Begin procedure if $M > 1$.] Let $BIN$ be the binary representation of $M$. Drop the first (left-most) digit from $BIN$. Let $\boldsymbol{FSI}$ and $\boldsymbol{FS1}$ be the unpacked DFT of $\boldsymbol{PS1}$.

5.1. [Convolve $\boldsymbol{PSI}$ with itself or $\boldsymbol{PS1}$ with itself.] Let $\boldsymbol{FSI}$ be $\boldsymbol{FSI}$ times itself. Let $\boldsymbol{PSI}$ be the inverse DFT of the packed $\boldsymbol{FSI}$. Set the last $n/2$ elements of $\boldsymbol{PSI}$ to 0. If the first digit of $BIN$ is 0, go to Step 5.3.

5.2. [Convolve $\boldsymbol{PSI}$ with $\boldsymbol{PS1}$.] Let $\boldsymbol{FSI}$ be the unpacked DFT of $\boldsymbol{PSI}$. Let $\boldsymbol{FSI}$ be $\boldsymbol{FSI}$ times $\boldsymbol{FS1}$. Let $\boldsymbol{PSI}$ be the inverse DFT of the packed $\boldsymbol{FSI}$. Set the last $n/2$ elements of $\boldsymbol{PSI}$ to 0.

5.3. [Check whether finished.] Drop the first digit from $BIN$. If there are no digits left, go to Step 5.4 Otherwise, let $\boldsymbol{FSI}$ be the unpacked DFT of $\boldsymbol{PSI}$. Go to Step 5.1.

5.4. [Initialize ***AI*** in the case $M > 1$.] Use the formula:

$$a_0^i = 1/i! \qquad \text{for } i \geq 1,$$

$$a_j^1 = 0 \qquad \text{for } j \geq 1,$$

$$a_j^i = \left(\frac{1}{i}\right)\left[(i-j)\,a_{j-1}^{i-1} + (j+1)\,a_j^{i-1}\right] \qquad \text{for } i \geq 2,\ j \geq 1$$

to compute $\boldsymbol{AI} = \boldsymbol{A}^M = \left(a_0^M, a_1^M, \ldots, a_{M-1}^M\right)$. Let $i$ equal $M$.

6. [Start main loop.] (When this step is reached for the first time, ***FSI*** has been computed, ***PSI*** and ***AI*** have been computed for some $i$ at least 2, and ***AGG*** has been initialized.) Let *FSIFLAG* be 0 (will be used later to determine whether ***FSI*** has been computed). If $i$ is greater than or equal to 100 go to Step 8.

7. [Convolve ***AI*** and ***SI*** without using DFTs. (***SI*** is the severity distribution convolved with itself $i$ times.)] Unpack ***PSI*** and let ***X*** be the first $n$ elements of the result. Let $j$ be 0 and ***Y*** be a (real) vector of length $n$ with all elements 0.

7.1 [Loop.] Let ***Y*** be ***Y*** plus ***AI***[$j$] times ***X***. Add 1 to $j$. If $j$ is greater than $i$ minus 1 go to Step 7.2. Drop the last element of ***X*** and add a 0 as the first element. Return to the start of this Step (7.1).

7.2 [Exit Step 7.1.] Add $n$ zeroes to the end of ***Y***, pack and call the result ***PY***. Go to Step 9.

8. [Convolve ***AI*** and ***SI*** using FFTs.] If ***AI*** is of length less than $n$, add zeroes until a vector of length $2n$ is achieved; otherwise take the first $n$ elements of ***AI*** and append $n$ zeroes. Pack this vector, compute the DFT, unpack the result and assign it to ***FAI***. Compute the DFT of ***PSI***, unpack, and assign the result to ***FSI***. Set *FSIFLAG* to 1. Let ***FY*** be ***FSI*** times ***FAI***. Pack ***FY***, apply the inverse DFT, and assign the result to ***PY***. Set the last $n/2$ (complex) elements of ***PY*** to 0.

9. [Add new packed ***AI*** $*$ ***SI*** ( = ***PY***) to ***AGG***.] Let ***AGG*** be ***AGG*** plus ***P***[ $i - M$ ] times ***PY***.

10. [Increment $i$.] Set $i$ to $i$ plus 1. If $i$ is greater than $N$, go to Step 12.

11. [Compute next ***AI***, ***PSI***.] If *FSIFLAG* is 0, compute the DFT of ***PSI***, unpack, and assign the result to ***FSI***. (If *FSIFLAG* is 1, it is because ***FSI*** was computed in Step 8.) Let ***FSI*** be ***FSI*** times ***FS1***. Pack ***FSI***, compute the inverse DFT, and assign this to ***PSI***. Let the last $n/2$ elements of ***PSI*** be 0. Compute the next ***AI*** using the formula in Step 5.4. Go to Step 6.

12. [The end.] Let ***AGG*** be the first $n$ elements of ***AGG*** unpacked. Add *PZERO* to ***AGG***[*0*].

The first two steps do some initializations and branch depending on the value of $M$. If $M$ is 0, the algorithm essentially converts to the case where $M$ is 1. If $M$ is 1 (or 0) the first steps set ***AGG*** to be ***PS1***, compute ***FS1***, and compute ***PSI*** and ***AI*** for $i = 2$. Then comes the main loop. If $M$ is greater than 1, the binary exponentiation tactic is used to compute ***PSI*** for $i = M$. ***AI*** is also computed for $i = M$.

The main loop repeatedly computes the convolution of ***AI*** with ***SI***, multiplies this by ***P***[$i$ - $M$], and adds the result to ***AGG*** (actually, the untransformed packed result is added to ***AGG***). Note that the convolution of ***AI*** with ***SI*** is the distribution of exactly $i$ claims. If $i$ is less than or equal to $N$ the next ***PSI*** and ***AI*** are computed and one continues with the main loop. (Note that each ***PSI*** is computed from the ***PSI*** for the previous $i$.) If $i$ is greater than $N$, one exits the main loop, reformats ***AGG***, and folds in *PZERO*. Observe that ***FSI*** is always recomputed from a current ***PSI*** which has had its "tail" (the last $n/2$ elements) set to zero. This gives the no-wrap convolutions that are needed, instead of regular convolutions.

Sometimes the algorithm computes the convolution of ***AI*** with ***PSI*** through use of the convolution theorem and FFTs, and sometimes it performs this computation directly. For "short" ***AI***, implicitly defined above as having 100 or fewer nonzero terms, it is faster to compute the convolution directly. Once ***AI*** becomes "long," it is faster to use the convolution theorem. In another implementation (using different hardware or software), it might be more efficient to set this cut-off of 100 to a higher or lower number.

Note that if there are fewer than $n$ nonzero terms in $\boldsymbol{AI}$ only the nonzero terms are kept, and one pads to the right with zeroes when a vector of length $n$ is needed. This differs slightly from definitions of $\boldsymbol{AI}$ given earlier.

APL functions implementing the complete algorithm are given in Appendix H.

## 4. EXAMPLES AND ADDITIONAL DISCUSSION

This section will give some examples of the use of the algorithm, show how parameter uncertainty can be reflected in the aggregate distribution, and discuss the computation of aggregate excess distributions. Comments in the first two areas are specific to this fast Fourier transform algorithm, but comments on the third topic apply generally.

### *Examples*

Three examples of use of this algorithm will be given. The first is simple and is intended to be easy to reproduce in order to test an actual implementation. It is not meant to be realistic. The second example is more typical of actual distributions that arise in practice. The third example is reasonably realistic, but is really meant to illustrate the flexibility of the algorithm.

The first example will compute the distribution of *exactly* five claims, with each claim following a uniform severity distribution. Let $k = 5$, so $n = 32$. (That $k = 5$ has nothing to do with the fact that the distribution of five claims is being computed; this is a coincidence.) The claim count distribution is defined by setting $M = 5$, and making $\boldsymbol{P}$ be a vector with one element, (1). The severity distribution, $\boldsymbol{S}$, is a vector of length 32 (= $n$), and $L$ is set to 6.4, so each element of $\boldsymbol{S}$ covers a range of 0.2 (= 6.4/32). Letting $\boldsymbol{S}$ be the uniform distribution on [0,1], gives:

$$\boldsymbol{S} = (0.2, 0.2, 0.2, 0.2, 0.2, 0, 0, \ldots, 0).$$

The output is a vector of length 32, $\boldsymbol{AGG}$, a complete listing of which is given in Table 4. Since the sum of five claims, each no greater than 1, cannot exceed 5, only the first 25 elements of the output are nonzero. It is

# TABLE 4

## AGGREGATE DISTRIBUTION FOR EXAMPLE 1

| Index | Range | $AGG[i]$ | Cumulative Distribution |
|---|---|---|---|
| 0 | 0.0 - 0.2 | 0.000003 | 0.000003 |
| 1 | 0.2 - 0.4 | 0.000083 | 0.000085 |
| 2 | 0.4 - 0.6 | 0.000563 | 0.000648 |
| 3 | 0.6 - 0.8 | 0.002083 | 0.002731 |
| 4 | 0.8 - 1.0 | 0.005603 | 0.008333 |
| 5 | 1.0 - 1.2 | 0.012389 | 0.020723 |
| 6 | 1.2 - 1.4 | 0.023669 | 0.044392 |
| 7 | 1.4 - 1.6 | 0.039749 | 0.084141 |
| 8 | 1.6 - 1.8 | 0.059669 | 0.143811 |
| 9 | 1.8 - 2.0 | 0.081189 | 0.225000 |
| 10 | 2.0 - 2.2 | 0.100816 | 0.325816 |
| 11 | 2.2 - 2.4 | 0.114496 | 0.440312 |
| 12 | 2.4 - 2.6 | 0.119376 | 0.559688 |
| 13 | 2.6 - 2.8 | 0.114496 | 0.674184 |
| 14 | 2.8 - 3.0 | 0.100816 | 0.775000 |
| 15 | 3.0 - 3.2 | 0.081189 | 0.856189 |
| 16 | 3.2 - 3.4 | 0.059669 | 0.915859 |
| 17 | 3.4 - 3.6 | 0.039749 | 0.955608 |
| 18 | 3.6 - 3.8 | 0.023669 | 0.979277 |
| 19 | 3.8 - 4.0 | 0.012389 | 0.991667 |
| 20 | 4.0 - 4.2 | 0.005603 | 0.997269 |
| 21 | 4.2 - 4.4 | 0.002083 | 0.999352 |
| 22 | 4.4 - 4.6 | 0.000563 | 0.999915 |
| 23 | 4.6 - 4.8 | 0.000083 | 0.999997 |
| 24 | 4.8 - 5.0 | 0.000003 | 1.000000 |
| 25 | 5.0 - 5.2 | 0.000000 | 1.000000 |
| 26 | 5.2 - 5.4 | 0.000000 | 1.000000 |
| 27 | 5.4 - 5.6 | 0.000000 | 1.000000 |
| 28 | 5.6 - 5.8 | 0.000000 | 1.000000 |
| 29 | 5.8 - 6.0 | 0.000000 | 1.000000 |
| 30 | 6.0 - 6.2 | 0.000000 | 1.000000 |
| 31 | 6.2 - 6.4 | 0.000000 | 1.000000 |

straightforward to check that the result is the sum of five uniform distributions. (Recall that the $A^i$ summarize the sums of uniform distributions.) For example, the sum of the first five elements of the result is .008335, which (up to rounding error) is 1/120, and this agrees with the theoretical sum. Similarly, the sums of the second through fifth sets of five elements are 26/120, 66/120, 26/120, and 1/120. (Ambitious readers can check that the intermediate values are also correct. Feller [11, p. 27] gives the needed formulae.)

The second example is more in line with distributions that arise in practice. In this example, $k = 10$, so $n = 1{,}024$. The claim count distribution is negative binomial with mean 10 and variance 12. For input, $M$ equals 0 and $\boldsymbol{P}$ is a vector of length 42, giving the probabilities of 0 through 41 claims. Actually, for the probability of 41 claims the probability of 41 or more claims is used, so the total of the elements of $\boldsymbol{P}$ is exactly 1.0. The probability of there being 42 or more claims is less than $10^{-10}$, so including this in the probability of there being exactly 41 claims is not significant. The values of $\boldsymbol{P}$ are shown in Table 5. This distribution is shown in detail for the benefit of readers who want to reproduce this example.

The severity distribution is a two-parameter Weibull distribution with mean \$10,000 and coefficient of variation 8 (standard deviation divided by mean). The mean and coefficient of variation completely define the Weibull distribution. For the interested reader, note that the parameterization of the Weibull distribution used has distribution function $F(x) = 1 - e^{-(x/b)^c}$ with $b = 454.82609$ and $c = 0.25371$. Any other severity distribution could be used in place of the Weibull, including lognormal, Pareto, or an empirical fit to data. Losses are capped at \$250,000 per loss. An $L$ of \$1,000,000 is chosen to be large enough to cover the highest values needed in the aggregate distribution. $\boldsymbol{S}$ is then a vector of length 1,024 with each element covering a range of \$1,000,000/1,024, or about \$977. As losses are capped at \$250,000, only the first 256 elements of $\boldsymbol{S}$ will be nonzero.

To determine the values of $\boldsymbol{S}$, a variant of a method of Venter [20, p. 21] is used. $\boldsymbol{S}$ is to be a piecewise uniform approximation to the Weibull

# TABLE 5

## CLAIM COUNT DISTRIBUTION FOR EXAMPLE 2

| Number of Claims | Probability of Given Number of Claims | Cumulative Distribution |
|---|---|---|
| 0 | 1.09885E-4 | 1.09885E-4 |
| 1 | 9.15707E-4 | 1.02559E-3 |
| 2 | 0.00389175 | 0.00491735 |
| 3 | 0.01124284 | 0.01616019 |
| 4 | 0.02482795 | 0.04098814 |
| 5 | 0.04469031 | 0.08567845 |
| 6 | 0.06827686 | 0.15395531 |
| 7 | 0.09103581 | 0.24499112 |
| 8 | 0.10810503 | 0.35309615 |
| 9 | 0.11611281 | 0.46920896 |
| 10 | 0.11417760 | 0.58338656 |
| 11 | 0.10379781 | 0.68718437 |
| 12 | 0.08793981 | 0.77512418 |
| 13 | 0.06990088 | 0.84502506 |
| 14 | 0.05242566 | 0.89745072 |
| 15 | 0.03728047 | 0.93473119 |
| 16 | 0.02524198 | 0.95997317 |
| 17 | 0.01633305 | 0.97630622 |
| 18 | 0.01013254 | 0.98643876 |
| 19 | 0.00604397 | 0.99248273 |
| 20 | 0.00347528 | 0.99595801 |
| 21 | 0.00193071 | 0.99788873 |
| 22 | 0.00103849 | 0.99892722 |
| 23 | 5.41821E-4 | 0.99946904 |
| 24 | 2.74673E-4 | 0.99974371 |
| 25 | 1.35505E-4 | 0.99987922 |
| 26 | 6.51468E-5 | 0.99994436 |
| 27 | 3.05627E-5 | 0.99997492 |
| 28 | 1.40079E-5 | 0.99998893 |
| 29 | 6.27940E-6 | 0.99999521 |
| 30 | 2.75596E-6 | 0.99999797 |
| 31 | 1.18536E-6 | 0.99999915 |
| 32 | 5.00074E-7 | 0.99999965 |
| 33 | 2.07101E-7 | 0.99999986 |
| 34 | 8.42617E-8 | 0.99999994 |
| 35 | 3.37047E-8 | 0.99999998 |
| 36 | 1.32634E-8 | 0.99999999 |
| 37 | 5.1381E-9 | 1.00000000 |
| 38 | 1.9606E-9 | 1.00000000 |
| 39 | 7.373E-10 | 1.00000000 |
| 40 | 2.734E-10 | 1.00000000 |
| 41 | 1.560E-10 | 1.00000000 |

distribution. To achieve this, pairs of consecutive elements $(s_{2i}, s_{2i+1})$ of $\boldsymbol{S}$ are chosen subject to two constraints:

- over each interval $\left[2i \cdot \frac{L}{n}, (2i+2) \cdot \frac{L}{n}\right]$ the integral of $\boldsymbol{S}$ and the integral of the Weibull density are the same, and
- over the same intervals the integral of the first moment distribution of $\boldsymbol{S}$ and the first moment distribution of the Weibull are the same.

If $RR$ and $SS$ are the integral of the density function and the integral of the first moment distribution of the Weibull (or whatever distribution is being approximated) over the above interval then

$$s_{2i} = \left(2i + \frac{3}{2}\right)RR - \frac{SS \cdot n}{L}$$

$$s_{2i+1} = RR - s_{2i}$$

No properties of the Weibull are used in the formulae above; they apply to any distribution, including empirical distributions.

When this method is applied in this particular case, the second element of $\boldsymbol{S}$ becomes negative. Thus, some additional fiddling is done on the first 12 elements of $\boldsymbol{S}$ so that the two constraints are satisfied for these 12 elements taken together, but not pair-wise. Selected values for $\boldsymbol{S}$ (including the first 12 elements) are shown in Table 6. Note in particular that the value of .007072 is the probability of the severity being in the interval \$249,023 to \$250,000 plus the probability of the severity being over \$250,000. This latter probability has been spread over the interval.

Selected values of the resulting aggregate distribution are shown in Table 7. Column (3) is selected elements of the vector $\boldsymbol{AGG}$ output by the algorithm. Each element, $t_i$, of $\boldsymbol{AGG}$ is the (exact) integral of the density function for the aggregate distribution over the interval $\left[i \cdot \frac{L}{n}, (i+1) \cdot \frac{L}{n}\right]$. For the purpose of interpolating values between integral multiples of $L \div n$ it is assumed that the density at each point in an interval is $t_i \div \left(\frac{L}{n}\right)$. Column (4) is the distribution function,

## TABLE 6

DISTRIBUTION OF SEVERITY OF SINGLE CLAIM FOR EXAMPLE 2

| | | Probability Distribution of a Single Claim | |
|---|---|---|---|
| Index | High End of Range* | $S[i]$ | Distribution |
| 0 | $977 | 0.716463 | 0.716463 |
| 1 | 1,953 | 0.114281 | 0.830745 |
| 2 | 2,930 | 0.015000 | 0.845745 |
| 3 | 3,906 | 0.015000 | 0.860745 |
| 4 | 4,883 | 0.010000 | 0.870745 |
| 5 | 5,859 | 0.005000 | 0.875745 |
| 6 | 6,836 | 0.005000 | 0.880745 |
| 7 | 7,813 | 0.005000 | 0.885745 |
| 8 | 8,789 | 0.003000 | 0.888745 |
| 9 | 9,766 | 0.003000 | 0.891745 |
| 10 | 10,742 | 0.003000 | 0.894745 |
| 11 | 11,719 | 0.003000 | 0.897745 |
| 12 | 12,695 | 0.004750 | 0.902494 |
| 13 | 13,672 | 0.004140 | 0.906634 |
| 14 | 14,648 | 0.003884 | 0.910518 |
| 15 | 15,625 | 0.003441 | 0.913959 |
| 74 | 73,242 | 0.000331 | 0.973485 |
| 75 | 74,219 | 0.000321 | 0.973806 |
| 76 | 75,195 | 0.000316 | 0.974122 |
| 77 | 76,172 | 0.000307 | 0.974429 |
| 78 | 77,148 | 0.000302 | 0.974732 |
| 101 | 99,609 | 0.000194 | 0.980249 |
| 102 | 100,586 | 0.000192 | 0.980440 |
| 103 | 101,563 | 0.000187 | 0.980628 |
| 123 | 121,094 | 0.000137 | 0.983819 |
| 124 | 122,070 | 0.000136 | 0.983955 |
| 125 | 123,047 | 0.000133 | 0.984088 |
| 235 | 230,469 | 0.000041 | 0.992213 |
| 236 | 231,445 | 0.000041 | 0.992254 |
| 237 | 232,422 | 0.000040 | 0.992294 |
| 253 | 248,047 | 0.000035 | 0.992893 |
| 254 | 249,023 | 0.000035 | 0.992928 |
| 255 | 250,000 | 0.007072 | 1.000000 |
| 256 | 250,977 | 0.000000 | 1.000000 |
| 1,023 | 1,000,000 | 0.000000 | 1.000000 |

*Each range has width of about 977.

# TABLE 7

## DISTRIBUTION OF AGGREGATE LOSS AND FIRST MOMENT DISTRIBUTION FOR EXAMPLE 2

| | | Aggregate Loss Distribution | | First Moment Distribution | |
|---|---|---|---|---|---|
| Index | High End of Range* | *AGG*[*i*] | Distribution | Density** | Distribution** |
| (1) | (2) | (3) | (4) | (5) | (6) |
| 0 | $977 | 0.002812 | 0.002812 | 1.373 | 1.373 |
| 1 | 1,953 | 0.010576 | 0.013387 | 15.492 | 16.865 |
| 2 | 2,930 | 0.021673 | 0.035061 | 52.913 | 69.778 |
| 3 | 3,906 | 0.031214 | 0.066275 | 106.689 | 176.467 |
| 4 | 4,883 | 0.036124 | 0.102399 | 158.750 | 335.217 |
| 5 | 5,859 | 0.036358 | 0.138758 | 195.284 | 530.501 |
| 6 | 6,836 | 0.033510 | 0.172268 | 212.712 | 743.213 |
| 7 | 7,813 | 0.029312 | 0.201580 | 214.687 | 957.900 |
| 12 | 12,695 | 0.013517 | 0.294650 | 165.005 | 1,884.273 |
| 13 | 13,672 | 0.012153 | 0.306802 | 160.216 | 2,044.489 |
| 14 | 14,648 | 0.011327 | 0.318129 | 160.388 | 2,204.877 |
| 15 | 15,625 | 0.010936 | 0.329065 | 165.539 | 2,370.416 |
| 74 | 73,242 | 0.003134 | 0.691222 | 228.013 | 16,154.978 |
| 75 | 74,219 | 0.003082 | 0.694304 | 227.219 | 16,382.197 |
| 76 | 75,195 | 0.003031 | 0.697335 | 226.418 | 16,608.615 |
| 77 | 76,172 | 0.002981 | 0.700316 | 225.611 | 16,834.226 |
| 78 | 77,148 | 0.002932 | 0.703248 | 224.797 | 17,059.023 |
| 123 | 121,094 | 0.001553 | 0.798766 | 187.309 | 26,306.223 |
| 124 | 122,070 | 0.001534 | 0.800300 | 186.519 | 26,492.741 |
| 125 | 123,047 | 0.001515 | 0.801816 | 185.732 | 26,678.473 |
| 150 | 147,461 | 0.001138 | 0.834470 | 167.189 | 31,076.113 |
| 151 | 148,438 | 0.001125 | 0.835595 | 166.494 | 31,242.606 |
| 152 | 149,414 | 0.001113 | 0.836709 | 165.802 | 31,408.408 |
| 153 | 150,391 | 0.001101 | 0.837810 | 165.114 | 31,573.522 |
| 154 | 151,367 | 0.001090 | 0.838900 | 164.429 | 31,737.951 |
| 235 | 230,469 | 0.000520 | 0.899889 | 119.608 | 43,086.466 |
| 236 | 231,445 | 0.000516 | 0.900405 | 119.168 | 43,205.634 |
| 237 | 232,422 | 0.000512 | 0.900916 | 118.731 | 43,324.364 |
| 253 | 248,047 | 0.000453 | 0.908589 | 112.035 | 45,166.396 |
| 254 | 249,023 | 0.000449 | 0.909038 | 111.635 | 45,278.031 |
| 255 | 250,000 | 0.000507 | 0.909546 | 126.564 | 45,404.595 |
| 256 | 250,977 | 0.000830 | 0.910376 | 207.973 | 45,612.569 |
| 283 | 277,344 | 0.000836 | 0.949716 | 231.421 | 55,893.081 |
| 284 | 278,320 | 0.000813 | 0.950530 | 225.980 | 56,119.061 |
| 285 | 279,297 | 0.000792 | 0.951322 | 220.846 | 56,339.908 |
| 419 | 410,156 | 0.000097 | 0.989983 | 39.907 | 68,828.765 |
| 420 | 411,133 | 0.000096 | 0.990080 | 39.546 | 68,868.312 |
| 1,022 | 999,023 | 0.000000 | 0.999996 | 0.052 | 73,804.421 |
| 1,023 | 1,000,000 | 0.000000 | 0.999996 | 0.058 | 73,804.479 |

* Each range has width of about 977.

** These columns have been multiplied by the aggregate mean. This table gives selected values of the distributions.

$t_0 + t_1 + ... + t_i$, i.e., the cumulative sum of Column (3). Column(4) readily gives the dollar amounts (frequently called *confidence levels* in the context of aggregate loss distributions) associated with given probability levels, and vice versa. For instance, the probability that aggregate losses will be less than or equal to \$250,000 is 91.0%. By simple interpolation, it is seen that \$75,000 corresponds to the 69.7% confidence level. Again through interpolation, the 80% confidence level is \$121,879.

The expected dollars of loss above and below given aggregate limits can be quickly determined. Suppose, for example, an insurer has purchased reinsurance that covers all loss amounts beyond a total of \$250,000. That is, the insurer pays the first \$250,000 of losses (which could be one claim or a number of claims), and the reinsurer pays any losses after the first \$250,000. The insurer's expected loss is:

$$\int_0^{250,000} x f(x)\, dx + 250,000 \int_{250,000}^{\infty} f(x)\, dx$$

where $f(x)$ is the density function of the aggregate distribution. As Columns (5) and (6) in Table 7 will help calculate this integral, these columns are described next.

Each entry in Column (5) is $\int x f(x)\, dx$ over its interval. For instance, under the above assumption that the density function is a constant $0.036358 \div 976.5625$ across the fifth interval, $\int x f(x)\, dx$ over the fifth interval is

$$\frac{(5,859.375)^2 - (4,882.8125)^2}{2} \times \frac{0.036358}{976.5625}$$

or 195.282. (This is slightly different from 195.284 shown in Column (5) because the Column (3) entry of 0.036358 is used in the above calculation, while more significant digits were used in the calculation of Table 7.) Column (6) is the cumulative sum of Column (5). Thus, Column (6) gives

$$\int_{0}^{(i+1)\frac{L}{n}} x f(x)\, dx$$

where $i$ is the index of the interval.

Returning to the original question, if losses are capped at an aggregate of \$250,000, Columns (6) and (4) show that expected losses are $45{,}405 + 250{,}000 \times [1 - 0.909546]$, or \$68,019.

The reinsurer is taking both occurrence and aggregate excess of \$250,000, and total expected losses are \$100,000 (10 expected claims times \$10,000 expected loss per claim), so the reinsurer's expected losses are \$31,981 (100,000 - 68,019).

Generally, to compute expected losses for an insurer that retains a given amount per occurrence and retains a given aggregate, use a severity distribution capped at the per occurrence limit, compute the aggregate distribution, and compute the expected retained losses up to the aggregate limit as was just done above.

The third example is a variation on the second example. The main purpose is to show how easy it is to use an arbitrary frequency distribution in the algorithm. For this example, the above frequency distribution is modified to assume that there is a 90% probability that claims will follow the distribution in example 2, and an additional 10% probability there will be exactly 20 claims. The same $\boldsymbol{S}$ as above is used. The modified $\boldsymbol{P}$ is shown in Table 8, and some of the output is given in Table 9.

Note that the severity distribution, $\boldsymbol{S}$, used in examples 2 and 3 is only an approximation to the Weibull distribution. Essentially, having to find a piecewise uniform function to approximate the true severity distribution, with each interval being the same size, $L \div n$, means that $\boldsymbol{S}$ is not going to be precisely the same as the original continuous function. Since $\boldsymbol{S}$ is an approximation to the true severity distribution, the output is an approximation to the true aggregate distribution. Comparisons of aggregate loss distributions computed using this algorithm to aggregate loss distributions

# TABLE 8
## CLAIM COUNT DISTRIBUTION FOR EXAMPLE 3

| Number of Claims | Probability of Given Number of Claims | Cumulative Distribution |
|---|---|---|
| 0 | 9.88963E-5 | 9.88963E-5 |
| 1 | 8.24136E-4 | 9.23032E-4 |
| 2 | 0.00350258 | 0.00442561 |
| 3 | 0.01011856 | 0.01454417 |
| 4 | 0.02234515 | 0.03688933 |
| 5 | 0.04022128 | 0.07711060 |
| 6 | 0.06144917 | 0.13855978 |
| 7 | 0.08193223 | 0.22049201 |
| 8 | 0.09729453 | 0.31778654 |
| 9 | 0.10450153 | 0.42228806 |
| 10 | 0.10275984 | 0.52504790 |
| 11 | 0.09341803 | 0.61846593 |
| 12 | 0.07914583 | 0.69761177 |
| 13 | 0.06291079 | 0.76052256 |
| 14 | 0.04718309 | 0.80770565 |
| 15 | 0.03355242 | 0.84125807 |
| 16 | 0.02271779 | 0.86397586 |
| 17 | 0.01469974 | 0.87867560 |
| 18 | 0.00911929 | 0.88779488 |
| 19 | 0.00543957 | 0.89323446 |
| 20 | 0.10312775 | 0.99636221 |
| 21 | 0.00173764 | 0.99809985 |
| 22 | 9.34641E-4 | 0.99903449 |
| 23 | 4.87639E-4 | 0.99952213 |
| 24 | 2.47206E-4 | 0.99976934 |
| 25 | 1.21955E-4 | 0.99989129 |
| 26 | 5.86321E-5 | 0.99994993 |
| 27 | 2.75064E-5 | 0.99997743 |
| 28 | 1.26071E-5 | 0.99999004 |
| 29 | 5.65146E-6 | 0.99999569 |
| 30 | 2.48036E-6 | 0.99999817 |
| 31 | 1.06682E-6 | 0.99999924 |
| 32 | 4.50066E-7 | 0.99999969 |
| 33 | 1.86391E-7 | 0.99999987 |
| 34 | 7.58356E-8 | 0.99999995 |
| 35 | 3.03342E-8 | 0.99999998 |
| 36 | 1.19371E-8 | 0.99999999 |
| 37 | 4.6243E-9 | 1.00000000 |
| 38 | 1.7645E-9 | 1.00000000 |
| 39 | 6.636E-10 | 1.00000000 |
| 40 | 2.461E-10 | 1.00000000 |
| 41 | 1.404E-10 | 1.00000000 |

# TABLE 9

## DISTRIBUTION OF AGGREGATE LOSSES FOR EXAMPLE 3

| | | Probability Distribution of Aggregate Losses | |
|---|---|---|---|
| Index | High End of Range* | *AGG*[*i*] | Distribution |
| 0 | $977 | 0.002530 | 0.002530 |
| 1 | 1,953 | 0.009518 | 0.012049 |
| 2 | 2,930 | 0.019506 | 0.031555 |
| 3 | 3,906 | 0.028093 | 0.059647 |
| 4 | 4,883 | 0.032512 | 0.092159 |
| 82 | 81,055 | 0.002900 | 0.684594 |
| 83 | 82,031 | 0.002857 | 0.687451 |
| 84 | 83,008 | 0.002815 | 0.690266 |
| 85 | 83,984 | 0.002773 | 0.693039 |
| 86 | 84,961 | 0.002733 | 0.695772 |
| 87 | 85,938 | 0.002694 | 0.698466 |
| 88 | 86,914 | 0.002655 | 0.701121 |
| 89 | 87,891 | 0.002617 | 0.703738 |
| 138 | 135,742 | 0.001420 | 0.797823 |
| 139 | 136,719 | 0.001405 | 0.799228 |
| 140 | 137,695 | 0.001390 | 0.800617 |
| 141 | 138,672 | 0.001375 | 0.801992 |
| 142 | 139,648 | 0.001360 | 0.803352 |
| 256 | 250,977 | 0.000847 | 0.898052 |
| 257 | 251,953 | 0.001418 | 0.899470 |
| 258 | 252,930 | 0.002059 | 0.901529 |
| 259 | 253,906 | 0.002517 | 0.904046 |
| 260 | 254,883 | 0.002688 | 0.906734 |
| 297 | 291,016 | 0.000691 | 0.948843 |
| 298 | 291,992 | 0.000679 | 0.949521 |
| 299 | 292,969 | 0.000666 | 0.950187 |
| 300 | 293,945 | 0.000654 | 0.950841 |
| 301 | 294,921 | 0.000642 | 0.951482 |
| 451 | 441,406 | 0.000091 | 0.989829 |
| 452 | 442,383 | 0.000090 | 0.989919 |
| 453 | 443,359 | 0.000089 | 0.990009 |
| 454 | 444,336 | 0.000088 | 0.990097 |
| 455 | 445,313 | 0.000087 | 0.990185 |
| 1,021 | 998,047 | 0.000000 | 0.999991 |
| 1,022 | 999,023 | 0.000000 | 0.999991 |
| 1,023 | 1,000,000 | 0.000000 | 0.999991 |

* Each range has a width of about 977. This table gives selected values of the aggregate distribution.

computed using other methods indicate that the algorithm presented here gives very accurate answers.

Apart from the need to use a severity distribution that is an approximation to the true distribution, the algorithm here is precise in the following sense. Each element of the aggregate loss vector is the exact difference of the distribution function for the exact aggregate loss distribution over the interval that corresponds to the element. In particular, this algorithm is not subject to the convergence difficulties sometimes encountered in certain characteristic function methods when the probability of a maximum loss is high. Of course, there is some potential for rounding error, but most computer languages have a provision for doing calculations to at least 17 decimal place accuracy, and as each element of the result (for $n = 1{,}024$; 20 expected claims) is affected by about 100,000 calculations, the result should be accurate to at least 12 places.

Note also, once Column (3) of Table 7 has been calculated, how simple and fast it is to calculate Columns (4), (5), and (6). Once ***AGG*** has been computed, there is very little computation time needed to determine confidence levels, expected losses subject to an aggregate, or expected losses excess of an aggregate. Also, since the entire aggregate loss distribution (up to some limit) is computed, the computation of any quantity that is related to the aggregate distribution (e.g., expected sliding scale commission for a reinsurance contract) is straightforward and fast.

*Computational Considerations*

The computational time for this algorithm seems to be roughly proportional to the number of elements in the vector ***P*** that gives the probabilities of the claim counts. The minimum number of claims for which there is a nonzero probability also has some effect on the computing time. But, due to the binary exponentiation tactic, the added computing time increases only as the logarithm to the base 2 of the minimum claim count.

Using APL 9 on a 386SX computer with 2 megabytes of RAM, this algorithm will run with $k$ as high as 10. This makes the maximum length of certain vectors $2^{11}$, or 2,048. Using APL 9, adding memory will not allow higher values of $k$ because all arrays active at any given moment must fit in the workspace available in the first 640K of memory (and this is about 400K because the APL system occupies about 200K).

It is likely that, compared to the computer programs presented in the appendices, the computations can be made more efficient in terms of the amount of memory used. References discuss computing the fast Fourier transform "in-place," which would use less memory than the programs given in the appendices. APL II, or other languages, might allow higher values of $k$ due to better use of memory above the first 640K.

Increasing $k$ by 1 roughly doubles the amount of memory needed, because the longest vectors double in length. Computational time is dominated by the time to compute the fast Fourier transforms, and this time increases by a factor of a bit more than 2 when $k$ is increased by 1. See any of the references given above on the fast Fourier transform for a more precise discussion of the relationship between $k$ and the time of computation.

To capture the distribution of the sum of $i$ claims for any $i$ with nonzero probability in the claim count distribution, just capture the ***PY*** for that $i$ from Step 9 of the main algorithm, given above. When using the same severity distribution but differing claim count distributions to compute several aggregate distributions, the following method might save some time. Capture all the distributions of the sum of exactly $i$ claims that will be needed (the ***PY***s above), and then just apply the probabilities given by the several claim count distributions and add. This can be much faster than recomputing each aggregate distribution from scratch.

*Parameter Uncertainty*

Patrik and John [17] distinguish *process risk* from *parameter risk* in estimating the distribution of final actual results relative to the estimated results. Essentially, if the frequency and severity distributions used are the best estimates of these distributions, then the calculated aggregate distribution reflects the inherent *process risk* or *process uncertainty*.

The extent to which the correct frequency and severity distributions are not known is termed *parameter risk* or *parameter uncertainty*. Some authors add *specification error* to the list of sources of potential difference between actual and expected results. *Specification error* refers to the fact that the model being used might not be appropriate. For instance, if it is known that the claim count distribution is Poisson, but the parameter of the Poisson distribution is not known exactly, then estimates of the aggregate distribution are subject to parameter uncertainty. If it is not known whether the claim count distribution is Poisson or some other distribution, then estimates are subject to specification error. Heckman and Meyers [2] discuss incorporation of parameter uncertainty into estimates of aggregate loss distributions.

*Parameter Uncertainty for the Claim Count Distribution*

To reflect parameter uncertainty in the claim count distribution, one could proceed as follows. First, identify all claim count distributions that might apply, and assign to each claim count distribution the probability that it is the correct distribution. Then, for each claim count distribution (and using some severity distribution), compute the aggregate distribution. Finally, take the weighted average of all these aggregate distributions, according to the probabilities of the claim count distributions. The resulting aggregate distribution reflects the various claim count distributions and the probabilities of those distributions.

For example, one might estimate there is a 20% probability that the claim count distribution is Poisson with mean 10; there is a 50% probability that the claim count distribution is Poisson with mean 20; and there is a 30% probability that the claim count distribution is negative binomial with mean 15 and variance 30. (This is not necessarily a realistic example.) Then, for instance, the probability of total losses being less than $\$X$ in the combined aggregate distribution would be 20% of the probability of losses being less than $\$X$ in the aggregate distribution generated by the Poisson claim count distribution with mean 10, plus 50% of the corresponding probability resulting from the Poisson distribution with mean 20, plus 30% of the corresponding probability from the negative binomial distribution with mean 15 and variance 30.

Fortunately, there is a shortcut that makes it possible to compute the aggregate distribution that reflects the uncertainty regarding the claim count distribution without computing a great number of aggregate distributions. Simply compute the weighted average of the claim count distributions, and then use this distribution in the computation of the aggregate loss distribution. For instance, using the above example, in the claim count distribution used as input to the main algorithm, the probability of $i$ claims would be $0.2 \times f(i) + 0.5 \times g(i) + 0.3 \times h(i)$, where $f$, $g$, and $h$ are the probability density functions for the Poisson distribution with mean 10, the Poisson distribution with mean 20, and the negative binomial distribution with mean 15 and variance 30. This combined distribution would be used as the claim count distribution in the algorithm to compute the aggregate loss distribution.

Generally, one will select a family of claim count distributions, and associated probabilities, so that the mean of the combined claim count distribution will be the expected number of claims. The variance of the combined claim count distribution usually will be greater than the variance of the best estimate claim count distribution. The effect of the combined claim count distribution on the variance of the new aggregate distribution can be computed by using the formula for the variance of the aggregate distribution:

$$\mu_N \sigma_S^2 + \mu_S^2 \sigma_N^2 .$$

Here $\mu_N$ and $\sigma_N^2$ are the mean and variance of the claim count distribution and $\mu_S$ and $\sigma_S^2$ are the mean and variance of the severity distribution (Mayerson, Jones, and Bowers [18, p. 179]).

A particularly simple situation results if it assumed that the possible claim count distributions are Poisson and that the parameters of these Poisson distributions are distributed according to a gamma distribution with mean $\lambda$ and variance $\sigma^2$. In this case, the resulting overall claim count distribution will be negative binomial with mean $\lambda$ and variance $\lambda + \sigma^2$. This is discussed in Beard, Pentikäinen, and Pesonen [16, p. 40] and in Heckman and Meyers [2].

As a practical matter, one frequently has a binomial, Poisson, or negative binomial distribution as the best estimate of the claim count distribution. To reflect parameter uncertainty in the claim count distribution used as input to the aggregate loss distribution algorithm, one might, where appropriate, simply use a claim count distribution with the same mean and a larger variance than the best estimate distribution. For instance, if one's best estimate of the claim count distribution is Poisson with parameter $\lambda$ then, to reflect parameter uncertainty, one might use a negative binomial distribution with mean $\lambda$ and variance larger than $\lambda$.

The three families of claim count distributions mentioned above are related. For the Poisson distribution, the variance and the mean are the same. The negative binomial has variance greater than the mean. The binomial has variance less than the mean. The Poisson is a limiting case of the negative binomial in that, as the variance of the negative binomial approaches the mean, the negative binomial approaches the Poisson. The Poisson is also a limiting case of the binomial.

In conclusion, parameter uncertainty for the claim count distribution can often be reflected in the computed aggregate loss distribution by choosing an appropriate claim count distribution with the same mean and larger variance than the best estimate distribution. This allows one to reflect parameter uncertainty while computing only one aggregate loss distribution.

*Parameter Uncertainty for the Severity Distribution*

To reflect parameter uncertainty for the severity distribution, one can proceed in the manner first discussed for the claim count distribution. That is, delineate all the severity distributions that might apply; assign to each a probability; compute the aggregate distribution using each severity distribution; and combine all of these aggregate distributions according to the probabilities of the severity distributions.

Unfortunately, when estimating the effect on aggregate distributions of parameter uncertainty in the severity distribution there is no shortcut quite as efficient as the one for claim count distributions. That is, to reflect parameter uncertainty for the severity distribution, it is not sufficient to use a severity distribution that is the combination of the various severity distributions in the same way that it is possible to use a claim count distribution that is the combination of the several claim count distributions. Later, it will be shown why this last statement is true, but methods of reflecting parameter uncertainty for the severity distribution will be covered first.

Some simplification in the reflection of parameter uncertainty for the severity distribution results if all the possible severity distributions are (or are assumed to be) multiples of some base distribution. More precisely, this assumption is that if $F_B$ is the distribution function of the base severity distribution, $B$, and if $F_Y$ is the distribution function of any other severity distribution in the family, $Y$, then there is a constant, $c$, such that $F_B(cx) = F_Y(x)$. Normally, this is written $B = cY$; it follows that $E(Y) = (1/c)\,E(B)$, and $\mathrm{Var}(Y) = (1/c^2)\,\mathrm{Var}(B)$. Let the constants $c$ be distributed according to a probability distribution with distribution function $H$ and density function $h$.

Let $F_A$ be the distribution function of the aggregate distribution computed using the base severity distribution $B$ (corresponding to $F_B$). Then the aggregate distribution reflecting parameter uncertainty, $T$, is given by

$$T(x) = \int_0^\infty F_A(cx)\,h(c)\,dc.$$

If $h$ has a form such that $h(t)$ and $th(t)$ are easily integrated over arbitrary intervals, and if $F_A$ is piecewise linear, then $T$, above, is easily computed. For $t$ in the interval $[l_i, u_i]$ let $F_A(t) = a_i + b_i t$. Then

$$\int_0^\infty F_A(cx)\,h(c)\,dc = \sum_{i=0}^\infty \left[\int_{l_i/x}^{u_i/x} (a_i + b_i cx)\,h(c)\,dc\right]$$

$$= \sum_{i=0}^\infty \left[a_i \int_{l_i/x}^{u_i/x} h(c)\;dc + b_i x \int_{l_i/x}^{u_i/x} ch(c)\,dc\right].$$

As a practical matter, the sums above are not taken to infinity, but rather to a high enough value that sufficient accuracy is achieved. If $h$ has been chosen so that the integrals are easy to compute, $T$ is also easy to compute.

Next is the demonstration, promised above, that to reflect the effect of parameter uncertainty in the severity distribution, it is not sufficient to simply use a severity distribution with a larger variance. To see this, consider one way a simulation model could be used to estimate the aggregate distribution. Choose a claim count, $n$, at random from the claim count distribution, $N$. Then $n$ times draw a random claim severity, $s_i$, from the severity distribution, $S$. Compute $s_1 + s_2 + ... + s_n$. This sum gives one "draw" from the aggregate distribution; that is, it gives one observation selected at random from the aggregate distribution. Repeat this process, i.e., make draws from the aggregate distribution, until the statistics of interest for the aggregate distribution are known with sufficient accuracy.

There are two methods one might use to reflect parameter uncertainty for the severity distribution when performing the above simulation. The first method is to choose a severity distribution at random each time a severity is needed. Within a given draw, $s_{i+1}$ would potentially be drawn from a different distribution than the preceding $s_i$. A second method is to fix a severity distribution each time an $n$ is chosen from $N$. This one severity distribution is used for all $s_i$ in the sum $s_1 + s_2 + ... + s_n$ corresponding to one draw. Then another $n$ is selected from $N$, and another severity distribution, possibly different from the severity distribution used in the previous draw, is used, and the process continues.

It is the second method that best reflects parameter uncertainty for the severity distribution. Under this method, only one severity distribution is used for each draw from the aggregate distribution. In contrast, under the first method, in many of the draws from the aggregate distribution, some claim amounts will come from severity distributions with larger-than-average means and some claim amounts will come from severity distributions with smaller-than-average means. The effects of severity distributions with larger-than-average means and severity distributions with smaller-than-average means will tend to cancel each other to some degree. Thus, the first method will tend to produce an aggregate distribution with a smaller variance than is correct. Under the second method, each draw is influenced by only one severity distribution.

The first simulation method corresponds to using a severity distribution that is the composite of the family of severity distributions being used to reflect parameter uncertainty. It is the second simulation method, where each draw is influenced by only one severity distribution, that corresponds to the methods discussed above for reflecting parameter uncertainty for the severity distribution.

The first and second methods differ fundamentally in the independence assumptions among samples from the severity distribution. A more mathematical discussion of the differences between the two methods, including a more precise discussion of the difference in independence assumptions, is given in Appendix I.

*Capped Severity Distributions and Parameter Uncertainty*

If a capped severity distribution is being used, e.g., losses are capped at $250,000 per claim, and if parameter uncertainty for the severity distribution is reflected using the method that assumes that all distributions are multiples of each other, then the loss cap becomes variable. In some cases, e.g., where the aggregate distribution of a self-insurance program with a given retention is being computed, it may not be appropriate to allow the loss cap to vary. There does not seem to be a simple way to reflect parameter uncertainty for the severity distribution in such a case.

One approach is to increase the degree of parameter uncertainty reflected in the claim count distribution to a level above that which would otherwise be used, and to not reflect parameter uncertainty in the severity distribution. This approach is not theoretically correct, but, as a practical matter, might be sufficiently accurate. Another approach is to let the cap be essentially variable, and perform tests to determine whether this significantly distorts the results. Finally, and with the greatest accuracy, one can compute a number of aggregate distributions, each using a different severity distribution with the correct cap, and take the weighted average.

*Excess Loss Distributions*

The results of this subsection apply to all methods used to calculate aggregate loss distributions, not to just the algorithm presented herein. The main results of this subsection appear to be well known, but have not previously appeared directly in actuarial literature. Schumi [21, 22] has presented material similar to this result. Bear and Nemlick [23] present the result in terms of the negative binomial distribution.

Suppose a claim count distribution, a severity distribution, and the corresponding aggregate distribution are specified. In regard to the severity distribution, suppose further that the probability of any given claim being excess of a given attachment point $A$ is $\alpha$. Suppose it is desired to compute the aggregate distribution for claims excess of $A$ (this $A$ has nothing to do with the vector of spreads $\mathbf{A}$ used previously). For example, the aggregate distribution might be based on a Poisson claim count distribution with parameter 1,000 (i.e., the number of expected claims is 1,000) and a Weibull severity distribution with mean \$10,000 and coefficient of variation 8. If $A$ is \$100,000 then $\alpha$ is 0.0197.

One way to compute the excess aggregate distribution (the aggregate distribution for the amount of claims excess of $A$ per claim) is to keep the same claim count distribution (e.g., Poisson with parameter 1,000 in the example) and adjust the severity distribution so that claims less than $A$ become 0 and claims, $x$, greater than or equal to $A$ become $x - A$. This gives a severity distribution that generally assigns a large probability, namely $1 - \alpha$, to claims being exactly 0.

Another way is to work directly with the excess claim count and severity distributions. The excess claim count distribution is the distribution of the number of excess claims (the distribution of the number of claims exceeding $A$). The excess severity distribution is the claim severity distribution for the amount of individual claims excess of $A$, given that a claim is excess.

The main purpose of this subsection is to note that, for certain claim count distributions, the excess claim count distribution is easily determined. Assume the claim count distribution is binomial, Poisson, or negative binomial, respectively, with mean $\lambda$ and variance $\sigma^2$. Suppose the probability of a given claim being excess of the attachment point $A$ is $\alpha$. Then the excess claim count distribution is binomial, Poisson, or negative binomial, respectively, with mean $\alpha\lambda$ and variance $\alpha\lambda + \alpha^2(\sigma^2 - \lambda)$.

The excess severity distribution is easy to determine if the total distribution and the attachment point are known. Suppose the severity distribution has distribution function $F$, and the attachment point for excess claims is $A$. Then the excess severity distribution has distribution function, $H$, defined by:

$$H(x) = \frac{F(x+A) - F(A)}{1 - F(A)}, \quad \text{for } x \geq 0.$$

That is, the portion of the severity distribution function below $A$ is eliminated, and the remaining distribution is rescaled so that $H(0)$ is 0 and $H(x)$ has limit 1 as $x$ tends to infinity.

In the example, the excess claim count distribution is Poisson with parameter 19.7 (= 0.0197 × 1,000). The excess severity distribution is the above Weibull distribution restricted to claims exceeding \$100,000. In particular, the probability of a claim being 0 is 0 (not 0.9803).

It should be clear that the excess severity distribution is as claimed above. Appendix J has a proof that the excess claim count distribution is as claimed. An interesting open problem is to find other claim count distributions for which the excess claim count distribution is of the same form as the original claim count distribution, or the excess claim count distribution is otherwise easy to compute.

For readers familiar with the notation in Heckman and Meyers, recall that they parameterize claim count distributions with $\lambda$ and $c$. In their method, the parameters for the excess claim count distribution are $\alpha\lambda$ and $c$.

The above formulae for the mean and variance of the excess claim count distribution hold only if the parameters of the severity distribution are known with certainty. Venter provided the following formulas for the mean and variance of the excess claim count distribution $N$ when $\alpha$ is uncertainly known:

$$\mathrm{E}(N) = \lambda \mathrm{E}(\alpha),$$

$$\mathrm{Var}(N) = \lambda \mathrm{E}(\alpha) + (\sigma^2 - \lambda)\mathrm{E}(\alpha)^2 + (\sigma^2 + \lambda^2 - \lambda)\mathrm{Var}(\alpha).$$

Proofs are as follows:

$$\mathrm{E}(N \mid \alpha) = \alpha\lambda, \text{ so } \mathrm{E}(N) = \mathrm{E}(\mathrm{E}(N \mid \alpha)) = \lambda \mathrm{E}(\alpha).$$

$$
\begin{aligned}
\mathrm{Var}\,(N \mid \alpha) &= \alpha\lambda + \alpha^2(\sigma^2 - \lambda)\text{ , so} \\
\mathrm{Var}\,(N) &= \mathrm{E}(\mathrm{Var}\,(N \mid \alpha)) + \mathrm{Var}\,(\mathrm{E}(N \mid \alpha)) \\
&= \lambda \mathrm{E}(\alpha) + (\sigma^2 - \lambda)\mathrm{E}(\alpha^2) + \lambda^2\mathrm{Var}\,(\alpha) \\
&= \lambda \mathrm{E}(\alpha) + (\sigma^2 - \lambda)[\mathrm{Var}\,(\alpha) + \mathrm{E}(\alpha)^2] + \lambda^2\mathrm{Var}\,(\alpha) \\
&= \lambda \mathrm{E}(\alpha) + (\sigma^2 - \lambda)\mathrm{E}(\alpha)^2 + (\sigma^2 + \lambda^2 - \lambda)\mathrm{Var}\,(\alpha)\ .
\end{aligned}
$$

These formulae are useful either if $\alpha$ varies from one claim to the next (for example, if the excess distribution is for a set of reinsurance contracts with attachment points that vary by contract), or if it is desired to reflect parameter uncertainty with regard to $\alpha$.

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# APPENDIX A

## COMPLEX NUMBERS

This is a brief summary of the properties of complex numbers used earlier. More extensive treatments are in Baase [3, p.279], and Aho, Hopcraft, and Ullman [4, p. 252].

In this Appendix, $i$ is $\sqrt{-1}$. Given two complex numbers, $a + bi$ and $c + di$, their sum, difference, product, and quotient are given as:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$$

$$(a + bi) \div (c + di) = \left(\frac{1}{c^2 + d^2}\right)[(ac + bd) - (ad - bc)i]$$

The *complex conjugate* of $a + bi$ is $a - bi$, sometimes denoted $(a + bi)^*$.

A number $\omega$ is a *primitive* $n^{th}$ *root of unity* if $\omega^n = 1$ and $\omega^j \neq 1$ for any positive $j$ less than $n$. If $\omega$ is a primitive $n^{\text{th}}$ root of unity, then:

$$\omega^{j*} = 1 \div \omega^j$$

$$\sum_{k=0}^{n-1} (\omega^j)^k = \begin{cases} 0 & \text{for } j \text{ not a multiple of } n \\ n & \text{for } j \text{ a multiple of } n \end{cases}$$

Let $\boldsymbol{F}$ be the $n \times n$ *matrix with entries* $\boldsymbol{F}_{jk} = \omega^{jk}$. This matrix $\boldsymbol{F}$ plays a key role in the discrete Fourier transform (DFT). $\boldsymbol{F}^2$ is:

$$\begin{bmatrix} n & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & & 0 & n \\ 0 & 0 & 0 & & n & 0 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & \cdot & & \cdot \\ 0 & 0 & n & & 0 & 0 \\ 0 & n & 0 & \dots & 0 & 0 \end{bmatrix}$$

Appendix B shows why this makes the inverse of the DFT so simple.

# APPENDIX B

## CONVOLUTION EXAMPLE

This Appendix provides an example of the use of the convolution theorem to compute the sum of two severity distributions. Two vectors are used to represent severity distributions, and the convolution of these vectors represents the sum of the two severity distributions.

*Example*

The first severity distribution, $U$, has probability $4/5$ of a claim amount of \$0, probability $1/10$ of a claim amount of \$100, probability $1/20$ of a claim amount of \$200, and probability $1/20$ of a claim amount of \$300. The second severity distribution, $V$, has probability $3/5$ of a claim amount of \$0, probability $1/5$ of a claim amount of \$100, probability $1/10$ of a claim amount of \$200, and probability $1/10$ of a claim amount of \$300. These are represented as vectors as follows:

$$U = [4/5, 1/10, 1/20, 1/20, 0, 0, 0, 0],$$

$$V = [3/5, 1/5, 1/10, 1/10, 0, 0, 0, 0]\,.$$

These representations have been padded with zeroes to the right so that no-wrap convolutions can be computed. (They are not what is used in the body of the paper for the main algorithm. These representations are being used only to give an example of the use of the convolution theorem.)

As $U$ and $V$ are vectors of length 8, $\omega$ must be a primitive eighth root of unity. Let $\omega$ be $\cos(\pi/4) + i \sin(\pi/4)$. This $\omega$ can also be written $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ or, approximately, $0.7071067810 + 0.7071067810i$. (Here $i$ is $\sqrt{-1}$.) The matrix $F$, with entries $\omega^{jk}$ for $j, k$ from 0 to 7, is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \%1 & i & \%2 & -1 & \%3 & -i & \%4 \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \%2 & -i & \%1 & -1 & \%4 & i & \%3 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \%3 & i & \%4 & -1 & \%1 & -i & \%2 \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \%4 & -i & \%3 & -1 & \%2 & i & \%1 \end{bmatrix}$$

where $\%1$ is $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ , $\%2$ is $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ , $\%3$ is $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ , and $\%4$ is $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ . These are $\omega$, $\omega^3$, $\omega^5$, and $\omega^7$, respectively.

The convolution theorem states that

$$\boldsymbol{U} * \boldsymbol{V} = \text{INVDFT}(\text{DFT}(\boldsymbol{U}) \times \text{DFT}(\boldsymbol{V}))$$

where DFT is the discrete Fourier transform, and INVDFT is the inverse DFT.

The discrete Fourier transform of $\boldsymbol{U}$ is the matrix product $\boldsymbol{F} \cdot \boldsymbol{U}$ where $\boldsymbol{U}$ is treated as a column vector. Thus DFT($\boldsymbol{U}$), or $\boldsymbol{F} \cdot \boldsymbol{U}$ , is approximately:

[1.0, .8353553391 + .1560660172$i$, .7500000000 + .05000000000$i$, .7646446610 + .05606601720$i$, .7000000000, .7646446610 – .05606601720$i$, .7500000000 – .05000000000$i$, .8353553391 – .1560660172$i$] .

Similarly, DFT($\boldsymbol{V}$), or $\boldsymbol{F} \cdot \boldsymbol{V}$ , is approximately:

[1.0, .6707106781 + .3121320343$i$, .5000000000 + .1000000000$i$, .5292893219 + .1121320343$i$, .4000000000, .5292893219 -.1121320343$i$, .5000000000 - .1000000000$i$, .6707106781 - .3121320343$i$].

DFT($\boldsymbol{U}$) $\times$ DFT($\boldsymbol{V}$) is

[1.0, .5115685425 + .3654163056$i$, .3700000000 + .1000000000$i$, .3984314575 + .1154163056$i$, .2800000000, .3984314575 -.1154163056$i$, .3700000000 - .1000000000$i$, .5115685425 - .3654163056$i$].

For example, the second element of the vector just above is .5115685425 + .3654163056$i$, which is

$$(.8353553391 + .1560660172i) \times (.6707106781 + .3121320343i).$$

To compute the inverse DFT of DFT($\boldsymbol{U}$) $\times$ DFT($\boldsymbol{V}$), one first computes the DFT of DFT($\boldsymbol{U}$) $\times$ DFT($\boldsymbol{V}$); divides each term of the result by 8; and inverts the order of the last seven terms. The DFT of DFT($\boldsymbol{U}$) $\times$ DFT($\boldsymbol{V}$), or $\boldsymbol{F}\cdot$ (DFT($\boldsymbol{U}$) $\times$ DFT($\boldsymbol{V}$) ) , is

[3.840000000, 0.0, .0400000000, .0800000000, .2000000000, 1.040000000, 1.040000000, 1.760000000].

Dividing by 8 gives

[0.480, 0.0, 0.005, 0.010, 0.025, 0.130, 0.130, 0.220].

Reversing the order of the last 7 terms gives

[0.480, 0.220, 0.130, 0.130, 0.025, 0.010, 0.005, 0.0].

Thus, for the sum of the distributions $\boldsymbol{U}$ and $\boldsymbol{V}$ there is a probability of 0.480 of a total claim amount of \$0, a probability of 0.220 of a total claim amount of \$100, a probability of 0.130 of a total claim amount of \$200, etc. This can be readily verified by direct computation of these probabilities.

*The Inverse DFT*

This subsection will justify the method used above to compute the inverse DFT. Suppose we have computed DFT($\boldsymbol{W}$), which is $\boldsymbol{F}\cdot\boldsymbol{W}$, for some vector $\boldsymbol{W}$. Then DFT(DFT($\boldsymbol{W}$)) is $\boldsymbol{F}\cdot(\boldsymbol{F}\cdot\boldsymbol{W})$. Matrix multiplication is associative, so this is the same as $(F \cdot F) \cdot W$ or $F^2 \cdot W$. But $F^2$ (for the example above) is

$$\begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is just 8 times the matrix $R$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix reverses the order of the last seven elements of any vector to which it is applied.

Thus $F^2 \cdot W$ is just 8 times $W$ with the last seven terms reversed. Dividing by 8 and reversing the last seven terms restores $W$.

Alternatively, $F^{-1}$ is just $(1/8) \times R \times F$, where $R$ is the matrix of 1s and 0s above.

## APPENDIX C

### THE FAST FOURIER TRANSFORM AND INVERSE

The fast Fourier transform is presented by Baase [3, p. 273] as follows[2]. Using a language similar to Modula-2 and Pascal:

*Input:* The $n$-vector $\boldsymbol{P} = (p_0, p_1, \ldots, p_{n-1})$, where $n = 2^k$ for some $k > 0$.

*Output:* *transform*, the discrete Fourier transform of $\boldsymbol{P}$.

*Comment:* We assume that *omega* is an array containing the $n$th roots of 1: $\omega^0, \omega, \ldots, \omega^{(n/2)-1}$. $\pi_k$ is a permutation on $\{0, 1, \ldots, n-1\}$ (described below).

```
procedure FFT (P: RealArray; n: integer;
              var transform: Complex Array);

var       l: integer;     {the level number}
          num: integer;   {the number of values to be computed at
                            each node at level l }
          t: integer;     {the index in transform for the first of
                           these values for a particular node}
          j: integer;     {counts off the pairs of values to be
                          computed for that node }
          m: integer;     {used to pick out the correct entry from
                          omega }
begin
          for t := 0 to n - 2 by 2 do
                          transform [t] := p[π_k(t)] + p[π_k(t + 1)];
                          transform [t + 1] := p[π_k(t)] - p[π_k(t + 1)]
end       {for t };

          {The main computation}
            m := n/2; num := 2;
          {Begin triply-nested loop}
          for l := k-2 to 0 by -1 do
```

[2] Reprinted with permission of the publisher.

```
        m := m/2; num := 2*num;
      for t := 0 to (2^l - 1)num by num do
      for j := 0 to (num/2)-1 do
          xPOdd := omega[mj]*transform[t+num/2+j];
          transform[t + num/2 + j] := transform[t + j] - xPOdd;
          transform[t + j] := transform[t + j] + xPOdd;
        end { for j }
        end { for t }
      {end of body of outer for loop }
      end { for l }
end { FFT }
```

Now, what is $\pi_k$? Let $t$ be an integer between 0 and $n-1$, where $n = 2^k$. Then $t$ can be represented in binary by $[b_0\, b_1 \ldots b_{k-1}]$, where each $b_j$ is 0 or 1. Let $rev_k(t)$ be the number represented by these bits in reverse order, i.e., by $[b_{k-1} \ldots b_1\, b_0]$. Then $\pi_k(t) = rev_k(t)$. (As an example, $\pi_3(3) = 6$ because 011 reversed is 110.)

The inverse fast Fourier transform is computed as follows. Apply the regular (forward) fast Fourier transform to the vector. Divide each element of the result by $n$. Reverse the order of the last $n-1$ elements (i.e., the first element stays in place and the order of the other elements is reversed).

Baase gives further discussion of the fast Fourier transform, including some analysis of the number of computations needed.

# APPENDIX D

## APL PROGRAMS FOR FFT AND INVFFT ALGORITHMS

This Appendix contains the functions FFT and INVFFT. Before running these, INIT must be run to initialize certain global variables. FFT and INVFFT do not modify these global variables, so INIT needs to be run only when the global variables have to be changed. INIT calls INITOMEGA and INITPIK, also listed here.

All the APL functions presented in this paper assume ⎕IO has been set to zero.

Complex vectors of length $n$ are represented as two-by-$n$ arrays; e.g., the vector $(a + bi, c + di, \ldots)$ is represented by:

$$\begin{pmatrix} a & c & \ldots \\ b & d & \ldots \end{pmatrix}$$

FFT:

```
  [0]  R←FFT P;Z;INDX;INDXP1;T1;T2;M;NUM;L;
          OMIND
  [1]  ⍝ REFERENCE SARA BAASE, COMPUTER
          ALGORITHMS, P 273 FF
  [2]  ⍝ ASSUMES YOU HAVE RUN INIT
  [3]  ⍝ INPUT IS P PER BAASE, N PER BAASE IS
          GLOBAL VARIABLE
  [4]  ⍝ OUTPUT IS TRANSFORM PER BAASE
  [5]  R←INITR
  [6]  R[;INDXE]←P[;PIK[INDXE]]+P[;PIK[INDXO]]
  [7]  R[;INDXO]←P[;PIK[INDXE]]-P[;PIK[INDXO]]
  [8]  ⍝ NOW HAVE INITIALIZED R (= TRANSFORM)
  [9]  M←⌊0.5+N÷2 ⋄ NUM←2 ⋄ L←K-2
 [10]  LLOOP:M←⌊0.5+M÷2 ⋄ NUM←⌊0.5+2×NUM
 [11]  T2←,⍉(⍳NUM÷2)∘.+NUM×⍳N÷NUM
 [12]  T1←T2+NUM÷2
 [13]  OMIND←(N÷2)⍴M×⍳N÷2×M
 [14]  Z←(2,N÷2)⍴(,-⌿OMEGA[;OMIND]×R[;T1]),
          ,+⌿OMEGA[;OMIND]×⊖R[;T1]
 [15]  R[;T1]←R[;T2]-Z ⋄ R[;T2]←R[;T2]+Z
 [16]  L←L-1
 [17]  →(L≥0)⍴LLOOP
```

INVFFT:

```
  [0]  R←INVFFT X
  [1]  R←FFT X
  [2]  R[;INVINDX]←⌽R[;INVINDX] ⍝ REVERSE
          ORDER OF LAST N-1 ELEMENTS
  [3]  R←R÷N
```

INIT:

```
  [0]  INIT KK
  [1]  K←KK
  [2]  N←⌊0.5+2*K
  [3]  ⎕IO←0
  [4]  INITOMEGA K
  [5]  INITPIK K
  [6]  INITR←(2,N)⍴0
  [7]  INDXE←2×⍳N÷2
  [8]  INDXO←INDXE+1
  [9]  INVINDX←1+⍳N-1
 [10]  TAIL←(N÷2)+⍳N÷2
```

INITOMEGA:

```
  [0]  INITOMEGA K
  [1]  N←⌊0.5+2*K
  [2]  OMEGA←(2,N÷2)⍴○(⍳N÷2)×2÷N
  [3]  OMEGA[0;]←2○OMEGA[0;]
  [4]  OMEGA[1;]←1○OMEGA[1;]
  [5]  OMEGA2←(2,N+1)⍴○(⍳N+1)×2÷2×N
  [6]  OMEGA2[0;]←2○OMEGA2[0;]
  [7]  OMEGA2[1;]←1○OMEGA2[1;]
```

INITPIK:

```
  [0]  INITPIK K;N
  [1]  N←⌊0.5+2*K
  [2]  PIK←2⊥⊖((K⍴2)⊤⍳N)
```

# APPENDIX E

## PROOF THAT $A^i * S^i$ IS THE DISTRIBUTION OF $i$ CLAIMS

This Appendix gives a proof sketch that $A^i * S^i$ is the probability distribution of the sum of exactly $i$ claims. More precisely, it shows that if $A^i$, $S^i$, $n$, and $L$ are defined as in the main body of this paper, and $X$ is the probability distribution of exactly $i$ claims, then the probability that $X$ is between $j \times \left(\frac{L}{n}\right)$ and $(j+1) \times \left(\frac{L}{n}\right)$ is $(A^i * S^i)\,[j]$.

Consider first the case where $L = n$ and $S = (1, 0, 0, ..., 0)$. This makes $S$ a uniform distribution on the unit interval $[0, 1]$. In this case the first $n$ terms of $A^i * S^i$ are:

$$a_0^i, a_1^i, a_2^i, \ldots a_{n-1}^i .$$

Let $F^i$ be the distribution function for the sum of $i$ mutually independent random variables uniformly distributed over $[0, 1]$. Let $b_j^i = F^i\,(j+1) - F^i\,(j)$. It needs to shown that $a_j^i = b_j^i$.

Combining equation 9.1 and Theorem 1 of Section I.9 of Feller [11, p. 27],

$$b_j^i = F^i\,(j+1) - F^i\,(j) = \frac{1}{i\,!} \sum_{\nu=0}^{i+1} (-1)^\nu \binom{i+1}{\nu} (j+1-\nu)_+^i ,$$

where $x_+^i$ is $x^i$ if $x \geq 0$ and $x_+^i$ is zero if $x < 0$. This yields:

$$b_j^i = \frac{1}{i\,!} \sum_{\nu=0}^{j} (-1)^\nu \binom{i+1}{\nu} (j+1-\nu)^i$$

because if $j < i + 1$ then $(j+1-\nu)_+$ is zero for terms with $\nu > j$ , and if $j > i + 1$ then $\binom{i+1}{\nu}$ is zero for $\nu > i + 1$.

It is easy to see that $a_0^i = b_0^i = 1/i!$ for $i > 1$, and $a_j^1 = b_j^1 = 0$ for $j \geq 1$. To complete the proof that $a_j^i = b_j^i$ it suffices to show that the $b_j^i$ satisfy the recursion relation used to define the $a_j^i$.

To this end, for $i > 1$ and $j > 0$ consider:

$$z = \left(\frac{1}{i}\right)\left[(i-j)\, b_{j-1}^{i-1} + (j+1)\, b_j^{i-1}\right].$$

It needs to be shown that $z$ equals $b_j^i$. This is done by plugging into the above formula the expression for $b_j^i$ as a sum, and rearranging terms:

$$i!\, z = (i-j)\,(i-1)!\, b_{j-1}^{i-1} + (j+1)\,(i-1)!\, b_j^{i-1}$$

$$= (i-j)\sum_{\tau=0}^{j-1}(-1)^{\tau}\binom{i}{\tau}(j-\tau)^{i-1} + (j+1)\sum_{v=0}^{j}(-1)^{v}\binom{i}{v}(j+1-v)^{i-1}$$

$$= (i-j)\sum_{v=0}^{j}(-1)^{v-1}\binom{i}{v-1}(j+1-v)^{i-1} +$$

$$(j+1)\sum_{v=0}^{j}(-1)^{v}\binom{i}{v}(j+1-v)^{i-1}$$

$$= (j+1)\,(1)\,(1)\,(j+1)^{i-1} +$$

$$\sum_{v=1}^{j}\left[(-1)\,(i-j)\binom{i}{v-1} + (j+1)\binom{i}{v}\right](-1)^{v}\,(j+1-v)^{i-1}$$

$$= (-1)^0\binom{i+1}{0}(j+1-0)^i +$$

$$\sum_{v=1}^{j}\left[\frac{(j-1)\,v}{i+1} + \frac{(j+1)\,(i+1-v)}{i+1}\right]\left(\frac{(i+1)\,!}{v\,!\,(i+1-v)\,!}\right)(-1)^{v}\,(j+1-v)^{i-1}$$

$$= (-1)^0 \binom{i+1}{0} (j+1-0)^i + \sum_{v=1}^{j} (j+1-v) \binom{i+1}{v} (-1)^v (j+1-v)^{i-1}$$

$$= \sum_{v=0}^{j} (-1)^v \binom{i+1}{v} (j+1-v)^i .$$

Thus, $z = b_j^i$. This establishes the main result for this case.

For the general case, consider the positive "quadrant" of $\mathbb{R}^i$, i.e, the points $(x_0, x_1, ..., x_{i-1})$ such that each $x_j$ is greater than or equal to zero. Divide this space into cubes with edge length $L \div n$ in the obvious way. Assign a density to each cube as follows. If the cube's vertex closest to the origin is $\left( v_0 \frac{L}{n}, v_1 \frac{L}{n}, \ldots, v_{i-1} \frac{L}{n} \right)$, assign a density of $( s_{v_0} s_{v_1} s_{v_2} \ldots s_{v_{i-1}} ) \div \left( \frac{L}{n} \right)^i$ where the $s_{v_k}$ are elements of the vector representing the severity distribution if $v_k \le n-1$, and $s_{v_k} = 0$ if $v_k \ge n$. As the volume of every cube is $\left( \frac{L}{n} \right)^i$, the integral of this density over the cube is $s_{v_0} s_{v_1} s_{v_2} \ldots s_{v_{i-1}}$. Now consider the integral of these densities between the parallel $(i-1)$-planes:

$$x_0 + x_1 + \ldots + x_{i-1} = k \frac{L}{n}, \text{ and}$$

$$x_0 + x_1 + \ldots + x_{i-1} = (k+1) \frac{L}{n} .$$

This integral is the probability that the sum of $i$ claims will have a value between $k \frac{L}{n}$ and $(k+1) \frac{L}{n}$.

This is also the $k^{\text{th}}$ term of $\boldsymbol{A}^i * \boldsymbol{S}^i$, as will now be shown. The $m^{\text{th}}$ element of $\boldsymbol{S}^i$ is the sum of all $s_{j_0} \times s_{j_1} \times \ldots \times s_{j_{i-1}}$ such that $j_0 + j_1 + \ldots + j_{i-1} = m$. For instance, if $i$ is 3, $s_2^3$ is:

$$s_2 s_0 s_0 + s_0 s_2 s_0 + s_0 s_0 s_2 + s_1 s_1 s_0 + s_1 s_0 s_1 + s_0 s_1 s_1 = 3s_0^2 s_2 + 3s_0 s_1^2.$$

Each of the cubes associated with the $m^{\text{th}}$ element of $\boldsymbol{S}^i$ (under the inverse of the above association of cubes with densities) has its vertex closest to the origin on the plane $x_0 + \ldots + x_{i-1} = m\frac{L}{n}$. Each cube also has its vertex farthest from the origin on the plane $(m+i)\frac{L}{n}$. The planes "$m$," "$m+1$," ..., "$m+i$" divide each of these cubes into the proportions given by the $\boldsymbol{A}^i$. Getting back to the planes "$k$" and "$k+1$", considering all the cubes that have some portion between these two planes, the integral of the density between these two planes is $(\boldsymbol{A}^i * \boldsymbol{S}^i)\,[k]$. The probability that the sum of the $i$ distributions will be between $k$ and $k+1$ is given by the same integral. This establishes the overall result.

# APPENDIX F

## PACKING AND UNPACKING

This Appendix provides the formulae for packing and unpacking transformed vectors. This treatment essentially follows that of Press, Flannery, Teukolsky, and Vetterling [6, p. 398]. APL programs to implement these routines are in Appendix G.

*Unpacking* a transformed vector is discussed first. Assume that one starts with a real (untransformed) vector, $\boldsymbol{U}$, of length $2n$ and packs it into a complex vector, $\boldsymbol{PU}$, of length $n$, as discussed in the main body of the paper. Then the FFT is applied to $\boldsymbol{PU}$ to obtain a vector $\boldsymbol{PH}$ that is the packed transformation of $\boldsymbol{PU}$. The next step is to unpack $\boldsymbol{PH}$ to obtain the FFT of $\boldsymbol{U}$.

The result of unpacking $\boldsymbol{PH}$ will be a complex vector of length $n + 1$. One might think the result would be a complex vector of length $2n$ since the goal is to obtain the FFT of $\boldsymbol{U}$ which is of length $2n$. If $\boldsymbol{R}$ is the (length $2n$ for the moment) FFT of $\boldsymbol{U}$, then:

$$r_{2n-j} = r_j^* \text{ for } 1 \le j \le n ,$$

where * denotes complex conjugation. Thus, from the first $n + 1$ terms (0 to $n$) it is easy to derive the remaining terms of $\boldsymbol{R}$.

Append to the end of $\boldsymbol{PH}$ the first element of $\boldsymbol{PH}$, making $\boldsymbol{PH}$ a complex vector of length $n + 1$. Let $\boldsymbol{PH2}$ be the complex conjugate of the "reverse" of $\boldsymbol{PH}$; i.e., $\boldsymbol{PH2}[j] = \boldsymbol{PH}[n - j]^*$ for $0 \le j \le n$ . Define $\boldsymbol{PH3}$ by:

$$\boldsymbol{PH3}[j] = -i(\boldsymbol{PH}[j] - \boldsymbol{PH2}[j]) \times \omega^j,$$

where $i$ is $\sqrt{-1}$ and $\omega$ is a $2n^{\text{th}}$ root of unity (such that $\omega^2$ is the $n^{\text{th}}$ root of unity used in the FFT). Finally, $\boldsymbol{R}$ is half the sum of $\boldsymbol{PH}$, $\boldsymbol{PH2}$, and $\boldsymbol{PH3}$.

The steps for packing a transformed vector $\boldsymbol{R}$ (of length $n+1$), to ready it for application of the inverse FFT, are almost the same steps as for unpacking. Let $\boldsymbol{R2}$ be the complex conjugate of the "reverse" of $\boldsymbol{R}$. Define $\boldsymbol{R3}$ by:

$$\boldsymbol{R3}[j] = i(\boldsymbol{R}[j] - \boldsymbol{R2}[j]) \times \omega^{-j},$$

where $i$ and $\omega$ are as immediately above. The final result is the first $n$ terms of half the sum of $\boldsymbol{R}$, $\boldsymbol{R2}$, and $\boldsymbol{R3}$.

# APPENDIX G

## APL ROUTINES FOR PACKING AND UNPACKING

UNPACK:

```
  [0]   R←UNPACK H;H2;H3
  [1]   ⍝ UNPACKS TRANSFORMED DATA.  ASSUMES X
           IS THE RESULT OF
  [2]   ⍝ APPLYING THE FFT TO A LENGTH 2N REAL
           VECTOR WHICH HAD BEEN
  [3]   ⍝ PACKED INTO A 2×N COMPLEX ARRAY.
  [4]   ⍝ RESULT IS A 2 × N+1 ARRAY
  [5]   H←H,H[;0]
  [6]   H2←⌽H
  [7]   H2[1;]←-H2[1;]
  [8]   H3←H-H2
  [9]   H3←(2,N+1)⍴(,-/H3×OMEGA2),,+/H3×⊖OMEGA2
 [10]   H3←⊖H3
 [11]   H3[1;]←-H3[1;]
 [12]   R←0.5×H+H2+H3
```

PACK:

```
  [0]   R←PACK X;X2;X3
  [1]   ⍝ PACKS TRANSFORMED VARIABLE
  [2]   X2←⌽X
  [3]   X2[1;]←-X2[1;]
  [4]   X3←X-X2
  [5]   X3←(2,N+1)⍴(,+/X3×OMEGA2),,-/OMEGA2×⊖X3
  [6]   X3←⊖X3
  [7]   X3[0;]←-X3[0;]
  [8]   R←0.5×X+X2+X3
  [9]   R← 0 ¯1 ↓R
```

OMEGA2 (the $2n^{th}$ roots of unity) is generated by INIT and INITOMEGA in Appendix D.

# APPENDIX H

## APL FUNCTIONS FOR THE COMPLETE ALGORITHM

This Appendix gives an APL function, AGGDISTR, that implements the full algorithm. AGGDISTR calls a number of subroutines. The subroutines FFT and INVFFT are listed in Appendix D, and the subroutines PACK and UNPACK are listed in Appendix G. The only other subroutine needed is MULT, listed below. Before running AGGDISTR it is necessary to run INIT, which sets certain global variables. INIT calls INITOMEGA and INITPIK; these three programs are given in Appendix D.

AGGDISTR:

```
AGGDISTR;M;M2;P;S;PS1;PSI;PZERO;FS1;
   FSI;AI;I;BIN;FSIFLAG;X;J;Y;PY;FAI;FY
⎕IO←0
⍝ ----
'Input the smallest number of claims
   with non-zero probability,'
'M:'
' '
M←⎕
' '
→((M≥0)∧(M=⌊0.5+M))⍴SKIP1 ⍝ M must be a
   non-negative integer.
'M, ',(⍕M),' is not a non-negative
   integer.  Stopped.'
→0
SKIP1:' '
'Input densities of claim frequency
   distribution.  These should'
'be the probabilities of M, M+1, M+2,
   ... claims.'
' '
P←,⎕
→(((⍴P)≠1)∨(P[0]≠0))⍴SKIP2 ◇ ⍝ IF ONLY
   ONE NUMBER IS INPUT AND IT IS ZERO,
   THEN EXIT.
'Only one number was input, and it is '
   ,(⍕P),'.  Stopped.'
→0
SKIP2:' '
'Input vector for severity distribu-
   tion.  Must be of length ',(⍕N),'.'
' '
S←,⎕
```

```
[23]  →((ρS)=N)ρSKIP3 ◇ ⍝ IF S IS NOT OF
         LENGTH N, THEN EXIT.
[24]  'Length of S is ',(⍕ρS),'.  Should be '
         ,(⍕N),'.  Stopped.'
[25]  →0
[26]  SKIP3:' '
[27]  ⍝ ----
[28]  AGG←(2,N)ρ0
[29]  PS1←⍉(N,2)ρS,(Nρ0) ◇ ⍝ 'PACK' S.
[30]  S←0 ◇ ⍝ FREE UP SPACE
[31]  PZERO←0 ◇ ⍝ INITIALIZE - WILL BE RESET
         IF M=0.
[32]  ⍝ ----
[33]  ⍝ THREE CASES ARE CONSIDERED, M=0, M=1,
         OR M>1.
[34]  →(M=0)ρMEQ0
[35]  →(M=1)ρMEQ1
[36]  →(M>1)ρMGT1
[37]  MEQ0:PZERO←P[0] ◇ P←,1↓P ◇ M←1
         ⍝ CONVERT TO CASE M=1
[38]  MEQ1:M2←ρP
[39]  AGG←P[0]×PS1
[40]  →(M2=1)ρENDIT
[41]  FS1←UNPACK FFT PS1
[42]  PSI←INVFFT PACK FS1 MULT FS1
[43]  PSI[;TAIL]←0
[44]  AI← 0.5 0.5 ◇ I←2 ◇ PS1←0
[45]  →MAINLOOP
[46]  ⍝ ----
[47]  ⍝ ----
[48]  MGT1: ⍝ START BINARY POWER TRICK.

[49]  BIN←,1↓((1+⌊2⍟M)ρ2)⊤M ⍝ EXPRESS M AS
         BINARY VECTOR, DROP FIRST TERM
[50]  FSI←FS1←UNPACK FFT PS1
[51]  BINLOOP:FSI←FSI MULT FSI ◇ PSI←INVFFT
         PACK FSI ◇ PSI[;TAIL]←0
[52]  'BINLOOP ',(⍕BIN),' ',⍕⎕TS
[53]  →((1↑BIN)=0)ρSKIP
[54]  FSI←UNPACK FFT PSI ◇ FSI←FSI MULT FS1 ◇
         PSI←INVFFT PACK FSI ◇ PSI[;TAIL]←0
[55]  SKIP:BIN←,1↓BIN ◇ →(0=ρBIN)ρEXIT
[56]  FSI←UNPACK FFT PSI
[57]  →BINLOOP
[58]  EXIT: ⍝ THIS IS THE EXIT FROM THE
         BINARY POWER TRICK.
[59]  ⍝ ----
[60]  AI←,1 ◇ I←1 ⍝ SET AI, INDEX
[61]  ALOOP:I←I+1 ◇ AI←(1÷I)×(⌽AI)+AI←(1+⍳I)×
         AI,0
[62]  →(I<M)ρALOOP
[63]  M2←¯1+M+(ρP) ◇ PS1←0
[64]  ⍝ ----
[65]  ⍝ ----
[66]  MAINLOOP:'MAINLOOP ',(⍕I),' ',⍕⎕TS
[67]  FSIFLAG←0
```

```
  [68]   ⍝ ----
  [69]   →(I≥100)ρBETA
  [70]   ⍝ IF HERE, WANT TO CONVOLUTE AI WITH
            PSI WITHOUT FFT'S
  [71]   X←N↑,⍉PSI
  [72]   J←0 ◇ Y←0
  [73]   LOOPB:Y←Y+AI[J]×X
  [74]   J←J+1 ◇ →((J>I-1)∨(J>N-1))ρENDLOOPB
  [75]   X←0,¯1↓X ⍝ DROP LAST ELEMENT OF X, ADD
            A ZERO TO THE FRONT.
  [76]   →LOOPB
  [77]   ENDLOOPB:PY←⍉(N,2)ρY,Nρ0 ◇ X←Y←0
  [78]   →GAMMA
  [79]   ⍝ ----
  [80]   BETA:FAI←UNPACK FFT⍉(N,2)ρ(N↑AI,Nρ0),
            Nρ0
  [81]   FSIFLAG←1
  [82]   FSI←UNPACK FFT PSI
  [83]   FY←FSI MULT FAI
  [84]   PY←INVFFT PACK FY
  [85]   PY[;TAIL]←0 ◇ FY←0
  [86]   ⍝ ----
  [87]   GAMMA:AGG←AGG+P[I-M]×PY ◇ PY←0
  [88]   I←I+1 ◇ →(I>M2)ρENDIT
  [89]   ⍝ ----
  [90]   →(FSIFLAG=1)ρSKIP4
  [91]   FSI←UNPACK FFT PSI
  [92]   SKIP4:FSI←FSI MULT FS1
  [93]   PSI←INVFFT PACK FSI
  [94]   PSI[;TAIL]←0
  [95]   ⍝ ----
  [96]   AI←(1÷I)×(⌽AI)+AI←(1+⍳I)×AI,0
  [97]   →MAINLOOP
  [98]   ENDIT:⎕TS ◇ AGG←N↑,⍉AGG
  [99]   AGG[0]←AGG[0]+PZERO
 [100]   ' '
 [101]   '***  REMEMBER, RESULT IS IN AGG.  ***'
 [102]   ' '
```

MULT:

```
   [0]   Z←X MULT Y
   [1]   Z←(ρX)ρ(,-⌿X×Y),,+⌿X×⊖Y
```

# APPENDIX I

## PARAMETER UNCERTAINTY FOR THE SEVERITY DISTRIBUTION

We will show mathematically that it is fundamentally impossible to reflect parameter uncertainty for the severity distribution by computing the aggregate distribution using a severity distribution with a larger variance than the best-estimate severity distribution. This discussion is based on suggestions by Venter.

Parameter uncertainty for the severity distribution is reflected by choosing a distribution $a$ with mean 1 and variance greater than 0, and computing the aggregate distribution:

$$AGG = aS_1 + aS_2 + \ldots + aS_T.$$

Here $AGG$, $S_i$, and $T$ are the aggregate distribution, the severity distribution, and the claim count distribution, defined earlier. The above equation is written to indicate that one sample from $T$ is associated with one sample from $a$ and multiple samples from $S$. The above equation could also be written

$$AGG = a(S_1 + S_2 + \ldots + S_T).$$

A general fact about variance (for arbitrary/independent distributions $X$ and $Y$) that will be used is:

$$\mathrm{Var}(XY) = \mathrm{Var}(X)\,\mathrm{Var}(Y) + (\mathrm{E}(X))^2\,\mathrm{Var}(Y) + (\mathrm{E}(Y))^2\,\mathrm{Var}(X). \quad \text{(I.1)}$$

Also, recall that

$$\mathrm{Var}(S_1 + S_2 + \ldots + S_T) = \mu_T\sigma_S{}^2 + \mu_S{}^2\sigma_T^2 \quad \text{(I.2)}$$

Here $\mu_T$ and $\mu_S$ are the means of the claim count and severity distributions, and $\sigma_T^2$ and $\sigma_S^2$ are the variances of the respective distributions.

Set

$$X = a, \; Y = S_1 + S_2 + \ldots + S_T, \text{ and } XY = AGG$$

and substitute in equation I.1 above. This gives

$$\begin{aligned}\mathrm{Var}(AGG) = {} & \mathrm{Var}(a)\,\mathrm{Var}(S_1 + S_2 + \ldots + S_T) + \\ & (\mathrm{E}(a))^2\,\mathrm{Var}(S_1 + S_2 + \ldots S_T) + (\mathrm{E}(S_1 + S_2 + \ldots + S_T))^2\,\mathrm{Var}(a).\end{aligned}$$

Denote $\mathrm{Var}(a)$ by $\sigma_a^2$, note that $\mathrm{E}(S_1 + S_2 + \ldots + S_T)$ is $\mu_T\mu_S$, recall that $\mathrm{E}(a)$ is 1, and substitute using equation I.2 to obtain

$$\mathrm{Var}(AGG) = \left(1 + \sigma_a^2\right)\left(\mu_T\,\sigma_S^2 + \mu_S^2\,\sigma_T^2\right) + \left(\mu_T\,\sigma_S\right)^2\sigma_a^2.$$

Dividing by $(\mu_T\,\mu_S)^2$ gives a formula for the square of the coefficient of variation for the aggregate distribution:

$$\frac{\mathrm{Var}(AGG)}{(\mu_T\,\mu_S)^2} = (1 + \sigma_a^2)\left[\frac{\sigma_S^2}{\mu_T\,\mu_S^2} + \frac{\sigma_T^2}{\mu_T^2}\right] + \sigma_a^2\ . \tag{I.3}$$

Now consider what happens as the mean of the claim count distribution, i.e., the expected number of claims, increases towards infinity. For the moment, assume there is no parameter uncertainty for the claim count distribution. The term $\dfrac{\sigma_S^2}{\mu_T\,\mu_S^2}$ tends to zero as $\mu_T$ increases. If the claim count distribution is Poisson, or is negative binomial with a fixed "probability of success" parameter, then the term $\dfrac{\sigma_T^2}{\mu_T^2}$ also tends to zero. (If the negative binomial is parameterized so the density function is $f(x) = \binom{y + x - 1}{x} p^y (1 - p)^x$ with $y > 0$ and $0 < p < 1$, then $p$ is the probability of success parameter.)

