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PRICING THE IMPACT OF ADJUSTABLE FEATURES AND LOSS SHARING PROVISIONS OF REINSURANCE TREATIES

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Abstract

Excess-of-loss reinsurance contracts often contain loss sharing provisions, such as aggregate deductibles, loss ratio caps or limited reinstatements, and loss corridor provisions. They also frequently contain adjustable premium or commission features, such as retrospective rating plans, profit commission plans, and sliding scale commission plans. Pro rata treaties frequently contain adjustable commission features.

This paper presents an overview of two approaches to pricing aggregate loss distribution problems: the lognormal model and the Heckman-Meyers Collective Risk Model. Applications to reinsurance pricing are then presented. Finally, the paper compares results of applying these approaches to representative working excess-of-loss treaties.

These comparisons suggest that the lognormal model can provide satisfactory approximations to the theoretically more appropriate Collective Risk Model when use of the latter more sophisticated procedure is not necessary due either to data *limitations or to the influence of market conditions and negotiations. The increased efficiency of the lognormal model can lead to greater accuracy by making judgmental estimates unnecessary in many situations.*

The basic lognormal model is generally applicable to pro rata treaties and working excess-of-loss treaties. A mixture of lognormal and discrete distributions is presented that may be applicable in many low mean frequency situations. Cash flow modelling is also discussed.

1. INTRODUCTION

Working excess-of-loss reinsurance contracts, where significant loss frequency is expected, often contain nonproportional coinsurance clauses. These involve provisions where the ceding company is to pay a nonproportional share of losses without receiving a commensurate share of the reinsurance premium. Such clauses include aggregate deductibles, aggregate limits such as loss ratio caps or limited reinstatements, and loss corridor provisions. Quite frequently in the broker market, and less frequently in the direct market, working excess-of-loss treaties contain adjustable premium or commission features. These adjustable features include retrospective rating plans, profit commission or profit sharing plans, and sliding scale commission plans. A relatively small number of excess-of-loss treaties contain both adjustable premium or commission features and nonproportional coinsurance clauses. Pro rata treaties also frequently contain adjustable commission features.

This paper will first present an overview of two approaches to pricing aggregate loss distribution problems: the lognormal model and the Heckman-Meyers Collective Risk Model. Six examples are then presented of how aggregate loss distributions are used in reinsurance pricing. Results of applying these two aggregate loss distribution approaches are compared. Finally, several enhancements of the basic model are discussed. The focus in the paper is on concepts, with technical details and proofs presented in the appendices. Appendices summarize important excess-of-loss pricing methodologies and provide an expanded lognormal table.

The list of references presented at the end of the paper contains several sources for those wishing to delve into reinsurance and excess pricing concepts in greater depth.

The authors' overall purpose is to determine if the lognormal model provides a suitable approximation for reinsurance price monitoring purposes and for pricing situations where limited information is available or a highly precise answer is not required. If the lognormal model provides a satisfactory approximation to the Collective Risk Model results, significant efficiency gains are achievable. A more sophisticated three-parameter alternative to the lognormal is not tested. The reason for this is that the Collective Risk Model or an equivalent approach would be employed if the data and other resources would permit a more sophisticated approach.

2. AGGREGATE LOSS DISTRIBUTIONS

In order to price the impact of adjustable features and nonproportional coinsurance clauses, it is necessary to estimate the aggregate loss distribution. Two methods of estimating this distribution are employed:

(a) *The Lognormal Model*
If the aggregate loss random variable is viewed as the product of a large number of independent, identically distributed random variables, then the logarithm is approximately normally distributed by the Central Limit Theorem. (The stringent condition that the factors be identically distributed may be relaxed [1].) By definition, the aggregate loss random variable is lognormally distributed. In Appendix A, standard formulas based on the Patrik-John Collective Risk Model are employed to estimate the aggregate mean and coefficient of variation from the assumed frequency and severity distributions [2]. An expanded lognormal table with excess pure premium ratios for coefficients of variation between 0.1 and 5 was programmed based on the formulas in Mr. Finger's paper "Estimating Pure Premiums by Layer" [3]. Mr. Finger developed the lognormal model for severity applications, although it is being tested here as an aggregate loss model. Appendix B summarizes the lognormal model and presents the

expanded lognormal table. Parameter uncertainty can be modelled by subjectively weighting indications based on alternative parameter values. (The subjective weights reflect degrees of belief in alternative scenarios, each of which yields a mean and a coefficient of variation of a particular lognormal model.)

(b) *The Collective Risk Model*
This model involves the estimation of parameters for the frequency and severity distributions, along with the judgmental selection of parameters to reflect the degree of uncertainty in the estimated frequency and severity means. If the shape of these distributions is also uncertain, one could assign subjective probabilities reflecting degrees of belief to several scenarios and compute a weighted average of the resulting cumulative probabilities and excess pure premium ratios. These quantities are computed using the Heckman-Meyers algorithm [4], which uses piecewise linear approximations of the cumulative severity distributions together with the assumed frequency distributions to generate the characteristic functions of the severity and aggregate loss distributions. As the characteristic function uniquely determines a probability distribution, numerical methods are employed to evaluate the rather complicated formulas that accomplish this inverse transformation, yielding the aggregate loss cumulative probability distribution function and excess pure premium ratios needed to price the reinsurance conditions that are the focus of this paper. Technical details are summarized in Appendix C.

Appendix D shows that if the ground-up occurrence count distribution for an insured selected at random is negative binomial, then the excess occurrence count distribution for a randomly selected insured is also negative binomial. Based on this result, the formula is derived for calculating the excess occurrence count variance-to-mean ratio for an individual insured selected at random, and it is shown that this formula also applies to the class as a whole. This latter result is then used to demonstrate that, if the proportion of occurrences exceeding the retention is small and the excess frequency mean is known, then the excess occurrence count distribution for the class as a whole is approximately Poisson.[1]

In particular, it is established that

$$VMR_E = (1 - p) + p(VMR_G),$$

where VMR_G and VMR_E are the variance-to-mean ratios for the ground-up and excess occurrence count distributions, respectively, and where p is the probability that a claim will exceed the retention. If VMR_G is two or three (as in the ISO increased limits reviews), and p (a value that may also be calculated via ISO increased limits parameters) is less than .02, then VMR_E is close to unity. This implies that the excess occurrence count distribution for an insured selected at random and for the class as a whole will be approximately Poisson under conditions of parameter certainty. In the Collective Risk Model, uncertainty in the mean frequencies for the various classes of business is reflected through selection of the contagion parameters. This results in negative binomial frequency distributions for the classes under consideration.

The Single Parameter Pareto (SPP) distribution is used to model occurrence severity. Mr. Philbrick's paper "A Practical Guide to the Single Parameter Pareto Distribution" and the discussion by Messrs. Reichle and Yonkunas [6] provide an excellent discussion of this distribution, which is widely used in excess pricing. Ms. Rytgaard recently presented a paper [7] that compares alternative estimates of the SPP parameter and applies credibility theory to obtain more stable estimators of this parameter for portfolios of excess-of-loss treaties with similar characteristics. Appendix E summarizes some of the key properties of the SPP distribution. In particular, it is shown that if ground-up loss occurrences in excess of a particular truncation point are distributed according to the SPP distribution with parameter q, then the excess portions of these occurrences are distributed according to the shifted Pareto distribution (used by Insurance Services Office in increased limits

[1] The proof given in Appendix D is a direct application of the Gamma-Poisson model frequently encountered in the actuarial literature. The authors acknowledge that these results have previously been established elsewhere, and note that Joseph Schumi has established these results using recursive relationships [5].

pricing), where the scale parameter is equal to the truncation point and the shape parameter is equal to q. In the Collective Risk Model, uncertainty in the mean severities is reflected through the selection of the mixing parameter.

Theoretically, if the SPP is appropriate for loss occurrences in excess of a particular attachment, it should be appropriate above all higher attachments, and the parameter should remain constant. Fits to industry data have led the authors to conclude that the SPP parameter varies with the truncation point used in the fitting procedure. Moreover, if the truncation point used in the fitting procedure is less than 50% of the attachment for a particular pricing analysis, the errors become unacceptably large. In order to calculate these parameters, development triangles of SPP parameter estimates for various truncation points were constructed, from which projections were made of the ultimate values of this parameter by class of business and truncation point. In the examples discussed in this paper, the class of business is not identified, because the intent is only to discuss actuarial methodology. Although alternative two- and three-parameter distributions should be tested when data permits, the SPP distribution with these qualifications can be a satisfactory severity model for reinsurance price monitoring work and in pricing situations where limited information is available.

3. EXAMPLES OF TREATIES WITH ADJUSTABLE FEATURES AND LOSS SHARING PROVISIONS

This section discusses the pricing of excess-of-loss treaties containing common types of nonproportional coinsurance clauses and adjustable premium or commission plans. This is accomplished through the examination of six hypothetical treaties, the key provisions of which are summarized at the start of Appendices F through K, respectively. The analysis of each example involves two major steps. First, various parameters (such as the expected claim count, mean severity, and aggregate coefficient of variation) which underlie the distribution of losses in the reinsured layer are calculated. This allows one to obtain an appropriate set of excess pure premium ratios, either by reference to an appropriate lognormal table (via coefficient of variation matching) or by direct generation via the Heckman-Meyers Collective Risk Model. The second step involves the use of the set of excess pure premium ratios derived in the first step in order to determine the expected impact of the particular nonproportional coinsurance clause or adjustable feature being evaluated. For the sake of clarity, excess pure premium ratios (which are called insurance charges in the examples) based on the lognormal assumption are initially used to analyze the six treaty examples. In Section 4, a comparison is made to the results obtained when excess pure premium ratios generated by direct applications of the Collective Risk Model are employed.