Thus, for any fixed severity distribution, the limit of the square of the coefficient of variation of the aggregate distribution, as $\mu_T$ goes to infinity, is equal to $\sigma_a^2$. In a practical sense, this means that, as the expected number of claims becomes large, the effect of the claim count distribution and the severity distribution on the coefficient of variation of the aggregate distribution becomes minimal. The coefficient of variation of the aggregate distribution is determined by the parameter uncertainty for the severity distribution.

In particular, this shows that there is a fundamental difference between the effect on the aggregate distribution of parameter uncertainty in the severity distribution and the effect of using a severity distribution with a greater variance. If a severity distribution, $S'$, with a larger variance is substituted for the best-estimate severity distribution, $S$, and $AGG = S'_1 + S'_2 + \ldots + S'_T$ is computed (this is the same as setting $a$ to a constant 1), then as $\mu_T$ goes to infinity, the coefficient of variation of the aggregate distribution tends to zero, and not to some positive value as above.

The two approaches differ in the independence assumptions regarding the samples from the severity distribution. If the severity distribution is diffused, then each draw from the severity distribution is a combination of an independent draw from the best-estimate severity distribution and an independent draw from the distribution used to reflect parameter uncertainty for the severity distribution. Under the correct method of reflecting parameter uncertainty, each draw is still an independent draw from the severity distribution, but there is only one draw from the distribution reflecting parameter uncertainty for each draw from the claim count distribution.

Equation I.3 shows that diffusing the severity distribution is not an adequate method for the recognition of parameter uncertainty for the severity distribution. Equation I.3 can also show the effect of parameter uncertainty for the claim count distribution, and the remainder of this appendix gives a brief discussion of that effect.

Apart from one quick comment at the end, only one method of representing parameter uncertainty for the claim count distribution will be considered here. It will be assumed that the best-estimate claim count distribution is Poisson with parameter $\lambda$. That is, the best-estimate claim count distribution, $N$, has probability density function

$$\mathrm{P}\,(N = x \mid \lambda) = \frac{\lambda^x}{x\,!}\, e^{-\lambda}\,.$$

Parameter uncertainty is incorporated by assuming that $\lambda$ follows a gamma distribution, denoted $\Lambda$, with probability function

$$P(\Lambda = \lambda) = \frac{(\lambda/b)^{c-1} e^{-\lambda/b}}{b\Gamma(c)}.$$

Above, $\Gamma$ is the gamma function and $b$ and $c$ are parameters. The distribution $\Lambda$ has mean $bc$ and variance $b^2c$. The parameter $b$ is the scale parameter because changing $b$ by some factor has the effect of multiplying the distribution uniformly by that same factor. The parameter $c$ is the shape parameter.

Straightforward computations show that the claim count distribution, $T$, that results from compounding the Poisson distribution with the gamma distribution is a negative binomial distribution with probability function:

$$P(T = t) = \binom{t + c - 1}{t} \left(\frac{1}{b + 1}\right)^{c} \left(\frac{b}{b + 1}\right)^{t}.$$

Here $b$ and $c$ are the parameters from the gamma distribution. This negative binomial distribution has mean $bc$ and variance $(bc)(b+1)$. Thus $\sigma_T^2/\mu_T^2$ is $\frac{1}{c}\left(1 + \frac{1}{b}\right)$.

One way to reflect a constant degree of parameter uncertainty in the claim count distribution while increasing the mean is to allow $b$ to increase but to hold $c$ constant. This maintains a constant percentage of uncertainty regarding the mean of the claim count distribution. In the Equation I.3, the term $\frac{\sigma_S^2}{\mu_T\mu_S^2}$ tends to zero as $\mu_T$ increases (as it did above). But now the term $\frac{\sigma_T^2}{\mu_T^2}$ tends to a positive limit, namely $1/c$. Thus, Equation I.3 shows that the coefficient of variation of the aggregate distribution has a component due to the parameter uncertainty in the claim count distribution that does not drop below a certain minimum no matter how large the mean of the claim count distribution becomes.

A little more generally, as $\mu_T$ goes to infinity, the limit of the square of the coefficient of variation of the aggregate distribution is

$$\left(1+\sigma_a^2\right)\left(\lim_{\mu_T\to\infty}\frac{\sigma_T^2}{\mu_T^2}\right)+\sigma_a^2 .$$

The severity distribution and the best-estimate claim count distribution have no effect on this limit. This limit depends only on the amount of parameter uncertainty reflected in each of these distributions.

# APPENDIX J

## PROOFS OF FORMULAE FOR EXCESS CLAIM COUNT DISTRIBUTIONS

This appendix proves that if a claim count distribution is binomial, Poisson, or negative binomial, with mean $\lambda$ and variance $\sigma^2$, and the probability of a claim being excess of some attachment point is $\alpha$, then the excess claim count distribution is binomial, Poisson, or negative binomial, respectively, with mean $\alpha\lambda$ and variance $\alpha\lambda + \alpha^2(\sigma^2 - \lambda)$. For each of the three types of claim count distributions, it is shown that the selection of claim counts from the given distribution with mean $\lambda$ and variance $\sigma^2$, followed by selection of excess claims with probability $\alpha$ (under a binomial process) gives a distribution of the same type with mean $\alpha\lambda$ and variance $\alpha\lambda + \alpha^2(\sigma^2 - \lambda)$.

The Poisson case is considered first. Here $\lambda = \sigma^2$ , so $\sigma^2$ will not appear. The total distribution will be $T$ and the excess distribution will be $X$. For the total distribution:

$$P(T = t) = \frac{\lambda^t}{t!} e^{-\lambda}.$$

Given $t$ claims in the total distribution, the distribution of excess claims is:

$$P(X = x \mid T = t) = \binom{t}{x} \alpha^x (1 - \alpha)^{t-x}.$$

$P(X = x)$ is the sum over all $t$ of $P(T = t) \times P(X = x \mid T = t)$ ; i.e.,

$$P(X = x) = \sum_{t=x}^{\infty} \frac{\lambda^t}{t!} e^{-\lambda} \binom{t}{x} \alpha^x (1 - \alpha)^{t-x}.$$

Letting $i = t - x$ so $t = x + i$ this becomes:

$$P(X = x) = \sum_{i=0}^{\infty} \frac{\lambda^{x+i}}{(x+i)!} e^{-\lambda} \binom{x+i}{x} \alpha^x (1 - \alpha)^i.$$

This is a "ground up" computation of the claim count distribution for aggregate claims. The starting point was the distribution of total claims, $T$, and then excess claims were selected according to a binomial process to get the distribution of excess claims.

It is necessary to show that this sum is $\frac{(\alpha\lambda)^x}{x!} e^{-\alpha\lambda}$.

But:

$$e^{-\alpha\lambda} e^{-\lambda+\lambda-\alpha\lambda} = e^{-\lambda} e^{\lambda(1-\alpha)} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i (1-\alpha)^i}{i!}$$

$$\Rightarrow \frac{\alpha^x \lambda^x}{x!} e^{-\alpha\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^x \lambda^i}{(x+i)!} e^{-\lambda} \frac{(x+i)!}{x!\, i!} \alpha^x (1-\alpha)^i$$

$$\Rightarrow \frac{(\alpha\lambda)^x}{x!} e^{-\alpha\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^{x+i}}{(x+i)!} e^{-\lambda} \binom{x+i}{x} \alpha^x (1-\alpha)^i$$

as was to be shown.

Now much the same is done for the negative binomial. According to Hastings and Peacock [24, p. 92], the negative binomial has density function:

$$\mathrm{P}(Y=y) = \binom{x+y-1}{y} p^x q^y$$

where $x$ and $p$ are parameters and $q = 1 - p$. This distribution has mean $xq/p$ and variance $xq/p^2$. If parameters $x$ and $p$ result in a mean of $\lambda$ and a variance of $\sigma^2$, then parameters of $x$ and $p/(p + \alpha - \alpha p)$ result in mean $\alpha\lambda$ and variance $\alpha\lambda + \alpha^2(\sigma^2 - \lambda)$. Let $T$ be the total claim count distribution, and let $Y$ be the excess claim count distribution. Then:

$$\mathrm{P}(T=t) = \binom{x+t-1}{t} p^x q^t$$

$$P(Y = y \mid T = t) = \binom{t}{y} \alpha^y (1 - \alpha)^{t - y}$$

$$P(Y = y) = \sum_{t = y}^{\infty} \binom{x + t - 1}{t} p^x q^t \binom{t}{y} \alpha^y (1 - \alpha)^{t - y}$$

$$P(Y = y) = \sum_{i = 0}^{\infty} \binom{x + y + i - 1}{i + y} p^x q^{i + y} \binom{i + y}{y} \alpha^y (1 - \alpha)^i$$

where $i + y$ was substituted for $t$ to get from the next-to-last equation to the last equation.

It is necessary to show that the right-hand side of this equation is equal to:

$$\binom{x + y - 1}{y} \left( \frac{p}{p + \alpha - \alpha p} \right)^x \left( \frac{\alpha - \alpha p}{p + \alpha - \alpha p} \right)^y$$

which is the density function for the negative binomial with the parameters for the excess distribution.

Now note that the Maclaurin series for $1/(1 - z)^a$ is given by:

$$\frac{1}{(1 - z)^a} = \sum_{i = 0}^{\infty} \binom{a + i - 1}{i} z^i .$$

Substitute $a = x + y$ and $z = (1 - p)(1 - \alpha) = q(1 - \alpha)$ to get $1 - z = p + \alpha - \alpha p$ and:

$$\left( \frac{1}{p + \alpha - \alpha p} \right)^{x + y} = \sum_{i = 0}^{\infty} \binom{x + y + i - 1}{i} q^i (1 - \alpha)^i .$$

Multiply the left-hand side of the above equation by:

$$K = \frac{p^x \alpha^y q^y (x+y-1)!}{(x-1)!\, y!}$$

and multiply the right-hand side by $K\dfrac{(i+y)!}{(i+y)!}$, rearrange, and obtain:

$$\binom{x+y-1}{y}\left(\frac{p}{p+\alpha-\alpha p}\right)^x\left(\frac{\alpha-\alpha p}{p+\alpha-\alpha p}\right)^y$$
$$= \sum_{i=0}^{\infty}\binom{x+y+i-1}{i+y} p^x\, q^{i+y}\binom{i+y}{y}\alpha^y\,(1-\alpha)^i,$$

as was to be shown.

Finally, consider the binomial. According to Hastings and Peacock [24, p. 36], the binomial has density function:

$$P(X = x) = \binom{n}{x} p^x\, q^{n-x},$$

where $n$ and $p$ are parameters and $q = 1 - p$. This distribution has mean $np$ and variance $npq$. If parameters $n$ and $p$ result in a mean of $\lambda$ and a variance of $\sigma^2$, then parameters of $n$ and $\alpha p$ result in mean $\alpha\lambda$ and variance $\alpha\lambda + \alpha^2(\sigma^2 - \lambda)$. As above, let $T$ be the total claim count distribution, and let $X$ be the excess claim count distribution. Then:

$$P(T = t) = \binom{n}{t} p^t\, q^{n-t},$$

$$P(X = x \mid T = t) = \binom{t}{x}\alpha^x\,(1-\alpha)^{t-x},$$

$$P(X = x) = \sum_{t=x}^{n}\binom{n}{t} p^t\, q^{n-t}\binom{t}{x}\alpha^x\,(1-\alpha)^{t-x}.$$

Let $t = x + i$, so:

$$P(X = x) = \sum_{i=0}^{n-x} \binom{n}{x+i} p^{x+i} q^{n-x-i} \binom{x+i}{x} \alpha^x (1-\alpha)^i$$

It is necessary to show that the right-hand side of the above equation is equal to:

$$\binom{n}{x} (\alpha p)^x (1 - \alpha p)^{n-x}.$$

Begin with an equality due to the binomial theorem:

$$(z + 1)^{n-x} = \sum_{i=0}^{n-x} \binom{n-x}{i} z^i.$$

Now let $z = \dfrac{p(1-\alpha)}{1-p}$, so $z + 1 = \dfrac{1 - \alpha p}{1-p}$. Substituting gives:

$$\left(\frac{1-\alpha p}{1-p}\right)^{n-x} = \sum_{i=0}^{n-x} \binom{n-x}{i} \left(\frac{p(1-\alpha)}{1-p}\right)^i.$$

$$\frac{(1-\alpha p)^{n-x}}{(n-x)!} = \sum_{i=0}^{n-x} \frac{p^i (1-\alpha)^i (1-p)^{n-x}}{i!\,(n-x-i)!\,(1-p)^i}.$$

Multiply the left side by $K = \dfrac{n!\,p^x \alpha^x}{x!}$ and the right side by $K \cdot \dfrac{(x+i)!}{(x+i)!}$ and rearrange to get:

$$\frac{n!}{x!\,(n-x)!} p^x \alpha^x (1-\alpha p)^{n-x} =$$

$$\sum_{i=0}^{n-x} \frac{n!}{(x+i)!\,(n-x-i)!} p^x p^i (1-p)^{n-x-i} \frac{(x+i)!}{x!\,i!} \alpha^x (1-\alpha)^i$$

or:

$$\binom{n}{x}(\alpha p)^x (1-\alpha p)^{n-x} = \sum_{i=0}^{n-x} \binom{n}{x+i} p^{x+i} q^{n-x-i} \binom{x+i}{x} \alpha^x (1-\alpha)^i$$

which completes the proof.

# DISCUSSION OF PAPER PUBLISHED IN VOLUME LXXIV

# AN ANALYSIS OF EXCESS LOSS DEVELOPMENT

EMANUEL PINTO AND DANIEL F. GOGOL

DISCUSSION BY ROBERT A. BEAR

*Abstract*

*Messrs. Pinto and Gogol have made a valuable contribution to actuarial literature through their analyses of industry excess loss development patterns. Based upon application of a theoretical model to industry data, the authors have convincingly demonstrated that paid and incurred loss and ALAE development patterns increase significantly as the retention increases. This is due to the phenomenon that the severity distribution becomes thicker-tailed as claims mature. This review presents a generalization of the Pinto-Gogol formula that shows how the authors' methodology can be applied to estimate account-specific development patterns for relatively high excess layers.*

This reviewer would like to thank Kurt A. Reichle for encouraging him to write this discussion, and Daniel F. Gogol for his helpful suggestions.

## 1. INTRODUCTION

Messrs. Pinto and Gogol [1] have made a valuable contribution to actuarial literature through their analyses of industry excess incurred loss development patterns. They have convincingly demonstrated that incurred loss and allocated loss adjustment expense (ALAE) development increases significantly as the retention increases. The same is likely to be true of paid loss and ALAE development, as the following argument shows.

The authors note that estimates of paid excess loss development factors can be computed by multiplying each incurred excess loss development factor by the quotient of the paid-to-reported ratios for the later and earlier valuations. The paid-to-reported ratios are simply ratios of excess paid losses and ALAE to excess incurred losses and ALAE. These ratios are computed at all valuations for a representative retention, since the authors found that they do not vary substantially as a function of the retention. Thus, paid excess loss development factors may be expressed as the product of incurred excess loss development factors (which increase with the retention) and a quantity which does not vary substantially with the retention. This implies that paid loss and ALAE development factors can be expected to increase significantly as the retention increases. Thus, the retention should be appropriately reflected in the estimation of discounted excess losses using paid development factors.

## 2. COMMENTS ON THE UNDERLYING MODEL

The function $y = ax^b$ was used by the authors to fit excess development factors as a function of the retention, based on Insurance Services Office (ISO) data. Basic properties of the underlying Single Parameter Pareto (SPP) severity distribution [2] are summarized in Appendix A. As the retention, $x$, was normalized through division by \$10,000, the parameter $a$ represents the factor for development excess of \$10,000. The incremental factors, $a - 1$, are then fitted to the inverse power function $y = cx^d$ as recommended by Sherman [3]. The inverse power function is used for interpolation and to yield tail factors for development beyond 99 months. The use of this same functional form to extrapolate $b$-parameters beyond 99 months appears to have been based on goodness-of-fit tests rather than on theoretical considerations, because the parameter $b$ represents the decline in the SPP $q$-parameter between the valuations underlying the age-to-age factor.

Philbrick [2] and Reichle and Yonkunas [4] noted that the tails of fitted SPP severity distributions are thicker than the tails of empirical casualty loss distributions at very large loss sizes. This implies that empirical average claim sizes in excess of high retentions will be less than those implied by the SPP distribution. Fits to more recent ISO data led Bear and Nemlick [5] to conclude that the SPP $q$-parameter varies with the truncation point used in the fitting procedure. This increase in the estimated $q$-parameter as a function of the truncation point supports the earlier findings of Philbrick and Reichle-Yonkunas. Pinto and Gogol note that the impact of this error will be reduced by using a ratio to estimate a development factor if the error is of comparable magnitude in the numerator and denominator.

Bear and Nemlick found that if the truncation point used in the fitting procedure is less than 50% of the attachment point for a particular analysis, the errors due to the redundant estimates of excess severities from the SPP distribution become unacceptably large. They used development triangles of SPP parameter estimates to derive the shape parameter $q$ at various stages of development and to project ultimate estimates of this parameter by class of business and truncation point. For the casualty classes of business analyzed, their fits of more recent ISO data confirmed the result noted by Philbrick and Reichle-Yonkunas; i.e., the $q$-parameter tends to decline as a function of the stage of development. This implies that the severity distribution becomes thicker-tailed as claims mature. (See Appendix A.) This is also confirmed by the fact that the $b$-parameters estimated by Pinto and Gogol were positive.

## 3. ESTIMATING ACCOUNT-SPECIFIC DEVELOPMENT

The authors concentrated on the estimation of industry loss development patterns for unbounded layers (with losses capped by policy limits), and suggested a reasonable approach for estimating industry incurred loss development patterns for reinsurance layers. This same approach can be applied to industry paid loss development patterns for unbounded layers to estimate paid loss development patterns for reinsurance layers.

This reviewer observes that the basic Pinto-Gogol formula for computation of industry loss development factors for unbounded layers as a function of the retention can be applied in large account primary pricing and in account-specific reinsurance pricing. This formula can also be applied to estimate account-specific development patterns for reinsurance layers and for large account primary excess layers.

A generalization of the Pinto-Gogol formula is presented below and proven in Appendix B. This generalization permits one to estimate the account-specific development pattern for a relatively high layer as a function of the development pattern for a lower layer, assuming the ratios of the gross limit to the retention for both layers are equal.

*Proposition*

Let $d$ represent the incurred loss development factor from valuation $i$ to valuation $j$ for losses in the layer from $k_1$ to $k_2$. Assume the SPP distribution is an appropriate severity model for claims in excess of $k_1$. Let $q_i$ and $q_j$ represent the estimated values of the SPP parameter based on claims at valuations $i$ and $j$, respectively, and let $e = q_i - q_j$. Then the incurred loss development factor from valuation $i$ to valuation $j$ for losses in the layer from $x_1$ to $x_2$ is given by

$$dc^e ,$$

$$\text{where } c = \frac{x_1}{k_1} \geq 1$$

$$\text{and } \frac{x_2}{x_1} = \frac{k_2}{k_1} = b .$$

This result also holds for unbounded layers (i.e., $k_2$ and $x_2$ are infinite) if the SPP parameters exceed one. If $q_j$ represents the projected value of $q$, the SPP parameter for fully developed claims, this result may be used to estimate age-to-ultimate development factors.

*Applications*

The SPP parameters can be estimated from account-specific data in large account primary pricing and in reinsurance pricing. The parameters estimated from account-specific data can be credibility weighted with parameters estimated from industry data ([4],[6]).

A key assumption in the above proposition is that the ratio of the gross limit to the retention (in reinsurance pricing), or self-insured retention (in primary pricing), are equal for both layers:

$$b = \frac{x_2}{x_1} = \frac{k_2}{k_1} .$$

Thus, one would want to select $k_1$ to be sufficiently high so that the SPP distribution is an appropriate severity model for claims in excess of $k_1$. On the other hand, one would want to select $k_1$ to be sufficiently low so that credible development patterns can be estimated for a layer in excess of $k_1$ . One would select $k_2$ so that

$$k_2 = bk_1 ,$$

where $b = x_2/x_1$ .

The proposition could then be applied to estimate the development pattern for a relatively high layer (where the account-specific data are not sufficiently credible) from the development pattern of a relatively low layer (where the account-specific data are more credible).

For example, suppose that the SPP parameter for a particular account and line of business after 24 months has been estimated to be 1.25, and the projected value of this parameter for fully mature claims is 1.10. These parameters have been estimated based on the account's claims in excess of \$100,000. The incurred loss development factor from 24 months to ultimate, for the layer from \$100,000 to \$300,000, has been estimated to be 3.5 based upon the account's historical development pattern. The development factor from 24 months to ultimate for the layer from \$200,000 to \$600,000 is given by

$$3.5\,(2.0)^{.15} = 3.88.$$

Note that $d = 3.5$, $c = 200{,}000/100{,}000 = 2.0$, $e = 1.25 - 1.10 = .15$, and $b = 600{,}000/200{,}000 = 300{,}000/100{,}000 = 3$. In fact, the gross limit for the lower layer was selected to be three times the retention of \$100,000 because this is the ratio of the gross limit to the retention for the layer for which we wished to estimate the development pattern.

Development patterns for layers with retentions in excess of $200,000 (more than twice the $100,000 truncation point used in estimating the SPP parameters) could be estimated with reasonable confidence using this procedure, if one had reason to believe the SPP parameters remained relatively stable as higher truncation points were used in the fitting procedure. (Recall that errors arising from this source may be reduced by using a ratio to estimate a development factor.)

Finally, it should be noted that paid loss development factors for bounded layers may be estimated by applying the Pinto-Gogol approach of multiplying each incurred loss development factor by the quotient of the paid-to-reported ratios for the later and earlier valuations. This reviewer suggests that the paid-to-reported ratios be estimated for the particular layer of interest (or at least for a similar layer), but possibly from a broader data source than was used to estimate the incurred loss development factors.

## 4. SUMMARY

Based upon application of a theoretical model to industry data, the authors have convincingly demonstrated that paid and incurred loss and ALAE development patterns increase significantly as the retention increases. This is due to the phenomenon (confirmed by recent ISO casualty data) that the severity distribution becomes thicker-tailed as claims mature. The proposition presented above shows how the Pinto-Gogol methodology can be applied to estimate account-specific development patterns for relatively high excess layers.

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# APPENDIX A

## SINGLE PARAMETER PARETO SEVERITY DISTRIBUTION

### 1. MODEL DEFINITION*

Assume ground-up loss occurrences, $W$, above the truncation point, $k$, are distributed according to the following cumulative distribution function:

$$F(w) = 1 - \left( \frac{k}{w} \right)^q , \text{ where } k > 0, q > 0, w \geq k .$$

Note that

$$F(w) = 1 - \left( \frac{k}{k + (w - k)} \right)^q .$$

Let $Y = W - k$ represent the occurrence size excess of $k$.

Then

$$F(y) = 1 - \left( \frac{k}{k + y} \right)^q , \text{ where } y \geq 0 .$$

Thus, occurrence losses excess of the truncation point $k$ are distributed according to the two-parameter shifted Pareto distribution, with scale parameter equal to $k$ and shape parameter equal to $q$ [5].

If we "normalize" the losses $W$ (which are all greater than or equal to $k$) by dividing each loss by the truncation point $k$, we have the well-known Single Parameter Pareto (SPP) severity distribution [2]:

$$F(z) = 1 - \frac{1}{z^q} = 1 - z^{-q}$$

where $Z = W/k \geq 1$ and $q > 0$.

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* Using standard statistical notation, capital letters in this appendix are used to represent random variables. Lower case represents actual values.

## 2. CLAIM FREQUENCIES

As $F(z)$ represents the proportion of normalized occurrence losses which are less than or equal to $z$, $G(z) = 1 - F(z) = z^{-q}$ represents the proportion of normalized losses which exceed $z$. Let $n_i$ represent the expected number of claims in excess of $k$ at valuation $i$, and let $m_i$ represent the expected number of claims in excess of $x$ at valuation $i$. Let $c$ represent the normalized value of $x$, $c = x/k$. Because $c^{-q}$ represents the proportion of normalized losses which exceed $c$, $c^{-q}$ is also the proportion of claims in excess of $k$ which are also larger than $x$. Then $m_i = n_i c^{-q}$ is the expected number of claims in excess of $x$ at valuation $i$, given that $n_i$ claims are expected to be in excess of $k$ at valuation $i$.

For example, the proportion of claims excess of \$500,000 that also exceed \$1,000,000 is $c^{-q} = 2^{-2} = 0.25$ if

$$q = 2 \ (c = 1{,}000{,}000/500{,}000 = 2).$$

Thus, if $n_3 = 400$ claims are expected to exceed \$500,000 at the third valuation, then $m_3 = n_3 c^{-q} = 400(0.25) = 100$ claims are expected to exceed \$1,000,000 at the third valuation. Note that if $q = 1.5$, then $c^{-q} = 0.354$. The proportion $c^{-q}$ becomes 0.5 if $q = 1$, and 0.707 if $q = 0.5$. Thus, the proportion of claims in excess of \$500,000 that also exceed \$1,000,000 increases as the $q$-parameter declines. Thus, the tail of the SPP distribution is thicker for lower values of the $q$-parameter.

## 3. MEAN SEVERITIES

The formula for the average ground-up unlimited claim which exceeds $x_1$ is given in [2]:

$$x_1 \left( \frac{q}{q-1} \right), \quad \text{if } q > 1 \ .$$

The average unlimited claim in excess of $x_1$ is given by

$$s = \mathrm{E}\,(Y) = x_1 \left( \frac{q}{q-1} - 1 \right) = \frac{x_1}{q-1}\ , \quad \text{if } q > 1\ ,$$

where $s$ represents the expected value of $Y = W - x_1$ and $W \geq x_1$ .

The formula for the average ground-up claim which exceeds $x_1$ but is limited to $x_2$ is given in [2]:

$$x_1 \left( \frac{q - b^{1-q}}{q-1} \right), \text{ if } q \neq 1,$$

$$\text{and } x_1 \,(1 + \ln(b)\,)\,, \text{ if } q = 1\,,$$

where $b = x_2/x_1$ and $\ln(b)$ represents the natural logarithm of $b$. The formula for the average claim in the layer from $x_1$ to $x_2$ (total losses in the layer divided by the number of claims in excess of $x_1$) is given by

$$s = \mathrm{E}(\min\,(Y, x_2 - x_1)) = x_1 \left( \frac{q - b^{1-q}}{q-1} - 1 \right) = x_1 \left( \frac{b^{1-q} - 1}{1-q} \right),$$

$$\text{if } q \neq 1 \text{ , and}$$

$$s = x_1\,((1 + \ln(b)\,) - 1) = x_1 \ln(b), \text{ if } q = 1, \text{ where } b = x_2/x_1 \text{ .}$$

Note that $s$ represents the expected value of $Y = W - x_1$, where $Y$ is capped by the layer limit $x_2 - x_1$ and $Y \geq 0$ .

For example, the average claim in the layer from \$500,000 to \$1,000,000 is calculated as follows, assuming $q = 2$:

$$s = (500{,}000) \left( \frac{2^{1-2} - 1}{1-2} \right) = \$250{,}000,$$

$$\text{since } b = 1{,}000{,}000/500{,}000 = 2.$$

If $q = 1.5$, then $s$ is similarly calculated to be \$292,893.

If $q = 1$, then $s = (500{,}000)(\ln(2)) = \$346{,}574$.

If $q = 0.5$, then

$$s = (500{,}000) \left( \frac{2^{1-0.5} - 1}{1 - .5} \right) = \$414{,}214.$$

This example illustrates the property of the SPP distribution that lower values of the $q$-parameter are associated with higher mean severities. This is because the distribution becomes thicker-tailed (more probability in excess of any large value) as the $q$-parameter declines.

For casualty classes of business, the $q$-parameter tends to decline as a function of the stage of development ([2], [4], [5]). This implies that casualty severity distributions tend to become thicker-tailed as claims mature, and so the average claim in any layer (where the SPP distribution is an appropriate model) will increase as claims mature.

# APPENDIX B

## PROOF OF PROPOSITION

The proof of the proposition is based upon estimating the incurred losses in a layer as the product of the expected number of claims above the retention and the average claim in the layer. The incurred loss development factors for both the relatively low and high layers are computed as ratios of layer incurred losses at the appropriate valuations. Simple algebra leads to the formula in the proposition when one assumes that the ratios of the gross limit to the retention for both layers are equal.

Recall that the SPP distribution is assumed to be an appropriate severity model for claims in excess of $k_1$. If $n_i$ represents the expected number of claims in excess of $k_1$ at valuation $i$, then $m_i = n_i c^{-q_i}$ represents the expected number of claims in excess of $x_1$ at valuation $i$, where $c = x_1/k_1$. The average claim in the layer from $x_1$ to $x_2$ (total losses in the layer divided by the number of claims in excess of $x_1$) at valuation $i$ is given by

$$s_i = x_1\left(\frac{b^{1-q_i} - 1}{1 - q_i}\right), \text{ if } q_i \neq 1,$$

$$\text{where } b = x_2/x_1.$$

The formulas for $m_i$ and $s_i$ follow from the properties of the SPP distribution and are proven in Appendix A.

Incurred losses in the layer from $x_1$ to $x_2$ at valuation $i$ are given by $m_i\, s_i$. Similarly, incurred losses in the layer from the retention $x_1$ to the gross limit $x_2$ at valuation $j$ are given by $m_j s_j$, where $m_j = n_j c^{-q_j}$ and

$$s_j = x_1\left(\frac{b^{1-q_j} - 1}{1 - q_j}\right), \text{ if } q_j \neq 1.$$

(Note that $n_j$ represents the expected number of claims in excess of $k_1$ at valuation $j$, and $m_j$ represents the expected number of claims in excess of $x_1$ at valuation $j$.)

The incurred loss development factor from valuation $i$ to valuation $j$ for losses in the layer from $x_1$ to $x_2$ is given by

$$f = \frac{m_j s_j}{m_i s_i} = \left( \frac{n_j (1 - q_i) (b^{1 - q_j} - 1)}{n_i (1 - q_j) (b^{1 - q_i} - 1)} \right) c^{q_i - q_j}.$$

Recall that $x_2/x_1 = k_2/k_1 = b$ and $c = x_1/k_1$. If $c = 1$ then $x_1 = k_1$ and $x_2 = k_2 = bk_1$. The formula for $f$ then yields the incurred loss development factor from valuation $i$ to valuation $j$ for losses in the layer from $k_1$ to $k_2$, which is denoted $d$:

$$f = d = \frac{n_j (1 - q_i) (b^{1 - q_j} - 1)}{n_i (1 - q_j) (b^{1 - q_i} - 1)}.$$

Hence, the formula for the incurred loss development factor from valuation $i$ to valuation $j$ for losses in the layer from $x_1$ to $x_2$ simplifies to

$$f = dc^{q_i - q_j} = dc^e,$$

$$\text{if } q_i \neq 1 \text{ and } q_j \neq 1.$$

If $j$ represents the valuation at which claims are fully developed, then $q_j = q$ and $f$ represents the development factor from valuation $i$ to ultimate.

In the case of unbounded layers (i.e., $k_2$ and $x_2$ are infinite), $m_i$ and $m_j$ do not change but $s_i$ and $s_j$ are as given below (see Appendix A):

$$s_i = \frac{x_1}{q_i - 1}, \text{ if } q_i > 1,$$

$$\text{and } s_j = \frac{x_1}{q_j - 1}, \text{ if } q_j > 1.$$

Then $f$ is given by

$$f = \frac{m_j s_j}{m_i s_i} = \left( \frac{n_j (q_i - 1)}{n_i (q_j - 1)} \right) c^{q_i - q_j}.$$

If $c = 1$, then $x_1 = k_1$ and so

$$f = d = \frac{n_j (q_i - 1)}{n_i (q_j - 1)} .$$

This implies that the incurred loss development factor from valuation $i$ to valuation $j$ for losses in the unbounded layer above $x_1$ is given by

$$f = dc^{q_i - q_j} = dc^e,$$

where $d$ is the incurred loss development factor from valuation $i$ to valuation $j$ for losses in the unbounded layer above $k_1$, $c = x_1/k_1$, and the SPP parameters are assumed to exceed one.

For bounded layers with $q_i = q_j = 1$, the averages of the claims in the layer from $x_1$ to $x_2$ at valuations $i$ and $j$ ($s_i$ and $s_j$) are as follows (see Appendix A):

$$s_i = s_j = x_1 \ln(b), \text{ where } b = x_2/x_1.$$

The expected numbers of claims in excess of $x_1$ at valuations $i$ and $j$ are given respectively by

$$m_i = n_i c^{-1} \text{ and } m_j = n_j c^{-1}.$$

This implies that

$$f = \frac{m_j s_j}{m_i s_i} = \frac{n_j}{n_i} .$$

However, the averages of the claims in the layer from $k_1$ to $k_2$ at valuations $i$ and $j$ ($t_i$ and $t_j$) are given by

$$t_i = t_j = k_1 \ln(b), \text{ where } b = k_2/k_1.$$

This implies that the development factor $d$ for the layer from $k_1$ to $k_2$ is given by

$$d = \frac{n_j t_j}{n_i t_i} = \frac{n_j}{n_i}\,.$$

Hence, $f = d$, which is in agreement with the formula in the proposition due to the unchanging $q$-parameter.

If $q_i \neq 1$ and $q_j = 1$, then the averages of the claims in the layer from the retention $x_1$ to the gross limit $x_2$ at valuations $i$ and $j$ are proven in Appendix A to be

$$s_i = x_1 \left( \frac{b^{1-q_i} - 1}{1 - q_i} \right),$$

$$\text{and } s_j = x_1 \ln(b)\,.$$

The incurred loss development factor from valuation $i$ to valuation $j$ for losses in the layer from $x_1$ to $x_2$ is given by

$$f = \frac{m_j s_j}{m_i s_i} = \frac{n_j}{n_i} \left( \frac{\ln(b)}{(b^{1-q_i} - 1)/(1 - q_i)} \right) c^{q_i - q_j}.$$

If $c = 1$, then $x_1 = k_1$ and so $x_2 = k_2 = bk_1$. The formula for $f$ then reduces to the formula for $d$,

$$d = \frac{n_j}{n_i} \left( \frac{\ln(b)}{(b^{1-q_i} - 1)/(1 - q_i)} \right).$$

Substituting into the formula for $f$,

$$f = dc^{q_i - q_j} = dc^e.$$

An analogous proof would hold if $q_i = 1$ and $q_j \neq 1$. Q.E.D.

# ADDRESS TO NEW MEMBERS—MAY 11, 1992

## THOUGHTS ON MY CAREER

RUTH E. SALZMANN

I am happy to be here today to add my congratulations to the new Associates and Fellows. It was 45 years ago that I received my Fellowship, and I remember I was very happy, very proud, and *greatly relieved!*

Back then, exams were given only once a year, and therefore diplomas were awarded only once a year. This recognition occurred at the Annual Meeting in November, which was always held at the Biltmore Hotel in New York City (and it took me 33 hours by train to get there from Stevens Point, Wisconsin).

At that meeting 45 years ago, there were a total of 21 new Associates and Fellows—and that made 1947 a record year! In the year before, as well as in the year after, there were only 7 new designations.

The entire attendance at the meeting was certainly low by today's standards. There were only 84 members present, but the 84 represented 30 percent of the total membership. That would be comparable to an attendance of approximately 550 members today.

Our Society has really grown over those 45 years, and I know that salaries have risen just as dramatically. My annual salary in 1947 was $4,520.45. However you look at it, "We've come a long way, Baby!"

I really wondered about what I could say today that might have some "take-home" value. It then occurred to me that with the benefit of considerable hindsight, I could pass along some thoughts about what made my career all the more worthwhile.

There were, of course, the normal pursuits of job satisfaction and promotions, but there was much more. I particularly enjoyed being involved in solving industry problems, and I encourage each of you to become involved in those areas that interest you most. My interests were primarily in the financial and Annual Statement areas.

Your participation broadens your outlook, but your contributions give you the most satisfaction. When an idea of yours is accepted by your peers, it is a great feeling. And even when your idea is not that successful, you feel good just because you entered the fray. As the saying goes, "It's not whether you win, but how you played the game!" The enjoyment comes in knowing that your work was worthy of your best efforts.

The frequency of acceptance and non-acceptance is low. Most often, the work is a team effort. If I grade my recommendations over the years, I get a count of four winners and four losers. And, believe it or not, I haven't given up on those four losers yet. I keep thinking that maybe someday...

The second and last thought I would like to address has to do with "how ye are known," or your professional reputation. You do not start out with a professional reputation, but you surely end up with one. By passing the exams, you have demonstrated your actuarial knowledge and skills. From here on in, it is how you employ this expertise that affects your professional reputation.

With this in mind, you will find that accountability takes on a new significance.

With this in mind, you will find that it is prudent to acknowledge the limitations of your expertise.

With this in mind, you will find that it is prudent to let the data dictate the methodology rather than to superimpose one method or program on all data.

With this in mind, you will find that better interprofessional relationships are achieved with the use of language that is clear, simple, and as non-technical as possible.

With this in mind, you will find that the ethical imperative is to use your mathematical capabilities to promote objectivity—the nemesis of bias, rationalization, and wishful thinking.

And I could go on, but you get the idea.

Be mindful that a professional reputation is the end-product of any professional career. Let yours be one to be proud of.

In closing, I wish you good luck and fortune. In this era when Congresspersons can't even balance their checkbooks, there has to be a great demand for your expertise.

# MINUTES OF THE 1992 SPRING MEETING

May 10-13, 1992

THE PALMER HOUSE HILTON, CHICAGO

*Sunday, May 10, 1992*

The Board of Directors held their regular quarterly meeting from 1:00 p.m. to 5:00 p.m.

Registration was held from 4:00 p.m. until 6:00 p.m.

From 5:30 p.m. until 6:30 p.m., there was a special presentation to new Associates and their guests. The session included an introduction to the standards of professional conduct and the CAS committee structure.

A welcome reception for all members and guests was held from 6:30 p.m. until 7:30 p.m.

*Monday, May 11, 1992*

Registration continued from 7:30 a.m. until 8:30 a.m.

CAS President Michael Toothman introduced James W. Schacht, Chief Deputy Director, Illinois Department of Insurance, who gave the welcoming address from 8:30 a.m. until 9:00 a.m.

At 9:00 a.m., Mr. Toothman opened the business session which included the ceremony for new members. There were 88 new Associates and 17 new Fellows. The names of these individuals follow:

FELLOWS

| | | |
|---|---|---|
| Karin H. Beaulieu | Catherine E. Eska | Julia L. Perrine |
| Allan R. Becker | William G. Fitzpatrick | Jennifer A. Polson |
| Robert G. Blanco | Nancy G. Flannery | Stephen D. Stayton |
| Patrick J. Burns | Brian A. Hughes | William Vasek |
| Kenneth E. Carlton III | Bruce E. Ollodart | Elizabeth A. Wellington |
| Daniel J. Czabaj | Brian G. Pelly | |

ASSOCIATES

Kristen M. Albright
Todd R. Bault
Herbert S. Bibbero
Wayne E. Blackburn
Annie Blais
Daniel D. Blau
Betsy L. Blue
John P. Booher
Christopher K. Bozman
J. Eric Brosius
David S. Cash
Dennis K. Chan
Bryan C. Christman
Wei Chuang
Kasing L. Chung
Gary T. Ciardiello
Peter J. Collins
Thomas P. Conway
Gregory L. Cote
Brian K. Cox
Timothy J. Cremin
Gregory A. Cuzzi
Michael K. Daly
Manon Debigare
Michael L. DeMattei
Herbert G. Desson
Stephen R. DiCenso
Michel Dionne
Jeffrey E. Doffing
Michael C. Dubin
Francois Dumas
Denise A. Feder
Charles C. Fung
Kim B. Garland
Jeffrey C. Gendron
Odile Goyer
Steven J. Groeschen
Farrokh Guiahi
Terry D. Gusler
Leigh J. Halliwell
David L. Homer
Paul R. Hussian
Hou-Wen Jeng
Susan E. Kent
Deborah E. Kenyon
Kevin A. Kesby
Gerald S. Kirschner
Timothy F. Koester
Gilbert M. Korthals
Benoit Laganiere
Alan E. Lange
Christopher Lattin
Marc-Andre Lefebvre
Paul R. Livingstone
Richard Maguire
Katherine A. Mann
Leslie R. Marlo
Suzanne Martin
Keith A. Mathre
Maria Mattioli
Thomas S. McIntyre
John H. Mize
Russell E. Moore
Francois Morin
Francois L. Morissette
David A. Murray
Victor A. Njakou
Kathleen C. Nomicos
Stephen R. Noonan
Robert C. Phifer
Mark W. Phillips
Karin L. Reinhardt
Lisa M. Ross
Daniel G. Roth
Michael R. Rozema
David O. Schlenke
Peter Senak
Robert D. Share
David B. Sommer
Barbara H. Thurston
Thomas C. Toce
Michael Toledano
Therese M. Vaughan
Jennifer A. Violette
Bryan C. Ware
John P. Welch
Robert J. White
Windrie Wong

Mr. Toothman introduced Ruth E. Salzmann, a past President of the Society, who addressed the new members from 9:30 a.m. until 9:45 a.m.

Mr. Toothman then recognized special guests in the audience, including Terry Clarke, Vice President of the British Institute of Actuaries and Chairman of the General Insurance Study Group, as well as W. Paul McCrossan, current President of the Canadian Institute of Actuaries, Jim Murphy, Executive Director of the American Academy of Actuaries, and Tim Tinsley, CAS Executive Director.

Highlights of the program were presented by Albert J. Beer, Vice President-Programs and Communications.

Gary R. Josephson, a member of the Continuing Education Committee, presented a summary of the Discussion Paper Program and the 23 discussion papers to be presented at the meeting.

A summary of the four new *Proceedings* papers was given by Irene K. Bass, Vice President-Continuing Education.

Vice President-Admissions Steven G. Lehmann provided an overview of the Course on Professionalism.

John M. Purple, Vice President-Administration, announced the recipient of the Harold W. Schloss Memorial Scholarship, Jennifer Bunker.

The business session was adjourned at 10:30 a.m.

After a refreshment break, Mr. Toothman introduced Leo McManus, President of L.F. McManus Co., Inc., who delivered an address on personality types and the need to manage your own personality in order to succeed in management. Mr. McManus spoke from 11:00 a.m. until noon.

A luncheon followed from noon until 1:30 p.m.

The afternoon was devoted to concurrent sessions from 1:30 p.m. until 5:00 p.m., with a break from 3:00 p.m until 3:30 p.m.

The concurrent sessions included presentations of the discussion papers, the *Proceedings* papers, the CAS-funded research paper, and a limited-attendance workshop.

The new *Proceedings* papers were:

1. "The Computation of Aggregate Loss Distributions"

   Author: John P. Robertson
   CIGNA Property & Casualty Co.

2. "Testing for Shifts in Reserve Adequacy"

   Author: Richard M. Duvall
   Sedgwick James, Inc.

3. "Parameterizing the Workers' Compensation Experience Rating Plan"

   Author: William R. Gillam
   National Council on Compensation Insurance

The CAS-funded research paper was:

"The Profit Provision in the Ratemaking Formula"

Author: Stephen D'Arcy
Department of Finance, University of Illinois

The Discussion Papers presented were:

1. "An Evaluation of Surplus Allocation Methods Underlying Risk-Based Capital Applications"

   Authors: Michael J. Miller
   Tillinghast/Towers Perrin

   Jerry W. Rapp
   Tillinghast/Towers Perrin

2. "Surplus Allocation: An Oxymoron"

   Authors: Irene K. Bass
   William M. Mercer, Inc.

   C.K. Khury
   William M. Mercer, Inc.

3. "Surplus—Concepts, Measures of Return, and Its Determination"

   Author: Russell Bingham
   ITT/Hartford Insurance Group

4. "Analysis of Surplus and Rate of Return Without Using Leveraged Ratios"

   Author: Richard J. Roth, Jr.
   State of California Department of Insurance

5. "The Implications of Market Return Pricing Strategies Upon Profit and Required Surplus"

   Author: Brian E. MacMahon
   Industrial Indemnity

6. "Simplified Confidence Boundaries Associated With Calendar Year Projections"

   Author: James P. McNichols
   Ernst & Young

7. "Practical Loss Reserving Method With Stochastic Development Factors"

   Author: Mary V. Kelly
   The Co-operators General Insurance Co.

8. "Modelling Asset Variability in Assessing Insurer Solvency"

   Author: Louise A. Francis
   Tillinghast/Towers Perrin

9. "Surplus in Investment Strategy Due to Mismatch With Liabilities"

   Author: John C. Burville
   ACE Limited

10. "The Value of Junk"

   Author: Louise A. Francis
   Tillinghast/Towers Perrin

11. "European Approaches to Insurance Solvency"

    Author: Sholom Feldblum
    Liberty Mutual Insurance Co.

12. "An Application of Risk Theory to Control Solvency and Financial Strength"

    Author: Heikki Bonsdorff
    Ministry of Social Affairs
    and Health of Finland

13. "Solvency Regulation in Canada"

    Authors: Canadian Institute of Actuaries
    P&C Solvency Sub-Committee
    Richard Gauthier, Chairperson
    Barbara Addie
    Jean Côte
    Alain Lessard
    Christopher J. Townsend

14. "Capitalization of Property/Casualty Insurance Companies"

    Author: Paula R. Federman
    Standard & Poor's Rating Group

15. "A New Look at Evaluating the Financial Condition of Property and Casualty Insurance and Reinsurance Companies"

    Authors: Thomas M. Redman
    John Hancock Management Co.

    Christopher E. Scudellari
    Ernst & Young

A limited-attendance workshop was also held:

Personality Profile Assessment

Presenter: Leo F. McManus
L.F. McManus Co., Inc.

There was an officers' reception for new Fellows and their guests from 5:30 p.m. until 6:30 p.m.

The general reception for all members and their guests was held from 6:30 p.m. until 7:30 p.m.

*Tuesday, May 12, 1992*

Concurrent sessions, which included both the presentations of papers and panel sessions, were held from 8:30 a.m. until noon, with a refreshment break from 10:00 a.m. until 10:30 a.m.

One *Proceedings* paper was presented:

A Discussion of "An Analysis of Excess Loss Development"

| | |
|---|---|
| Author: | Robert A. Bear<br>North Star Reinsurance Corp. |

The panel presentations covered the following topics:

1. Actuaries in Regulation

| | |
|---|---|
| Moderator: | Anne Kelly<br>Chief Casualty Actuary<br>New York State Insurance Department |
| Panelists: | Richard J. Roth, Jr.<br>Assistant Commissioner<br>California Department of Insurance |
| | Robert W. Gossrow<br>Casualty Actuary<br>Illinois Department of Insurance |
| | Kevin J. Conley<br>Actuarial Administrator<br>Iowa Insurance Division. |

2. Questions and Answers with the CAS Board of Directors

| | |
|---|---|
| Moderator: | David P. Flynn (President-Elect)<br>Senior Vice President and Chief Actuary<br>Crum and Forster Corp. |
| Panelists: | Charles A. Bryan (Chairman and Immediate Past President)<br>Partner<br>Ernst & Young |
| | Linda L. Bell (Elected 1990)<br>Senior Vice President and Chief Actuary<br>Transamerica Insurance Group |
| | W. James MacGinnitie (Elected 1990)<br>Consulting Actuary<br>Tillinghast/Towers Perrin |
| | Heidi E. Hutter (Elected 1991)<br>Executive Vice President<br>Atrium Corp. |

The Discussion Papers presented included:

1. "NAIC Risk-Based Capital Efforts in 1990-1991"

| | |
|---|---|
| Authors: | Allan M. Kaufman<br>Milliman & Robertson, Inc. |
| | Elise C. Liebers<br>New York State Insurance Department |

2. "Solvency Measurement for Property-Liability Risk-Based Capital Applications"

| | |
|---|---|
| Author: | Robert P. Butsic<br>Fireman's Fund Insurance Companies |

3. "A Method for Risk Quantification for Surplus Requirements"

| | |
|---|---|
| Author: | Anthony Iafrate<br>General Reinsurance Corp. |

4. "An Analysis of Variations in Leverage Ratios Among Insurers"

   Author: Chester J. Szczepanski
   Pennsylvania Insurance Department

5. "The Schedule F Penalty—Effective or Evaded?"

   Authors: LeRoy J. Simon
   Coopers & Lybrand

   Steven M. Visner
   Coopers & Lybrand

6. "Underwriting Cycles and Insurance Solvency"

   Author: Sholom Feldblum
   Liberty Mutual Insurance Co.

7. "Self-Insurer Solvency and Estimating the Collectibility of the Retrospective Premium Reserve"

   Author: Brian Z. Brown
   Milliman & Robertson, Inc.

Certain discussion papers presented on Monday afternoon were repeated. A limited-attendance workshop was also offered:

CAS Professionalism Case Studies

Facilitators: Roger A. Schultz
Senior Actuary
Allstate Insurance Co.

James E. Buck
Actuary
The Wyatt Co.

Alfred O. Weller
Senior Consulting Actuary
Ernst & Young

Gary LaRose
Senior Consulting Actuary
Ernst & Young

The afternoon was reserved for the various CAS committees to convene from noon until 5:00 p.m.

Dinner and entertainment was a dinner cruise aboard the "Spirit of Chicago" from 7:00 p.m. until 10:00 p.m.

*Wednesday, May 13, 1992*

Certain discussion papers that had been presented earlier in the meeting were repeated this morning from 8:00 a.m.-9:30 a.m., along with the "Actuaries in Regulation" panel discussion.

Following a break from 9:30 a.m. until 10:00 a.m., a General Session on "Solvency: A Confluence of Perspectives," was conducted. The panel was moderated by Ronald E. Ferguson, Chairman and Chief Executive Officer, General Reinsurance Corp. Members of the panel were Michael A. Goldberg, Vice President, Berkshire Hathaway Inc.; Vincent Laurenzano, Assistant Deputy Superintendent and Chief Examiner, State of New York Insurance Department; W. Paul McCrossan, Partner, Eckler Partners Ltd.; and Myron Picoult, Managing Director, Oppenheimer, Inc. The general session ran from 10:00 a.m. until 11:45 p.m.

The business session resumed at 11:45 p.m. with the presentation of the Michelbacher Award to Robert P. Butsic by Irene K. Bass for his call paper, "Solvency Measurement for Property-Liability Risk-Based Capital Applications."

The meeting was adjourned at noon after closing remarks.

*May 1992 Attendees*

In attendance, as indicated by the registration records, were 226 Fellows, and 143 Associates. The list of their names follows:

FELLOWS

| | | |
|---|---|---|
| Greg Alff | Allan Becker | Regina Berens |
| Bob Anker | Al Beer | James R. Berquist |
| Donald Bashline | Linda Bell | David Bickerstaff |
| Irene Bass | Robert Bender | Ralph Blanchard |
| Bob Bear | Abbe Bensimon | Robert Blanco |
| Karin Beaulieu | Phil Ben-Zvi | Bill Bland |

FELLOWS

Mike Blivess
Steven Book
Ron Bornhuetter
Chuck Boucek
Roger Bovard
Paul Braithwaite
James Brannigan
Bob Brian
Dale Brooks
Brian Brown
Randall Brubaker
Chuck Bryan
James Buck
Patrick Burns
George Busche
Jeanne Camp
John Captain
Kenneth Carlton
Sandy Cathcart
Joel Chansky
Jim Christie
Mark Cis
David Clark
Kevin Conley
Chap Cook
Alan Crowe
Patrik Crowe
Alan Curry
Janice Cutler
Daniel J. Czabaj
Stephen D'Arcy
Ronald Dahlquist
Bob Daino
Jerome Degerness
Tony Didonato
Robert Downer
Paul Dyck
Kenneth Easlon
Richard Easton
Douglas Eland
Catherine Eska
John Ewert
Janet Fagan
Steven Fallon
Bill Fanning
Sholom Feldblum
Ron Ferguson
Russell Fisher
Walt Fitzgibbon
Bill Fitzpatrick
Kirk Fleming
David Flynn
Louise Francis
Kenneth Frohlich
Mike Fusco
Alice Gannon
Chris Garand
Bob Gardner
Richard Gauthier
Judy Gillam
W. Robin Gillam
Jim Gillespie
Steve Goldberg
James Golz
Karen Gorvett
Tim Graham
Nancy Graves
Ronald Greco
Ann Griffith
Christy Gunn
Sam Gutterman
Larry Haefner
David Hafling
Malcolm Handte
Don Hanson
Jeff Hanson
Dave Hartman
Gayle Haskell
Paul Henzler
Barbara Higgins
Anthony Hill
Randy Holmberg
Mark Homan
Carl Honebein
Brian Hughes
Heidi Hutter
Ron Jean
Dick Johe
Russ John
Andy Johnson
Jeff Jordan
Gary Josephson
Bob Kaplan
Allan Kaufman
Anne Kelly
Stan Khury
Joel Kleinman
Leon Koch
Mikhael Koski
Gary Koupf
Eric Johnson
Rodney Kreps
Jeff Kucera
Ken Krissinger
Mike LaMonica

FELLOWS

Gary LaRose
Bob Daino
Gus Krause
Mike Larsen
Fran Lattanzio
Robert Lee
Steve Lehmann
Joseph Levin
Peter Licht
Aileen Lyle
James MacGinnitie
Jan Lommele
Brian MacMahon
Howard Mahler
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# PROCEEDINGS
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# CREDIBILITY BASED ON ACCURACY

JOSEPH A. BOOR

*Abstract*

*This paper shows that the optimal credibility split between two estimators is related to how well each estimator predicts the underlying experience. First, an equation is shown which expresses the credibility assigned to each estimator in terms of the average prediction error of the other estimator and the average squared difference between the two estimators. That equation is verified using the classic Bayesian credibility* $n/(n+k)$ *formula and a formula for weighting prior observations of time series that was developed by the author. An enhancement to the classic Bayesian method for class credibilities is shown. Finally, the author shows that optimal credibility is proportional to the accuracy of each estimator, less the extent to which both estimators make the same errors.*

## 1. INTRODUCTION

Much of historical credibility emanates from one of three philosophies:

1. The square root rule and its cousins.
2. The Bayesian $n/(n+k)$ formula.
3. Alternative Bayesian philosophies that assume that losses are distributed according to some member of a family of distributions, and assign some judgmental probabilities to the distributions within the family.

Each of these approaches has its own set of problems.

Approach 1, the square root rule, apparently is little more than an ad hoc formula to graduate credibility from 0 to 1 in a fashion that:

a) tends to assign relatively high credibilities to small samples; and

b) achieves full credibility at some point.

Approach 2, the $n/(n+k)$ formula, is based upon a presumption that the sample mean is from a distribution chosen from a set of distributions. The complement of the credibility is to be assigned to the grand mean of all possible distributions. However, many credibility situations are not characterized by the process of first choosing a distribution randomly and then sampling from that distribution. For example, consider the case where a rate change indicated by a state's data is credibility-weighted against straight trend. While some might argue that the choice between the state's data and straight trend is just such a "distribution of distributions," clearly straight trend applied to the last rate indication is not the grand mean of that family of distributions. Further, when those assumptions do apply, such as in the class ratemaking problem, the grand mean must also be estimated.

Approach 3, the alternative Bayesian approach, relies on a presumption that the distribution of potential losses is a member of some family of distributions. The major problem with this approach is that the typical real-world loss distribution is not a precise mathematical curve. Further, this approach usually imposes some second probability distribution upon the choice of a distribution. An ideal method should be distribution-free.

To avoid these difficulties, it is worthwhile to list some of the attributes of a good approach to credibility.

1. Since the purpose of credibility is to hone an estimate of losses, it should do so in the best fashion possible. Specifically, it should produce optimum estimators of the unknown mean expected loss.
2. It should work in a wide variety of situations; e.g., when the complement of the credibility is assigned to trend, econometric projections, alternative methods of estimating the sample mean, or a sample of a larger but related distribution.
3. It should be distribution-free. It should not work solely when losses approximate some specific mathematical probability distribution.
4. Intuitively, it seems that the credibility should, in some sense, be related to how effectively the observed sample losses approximate the underlying propensity for loss. Further, whatever statistic receives the complement of credibility should receive greater weight as its effectiveness in estimating the underlying propensity for loss increases.

One method that meets all these criteria involves minimizing the expected squared error in estimating the propensity for loss. Specifically, if we seek to credibility-weight two statistics $x_1$ and $x_2$ to approximate $Y$, and if:

$\tau_1^2 = \mathrm{E}[\ (x_1 - Y)^2\ ]$ ; i.e., $\tau_1^2$ is the expected error of $x_1$ as an estimator;

$\tau_2^2 = \mathrm{E}[\ (x_2 - Y)^2\ ]$ ; i.e., $\tau_2^2$ is the expected error of $x_2$ as an estimator;

$\delta_{12}^2 = \mathrm{E}[\ (x_1 - x_2)^2\ ]$ ; i.e., $\delta_{12}^2$ is their expected squared difference;

then

$$Z(x_1, x_2) = \frac{\tau_2^2 - \tau_1^2 + \delta_{12}^2}{2\delta_{12}^2}$$

produces the estimate

$$Zx_1 + (1 - Z)x_2 ,$$

which minimizes the error in estimating $Y$; i.e.,

$$\mathrm{E}[\ (Zx_1 + (1 - Z)x_2 - Y)^2\ ] = \min.$$

The proof is provided in Appendix A.

A quick review of the above criteria will show that this approach does indeed fulfill all four: it is optimal by design, works with a wide variety of estimators, is distribution-free, and assigns credibility in accordance with predictive accuracy.

## 2. USING OBSERVED ACCURACY

Before going further, it might be worthwhile to note that this method produces the same value of $Z$ when we attempt to predict an observed value $Y'$ rather than the actual propensity for losses Y.

In particular, if Y is the propensity for loss, then let $Y' = Y + \Delta$ be the observed losses. In this case, $\Delta$ would be independent of $Y, x_1$, and $x_2$, and have a mean of 0 and a variance of $S^2$.

Then, since the x and $\Delta$ are independent,

$$\tau_1'^2 = \tau_1^2 + S^2,$$

$$\tau_2'^2 = \tau_2^2 + S^2,$$

and

$$Z(x_1, x_2, Y') = \frac{\tau_2'^2 - \tau_1'^2 + \delta_{12}^2}{2\delta_{12}^2}$$

$$= \frac{\tau_2^2 + S^2 - \tau_1^2 - S^2 + \delta_{12}^2}{2\delta_{12}^2}$$

$$= \frac{\tau_2^2 - \tau_1^2 + \delta_{12}^2}{2\delta_{12}^2}$$

$$= Z(x_1, x_2, Y) \; .$$

Hence, computing Z for actual observed losses produces the same Z as that appropriate for predicting the underlying propensity for loss. Further, since $\tau_1'^2$, $\tau_2'^2$, and $\delta_{12}^2$ may then be estimated from historical observed losses, $Z$ may be estimated from actual observed losses.

*The Pitfalls of Skew*

One aspect of using historic predictive accuracy must be noted—it is impractical with the highly-skewed distributions typically associated with individual risks and small pools of losses.

The classic example is the credibility of a medium-size commercial insured's own experience relative to industry experience. Even when the insured has average exposure (i.e., the unmodified manual rate is right for the insured), the insured will typically experience loss ratios in the 40% to 50% range year after year. However, every five to 10 years it will experience a loss ratio in excess of 100%. That is because the loss size distribution, and hence the insured's aggregate loss distribution, is highly skewed. Simply stated, the insured is exposed to very large, but relatively infrequent, losses. Because those losses are so large, they represent a disproportionately large part of the insured's exposure to loss. Because they are so infrequent, they do not show up in the insured's loss experience every year.

The naive observer might conclude after viewing several years of low loss ratios that the insured's own low loss experience is a much better predictor of that insured's loss experience than the manual rate. The previous section of this paper would seem to support that statement.

Actual observed accuracy is misleading, however, because of the partial observation of prediction error. Over the last four years or so, only a portion of the distribution of prediction errors (specifically the low side) has been observed. When a large loss occurs, the full range of prediction error can be glimpsed. Depending on how long it takes for that large loss to occur, the manual rate may then appear to be either too high or too low.

Many of the standard actuarial treatments for skewed distributions correct this problem. For instance, one could compare an insured's historical loss experience with the following year's basic limits losses and an industry provision for excess losses. One could also replace the industry excess loss provision with, say, 30 years of the insured's own trended excess losses. Alternatively, one could compare the estimating accuracy of a large body of similar insureds. Although each member insured's observed losses may not be fully representative of the full error distribution, a large enough body of insureds should approximate all probable prediction errors.

Before proceeding further with an analysis of how this method may be applied in practice, it is worthwhile to investigate whether it reproduces some of the credibility formulae that are already known.

*Example 1. Classic Bayesian Credibility*

Let $x$ be the result of a two-stage process. First, a mean $\theta \in \omega$ is selected randomly from a set of means $\omega$ with (grand) mean $M$. Then $x_1$, $x_2$, ..., $x_n$ are selected from a distribution with mean $\theta$ and variance $S^2$. Their mean $\bar{x}$ is, of course, dependent on $\theta$, but each $x_i - \theta$ is independent, not only of the other $x_j - \theta$, but also of $\theta$. Further, let the $\theta \in \omega$ have variance $\sigma^2$. Then, classic credibility [1] says

$$Z(\bar{x}, M, \theta) = n/[n + (S^2/\sigma^2)].$$

To prove this, note that independence of the $x_i$ and $\theta$ implies:

$$\tau_{\bar{x}}^2 = (S^2/n),$$

$$\tau_M^2 = \sigma^2, \text{ and}$$

$$\delta_{\bar{x},M}^2 = \sigma^2 + S^2/n.$$

Then, the formula states:

$$Z(\bar{x}, M, \theta) = \frac{\sigma^2 - (S^2/n) + \sigma^2 + (S^2/n)}{2\,(\sigma^2 + (S^2/n))}$$

$$= \frac{2\sigma^2}{2\,[\sigma^2 + (S^2/n)\,]}$$

$$= \frac{n\sigma^2}{n\sigma^2 + S^2} = \frac{n}{n + (S^2/\sigma^2)}\,.$$

In other words, the classic Bayesian credibility is reproduced.

*Example 2: Weighting of Trended Time Series*

Let $x(t)$ begin at some $x(0)$ and then be changed infinitesimally often by infinitesimally small, but variable, perturbations. In other words, the change in $x(t)$ from time to time is a result of a very large number of very slight random occurrences, just like those affecting most econometric time series. The random nature of the perturbations assures us that $x(t)$ will be distributed normally about $x(0) + t\mu$ (where $\mu$ is the average rate of change), and the variance of $x(t)$ is $t\sigma^2$ (where $\sigma^2$ is the unit variance).[1] In fact,

$$\mathrm{E}[x(a) - x(b)\,] = (a - b)\mu$$

and

$$\mathrm{Var}\,[x(a) + (b - a)\mu - x(b)\,] = |\,a - b\,|\,\sigma^2.$$

To simplify matters, let us consider the case where $\mu = 0$. Further, as in most practical problems, when x(1), x(2), ..., $x(n)$ are estimated by $\hat{x}(1)$, $\hat{x}(2)$, ..., $\hat{x}(n)$, there should be some estimation errors $\Delta_i$. So, $\hat{x}(i) = x(i) + \Delta_i$, and the $\Delta_i$ are independent and identically distributed with mean 0 and variance $S^2$. Then, when weighting $\hat{x}(1)$, $\hat{x}(2)$, $\hat{x}(3)$, etc. to estimate $x(n)$, the weights

[1] The limited reading of this author on Poisson processes suggests that some of the conclusions on the mean and variance may be common knowledge of statisticians.

$$W_i = S^{2(n-i)} F_i ,$$

where the $F_i$ are defined recursively by

$$F_1 = 1;$$

$$F_2 = S^2 + \sigma^2 \text{ ; and}$$

$$F_{i+1} = (2S^2 + \sigma^2)\, F_i - S^4 F_i - 1$$

produce the optimal estimate. To test this, only the two-point estimate will be verified. Unfortunately, similar credibility estimators for three or more estimators are very unwieldy. The two-point estimate involves an estimate for $\hat{x}(3)$ given $\hat{x}(2)$ and $\hat{x}(1)$.

According to our formula, the result should be

$$\hat{x}(3) \approx \frac{S^2\hat{x}(1) + (\sigma^2 + S^2)\,\hat{x}(2)}{\sigma^2 + 2S^2} ;$$

i.e., it should be true that

$$Z\,[\hat{x}(2), \hat{x}(1), \hat{x}(3)\,] = \frac{S^2 + \sigma^2}{\sigma^2 + 2S^2} .$$

Note that, since $\hat{x}_i - x_i$ is orthogonal to $\hat{x}_j - x_j$ and $x_i - x_j$:

$$\begin{aligned}\tau_1^2 &= \mathrm{E}[\,(\hat{x}(2) - \hat{x}(3)\,)^2\,] \\ &= \mathrm{E}[\,(\hat{x}(2) - x(2)\,)^2\,] + \mathrm{E}[\,(x(2) - x(3)\,)^2\,] + \mathrm{E}[\,(x(3) - \hat{x}(3)\,)^2\,] \\ &= S^2 + \sigma^2 + S^2 = 2S^2 + \sigma^2 .\end{aligned}$$

$$\begin{aligned}\tau_2^2 &= \mathrm{E}[\,(\hat{x}(1) - \hat{x}(3)\,)^2\,] \\ &= \mathrm{E}[\,(\hat{x}(1) - x(1)\,)^2\,] + \mathrm{E}[\,(x(1) - x(3)\,)^2\,] + \mathrm{E}[\,(x(3) - \hat{x}(3)\,)^2\,] \\ &= S^2 + 2\sigma^2 + S^2 = 2S^2 + 2\sigma^2 .\end{aligned}$$

$$\begin{aligned}\delta_{12}^2 &= \mathrm{E}[\ (\hat{x}(1) - \hat{x}(2)\ )^2\ ] \\ &= \mathrm{E}[\ (\hat{x}(1) - x(1)\ )^2\ ] + \mathrm{E}[\ (x(1) - x(2)\ )^2\ ] + \mathrm{E}[\ (x(2) - \hat{x}(2)\ )^2\ ] \\ &= S^2 + \sigma^2 + S^2 = 2S^2 + \sigma^2\,.\end{aligned}$$

Thus,

$$Z\,(\hat{x}(2), \hat{x}(1), \hat{x}(3)\ ) = \frac{2S^2 + 2\sigma^2 - (2S^2 + \sigma^2) + 2S^2 + \sigma^2}{2\,(2S^2 + \sigma^2)} = \frac{S^2 + \sigma^2}{2S^2 + \sigma^2}$$

So, in at least two cases, this approach does reproduce credibilities already known from other analyses.

To be truly useful, an approach should yield new methods. The true classification ratemaking problem will now be addressed.

## 3. CLASS CREDIBILITIES

The objective is to produce a rate for a subgroup $\alpha$ of a large group $\Gamma$. The means of $\alpha$ and $\Gamma$ are unknown, but have been estimated using:

$$\bar{a} = (1/n)\sum_{i}^{n} a_i\,;\ \ a_i \in \alpha \ \text{ for } \mu_\alpha \ \text{ and}$$

$$\bar{g} = (1/(m+n)\,)\left(\sum_{i}^{n} a_i + \sum_{j}^{m} b_j\right);\ \ b_j \in \beta = \Gamma - \alpha\,.$$

Further, Var $(a_i) = \sigma_\alpha^2$ and Var $(b_j) = \sigma_\beta^2$ have been estimated from actual data.

Before proceeding, note that weighting $\bar{a}$ with $\bar{g}$ using $Z\,(\bar{a}, \bar{g}, \mu_\alpha)$ effectively assigns a portion of the complement of the credibility back to $\bar{a}$ since $\bar{g} = (n\bar{a} + m\bar{b})/(n+m)$. So, it may be more worthwhile to evaluate

$$Z\,(\bar{a}, \bar{b}, \mu_\alpha) = \frac{mZ\,(\bar{a}, \bar{g}, \mu_\alpha) + n}{n+m}\ ;$$

and then use

$$Z(\bar{a}, \bar{g}, \mu_\alpha) = \frac{Z(\bar{a}, \bar{b}, \mu_\alpha)(n+m) - n}{m}.$$

Note that

$$\mathrm{Var}(\bar{a}) = (\sigma_\alpha^2)/n\ ;\ \ \mathrm{Var}(\bar{b}) = (\sigma_\beta^2)/m\ ;\ \text{and}$$

$$\tau_1^2 = \mathrm{E}[\,(\bar{a} - \mu_\alpha)^2\,] = \sigma_\alpha^2/n\ .$$

Since the $b_j$ are independent, and the $b_j - \mu_\beta$ have mean 0,

$$\tau_2^2 = \mathrm{E}[\,(\bar{b} - \mu_\alpha)^2\,] = \sigma_\beta^2/m + (\mu_\beta - \mu_\alpha)^2\ ;$$

and

$$\begin{aligned}\delta_{12}^2 &= \mathrm{E}[\,(\bar{a} - \bar{b})^2] = \mathrm{E}[\,(\bar{a} - \mu_\alpha)^2] + \mathrm{E}[\,(\mu_\alpha - \mu_\beta)^2] + \mathrm{E}[\,(\bar{b} - \mu_\beta)^2] \\ &= (\sigma_\alpha^2/n) + (\mu_\alpha - \mu_\beta)^2 + (\sigma_\beta^2/m)\ .\end{aligned}$$

Hence

$$\begin{aligned}&Z(\bar{a}, \bar{b}, \mu_\alpha) \\ &= \frac{(\sigma_\beta^2/m) + (\mu_\alpha - \mu_\beta)^2 - (\sigma_\alpha^2/n) + (\sigma_\alpha^2/n) + (\mu_\alpha - \mu_\beta)^2 + (\sigma_\beta^2/m)}{2\,(\,(\sigma_\alpha^2/n) + (\mu_\alpha - \mu_\beta)^2 + (\sigma_\beta^2/m)\,)} \\ &= \frac{\sigma_\beta^2/m + (\mu_\alpha - \mu_\beta)^2}{(\sigma_\alpha^2/n) + (\sigma_\beta^2/m) + (\mu_\alpha - \mu_\beta)^2} \\ &= \frac{n}{n + [\sigma_\alpha^2/(\,(\sigma_\beta^2/m) + (\mu_\alpha - \mu_\beta)^2\,)\,]}\ .\end{aligned}$$

Or, if $m$ is much larger than $n$

$$= \frac{n}{n + [\sigma_\alpha^2/(\mu_\alpha - \mu_\Gamma)^2]}\ .$$

Of course, $\sigma_\alpha^2$ and $(\mu_\alpha - \mu_\Gamma)^2$ are unknown, but they may be estimated using the sample variance of $a_i \in \alpha$ and the difference between the existing rate for $\alpha$ and the overall average rate.

This formula illustrates several key points:

1. Highly heterogeneous classes (high $\sigma_\alpha^2$) should receive lower credibilities.
2. Extremely high or low rates [high $(\mu_\alpha - \mu_\Gamma)^2$] should be based more heavily on their own experience than on the overall average rate.
3. For classes that form a statistically large proportion of the group, the complement of the credibility should be assigned to the mean of the remainder of the group, not the group mean.

## 4. A NUMERICAL EXAMPLE

Suppose that one is attempting to find the underlying mean, $\mu_x$, of a distribution given last year's observation $X_{1,i-1}$ of the distribution, and last year's observation of $X_{2,i-1}$, a related statistic. Further, assume $X_{2,i}$ is thought to be cyclic and biased as a predictor of $X_{1,i}$, and its year-to-year variations are thought to be independent of those of $X_1$. Whether $X_1$ is cyclic or stationary is not known. The observations are shown in Table 1. Of course, the values $X_{1,i+1}$ and $X_{2,i+1}$ are unknown at time $i$, but the goal is to find the $Z_i$ such that $Z_i X_{1,i} + (1 - Z_i) X_{2,i}$ is an optimum estimator of $\mu_x$ at time $i + 1$.

In this case, since $X_1$ and $X_2$ are independent predictors of $\mu_x$, $E[(X_1 - X_2)^2]$ reduces to $\tau_{x,1}^2 + \tau_{x,2}^2$, so $Z$ becomes $Z = \tau_{x,2}^2/(\tau_{x,1}^2 + \tau_{x,2}^2)$. $X_1$'s error, $\tau_{x,1}^2$, may be estimated by $S_{x,1}^2$, the sample variance of the $X_{1,i}$ seen to date. Since $X_{2,i}$ is biased, $\tau_{x,2}^2$ will be estimated by $S_{x,2}^2 + (X_{2,i+1} - \hat{Y}_i)^2$, where $\hat{Y}_i$ is the last estimate of $\mu_x$. This method recognizes both the cyclic nature of $X_2$ (by using $X_{2,i+1} - \hat{Y}_i$) and a potential cyclic pattern in $X_1$ because it considers just the last observed values, not the history of $X_1$ and $X_2$. Arbitrarily, the first credibility was chosen at 50%. Results are shown in Table 2.

TABLE 1

| Year | $X_{1,i}$ | $X_{2,i}$ |
|---|---|---|
| 1 | 72.44 | 104.15 |
| 2 | 79.06 | 114.66 |
| 3 | 72.98 | 112.75 |
| 4 | 79.74 | 99.01 |
| 5 | 66.69 | 103.04 |
| 6 | 86.38 | 102.80 |
| 7 | 68.97 | 106.23 |
| 8 | 78.61 | 97.79 |
| 9 | 88.97 | 101.63 |
| 10 | 74.97 | 102.83 |

TABLE 2

OBSERVED HISTORY

| Year | $X_1$ | $X_2$ | $\tau^2_{x,1} = S^2_{x,1}$ | $S^2_{x,2}$ | $(X_2 - \mu_x)^2$ | $\tau^2_{x,2}$ | $Z$ | $\hat{Y}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 72.44 | 104.15 | — | — | — | — | 50% | 88.30 |
| 2 | 79.06 | 114.66 | 43.82 | 110.46 | 694.85 | 805.31 | 95 | 80.84 |
| 3 | 72.98 | 112.75 | 13.51 | 31.34 | 1018.25 | 1049.59 | 99 | 73.38 |
| 4 | 79.74 | 99.01 | 15.04 | 54.02 | 656.90 | 710.92 | 98 | 80.07 |
| 5 | 66.69 | 103.04 | 28.82 | 44.75 | 527.70 | 572.45 | 95 | 68.51 |
| 6 | 86.38 | 102.80 | 47.86 | 38.36 | 1175.98 | 1214.34 | 96 | 87.04 |
| 7 | 68.97 | 106.23 | 47.38 | 31.97 | 368.38 | 400.35 | 89 | 73.07 |
| 8 | 78.61 | 97.79 | 42.08 | 36.02 | 611.08 | 647.10 | 94 | 79.76 |
| 9 | 88.97 | 101.63 | 56.66 | 32.82 | 478.30 | 511.12 | 90 | 90.24 |
| 10 | 74.97 | 102.83 | 50.81 | 29.51 | 158.61 | 188.12 | 79 | 80.82 |

In this case $\hat{Y}_i$ looks like a poor predictor of $\mu_x$ at $i+1$.

However, when the true $\mu_x$ is considered, $\hat{Y}$ is a very good estimator of $\mu_x$. The $X_{1,i}$ were generated using a normal distribution with mean $\mu_x = 80$ and variance $\sigma_x^2 = 100$. The $X_{2,i}$ were generated using a normal distribution with mean 100 and variance 100 for $X_{2,1}$, and successively generating each new $X_{2,i+1}$ using a normal distribution with mean $100 + .8\,(X_{2,i} - 100)$ and variance 36. One can show that the resulting $X_{2,i}$ all have a marginal distribution that has mean 100 and variance 100 since $36 = 100\,(1-(.8)^2)$. Therefore, a priori, the credibility of $X_i$ should be $(100+400)/(100+400+100) = .83333$. However, that credibility should vary with where $X_2$ happens to be in its cycle.

It just happened that the $X_{1,i}$ tended to fall on the low side of the distribution, and that the $X_{2,i}$ began on the high side of the distribution and tended to stay there.

In any event, the average squared prediction error of $\hat{Y}$ is 44.86, roughly a 20% reduction in the prediction error of $X_1$ alone (55.46 as a predictor of the value $\mu_x = 80$). In fact, the error of $\hat{Y}$ is even below 45.6, which results from what retrospectively turns out to be the best possible fixed credibility (96%). That is because this method gives greater weight to $X_2$ when it is close to $\hat{Y}$ in the cycle, and less weight when it is further away. So, in this example, the theory works.

## 5. CREDIBILITY DEMYSTIFIED

One of the side benefits of this approach is that it offers an explanation of credibility that can be understood by laymen. The credibility of each estimator is proportional to its accuracy as an estimator, less the extent to which the two estimators say the same thing. Clearly this explanation is much more desirable than "we've always done it this way," and more understandable to lay people than "we look at the process variance and the variance of the hypothetical means." But it has yet to be shown that the above explanation is true.

Note that

$$Z(x_1, x_2, Y) = \frac{\tau_2^2 - \tau_1^2 + \delta_{12}^2}{2\delta_{12}^2}.$$

Further

$$\begin{aligned}\delta_{12}^2 &= \mathrm{E}[\ (x_1 - x_2)^2\ ] \\ &= \mathrm{E}[\ (\ (x_1 - Y) - (x_2 - Y)\ )^2\ ] \\ &= \mathrm{E}[\ (x_1 - Y)^2\ ] + \mathrm{E}[\ (x_2 - Y)\ )^2\ ] - 2\mathrm{E}[\ (x_1 - Y)\,(x_2 - Y)\ ]\ .\end{aligned}$$

Further, assuming that $x_1$ and $x_2$ are unbiased estimators, $\mu_1 = \mu_2 = Y$, so

$$\delta_{12}^2 = \tau_1^2 + \tau_2^2 - 2\mathrm{Cov}(x_1, x_2)\ .$$

And when that is included in the formula for $Z$,

$$\begin{aligned}Z(x_1, x_2, Y) &= \frac{\tau_2^2 - \tau_1^2 + \tau_1^2 + \tau_2^2 - 2\mathrm{Cov}(x_1, x_2)}{2\tau_1^2 + 2\tau_2^2 - 4\mathrm{Cov}(x_1, x_2)}\ ; \\ &= \frac{2\tau_2^2 - 2\mathrm{Cov}(x_1, x_2)}{2\tau_1^2 + 2\tau_2^2 - 4\mathrm{Cov}(x_1, x_2)}; \\ &= \frac{\tau_2^2 - \mathrm{Cov}(x_1, x_2)}{\tau_2^2 + \tau_1^2 - 2\mathrm{Cov}(x_1, x_2)}\ .\end{aligned}$$

Dividing top and bottom by $\tau_1^2\tau_2^2$ yields

$$Z(x_1, x_2, Y) = \frac{(1/\tau_1^2) - (R_{12}^2/\tau_1\tau_2)}{(1/\tau_1^2) + (1/\tau_2^2) - 2\,(R_{12}^2/\tau_1\tau_2)}\ ;$$

where $R_{12}^2$ is the correlation of $x_1$ and $x_2$. Clearly, if $\tau_1^2$ is the error of $x_1$, then $1/\tau_1^2$ must be $x_1$'s accuracy. Further, $R_{12}^2$ is the extent to which $x_1$ and $x_2$ vary together, and the division by $\tau_1^2\tau_2^2$ normalizes it relative to the inverse squared errors. Hence, the credibility of each estimator is proportional to its accuracy, less the extent to which the estimators say the same thing.

## 6. PRACTICAL APPLICATIONS AND FORMULAE

One criticism of this approach is that, like optimum credibility, the appropriate credibility formula depends on the circumstances. For instance, if the two estimators are not heavily skewed, their historic accuracy $n/\sum(x_{1,j}-Y')^2$ and $n/\sum(x_{2,j}-Y')^2$ may be used to derive the optimum credibility as shown in Section 2. Per Example 2, the formula $(1+(\sigma^2/S^2))/(2+(\sigma^2/S^2))$ may be used with $\sigma^2/S^2$ estimated using the historic year-to-year changes in $\hat{x}$. As shown in Section 3, the credibility of a class's own experience should be $n/(n+(S_\alpha^2/(r_\alpha-r_\Gamma)^2))$, where $n$ is the number of exposure units, $S_\alpha^2$ is estimated by comparing each year's class experience to a long-term average, and $r_\alpha - r_\Gamma$ is the difference between the current rate for class $\alpha$ and the current average rate. Alternatively, $S_\alpha^2$ could be presumed to be constant across all classes and one could then find the $S_\alpha^2$ that minimized the average squared error (weighted by exposures) when such a formula uses last year's experience to predict this year's data.

The most important results are:

1. The credibility of a piece of data and the formula used to derive it vary with the specific situation.
2. Using the formulae in this paper, one may derive credibility formulae that, up to determining a constant or two, represent the best credibility formulae. The constants can then be determined using historic data to find the constants that minimize the average squared error.
3. The fundamental truth of this paper, that credibility should be based on accuracy, makes intuitive sense and can be understood by laymen.

## 7. SUMMARY AND CONCLUSIONS

In summary, this approach seems to hold promise and appears to offer opportunities to improve the accuracy of loss estimates. However, it will only truly be useful when the estimation errors $\tau_1^2$ and $\tau_2^2$ are evaluated. Whether one believes in this approach or not, this author believes that the large ratemaking organizations should collect statistics on the effectiveness of the various loss estimators they use. Even if other credibility procedures are used, it only makes sense that their effectiveness be monitored. Futher, this author believes that greater understanding of how credibility should work can only improve the actuarial work product.

## REFERENCES

[1] Venter, Gary G., et al, *Foundations of Casualty Actuarial Science,* New York, Casualty Actuarial Society, 1990, pp. 375-483.

[2] Efron, Bradley, "Controversies in the Foundations of Statistics," *American Mathematical Monthly,* Vol. 85, No. 4, April 1978, pp. 231-246; reprinted in the *Casualty Actuarial Society Forum,* Fall 1991, pp. 259-275.

# APPENDIX A

## PROOF OF $Z(x_2, x_1, Y) = \dfrac{\tau_2^2 - \tau_1^2 + \delta_{12}^2}{2\delta_{12}^2}$

If $\tau_1^2 = \mathrm{E}[\ (x_1 - Y)^2\ ]$, $\tau_2^2 = \mathrm{E}[\ (x_2 - Y)^2\ ]$, and $\delta_{12}^2 = \mathrm{E}[\ (x_1 - x_2)^2\ ]$,

the goal is to set

$$\Phi(Z) = \mathrm{E}[\ (Zx_1 + (1 - Z)\, x_2 - Y)^2\ ] = \ \min\,.$$

But algebra gives

$$\begin{aligned}
&\mathrm{E}[\ (Zx_1 + (1 - Z)\, x_2 - Y)^2\ ] \\
&\quad = \mathrm{E}[\ (Z\,(x_1 - Y) + (1 - Z)\,(x_2 - Y)\,)^2\ ] \\
&\quad = Z^2\, \mathrm{E}[\ (x_1 - Y)^2\ ] + (1 - Z)^2\, \mathrm{E}[\ (x_2 - Y)^2\ ] \\
&\qquad + 2Z\,(1 - Z)\, \mathrm{E}[\ (x_1 - Y)\,(x_2 - Y)\ \ ] \\
&\quad = Z^2\tau_1^2 + (1 - Z)^2\tau_2^2 + 2Z\,(1 - Z)\, \mathrm{E}[\ (\ x_1 - Y)\,(x_2 - Y)\ ]\,.
\end{aligned}$$

Using $2(A - C)\,(B - C) = (A - C)^2 + (B - C)^2 - (A - B)^2$,

$$\begin{aligned}
&2\ \mathrm{E}[\ (x_1 - Y)\,(x_2 - Y)\ ] \\
&\quad = \mathrm{E}[\ (x_1 - Y)^2\ ] + \mathrm{E}[\ (x_2 - Y)^2\ ] - \mathrm{E}[\ (\ x_1 - x_2)^2\ ] \\
&\quad = \tau_1^2 + \tau_2^2 - \delta_{12}^2\ .
\end{aligned}$$

Substituting that back in the overall squared error ($\Phi$) equation,

$$\begin{aligned}
\Phi\,(Z) &= \mathrm{E}[\ (Zx_1 + (1 - Z)\, x_2 - Y)^2\ ] \\
&= Z^2\tau_1^2 + (1 - Z)^2\tau_2^2 + Z(1 - Z)\ [\tau_1^2 + \tau_2^2 - \delta_{12}^2] \\
&= Z\tau_1^2 + (1 - Z)\tau_2^2 + (Z^2 - Z)\ \delta_{12}^2\,.
\end{aligned}$$

This is minimized when

$$\frac{d\Phi}{dZ} = 0 = \tau_1^2 - \tau_2^2 + (2Z - 1)\delta_{12}^2$$

or

$$\tau_2^2 - \tau_1^2 + \delta_{12}^2 = 2Z\,\delta_{12}^2\,,$$

or

$$Z = \frac{\tau_2^2 - \tau_1^2 + \delta_{12}^2}{2\delta_{12}^2}\,.$$

Now, this is only a minimum when

$$\frac{d^2\Phi}{dZ^2} = 2\delta_{12}^2 \geq 0\,.$$

However $\delta_{12}^2$ is always non-negative.

Note that $Z$ may be negative; i.e., when

$$\tau_1^2 \geq \tau_2^2 + \delta_{12}^2\,.$$

But, fortunately, that occurs only where $(x_2 - Y)$ and $(x_1 - x_2)$ tend to have the same sign overall; i.e., $x_2$ is generally between $x_1$ and $Y$. Thus, there may be cases where zero credibility is warranted; i.e., $x_1$ is not a useful predictor. Alternatively, where

$$\tau_2^2 \geq \tau_1^2 + \delta_{12}^2\,,$$

full credibility should be assigned to $x_1$.

# APPENDIX B

## JUST HOW DISTRIBUTION-FREE IS DISTRIBUTION-FREE?

As noted in the Section 1 of this paper, the credibility, $Z$, produced by this method produces the lowest expected squared error attainable using the $Zx_1 + (1 - Z)\,x_2$ (additive weighting) formula. That credibility does not depend on the particular characteristics of each estimator's distribution, but only on how well each estimator predicts the unknown quantity $Y$, and on the average squared difference between the two estimators. However, it does assume that the best estimator of the unknown $Y$ is the one that minimizes the expected squared estimating error, and that it uses a $Zx_1 + (1 - Z)\,x_2$ formula. One should consider whether each of these implicit assumptions really produces the best estimator of the unknown $Y$.

Aside from the fact that the expected squared error function is ubiquitous in mathematics and related disciplines, there is a practical reason for using it as a penalty function whose minimum defines the best estimator. Conceptually, one might begin by viewing the expected absolute error $\mathrm{E}\,[\,|\,\text{estimator} - Y\,|\,]$ as the best penalty function, since it measures the actual error of the estimator. That approach, however, has one considerable drawback. An extremely large error receives the same weight as a small error, even though extremely large errors may have catastrophic consequences. For example, if a rate would produce precisely the required profit 19 years out of 20 but threaten the company's solvency one year out of 20, prudence would dictate that the one year out of 20 receives a disproportionate share of attention. One logical approach is to weight each absolute error with the size of that absolute error—in effect to use $\mathrm{E}[\,|\,\text{estimator} - Y\,| \cdot |\,\text{estimator} - Y\,|\,]$ or $\mathrm{E}[(\text{estimator} - Y)^2\,]$.

The use of an additive weighting is less supportable. This author knows of no reason why an estimator of the form, say $x_1^Z \cdot x_2^{(1-Z)}$ would not be a better estimator. It is clear that if $x_2$ is biased, some $Zx_1 + (1 - Z)\,(x_2 - C)$ formula is better. Superficially, it appears that a rigorous analysis, perhaps using calculus of variations, could produce different formulae for different families of distributions.

On a more positive note, there are two reasons for using an additive weighting formula. When the two estimators $x_1$ and $x_2$ are known to be normally distributed about unknown means, but $\tau_1^2$, $\tau_2^2$, and $S_{12}^2$ can be estimated, the additive weighting formula is best.[2] Also, additive weighting has a long history in ratemaking.

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[2] The author doubts he truly discovered this. Witness the argument in [2].