Aggregate Deductible Example

Treaty I is an example of a contract containing an annual aggregate deductible provision. The calculation of the treaty's aggregate loss coefficient of variation (CV), which is displayed on Appendix F Exhibit 1, is based on the theory and formulas presented in the second section of this paper as well as in Appendices A through E. The computation of the impact of the aggregate deductible is shown in Appendix F Exhibit 2. The deductible amount is compared to the expected losses in the reinsured layer in order to obtain a corresponding entry ratio, which allows one to look up the appropriate insurance charge from the lognormal tables in Appendix B. (Linear interpolation is used to calculate excess pure premium ratios for CV and entry ratio combinations not explicitly listed in these tables.) Since the insurance charge (29.33% in this case) represents the expected proportion of aggregate losses above the deductible amount, it is easy to see that the complement of this value (70.67%) is the expected percentage of treaty losses eliminated by the aggregate deductible. Thus, if a burning cost or similar study shows that the expected loss cost for the entire layer is 3.75% of subject premium, then the introduction of an aggregate deductible provision reduces this loss cost to 3.75% × [100% − 70.67%], or about 1.10% of subject premium. As shown in Appendix F, this loss cost can easily be converted to an indicated treaty rate through the application of an appropriate expense, profit, and risk loading factor. It should be noted that the factor selected for this purpose should include a provision for risk commensurate with the degree of variability in layer losses after application of the deductible. The degree of variability in this case, and hence the risk load, is higher than that for losses prior to the reflection of this provision.

Aggregate Limit Example

Treaty II contains a limited reinstatement clause. The contract allows three free reinstatements of coverage during the treaty year, which means that the ceding company is covered for losses in the specified layer until those losses exceed four times the width of that layer. After that point, no coverage is provided. (This type of reinstatement clause should be contrasted with the kind that reinstates coverage after a certain number of losses have occurred only if an additional premium is paid. This latter type is really a separate cover, rather than a form of coinsurance on the original treaty.)

The pricing of this treaty is summarized in Appendix G. As was done in the previous example, an entry ratio is calculated by dividing the dollar value of the limited reinstatement provision ($2,800,000 in this case) by the expected losses in the layer prior to all forms of coinsurance. The insurance charge corresponding to this entry ratio (2.37% in this example) is equivalent to the expected percentage of losses eliminated by the limited reinstatement clause. Combining this quantity with the treaty's 20% proportional coinsurance provision yields a 21.89% overall coinsurance percentage. This latter coinsurance percentage is then applied to the expected layer loss cost prior to all coinsurance in order to obtain an expected loss cost and an indicated rate for the treaty. As in the previous example, the loading to convert the expected loss cost to a rate includes a provision for risk that reflects the potential volatility in treaty losses after the limited reinstatement is taken into account. Note that this risk provision should be somewhat lower than that for a similar treaty with no limited reinstatement clause. This is due to the fact that limited reinstatements, along with most other kinds of mechanisms that place a cap on losses, tend to reduce loss variability.

Loss Corridor Example

Treaty III is an example containing a loss corridor provision. Under a loss corridor provision, the reinsurer pays all losses falling in the reinsured layer up to a certain aggregate amount (called the lower bound of the loss corridor interval). Once this amount is reached, the reinsurer stops paying all losses until the total losses in the layer exceed a second threshold amount (the upper bound of the loss corridor interval). After this, the reinsurer resumes payment for all losses in the reinsured layer. The bounds of the loss corridor interval may be expressed in terms of dollar amounts, percentages of expected layer losses, or ratios to treaty premium.

In the example presented in Appendix H, the loss corridor bounds are stated as percentages of expected losses in the layer. This makes the analysis extremely straightforward, since these percentages are directly equivalent to the corresponding entry ratios. The difference between the insurance charges at the lower and upper bounds, respectively, results in the expected percentage of layer losses eliminated by the loss corridor provision. The computation of the expected layer loss cost after coinsurance and the indicated treaty rate is analogous to the calculations presented in the first two examples. Unlike the previous examples, however, there is no definite rule concerning the proper risk load to be included in the factor used to convert the loss cost into a rate. This is due to the fact that the loss corridor provision may either reduce or increase the variability of layer losses, depending on both the location and the size of the eliminated loss interval.

While the simplicity of the loss corridor analysis is not altered very much when the interval bounds are expressed in terms of dollars, the analysis does get complicated when the bounds are stated as ratios to treaty premium. This is due to the fact that the treaty premium is dependent on the treaty rate, which should already reflect the effect of the loss corridor. It is clear that the solution to this problem requires an iterative procedure in which the algorithm presented in Appendix H is repeated until the rate used to compute the loss corridor bounds (expressed as percentages of expected losses) equals the rate indication for the treaty with the loss corridor provision.

Having covered three common types of nonproportional loss sharing plans, the remainder of this section will discuss the analysis of accounts containing adjustable premium or commission plans.

Retrospective Rating Plan Example

Treaty IV is an example of an account with a one-year retrospective rating plan. Similar to the plans encountered in primary insurance, the adjusted treaty rate (and hence the adjusted premium) is based on the account's actual loss experience during the period subject to the plan. This rate is determined by loading the ratio of the treaty's actual losses to subject premium by a multiplicative loss conversion factor and/or an additive flat margin. The computed rate is further subject to a maximum and a minimum as specified in the treaty. (The loss conversion factor or flat margin accounts for the reinsurer's expenses, risk, and profit, and may also contain some provision, subjective or otherwise, to reflect the effect of the plan's maximum and minimum rates.) The main goal of this analysis is to determine the expected rate to be received on this treaty after all retrospective adjustments have been completed. This will enable one to assess the adequacy of the retro plan.

The calculation of the expected treaty rate for this example is outlined in Appendix I. As in the analysis of primary plans, the major step in this calculation is the determination of the true effect of the retro plan's maximum and minimum rates on the expected layer loss cost to be charged to the reinsured (which may differ from any subjective estimates of this effect included in the plan's loss loading factors). This is accomplished by dividing the loss costs that are consistent with the maximum and minimum rates, respectively, by the expected layer loss cost, in order to obtain entry ratios at these two points. These entry ratios enable one to look up the associated excess pure premium ratios, so that the insurance charge at the maximum and the insurance savings at the minimum may be computed. The difference between these latter two quantities is the net insurance charge. Applying the complement of the net insurance charge to the expected layer cost yields the adjusted expected layer cost, which is the loss expected to be charged to the reinsured. This latter quantity is loaded with the retro plan's loss conversion factor and any flat margin in order to obtain the expected treaty rate after retro adjustments. Note that the net insurance charge in this example is negative, indicating that the premium the reinsurer expects to lose because of the maximum rate provision is more than offset by the additional premium expected to be received due to the minimum provision.

The degree of adequacy of the retro plan can be measured by calculating the ratio of the guaranteed cost rate (which is the equivalent treaty rate if the contract were flat rated) to the expected treaty rate after retro adjustments. (To be comparable, the guaranteed cost rate contains the same amount of risk load as that contained in the retro plan parameters but with any insurance charge removed.) As shown on the bottom of the Appendix I Exhibit, the resulting ratio of 0.996 indicates a very slight redundancy in the retro plan.

Profit Sharing Example

Treaty V contains a three-year profit commission plan, in which the profit commission ratio (to treaty premium) is computed via the following formula:

Profit Commission Ratio =
25% × [100% − (Actual 3-Year Treaty Loss Ratio)
− (20% Reinsurer's Overhead Provision)].

Note that the same formula could be used to compute a profit sharing adjustment that is treated as return premium.

On the surface, the calculation of the expected profit commission ratio for the three-year period (1/1/90–12/31/92 in this case) may seem trivial (i.e., simply plug the three-year expected loss ratio into the formula). It is really not, however, since a three-year loss ratio above 80% (the breakeven point) is implicitly capped at 80% to yield a 0% profit commission for the period. Hence, one must determine the effect of this capping on the expected loss ratio used in the profit sharing formula in order to estimate the expected commission. As in the previous examples, this involves the use of excess pure premium ratios for a lognormal distribution with an appropriate CV.

Appendix J Exhibit 1 displays the calculation of the CV for the distribution of one year's worth of aggregate losses in the reinsured layer. Since this is a three-year profit commission plan, the CV appropriate for aggregate treaty losses for three years combined needs to be determined. This is accomplished on Appendix J Exhibit 2, using the formulas discussed in the second section of the paper and in the related appendices. In reviewing this exhibit, it should be assumed that the subject premium and expected layer cost given for 1990 are values based on ceding company projections and rating analyses, respectively. The numbers shown for 1991 and 1992 are simply copied from 1990, since the information needed to make independent projections for these years is not presently available.

The calculation of the expected profit commission is shown on Appendix J Exhibits 3A and 3B. The expected treaty loss ratio of 48% is computed by reducing the expected loss cost for the entire layer by the 20% proportional coinsurance provision and then dividing the result by the treaty rate. By relating the 80% breakeven loss ratio to the expected loss ratio, an entry ratio is obtained from which the corresponding net insurance charge (*NIC*) is determined. Since the net insurance charge represents the percentage of expected losses eliminated from the profit commission formula by the implicit cap at the breakeven loss ratio, the expected profit commission ratio can be calculated via the following formula:

Expected Profit Commission Ratio =

$$P \times [100\% - ELR \times (100\% - NIC) - EXP],$$

where P = The proportion of profits to be paid to the reinsured;
ELR = Expected treaty loss ratio;
NIC = Net insurance charge;
EXP = Reinsurer's overhead provision.

In the Appendix J exhibits, the expected profit commission based on the formula above is called the "actuarial view," while that obtained by simply plugging the expected loss ratio into the profit commission formula is labelled the "simplistic view." Based on these definitions, it is clear that the expected profit commission based on the actuarial view should generally exceed that based on the simplistic view, as it does in this example.