# PRICING FOR CREDIT EXPOSURE

BRIAN Z. BROWN

*Abstract*

*This paper incorporates financial theory with insurance pricing. A general procedure to price for credit exposure has been developed and extended to several insurance products. For retrospectively rated insureds with a below-investment-grade rating from a credit rating agency, the credit exposure is significant to the insurer. If this exposure is ignored, operating results may be negatively affected, as it is likely that some additional premium amounts will not be collected. The principles of credit exposure pricing can be extended to the pricing of surety bonds as outlined in Section 6. In addition, the concepts outlined in this paper may be used to: 1. establish a bad debt reserve for GAAP statements; 2. provide insurance regulators with an additional method to determine collateral requirements; 3. estimate surety bond loss ratios; and 4. provide banks with an additional methodology to price letters of credit which collateralize unpaid claim liabilities.*

## 1. INTRODUCTION

The term "credit exposure" is defined as the possibility that one entity will suffer a financial loss due to the inability of a second entity to satisfy its contractual obligations (as a result of the poor financial health of the second entity). Credit exposure, as it relates to underwriting activities of property-casualty insurance companies, refers to the possibility that the insurance company may not be able to collect premiums, deductibles, and other charges when due. This paper will focus on the credit risk associated with issuing retrospectively rated policies and will outline a procedure to price this credit exposure. In addition, the credit exposure pricing model will be extended to other areas of credit risk exposure, such as surety bond pricing.

For retrospectively rated policies, the insurance company collects premium from the insured based on paid or incurred losses until all claims are closed, or until the insured and insurance company agree on a price to end the retrospective accountings. Thus, if the insured files for bankruptcy, the insurance company may not be able to collect all the additional retrospective premiums due. These additional premiums represent a credit exposure to the insurance company. The insurance company may be obligated to pay claims associated with the expired policy even though additional premiums are not forthcoming.

The discussion below is divided into six sections. Sections 2 through 5 deal with the credit exposure associated with retrospectively rated policies. Section 6 describes a pricing model for surety bonds. Section 7 discusses additional applications of the credit exposure pricing model.

## 2. MAGNITUDE AND QUALITY OF THE CREDIT EXPOSURE

The magnitude of the credit exposure refers to the absolute amount of the expected additional premium due the insurance company on expired policies. Credit exposure results from the nature of retrospectively rated plans. For incurred loss plans, an insured typically receives a return premium at the first retrospective accounting, and thereafter the insured pays additional premium to the insurance company as losses develop. This procedure is similar to the insurance company "lending" the insured the difference between ultimate premium and the collected retrospective premium. This difference between the ultimate premium and the collected premium defines the magnitude of the credit exposure.

For an incurred loss retrospectively rated plan, the insurance company collects standard premium during the first 12 months of the policy period. At 18 months and annually thereafter, a retrospective accounting is performed via the following formula:

$$R_t = (B + (C \times E) + (C \times L_t)) \times T . \qquad (2.1)$$

$R_t$ is the retrospective premium at time period $t$ and is subject to a maximum and minimum premium. The maximum and minimum premiums are factors of the standard premium.

$B$ is the "basic charge" and covers expenses, profit, and the insurance charge. The basic charge is determined by a factor multiplied by standard premium. The factor $C$ is the loss conversion factor and covers a load for loss adjustment expenses. $E$ is the charge for limiting losses for individual claims that enter the retrospective rating formula. $T$ is the tax multiplier and covers premium tax and some premium-based assessments. The tax multiplier and basic charge may also include a provision for residual market assessments.

$L_t$ is the aggregate case incurred loss amount at time $t$, limited by the per claim amount. Thus, there is no provision for incurred but not reported (IBNR) losses in the standard incurred loss retrospective rating formula.[1] The magnitude of the credit exposure is proportional to the IBNR.

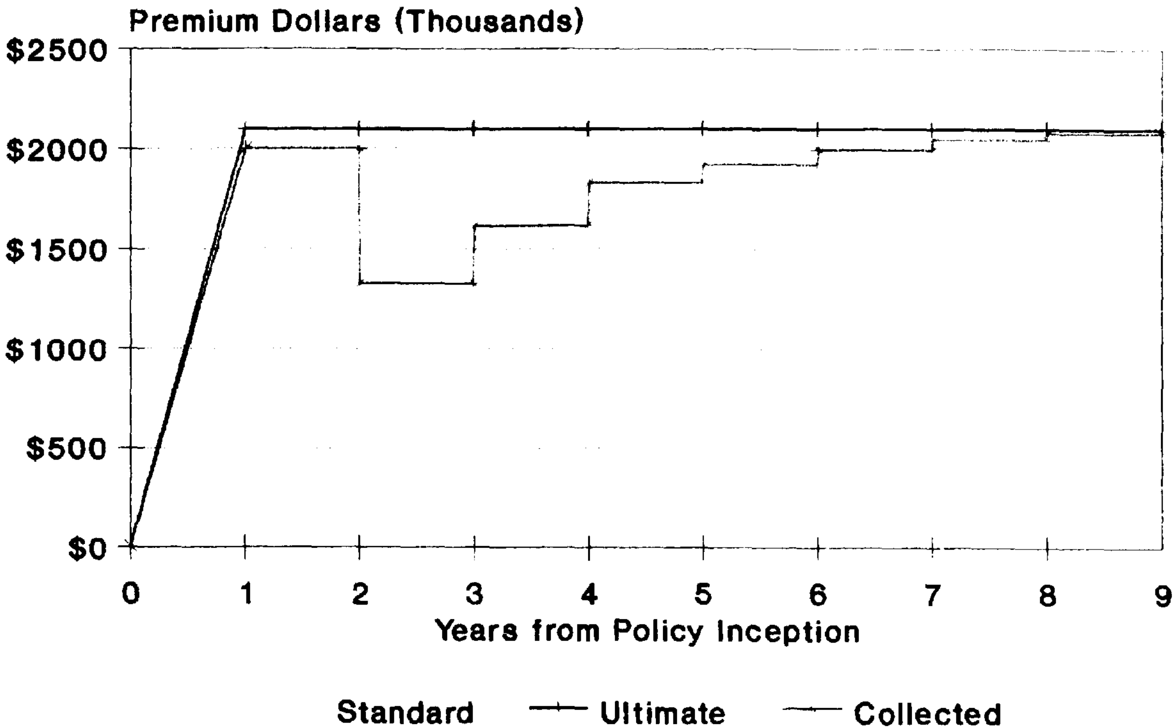
Figure 1 displays the projected transfer of funds for a standard incurred loss retrospectively rated plan. In this example, it is assumed that the insured is written for a one-year term and does not renew. The retrospective rating parameters and incurred loss development pattern are shown on Exhibit 1. For simplicity, the tax multiplier and loss conversion factor are both set equal to one. These assumptions do not affect the conclusions of the paper.

Exhibit 2 displays the amounts underlying the graph on Figure 1. The insurance company collects \$2.0 million in standard premium during the first year. The first retrospective accounting is performed with incurred losses evaluated as of 18 months and \$674,000 is returned to the insured.[2] Over the next seven years, the insurance company will collect the difference between the ultimate retrospective premium (\$2.1 million) and the first retrospective accounting premium (\$1.326 million), or \$774,000. The

[1] IBNR is defined to include both incurred but not reported losses as well as development on existing case reserves.

[2] The example in this paper assumes that the billing and collection process takes six months. Therefore, the insurance company will return funds to the insured as of 24 months.

FIGURE 1
STANDARD INCURRED LOSS RETRO PLAN
INSURED WRITTEN FOR ONE YEAR



$774,000 is the credit exposure as of 24 months and, in essence, represents a loan by the insurance company to the insured. If the insured filed for bankruptcy 25 months after the inception of the retrospectively rated policy (one month after the insurance company returned funds to the insured), the insurance company could suffer a future earnings loss equal to $774,000.

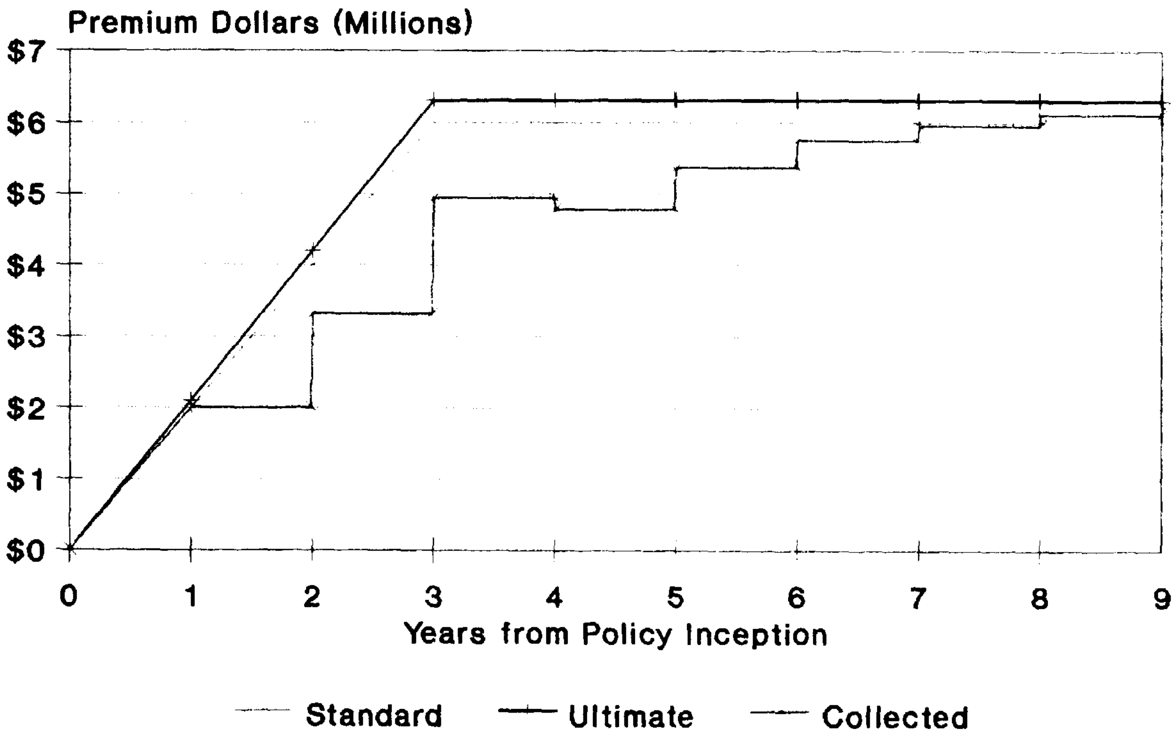
The credit exposure decreases as losses develop (assuming the insured pays the premium when due). The exposure equals zero when the insured pays the ultimate premium.

The magnitude of the credit exposure for a specific insured at a specific point in time depends upon how much premium is yet to be collected. In general, the quicker the premium is collected, the smaller the credit exposure. Thus, plans with low loss limits will have a smaller credit exposure.

Figure 2 displays the situation where the insurance company insures the above-mentioned risk for three years at a standard premium of $2.0 million a year, and then the insured non-renews. Exhibit 3 displays the numerical backup for Figure 2. The collected premium at Year Two, $3.326 million, equals the first retrospective accounting premium of $1.326 million for the first year of the three-year period plus the $2.0 million in standard premium for the second year of the three-year period. The credit exposure is largest at Year Four, after all policies have had a retrospective accounting. The credit exposure at Year Four (four years after the inception of the first policy) is $1.53 million or 76.5% of the annual standard premium.

The National Association of Insurance Commissioners (NAIC) has recognized the significance of credit exposure and has altered the statutory Annual Statement twice in the last four years in order to provide more information on credit exposure. The 1988 Statement was altered to display the amount of additional premium that an insurance company anticipates collecting on retrospectively rated plans. Line 9.3 on page 2 of the Statement or line 33 on page 8 displays the additional premium amount (Accrued Retrospective Premium). Beginning in 1988, the Statement also required companies to display the amount of letters of credit,

FIGURE 2
STANDARD INCURRED LOSS RETRO PLAN
INSURED WRITTEN FOR THREE YEARS



collateral, and other funds that secure the accrued retrospective premiums (General Interrogatory # 31). In 1991, the Statement required that 10% of the amount of unsecured accrued retrospective premium be recorded as a non-admitted asset. These changes show that the NAIC recognizes the possibility that the additional premium due may not be collectible. The collectibility of the additional premium is a function of the quality of the credit exposure.

Francis Hope stated that "negative reserves; i.e., the anticipation of additional premium due the company, [should] be included in the Annual Statement, provided that one is fully confident that the money is truly forthcoming." [1] Hope probably intended this statement to refer to the accuracy of the retrospective premium reserve; however, it is equally important that the monies due the insurance company be collectible. Amounts due the insurance company may be uncollectible due to the insured's financial condition.

The quality (collectibility) of the credit exposure is largely determined by the insured's financial condition. If the insured files for bankruptcy, the insurance company may not be able to collect additional premium. Clearly, an insured in a strong financial position at policy inception is less likely to file for future bankruptcy than an insured in a poor financial position at policy inception. Section 3 outlines several methods to determine the financial condition of an insured.

Nationally, credit exposure has increased as the financial strength of American companies has decreased. The middle 1980s saw a large increase in the number of leveraged-buyouts (LBOs). The LBO activity resulted in companies replacing equity with debt, and, therefore, the financial strength of these companies has decreased. Bonds rated Caa by Moody's—the bonds closest to default—increased from $7.2 billion at December 1987 to $23.7 billion by March 1990. [2] For non-financial corporations, net interest expense as a percentage of earnings before interest and taxes rose from 18.2% in 1979 to 42.9% by year-end 1989. These statistics display the decrease in the credit quality of American corporations. Thus, insurance companies may not be able to collect as much retrospective premium on policies of the 1980s because relatively more insureds may file for bankruptcy.

## 3. ANALYZING THE QUALITY OF THE CREDIT EXPOSURE

This section outlines two approaches that can be used to determine the financial strength of a commercial insured:

1. ratio analysis; and
2. independent rating agencies' ratings.

### *Ratio Analysis*

Ratio analysis depicts relationships between accounts on a firm's financial statement. The ratios can be used to measure a firm's financial health. Ratios can be classified as liquidity ratios, equity ratios, and profitability ratios.[3]

Liquidity ratios measure a firm's short-term debt paying ability. One commonly used liquidity ratio is the *current ratio*. The current ratio is a firm's current assets divided by its current liabilities. Short-term creditors use the current ratio to financially underwrite accounts. All other things being equal, the higher the current ratio, the more likely a firm will be able to pay its short-term debts. Even though credit exposure is similar to long-term debt, an insurance company will be interested in the liquidity ratios because if a firm cannot pay short-term debts it may not be able to pay long-term debts.

A commonly used equity ratio is the *debt-to-equity* ratio which measures the amount of relative debt of a firm. The higher the debt-to-equity ratio, the more financially leveraged the company. Since the company must eventually repay the debt, a company with a relatively high debt-to-equity ratio may not be able to pay additional insurance premium if the company falls on hard times.

A commonly used profitability ratio is the *times-interest-earned ratio*. The ratio is:

$$\text{Times Interest Earned} = \frac{\text{Income Before Interest and Taxes}}{\text{Interest Expense}}.$$

[3] Another classification of ratios is market ratios; however, these ratios are more relevant for investment decisions than for credit quality analysis.

A firm that cannot pay its interest payments when due will eventually have financial problems. The higher the times-interest-earned ratio, the more money a firm will have available for general corporate purposes. Therefore, a firm with a high times-interest-earned ratio is not likely to default on premium payments in the near future.

A financial analyst can use these ratios as tools to evaluate the credit exposure of a prospective or current insured. These ratios have "rule-of-thumb" levels at which the firm is considered to be potentially deficient. As a technical note, the rule-of-thumb levels vary considerably by industry, and financially underwriting accounts is a complex procedure which often cannot be based solely on financial ratios. Therefore, for accounts that possess a significant credit exposure, the analyst should review all of the firm's published financial information (annual reports, 10-Ks, etc.) and meet with a financial officer of the firm. A firm's ratios may be acceptable; however, the firm may have a significant receivable from another firm in poor financial condition. Only the notes to the financial statements will highlight such a problem.

*Independent Rating Agency Ratings*

Several independent rating agencies rate the credit quality of companies. Insurance companies can use the agencies' ratings to supplement ratio analysis in order to determine the quality of the credit exposure. If the retrospective rating plan for a particular insured has a small credit exposure, it may be more cost effective for the insurance company to rely solely on the agency ratings. Three possible agency ratings are those of Moody's, Dun & Bradstreet, and Standard & Poor's.

Moody's publishes corporate bond ratings indicating the degree of default risk associated with a bond. The default risk on bonds may be similar to the default risk for retrospective premium payments because both are long-term obligations. Moody's bond ratings vary from Aaa (where interest payments are protected by a large and stable margin and the bond principal is secure) to C (the lowest class of bonds where the prospect of the bonds ever attaining any real investment standing is extremely poor).

Dun & Bradstreet provides credit ratings on three million domestic and foreign firms. Dun & Bradstreet's *Business Information Report* contains a summary section on each firm. The summary section includes a review of the company's financial condition and its sales and earnings trend along with the firm's payment record, as reported by suppliers. The report also includes other financial and operating information. Dun & Bradstreet supplies a composite credit approval rating from 1 to 4, with 1 being the highest rating.

Standard & Poor's publishes bond credit quality ratings. Standard & Poor's bond ratings vary between AAA and D. AAA is the highest rating and signifies that the capacity to pay interest and principal is extremely strong. A rating of D indicates that the firm has defaulted on interest or principal payments.

## 4. CURRENT INDUSTRY PRACTICES FOR MINIMIZING CREDIT EXPOSURE

The insured's financial condition will determine the quality of the credit exposure. For firms in excellent financial condition, the insurance company may believe it is not necessary to recognize the credit exposure. I will show later that the risk of default for firms in excellent financial health is minimal, so that it may be reasonable to ignore credit exposure for these accounts. However, for firms in average or below-average financial condition, the insurance company may want to take steps to reduce its credit exposure.

The insurance company can reduce credit exposure by:

1. not offering retrospectively rated plans;
2. requiring collateral for anticipated additional retrospective premiums; and
3. altering the cash flow parameters of retrospectively rated policies.

For very large accounts, the first option of not offering retrospectively rated plans and still writing the account is not feasible. Most large insureds believe that their loss experience should determine their premium payments. Even though the experience rating plan reflects loss experience, the insureds recognize that retrospectively rated plans will reflect favorable loss experience earlier and more directly.

The second option to reduce the credit exposure is for the insurance company to require collateral from the insured. A letter of credit (LOC) or surety bond may be obtained to secure additional premium amounts. This, in essence, transfers some or all of the credit exposure to the bank or surety.

For example, the insurance company may request collateral equal to the ultimate premium less premium paid to date. Thus, if the insured defaulted at any time, the insurance company's credit exposure would be fully collateralized. The insurance company could exercise the collateral and collect all expected future amounts due. However, if losses developed higher than anticipated, the insurance company may suffer an earnings loss.

For firms in below-average financial condition, the insurance company may require collateral equal to the maximum premium less premium payments to date. The additional collateral could be required due to the greater variability of ultimate losses for firms in poor financial condition. For example, firms in poor financial condition may be more likely to lay off workers and overlook loss control in order to cut costs. These measures may increase the variability of ultimate losses.

Finally, insurance companies may alter the cash flow of the retrospectively rated plan in order to minimize the credit exposure. Two ways to alter the plan are:

1. modify the funding of the retrospectively rated plan; and
2. use loss development factors to project losses used in the retrospective rating formula.

The funding for incurred loss retrospective rating plans could be modified in order to minimize the credit exposure. For example, a plan could be designed such that the insured pays full standard premium the first year, with no funds changing hands until 42 months, even though retrospective accountings are performed at 18 and 30 months. Using the figures on Exhibit 2, this plan would reduce the credit exposure from

$774,000 at Year Two to $100,000, as collected premium would equal $2 million until Year Four. At Year Four and subsequent, the credit exposure under the revised program would be identical to the credit exposure in Exhibit 2. While this approach does not eliminate the credit exposure, it substantially reduces the exposure for the first three years after policy inception. In return for holding onto the insured's cash for a longer period of time, the insurance company could increase the dividend amount. The dividend payment could be increased by the amount of additional investment income earned on the incremental funds from 18 to 42 months. The incremental funds are the standard premium less the normally calculated retrospective premiums from 18 to 42 months.

The retrospective rating plan could incorporate incurred loss development factors to minimize the credit exposure. The development factors would be applied to the incurred losses at 18, 30, and 42 months. These factors would be used to develop the losses that enter the retrospective rating calculation to an estimated ultimate level. Therefore, the expected credit exposure would be zero at 18, 30, and 42 months because collected premium will equal ultimate premium. This analysis ignores any variation in development factors by account. As in the preceding option, the dividend amount could be increased for the additional investment income earned by the insurance company.

## 5. PRICING THE CREDIT EXPOSURE

This section outlines a new method to price the credit exposure on retrospectively rated policies utilizing a financial risk charge (herein referred to as Charge). The Charge is intended to provide sufficient funds to the insurance company to offset the expected loss of funds (on a present value basis) due to premium defaults. This section assumes that the Charge will be an addition to the basic premium. Alternatively, the Charge could be incorporated through a dividend reduction. The dividend reduction Charge calculation is similar to the additional premium calculation. However, the present value analysis would be different because policyholder dividends typically are first paid 18 months after policy inception, as opposed to the basic premium which is usually collected at policy inception.

The Charge is largely a function of two factors:

1. the insured's financial condition; and
2. the nature of the retrospectively rated plan.

Insureds in poor financial condition at policy inception are more likely to default on future premium payments. In addition, the slower the loss development pattern, the more premium there is outstanding at any point in time. Hence, the relative magnitude of the credit exposure is larger in either case. In addition, there is a relationship between the development pattern and financial condition of the insured. The longer it takes to collect the ultimate premium, the more time there is for a good financial risk to turn bad. This is similar to the phenomenon in life insurance, where "select" risks deteriorate over time to average risks (in the aggregate). Thus, plans with low loss limits or plans that contain lines of insurance which develop quickly, will have a lower Charge. A paid loss retrospectively rated plan will have a relatively large Charge.

A formula for the Charge is developed below based on the insured's financial condition at policy inception for a three-line incurred loss retrospectively rated policy. The illustrative plan includes workers compensation, general liability, and automobile liability coverages. The parameters of the plan are displayed on Exhibit 1.

Moody's bond default probabilities were used to determine the probability of the insured defaulting on additional retrospective premium payments. The probabilities are published in "Corporate Bond Default and Default Rates." [2] As both bonds and additional retrospective premium payments are long-term obligations, bond default rates are used as a proxy for premium payment defaults. Moody's studied bonds by initial rating over a 20-year period in order to determine default probabilities by year and initial rating. A portion of Moody's table is reproduced in Table 1.

TABLE 1

AVERAGE CUMULATIVE DEFAULT RATES
(YEARS)

| Bond Rating | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Baa | 0.2% | 0.5% | 0.9% | 1.3% | 1.7% | 2.2% |
| Ba | 1.7% | 3.7% | 5.5% | 7.2% | 8.9% | 10.4% |
| B | 7.0% | 11.8% | 15.9% | 18.9% | 21.1% | 23.0% |

The interpretation of the six-year value for B-rated bonds is that 23.0% of the bonds rated B at time period $t$ will default by time period $t+6$. Moody's has defined default as any missed or delayed interest or principal payment. Thus Moody's default rates are a conservative estimate of retrospective premium defaults.

Moody's default probabilities are used for illustrative purposes. The Charge could be based on default probabilities from other rating agencies or other studies. All of the tools referenced in Section 3 can be used to determine the insured's financial condition and default probability.

The Charge can be calculated from Table 1 and an estimate of the incremental retrospective premium at each time period. On Exhibit 2, we can see an estimate of the incremental retrospective premium at specific points in time. For example, as of the second retrospective accounting, the insured will be billed \$288,000 (this is the estimated retrospective premium of \$1,614,000 at 30 months less the collected premium of \$1,326,000 at 18 months).

The financial risk Charge can be calculated as follows:

$$\text{Financial Risk Charge} = \sum_{t=3}^{\infty} P_t \times R_t/(1+i)^t, \tag{5.1}$$

where $P_t =$ cumulative default probability through period $t$,

$R_t =$ incremental retrospective premium at time $t$, and

$i =$ an appropriate discount rate.

This calculation assumes that the basic premium will be increased by the amount of the Charge. Therefore, the calculation incorporates present value concepts (discounting) because the Charge is collected at policy inception, whereas the estimated premium defaults will occur over an extended period of time.

The time index begins three years after policy inception, as the financial exposure prior to the third year for standard incurred loss retrospectively rated plans is usually small. If the insurance company financially underwrites accounts, it is unlikely that the insured would file for bankruptcy in the first year-and-a-half after policy inception. In addition, at the first retrospective accounting, the insurance company typically returns funds to the insured because the first retrospective accounting premium is less than the standard premium.

The Charge calculation for an account with a Baa bond rating by Moody's and a premium payment vector as on Exhibit 2 is outlined on Exhibit 4. Column 1 on Exhibit 4 is the collected premium from Exhibit 2. Column 2 is the incremental collected premium, the additional premium the insurance company anticipates collecting at each accounting. Column 3 is the cumulative default probability based on Moody's default rates. Column 4 is the expected default amount which is Column 2 multiplied by Column 3. Column 5 is the present value of the expected default amount. For illustrative purposes, a 6% discount rate is used.

As displayed on Exhibit 4, the Charge for an insured with a senior unsecured bond rating of Baa and an anticipated premium collection vector as on Exhibit 2 is only 0.43% of standard premium (the sum of Column 5 divided by standard premium). Thus, the current industry practice, of not collateralizing the additional retrospective premium of insureds in good financial condition, may not be unreasonable. Insureds rated above Baa will have a lower Charge.

However, for an account with a premium vector as contained in Exhibit 2 and a bond rating of B (which is below Baa), the Charge is 5.7% of standard premium (Exhibit 5). Moody's report did not calculate default rates for firms with ratings below B; however, these companies would probably have a Charge significantly above 5.7%. Exhibit 6 displays the

Charge calculation for a *paid* loss retrospectively rated plan for a firm whose bonds are rated B. Thus, the only difference between Exhibits 5 and 6 is the type of plan. As discussed previously, the Charge for a paid loss plan will be greater than the Charge for an incurred loss plan due to the slower premium collection pattern for the paid loss plan. For the paid loss plan displayed on Exhibit 6, the Charge is 10.2% of standard premium. The magnitude of this change indicates that ignoring the credit exposure on certain accounts may result in loss of earnings. This analysis assumes that the insurance company does not require any collateral (e.g., LOC) for the paid loss plan. If the insurance company receives collateral, the credit exposure would be reduced and, therefore, the Charge would be lower.

## 6. PRICING SURETY BONDS

A surety bond is an agreement by one party (the surety) to be responsible to another party (the obligee) for the conduct of a third party (the principal). If the principal fails to fulfill its obligation under the bond, the bond indemnifies the obligee for loss sustained as a result of such default, up to the amount of the bond.

Several states require employers who self-insure their workers compensation exposure to provide the state with a surety bond. If the self-insured employer files for bankruptcy, the surety is required to make loss payments in place of the self-insured employer. The surety's payments are limited by the face value of the bond. Some bonds are written such that the obligee can require the surety to make payment for the full face of the bond.

For self-insured workers compensation bonds, the bond price is a function of both the financial health of the self-insured employer and the payment pattern for losses. The concepts used to price credit exposure are directly applicable to pricing surety bonds. This section outlines a procedure to price surety bonds based on Moody's bond default probabilities and the workers compensation payment pattern displayed on Exhibit 7.

Exhibit 8 displays the pricing of a workers compensation self-insured bond, for a one-year period, for a company with a Moody's senior unsecured bond rating of B. The first column of Exhibit 8 displays the number of years from the time the bond was written. Column 2 displays the liability, or expected total needed reserves, by year. The total needed reserves equal case reserves plus IBNR reserves. Column 3 is the *incremental* default probability based on Moody's bond default rates. The incremental probabilities are utilized because it is assumed that the bond would be utilized to the full extent of the liability. Column 4 is the expected default in the year (Column 2 multiplied by Column 3). Column 5 is the present value (at a 6% discount rate) of the expected default stream. The sum of Column 5 is the discounted loss cost for this bond.

Based on this analysis, the loss cost for an insured with a Moody's bond rating of B purchasing a $1.8 million bond is $267,600 or 15% of the face of the bond. Insurance companies typically require collateral on surety bonds (e.g., an LOC) and the collateral will reduce the credit exposure and, therefore, the price of the bond. The bond price should include a provision for company expenses, loss adjustment expenses, and profit. Therefore, without collateral, the bond price would exceed 15% of the face value of the bond. This analysis ignores any recovery from the principal in a bankruptcy proceeding; therefore, the loss cost is conservative.

## 7. OTHER APPLICATIONS

The financial methodology outlined in this paper has several additional applications including:

1. Providing insurance companies a method to establish a bad premium debt reserve for GAAP statements;
2. Providing insurance regulators a method to determine collateral requirements for self-insured employers;
3. Providing insurance companies a method of estimating surety bond loss ratios; and
4. Providing banks an additional method to price letters of credit.

*Estimation of Premium Bad Debt Reserve*

The procedures outlined in Section 6 can be utilized to establish a bad debt reserve for expected premium defaults on retrospectively rated policies for GAAP statements. Referring to Exhibit 5, the expected default amount for this insured at policy inception is $149,200 (the sum of Column 4). The bad debt reserve for this account would then be the "earned" (based on pro rata earnings) portion of the $149,200. The expected default amount for the insured three years after policy inception is $59,140 (the calculation is displayed on Exhibit 9). The expected default amount should be calculated based on the insured's current bond rating and estimated premium collection pattern. The bad debt calculation should be performed by account and policy year and the results summed in order to estimate the bad debt reserve.

*Collateral Requirements for Self-Insured Employers*

The methodology outlined in this paper may help insurance regulators in determining collateral requirements for self-insured employers. As displayed in Exhibits 4 and 5, the Charge for an insured with a Baa bond rating (investment grade) is 0.43%, while the charge for an insured with a B rating (non-investment grade) is 5.7%, of standard premium. Insurance departments may want to institute a procedure whereby the face value of the surety bond as a percentage of total liabilities varies with the self-insured employer's bond rating. The Iowa workers compensation self-insurance statute [3] incorporates a somewhat related concept. In Iowa, an estimate of the self-insured employer's unpaid claim liability is multiplied by a ratio which varies from 0.00 to 1.00 in order to determine the indicated amount of collateral for the self-insured employer. The ratio is computed based on three financial ratios. A ratio of 0.00 indicates that the firm is in strong financial condition. The ratio gradually increases to 1.00 as the indicated financial condition of the firm deteriorates. As a technical note, Iowa does not rely solely on this procedure to establish collateral requirements. For example, there is a minimum bond requirement of $200,000.

State insurance departments will need to estimate default probabilities for employers not rated by Moody's or use a procedure which is similar to the methodology used in Iowa. In addition, state insurance departments may want to require an actuarial opinion for an employer's self-insured liabilities. Insurance departments cannot determine an appropriate surety bond amount without an accurate estimate of the amount at risk, which frequently requires an actuarial opinion.

*Surety Bond Loss Ratio*

The procedures outlined in Section 6 can be utilized to determine a priori expected loss ratios for surety bonds. Referring to Exhibit 8, this insured has an expected loss of $286,000. Dividing by the premium charged gives the loss ratio. A similar calculation could be performed for all bonds written during the year and the results aggregated. A Bornhuetter-Ferguson projection method could then be used as one method to establish unpaid claim liabilities.

*Pricing LOCs*

The procedures utilized in Section 6 to price surety bonds can be used by banks to price LOCs collerateralizing insurance products. Typically, the insurance company will only draw down on the LOC in a bankruptcy situation and will draw the full amount of the LOC. Referring to Exhibit 8, the discounted loss cost of a $1.8 million LOC (supporting, for example, a deductible program) for an insured with a bond rating of B would be $267,600, or 15% of the LOC amount. This analysis assumes that the bank does not request collateral backing the LOC, and this analysis ignores expenses, profit, and any funds received in a bankruptcy proceeding.

## 8. CONCLUSION

As the number of financially related insurance products increases, it is important that financial theory be blended with actuarial pricing concepts. If the credit exposure associated with financially oriented insurance products is ignored, insurance company operating results will be negatively affected as insureds may default on some financial obligations (e.g., retrospective premium payments).

## REFERENCES

[1] Hope, Francis J., Discussion of Fitzgibbon, "Reserving for Retrospective Returns," *PCAS* LIII, 1966, p. 185.

[2] Moody's Special Report: "Corporate Bond Default and Default Rates," April 1990.

[3] Workers Compensation Self-Insurance for Individual Employers, Iowa Administrative Code, Title 191, Chapter 57, pp. 1-6.

# EXHIBIT 1

## STANDARD INCURRED LOSS RETROSPECTIVE PLAN
## REPORTING PATTERN

Assumptions:

Expected Losses = $1,800,000
Standard Premium = $2,000,000
Loss Conversion Factor ($C$) = 1.0
Tax Multiplier ($T$) = 1.0
Basic Charge ($B$) = 0.15
Lines of Insurance = Workers Compensation
General Liability
Automobile Liability
No Loss Limit

### REPORTING PATTERN

| Quarter | Incurred Losses as Percentage of Ultimate Losses |
|---|---|
| 4 | 38% |
| 6 | 57 |
| 10 | 73 |
| 14 | 85 |
| 18 | 90 |
| 22 | 94 |
| 26 | 97 |
| 30 | 99 |
| 34 | 100 |
| 38 | 100 |
| 42 | 100 |

## EXHIBIT 2

### STANDARD INCURRED LOSS RETROSPECTIVE RATING PLAN INSURED WRITTEN FOR A ONE-YEAR TERM*

| Number of Years from Inception of Policy | (Amounts in Thousands) Ultimate Premium | Collected** Premium | Credit Exposure |
|---|---|---|---|
| 1 | $2,100 | $2,000 | $100 |
| 2 | 2,100 | 1,326 | 774 |
| 3 | 2,100 | 1,614 | 486 |
| 4 | 2,100 | 1,830 | 270 |
| 5 | 2,100 | 1,920 | 180 |
| 6 | 2,100 | 1,992 | 108 |
| 7 | 2,100 | 2,046 | 54 |
| 8 | 2,100 | 2,082 | 18 |
| 9 | 2,100 | 2,100 | 0 |

* Assuming insured non-renews after the one-year period.

** Assumes a two-quarter lag in collecting/returning funds. Standard Premium equals $2,000; Basic Charge equals 15%; Expected Losses equals $1,800; Loss Conversion Factor and Tax Multiplier equal 1.00.

## EXHIBIT 3

### STANDARD INCURRED LOSS RETROSPECTIVE RATING PLAN
### INSURED WRITTEN FOR A THREE-YEAR TERM*
### (AMOUNTS IN THOUSANDS)

| Number of Years from Inception of Policy | Ultimate Premium | Collected** Premium | Credit Exposure |
|---|---|---|---|
| 1 | $2,100 | $2,000 | $ 100 |
| 2 | 4,200 | 3,326 | 874 |
| 3 | 6,300 | 4,940 | 1,360 |
| 4 | 6,300 | 4,770 | 1,530 |
| 5 | 6,300 | 5,364 | 936 |
| 6 | 6,300 | 5,742 | 558 |
| 7 | 6,300 | 5,958 | 342 |
| 8 | 6,300 | 6,120 | 180 |
| 9 | 6,300 | 6,228 | 72 |

* Assumes insured is written for three consecutive annual policy periods and then non-renews.

** Assumes a two-quarter lag in collecting/returning funds. Standard Premium equals $2,000; Basic Charge equals 15%; Expected Losses equals $1,800; Loss Conversion Factor and Tax Multiplier equal 1.00.

## EXHIBIT 4

### FINANCIAL RISK CHARGE
### FIRM'S BOND RATING Baa
### (AMOUNTS IN THOUSANDS)

| Number of Years from Inception of Policy | (1) Collected Premium | (2) Incremental Collected Premium | (3) Cumulative Default Probability | (4) Expected Default Amount | (5) Economic* Cost of the Default |
|---|---|---|---|---|---|
| 1 | $2,000 | — | — | — | — |
| 2 | 1,326 | $(674) | — | — | — |
| 3 | 1,614 | 288 | .009 | $2.59 | $2.17 |
| 4 | 1,830 | 216 | .013 | 2.81 | 2.23 |
| 5 | 1,920 | 90 | .017 | 1.53 | 1.14 |
| 6 | 1,992 | 72 | .022 | 1.58 | 1.11 |
| 7 | 2,046 | 54 | .026 | 1.40 | .93 |
| 8 | 2,082 | 36 | .031 | 1.12 | .70 |
| 9 | 2,100 | 18 | .035 | .63 | .37 |
| Total | | | | $11.66 | $8.65 |

Financial risk charge as a percentage of standard premium =

$$\frac{8.65}{2,000} = 0.43\%$$

* At a 6% discount rate. For example, $2.17 = 2.59/(1.06)^3$

## EXHIBIT 5

FINANCIAL RISK CHARGE
FIRM'S BOND RATED B
(AMOUNTS IN THOUSANDS)

| Number of Years from Inception of Policy | (1) Collected Premium | (2) Incremental Collected Premium | (3) Cumulative Default Probability | (4) Expected Default Amount | (5) Economic* Cost of the Default |
|---|---|---|---|---|---|
| 1 | $2,000 | — | — | — | — |
| 2 | 1,326 | $(674) | — | — | — |
| 3 | 1,614 | 288 | .159 | $45.79 | $38.45 |
| 4 | 1,830 | 216 | .189 | 40.82 | 32.33 |
| 5 | 1,920 | 90 | .211 | 18.99 | 14.19 |
| 6 | 1,992 | 72 | .230 | 16.56 | 11.67 |
| 7 | 2,046 | 54 | .244 | 13.18 | 8.77 |
| 8 | 2,082 | 36 | .255 | 9.18 | 5.76 |
| 9 | 2,100 | 18 | .260 | 4.68 | 2.77 |
| Total | | | | $149.20 | $113.94 |

Financial Risk Charge as a percentage of Standard Premium =

$$\frac{113.94}{2{,}000} = 5.7\%$$

* At a 6% discount rate. For example, $38.45 = 45.79/(1.06)^3$

## EXHIBIT 6

FINANCIAL RISK CHARGE
PAID LOSS RETROSPECTIVELY RATED PLAN*
FIRM'S BONDS RATED B
(AMOUNTS IN THOUSANDS)

| Number of Years from Inception of Policy | (1) Collected Premium | (2) Incremental Collected Premium | (3) Cumulative** Default Probability | (4) Expected Default Amount | (5) Economic*** Cost of the Default |
|---|---|---|---|---|---|
| 1 | $606 | $606 | — | — | — |
| 2 | 1,056 | 450 | .118 | $53.10 | $47.26 |
| 3 | 1,398 | 342 | .159 | 54.38 | 45.66 |
| 4 | 1,632 | 234 | .189 | 44.23 | 35.03 |
| 5 | 1,758 | 126 | .211 | 26.59 | 19.87 |
| 6 | 1,992 | 234 | .230 | 53.82 | 37.94 |
| 7 | 2,046 | 54 | .244 | 13.18 | 8.77 |
| 8 | 2,082 | 36 | .255 | 9.18 | 5.76 |
| 9 | 2,100 | 18 | .260 | 4.68 | 2.77 |
| Total | | | | $259.16 | $203.06 |

Financial Risk Charge as a percentage of Standard Premium =

$$\frac{203.06}{2,000} = 10.2\%$$

* Assumes that the plan converts to an incurred loss plan at 66 months and that all premium is collected at the end of the fiscal year.

** Assumes that due to financial underwriting the insured will not default in the first year. As a related point, if the insured defaulted during the first year, the insurance company may not be responsible for the full year's claim liability.

*** At a 6% discount rate.

Note: Exhibit 7 displays the payment pattern.

## EXHIBIT 7

### WORKERS COMPENSATION
### LOSS PAYMENT PATTERN
### (AMOUNTS IN THOUSANDS)

| Number of Years from Inception of Exposure | Ultimate Losses | Payout Pattern | Percentage Outstanding | Amount Outstanding |
|---|---|---|---|---|
| 1 | $1,800 | 17% | 83% | $1,494 |
| 2 | 1,800 | 42 | 58 | 1,044 |
| 3 | 1,800 | 61 | 39 | 702 |
| 4 | 1,800 | 74 | 26 | 468 |
| 5 | 1,800 | 81 | 19 | 342 |
| 6 | 1,800 | 85 | 15 | 270 |
| 7 | 1,800 | 88 | 12 | 216 |
| 8 | 1,800 | 91 | 9 | 162 |
| 9 | 1,800 | 94 | 6 | 108 |
| 10 | 1,800 | 97 | 3 | 54 |
| 11 | 1,800 | 100 | 0 | 0 |

## EXHIBIT 8

### Pricing a Surety Bond
### Firm's Bonds Rated B
### (Amounts in Thousands)

| (1) Number of Years from Inception of Exposure | (2) Liability* | (3) Incremental Default Probabilities | (4) Expected Default Amount | (5) Economic Cost of the Default** |
|---|---|---|---|---|
| 0 | $1,800 | .070 | $126.0 | $126.0 |
| 1 | 1,494 | .048 | 71.7 | 67.6 |
| 2 | 1,044 | .041 | 42.8 | 38.1 |
| 3 | 702 | .030 | 21.1 | 17.7 |
| 4 | 468 | .022 | 10.3 | 8.2 |
| 5 | 342 | .019 | 6.5 | 4.9 |
| 6 | 270 | .014 | 3.8 | 2.7 |
| 7 | 216 | .011 | 2.4 | 1.6 |
| 8 | 162 | .005 | 0.8 | 0.5 |
| 9 | 108 | .004 | 0.4 | 0.2 |
| 10 | 54 | .004 | 0.2 | 0.1 |
| 11 | 0 | .004 | 0.0 | 0.0 |
| Total | | | $286.0 | $267.6 |

* Assuming a one-year exposure period. The liability is equal to the total needed reserve. Total needed reserve equals case reserves plus IBNR, or ultimate losses less paid losses. The total needed reserves can be derived from Exhibit 7. This pricing is conservative as it assumes the insured goes bankrupt when the exposure is the largest.

** At a 6% discount rate. For example, $38.1 = 42.8/(1.06)^2$

## EXHIBIT 9

### Bad Debt Reserve Three Years After Policy Inception
### Firm's Bonds Rated B
### (Amounts in Thousands)

| Number of Years from Inception of Policy | (1) Collected Premiums | (2) Incremental Collected Premium | (3) Cumulative Default Probability | (4) Expected Default Amount |
|---|---|---|---|---|
| 3 | $1,614 | — | — | — |
| 4 | 1,830 | $216 | 0.070* | $15.12 |
| 5 | 1,920 | 90 | 0.118 | 10.62 |
| 6 | 1,992 | 72 | 0.159 | 11.45 |
| 7 | 2,046 | 54 | 0.189 | 10.21 |
| 8 | 2,082 | 36 | 0.211 | 7.60 |
| 9 | 2,100 | 18 | 0.230 | 4.14 |
| Total | | | | $59.14 |

* This is the probability that the insured will default in the next fiscal year.

Note: Assumes that the insured has not defaulted through Year Three. As a technical note, the firm's current bond rating, not the firm's bond rating at policy inception, is used in the calculation.

# WORKERS COMPENSATION EXPERIENCE RATING: WHAT EVERY ACTUARY SHOULD KNOW

WILLIAM R. GILLAM

*Abstract*

*Workers Compensation experience rating affects the distribution of billions of dollars of insurance premium. It is a large-scale application of actuarial science, one which has evolved since the very first days of the Casualty Actuarial Society. A fair amount of material exists on the theory underlying the plan, and some of that material is required reading for actuarial students. This paper tries to bridge the gap between theory and practice. Most of the insureds eligible for experience rating do not think in terms of overall performance of the plan. They may agree individual risk equity is fine—as long as it does not affect them adversely. Actuaries can help relate the theoretical underpinnings of the plan to the bottom-line effect on the individual insured.*

*The exposition begins with a discussion of performance testing. It then puts experience rating into the context of the premium transaction for a policy, then turns to a discussion of overall financial impact. The latter part of the paper details the activity necessary to administer experience rating, including the calculation of plan parameters, the assembly of experience data, and the promulgation of rating forms.*

*The paper is intended to be expository, describing concepts, vocabulary, and details of the plan from an actuarial perspective. Italics are used for emphasis, which may include introduction of a new word. Phrases that demonstrate usage, as well as honest-to-goodness quotations, are shown in quotes.*

## 1. PURPOSE OF EXPERIENCE RATING

The primary goal of experience rating is individual risk equity. This equity is not just between individual risks, but equity that pervades the relationship of insureds to their insurers and to society at large. Experience rating engenders an incentive for safety and enhances market competition. Even if not the primary goal of experience rating, these desirable results should be touted.

Equity is an actuarial concept: each insured should pay a rate correct for its inherent potential for loss. It is important to stress the word *potential*, for the experience rating modification (the *mod*) provides the means to adjust manual premium prospectively so that next year's rate will be tailored to fit the particular employer. This point is developed later.

The following passage from Dorweiler's prescient presidential address in 1934 explains in practical terms why consideration of risk experience can accomplish this goal:

> "The object of experience rating is to determine a more equitable rate for the individual risk based [to] a degree on the evidence presented by its own experience. It is recognized that individual risks within a classification are not alike and that there exist inherent differences due, for example in compensation, to variations in plants and premises, in operating processes, in the materials involved, in the management, in the morale of employees, in claim consciousness, and in the relation to the community. These differences are of such a nature that it is difficult to label them definitely and they cannot be associated with conditions measurable in advance. It is known, however, that variations in experience do exist in a way that definitely precludes ascribing all of them to chance. Experience rating is considered by many as the most practical method yet devised, or even suggested, of giving recognition to variations produced by such factors."[1]

In the parlance of risk theory, Dorweiler is talking about the *variance of the hypothetical means* or the *between variance* of risks within a class.

The greater the between variance, the greater the credibility of individual risk experience.

The original idea of applying experience rating to workers compensation was a good one. Quoting Dorweiler:

> "Compensation insurance, particularly, is subject to experience rating, for to a considerable degree the losses may be controlled and individuality of management reflected in the experience through the employer's ability to correct defective conditions and to enforce safe practices among employees by his potential power to dismiss or to withhold promotions. There are a few other lines, like employers' liability, workmen's collective, and automobile fleet collision, where the assured has similar power to affect losses." [1, p. 4]

## 2. MEASURING HOW WELL EXPERIENCE RATING WORKS

This question has been the subject of much study in the last 10 years, and the science has advanced far beyond what it was in the early years of the CAS.

The first principle is that experience rating be an accurate predictor of an individual insured's future experience. This is the basis of the "credibility conditions" and the empirical performance tests which are described below. It has some market implications, too, and these are now noted.

The mod is a prospective measure only. There is no intention to recoup past losses or rebate savings. The debit mod, for instance, gives an indication that more than manual premium will be needed to cover the expected losses of the particular employer next year.

The debit mod should not be thought of as a stigma. To decide between contractors bidding on a project, some owners erroneously eliminate those with mods higher than some threshold. The bid itself is far more relevant. Rating bureaus and regulators, who ought to know better, sometimes unfairly attach penalty programs to only those insureds with debits. These insureds have already paid their debt to society, so to speak.

The rationale for these programs is that a debit mod is an indicator of an unsafe operation. Although a debit mod may reflect poor safety habits, it just as often may be applicable to an employer with a good safety program who is a poor fit into the manual classifications. Contractors are especially difficult to classify. Each has a unique set of skills and an equally unique set of projects. Further, even though statistics show that poor prior experience is an indication of poor future experience, any single accident is probably a matter of pure chance.

Many aspects of a rating plan may affect its performance, but in the early days, the quality of the plan was thought to depend largely on "proper" credibility. Dorweiler proposed two conditions for correct credibility: *necessary* and *sufficient*. These are his words:

> "A necessary condition for proper credibility is that the credit risks and debit risks equally reproduce the permissible loss ratio. Also, if the proper credibility has been attained, each [random] subgroup of the credit and debit risks, provided it has adequate volume, should give the permissible loss ratio. While these conditions are necessary for a proper credibility of the experience rating plan, it does not follow that they are also sufficient. For a sufficient condition it would be required to establish that the risks within a group cannot be subdivided on any experience basis so as to give different loss ratios for the subdivisions, assuming the latter have adequate volume." [1, p. 11]

By "reproduce," Dorweiler was referring to the risk experience that would emerge in the prospective period; i.e., the losses during the time when the mod is applicable to risk premium. Given that rates today are rarely adequate, it is too much to ask that the two subgroups of risks equally reproduce the permissible loss ratio. So the necessary condition is that the two groups show equal loss ratios to standard premium in the prospective period. Since credibility is a function of risk size, the question must be posed for each size group: does the plan satisfy the necessary condition? The result of a plan satisfying this necessary condition is that insurers would find credit risks and debit risks equally desirable as insureds.

Even though Dorweiler calls the condition necessary, it may be too much to require of a plan. Early in the study leading to the 1990s revisions of the *Workers Compensation Experience Rating Plan*, it was seen that the existing plan did not perform well in simple tests (described below) based on the necessary condition, at least among the smallest and largest risks. This testing suggested the need for the changes that have recently been put into effect.

The sufficient condition is that there be no way to select subgroups of risks based on their experience that will produce significantly different loss ratios in the prospective period. It should be clear that it would be difficult, if not impossible, to test all possible a priori subdivisions of risks.

Dorweiler characterizes the sufficient condition as a goal, not a requirement. It can be used as a relative measure for judging how well a plan is working. He documented a simple method for testing credibility under this condition using the experience of 1931 experience-rated risks in New York [1, p. 12].

He first grouped risks by size. Within each size group, risks were stratified by the value of their modification. This is a "subdivision on an experience basis." Following the risks in each subdivision, or stratum, to the effective period of the modifications, he calculated the loss ratio of each stratum, first to modified (actual) premiums, then to manual premiums. The ideal result was that the loss ratios to manual premiums would track with the value predicted by the mod, but those to actual premiums would be nearly flat. A trend upward or downward in the loss ratios to actual premium across risks grouped by increasing value of the mod would show too little or too much credibility, respectively.

This author documents a refinement to this test used in developing the revised *Workers Compensation Experience Rating Plan* [2]. This is the *quintiles test* in which risks within an expected loss size range are grouped into five equal strata (quintiles) by the value of their modification. The first stratum contains the 20% of risks with the lowest mods, and so on. The subsequent ratios of actual to expected loss and actual to modified expected loss, are evaluated for each stratum. The test statistic for each size group is the variance of the modified ratios divided by the variance of the unmodified ratios. A low test statistic indicates a plan that has eliminated much of the between variance (in risk theoretic terms) or made risks of differing experience more equally desirable.

The use of this test in the development of the revised plan demonstrated that experience rating is very effective. Test statistics for the old plan were far lower than unity, showing that, even without proper calibration, experience rating is valuable. The test statistics for the smallest and the largest insureds indicated greater credibilities for the former and smaller for the latter.

Division into quintiles represents only one "subdivision on an experience basis," so the test does not check all possible subdivisions. Another test of experience rating, which is grounded in modern risk theory (thus avoiding conflict with Dorweiler), is the *efficiency test* as documented by Meyers [3]. In this test, the sample variance in loss ratios across all risks in a size group is compared for modified versus unmodified loss ratios. The statistic is the ratio of the former to the latter sample variances: lower statistics indicate better reduction in risk loss ratio variance.

All of these tests have been used to good effect in their historical contexts. Although they have been presented in a progression, each must be considered superior to pure judgment as a means of testing plan performance. Judgment is still essential, however, in the choice of data and interpretation of results.

## 3. PAYING FOR WORKERS COMPENSATION INSURANCE

### *Introduction*

Experience rating is an essential part of the workers compensation pricing system. All insureds over a certain premium size are eligible for experience rating or are *ratable*. "Eligible" is an euphemism, as there is no choice: the insured must accept its mod, be it a credit or debit (i.e., premium reduction or increase). The manual premium, based on the classification of the employer, is tailored by the mod so that it will better reflect the hazard inherent in the insured's operation.

*The Premium Transactions*

Figure 1 shows a simplified flow chart of the premium determination and bureau reporting process for a rated insured with a policy effective in July 1990. Loosely speaking, in the left columns are statistical plan data items generated by the transaction; in the center is the financial information pertaining to the insurance contract; on the right is an approximate time line.

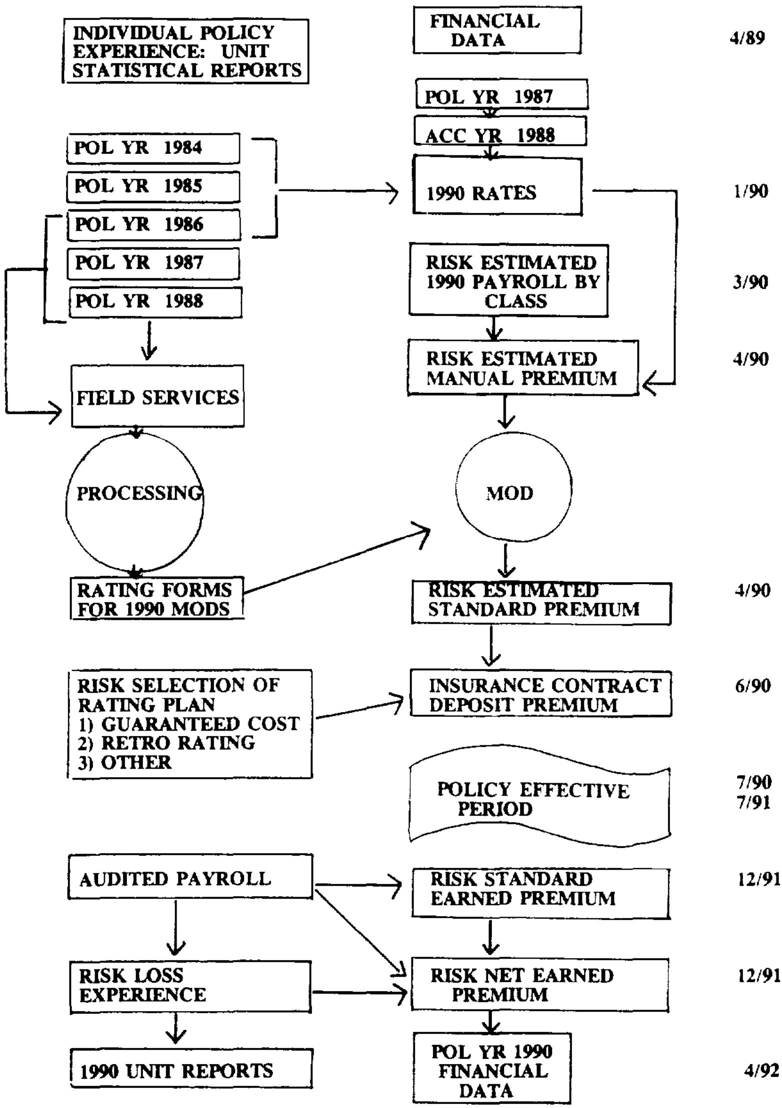
During the effective period of a compensation policy, the insured generates loss and payroll experience. This is coded by the insurer and submitted to the rating bureau according to its statistical plan. At the time of quotation for a policy beginning July 1990, experience from the policies incepting July 1986, July 1987, and July 1988 has been coded on unit report cards of the Workers Compensation Statistical Plan, commonly called the *Unit Plan*. A sample unit card for a 1988 policy is attached as Figure 2. More detail on the unit cards is found below.

The employer provides his/her prospective 1990 exposure data to the agent/broker to obtain insurance. The agent can compute the insured's *manual premium* as the extension of manual rates on estimated payroll (in hundreds of dollars). The agent can obtain a 1990 rating form, such as the one in Figure 3, apply the insured's modification to manual premium, and add certain other elements to compute the *standard premium*. Note that the form shows the expected loss rates (ELRs) and discount ratios (D-ratios), defined below, for the 1990 mods. Their derivation and use is explained in Sections 5 and 6.

The standard premium is a benchmark. It gives the best indication of the true underlying expected loss of the insured. It is the basis of any premium calculation plan. From it, the agent can compute the *guaranteed cost premium*, if the insured elects the simplest plan of paying for coverage up front. The insured may also elect retrospective rating, if an agreement can be reached with an insurer on a specific plan. There may be plans other than retrospective rating wherein the risk premium is affected by the emerged losses during the policy period. For example, losses of the particular insured or of some larger group of insureds can often be the basis of dividend plans on participating policies.

FIGURE 1

POLICY TRANSACTION FLOWCHART



# FIGURE 2

## SAMPLE UNIT CARD

### STATISTICAL PLAN UNIT REPORT

7-01-88 TO 7-01-89 Hypothetical Inc.

| CLASS CODE | EXPOSURE AMT | MANUAL RATE | PREMIUM | CLAIM # | ACCIDENT DATE | CLASS CODE | INJ | INDEM. | MED. | O/F | L |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3030 | 2655246 | 7.12 | 189054 | | | | | | | | |
| 8810 | 1256233 | .21 | 2638 | | | | | | | | |
| 8742 | 268048 | .50 | 1340 | | | | | | | | |
| 9807* | | .00 | 3668 | | | | | | | | |
| | | | | 140927 | 03-08-89 | 3030 | 5 | 1696 | 2907 | 0 | 1 |
| | | | | 138365 | 07-19-88 | 3030 | 5 | 2849 | 4120 | 1 | 1 |
| | | | | 044319 | 06-14-89 | 3030 | 9 | 389000 | 325000 | 0 | 4 |
| | | | | 039854 | 11-29-88 | 3030 | 9 | 25500 | 18000 | 0 | 1 |
| | | | | 039646 | 11-03-88 | 3030 | 9 | 18675 | 10332 | 1 | 1 |
| | | | | 039253 | 09-26-88 | 3030 | 6 | 0 | 2169 | 1 | 1 |
| | | | | 038253 | 07-18-88 | 3030 | 5 | 1783 | 2410 | 0 | 1 |
| | | | | | 2 | 3030 | 5 | 816 | 1142 | 1 | 1 |
| | | | | | 34 | 3030 | 6 | 0 | 6949 | 1 | 1 |
| TOTAL SUBJECT PREM. | | | 196700 | | | | | | | | |
| EXPERIENCE MOD. | | | .760 | | | | | | | | |
| TOTAL MODIFIED PREM. | | | 149492 | | | | | | | | |
| STD. 4179527 OTHR. 0 | | | 149492 | | | | | | | | |
| | | | | | | | | | | | |
| 0900 | | | 140 | TOTALS | 43 | | | 440119 | 373029 | | |

*CLASS CODE 9807 DENOTES EMPLOYERS LIABILITY EXCESS LIMITS PREMIUM

# FIGURE 3

## SAMPLE RATING FORM

### WORKERS COMPENSATION EXPERIENCE RATING

Risk ID #: 12345 Effective Date: 7/1/90
Name: Hypothetical, Inc. State: N

| | CLASS | ELR | D RATIO | PAY ROLL | EXP TOT LOSS | EXP PRM LOSS | EXP XS LOSS | ACT TOT LOSS | ACT PRM LOSS | ACT XS LOSS | STAT |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 07/01/86 | 3030 | 2.43 | 0.39 | 1,704,505 | 41,419 | 16,154 | 25,266 | 16,493 | 16,493 | 0 | * |
| 06/30/87 | 8742 | 0.25 | 0.34 | 62,400 | 156 | 53 | 103 | | | | |
| | 8810 | 0.11 | 0.41 | 1,670,166 | 1,837 | 753 | 1,084 | | | | |
| 07/01/87 | 3030 | 2.43 | 0.39 | 2,119,627 | 51,507 | 20,088 | 31,419 | 23,500 | 5,000 | 18,500 | O |
| 06/30/88 | 8742 | 0.25 | 0.34 | 86,389 | 216 | 73 | 143 | 13,000 | 5,000 | 8,000 | O |
| | 8810 | 0.11 | 0.41 | 1,346,461 | 1,481 | 607 | 874 | 10,686 | 10,686 | 0 | * |
| 07/01/88 | 3030 | 2.43 | 0.39 | 2,655,246 | 64,522 | 25,164 | 39,359 | 6,969 | 5,000 | 1,969 | F |
| 06/30/89 | 8742 | 0.25 | 0.34 | 268,048 | 670 | 228 | 442 | 714,000 | 5,000 | 28,500 | @O |
| | 8810 | 0.11 | 0.41 | 1,256,233 | 1,382 | 567 | 815 | 43,500 | 5,000 | 28,500 | @O |
| | | | | | | | | 29,007 | 5,000 | 24,007 | F |
| | | | | | | | | 4,603 | 4,603 | 0 | O |
| | | | | | | | | 2,169 | 2,169 | 0 | F |
| | | | | | | | | 4,193 | 4,193 | 0 | O |
| | | | | | | | | 8,707 | 8,707 | 0 | * |

| SRP | | WT (A) | BALLAST (B) | | (C) | (D) | | (E) | (F) |
|---|---|---|---|---|---|---|---|---|---|
| 335 | | 0.34 | 19,575 | 163,191 | 63,686 | 99,505 | 186,327 | 76,851 | 109,476 |

| CALCULATION | | PRIMARY | STAB. VALUE | EXCESS | TOTALS |
|---|---|---|---|---|---|
| | | (E) | (1−A)(D)+(B) | (A)(F) | (G) |
| | ACTUAL | 76,851 | 85,248 | 37,222 | 199,321 |
| | | (C) | (1−A)(D)+(B) | (A)(D) | (H) |
| | EXPECTED | 63,686 | 85,248 | 33,832 | 182,766 |

| EXP MOD |
|---|
| (G)/(H) |
| 1.09 |

Page: 1
Plan: RERP
Date: 04/30/90

* Total by policy year of all cases $2000 or less
@ Limited loss.
O Open claim
F Closed claim

In this case, the actual 1990 premium cannot be computed until after the policy period has elapsed. The first adjustment usually occurs 18 months after inception, or six months after the expiration of the policy. Payroll is audited so that authorized rates can be extended on payroll (in hundreds of dollars) by class. After the mod is applied, premium discounts are applied or the retrospective premium is calculated for the first (but not the last) time. The expense constant and other non-ratable elements are added. The end result is referred to as the net earned premium. At this point, the insured pays (or receives) the difference between this total and what has already been paid on deposit.

## 4. PREMIUM IMPACT OF EXPERIENCE RATING

A phenomenon of the 1980s has been the repetitive need for substantial workers compensation premium level increases. It is unusual that the need never seems to be satisfied. A state may grant a large rate increase as indicated in one year, only to face a filing for as large an increase the following year. The socioeconomic forces underlying this trend are the subject of much discussion and lie outside the scope of this paper.

In workers compensation, the phrase *premium level* should be understood as something apart from *rate level*. The rate level in workers compensation is a function solely of the manual rates before application of experience rating, schedule rating, premium discounts, retrospective rating, or dividend plans. A change in the premium discount plan, for instance, could engender a premium level change with no rate level change, or alternatively, could be accompanied by an offsetting rate level change to assure no premium level change.

Of particular relevance to this article is the nexus of manual premium and standard premium in ratemaking. Overall premium level needs are determined by the adequacy of total standard premium in a state. Inherent in standard premium is the experience rating *off-balance*, a term used to mean standard premium divided by manual premium or, put another way, manual premium weighted average modification.[1] In spite of the fact that

[1] California uses the term "off-balance" to denote a factor applied to rates (usually) to correct for the change in the average modification associated with a rate change.

the term "off-balance" sounds like it describes a *discrepancy* between manual premium and standard premium, it refers to the relation. Even when there is no discrepancy, we say there is "a unity off-balance."

The average modification is a function of the adequacy of rating values: ELRs and D-ratios. These values are calculated at the same time as new rates, but pertain to the experience period used in the associated ratings. Technical details on the computation of rating values may be found in Sections 5 and 6. In any case, the accuracy of these values should be judged on the aggregate totals by class (or at least by state) of emerged versus predicted loss. When the ratemakers are on target and rating values are accurate, the off-balance is usually near unity or a slight credit. The slight credit frequently results because insured risks large enough to be rated tend to have better experience than smaller risks.

Interestingly enough, even if the rating values are geared to be correct for the subpopulation of rated insureds, the off-balance is still frequently a credit. Testing at the National Council on Compensation Insurance (NCCI) has shown this to be the case. Researchers have been able to derive pairs of adjustment factors applicable to ELRs and D-ratios, in several rating years in several states, so that expected losses (total and primary) match, in aggregate, the actual emerged loss experience of rated risks. Recomputing all modifications with these adjustments, the (weighted) averages for rated risks are near unity, but definitely a credit. This suggests that the largest rated risks, those with the most credibility, have relatively better experience than smaller rated risks, at least at the first, second, and third reports. Dorweiler observed the same phenomenon in the larger risks of 1931:

> "These have more favorable experience and by virtue of their size under the experience rating plan receive larger credibility and therefore obtain credits which cannot be expected to be offset by an equal volume of less favorable experience on the smaller experience rated risks whose credibility is less." [1, p. 7]

In the NCCI committees, there is always some discussion about the effect of experience rating on premium, but no clear indication of what to do about it, if anything.

Contrary to suspicions, there is no ongoing scheme to use experience rating to assure premium adequacy. The calculation of needed premium level change at NCCI uses a loss-to-standard-premium ratio. This produces an indication that contemplates no change in off-balance between the experience period and the prospective period.

Our investigations show that, in most states, over the years the off-balance does not stray more than one or two points from unity. The all-too-common exceptions occur in states that allow rate adequacy to deteriorate for several successive years in times of increasing costs. The increasing off-balance reduces premium inadequacy. Since needed rate level changes are based on standard premium, indicated increases will be lower when the off-balance moves above unity. If adequate rates are approved, and the off-balance moves back toward unity, premium income may remain inadequate.

The production of rates and rating values is carefully monitored in the hope that changes in the off-balance will be minimal. Unfortunately, estimations of needed rates and rating values have proven to be difficult even if rate regulation is not a factor. The tendency is for too high a rate level (admittedly a rare occurrence) to result in a credit off-balance and too low a rate level (somewhat more common) to produce a debit off-balance. This is because ELRs are proportional to rates, and rates are presumed accurate. Without specific intent, then, experience rating can partially correct errors in class relativities or a rate level that is too low or too high. In the long run, there can be no net gain or loss but more stability results from experience rating.

## 5. CALCULATION OF EXPERIENCE RATING VALUES AND PLAN PARAMETERS

This section describes some of the actuarial tasks necessary to keep the plan functioning properly. The description is technical in nature, aimed primarily at the actuarial student.

Besides experience rating, another essential part of pricing workers compensation is ratemaking. There is ample material available documenting this part of the process that should be reviewed before entering the special realm described here. See Kallop [5] or Harwayne [6] for more detail.

We begin with a description of the data elements that underlie the pricing process.

*A. Elements of Ratemaking Data*

This section provides a brief background on the NCCI ratemaking procedures, especially those relevant to calculation of experience rating values.

The most basic element of the process is the *Workers Compensation Statistical Plan* (called Unit Plan here) of NCCI. The term "unit" refers to the fact that there is a separate report for experience on every policy and every state, evaluated annually to the fifth report. It is on this basis that the members of the NCCI report the data for experience rating and class ratemaking. NCCI summarizes the Unit Plan payroll and losses by risk for experience rating, and by classification within state for class ratemaking. Claims less than $2,000 may be (and usually are) summarized, but claims of a greater amount must be listed individually and categorized by injury type. Table 1 displays the codes for the types of injuries reported under the Unit Plan. For each injury type, medical and indemnity portions of a claim are reported separately. (Some states have modified this list.)

There are two compressions of this data made by the NCCI for ratemaking purposes. First, contract medical amounts are added to medical-only losses for use in most calculations. The second adjustment is a bit more complex. The permanent partial (PP) category, injury type 9, includes claims covering a wide range of values. For example, some claims coded as PP turn into life annuities not unlike permanent total (PT) cases. Other PP claims may be of short duration. Consequently, PP claims are separated into two categories: *major*, which becomes injury type 3, and *minor*, which becomes injury type 4.

TABLE 1
UNIT PLAN INJURY TYPE CODES

| Injury Type Code | Injury Type Description |
|---|---|
| 1 | (F) Fatal—Medical |
| | Fatal—Indemnity |
| 2 | (PT) Permanent Total—Medical |
| | Permanent Total—Indemnity |
| 9 | (PP) Permanent Partial—Medical |
| | Permanent Partial—Indemnity |
| 5 | (TT) Temporary Total—Medical |
| | Temporary Total—Indemnity |
| 6 | (M) Medical Only |
| 7 | Contract Medical |

The split is made by reference to a dollar amount called the *critical value*, which varies by state and over time. PP claims whose indemnity amount exceeds this value are considered major. The critical value is normally calculated as a part of the annual ratemaking process.

After these adjustments are made, the loss data is summarized for ratemaking purposes into three categories: 1) the indemnity portions of fatal, PT, and major are summed to one *serious* indemnity loss total; 2) the indemnity portions of minor and temporary total (TT) claims are summed to a *non-serious* indemnity loss total; and 3) the medical portions of all claims are summed to one medical loss total. Table 2 displays the groupings.

This categorization is central to the calculation of both rates and rating values. Actuaries perform the many loss manipulations associated with ratemaking (loss development, trend, law changes, multi-dimensional credibility) and compare the results with payroll by class to calculate loss costs by type for the serious, non-serious, and medical categories. These ratios are the projected partial pure premiums by category which underlie rates filed for the prospective period. Each class rate, then, has serious, non-serious, and medical components.

TABLE 2

CATEGORIZATION OF INJURY DATA FOR CLASSIFICATION RATEMAKING

| | |
|---|---|
| Fatal Indemnity (1) | *Serious* |
| Permanent Total Indemnity (2) | *Serious* |
| Major Permanent Partial Indemnity (3) | *Serious* |
| Minor Permanent Partial Indemnity (4) | *Non–Serious* |
| Temporary Total Indemnity (5) | *Non–Serious* |
| All Associated Medical (1,2,3,4,5) | *Medical* |
| Medical Only (6) | *Medical* |
| Contract Medical (7) | *Medical* |

*B. Elements of Experience Rating*

Several sets of values used in the NCCI Experience Rating Plan are revised as part of the regular rate filing process. *Plan parameters*, which vary by state and by size of the insured, are the *state reference point* (SRP), *weighting* (W), and *ballast* (B) values used in the rating formula. *Rating values*, applicable to individual insureds, vary by state and by classification. These are the expected loss rates (ELRs), and the discount ratios (D-ratios). Figure 3 shows a replica rating form. These values fit into the modification formula as follows:

$$M = \frac{A_p + W A_x + (1 - W) E_x + B}{E + B} .$$

where:

$M$ = the risk modification (mod);

$A$ = actual losses of the insured being rated ( $p$ = primary, $x$ = excess);

$E$ = expected losses of the insured being rated ( $p$ = primary, $x$ = excess);

$W$ = weighting value; and

$B$ = ballast value.

$E$ is calculated as the sum of expected losses by class:

$$E = \sum_{\text{all classes } i} E_i$$

$$= \sum_{\text{all classes } i} (\text{Payroll}_i \div 100) \times (\text{ELR}_i)$$

where $\text{ELR}_i$ is the expected loss rate for class $i$.

Then

$$E_p = \sum_{\text{all classes } i} D_i \times E_i$$

$$E_x = E - E_p$$

where $D_i$ is the discount ratio (D-ratio) for class $i$. The D-ratio is the estimated portion of ratable losses that will be primary.

The actual ratable losses, $A$, is the sum of the individual losses, indexed by $n$, each limited as described in the next section.

$$A = \sum_n A_n .$$

Each loss (occurrence) has a primary component:

$$A_{np} = \begin{cases} A_n \text{ if } A_n \leq L \\ L \text{ if } A_n > L \end{cases}$$

where $L$ is the primary loss limit. ($L$ is \$5,000 today.)

The actual primary and excess losses of the insured being rated are as follows:

$$A_p = \sum_n A_{np} \, .$$

$$A_x = A - A_p \, .$$

*C. Interrelationships in Calculations of Rates and Rating Values*

Current practice is to calculate the experience rating values at the same time and using the same data as used in the filed rates. It is important to point out that ELRs are usually quite different from the pure loss costs underlying prospective rates. This is discussed further in Subsection E. Some of the similarities are discussed in the rest of this section.

The filed loss costs (rates) provide a best estimate of the amounts necessary by class to cover losses (and expenses) for the future period when the loss costs (rates) will actually be used. Experience rating values pertain to losses that occur a year or more before the time when loss costs (rates) and ratings will be effective. Of interest here are the relative time frames of: 1) the underlying experience used in the rate filing; 2) the prospective effective period of the rates; and 3) the associated experience period to be used in the experience ratings applicable to the prospective period. These three time frames are not the same; reference to Figure 1 will help in visualizing the differences.

A key aspect of ratemaking is the practice of limiting individual losses to minimize volatility in rates and rating values. In ratemaking, the overall change in premium is estimated in one step (overall rate level) and, in a second step, is distributed among the various classes (class ratemaking). Capped losses are used in this second step to avoid distortions in class relativities due to the effect of unusually large losses. Loss dollars excluded by these caps must be spread back to all classes, respective of industry group, because rates are designed to be adequate for unlimited losses. As of this writing, NCCI uses a multiple of the average serious loss for limitations to single losses in class ratemaking.

In experience rating, losses that are similarly limited (ratable losses) enter the calculation of the experience modification. The limit applied to a single claim is 10% of the state reference point (SRP) which is defined below. This limiting value is called the state accident limit (SAL). There is a secondary cap on multiple claim occurrences of twice the SAL, or 20% of the SRP. There are special caps for losses incurred under the U.S. Longshoremen's and Harbor Workers Act and losses that are strictly employer's liability. The total disease losses for a policy are also capped at three times the SAL, plus 120% of the risk's total expected losses for the experience period. (There are specific rules in the *Experience Rating Plan Manual* [4] defining these experience periods.)

Another procedure lending stability to ratemaking, as well as to the calculation of experience rating values, is the imposition of swing or change limits. The rate (or rating value) for each class can change only by a specified percentage from one rate filing to the next. The average effect of the loss limits and change limits is spread to all classes in such a way that the selected rate level change is achieved.

### *D. Calculation of Plan Parameters*

#### *1. State Reference Point*

The SRP is an index of state benefits. It is used to calculate a value $G$ that is a scale factor for credibilities varying by state and is updated annually as part of the annual rate revision. The SRP is also used to calculate the SAL, as mentioned above.

The SRP is based on the *state average cost per claim* (SACC) for all types of claims. There is no per-claim limit on losses in this calculation, except that on employer's liability claims, which currently is $100,000. The SACC is calculated from the latest three years of undeveloped Unit Plan data. This data set is at the same maturity level as the experience period which will be used in the ratings. However, it is necessary to trend the average value from those data, since ratings will be using slightly more recent data. (In hypothetical State N, the length of the trending period is two years. Usually it would be between one year and 18 months. However, statutes in State N require that new rates be filed well in advance of the proposed effective date.) The trend rate is taken from the most recent countrywide Retrospective Rating Expected Loss Size Ranges update filing.

$$\text{SRP} = 250 \times \text{SACC}, \text{ rounded to the nearest } \$5{,}000.$$

$G = \text{SRP}/250{,}000.$

$G$ is rounded to the nearest 0.05.

Because of the potential for volatility in the data, and the normal effect of inflation, it is further stipulated that $G$ and SRP not be allowed to decrease from one year to the next, unless there is a significant benefit reduction. There also is a reasonability limit on the upward change, so that any changes over +20% will be investigated.

Exhibit 1 shows the calculation of the 1990 SRP for State N.

*2. Calculation of the* W *and* B *Values by the NCCI*

The $B$ and $W$ values are functions of $G$ and the expected losses, $E$, of a particular insured. First,

$$B = E\,[\ (0.1E + 2{,}570G)/(E + 700G)\ ],$$

subject to a minimum \$7,500.

Also, we define the intermediate value

$$C = E\,[\ (0.75E + 203{,}825G)/(E + 5{,}100G)\ ],$$

subject to a minimum of \$150,000.

$B$ and $C$ are the respective credibility constants $K_p$ and $K_e$ documented by Mahler [7].

Then

$$W = \frac{E + B}{E + C}\ .$$

$W$ is rounded to the nearest 0.01, with the requirement that it never increase for decreasing $E$. (This turns out to be a non-trivial programming challenge, although the effect on $W$ is at most a point or two for small risk sizes and certain $G$ values.)

# EXHIBIT 1

## Calculation of State Reference Point—State N

| Report | Policy Period | (1) Total Cases | (2) Total Incurred Losses | (3) Average Cost Per Case (2)/(1) |
|---|---|---|---|---|
| 1st | 1/86 - 12/86 | 165,250 | $195,722,802 | 1,184 |
| 2nd | 1/85 - 12/85 | 189,629 | 206,805,713 | 1,091 |
| 3rd | 1/84 - 12/84 | 188,074 | 196,806,051 | 1,046 |
| Total | | 542,953 | 599,334,566 | 1,104 |

| | |
|---|---|
| (4) Indicated State Reference Point = ( Total (3) $\times$ 250) | 276,000 |
| (5) Average Annual Trend = (exp ((0.09833) $\times$ (1.000))) | 1.103 |
| (6) Length of Trending Period in Years | 2.000 |
| (7) Trend Factor = (exp ((0.098333) $\times$ (6))) | 1.217 |
| (8) Trended State Reference Point = (4) $\times$ (7) | 335,892 |
| (9) Proposed State Reference Point (Rounded to the nearest 5,000) | 335,000 |
| (10) $G$ = (9)/250,000 (Rounded to the nearest 0.05) | 1.35 |

The above formulae are valid for all rated risks, with appropriate rounding for tabular presentation. $B$ is rounded to the nearest $500 \times G$ in the tables that apply to values of $E < 477{,}500 \times G$. For higher values of $E$, $B$ is rounded to the nearest dollar.

In particular, it should be noted that in all cases, $0 < W < 1$ and $B > 0$. Hence, no insured's rate is completely determined by its own experience.

The derivation of these formulae is explained elsewhere in the literature [2].

*E. Calculation of Rating Values—Major Steps*

*1. Calculation of Expected Loss Rates—Overview*

ELRs are used to calculate the insured's total expected loss, $E$, in the experience rating plan. The exposure base for the ELRs is \$100 of payroll, just as for rates. As is the case with manual rates, ELRs are calculated by the bureau for each class at the time a rate filing is made.

The class ELR should be proportional to the loss cost underlying the manual rate, but should be adjusted to the same level as the actual experience to be used in the calculation of the modification. Three major adjustments that must be made to rates to obtain ELRs stem from: 1) the loadings (if any) for expense, profit, tax and loss assessment; 2) differences in time frames; and 3) the fact that claims covered by the policy have no limit on size, but claims used in experience rating do.

The first adjustment is simple. Rates can be stripped of taxes, expenses, and profit by a single factor, the *permissible loss ratio* (PLR) in the filing. Even in states where loss costs are filed, there is a PLR, although it is close to unity.

The third adjustment is non-trivial. Its explanation is left to the detail described in the second step of the ELR calculation, below.

To better understand the second adjustment, hypothesize rating an individual policy effective 7/1/90, using rates and rating values effective 1/1/90. At the time the insured's prospective premium is being quoted, its experience for the policy effective 7/1/89 is not yet available. That policy is still in effect. Thus the policy effective 7/1/88 is the most recent one completed. In order to increase the credibility of the individual insured's experience, three years of data are used in the rating calculation, from the policies effective 7/1/86, 7/1/87, and 7/1/88. The manual rates used for the insured's 7/1/90 policy are intended to reflect ultimate costs in the 1990 policy year. In order to calculate ELRs which can be compared to the actual losses from the experience period, the prospective rates must be adjusted to loss levels prevailing during the three expired policy years (1986, 1987, and 1988). The rate filing contains well-documented analyses of trend, loss development, and benefit changes. This information is used to derive factors to adjust the rates.

In the example, suppose that from 1/1/86 through 12/31/89 the manual rate for our hypothetical insured's class was $4, and at 1/1/90 it went up to $5 due to a benefit change. Oversimplifying, suppose that losses do not develop after first report and there has been no trend in loss experience. Assume 70% of the premium is allocated for the payment of claims, the remainder being for expenses (27.5%) and profit (2.5%). Thus, $2.80 in claims are expected for every $100 of payroll for the policy periods used in the calculation of the experience modification; i.e., 7/1/86 through 6/30/89.

It is erroneous to compare the $2.80 ELR with the $5 rate and infer that only 56% of the premium is allocated to payment of claims. Actually $3.50, or 70% of the $5 rate, is necessary to pay for claims occurring under the 7/1/90 policy, because the claims under this policy will be paid at the new higher benefit level. But, since the insured's actual claims experience used in the calculation of its experience modification is at the old benefit level, the class expected loss rate used in the calculation must also be at the old level, namely $2.80, so that a fair comparison can be made.

While the foregoing illustration is a benefit change, the concept applies to anything that would make the past class average experience, reflected in the ELR, different from the future projected average experience underlying the manual rate. Loss development and trend can both be quite significant.

All of the above elements can make the manual rate considerably higher than the ELR. An ELR of 35% of the manual rate, for instance, is not unusual.

The three major steps of the actual ELR calculation are as follows. First, calculate a factor to reduce manual rates to pure premiums (at second report) on the same benefit level as the experience period used in ratings. This is called the ELR level factor.

Second, calculate the Hazard Group ELR factors and the ELRs by class. *Hazard Groups* are classes grouped according to relative severity. For each Hazard Group, calculate the average cost per case (indemnity and medical combined) for the three serious injury types: fatal, PT, and major. These average costs are used to remove the expected loss above the SAL from the ELR, so that the expected losses correspond to the limited or *ratable* losses used as the actual experience in the rating.

Using the ratios of the SAL to the average cost per case by serious injury type and Hazard Group, find the respective *excess ratios* (ratios of expected excess losses to total losses) from the former ELPF calculation. (See Harwayne [8] for details of this calculation.) There are three excess ratios for each Hazard Group. Using injury weights for the three serious types, also varying by Hazard Group, find a single weighted excess ratio for each Hazard Group. Multiply the ELR level factor from Step 1 by the Hazard Group adjustment factors, which are the complements of the weighted excess ratios.

The resulting four Hazard Group ELR factors are applied to rates respective of Hazard Group to produce the ELRs by class.

The third, and last, step is to check the ELRs for reasonableness. The technicians use checksheets to look for unreasonable changes. These checksheets are described in Section 6.

*2. Calculation of D-ratios—Overview*

D-ratios currently are calculated using the most recent single policy year of statistical plan data available. (Subsequent to this writing, a change to use of three years' data has been made. There are some associated changes, noted parenthetically below.) A policy year is labeled by the year in which the policies were written but extends over two calendar years, and the reporting, verification, and processing of unit data takes some months to complete. Consequently, a rate filing effective 1/1/90 would generally contain D-ratios based on statistical plan data from the 1987 policy year. Occasionally, 1986 data would be used.

*D-ratio Factors* (sometimes called partial D-ratios) are calculated for serious, non-serious, and medical losses. These factors are then weighted by the corresponding pure premium components of the class rates to produce D-ratios by class. The results are then checked for unusual changes in the average D-ratio.

## 6. CALCULATION OF RATING VALUES—DETAIL

*STEP 1—Calculation of the ELR Level Factor*

Exhibit 2 shows the worksheets for calculation of the ELR Level Factor. The explanation of the columns on Exhibit 2 follows:

Column 1: The three policy years of the experience rating period. The experience period ends one year before the prospective period of the new rates. This time period is usually later than the periods of statistical data actually available at the time rates are made.

Column 2: A factor to correct for the natural off-balance produced by experience rating. This factor compensates for the fact that, on average, insureds large enough to be eligible for experience rating have better loss experience than the average of the total population, including non-rated risks. This factor is the result of a broad-based analysis of data, but may well be subject to a more state specific procedure in the future.

Column 3: The factors necessary to take the third, second, and first reports that will be available for ratings to the benefit level of the proposed manual rates.

Column 4: Loss development factors to take third, second, and first reports to their ultimate level. These factors are calculated using statistical plan data to fifth report and financial data from fifth report to

# EXHIBIT 2

## CALCULATION OF ELR LEVEL FACTOR—STATE N

### STEP 1

### FACTORS DERIVED FROM LATEST RATE REVISION

| (1) Policy Year | (2) Off-Balance Adjustment | (3) Benefit Changes | (4) ELR Loss Development | (5) Composite Factor | (6) Expense Factor | (7) Product (2) × … × (6) | (8) Reciprocal 1 / (7) |
|---|---|---|---|---|---|---|---|
| 01/86-12/86 | 1.01 | 1.067 | 1.072 | 1.145 | 1.574 | 2.082 | 0.480 |
| 01/87-12/87 | 1.01 | 1.047 | 1.122 | 1.145 | 1.574 | 2.138 | 0.468 |
| 01/88-12/88 | 1.01 | 1.012 | 1.216 | 1.145 | 1.574 | 2.240 | 0.446 |
| | | | | | | ELF Level Factor | 0.465 |

ultimate. In using the financial data, it is assumed that all development beyond the fifth report is due to serious claims only.

Column 5: The composite factor for miscellaneous changes in the rates. Particularly important is the ratio of the proposed financial data loss ratio to that of the Unit Plan. This ratio includes the impact of trend between the dates of the statistical plan data and the effective period of the proposed rates.

Column 6: The reciprocal of the PLR.

Column 7: The product of Columns 2 through 6.

Note that Column 7 has three factors necessary to take the third, second, and first report loss costs to the same level as the proposed manual rates. Since we wish to perform the reverse operation, we take the reciprocals of the three values and record them in Column 8. The arithmetic average of the three reciprocals is at the foot of Column 8. This average is the ELR Level Factor which is carried into subsequent steps of the calculation.

Columns 3, 4, 5, and 6 are based on analysis of the actual data periods used in ratemaking. Exhibit 3 is the worksheet for these factors and shows how law amendment and loss development factors by injury type are weighted by policy period losses. The data used for weighting generally are not of the same policy period as the ones used for ratings. They are, however, put at the same stage of development. The development factors used in this exhibit were derived as part of the regular ratemaking procedure.

It should be noted that the filing schedule of State N has led to a minor inconsistency. The use of latest second, first, and first reports as weights usually matches the policy year of the experience rating period for two of the three years. In State N, however, 1985, 1986, and 1986 are used to weight experience period years 1986, 1987, and 1988. Usually the weights would be based on 1986, 1987, and 1987.

# EXHIBIT 3

## CALCULATION OF ELR LEVEL FACTOR—STATE N EFFECTIVE 01/01/90

| | |
|---|---|
| 1. (a) Financial Data Loss Ratio | 0.8250 |
| (b) Unit. Stat. Plan Loss Ratio | 0.7170 |
| (c) = (a) / (b) | 1.151 |
| 2. Other Adjustments+ 0.995 × 0.9997 | 0.005 |
| 3. ELR Composite Factor (1c) × (2) | 1.145 |
| 4. (a) Target Cost Ratio | .7115 |
| (b) Loss Adjustment Expense | 1.12 |
| (c) Permissible Loss Ratio 4a/4b | 0.6353 |
| (d) Reciprocal | 1.574 |

| 1/85-12/85 Loss Weights | A.F.* | | For: Policy Year 01/86-12/86 | | | |
|---|---|---|---|---|---|---|
| | | | 2nd/3rd Dev. Fac. | | 3rd to Ult. Dev. Fac. | |
| Death | 6,281,433 × 1.056 | | | | | |
| PT | 3,794,997 × 1.041 | | | | | |
| Major | 54,973,297 × 1.045 | 68,030,880 × | 1.131 | 76,942,925 | 1.130 | 86,945,505 |
| Minor | 23,816,205 × 1.048 | | | | | |
| TT | 23,155,932 × 1.057 | 49,435,203 × | 1.005 | 49,682,379 | 0.999 | 49,632,697 |
| Ser. Med. | 34,533,326 × 1.088 | 37,572,259 × | 1.036 | 38,924,860 | 1.180 | 45,931,335 |
| N. Ser. Med. | 60,250,523 × 1.088 | 65,552,569 × | 1.036 | 67,912,461 | 0.998 | 67,776,636 |
| Total | 206,805,713 | 220,590,911 | | 233,462,625 | | 250,286,173 |
| Benefit Charge = | 1.067 | | Loss Development = 1.072 | | | |

| 1/86-12/86 Loss Weights | A.F.* | | For: Policy Year 01/87-12/87 | | | |
|---|---|---|---|---|---|---|
| | | | 1st/2nd Dev. Fac. | | 2nd to Ult. Dev. Fac. | |
| Death | 7,187,873 × 1.028 | | | | | |
| PT | 2,920,892 × 1.026 | | | | | |
| Major | 38,424,936 × 1.026 | 49,809,952 × | 1.247 | 62,113,010 | 1.278 | 79,380,427 |
| Minor | 26,561,811 × 1.027 | | | | | |
| TT | 25,482,907 × 1.030 | 53,526,374 × | 0.966 | 51,706,477 | 1.004 | 51,913,303 |
| Ser. Med. | 27,226,941 × 1.067 | 29,051,146 × | 1.065 | 30,939,470 | 1.222 | 37,808,032 |
| N. Ser. Med. | 67,917,442 × 1.067 | 72,467,911 × | 1.065 | 77,178,325 | 1.034 | 79,802,388 |
| Total | 195,722,802 | 204,855,383 | | 221,937,282 | | 248,904,150 |
| Benefit Charge = | 1.047 | | Loss Development = 1.122 | | | |

| 1/85-12/85 Loss Weights | A.F.* | | For: Policy Year 01/88-12/88 |
|---|---|---|---|
| | | | 1st to Ult. Dev. Fac. |
| Death | 7,187,873 × 1.012 | | |
| PT | 2,920,892 × 1.012 | | |
| Major | 38,424,936 × 1.012 | 49,116,105 × | 1.594 |
| Minor | 26,561,811 × 1.012 | | |
| TT | 25,482,907 × 1.012 | 52,669,255 × | 0.970 |
| Ser. Med. | 27,226,941 × 1.013 | 27,580,891 × | 1.301 |
| N. Ser. Med. | 67,917,442 × 1.013 | 68,800,369 × | 1.101 |
| Total | 197,722,802 | 198,166,620 | |
| Benefit Charge = | 1.012 | | Loss Development = 1.216 |

+ Includes change in trend, minimum premium multiplier change, C&R decision, etc.

* To latest law level effective 1/1/90

*STEP 2—Calculation of Hazard Group Adjustment Factors*

Exhibit 4 is the start of the calculations underlying Step 2. The most recent first, second, and third reports from statistical plan data are used. The average cost per case is calculated for fatal, PT, and major claims. These three serious injury types are the likely source for claims exceeding the SAL. Medical and indemnity losses of three policy periods are added for each of the three injury types. The number of cases for the policy periods is also added by type. The average cost per case is thus calculated for three years of claims (medical and indemnity) at their respective maturities. An adjustment for trend similar to that used for the SRP is made to the losses.

Exhibit 5 shows the final calculations of the Hazard Group ELR Factors. These final calculations adjust for the limitation of losses in the experience ratings.

Line 1: The SAL, which is 10% of the SRP as calculated in the proposed rate filing.

Lines 2, 5, and 8: The average cost per case for fatal, PT, and major claims by Hazard Group. These are from Exhibit 4 and are calculated as part of the rate review.

Lines 3, 6, and 9: The ratio of the SAL from Line 1 by type to the average cost per case by type from Lines 2, 5, and 8.

Lines 4, 7, and 10: The excess ratios. These are the fractions of the pure premium for the portion(s) of individual loss(es) above the entry ratios on Lines 3, 6, and 9. The excess ratio tables are those used in the former ELPF calculation as described in Harwayne [8].

Line 11: The weights for fatal, PT, and major claims by Hazard Group.

Line 12: The fraction of the total pure premium expected to be above the SAL. It uses Line 11 to calculate a weighted average of Lines 4, 7, and 10.

Line 13: The Hazard Group adjustment factors.

Line 14: The ELR Level Factor from Step 1.

## EXHIBIT 4

### ELR Factor Worksheet—State N

#### Average Cost Per Case

| Injury Type | Statewide |
|---|---|
| Fatal | $ 89,073 |
| PT | 245,992 |
| Major | 45,537 |

Based on total losses and total claims, each by type, undeveloped, from the three-year experience period used for rates.

#### Hazard Group

| | I | II | III | IV |
|---|---|---|---|---|
| Fatal | $ 83,036 | $ 98,104 | $117,074 | $132,575 |
| PT | 249,377 | 292,786 | 302,965 | 381,999 |
| Major | 50,210 | 51,983 | 58,190 | 63,233 |

Uses countrywide Hazard Group Severity Relativities, adjusted to balance to state total, and appropriate severity trend from the ratemaking experience period to the experience rating experience period.

# EXHIBIT 5

## ELR FACTOR WORKSHEET—STATE N

| | | Hazard Group | | | |
|---|---|---|---|---|---|
| | | I | II | III | IV |
| 1. | 10% of Proposed State Reference Point | 33,500 | 33,500 | 33,500 | 33,500 |
| 2. | Average Fatal Cost | 83,036 | 98,104 | 117,074 | 132,575 |
| 3. | Ratio to Average for Fatal (1) / (2) | 0.40 | 0.34 | 0.29 | 0.25 |
| 4. | Excess Ratio for Fatal | 0.693 | 0.752 | 0.801 | 0.840 |
| 5. | Average PT Cost | 249,377 | 292,786 | 302,965 | 381,999 |
| 6. | Ratio to Average for PT (1) / (5) | 0.13 | 0.11 | 0.11 | 0.09 |
| 7. | Excess Ratio for PT | 0.941 | 0.954 | 0.954 | 0.966 |
| 8. | Average Major PP Cost | 50,210 | 51,983 | 58,190 | 63,233 |
| 9. | Ratio to Average for Major PP (1) / (8) | 0.67 | 0.64 | 0.58 | 0.53 |
| 10. | Excess Ratio for Major PP | 0.399 | 0.417 | 0.457 | 0.494 |
| 11. | (A) Fatal Weight Factor | 0.014 | 0.022 | 0.048 | 0.096 |
| | (B) PT Weight Factor | 0.022 | 0.030 | 0.040 | 0.058 |
| | (C) Major PP Weight Factor | 0.328 | 0.344 | 0.432 | 0.433 |
| 12. | Weighted Average Excess Ratio | 0.161 | 0.189 | 0.274 | 0.351 |
| 13. | Adjustment Factor = 1.0 − (12) | 0.839 | 0.811 | 0.726 | 0.649 |
| 14. | ELR Level Factor | 0.465 | 0.465 | 0.465 | 0.465 |
| 15. | Hazard Group ELR Factors | 0.390 | 0.377 | 0.338 | 0.302 |

Line 15: The Hazard Group ELR Factors. This line is the product of (3) × (14). One of the factors is applied to the rate of each class, depending on the Hazard Group assignment of the class, to produce the final class ELR.

*STEP 3—Expected Loss Rate Checksheet*

In addition to the standard calculation of the ELR as described above, the NCCI also has checksheets to identify cases where the ELR Factor (averaged over the Hazard Groups) changes significantly from the previous year. A more detailed investigation is conducted if the change is more than 10%. These checksheets are included in Exhibit 6.

Exhibit 6, Part 1 examines ELRs as a function of macroscopic changes in rates. Exhibit 6, Part 2 considers the microscopic changes by component to provide insight into the cause of ELR changes.

The checksheet in this exhibit shows a significant decrease in ELR factors over the previous year, which would normally result in an investigation of changes in State N. In this case, it was determined that the shift was due to a change to the experience rating plan formula, so that the change in ELR factors was justified. This can plainly be seen on Part 2, where the change in the excess ratio factor is 0.814, explaining most of the decrease.

*STEP 4—Calculation of D-ratio Factors*

The worksheet for this calculation can be found in Exhibit 7.

Line 1: Total Indemnity Losses (unlimited on a per-claim basis).

Line 2: Total Medical Losses (unlimited on a per-claim basis). Even though ratable losses are limited as described above, use of unlimited losses in the calculation of D-ratio factors provides a measure of conservatism. This is offset to some degree by the use of losses at first report, when severities are likely to be less skewed and D-ratios too high. (At the same time as the NCCI changes to a three-year experience period, it will begin using limited losses in this part of the calculation.)

# EXHIBIT 6
## PART 1

### EXPECTED LOSS RATE CHECKSHEET—STATE N
### EFFECTIVE DATE: 1/1/90

If amount on Line 8 is greater than 1.100 or less than 0.900, the underlying cause of the large change should be determined and brought to the attention of Rates Department Supervisor.

| | | | | |
|---|---|---|---|---|
| 1. | Effective Date of Last Change in ELRs | | | 01/01/88 |
| 2. | Rate Change Approved Effective on (1) | | | 1.159 |
| 3. | Rate Change Proposed Effective on (1) | | | 1.168 |
| 4. | Average ELR Factor Underlying Rate Proposal on Line 3<br>(Proposed ELR HGII + Proposed ELR HGIII) / 2 | | | 0.447 |
| 5. | Interim Rate Changes Approved | | | |
| | | Eff. | a. | 1 |
| | | Eff. | b. | 1 |
| | | Eff. | c. | 1 |
| | | Eff. | d. | 1 |
| 6. | CurrentAverage ELR Factor<br>$((4) \times ((3)/(2)))/((5a) \times (5b) \times (5c) \times (5d))$ | | | 0.451 |
| 7. | Proposed Average ELR Factor<br>(Proposed ELR HGII + Proposed ELR HGIII) / 2 | | | 0.358 |
| 8. | Change in ELR Factors (7) / (6) | | | 0.794 |
| 9. | Proposed Rate Change | | | 1.164 |
| 10. | Indicated Change in Expected Losses $(8) \times (9)$ | | | 0.924 |

# EXHIBIT 6

## PART 2

### EXPECTED LOSS RATE CHECKSHEET—STATE N
### EFFECTIVE DATE: 1/1/90

| | | (1) Last Approved Filing | (2) Proposed Filing | (3) Change (2) / (1) |
|---|---|---|---|---|
| 1. | Benefit Changes | | | |
| | a. 3rd Report | 1.084 | 1.067 | xx |
| | b. 2nd Report | 1.064 | 1.047 | xx |
| | c. 1st Report | 1.031 | 1.012 | xx |
| | d. Average | 1.060 | 1.042 | xx |
| 2. | Loss Development | | | |
| | a. 3rd Report | 1.052 | 1.072 | xx |
| | b. 2nd Report | 1.080 | 1.122 | xx |
| | c. 1st Report | 1.175 | 1.216 | xx |
| | d. Average | 1.102 | 1.137 | xx |
| 3. | Off-Balance | 1.01 | 1.01 | xx |
| 4. | Composite Factor All Reports | 1.144 | 1.145 | xx |
| 5. | Financial Data Loss Ratio | 0.8115 | 0.825 | xx |
| 6. | USP Loss Ratio | 0.720 | 0.717 | xx |
| 7. | Loss Ratio Factor (7) / (8) | 1.127 | 1.151 | xx |
| 8. | Profit and Exp. Factor All Reports | 1.570 | 1.574 | xx |
| 9. | Reciprocal of the Combined Effect of these Factors** | | | |
| | a. 3rd Report | 0.483 | 0.480 | xx |
| | b. 2nd Report | 0.480 | 0.468 | xx |
| | c. 1st Report | 0.455 | 0.446 | xx |
| 10. | Comparable ELR Level Factors Average (9) | 0.473 | 0.465 | 0.983 |
| 11. | Excess Ratio Factor | 0.945 | 0.769# | 0.814 |
| 12. | Overall Change in ELR Factors (10) × (11) | xx | xx | 0.800 |

** $1/[((1) \times (2) \times (3) \times (4) \times (8)]$

# From Exhibit 5, Line 13, (HGII + HGIII) / 2

# EXHIBIT 7

## CALCULATION OF DISCOUNT RATIO FACTORS—STATE N

| | | (A) Serious | (B) Non-Serious | (C) Medical | (D) Total |
|---|---|---|---|---|---|
| 1. | Total Indemnity Losses | 49,351,958 | 52,329,922 | xxx | xxx |
| 2. | Total Medical Losses | 27,437,361 | 45,660,200 | 22,648,393 | 95,745,954 |
| 3. | Total Losses (1) +(2) | 76,789,319 | 97,990,122 | 22,648,393 | 197,427,834 |
| 4. | Total Primary Losses | 7,124,224 | 55,450,538 | 22,028,546 | xxx |
| 5. | Estimated Indemnity Primary (4) × ((1)/(3)) | 4,580,876 | 29,610,587 | xxx | xxx |
| 6. | Estimated Medical Primary (4) - (5) | 2,543,348 | 25,839,951 | 22,028,546 | 50,411,845 |
| 7. | Primary for D-ratios A & B = (5), C = (6D) | 4,580,876 | 29,610,587 | 50,411,845 | xxx |
| 8. | Total Losses for D-ratios A & B = (1), C = (2D) | 49,351,958 | 52,329,922 | 95,745,954 | 197,427,834 |
| 9. | First Report Partial D-ratios (7) / (8) | 0.093 | 0.566 | 0.527 | xxx |
| 10. | First Report Loss Distribution (8) / Sum of (8) | 0.250 | 0.265 | 0.485 | 1.000 |
| 11. | WCSP Experience Adjusted, On Level | 290,981,723 | 178,348,670 | 390,464,152 | 859,794,545 |
| 12. | Adjusted Experience Distribution (11) / Sum (11) | 0.338 | 0.208 | 0.454 | 1.000 |
| 13. | Final D-ratio Factors (9) × (10)/(12) | 0.069 | 0.721 | 0.563 | xxx |

Line 3: Total serious and non-serious losses. These include associated medical amounts.

Line 4: Primary Losses. These are the first \$5,000 of each claim. (When other changes described above are made, the split point used in this calculation will be deflated over the appropriate year or two by an appropriate severity trend.)

Lines 5, 6, 7, 8: The denominators of the D-ratio factors for serious and non-serious losses will be indemnity losses only. Medical will be all medical, as can be seen in Line 8. This is appropriate because the pure premium weights are serious indemnity, non-serious indemnity, and total medical. Lines 5 through 7 adjust the primary losses in Line 4, which are on a combined basis, to a more proper basis.

Line 9: The first report D-ratio factors. The ratios are, from left to right, primary to serious indemnity, primary to non-serious indemnity, and primary to total medical.

Lines 10, 11, 12: An adjustment is necessary because the pure premiums used to weight the partial D-ratios contemplate a future distribution of losses into serious, non-serious, and medical. Rather than compute component pure premiums by class for the earlier time period, it works well to put the distribution change adjustment in the partial D-ratios.

Line 13: The final D-ratio factors. These are the partial D-ratios from Line 9, adjusted by the distribution change Line 10 / Line 12.

The D-ratio for class XXXX in State N is:

$$\frac{\text{Serious Pure Premium (class XXXX)}}{\text{Total Pure Premium (class XXXX)}}$$

$\times$ Serious D–ratio Factor

$$+\ \frac{\text{Non–Ser. Pure Premium (class XXXX)}}{\text{Total Pure Premium (class XXXX)}}$$

$\times$ Non–Serious D–ratio Factor

$$+\frac{\text{Medical Pure Premium (class XXXX)}}{\text{Total Pure Premium (class XXXX)}}$$

$$\times \text{ Medical D–ratio Factor}$$

*STEP 5—D-ratio Checksheet*

Exhibit 8 shows the D-ratio checksheet. The average D-ratio for all classes should not decrease from the past one by more than 10 points, or increase at all. The normal change expected from inflation is a decrease. Greater changes would be investigated. The maximum D-ratio is 0.90 and the minimum is 0.25 for the revised experience rating plan.

Once calculated, these D-ratios are included with the rate filing and go into effect if and when the new rates are approved.

## 7. ADMINISTRATION OF EXPERIENCE RATING

The accumulation and processing of the experience data for every individual insured is a remarkable undertaking and, when it goes smoothly, it is an often-forgotten function of the NCCI and other bureaus. Promulgation of a mandatory rate modification for every insured of qualifying size is also a monumental task, though it is seldom forgotten. The players in this piece—insurers, insureds, workers, bureaus, and regulators—all have self-interested points of view about the process. Actuaries should have a belief in the objective intent of this measurement of individual risk quality and its promotion of a correctly functioning market.

In order to calculate the modification factor for an individual employer, data from three annual policies is usually required. Data may come from more than one state, more than one insurer, and more than one medium. The move to electronic media has been slow, and many reports are still collected on unit cards. For each insured there is one card per state, per insurer, per year, and per evaluation. All these cards must be organized so that the data for each insured are in one place to do the rating.

A sample unit card for the first evaluation of a policy is shown in Figure 2. The card shows payroll, rates, premium, and loss for a single

# EXHIBIT 8

## DISCOUNT RATIO (D-RATIO) CHECKSHEET—STATE N

If the value of line (9) is ≥ 1.000 or ≤ 0.900, the underlying cause should be determined and brought to the attention of the supervisor.

| | A. Current Values Effective 01/01/88 | | | | B. Proposed Values Effective 01/01/90 | | | |
|---|---|---|---|---|---|---|---|---|
| | Serious | Non-Serious | Medical | Total | Serious | Non-Serious | Medical | Total |
| 1. D-ratio Factors | 0.271 | 1.175 | 0.253 | xx | 0.069 | 0.721 | 0.563 | xx |
| 2. Total Adjusted Losses For All Industry Groups | 204,002,232 | 133,319,839 | 345,267,373 | xx | 282,506,527 | 173,154,049 | 401,299,231 | xx |
| 3. Payroll/$100 | xx | xx | xx | 682,220,187 | xx | xx | xx | 811,960,370 |
| 4. Average Pure Premium (2) / (3)* | 0.299027 | 0.195421 | 0.506094 | xx | 0.347931 | 0.213254 | 0.494235 | xx |
| 5. Effect by Parts Used in Filing | | | | | | | | |
| a. Law | 1.025 | 1.030 | 1.000 | xx | 1.014 | 1.014 | 1.000 | xx |
| b. Trend | 1.071 | 1.071 | 0.960 | xx | 1.016 | 1.016 | 0.973 | xx |
| c. Assessment | 1.000 | 1.000 | 1.000 | xx | 1.000 | 1.000 | 1.000 | xx |
| d. Total | 1.098 | 1.103 | 0.960 | xx | 1.030 | 1.030 | 0.973 | xx |
| 6. Adjusted Pure Premium (4) × (5d) | 0.328332 | 0.215549 | 0.48585 | 1.029731 | 0.358369 | 0.219652 | 0.480891 | 1.058912 |
| 7. Average D-ratio Sum ((1) × (6))/Total of (6) | 0.086409 | 0.245958 | 0.119371 | 0.451738 | 0.023352 | 0.149558 | 0.255679 | 0.428589 |
| 8. D-ratios for: | | | | | | | | |
| a. Code 2041 (HGI) | xx | xx | xx | 0.42 | xx | xx | xx | 0.53 |
| b. Code 7380 (HGIII) | xx | xx | xx | 0.43 | xx | xx | xx | 0.38 |
| c. Code 7405 (HGIV) | xx | xx | xx | 0.54 | xx | xx | xx | 0.43 |
| d. Code 8742 (HGIII) | xx | xx | xx | 0.43 | xx | xx | xx | 0.40 |
| e. Code 8810 (HGII) | xx | xx | xx | 0.42 | xx | xx | xx | 0.44 |
| 9. Expected Average Change in D-ratios: (7B) / (7A) = | 0.948756 | | | | | | | |

*These pure premiums reflect the average only if each class code in a state is 100% credible, but they can be used for comparative purposes.

state. The modification shown is the one applicable to the premium of the policy reported. The experience of that policy will be used to calculate a modification applicable to subsequent policies.

A somewhat simplified form is used for subsequent evaluations, so that only those items that change, such as loss reserves, need to be updated. This form makes things easier for the insurer, but requires careful processing by the administrative personnel to assure that the updated totals are correct. In addition to annual update cards, the staff handles numerous off-anniversary corrections and replacements for cards already submitted.

In any case, there could easily be enough activity on the account of a large intrastate insured so that six unit cards would be required to do a rating. A 1990 rating would need one card for the first report (1988 as of 18 months); two cards for the next most recent policy year at second report (1987 as of 18 months, updated at 30 months); and three cards for the most mature policy year. In practice, there are seldom so many cards used to do an intrastate rating, so that the average number of cards is less than four. As a rule, these insureds tend to be smaller and enjoy a large number of loss-free years, which do not need to be updated. In addition, many newly formed businesses grow so fast that their first rating is based on only two completed policy years.

When rating an interstate insured, the potential number of cards is multiplied by the number of states with subject payroll. There also may be a variety of subsidiary operations, each with its own compensation insurer. The average number of cards for these insureds is more than 15, with some insureds requiring many more than average. Collecting these cards and determining if all states and all years are in hand is a non-trivial activity, occasionally causing delay in the release of modifications. This delay may be due in part to the size of the clerical undertaking. It is also caused by the slowness of some insurers, which in some cases may stem from the low incentive to submit unit cards for non-renewing insureds.

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# STOCHASTIC CLAIMS RESERVING WHEN PAST CLAIM NUMBERS ARE KNOWN

THOMAS S. WRIGHT

*Abstract*

*This paper addresses the problem of estimating future claim payments when two run-off triangles are available: one of the number of claims, the other of total amounts. Each single claim can have partial payments included in the total for several development periods. The method does not require additional information, such as measures of exposure and claims inflation. The approach adopted is to model the mean claim amount as a function of operational time, using generalized linear models. Techniques are described for fitting and comparing a number of models of this type, and for predicting the total of future claims from the best fitting model. Formal statistical tests are used for comparing models. It is shown how the root-mean-square (RMS) error of prediction can be calculated, making due allowance for modelling error and random variation in both the number and amounts of future payments. Models are formulated to make explicit allowance for claims inflation and partial payments. Assumptions are minimal, and diagnostic techniques are described for checking their validity in each application. Numerical examples are given.*

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## 1. INTRODUCTION

*Background, Notation, and Overview*

This paper complements a previous work by the author [10]. That paper, like many others on stochastic claims reserving in property/casualty insurance, deals with methods applicable when the past run-off of the number of claims is not known (a common situation for actuaries in the U.K.). This paper addresses the problem of claims reserving when at least two run-off triangles are available: one of the number of claims, the other of claim amounts. These primary triangles may be:

(a) the number of claims closed, and the total of all payments on all claims closed (partial payments assigned to the development period of settlement);

(b) the total number of payments, including partial payments, and the usual paid claims triangle (with each partial payment assigned to the development period in which it was made); or

(c) the number of claims closed, and the usual paid claims triangle (with each partial payment assigned to the development period in which it was made).

Of these possibilities, (a) and (b) are the simplest to model, and are considered first. They are equivalent to each other as far as the modelling and prediction methods proposed in this paper are concerned. Later, it is shown how basically the same methods can be applied in situation (c), which is more common in practice.

If $w$ is used to label origin (i.e., accident, report, or policy) years, and $d$ to label development periods, the two run-off triangles can be denoted $Y_{wd}$ and $N_{wd}$, respectively, where $w$ runs from 1 to $W$, and $d$ runs from 0 to $T-1$. This notation is used for the incremental, rather than the cumulative, run-off. For example, in case (a), $N_{wd}$ is the number of claims closed in development period $d$ of origin year $w$, and $Y_{wd}$ is the total of payments on these claims made in development period $d$ and previous development periods. In case (c), $N_{wd}$ is the number of claims closed as in (a), but $Y_{wd}$ is the total amount of all partial payments made in development period $d$ of origin year $w$, including partial payments on claims not yet closed. In both cases (a) and (c), $N_{wd}$ should exclude claims settled with no payment if possible. Such claims do not contribute to $Y_{wd}$, so their inclusion in $N_{wd}$ introduces an undesirable element of additional random variation.

As in [10], the methods described here do not involve an assumption that the run-off pattern has been the same for all origin years; indeed, the shape of the run-off may be different for each origin year. Similarly, there is no assumption that the claim size distribution is the same for all development periods. It is common for larger claims to take longer to settle so that the mean claim size increases with $d$ . Higher moments of the distribution may also depend on $d$. The methods allow projection as far into the future as is necessary, not limited by the extent of the data. The data triangles may have missing values. This does not cause any problems provided the total number of data points is sufficient to fit an adequate model. The occasional negative values which occur in real data can also be handled without special treatment.

The approach used in [10] is to derive a model for the known data $Y_{wd}$ from more basic models for the unknown quantities $N_{wd}$ and the individual claim amounts $X_{wd}$ (also unknown). The assumptions of the models for $N_{wd}$ and $X_{wd}$ are then checked indirectly by applying diagnostic tests to the resulting model for $Y_{wd}$. If satisfactory, the model that is fitted to the $Y_{wd}$ is used to project into the future. The logical progression of this approach to situations where $N_{wd}$ is known would be to formulate models for $N_{wd}$ and $X_{wd}$ separately (as before), but then to test each of these models directly from the data. This should allow good models to be found for each of these components. These models could then be used to project $N_{wd}$ and $X_{wd}$ separately, and the projections combined into projections for the total payments $Y_{wd}$. However, the calculation of standard errors for predictions of $Y_{wd}$ obtained in this way is complex. Hayne [3] deals with the case when, for each origin year, the distribution of future claim amounts $X_{wd}$ does not depend on the development period $d$. The intention in this paper is to remove this restriction (as, for example, when larger claims tend to take longer to settle than smaller claims). In this case, the calculation of standard errors for the predictions would be extremely complex using real-development time, because the precise time of settlement of each future claim (hence, the appropriate claim size distribution) is uncertain. The problem is simplified in this paper by making use of the concept of *operational time*. This concept seems to have been used first in claims reserving by Reid [7] and later taken up by Taylor [8, 9], but a fresh approach, including a number of innovations, is proposed in this paper.

Operational time, $\tau$, is defined as the proportion of all claims closed to date. Thus, for each origin year, operational time starts at 0, and increases ultimately to 1. If the individual claim amounts $X$ can be modelled as a function of operational time $\tau$ rather than development time $d$, then there is no need for a separate model of the number of claims. This is because the dependence of the number of claims on operational time is known exactly: it follows trivially from the definition of operational time. Projections of future payments $Y$ can therefore be obtained from the model of claim size $X_{w\tau}$ alone, and the problem is an order of magnitude simpler than when $X_{wd}$ and $N_{wd}$ are both projected.

The data $N_{wd}$ is used at three points in the operational time approach:

- to estimate the ultimate number $M_w$ of claims for each origin year $w$ (obviously, numbers of claims reported are also useful for this estimation, if available);
- to calculate a triangle of operational times (for use as the explanatory variable in the claim size model); and
- to calculate the observed mean claim sizes $Y_{wd}/N_{wd}$ (for use as the dependent variable in the claim size model).

There are often substantive reasons for expecting the size of individual claims to depend more on operational time than on development time. The main reason is that changes in claim handling procedures may affect the actual delay to settlement but should not affect the size of claims. The plausibility of such arguments need not be left entirely to judgment. It is possible to use the figures themselves to verify this basic hypothesis of operational time methods. This is shown in Appendix B.

*Summary of Later Sections*

Sections 2 through 6 deal with circumstances (a) or (b); that is, when the claim counts triangle gives the number of individual components of each element of the claim amounts triangle. Section 7 describes special procedures and enhancements to the method of earlier sections which may be necessary for case (c). All sections conclude with a numerical example. The data for the examples have been taken from Berquist and Sherman [1], and are reproduced in Appendix A. The data are actually of type (c), so the methods of Sections 2 through 6 are not wholly appropriate. They are applied purely for illustrative purposes. Section 7 also contains an analysis of the data used by Taylor [9]. They are also of type (c), and are given in Appendix A.

Section 2 gives a complete account of the method applicable in cases (a) or (b), under several simplifying assumptions. The assumptions are unrealistic but are made initially in order to simplify the presentation. Sections 3 through 5 show how the assumptions can be relaxed. The assumptions used in Section 2 are that:

1. The expected claim size at each operational time $\tau$ is the same for all origin years, after allowing for claims inflation. In other words, the mean claim amount in real terms is a function of $\tau$ but not $w$. It can therefore be denoted $m_\tau$. (In the presence of inflation, the mean claim amount will depend on $w$ also. See Section 4.)

2. The coefficient of variation of individual claim amounts is the same for all operational times $\tau$, that is:

$$\text{Var}\,(X_\tau) = \varphi^2 \cdot m_\tau^2\,, \tag{1.1}$$

   where $X_\tau$ is the size of an individual claim at operational time $\tau$, and $\varphi$ is the coefficient of variation.

3. The data $Y_{wd}$ have been adjusted for inflation so the triangle is in constant money terms.

4. The ultimate number of claims $M_w$ is fully known (that is, there is no uncertainty) for each origin year $w$.

Assumption 1 is the only condition that must hold in order to predict future claim payments using methods proposed in this paper. Even Assumption 1 needs not be an assumption in the sense that its validity can be checked using the data themselves. (This is the subject of Appendix B.) Section 3 describes how Assumption 2 can be tested and relaxed if necessary. Assumption 3 cannot often be valid in practice because the rate of claims inflation is usually unknown. Section 4 shows how the rate of inflation can be estimated and removed from the data at the same time as fitting the claim size model, rendering preadjustment unnecessary. Assumption 4 only holds in practice if the origin years are report years. Often with accident or policy years, there will be considerable uncertainty in the estimates of ultimate numbers $M_w$. Section 5 describes how this uncertainty can be taken into account.

The main point of Sections 2 through 4 is to discover how the mean and the variance of an individual claim $X_\tau$ depend on the operational time $\tau$. When this has been achieved, since we know the operational time $\tau$ of every future claim (from the definition of operational time), we can find the expected value and the variance of every future claim. This, in turn, can be used to find the expected value and the variance of the total of all future claims.

A broad outline of how predictions can be made from a fitted operational time model was provided in the previous paragraph. (Here the term "model" refers to the mathematical representation of the relationship between operational time and the mean and variance of $X_\tau$.) The details of prediction are given in Appendices D, E, and F. These are more complex than would be expected from the comments above: first, because of parameter uncertainty (that is, the fitted model will not be exactly right); second, because uncertainty in the ultimate numbers $M_w$ implies uncertainty in the operational time of each future claim; and, third, because of uncertainty about future claims inflation. Section 6 shows how uncertain future claims inflation can be included in the predictions obtained from an operational time model. This is necessary to comply with standard reserving practices.

All the models proposed in this paper are *generalized linear models*. Such models can be fitted using an algorithm known as Fisher's scoring method. This is the algorithm used in the well-known statistical package GLIM, which was used for all the numerical examples in this paper. Fisher's scoring method maximizes the so-called *quasi-likelihood*, or equivalently, minimizes the *deviance*. The deviance can be regarded as a generalization of the weighted sum of squared differences between observed and fitted values. The weights are determined from the assumed variances of the observations. The generalization is that the variance of each observation may be a function of its mean, which, of course, is not known. The purpose of fitting the model is to estimate the mean. Fisher's scoring method sometimes gives conventional maximum likelihood estimates. In other cases, it gives estimates which have all the desirable properties of maximum likelihood estimates (asymptotically unbiased, efficient, and Normal) although they may not actually be maximum likelihood estimates. An approximate variance/covariance matrix for the parameter estimates is also produced by the algorithm. Further details are not given here as they are well documented elsewhere: the theory in McCullagh and Nelder [5], briefly in Hogg and Klugman [4], and practical aspects in the GLIM manual [6]. The application of GLIM in actuarial work has previously been advocated by Brown [2].

## 2. SIMPLIFIED SCENARIO

### *Assumptions*

Throughout Section 2, the four assumptions listed in Section 1 are made. These assumptions are not thought to be realistic, but are made at this stage to simplify the presentation. Assumptions 2, 3, and 4 are relaxed in later sections.

### *Transformation of the Data*

In order to model the dependence of claim size on operational time, the original data triangles $Y_{wd}$ and $N_{wd}$ must first be transformed into a triangle $\tau_{wd}$ of operational times, and a triangle $S_{wd}$ of observed mean claim amounts. In the subsequent modelling, $\tau$ will be the explanatory variable, and $S$ will be the dependent variable.

Operational time, $\tau$, which has previously been defined as the proportion of claims closed, is an alternative to development time, $d$. This definition gives the value of operational time *between* claim settlements. In this paper, the value of operational time *at* each claim settlement is defined to be the mean of the values immediately before and after settlement. So, for example, if there are $M$ claims for a certain origin year, the operational time of settlement of the $N^{\text{th}}$ claim is given by $\tau = (N - \frac{1}{2})/M$. The values of operational time for each claim settlement are $(\frac{1}{2})/M, (\frac{3}{2})/M, \ldots, (M - \frac{1}{2})/M$. These values are shown as crosses in Figure 1, which illustrates a typical relationship between operational time and true development time. The mean operational time of the $N_{wd}$ claims in development period $d$ of origin year $w$ can be calculated as:

$$\tau_{wd} = (N_{w,1} + N_{w,2} + \ldots + N_{s,d-1} + \tfrac{1}{2} \cdot N_{w,d})/M_w \,. \tag{2.1}$$

Note that only half of $N_{wd}$ is included in the numerator in order to give the *mean* operational time for these claims.

The sample mean size $S_{w\tau}$ of the $N_{wd}$ claims from origin year $w$ observed at mean operational time $\tau$ can be calculated as $S_{w\tau} = Y_{wd}/N_{wd}$. As $S_{w\tau}$ is a sample mean, its expected value is equal to the mean of the underlying population:

$$\mathrm{E}(S_{w\tau}) = m_\tau \,. \tag{2.2}$$

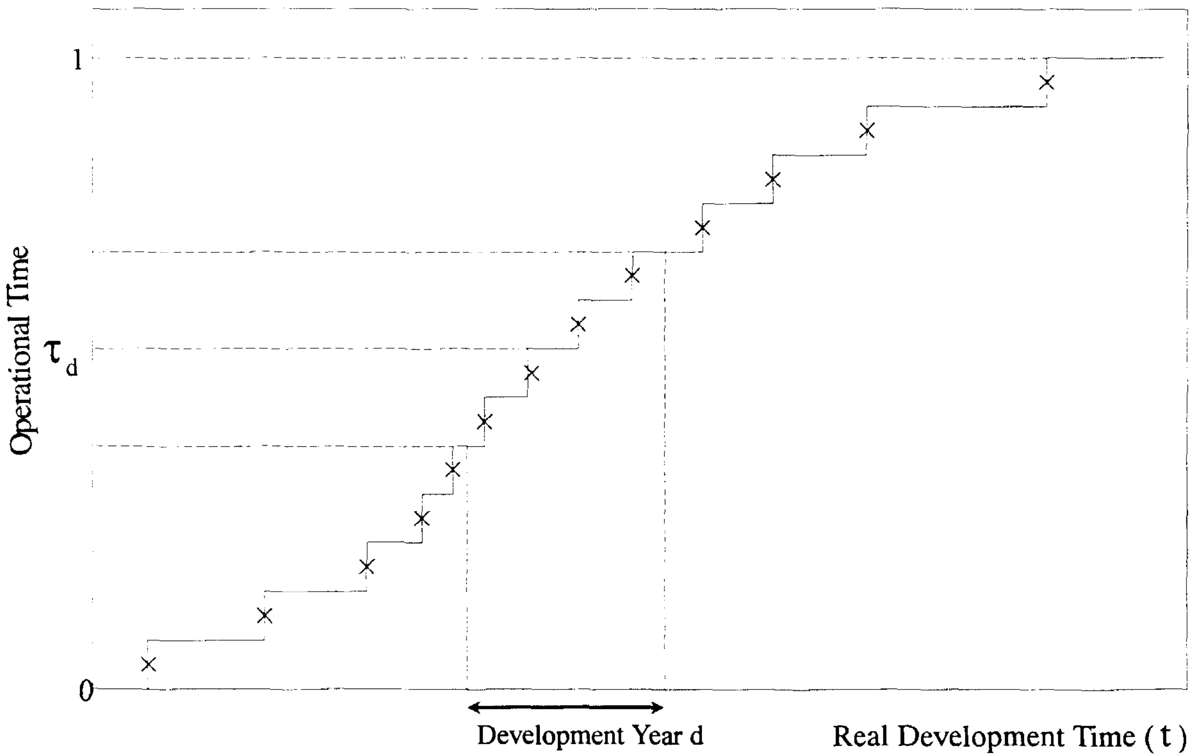
The variance of $S_{w\tau}$ is the population variance divided by the sample size. Using the population variance of Assumption 2 (Equation 1.1) gives:

$$\mathrm{Var}\,(S_{w\tau}) = \varphi^2 \cdot m_\tau^2/N_{wd} \tag{2.3}$$

Equations 2.2 and 2.3 are actually approximations in general because the $N_{wd}$ claims do not have exactly the same mean and variance. Equation 2.2 is exact if $m_\tau$ is linear in $\tau$, and equation 2.3 is exact if $m_\tau^2$ is linear in $\tau$. Both are good approximations if $m_\tau$ does not vary greatly within each development period.

FIGURE 1

TYPICAL RELATIONSHIP BETWEEN OPERATIONAL TIME AND REAL DEVELOPMENT TIME



The form of Equation 2.3 for the variance of $S_{w\tau}$ can be:

- tested (as described below) and if not true, modified (as described in Section 3), and
- used to test and compare alternatives for the systematic component of the model; that is, the dependence of $m_\tau$ on $\tau$ (also described below).

It is not necessary to have any further knowledge about the distribution of the data $S$ in order to fit models of "generalized linear" type for the $m_\tau$ ; the variance alone is sufficient.

Equation 2.1 defines a relationship between $\tau$, $w$, and $d$ for the observed data. Given any two, the third can be found. By virtue of this known relationship, $N_{wd}$ can alternatively be expressed as $N_{w\tau}$ , and this is done for the remainder of the paper.

*Models for the Mean Claim Size*

An expression is needed to describe how expected severity varies as a function of the length of time a claim is open. A number of possible relationships between $m_\tau$ and $\tau$ are considered:

1. $m_\tau = \exp\left(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \ln(\tau)\right)$
2. $m_\tau = \exp\left(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2\right)$
3. $m_\tau = (\beta_0 + \beta_1 \cdot \tau)^2$
4. $m_\tau = 1/(\beta_0 + \beta_1/\tau)$ .

All these models are of generalized linear form; that is, the mean $m_\tau$ of the data $S_{w\tau}$ is some function of a known linear form of the unknown parameters $\beta$:

$$h(m_\tau) = \boldsymbol{x}_\tau \cdot \beta,$$

where

$h(m_\tau)$ is a known function,

$x_\tau$ is a known vector, and

$\beta = (\beta_0, \beta_1, \beta_2)$.

Table 1 gives $h(m_\tau)$ and $x_\tau$ for each of the models.

TABLE 1

| Model | $m_\tau$ | $h(m_\tau)$ | $x_\tau$ |
|---|---|---|---|
| 1 | $\exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \ln(\tau))$ | $\ln(m)$ | $(1, \tau, \ln(\tau))$ |
| 2 | $\exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2)$ | $\ln(m)$ | $(1, \tau, \tau^2)$ |
| 3 | $(\beta_0 + \beta_1 \cdot \tau)^2$ | $\sqrt{m}$ | $(1, \tau)$ |
| 4 | $1/(\beta_0 + \beta_1/\tau)$ | $1/m$ | $(1, 1/\tau)$ |

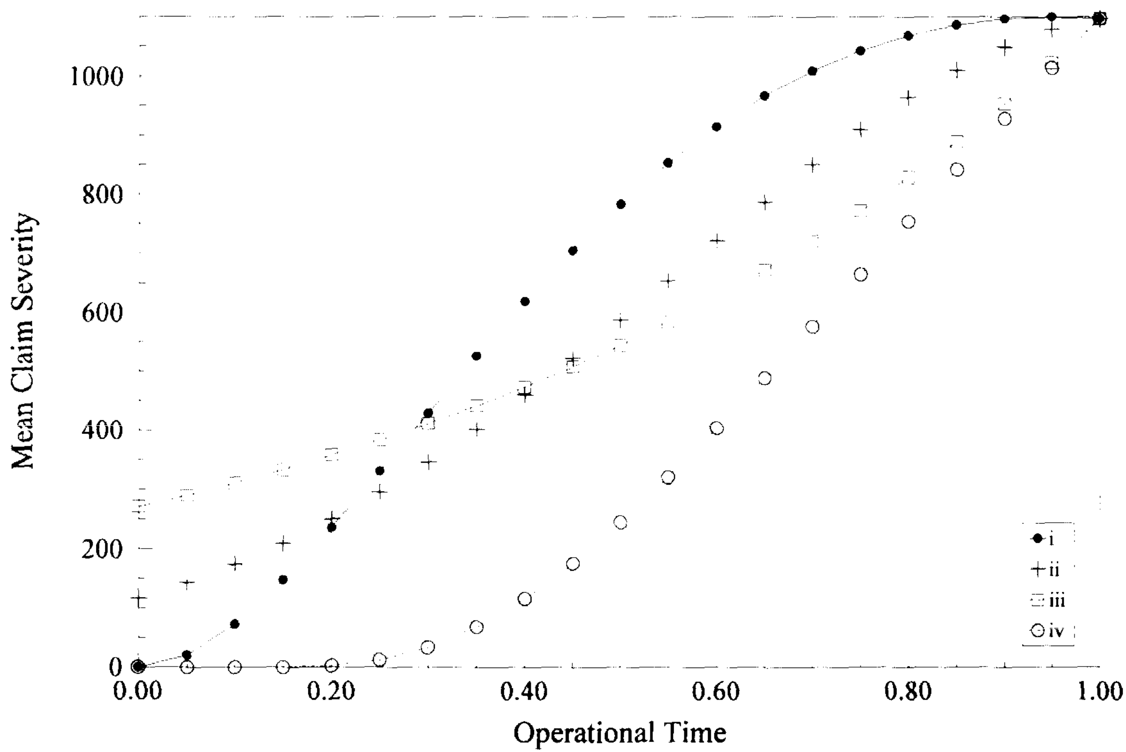
Appendix C shows how models for the mean claim size as a function of operational time can be interpreted in terms of real development time. Such models often correspond to simple relationships between the mean claim size and the distribution function of the delay. A graph of $m_\tau$ for each of the models is given in Figure 2. Although Figure 2 shows typical shapes, each model embodies a family of curves, and different shapes can be obtained within each family by varying the $\beta$-parameters. Of course, many other generalized linear models for $m_\tau$ could be formulated. All such models can be fitted, tested, and projected using the methods described below. The four models considered here have been chosen arbitrarily, for illustrative purposes.

*Testing the Variance Assumption*

As all the proposed models for $m_\tau$ are of generalized linear form, they can be fitted efficiently, given a second moment assumption, using Fisher's scoring method. First, it is necessary to test the proposed second moment assumption (Equation 2.3). This can be done by fitting a model (referred to as "Model 0") which makes minimal assumptions about the form of $m_\tau$.

FIGURE 2

TYPICAL SHAPES OF VARIOUS MODELS RELATING MEAN SEVERITY TO OPERATIONAL TIME



A suitably minimal assumption is that the expected claim size $m_\tau$ is a piecewise exponential function of $\tau$; that is, $\ln(m_\tau)$ is a piecewise linear function of $\tau$. The important point is that this form of model is very flexible. Any reasonable function $m_\tau$ can be well approximated in this way if the intervals are sufficiently small. It is probably sufficient to take a number of sub-intervals of equal width, the number being equal to the observed number, $T$, of development periods. A subscript, $j$, is used to label these sub-intervals of the observed operational time range.

Model 0 can be expressed as:

$$m_\tau = \exp\,(\beta_0 + \sum_j \beta_j \cdot \tau_j)\,, \tag{2.4}$$

where each $\tau_j$ is the amount of $\tau$ lying in each of the sub-intervals of the operational time scale such that $\tau = \sum_j \cdot \tau_j$. This gives a continuous piecewise linear function in the exponent of Equation 2.4. The $\beta_j$ are the slopes of the line segments. See Figure 3. An example of this piecewise exponential form for $m_\tau$ is shown in Figure 4. No assumption is made about the relationship between the $\beta_j$ values as $j$ varies from zero to $T$.

In terms of $h(m)$ and $\boldsymbol{x}_\tau$, Model 0 is:

$$h(m) = \ln(m)$$

$$\boldsymbol{x}_\tau = (1, \tau_1, \tau_2, \tau_3, \ldots, \tau_T)\,.$$

If all the sub-intervals of operational time have the same width $u$, then $\boldsymbol{x}_\tau$ is of the form:

$$\boldsymbol{x}_\tau = (1, u, u, \ldots, u, \tau_k, 0, 0, 0),$$

where $(k-1)u < \tau < k \cdot u$ and $\tau_k$ is the fractional part in this sub-interval. Of course, $k$ and $\tau_k$ may differ for each data point $S_{w\tau}$, but since $\tau$ is known for each data point, $\boldsymbol{x}_\tau$ can be determined and the model fitted to estimate the parameters $\beta_j$ for $j = 1$ to $T$.

FIGURE 3

TYPICAL MODEL ZERO (ON LOG SCALE )

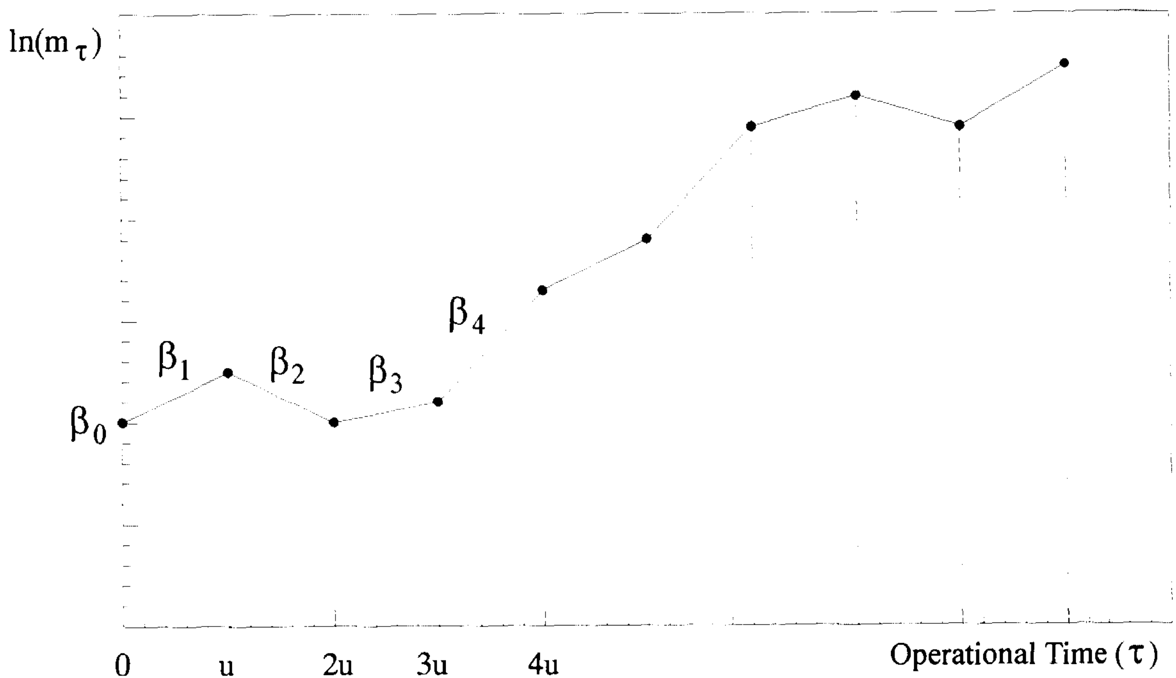
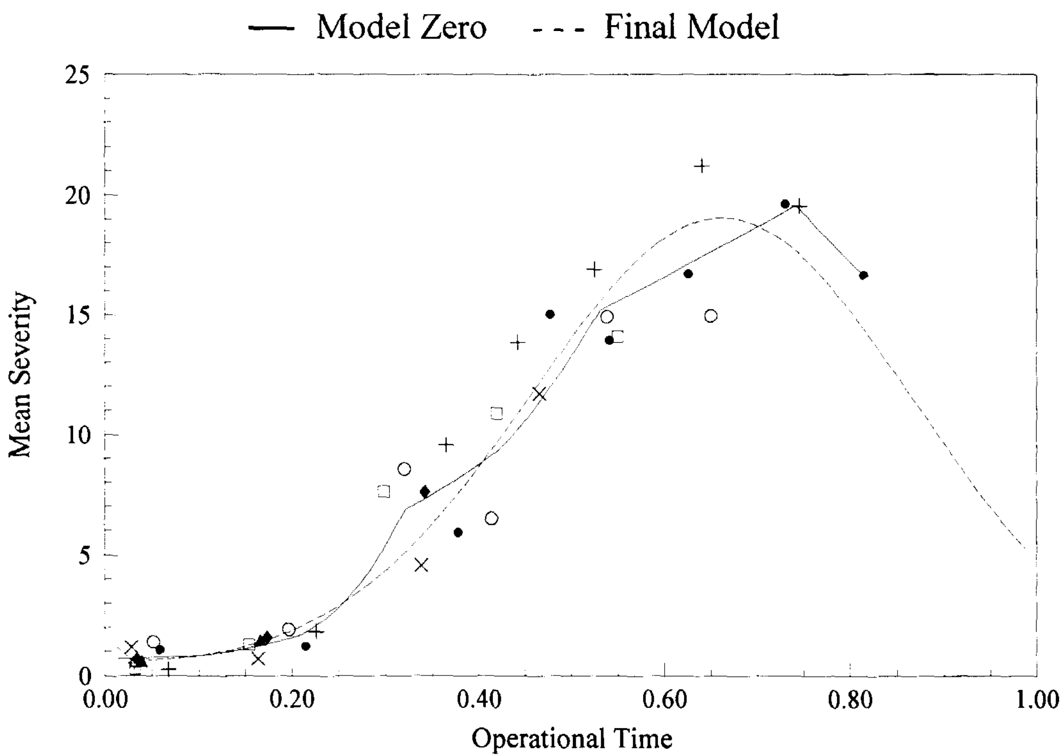


FIGURE 4

FITTED MEAN SEVERITIES FOR BERQUIST AND SHERMAN DATA



Consider the quantities:

$$R_{w\tau} = (S_{w\tau} - m_\tau) \cdot \sqrt{N_{w\tau}}/m_\tau . \qquad (2.5)$$

If the variance of $S$ is indeed as specified by Equation 2.3 then these quantities have $\mathrm{E}(R_{w\tau}) = 0$ and $\mathrm{Var}\,(R_{w\tau}) = \varphi^2$. After fitting Model 0 (by Fisher's scoring method), the $R_{w\tau}$ can be estimated by using the fitted values for the $m_\tau$ (these estimated $R_{w\tau}$ are the *standardized residuals*).

The variance assumption can be tested by plotting the $R_{w\tau}$ against $\tau$. The variance should be constant; that is, it should not depend on $\tau$. In such a case, $\varphi^2$ can be estimated as follows:

$$\varphi_0^2 = \Big( \sum_{w\tau} R_{w\tau}^2/(n - T - 1)\Big), \qquad (2.6)$$

where

$n$ is the total number of points in the triangle, and

$T + 1$ is the number of $\beta$-parameters.

If the residual plot shows heteroscedasticity (that is, the variance appears to depend on $\tau$), then the variance assumption (Assumption 2 of Section 1) should be modified. (See Section 3.)

*Testing Models for the Mean Claim Size*

Model 0 is so flexible that we can be fairly confident it will provide a good fit. The quality of fit of other models can therefore be assessed by comparison with the fit of Model 0. When the variance assumption has been validated, any other model for $m_\tau$ of generalized linear form can be formally tested as follows. After fitting by Fisher's scoring method, the standardized residuals can be calculated from:

$$R_{w\tau} = (S_{w\tau} - m_\tau) \cdot \sqrt{N_{w\tau}}/m_\tau \quad \text{(as for Model 0)}.$$

From these, another estimate of $\varphi^2$ is given by:

$$\varphi_1^2 = (\sum_{w\tau} R_{w\tau}^2)/(n-p), \tag{2.7}$$

where

$n$ is the total number of points in the triangle, and

$p$ is the number of parameters in the model (the $\beta$s), either two or three for each model listed in this section.

The following statistic can then be calculated:

$$F = [\varphi_1^2 \cdot (n-p)/(T+1-p) - \varphi_0^2 \cdot (n-T-1)/(T+1-p)]/\varphi_0^2\,, \tag{2.8}$$

where $\varphi_0^2$ is the estimate of $\varphi^2$ obtained from Model 0.

This should be compared against the theoretical $F$-distribution with $(T+1-p)$ and $(n-T-1)$ degrees of freedom. If the $F$-statistic is too large, then the current model for $m_\tau$ cannot be accepted. In such a case, the lack of fit may well be apparent from the plot of residuals against $\tau$. For some values of $\tau$, the mean may appear to be significantly different from zero. If the $F$-statistic could reasonably have come from the theoretical $F$-distribution, then the fitted means $m_\tau$ obtained using Model 0 do not vary significantly from the form assumed in the current model. Therefore, the current model can be accepted. Several of the models proposed in this section may give reasonably small $F$-statistics. If so, tables will indicate which $F$-statistic corresponds to the largest tail probability, but it may be safer to use a more general model, of which all acceptable models are special cases.

Estimates of $\varphi^2$ (hence $F$-tests) alternatively may be based on the minimized deviance rather than the sum of squares of the standardized residuals. This is more satisfactory in view of the likely skewness of the data. The deviance is less sensitive to the incidence of large claims than the residual sum of squares, so it will be more stable. The deviance is:

$$Q = 2 \cdot \sum_{w\tau} N_{w\tau} \cdot [-\ln(S_{w\tau}/m_\tau) + (S_{w\tau} - m_\tau)/m_\tau], \qquad (2.9)$$

from which:

$$\varphi_0^2 = Q_0/(n - T - 1),$$

$$\varphi_1^2 = Q_1/(n - p),$$

hence an $F$-statistic from Equation 2.8.

The choice between using the residual sum of squares or the deviance to construct $F$-statistics arises because in neither case is the distribution truly the $F$-distribution. With an infinite number of data points, and models which were restricted cases of Model 0, both alternatives would have the true $F$-distribution. Neither of these conditions is satisfied, but the $F$-statistic based on the minimized deviance provides an effective, pragmatic technique for testing and comparing models. It is of no practical consequence that precise probability levels cannot be assigned to the $F$-statistics. Further details on the relevant theory are given by McCullagh and Nelder [5].

*Prediction*

If a simple model is found with an acceptably small $F$-statistic (not much greater than one), then it can be used for predicting future payments. The expected value of each future claim is obtained by evaluating the fitted mean $m_\tau$ at the operational time $\tau$ as defined earlier. Similarly, the variance of each future claim is obtained by evaluating Equation 1.1, using the fitted mean $m_\tau$, and the estimate of $\varphi^2$ given by the minimized deviance as described in the preceding paragraphs. Assuming the amounts of future claims are stochastically independent, the mean and variance of the total can be obtained as the sum of the figures for the individual claims. The resulting variance must then be augmented to allow for estimation error in the fitted means $m_\tau$. Details are given in Appendix D.

*Numerical Example*

The data used in the examples are the medical malpractice data published in Berquist and Sherman [1]. They are given in Appendix A. To satisfy Assumption 3 of Section 1, the $Y_{wd}$ values of Appendix A were brought up to 1976 terms using an assumed inflation rate of 15% (the rate used by Berquist and Sherman) before calculating the sample means $S_{w\tau} = Y_{wd}/N_{wd}$. The triangle of operational times was calculated using Equation 2.1 and is given in Table A.4. A plot of the sample means $S_{w\tau}$ against operational time $\tau$ is given in Figure 5.

Model 0, which has nine parameters (one intercept parameter, and a slope parameter for each of eight subintervals of the observed range [0.0, 0.85] of operational time) gave a minimum deviance of 1,803. The plot of standardized residuals against $\tau$ is shown in Figure 6. This shows clear evidence of heteroscedasticity. The spread of the points decreases as $\tau$ increases. This indicates that the variance assumption (Assumption 2, Equations 1.1 and 2.3) is false. Consequently, all results obtained using this variance assumption are invalid. The minimized deviance, the number of residual degrees of freedom, and the $F$-statistic for each mean claim size model are given in Table 2. The number of degrees of freedom, *df*, is the number of data points less the number of model parameters. It appears in the denominator of Equation 2.7.

TABLE 2

| Model | Deviance | *df* | $F$ |
|---|---|---|---|
| 1 | 3,417 | 33 | 4.0 |
| 2 | 2,685 | 33 | 2.2 |
| 3 | 3,521 | 34 | 3.7 |
| 4 | 4,521 | 34 | 5.8 |

It is stressed that the clear falseness of the variance assumption renders the above figures meaningless. They are presented here merely to illustrate orders of magnitude, and to show how the $F$-statistic relates to the deviance. The example is continued in Section 3.

FIGURE 5

OBSERVED MEAN SEVERITY AGAINST MEAN OPERATIONAL TIME
(DATA FROM BERQUIST AND SHERMAN)

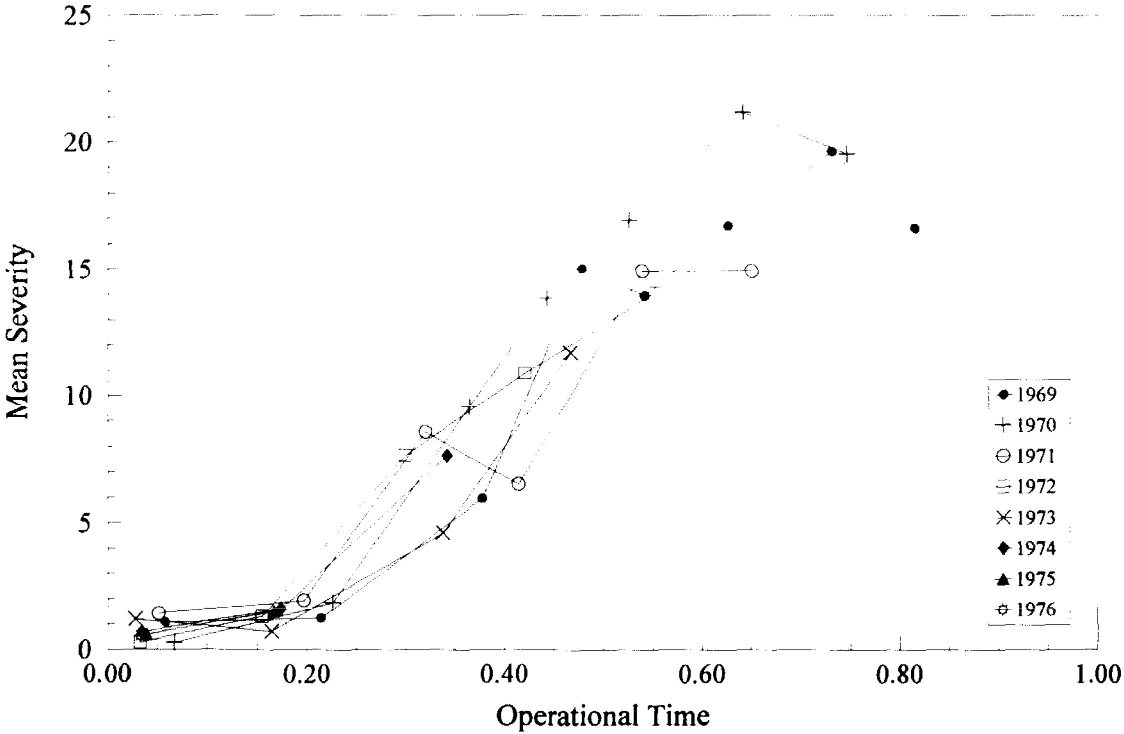
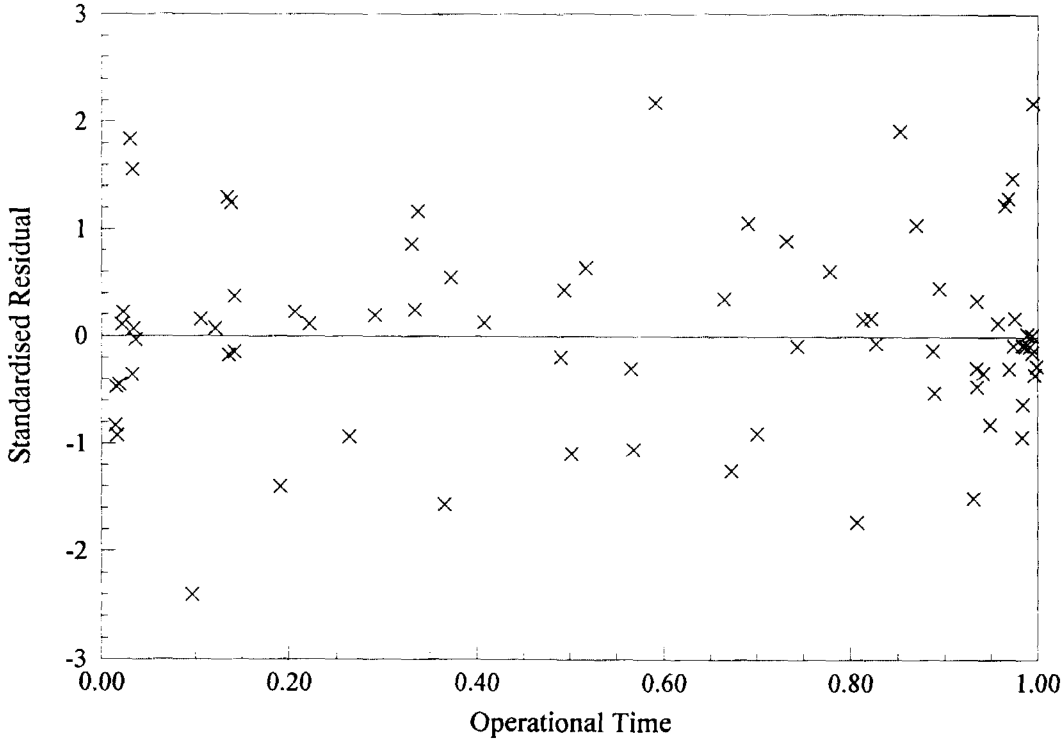


FIGURE 6

RESIDUAL PLOT FOR MODEL ZERO WITH $\alpha = 2$



## 3. RELAXING THE VARIANCE ASSUMPTION

*Theory*

If the initial variance assumption (Assumption 2, Equations 1.1 and 2.3) is found to be incorrect when tested as described in Section 2, then an alternative must be tried. The coefficient of variation of individual claims may depend on the mean claim size $m_\tau$. Since this is usually an increasing function of $\tau$, the nature of the dependence should be apparent from the plot of standardized residuals against $\tau$ for Model 0. For example, if this plot suggests that the variance is decreasing as $\tau$ increases, then the coefficient of variation decreases as the mean $m_\tau$ increases. Such a case can probably be modelled adequately by replacing Assumption 2 from Section 1 with:

$$\mathrm{Var}\,(X_\tau) = \varphi^2 \cdot m_\tau^\alpha \text{ , for some } \alpha < 2. \tag{3.1}$$

In terms of the sample mean $S_{w\tau}$, this is:

$$\mathrm{Var}\,(S_{w\tau}) = \varphi^2 \cdot m_\tau^\alpha / N_{w\tau} \text{ .} \tag{3.2}$$

Model 0 can be refitted on this basis (details are given below) and the standardized residuals examined to determine whether $\alpha$ needs to be further adjusted. The standardized residuals are given by:

$$R_{w\tau} = (S_{w\tau} - m_\tau) \cdot \sqrt{(N_{w\tau}/m_\tau^\alpha)} \text{ .} \tag{3.3}$$

Similarly, if after fitting Model 0 using the initial assumption the standardized residuals fan out, then the model should be refitted with $\alpha > 2$.

When the variance has been satisfactorily modelled in this way for Model 0, the other models can be fitted using the variance function defined by Equation 3.1, with $\alpha$ taking the value found using Model 0.

The deviance to be minimized when $\alpha$ is not equal to one or two is given by:

$$Q = 2 \cdot \sum_{u\tau} N_{u\tau} \cdot [S_{u\tau} \cdot (S_{u\tau}^{1-\alpha} - m_\tau^{1-\alpha})/(1-\alpha) - (S_{u\tau}^{2-\alpha} - m_\tau^{2-\alpha})/(2-\alpha)\ ]. \tag{3.4}$$

This is the quantity which is used to calculate $F$-statistics for testing and comparing the different models for the mean claim size $m_\tau$. (See Section 2.)

*Numerical Example*

The example of Section 2 has been rerun using an index $\alpha = 1.5$ in the variance function, instead of $\alpha = 2$. The minimized deviance for Model 0 (which has nine parameters) is now 2,404. The plot of standardized residuals against operational time $\tau$ is given in Figure 7. It shows no evidence of heteroscedasticity, so the variance assumption (Equations 3.1 and 3.2 with $\alpha = 1.5$) is acceptable, and the results of modelling under this assumption are valid. The minimized deviance and the $F$-statistic for each of the models of Section 2 are listed in Table 3.

TABLE 3

| Model | Deviance | *df* | *F* |
|---|---|---|---|
| 1 | 5,829 | 33 | 6.41 |
| 2 | 3,567 | 33 | 2.18 |
| 3 | 5,053 | 34 | 4.25 |
| 4 | 6,568 | 34 | 6.68 |

Table 3 shows that none of these models fit the data very well. For a model to be acceptable, the $F$-statistic must be much closer to one.

An $F$-value of 1.22 is achieved by the following four-parameter model, which is a generalization of Models 1 and 2:

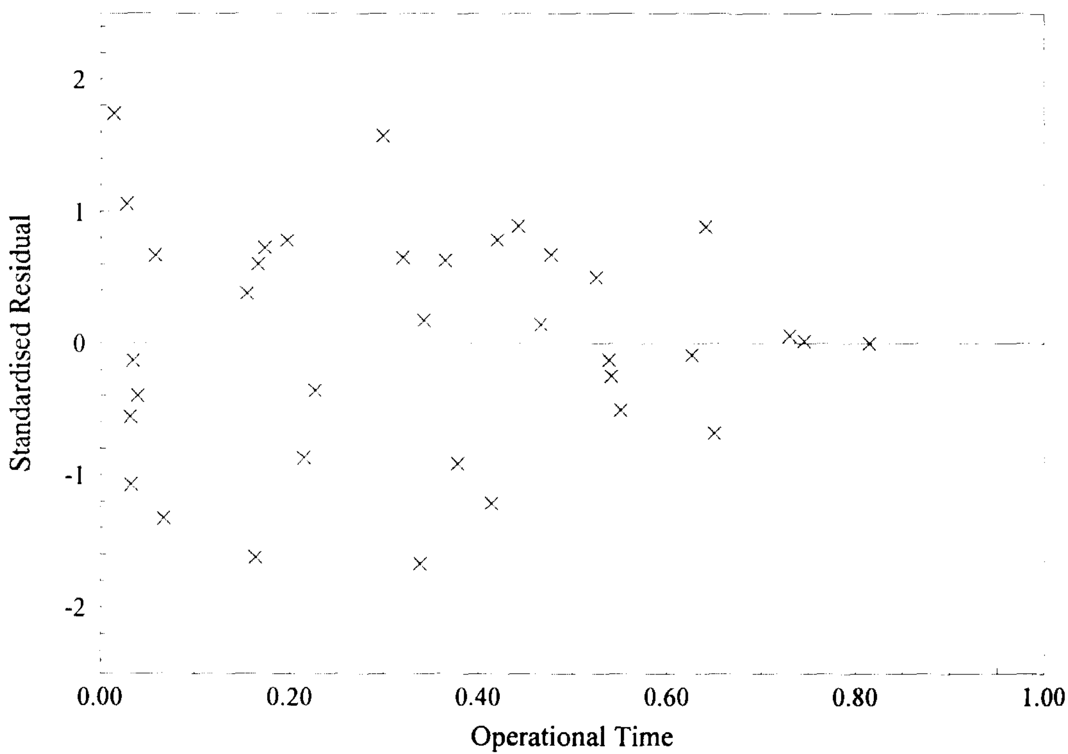
$$h(m) = \ln(m)$$

$$x_\tau = (1,\ \tau,\ \tau^2,\ \ln(\tau))\ .$$

The estimated parameters (with their standard errors) are:

FIGURE 7

RESIDUAL PLOT FOR MODEL ZERO WITH $\alpha = 1.5$



| | | | |
|---|---|---|---|
| $\beta_0$ | -3.90 | (1.08) | |
| $\beta_1$ | 18.3 | (2.87) | |
| $\beta_2$ | -12.8 | (2.29) | (coefficient of $\tau^2$) |
| $\beta_3$ | -0.87 | (0.33) | (coefficient of $\ln(\tau)$) . |

This model has five fewer parameters than model zero, so the $F$-statistic has five and 27 degrees of freedom. Statistical tables indicate a greater than one-in-three chance of an $F$-value as large as 1.22, if the model is true. In other words, the variation of the fitted values $m_\tau$, obtained using Model 0, around the curve obtained under the present model could well be purely random. So, the present model gives a good representation of the underlying pattern in the data. This is confirmed by Figure 4, which shows the fitted values of $m_\tau$ under both models. The difference between the two curves is insignificant compared to the random variation in the data.

Figure 4 also shows that the fitted curve for $m_\tau$ decreases for $\tau$ greater than about 0.66. This decrease exists in the data (Figure 5), but there are no data for operational times greater than 0.82. It is reasonable to question whether a decreasing curve for $m_\tau$ should be projected beyond this value. It is shown in Section 7 that the decrease in the observed mean claim amounts is caused largely by the presence of partial payments. In the terms of the primary triangles in Section 1, the data is actually type (c), not (a) or (b). It is analyzed here as if it were type (a) or (b) purely to illustrate the method. In practice, one should be very wary of projecting a decreasing curve for $m_\tau$ beyond the observed range of operational times, in either case (a) or (b).

The fitted model also has a minimum at $\tau = 0.05$ and $m_\tau$ tending to infinity as $\tau$ tends to zero. Although unrealistic, this is not important because projections are required only for operational times greater than 0.064 (the present operational time for the latest year of origin, 1976).

Table 4 gives the following quantities for each origin year: estimates of expected total of future payments, approximate standard errors of these estimates, estimates of standard deviations of total future payments, and approximate root-mean-square (RMS) errors of prediction.

Columns 1 and 3 have been calculated by totalling the estimated mean and variance for all future claims, as described in Section 2. Column 2 is the standard error of Column 1 arising from uncertainty in the estimated $\beta$-parameters of the mean $m_\tau$. It has been calculated using the formulae derived in Appendix D. Column 4 is the combination of Columns 3 and 4, calculated as the square root of the sum of their squares. This is appropriate because the uncertainty represented by the standard errors in Column 2 is independent of the uncertainty represented by Column 3. Column 2 arises from random variation in past claims, whereas Column 3 arises from random variation in future claims (as described in Appendix D).

TABLE 4

| Year | (1) Expected Total Future Payments | (2) Standard Error | (3) Standard Deviation | (4) Root-Mean-Square Error |
|---|---|---|---|---|
| 1969 | 3,350 | 1,209 | 959 | 1,543 |
| 1970 | 6,260 | 1,875 | 1,382 | 2,329 |
| 1971 | 14,835 | 3,422 | 2,239 | 4,089 |
| 1972 | 25,177 | 4,497 | 2,999 | 5,405 |
| 1973 | 35,842 | 5,120 | 3,607 | 6,263 |
| 1974 | 40,098 | 4,642 | 3,779 | 5,985 |
| 1975 | 47,265 | 4,921 | 4,032 | 6,362 |
| 1976 | 59,001 | 5,989 | 4,461 | 7,467 |
| All | 231,828 | 31,270 | 8,960 | 32,528 |

The final row (labelled "All") is for all origin years combined. The predicted total of future payments for all origin years combined is $232 million. This is simply the sum of the figures in Column 1. The uncertainty represented by Column 2 is highly correlated between origin years, because the same set of parameter estimates ($\beta_j$, given earlier in this section) is used for all origin years. Therefore, the standard error representing this source of uncertainty for all years combined (31,270) is only slightly less than the sum of the standard errors for each origin year (the sum of Column 2 is 31,675). Full details of this calculation are given in Appendix D. In contrast, the uncertainty represented by Column 3 is stochastically independent between origin years, because the future claims for each origin year are mutually disjoint sets. Therefore, the standard deviation for all years combined (8,960), is simply the square-root of the sum of the squares of the figures in Column 3. The RMS error of prediction for all origin years combined can be calculated in the same way as for a single origin year; i.e., $32{,}528 = \sqrt{31{,}270^2 + 8{,}960^2}$, because the first component represents uncertainty arising from random variation in past claims, and the second component represents uncertainty arising from random variation in future claims. A reasonably safe reserve (for all origin years combined) can be calculated by adding one RMS error ($32.5 million) to the best estimate ($231.8 million) to give $264 million. However, since the data were adjusted to remove claims inflation, this is in 1976 terms. Section 6 shows how future claims inflation can be included in the predictions. Also, the assumed past inflation rate of 15% may not be correct, and no allowance has been made for the uncertainty in the estimates of ultimate claim numbers, $M_w$, used in the calculations. These two matters are dealt with in Sections 4 and 5, respectively.

## 4. SIMULTANEOUS ESTIMATION OF INFLATION

### *Basic Assumptions*

This section describes techniques that can be applied to data $Y_{wd}$ that has not been adjusted for inflation. For some of the models specified in Section 2, the force of claims inflation can be estimated from the data at the same time as estimating the other parameters.

The sample mean payment amounts $S_{w\tau}$ are now calculated as $S_{w\tau} = Y_{wd}/N_{wd}$ using the unadjusted $Y_{wd}$. The expected value of $S_{w\tau}$ will now depend on the origin year $w$ (as well as $\tau$ ) because of inflation, so is denoted $m_{w\tau}$. However, by assumption 1 of Section 1, the mean in real terms is the same for all origin years. Thus, $m_{w\tau}$ is of the form:

$$m_{w\tau} = \exp\left[(w + d/P) \cdot i\right] \cdot m_\tau \tag{4.1}$$

for some $m_\tau$ which are the same for all origin years. Here, $i$ represents the annual force of claims inflation; $P$ represents the number of development periods per year; and $w + d/P$ is, therefore, the calendar time (in years) of each data point.

The initial variance assumption is that the coefficient of variation of individual claims is constant, which implies:

$$\text{Var}\,(S_{w\tau}) = \varphi^2 \cdot m_{w\tau}^2 / N_{w\tau}\,.$$

As before, this can be generalized, if necessary, to:

$$\text{Var}\,(S_{w\tau}) = \varphi^2 \cdot m_{w\tau}^{\alpha} / N_{w\tau}\,. \tag{4.2}$$

*Models for the Mean Claim Size*

Given a model for $m_\tau$, Equation 4.1 yields a model for the mean $m_{w\tau}$ of the data $S_{w\tau}$. If $i$ is to be treated as a parameter to be estimated, then the model for $m_\tau$ must have $\ln(m_\tau)$ linear in the unknown parameters in order for $m_{w\tau}$ to be of generalized linear form:

$h(m_{w\tau}) = \boldsymbol{x}_{w\tau} \cdot \beta$, for some known vector $\boldsymbol{x}_{w\tau}$.

Thus, of the models for $m_\tau$ proposed in Section 2, only Models 0, 1, and 2 can be fitted directly using Fisher's scoring method. These all have $h(m) = \ln(m)$.

If $\beta = (i, \beta_0, \beta_1, \ldots)$ where the $\beta_j$ are the same as in Section 2, then the vector $\boldsymbol{x}_{w\tau}$ is given by:

Model 0: $\boldsymbol{x}_{w\tau} = (w + d/P, u, \ldots u, \tau_j, 0, \ldots 0)$

Model 1: $\boldsymbol{x}_{w\tau} = (w + d/P, 1, \tau, \ln(\tau))$

Model 2: $\boldsymbol{x}_{w\tau} = (w + d/P, 1, \tau, \tau^2)$.

$\boldsymbol{x}_{w\tau}$ is known for each data point as $w$, $d$, and $\tau$ are known.

Model fitting and testing can proceed with these models exactly as described in Sections 2 and 3, except that the number of parameters in each model has increased by one. If $T$ is the number of operational time intervals used in Model 0, the number of parameters of the model is now $T+2$. The number of residual degrees of freedom is therefore $n-T-2$. This should replace $n-T-1$ in Equations 2.6 and 2.8. Similarly, the number of parameters $p$ of Models 1 and 2 is now one greater than previously.

The question remains of how to fit models such as Models 3 and 4 which do not have $h(m) = \ln(m)$, when the rate of claims inflation is not known. The following procedure can be employed. First, fit Models 0, 1, and 2 as described above (generalizing the variance assumption if necessary). If none of these models gives an acceptable fit when compared to Model 0 (using $F$-tests as in Sections 2 and 3), then use the force of inflation estimated using Model 0 to adjust the data $Y_{wd}$ into constant money terms. All models can then be fitted to the inflation adjusted data as described in Sections 2 and 3, and the best model determined. If the best model is one such as Models 1 or 2, then the version fitted to the unadjusted data should be used.

Although $x_{w\tau}$ can be determined from the data for each cell of the triangle, it is not fully known for cells corresponding to the future. The relationship between $\tau$, $d$, and $w$ for the future depends on the rate at which claims will be settled, which is uncertain. Having fitted a model, the formulae of Appendix D apply only to the factor $m_\tau$ of Equation 4.1 so the predictions are in constant prices. A further stage of estimation is necessary before claims inflation can be incorporated in projections. This approach is illustrated in Section 6.

Note that the methods described here assume that past claims inflation has been at a constant rate. In cases where this is considered to be a poor approximation, the data $Y_{wd}$ should be preadjusted to remove any non-constant elements of claims inflation believed to be present.

*Numerical Example*

The methods of Section 4 are illustrated by repeating the example of Sections 2 and 3, this time with the sample means $S_{w\tau}$ calculated from the unadjusted data $Y_{wd}$ and $N_{wd}$ from Appendix A. As in Section 2, the residual plot from Model 0 with $\alpha = 2$ shows that this value is incorrect, and as in Section 3, the value $\alpha = 1.5$ is found to be acceptable.

Table 5 gives the minimized deviance and the $F$-statistic for comparing each of Models 1 and 2 to Model 0. These $F$-statistics each have six and 26 degrees of freedom. The table also gives the estimated force of claims inflation (and its standard error) obtained from each of the models.

TABLE 5

| Model | Deviance | *df* | $F$ | Inflation |
|---|---|---|---|---|
| 0 | 1,961 | 26 | — | 0.132 (0.035) |
| 1 | 4,896 | 32 | 6.49 | 0.141 (0.047) |
| 2 | 2,865 | 32 | 2.00 | 0.138 (0.036) |

From statistical tables, there is only about a one-in-10 chance that an $F$-variate with six and 26 degrees of freedom is as large as 2. Therefore, neither Model 1 nor 2 adequately represents the data. This implies that all results obtained from these models are invalid, including the estimates of the force of claims inflation given above.

However, the model used in Section 3 still gives a good fit when applied to the unadjusted data with an additional parameter for inflation. The minimized deviance is 2,402, which gives an $F$-value of 1.17 on five and 26 degrees of freedom. The estimated parameters (with their standard errors) are:

| | | | |
|---|---|---|---|
| $i$ | 0.135 | (0.034) | |
| $\beta_0$ | -3.71 | (1.06) | |
| $\beta_1$ | 17.8 | (2.80) | |
| $\beta_2$ | -12.5 | (2.20) | (coefficient of $\tau^2$) |
| $\beta_3$ | -0.80 | (0.33) | (coefficient of $\ln(\tau)$) . |

The figure 0.135 for the force of claims inflation corresponds to a 14.5% annual rate.

The final results in 1976 terms are:

TABLE 6

| Year | (1) Expected Total Future Payments | (2) Standard Error | (3) Standard Deviation | (4) Root-Mean-Square Error |
|---|---|---|---|---|
| 1969 | 3,450 | 1,169 | 898 | 1,475 |
| 1970 | 6,397 | 1,800 | 1,287 | 2,213 |
| 1971 | 15,034 | 3,261 | 2,071 | 3,863 |
| 1972 | 25,360 | 4,271 | 2,761 | 5,086 |
| 1973 | 35,962 | 4,873 | 3,312 | 5,892 |
| 1974 | 40,132 | 4,464 | 3,464 | 5,651 |
| 1975 | 47,279 | 4,796 | 3,696 | 6,055 |
| 1976 | 59,015 | 5,876 | 4,089 | 7,158 |
| All | 232,630 | 29,988 | 8,229 | 31,096 |

Adding one standard error to the best estimate gives a reserve for all origin years combined of $264 million, in 1976 terms. Although these results hardly differ from those obtained in Section 3, more confidence can be placed in them now, because the inflation rate has been estimated from the data themselves, and not based on any prior assumptions. However, no allowance has yet been made for the uncertainty in the ultimate numbers of claims $M_w$.

## 5. ALLOWING FOR UNCERTAINTY IN ULTIMATE NUMBER OF CLAIMS

*Theory*

In previous sections, it has been assumed that the ultimate number of claims, $M_w$, is accurately known for each origin year $w$ (Assumption 4, Section 1). This assumption is realized in practice only if the origin years are report years. For any other definition of origin year, there will be an unknown number (possibly zero) of IBNR claims. This number has to be estimated in order to arrive at an estimate of $M_w$. It is shown in this section how the uncertainty in the $M_w$ estimates can be taken into account in calculating standard errors of the final results. First, the source of the $M_w$ estimates is briefly considered.

For reserving purposes, the origin years must usually be either accident years or policy years. In such cases, in addition to the triangle of the number of settled claims, $N_{wd}$, it may also be possible to obtain a triangle of the number of reported claims. Such a triangle will often give more information about the ultimate number of claims $M_w$ than does $N_{wd}$, because claims are reported before being settled. However, the reported claims triangle will include those claims eventually settled with no payment, whereas $N_{wd}$ and $M_w$ should not. These matters should be considered carefully when estimating the ultimate number $M_w$ of non-zero claims. Whether reported claims, settled claims, or both, are available for estimating $M_w$, the stochastic method previously developed by the author [10] can be used. As well as estimates of $M_w$, this method gives standard errors of the estimates, $v_w$. The following paragraphs describe how these standard errors can be used.

The quantities $M_w$ are used at two points in the methods described in previous sections: in calculating the triangle of operational times $\tau$ (Equation 2.1) and in calculating predictions from the fitted model (Appendix D).

In the following paragraphs, the effect of variability in $M_w$ is considered for each of these in turn.

If $M_w$ is overestimated for a particular origin year $w$, then the operational times for that origin year will all be underestimated by a certain factor. ($M_w$ appears in the denominator of Equation 2.1.) The observed average claim amounts $S_{w\tau}$ for that origin year, therefore, will be for later operational times than those calculated and will tend to overestimate the true mean claim amount for the calculated operational times. Conversely, if $M_w$ is underestimated, then the mean claim sizes will also be underestimated. However, estimation of the mean claim size $m_\tau$ is done by fitting a model to the data for all origin years simultaneously. Provided the estimates $M_w$ are unbiased and not highly correlated, the effects will tend to cancel out across origin years. There will be more variability in the data $S_{w\tau}$ across origin years $w$ than there would otherwise be, but this variability is already taken into account through the estimate of the scale parameter $\varphi^2$. The additional effect on the variability of the final results is therefore minimal and can reasonably be ignored.

Experience with a number of data sets has confirmed these comments. Ultimate counts $M_w$ may be estimated by a variety of methods, but the parameter estimates of the fitted model for $m_\tau$ are invariably very similar whichever set of estimates $M_w$ is used to calculate the operational times. Usually, it is only the last few origin years that have much uncertainty in the ultimate number $M_w$, and these origin years contribute only a few data points $(\tau, S)$ for the modelling. Therefore, the results of the modelling are relatively insensitive to the choice of estimates $M_w$.

Having estimated the parameters $\beta_j$ of a model relating mean claim size $m_\tau$ to operational time, the method described in Appendix D has been used in previous sections to project the fitted model and to calculate the mean-square-error of the projections. An estimate $\hat{\mu}$ of the expected total of future payments for a single origin year is calculated by summing the fitted mean $m_\tau$ over the operational times $\tau$ of each expected future claim. The values of $\tau$ for this summation are given in Equation D.1.

In Appendix E it is shown that, whatever the fitted model for $m_\tau$, each increment of one in the estimate $\hat{M}$ will cause the estimate $\hat{\mu}$ to increase by approximately $[\tau_0 \cdot m_0 + \hat{\mu}/\hat{M}]$, where $\tau_0$ is the operational time reached for the origin year, and $m_0$ is the fitted mean value corresponding to $\tau_0$. This implies that the additional uncertainty in $\hat{\mu}$ caused by the uncertainty in $M$ is represented by a standard error $u$ given by:

$$u = [\tau_0 \cdot m_0 + \hat{\mu}/\hat{M}] \cdot v\,, \tag{5.1}$$

where $v$ is the standard error of the estimate $\hat{M}$.

*Numerical Example*

Continuing with the example of Section 4, Table 7 gives the quantities in Equation 5.1 for each origin year.

TABLE 7

| Year | $\hat{\mu}$ | $\tau_0$ | $m_0$ | $\hat{M}$ | $v$ | $u$ |
|---|---|---|---|---|---|---|
| 1969 | 3,450 | 0.85 | 12.67 | 2,664 | 70 | 845 |
| 1970 | 6,397 | 0.79 | 15.94 | 2,896 | 102 | 1,505 |
| 1971 | 15,034 | 0.70 | 18.65 | 4,065 | 148 | 2,484 |
| 1972 | 25,360 | 0.62 | 18.41 | 4,771 | 215 | 3,580 |
| 1973 | 35,962 | 0.53 | 15.27 | 5,280 | 314 | 4,671 |
| 1974 | 40,132 | 0.41 | 8.87 | 4,837 | 461 | 5,481 |
| 1975 | 47,279 | 0.25 | 3.03 | 5,169 | 690 | 6,843 |
| 1976 | 59,015 | 0.06 | 0.66 | 6,257 | 1,097 | 10,393 |

$\hat{\mu}$ is the best estimate of future payments as given in Section 4. $\hat{M}$ and $v$ come directly from Appendix A. $\tau_0$ is the row total $N_0$ of the number-of-claims-settled triangle divided by $M$. $m_0$ is calculated from $\tau_0$ using the fitted model, which is:

$$m_\tau = \exp\left(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2 + \beta_3 \cdot \ln(\tau)\right),$$

with

$$\beta_0 = -3.71,\ \beta_1 = 17.8,\ \beta_2 = -12.5,\ \beta_3 = -0.80.$$

$u$ is given from the other quantities using Equation 5.1.

Table 8 gives the final results. Columns 1, 2, and 3 are as in Table 6 and Column 4 holds the new component of uncertainty.

TABLE 8

| Year | (1) Total Expected Future Payments | (2) Standard Error | (3) Standard Deviation | (4) Additional Uncertainty (Number of Claims) | (5) Root-Mean-Square Error |
|---|---|---|---|---|---|
| 1969 | 3,450 | 1,169 | 898 | 845 | 1,700 |
| 1970 | 6,397 | 1,800 | 1,287 | 1,505 | 2,676 |
| 1971 | 15,034 | 3,261 | 2,071 | 2,484 | 4,593 |
| 1972 | 25,360 | 4,271 | 2,761 | 3,580 | 6,220 |
| 1973 | 35,962 | 4,873 | 3,312 | 4,671 | 7,519 |
| 1974 | 40,132 | 4,464 | 3,464 | 5,481 | 7,872 |
| 1975 | 47,279 | 4,796 | 3,696 | 6,843 | 9,137 |
| 1976 | 59,015 | 5,876 | 4,089 | 10,393 | 12,620 |
| All | 232,630 | 29,988 | 8,229 | 15,122 | 34,578 |

The uncertainty in Column 4 for all years combined has been calculated on the assumption that the estimates $M_w$ are mutually independent. It is the square root of the sum of the squares of the separate origin year figures. If non-zero covariances for the $M_w$ were known, they could easily be brought into the calculation.

The three components of error (Columns 2, 3, and 4) are always mutually independent (to a good approximation), so the overall RMS error (Column 5) is simply the square root of the sum of the squares of these three columns.

Allowing for uncertainty in the number of claims outstanding has resulted in an increase in the overall standard error from \$31.1 million to \$34.6 million. The reserve based on best estimate plus one standard error has changed from \$264 to \$267 million, an increase of 1.01%.