Sliding Scale Commission Example

Treaty VI contains another kind of adjustable commission provision known as a sliding scale plan. Like the profit commission in the previous example, the commission that is ultimately paid by this plan depends directly on the reinsured's actual experience as measured by the treaty loss ratio. The major difference between these two plans lies in the structure of the formula used to compute the adjustable commission. Whereas the profit commission formula is essentially a straight linear function of the treaty loss ratio (at least up to the breakeven point), the typical sliding scale plan is best described as a piecewise linear function of the loss ratio.

Under a typical sliding scale plan, a minimum commission ratio C_{min} is paid if the treaty loss ratio exceeds a certain fixed value (call it L_1). If the actual loss ratio is less than L_1 but greater than a second fixed value L_2, b_2 points of commission are added to C_{min} for each point by which the actual loss ratio falls short of L_1. Similarly, if the actual loss ratio is below L_2 but greater than some third value L_3, the commission ratio corresponding to L_2 is increased by b_3 points for each point of difference between L_2 and the actual treaty loss ratio. The commissions corresponding to actual loss ratios falling into successively lower intervals (i.e., $[L_i, L_{i-1}]$, where $i = 4,\ldots,n$) are calculated in a manner similar to those for loss ratios falling in the previous two intervals. A maximum commission C_{max} is paid when the loss ratio is zero. It should be noted that the b_i's, which represent the commission slides on the various intervals, are generally less than unity, and some may be equal to zero. The sliding scale plan for Treaty VI (see the bottom of Appendix K Exhibit 1) is expressed in the format described above.

Since the typical sliding scale plan involves both a minimum and maximum commission as well as different commission slide percentages for the various loss ratio intervals, it is clear that the calculation of the expected commission ratio under such a plan requires more than simply looking up the commission that corresponds to the expected loss ratio. Appendix L outlines the derivation of a concise formula for computing this expected commission, which can be expressed as follows:

Expected Sliding Scale Commission Ratio =

$$C_{\max} - \sum_{i=1}^{n} b_i \,\{\text{Expected loss ratio points in the interval } L_i \text{ to } L_{i-1}\},$$

where: $C_{\max}$ is the maximum commission ratio;
b_i is the commission slide on the i^{th} loss ratio interval (b_1 is defined to be 0 and L_0 is infinity);
and Expected loss ratio points in the interval $[L_i, L_{i-1}]$ = (Expected loss ratio) $\times$ $[P_2(L_i) - P_2(L_{i-1})]$.
($P_2(L)$ is the excess pure premium ratio at loss ratio L.)

Appendix L also shows that the above formula is equivalent to saying that the expected commission ratio equals the maximum commission ratio minus the expected points of commission lost over the entire range of possible loss ratios. This interpretation provides a good intuitive justification for the formula stated above.

The above formula is used to calculate the expected commission ratio for the one-year plan given in Treaty VI, the details of which are provided in the Appendix K exhibit. As this exhibit shows, in order to determine the expected number of loss ratio points falling in each interval specified in the plan, it is necessary to multiply the treaty expected loss ratio by the difference between the insurance charges corresponding to both end points of the interval.

On the bottom of the Appendix K exhibit the expected sliding scale commission based on the above formula (the "actuarial view") is compared to the commission that corresponds to the expected loss ratio (the "simplistic view"). Although the actuarial estimate of the expected commission exceeds the simplistic estimate in this example, this is not a general rule. Both the magnitude and the direction of the difference between these two quantities depend on the minimum and maximum commission ratios as well as on the commission slides on the various loss ratio intervals.

4. MODEL COMPARISONS

For the examples presented above, Table 1 compares the key item of interest (either the adjusted rate or expected commissions) under the alternative models. The unadjusted rate is the loaded loss cost before all forms of coinsurance using the same expense and profit loading factor as that used to compute the adjusted rate. (In practice, the loadings for a treaty without coinsurance provisions or premium adjustments would generally not be considered appropriate for a treaty with such provisions.)

The alternative indications for the Heckman-Meyers version of the Collective Risk Model reflect varying levels of parameter uncertainty. The contagion parameter is represented by c and represents the level of parameter uncertainty in the estimated frequency mean. The mixing parameter is represented by b and represents the level of parameter uncertainty in the estimated severity mean.

Values of zero represent no parameter uncertainty, values of .05 represent a moderate level of parameter uncertainty, while values of .10 represent a higher level of parameter uncertainty. Please refer to Appendix C for further technical details. The lognormal model was run under the same assumptions that were used to generate the Collective Risk Model results without parameter uncertainty. Parameter uncertainty was not reflected here for the lognormal model (as it should be in practice using the methods presented in Appendices A and B) in an effort to simplify the presentation.

The comparisons above suggest that the lognormal model provides a satisfactory approximation to the theoretically more appropriate Collective Risk Model results, when use of the latter more sophisticated procedure is not necessary due either to data limitations or to the influence of market conditions and negotiations on final pricing. Application of the lognormal model can lead to significant efficiency gains in reinsurance price monitoring work and in many pricing situations, because it can easily be programmed in spreadsheets and applied efficiently by those with good quantitative skills. The increased efficiency achieved by

TABLE I

Comparison of Key Items

Example	Unadjusted Rate	Item Compared	Lognormal Model	Collective Risk Model $c = 0$ $b = 0$	Collective Risk Model $c = .05$ $b = .05$	Collective Risk Model $c = .05$ $b = .10$	Collective Risk Model $c = .10$ $b = .10$
I) Aggregate Deductible	5.00%	Adjusted Rate	1.47%	1.58%	1.68%	1.73%	1.77%
II) Limited Reinstatement	25.00	Adjusted Rate	19.53	19.89	19.72	19.55	19.52
III) Loss Corridor	5.00	Adjusted Rate	4.02	3.67	3.71	3.74	3.73
IV) Retro Rating Plan	5.00	Expected Rate After Retro Adjustments	5.02	5.20	5.18	5.14	5.14
V) Profit Commission		Expected Profit Commission	8.37	8.24	8.50	8.69	8.75
VI) Sliding Scale Commission		Expected Sliding Scale Commission	31.04	30.31	30.90	31.22	31.33

this model permits one to apply it much more frequently than more theoretically appropriate methods. Use of a satisfactory quantitative method is usually superior to judgment.

5. MODEL ENHANCEMENTS

The six treaty examples discussed in this paper illustrate methods for pricing common types of nonproportional coinsurance and adjustable features provisions in reinsurance contracts. Although the examples themselves were kept reasonably simple to allow the reader to focus on the basic pricing techniques, the authors recognize that a number of enhancements can be made to the models in order to make them more applicable to specific situations. Unless otherwise stated, the following potential enhancements apply to both the lognormal and Collective Risk Model approaches.

1. Layer Retention and Limits

All the multiline contracts presented in the paper assume that the same layer retention and limit apply to all the classes of business covered by the treaty. In practice, however, some excess-of-loss contracts have retentions and/or limits that vary by line of business (e.g., auto liability losses may be subject to a $200,000 per occurrence retention, while workers compensation losses have a $300,000 retention). In these situations, the excess claim severity mean and standard deviation would be calculated for each class of business based on the retention and limit applicable to that class. (The formulas given in Appendix E would be used if a Single Parameter Pareto severity distribution is assumed). Similarly, the expected number of claims in the layer and the excess frequency variance-to-mean ratio would be calculated for each class based on the applicable retention. Once these quantities are computed, the calculation of the aggregate loss distribution for all classes combined would follow the same sequence of steps as if the contract had a single retention and limit applicable to all lines.

A similar procedure could be employed to derive the aggregate loss distribution for a multi-year rating block on an adjustable features contract, if the covered layer of reinsurance varies between the years comprising the block. The method can even be used to reflect underlying primary policy limits. In this case, one would treat each group of insureds purchasing the same policy limit above the layer retention as a separate class of business. The layer limit applicable to each class would be the lesser of the primary policy limit and the reinsurance gross limit (i.e., the retention amount plus the width of the layer). This layer limit, together with the layer retention, would then be used to calculate the expected layer loss cost, as well as the severity mean and standard deviation, for the particular class of policyholders.

2. Severity Distribution Assumptions

The examples in Section 3 use the Single Parameter Pareto severity (SPP) distribution to model occurrence severities. As mentioned in Section 2, the SPP parameter applicable to a particular line of insurance and truncation point can be derived by fitting this curve to empirical claims data. When performing this procedure directly or when utilizing published parameters, it is important to note whether the underlying claims data include allocated loss adjustment expenses (ALAE). If the empirical data used to compute the SPP parameter include ALAE, and if the reinsurance contract handles ALAE as a part of loss, then the formulas presented in the paper for calculating the aggregate loss (and ALAE) distribution can be applied without any modifications. The same is true if the SPP distribution is based on pure losses only and if the reinsurance contract does not cover ALAE.

In most cases, however, the reinsurance contract covers, to some extent, the ALAE associated with layer losses, but the assumed severity distribution describes pure losses only. These situations require one to make minor modifications to the methods presented earlier in the paper.

For the lognormal model, one should use expected layer loss costs excluding ALAE in order to determine the aggregate coefficient of variation (CV). If the ALAE covered by the contract is a constant multiple of layer losses (or nearly so), then the same CV describes the distribution of aggregate losses and ALAE in the layer. One instance when this would be true is where ALAE is a fixed percentage of ground-up losses and where ALAE is shared pro rata between the reinsurer and the ceding company. The particular loss sharing or adjustable feature provision would then be priced, using the expected layer loss cost including ALAE to compute the entry ratios needed to determine the appropriate excess pure premium ratios from the lognormal table. The adjustments to the Collective Risk Model approach for determining the aggregate loss distribution entail adjusting the parameters of the underlying frequency and severity distributions appropriately to reflect the relationship of ALAE to loss and the particular contractual provision concerning the manner by which ALAE will be shared.

Although the SPP distribution was chosen to model claim severities in the treaty examples, it is important to note that other severity distributions could have been used to derive the aggregate loss distribution under either the lognormal or Collective Risk Model approaches. The relaxation of this restriction allows one to use these models to determine the aggregate loss distributions for pro rata reinsurance contracts. (Recall that the SPP distribution is appropriate only above a sufficiently large truncation point, and hence it cannot be used to price pro rata treaties.) Once the aggregate loss distribution has been determined, the particular coinsurance clause or adjustable feature can be priced using the methods presented in the treaty examples.

3. Treaties with Both Coinsurance Provisions and Adjustable Features

The first three treaty examples presented in Section 3 illustrate methods for pricing common types of nonproportional coinsurance provisions, while the latter three examples involve the analysis of treaties with adjustable premium or commission plans. The case in which a treaty contains both a nonproportional coinsurance clause and an adjustable feature has not been considered. In such a situation, one needs to determine not only the effect that the nonproportional coinsurance clause has on expected treaty losses (which can be accomplished using the techniques discussed above) but also the distribution of aggregate losses after the effect of the nonproportional coinsurance has been taken into account. The latter item is necessary in order to compute the expected impact of the adjustable premium or commission plan, since these plans generally operate on actual treaty experience after all coinsurance.

The calculation of the aggregate distribution after nonproportional coinsurance can be accomplished by making direct modifications to the aggregate loss distribution prior to coinsurance (e.g., truncate it at the aggregate deductible amount or censor it at the aggregate limit). The Collective Risk Model would be run again to compute the needed insurance charges, assuming that there will be one claim with a severity distribution equal to the aggregate loss distribution after all forms of nonproportional coinsurance. Another approach is to determine the effects that the nonproportional coinsurance feature has on both the occurrence count and the occurrence severity distributions that underlie the aggregate distribution. The adjusted count and severity distributions can then be combined (using either method discussed in this paper or the alternative recursive or simulation techniques) in order to obtain an aggregate loss distribution that reflects the effects of the nonproportional coinsurance provision.

4. *Aggregate Losses of Zero*

When working with the models presented in the paper, one must consider the probability that no treaty losses will occur during a particular year. Although the chance of this occurring on pro rata or working layer excess-of-loss treaties may be sufficiently small that it can be ignored, treaties reinsuring rare events or high layers could have many loss-free years. One needs to estimate the probability of a loss-free year occurring on the treaty, either subjectively or by examining past treaty experience (if credible), in order to properly estimate the aggregate loss distribution.

If the Collective Risk Model is used to generate the aggregate distribution, the probability of a loss-free year could be reflected directly through the choice of the claim count distribution. A problem arises, however, when one attempts to use the lognormal aggregate loss distribution assumption to price a treaty with a positive probability of having no losses during a particular year. This is due to the fact that the lognormal distribution is not defined at the value zero. One solution to this problem involves the use of a mixture of a lognormal and a discrete distribution to model aggregate losses. This enhanced model may be applicable in many low mean frequency situations. Technical details are summarized in Appendix M.

5. *Investment Income*

The time value of money also has not been considered in the examples presented above, even though it is a legitimate underwriting consideration in evaluating alternative proposals. One way of handling this item would be to develop aggregate loss distributions for the lines of business subject to the treaty prior to all forms of nonproportional coinsurance. (Either the lognormal or Collective Risk Model may be used for this purpose.)

The analysis then becomes a simulation problem. One would simulate annual losses before coinsurance for each line, apply payout patterns to estimate future loss payments by line, apply the nonproportional coinsurance provisions, and finally discount the future treaty losses. (To accomplish this, one might develop and apply stochastic loss reporting, loss payout and interest rate models. Alternatively, one could develop a range of scenarios concerning these parameters and subjectively weight the final results derived from these alternative scenarios.) One would also need to estimate when future premium or commission adjustments would be made and when brokerage and other reinsurance expenses (including taxes [8]) would be paid. The economic value of the proposed treaty would be the difference between discounted reinsurance premium and the sum of the discounted values of all expense items. This economic value should be adjusted for risk considerations, possibly through the selection of the interest rates used in the discounting procedure [9].

A second approach is to estimate the ultimate loss ratio after all coinsurance as a percentage of provisional treaty premium using the methods of Section 3. Payout and loss reporting scenarios that approximately reflect the impact of the coinsurance provisions could then be selected. The loss reporting pattern would be used to estimate both IBNR reserves and the emergence of reported losses. Contractual formulas would be applied to estimate the magnitude of premium or commission adjustments to occur at specified points in time. The remainder of the analysis would proceed as in the first approach.

In this second approach, one item that needs to be considered in calculating premium or commission adjustments at various points in time is the impact of the insurance charges. For a retrospective rating or profit sharing formula, expected reported losses at various stages of development should be multiplied by the complement of the net insurance charge to approximate expected losses subject to the adjustment formula. In sliding scale commission plans, the commission ratio computed by plugging the expected loss ratio into the formula should be adjusted by the difference between the expected commission ratio (the actuarial view) and this formula estimate (the simplistic view), using the methods presented in Section 3. Sliding scale commission adjustments at various points in time would be computed by applying the contractual formula to the expected reported loss ratio and reflecting this commission adjustment gradually.

APPENDIX A
COMPUTATION OF AGGREGATE MEAN AND COEFFICIENT OF VARIATION (PATRIK-JOHN [2] VERSION OF COLLECTIVE RISK MODEL)

Let L represent the random variable of aggregate loss to be paid on a given contract for a particular coverage period.

$$L = L_1 + L_2 + \ldots + L_k,$$

where L_i represents the aggregate loss random variable for group i, $i = 1,2,\ldots,k$.

The groupings may represent distinct groups of classes of insureds or coverages, similar insureds grouped by distinct policy limits, or the overall coverage time period split into sub-periods.

$$L_i = X_{i1} + X_{i2} + \ldots + X_{iN_i},$$

where N_i is the random variable of the number of loss occurrences for group i and X_{ij} is the random variable of loss size of the j^{th} loss for group i.

Let v represent the parameter vector containing all parameters necessary to specify the particular cumulative probability distribution functions (c.d.f.'s) for the L_i's, N_i's, and X_{ij}'s.

The following three assumptions guarantee that the total coverage has been split into independent, homogeneous coverage groups:

Assumption 1: Given v, the L_i's are stochastically independent.

Assumption 2: Given v, the X_{ij}'s are stochastically independent of the N_i's.

Assumption 3: Given v and fixed i (i.e., a particular group), the X_{ij}'s are stochastically independent and identically distributed.

Let $F(x|v)$ represent the c.d.f. of L and let $F_i(x|v)$ represent the c.d.f. of L_i, $i = 1,2,\ldots,k$.

Properties of Model with Known Parameters

(1) The c.d.f. of the aggregate loss L is the convolution of the aggregate loss c.d.f.'s for the individual groups:

$$F(x|v) = P(L \leq x|v) = F_1(x|v) * F_2(x|v) * \ldots * F_k(x|v),$$

where $F_i(x|v) = P(L_i \leq x|v)$ and $*$ denotes the convolution operation.

(2) The cumulants of L given v are sums of the corresponding cumulants of the L_i's given v. This implies that

(a) $\mathrm{E}(L|v) = \sum_i \mathrm{E}(L_i|v)$ (the means are additive).

(b) $\mathrm{Var}(L|v) = \sum_i \mathrm{Var}(L_i|v)$ (the variances are additive).

(3) The aggregate loss c.d.f. of the i^{th} group, $F_i(x|v)$, can be expressed in the form

$$F_i(x|v) = \sum_n P(N_i = n|v) \cdot G_i^{*n}(x|v),$$

where $G_i(x|v) = P(X_i \leq x|v)$ is the loss amount c.d.f. for the i^{th} group, and G_i^{*n} is the convolution of the n G_i's and represents the c.d.f. of the total amount of exactly n loss occurrences.

(4) The above properties imply that

(a) $\mathrm{E}(L_i|v) = \mathrm{E}(N_i|v) \cdot \mathrm{E}(X_i|v)$.

The mean aggregate loss for the i^{th} group is the product of the mean frequency and mean severity.

(b) $\mathrm{Var}(L_i|v) = \mathrm{E}(N_i|v) \cdot \mathrm{Var}(X_i|v) + \mathrm{Var}(N_i|v) \cdot \mathrm{E}(X_i|v)^2$.

The variance of the i^{th} group's aggregate loss is the sum of the mean frequency times the variance of severity and the variance of frequency times the square of the mean severity. Substitution into the formulas in (2) above yields the mean and variance of the aggregate loss distribution.

Collective Risk Model

Now delete the restriction that the parameter vector v is known. Assume that the set V of possible parameters is finite and known and that one can specify the subjective likelihood of each element v of V. The structure function $U(v)$ is a discrete probability function that specifies the observer's uncertainty regarding the "best" parameter.

The unconditional c.d.f. $F(x)$ of the aggregate loss L has the following properties:

(1) $F(x) = \sum_{v} F(x|v) \cdot U(v)$.

The c.d.f. $F_i(x)$ of L_i is computed similarly.

(2) $E(L^m) = \sum_{v} E(L^m|v) \cdot U(v)$.

The m^{th} moment of L_i about the origin is computed similarly.

(3) With v unknown, assumptions (1)–(3) above may no longer hold, for the uncertainty regarding v may simultaneously affect the model at all levels. With v unknown, only the first cumulant is additive:

$E(L) = \sum_{i} E(L_i)$,

but $\mathrm{Var}(L) \neq \sum_{i} \mathrm{Var}(L_i)$.

However, $\mathrm{Var}(L) = E(L^2) - E(L)^2$,

and $E(L^2) = \sum_{v} E(L^2|v) \cdot U(v) = \sum_{v} \{\mathrm{Var}(L|v) + E(L|v)^2\} \cdot U(v)$.

$\mathrm{Var}(L|v)$ and $E(L|v)$ are evaluated using the formulas above for the model with known parameters.