To demonstrate the validity of Equation 5.1, the variation of $\hat{\mu}$ with $\hat{M}$ has been investigated empirically. In Table 9, the first column gives the theoretical rate of change of $\hat{\mu}$ with $\hat{M}$ for each origin year; that is, the quantity in square brackets in Equation 5.1. The remaining columns show the actual changes in $\hat{\mu}$ per unit change in $M$, when $M$ is changed by the amount shown at the head of each column. A dash indicates that a result could not be calculated because the changed value for $M$ was less than the number of claims paid to date, $N_0$.

For example, in Table 9, the figure 16.56 in the fifth column for 1972 was obtained as follows: The best estimate 4,771 of the ultimate number of claims $M$ was increased by 100 to 4,871. Since the number $N_0$ of claims to date is 2,938, this implies 1,933 claims remaining. The fitted model $m_\tau$ of Section 5 was summed over the 1,933 different values $\tau = 2{,}938.5/4{,}871$ to $\tau = 4{,}870.5/4{,}871$. This gave the result 27,016. This is 1,656 greater than the best estimate of 25,360; and, since $M$ was increased by 100, the mean rate of change is 16.56.

TABLE 9

| | | Change in $M$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Year | (1) Theoretical Value | (2) -10 | (3) 10 | (4) -100 | (5) 100 | (6) -500 | (7) 500 | (8) -1,000 | (9) 1,000 |
| 1969 | 12.08 | 12.00 | 12.15 | 11.29 | 12.79 | — | 14.82 | — | 15.85 |
| 1970 | 14.75 | 14.70 | 14.80 | 14.22 | 15.20 | 11.36 | 16.26 | — | 16.48 |
| 1971 | 16.78 | 16.78 | 16.79 | 16.67 | 16.86 | 15.80 | 16.91 | 13.64 | 16.59 |
| 1972 | 16.65 | 16.66 | 16.64 | 16.73 | 16.56 | 16.89 | 16.15 | 16.55 | 15.56 |
| 1973 | 14.88 | 14.89 | 14.86 | 15.00 | 14.75 | 15.51 | 14.27 | 16.06 | 13.74 |
| 1974 | 11.89 | 11.90 | 11.88 | 11.98 | 11.81 | 12.38 | 11.51 | 13.02 | 11.22 |
| 1975 | 9.92 | 9.92 | 9.92 | 9.93 | 9.90 | 10.01 | 9.85 | 10.14 | 9.80 |
| 1976 | 9.47 | 9.47 | 9.47 | 9.47 | 9.47 | 9.47 | 9.47 | 9.48 | 9.47 |

These results show that, for all origin years, the rate of change is almost constant within the range $M \pm v$, and is close to the theoretical value, so Equation 5.1 is a good approximation.

## 6. FUTURE INFLATION

*Theory*

Previous sections have been concerned with finding a model $m_\tau$ of the mean claim amount in constant money terms, and using the fitted model to calculate predictions in constant money terms. This section is concerned with the inclusion of future claims inflation in predictions, with due allowance for the inevitable uncertainty. This is necessary if the predictions are to be used as a basis for setting reserves, because reserves are conventionally in current money terms (not discounted).

Uncertainty in future claims inflation arises from two sources:

1. uncertainty in the future rate of claims inflation, and
2. uncertainty in the timing of the run-off of future payments.

Appendix F shows how both these elements of uncertainty can be taken into account simultaneously. Obviously, if the run-off of future claim payments is expected to take many years, moderate uncertainty in the future rate of claims inflation may lead to substantial uncertainty in current price predictions, because of the exponential effect of inflation.

*Numerical Example*

To illustrate the method of Appendix F, future inflation is introduced into the predictions obtained in Section 5. An exponential run-off of the remaining claim settlements over development time is used for all origin years. The time scale of the run-off can be estimated by examining the triangle of operational times (Table A.4).

Since the origin years are accident years, the mean delay to settlement is approximately $d$ years for claims closed in development year $d$ (except $d=0$, for which the mean delay is about 0.33 years). The triangle of operational times indicates a "half life" of just over three years. The 95% confidence range for the half life is judged to be 2.8 to 3.6 years. This corresponds to a best estimate of 3.2, and a coefficient of variation of about 0.06. In the notation of Appendix F: $U_\varphi = 0.06^2 = 0.0036$.

From Equation F.13, the best estimate of the parameter $\beta$ of the exponential distribution is:

$$\beta = 3.2/\ln(2) = 4.6 \text{ years.}$$

Using Equation F.14, the remaining real delay $t$ corresponding to future operational time $\tau$ is estimated (in years) to be:

$$t = H(\tau) = -4.6 \cdot \ln[\,(1-\tau)/(1-\tau_0)\,]\,.$$

For this example, it is assumed that the estimate of the average force of future claims inflation (from mid-1976 onwards) is 0.1 with a standard error of 0.02. Thus, it is expected that inflation will be less in the future than in the past, but the 95% confidence range of 0.06 to 0.14 contains the best estimate 0.135 of the average past force of inflation (Section 4). In the notation of Appendix F, $i = 0.1$ and $U_i = 0.02^2$. Equation F.7 gives $0.021^2$ for the variance $U_\theta$ due to both uncertainty in the future force of inflation and uncertainty in the future time scale. This is only slightly greater than $U_i$, indicating that the second element of uncertainty is relatively minor.

The current price predictions for each origin year are given in Table 10.

TABLE 10

| Year | (1) Expected Total Future Payments | (2) Standard Error | (3) Additional Uncertainty (Inflation) | (4) Standard Deviation | (5) Additional Uncertainty (Number of Claims) | (6) Root-Mean-Square Error |
|---|---|---|---|---|---|---|
| 1969 | 5,531 | 2,056 | 735 | 1,306 | 900 | 2,699 |
| 1970 | 9,934 | 3,202 | 1,230 | 1,801 | 1,629 | 4,203 |
| 1971 | 22,794 | 6,027 | 2,657 | 2,819 | 2,767 | 7,680 |
| 1972 | 38,233 | 8,374 | 4,345 | 3,735 | 4,160 | 10,966 |
| 1973 | 54,798 | 10,254 | 6,299 | 4,530 | 5,791 | 14,103 |
| 1974 | 63,436 | 10,329 | 7,752 | 4,917 | 7,702 | 15,821 |
| 1975 | 79,899 | 12,211 | 10,903 | 5,580 | 11,197 | 20,603 |
| 1976 | 109,297 | 16,480 | 17,054 | 6,658 | 19,209 | 31,236 |
| All | 383,922 | 68,658 | 50,976 | 12,124 | 24,812 | 89,861 |

Columns 1, 2, 3, 4, and 5 are as in Table 8, except that each expected claim amount $m_\tau$ has been inflated using the factor $\exp(i \cdot H(\tau))$ before finding the total for each origin year. The new Column 3 gives the additional element of uncertainty calculated from Equation F.11. The figure for all years combined (50,976) is simply the sum of the figures for the separate origin years (this comes from repeating the argument given in Appendix D using the variance-covariance matrix of Appendix F). Intuitively, it is clear that this new component of uncertainty will be highly correlated between origin years, because the projections for all origin years are based on the same estimate of future inflation. If we overestimate future inflation, then we overestimate the reserve for all origin years simultaneously. The apparent *perfect* correlation (additivity of Column 3) is an approximation resulting from the use of first order Taylor series for these standard errors (Appendices D and F). The use of Taylor series approximations does not induce apparent perfect correlation in Column 2 (68,658 is less than the column total of 68,933) because these standard errors represent uncertainty in more than one parameter estimate (the $\beta$s) and the estimation errors are not perfectly mutually correlated.

Column 6 gives the overall mean-square-error calculated as the square root of the sum of the squares of Columns 2, 3, 4, and 5.

It is interesting to look at the delays and inflation factors of the last claims as given by the estimated function $H(\tau)$. The expected ultimate number of claims is 6,257 for origin year 1976. The operational times of the last three claims for this origin year are therefore 0.99960, 0.99976, and 0.99992. The expected delays from accident to settlement of these claims (calculated from $t = -4.6 \cdot \ln(1 - \tau)$) are 36.0, 38.3, and 43.4 years, respectively. Using the estimated force of inflation $i = 0.1$, the estimated inflation factors are 36.6, 46.1, and 76.7. These factors are obviously very sensitive to the estimate $i$, which is why the extra element of uncertainty can be substantial.

In practice, the function $H(\tau)$ would be estimated more carefully than in this example, making use of any additional information on likely delays. The estimated time scale parameter (4.6 in the example) need not be the same for all origin years. The method detailed in [10] can be applied to the number of claims triangle to obtain an estimate and a standard error (hence a value for $U_\varphi$) for each origin year. A Gamma, rather than an exponential run-off can be used to construct $H(\tau)$ if necessary, but this is unlikely to make much difference to the results except for the last one or two origin years. The use of a Gamma run-off is illustrated in the example of Section 7.

## 7. PARTIAL PAYMENTS

*Theory*

Paid claims run-off triangles are usually of type (c) (see Section 1) in practice. That is, the counts triangle $N_{wd}$ is the number of claims closed in each development year $d$ of each origin year $w$, but the paid amounts triangle $Y_{wd}$ is the total of all payments made in development period $d$ of origin year $w$. Each $Y_{wd}$ includes partial payments on claims settled at some later development period, as well as the settlement payments counted in $N_{wd}$. The following paragraphs describe special procedures that may be necessary when the partial-payment component of $Y_{wd}$ is substantial.

Dropping the subscripts $w$ and $d$ temporarily, each $Y$ has two components:

$$Y = Y_1 + Y_2 ,$$

where

$Y_1$ is the total of payments made on claims closed, and

$Y_2$ is the total of payments made on claims not closed.

$N_1$ will denote the number of settlement payments; that is, the number of individual payments making up $Y_1$. Similarly, $N_2$ will denote the number of prepayments on claims not yet closed: the number of individual payments in $Y_2$. These quantities are not all known: the data consists only of $Y$ and $N_1$, for each $w$, $d$ combination. The mean claim amount which can be calculated from the data is:

$$S = (Y_1 + Y_2)/N_1 . \tag{7.1}$$

Since the expected values of $Y_1$ and $Y_2$ may follow two different patterns as operational time $\tau$ varies from 0 to 1, the form of the expected value of $S$ (as a function of $\tau$) is likely to be more complex than when the component $Y_2$ is not included (situation (a) from Section 1). Furthermore, random variation of $S$ around its expected value will be negatively correlated with random variation in $N_1$. This is explained further in the following sections.

Both $N_1$ and $N_2$ are subject to random variation. Initially it is assumed that they are stochastically independent. This will be discussed further below. If $N_1$ is higher than expected (it just happens that a large number of claims reach the settlement stage at about the same time), $Y_1$ will be correspondingly high, as it is the total of the $N_1$ settlement payments. But, $N_2$ (hence $Y_2$) will not be affected, so $S$ will tend to be lower than expected. Conversely, a low value for $N_1$ will tend to give a high value for $S$, because $Y_1$ will be proportionately low, but $Y_2$ will not.

There is an argument which suggests that $N_1$ and $N_2$ may be positively correlated. This would limit the negative association between $N_1$ and $S$ described above, and would eliminate it completely if the expected value of $N_2$, given $N_1$, were proportional to $N_1$. The argument is that both $N_1$ and $N_2$ may be affected in the same direction by a common cause; namely, increased activity by the insurance company on claim payment procedures, regardless of whether the payments are settlements or prepayments. This is discussed further in Appendix G. There are also arguments which suggest that $N_1$ and $N_2$ may be negatively correlated. This would substantially increase the negative association between $N_1$ and $S$ described above. First, if the number of claims closed in a certain development period is unusually large, the number of claims left outstanding at the end of the period will be correspondingly small, so the number of partial payments on such claims will also tend to be small. Second, if many claims are ready for settlement at about the same time, the demand on resources made by these settlements may reduce the resources available to deal with prepayments on outstanding claims.

Previous sections have been concerned with modelling the expected value $m_\tau$ of the sample means calculated from $S = Y_1/N_1$. (In the terms of Section 1, this is situation (a).) It is shown in Appendix G that, under certain assumptions, the effect of including partial payments $Y_2$ in the numerator of $S$ is approximately the same as increasing the mean $m_\tau$ by a factor of $\exp(c \cdot R_{wd})$, where $c$ is a constant, and $R_{wd}$ is the ratio of the number of claims outstanding in development period $d$ to the number settled during development period $d$. That is, if $L_{wd}$ is the number of claims outstanding, then $R_{wd} = L_{wd}/N_{wd}$. The coefficient $c$ represents the expected partial payment per outstanding claim (including those with no partial payments), as a proportion of the mean size of settlement payments. For example, if the average number of partial payments in any development period is one for every five outstanding claims, and the mean size of these partial payments is half the mean size of settlement payments, then $c = 0.2 \times 0.5$.

If the model for $m_\tau$ has a linear form for $\ln(m_\tau)$, then the factor $\exp(c \cdot R_{wd})$ simply introduces a further term to the linear exponent. If $R_{wd}$ is known, $c$ can be estimated in the same way as the other parameters of the model. For example, in the models of Section 4, the parameter vector becomes:

$$\beta = (c, i, \beta_0, \beta_1, \ldots)$$

and the vector $\boldsymbol{x}_{w\tau}$ of known explanatory variables becomes:

Model 0: $\boldsymbol{x}_{w\tau} = (R_{wd}, w + d/P, u \ldots u, \tau_j, 0 \ldots 0)$

Model 1: $\boldsymbol{x}_{w\tau} = (R_{wd}, w + d/P, 1, \tau, \ln(\tau))$

Model 2: $\boldsymbol{x}_{w\tau} = (R_{wd}, w + d/P, 1, \tau, \tau^2)$.

Other models (such as Models 3 and 4) can be fitted by first dividing each observation $S$ by $\exp(c \cdot R_{wd})$, using the value of $c$ estimated from Model 0. This procedure is similar to the preadjustment for inflation described in Section 4.

Appendix G also shows that if $m_\tau$ represents the mean with the factor $\exp(c \cdot R_{wd})$ included, then Equation 2.3 for the variance of $S$ becomes approximately:

$$\mathrm{Var}(S_{w\tau}) = \varphi^2 \cdot m_\tau^2/(\exp(c \cdot R_{wd}) \cdot N_{wd}). \tag{7.2}$$

Therefore, to fit the models above, the factor $N_{w\tau}$ must be replaced by $\exp(c \cdot R_{wd}) \cdot N_{wd}$ in the deviance (Equation 2.9). An initial estimate of $c$ is required for this purpose. If the estimate of $c$ obtained by fitting Model 0 differs significantly from the initial estimate, then other results should be disregarded and the model fitting should be repeated using the new estimate of $c$ in the deviance.

A number of assumptions are made in Appendix G, leading to the results quoted above. There is no need to consider too carefully how realistic these assumptions are in each application. The purpose of the mathematics in Appendix G is to find a broad model of which can be tested against the data. Standard statistical techniques, such as residual plots and $F$-tests can be used to determine whether or not the models adequately represent any particular data set. In a similar vein, although the mathematics of Appendix G deal only with the case $\alpha = 2$ in the variance function, other values of $\alpha$ can be used in fitting the models of this section exactly as described in Section 3, if the data indicate that this is necessary.

If the coefficient $c$ is found to be significant, then forecasting is not as simple as in the pure operational time models of Sections 2 and 3. In order to include the partial payment effect in the forecasts, values of $R = L/N$ must be projected for future operational times so that $x_{w\tau}$ is known. In some cases, $R$ can be modelled as a function of operational time. Projections can then be obtained as described in earlier sections, with just one change. The expression at Equation D.2 for the variance of an origin year total (standard deviation columns in the results tables) is replaced by:

$$\sigma^2 = \varphi^2 \cdot \sum_{\tau} (m_\tau^\alpha / \exp(c \cdot R_\tau)). \tag{7.3}$$

This follows from Equation 7.2. The quantity $\varphi^2 \cdot m_\tau^\alpha / \exp(c \cdot R_\tau)$ is the variance of the total of payments made over a single increment $1/M$ in operational time. $m_\tau$ is the expected value. More generally, separate projections of $R$ can be made for each origin year. The situation is much the same as for claims inflation.

If the reported number of claims triangle is not known, then $L$, the number of claims outstanding, is not known. In such a case, $L$ can be defined as the number of claims not yet settled (whether reported or not). If the proportion not yet reported is approximately constant over development time, then $L$ is increased by some constant factor under this alternative definition, so the parameter $c$ will be decreased by the same factor. If the run-off of claims closed is approximately exponential from some point onwards, then $R = L/N$ is approximately constant. This can simplify projections. If the run-off is exponential from operational time 0 onwards, then the parameter $c$ will probably be insignificant because $c \cdot R$ will be subsumed into the parameter $\beta_0$ of $m_\tau$. In such a case, the models of Sections 2, 3, and 4 can be used even if partial payments are substantial.

*Numerical Example*

To illustrate, the methods described above are applied to the data from Berquist and Sherman [1] used in previous examples. These methods are more appropriate for this data set than the methods of earlier sections because the claim amounts triangle includes partial payments. In the terminology of Section 1, the data set is type (c).

The variable $R_{wd}$ was calculated using the numbers $L_{wd}$ of claims outstanding given in Table A.5. Table 11 gives the minimized deviance and $F$-statistics for the models described in this section. The $F$-statistic for Model 0 compares Model 0 to the less restrictive model of Appendix B. It has eight and 17 degrees of freedom. The $F$-statistics for Models 1 and 2 compare each of these models to Model 0. They have six and 25 degrees of freedom. The models were fitted using the variance function given in Equation 7.2, with a prior estimate of 0.1 for $c$ and an index $\alpha = 1.5$, instead of 2. This gave satisfactory residual plots. Figure 8 shows the standardized residuals from Model 0 plotted against operational time. The same residuals are plotted against $R_{wd}$ in Figure 9 (for the reasons given in Appendix G).

FIGURE 8

RESIDUAL PLOT FOR MODEL ZERO

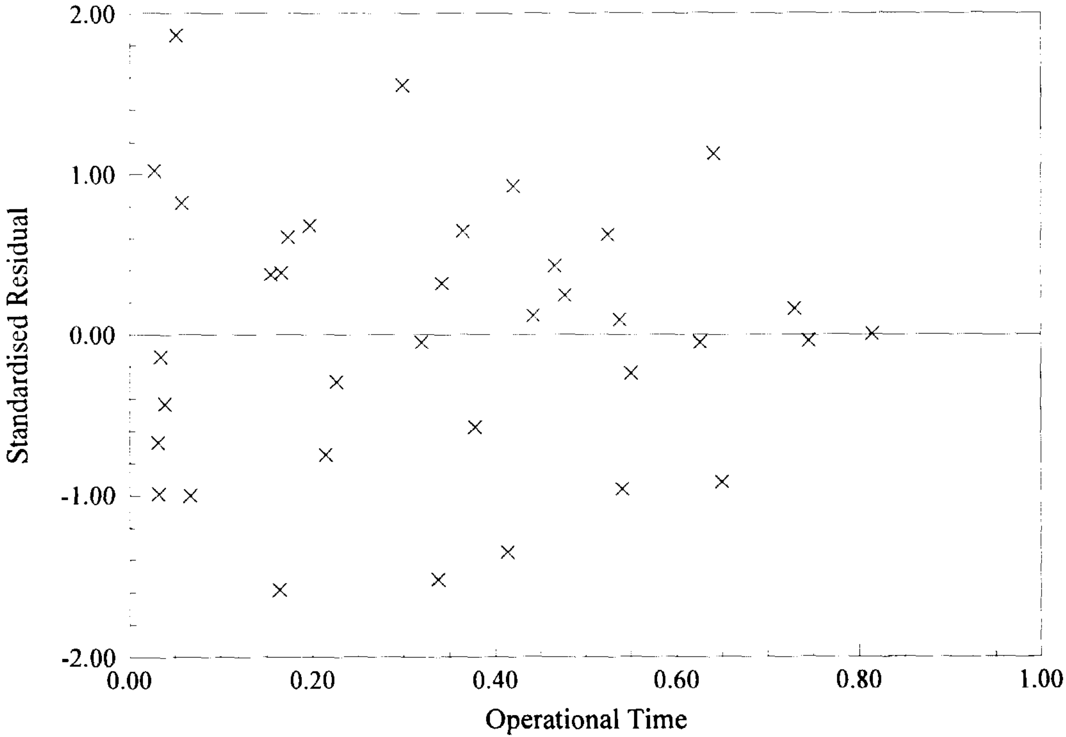


FIGURE 9

RESIDUAL PLOT FOR MODEL ZERO

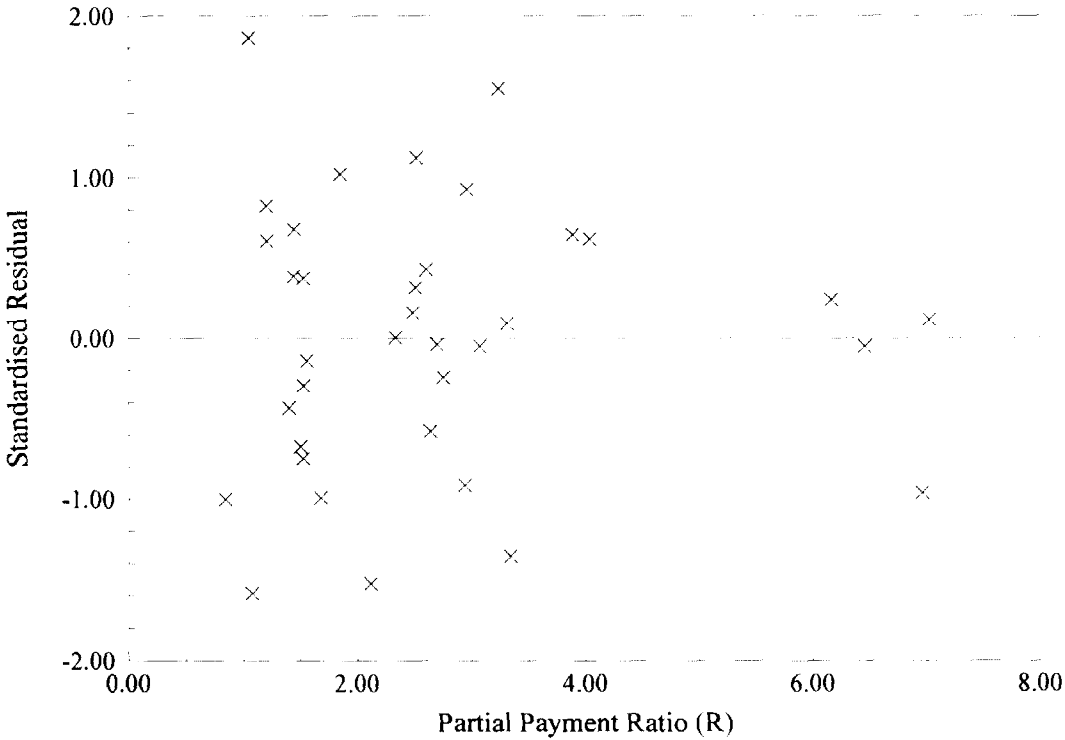


TABLE 11

| Model | Deviance | $df$ | $F$ |
|---|---|---|---|
| 0 | 2,143 | 25 | 0.29 |
| 1 | 3,909 | 31 | 3.43 |
| 2 | 2,773 | 31 | 1.23 |

Model 2 appears to fit reasonably well. A direct comparison of the deviance with that obtained in Table 5 is not valid, because the prior weights have been changed but, compared to Model 0, the fit is considerably better than in Table 5. The parameter estimates (and standard errors) for Model 2 are:

| | | | |
|---|---|---|---|
| $c$ | 0.127 | (0.044) | |
| $i$ | 0.176 | (0.036) | |
| $\beta_0$ | -1.04 | (0.21) | |
| $\beta_1$ | 9.56 | (1.16) | |
| $\beta_2$ | -6.24 | (1.26) | (coefficient of $\tau^2$) . |

The magnitude of $c$ is consistent with its theoretical interpretation and is not significantly different from the prior estimate of 0.1 used in the weights.

In the examples of previous sections, the fitted models $m_\tau$ have always been decreasing for large values of $\tau$ (see Exhibit 4, for example). In Section 3, this decrease was attributed to the presence of partial payments in the data. In the present example, the partial payments have explicitly been taken into account by including the factor $\exp(c \cdot R)$ in the model. Figure 10 shows that $R$ tends to decrease from about $\tau = 0.5$ onwards, so the partial payment factor $\exp(c \cdot R)$ also decreases. However, the other factor of the fitted model, $\exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2)$, also decreases for large values of $\tau$. This is shown in Figure 11. The next paragraph describes how to test whether this remaining decrease is genuine or is due to estimation error.

FIGURE 10

FITTED MODEL FOR PARTIAL PAYMENT RATIO

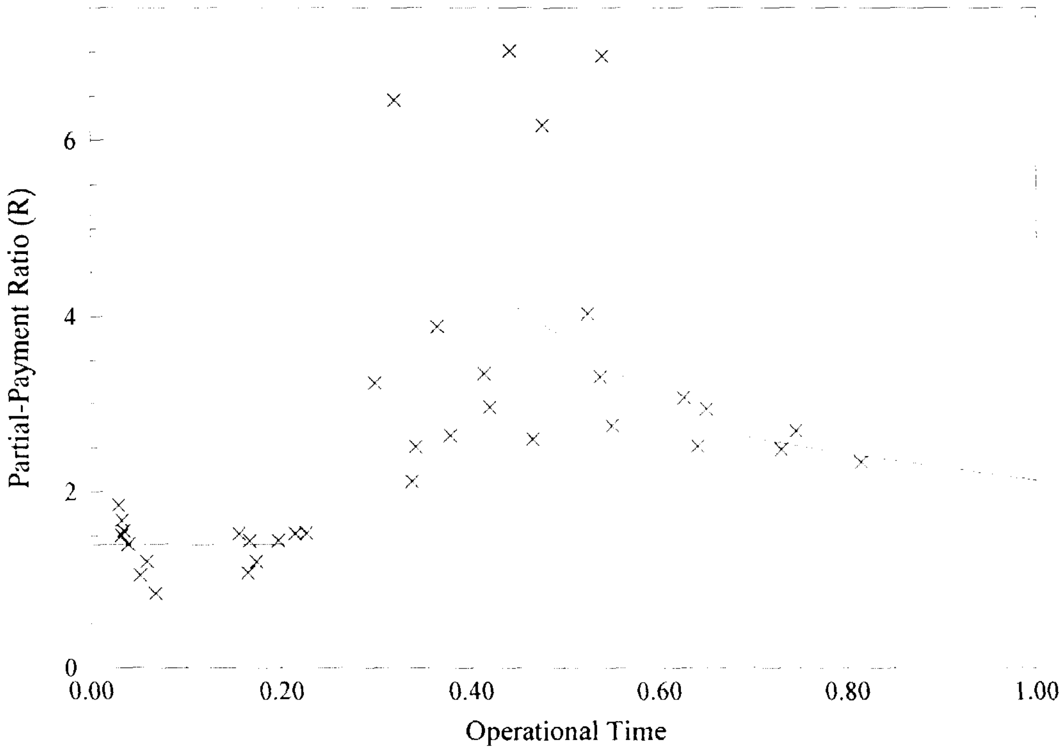
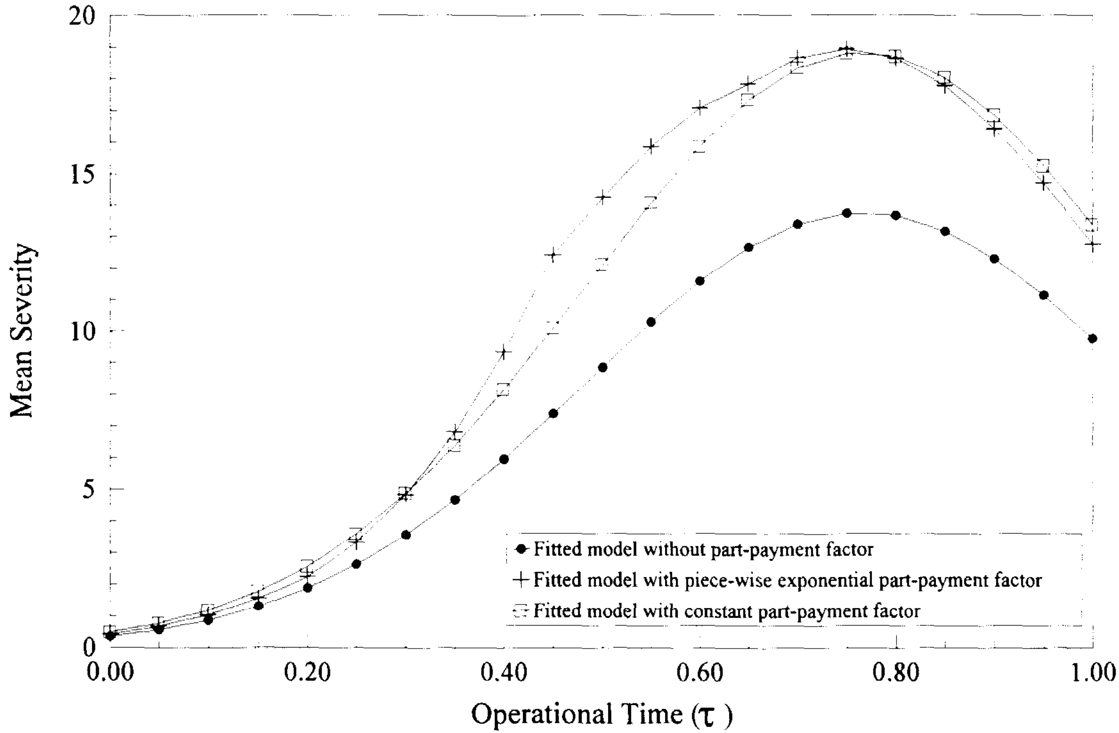


FIGURE 11

FITTED MODELS FOR MEAN SEVERITY



The slope of the exponent $\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2$ is $\beta_1 + 2 \cdot \beta_2 \cdot \tau$, so the function does not decrease for $\tau$ in the range $(0, 1)$ if, and only if, $\beta_1$ is not negative and $\beta_2$ is not less than $-\beta_1/2$. If $\beta_3$ is defined by $\beta_3 = \beta_2 + \beta_1/2$, the exponent can be expressed as $\beta_0 + \beta_1 \cdot (\tau - \tau^2/2) + \beta_3 \cdot \tau^2$, and the condition for the function to be non-decreasing is that $\beta_1$ and $\beta_3$ should both be non-negative. This condition can be tested by refitting the model using the new explanatory variable $(\tau - \tau^2/2)$ instead of $\tau$. This is essentially the same model as before so the parameter estimates are unchanged. $\beta_3$ is estimated to be -1.46, which follows from the estimates of $\beta_1$ and $\beta_2$ given above. However, fitting the model in this new form gives a standard error for $\beta_3$ which cannot be calculated from the previous parameter estimates and standard errors. The value is 0.72—less than half the absolute value of the estimate itself, indicating that $\beta_3$ is significantly negative. This is confirmed by refitting with $\beta_3$ set to zero. The minimized deviance becomes 3,154, an increase of 381, giving an $F$-statistic of 4.3 on one and 31 degrees of freedom.

The analysis described in the earlier paragraph shows that, for this data set, the decrease in the mean claim amount for large values of $\tau$ is not fully explained by the partial payment factor $\exp(c \cdot R)$. However, this does not imply that the mean settlement payment decreases with $\tau$. A more plausible explanation is that the factor $\exp(c \cdot R)$ only partly accounts for the effects of partial payments. Full details of how this might occur are given Appendix G. Briefly, the explanation is that the rate at which partial payments are made on an open claim tends to decrease the longer the claim remains open.

As the coefficient $c$ is significant, it is necessary to estimate a value of $R$ for each future operational time in order to project the fitted model. Experience with other data sets suggests that $R$ shows little variation as $\tau$ approaches one. With this in mind, a continuous piecewise linear approximation for $R$ has been estimated by eye from the observed values:

$$
\begin{aligned}
R_\tau &= 1.40 & &\text{for} & \tau &< 0.21 \\
&= -0.96 + 11.25 \times \tau & &\text{for } 0.21 < & \tau &< 0.45 \\
&= 7.25 - 7.00 \times \tau & &\text{for } 0.45 < & \tau &< 0.65 \\
&= 3.80 - 1.70 \times \tau & &\text{for } 0.65 < & \tau &.
\end{aligned}
$$

Figure 10 shows both the observed values of $R$ and the piecewise linear approximation $R_\tau$.

In 1976 money terms, the formula for $R_\tau$ gives the results in Table 12. Column 3 has been calculated using Equation 7.3.

TABLE 12

| Year | (1) Expected Total Future Payments | (2) Standard Error | (3) Standard Deviation | (4) Additional Uncertainty | (5) Root-Mean-Square Error |
|---|---|---|---|---|---|
| 1969 | 6,187 | 1,711 | 1,323 | 1,224 | 2,485 |
| 1970 | 10,119 | 2,434 | 1,710 | 1,869 | 3,514 |
| 1971 | 20,757 | 4,138 | 2,467 | 2,705 | 5,525 |
| 1972 | 31,795 | 5,287 | 3,049 | 3,741 | 7,159 |
| 1973 | 42,811 | 6,029 | 3,506 | 5,074 | 8,625 |
| 1974 | 46,865 | 5,554 | 3,598 | 6,277 | 9,121 |
| 1975 | 54,903 | 5,913 | 3,828 | 7,930 | 10,607 |
| 1976 | 68,591 | 7,197 | 4,240 | 12,079 | 14,686 |
| All | 282,027 | 37,839 | 8,828 | 17,327 | 42,544 |

As no allowance has been made for uncertainty in the projected values of $R$, it is interesting to examine the sensitivity of the results to these projections. The piecewise linear function defined earlier implies an average value of $R$ over the entire range (0, 1) of $\tau$ of 2.47. Table 13 was obtained using this constant value for $R$.

TABLE 13

| Year | (1) Expected Total Future Payments | (2) Standard Error | (3) Standard Deviation | (4) Additional Uncertainty | (5) Root-Mean-Square Error |
|---|---|---|---|---|---|
| 1969 | 6,372 | 1,778 | 1,337 | 1,244 | 2,549 |
| 1970 | 10,345 | 2,512 | 1,723 | 1,878 | 3,578 |
| 1971 | 21,017 | 4,234 | 2,477 | 2,679 | 5,589 |
| 1972 | 31,867 | 5,373 | 3,051 | 3,611 | 7,156 |
| 1973 | 42,180 | 6,080 | 3,489 | 4,710 | 8,446 |
| 1974 | 45,069 | 5,541 | 3,555 | 5,864 | 8,816 |
| 1975 | 52,742 | 5,873 | 3,781 | 7,687 | 10,386 |
| 1976 | 66,257 | 7,154 | 4,190 | 11,677 | 14,321 |
| All | 275,850 | 38,159 | 8,767 | 16,653 | 42,547 |

Between the two sets of results given in Tables 12 and 13 the estimate for the entire triangle differs by just over $7 million, which is quite small compared to the RMS error of about $43 million. Thus, the uncertainty in future values of $R$ appears to be relatively unimportant. Experience with other data sets suggests this is true quite generally. Figure 11 shows the fitted model for $m_\tau$ obtained using both the piecewise linear model for $R$ (Curve 2) and the constant model (Curve 3). The difference between these two curves is slight, which explains the similarity in the two sets of results. Curve 3 is simply a scaled up version of curve 1 which shows the fitted model with the partial payment factor exp $(c \cdot R)$ excluded.

Table 14 gives results based on the piecewise linear model for $R$ in current money terms. Future inflation has been included using the methods described in Section 6. The run-off of settlements over real development time was taken to be exponential with the same parameters as in Section 6, and the 95% confidence interval for the future force of inflation was taken as 0.10 to 0.18.

TABLE 14

| Year | (1) Expected Total Future Payments | (2) Standard Error | (3) Additional Uncertainty (Inflation) | (4) Standard Deviation | (5) Additional Uncertainty (Number of Claims) | (6) Root-Mean-Square Error |
|---|---|---|---|---|---|---|
| 1969 | 15,193 | 4,796 | 3,317 | 3,176 | 1,461 | 6,799 |
| 1970 | 24,570 | 7,202 | 5,370 | 4,097 | 2,378 | 10,156 |
| 1971 | 50,298 | 13,404 | 11,132 | 5,994 | 3,780 | 18,810 |
| 1972 | 77,679 | 18,976 | 17,392 | 7,550 | 5,809 | 27,447 |
| 1973 | 106,385 | 24,112 | 24,200 | 8,932 | 8,855 | 36,404 |
| 1974 | 121,292 | 25,510 | 28,390 | 9,628 | 13,371 | 41,572 |
| 1975 | 155,292 | 31,440 | 38,697 | 11,172 | 21,330 | 55,369 |
| 1976 | 220,251 | 44,179 | 59,611 | 13,823 | 38,668 | 84,803 |
| All | 770,959 | 169,504 | 188,109 | 24,661 | 47,574 | 258,821 |

Berquist and Sherman produced several sets of projections, with totals ranging from \$430 million to \$750 million. The best estimate shown in Table 14 is comparatively high, but the RMS error is large. It would be interesting to see how these estimates compare to the actual experience.

*Another Numerical Example*

As a final example, the data set from Taylor [9] is analyzed using the methods developed in the present paper. This relates to a compulsory third party motor portfolio for the 12 accident years 1969 to 1980, broken down by development year. There were no claims closed in 1980 for origin year 1969, so the number of data points is 77. The triangles $Y_{wd}$ and $N_{wd}$ are given in Tables A.6 and A.7. The estimates of ultimate numbers $M$ given by Taylor have been used. For the purposes of this example, the standard errors of these estimates have been taken as 5% of the number of claims estimated to be remaining (not yet settled). These figures are given in Table A.8. As the reported counts triangle is not given in [9], values for $R = L/N$ have been calculated using the alternative definition of $L$ given earlier in Section 7. This triangle is given in Table A.10.

Figures 12 and 13 show the data $S$ and $R$ plotted against operational time $\tau$. Figure 14 shows the quantity $S/\exp(c \cdot R)$ plotted against $\tau$ using the prior estimate 0.1 for $c$. This quantity is the mean payment per claim closed adjusted to remove the partial payment effect (see Appendix G). One data point has been excluded from Figures 12 and 14 for the sake of clarity: the value of $(\tau, S)$ for $d = 11$ of origin year 1970 is (0.995, 108.2). This value for $S$ is more than double the largest value shown in Figure 12. However, it is based on only two settlements, and standardized residual plots show that it is not an outlier, therefore, it has not been excluded from the analysis.

Table 15 gives the results of fitting models with both a partial payment parameter and an inflation parameter to the unadjusted data shown in Figures 12 and 13. The variance index was taken to be $\alpha = 2$, and the prior estimate of the partial payment parameter for use in the weights was taken as $c = 0.1$. The plot of standardized residuals from Model 0 against operational time is shown in Figure 15. There is no evidence of heteroscedasticity, so the $F$-statistics are valid. The standardized residual for the point excluded from Figures 12 and 14 is included in Figure 15.

TABLE 15

| Model | Deviance | $df$ | $F$ |
|---|---|---|---|
| 0 | 618.6 | 63 | — |
| 1 | 860.4 | 72 | 2.74 |
| 2 | 848.2 | 72 | 2.60 |

The $F$-statistics indicate that neither Model 1 nor Model 2 fit the data very well. However, the more general model with both $\tau^2$ and $\ln(\tau)$ in the linear predictor (as in the numerical examples of Sections 3 and 4) fits well. The minimized deviance is 704.8, giving an $F$-statistic of 1.10 on eight and 63 degrees of freedom.

FIGURE 12

OBSERVED MEAN SEVERITY VS. OPERATIONAL TIME

(DATA FROM TAYLOR [9])

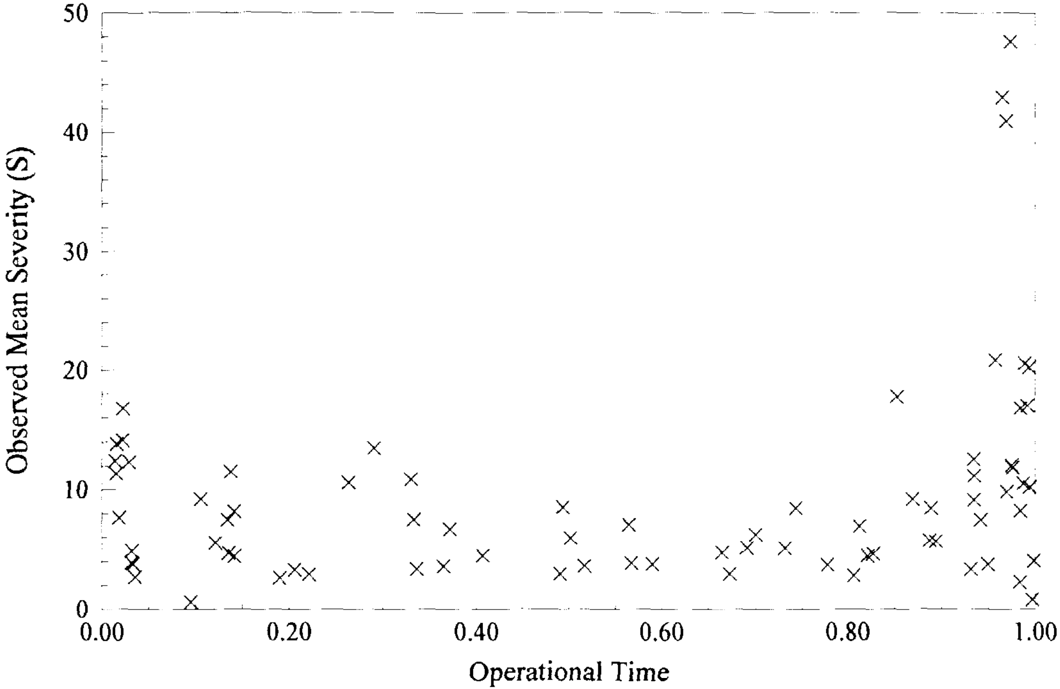


FIGURE 13

PARTIAL PAYMENT RATIO (R) VS. OPERATIONAL TIME

(DATA FROM TAYLOR [9])

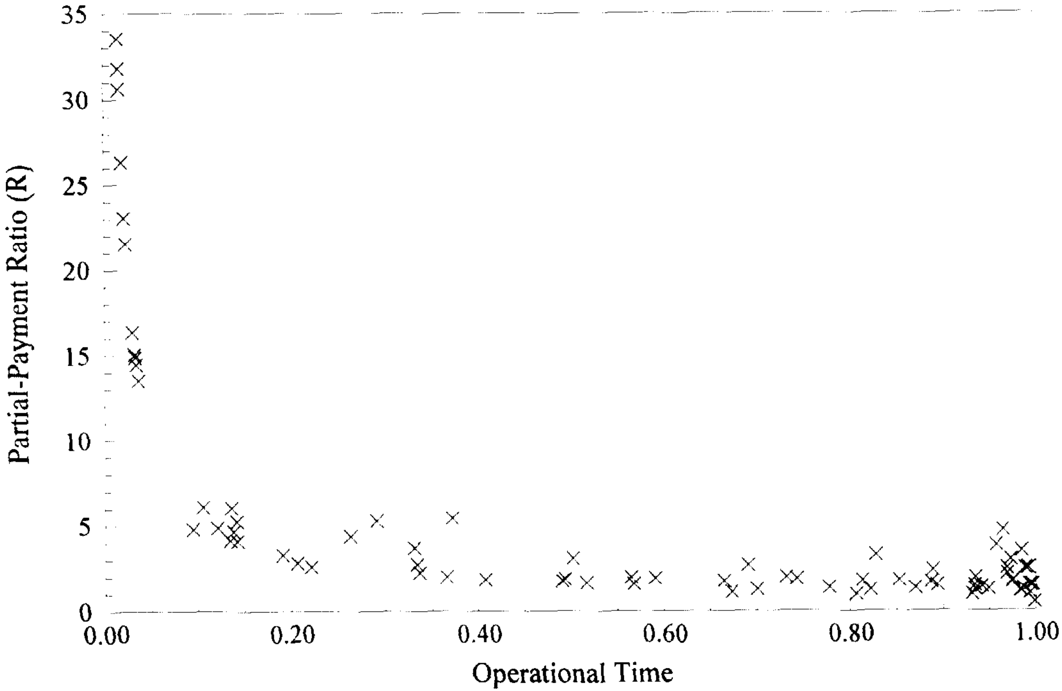


FIGURE 14

ADJUSTED MEAN SEVERITY VS. OPERATIONAL TIME

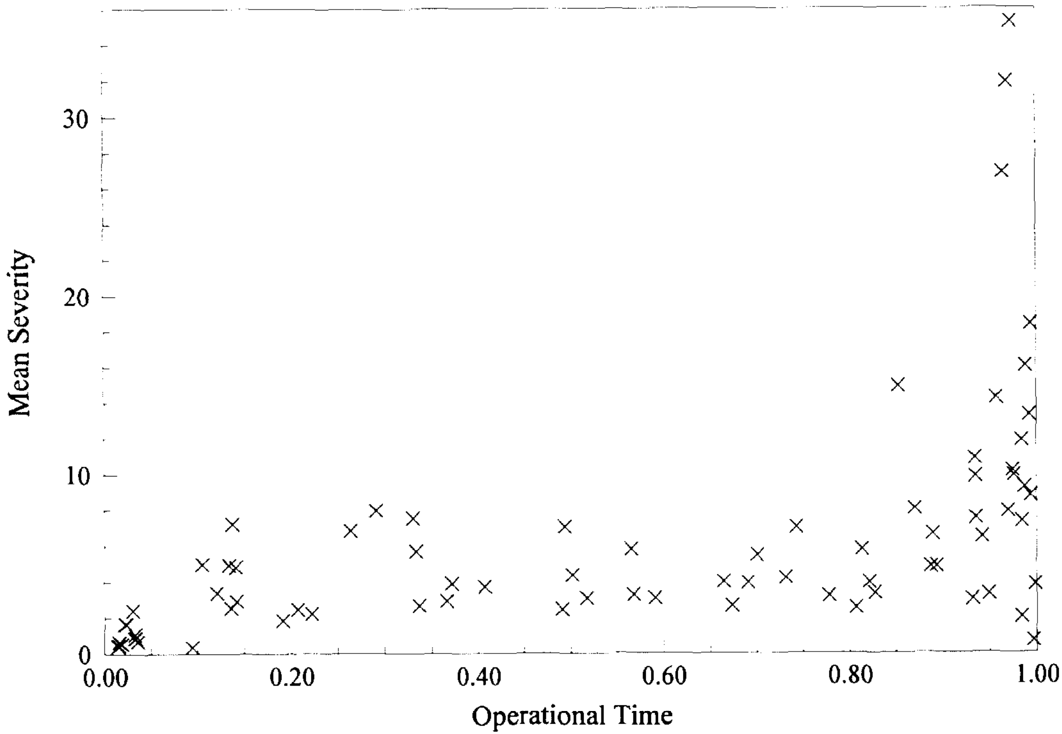
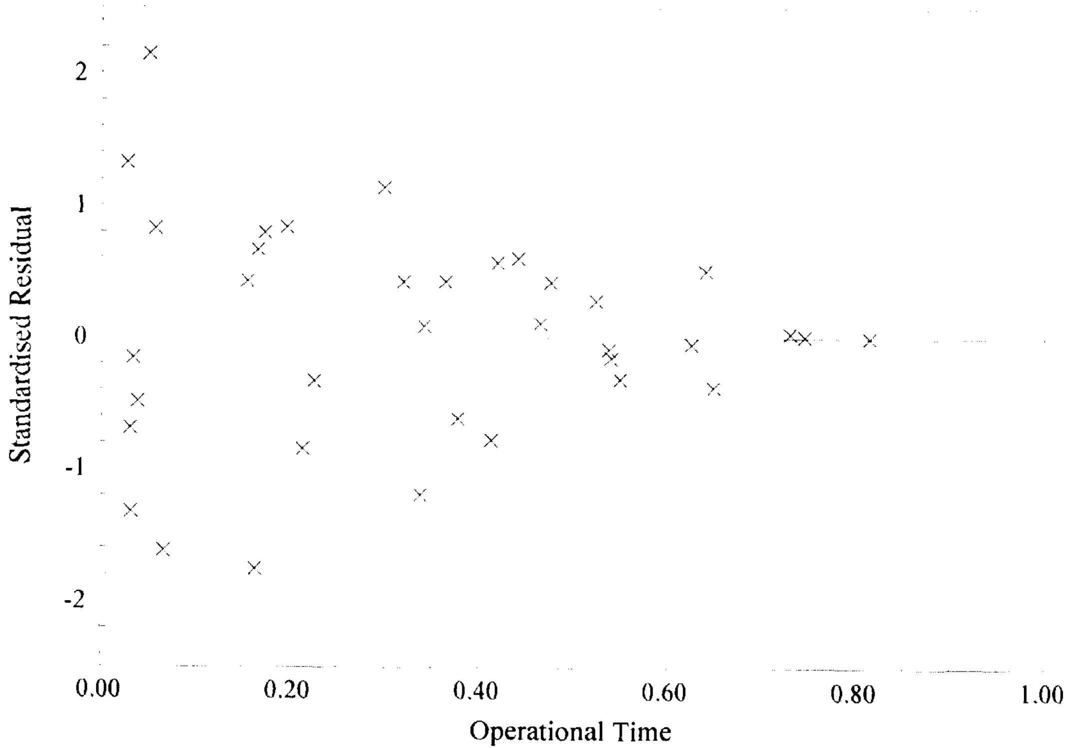


FIGURE 15

RESIDUAL PLOT FOR MODEL ZERO WITH $\alpha = 2$



The parameter estimates are:

| | | | |
|---|---|---|---|
| $c$ | 0.111 | (0.024) | |
| $i$ | 0.131 | (0.013) | |
| $\beta_0$ | 5.17 | (0.81) | |
| $\beta_1$ | -7.11 | (1.88) | |
| $\beta_2$ | 4.54 | (1.18) | (coefficient of $\tau^2$) |
| $\beta_3$ | 1.25 | (0.34) | (coefficient of $\ln(\tau)$) . |

Note that the estimate of $c$ is not significantly different from the value of 0.1 used in the weights. Also, the average force of inflation estimated from the data does not differ significantly from Taylor's prior estimate of 0.117 (derived from the Australian Capital Territory Average Weekly Earnings Index).

The function $\exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2 + \beta_3 \cdot \ln(\tau))$ represents the payment per claim closed in constant 1980 terms, with the partial payment factor excluded. This is shown in Figure 16. This should be compared to the adjusted data shown in Figure 14. The slight decrease from 7.0 to 6.6 over the range $\tau = 0.27$ to $\tau = 0.52$ is attributed to a declining partial payment rate on open claims, as explained in Appendix G.

Figure 13 shows the observed values of $R$ used in fitting the model. The form $R = \alpha_0 + \alpha_1/\tau$ fits reasonably well to these data. Least squares estimation gives: $\alpha_0 = 1.32$ and $\alpha_1 = 0.463$. The fitted curve is shown in Figure 17. Figure 18 shows the fitted mean payment per claim closed with the factor $\exp(c \cdot R)$ included, using the estimates $c = 0.111$ and $R = 1.32 + 0.463/\tau$. This should be compared to Figure 12. The fitted curve tends to infinity as $\tau$ tends to zero, but this is unimportant because projection is unnecessary for $\tau$ less than 0.043. Table 16 gives the forecasts obtained from this model, in constant terms; the units are thousands of 1980 Australian dollars.

FIGURE 16

FITTED MODEL WITH PARTIAL PAYMENT FACTOR EXCLUDED

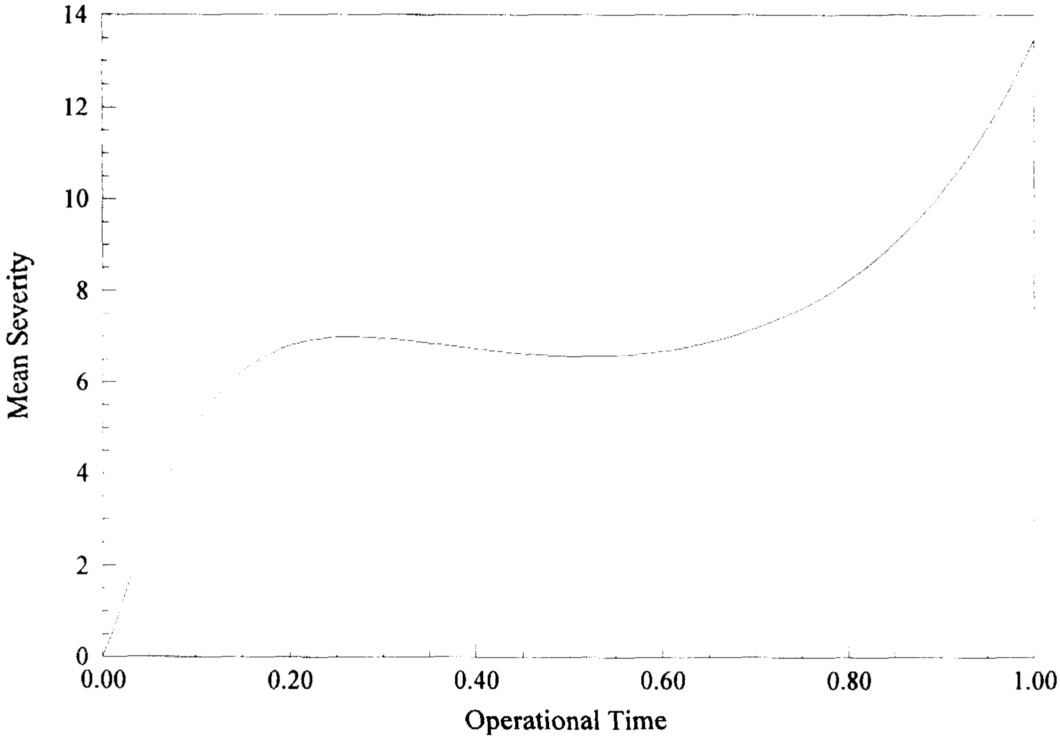


FIGURE 17

FITTED MODEL FOR PARTIAL PAYMENT RATIO

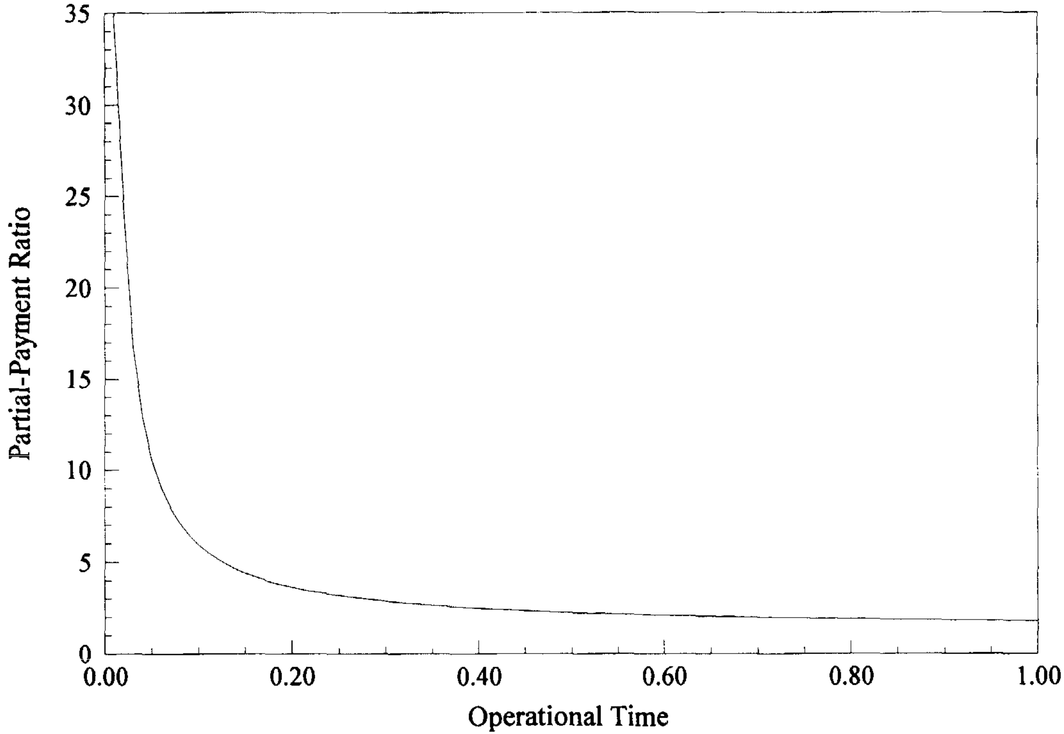


FIGURE 18

FINAL FITTED MODEL FOR DATA FROM TAYLOR [9]

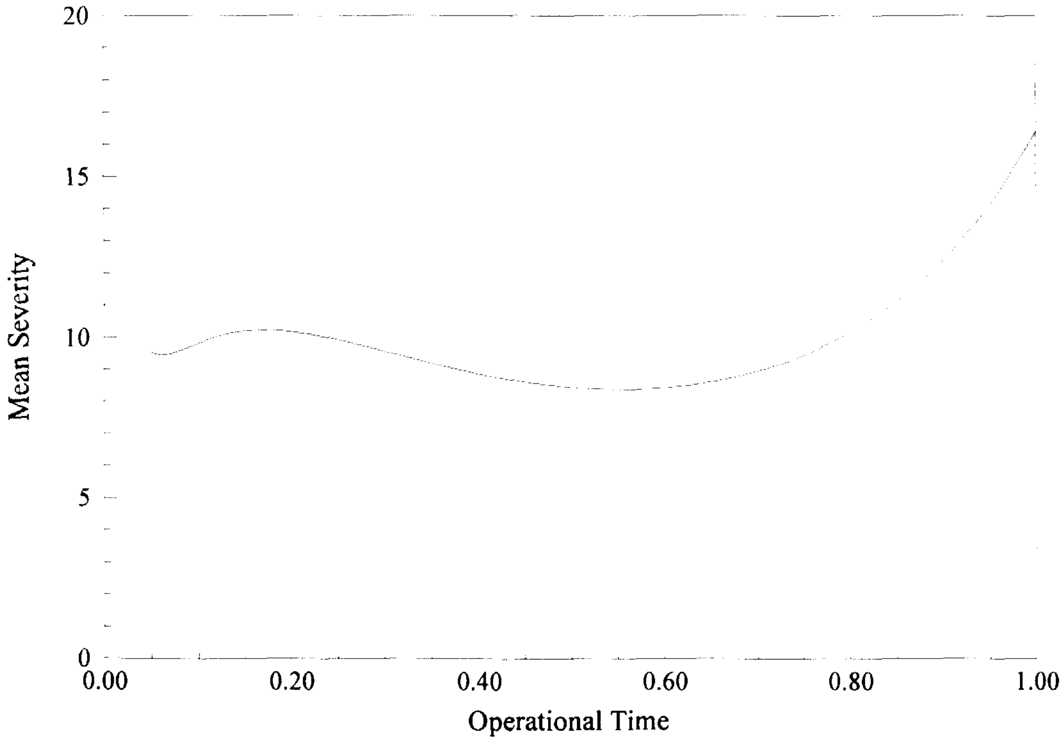


TABLE 16

| Year | (1) Expected Total Future Payments | (2) Standard Error | (3) Standard Deviation | (4) Additional Uncertainty | (5) Root-Mean-Square Error |
|---|---|---|---|---|---|
| 1969 | 0 | 0 | 0 | 0 | 0 |
| 1970 | 33 | 5 | 66 | 0 | 67 |
| 1971 | 33 | 5 | 66 | 0 | 67 |
| 1972 | 97 | 16 | 114 | 0 | 115 |
| 1973 | 315 | 49 | 203 | 15 | 209 |
| 1974 | 444 | 66 | 238 | 14 | 247 |
| 1975 | 774 | 102 | 301 | 37 | 320 |
| 1976 | 1,278 | 147 | 372 | 54 | 404 |
| 1977 | 2,550 | 227 | 491 | 113 | 553 |
| 1978 | 4,650 | 338 | 627 | 223 | 747 |
| 1979 | 5,041 | 349 | 645 | 249 | 775 |
| 1980 | 8,494 | 571 | 814 | 426 | 1,082 |
| All | 23,708 | 1,741 | 1,436 | 558 | 2,325 |

The RMS error (Column 5) for all years combined indicates that about 10% must be added to the best estimate (Column 1) for a reasonably safe reserve. This gives \$26 million in 1980 terms.

To introduce future claims inflation, the method given in [10] has been applied to the triangle of the number of settlements. This indicates that the run-off pattern does not differ significantly across origin years, the rate of settlement being proportional to $\exp(1.422 \times \ln(t) - 0.897 \times t)$, where $t$ is real development time. Operational times have been converted to expected real development times by numerically inverting the corresponding Gamma distribution function, and future claims inflation introduced using the method outlined in Appendix F. The best estimate of the force of inflation in the future was taken as 0.12, and the combined uncertainty of this estimate and the real time scale was taken to be $0.02^2$. The results are given in Table 17:

TABLE 17

| Year | (1) Expected Total Future Payments | (2) Standard Error | (3) Additional Uncertainty (Inflation) | (4) Standard Deviation | (5) Additional Uncertainty (Number of Claims) | (6) Root-Mean-Square Error |
|---|---|---|---|---|---|---|
| 1969 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1970 | 38 | 6 | 8 | 77 | 0 | 78 |
| 1971 | 38 | 6 | 8 | 77 | 0 | 78 |
| 1972 | 115 | 18 | 23 | 137 | 0 | 140 |
| 1973 | 380 | 59 | 65 | 250 | 15 | 265 |
| 1974 | 540 | 81 | 85 | 296 | 15 | 319 |
| 1975 | 959 | 129 | 129 | 388 | 38 | 430 |
| 1976 | 1,612 | 193 | 195 | 494 | 58 | 568 |
| 1977 | 3,329 | 317 | 343 | 692 | 128 | 844 |
| 1978 | 6,275 | 486 | 542 | 934 | 275 | 1,216 |
| 1979 | 6,884 | 505 | 552 | 976 | 318 | 1,271 |
| 1980 | 12,113 | 851 | 866 | 1,305 | 600 | 1,881 |
| All | 32,283 | 2,539 | 2,817 | 2,141 | 748 | 4,419 |

The additional uncertainty of future inflation means that the best estimate of 32,283 must be augmented by 13.7% for a reasonably safe reserve. This gives \$36.7 million.

## 8. CONCLUDING REMARKS

### *Origin Year Effects*

All the methods described in this paper are based on the hypothesis that, in real terms, the mean claim amount as a function of operational time is the same for all origin years. The plausibility of this hypothesis should be considered for each data set before proceeding to apply the methods. If there is a trend change in the mean claim amount over the origin years, for fixed operational time, the test of Appendix B should give a warning. However, the basic hypothesis could be violated in other ways. A change in the mix of claim types over origin years might cause a significant violation. For example, in property insurance, the proportion of land subsidence claims may be increased for a certain origin year because of hot dry weather. Since subsidence claims tend to be large, the mean claim amount in real terms would be higher for such an origin year. The approach of Appendix B can be modified to test for the presence of such phenomena. For example, Model 0 could be generalized by having a separate level parameter ($\beta_0$ in Appendix B) for certain origin years. Plots of residuals against origin year should help in deciding which origin years are affected.

If the basic hypothesis is violated because of a changing mix of claim types across origin years, there are two possible remedies:

1. if there are sufficient data points in each group of similar origin years, a separate "level parameter" for each group can be retained throughout the analysis, or
2. if the data are available, each claim type can be analyzed separately.

### *The Purpose of Modelling Partial Payments*

Frequently in practice, the only data available is of type (c) (Section 1), so the methods of Section 7 are appropriate. These methods are particularly valuable when the triangle does not contain data over the full range of operational times, so that projection is necessary. For the Berquist and Sherman data, the highest observed operational time is about 0.85. Figure

4 shows the curve for the mean payment per claim closed fitted with no allowance for the presence of partial payments. This should be compared to Curve 2 of Figure 11 which was obtained by the methods of Section 7. The two fitted curves are in close agreement over the range of the data, because both fit the data well. However, their projections into the range (0.85, 1) of operational time are very different, causing a substantial effect on the forecasts (compare the results in Table 12 to those in Table 8). The improvement in projections possible by considering the effect of partial payments does not, of course, negate the need for caution when projecting a fitted curve beyond the range of the data. An informal Bayesian approach is appropriate. The indications of the particular data set under analysis (via $F$-tests, etc.) should be tempered by experience of more fully developed triangles for similar lines of business.

Even when the full range of operational time is covered by the data, so that no projection of $m_\tau$ is necessary, the methods of Section 7 are recommended for data of type (c). The models of Section 7 usually explain more of the variation in such data than the models of earlier sections. This is indicated by a significant estimate of the partial payment parameter $c$. Consequently, the other parameters of $m_\tau$ will be more reliably estimated if allowance is made for the partial payment effect. As the observed values for $R = L/N$ differ between origin years, the models of Section 7 effectively allow a different function $m_\tau$ to be fitted for each origin year.

*Distribution of Settlements Over Real Development Time*

The similarity between projecting the partial payment ratio $R$ and projecting the run-off of settlements over real development time for the purpose of introducing future inflation, was mentioned in Section 7. For future inflation, the distribution function $F_t$ of settlement delays must be projected. The relationship between $\tau$ and $t$ can then be approximated using $\tau = F_t$. The partial payment ratio $R = L/N$ could similarly be approximated using $R = (1 - F_t)/f_t$ where $f_t$ is the probability density function of the delays ($f_t = dF_t/dt$). This is the reciprocal of the hazard function of the delay distribution. Thus, projection of $R$ could be based on the same model for the run-off of settlements over real development time as used to introduce future inflation. This possibility has not been fully explored but would probably require unusually accurate estimation of $F_t$ for reasonably reliable estimates of $R$.

For triangles which are not well developed (such as the Berquist and Sherman triangle used in the examples) experience with more fully developed triangles for similar lines of business would be very valuable in estimating the real time scale of the remaining run-off. The technique used in the example in Section 6 of estimating the "half life" by examining the triangle of operational times is not recommended for general use. Although not illustrated in the examples, the projected total of payments remaining for each origin year could, of course, be broken down by development year, given a projection for the remaining run-off of settlements.

*Integration into a Comprehensive Approach*

The question of how the methods proposed in this paper can be combined with results obtained by other methods and additional items of information can obviously not be answered definitively because every reserving problem is different. A few suggestions are given below.

If reliable case estimates are available for some of the outstanding claims, the estimate $M$ can be reduced by the number of claims concerned so that the fitted model $m_\tau$ is summed over those claims for which reliable case estimates are not available. This involves an assumption that the operational times of those claims with reliable case estimates are uniformly distributed over the remaining interval of operational time, and that the presence of a claim in this class does not depend on its size.

Because large claims are often assessed individually, and as accurately as possible, the entire method could be restricted to smaller claims only.

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## APPENDIX A

## DATA USED FOR THE EXAMPLES

The first data set consists of medical malpractice triangles taken from Berquist and Sherman [1]. The triangle below is $Y_{wd}$ , the total amount of all payments made in development year $d$ for each origin year $w$. This has been calculated from Exhibit E of Berquist and Sherman. The units are thousands of dollars, not adjusted for inflation.

TABLE A.1

| | $d$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1969 | 125 | 281 | 1,037 | 1,543 | 1,481 | 3,712 | 4,459 | 3,177 |
| 1970 | 43 | 486 | 1,487 | 1,625 | 3,882 | 6,772 | 4,688 | |
| 1971 | 295 | 852 | 1,332 | 2,592 | 6,328 | 6,308 | | |
| 1972 | 50 | 736 | 3,024 | 5,961 | 8,747 | | | |
| 1973 | 213 | 620 | 2,766 | 7,693 | | | | |
| 1974 | 172 | 1,415 | 4,680 | | | | | |
| 1975 | 210 | 1,355 | | | | | | |
| 1976 | 209 | | | | | | | |

The triangle below is $N_{wd}$ , the number of claims closed in development year $d$ for each origin year $w$. This has been derived from Exhibits C and E of Berquist and Sherman [1]. The final column is $N_0$, the total of the $N_{wd}$ for each origin year.

TABLE A.2

| | $d$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $N_0$ |
| 1969 | 311 | 521 | 349 | 179 | 161 | 293 | 261 | 191 | 2,266 |
| 1970 | 391 | 529 | 271 | 178 | 303 | 367 | 240 | | 2,279 |
| 1971 | 418 | 764 | 236 | 526 | 487 | 422 | | | 2,853 |
| 1972 | 311 | 854 | 523 | 629 | 621 | | | | 2,938 |
| 1973 | 294 | 1,146 | 691 | 657 | | | | | 2,788 |
| 1974 | 332 | 1,015 | 613 | | | | | | 1,960 |
| 1975 | 406 | 907 | | | | | | | 1,313 |
| 1976 | 398 | | | | | | | | 398 |

Table A.3 is the number of non-zero claims reported in each development year, from Exhibits C, D, and E of Berquist and Sherman [1]. The final two columns are the best estimate of the ultimate number of claims, and its standard error. These were obtained by applying the stochastic method detailed in [10] to the reported numbers triangle.

TABLE A.3

| | | | | *d* | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $M$ | $v$ |
| 1969 | 1,060 | 612 | 511 | 383 | -11 | 24 | 29 | 17 | 2,664 | 70 |
| 1970 | 1,051 | 826 | 463 | 379 | 48 | 27 | 24 | | 2,896 | 102 |
| 1971 | 1,296 | 1,215 | 627 | 605 | 116 | 50 | | | 4,065 | 148 |
| 1972 | 1,354 | 1,372 | 790 | 695 | 249 | | | | 4,771 | 215 |
| 1973 | 1,382 | 1,446 | 843 | 994 | | | | | 5,280 | 314 |
| 1974 | 1,365 | 1,400 | 858 | | | | | | 4,837 | 461 |
| 1975 | 1,544 | 1,241 | | | | | | | 5,169 | 690 |
| 1976 | 1,594 | | | | | | | | 6,257 | 1,097 |

The triangle below gives the mean operational times $\tau_{wd}$ calculated from the triangle $N_{wd}$ given in Table A.2 and the estimates $M$ given in Table A.3 using Equation 2.1.

TABLE A.4

| | *d* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1969 | 0.058 | 0.215 | 0.378 | 0.477 | 0.541 | 0.626 | 0.730 | 0.815 |
| 1970 | 0.068 | 0.226 | 0.364 | 0.442 | 0.525 | 0.641 | 0.746 | |
| 1971 | 0.051 | 0.197 | 0.320 | 0.414 | 0.538 | 0.650 | | |
| 1972 | 0.033 | 0.155 | 0.299 | 0.420 | 0.551 | | | |
| 1973 | 0.027 | 0.164 | 0.338 | 0.466 | | | | |
| 1974 | 0.034 | 0.174 | 0.342 | | | | | |
| 1975 | 0.039 | 0.166 | | | | | | |
| 1976 | 0.032 | | | | | | | |

Below is the triangle of average numbers of claims outstanding calculated from the triangles given in Tables A.2 and A.3. These figures are the $L_{wd}$ of Section 7.

TABLE A.5

| | $d$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1969 | 375 | 795 | 921 | 1,104 | 1,120 | 900 | 649 | 446 |
| 1970 | 330 | 809 | 1,053 | 1,250 | 1,223 | 925 | 647 | |
| 1971 | 439 | 1,104 | 1,525 | 1,760 | 1,614 | 1,242 | | |
| 1972 | 522 | 1,302 | 1,695 | 1,861 | 1,708 | | | |
| 1973 | 544 | 1,238 | 1,464 | 1,709 | | | | |
| 1974 | 517 | 1,226 | 1,541 | | | | | |
| 1975 | 569 | 1,305 | | | | | | |
| 1976 | 598 | | | | | | | |

The second data set is taken from Taylor [9]. The triangle below is $Y_{wd}$. The units are thousands of Australian dollars.

TABLE A.6

| | $d$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1969 | 57.4 | 514.1 | 418.0 | 305.7 | 208.8 | 77.8 | 82.8 | 52.6 | 20.4 | 0.8 | 4.0 | 0.0 |
| 1970 | 122.2 | 559.6 | 674.9 | 453.8 | 156.9 | 246.3 | 200.5 | 88.2 | 20.4 | 34.1 | 216.3 | |
| 1971 | 213.0 | 619.8 | 561.5 | 398.5 | 438.1 | 161.3 | 120.3 | 50.5 | 61.7 | 81.1 | | |
| 1972 | 168.3 | 426.5 | 518.5 | 705.6 | 378.7 | 400.2 | 214.7 | 285.6 | 73.9 | | | |
| 1973 | 201.4 | 536.5 | 912.8 | 707.7 | 717.0 | 246.3 | 187.8 | 409.4 | | | | |
| 1974 | 191.0 | 612.9 | 959.9 | 608.6 | 483.6 | 261.3 | 376.3 | | | | | |
| 1975 | 181.8 | 511.0 | 942.9 | 798.3 | 599.9 | 747.5 | | | | | | |
| 1976 | 198.7 | 850.1 | 1,074.8 | 502.5 | 796.1 | | | | | | | |
| 1977 | 455.9 | 830.6 | 1,129.4 | 1,426.5 | | | | | | | | |
| 1978 | 318.2 | 963.6 | 1,279.2 | | | | | | | | | |
| 1979 | 424.3 | 1,402.7 | | | | | | | | | | |
| 1980 | 638.0 | | | | | | | | | | | |

The triangle below is $N_{wd}$. The final column, $N_0$, is the total of each row.

TABLE A.7

| | *d* | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $N_0$ |
| 1969 | 99 | 154 | 112 | 84 | 37 | 21 | 7 | 5 | 2 | 1 | 1 | 0 | 523 |
| 1970 | 46 | 193 | 187 | 89 | 34 | 43 | 27 | 9 | 9 | 2 | 2 | | 641 |
| 1971 | 44 | 191 | 193 | 78 | 99 | 49 | 10 | 3 | 3 | 4 | | | 674 |
| 1972 | 45 | 166 | 78 | 185 | 136 | 36 | 5 | 6 | 9 | | | | 666 |
| 1973 | 52 | 115 | 256 | 240 | 78 | 27 | 9 | 10 | | | | | 787 |
| 1974 | 25 | 140 | 216 | 129 | 70 | 31 | 30 | | | | | | 641 |
| 1975 | 16 | 93 | 126 | 113 | 71 | 42 | | | | | | | 461 |
| 1976 | 16 | 114 | 99 | 88 | 128 | | | | | | | | 445 |
| 1977 | 37 | 102 | 84 | 168 | | | | | | | | | 391 |
| 1978 | 23 | 105 | 121 | | | | | | | | | | 249 |
| 1979 | 30 | 122 | | | | | | | | | | | 152 |
| 1980 | 38 | | | | | | | | | | | | 38 |

Table A.8 gives the estimated ultimate number of claims, $M$, for each origin year, and the standard error, $v$, of this estimate. $M$ has been taken directly from Taylor [9], and $v$ has been calculated (for the purposes of the second example in Section 7) as 5% of the number not yet settled ($M - N_0$).

TABLE A.8

| Year | $M$ | $v$ |
|---|---|---|
| 1969 | 523 | 0 |
| 1970 | 643 | 0 |
| 1971 | 676 | 0 |
| 1972 | 672 | 0 |
| 1973 | 807 | 1 |
| 1974 | 670 | 1 |
| 1975 | 516 | 3 |
| 1976 | 544 | 5 |
| 1977 | 622 | 12 |
| 1978 | 715 | 23 |
| 1979 | 660 | 25 |
| 1980 | 894 | 43 |

The triangle below gives the mean operational times $\tau_{wd}$ calculated using Equation 2.1 from the data in Table A.8.

TABLE A.9

| | $d$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1969 | 0.095 | 0.337 | 0.591 | 0.778 | 0.894 | 0.949 | 0.976 | 0.988 | 0.994 | 0.997 | 0.999 | 1.000 |
| 1970 | 0.036 | 0.222 | 0.517 | 0.732 | 0.827 | 0.887 | 0.942 | 0.970 | 0.984 | 0.992 | 0.995 | |
| 1971 | 0.033 | 0.206 | 0.490 | 0.691 | 0.822 | 0.931 | 0.975 | 0.984 | 0.989 | 0.994 | | |
| 1972 | 0.033 | 0.190 | 0.372 | 0.568 | 0.807 | 0.935 | 0.965 | 0.973 | 0.984 | | | |
| 1973 | 0.032 | 0.136 | 0.366 | 0.673 | 0.870 | 0.935 | 0.957 | 0.969 | | | | |
| 1974 | 0.019 | 0.142 | 0.407 | 0.665 | 0.813 | 0.889 | 0.934 | | | | | |
| 1975 | 0.016 | 0.121 | 0.333 | 0.565 | 0.743 | 0.853 | | | | | | |
| 1976 | 0.015 | 0.134 | 0.330 | 0.502 | 0.700 | | | | | | | |
| 1977 | 0.030 | 0.141 | 0.291 | 0.494 | | | | | | | | |
| 1978 | 0.016 | 0.106 | 0.264 | | | | | | | | | |
| 1979 | 0.023 | 0.138 | | | | | | | | | | |
| 1980 | 0.021 | | | | | | | | | | | |

The triangle in Table A.10 gives the mean number of claims not yet settled, calculated as the difference between $M$ (from Table A.8) and the cumulative values of $N_{wd}$ (from Table A.7). These figures were used for $L_{wd}$ in the second example of Section 7.

TABLE A.10

| | d | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1969 | 473.5 | 347.0 | 214.0 | 116.0 | 55.5 | 26.5 | 12.5 | 6.5 | 3.0 | 1.5 | 0.5 |
| 1970 | 620.0 | 500.5 | 310.5 | 172.5 | 111.0 | 72.5 | 37.5 | 19.5 | 10.5 | 5.0 | 3.0 |
| 1971 | 654.0 | 536.5 | 344.5 | 209.0 | 120.5 | 46.5 | 17.0 | 10.5 | 7.5 | 4.0 | |
| 1972 | 649.5 | 544.0 | 422.0 | 290.5 | 130.0 | 44.0 | 23.5 | 18.0 | 10.5 | | |
| 1973 | 781.0 | 697.5 | 512.0 | 264.0 | 105.0 | 52.5 | 34.5 | 25.0 | | | |
| 1974 | 657.5 | 575.0 | 397.0 | 224.5 | 125.0 | 74.5 | 44.0 | | | | |
| 1975 | 508.0 | 453.5 | 344.0 | 224.5 | 132.5 | 76.0 | | | | | |
| 1976 | 536.0 | 471.0 | 364.5 | 271.0 | 163.0 | | | | | | |
| 1977 | 603.5 | 534.0 | 441.0 | 315.0 | | | | | | | |
| 1978 | 703.5 | 639.5 | 526.5 | | | | | | | | |
| 1979 | 645.0 | 569.0 | | | | | | | | | |
| 1980 | 875.0 | | | | | | | | | | |

# APPENDIX B

## TESTING MODEL ZERO

This appendix is concerned with testing the hypothesis that the mean claim amount in real terms is a function of operational time only. This hypothesis underlies all the methods proposed in this paper, and should be checked for each data set before applying these methods. Intuitively, the most likely violation is that there may be a trend across the origin years in the mean claim amount at a certain operational time. Such a trend would give a different mean claim size for the earlier origin years than for the later origin years, for a certain operational time. This can be tested as follows.

The run-off triangle is bisected into Regions A and B as shown in Figure 19. If $P$ is the number of development periods per annum (for example, $P = 4$ for quarterly development) then calendar time $t'$ is given by:

$$t' = d + P \cdot (w - 1). \tag{B.1}$$

If $d$ runs from 0 to $T - 1$, then $t'$ also varies from 0 to $T - 1$, so data for the last calendar period (represented by the hypotenuse of the run-off triangle) is given by $T - 1 = d + P \cdot (w - 1)$, which is equivalent to:

$$w = 1 + (T - 1 - d)/P \, . \tag{B.2}$$

Note that data exists for the latest calendar period only for those $d$-values that give an integer value for $w$.

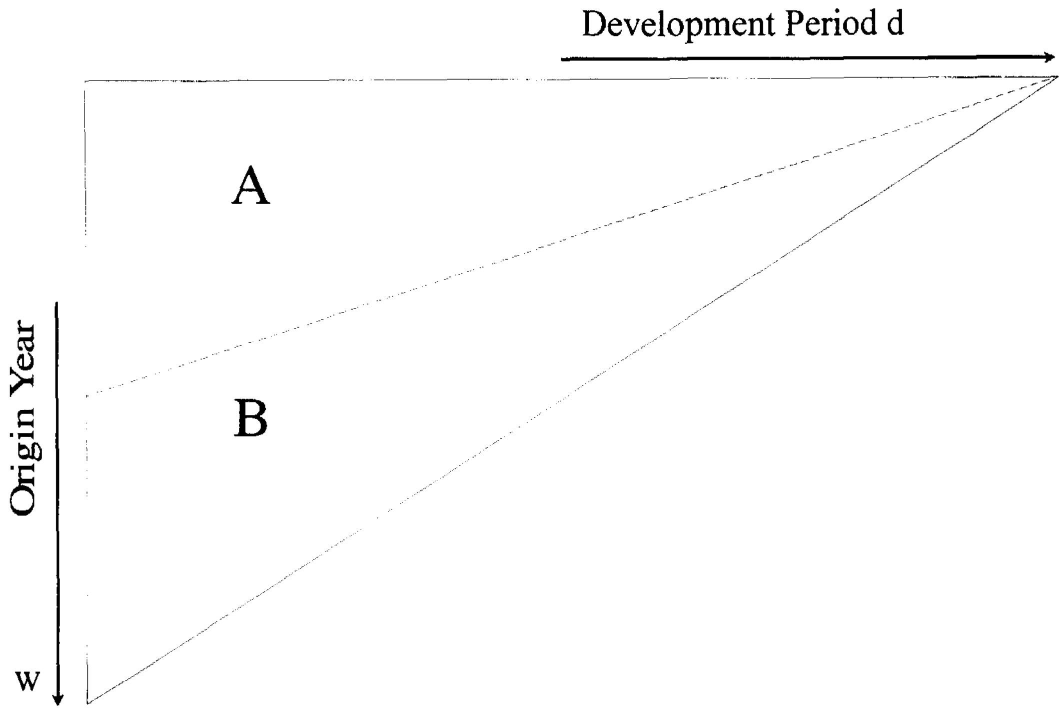
The boundary between regions A and B is therefore given by:

$$w = 0.5 \times \{ 1 + (T - 1 - d)/P \},$$

so region B is defined as those data points satisfying:

$$w > 0.5 \times \{ 1 + (T - 1 - d)/P \}.$$

FIGURE 19



Model 0 of Section 2 is generalized to allow the mean claim amount as a function of operational time to differ between the Regions A and B. This less restrictive model can be expressed as:

$$m_\tau^A = \exp\,(\beta_0^A + \sum_j \beta_j^A \cdot \tau_j) \tag{B.3}$$

$$m_\tau^B = \exp\,(\beta_0^B + \sum_j \beta_j^B\, \tau_j)\,, \tag{B.4}$$

where the explanatory variables $\tau_j$ are the same simple functions of $\tau$ used in Model 0. Model 0 is the special case $\beta_j^A = \beta_j^B$ for all $j$, and can be tested against the more general model using an $F$-test in the usual way. However, Model 0 could appear to be unacceptable when tested in this way if an incorrect inflation rate has been used to preadjust the data. For this reason it is better to use unadjusted data and include an inflation parameter. Model 0 of Section 4 is generalized to:

$$m_{w\tau}^A = \exp\,(i \cdot (w + d/P) + \beta_0^A + \sum_j \beta_j^A \cdot \tau_j) \tag{B.5}$$

$$m_{w\tau}^B = \exp\,(i \cdot (w + d/P) + \beta_0^B + \sum_j \beta_j^B \cdot \tau_j)\,. \tag{B.6}$$

In the example of Section 4, Model 0 has 10 parameters (including $i$) and gives a minimized deviance of 1,961 using an index $\alpha = 1.5$ in the variance function. The less restrictive model specified in Equations B.3 and B.4 has 19 parameters, but only 18 can be estimated due to an absence of data in Region A for the last operational time band. The minimized deviance (using the variance function with $\alpha = 1.5$) is 1,676. Residual plots confirm the assumption $\alpha = 1.5$. As there are 36 data points, the model has 18 degrees of freedom, giving an estimate of 93.1 for the scale parameter. The mean increase in the deviance per degree of freedom under Model 0 is (1,961-1,676)/8 = 35.6, so the $F$-statistic is 35.6/93.1 = 0.38 on eight and 18 degrees of freedom. The lack of significance indicates that Model 0 fits the data as well as the 18-parameter model, so the hypothesis that the mean $m_\tau$ does not vary across origin years for any $\tau$ is verified.

# APPENDIX C

## INTERPRETATION OF OPERATIONAL TIME MODELS FOR MEAN CLAIM AMOUNT

When formulating models for the mean claim amount as a function of operational time, it is helpful to discover what such models imply about the mean claim amount as a function of real development time. This Appendix describes how this can be done, and illustrates the techniques by giving the real-time interpretation of certain special cases and generalizations of the models proposed in Section 2. The following notation is used:

$m_\tau$ = mean claim amount as function of operational time $\tau$,

$\mu_t$ = mean claim amount as function of real development time $t$,

$M$ = ultimate number of claims closed,

$F_t$ = distribution function (over individual settlements) of delay $t$,

$N_t$ = cumulative number of claims closed by real development time $t$,

$C_t$ = expected cumulative amount paid by real development time $t$.

By definition of $\tau$, $\tau$ and $t$ are related by:

$$\tau = N_t/M\ . \tag{C.1}$$

Each of the $M$ claims has probability $F_t$ of being closed by time $t$ (by definition of $F_t$), so $N_t$ is binomially distributed with parameters $M$ and $F_t$. Therefore we have:

$$E(N_t) = M \cdot F_t \text{ and } \operatorname{Var}(N_t) = M \cdot F_t \cdot (1 - F_t)\ . \tag{C.2}$$

Hence, using Equation C.1:

$$E(\tau \mid t) = F_t \text{ and } \operatorname{Var}(\tau \mid t) = F_t \cdot (1 - F_t)/M. \tag{C.3}$$

So, if $M$ is reasonably large, to a good approximation we have:

$$\tau = F_t \ . \tag{C.4}$$

That is, operational time is simply the distribution function of the real delay.

By definition of $m$ and $\mu$, we have:

$$\mu_t = m_\tau \ .$$

Using Equation C.4, this gives:

$$\mu_t = m(F_t) \ . \tag{C.5}$$

Given a functional form for $m_\tau$, equation C.5 immediately gives a relationship between $\mu_t$ and $F_t$. For example, Model 1 of Section 2 in the case $\beta_1 = 0$ is:

$$\begin{aligned} m_\tau &= \exp\left(\beta_0 + \beta_2 \cdot \ln(\tau)\right) \\ &= k \cdot \tau^\beta. \end{aligned}$$

Using Equation C.5, this is equivalent to:

$$\mu_t = k \cdot F_t^\beta.$$

That is, in real time, the mean claim size is a power function of the delay distribution function.

By definition,

$$C_t = M \cdot \int_0^t \mu_s \, dF_s \ .$$

If $M$ is sufficiently large, then from Equation C.4 we have, to a good approximation,

$$C_t = M \cdot \int_0^t m_\sigma \, d\sigma. \tag{C.6}$$

For certain functional forms $m_\tau$, the integral on the right of Equation C.6 is analytically tractable and can be expressed simply in terms of $m_\tau$. In such cases the equation can be rearranged to express the mean claim amount in terms of $C_t$.