APPENDIX B
THE LOGNORMAL MODEL [3]

If the aggregate loss random variable is viewed as the product of a large number of independent, identically distributed random variables, the logarithm is then approximately normally distributed by the Central Limit Theorem. (The stringent condition that the factors be identically distributed may be relaxed [1].) This implies that the aggregate loss random variable is lognormally distributed.

The formulas in Appendix A for the model with known parameters are used to estimate the mean and variance of the aggregate loss distribution. It is assumed that the mean aggregate loss for each coverage of the excess-of-loss reinsurance contract has been estimated accurately using standard burning cost and/or exposure rating methods. A Single Parameter Pareto severity distribution is assumed for each coverage and is used to compute the mean and variance of the severity distribution (see Appendix E). The ratio of the mean aggregate loss to the mean severity is the mean number of loss occurrences for a given coverage. The variance of the excess frequency distribution is computed based on the assumptions and the formula developed in Appendix D. Thus, the mean and variance of the frequency and severity distributions for each coverage are specified and used to compute the variance of the aggregate loss distribution for each coverage. The sum of these variances for all of the coverages is the variance of the aggregate loss distribution for all coverages combined, since independence of aggregate losses for the individual coverages is assumed.

The Coefficient of Variation (CV) of the aggregate loss distribution is the ratio of the standard deviation to the mean of L, based on the frequency and severity distributions specified by the vector of parameters v or based on empirical methods applied to burning cost analyses:

$$\mathrm{CV}(L|v) = \frac{\{\mathrm{Var}(L|v)\}^{1/2}}{\mathrm{E}(L|v)} .$$

For simplicity, let $M = \mathrm{E}(L|v)$.

The Entry Ratio r is the ratio of the attachment A to the mean aggregate loss:

$$r = A/M.$$

The Excess Pure Premium (XSP) for a particular attachment A is the expected aggregate losses excess of A:

$$XSP(A|v) = \int_A^\infty (L - A)dP(L|v),$$

where P is the c.d.f. of L, given the vector of parameters v. The Excess Pure Premium Ratio P_2 at entry ratio r is the ratio of the corresponding Excess Pure Premium to the mean aggregate loss:

$$P_2(r|v) = XSP(A|v)/M.$$

Assume that the distribution of L is lognormal, given frequency and severity distributions specified by the vector of parameters v. If the parameters of this lognormal distribution are μ and σ^2, then

$$(1)\ M = \mathrm{E}(L|v) = \exp\{\mu + \frac{\sigma^2}{2}\}, \text{ and}$$

$$(2)\ \mathrm{CV} = \mathrm{CV}(L|v) = \{\exp(\sigma^2) - 1\}^{1/2}.$$

The first moment distribution P_1 is also lognormally distributed, but with parameters $\mu + \sigma^2$ and σ^2. P_1 is defined by

$$P_1(r|v) = \frac{1}{M}\int_0^A L \cdot dP(L|v).$$

The first moment distribution represents the percentage of total aggregate losses from coverage periods where the aggregate loss is less than the attachment. The Excess Pure Premium Ratio can be computed using

$$P_2(r|v) = \{1 - P_1(r|v)\} - r\{1 - P(r|v)\}.$$

Given that M and CV have been established as described above, the parameters of the assumed lognormal aggregate loss distribution can be estimated from formulas (1) and (2) above:

$$\sigma^2 = \ln(1 + \mathrm{CV}^2), \text{ and}$$

$$\mu = \ln(M) - \frac{\sigma^2}{2}.$$

As noted above, P_1 is also lognormally distributed with parameters $\mu' = \mu + \sigma^2$ and σ^2. The vector of parameters v determines M and CV through the formulas previously presented. While the Excess Pure Premium is a function of both M and CV, the Excess Pure Premium Ratio is solely a function of the CV. Thus, the Excess Pure Premium Ratios are computed using

$$P_2(r|\mathrm{CV}) = \{1 - P_1(r|\mathrm{CV})\} - r\{1 - P(r|\mathrm{CV})\}.$$

This formula was used to compute values for the expanded version of Mr. Finger's famous table which is displayed in Tables 1–3 of this Appendix.

The Excess Pure Premium for attachment A is given by

$$XSP(A|M,\mathrm{CV}) = M \cdot P_2(r|\mathrm{CV}), \text{ where } r = A/M.$$

Parameter uncertainty may be reflected using the method described under the Collective Risk Model section of Appendix A. For each element v of V, compute M and CV. Since $U(v) = U(M,\mathrm{CV})$, the unconditional Excess Pure Premium for attachment A may be computed using

$$XSP(A) = \sum_M \sum_{\mathrm{CV}} XSP(A|M,\mathrm{CV})\ U(M,\mathrm{CV}).$$

For the sake of simplicity, a probability of one is assigned to our most likely scenario for the examples in this paper.

TABLE B1

Excess Pure Premium Ratios
Lognormal Model

Entry Ratio	Coefficient of Variation .1	.2	.3	.4	.5
0	1.000	1.000	1.000	1.000	1.000
.1	.900	.900	.900	.900	.900
.2	.800	.800	.800	.800	.800
.3	.700	.700	.700	.700	.700
.4	.600	.600	.600	.601	.603
.5	.500	.500	.501	.504	.510
.6	.400	.400	.404	.413	.426
.7	.300	.302	.313	.331	.351
.8	.200	.211	.234	.260	.286
.9	.107	.135	.168	.200	.232
1.0	.040	.079	.117	.153	.187
1.1	.009	.042	.079	.115	.150
1.2	.001	.021	.052	.086	.120
1.3	.000	.010	.034	.064	.096
1.4		.004	.022	.048	.077
1.5		.002	.014	.035	.062
1.6		.001	.009	.026	.049
1.7		.000	.005	.019	.040
1.8			.003	.014	.032
1.9			.002	.010	.026
2.0			.001	.008	.021
2.2			.000	.004	.014
2.4				.002	.009
2.6				.001	.006
2.8				.001	.004
3.0				.000	.003

The Coefficient of Variation is the ratio of the standard deviation to the mean of the assumed lognormal distribution. The Entry Ratio is the ratio of the attachment to the mean. The Excess Pure Premium Ratios are ratios of excess pure premiums to the mean (i.e., ratios of expected excess losses to the total expected loss).

TABLE B2

EXCESS PURE PREMIUM RATIOS
LOGNORMAL MODEL

Entry Ratio	Coefficient of Variation .6	.7	.8	.9	1.0
0	1.000	1.000	1.000	1.000	1.000
.1	.900	.900	.900	.900	.900
.2	.800	.800	.801	.802	.804
.3	.702	.704	.707	.710	.714
.4	.607	.613	.619	.627	.634
.5	.519	.530	.541	.552	.563
.6	.441	.456	.472	.487	.502
.7	.371	.392	.412	.430	.447
.8	.312	.336	.359	.381	.400
.9	.261	.289	.314	.337	.359
1.0	.218	.248	.275	.300	.323
1.1	.183	.213	.241	.267	.291
1.2	.153	.183	.212	.239	.263
1.3	.128	.158	.187	.213	.238
1.4	.107	.137	.165	.192	.216
1.5	.090	.118	.146	.172	.197
1.6	.076	.103	.130	.155	.180
1.7	.064	.089	.115	.140	.164
1.8	.054	.078	.103	.127	.150
1.9	.046	.068	.092	.115	.138
2.0	.039	.060	.082	.105	.127
2.2	.028	.046	.066	.087	.108
2.4	.020	.036	.053	.073	.092
2.6	.015	.028	.044	.061	.079
2.8	.011	.022	.036	.052	.068
3.0	.008	.017	.030	.044	.059
3.2	.006	.014	.025	.037	.052
3.4	.005	.011	.021	.032	.045
3.6	.004	.009	.017	.028	.040
3.8	.003	.007	.015	.024	.035
4.0	.002	.006	.012	.021	.031
5.0	.001	.001	.006	.011	.018
10.0	.000	.000	.000	.001	.002

The Coefficient of Variation is the ratio of the standard deviation to the mean of the assumed lognormal distribution. The Entry Ratio is the ratio of the attachment to the mean. The Excess Pure Premium Ratios are ratios of excess pure premiums to the mean (i.e., ratios of expected excess losses to the total expected loss).

TABLE B3

Excess Pure Premium Ratios
Lognormal Model

Entry Ratio	Coefficient of Variation 1.5	2.0	2.5	3.0	4.0	5.0
0	1.000	1.000	1.000	1.000	1.000	1.000
.1	.902	.905	.908	.911	.916	.921
.2	.813	.824	.834	.842	.855	.864
.3	.736	.756	.772	.786	.805	.820
.4	.670	.699	.721	.738	.764	.782
.5	.612	.649	.676	.697	.728	.750
.6	.562	.605	.637	.661	.697	.721
.7	.518	.567	.602	.630	.669	.696
.8	.479	.532	.571	.601	.644	.673
.9	.444	.502	.544	.575	.621	.652
1.0	.413	.474	.518	.552	.600	.633
1.2	.360	.426	.474	.511	.563	.599
1.4	.316	.386	.437	.476	.531	.570
1.6	.280	.352	.405	.445	.503	.544
1.8	.250	.323	.377	.418	.478	.521
2.0	.224	.297	.352	.394	.456	.500
2.2	.202	.275	.330	.373	.436	.481
2.4	.183	.255	.311	.354	.418	.464
2.6	.167	.238	.293	.337	.402	.448
2.8	.152	.222	.277	.321	.386	.433
3.0	.139	.208	.263	.307	.372	.419
3.5	.114	.179	.232	.275	.341	.389
4.0	.094	.155	.207	.250	.315	.364
4.5	.079	.136	.186	.228	.293	.342
5.0	.067	.120	.168	.209	.274	.322
5.5	.057	.107	.153	.193	.257	.305
6.0	.049	.096	.140	.179	.242	.290
6.5	.043	.087	.129	.167	.229	.276
7.0	.037	.079	.119	.156	.216	.264
7.5	.033	.072	.111	.146	.206	.252
8.0	.029	.066	.103	.137	.196	.242
10.0	.019	.047	.079	.110	.164	.208
20.0	.004	.015	.031	.049	.087	.120
30.0	.001	.007	.016	.029	.056	.083
50.0	.000	.002	.007	.013	.030	.049
100.0	.000	.000	.000	.004	.012	.022

The Coefficient of Variation is the ratio of the standard deviation to the mean of the assumed lognormal distribution. The Entry Ratio is the ratio of the attachment to the mean. The Excess Pure Premium Ratios are ratios of excess pure premiums to the mean (i.e., ratios of expected excess losses to the total expected loss).

APPENDIX C

HECKMAN-MEYERS VERSION OF COLLECTIVE RISK MODEL [4]

This appendix uses the same notation as presented in Appendix A. Let N_i represent the number of loss occurrences for group i and let m_i represent the unconditional mean number of occurrences,

$$m_i = \mathrm{E}(N_i).$$

Let C represent a random variable with $\mathrm{E}(C) = 1$ and $\mathrm{Var}(C) = c$. In this paper, C is assumed to be Gamma distributed. The parameter c is used to model parameter uncertainty in the frequency mean and is called the contagion parameter. Let X_{ij} represent the loss size of the j^{th} loss for group i. L_i is the aggregate loss of the i^{th} group:

$$L_i = X_{i1} + X_{i2} + \ldots + X_{iN_i}.$$

Parameter uncertainty in the severity mean is modelled through a random variable B with $\mathrm{E}(1/B) = 1$ and $\mathrm{Var}(1/B) = b$. B is assumed to be Gamma distributed so $1/B$ is Inverse Gamma distributed. The parameter b is called the mixing parameter.

The Algorithm

(1) Select C at random from the assumed distribution.

(2) Select the number of loss occurrences N_i at random from a Poisson distribution with mean $C \cdot m_i$.

(3) Select B at random from the assumed distribution.

(4) Select the loss occurrence amounts $X_{i1}, X_{i2}, \ldots, X_{iN_i}$ at random from the assumed occurrence severity distribution.

(5) Compute the aggregate loss L_i as the sum of all loss occurrence amounts divided by the scaling parameter B.

Since C is assumed to be Gamma distributed, the frequency distribution generated by the above algorithm will be negative binomial. If the conditions in Appendix D are satisfied, the excess frequency distribution for each group will be approximately Poisson under conditions of parameter certainty, and the excess frequency distribution for all groups combined will also be approximately Poisson due to the independence assumptions. The negative binomial frequency distribution is used to model uncertainty in the mean frequencies.

It is assumed that the shape of the severity distribution is known, and so the mixing parameter b models uncertainty in the severity means for the various groups. If uncertainty exists concerning the shape of the severity distribution, the approach to parameter uncertainty discussed in Appendix A may be applied through assignment of subjective probabilities to alternative scenarios concerning the shape parameter. In this paper, a Single Parameter Pareto severity distribution, as discussed in Appendix E, is assumed. The examples in this paper are evaluated for the following combinations of b and c: $b = c = 0$, $b = c = .05$, $b = .10$ and $c = .05$, and $b = c = .10$. These combinations represent no parameter uncertainty, moderate parameter uncertainty, higher uncertainty concerning the mean severity but moderate uncertainty concerning the mean frequency, and higher parameter uncertainty. Although many other combinations may be appropriate for particular circumstances, these values are used in this paper to illustrate the impact of modelling parameter uncertainty.

The reader may presume that a simulation is performed by running the above algorithm a sufficiently large number of times for each group to generate an accurate estimate of its aggregate loss distribution. Once aggregate loss distributions for each group are obtained in this manner, the aggregate loss distribution for all groups combined can be estimated by conducting a second simulation as follows:

(1) For group i, select L_i at random from the aggregate loss distribution already estimated.

(2) Compute the aggregate loss L for all groups combined by summing the L_i's, $i = 1, 2, \ldots, k$.

This second simulation is performed a sufficiently large number of times to generate an accurate estimate of the aggregate loss distribution for all groups combined. (Note that aggregate limits or deductibles may be applied to individual groups before the second simulation is performed.)

Instead of performing the above simulations, the Heckman-Meyers algorithm computes the aggregate loss distribution directly through application of the characteristic function method briefly summarized in

TREATY IV
COLLECTIVE RISK MODEL

Line	Expected Loss	Claim Severity Distribution	Contagion Parameter	Claim Count Mean	Claim Count Std. Dev.
1	359,995	class1.sev	$0.0500 = c_1$	5.154	2.546
2	90,033	class2.sev	$0.0500 = c_2$	1.343	1.197

Mixing Parameter	$0.1000 = b$
Aggregate Mean	450,028
Aggregate Std. Dev.	297,472

Aggregate Loss Amount	Entry Ratio	Cumulative Probability	Excess Pure Premium	Excess Pure Premium Ratio
0.00	0.0000	0.0015	450,028.21	1.0000
90,005.64	0.2000	0.0572	362,454.04	0.8054
180,011.28	0.4000	0.1577	281,663.70	0.6259
270,016.93	0.6000	0.2988	212,038.72	0.4712
360,022.57	0.8000	0.4477	155,661.12	0.3459
450,028.21	1.0000	0.5832	112,206.06	0.2493
540,033.85	1.2000	0.6949	79,912.14	0.1776
630,039.49	1.4000	0.7811	56,513.45	0.1256
720,045.14	1.6000	0.8450	39,840.27	0.0885
810,050.78	1.8000	0.8911	28,079.72	0.0624
900,056.42	2.0000	0.9237	19,828.73	0.0441

Section 2. The reader is referred to the paper and to the excellent review by Gary Venter for technical details [4]. The alternative recursive method, which is discussed in Mr. Venter's review and in his recent CAS Forum contribution [10], is simpler and in some circumstances more accurate [5], but in other circumstances it is less efficient than the characteristic function method and requires the structure function method discussed in Appendix A to model parameter uncertainty. A sample run of the model is presented in the charts below.

TREATY IV
COLLECTIVE RISK MODEL

Line	Expected Loss	Claim Severity Distribution	Contagion Parameter	Claim Count Mean	Claim Count Std. Dev.
1	359,995	class1.sev	0.1000 = c_1	5.154	2.795
2	90,033	class2.sev	0.1000 = c_2	1.343	1.234

Mixing Parameter	0.1000 = b
Aggregate Mean	450,028
Aggregate Std. Dev.	309,940

Aggregate Loss Amount	Entry Ratio	Cumulative Probability	Excess Pure Premium	Excess Pure Premium Ratio
0.00	0.0000	0.0023	450,028.21	1.0000
90,005.64	0.2000	0.0667	363,013.42	0.8066
180,011.28	0.4000	0.1716	283,301.51	0.6295
270,016.93	0.6000	0.3120	214,944.81	0.4776
360,022.57	0.8000	0.4562	159,564.03	0.3546
450,028.21	1.0000	0.5861	116,620.91	0.2591
540,033.85	1.2000	0.6933	84,374.46	0.1875
630,039.49	1.4000	0.7767	60,690.76	0.1349
720,045.14	1.6000	0.8392	43,547.09	0.0968
810,050.78	1.8000	0.8850	31,246.92	0.0694
900,056.42	2.0000	0.9180	22,462.93	0.0499

APPENDIX D

DERIVATION OF EXCESS OCCURRENCE COUNT VARIANCE-TO-MEAN RATIO

This appendix shows that if the ground-up occurrence count distribution for an insured selected at random is negative binomial, then the excess occurrence count distribution for a randomly selected insured is also negative binomial. Based on this result, the formula for calculating the excess occurrence count variance-to-mean ratio for an individual insured selected at random is derived, and it is shown that this formula also applies to the class as a whole. This latter result is then used to demonstrate that, if the proportion of occurrences exceeding the retention is small and the excess frequency mean is known, then the excess occurrence count distribution for the class as a whole is approximately Poisson.

Assume

(1) An individual policy's distribution of ground-up occurrence counts over a given period of time is Poisson with parameter λ_i.

(2) The policies in the given class are independent and of identical size.

(3) The distribution of the individual policy expected occurrence counts (i.e., the λ_i's) over the class is Gamma with parameters a,r.

(4) The probability of a given occurrence being an excess occurrence (i.e., the probability that it exceeds a fixed retention R) is p. This probability is applicable to all policies and may be calculated from the parameters given in the ISO increased limits reviews.

Given (1) and (3) above, it follows [11] that the distribution of the observed ground-up occurrence counts for an individual policy selected at random is negative binomial with a mean $\mu_G = r/a$ and variance $\sigma_G^2 = (r/a)((a + 1)/a)$. This implies a variance-to-mean ratio $VMR_G = \sigma_G^2/\mu_G = (a + 1)/a = 1 + (1/a)$. Assuming that VMR_G is known (from the ISO increased limits reviews or elsewhere), one can easily solve for a.

It follows [1,2] from the assumptions of a Poisson process that if an individual policy's distribution of ground-up occurrence counts is Poisson with parameter λ_i, then the distribution of excess occurrence counts (claims above R) for the individual policy is also Poisson but with parameter $p\lambda_i$.

The Gamma Distribution has the property [12] that if λ has the distribution $\Gamma\ (a,r)$, then $p\lambda$ has a $\Gamma\ (a/p,r)$ distribution. Hence, the distribution of the individual policy expected excess occurrence counts over the class is $\Gamma\ (a/p,r)$.

Thus, the distribution of observed excess occurrence counts for an individual policy selected at random from the class of policies is negative binomial with a mean $\mu_E = r/[a/p] = pr/a$ and variance

$$\sigma_E^2 = \{r/[a/p]\}\ \{[a/p + 1]/[a/p]\} = [pr/a][1 + p/a].$$

This implies a variance-to-mean ratio $VMR_E = \sigma_E^2/\mu_E = 1 + p/a$. Note that since $p < 1$, $VMR_E < VMR_G$.