For example, consider Model 1 or 2 of Section 2 in the case $\beta_2 = 0$:

$$m_\tau = \exp\,(\beta_0 + \beta_1 \cdot \tau) \Rightarrow \int_0^\tau m_\sigma \, d\sigma = (m_\tau - m_0)/\beta_1 \, ,$$

and, using Equation C.6, we have:

$$C_t = M \cdot (m_\tau - m_0)/\beta_1 \, .$$

Rearranging gives:

$$\mu_t = \mu_0 + \beta_1 \cdot C_t \, /M. \tag{C.7}$$

As a second example, consider the generalization of Model 3 of Section 2:

$$m_\tau = (\beta_0 + \beta_1 \cdot \tau)^\delta \, .$$

It is straightforward to show that this implies:

$$\int_0^\tau m_\sigma \, d\sigma = [m_\tau^{(\delta+1)/\delta} - m_0^{(\delta+1)/\delta}]/[\beta_1 \cdot (\delta + 1)\,].$$

Hence, using Equation C.6 and rearranging:

$$\mu_t = [\beta_0^{(\delta+1)} + (\delta + 1) \cdot \beta_1 \cdot C_t/M]^{\delta/(\delta+1)}.$$

Model 3 is the case $\delta = 2$, so is equivalent to the real-time relationship:

$$\mu_t = [\ \beta_0^3 + 3 \cdot \beta_1\ \ C_t/M\ ]^{2/3}\ .$$

# APPENDIX D

## PREDICTION WHEN ULTIMATE NUMBERS ARE KNOWN

We have $E(X_\tau) = m_\tau$ and Var $(X_\tau) = \varphi^2 \cdot m_\tau^\alpha$ where:

$X_\tau$ is the size of the individual claim payment (in constant money terms) made at operational time $\tau$, and

$m_\tau$ is a function of several parameters which can be expressed as a vector $\beta$.

Sections 2 through 4 of the paper describe how the data triangles can be used to decide on the functional form of $m_\tau$, and to estimate the parameters $\beta$, $\alpha$, and $\varphi$. The estimation algorithm (Fisher's scoring method) also gives the variance-covariance matrix $V$ for the estimates of $\beta$. This Appendix describes how the fitted model can be used to predict totals of future claims, under the assumption that the ultimate number of claims $M$ is fully known for each origin year.

Consider a single origin year. If $M$ is the ultimate number of claims, and $N_0$ is the number to date, then there are $M - N_0$ claims in the future. The operational times of these future claims are:

$$\tau = (N_0 + 0.5)/M, (N_0 + 1.5)/M, ..., (M - 0.5)/M. \qquad \text{(D.1)}$$

If $R$ represents the total of future claims, and $\mu$ and $\sigma^2$ denote the mean and variance of $R$, respectively, then since separate claim amounts are mutually independent we have:

$$\mu = \sum_\tau m_\tau \text{ and } \sigma^2 = \varphi^2 \cdot \sum_\tau m_\tau^\alpha, \qquad \text{(D.2)}$$

the summation being over the values of $\tau$ given at Equation D.1.

Hence, an estimate $\hat{\mu}$ can be obtained using the fitted model:

$$\hat{\mu} = \sum_\tau \hat{m}_\tau, \qquad \text{(D.3)}$$

where $\hat{m}_\tau$ means $m_\tau$ evaluated using the estimated values of β.

This is the "best estimate" of $R$ given in Columns 1 of the example result tables in Sections 3 through 7.

Now consider the mean-square-error of the estimate for a single origin year given by Equation D.3:

$$\begin{aligned} E(R - \hat{\mu})^2 &= E[(R - \mu) + (\mu - \hat{\mu})]^2 \\ &= E(R - \mu)^2 + E(\hat{\mu} - \mu)^2 - 2 \cdot E(R - \mu)(\hat{\mu} - \mu) . \end{aligned} \tag{D.4}$$

The last term of Equation D.4 is zero because $R$ and $\hat{\mu}$ are stochastically independent. The randomness in $R$ comes from future claims, and the randomness in $\hat{\mu}$ comes from past claims.

The first term of Equation D.4 is simply $\sigma^2$ and can be estimated by using the estimated parameters in the expression at Equation D.2. This gives the quantity in the standard deviation columns of the examples.

For the middle term of equation D.4, note that μ is a known function of the parameters β. See Equation D.2. Using a first order Taylor series, we have:

$$\hat{\mu} \approx \mu + (\hat{\beta} - \beta)^T \cdot \delta,$$

where δ is the vector of first derivatives of μ with respect to the βs.

Hence:

$$(\hat{\mu} - \mu)^2 \approx \delta^T \cdot (\hat{\beta} - \beta) \cdot (\hat{\beta} - \beta)^T \cdot \delta.$$

Taking expected values,

$$E(\hat{\mu} - \mu)^2 \approx \delta^T \cdot V \cdot \delta, \tag{D.5}$$

where $V$ is the variance-covariance matrix of the estimates $\hat{\beta}$.

The vector of derivatives δ can be calculated from Equation D.2:

$$\delta_i = d\mu/d\beta_i = \sum_{\tau} dm_\tau/d\beta_i\,. \tag{D.6}$$

Each term $dm_\tau/d\beta_i$ can be evaluated using the estimated $\beta$-values.

The standard error for a single origin year is given by the square root of the estimated right-hand side of Equation D.4:

$$SE = \sqrt{(\hat{\sigma}^2 + \hat{\delta}^T \cdot \boldsymbol{V} \cdot \hat{\delta})}\ .$$

This is the quantity given in the RMS columns of Tables 4 and 6. The standard error columns of these tables are from:

$$\sqrt{(\hat{\delta}^T \cdot \boldsymbol{V} \cdot \hat{\delta})}\,.$$

Now consider the prediction of total future payments for all origin years combined. The best estimate is obtained simply by summing the estimates given by Equation D.3 for each origin year. The mean-square-error of this estimate cannot be obtained so simply. The estimate of $\sigma^2$ (given by using the estimated parameters in Equation D.2) can be summed over all origin years, because all future claims are mutually independent. However, the "estimation error" component (given by Equation D.5 for a single origin year) cannot be summed over all origin years, because the same estimates $\hat{\beta}$ are used in equation D.3 for each origin year, so the estimates are not mutually independent.

Corresponding to the middle term of Equation D.4, we need to evaluate E[ $(\Sigma\hat{\mu}) - (\Sigma\mu)$ ]$^2$, where the summation is over all origin years.

An analysis similar to that in Equation D.5 shows that:

$$\mathrm{E}[\ (\Sigma\hat{\mu}) - (\Sigma\mu)\ ]^2 \approx \boldsymbol{D}^T \cdot \boldsymbol{V} \cdot \boldsymbol{D}, \tag{D.7}$$

where $\boldsymbol{V}$ is the variance-covariance matrix of the estimates $\beta$, and $\boldsymbol{D}$ is the vector of first derivatives of $\Sigma\mu$ with respect to the $\beta$s. That is,

$$\boldsymbol{D} = \Sigma\ \delta,$$

where, again, summation is over all origin years.

The figure for the total over all origin years, in the second columns of the examples, is the square root of the right-hand side of Equation D.7.

If the number of outstanding claims $M - N_0$ is large for some origin years, the amount of computation involved in evaluating Equations D.2, D.3, and D.6 may be substantial. However, when $M$ is large the variation in the summands from one point in operational time to the next is usually so small that the sums can be well approximated by integrals. That is, Equation D.2 can be replaced by:

$$\mu = M \cdot \int_{N_0/M}^{1} m_\tau \, d\tau \text{ and } \sigma^2 = \varphi^2 \cdot M \cdot \int_{N_0/M}^{1} m_\tau^\alpha \, d\tau. \tag{D.8}$$

If the form of $m_\tau$ is such that either or both of these definite integrals are analytically tractable, the burden of computation can be substantially reduced. If the first of these is tractable, the $\delta_i$ can easily be obtained from the first part of Equation D.6:

$$\delta_i = d\mu/d\,\beta_i,$$

after finding $\mu$ as a function of $\beta$ by analytic integration.

Otherwise, the integration corresponding to the second part of Equation D.6 may be tractable for some $i$ :

$$\delta_i = M \cdot \int dm_\tau/d\,\beta_i \, d\tau.$$

Note that for models $m_\tau$ with a log link function, the partial derivatives required for calculating the $\delta_i$ have a particularly simple form:

$$m_\tau = \exp\,(\boldsymbol{x}_\tau \cdot \beta) \Rightarrow dm_\tau/d\,\beta_i = x_{\tau i} \cdot m_\tau\,.$$

# APPENDIX E

## PREDICTION WHEN ULTIMATE NUMBERS ARE UNCERTAIN

Appendix D deals with the distributions of future claim totals conditional on the ultimate number of claims $M$. These conditional distributions are relevant for forecasting when $M$ is known. In practice, $M$ is rarely fully known. This appendix shows how uncertainty in $M$ can be allowed for in forecasting. Briefly, the problem is that uncertainty in $M$ for a particular origin year implies uncertainty in the operational times of future claims, given at Equation D.1. The uncertainty affects both the number of future values of $\tau$ and the values themselves.

Replacing the summation in Equation D.2 by integration (as suggested in Appendix D), and making the conditioning on $M$ explicit, we have:

$$\mathrm{E}(R \mid M) = M \cdot \int_{\tau_0}^{1} m_\tau \, d\tau, \tag{E.1}$$

and

$$\mathrm{Var}\,(R \mid M) = \varphi^2 \cdot M \cdot \int_{\tau_0}^{1} m_\tau^\alpha \, d\tau, \tag{E.2}$$

where the integration is from the present operational time $\tau_0$ $(= N_0/M)$ to 1.

It is shown below that both these functions are sufficiently near linear in $M$ (compared to the magnitude of the uncertainty in $M$) to ensure that good approximations of their expected values are given by replacing $M$ by its expected value. That is,

$$\mathrm{E}(\mathrm{E}(R \mid M)) \approx \mathrm{E}(R \mid M = \mathrm{E}(M)), \tag{E.3}$$

and

$$\mathrm{E}(\mathrm{Var}(R \mid M)) \approx \mathrm{Var}(R \mid M = \mathrm{E}(M)).$$

In this appendix, $\mu$ and $\sigma^2$ are used to denote the unconditional mean and variance of the total $R$ of future claims for an origin year. Thus,

$$\mu = \mathrm{E}(R) = \mathrm{E}(\mathrm{E}(R \mid M)) \approx \mathrm{E}(\mathrm{R} \mid M = \mathrm{E}(M)) \tag{E.4}$$

using Equation E.3, and

$$\sigma^2 = \mathrm{Var}(R) = \mathrm{E}(\mathrm{Var}(R \mid M)) + \mathrm{Var}(\mathrm{E}(R \mid M))$$

$$\approx \mathrm{Var}(R \mid M = \mathrm{E}(M)\ ) + \mathrm{Var}(\mathrm{E}(R \mid M))\,. \tag{E.5}$$

Equation E.4 implies that the best estimate of $R$ is given by evaluating the right-hand side of Equation E.1 using the estimate $\hat{M} = \mathrm{E}(M)$ in place of $M$. Similarly, the first term on the right of Equation E.5 is simply the right-hand side of Equation E.2 evaluated using the estimate $\hat{M} = \mathrm{E}(M)$. Thus, the estimates of the unconditional $\mu$ and $\sigma^2$ are exactly the same as the estimates of the conditional $\mu$ and $\sigma^2$ given in Appendix D, except for the addition of the second term on the right of Equation E.5. Equation D.4 remains valid, and as the estimate $\hat{\mu}$ is unchanged, the middle term of Equation D.4 (the estimation error) can be evaluated exactly as described in Appendix D. Therefore, the only change necessary to the mean-square-error given in Appendix D is the addition of the second term on the right of Equation E.5.

Using the usual approximation derived from a first order Taylor series:

$$\mathrm{Var}(\mathrm{E}(R \mid M)) \approx [d\mathrm{E}(R \mid M)/dM]^2 \cdot \mathrm{Var}(M).$$

From Equation E.1:

$$d\mathrm{E}(R \mid M)/dM = \int_{\tau_0}^{1} m_\tau \, d\tau + M \cdot d/dM \int_{\tau_0}^{1} m_\tau \, d\tau\,.$$

But,

$$d/dM \int_{\tau_0}^{1} m_\tau \, d\tau = -m_0 \cdot d\tau_0/dM = m_0 \cdot \tau_0/M ,$$

where

$$m_0 = m_\tau \text{ evaluated at } \tau = \tau_0 ,$$

and

$$\int_{\tau_0}^{1} m_\tau \, d\tau = \mathrm{E}(R \mid M)/M.$$

Hence,

$$\frac{d\,\mathrm{E}(R \mid M)}{dM} = \frac{\mathrm{E}(R \mid M)}{M} + m_0 \cdot \tau_0 .$$

Using the estimate $\hat{M} = \mathrm{E}(M)$ to evaluate the first term gives:

$$\frac{d\,\mathrm{E}(R \mid M)}{dM} \approx \hat{\mu}/\hat{M} + m_0 \cdot \tau_0 \ . \tag{E.6}$$

The accuracy of this approximation is demonstrated in the example of Section 5.

The approximations quoted at Equation E.3 are derived in this section. For any analytic function $h(M)$, a second order Taylor series about $\mathrm{E}(M)$ gives:

$$\mathrm{E}(h(M)) \approx h(\mathrm{E(M)}) + ½ \cdot h''(\mathrm{E(M)}) \cdot \mathrm{Var}\,(M),$$

so

$$\mathrm{E}(h(M)) \approx h(\mathrm{E}(M)) \tag{E.7}$$

$$\text{if } ½ \cdot | h''(\mathrm{E}(M)) | \cdot \mathrm{Var}\,(\mathrm{E}(M)) << h(\mathrm{E}(M)).$$

In the remainder of this section, E($M$) is shortened to $\hat{M}$, because the best estimate of $M$ is its prior expected value.

It is straight-forward to show that if:

$$h(M) = M \cdot \int_{N/M}^{1} g(\tau)\, d\tau, \tag{E.8}$$

where $g$ is any analytic function, then:

$$h''M = -(N^2/M^3) \cdot g'(N/M).$$

Also, if $g$ is increasing,

$$h(M) > (M - N) > g(N/M)\,, \tag{E.9}$$

so for functions $h(M)$ of the form in Equation E.8 with $g$ increasing, a sufficient condition for Equation E.7 is:

$$\tfrac{1}{2} \cdot (N^2/\hat{M}^3) \cdot g'(N/\hat{M}) \cdot \mathrm{Var}(M) << (\hat{M} - N) \cdot g(N/\hat{M}).$$

Writing $f(\tau)$ for $\ln(g(\tau))$ so that $f'(\tau) = g'(\tau)/g(\tau)$, and assuming $g(\tau) > 0$, this becomes:

$$f'(N/\hat{M}) << 2 \cdot \hat{M}^3 \cdot (\hat{M} - N)/(N^2 \cdot \mathrm{Var}(M)).$$

Writing $\tau$ for $N/M$ and $\Omega$ for the coefficient of variation of $M$ ($\Omega^2 = \mathrm{Var}(M)/\hat{M}^2$) this becomes:

$$f'(\tau) << 2 \cdot (1 - \tau)/(\Omega \cdot \tau)^2\,. \tag{E.10}$$

The functions in Equations E.1 and E.2 are of the form given in Equation E.8 with $g(\tau)$ positive and increasing. In terms of $f(\tau) = \ln(g(\tau))$, the mean (Equation E.1) is the case: $f(t) = \ln(m_\tau)$, and the variance (Equation E.2) is the case: $f(\tau) = 2 \cdot \ln(\varphi) + \alpha \cdot \ln(m_\tau)$.

Therefore, using Equation E.10, a sufficient condition for Equation E.3 is:

$$\alpha \cdot d\ln(m_\tau)/d\tau << 2 \cdot (1-\tau)/(\Omega \cdot \tau)^2 . \tag{E.11}$$

This is invariably satisfied by fitted models $m_\tau$. This appendix demonstrates this for the example of Section 5.

Although the fitted model $m_\tau$ of Section 5 is not strictly increasing (because of partial payments), it is mostly increasing and the inequality in Equation E.9 remains true for most $\tau$. The fitted model has $\alpha = 1.5$, and

$$\ln(m_\tau) = -3.71 + 17.8 \cdot \tau - 12.5 \cdot \tau^2 - 0.80 \cdot \ln(\tau) .$$

Therefore,

$$d\ln(m_\tau)/d\tau = 17.8 - 25 \cdot \tau - 0.8/\tau .$$

Table E.1 gives both sides of Equation E.11 evaluated for each origin year using figures $\tau = \tau_0$ and $\Omega = \sqrt{\text{Var}\,(M)}/M$ from Table 7.

TABLE E.1

| Year | $\tau_0$ | $\Omega$ | LHS | RHS |
|---|---|---|---|---|
| 1969 | 0.85 | 0.026 | -6.6 | 614 |
| 1970 | 0.79 | 0.035 | -4.4 | 549 |
| 1971 | 0.70 | 0.036 | -1.3 | 945 |
| 1972 | 0.62 | 0.045 | 1.5 | 976 |
| 1973 | 0.53 | 0.060 | 4.6 | 930 |
| 1974 | 0.41 | 0.095 | 8.4 | 778 |
| 1975 | 0.25 | 0.134 | 12.5 | 1,337 |
| 1976 | 0.06 | 0.175 | 4.5 | 17,052 |

These figures indicate that, for this example, the exclusion of non-linear terms at Equations E.4 and E.5 leads to an error of no more than about 1% in the mean and variance of total future payments for each origin year. The near-linearity of $E(R \mid M)$ is demonstrated more directly in Section 5.

# APPENDIX F

## PREDICTION WITH UNCERTAIN FUTURE INFLATION

From Appendix D, the expected constant-price total of future claims for a single origin year is given by:

$$\hat{\mu} = \sum_{\tau} \hat{m}_{\tau}. \tag{F.1}$$

where $\hat{m}_{\tau}$ is the fitted mean claim amount in constant terms, and summation is over the operational times $\tau$ of all expected future claims (given in Equation D.1).

If $t$ represents the real calendar time corresponding to operational time $\tau$ (with the convention $t = 0$ when $\tau = \tau_0$), and if the force of future claims inflation is $i$, the current price total is obviously given by:

$$\hat{\mu}' = \sum_{\tau} \exp(i \cdot t) \cdot \hat{m}_{\tau}. \tag{F.2}$$

To evaluate the current price prediction $\hat{\mu}'$ from equation F.2, the relationship between $t$ and $\tau$ is needed. This can be approximated using a continuous curve. A typical shape is shown in Figure 20. The shape can usually be well approximated using an exponential distribution function, although a Gamma distribution function is sometimes necessary for later origin years. There is usually uncertainty about the real time scale.

Thus, we have $\tau = F(\varphi' \cdot t)$ for some known function $F$. The uncertainty in the real time scale is represented by the random variable $\varphi'$.

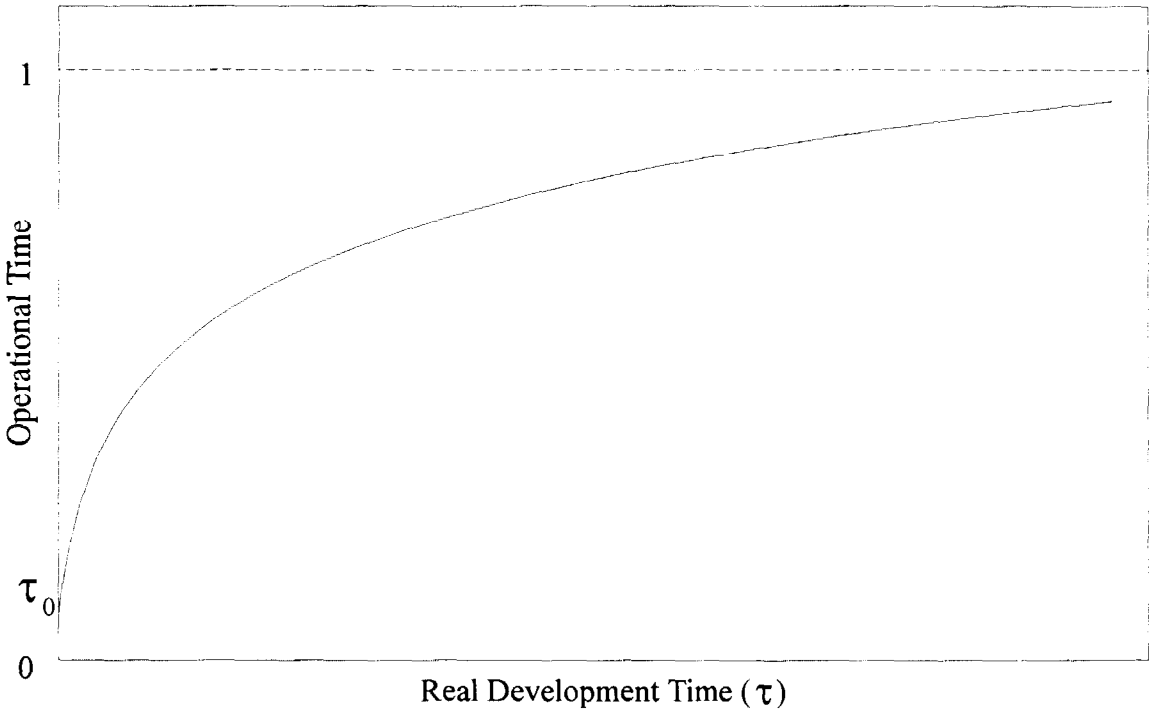
Rearranging:

$$t = \varphi \cdot H(\tau), \tag{F.3}$$

where $H$ is the inverse of $F$, and $\varphi = 1/\varphi'$. A following section will describe how the function $H$ can be found in cases where the run-off of settlements is approximately exponential.

FIGURE 20

TYPICAL TAIL RELATIONSHIP BETWEEN OPERATIONAL TIME AND REAL DEVELOPMENT TIME



Substituting from Equation F.3 into Equation F.2, and writing $\theta$ for $i \cdot \varphi$ gives:

$$\hat{\mu}' = \sum_{\tau} \exp(\theta \cdot \mathrm{H}(\tau)) \cdot \hat{m}_{\tau} . \tag{F.4}$$

In general, there is uncertainty in both the rate of future claims inflation and the real time scale, so both $i$ and $\varphi$ are random variables. The expected values are denoted $\hat{i}$ and $\hat{\varphi}$, and the variances (representing the uncertainty) are denoted $U_i$ and $U_\varphi$, respectively. Since $\theta$ is the product of these two random variables, its mean is given by $\hat{\theta} = \hat{i} \cdot \hat{\varphi}$ and its variance by:

$$U_\theta = U_i \cdot U_\varphi + \hat{i}^2 \cdot U_\varphi + \hat{\varphi}^2 \cdot U_i . \tag{F.5}$$

It is always possible to have $\hat{\varphi} = 1$ by scaling the function $H$ (Equation F.3). When this is done, we have $\hat{\theta} = \hat{i}$. Equations F.4 and F.5 become:

$$\hat{\mu}' = \sum_{\tau} \exp(\hat{i} \cdot H(\tau)) \cdot \hat{m}_{\tau} , \tag{F.6}$$

and

$$U_\theta = U_i \cdot U_\varphi + \hat{i}^2 \cdot U_\varphi + U_i . \tag{F.7}$$

Writing $A_\tau$ for the inflation factor $\exp(\hat{i} \cdot \mathrm{H}(\tau))$, we have:

$$\hat{\mu}' = \sum_{\tau} A_\tau \cdot \hat{m}_{\tau} . \tag{F.8}$$

Current price predictions can be calculated using the methods described in Appendices D and E with the following changes:

1. The estimate of the total of future payments is given by Equation F.8 instead of Equation D.3.
2. The "future process variance" (given in the standard deviation columns in the examples) is not given by Equation D.2, but by:

$$\hat{\sigma}^2 = \varphi^2 \cdot \sum_\tau A_\tau^2 \cdot \hat{m}_\tau^\alpha$$

if the model was fitted using inflation adjusted data (as in Sections 2 and 3), or:

$$\hat{\sigma}^2 = \varphi^2 \cdot \sum_\tau (A_\tau \cdot \hat{m}_\tau)^\alpha \tag{F.9}$$

if the model was fitted using unadjusted data (as in Section 4).

3. In calculating the estimation error as described in Appendix D, the variance-covariance matrix $\boldsymbol{V}$ of the estimated $\beta$-parameters is extended to:

$$\begin{pmatrix} U_\theta \; 0 \ldots 0 \\ 0 \\ . \quad \boldsymbol{V} \\ . \\ 0 \end{pmatrix}$$

and the vector $\delta$ is extended to have a first component $\delta_0$ given by:

$$\delta_0 = d\mu'/di = \sum_\tau H(\tau) \cdot A_\tau \cdot m_\tau. \tag{F.10}$$

The effect of this is simply to augment the mean-square-error by an amount:

$$(d\mu'/di)^2 \cdot U_\theta . \tag{F.11}$$

Note that $E(R' \mid i)$ and Var $(R' \mid i)$ are not nearly linear in $i$, (where $R'$ is the total of future claims for an origin year in current prices). Therefore, using the best estimate $i$ in these functions, as done in Equations F.8 and F.9, will not give such good approximations to the unconditional mean and variance as in the case of $M$ (Equations E.4 and E.5). Also, it may be noted that the additional element of variance was regarded as part of the "future process variance" $\sigma^2$ in Equation E.5, whereas it is regarded here as an element of the "estimation error." This is unimportant, but it seems more natural to regard Var $(E(R' \mid i))$ as estimation error because in cases where $m_\tau$ has a log link-function (such as Models 1 and 2 of Sections 3 and 4), $i$ is exactly like one of the β-parameters of the model $m_\tau$.

The exponential model for the run-off of the remaining $(1 - \tau_0) \cdot M$ claim settlements over real development time is:

$$F(t) = 1 - (1 - \tau_0) \cdot \exp(-t/\beta), \tag{F.12}$$

where

$$\beta \cdot \ln(2) = \text{expected time for half the remaining claims to be closed.} \tag{F.13}$$

Inverting this gives:

$$t = -\beta \cdot \ln[(1 - \tau)/(1 - \tau_0)].$$

Comparing with Equation F.3, in order to have $\hat{\varphi} = 1$, we must have:

$$H(\tau) = -\hat{\beta} \cdot \ln[1 - \tau)/(1 - \tau_0)], \tag{F.14}$$

where $\hat{\beta}$ is the best estimate given by Equation F.13. The uncertainty in the estimate $\hat{\beta}$ obtained from Equation F.13 can be used to give a value for the variance $U_\varphi$ of $\varphi$. A numerical example is given in Section 6.

# APPENDIX G

## MODELS FOR PARTIAL PAYMENTS

### *Introduction*

Each figure from the claim amounts triangle has two components:

$$Y_d = Y_{d\,1} + Y_{d\,2}\,,$$

where

$Y_{d\,1}$ = total of settlement payments, and

$Y_{d\,2}$ = total of partial payments on claims not yet closed.

In this appendix, the second subscript is used to distinguish between these two components. The origin year subscript $w$ is dropped to simplify the presentation. Thus:

$N_{d\,1}$ = number of claims closed, and

$N_{d\,2}$ = number of partial payments, in development period $d$.

This appendix is concerned with the distribution of $Y_{d\,1}$ and $Y_{d\,2}$ conditional on $N_{d\,1}$. This is relevant for the modelling of $Y_d$ when $N_{d\,1}$ is known.

### *Conditional Distribution of* $Y_{d\,1}$

If:

$m_\tau$ = expected size of settlement payment at operational time $\tau$, and

$\varphi$ = coefficient of variation of settlement payments (assumed the same for all $\tau$),

then:

$$\mathrm{E}(Y_{d\,1}) = N_{d\,1} \cdot m_\tau \text{ and } \mathrm{Var}(Y_{d\,1}) = N_{d\,1} \cdot \varphi^2 \cdot m_\tau^2\,, \tag{G.1}$$

where $\tau$ is the "mean" operational time of development period $d$.

*Conditional Distribution of $Y_{d2}$*

It is assumed that:

- the expected size of a non-zero partial payment $= \Gamma \cdot m_\tau$ (for some constant $\Gamma$, the same for all $\tau$), and
- the coefficient of variation of non-zero partial payments $= \varphi$ (the same as for settlement payments).

It is assumed that each open claim generates partial payments according to a Poisson process with parameter $p$ ($p$ is the expected number of partial payments per year of delay until the claim is closed). Therefore, if $L_d$ represents the mean number of claims outstanding at the end of development period $d$, the number of prepayments $N_{d2}$ is Poisson distributed with parameters $L_d \cdot p$. Initially it is assumed that $p$ is constant for all $w$ and $d$, that is, the partial payment Poisson process is homogeneous over real development time and the rate is the same for all claims. Alternatives are considered shortly.

Under the above assumptions, $Y_{d2}$ has a compound Poisson distribution. Using standard results from risk theory,

$$E(Y_{d2}) = p \cdot \Gamma \cdot L_d \cdot m_\tau \ ,$$

and

$$\mathrm{Var}(Y_{d2}) = (1+\varphi^2) \cdot p \cdot \Gamma^2 \cdot L_d \cdot m_\tau^2 \ . \tag{G.2}$$

*Conditional Distribution of Payment Per Claim Closed*

Conditional on $N_{d1}$ and $L_d$, $Y_{d1}$ and $Y_{d2}$ are mutually independent, so the variance (as well as the mean) is additive. Adding Equations G.1 and G.2 gives:

$$E(Y_d) = N_{d1} \cdot [1 + c \cdot R_d] \cdot m_\tau \ ,$$

and

$$\mathrm{Var}(Y_d) = N_{d1} \cdot [\varphi^2 + (1+\varphi^2) \cdot \Gamma \cdot c \cdot R_d] \cdot m_\tau^2 \ ,$$

where:

$$R_d = L_d/N_{d\,1} \text{ and } c = p \cdot \Gamma \,.$$

The payment per claim closed, $S_d = Y_d/N_{d\,1}$ therefore has:

$$\mathrm{E}(S_d) = [1 + c \cdot R_d] \cdot m_\tau \,, \tag{G.3}$$

and

$$\mathrm{Var}\,(S_d) = \varphi^2 \cdot [1 + (1 + 1/\varphi^2) \cdot \Gamma \cdot c \cdot R_d] \cdot m_\tau^2/N_{d\,1} \,.$$

If $c \cdot R_d < 1$ (as is usually the case) then we have:

$$\mathrm{E}(S_d) \approx \exp\,(c \cdot R_d) \cdot m_\tau \,. \tag{G.4a}$$

If $(1 + 1/\varphi^2) \cdot \Gamma \approx 1$ (this seems plausible; e.g., $\varphi = 1, \Gamma = 0.5$), then we have:

$$\begin{aligned}\mathrm{Var}\,(S_d) &\approx \varphi^2 \cdot \exp\,(c \cdot R_d) \cdot m_\tau^2/N_{d\,1} \\ &= \varphi^2 \cdot \mathrm{E}(S_d)^2/[\exp\,(c \cdot R_d) \cdot N_{d\,1}].\end{aligned} \tag{G.4b}$$

Equations G.4 can be used to estimate the parameter $c$ by Fisher's scoring method in the case of a log-link model for $m_\tau$.

Equation G.4b for the variance of $S$ is more approximate than the expression for the mean. This is acceptable because the variance is only of secondary importance. Its role is to determine the weights in fitting the model for the mean. The expression for the variance can be checked empirically via residual plots. The approximation $(1 + 1/\varphi^2) \cdot \Gamma \approx 1$ can be monitored by plotting standardized residuals against $R_d$. Fanning out indicates that $(1 + 1/\varphi^2) \cdot \Gamma > 1$.

If $c$ is known, the data $S$ can be preadjusted to remove the partial payment effect:

$$S_d' = S_d/\exp\,(c \cdot R_d) \,.$$

We then have:

$$E(S_d') = m_\tau = \text{ mean size of settlement payments},$$

and

$$\mathrm{Var}\,(S_d') = \varphi^2 \cdot m_\tau^2/[\exp\,(c \cdot R_d) \cdot N_{d\,1}]. \qquad \text{(G.5)}$$

If $c = 0$ (i.e., $p = 0$ or $\Gamma = 0$; no non-zero partial payments), Equations G.3 and G.4 reduce to Equations 2.2 and 2.3.

*Alternatives and Generalizations*

In previous paragraphs it was assumed that until a claim is closed, it yields partial payments at a constant rate $p$ per development year. This implies that the total number of partial payments (and hence the total claim amount) depends on the real time scale. An obvious generalization is to allow the partial payment rate to be a function of $d$, say $p_d$. Intuitively, one might expect the rate to decrease with $d$; for example, $p_d = k/d^{\alpha}$ or $p_d = k \cdot \exp(-\alpha \cdot d)$. The form of Equation G.4 is unchanged if $p_d$ is not constant. For example, $p_d = k/d^{\alpha}$ gives the same equations but with $c = k \cdot \Gamma$ and $R_d = L_d/(d^{\alpha} \cdot N_{d\,1})$. However, for any function $p_d$, the expected total number of partial payments, and hence the expected amount of each claim, will depend on the real time scale.

Earlier portions of this appendix have all been concerned with the partial payment rate over real development time. An alternative is to consider the partial payment rate over operational time (defined in terms of the number of claims closed). The simplest model of this type (analogous to the constant $p_d$ model presented earlier) is that the partial payment rate over operational time is constant. That is, each claim outstanding at operational time $\tau$ has a fixed probability of yielding a partial payment in the next increment of operational time, the probability not depending on $\tau$. In terms of the partial payment rate over real development time this can be expressed as $p_d = k \cdot N_{d\,1}/M$, for some constant $k$ (because $N_{d\,1}/M$ is the increase in operational time). This implies that the expected number of partial payments on a claim is proportional to the operational time of settlement of the claim, and does not depend on the real time scale. (This is invariance in Taylor's sense; see [9]).

Using this expression for $p_d$ in the development leads to equations of the same form as Equation G.4 but with $c = k \cdot \Gamma$ and $R_d = L_d/M$ instead of $R_d = L_d/N_{d1}$.

If the ratio of the number of claims outstanding (i.e., reported but not closed) to the total number not yet closed is approximately constant, then we have $L_d/M = \pi \cdot (1 - \tau)$, where $\pi$ is a constant. In such a case we have

$$\mathrm{E}(S_d) = \exp\,(c \cdot \pi \cdot (1 - \tau)) \cdot m_\tau\,, \tag{G.6a}$$

and

$$\mathrm{Var}\,(S_d) = \varphi^2 \cdot \mathrm{E}(S_d)^2/[\exp\,(c \cdot \pi \cdot (1 - \tau)) \cdot N_{d1}]\,, \tag{G.6b}$$

where $c = k \cdot \Gamma$. Note that the factor $\exp\,(c \cdot \pi \cdot (1 - \tau))$ decreases monotonically as $\tau$ increases from zero to one. Thus, although the mean settlement payment $m_\tau$ will usually increase, the mean payment per claim closed may decrease for some values of $\tau$. As operational time progresses, the number of claims which may contribute partial payments to the payment per claim closed decreases, so $\mathrm{E}(S_d)$ may decrease.

Throughout this appendix, $\mathrm{E}(S_d)$ denotes the expected value of $S_d$ conditional on $N_{d1}$. The absence of any dependence on $N_{d1}$ in Equation G.6a implies that the $\mathrm{Cov}(S_d, N_{d1}) = 0$, so the negative association between $S$ and $N_1$ described in Section 7 does not exist under the assumptions presented earlier. However, this is not usually plausible for the reasons given in Section 7. Empirical studies have confirmed the existence of a negative association between $S$ and $N_1$ (for example, Taylor [8], Section 6).

In previous paragraphs, the notation $p_d$ has been used for the rate (in real time) at which partial payments are made on each claim outstanding in development period $d$. Similarly, $p_\tau$ will be used to denote the rate (in operational time) at which partial payments are made on each claim outstanding in the development period corresponding to operational time $\tau$.

Thus,

$p_\tau/M$ = expected number of partial payments per $1/M$ increase in operational time (per outstanding claim).

Therefore:

$$p_d = N_{d\,1} \cdot p_\tau/M.$$

Previous sections have considered the cases:

$p_d$ = constant (above, in this appendix),

$p_\tau$ = constant, that is, $p_d = k \cdot N_{d\,1}/M$ .

In the real world, the truth probably lies somewhere between these two extremes. The arguments of Section 7 suggest that $p_d$ increases with the number of settlements $N_{d\,1}$ but not proportionately. In other words, $p_d$ increases but $p_\tau$ decreases as $N_{d\,1}$ increases.

This can be modelled using $p_d = p_0 + k \cdot N_{d\,1}/M$, for some constants $p_0$ and $k$. This implies that $p_d$ decreases from some value of $d$ onwards, as suggested earlier. By varying the ratio of $p_0$ to $k$, any situation between the extremes of $p_d$ constant and $p_\tau$ constant can be attained. Repeating the development and using this expression for $p_d$ leads to:

$$\mathrm{E}(S_d) = \exp\,(c_0 \cdot L_d/N_{d\,1} + c_1 \cdot L_d/M) \cdot m_\tau\,,$$

and

$$\mathrm{Var}\,(S_d) = \varphi^2 \cdot \mathrm{E}(S_d)^2/[\exp\,(c_0 \cdot L_d/N_{d\,1} + c_1 \cdot L_d/M) \cdot N_{d\,1}]\,, \quad (\mathrm{G.7})$$

where $c_0 = p_0 \cdot \Gamma$ and $c_1 = k \cdot \Gamma$.

If the ratio of the number of claims outstanding to the total number not yet closed is approximately constant at $\pi$, then we have $L_d/M \approx \pi \cdot (1 - \tau)$. If $\ln(m_\tau)$ includes a constant term and a term linear in $\tau$ (for example Models 1 and 2 of Section 2), the factor $\exp\,(c_1 \cdot L_d/M)$ can therefore be subsumed into $m_\tau$ and we have:

$$E(S_d) = \exp\,(c_0 \cdot L_d/N_{d\,1}) \cdot m_\tau\,, \tag{G.8}$$

which is the form obtained in Equations G.4a and G.4b under the assumption, $p_d$, constant. The only difference is that in Equation G.4a, $m_\tau$ is the mean settlement payment, so would normally be a monotonic increasing function of $\tau$, whereas, here, with $p_d$ not necessarily constant, $m_\tau$ includes the factor $\exp\,(c_1 \cdot \pi \cdot (1 - \tau))$ of the partial payment effect, and is not necessarily monotonic increasing.

Since $c_1$ is not estimated (it is confounded with other parameters of $m_\tau$), the factor $\exp\,(c_1 \cdot L_d/M)$ cannot easily be included in the denominator of Var $(S_d)$. Since $c_1$ is probably small, it seems reasonable to omit this factor from the prior weights in fitting the model. Residual plots will indicate if this is unreasonable.

# DISCUSSION OF PAPER PUBLISHED IN VOLUME LXXVII

# REINSURER RISK LOADS FROM MARGINAL SURPLUS REQUIREMENTS

RODNEY KREPS

DISCUSSION BY DANIEL F. GOGOL

I would like to thank Jim Higgins and Tony Iafrate for their suggestions on this discussion.

## 1. INTRODUCTION

Rodney Kreps's paper contains some useful formulae, and the central idea is an important one. That idea is to determine risk loads by estimating the additional surplus that is required to write an additional contract, and then requiring premium such that the return on additional surplus equals some rate selected by management. The amount of additional surplus is such that writing the contract does not change the probability that the losses from the book of business will cause surplus to fall below zero within the year. The required additional surplus is estimated using the formula

$$\mathrm{Var}(L_1 + L_2) = \mathrm{Var}(L_1) + \mathrm{Var}(L_2) + 2\mathrm{Cov}(L_1, L_2)\,, \tag{1.1}$$

where $L_1$ is a random variable equal to the ultimate losses from the contract, and $L_2$ is a random variable equal to the ultimate losses from the rest of the book of business for the next accident year.

I believe this approach could be useful, but it would require some modifications.

## 2. MARGINAL SURPLUS

If risk loads are estimated based on a required yield on marginal expected value lower than the required yield. An example may be the best way to point this out.

Suppose that a reinsurance company with surplus $S$ has 1,000 contracts on its books, each with standard deviation of losses $\sigma$. If the losses from each contract are independent, then the standard deviation of the total losses is $\sqrt{1{,}000}\,\sigma$. Suppose a new contract, independent of the others and with standard deviation $\sigma$, is added to the books. The standard deviation of total losses becomes $\sqrt{1{,}001}\,\sigma$. Therefore, if $r$ (in Kreps's terminology) and the desired yield on marginal surplus are small, the marginal surplus is approximately $((\sqrt{1{,}001} - \sqrt{1{,}000})/\sqrt{1{,}000})S$, i.e., about $(1/2{,}000)S$.

So, if Kreps's method is applied to each of the other 1,000 contracts as they are renewed, the sum of the marginal surplus amounts for each will equal approximately half the total surplus and the required yield on marginal surplus will be approximately twice the yield on total surplus.

The "marginal surplus required" as defined in Kreps's paper is related to the increase in the standard deviation of the book of business caused by the additional contract. However, if all the contracts in the book of business were ordered from first to $n^{\text{th}}$, then $\sum_{k=1}^{n} (\sigma_{1,k} - \sigma_{1,k-1}) = \sigma_{1,n}$, where $\sigma_{1,i}$ is the standard deviation of the set of the first $i$ contracts. Kreps's method, however, estimates the effect on the standard deviation of each contract as if it were added at the end of the list, when the marginal effect is less.

If $C_k$ is the $k^{\text{th}}$ contract in the above type of ordering and $\Delta_k\sigma$ is the increase in the total standard deviation caused by the addition of the $k^{\text{th}}$ contract, then $\sigma_k \geq \Delta_k\sigma \geq \sigma_k^1$ where $\sigma_k$ is the standard deviation of the $k^{\text{th}}$ contract and $\sigma_k^1$ is the effect of the $k^{\text{th}}$ contract if it is added at the end of the list, as in Kreps's method. Therefore, there is some $W$ such that, if $n$ is the total number of contracts,

$$\sum_{k=1}^{n} (W\sigma_k + (1 - W)\sigma_k^1) = \text{total standard deviation of the book.} \quad (2.1)$$

If the contribution of the $k^{th}$ contract to the total standard deviation of the book is estimated as $W\sigma_k + (1 - W)\sigma_k^1$, then the sum of the individual estimates equals the total standard deviation, as it should, and each estimate uses the same weighting.

## 3. VARIANCE OF LOSS RESERVE

A portion of surplus is needed to support the variance of the loss reserve, and, therefore, some amount may be required for many years to support a new contract. This affects the yield rate, but the paper's discussion of the yield rate on surplus does not address this complication. The method of allocating surplus to contracts based on their effect on total standard deviation could be used for allocating surplus to the various subdivisions of the loss reserves, as well as to contracts. The effect of the standard deviation of the run-off of reserves one year later on the total standard deviation of surplus could be used together with a method for contracts which will be suggested below.

## 4. DISCOUNTING OF LOSSES

For simplicity, the expected return is defined in the paper as premium less losses and expenses; but in practice, some decision has to be made on discounting losses to correctly reflect economic values in the risk loads. The yield rate on surplus in the paper is based on undiscounted losses without reflecting the time value of money.

## 5. SUGGESTED METHOD FOR SELECTING RISK LOADS

The two problems mentioned above, i.e., the need for discounted losses and the need for surplus to support loss reserves, can be dealt with simultaneously.

Butsic [1] explains the need to discount loss reserves at a rate lower than the risk-free rate. In this way, when the value of the liabilities is invested at the risk-free rate, there is an expected profit and thus a reward for risk. Myers and Cohn [2] use the Capital Asset Pricing Model (CAPM) to compute this discount rate.

The term "the value of the loss reserve" means that the value is discounted at the above rate. For contract $i$, let the random variable $X_i$ be the present value at the risk-free rate, on the effective date of the contract, of the losses to be paid in the next year plus the value of the loss reserve at the end of the year. Let $\sigma_i$ be the standard deviation of $X_i$ . Surplus could be allocated to each contract $i$ by using Kreps's method with the above formula $W\sigma_i + (1 - W)\sigma_i^1$ used in place of $\sigma_i^1$ .

The risk load which must be added to $E(X_i)$ to provide the required yield on surplus may then be determined. After the end of the year, the required yield on the portion of surplus allocated to the loss reserves of the contract is provided for by the rate at which loss reserves are discounted, as mentioned above. Also, a portion of surplus should be allocated to assets as well as loss reserves in order to reflect the fact that they are not risk-free. In addition, when surplus is allocated to contracts, loss reserves, and assets, the effect of each on the total risk of the company is considered. Therefore, the covariance between a contract's risk and the entire remainder of the insurer's risk must be considered in formula (1.1).

Kreps's approach is only a way of relating the required return on a new contract to its effect on total standard deviation. There is no consideration of systematic versus unsystematic risk (in the terminology of modern financial theory). The author states that market pricing is consistent with his approach, but financial theorists generally believe that the covariance of stock market returns and the rate of return on surplus must be considered in explaining market pricing. (See Myers and Cohn [2] and Cummins [3].)

The necessary risk load for a high layer may be much different for a small company than for a larger company, using Kreps's method. The larger company is able to diversify away much of the risk. If a company insures a high layer that it is going to reinsure, it would be reasonable for it to charge for that layer based on the actual cost of reinsurance rather than to apply Kreps's method to the gross losses.

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# DISCUSSION OF PAPER PUBLISHED IN VOLUME LXXVIII

# THE COMPETITIVE MARKET EQUILIBRIUM RISK LOAD FORMULA FOR INCREASED LIMITS RATEMAKING

GLENN G. MEYERS

DISCUSSION BY IRA ROBBIN

## 1. INTRODUCTION

Glenn Meyers has made a valuable contribution to actuarial literature with his well-written paper on how to load increased limits factors (ILFs) for risk. Given the complexity of the topic, he deserves special commendation for his coherent presentation. Meyers clearly states his fundamental assumptions and provides sufficient background for the reader to understand his results in context. His skill in composing mathematical derivations is also noteworthy. As to substance, Meyers uses the intuitively appealing market paradigm as the foundation for his risk load theory. This represents a conceptual step forward. Unfortunately, Meyers does not carry the theory far enough, and, in my opinion, ends up with an incorrect answer.

## 2. WHAT IS RISK LOAD?

Before explaining why his answer is wrong, it is necessary to set the stage by first defining risk load. What is risk load? I would define it as an extra component of indicated premium arising from the potential for possible deviations between expected and actual loss results. The indicated premium for a coverage is the sum of the expected loss cost, expense provisions, and usual profit load, plus the risk load.

Why is risk load needed? The most general answer is that the price for insurance coverage ought to somehow depend on the volatility of the actual losses covered by the insurance. In the pricing of assets, such as bonds, it is well accepted that the interest rate demanded by the market rises with the riskiness of the asset. With respect to increased limits, risk load is important because increasing the limit of coverage changes the relative volatility of actual results versus initial expectations.

## 3. OVERALL LEVEL VERSUS RELATIVITY BY LIMIT OR LINE

When considering risk loads for increased limits, it is useful to split the question into two parts by asking first, "What is the proper overall level of risk load?" and, second, "How should the risk load vary by limit and by line of coverage?" While I have some question about whether Meyers developed a logically consistent theory for setting an appropriate overall level of risk load, my major criticism is that his model produces risk loads that can rise too steeply by limit.

## 4. PROCESS AND PARAMETER RISK

Before detailing this criticism, I should note my agreement with much of what Meyers has done. In particular, I concur with Meyers that risk loads should reflect not only the stochastic variability of actual results versus expectation (process risk), but also the uncertainty about the loss expectation itself (parameter risk). This is generally accepted in principle by most actuaries knowledgeable in the subject. I also accept the collective risk model Meyers employed to incorporate parameter risk (see Section 4 of his paper). While I might want to quibble with the specific techniques Meyers used for quantifying parameter risk, I will not do so in this discussion.

## 5. MARKET EQUILIBRIUM THEORY AS A FOUNDATION FOR RISK LOAD

I agree with Meyers that market-based theory can provide a sensible foundation for risk load calculations. The basic idea in using a market-based theory is to apply the "supply-versus-demand" concept of general economics to the pricing of risk. A key advantage of this approach is that it explains the need for risk loads in ILFs using fundamental economic principles. I also believe it provides a basis for conceptually defining what risk loads should be included in ILFs filed by a rating bureau. Under my view, risk loads filed by a rating bureau should be those that would be theoretically charged by a rational market in equilibrium. Actual risk loads charged in the real market could, of course, be different since the market may be irrational or in disequilibrium. However, by relying on a theoretical market, one should obtain bureau risk loads that are sound benchmarks unaffected by market cycles or imperfections. Individual companies can then deviate up or down as they deem appropriate.

The theoretical market approach is not universally accepted as the logical foundation for a theory of risk load. Some, for instance, have insisted on determining what the risk load should really be and not what some hypothetical market says it should be. While this "just give me the real risk load" attitude sounds direct and practical, it leads nowhere. The problem is that there is no inherent notion of how to properly price for risk, either before or after the fact. In contrast, the right price for losses is an amount sufficient, on average, to cover the actual losses. After the fact, we know the actual loss costs and what we should have charged for losses. However, we have no actual "risk" costs to tell us what the charge for risk should have been. If there is a difference between expected and actual losses, we have evidence of volatility and thus proof that some risk load should have been charged. Nonetheless, this evidence alone does not tell us how to measure volatility, nor how to translate volatility, however measured, into a charge for risk.

## 6. UTILITY THEORY

Utility theory has been used as a conceptual framework for pricing risk. Under utility theory, the minimum premium an insurer is willing to accept is the lowest one for which the insurer's expected utility will not suffer if it provides the coverage. This means that the utility of initial wealth is the same as the expectation of utility of final wealth, where final wealth equals initial wealth plus premium less expenses and actual losses. A key point is that the resulting risk load is, to first approximation, proportional to the variance of losses.

I agree with Meyers that a "single insurer, single insured" implementation of utility theory is too simplistic. It produces risk loads an insurer might want to charge if there were no other insurers competing for the business and if there were no reinsurance market. With only one insurer in the model, there is no market and therefore no room for the forces of supply and demand to operate.

## 7. THE MEYERS RISK LOAD THEORY

In the Meyers model, there are many insurers and many insureds. So far, so good. Each insurer sets a constraint on the variance of losses it will tolerate on its individual book of business, and seeks to achieve the maximum profit subject to that constraint. The insureds are assumed to have an inelastic demand for insurance coverage by limit. For example, half the market may demand coverage at a $1 million limit and will pay the going rate to obtain it. The insurers then "bid" on the risks. Under the given assumption, and a further hypothesis about the total needed risk load over all lines and limits, the market in each subline and limit will be cleared at an optimal price. With some elegant mathematics, Meyers shows that the market clearing price can be viewed as the price that would have been charged by an average insurer acting alone. This result is shown in his Equation 5.6. Thus, Meyers effectively ends up with a variance-based risk load, and his theory could be regarded as another argument for variance. The central question, then, is whether his argument undercuts the serious criticisms made against variance-based risk loads.

## 8. THE "UNITS DON'T MATCH" OBJECTION TO VARIANCE-BASED RISK LOADS

Before presenting what I regard as valid criticisms of variance-based risk loads, I would like to switch sides for a moment and refute one set of common arguments made against variance. This set of arguments deals with the units of risk load and with currency translation. Consider that variance is in units of "dollars squared," while the desired cost is in units of "dollars." As any engineering or physics student knows, if the units don't match, there is a mistake somewhere in the derivation. Related to this is the criticism that variance-based risk load formulae lead to nonsensical results if one tries to switch from one currency to another.

Both these criticisms are an attack on the formula: $RL = \lambda \cdot \mathrm{Var}\,(L)$, where $RL$ is risk load, $L$ is loss, and $\lambda$ is a constant. If $L$ is in dollars, then $\mathrm{Var}\,(L)$ is in units of dollars squared. It is true that if $\lambda$ were a unitless constant, then there would be a mismatch of units. However, $\lambda$ should carry units of inverse dollars, so that the resulting risk load is, as it should be, in units of dollars.

This also refutes the currency translation paradox. Write $\$L$ for loss in dollars and £$L$ for loss in pounds and let £$L = \kappa \cdot \$L$, where $\kappa$ is the exchange rate constant. It follows that:

$$\mathrm{Var}\,(£L) = \kappa^2 \cdot \mathrm{Var}\,(\$L).$$

Ostensibly confounding results arise if $\lambda$ is viewed as a unitless constant and not adjusted for exchange rate. However, if £$RL = \lambda_{£} \cdot \mathrm{Var}\,(£L)$, where $\lambda_{£} = \lambda_{\$}/\kappa$, then £$RL = \kappa \cdot \$RL$. Thus, the problem disappears if $\lambda$ carries units of inverse currency and is properly adjusted to reflect exchange rates.

## 9. VARIANCE-BASED RISK LOADS MAY LEAD TO INCONSISTENT INCREASED LIMITS FACTORS

Having shown I do not find merit in every argument against variance-based risk loads, I will now turn to one I do regard as valid. In my view, the major flaw with variance-based risk loads is that they can lead to prices that may rise too rapidly by limit. For example, it might cost more to raise the limit from \$2 million to \$3 million than it does to raise the limit from \$1 million to \$2 million. With a variance-based risk load, it would not be impossible to have the following ILF table:

TABLE 1

| Limit (\$000) | ILF |
|---|---|
| 1,000 | 2.50 |
| 2,000 | 3.00 |
| 3,000 | 3.75 |

To see what is wrong here, consider matters from the point of view of a prospective insured whose basic limits premium is $10,000. This insured will be asked to pay $25,000 for $1 million of coverage, an extra $5,000 to increase coverage from $1 million to $2 million, and an extra $7,500 to increase coverage from $2 million to $3 million. Breaking this down by layer and using an "M" suffix to denote million(s), the insured would see the following:

TABLE 2

| Layer | Cost |
|---|---|
| 0 -$1M | $25,000 |
| $1M excess of $1M | $5,000 |
| $1M excess of $2M | $7,500 |

What is exceedingly strange about this is that the loss in the lower excess layer ($1M excess of $1M) must be $1 million before there is even a penny of loss in the upper excess layer ($1M excess of $2M). Further, loss in the upper layer can never exceed the loss in the lower layer. How can it make sense to charge more for the upper layer when it never has more loss?

This problem of inverted layer costs is referred to as "inconsistency by layer." Increased limits factors are consistent if they exhibit a pattern of declining marginal increases as the limit of coverage is raised. If the ILF formula is viewed as a function with respect to the coverage limit, and this function has a second derivative, then the ILFs are consistent if, and only if, the second derivative is negative. Consistency implies that excess layer costs decline as the attachment point is raised, assuming all layers carry the same limit of excess coverage, and assuming that excess layer costs are computed by taking differences in appropriate ILFs.

Meyers is well aware of the inconsistency problem. He tries to get around it by defining a new notion of consistency. Under the Meyers definition (see Meyers's Appendix E), one considers any two excess layers with identical limits, but different attachment points. Consistency, according to Meyers, exists if the layer with the higher attachment point always has a lower indicated price. Meyers calculates the indicated price for a layer as the sum of its expected loss plus risk load. His risk load is proportional to the variance of losses in the *layer*. In his paper, Meyers gave a proof that this calculation results in consistency under *his* definition. Subsequently, Meyers has told this reviewer that the proof may not be valid.

However, even if it were valid, it would prove much less than it may seem. Since Meyers does not calculate layer prices by taking the difference between ILFs, his notion of consistency does not apply to the ILFs, but rather to his premium calculation principle. In fact, taking differences of ILFs gives a larger indicated (variance-based) risk load for a layer than would calculating layer risk load based on the variance of layer losses. This happens because the variance in layer losses is always less than the difference in the variance of losses capped at the upper limit minus the variance of losses capped at the lower limit (see Appendix A). This is the essence of "risk reduction through layering." (See Miccolis [2].)

Therefore, Meyers is, at least implicitly, asserting that ILFs are not appropriate for pricing layers. Also inherent in his theory is the idea that the price for coverage up to a limit depends on how the coverage is layered. For example, under Meyers, coverage for the layer from zero to $2 million should have a price different than the sum of prices for coverage on the underlying $1 million plus coverage on the $1-million-excess-of-$1-million layer. Not only does this fail to produce a unique benchmark price for coverage, because it allows that different layerings of coverage could result in different prices, it also leaves open the question of what layering, if any, should be assumed when filing ILFs. Meyers believes that the bureau should file ILFs under the hypothesis that layering is not allowed, and yet also promulgate advisory factors for various possible layerings. I feel the publication of different costs for different layerings of the same coverage is no solution, and only adds to the confusion. Try as he might, Meyers is unsuccessful at defining away the consistency problem. ILFs produced with the variance-based risk load can be inconsistent.

## 10. VARIANCE REDUCTION THROUGH PRO RATA SHARING

It is also somewhat troubling that variance-based risk loads lead not only to "risk reduction through layering," but also to "risk reduction through sharing." To illustrate this, suppose two insurers decide to become "50-50" partners in writing a risk. Each will take half the premium and pay half the loss. If $\sigma^2$ is the variance of the total loss, then $\sigma^2/4$ is the variance of the loss covered by each insurer. If each uses a variance-proportional risk load with a common risk loading scalar, $\lambda$, then the total risk load demanded by their syndicate will be $\lambda \cdot \sigma^2/2$. If either insurer had written the risk on its own, then the risk load would have been $\lambda \cdot \sigma^2$. Due to syndication, the risk load has been cut in half. The syndicate operates, in effect, with a reduced risk load constant. More generally, if risk load is in proportion to variance and if expense considerations are neglected, then syndicates ought to be able to take advantage of "risk reduction through sharing" in order to reduce their risk loading constants.

It appears that quota share syndicates would not be allowed in the Meyers's Competitive Market Equilibrium (CME) model. I deduce this from Equation 5.3 in which Meyers derives the theoretical market risk loading constant as the (harmonic) *average* of the $\lambda$s for the individual insurance companies. If the companies were allowed to form a syndicate and quota share the business, the theoretical market $\lambda$ would be much lower than the average $\lambda$.

## 11. RESTRICTIONS ON THE "COMPETITIVE MARKET"

In the market posited by Meyers, risk load is proportional to variance and there is no excess layer or quota-share reinsurance. Yet, the variance principle implies that such reinsurance is decidedly advantageous. Implicitly, Meyers has prohibited insurers from entering into transactions that his theory says are beneficial.

Meyers must also have some hidden restrictions on the insureds to prevent them from taking advantage of analogous ways to reduce their variance-based risk loads. For example, instead of buying \$2 million of coverage, an insured could opt to save money by buying a primary policy with a limit of $1 million and an excess policy covering $1 million excess of $1 million. As well, the insured could save money by getting two policies with each covering half of the insured's losses.

If Meyers had a free market model, then the insurers would be allowed to reinsure and the insureds would be permitted to package coverage. In either event, the market would soon cease to operate under the original variance-based risk load. As Venter [3] has noted, variance-based risk loads create arbitrage possibilities. Yet one aspect of competitive market theory is that "arbitrage profit possibilities are quickly extinguished by market competition"[3]. Meyers started off to build a "Competitive Market Equilibrium" theory, but ended up with ILFs that could never exist in a free market.

## 12. PUTTING REINSURANCE INTO THE MODEL

In my opinion, the theory ought to be extended to arrive at the theoretical premium that would prevail in a competitive market that allowed reinsurance. Additional expenses associated with reinsurance must also be considered. Since reinsurance companies do exist, such an extended theory would be a better model of reality than the highly-restricted model proposed by Meyers.

By theoretically allowing coverage to be reinsured, one is not forcing any individual insurer to abandon use of a utility function to price the risk on its net coverage. Each insurer could still use a utility function and end up demanding a risk load proportional to the variance of loss it covers. However, as previously noted, variance reduction through layering or sharing can lead to a market risk load that is not proportional to the variance of the total (unlayered and undivided) loss.

The reduction of variance through layering will lead to a hypothetical market price for coverage by limit in which risk load rises less rapidly than the variance of limited losses. Indeed, if one neglects to consider expenses associated with layering, the continual subdivision into finer layers will drive the process risk component of theoretical market risk loads to zero. This is proven in Appendix A.

However, layering has its costs. At the very least, it leads to additional processing and billing as each reinsurer needs to receive its share of the premium and pay losses in its layer. While such costs may be small, they rise as the number of subdivisions is increased. Intuitively, it is clear that there will come a point when risk load reduction through further layering will be canceled out by the additional costs. At that point, one arrives at the best possible price the theoretical market could offer.

I do not have a general formula for this lowest theoretical price under arbitrary layerings that also reflects the expenses associated with layering. However, I can present to the reader a comparable formula derived by Fred Klinker for the risk-sharing case. As shown in Appendix B, if one allows a syndicate of insurers to take percentage shares of coverage, with each charging a risk load proportional to the *variance* of its share of loss, and add in expenses which rise with the number of syndicate members, then the total risk load is proportional to the *standard deviation* of the total loss. Since layering is more efficient than sharing in reducing process variance, it is likely that the lowest theoretical market price for coverage will increase less rapidly by limit than the standard deviation of limited loss. In summary, if Meyers were to complete his theoretical foundation and allow layering and reflect associated expenses, his theory would produce risk loads whose process risk load components would not increase in proportion to process variance, but to something that rises even less steeply by limit than does the process standard deviation.

Venter [3] has advocated calculation of the risk-loaded premium using the expected value of a risk-adjusted distribution. This distribution is obtained by transforming the original loss distribution. In his paper, Venter deduces some general principles from the requirement that the market be free of arbitrage possibilities. The transformation methodology satisfies these principles. Further, according to Venter, the only premium calculation principles with the desired properties are those that can be generated from transformed distributions. However, it is not clear if any transformation would be equivalent to an extended Competitive Market Equilibrium model which incorporated reinsurance and associated expenses. Once such a model is developed, its equivalence to some transformation of distributions should be investigated.

## 13. SUPPLY AND DEMAND

The supply assumption in the Meyers CME theory is that each company sets a constraint on the loss variance it will tolerate. While this reviewer knows of no company that actually does this in practice, it is likely that most insurers implicitly operate under such a constraint. Thus, the assumption seems reasonable, especially in the context of modelling a theoretical market.

However, the assumption about demand could be much improved. Recall that the Meyers CME theory effectively ignores the demand side of the "supply and demand" equation under the dubious assumption that demand is inelastic. I would assert on general grounds that demand by limit might well be influenced by the pattern of increase in a set of ILFs. While it is beyond the scope of this discussion to propose a theory incorporating an explicit demand function, future research in this direction would seem worthwhile. Such a theory would require an explicit demand function by limit. Estimation of the demand function might be rather difficult. One problem is that the only data currently available is on the limits of primary policies purchased by insureds. To obtain a true picture of the demand by limit, one would need data on the total coverage afforded by the combination of primary and excess policies purchased by insureds.

The "demand side" perspective also leads to the consideration of the impact of risk retention groups. Under one approach to risk retention financing where the goal is to minimize the probability of ruin, a risk retention group must assess the total risk load proportional to the standard deviation of losses for all members of the group. Within this framework, principles of game theory were used by Lemaire [1] to calculate a fair cost allocation for each member of the group. The resulting risk load assessment for a group member is not proportional to the variance of the member's losses. This strongly suggests that variance-based risk loads would not be theoretically tolerated in a market that allowed the formation of risk retention groups. Note also that proper consideration of risk retention groups would entail consideration of expenses. One might end up with a result not too dissimilar from what one would achieve by incorporating reinsurance layering and associated expenses in the CME theory.

## 14. CONCLUSION

Meyers has made a valuable contribution to the field of actuarial literature with a well-written and thought-provoking paper. He has laid down some of the foundation for a solid theory of risk load and has artfully applied mathematical techniques to get a result. Meyers was courageous enough to go beyond the "single insurer, single insured" paradigm to develop a competitive market approach. Unfortunately, he posited an artificially restricted market and ended up with variance-based risk load. This is the same answer produced by the "single insurer, single insured" utility theory model. The Meyers theory does nothing to dispel valid criticisms made against variance-based risk load. Most important, his variance-based risk load formula could lead to inconsistent ILFs, where "inconsistent" retains its original and appropriate meaning as given by Miccolis [2]. In conclusion, I believe Meyers did not succeed in obtaining a risk load formula appropriate to use in bureau increased limits ratemaking.

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[2] Miccolis, Robert S., "On the Theory of Increased Limits and Excess of Loss Pricing," *PCAS* LXIV, 1977, p. 27.

[3] Venter, Gary G., "Premium Calculation Implications of Reinsurance Without Arbitrage," *ASTIN Bulletin*, Vol. 21, No. 2, 1991, p. 223.

# APPENDIX A

## IMPACT OF LAYERING ON PROCESS VARIANCE

Suppose coverage up to limit $K$ is achieved by stacking $n$ layers, and assume the indicated risk-loaded pure premium for the coverage is the sum of the indicated risk-loaded pure premiums for the layers. Suppose the indicated risk-loaded pure premium for each layer is calculated as the sum of expected loss in the layer plus risk load, and assume risk load is proportional to variance. Consider the "process" variance component of the risk load. I will first show that this layering reduces the risk load compared to the risk load of the unlayered coverage. I will then show that the sum of process variance risk loads becomes smaller as the layer subdivisions grow ever finer. In other words, if expenses are not included in the analysis, infinite layering will drive process variance risk load to zero.

Let $0 = k_0 < k_1 < ... < k_n = K$ be the layer end points.

Let $\Delta k_i = k_i - k_{i-1}$ for $i = 1, 2, ..., n$.

Given loss severity random variable, $X$, define $X_i = \min(X, k_i)$ and let $\mu_i = E(X_i)$ and $\sigma^2{}_i = \mathrm{Var}\,(X_i)$.

Let $Y_i$ denote the loss in the $i^{\text{th}}$ layer so that $Y_i = X_i - X_{i-1}$.

Thus, $E(Y_i) = \Delta\mu_i = \mu_i - \mu_{i-1}$.

Now, set $\tau_i^2 = \mathrm{Var}\,(Y_i)$.

*Proposition*

$$\mathrm{Var}\,(X_n) = \sum_{i=1}^{n} \tau_i^2 + 2 \cdot \sum_{i=1}^{n-1} (k_i - \mu_i) \cdot \Delta\mu_{i+1} \tag{A.1}$$

Proof: Since $X_n = \sum Y_i$, it follows that:

$$\mathrm{Var}\,(\min(X, K)) = \sum_{i=1}^{n} \tau_i^2 + 2 \cdot \sum_{i<j} \mathrm{Cov}(Y_i, Y_j).$$

Now consider that, for $i < j$,

$$\mathrm{Cov}(Y_i, Y_j) = (\Delta k_i)(\Delta\mu_j) - (\Delta\mu_i)(\Delta\mu_j) = ((\Delta k_i) - (\Delta\mu_i)) \cdot (\Delta\mu_j) .$$

The desired result then follows since:

$$\Delta k_1 + \Delta k_2 + \ldots + \Delta k_i = k_i \text{ and } \Delta\mu_1 + \Delta\mu_2 + \ldots + \Delta\mu_i = \mu_i .$$

Since the latter sum in equation A.1 has only non-negative terms, one can immediately conclude that layering reduces total risk load if risk load is proportional to variance.

*Variance Reduction Through Layering*

$$\mathrm{Var}(\min(X, K)) \geq \sum_{i=1}^{n} \tau_i^2 \tag{A.2}$$

The inequality is strict if there is some non-zero probability of a non-zero loss strictly less than the upper limit, $K$.

If the subdivision process continues indefinitely, the process risk load will shrink to nothing.

*Infinite Layering Drives Process Variance to Zero*

$$\text{If } \max\{\Delta k_i : i = 1, 2, \ldots, n\} \to 0 \text{, then } \sum \tau_i^2 \to 0 . \tag{A.3}$$

Proof: Consider that $\tau_i^2 < \Delta k_i^2$ so that $\sum \tau_i^2 < \sum \Delta k_i^2$.

Further observe that $\sum \Delta k_i^2 < \max\{\Delta k_i\} \cdot \sum \Delta k_i = \max\{\Delta k_i\} \cdot K$.

The result follows since the latter expression approaches zero by assumption and by the fact that $K$ is unchanged by the layering.

# APPENDIX B

## RISK LOADS CHARGED BY A PRO RATA SHARING SYNDICATE

(Simplified from the unpublished work of Fred Klinker)

If there is a competitive insurance market with risk sharing among the insurers via pro rata reinsurance, and if the expenses associated with risk sharing are considered, then variance-based risk loads at the level of the individual firm become standard deviation-based at the level of the market. This seemingly paradoxical result stems from a trade-off between the increased expense and decreased "risk" associated with sharing.

Assume each insurance firm charges a risk load proportional to the variance of loss for the coverage it provides. Now suppose there is a syndicate of $n$ insurers in which each takes a fixed share, $\rho_i$ , of the total loss, where the $\rho_i$ are non-negative and sum to one.

The loss experienced by the syndicate as a whole, $L$, is a random variable with expectation $\mathrm{E}(L)$ and variance $\mathrm{Var}(L)$. The loss experienced by the $i^{\text{th}}$ insurer is $\rho_i L$ with expectation $\rho_i \mathrm{E}(L)$ and variance $\rho_i^2 \mathrm{Var}(L)$. Neglecting regular underwriting expenses and the usual profit load, the components of the net premium charged by this insurer are:

| | |
|---|---|
| Expected loss: | $\rho_i \mathrm{E}(L)$ |
| Fixed expense (transaction costs): | $\varepsilon$ |
| Risk load: | $\lambda_i \rho_i^2 \mathrm{Var}(L)$ |

The resulting net premium charged by this insurer is

$$P_i = \rho_i \mathrm{E}(L) + \varepsilon + \lambda_i \rho_i^2 \mathrm{Var}(L). \tag{B.1}$$

The resulting premium charged by the syndicate of $n$ insurers is, recalling that the $\rho_i$ are weights summing to unity,

$$P(n, \rho_i) = \mathrm{E}(L) + n\varepsilon + \mathrm{Var}(L) \sum_{i=1}^{n} \lambda_i \, \rho_i^2 \, . \tag{B.2}$$

Now minimize the premium given in equation B.2, holding $n$ fixed. Equation B.2 will be minimized by minimizing

$$\sum_{i=1}^{n} \lambda_i \, \rho_i^2 \, .$$

The minimization is constrained since the $\rho_i$ must sum to unity. The constrained minimization can be solved by the method of Lagrange multipliers.

$$\frac{\partial}{\partial \rho_i} \sum_{j=1}^{n} \lambda_j \, \rho_j^2 = \Lambda \frac{\partial}{\partial \rho_i} \sum_{j=1}^{n} \rho_j \;\; \text{for all } i \, .$$

$$2\lambda_i \, \rho_i = \Lambda \;\; \text{for all } i \, .$$

In other words, the optimal $\rho_i$ are proportional to the reciprocals of the $\lambda_i$ for all $i$ .

Imposing the constraint that the $\rho_i$ sum to one leads to

$$\rho_i = \frac{1}{\lambda_i \left( \sum_{j=1}^{n} \frac{1}{\lambda_j} \right)} \, . \tag{B.3}$$

It follows that

$$\sum_{i=1}^{n} \lambda_i \, \rho_i^2 = \frac{1}{\sum_{j=1}^{n} \frac{1}{\lambda_j}} = \frac{1}{n \left\langle \frac{1}{\lambda} \right\rangle} \, . \tag{B.4}$$

where the angled brackets denote the average value. The minimum premium, for fixed $n$, from equation B.2 and the above, is

$$P(n) = \mathrm{E}(L) + n\,\varepsilon + \frac{\mathrm{Var}(L)}{n\left\langle \frac{1}{\lambda} \right\rangle}. \tag{B.5}$$

Note the behavior of equation B.5 with respect to $n$, the number of insurers in the syndicate. There is a term due to fixed transaction costs per insurer which increases linearly with the number of insurers. Another term declines as the reciprocal of the number of insurers, which captures the declining aggregate of the variance-based risk loads of the individual insurers. The decline is due to the increasing spread of risk among more insurers.

Because of the above behavior, there is an optimal $n$ for which the premium, $P(n)$, is minimized. This occurs where the derivative of $P(n)$ with respect to $n$ vanishes.

$$0 = \frac{d}{dn} P(n) = \varepsilon - \frac{\mathrm{Var}(L)}{n^2\left\langle \frac{1}{\lambda} \right\rangle}.$$

Hence,

$$n = \sqrt{\frac{\mathrm{Var}[L]}{\varepsilon\left\langle \frac{1}{\lambda} \right\rangle}} \tag{B.6}$$

and the minimum value of $P(n)$ is

$$P = \mathrm{E}(L) + 2\left( \sqrt{\frac{\varepsilon\,\mathrm{Var}(L)}{\varepsilon\left\langle \frac{1}{\lambda} \right\rangle}} \right) \tag{B.7}$$

Equation B.6 provides the optimal number of insurers and equation B.7 provides the minimum premium over all possible numbers of insurers and all possible pro rata distributions of risk among insurers. The last term of equation B.7 can be identified as the market risk load. In the immediate context of increased limits factors, with increasing limit—hence increasing Var $(L)$—the optimal number of insurers on the contract will increase according to equation B.6. Also, the market risk load of equation B.7 will increase only as the square root of Var $(L)$; i.e., as the standard deviation of $L$, where $L$ is the aggregate loss across the syndicate, despite the fact that individual insurer risk loads are variance-based rather than standard deviation-based.

# DISCUSSION OF PAPER PUBLISHED IN VOLUME LXXIII

# THE COST OF MIXING REINSURANCE

RONALD F. WISER

DISCUSSION BY MICHAEL WACEK

*Abstract*

*In his 1986 paper, "The Cost of Mixing Reinsurance," Ron Wiser analyzed the consequences of mixing pro rata and excess of loss reinsurance. He concluded that such mixed reinsurance situations were always unfavorable to the ceding company, both in terms of financial cost and loss ratio stability. This paper refutes that conclusion and shows that Wiser proved his argument only for the special case of a particular, inefficient reinsurance structure. Some of his reported "cost of mixing" was actually a result of purchasing redundant reinsurance. The degree of inefficiency in a mixed reinsurance structure can be quantified as the "cost of overlap" between the types of reinsurance coverage. This should be measured separately from the cost of mixing per se. A case of negative cost of mixing is presented, and a relatively simple test of whether the mixing cost will be positive or negative is derived and demonstrated.*

## 1. INTRODUCTION

Ron Wiser's "The Cost of Mixing Reinsurance" [1] is an excellent presentation of many of the major issues involved in the analysis of the effect of mixing proportional and excess of loss reinsurance. It is a well-written paper that introduces the reader to several important concepts. Because I see that paper as so important, I believe it is necessary to comment on some aspects the author has omitted. My primary concern is that the Wiser paper may leave the casual reader with the impression that mixing excess of loss and proportional reinsurance *always* has adverse cost and stability consequences for the ceding company. That is not true, as I will show.

This discussion is best read with a copy of Wiser's paper close at hand, since it frequently refers in detail to the examples presented in that paper. An effort has been made to use the terminology and notation of the original paper, in order to make it easier to read the two together.

## 2. EFFICIENT MIXING OF REINSURANCE

The key point of this discussion is to make it clear that Wiser proved his Mixing Loss Ratio Rule *only* for the special case in which the ceding company buys excess of loss coverage all the way up to the top of the policy limits it has issued. In such circumstances, the purchase of proportional reinsurance is redundant and is thus inherently inefficient. That it has a cost should come as no surprise. This is understood by most insurers, and they do not usually structure their reinsurance in this way. As a result, Wiser's conclusion that mixed reinsurance situations are always costly to the ceding company is relevant only to reinsurance arrangements that are not often found in practice.

An insurer normally will determine the net retention it desires for a particular risk and use a mix of proportional and excess of loss reinsurance to absorb the exposure between the retention and the policy limit. In property insurance the proportional reinsurance is typically provided under "surplus share" treaties, which essentially provide the insurer with the capacity to write policy limits larger than it could with its excess of loss reinsurance alone. These treaties act like a sponge, soaking up the surplus exposure above its excess reinsurance coverage. Normally, if there is no "surplus" policy limit exposure, the insurer does not cede any exposure to the surplus share treaties. If it does, it is generally according to a line guide agreed to in advance with the surplus and excess reinsurers, and the price and terms of the reinsurance contracts will reflect the expected cessions implied by the line guide. Insurers also use a mixture of excess and proportional reinsurance to extend their capacity for casualty policy limits, though here, the use of surplus share treaties is less common than in property insurance.

Figure 1 is a graphical illustration of this common type of property reinsurance program. It shows: A) a net retention of $500,000; B) a $500,000 excess of $500,000 treaty; and C) a five-line surplus share treaty. This gives the insurer total gross line capacity of $6,000,000. Policies larger than $6,000,000 would require facultative reinsurance (either proportional or excess of loss).

If $P$ denotes the policy limit and $F$ denotes the facultative coverage limit, an insurer with the treaty reinsurance program shown in Figure 1 generally would keep a share, equal to $\$1{,}000{,}000/(P-F)$, of any "net and treaty" policy limit, $P-F$, greater than $1,000,000. The excess reinsurer remains exposed for its entire $500,000 limit, though the composition of that exposure is different after proportional reinsurance is introduced.

Figure 2 illustrates the allocation of loss exposure arising from a $5 million policy issued by an insurer that has the reinsurance program summarized in Figure 1.

## 3. ORDER OF REINSURANCE RECOVERIES

In mixed reinsurance situations, i.e., where a policy is protected by a number of reinsurance contracts, working out the correct loss recoveries can be fairly involved. In theory the interplay between the applicable coverages is negotiable but, in practice, the following principles are normally applied as standard unless otherwise agreed:

1. *More specific coverage responds before less specific coverage.* For example, facultative always responds before treaty. Per risk covers always respond before catastrophe (i.e., per occurrence) covers.
2. *Proportional coverage responds before excess of loss coverage, subject, however, to Principle 1.* For example, a property surplus treaty responds before a per risk excess cover but after a facultative excess cover.

This discussion assumes the application of these recovery principles. The conclusions will not be applicable to those relatively rare instances where a proportional treaty has been negotiated to respond *after* an excess of loss treaty.

FIGURE 1

COMMON PROPERTY REINSURANCE STRUCTURE

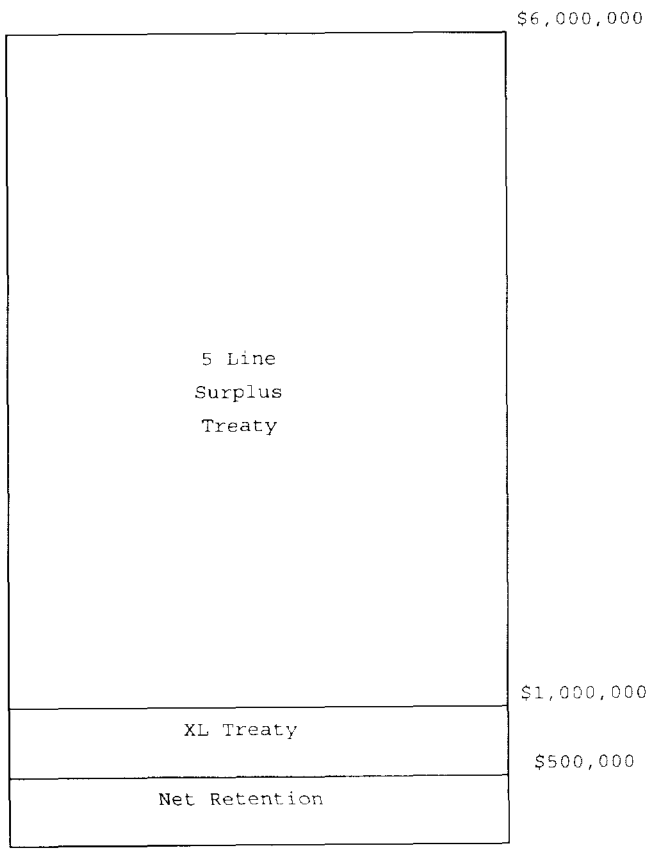
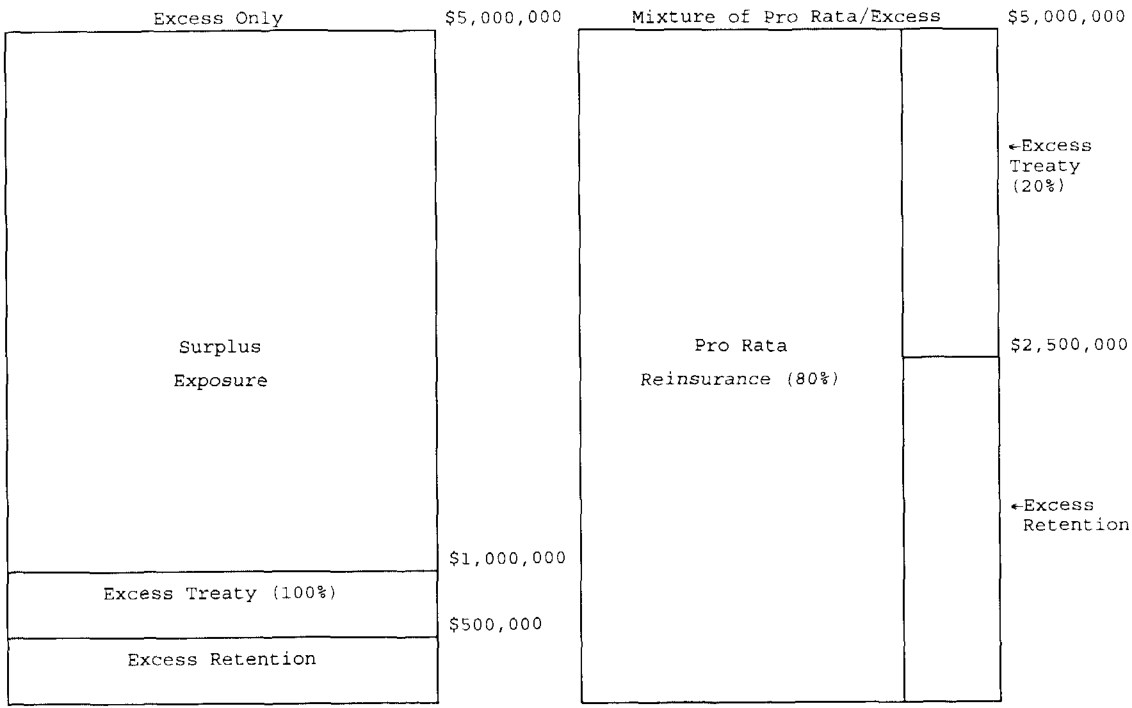


FIGURE 2
ALLOCATION OF LOSS EXPOSURE
COMMON PROPERTY REINSURANCE STRUCTURE / $5 MILLION POLICY



## 4. EXCESS LIMIT EXPOSURE

At least a portion of the "cost of mixing" demonstrated by Wiser is a result of the inefficient reinsurance structure he assumed, for both his examples and his general case. Consider his examples.

In the casualty case presented in Wiser's paper [1, pp. 175-187], the excess coverage is $2 million excess of $250,000. For some reason, the insurer buys 50% pro rata facultative coverage on a $1 million excess of $100,000 policy. Without the facultative placement, the excess reinsurer is exposed for $750,000 excess of $250,000. But with it, the excess exposure is only $250,000 excess of $250,000. We should not find it surprising that the excess reinsurer has a lower expected pure premium and the insurer a higher expected cost of reinsurance! The allocation of loss exposure is shown graphically in Figure 3, where it is evident that the proportional coverage has crowded out the excess of loss protection. The bar on the left shows the allocation of exposure in the pure excess case. The bar on the right illustrates the mixed case. The much smaller area corresponding to excess treaty exposure in the mixed case is indicative of the reduced exposure compared to the pure excess case.

The same is true of Wiser's property example [1, pp. 191-201], though the inefficiency of the reinsurance is less extreme than in the first case. The excess coverage is $2 million excess of $250,000. The insurer issues a $20 million policy and cedes 90% of the risk ($18 million) on a pro rata basis. The pro rata placement reduces the excess reinsurer's maximum exposure from $2 million excess of $250,000 to $1.75 million excess of $250,000. Again, it should be obvious that the excess reinsurer will benefit and the insurer's expected net cost of reinsurance will be increased. The allocation of exposure in the pure excess and mixed scenarios is shown graphically in Figure 4.

In both of these examples the insurer has bought *overlapping* reinsurance coverage. While there might be valid reasons for doing this under certain circumstances (e.g., "protecting the excess treaty"), it will always be more expensive than not doing so. In the casualty case, it is not clear why the insurer bought any proportional cover at all, since it had adequate coverage under its excess of loss treaty alone. In the property case, the

## FIGURE 3
## ALLOCATION OF LOSS EXPOSURE
## FIRST CASUALTY EXAMPLE $1 MILLION EXCESS OF $100,000 POLICY

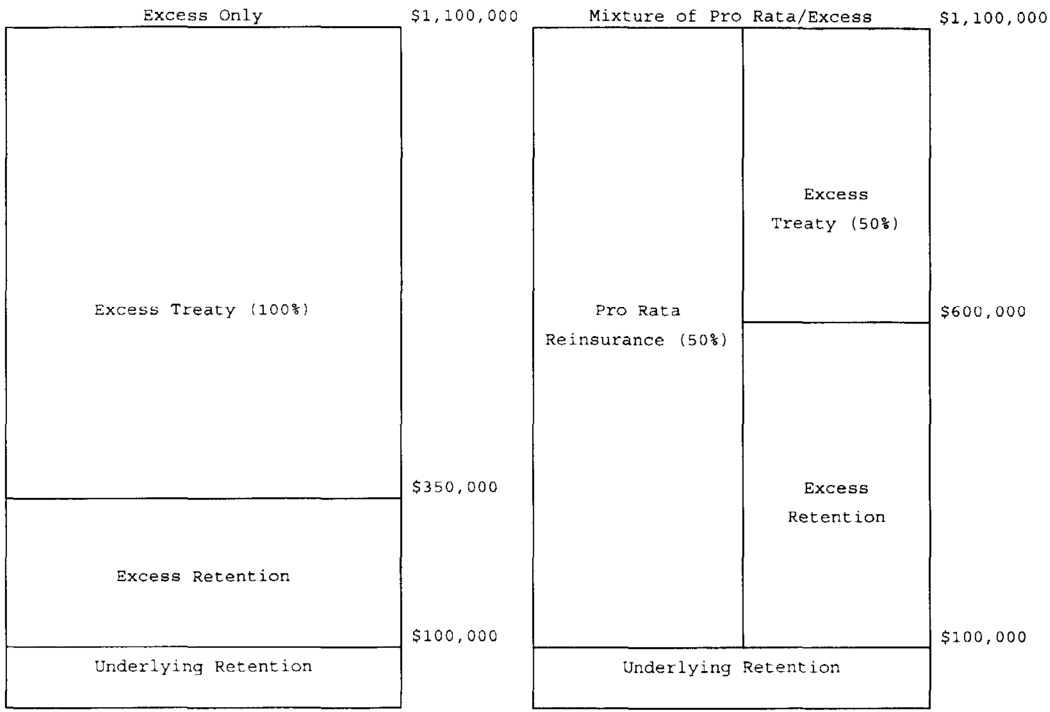
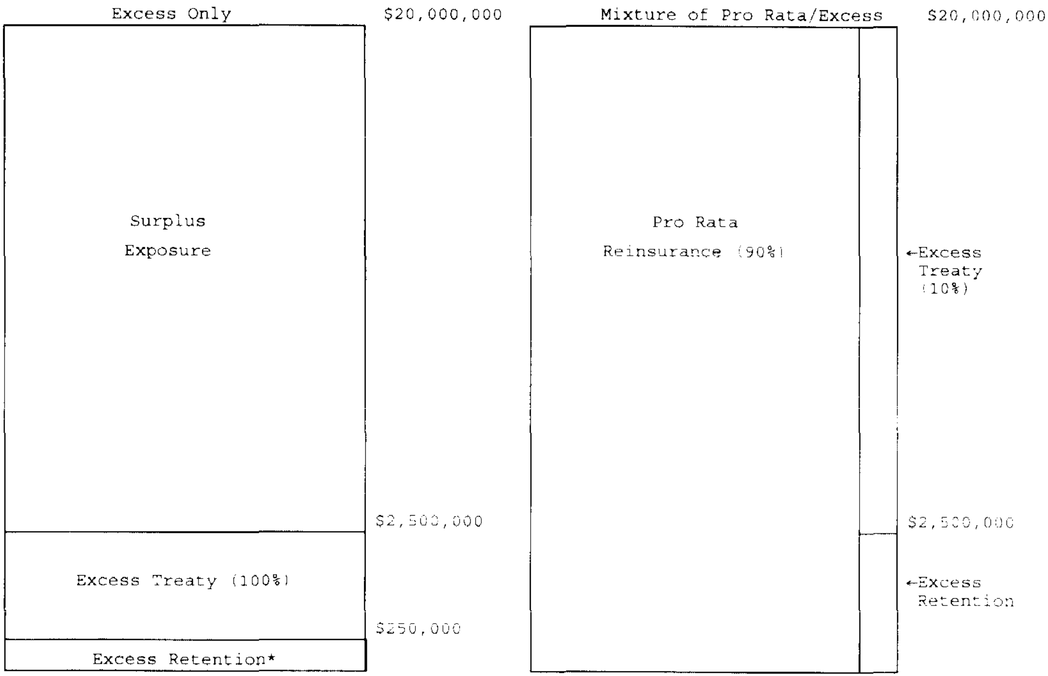


FIGURE 4
ALLOCATION OF LOSS EXPOSURE
PROPERTY EXAMPLE / $20 MILLION POLICY




* Area not to scale

correct pro rata purchase to avoid overlap would have been $17.75 million, or 88.75%, instead of $18 million (90%).