One can think of the group of policies covered by a particular excess reinsurance treaty as a statistical sample taken from the theoretically infinite population of all insureds belonging to the particular class [13]. Assuming that the sample is taken at random, the policies selected are independent of each other. From the above, each policy's excess occurrence count distribution has mean μ_E and variance σ_E^2. Given that n policies from the particular class are covered by the reinsurance treaty, the expected number of occurrences subject to the excess treaty is $n\mu_E$ and the variance of the number of occurrences subject to the treaty is $n\sigma_E^2$. This implies a variance-to-mean ratio of $(n\sigma_E^2)/(n\mu_E) = \sigma_E^2/\mu_E = VMR_E$ for the total number of occurrences subject to the treaty. Thus, the excess occurrence count variance-to-mean ratio for the entire group of policies covered by the reinsurance treaty equals the excess occurrence count variance-to-mean ratio for an individual policy selected at random from the class.

If VMR_G is known, a simple formula for calculating VMR_E can be easily derived using the following two relationships (which were proven above):

(1) $VMR_G = 1 + \frac{1}{a}$, and

(2) $VMR_E = 1 + \frac{p}{a}$.

Solving equation (1) for a, we get

(3) $a = \frac{1}{VMR_G - 1}$.

Substituting expression (3) into (2), we get

$$VMR_E = 1 + \frac{p}{\left[\frac{1}{VMR_G - 1}\right]}$$

$$= 1 + p(VMR_G - 1)$$

$$= (1 - p) + p(VMR_G).$$

Based on the above formula, if VMR_G is two or three, as in the ISO increased limits reviews, and p is small (say less than .02), VMR_E will be close to unity. This implies that the excess occurrence count distribution for an insured selected at random and for the class as a whole will be approximately Poisson, provided that the excess frequency mean is known. (Recall that the sum of independent Poisson random variables is also Poisson.)

APPENDIX E
SINGLE PARAMETER PARETO SEVERITY DISTRIBUTION [6]

General Properties of Model

Assume ground-up loss occurrences above the truncation point k are distributed according to the following cumulative distribution function (c.d.f):

$$F(w) = 1 - \left(\frac{k}{w}\right)^q, \text{ where } k > 0,\ q > 0,\ w \geq k.$$

Note that

$$F(w) = 1 - \left(\frac{k}{k + (w - k)}\right)^q.$$

Let $y = w - k$ represent the occurrence size excess of k. Then

$$F(y) = 1 - \left(\frac{k}{k + y}\right)^q, \text{ where } y \geq 0.$$

Thus, occurrence losses excess of the truncation point k are distributed according to the two-parameter shifted Pareto distribution with scale parameter equal to k and shape parameter equal to q.

Assume ground-up occurrences are censored at limit $k \cdot b$. Then

$$F(y) = 1 - \left(\frac{k}{k + y}\right)^q \text{ if } 0 \leq y < k\,(b - 1),$$

$$\text{and } F(y) = 1 \qquad \text{if } y \geq k(b - 1).$$

The mean censored excess occurrence is given by

$$E(y) = \frac{k\,(b^{1-q} - 1)}{1 - q} \quad \text{if } q \neq 1,$$

$$\text{and } E(y) = k \cdot \ln(b) \quad \text{if } q = 1.$$

The variance of the censored excess occurrences is given by

$$\mathrm{Var}(y) = k^2 \left[\frac{q - 2b^{2-q}}{q-2} - \left(\frac{q - b^{1-q}}{q-1}\right)^2\right] \quad \text{if } q \neq 1,\ q \neq 2,$$

$$\mathrm{Var}(y) = k^2 \left[2b - 1 - (1 + \ln(b))^2\right] \quad \text{if } q = 1, \text{ and}$$

$$\mathrm{Var}(y) = k^2 \left[1 + 2 \cdot \ln(b) - \left(\frac{2b-1}{b}\right)^2\right] \quad \text{if } q = 2.$$

Maximum Likelihood Estimation of q

Assume one wishes to compute the Maximum Likelihood Estimator (MLE) of q by fitting n loss occurrences ($W_1, W_2, \ldots, W_n$) above the truncation point k. Let X_i (for $i = 1, 2, \ldots, n$) represent the normalized losses, $X_i = W_i/k$. The c.d.f. of the normalized losses is $F(x) = 1 - x^{-q}$, which is the customary representation of the Single Parameter Pareto (SPP) distribution. Assume m_j occurrences have been censored at limit C_j and let $b_j = C_j/k$, $j = 1, 2, \ldots s$.

Let $u = n - \sum_{j=1}^{s} m_j$ represent the number of uncensored occurrences. Then the MLE of q is given by

$$\hat{q} = \frac{u}{\sum_{i=1}^{u} \ln(X_i) + \sum_{j=1}^{s} m_j \cdot \ln(b_j)},$$

where the X_i's are the uncensored normalized occurrences. If no occurrences have been censored, the MLE of q is

$$\hat{q} = \frac{n}{\sum_{i=1}^{n} \ln(X_i)}.$$

If cases are developing, q should be estimated for each accident year or policy year at each evaluation, and a triangulation approach should be used to project the ultimate estimate of q for losses in excess of the particular truncation point. If cases are not developing and q is to be estimated by pooling the losses from several years, they first need to be adjusted for trend if some of the losses have been censored.

Leveraged Impact of Inflation

Let n represent the number of loss occurrences above truncation k at time 0, and assume the annual loss inflation factor between time 0 and time t is $1 + i$. Based on the SPP distribution with parameter q, the projected number of loss occurrences excess of truncation k at time t is $n(1 + i)^{tq}$.

As long as inflation does not erode the real value of a retention to the point that the SPP distribution is no longer a satisfactory model above the retention, the parameter q and the average occurrence size in the layer of interest will theoretically remain constant over time. The leveraged impact of inflation over a fixed retention is quantified through the application of the adjustment factor $(1 + i)^{tq}$ to excess occurrence frequency.

Change in Layer

Assume that one has credibly estimated losses in the layer from a to b and wishes to estimate expected losses in the layer from c to d, where the SPP distribution with parameter q is appropriate above the lower of the two retentions. The change in expected losses due to the change in reinsurance layer is theoretically given by

$$\text{Change in Layer} = \frac{c^{1-q} - d^{1-q}}{a^{1-q} - b^{1-q}} \text{ if } q \neq 1, \text{ and}$$

$$\text{Change in Layer} = \frac{\ln (d/c)}{\ln (b/a)} \qquad \text{if } q = 1.$$

(The layer limits need not be normalized in the above formulas.) The Change in Layer factor is applied to expected losses in the layer from a to b to estimate expected losses in the layer from c to d.

APPENDIX F

TREATY I

SUMMARY OF KEY CONTRACT PROVISIONS

Treaty Period: 1/1/90–12/31/90

Layer Reinsured: $160,000 in excess of $40,000 per occurrence

Estimated Treaty Subject Premium: $12,000,000 for 1990,
distributed as follows:
Class 1—$9,000,000
Class 2—$3,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1—4.00%
Class 2—3.00%
Both Classes Combined—3.75%

Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Rate: 100/75

Proportional Coinsurance: None

Nonproportional Coinsurance: Aggregate Deductible equal to 3% of Subject Premium

APPENDIX F

EXHIBIT 1

DETERMINATION OF AGGREGATE LOSS DISTRIBUTION SPECIFICATION PARAMETERS FOR NONPROPORTIONAL LOSS SHARING PLANS

	Class of Business		
	Class 1	Class 2	All Classes Combined
(1) Actual or estimated subject premium for treaty period	9,000,000	3,000,000	12,000,000
(2) Expected layer loss cost for entire layer prior to the application of all forms of coinsurance (layer burning cost) [expressed as a percentage of subject premium]	4.0000%	3.0000%	3.7500%
(3) Expected losses for the entire reinsured layer for the treaty period [(1) × (2)]	360,000	90,000	450,000
(4) Single Parameter Pareto q values for severity distributions	0.900	0.950	
(5) Mean excess claim size in layer (μ_s)	69,848	67,039	69,267
(6) Standard deviation of excess claim sizes in layer (σ_s)	60,908	60,084	60,749
(7) Expected number of claims in layer prior to the application of nonproportional loss sharing provisions (μ_c) [(3)/(5)]	5.154	1.343	6.497
(8) Excess claim count variance-to-mean ratio (VMR_c)	1.032	1.067	1.039
(9) Standard deviation of distribution of aggregate losses in layer $[\sigma_s^2 \cdot \mu_c + (\mu_c \cdot VMR_c) \cdot \mu_s^2]^{1/2}$	212,298	106,228	237,391
(10) Coefficient of variation of distribution of aggregate losses in layer [(9)/(3)]	0.590	1.180	0.528
(11) Selected coefficient of variation of aggregate loss distribution for all classes combined			0.528

NOTES:

Lines (5) and (6): The mean excess claim size and the standard deviation of the excess claim sizes are based on a Single Parameter Pareto distribution assumption with the parameter (q value) given in item (4). (See Appendix E for formulas.) The all classes combined mean excess claim size is an average of the individual class mean claim sizes, weighted on the expected excess claim counts on line (7). The all classes combined claim size standard deviation is computed as follows:

(A) For each class of business, calculate the sum of the squares of items (5) and (6), respectively.
(B) Take a weighted average of the sums in (A), using the expected excess claim counts on line (7) as weights.
(C) Subtract the square of the all classes combined mean excess claim size from the result in (B).
(D) Take the square root of the result in (C) to obtain the all classes combined excess claim size standard deviation.

Line (8): The individual class excess claim count variance-to-mean ratios are calculated using the ISO increased limits parameters and the formulas in Appendix D. The all classes combined excess claim count variance-to-mean ratio is an average of the individual class variance-to-mean ratios, weighted on the expected claim counts on line (7).