The point is that the "costs of mixing" shown on Wiser's Exhibits 2 and 3 and implied by his Mixing Loss Ratio Rule include not only the effect of mixing but also the cost of buying down the excess reinsurer's limits exposure. Indeed, we cannot be certain that the effect of mixing per se is unfavorable without further analysis.

## 5. COST OF OVERLAP AND COST OF MIXING PER SE

In the casualty example presented by Wiser, the reinsurer's total policy limit exposure in the pure excess case is $750,000 excess of $250,000, which implies expected losses of $85,144, or 35.48% of the policy losses subject to the excess coverage. See Wiser's Exhibit 1.1 [1, p. 178]. This layer can be subdivided into two layers of $250,000 excess of $250,000 and $500,000 excess of $500,000. The expected losses in these two layers are 19.71% and 15.77% of subject losses, respectively. These ratios are easily computed using values from the limited mean function of the size of loss distribution. (See Exhibit 6, or Wiser's Exhibit 1.2, for a partial tabulation of this function.)

In the mixed reinsurance case, the excess reinsurer's total policy limit exposure is only $250,000 excess of $250,000 with respect to subject loss exposure of $500,000. (This is equivalent to 50% of $500,000 excess of $500,000 with respect to original policy exposure of $1 million.) The $500,000 excess of $500,000 layer now has no policy limit exposure at all. The expected excess losses in these two layers are 15.77% and 0% of subject losses, respectively.

The difference in $250,000 excess of $250,000 layer expected losses between the pure excess and mixed reinsurance cases is approximately 3.93% (19.71% - 15.77%) of subject losses, or about $4,720. This is the true cost of *mixing* excess and proportional reinsurance in this example.

See Exhibit 1 of this discussion for a summary of the key limits, limited means, and relative exposure for the first layer. This exhibit measures the cost of *mixing per se*.

Since the ceding company has the same net retention of $250,000 after buying the pro rata coverage as in the pure excess case, all it has achieved is to buy down the excess reinsurer's limit exposure. This is manifested by the exposure in the $500,000 excess of $500,000 layer going to zero in the mixed case. The cost of the limit buy-down is about $18,927, which is 15.77% of subject losses. This is the measure of the inefficiency of this particular mixed reinsurance structure, the "cost of *overlapping* reinsurance." See Exhibit 2, which is analogous to Exhibit 1, but summarizes the cost of *overlap* calculations.

Wiser reports the cost of mixing in this example to be $23,653. In fact, the cost of mixing per se is only $4,720, or 20%, of this total. The remaining 80% is due to buying unnecessary reinsurance.

In the property example, the excess coverage can be layered as $1.75 million excess of $250,000 and $250,000 excess of $2 million. In the pure excess case the expected losses in these two layers are 32.88% and 1.41% of subject policy losses, respectively. (This can be verified using values from Exhibit 7.)

Matching up these layers with their counterparts in the mixed case, the reinsurer's exposure in the first layer is 13.88% of subject losses and the second layer is not exposed at all. The cost of mixing is given by the difference in first layer expected losses, which is approximately 19% of $30,000, or $5,699. The cost of overlap is the difference in second layer expected losses. This is equal to 1.41% of $30,000, or $422. Thus, in this example, 93% of the cost of mixing reported by Wiser is due to mixing itself and 7% is due to the purchase of redundant coverage. The cost of overlap is much lower than in the casualty example because, here, the degree of overlap between excess and proportional reinsurance is much less.

Exhibits 3 and 4 summarize the cost of mixing and cost of overlap calculations, respectively, for this example.

## 6. NEGATIVE COST OF MIXING

To see that mixing pro rata and excess coverage is sometimes actually favorable to an insurer, suppose an insured having the same loss severity characteristics as in Wiser's casualty example buys total insurance coverage of $10 million.

Assume the $5 million excess of $5 million layer is written by an insurer that has excess of loss reinsurance of $1.75 million excess of $250,000. With no proportional reinsurance, the insurer has a total net retention of $3.25 million ($250,000 at the bottom and $3 million at the top). The left portion of Figure 5 illustrates this graphically.

Exhibit 5 shows the calculation for the cost of mixing per se for this example. With excess reinsurance only, the expected excess reinsurance recovery is 49.77% of subject losses. If expected subject losses are $50,000, the expected recovery amount is $24,884.

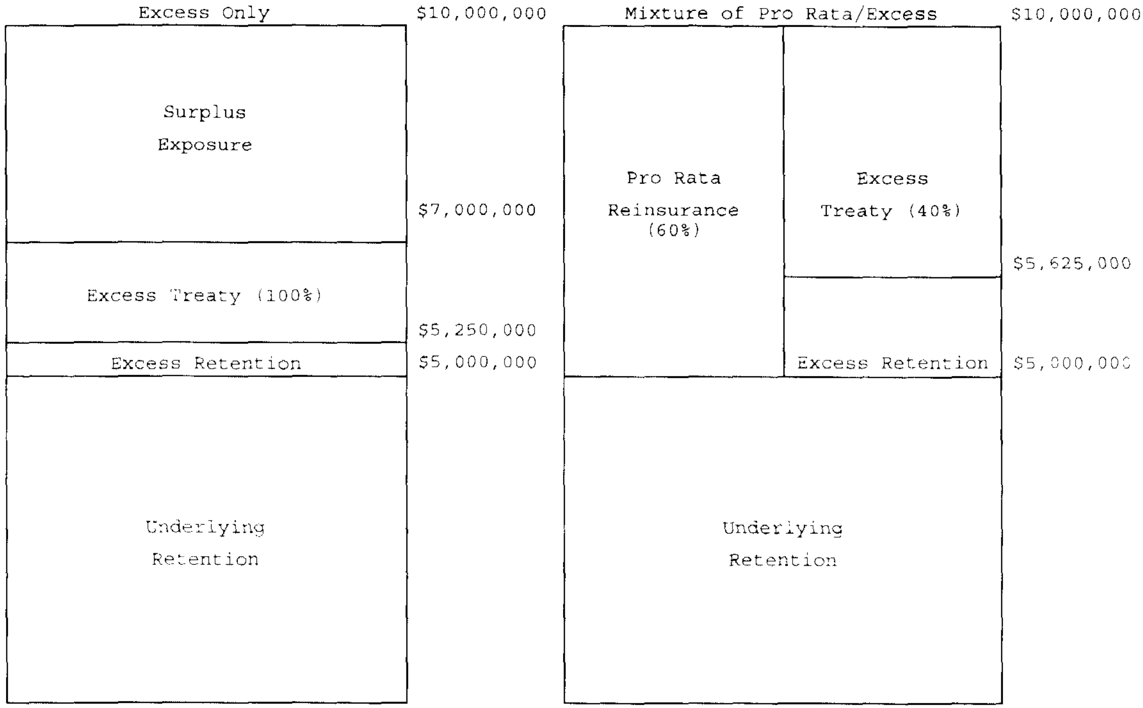
With proportional reinsurance of 60%, the insurer's net retention is reduced to $250,000. This eliminates all net exposure above the excess of loss protection, but leaves the excess reinsurer's maximum loss exposure unchanged at $1.75 million. This is illustrated by the right portion of Figure 5. The expected excess recovery is 40% of $37,695 (the expected losses in the layer $4.375 million excess of $5.625 million) or 75.35% of subject losses (40% of $50,000).

The cost of mixing is (49.77% – 75.35%) of $20,000, or –$5,116. The cost consequences of mixing are adverse to the reinsurer. The insurer sees a cost benefit. It should be clear on the face of it that the insurer also benefits in terms of stabilization: without mixing, it has a much higher net retention, most of it in the form of the unreinsured layer $3 million excess of $2 million.

## 7. DETERMINING COST OR BENEFIT OF MIXING REINSURANCE

Apart from a minor refinement or two, the following analysis uses Wiser's notation.

# FIGURE 5
## ALLOCATION OF LOSS EXPOSURE
## SECOND CASUALTY EXAMPLE / $5 MILLION EXCESS OF $5 MILLION POLICY



Let *RateXS*($a$, $M$, $L$) denote the excess pure premium rate for excess treaty retention, $M$, excess treaty limit, $L$, and pro rata treaty retention percentage, $a$. Wiser states that the most general characterization of the excess pure premium rate where there is no proportional reinsurance is:

$$RateXS(1, M, L) = \frac{\int_{M}^{M+L} (x - M) \cdot f(x)dx + L \cdot \int_{M+L}^{\infty} f(x)dx}{\text{Subject Premium}}, \qquad (7.1)$$

where $f(x)$ is the p.d.f. describing the distribution of policy losses by size.[1] (Note that the variable name on the left side of the equation has been modified slightly from Wiser's notation to record the reinsurance limit.)

The numerator of formula 7.1 can be expressed in terms of a difference of limited means of $f(x)$:

$$RateXS(1, M, L) = \frac{\mathrm{E}_{M+L}(x) - \mathrm{E}_{M}(x)}{\text{Subject Premium}}. \qquad (7.2)$$

The discerning reader will have noticed that, as the definition of the *pure premium* rate, formula 7.2 is incorrect—it reflects only claim severity. The subject claim count, E($N$), has been left out of the numerator. Correcting for this omission, formula 7.2 becomes:

$$RateXS(1, M, L) = \frac{\mathrm{E}(N) \cdot \left[\mathrm{E}_{M+L}(x) - \mathrm{E}_{M}(x)\right]}{\text{Subject Premium}}. \qquad (7.3)$$

Subsequent reference to the formulae in the Wiser paper will assume this correction.

In his analysis Wiser assumed that the insurer buys sufficient excess reinsurance to cover the largest policies issued. In such a case, policy limits always truncate excess exposure at or before the point where the

[1] The formula appearing in the paper shows the factor before the integral in the second term as ($L$ + $M$), but it clearly should be $L$ as shown here.

reinsurance limit does. That makes the second term of formula 7.1 unnecessary, and Wiser proceeds with the following simplification, which does not depend on $L$:

$$RateXS(1, M) = \frac{E(N) \cdot \left[ \int_M^\infty (x - M) \cdot f(x)dx \right]}{\text{Subject Premium}} \tag{7.4}$$

$$= \frac{E(N) \cdot [E(x) - E_M(x)]}{\text{Subject Premium}} .$$

However, because insurers normally do not structure their reinsurance in the way he assumes, the second term cannot be dropped for the analysis of a more realistic scenario, much less for the general case. Therefore, the following discussion rests on formula 7.3, which applies in the general case.

Let us investigate the relationship between expected excess losses in mixed and pure excess cases. In the mixed reinsurance case, the Mixed Pricing Rule tells us to divide the excess retention and limit by the pro rata retention ratio, $a$, to determine the effective excess layer in terms of the policy loss function, $f(x)$. The limited mean claim size for a net limit, $k$, reflecting a pro rata retention, $a$, is given by the following:

$$a \cdot E_k(x, a) = a \cdot \left[ \int_0^{k/a} x \cdot f(x)dx + (k/a) \cdot \int_{k/a}^\infty f(x)dx \right] = a \cdot E_{k/a}(x) . \tag{7.5}$$

The excess pure premium rate in the mixed case is given by:

$$RateXS(a, M, L) = \frac{a \cdot E(N) \cdot \left[ E_{M+L}(x, a) - E_M(x, a) \right]}{a \cdot (\text{Subject Premium})}$$

$$= \frac{E(N) \cdot \left[ E_{M+L}(x, a) - E_M(x, a) \right]}{\text{Subject Premium}} . \tag{7.6}$$

If $R$ is a relativity describing the relationship between $RateXS(a, M, L)$ and $RateXS(1, M, L)$, then:

$$R = \frac{RateXS(a, M, L)}{RateXS(1, M, L)} = \frac{\mathrm{E}_{M+L}(x, a) - \mathrm{E}_M(x, a)}{\mathrm{E}_{M+L}(x) - \mathrm{E}_M(x)} . \tag{7.7}$$

If $R < 1$, the ceding insurer's expected loss recoveries from the excess reinsurer will be lower in the mixed reinsurance case than in the pure excess case—the cost of mixing is positive. On the other hand, if $R > 1$, the insurer will recover more in the mixed case and the cost of mixing is negative.

Let $m$ and $m_a$ denote the slope of the limited mean function between 1) $\mathrm{E}_M(x)$ and $\mathrm{E}_{M+L}(x)$, and 2) $\mathrm{E}_M(x, a)$ and $\mathrm{E}_{M+L}(x, a)$, respectively. Then formula 7.7 can be restated as:

$$R = \frac{m_a \cdot L/a}{m \cdot L} = \frac{m_a}{a \cdot m} . \tag{7.8}$$

From formula 7.8, we can see that whether the cost of mixing is positive ($R < 1$) or negative ($R > 1$) depends on the shape of the limited mean function between the points that define the excess reinsurance coverage. If the slope on the portion of the curve that defines mixed coverage is less than $100 \cdot a\%$ of the slope in the pure excess case, then $R < 1$ and the cost of mixing is positive. But if the mixed coverage slope exceeds $100 \cdot a\%$ of the pure excess slope, then $R > 1$ and mixing has a negative cost.

A result proved by Miccolis [2] can be used to define a simple test of whether mixing has a positive or negative cost to the ceding insurer. Though he used a different notation invented for his discussion of increased limits factors, Miccolis showed that

$$d\,[\, E_k(x)/C \,]/dk = [1 - f\!\#(k)]/C , \tag{7.9}$$

where $C$ is a constant and $f\!\#(k)$ is the c.d.f. of policy losses by size (Wiser's notation). Since $1 - f\!\#(k) = \mathrm{Prob}\,(x > k)$, this tells us that the slope of the limited mean function at $k$ is the same as the probability, given a claim, that the claim exceeds $k$.

So for the infinitesimal layer $\Delta x$ excess of $k$, the following specifies a precise test for the cost to the ceding insurer of mixing per se:

$$\begin{array}{lll} \text{Prob}\,(x > k/a) < \text{Prob}\,(x > k) \cdot a & <==> & \text{Positive Cost} \\ \text{Prob}\,(x > k/a) = \text{Prob}\,(x > k) \cdot a & <==> & \text{No Cost} \\ \text{Prob}\,(x > k/a) > \text{Prob}\,(x > k) \cdot a & <==> & \text{Negative Cost} \end{array} \tag{7.10}$$

For excess layers of practical importance, Formula 7.10 is not a precise test, but it suggests a way of screening. There are numerous such screens of varying complexity that could be employed, but here is a simple one that is easy to apply: test the layer endpoints, denoted by $k_1$ and $k_2$.

$$\begin{array}{lll} \text{Prob}\,(x > k_1/a) < \text{Prob}\,(x > k_1) \cdot a & & \\ \text{and} & <==> & \text{Positive Cost} \\ \text{Prob}\,(x > k_2/a) < \text{Prob}\,(x > k_2) \cdot a & & \\ & & \\ \text{Prob}\,(x > k_1/a) > \text{Prob}\,(x > k_1) \cdot a & & \\ \text{and} & <==> & \text{Negative Cost} \\ \text{Prob}\,(x > k_2/a) > \text{Prob}\,(x > k_2) \cdot a & & \\ & & \\ \text{Other Combinations} & <==> & \text{Test Inconclusive} \end{array} \tag{7.11}$$

Refinements to this test would be possible, but at the cost of introducing additional complexity.

## 8. APPLYING THE TEST

To confirm that formula 7.11 correctly identifies the mixing cost characteristics of the examples discussed earlier, let us apply the test to those cases.

The first example involved a $1 million casualty policy that attached in excess of a $100,000 self-insured retention (SIR). Recall that despite

\$750,000 of loss exposure to the excess reinsurer before proportional reinsurance, after the 50% pro rata cession, the excess limit exposure is only \$250,000. The cost of mixing calculation was done in respect of this \$250,000 that is present in both the pure excess and mixed cases. Exhibit 1 shows that the excess reinsurer's exposure is higher in the pure excess case, demonstrating a positive cost of mixing.

Exhibit 6 tabulates various information about the loss distribution "from the ground up" (FGU), i.e., including the SIR, so in the pure excess case, the reinsurance retention of \$250,000 equates to \$350,000 on the FGU loss table. From Exhibit 6, Prob ($x >$ \$350,000) = 1.18%. Fifty percent of that (reflecting the pro rata retention $a = 50\%$) is 0.59%. This compares to a mixed case retention in FGU terms of \$600,000 (\$100,000 + \$250,000/.50). Since Prob ($x >$ \$600,000) = 0.52% < 0.59%, this is an indication that the cost of mixing may be positive.

Now test the upper end of common excess coverage. In the pure case, this is \$600,000 FGU (\$100,000 + \$250,000 + \$250,000). Prob ($x >$ \$600,000) = 0.52%. Fifty percent of this is 0.26%. For the mixed case, the upper end of coverage on an FGU basis is \$1,100,000 (\$100,000 + \$250,000/.50 + \$250,000/.50). Prob ($x >$ \$1,100,000) = 0.188% < 0.26%, which confirms the indication of a positive cost of mixing.

The second example involved a \$20 million property policy with no SIR. Available excess reinsurance coverage is \$2 million excess of \$250,000, but after a 90% pro rata cession, only \$1.75 million is used. The cost of mixing calculation done on Exhibit 3 in respect of this \$1.75 million shows a positive cost of mixing.

Exhibit 7 tabulates information about the loss distribution for this example. There is no SIR in this case, so it will not be necessary to make any special adjustments. The excess retention is \$250,000 in the pure excess case; according to Exhibit 7, Prob ($x >$ \$250,000) = 4.613%. Ten percent of this (reflecting the pro rata retention, $a = 10\%$) is 0.461%. The effective mixed case retention in FGU terms is \$2,500,000 (\$250,000/.10). Prob ($x >$ \$2,500,000) = 0.293%, which is less than 0.461%. This indicates a possible positive cost of mixing.

Testing the upper end of common excess coverage, which in the pure excess case is \$2 million (\$250,000 + \$1.75 million), yields Prob ($x >$ \$2 million) = 0.401%. Ten percent of this is 0.04%. The effective upper bound of coverage in the mixed case is \$20 million FGU. Prob ($x >$ \$20 million) = 0.01% < 0.04%, which confirms a positive cost of mixing.

The third example involved a \$5 million excess casualty policy attaching excess of \$5 million. There was excess coverage of \$1.75 million excess of \$250,000 and a proportional cession of 60%. The cost of mixing calculation for this example is summarized on Exhibit 5. It shows a negative cost of mixing.

Exhibit 6 is the source of loss information. Since the attachment point of the policy is \$5 million, in the pure excess case the reinsurance retention of \$250,000 equates to \$5.25 million on the FGU loss table. From Exhibit 6, Prob ($x >$ \$5.25 million) = 0.008%. Reflecting the pro rata retention, 40% of this is 0.003%. This compares to a mixed case retention in FGU terms of \$5.625 million (\$5 million + \$250,000/0.40). Since Prob ($x >$ \$5.625 million) = 0.007% > 0.003%, this is an indication that the cost of mixing may be negative.

Now examine the upper end of excess coverage. In the pure excess case this is \$7 million FGU (\$5 million + \$250,000 + \$1.75 million). Prob ($x >$ \$7 million) = 0.004%. Forty percent of this is 0.0016%. This compares to an upper end of \$10 million FGU in the mixed case (\$5 million + \$250,000/0.40 + \$1.75 million/0.40). Confirming that the cost of mixing is negative, Prob ($x >$ \$10 million) = 0.002% > 0.0016%.

## 9. CONCLUSION

I hope this discussion will be seen as building on the foundation of Wiser's original paper. Its purpose has been to aid the reader in understanding more clearly the effect of mixing proportional and excess of loss reinsurance, and the importance of distinguishing between the cost of overlap and the cost of mixing per se.

## REFERENCES


[1] Wiser, Ronald F., "The Cost of Mixing Reinsurance," *PCAS* LXXIII, 1986, p. 168.

[2] Miccolis, Robert S., "On the Theory of Increased Limits and Excess of Loss Pricing," *PCAS* LXIV, 1977, p. 27.

# EXHIBIT 1

## ANALYSIS OF REINSURED LOSS EXPOSURE VS. INSURED LOSS EXPOSURE*
## CASUALTY EXAMPLE
## REINSURED LAYER: $250,000 EXCESS OF $250,000**
## CALCULATION OF THE COST OF MIXING PER SE

| | | Critical Limits of Insured and Reinsured Loss Layers and the Corresponding Limited Means | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Subject Policies | | | | Reinsured Layer | | | Relative Exposure | | | |
| | | A | B | C | D | E | F | G | H | I | J | K |
| Reinsurance Structure | | Under-lying Retention | Policy Limit | Policy + Underlying Retention | Subject Share *a* | Effective Retention | Effective Rein-surance Limit | Effective Reten-tion + Limit | Reinsured Exposure as % of Subject Loss | Subject Loss | Reinsured Layer Loss | Adjusted Rein-sured Layer Loss # |
| Pure XS | Limits | 100,000 | 1,000,000 | 1,100,000 | 100.00% | 350,000 | 250,000 | 600,000 | | | | |
| | Lim Means | 18,034 | 9,916 | 27,950 | | 24,432 | 1,954 | 26,386 | 19.71% | 240,000 | 47,293 | 23,647 |
| Mixed | Limits | 100,000 | 1,000,000 | 1,100,000 | 50.00% | 600,000 | 500,000 | 1,100,000 | | | | |
| | Lim Means | 18,034 | 9,916 | 27,950 | | 26,386 | 1,564 | 27,950 | 15.77% | 120,000 | 18,927 | 18,927 |
| | | | | | | | Pure - Mixed | | 3.93% | | | 4,720 |

* Loss model is lognormal: $\mu = 8.6799043$; $\sigma = 1.8050198$. Calculations use limited means to precision displayed. See Exhibit 6 for corresponding loss distribution table.

** Both pure excess and mixed reinsurance structures fully expose the excess reinsurer in this layer.

\# Based on subject premiums from mixed case.

B = C - A
F = G - E
H = F / B
J = I * H

# EXHIBIT 2

## ANALYSIS OF REINSURED LOSS EXPOSURE VS. INSURED LOSS EXPOSURE*
## CASUALTY EXAMPLE
## REINSURED LAYER: $1,750,000 EXCESS OF $500,000**
## CALCULATION OF THE COST OF OVERLAP

| | | Critical Limits of Insured and Reinsured Loss Layers and the Corresponding Limited Means | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Subject Policies | | | | Reinsured Layer | | | Relative Exposure | | | |
| | | A | B | C | D | E | F | G | H | I | J | K |
| Reinsurance Structure | | Under-lying Retention | Policy Limit | Policy + Underlying Retention | Subject Share *a* | Effective Retention | Effective Rein-surance Limit | Effective Retention + Limit | Reinsured Exposure as % of Subject Loss | Subject Loss | Reinsured Layer Loss | Adjusted Rein-sured Layer Loss # |
| Pure XS | Limits | 100,000 | 1,000,000 | 1,100,000 | 100.00% | 600,000 | 500,000 | 1,100,000 | | | | |
| | Lim Means | 18,034 | 9,916 | 27,950 | | 26,386 | 1,564 | 27,950 | 15.77% | 240,000 | 37,854 | 18,927 |
| Mixed | Limits | 100,000 | 1,000,000 | 1,100,000 | 50.00% | 1,100,000 | 0 | 1,100,000 | | | | |
| | Lim Means | 18,034 | 9,916 | 27,950 | | 27,950 | 0 | 27,950 | 0.00% | 120,000 | 0 | 0 |
| | | | | | | | | Pure - Mixed | 15.77% | | | 18,927 |

* Loss model is lognormal: $\mu = 8.6799043$; $\sigma = 1.8050198$. Calculations use limited means to precision displayed. See Exhibit 6 for corresponding loss distribution table.

** Only pure excess structure exposes the excess reinsurer in this layer.

# Based on subject premiums from mixed case.

B = C - A
F = G - E
H = F / B
J = I * H

# EXHIBIT 3

## ANALYSIS OF REINSURED LOSS EXPOSURE VS. INSURED LOSS EXPOSURE*
## PROPERTY EXAMPLE
## REINSURED LAYER: $1,750,000 EXCESS OF $250,000**
## CALCULATION OF THE COST OF MIXING PER SE

| | | Critical Limits of Insured and Reinsured Loss Layers and the Corresponding Limited Means | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Subject Policies | | | | Reinsured Layer | | | Relative Exposure | | | |
| | | A | B | C | D | E | F | G | H | I | J | K |
| Reinsurance Structure | | Underlying Retention | Policy Limit | Policy + Underlying Retention | Subject Share *a* | Effective Retention | Effective Reinsurance Limit | Effective Retention + Limit | Reinsured Exposure as % of Subject Loss | Subject Loss | Reinsured Layer Loss | Adjusted Reinsured Layer Loss # |
| Pure XS | Limits | 0 | 20,000,000 | 20,000,000 | 100.00% | 250,000 | 1,750,000 | 2,000,000 | | | | |
| | Lim Means | 0 | 65,577 | 65,577 | | 33,205 | 21,559 | 54,764 | 32.88% | 300,000 | 98,628 | 9,863 |
| Mixed | Limits | 0 | 20,000,000 | 20,000,000 | 10.00% | 2,500,000 | 17,500,000 | 20,000,000 | | | | |
| | Lim Means | 0 | 65,577 | 65,577 | | 56,475 | 9,102 | 65,577 | 13.88% | 30,000 | 4,164 | 4,164 |
| | | | | | | | | Pure - Mixed | 19.00% | | | 5,699 |

* Loss model is lognormal: $\mu = 8.8123226$; $\sigma = 2.1482831$. Calculations use limited means to precision displayed. See Exhibit 7 for corresponding loss distribution table.

** Both pure excess and mixed reinsurance structures fully expose the excess reinsurer in this layer.

\# Based on subject premiums from mixed case.

B = C - A
F = G - E
H = F / B
J = I * H

# EXHIBIT 4

## ANALYSIS OF REINSURED LOSS EXPOSURE VS. INSURED LOSS EXPOSURE*
## PROPERTY EXAMPLE
## REINSURED LAYER: $250,000 EXCESS OF $2,000,000**
## CALCULATION OF THE COST OF OVERLAP

| | | Critical Limits of Insured and Reinsured Loss Layers and the Corresponding Limited Means | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Subject Policies | | | | Reinsured Layer | | | Relative Exposure | | | |
| | | A | B | C | D | E | F | G | H | I | J | K |
| Reinsurance Structure | | Underlying Retention | Policy Limit | Policy + Underlying Retention | Subject Share *a* | Effective Retention | Effective Reinsurance Limit | Effective Retention + Limit | Reinsured Exposure as % of Subject Loss | Subject Loss | Reinsured Layer Loss | Adjusted Reinsured Layer Loss # |
| Pure XS | Limits | 0 | 20,000,000 | 20,000,000 | 100.00% | 2,000,000 | 250,000 | 2,250,000 | | | | |
| | Lim Means | 0 | 65,577 | 65,577 | | 54,764 | 922 | 55,686 | 1.41% | 300,000 | 4,218 | 422 |
| Mixed | Limits | 0 | 20,000,000 | 20,000,000 | 10.00% | 20,000,000 | 0 | 20,000,000 | | | | |
| | Lim Means | 0 | 65,577 | 65,577 | | 65,577 | 0 | 65,577 | 0.00% | 30,000 | 0 | 0 |
| | | | | | | | | Pure - Mixed | 1.41% | | | 422 |

* Loss model is lognormal: $\mu = 8.8123226$; $\sigma = 2.1482831$. Calculations use limited means to precision displayed. See Exhibit 7 for corresponding loss distribution table.

** Only the pure excess structure exposes the excess reinsurer in this layer.

# Based on subject premiums from mixed case.

B = C - A
F = G - E
H = F / B
J = I * H

# EXHIBIT 5

## ANALYSIS OF REINSURED LOSS EXPOSURE VS. INSURED LOSS EXPOSURE*
## REVISED CASUALTY EXAMPLE
## REINSURED LAYER: $1,750,000 EXCESS OF $250,000**
## CALCULATION OF THE COST OF MIXING PER SE

| | | Critical Limits of Insured and Reinsured Loss Layers and the Corresponding Limited Means | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Subject Policies | | | | Reinsured Layer | | | Relative Exposure | | | |
| | | A | B | C | D | E | F | G | H | I | J | K |
| Reinsurance Structure | | Underlying Retention | Policy Limit | Policy + Underlying Retention | Subject Share *a* | Effective Retention | Effective Reinsurance Limit | Effective Retention + Limit | Reinsured Exposure as % of Subject Loss | Subject Loss | Reinsured Layer Loss | Adjusted Reinsured Layer Loss # |
| Pure XS | Limits | 5,000,000 | 5,000,000 | 10,000,000 | 100.00% | 5,250,000 | 1,750,000 | 7,000,000 | | | | |
| | Lim Means | 29,665 | 215 | 29,880 | | 29,687 | 107 | 29,794 | 49.77% | 50,000 | 24,884 | 9,953 |
| Mixed | Limits | 5,000,000 | 5,000,000 | 10,000,000 | 40.00% | 5,625,000 | 4,375,000 | 10,000,000 | | | | |
| | Lim Means | 29,665 | 215 | 29,880 | | 29,718 | 162 | 29,880 | 75.35% | 20,000 | 15,070 | 15,070 |
| | | | | | | | | Pure - Mixed | -25.58% | | | (5,116) |

* Loss model is lognormal: $\mu = 8.6799043$; $\sigma = 1.8050198$. Calculations use limited means to precision displayed. See Exhibit 6 for corresponding loss distribution table.

** Both pure excess and mixed reinsurance structures fully expose the excess reinsurer in this layer.

\# Based on subject premiums from mixed case.

B = C - A    F = G - E

H = F / B    J = I * H

# EXHIBIT 6

## LOSS DISTRIBUTION TABLE—CASUALTY EXAMPLE

## LOG NORMAL MODEL*

| Limit $k$ | Limited Mean $LM(k)$ | Limit Factor $LF(k)$ | $f\#(k)$ | $f\$(k)$ | Prob $(x > k)$ |
|---|---|---|---|---|---|
| 100,000 | 18,034 | 0.60112 | 0.94174 | 0.40691 | 0.05826 |
| 350,000 | 24,432 | 0.81439 | 0.98820 | 0.67672 | 0.01180 |
| 600,000 | 26,386 | 0.87954 | 0.99480 | 0.77552 | 0.00520 |
| 1,000,000 | 27,745 | 0.92485 | 0.99778 | 0.85087 | 0.00222 |
| 1,100,000 | 27,950 | 0.93165 | 0.99812 | 0.86280 | 0.00188 |
| 1,250,000 | 28,201 | 0.94003 | 0.99851 | 0.87774 | 0.00149 |
| 1,500,000 | 28,517 | 0.95058 | 0.99893 | 0.89703 | 0.00107 |
| 2,000,000 | 28,926 | 0.96419 | 0.99938 | 0.92280 | 0.00062 |
| 2,100,000 | 28,985 | 0.96616 | 0.99944 | 0.92664 | 0.00056 |
| 2,350,000 | 29,111 | 0.97037 | 0.99955 | 0.93493 | 0.00045 |
| 3,000,000 | 29,341 | 0.97805 | 0.99972 | 0.95041 | 0.00028 |
| 4,600,000 | 29,624 | 0.98748 | 0.99989 | 0.97033 | 0.00011 |
| 5,000,000 | 29,665 | 0.98884 | 0.99991 | 0.97331 | 0.00009 |
| 5,250,000 | 29,687 | 0.98958 | 0.99992 | 0.97493 | 0.00008 |
| 5,625,000 | 29,718 | 0.99061 | 0.99993 | 0.97722 | 0.00007 |
| 7,000,000 | 29,794 | 0.99312 | 0.99996 | 0.98292 | 0.00004 |
| 10,000,000 | 29,880 | 0.99601 | 0.99998 | 0.98972 | 0.00002 |
| 100,000,000 | 29,998 | 0.99995 | 1.00000 | 0.99984 | 0.00000 |

* $\mu = 8.6799043$ $\sigma = 1.8050198$

# EXHIBIT 7

## Loss Distribution Table—Property Example

### Log Normal Model*

| Limit $k$ | Limited Mean $LM(k)$ | Limit Factor $LF(k)$ | $f\#(k)$ | $f\$(k)$ | Prob $(x > k)$ |
|---|---|---|---|---|---|
| 250,000 | 33,205 | 0.49193 | 0.95387 | 0.32108 | 0.04613 |
| 500,000 | 41,145 | 0.60956 | 0.97759 | 0.44353 | 0.02241 |
| 1,000,000 | 48,520 | 0.71881 | 0.99007 | 0.57168 | 0.00993 |
| 1,500,000 | 52,336 | 0.77534 | 0.99409 | 0.64408 | 0.00591 |
| 2,000,000 | 54,764 | 0.81132 | 0.99599 | 0.69262 | 0.00401 |
| 2,222,000 | 55,590 | 0.82356 | 0.99654 | 0.70963 | 0.00346 |
| 2,250,000 | 55,686 | 0.82498 | 0.99660 | 0.71162 | 0.00340 |
| 2,500,000 | 56,475 | 0.83667 | 0.99707 | 0.72813 | 0.00293 |
| 3,000,000 | 57,760 | 0.85570 | 0.99775 | 0.75554 | 0.00225 |
| 4,000,000 | 59,577 | 0.88262 | 0.99853 | 0.79558 | 0.00147 |
| 5,000,000 | 60,813 | 0.90093 | 0.99896 | 0.82377 | 0.00104 |
| 8,889,000 | 63,347 | 0.93848 | 0.99959 | 0.88447 | 0.00041 |
| 10,000,000 | 63,760 | 0.94459 | 0.99966 | 0.89480 | 0.00034 |
| 20,000,000 | 65,577 | 0.97151 | 0.99990 | 0.94239 | 0.00010 |
| 100,000,000 | 67,208 | 0.99567 | 1.00000 | 0.98995 | 0.00000 |

*$\mu = 8.8123226 \quad \sigma = 2.1482831$

# ADDRESS TO NEW MEMBERS—NOVEMBER 16, 1992

PAUL S. LISCORD

The theme of this brief address comes from one of your "confreres" who will be writing his last exam this coming Spring. When I asked what he would like to hear upon receiving his Fellowship diploma, he spoke with some apprehension of the ever-increasing expectations of the actuarial profession in the form of discipline committees, standards boards, etc., and, perhaps more importantly, the apparent increased probability of malpractice litigation against actuaries by the publics we serve.

In order to discuss this subject, I think it useful to briefly review how we got where we are as the Casualty Actuarial Society. Then, hopefully, we'll try to draw some observations from my experiences as an expert witness in a number of litigations involving actuaries.

I'm sure you won't be surprised that the CAS did not even have a code of professional conduct during its first 50 years of existence. Life was a lot simpler when you only had the Workers Compensation and Automobile lines to worry about, and the number of lawyers per capita was at a more reasonable level. It was my generation of actuaries who began composing and subsequently enacting guidelines. We established a code of professional conduct in the 1960s and a series of Statements of Principles for our more important actuarial functions in the 1970s and 1980s. Now, our successor generation is taking the next step by writing professional standards for a whole variety of actuarial functions, and by creating with our sister organizations the Actuarial Board for Counseling and Discipline (ABCD). In some ways, I don't blame any new member for wondering what he or she has gotten into.

As for my observations from serving as an expert witness, first and foremost is that allegations of actuarial malpractice usually stem from the plaintiff's disappointment that actuarial forecasts were not positively fulfilled. While the nature of our profession cannot always guarantee success, all too often, plaintiffs were simply not adequately apprised of what actuaries do, or better still, what they cannot do. Often, they were oversold on actuarial science. For actuaries in our competitive world, that's an easy trap to fall into.

Second, a lot of actuarial problems associated with malpractice cases involve what I call the accounting approach to actuarial science. This usually means that a particular methodology is either misapplied or is applied without adequate judgment. Often, the actuary merely applies a formula and accepts the answer because the formula is contained in some actuarial text, or even worse, because it's been programmed into the black box known as a computer. Further, one of the most important tenets contained in the various CAS Principles is that the actuary must *personally* investigate any changes in the insurance operation being studied so as to modify the data judgmentally as required. It's these subjective steps that are most often overlooked and the lack of which usually causes most poor forecasts.

Third, in my opinion, such personal investigation also requires a hands-on knowledge of the business most of us serve, the insurance business. The kind of knowledge I'm talking about is not usually found in actuarial textbooks. For example, most underwriters are optimists; they have to be or else they could not accept risk. However, this attribute is often exacerbated by management's insistence on growth, which inevitably reduces underwriting selection to the point of lunacy. Actuaries then find themselves in a catch-up posture as to both rates and reserves. Depending upon the availability of recent data to uncover changing trends, such circumstances can hamper their ability to "prove" their recommendations either to management, or to underwriting, or to both.

Another example of this kind of knowledge would be in the claims operation. Here, the interplay between the adjusters and management is crucial to the understanding of claims data, particularly claim development. Often management attempts to intervene, usually by strengthening case reserves, which unfortunately distorts, if not completely destroys, historical claim development patterns. At the same time, this strengthening usually results in the company settling for higher claim amounts. Incidentally, have you ever noticed how adjusters inevitably raise their case estimates just prior to settlement to show management that they have settled for a lower amount than estimated?

One last observation: actuarial haste makes for expensive waste in terms of malpractice settlements. Inevitably, most of the actuarial mistakes I have observed could have been avoided by the actuary's taking the time to step back to observe the reasonableness of his or her forecast before signing off. The old adage still rings true.

Congratulations on achieving the actuarial successes being recognized today. I'm confident that as you expand your actuarial horizons you will more fully appreciate the guidelines that have been, and are being, established for the benefit of all actuaries as well as their employers. You certainly have nothing to fear from these constraints. In fact, by observing them to the fullest you will be very well protected against any allegations of wrongdoing that might be directed against you.

# PRESIDENTIAL ADDRESS—NOVEMBER 16, 1992

## AN INVESTMENT IN OUR FUTURE

MICHAEL L. TOOTHMAN

One of the highlights of my year as President, on a personal basis, was getting Ruth Salzmann back to a CAS meeting, this last May in Chicago, and beyond that having Ruth back at the CAS podium again to provide the address to new members. In her 1979 Presidential Address, Ruth referred to the presidential address as the "last rite" in the term of office of the president.

Then last year, in his address to new members, Dan McNamara indicated that CAS past presidents essentially serve the same function as the deceased at an Irish wake—where all the relatives and friends are convinced that their beloved has gone on to a better life and are in a mood to celebrate the event. They need a body to legitimize the party, but nobody expects the body to say very much. Such morbid thoughts arise on the occasion when our presidents attempt to provide some last pearls of wisdom as they slip off into actuarial obscurity.

Fortunately, we at the CAS no longer allow our past presidents to become total "has beens." We now recycle them in several different ways. One way is the tradition of having a past president address the new Associates and Fellows during the business meeting, but we also now utilize our past presidents on committees, task forces, and on the Board of Directors. This past year, we have been fortunate to have two past presidents serve on the Board of Directors, Ron Bornhuetter and Jim MacGinnitie, and their counsel, wisdom, and experience has been of great benefit to the Board. Another past president, Phil Ben-Zvi, has just been elected to a three-year Board term.

Over a year ago, I began to get inquiries from several of our past presidents as to whether I had begun my Presidential Address. Jerry Scheibl was particularly persistent in reminding me of this duty, and I had to continue to reply that I had not yet begun it. Actually, I did have an outline of my Presidential Address done well over a year ago, but I have now decided not to use it. The title was "Winning the Rat Race," an appropriate description of what goes on during a president's term. The outline consisted of five points:

1) Worship God in whatever way you know Him—or Her.
2) Nurture and enjoy the personal relationships.
3) Know and understand yourself as a professional and the profession that you are a part of, and commit some time to improving it.
4) Understand the dynamics of the business you are in.
5) Dream BIG dreams.

As I said, I have decided not to use that address. Perhaps I will use it some time in the future. On the other hand, if any of you like the ideas in that outline, feel free to use it yourself. Still, I would like to spend just a minute on the second point: the personal relationships that one establishes within the CAS over the years.

It is easy for any one of us to get wrapped up in our own individual rat races and not really appreciate all of the opportunities for personal relationships and friendships that present themselves as part of our day-to-day activities. The CAS is filled with wonderful people. What remains, years after the Committee work is completed, are the personal relationships that were developed in the performance of that work. Even after business relationships are terminated, the bonds that we share as friends and as members of the actuarial profession remain. So I hope each of you will take advantage of the opportunities you have to serve the CAS and, more importantly, take advantage of the opportunities you have to develop those friendships and bonds that will prove to be of lasting value. But, as I said, that is another speech.

Today, I would like to talk about an investment in our future: an investment in the future of the CAS and in the futures of all of us collectively as casualty actuaries.

Investments are not always made knowing just where they will lead. By far, the majority of incoming college freshmen either have not yet decided upon their major field of study or will change that selection on at least one occasion before graduation. I was in the minority on that score. I entered a five-year dual degree program in Applied Mathematics and completed that program, yet my career direction changed drastically over the course of those five years. Many individuals, once they have completed their degree programs, do not work in the field covered by their degree, yet that education is not wasted. Those years of study represent an investment in one's future, the acquisition of specific knowledge and the development of an ability to think and analyze critically. Those investments almost always prove to be worthwhile, even though we do not know precisely where they will lead at the time that we make them.

The actuarial exams, though much more specific, can be described in much the same way. Our education as actuaries is not completed at the time that we complete our actuarial exams. Indeed, it has just begun. Much of the knowledge that I use in my professional career these days was not even known at the time that I was taking actuarial exams. Environmental liabilities were not an issue back in the early 1970s. Captive insurance companies existed, but were such a small factor in the marketplace that they were hardly mentioned anywhere in the *Syllabus*. Financial reinsurance is a whole new area since then, and I do not believe that the term commutation was used anywhere in the readings on the *Syllabus* that I had. The list can go on and on.

Similarly with on-the-job training. Successful careers usually are the accumulation of one successful task after another or the successful completion of one set of responsibilities after another. It is the rare individual that can lay out his future career progression while still in his 20s—perhaps in very broad strokes—but the point is that it is much more common to really not know what the future may hold. But if you do your current job well and learn as much about your business as possible, good opportunities will become available to you.

So it is with the CAS as an organization and with our profession as a whole.

Today, I would like to focus on investments that we can make to expand our vision in three areas.

First, I have the vision of the CAS in the future becoming a resource for developing casualty actuaries throughout the world.

Second, I have the vision of us as casualty actuaries moving from the realm of assessing risks associated with the insurance transaction to the broader area of risk financing. That is, quantifying the financial impact of future contingent events wherever they occur, not just within the insurance transaction. The most obvious area is the whole field of self-insurance, which, in its myriad forms, annually makes up a larger and larger percentage of the commercial insurance market.

Third, we must move from being experts in particular liabilities to being experts on the total balance sheet.

I believe it is time for us to make investments as an organization and as professionals that will move us in these directions, yet I cannot tell you what all the implications of those investments will be. I do firmly believe, however, that these investments will prove to be beneficial to the CAS and to the actuarial profession.

In his 1976 Presidential Address, Ron Bornhuetter said that the CAS had been concentrating all its efforts in the continental United States and to this degree had been selfish. He said that there was an acute need, a search for knowledge in the property and casualty business worldwide, and that we were the owners of the biggest bank. Ron's address, entitled "Challenges," included the challenge of expanding our horizons beyond the United States. We have accomplished much since 1976, but our horizon as an organization has still been limited largely to North America. It is time for us to become active participants in the worldwide actuarial community.

It made me feel very good when I was described once as a "world class" actuary. Whether that description was accurate or not, it is my belief that the CAS has many world class actuaries, but we need a global vision. Over the last few years, I believe we have begun to develop that vision, and we have made some of the initial investments that we need to make, but the first step is one of perspective and attitude. Individually and collectively, we must begin to see ourselves as part of the worldwide actuarial community, and we must begin to view insurance as only one form of risk financing. At the same time, we must take a more holistic view of our role as financial experts, particularly with regard to insurance companies, as we expand our role from being experts on loss and loss adjustment expense liabilities to being experts on the total financial condition of insurance companies.

What, then, are some of these investments?

1. The Academic Base

First, we need a broader academic base. In this regard, it could be argued that the profession in Europe is further advanced than it is here in North America. We refer to our field as actuarial science, yet we spend relatively little energy in pushing the frontiers of that science outwards, and we can hardly claim that the science is fully developed.

I am not one to believe that the best actuaries necessarily come from actuarial science programs. Indeed, I believe that there is much more to being a good actuary than being technically competent, but we must, as a profession, maintain and improve upon our technical competence. Other skills need to be developed in addition to that but not at the expense of it. I am convinced that an investment in developing a stronger academic base will strengthen our profession.

2. The Knowledge Base

The idea of improving and expanding upon our collective knowledge base goes hand in hand with strengthening the academic base. Our relatively new philosophy of managed research within the CAS is now beginning to pay dividends, but even this improved flow of productive research does not begin to fulfill the future potential for our profession. A strong academic base will improve the amount of productive research by orders of magnitude. I don't believe we will ever want to limit our research efforts to academia. But, to believe that we can fulfill the research needs of our profession without a strong contribution from academia is naive and faulty reasoning in my view. The global vision is important here, in many ways. We have implemented a literature exchange with many foreign actuarial societies, and I hope our membership will start to take advantage of the easier access to this literature. Other fields are making advances that are either directly or indirectly applicable to risk financing, and we need to expand our collective knowledge base to encompass those developments. No one can foretell where this investment will lead us, but I am convinced that investments in expanding our collective knowledge base can't help but benefit our profession.

3. Basic Education

I believe it is now time for us to begin the process of making a major investment in the design of our basic education system and in its implementation. The implementation issue involves accelerating the long-term gradual trend in our exams from an emphasis on specific, detailed knowledge to an understanding of principles and development of the individual's ability to apply those principles. The structural investment we must make is to restructure our *Syllabus* so as to separate the actuarial principles that apply to casualty insurance anywhere in the world from the knowledge that is important for someone if they are to practice in the United States or in Canada. Once such a syllabus is achieved, conceptually it would be possible for us to qualify actuaries who wish to practice in the United States and who are fully qualified abroad by having them pass simply the exams that focus on the nation-specific material. I believe it is appropriate for the CAS to make such an investment. Once we have done so, we will be in a better position to participate in the development of casualty actuaries throughout the world.

In all of this, it is not my suggestion that the CAS try to impose itself upon existing actuarial organizations abroad. Rather, I believe that the CAS can serve as an important resource that will add value to a partnership involving the CAS and other existing actuarial organizations around the world. Our members will benefit from such an expansion of our vision, and I believe we bring enough value to the table that the members of the actuarial profession abroad will benefit as well. But, in all our endeavors, we must act as true and equal partners and treat all of our potential partners as we would wish to be treated.

4. Continuing Education

Our continuing education program has been successful by almost any standard, yet we must make a significant short-term investment here as we move to prepare our members to function as appointed actuaries opining on an insurer's current and future solvency prospects. In the longer term, as we adopt a more global perspective, our continuing education program will have to expand to provide many more opportunities than are currently being provided, in order to allow each individual to find those continuing education opportunities that will best assist that individual in the development of his or her career. This investment will require an expansion of our vision and will require us to leverage our resources much more than we do today. It will require us to use resources from outside the actuarial community, where appropriate, but it is an investment that will benefit our profession.

5. Standards of Practice and Discipline Procedures

The final investment is one that we have already started to make—an investment in the development of standards of practice and in the establishment of a well-functioning, self-policing procedure. I will combine these two areas because they are so closely related and because we have made the initial investments in both areas, both in the United States and in Canada. These issues are also very high priority issues for the actuarial profession around the world. Indeed, virtually every actuarial organization is either actively involved or is highly concerned with how to become more involved in these two areas. The initial investments in North America have been made with the establishment of the Actuarial Standards Board and the development of Standards of Practice within Canada, and with the establishment of the ABCD and development of stronger disciplinary procedures within the CIA. Yet these investments are just the beginning. Ongoing investments will be required as further standards of practice are developed and as we, as a profession, now face up to the questions of whether we are really prepared to police ourselves. As in all of the other investments, we do not know just where these investments will lead, but I am convinced that investments in these areas are necessary to the further development of our profession.

What will these investments mean to us organizationally? I do not know. I believe that in public interface issues it is important for the actuarial profession within each country to be able to speak with one voice. It was for this reason that we established the American Academy of Actuaries within the United States. That organization is important to our collective future. However, the Casualty Actuarial Society and the Society of Actuaries are basically research and educational organizations. The research and educational functions are, for the most part, not nation-specific, and I strongly believe that it is therefore beneficial for the research and educational functions to be organizationally separate from the public interface function.

Some leaders of our profession decry the separation of the casualty branch and have suggested that actuaries ought to be qualified to practice in all practice areas. I believe that that philosophy is misguided. I am persuaded that the casualty branch of the profession is stronger in the United States than it is anywhere else in the world. As Ron Bornhuetter said, we have the biggest bank of property/casualty actuarial knowledge. I believe that it is highly unlikely that our bank would be anywhere near the size that it is if it were not for the founding of the Casualty Actuarial and Statistical Society of America, now the Casualty Actuarial Society, in November 1914.

I believe it *is* appropriate for all actuaries to have some basic understanding of the underlying concepts and principles in the other branches of the actuarial profession—what I would call "conversational competence"—but I believe that it is far more important for us to maintain the depth of knowledge that we provide to members of the Casualty Actuarial Society than it is to dilute that effort over several branches of the actuarial profession. For this reason, we will continue to resist any attempts to undermine a separate casualty organization. Rather our focus needs to be on cooperation, not consolidation, with the other North American actuarial organizations wherever that makes sense. Our focus must also be on creating new partnerships with actuarial organizations abroad, as we expand our view as casualty actuaries and assist in the development of the actuarial profession on a global basis.

Investments in our future—Big dreams!

This is a wonderful profession. I hope that each of you will involve yourselves in it deeply. Debate the vision and the investments. Work to shape them, and then make them happen.

In closing, I would like to share a poem entitled, "Take Time." It clearly is written with the intention that it be applied to your individual lives, but I would ask that you listen to it and think of how you might apply it in the context of the CAS—

TAKE TIME

Take time to WORK:
it is the Price of Success.
Take time to THINK:
it is the Source of Power.
Take time to PLAY:
it is the Secret of Perpetual Youth.
Take time to READ:
it is the Fountain of Wisdom.
Take time to WORSHIP:
it is the Highway to Reverence.
Take time to be FRIENDLY:
it is the Road to Happiness.
Take time to LAUGH:
it is the Music of the Soul.
Take time to DREAM:
it is Hitching Your Wagon to a Star.
TAKE TIME TO LIVE.

In two days, it will be time for me to take some time. Thank you for the honor of serving as your President.

# MINUTES OF THE 1992 ANNUAL MEETING

November 15-18, 1992

THE BOCA RATON RESORT & CLUB
BOCA RATON, FLORIDA

*Sunday, November 15, 1992*

The Board of Directors held their regular quarterly meeting from noon until 5:00 p.m.

Registration was held from 4:00 p.m. until 6:30 p.m.

From 5:30 p.m. until 6:30 p.m., there was a special presentation to new Associates and their guests. The session included an introduction to the standards of professional conduct and the CAS committee structure.

A welcome reception for all members and guests was held from 6:30 p.m. until 7:30 p.m.

*Monday, November 16, 1992*

Registration continued from 7:00 a.m. until 8:00 a.m.

CAS President Michael Toothman opened the meeting at 8:00 a.m. with the results from the recent election of officers. The members of the 1993 Executive Council will be Vice President-Administration, John M. Purple; Vice President-Admissions, Steven G. Lehmann; Vice President-Continuing Education, David N. Hafling; Vice President-Programs and Communications, Alice H. Gannon; Vice President-Research and Development, Allan M. Kaufman. President-Elect will be Irene K. Bass. New Board members will be Albert J. Beer, Phillip N. Ben-Zvi, Michael J. Miller, and Susan T. Szkoda.

Toothman thanked the outgoing Board members and welcomed esteemed guests in the audience, including: Tim Tinsely, executive director of the CAS; Takehisa Kikuchi, vice president, Institute of Actuaries, Japan; James Murphy, executive vice president, American Academy of Actuaries; Donald Sondergeld, past president of the Society of Actuaries; Pablo Noriega, president of the Instituto Nacional de Estadistica (Mexico); and Oliva Sanchez, director of the Escuela de Actuaria, Universidad Anahuac (Mexico).

John Purple read the Secretary/Treasurer's Report.

Steven Lehmann and Allan Kaufman introduced the 52 new Associates and David Flynn introduced the 62 new Fellows at the meeting. The names of these individuals follow.

FELLOWS

Rebecca C. Amoroso
Anthony J. Balchunas
Douglass L. Beck
Nathalie Bégin
Martin Cauchon
Denis Cloutier
Jeffrey R. Cole
Charles Cossette
Robert J. Curry
Francois Dagneau
Patrick K. Devlin
John W. Ellingrod
James Ely
David A. Foley
France Fortin
John F. Gibson
Susan M. Gozzo
Eric L. Greenhill
Diane K. Hausserman
Todd J. Hess
James S. Higgins
Keith D. Holler
George A. Hroziencik
Kathleen M. Ireland
Peter H. James
Brian J. Kincaid
Richard O. Kirste
Ronald T. Kozlowski
David J. Kretsch
John A. Lamb
Jean-Marc Léveillé
Blaine C. Marles
Burton F. Marlowe
Steven E. Math
Liam M. McFarlane
William T. Mech
Charles B. Mitzel
Richard B. Moncher
Todd B. Munson
Margaret M. O'Brien
Jacqueline E. Pasley
Susan J. Patschak
Robert Potvin
Richard W. Prescott
Kenneth P. Quintilian
Kay K. Rahardjo
Srinivasa Ramanujam
Scott E. Reddig
Sharon K. Robinson
Diane R. Rohn
Edmund S. Scanlon
Margaret E. Seiter
Marie Sellitti
Vincent M. Senia
Christy L. Simon
Christopher M. Smerald
Douglas N. Strommen
Michael A. Visintainer
Sebastian Vu
Christopher P. Walker
Marjorie C. Weinstein
Gregory S. Wilson

ASSOCIATES

| | | |
|---|---|---|
| Daniel N. Abellera | Jeffrey R. Hughes | Jean Roy |
| Nancy L. Arico | Daniel R. Keddie | Maureen S. Ruth |
| George P. Bradley | Ann L. Kiefer | Melodee J. Saunders |
| Donna D. Brasley | Stephen E. Lehecka | Jeffery J. Scott |
| Ward M. Brooks | Stephanie J. Lippl | Gregory R. Scruton |
| Linda J. Burrill | James M. Maher | James J. Smaga |
| Anthony E. Cappelletti | Galina Margulis | Scott J. Swanay |
| Mary L. Corbett | Heather L. McIntosh | Charles F. Toney, II |
| Jeffrey L. Dollinger | Van A. McNeal | Michael J. Toth |
| Jeffrey D. Donaldson | Raymond D. Muller | James F. Tygh |
| Paul E. Ericksen | Mary Beth O'Keefe | David B. Van Koevering |
| Lynne W. Faucher | Douglas W. Oliver | Phillip C. Vigliaturo |
| Judith M. Feldmeier | Richard A. Olsen | William E. Vogan |
| Russell Frank | William Oostendorp | Stephen D. Warfel |
| Laurence B. Goldstein | Todd F. Orrett | Joyce A. Weisbecker |
| Sandra K. Halpin | Joseph M. Palmer | Ralph T. Zimmer |
| Bradley A. Hanson | Jennifer J. Palo | |
| Renee Helou | Daniel A. Powell | |

Paul S. Liscord was introduced and gave the Address to New Members.

Mr. Toothman presented the 1992 Matthew Rodermund Service Award to Norman J. Bennett. As part of the presentation ceremony, Toothman read a letter from Matt Rodermund.

A moment of silence was held to mark the passing of six members of the CAS during the past year.

Mr. Beer gave the highlights of the program, and Chairman of the Committee on Review of Papers Richard Biondi summarized the six *Proceedings* papers being presented. The Woodward-Fondiller Prize was awarded to William R. Gillam and the Dorweiler Prize was awarded to John P. Robertson. Mr. Toothman made a call for reviews of previously presented *Proceedings* papers and received no responses.

A General Session panel discussion was held from 10:00 a.m. until 10:45 a.m. on "The Role of the American Academy of Actuaries." The session was led by David G. Hartman, with Ronald L. Bornhuetter and Michael A. Walters as panelists.

The Featured Speakers, Fred Barnes and Morton Kondracke, senior editors of *The New Republic* and frequent panelists on "The McLaughlin Group," spoke from 10:45 a.m. until noon.

Their presentation was followed by a luncheon with the Presidential Address by Michael Toothman. Lunch was from noon until 1:30 p.m.

The afternoon was devoted to concurrent sessions which consisted of various panels and papers.

The panel presentations covered the following topics:

1. Finite Risk Reinsurance—Three Perspectives

| | |
|---|---|
| Moderator: | Heidi E. Hutter<br>Executive Vice President<br>Atrium Corporation |
| Panelists: | Gregory E. Leonard<br>President and Chief Executive Officer<br>Pegasus Advisors, Inc. |
| | Richard J. Roth, Jr.<br>Assistant Commissioner<br>California Department of Insurance |
| | David Holman<br>Partner<br>Ernst & Young |

2. The Alphabet Soup of Professionalism: AAA, ABCD, ASB, and CIA

| | |
|---|---|
| Moderator: | Jerome A. Scheibl<br>(Chairman CAS Discipline Committee and Member Actuarial Board for Counseling and Discipline)<br>Vice President-Industry Affairs<br>Wausau Insurance Companies |

| | |
|---|---|
| Panelists: | W. James MacGinnitie<br>(Chairman American Academy of Actuaries<br>Committee on Professional Responsibility)<br>Consulting Actuary<br>Tillinghast/Towers Perrin |
| | Michael J. Miller<br>(Chairman Actuarial Standards Board<br>Casualty Committee)<br>Consulting Actuary<br>Tillinghast/Towers Perrin |

3. Loss Distributions and the Collective Risk Model

| | |
|---|---|
| Speaker: | Stuart A. Klugman, F.S.A., Ph.D.<br>Principal Financial Group<br>Professor of Actuarial Science<br>College of Business and Public Administration<br>Drake University |

4. What an Investment Department Needs to Know from an Actuary

| | |
|---|---|
| Moderator: | Sheldon Rosenberg<br>Vice President and Chief Actuary<br>Continental Insurance |
| Panelists: | Walter Blasberg<br>President<br>Continental Asset Management |
| | Sar Kun Moy<br>Vice President<br>Continental Asset Management |

5. New ISO Risk Load in ILFs—What Does it Mean?

| | |
|---|---|
| Moderator: | Gary S. Patrik<br>Senior Vice President and Actuary<br>North American Reinsurance Corporation |

| | |
|---|---|
| Panelists: | Glenn G. Meyers<br>Assistant Vice President and Actuary<br>Insurance Services Office, Inc. |
| | Ira Robbin, Ph.D.<br>Assistant Vice President<br>Director of Actuarial Research<br>CIGNA Property & Casualty Companies |
| | Edward W. Weissner<br>Vice President, Actuarial<br>Prudential Reinsurance Company |

6. The Oakland Fires—What Have We Learned?

| | |
|---|---|
| Moderator: | Karen F. Terry<br>Actuary<br>State Farm Fire and Casualty Company |
| Panelists: | Joseph Meyer<br>Senior Vice President<br>United Services Automobile Association<br>Western Region |
| | Tom Morrison<br>Property Lines Director-Claims<br>Allstate Insurance Companies |

7. Risk Classification Principles

| | |
|---|---|
| Panelists: | Cecily A. Gallagher<br>(Chairperson-Committee on Risk Classification)<br>Consulting Actuary<br>Tillinghast/Towers Perrin |
| | Frederick F. Cripe<br>Assistant Vice President<br>Allstate Insurance Company |
| | Kevin J. Conley<br>Actuarial Administrator<br>Iowa Insurance Division |

8. Questions and Answers with the CAS Board of Directors

| | |
|---|---|
| Moderator: | David P. Flynn (President-Elect)<br>Senior Vice President and Chief Actuary<br>Crum & Forster Corporation |
| Current Board Members: | Janet L. Fagan (Elected 1989)<br>Director of Casualty Actuarial Services<br>Coopers & Lybrand |
| | Gary S. Patrik (Elected 1991)<br>Senior Vice President and Actuary<br>North American Reinsurance Corporation |

The new *Proceedings* papers presented were:

1. "Workers' Compensation Experience Rating: What Every Actuary Should Know"

   Author: William R. Gillam<br>National Council on Compensation Insurance

2. "Stochastic Claims Reserving When Past Claim Numbers Are Known"

   Author: Thomas Wright<br>Consultant

3. "Pricing for Credit Exposure"

   Author: Brian Z. Brown<br>Milliman & Robertson, Inc.

The officers held a reception for new Fellows and their guests from 5:00 p.m. until 6:00 p.m. There was a general reception for all members from 6:00 p.m. until 7:00 p.m. The musical comedy, "The Sting," was presented from 7:00 p.m. until 8:30 p.m.

*Tuesday, November 17, 1992*

A General Session on a "Mock Trial on Actuarial Professional Standards" was held from 8:30 a.m. to 10:00 a.m. David S. Powell, Consulting Actuary with Tillinghast/Towers Perrin, and Ralph B. Levy, Esq., of King & Spalding, were the presenters.

From 10:00 a.m. until noon, several concurrent sessions were held. The panel presentations, in addition to some of the subjects covered on Monday, covered the topics of:

1. Malpractice Avoidance for Actuaries

| | |
|---|---|
| Moderator: | David S. Powell<br>Consulting Actuary<br>Tillinghast/Towers Perrin |
| Panelists: | Ralph B. Levy<br>King & Spalding |
| | Jack M. Turnquist<br>(Chairman, Actuarial Standards Board)<br>Totidem Verbis |
| | Lauren M. Bloom<br>General Counsel<br>American Academy of Actuaries |

2. CAS-Sponsored Research

| | |
|---|---|
| Moderator: | Allan M. Kaufman<br>(Vice President-Research and Development)<br>Principal<br>Milliman & Robertson, Inc. |
| Panelists: | LeRoy A. Boison, Jr.<br>(Liaison to SOA Research Policy Committee)<br>Vice President<br>Insurance Services Office, Inc. |
| | Gregory L. Hayward<br>(Chairperson, Ratemaking Committee)<br>Actuary<br>State Farm Mutual Automobile Insurance Co. |
| | James W. Yow<br>(Chairperson<br>Valuation & Financial Analysis Committee)<br>Vice President and Corporate Actuary<br>Aetna Life & Casualty |

3. Catastrophe Insurance Futures

Moderator: Joseph A. Herbers
Consulting Actuary
Tillinghast/Towers Perrin

Panelists: Gregory Samorajski
Group Manager-Financial Instruments
Economic Analysis and Planning
Chicago Board of Trade

Stephen P. D'Arcy
Department of Finance, University of Illinois

Jonathan Lewis
Manager, Fixed Income Portfolio
Skandia Investment Managers

4. Considerations Regarding Catastrophic Medical Claims and Automobile Insurance

Panelists: William J. VonSeggern
Chief Actuary
AAA Michigan

Roger M. Hayne
Consulting Actuary
Milliman & Robertson, Inc.

5. Experience Rating Overview

Moderator: Mark W. Mulvaney
Consulting Actuary
Milliman & Robertson, Inc.

Panelists: William R. Gillam
Assistant Vice President and Actuary
National Council on Compensation Insurance

David M. Bellusci
Senior Vice President and Actuary
Workers' Compensation Insurance Rating
Bureau of California

One new *Proceedings* paper presented was:

1. A Review of "The Competitive Market Equilibrium Risk Load Formula for Increased Limits Ratemaking" (Glenn Meyers, *PCAS* LXXVIII)

| | |
|---|---|
| Author: | Ira Robbin, Ph.D.<br>Assistant Vice President<br>Director of Actuarial Research<br>CIGNA Property & Casualty Companies |

The afternoon was reserved for committee meetings.

A clambake was held from 6:30 p.m. until 10:30 p.m.

*Wednesday, November 18, 1992*

From 8:00 a.m. to 9:30 a.m. several concurrent sessions were held and two *Proceedings* papers were presented. They were:

1. Credibility Based on Accuracy"

| | |
|---|---|
| Author: | Joseph A. Boor<br>Director-Actuarial<br>Motors Insurance Corporation |

2. A Review of "The Cost of Mixing Reinsurance" (Ronald F. Wiser, *PCAS* LXXIII)

| | |
|---|---|
| Author: | Michael Wacek<br>General Manager<br>St. Paul Fire & Marine Insurance Company |

Following a refreshment break, a General Session was held on the topic of "The NAIC's Examination of the NCCI."

| | |
|---|---|
| Moderator: | E. Frederick Fossa<br>Consulting Actuary<br>Milliman & Robertson, Inc. |
| Panelists: | Margaret C. Spencer<br>Arthur Andersen & Co. |
| | Patrick J. Grannan<br>Consulting Actuary<br>Milliman & Robertson, Inc. |

Ronald C. Retterath
Senior Vice President and Actuary
National Council on Compensation Insurance

James D. Watford
Actuary
Florida Insurance Department

After the transfer of the presidency, Michael Toothman gave the closing remarks. The meeting adjourned at 11:45 a.m.

*November 1992 Attendees*

In attendance, as indicated by the registration records, were 363 Fellows, 193 Associates, and 64 guests, subscribers, and candidates. The list of members' names follows.

FELLOWS

Abell, Ralph
Alfuth, Terry
Amoroso, Rebecca
Anker, Robert
Apfel, Kenneth
Aquino, John
Artes, Lawrence
Asch, Nolan
Atkinson, Richard
Balchunas, Anthony
Barnes, W. Brian
Bass, Irene
Bassman, Bruce
Beck, Douglas
Beer, Albert
Bégin, Nathalie
Bell, Linda
Bellusci, David
Bennett, Norman
Ben-Zvi, Phillip
Berens, Regina
Biegaj, William
Bill, Richard
Biller, James
Biondi, Richard
Blair, Gavin
Blakinger, Jean
Blanco, Robert
Blodget, Hugh
Boison, LeRoy
Boor, Joseph
Bornhuetter, Ronald
Boyd, Wallis
Bradley, Scott
Braithwaite, Paul
Brannigan, James
Brehm, Paul
Brown, Brian
Bryan, Charles
Buck, James
Captain, John
Cardoso, Ruy
Carlson, Christopher
Carpenter, William
Carroll, Lynn
Cascio, Michael
Cauchon, Martin
Caulfield, Michael
Charest, Danielle
Chernick, David
Chiang, Jeanne
Christie, James
Chuck, Allan
Cieslak, Walter
Ciezadlo, Gregory
Cis, Mark
Clark, Brooks
Cloutier, Denis
Cole, Jeffrey
Cook, Charles
Corr, Francis
Cossette, Charles
Cripe, Frederick

FELLOWS

Crowe, Alan
Cundy, Richard M.
Curran, Kathleen
Currie, Diana
Currie, Ross
Curry, Alan
Curry, Robert
Dagneau, Francois
Daino, Robert
D'Arcy, Stephen
Dean, Curtis G.
Dekle, James
DeLiberato, Robert
Dembiec, Linda
Dempster,
HowardDevlin, Patrick
Dodd, George
Dolan, Michael
Dornfeld, James
Drennan, John
Duffy, Timothy
Dukatz, Judy
Dye, Myron
Earwaker, Bruce
Edlefson, Dale
Effinger, Jr., Bob
Ellefson, Thomas
Ellingrod, John
Ely, James
Englander, Jeffrey
Engles, David
Ericson, Janet
Faga, Doreen
Fagan, Janet
Fein, Richard
Finger, Robert
Fitzgerald, Beth
Fitzpatrick, William
Flynn, David
Foley, David
Forker, David
Fortin, France
Fossa, E. Frederick
Frohlich, Kenneth R.
Furst, Patricia
Fusco, Michael
Gallagher, Cecily
Gallagher, Thomas
Gannon, Alice
Garand, Christopher
Gardner, Robert
Gebhard, James
Gelinne, David
Gibson, Richard
Gibson, John A.
Gibson, John F.
Gill, Bonnie
Gillam, Judy
Gillam, William R.
Gilles, Joseph
Gillespie, Bryan C.
Girard, Gregory
Gluck, Spencer
Goddard, Daniel
Goldfarb, Irwin
Gottlieb, Leon
Gozzo, Susan
Grady, David
Grannan, Patrick
Grant, Gary
Graves, Gregory
Greco, Ronald
Greenhill, Eric
Griffith, Ann
Groh, Linda
Hafling, David
Hale, Kyleen
Hall, James
Harrison, David
Hartman, David
Hausserman, Diane
Hayne, Roger
Hayward, Gregory
Hennessy, Mary
Heer, E. LeRoy
Henry, Dennis
Hermes, Thomas
Hess, Todd
Heyman, David
Higgins, James
Hines, Alan
Holler, Keith
Hough, Paul
Howald, Ruth
Hroziencik, George
Hutter, Heidi
Ingco, Aguedo
Ireland, Kathleen
Irvan, Robert
James, Peter
Jameson, Stephen
John, Russell
Johnson, Marvin
Johnson, Warren
Jones, Bruce

FELLOWS

Kane, Adrienne
Karlinski, Frank
Kaufman, Allan
Keen, Eric
Keller, Wayne
Kincaid, Bryan
Kirste, Richard
Kist, Fredrick
Koupf, Gary
Kozlowski, Ronald
Krakowski, Israel
Kreps, Rodney
Lalonde, David
Lamb, Dean
Lamb, John
LaMonica, Michael
Lebens, Joseph
Lehmann, Steven
Leonard, Gregory
Leong, Winsome
Léveillé, Jean-Marc
Levin, Joseph
Lew, Allen
Lipton, Barry
Liscord, Paul
Lo, Richard
Loisel, Andre
Lotkowski, Edward
Lowe, Stephen
Ludwig, Stephen
MacGinnitie, James
Marks, Steven
Marles, Blaine
Marlowe, Burton
Mashitz, Isaac
Math, Steven
Mathewson, Stuart
McAllister, Kevin
McClure, John
McCoy, Mary
McDonald, Gary
McFarlane, Liam
McManus, Michael
Mealy, Dennis
Mech, William
Menning, David
Meyers, Glenn
Miccolis, Robert
Miller, David
Miller, David
Miller, Mary Frances
Miller, Michael
Miller, Philip
Miller, Susan
Miller, William
Mitchell, H. Elizabeth
Mitzel, Charles
Moncher, Richard
Morrow, Jay
Mucci, Robert
Mueller, Nancy
Mulder, Toni
Muleski, Robert
Mulvaney, Mark
Munro, Richard
Munson, Todd
Murphy, Daniel
Murrin, Thomas
Muza, James
Myers, Thomas
Nelson, Chris
Nelson, Janet
Nemlick, Kenneth
Nester, Karen
Newell, Richard
Nichols, Richard
Nickerson, Gary
Norton, Jonathan
O'Brien, Margaret
O'Connell, Paul
Oien, R. Gus
Onufer, Layne
Overgaard, Wade
Pagnozzi, Richard
Palm, Robert
Parker, Curtis
Pasley, Jacqueline
Patrik, Gary
Patschak, Susan
Perreault, Stephen
Peterson, Steven
Petit, Charles
Philbrick, Stephen
Pinto, Emanuel
Post, Jeffrey
Potvin, Robert
Pratt, Joseph
Prescott, Richard
Prevosto, Virginia
Pridgeon, Ronald
Purple, John
Quintano, Richard
Quintilian, Kenneth
Quirin, Albert
Racine, Andre

FELLOWS

Rahardjo, Kay
Ramanujam, Srinivasa
Reale, Pamela Sealand
Reddig, Scott
Retterath, Ronald
Richardson, James
Robinson, Richard
Robinson, Sharon
Rominske, Steven
Rosenberg, Sheldon
Ross, Gail
Roth, Jr., Richard
Salton, Jeffrey
Salton, Melissa
Sandman, Donald
Scanlon, Edmund
Scheibl, Jerome
Scheuing, Jeffrey
Schmidt, Jeffrey
Schmidt, Neal
Schultheiss, Peter
Schultz, Roger
Schwartzman, Joy
Seiter, Margaret
Sellitti, Marie
Shepherd, Linda
Sherman, Harvey
Shoop, Edward
Shrum, Roy
Silver, Melvin
Simon, Christy
Simon, LeRoy
Skurnick, David
Smerald, Christopher
Sobel, Mark
Spidell, Bruce
Stanard, James
Steeneck, Lee
Steinen, Phillip
Steinert, Lawrence
Stergiou, James
Stewart, C. Walter
Strommen, Douglas
Suchoff, Stuart
Sutter, Russel
Svendsgaard, Christian
Szkoda, Susan
Taylor, Angela
Taylor, Frank
Taylor, Jane
Terrill, Kathleen
Terry, Karen
Thompson, Kevin
Tistan, Ernie
Toothman, Michael
Treitel, Nancy
Trudeau, Michel
Venter, Gary
Verges, Ricardo
Visintainer, Michael
Visintine, Gerald
Visner, Steven
VonSeggern, William
Votta, James
Vu, Sebastian
Wacek, Michael
Walker, Christopher
Walker, Glenn
Walters, Michael
Walters, Mavis
Walton, Patrick
Wargo, Kelly
Weber, Dominic
Weinstein, Marjorie
Weissner, Edward
White, David
White, Jonathan
Whitlock, Robert
Whitman, Mark
Wildman, Peter
Willsey, Robert
Wilson, Gregory
Winslow, Martha
Wulterkens, Paul
Yocius, Richard
Yow, James W.
Zatorski, Richard

ASSOCIATES

Abellera, Daniel
Allard, Jean-Luc
Allison, Kerry
Anderson, Bruce
Andler, James
Applequist, Virgil
Arico, Nancy
Ashman, Martha
Bauer, Bruno
Boardman, Thomas
Bradley, George
Brasley, Donna
Brauner, Yaakov
Brooks, Ward
Burns, William
Burrill, Linda
Cappelletti, Anthony
Casale, Kathleen
Cellars, Ralph
Chabarek, Paul
Charbonneau, Scott
Chen, Chyen
Chorpita, Fred
Closter, Donald
Coca, Michael
Connor, Vincent
Conway, Thomas
Corbett, Mary
Covitz, Burton
Creighton, Kenneth
Curry, Michael
Daly, Michael
Davenport, Edgar
Davis, Brian
Davis, James
DeConti, Michael
Der, William
Diss, Gordon
Dollinger, Jeffrey
Donaldson, Jeffrey
dos Santos, Victor
Douglas, Frank
Dubin, Michael
Ebert, Maribeth
Elia, Dominick
Emmons, William
Ericksen, Paul
Faucher, Lynne
Feldmeier, Judith
Fields, David
Fonticella, Ross
Frank, Russell
Gaillard, Mary
Gerard, Felix
Gerlach, Scott
Gidos, Peter
Godbold, Nathan Terry
Goldberg, Steven B.
Goldberg, Terry
Goldstein, Laurence
Granoff, Gary
Griffith, Roger
Grose, Carleton
Gusler, Terry
Gutman, Ewa
Haidu, James
Halliwell, Leigh
Hanson, Bradley
Harbus, Jonathan
Hay, Randolph
Head, Thomas
Helou, Renee
Henry, Thomas
Herbers, Joseph
Hinds, Kathleen
Hobart, Gary
Horovitz, Deborah
Hofmann, Richard
Hughes, Jeffrey
Ikeda, Joanne
Ill, Jeffrey
Jeng, Hou-Wen
Johnston, Daniel
Jonske, James W.
Kaufman, David
Kavacky, Trina
Keddie, Daniel
Kiefer, Ann
Kreuser, Adam
Kuo, Chung-Kuo
Lannutti, Nicholas
Larkin, James
Larson, Michael
Lehecka, Stephen
Leo, Carl
Lippl, Stephanie
Llewellyn, Barry
Macesic, David
Maguire, Brian
Maher, James
Main, William
Malik, Sudershan
Mango, Donald
Manis, Donald
Manktelow, Blair

Margulis, Galina
Marks-Samuelson, Rosemary
McCreesh, James
McCutcheon, John
McGee, Stephen
McIntosh, Heather
Mckay, Donald
McNeal, Van A.
McPadden, Sean
Miller, Brett
Mittal, Madan
Muller, Raymond
Musante, Donald
Neghaiwi, Antoine
Newman, Henry
Nystrom, Keith
O'Keefe, Mary Beth
Oliver, Douglas
Olsen, Richard
Olszewski, Laura
Oostendorp, William
Orrett, Todd
Ostergren, Gregory
Paddock, Timothy
Paffenback, Teresa
Palmer, Joseph
Peacock, Willard
Perr, Timothy
Poole, Brian
Powell, Daniel
Pulis, R. Stephen
Rabenold, Eric
Raws, Alfred
Rech, James
Reed, Donna
Rhoads, Karin
Rosenbach, Allen
Rosenstein, Kevin
Roth, Scott
Rowe, Bradley
Roy, Jean
Ruth, Maureen
Ryan, John
Samson, Sandra
Sandler, Robert
Saunders, Melodee
Schadler, Thomas
Schlenker, Sara
Schmidt, Lisa
Schoenberger, Susan
Scott, Jeffrey
Scruggs, Michael
Scruton, Gregory
Shannon, Derrick
Shepherd, David
Silverman, Janet
Smaga, James
Smith, David
Snow, David
Strauss, Frederick
Swanay, Scott
Taylor, R. Glenn
Thompson, Eugene
Thompson, Robert
Toledano, Michael
Torgrimson, Darvin
Treskolasky, Susan
Tygh, James
Valentine, Peter
Van De Water, John
Vigliaturo, Philip
Vogan, William
Wachter, Christopher
Walker, Patricia
Warfel, Stephen D.
Washburn, Monty
Watford, James D.
Weinstein, Scott
Weisbecker, Joyce
Wenitsky, Russell
Werland, Debra
Wills, Mary
Wolter, Kathy
Yezzi, Vincent
Yu, Sheng
Yunque, Mark
Zaleski, Ronald
Zimmer, Ralph

# REPORT OF THE VICE PRESIDENT-ADMINISTRATION

As I complete my first full year as Vice President-Administration, I am amazed, maybe even overwhelmed, by the scope, breadth, and sheer volume of activities in which the CAS is involved. Fulfilling the objective of this report, which is to provide the membership with a brief summary of CAS activities since the last annual meeting, is a real challenge! My apologies, in advance, for any sins of omission.

I will summarize the activities of the last 12 months into three categories: major initiatives, ongoing significant activities, and an update on the "state of the CAS."

## MAJOR INITIATIVES

### *Professionalism*

The CAS Board of Directors approved the Code of Professional Conduct which became effective on January 1, 1992. This Code replaced the Guides to Professional Conduct and Opinions in their entirety and provides for a greater degree of uniformity with the Codes of other actuarial organizations. The Code obligates CAS members to abide by applicable qualification and practice standards.

In January, the membership voted to approve revisions to the CAS Constitution and Bylaws that recognize the new Code of Conduct and establish the Actuarial Board for Counseling and Discipline (ABCD) as the counseling and investigatory arm of the profession. The ABCD officially became operational on July 1, 1992, and has developed rules of procedure which have been circulated to members for comments. At the same time, the CAS Discipline Committee is revising its operating rules and administrative guidelines in order to handle ABCD referrals. These are expected to be reviewed by the CAS Board in March 1993.

The Course on Professionalism was given six times in five different locations during 1992, with 285 students attending. A mock actuarial trial has been introduced in the course and other enhancements are under consideration. A limited attendance workshop was held at the May Meeting in Chicago to expose Fellows to case studies from the course, and, in addition, a mock trial is being presented at the November 1992 Annual Meeting.

The Long Range Planning Committee has identified the issue of "actuarial integrity" as one of its highest priority issues for 1993. Further initiatives to increase the membership's awareness of professionalism will be developed. A workshop on this topic at the November Meeting will focus on current activities and concerns about development and compliance.

*International Activities*

The Task Force on International Policy was established by CAS President Toothman to develop views and alternatives on the potential future role for the CAS outside of North America. A draft report of the Task Force was presented for discussion at the September Board meeting. The general direction from the Board focused on the following areas:

- We should continue to move forward in establishing diplomatic relations. We have already contacted 31 international actuarial organizations and have established publication exchange agreements with 18. We will continue to invite the leaders of other actuarial organizations to CAS meetings and seminars.
- Both the Syllabus and Continuing Education Committees should explore opportunities for including international content in their readings and seminars.
- We should continue high-level counterpart discussions. The Council of Presidents is working to establish four new seats on the International Actuarial Association for the Presidents and Presidents-Elect of the CAS and SOA. Mike Toothman recently represented the CAS at the Groupe Consultatif meetings in Dublin, Ireland, as reported in the November *Actuarial Review*. Many such contacts are planned for 1993.
- The CAS should take a more proactive posture in the international arena. The charge of the International Relations Committee under the

Vice President-Programs and Communications will be revised in 1993 in accordance with this direction.

*Appointed Actuary*

The September CAS Board of Directors meeting included a lengthy discussion of what the actuary's involvement should be regarding the asset side of the balance sheet. The discussion included reference to the AAA's Solvency Position Statement recently released by its Board of Directors and the implications for the CAS of this expanded role.

At its September meeting, the CAS Board directed the Executive Council to formulate a plan to educate current and future members about the implications of interlinking the asset and liability sides of the balance sheet. A Task Force on the Appointed Actuary has been created by President Toothman. This Task Force, chaired by Robert A. Miller III, is charged with developing a plan to cover those activities the CAS needs to undertake in order to prepare its members to function in the role of appointed actuary in North America opining on an insurer's current and future solvency.

## ONGOING SIGNIFICANT ACTIVITIES

The CAS Office completed its first full year of operation in its Arlington, Virginia, location. I'm pleased to report that Tim Tinsley and the entire office staff are focused on delivering quality service to our members and candidates. Numerous administrative functions have been absorbed by the office, freeing up CAS committees to focus on substantive issues. A staff editor, Brenda Huber, was hired in December. Brenda's addition has led to savings in printing costs, more timely release of publications, and other enhancements to our various publications.

On a personal note, it has been a pleasure for me to work closely this past year with all the office staff. I encourage you to avail yourselves of this valuable resource. I can assure you that you'll be pleased with the results.

During the year, we entered into a contract with Morant Data Company to install and activate a database system that will provide on-line capabilities for all of our office information needs. Much of the work is already done and the system should be operational by year-end 1992. When completed, we will have a robust, integrated membership and candidate database at the CAS office.

A complete legal audit of the CAS was completed on October 2. The Washington law firm of Jenner & Block undertook a comprehensive review of appropriate CAS documents, files, and practices. The final audit is currently under review by the Executive Council with a goal of presenting the necessary legal and antitrust compliance policy recommendations to the Board in early 1993.

The concept of managed research, where funded programs target specific areas, has borne fruit during the year. Projects underway or completed include:

- A Financial Analysis survey of solvency literature presented at the Valuation Seminar in April 1992.
- Fourteen completed papers on ratemaking to be published in the February 1993 *Forum*. The material will support four concurrent sessions at the March 1993 Ratemaking Seminar.
- A Theory of Risk prize paper program is on schedule for papers to be completed by January 1993.
- Work continues on the request for proposal for practical methods to implement risk margins. In addition, proposals are being reviewed for the survey of valuation literature RFP.
- Additional research efforts are being considered for next year as well. In that regard, a concurrent session on CAS-sponsored research is being held at the November Meeting. The panel will brief members on current research and solicit input from members for future efforts.

Continuing education opportunities, with additional focus due to the AAA requirement, were numerous and well-attended during 1992. Meetings and seminars for the year have included:

- The Spring Meeting in Chicago, attended by 379 members and 132 non-members.

- The Annual Meeting in Boca Raton, with advance registration of 569 members and 63 non-members.
- The Ratemaking Seminar in Dallas attended by 524.
- A special interest seminar on valuation issues in April; meeting attendance was 156.
- A special interest seminar on reinsurance held in June with 122 members and 75 non-members in attendance. This was a joint effort with the CAS special interest section, Casualty Actuaries in Reinsurance (CARe).
- The Casualty Loss Reserve Seminar, jointly sponsored with the AAA and Conference of Consulting Actuaries, held in September with 731 in attendance.
- A special interest seminar on rate of return topics was held October 15-16 in Seattle. Attendance was 90 members and 34 non-members.
- The P&C Insurance Liabilities Seminar, which was combined with the Canadian Institute of Actuaries' Appointed Actuary Seminar, was held October 1-2 in Toronto. A total of 315 attended.

A number of our regional affiliates also sponsored continuing education sessions as part of their regular meetings.

Efforts were also continued to increase public awareness of the actuarial profession. A student newsletter, scheduled for publication in January 1993, was jointly developed by the CAS and SOA. An external informational brochure has been developed, and a video presentation on the actuarial career is under development.

## STATE OF THE CAS

We continue to grow! A total of 5,627 candidates registered for exams in May and November. This represents a 19% increase over 1991. During the year, 141 new Associates joined the CAS. With the addition of 76 new Fellows, our membership now stands at 1,121 Fellows and 816 Associates.

The Board of Directors met four times during 1992. New members elected to the Board for next year include Al Beer, Phil Ben-Zvi, Mike Miller, and Susan Szkoda. The membership elected Irene Bass to the position of President-Elect, while Dave Flynn assumed the Presidency.

The Executive Council, with primary responsibility for day-to-day operations, met either by teleconference or in person at least once a month during the year. The Board of Directors elected the following Vice Presidents for the coming year:

Vice President-Administration John Purple
Vice President-Admissions Steven Lehmann
Vice President-Continuing Education Dave Hafling
Vice President-Programs and Communications Alice Gannon
Vice President-Research and Development Allan Kaufman

In closing, here are some comments on our financial status. The Audit Committee examined the CAS books for fiscal year 1992 and found the accounts to be properly stated. The fiscal year ended with an increase in surplus of $48,899.76 which compares favorably to a budgeted increase of approximately $26,000. Members' equity now stands at $648,794.07, subdivided as follows:

| | |
|---|---|
| Michelbacher Fund | $83,895.16 |
| Dorweiler Fund | 7,402.26 |
| CAS Trust | 3,114.76 |
| Scholarship Fund | 7,976.27 |
| Rodermund Fund | 15,551.14 |
| CLRS Fund | 5,000.00 |
| CAS Surplus | 525,854.48 |
| TOTAL MEMBERS' EQUITY | $648,794.07 |

For 1992-93, the Board of Directors has approved a budget of approximately $2.0 million. This is not directly comparable to last year's expenses, due to an accounting change to no longer reflect meetings and seminars on a net basis.

Member's dues for next year will be $230, an increase of $15, while fees for the invitational program will increase by $45 to $275. Examination fees for Parts 4-10 will remain the same.

Respectfully submitted,

JOHN M. PURPLE
*Vice President-Administration*
November 16, 1992

# TREASURER'S REPORT
## Fiscal Year Ended 9/30/92

OPERATING RESULTS BY FUNCTION:

| FUNCTION | INCOME | EXPENSE | DIFFERENCE |
|---|---|---|---|
| Exams | $653,069.35 | $533,253.37(a) | $119,815.98 |
| Member Services (b) | 413,440.05 | 681,178.49 | (267,738.44) |
| Programs | 268,068.68 | 118,267.45(c) | 149,801.23 |
| Other (d) | 47,020.99 | 0.00 | 47,020.99 |
| TOTAL | $1,381,599.07 | $1,332,699.31 | $48,899.76(e) |

Notes: (a) Does not include exam-related expenses incurred by the Research & Development Function. (b) Areas under the supervision of VP-Administration & VP-Research and Development. (c) Does not include program-related expenses incurred by the Research & Development Function. (d) Investment income less foreign exchange and miscellaneous bank debits. (e) Change in CAS Surplus.

BALANCE SHEET

| *ASSETS* | 9/30/91 | 9/30/92 | DIFFERENCE |
|---|---|---|---|
| Checking Account | $278,635.93 | $255,199.23 | ($23,436.70) |
| U.S. Treasury Bills | 767,724.58 | 972,768.01 | 205,043.43 |
| Accrued Interest | 15,157.44 | 12,348.41 | (2,809.03) |
| Prepaid Meeting Exp. | 10,212.23 | 9,912.01 | (300.22) |
| Prepaid Room Rentals | 0.00 | 2,400.00 | 2,400.00 |
| Prepaid Seminar Exp. | 490.00 | 306.24 | (183.76) |
| CLRS Receivable | 40,000.00 | 50,000.00 | 10,000.00 |
| CLRS Fund | 5,000.00 | 5,000.00 | 0.00 |
| MIS | 0.00 | 36,322.37 | 36,322.37 |
| *TOTAL ASSETS* | $1,117,220.18 | $1,344,256.27 | $227,036.09 |
| *LIABILITIES* | | | |
| Exam Fees Deferred | 244,353.20 | 278,507.00 | 34,153.80 |
| Printing Expense | 185,700.00 | 208,000.00 | 22,300.00 |
| Research Grant | 50,000.00 | 58,664.70 | 8,664.70 |
| Office Expense | 21,629.00 | 0.00 | (21,629.00) |
| Nov. Mtg. Fees Deferred | 5,591.00 | 84,331.50 | 78,740.50 |
| Seminar Fees Deferred | 12,600.00 | 40,434.00 | 27,834.00 |
| Mtg. & Sem. Expenses | 1,402.55 | 0.00 | (1,402.55) |
| ABCD Payable | 0.00 | 11,600.00 | 11,600.00 |
| Legal Audit Payable | 0.00 | 10,000.00 | 10,000.00 |
| Inv. Program Deferred | 0.00 | 3,530.00 | 3,530.00 |
| Academic Corr. Deferred | 0.00 | 225.00 | 225.00 |
| Act. Review Deferred | 0.00 | 170.00 | 170.00 |
| *TOTAL LIABILITIES* | $521,275.75 | $695,462.20 | $174,186.45 |
| *MEMBERS' EQUITY* | | | |
| Michelbacher Fund | $80,171.51 | $83,895.16 | $3,723.65 |
| Dorweiler Fund | 6,983.26 | 7,402.26 | 419.00 |
| CAS Trust | 2,938.45 | 3,114.76 | 176.31 |
| Scholarship Fund | 7,996.49 | 7,976.27 | (20.22) |
| Rodermund Fund | 15,900.00 | 15,551.14 | (348.86) |
| CLRS Fund | 5,000.00 | 5,000.00 | 0.00 |
| CAS Surplus | 476,954.72 | 525,854.48 | 48,899.76 |
| *TOTAL EQUITY* | $595,944.43 | $648,794.07 | $52,849.64 |

John M. Purple, Vice President-Administration

*This is to certify that the assets and accounts shown in the above financial statement have been audited and found to be correct.*

Audit Committee: Lee M. Smith, Chairman, Anthony J. Grippa, Albert J. Quirin, William J. Rowland, Charles Walter Stewart, Russel L. Sutter

## 1992 EXAMINATIONS—SUCCESSFUL CANDIDATES

Examinations for Parts 3B, 4A, 4B, 6, 8, 8C, and 10 of the Casualty Actuarial Society were held on May 4, 5, 6, 7, and 8. Examinations for Parts 3B, 5, 5A, 5B, 7, and 9 were held on November 4, 5, and 6.

Examinations for Parts 1, 2 and 3 (SOA courses 100, 110, 120, 130 and 135) are jointly sponsored by the Casualty Actuarial Society and the Society of Actuaries. Parts 1 and 2 were given in February, May, and November of 1992 and Part 3 was given in May and November of 1992. Candidates who were successful on these examinations were listed in the joint releases of the two societies.

The Casualty Actuarial Society and the Society of Actuaries jointly awarded prizes to the undergraduates ranking the highest on the Part 1 examination.

For the February 1992 examination, the $200 first prize was awarded to Mark Krosky. The $100 prize winners were Louis Charbonneau, Vincent Ha, Gary C. Mei, and Ronnie Y. Tan.

For the May 1992 examination, the $200 first prize was awarded to Hans K. VanDelden. The $100 prize winners were Igor Ioppe, Eve Rainville, Shing-Wei Shih, and Christopher James Wiggenhold.

For the November 1992 examination, the $200 first prize was awarded to Chen Hui Su. The $100 prize winners were Levi M. Askovitz, Henry J. Elliott, Yue Wu, and Bo Yao.

The following candidates were admitted as Fellows and Associates at the May 1992 meeting as a result of their successful completion of the Society requirements in the November 1991 examinations.

### FELLOWS

Karin H. Beaulieu
Allan R. Becker
Roberto G. Blanco
Patrick J. Burns
Kenneth E. Carlton III
Daniel J. Czabaj
Catherine E. Eska
William G. Fitzpatrick
Nancy G. Flannery
Brian A. Hughes
Bruce E. Ollodart
Brian G. Pelly
Julia L. Perrine
Jennifer A. Polson
Stephen D. Stayton
William Vasek
Elizabeth A. Wellington

ASSOCIATES

Kristen M. Albright
Todd R. Bault
Herbert S. Bibbero
Wayne E. Blackburn
Annie Blais
Daniel D. Blau
Betsy L. Blue
John P. Booher
Christopher K. Bozman
J. Eric Brosius
David S. Cash
Dennis K. Chan
Bryan C. Christman
Wei Chuang
Kasing L. Chung
Gary T. Ciardiello
Peter J. Collins
Thomas P. Conway
Gregory L. Cote
Timothy J. Cremin
Gregory A. Cuzzi
Michael K. Daly
Manon Debigare
Michael L. DeMattei
Herbert G. Desson
Stephen R. DiCenso
Michel Dionne
Jeffrey E. Doffing
Michael C. Dubin
Francois Dumas
Denise A. Feder
Charles C. Fung
Kim B. Garland
Jeffrey C. Gendron
Odile Goyer
Steven J. Groeschen
Farrokh Guiahi
Terry D. Gusler
Leigh J. Halliwell
David L. Homer
Paul R. Hussian
Hou-Wen Jeng
Susan E. Kent
Deborah E. Kenyon
Kevin A. Kesby
Gerald S. Kirschner
Timothy F. Koester
Gilbert M. Korthals
Benoit Laganiere
Alan E. Lange
Christopher Lattin
Marc-Andre Lefebvre
Paul R. Livingstone
Richard Maguire
Katherine A. Mann
Leslie R. Marlo
Suzanne Martin
Keith A. Mathre
Maria Mattioli
Thomas S. McIntyre
John H. Mize
Russell E. Moore
Francois Morin
Francois L. Morissette
David A. Murray
Victor A. Njakou
Kathleen C. Nomicos
Stephen R. Noonan
Robert C. Phifer
Mark W. Phillips
Karin L. Reinhardt
Lisa M. Ross
Daniel G. Roth
Michael R. Rozema
David O. Schlenke
Peter Senak
Robert D. Share
David B. Sommer
Barbara H. Thurston
Thomas C. Toce
Michael Toledano
Therese M. Vaughan
Jennifer A. Violette
Bryan C. Ware
John P. Welch
Robert J. White
Windrie Wong

The following is a list of successful candidates in examinations held in May 1992.

*Part 3B*

Tamela Alamo
Mark K. Altschuler
Kristine M. Anderson
Michael J. Anstead
Martin S. Arnold
Kathleen J. Atkinson
Larina D. Baird
Patricia A. Baiyor
Jaclyn W. Ballin
Kelly A. Barratt
Rose D. Barrett
Claudia M. Barry
Philip A. Baum
Mary P. Bayer
Martin Beaulieu
Michael J. Bednarick
Bruce J. Bergeron
Steven L. Berman
Abbe Binkowitz
Darrin W. Birtciel
Mariano R. Blanco
Karen A. Blaszczak
Raju Bohra
Carl M. Bolstad
Hobart F. Bond
Brian K. Bouvier
Richard A. Brassington
Jeffrey M. Brobjorg
Bruce D. Browning
Michael L. Bruess
Hayden Burrus
Sandra L. Cagley
Donna L. Callison
James E. Calton
Jacqueline M. Campbell
Francine Cardi
Mary Ellen Carino
Sondra A. Cavanaugh
Elina L. Chachkes
Bina S. Cherian
Gary C. Cheung
Wanchin W. Chou
Jennifer A. Cincola
Frank S. Conde
Greg E. Conklin
Christopher G. Cunniff
David F. Dahl
Kenneth S. Dailey
David B. Dalton
Jill A. Davis
Robert E. Davis
Willie L. Davis
Raymond V. DeJaco
Emily Y. Deng
Kenneth R. Dipierro
Gayle L. Dittrich
William A. Dowell, Jr.
Annette M. Eckhardt
Dana L. Eisenberg
Ken B. Eliezer
Ellen E. Evans
James G. Evans
Michael A. Falcone
Sharon R. Farmer
Renee L. Feathers
Alan E. Feldman
Brian M. Fernandes
Daniel B. Finn
Annette L. Fischer
William J. Fogarty
Lloyd A. Foster
Brian A. Franklin
Mauricio Freyre
Beverly J. Frickel
Amy A. Gadsden
Micah R. Gentile
Barry A. Gertschen
Eric J. Gesick
David E. Gill
Nicholas P. Giuntini
Lindsey C. Gleadall
Stewart H. Gleason
Michael J. Goetsch
Matthew L. Gossell
Jennifer Graunas
John E. Green
Paul E. Green
Steven A. Green
David T. Groff
David J. Gronski
David W. Haaf
Daniel J. Hayes
Mary K. Hays

*Part 3B (cont'd)*

Martin G. Heagen
Sara L. Helgeson
Glenn S. Hochler
Jason N. Hoffman
Margaret M. Hook
Bernard R. Horovitz
Marguerite M. Hunt
Chaudhry M. Ishaq
Vincent H. Jackson
Jean-Claude J. Jacob
Deborah M. Jasper
Jill C. Johnson
Aline Kafafian
Kristie L. Kantowski
Deborah M. King
Diane L. Kinner
Wendy A. Knopf
Anne Kochendorfer
Kevin C. Kolakowski
Kimberly A. Kracht
Susan M. Krawiec
Brian S. Krick
Joseph J. LaBella
Christine L. Lacke
Steven M. Lacke
William C. Lawson
Betty F. Lee
Joan K. Lee
Frederic A. Leederman
Claude Lefebvre
Daniel Leff
Sidney Leffler
Paul B. LeStourgeon
Katherine E. Lewis
Cheng-Te Liang
Paul Liu
Lee C. Lloyd
John Lum
Kyra D. Lynn
Sally A. MacFadden
Rita M. MacIntyre
Jeffrey S. Magrane
Leodivini Q. Magtoto
Vahan A. Mahdasian
John T. Maher
Elaine J. Malupa
Richard W. Malus
Eileen M. McGaheran
Patricia McGeeney
Peter B. McOrmond
Lawrence J. McTaggart III
Hernan L. Medina
Constance M. Mika
David J. Miller
Scott M. Miller
Michael J. Miraglia
Kimberly A. Moran
Michael W. Morro
Janice C. Moskowitz
Matthew S. Mrozek
Timothy O. Muzzey
Jarow G. Myers
Jennifer A. Nelson
Kari S. Nelson
Robert P. Nelson
Thomas E. Newgarden
Norman Niami
Eric C. Nordman
Lauren E. Norton
Marc F. Oberholtzer
Matthew J. Odendahl
Richard D. Olsen
Denise R. Olson
Milary N. Olson
Stanley T. Olszewski
James A. Partridge
Anna M. Paul
David M. Pfahler
David R. Picking
Karen J. Poklikuha
Deborah J. Pomerantz
Kathy A. Poppe
Elisabeth N. Power
Stephen H. Press
Julie Privman
Rhonda A. Puda
Lovely G. Puthenveetil
Jacqueline M. Ramberger
Eula A. Rath
Frank S. Rau
Jaak M. Raudsepp
Joanne E. Reitz
Michael T. Reitz
Jennifer L. Reller
David C. Riek
Brad M. Ritter

*Part 3B (cont'd)*

Christopher R. Ritter
Linda L. Roberts
Pamela L. Rose
William P. Rudolph
Douglas A. Rupp
Anthony N. Sammur
Rachel Samoil
James C. Sandor
Teresa M. Scharn
Catherine I. Schedlbauer
Michael K. Schepak
Steven M. Schienvar
Matt J. Schmitt
Lisa M. Schultz
Joyce E. Segall-Lopez
Anastasios Serafim
Alan J. Sexter
Lisa A. Sgaramella
Paul E. Shangold
Scott A. Sheldon
Theodore J. Shively
Faith E. Signorile
Daniel D. Smalley
Cindy W. Smith
Gene W. Smith
Robert K. Smith
Halina H. Smosna
Scott G. Sobel
Jay M. South
Tracey A. Stark-Beldere
Christopher M. Steinbach
Lori E. Stoeberl
Judy L. Stolle
Heather L. Strinmoen
Christopher S. Strohl
Mark R. Strona
Elizabeth A. Strong
Brian J. Sullivan
Siu Cheung S. Szeto
Rachel R. Tallarini
Hung K. Tang
Michael J. Tempesta
Diane R. Thurston
Myles J. Tilley
Tammy M. Titus
Jennifer M. Tornquist
Joseph S. Tripodi
Nicole G. Trotman
Turgay F. Turnacioglu
Nicholas Turville
Randall H. Tweedt
Steven D. Umansky
Jeffrey A. VanKley
Amy R. Waldhauer
Robert J. Walling III
Jon S. Walters
David W. Warren
Karen E. Watson
Erica L. Weida
Robert F. Wolf
Gretchen L. Wolfer
Cheng-Sheng P. Wu
George H. Zanjani

*Part 4A*

Shawna S. Ackerman
Jonathan D. Adkisson
Vicki L. Agerton
Michael W. Allard
Fred S. Allsbrook
John P. Alltop
K. Athula P. Alwis
Timothy P. Aman
Donald W. Anson
Marc D. Archambault
Steven D. Armstrong
Martin S. Arnold
Kevin J. Bakken
Daniel M. Bankson
Kendra D. Barnes
Dana Barre
Robert S. Beatman
Corey J. Bilot
Gina S. Binder
Abbe Binkowitz
Lisa A. Bjorkman
Daniel E. Block
David Bockol
Donna M. Bono
Lesley R. Bosniack
Michael D. Brannon
Donna D. Brasley
Kevin J. Brazee
Laura G. Brill
Jeffrey H. Brooks

*Part 4A (cont'd)*

Ward M. Brooks
Tracy L. Brooks-Szegda
Conni J. Brown
Robert L. Brown
David A. Bulin
Mark E. Burgess
Hayden Burrus
Sharon L. Cage
Linda E. Callas
Julia C. Causbie
Maureen A. Cavanaugh
Kevin J. Cawley
Joseph G. Cerreta
Randall A. Chaffinch
Jean-Francois Chalifoux
Andrea L. Chan
Edward L. Chan
Whye-Loon Chan
Debra S. Charlop
Soo H. Choo
Rene Chouinard
Normand R. Chretien
Gregory C. Christensen
Alan R. Clark
Derek A. Clark
Jo Ellen Cockley
Maryellen J. Coggins
Christopher C. Coleman
David G. Cook
Sheila C. Cooley
William F. Costa
Kirsten J. Costello
Jose R. Couret
Paul T. Cucchiara
Kendra S. Cupp
Marie-Claude Cyr
Charles A. Dal Corobbo
Francis L. Decker
Kris D. DeFrain
Romulo Deo-Campo Vuong
Sean R. Devlin
Behram M. Dinshaw
Patricia J. Donnelly
Kevin G. Donovan
John P. Doucette
Robert G. Downs
Martin W. Draper
Kimberly J. Drennan
Barry P. Drobes
Raymond S. Dugue
Anthony D. Edwards
Jeffrey S. Ellis
Dawn E. Elzinga
Martin A. Epstein
Dianne L. Estrada
Michael A. Falcone
Brian M. Fernandes
Jeffrey M. Forden
Sally M. Forsythe
David I. Frank
Kirsten A. Frantom
Bethany L. Fredericks
Mauricio Freyre
Shina N. Fritz
Brad P. Gardner
Charles E. Gegax
Micah R. Gentile
Thomas P. Gibbons
Julie T. Gilbert
Mary K. Gise
Nicholas P. Giuntini
Stewart H. Gleason
John T. Gleba
Ronald E. Glenn
Laurence B. Goldstein
Jie Gong
Annette J. Goodreau
Karl Goring
John W. Gradwell
Michael D. Green
Joseph P. Greenwood
Russell H. Greig
Christine M. Grey
Francis X. Gribbon
Gary J. Griesmeyer
Monica A. Grillo
Richard J. Haines
Barbara Hallock
Sandra K. Halpin
Elizabeth E. Hansen
William D. Hansen
Scott W. Hanson
Christopher L. Harris
Adam D. Hartman
Michelle L. Hartrich

*Part 4A (cont'd)*

Michael B. Hawley
Ronald J. Herrig
Betty-Jo Hill
Laura K. Hobart
Wayne Hommes
David B. Hostetter
Melissa K. Houck
Alex Y. Hsiao
Marie-Josee Huard
Gloria A. Huberman
Sandra L. Hunt
Caleb E. Huntington
Randall A. Jacobson
Brian J. Janitschke
Fong-Yee J. Jao
Brian E. Johnson
Jill C. Johnson
Steven J. Jordan
Mark J. Kaufman
Kimberly S. Kaune
Lowell J. Keith
Scott A. Kelly
Rebecca A. Kennedy
William J. Keros
Michael B. Kessler
He-Jung Kim
John H. Kim
Ung M. Kim
Jill E. Kirby
Russell G. Kirsch
Omar A. Kitchlew
Michael F. Klein
Jonelle A. Kohne
Gary R. Kratzer
Kenneth A. Kurtzman
Edward M. Kuss
Blair W. Laddusaw
Jean-Sebastien Lagarde
Jin-Mei J. Lai
Yaohsien Lai
Mai B. Lam
John B. Landkamer
John P. Lebens
Helen P. LeClair
Robin R. Lee
Claude Lefebvre
Stephen E. Lehecka
Glen A. Leibowitz
Terry Lem
Charles R. Lenz
David R. Lesieur
Kuen-Shan Ling
Timothy D. Logie
Nora J. Lovall
Michelle Luneau
Allen S. Lynch, Jr.
John B. Mahon
Barbara S. Mahoney
Gary P. Maile
Donna M. Marazzo
Lawrence F. Marcus
Joseph Marracello
Leslie A. Martin
Peter R. Martin
Michael B. Masters
Robert F. Maton
Lynda L. Mattes
Marci A. Maxwell
Deann M. Mays
Camley A. Mazloom
Angela T. Mazzaferro
Timothy C. McAuliffe
Charles L. McGuire
Kelly S. McKeethan
Michael B. McKnight
David W. McLaughry
Mary Jo Meeks
Timothy J. Melvin
Mitchel Merberg
Daniel J. Merk
Constance M. Mika
Michael J. Miller
Camille D. Minogue
Kenneth B. Morgan
Turhan E. Murguz
Kevin T. Murphy
Amirul Z. Musanif
Giovanni A. Muzzarelli
Jennifer A. Naehr
Melissa J. Neidlinger
Ronald T. Nelson
Khanh K. Nguyen
Anne H. Nimick
Dianna C. Norman
Lauren E. Norton
Steven B. Oakley
Kathleen C. Odomirok
Denise R. Olson
Joseph M. Palmer
Genevieve Pare

*Part 4A (cont'd)*

Nicholas H. Pastor
Abha B. Patel
Rick S. Pawelski
Wende A. Pemrick
Luba Pesis
Mark A. Piske
Anthony K. Postert
Thomas M. Potter
Tracey S. Powers
Arlie J. Proctor
Regina M. Puglisi
John K. Punzak
Gene Z. Qian
Karen L. Queen
Kathleen M. Quinn
Darin L. Rasmussen
Peter S. Rauner
Brenda L. Reddick
Nathalie J. Rekittke
Donald A. Riggins
Christopher R. Ritter
Marn Rivelle
Jeremy Roberts
Ronald J. Robinson
Dave H. Rodriguez
John W. Rollins
Joseph F. Rosta
Jean Roy
Jean-Denis Roy
David A. Russell
Sean W. Russell
John C. Ruth
Mark T. Rutherford
Cheryl Y. Sabiston
Rajesh V. Sahasrabuddhe
Michael J. Scholl
Sharon M. Schorge
Michael F. Schrah
Michael R. Schummer
Craig J. Scukas
William H. Scully III
Patrick J. Seaman
Huidong Shang
Thomas J. Sheppard
Andrea W. Sherry
Laura E. Siegel
Michael N. Singer
Helen A. Sirois
Charles L. Sizer
Larry K. Smith
Theodore S. Spitalnick
Beth A. Stahelin
Christina L. Staudhammer
Nathan R. Stein
Susan D. Stieg
Richard A. Stock
Lori E. Stoeberl
Kevin D. Strous
Hung K. Tang
Robert D. Taylor
David J. Tenembaum
Josephine T. Teruel
Santy S. Thalappillil
Patricia Therrien
Sadhana Tiwari
Nathalie L. Tremblay
Joseph D. Tritz
Scotty M. Tucker
Arthur J. Turner
Robert C. Turner
James F. Tygh
Eric Vaith
Mark D. VanZanden
Robert J. Vogel
Lynne K. Wehmueller
Shu-Mei Wei
Mark S. Wenger
Elizabeth A. Wentzien
Geoffrey T. Werner
James C. Whisenant
Wyndi S. White
David L. Whitley
Michael J. Williams
David S. Wolfe
Barbara A. Wolinski
Denise Y. Wright
Cheng-Sheng P. Wu
Xuening Wu
Edward J. Yorty
Timothy W. Young
Guangjian Zhu

*Part 4B*

Jonathan D. Adkisson
Elise M. Ahearn
John S. Alexander
Anthony L. Alfieri
Michael W. Allard
Kay L. Allen
Fred S. Allsbrook
John P. Alltop
K. Athula P. Alwis
Jennifer J. Andrzejewski
Martin S. Arnold
Mark E. Austin
Kevin J. Bakken
Phillip W. Banet
Daniel M. Bankson
Michael W. Barlow
Claudia M. Barry
James C. Berger
Claire Bilodeau
Corey J. Bilot
Lisa A. Bjorkman
Jean-Francois Blais
Barry E. Blodgett
Rejean Boivin
Elizabeth S. Borchert
Lesley R. Bosniack
Scott A. Bowser
Lori M. Bradley
Betsy A. Branagan
Donna D. Brasley
Kevin J. Brazee
Odile S. Brock
Ward M. Brooks
Tracy L. Brooks-Szegda
Conni J. Brown
Robert L. Brown
David A. Bulin
Anthony R. Bustillo
Steven M. Byam
Sharon L. Cage
Douglas A. Carlone
Julia C. Causbie
Maureen A. Cavanaugh
Kevin J. Cawley
Francis D. Cerasoli
Joseph G. Cerreta
Randall A. Chaffinch
Jean-Francois Chalifoux
Andrea L. Chan
Debra S. Charlop
Sigen Chen
Shu-Fang Cheng
Ching-Ping Chi
Hsiao-Yuan Ching
Rene Chouinard
William Chow
Normand R. Chretien
Liping Chu
Jo Ellen Cockley
Christopher C. Coleman
Craig A. Cooper
David C. Coplan
William F. Costa
Kirsten J. Costello
Jose R. Couret
Sandra Creaney
Kendra S. Cupp
Andrew S. Dahl
Charles A. Dal Corobbo
Rachel Dass
Smitesh Dave
Robin M. Davis
Shaila De Leede
Francis L. Decker
Karen D. Derstine
Kevin G. Donovan
John P. Doucette
Michael J. Doviak
Robert G. Downs
Martin W. Draper
Lucy Drozd
David L. Drury
Nathalie Dufresne
Raymond S. Dugue
Jeffrey S. Ellis
Sarkis M. El-Zein
Martin A. Epstein
Dianne L. Estrada
James G. Evans
Linda S. Eveland
Michael A. Falcone
William P. Fisanick
Stephanie Fontaine
David I. Frank
Kirsten A. Frantom
Mark J. Franzen

*Part 4B (cont'd)*

Ramon D. Galanes
Isabelle Gaumond
Hannah Gee
Charles E. Gegax
Justin G. Gensler
Julie T. Gilbert
Serge Girard
Stewart H. Gleason
John T. Gleba
Ronald E. Glenn
Lynn E. Golas
Jennifer L. Goldberg
Laurence B. Goldstein
Judith M. Gottesman
John W. Gradwell
Russell H. Greig
Christine M. Grey
Francis X. Gribbon
Monica A. Grillo
Barbara Hallock
Sandra K. Halpin
Elizabeth E. Hansen
William D. Hansen
Joel D. Hanson
Claude Harnois
Christopher L. Harris
Adam D. Hartman
Michelle L. Hartrich
Shohreh Heshmati
Anne M. Hoban
Brett D. Hodgson
D. Kent Holbrook
Robert J. Hopper
Melissa K. Houck
Pi-Chun Hsu
Gloria A. Huberman
Sandra L. Hunt
Tina T. Huynh
Sadagopan S. Iyengar
Brian J. Janitschke
John F. Janssen
Vibha N. Jayasinghe
Donald T. Jemison
Christopher P. Johnson
Daniel J. Judd
Robert B. Katzman
Mark J. Kaufman
Jack S. Keck
Lowell J. Keith
Mark R. Kelbaugh
Thomas P. Kenia
Cynthia A. Keyes
Tricia M. Keyes
Elizabeth A. Kinney
Joseph P. Kirley
Michael F. Klein
Louis K. Korth
Suzanne D. Kuntz
Edward M. Kuss
Kirk L. Kutch
Bertrand J. LaChance
Blair W. Laddusaw
Jean-Sebastien Lagace
Sao-Kun "Connie" Lam
John M. Lamendola
Matthew G. Lange
Douglas W. Latimer
Khanh M. Le
Manuel Alberto T. Leal
Claude Lefebvre
Stephen E. Lehecka
David R. Lesieur
Isabelle Letourneau
Aaron S. Levine
Guey-Ru Lieu
Chinlong Lin
Frank K. Ling
Pascal Longpre
Laura J. Lothschutz
Robb W. Luck
Kyra D. Lynn
Emma B. Macasieb
Kenneth W. Macko
Barbara S. Mahoney
David K. Manski
Yishi Mao
Donna M. Marazzo
Lawrence F. Marcus
Albert Maroun
Michael B. Masters
Robert F. Maton
Lynda L. Mattes
Tracey L. Matthew
Bonnie C. Maxie
Laura A. Maxwell
Deann M. Mays
Camley A. Mazloom
Deborah L. McCrary
Charles L. McGuire

*Part 4B (cont'd)*

Kelly S. McKeethan
Phillip E. McKneely
David W. McLaughry
William E. McWithey
Frederic N. Michaud
Stephen J. Mildenhall
Michael J. Miller
Catherine E. Moody
Kenneth B. Morgan
Kimberly J. Mullins
Peter J. Murdza, Jr.
Joseph L. Murdzek
Kevin T. Murphy
Giovanni A. Muzzarelli
Karen E. Myers
Kathleen V. Najim
Dianna C. Norman
Andres F. Ochoa-Gomez
Kathleen C. Odomirok
Helen S. Oliveto
Kevin J. Olsen
Kory J. Olsen
Lowell D. Olson
Stanley T. Olszewski
Krzysztof M. Ostaszewski
Michael G. Owen
Gerard J. Palisi
Joseph M. Palmer
Charles Pare
Genevieve Pare
Thomas Passante
Christos G. Patsalides
Lisa M. Pawlowski
Jeremy P. Pecora
Wende A. Pemrick
Claude Penland IV
Miriam E. Perkins
Peter P. Perrotti
Anne M. Petrides
Aleksey Popelyukhin
Anthony K. Postert
Thomas M. Potter
Tracey S. Powers
Michael D. Price
Armand Principato
Doug L. Pryor
Regina M. Puglisi
Patricia A. Pyle
Gene Z. Qian
Mark S. Quigley
Kathleen M. Quinn
Kelly L. Raasch
Robin A. Rabideau
Patrice Raby
Kiran Rasaretnam
Darin L. Rasmussen
Peter S. Rauner
Brenda L. Reddick
Jennifer L. Reller
Ellen J. Respler
Donald A. Riggins
Dennis L. Rivenburgh, Jr.
Dominique Robert
John W. Rollins
Joseph F. Rosta
Jean Roy
Jean-Denis Roy
David A. Russell
Sean W. Russell
Laura A. Ryan
Charles J. Ryherd
Samuel E. Sackey
Cindy R. Schauer
Lori A. Scheirer
Michael J. Scholl
Annmarie Schuster
William H. Scully III
Kelvin B. Sederburg
Joyce E. Segall-Lopez
Vinaya K. Sharma
Thomas J. Sheppard
Craig T. Shigeno
Jeffrey P. Shirazi
Charles L. Sizer
Raleigh R. Skaggs, Jr.
Klayton N. Southwood
Calvin C. Spence, Jr.
Theodore S. Spitalnick
Michael P. Spurbeck
Christina L. Staudhammer
Scott T. Stelljes
Richard A. Stock
Brian K. Sullivan
Steven W. Sun
Siu Cheung S. Szeto
Duc M. Ta

*Part 4B (cont'd)*

Ken Seng Tan
Robert D. Taylor
Judith D. Teglas
Josephine T. Teruel
Daniel A. Tess
Patricia Therrien
John L. Timmerberg
Amy B. Treciokas
Joseph S. Tripodi
Chuan-Shin D. Tu
Robert C. Turner
James F. Tygh
Pamela J. VanLeirsburg
Mark D. VanZanden
Trent R. Vaughn
Wittie O. Wacker
Heather A. Waldron
Lisa M. Walsh
Yu-Hwa Wang
Kimberley A. Ward
Linda F. Ward
Jamil Wardak
Jennifer M. Webb
Petra L. Wegerich
Christopher B. Wei
Deyue Wei
Geoffrey T. Werner
James C. Whisenant
Wyndi S. White
David L. Whitley
Tammy L. Wimer
David S. Wolfe
Terry C. Wolfe
Milton K. Wong
Stephen K. Woodard
Mark L. Woods
Cheng-Sheng P. Wu
Fenghua H. Wu
Hsuli Wu
Xuening Wu
Jacinthe Yelle
Robert S. Yenke
Edward J. Yorty
Kong Hung Yu
Jeffery M. Zacek
George H. Zanjani
Guangjian Zhu
Eric E. Zlochevsky

*Part 6*

Daniel N. Abellera
Mark A. Addiego
Craig A. Allen
Ann L. Alnes
Michael J. Andring
Nancy L. Arico
Barry L. Bablin
Timothy J. Banick
Jack Barnett
John A. Beckman
Cynthia Bentley
Lyne Bergeron
Eric D. Besman
Michael G. Blake
Gina L. Blakeney
Gary Blumsohn
George P. Bradley
James L. Bresnahan
Louis M. Brown
Michelle M. Bull
Linda J. Burrill
John F. Butcher II
Robert N. Campbell
Anthony E. Cappellitti
Michael E. Carpenter
Daniel G. Carr
Michael W. Cash
Hsiu-Mei Chang
Sigen Chen
Peggy Cheng
Kuei-Hsia R. Chu
Christopher J. Claus
Mary L. Corbett
Deanna L. Crist
Michael T. Curtis
Mujtaba H. Datoo
Anne M. DelMastro
Lisa A. Doedtman
Andrew J. Doll
Jeffrey L. Dollinger
Jeffrey D. Donaldson
Peter F. Drogan
Pierre Drolet
David M. Elkins
Paul E. Ericksen
Lynne W. Faucher
Matthew G. Fay
Judith M. Feldmeier

*Part 6 (cont'd)*

Kelly F. Fogarty
Russell Frank
Douglas E. Franklin
Cynthia J. Friess
Nathalie Gamache
Christopher H. Geering
Donna L. Glenn
Marc C. Grandisson
William A. Guffey
Timothy J. Hansen
Bradley A. Hanson
Lise A. Hasegawa
Matthew T. Hayden
Barton W. Hedges
Noel M. Hehr
Renee J. Helou
Mary B. Hemerick
Thomas E. Hinds
Thomas A. Huberty
John F. Huddleston
Jeffrey R. Hughes
Jason Israel
Patrick C. Jensen
Anita J. Johnson
Daniel K. Johnson
Kurt J. Johnson
Mark R. Johnson
Marvin F. Johnson
Jean M. Karnick
Charles N. Kasmer
Anthony N. Katz
Daniel R. Keddie
Steven A. Kelner
Timothy P. Kenefick
George A. Kish
Terry A. Knull
Elizabeth Kolber
Debra K. Kratz
Terri C. Kremenski
Mary C. Kroggel
Cheung S. Kwan
Elaine Lajeunesse
Rita Ann B. Lamb
Marc LaPalme
David L. Larson
Doris Lee
Scott J. Lefkowitz
Louise L. Legros
Elizabeth A. Lemaster
Julie Lemieux-Roy
Deanne C. Lenhardt
Richard S. Light
Edward A. Lindsay
Stephanie J. Lippl
Ronald P. Lowe, Jr.
Christopher J. Luker
James M. Maher
Daniel J. Mainka
Betsy F. Maniloff
Anthony L. Manzitto
Gabriel O. Maravankin
Galina Margulis
Anthony G. Martella, Jr.
Kelly J. Mathson
Robert D. McCarthy
Richard T. McDonald
Heather L. McInstosh
Kathleen A. McMonigle
Van A. McNeal
Lynne S. McWithey
Brian J. Melas
Timothy Messier
Paul A. Mestelle
Paul W. Mills
Raymond D. Muller
David Y. Na
Donna M. Nadeau
Douglas W. Oliver
Richard A. Olsen
Melinda H. Oosten
Todd F. Orrett
Paul S. Osborn
Nathalie Ouellet
Mary Beth O'Keefe
Maureen D. O'Keefe
Jennifer J. Palo
Charles C. Pearl, Jr.
Edward F. Peck
Daniel B. Perry
William Peter
Beverly L. Phillips
Daniel C. Pickens
Glen-Roberts Pitruzzello
Joseph W. Pitts
Daniel A. Powell
Charlene M. Pratt
Mark Priven

*Part 6 (cont'd)*

Richard B. Puchalski
Eduard J. Pulkstenis
Robert E. Quane III
John F. Radwanski
Frank J. Rau, Jr.
Yves Raymond
Al J. Rhodes
Andrew T. Rippert
Douglas S. Rivenburgh
Sallie S. Robinson
Paul J. Rogness
James J. Romanowski
James B. Rowland
Jean Roy
David L. Ruhm
Kenneth W. Rupert, Jr.
James V. Russell
Maureen S. Ruth
Melodee J. Saunders
Letitia M. Saylor
Marilyn E. Schafer
Michael B. Schenk
Peter R. Schwanke
Jeffery J. Scott
Gregory R. Scruton
Jonathan N. Shampo
James J. Smaga
Douglas W. Stang
Katie Suljak
Colleen M. Sullivan
Collin J. Suttie
Scott J. Swanay
Jeanne E. Swanson
Todd D. Tabor
Joy Takahashi
Charles F. Toney
Michael J. Toth
Son T. Tu
Patrick N. Tures
Charles E. VanKampen
David B. Van Koevering
Phillip C. Vigliaturo
William E. Vogan
Joseph W. Wallen
Stephen D. Warfel
Joyce A. Weisbecker
Carol B. Werner
William R. Wilkins
Marcia C. Williams
William M. Wilt
John S. Wright
Gerald T. Yeung
Claude D. Yoder
Benny S. Yuen
Barry Zurbuchen
Ralph T. Zimmer

*Part 8*

Kristen M. Albright
Rebecca C. Amoroso
Richard R. Anderson
Katherine Barnes
Alicia E. Bowen
Mark L. Brannon
J. Eric Brosius
Anthony J. Burke
Scott K. Charbonneau
John S. Chittenden
Peter J. Collins
Raymond V. Debs
Victor G. dos Santos
William F. Dove
Michael C. Dubin
Maribeth Ebert
Charles C. Emma
Philip A. Evensen
Kerry L. Fitzpatrick
Scott F. Galiardo
Andrea Gardner
Bruce R. Gifford
Richard S. Goldfarb
Charles T. Goldie
Edward M. Grab
Steven J. Groeschen
Carleton R. Grose
George M. Hansen
Steven T. Harr
Thomas G. Hess
Todd J. Hess
Keith D. Holler
Beth M. Hostager
Laura A. Johnson
Gerald S. Kirschner
Gilbert M. Korthals
Nancy E. Kot
David J. Kretsch
James W. Larkin
Eric F. Lemieux
Giuseppe F. LePera
William G. Main
Donald E. Manis
Suzanne Martin
Steven E. Math
Jeffrey F. McCarty
Robert L. Miller
John H. Mize
Kelly L. Moore
Michelle M. Morrow
David A. Murray
Antoine A. Neghaiwi
Kwok C. Ng
Keith R. Nystrom
Margaret M. O'Brien
Laura A. Olszewski
Chandrakant C. Patel
Susan J. Patschak
Sarah L. Petersen
Mark W. Phillips
Lisa M. Ross
John M. Ruane, Jr.
Stephen P. Sauthoff
Margaret E. Seiter
Derrick D. Shannon
Rial R. Simons
Steven A. Skov
David A. Smith
Elizabeth L. Sogge
Lisa Steenken-Dennison
Leslie D. Svoboda
Eileen M. Sweeney
Barbara H. Thurston
Susan M. Treskolasky
Peter S. Valentine
Jennifer A. Violette
Christopher P. Walker
Monty J. Washburn
Scott P. Weinstein
Leigh F. Wickenden
Teresa J. Williams
John M. Woosley
Chung-Ye S. Yen
Sheng H. Yu

*Part 8C*

Annie Blais
Benoit Carrier
Jean Cloutier
Marie-Julie Demers
Shawn F. Doherty
Odile Goyer
Bradley A. Granger
Mathieu Lamy
France LeBlanc Heidi
J. McBride
Andre Perez
Denis Poirier
Robert Potvin

*Part 10*

Guy A. Avagliano
Anthony J. Balchunas
Douglas L. Beck
Nathalie Bégin
Xavier Benarosch
Lloyd J. Bouchard
Christopher K. Bozman
Paul A. Bukowski
Martin Cauchon
Denis Cloutier
Michael A. Coca
Jeffrey R. Cole
Charles Cossette
Martin L. Couture
Robert J. Curry
Francois Dagneau
Patrick K. Devlin
Stephen R. DiCenso
Kevin G. Dickson
Michel Dionne
Francois Dumas
Bradley C. Eastwood
John W. Ellingrod
James Ely
George Fescos
David A. Foley
France Fortin
Louis Gariepy
John F. Gibson
Susan M. Gozzo
Eric L. Greenhill
Farrokh Guiahi
Diane K. Hausserman
James S. Higgins
Keith D. Holler
George A. Hroziencik
Anthony Iafrate
Joanne K. Ikeda
Kathleen M. Ireland
Peter James
Changseob J. Kim
Bryan J. Kincaid
Richard O. Kirste
Ronald T. Kozlowski
John A. Lamb
Jean-Marc Léveillé
Paul R. Livingstone
Mark J. Mahon
Blaine C. Marles
Burton F. Marlowe
Liam M. McFarlane
Cassandra M. McGill
William T. Mech
Charles B. Mitzel
Richard B. Moncher
Todd B. Munson
Kathleen C. Nomicos
Charles P. Orlowicz
Donald D. Palmer
Jacqueline E. Pasley
Karen L. Pehrson
Timothy B. Perr
Richard W. Prescott
Kenneth P. Quintilian
Kay K. Rahardjo
Srinivasa Ramanujam
Scott E. Reddig
Sharon K. Robinson
Diane R. Rohn
Allen D. Rosenbach
Stuart G. Sadwin
Yves Saint-Loup
Leigh Saunders Oates
Edmund S. Scanlon
Gordon L. Scott
Marie Sellitti
Vincent M. Senia

*Part 10 (cont'd)*

Christy L. Simon
Christopher M. Smerald
Tom A. Smolen
David B. Sommer
Douglas N. Strommen
Marianne Teetsel
Michael A. Visintainer
Sebastian Vu
Bryan C. Ware
Marjorie C. Weinstein
John P. Welch
Kevin Wick
Gnana K. Wignarajah
Gregory S. Wilson

The following candidates were admitted as Fellows and Associates at the November 1992 meeting as a result of their successful completion of the Society requirements in the May 1992 examinations.

FELLOWS

Rebecca C. Amoroso
Anthony Joseph Balchunas
Douglass L. Beck
Nathalie Bégin
Martin Cauchon
Denis Cloutier
Jeffrey Roger Cole
Charles Cossette
Robert John Curry
Francois Dagneau
Patrick K. Devlin
John William Ellingrod
James Ely
David Alan Foley
France Fortin
John Foster Gibson
Susan Marie Gozzo
Eric L. Greenhill
Diane Kae Hausserman
Todd J. Hess
James S. Higgins
Keith Douglas Holler
George Alan Hroziencik
Kathleen Marie Ireland
Peter H. James
Brian Joseph Kincaid
Richard Owen Kirste
Ronald Thaddeus Kozlowski
David Jon Kretsch
John A. Lamb
Jean-Marc Léveillé
Blaine C. Marles
Burton F. Marlowe
Steven E. Math
Liam Michael McFarlane
William Theodore Mech
Charles Bradley Mitzel
Richard B. Moncher
Todd Burton Munson
Margaret M. O'Brien
Jacqueline Edith Pasley
Susan Jean Patschak
Robert Potvin
Richard Warren Prescott
Kenneth Paul Quintilian
Kay Kellogg Rahardjo
Srinivasa Ramanujam
Scott Edward Reddig
Sharon K. Robinson
Diane Renee Rohn
Edmund Sean Scanlon
Margaret Elizabeth Seiter
Marie Sellitti
Vincent M. Senia
Christy L. Simon
Christopher Michael Smerald
Douglas Nelson Strommen
Michael A. Visintainer
Sebastian Vu
Christopher Patrick Walker
Marjorie Cindy Weinstein
Gregory S. Wilson

## ASSOCIATES

Daniel Navarro Abellera
Ann Louise Alnes
Nancy Lee Arico
George Peter Bradley
Donna Dionne Brasley
Ward M. Brooks
Linda Jean Burrill
Anthony E. Cappelletti
Mary Laurene Corbett
Jeffrey Lawrence Dollinger
Jeffrey David Donaldson
Paul Edmund Ericksen
Lynne Woody Faucher
Judith Michalojko Feldmeier
Russell Frank
Laurence B. Goldstein
Sandra Kathleen Halpin
Bradley A. Hanson
Renee Helou
Jeffrey Robert Hughes
Daniel Robert Keddie
Stephen E. Lehecka
Stephanie Jean Lippl
James Michael Maher
Galina Margulis
Heather Lynn McIntosh
Van Allen McNeal
Raymond D. Muller
Mary Beth O'Keefe
Douglas W. Oliver
Richard Alan Olsen
William Oostendorp
Todd Franklin Orrett
Joseph Martin Palmer
Jennifer Joan Palo
Daniel Anthony Powell
Jean Roy
Maureen Schaller Ruth
Melodee Jane Saunders
Jeffery Jay Scott
Gregory R. Scruton
James J. Smaga
Scott Jay Swanay
Charles French Toney II
Michael J. Toth
James F. Tygh
David Brad Van Koevering
Phillip C. Vigliaturo
William E. Vogan
Stephen Douglas Warfel
Joyce Ann Weisbecker
Ralph T. Zimmer

The following is the list of successful candidates in examinations held in November 1992.

*Part 3B*

Bruce J. Adams
Elise M. Ahearn
Michael J. Alexander
Denise M. Ambrogio
Bijoy Anand
Julie A. Anderson
Mark B. Anderson
Adeline T. Anggelico
Steven D. Armstrong
Aeran A. Atlas
Evan C. Ayala
Karen L. Babitt
Melissa M. Bados
Phillip W. Banet
Amy L. Baranek
Michael W. Barlow
Karen L. Barrett
Karen E. Bashe
Linda S. Baum
Andrew S. Becker
David J. Belany
Joseph M. Bernardi
Robert C. Birmingham
Nicole P. Bitros
Linda J. Bjork
Carol A. Blomstrom
Douglas J. Bradac
David J. Braza
Charles Brindamour
Audrey W. Broderick
Martha E. Bronson
Robert F. Brown
Stephen J. Bruce
Kirsten R. Brumley
Joanne E. Burk
Elise S. Burns
Sharon L. Cage
Sandra J. Callanan
Michael V. Campbell
Janet P. Cappers
Douglas A. Carlone
Francis D. Cerasoli
Joseph G. Cerreta
Whye-Loon Chan
Daniel G. Charbonneau
Lisa C. Chen
Stephen D. Clapp
Scott R. Clark
Lisa V. Clarke
Susan M. Cleaver
Jeffrey A. Clements
Robert A. Clyborne
Brian R. Coleman
Pamela M. Corey
Dino F. Costabile
Brenda K. Cox
Gregory E. Daggett
Ted W. Daniel
Smitesh Dave
Timothy M. DiLellio
Behram M. Dinshaw
Melodee S. Dixon
Patricia L. Drajin
Martin W. Draper
Sara P. Drexler
Stephanie S. Dubose
Patrick W. Duncan
Denis Durand
Jeffrey S. Ellis
Lisa D. Ely
Keith A. Engelbrecht
Juan Espadas
Todd E. Fansler
Daniel J. Flick
Richard Y. Fong
Donovan M. Fraser
Bethany L. Fredericks
Kevin J. Fried
Michelle R. Garnock
Kathy H. Garrigan
Carina F. Glasgow
Mary T. Glaudell
Mark A. Gorham
Allen J. Gould
Jeffrey S. Goy
Kristin J. Grimshaw
Denise H. Guluk
John A. Hagglund
Francis G. Hall, Jr.
Faisal O. Hamid

*Part 3B (cont'd)*

Alessandrea C. Handley
William D. Hansen
Harry K. Hariharan
David S. Harris
Scott E. Haskell
Michael B. Hawley
Matthew T. Hayden
Daniel J. Henderson
Tina M. Henninger
Christopher T. Hochhausler
David E. Hodges
Amy L. Hoffman
Tracy L. Hoffman
Po-Wo Hsieh
Joseph M. Izzo
Randall A. Jacobson
David R. James
Suzanne M. James
Kelly A. Jensen
Jean M. Karnick
Claudine H. Kazanecki
Dennis J. Keegan
Lowell J. Keith
Thomas P. Kenia
Lisa M. Kerns
Michael B. Kessler
Robert W. Kirklin
James J. Konstanty
Dawna L. Koterman
Robert E. Krulish
Leonard L. Kruse
Cheung S. Kwan
Bertrand J. LaChance
Douglas H. Lacoss
Stephanie J. Ladiana
Salvatore T. LaDuca
Julia M. Lavolpe
Lawrence K. Law
Thomas V. Le
Thomas C. Lee
Neal M. Leibowitz
James P. Leise
Isabelle Lemay
Adam M. Lesser
Teresa L. Lett
Michael Leybov
Michael Lipkin
Linda R. Lisi
Timothy D. Logie
Nora J. Lovall
Cara M. Low
Susan I. Lynch
Kelly A. Lysaght
James M. MacPhee
Barbara D. Majcherek
Robert G. Mallison, Jr.
Betsy F. Maniloff
Kelly J. Mathson
Claudia A. McCarthy
Deborah L. McCrary
Smith W. McKee
Kelly S. McKeethan
Leslie A. McMahon
Lisa R. McNeal
James C. McPherson
Stephanie J. Michalik
Stephen J. Mildenhall
Michael J. Miller
Beth K. Millington
Scott P. Monard
Robert J. Moser
Kevin T. Murphy
Jennifer L. Nelson
Heather L. Nemeth
Denis P. Neumann
Khanh K. Nguyen
Steven A. Nichols
Douglas K. Nishimura
Chris M. Norman
Sandra F. Norowitz
Steven B. Oakley
Marie A. Olon
Stacey L. Otterson
Jill E. O'Dell
Marianne E. Papay
Thomas Passante
Javanika Patel
Tracie L. Pencak
Jennifer L. Pepin
David J. Persik
Mark A. Piske
Joseph W. Pitts
Matthew H. Price
Cindy Q. Qiu
Brentley J. Radeloff
William D. Rader, Jr.
Kimberly E. Ragland
Alicia M. Ransom

*Part 3B (cont'd)*

Charles J. Reichardt, Jr.
Meredith G. Richardson
Janelle P. Ridder
Brad E. Rigotty
Gail S. Rohrbach
Luis Romero
Jay A. Rosen
Jean-Denis Roy
David L. Ruhm
Sean W. Russell
Shama S. Sabade
Asif M. Sardar
Barbara A. Satsky
Rebecca A. Schafer
Christine E. Schindler
Annmarie Schuster
Terri L. Schwomeyer
John P. Scott
Terry M. Seckel
Michael Shane
George L. Shields
Nathan I. Shpritz
Michael N. Singer
Laura M. Smith
Michele L. Spale
Jonathan C. Stavros
D. Gregory Stitts
Shelley A. Stone
Lauren M. Stump
Elizabeth A. Sullivan
Patricia A. Sullivan
Roxann P. Swenson
Ming Tang
Judith D. Teglas
Steve D. Tews
Amy B. Treciokas
Beth S. Tropp
Mary H. Vale
Tim A. Vargo
Leslie A. Vernon
Edward H. Wagner
Robert A. Walsh
Edith A. Wendell
Jeffrey D. White
Steven B. White
Wyndi S. White
Ellen G. Wiener
Bruce P. Williams
Jennifer N. Williams
William M. Wilt
Curtis W. Withers
Kah-Leng Wong
John S. Wright
Nancy E. Yarnall
Virginia R. Young
Philip A. Zakas
Michael R. Zarember
Darci L. Zelenak

*Part 4A*

Elise M. Ahearn
Joseph J. Allard
Kay L. Allen
Jennifer A. Andrzejewski
John A. Annino
Robert C. Birmingham
Barry E. Blodgett
Carol A. Blomstrom
Jodi L. Bohac
Josee Bolduc
Edmund L. Bouchie
Lori M. Bradley
Tobias E. Bradley
Betsy A. Branagan
Glen R. Bratty
Robert F. Brown
Michelle L. Busch
Tara E. Bush
Douglas A. Carlone
Francis D. Cerasoli
Elina L. Chachkes
Joyce Chen
Gary C. Cheung
Sally M. Cohen
Kevin A. Cormier
Brian C. Cornelison
Sandra Creaney
Michael T. Curtis
Smitesh Dave
Jeffrey W. Davis
Catherine L. DePolo
Sharon D. Devanna
David L. Drury
Stephanie S. Dubose
Nathalie Dufresne
Stephen C. Dugan
Sarkis M. El-Zein
Bruce D. Fell
John R. Ferrara
Stephen C. Fiete
Daniel B. Finn
William P. Fisanick
Daniel J. Flick
Sy Foguel
Heather A. Ford
Lilane L. Fox
Keith E. Friedman
Gary J. Ganci
Christian Gaouette
Robert J. Garbus
Perriann R. Garcia
Justin G. Gensler
Eric J. Gesick
Barbara B. Glasbrenner
Lynn E. Golas
Chris D. Goodwin
Patricia A. Gorski
John E. Green
Steven A. Green
Daniel E. Greer
Scott J. Hartzler
Curtis D. Harvey
Douglas J. Hatlestad
Lisa K. Hiatt
Anne M. Hoban
Jason N. Hoffman
Dave R. Holmes
Eric A. Hoppe
Robert J. Hopper
Chaudhry M. Ishaq
Paul A. Johnson
Ira M. Kaplan
Brian D. Kemp
Martin T. King
Craig W. Kliethermes
Louis K. Korth
Kimberly A. Kracht
Karen M. Kulchyski
Cheung S. Kwan
Patrick P. Lacasse
Bertrand J. LaChance
Matthew G. Lange
Steven W. Larson
Khanh M. Le
Betty F. Lee
Paul B. LeStourgeon
Philip Lew
Yuan Long Liu
Lee C. Lloyd
William R. Maag
Barbara D. Majcherek
David K. Manski
Christopher M. Mariani
Albert Maroun
Meredith J. Martin
Tracey L. Matthew
Archibald G. Mattis
Bonnie C. Maxie

*Part 4A (cont'd)*

Phillip E. McKneely
Jeffrey A. Mehalic
Scott M. Miller
Stephen A. Moffett
Lisa J. Moorey
Jonathan M. Moss
Robert J. Moss
Karen E. Myers
Milary N. Olson
Thomas Passante
Lisa M. Pawlowski
Chantal Pelletier
Jeremy P. Pecora
Claude Penland IV
Anne M. Petrides
Michael W. Phillips
Michael L. Pisula
Thomas L. Poklen, Jr.
Ni Qin Feng
Mark S. Quigley
Thomas O. Rau
Ellen J. Respler
Dennis L. Rivenburgh, Jr.
Sallie S. Robinson
Rita L. Rogers
David M. Savage
Daniel V. Scala
Cindy R. Schauer
Suzanne E. Schoo
Annmarie Schuster
Steven G. Searle
Jerelyn K. Seeger
Jill C. Sidney
M. Kate Smith
Gregory T. Snider
Sandra L. Spiroff
Scott D. Spurgat
Kenneth W. Stam
Tracey A. Stark-Beldere
Scott T. Stelljes
Michael J. Steward II
Stephen J. Streff
Brian K. Sullivan
Frank Tancredi
Judith Teglas
Jennifer M. Tornquist
Turgay F. Turnacioglu
Michael O. VanDusen
Elayne M. Vargo Moxon
Trent R. Vaughn
Jennifer A. VonSchaven
Wittie O. Wacker
Galen L. Wadzinski
Michael A. Wallace
Lisa Marie Walsh
Caroline Ward
Erica L. Weida
Kirby W. Wisian
Stephen K. Woodard
Jeffrey F. Woodcock
Linda Yang
Mindy Yu
Jeffery M. Zacek
Fengming Zhang
Steven B. Zielke
Robin Zinger

*Part 4B*

Vicki L. Agerton
Timothy K. Allen
Jason C. Alleyne
Timothy P. Aman
Steven D. Armstrong
John J. Ascencio
Bruce E. Bach
Kendra D. Barnes
Rajesh K. Barnwal
David M. Baxter
Martin Bilodeau
Gina S. Binder
Latisha L. Boothe
Edmund L. Bouchie
Lee M. Bowron
Tobias E. Bradley
David J. Braza
Laura G. Brill
Linda M. Brockmeier
James A. Bull
Michelle L. Busch
Tara E. Bush
Linda E. Callas
Victoria J. Carter

*Part 4B*

Thomas E. Cerulli
Peter T. Chang
Wei Fun W. Chang
Sharon L. Chapman
Joyce Chen
Gary C. Cheung
Ling-Jyh Chiou
Soo H. Choo
Gregory C. Christensen
Kathy A. Christensen
Juite Chuang
Clark R. Chumley
Frank S. Conde
David G. Cook
Sharon A. Crosson
Malcolm H. Curry
Michael T. Curtis
Kenneth S. Dailey
Jeffrey W. Davis
Raymond V. DeJaco
Catherine L. DePolo
Dina M. Deschino
Sean R. Devlin
Behram M. Dinshaw
Ronald R. Dionne
Patricia J. Donnelly
Kimberly J. Drennan
Barry P. Drobes
Mary F. Drueke
Denis Dubois
Louis Durocher
Anthony D. Edwards
Ellen E. Evans
Bruce D. Fell
John R. Ferrara
Daniel B. Finn
Ginda K. Fisher
Daniel J. Flick
Jeffrey M. Forden
Walter H. Fransen
Keith E. Friedman
Richard A. Fuller
Scott F. Fuller
Wang-Tat D. Fung
Serge Gagne
Gary J. Ganci
Carol Ann Garney
Micah R. Gentile
Thomas P. Gibbons
Bernard H. Gilden
Mary K. Gise
Barbara B. Glasbrenner
Yethun Goh
Annette J. Goodreau
Patricia A. Gorski
George J. Gourdourakos
Michael D. Green
Steven A. Green
John E. Green
Joseph P. Greenwood
Michael K. Griffin
Robert W. Guth
Steven K. Haine
Richard J. Haines
Scott W. Hanson
David S. Harris
Michael A. Harris
Scott J. Hartzler
Curtis D. Harvey
Gary M. Harvey
Michael B. Hawley
Peter A. Heinrichs
Daniel F. Henke
Betty-Jo Hill
John D. Hill
Jason N. Hoffman
Todd H. Hoivik
Keith D. Holler
Dave R. Holmes
Wayne Hommes
Yu-Hui A. Huang
Marie-Josee Huard
Corine Huey
Brian L. Ingle
Randall A. Jacobson
Fong-Yee J. Jao
Scott L. Johnson
Gregory K. Jones
Steven J. Jordan
Ira M. Kaplan
Joseph P. Karlovich
Michael B. Kessler
Hyung-Shim Kim
Deborah M. King
Martin T. King
Omar A. Kitchlew
Craig W. Kliethermes
Robert J. Knadler
Rachna Kohli

*Part 4B (cont'd)*

Jonelle A. Kohne
Paul W. Kollner
Andrew M. Koren
Eleni Kourou
Gary R. Kratzer
Dean F. Kruger
Ignace Y. Kuchazik
Patrick P. Lacasse
Jin-Mei J. Lai
Kunjung Lai
Siu-Wai Lai
John B. Landkamer
Frank A. Laterza
Robin R. Lee
Paul B. LeStourgeon
John N. Levy
Pak-Chuen Li
Karen A. Liholt
Yachin Lin
Kim D. Litwack
William M. Londeree
Richard B. Lord
Sak-Man Luk
Xinhong Luo
Allen S. Lynch, Jr.
James P. MacDougall
Gary P. Maile
Keith M. Marcus
Laura S. Marin
Joseph Marracello
Leslie A. Martin
Peter R. Martin
Michele A. Mathieu
Angela T. Mazzaferro
Wendy M. Mazzarella
Robert D. McCarthy
Cassandra M. McGill
Sean L. McIntosh
Christopher P. McMann
Arthur J. Mees, Jr.
James R. Merz
Marci A. Meyer
Scott M. Miller
Susan A. Minnich
Quynh-Nhu T. Morse
Matthew S. Mrozek
Turhan E. Murguz
Iris A. Nance
Melissa J. Neidlinger
Gary R. Nidds
Lauren E. Norton
Steven B. Oakley
Denise R. Olson
Milary N. Olson
Ajay Pahwa
Dmitry Papush
Nicholas H. Pastor
Rick S. Pawelski
Mark Paykin
Priyantha L. Perera
John M. Pergrossi
Luba Pesis
Kevin W. Picone
Donna M. Pinetti
Mark A. Piske
Alan D. Potter
Matthew H. Price
Arlie J. Proctor
John K. Punzak
Ni Qin Feng
Mary S. Rapp
Thomas O. Rau
Patrick J. Reilly
Mary B. Rios-Gandara
Christopher R. Ritter
Jeremy Roberts
Michelle N. Rodriguez
Constance D. Rogers
Hal D. Rubin
John C. Ruth
Rajesh V. Sahasrabuddhe
Romel G. Salam
Sujata S. Sanghvi
Matt J. Schmitt
Suzanne E. Schoo
Michael F. Schrah
Michael R. Schummer
Gordon L. Scott
Patrick J. Seaman
Jerelyn K. Seeger
Michael Shane
Jennifer M. Shantz
Stacy L. Shimizu-Hall
Laura E. Siegel
Cindy W. Smith
Jeffery J. Smith
Katherine R. Smith
L. Kevin Smith
Halina H. Smosna

*Part 4B (cont'd)*

David E. Sowers
Scott D. Spurgat
Beth A. Stahelin
Robert P. Stahnke
Glenda M. Stalkfleet
P.J. Eric Stallard
Michael J. Steward II
Curt A. Stewart
Lori E. Stoeberl
William J. Stone
Stephen J. Streff
Steven J. Symon
Jinhua Tao
Sandra L. Theile
Sadhana Tiwari
Jennifer M. Tornquist
Kai L. Tse
Scotty M. Tucker
Jeffrey R. Turcotte
Arthur J. Turner
Michael O. VanDusen
Charles J. Veres
Michel F. Viau
Benjamin A. Walden
Isabelle T. Wang
Shaun Wang
Lynne K. Wehmueller
Erica L. Weida
Elizabeth A. Wentzien
Scott Werfel
L. Alicia Williams
Ronald J. Williams
Kirby W. Wisian
Trevar K. Withers
Brandon L. Wolf
Jeffrey F. Woodcock
Rick A. Workman
John Yannis Yatracos
Xiang Zhang
Robin Zinger
Edward J. Zonenberg

*Part 5*

John S. Alexander
John P. Alltop
Larry D. Anderson
Martin S. Arnold
Claudia M. Barry
Gina S. Binder
Lisa A. Bjorkman
Christina M. Bond
Conni J. Brown
Mark E. Burgess
Sandra L. Cagley
Pamela J. Cagney
Mark W. Callahan
Maureen A. Cavanaugh
Jean-Francois Chalifoux
Hong Chen
Suhui Chen
Bina S. Cherian
Christopher J. Claus
William B. Cody
Frank S. Conde
Kirsten J. Costello
Christopher G. Cunniff
Raymond V. DeJaco
Joseph A. DiBiase
Patricia J. Donnelly
Kevin G. Donovan
Robert G. Downs
Sara P. Drexler
David L. Drury
Martin A. Epstein
Dianne L. Estrada
James G. Evans
Joseph G. Evleth
Michael A. Falcone
Richard B. Federman
John D. Ferraro
Daniel B. Finn
Jeffrey M. Forden
Mark R. Frank
J'ne E. Furrow
Paul Gauthier
Justin G. Gensler
Margaret W. Germani
Nicholas P. Giuntini
Michael F. Glatz
Stewart H. Gleason
Karl Goring
John W. Gradwell
Michael K. Griffin
Lynne M. Halliwell

*Part 5*

Brian D. Haney
Elizabeth E. Hansen
Anne M. Hoban
Brook A. Hoffman
Eric J. Hornick
Linda M. Howell
Jill C. Johnson
Gregory K. Jones
Steven J. Jordan
Barbara L. Kanigowski
John P. Kannon
Anthony N. Katz
Mark J. Kaufman
Brandelyn C. Klenner
Brian R. Knox
Louis K. Korth
Debra K. Kratz
Thomas F. Krause
Brian S. Krick
Kenneth A. Kurtzman
Edward M. Kuss
Kirk L. Kutch
Steven M. Lacke
Blair W. Laddusaw
John M. Lamendola
Robert J. Larson
Thomas V. Le
John P. Lebens
Helen P. LeClair
Chi Hei Lee
Daniel Leff
Glen A. Leibowitz
Charles R. Lenz
Paul Liu
Nora J. Lovall
Barbara S. Mahoney
Michael B. Masters
Robert F. Maton
Tracey L. Matthew
Camley A. Mazloom
William R. McClintock
David W. McLaughry
Stephen J. Mildenhall
Mark J. Moitoso
Kimberly A. Moran
Benoit Morissette
Michael J. Moss
Turhan E. Murguz
Jarow G. Myers
Karen E. Myers
Kathleen V. Najim
Marc F. Oberholtzer
Leo M. Orth, Jr.
Mark A. O'Brien
James D. O'Malley
James A. Partridge
Wende A. Pemrick
Claude Penland IV
Michael C. Petersen
Glen-Roberts Pitruzzello
Dale S. Porfilio
Kiran Rasaretnam
Natalie J. Rekittke
Cynthia L. Rice
Donald A. Riggins
Brad M. Ritter
Christopher R. Ritter
John W. Rollins
Thomas A. Ryan
Christina L. Scanell
Lawrence M. Schober
Michael J. Scholl
Jay M. Schwartz
Craig J. Scukas
Kevin H. Shang
Jeffrey P. Shirazi
Jeffery J. Smith
Caroline B. Spain
Christina L. Staudhammer
Nathan R. Stein
Christopher M. Steinbach
Scott T. Stelljes
Kevin D. Strous
Josephine T. Teruel
Daniel A. Tess
Marie-Claire Turcotte
Laura M. Turner
Robert C. Turner
Eric Vaith
Mark D. van Zanden
Martin Vezina
Lisa Marie Walsh
Petra L. Wegerich
David L. Whitley
Michael J. Williams
Tammy L. Wimer
Mark L. Woods
Guangjian Zhu

*Part 5A*

Michael J. Anstead
Mohammed Q. Ashab
Nathalie J. Auger
Andre Beaulieu
Bruce J. Bergeron
Lisa A. Brown
Bruce D. Browning
Peter V. Burchett
Randall A. Chaffinch
Henry H. Chen
Richard M. Chiarini
Thomas J. Chisholm
Michael J. Christian
Theresa A. Christian
Darrel W. Chvoy
Danielle G. Comtois
Sheila C. Cooley
Beverly E. Cordner
Robert E. Davis
Ronald M. Dennis
Sean R. Devlin
Annette M. Eckhardt
Charles V. Faerber
Alexander Fernandez, Jr.
Jean-Pierre Gagnon
Nathalie Gamache
Michael J. Goetsch
M. Harlan Grove
Julie K. Halper
Shohreh Heshmati
Anna M. Hnateyko
David E. Hodges
Wayne Hommes
Alex Y. Hsiao
Po-Wo Hsieh
Thomas A. Huberty
Vincent H. Jackson
Paul J. Johnson
Kathryn A. Karoski
William J. Keros
W. Keith Landry
Matthew G. Lange
Manuel Alberto T. Leal
Betty F. Lee
Thomas L. Lee
Marc E. Levine
Hsin-Hui G. Lin
Steven C. Lin
Kevin E. Litton
John Lum
Victoria S. Lusk
Tai-Kuan Ly
Lennette U. Maala
Joseph A. Malsky
Jay E. McClain
Camille D. Minogue
Thomas M. Mount
Hiep T. Nguyen
Russell R. Oeser
Pierre Parenteau
Leslie C. Pelecovich
Armand Principato
Jennifer L. Reisig
Jeremy Roberts
Paul J. Rogness
Joseph F. Rosta
Caroline Roy
Peter A. Royek
Anthony V. Rizzuto
Asif M. Sardar
Matt J. Schmitt
Lisa A. Sgaramella
Gena A. Shangold
Linda M. Sowter
Calvin C. Spence, Jr.
Scott D. Spurgat
Anthony T. Stanford
Brian J. Sullivan
Siu Cheung S. Szeto
Ngoc H. Tran
Steven D. Umansky
Mary E. Waak
Robert J. Walling III
Joel D. Whitcraft
Thomas J. White
David S. Wolfe
Doug A. Zearfoss

*Part 5B*

Cheng-Te Liang

*Part 7*

Shawna S. Ackerman
Jonathan D. Adkisson
Rhonda K. Aikens
Christopher R. Allan
Craig A. Allen
Scott C. Anderson
Michael E. Angelina
William P. Ayres
Barry L. Bablin
Robert S. Ballmer
Timothy J. Banick
Philip A. Baum
John A. Beckman
Douglas S. Benedict
Gary Blumsohn
Ann M. Bok
Elizabeth S. Borchert
Maurice P. Bouffard
Kevin M. Brady
Richard A. Brassington
Tracy L. Brooks-Szegda
Pamela A. Burt
Richard F. Burt, Jr.
John F. Butcher II
Mark W. Callahan
Michael E. Carpenter
Kristi I. Carpine-Taber
Benoit Carrier
Michael W. Cash
Tania J. Cassell
Kevin J. Cawley
Debra S. Charlop
Jo Ellen Cockley
Thomas V. Daley
Joyce A. Dallessio
David J. Darby
Karen L. Davies
Marie-Julie Demers
Shawn F. Doherty
Dean P. Dorman
Ronald R. Earls
Gregg Evans
Matthew G. Fay
George Fescos
David I. Frank
Douglas E. Franklin
Kai Y. Fung
James E. Gant
Donna L. Glenn
Ronald E. Glenn
Marc C. Grandisson
Bradley A. Granger
Russell H. Greig
William A. Guffey
Marc S. Hall
Paul J. Hancock
Timothy J. Hansen
Robert L. Harnatkiewicz
Christopher L. Harris
Lise A. Hasegawa
Lisa A. Hays
Barton W. Hedges
Noel M. Hehr
Mary B. Hemerick
Paul D. Henning
Thomas H. Highet
Robert J. Hopper
Bernard R. Horovitz
Sandra L. Hunt
John F. Janssen
Patrick C. Jensen
Kurt J. Johnson
Mark R. Johnson
Charles N. Kasmer
Janet S. Katz
Steven A. Kelner
Timothy P. Kenefick
Joseph P. Kilroy
Bradley J. Kiscaden
Michael F. Klein
Terry A. Knull
Elizabeth Kolber
Howard A. Kunst
David L. Larson
Michel Laurin
Lewis Y. Lee
Thomas L. Lee
Scott J. Lefkowitz
Elizabeth A. Lemaster
Deanne C. Lenhardt
David R. Lesieur
Aaron S. Levine
Richard S. Light
Shu C. Lin
Daniel J. Mainka
Stephen N. Maratea
Richard J. Marcks
Lawrence F. Marcus
Sharon L. Markowski

*Part 7 (cont'd)*

Kelly J. Mathson
Deann M. Mays
Robert D. McCarthy
Kathleen A. McMonigle
Conrad O. Membrino
Stephen V. Merkey
Paul A. Mestelle
Anne C. Meysenburg
Douglas H. Min
Michelle M. Morrow
Matthew C. Mosher
Giovanni A. Muzzarelli
Timothy O. Muzzey
David Y. Na
Mark Naigles
James F. Neary
Peter M. Nonken
Douglas J. Onnen
Melinda H. Oosten
Paul S. Osborn
Nathalie Ouellet
Charles C. Pearl, Jr.
Edward F. Peck
John R. Pedrick
Karen L. Pehrson
Miriam E. Perkins
Daniel C. Pickens
Janice L. Polofsky
Mark Priven
Cathy A. Puleo
Eduard J. Pulkstenis
Steven J. Regnier
Ellen J. Respler
Victor U. Revilla
Al J. Rhodes
Andrew T. Rippert
James J. Romanowski
David A. Rosenzweig
James B. Rowland
Kenneth W. Rupert, Jr.
David A. Russell
James V. Russell
Stephen P. Sauthoff
Letitia M. Saylor
Michael B. Schenk
Peter R. Schwanke
Nathan I. Shpritz
Jeffrey S. Sirkin
Carl J. Sornson
Victoria G. Stachowski
Douglas W. Stang
Michael J. Steward II
Richard A. Stock
Brian M. Stoll
Katie Suljak
Todd D. Tabor
Christopher Tait
Yuan-Yuan Tang
Dom M. Tobey
Glenn A. Tobleman
Thomas A. Trocchia
Patrick N. Tures
Robert W. VanEpps
Charles E. VanKampen
Robert J. Vogel
David M. Vogt
Joseph W. Wallen
Marcia C. Williams
Calvin Wolcott
Tad E. Womack
Cheng-Sheng P. Wu
Gerald T. Yeung
Claude D. Yoder
Edward J. Yorty
Jeffery M. Zacek
George H. Zanjani

*Part 9*

Kristen M. Albright
Karen L. Ayres
Bruno P. Bauer
Todd R. Bault
Lisa M. Besman
Betsy L. Blue
Lloyd J. Bouchard
Alicia E. Bowen
Christopher K. Bozman
Mark L. Brannon
Mark D. Brissman
J. Eric Brosius
David S. Cash
Janet L. Chaffee
Kasing L. Chung
Jean Cloutier
Peter J. Collins
Martin L. Couture
David B. Cox
Michael K. Daly
Guy R. Danielson
Manon Débigaré
Kevin G. Dickson
Michel Dionne
Norman E. Donelson
Michael C. Dubin
Francois Dumas
Bradley C. Eastwood
Maribeth Ebert
Charles C. Emma
Madelyn C. Faggella
James E. Fletcher
Ross C. Fonticella
Russell Frank
Louis Gariepy
Kim B. Garland
Bruce R. Gifford
Bradley J. Gleason
Richard S. Goldfarb
Linda M. Goss
Odile Goyer
Edward M. Grab
James W. Haidu
Leigh J. Halliwell
Renee Helou
Kathleen A. Hinds
Joanne K. Ikeda
Laura A. Johnson
Deborah E. Kenyon
Kevin A. Kesby
Gerald S. Kirschner
Timothy F. Koester
David R. Kunze
Frank O. Kwon
Benoit Laganiere
Alan E. Lange
Christopher Lattin
Marc-Andre Lefebvre
Stephanie J. Lippl
Katherine A. Mann
Galina Margulis
Leslie R. Marlo
James P. McNichols
Linda K. Miller
John H. Mize
Russell E. Moore
Francois Morin
Antoine A. Neghaiwi
William L. Oostendorp
Ann E. Overturf
Timothy A. Paddock
Chandrakant C. Patel
Sarah L. Petersen
Denis Poirier
Stuart Powers
Donna J. Reed
Daniel G. Roth
Jean Roy
Leigh Saunders Oates
Thomas E. Schadler
Gregory R. Scruton
Peter Senak
David M. Shepherd
David A. Smith
Elizabeth L. Sogge
David B. Sommer
Lisa N. Steenken-Dennison
John P. Stefanek
Edward D. Thomas
Richard D. Thomas
Michael Toledano
Charles F. Toney II
Susan M. Treskolasky
Stacy L. Trowbridge
Peter S. Valentine
Jennifer A. Violette
Bryan C. Ware
Scott P. Weinstein
Peter A. Weisenberger

*Part 9 (Cont'd)*

John P. Welch
L. Nicholas Weltmann, Jr.
Russell B. Wenitsky
Kevin Wick
Vincent F. Yezzi, Jr.
Sheng H. Yu
Ralph T. Zimmer

# NEW FELLOWS

(Admitted in May 1992)

# NEW ASSOCIATES

(Admitted in May 1992)
Group 1

# NEW ASSOCIATES

(Admitted in May 1992)
Group 2

# NEW FELLOWS

(Admitted in November 1992)

# NEW ASSOCIATES

(Admitted in November 1992)

# OBITUARIES

A. EDWARD ARCHIBALD
GEORGE Y. CHERLIN
JOHN W. CLARKE
MILES R. DROBISH
JAMES F. GILL
WILLIAM H. MAYER, JR.
JOSEPH M. MUIR
STEFAN PETERS
ALBERT Z. SKELDING

## A. EDWARD ARCHIBALD
1903—1992

A. Edward Archibald, an Associate of the Casualty Actuarial Society since 1930, a Fellow of the Society of Actuaries since 1931, and a member of the American Academy of Actuaries since 1965, died on May 8, 1992, at the age of 89.

When he received his Associateship designation, Mr. Archibald was working for Woodward, Fondiller & Ryan in New York City as an Associate Actuary. In 1933, he became an Actuary with the Volunteer State Life Insurance Company in Chattanooga, Tennessee, where he was promoted to Vice President and Actuary in 1944. He joined Investors Diversified Services, Inc. (IDS) of Minneapolis as Director of Management Controls, and was promoted to Vice President of the firm in 1958. In 1966, he was named President of Investors Syndicated Life Insurance and Annuity Corp. in Minneapolis. Beginning in 1967, he served as a director for several years. In 1969, Mr. Archibald returned to Lookout Mountain, Tennessee, and retired in 1972.

A native of Huron, Ontario, Canada, Mr. Archibald was a graduate of the University of Toronto. He was a member of the Lookout Mountain Presbyterian Church, the Downtown Kiwanis Club, Lookout Mountain Golf Club, and Fairyland Club.

Survivors include his wife, Dorothy Burns Archibald, of Lookout Mountain; a daughter, Ellen R. Archibald, of Charleston, West Virginia; and a brother, F.R. Archibald, of Naples, Florida.

## DR. GEORGE Y. CHERLIN
1924—1992

George Yale Cherlin, an Associate of the Casualty Actuarial Society since 1961, a Fellow of the Society of Actuaries since 1955, and a Member of the American Academy of Actuaries since 1965, died on August 5, 1992, at the age of 68.

Dr. Cherlin was also a Fellow of the Conference of Consulting Actuaries since 1982, and an Enrolled Actuary since 1976. In addition to his memberships in actuarial organizations, Dr. Cherlin also was a member of the American Mathematical Society and the Association for Computing Machinery Special Interest Group.

A former instructor in Mathematics at Rutgers University and the first individual to be awarded a Ph.D. in mathematics from the university (1951). Dr. Cherlin also received his bachelor and master of science degrees from Rutgers. He was active as a fund raiser for the university, served as president of the Class of 1946, and was a member of Phi Beta Kappa, Sigma Chi, and Beta Iota Lambda. During World War II he served in the United States Navy.

Dr. Cherlin joined Mutual Benefit Life in 1951 as an actuarial student. He served the company during the years 1951 to 1962, attaining the position of Assistant Mathematician. In 1962, Dr. Cherlin joined the National Health and Welfare Retirement Association, in New York City, as an actuary. He was promoted to Vice President and Actuary in 1964. He returned to Mutual Benefit Life in 1972 where he assumed the position of Associate Mathematician until 1976 when he was named Second Vice President and Actuary. In 1978, Dr. Cherlin became President of APL Business Consultants, Inc., in Newark, New Jersey. He continued his consulting career with STSC, Inc., in New York City, where he worked as a consulting actuary from 1982 until 1983.

Dr. Cherlin returned Mutual Benefit Life in 1983 as an Applications Systems Consultant. He was promoted to Actuary in 1986, a position he held until 1987. He then relocated to the Chicago area and joined the United Insurance Company of America. In 1988, he moved to Milwaukee, Wisconsin, and joined General Life Insurance as Vice President and Actuary. In 1992, he retired and moved to Mount Shasta, California.

A resident of Newark, New Jersey, for 24 years, Dr. Cherlin was born in New Haven, Connecticut, in 1924. Active in Newark civic affairs, Dr. Cherlin was a former chairman of the Professional Division of the United Appeals and a former president of the Newark Jaycees. He served in numerous volunteer capacities with the Robert Treat Council of the Boy Scouts. He was also associated with the New Jersey Symphony Chorus.

Dr. Cherlin is survived by his wife, Mary Elizabeth Monroe Cherlin, of Mt. Shasta; two sons, Gary Cherlin of Weed, California, and Gregory Cherlin of Princeton, New Jersey; four grandchildren; three sisters, Lillian Heimberg of Edison, New Jersey, Judith Sagotsky of Freehold, New Jersey, and Harriet Hunt of San Antonio, Texas.

## JOHN W. CLARKE
1916—1992

John W. Clarke, a Fellow of the Casualty Actuarial Society since 1949, died on April 10, 1992, in Missouri City, Texas. He was 76. He was born on February 24, 1916 in Kingston, New York.

Mr. Clarke earned an A.B. degree from Cornell University, a graduate degree from the Massachusetts Institute of Technology, and an LL.D. with high honors from the University of Connecticut. In addition, Mr. Clarke spent five years with the Army Air Corps.

Mr. Clarke began his career with the Travelers Insurance Company in 1937 and held the position of Associate Actuary when he left the firm in 1954. That year he moved to the Pan American Life Insurance Company where he rose to the position of Vice President, Actuary, and Controller. From 1956 until 1961, he worked for the Gulf Life Insurance Company and was named Senior Vice President. Mr. Clarke joined the General Reinsurance Life Corp. in 1962 and held the post of President. In 1966, he joined Hartford Life as Executive Vice President and was named President of the firm in 1967. He retired in 1972.

Mr. Clarke was a member of several professional and social organizations including the Casualty Actuarial Society, the Connecticut Bar Association, the Conference of Actuaries in Public Practice (now the Conference of Consulting Actuaries), and the American Academy of Actuaries. He was also a director of the Hartford Fire Insurance Company and its major subsidiaries.

## MILES R. DROBISH
1918—1991

Miles R. Drobish, a Fellow of the Casualty Actuarial Society since 1957 and a member of the American Academy of Actuaries since 1965, died on January 8, 1991, at the age of 73.

A native of San Francisco, California, Mr. Drobish attended the University of California, Los Angeles, and the University of California, Berkeley, where he received his bachelor's degree. He served in the United States Army for three years during World War II.

At the time he received his Fellowship designation, Mr. Drobish was working as a Statistician with the California Inspection Rating Bureau (now known as the Workers' Compensation Insurance Rating Bureau of California) in San Francisco. He was promoted to Superintendent in 1960, and Assistant Actuary in 1961, a position he held until his retirement in 1980.

Mr. Drobish was active in the Casualty Actuarial Society, serving on the Society's Public Relations Committee from 1976 through 1979, and on its Editorial Committee, Special Publications in 1980.

He is survived by a son, Denman Drobish, of San Francisco, and two grandchildren.

## JAMES F. GILL
## 1909—1992

James F. Gill, an Associate of the Casualty Actuarial Society since 1963, an Associate of the Conference of Consulting Actuaries since 1959, and a Member of the American Academy of Actuaries since 1966, died at home on September 27, 1992. He was 82 years old.

A graduate of the University of Pittsburgh, Mr. Gill worked for the National Association of Independent Insurers in Chicago, Illinois, for 25 years, beginning in 1948. He rose to the position of Vice President and Actuary, a position he held from 1967 until 1973.

In 1973, Mr. Gill joined the Westfield Companies in Westfield Center, Ohio, as Actuary. He retired from the firm in 1980. He then moved to Portland, Oregon, and worked as a consultant before retiring from actuarial practice in 1982.

Mr. Gill served on two committees in the Society, the Financial Review Committee in 1966, and the Committee on Annual Statement from 1967-1971. He authored two papers published in the *Proceedings.* His first paper, "An Approximation for the Testing of Private Passenger Liability Territorial Rate Levels Using Statewide Distribution of Classification Data," appeared in the 1964 *Proceedings* and his review of "Ratemaking Procedures for Automobile Liability Insurance" was published in the 1966 edition. Mr. Gill also moderated a panel discussion on "Loss Reserve Problems—Financial and Ratemaking" at the 1969 CAS Spring Meeting.

Mr. Gill is survived by his wife, Margaret, of Portland; a daughter, Mary Donevan; and a sister, Kathleen Gill.

## WILLIAM H. MAYER, JR.
1910—1990

William H. Mayer, Jr., an Associate of the Casualty Actuarial Society since 1936, died on December 27, 1990, of the complications from Parkinson's disease. He was 80.

Mr. Mayer worked his entire career at the Metropolitan Life Insurance Company, in New York City. When he became an Associate in 1936 he was working in the actuarial department. He was promoted in 1949 to group contract referee. He became the associate manager of Met Life's Group Contract Bureau in 1951, and was appointed manager in 1958.

In 1979, Mr. Mayer retired and resided in Dix Hills, New York, until his death. He is survived by his wife, Marilyn Mayer, of Dix Hills; and four sons: William, Peter, Richard, and Thomas; and an aunt.

## JOSEPH M. MUIR
1904—1989

Joseph M. Muir, an Associate of the Casualty Actuarial Society since 1957, died on September 13, 1989. He was 85.

In 1927, Mr. Muir joined the Mutual Insurance Rating Bureau and the Insurance Advisory Association in New York City where he worked for 42 years until his retirement in 1969. For more than 20 years he held the position of general manager for both organizations.

He earned his Bachelor of Science degree from Cooper Union in New York City, going to night school for seven and a half years.

In 1973, Mr. Muir moved from Haworth, New Jersey, to Boca Raton, Florida, where he lived until his death.

Mr. Muir is survived by his wife, Mabel J. Muir, of St. Augustine, Florida; a daughter, Grace E. Emden, also of St. Augustine; and a granddaughter, Debra J. Emden, of Metairie, Louisiana.

# DR. STEFAN PETERS
## 1909—1990

Dr. Stefan Peters, a Fellow of the Casualty Actuarial Society since 1941 and a Fellow of the Society of Actuaries since 1951, died on August 16, 1990. He was 81.

Dr. Peters was born in Erlangen, Bavaria, Germany, and received his degree in mathematics from the University of Berlin. After graduation, he worked in Berlin until the political climate provoked his emigration. He moved to Trieste, Italy, and spent some time with the Assicurazione Generali as a life actuary. It was in Italy that he met his wife, Renata. In a few years, he was again forced to move, this time to the United States.

Dr. Peters became an Associate in the Society in 1940 while he was working for the Compensation Insurance Rating Board in New York City as an assistant actuary. From this position, he was inducted into the Army of the United States and served his adopted country in enlisted status both at home and overseas in the Mediterranean Theater. This service qualified him for citizenship.

In 1945, Dr. Peters moved to Berkeley, California, and in 1948, he assumed the position of Lecturer with the Department of Statistics, then headed by Jerzy Neyman, at the University of California. In 1949, after the sudden death of Professor Albert Mowbray, FCAS, Dr. Peters took over the insurance classes in the School of Business Administration and was appointed Associate Professor. During this period, he also served as consulting actuary to West Coast Life Insurance Company.

His experience under fascist governments led Dr. Peters to resist imposition of the loyalty oath required of its faculty by the University of California in the early 1950s. After losing a courageous legal battle that went to the California Supreme Court, he moved back to the East Coast in 1952, and took a position with Morse and Seal in New York City as Actuary. He moved to Boston in 1953 and joined the firm of Connell, Price and Company. In 1959, he started several years' service with Arthur D. Little Consultants in Cambridge, Massachusetts. In 1974, Dr. Peters became the Chief Actuary for the Department of Insurance for the Commonwealth of Massachusetts, a position he held until his retirement in 1986.

Dr. Peters published three papers in the *Proceedings*, all on workers compensation insurance. In 1941, his paper, "A Method of Testing Classification Relativities," introduced to the *Proceedings* what R.A. Fisher called the "analysis of variances." The paper covered a specific application of the then relatively new formal testing of hypotheses and making of decisions under uncertainty.

Stefan Peters was admired by his many friends for successfully and gracefully overcoming severe problems in his long life. He was a quiet person whose warmth and friendliness were sometimes apparent only to the close observer, but were always clear to the many students and friends to whom he was ever pleased to give help. We who knew him warmly remember a good friend.

Dr. Peters is survived by his wife; a son, Matthew Peters, of Cambridge; and a daughter, Clara Simon, of Albany, New York.

## ALBERT Z. SKELDING
## 1897—1992

Albert Z. Skelding, a Fellow of the Casualty Actuarial Society since 1929 and a member of the American Academy of Actuaries since 1965, died on March 31, 1992, at the age of 95.

In 1925, Mr. Skelding became an Associate of the Society. He spent his entire career working for the National Council on Compensation Insurance (NCCI) in New York City. He started as an Associate Actuary and in 1934 he was promoted to Actuary. In 1950 he was named Assistant Manager, and in 1957 he was promoted to Associate General Manager of NCCI, a position he held until his retirement in 1961.

At NCCI, Mr. Skelding worked to gain approval of workers' compensation rate filings throughout the country. At his retirement party, he delighted well-wishers with his reminiscences of those attempts. Since the methods used to win approval from the state insurance departments were not always limited to logical and statistical presentations, his account was both historically interesting and hilarious.

Mr. Skelding held several volunteer positions within the Casualty Actuarial Society. He was elected to the Council of the CAS for a three-year term ending in 1942. He was elected Vice President of the Society from 1942 until 1944 and served as Secretary-Treasurer for 20 years, from 1953 until 1973. He was also active on the committee level, serving on the Examinations Committee, the Program Committee, and the Committee on Publications.

Mr. Skelding also contributed many important book reviews to the *Proceedings*. He reviewed the following publications: *Reversions and Life Interests; An Introduction to the Mathematics of Life Insurance; Compound Interest Tables; Tables for Calculating by Machine; Logarithms to 13 Places of Decimals;* and the *Cost of Compensation for the Year Ended June 30, 1928* (a Special Bulletin of the New York Department of Labor).

A lifelong resident of New York, Mr. Skelding was born in Brooklyn, and graduated from City College of New York in 1917. He was named valedictorian of his class, but was unable to attend the ceremonies because of his enlistment in the United States Navy three weeks before he graduated. It is rumored that Mr. Skelding was the youngest ensign to serve in World War I.

In 1990, he moved from Massapequa, New York, to Lynchburg, Virginia. Until a week before his death, Mr. Skelding was doing calculus problems, playing contract bridge, and handling his own stock portfolio.

He is survived by a son, Robert Skelding, of Lynchburg; two daughters, Kathleen Wennerstrom and Lois MacFarland; and ten grandchildren.

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