Line (9): The standard deviation of the aggregate loss distribution for all classes combined is obtained by summing the squares of the aggregate loss distribution standard deviations for the individual classes and then taking the square root of the result.

APPENDIX F

EXHIBIT 2

AGGREGATE DEDUCTIBLES

(1) Actual or estimated subject premium for treaty period	12,000,000
(2) Expected layer loss cost for entire layer prior to the application of all coinsurance (layer burning cost) [expressed as a percentage of subject premium]	3.7500%
(3) Coinsurance percentage (cedant's participation in layer losses not corresponding to an explicit share of the reinsurance premium, excluding the presumed effect of the aggregate deductible).	0.00%
(4) Expected dollars of loss for the entire layer prior to the application of all coinsurance [(1) × (2)]	450,000
(5) Loading to convert expected layer loss cost after all forms of coinsurance into a loaded rate (expressed as a multiplicative factor to be applied to the expected layer loss cost)	1.333 = 100/75
(6) Aggregate deductible amount in dollars applicable to the entire layer [3% × $12,000,000]	360,000
(7) Entry ratio corresponding to the aggregate deductible amount [(6)/(4)]	0.800
(8) Insurance charge at entry ratio corresponding to the aggregate deductible amount*	29.33%
(9) Expected percentage of treaty losses eliminated by the aggregate deductible [100% − (8)]	70.67%
(10) Composite coinsurance percentage 100%-{[100% − (3)] × [100% − (9)]}	70.67%
(11) Expected layer loss cost for the entire reinsured portion of layer, after the application of the aggregate deductible (expressed as a percentage of subject premium) (2) × [100% − (10)]	1.0998%
(12) Indicated treaty rate after the application of the aggregate deductible and any proportional coinsurance (expressed as a percentage of subject premium) [(5) × (11)]	1.4664%

*The insurance charge appearing in item (8) above is based on a lognormal distribution with a 0.528 coefficient of variation. The insurance charge is obtained via linear interpolation of the table of insurance charges given on Tables B1–B3 of Appendix B. See Appendix F Exhibit 1 for the derivation of the 0.528 coefficient of variation.

APPENDIX G

TREATY II

SUMMARY OF KEY CONTRACT PROVISIONS

Treaty Period: 1/1/90 – 12/31/90

Layer Reinsured: $700,000 in excess of $300,000 per occurrence

Estimated Treaty Subject Premium: $6,000,000 for 1990,
distributed as follows:
Class 1 – $2,000,000
Class 2 – $2,000,000
Class 3 – $2,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1 – 10.0%
Class 2 – 14.0%
Class 3 – 21.0%
All Classes Combined – 15.0%

Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Rate: 100/60

Proportional Coinsurance: 20%

Nonproportional Coinsurance: Three (3) full free reinstatements permitted under treaty.

APPENDIX G
EXHIBIT 1

AGGREGATE LIMITS

(1) Actual or Estimated Subject Premium for Treaty Period	6,000,000
(2) Expected Layer Loss Cost for Entire Layer Prior to the Application of All Coinsurance (Layer Burning Cost) [Expressed as a Percentage of Subject Premium]	15.000%
(3) Coinsurance Percentage (Cedant's Participation in Layer Losses Not Corresponding to an Explicit Share of the Reinsurance Premium, Excluding the Presumed Effect of the Aggregate Limit Provision)	20.00%
(4) Expected Dollars of Loss for the Entire Layer Prior to the Application of All Coinsurance [(1) × (2)]	900,000
(5) Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Loaded Rate (Expressed as a Multiplicative Factor to be Applied to the Expected Layer Loss Cost)	1.667 = 100/60

Complete Item (6) if the aggregate limit is expressed as a percentage of treaty losses, or Item (7) if the aggregate limit is expressed in terms of limited reinstatements.

(6) Aggregate Limit Amount (Expressed as a Percentage of the Expected Losses for the Treaty Prior to the Application of this Provision)	N/A
(7) (A) Number of Free Reinstatements Allowed Under Treaty	3
(B) Layer Retention	300,000
(C) Layer Gross Limit	1,000,000
(D) Layer Width [(7C) − (7B)]	700,000
(E) Effective Aggregate Limit for the Entire Layer Prior to All Coinsurance (Expressed in Dollars) [1 + (7A)] × (7D)	2,800,000
(F) Effective Treaty Aggregate Limit (Expressed as a Percentage of Treaty Expected Losses) [(7E)/(4)]	311.11%
(8) *Entry Ratio Corresponding to the Aggregate Limit [(6) or (7F). Expressed as a Decimal]*	3.111
(9) Insurance Charge at the Entry Ratio Corresponding to the Aggregate Limit* (This Percentage Is Equivalent to the Expected Percentage of Treaty Losses Eliminated by the Aggregate Limit Provision)	2.37%
(10) Composite Coinsurance Percentage 100% − {[(100% − (3)] × [100% − (9)]}	21.89%
(11) Expected Layer Cost for the Entire Reinsured Portion of Layer, After the Application of the Aggregate Limit Provision (Expressed as a Percentage of Subject Premium) (2) × [100% − (10)]	11.7161%
(12) Indicated Treaty Rate After the Application of Aggregate Limits and Any Proportional Coinsurance (Expressed as a Percentage of Subject Premium) [(5) × (11)]	19.5268%

*The insurance charge appearing in Item (9) above is based on a lognormal distribution with a 0.770 coefficient of variation. The insurance charge is obtained via linear interpolation of the table of insurance charges given on Tables B1–B3 of Appendix B. A procedure similar to that employed in the Treaty I example (see Appendix F Exhibit 1) is used to derive the 0.770 coefficient of variation for aggregate losses in the reinsured layer on this treaty.

APPENDIX H
TREATY III
SUMMARY OF KEY CONTRACT PROVISIONS

Treaty Period: 1/1/90 – 12/31/90

Layer Reinsured: $400,000 in excess of $100,000 per occurrence

Estimated Treaty Subject Premium: $10,000,000 for 1990,
distributed as follows:
Class 1 – $4,500,000
Class 2 – $4,500,000
Class 3 – $1,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1 – 3.20%
Class 2 – 3.80%
Class 3 – 3.50%
All Classes Combined – 3.50%

Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Rate: 100/70

Proportional Coinsurance: None

Nonproportional Coinsurance: Loss Corridor – Reinsurer stops paying losses that fall in the reinsured layer when the ratio of actual losses in the layer to expected losses in the layer reaches 100%, but he resumes full payment of losses in the layer if this ratio goes above 200%.

APPENDIX H

EXHIBIT 1

LOSS CORRIDORS

(1) Actual or Estimated Subject Premium for Treaty Period	10,000,000
(2) Expected Layer Loss Cost for Entire Layer Prior to the Application of All Coinsurance (Layer Burning Cost) (Expressed as a Percentage of Subject Premium)	3.500%
(3) Coinsurance Percentage (Cedant's Participation in Layer Losses Not Corresponding to an Explicit Share of the Reinsurance Premium, Excluding the Presumed Effect of the Loss Corridor Provision)	0.00%
(4) Expected Dollars of Loss for the Entire Layer Prior to the Application of All Coinsurance [(1) × (2)]	350,000
(5) Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Loaded Rate (Expressed as a Multiplicative Factor to be Applied to the Expected Layer Loss Cost)	1.429 = 100/70
(6) Lower Bound of Loss Corridor Interval (Expressed as a Percentage of Expected Losses for the Treaty Prior to the Application of the Loss Corridor Provision)	100.00%
(7) Upper Bound of Loss Corridor Interval (Expressed as a Percentage of Expected Losses for the Treaty Prior to the Application of the Loss Corridor Provision)	200.00%
(8) Reinsurer's Participation Percentage in Loss Corridor Interval (If Any)	0.00%
(9) Entry Ratio Corresponding to Lower Bound of Interval [(6) Expressed as a Decimal]	1.000
(10) Insurance Charge at Entry Ratio Corresponding to Lower Bound of Interval*	30.11%
(11) Entry Ratio Corresponding to Upper Bound of Interval [(7) Expressed as a Decimal]	2.000
(12) Insurance Charge at Entry Ratio Corresponding to Upper Bound of Interval*	10.61%
(13) Percentage of Expected Treaty Losses Eliminated by the Loss Corridor Provision [(10) − (12)] × [100% − (8)]	19.51%
(14) Composite Coinsurance Percentage 100% − {[(100% − (3)] × [100% − (13)]}	19.51%
(15) Expected Loss Cost for the Entire Reinsured Portion of Layer, After the Application of the Loss Corridor Provision (Expressed as a Percentage of Subject Premium) (2) × [100% − (14)]	2.8173%
(16) Indicated Treaty Rate After the Application of the Loss Corridor Provision and Any Proportional Coinsurance (Expressed as a Percentage of Subject Premium) [(5) × (15)]	4.0248%

*The insurance charges appearing in Items (10) and (12) above are based on a lognormal distribution with a 0.905 coefficient of variation. The insurance charges are obtained via linear interpolation of the table of insurance charges given on Tables B1–B3 of Appendix B. A procedure similar to that employed in the Treaty I example (see Appendix F Exhibit 1) is used to derive the 0.905 coefficient of variation for aggregate losses in the reinsured layer on this treaty.

APPENDIX I
TREATY IV
Summary of Key Contract Provisions

Treaty Period: 1/1/90–12/31/90

Layer Reinsured: $160,000 in excess of $40,000 per occurrence

Estimated Treaty Subject Premium: $12,000,000 for 1990, distributed as follows:
Class 1—$9,000,000
Class 2—$3,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1—4.00%
Class 2—3.00%
Both Classes Combined—3.75%

Indicated Flat Treaty Rate After the Application of All Forms of Coinsurance (Expressed as a Percentage of Subject Premium): 5.00%

Proportional Coinsurance: None

Nonproportional Coinsurance: None

Retrospective Rating Plan:
Adjustment Period—1/1/90 through 12/31/90 (1 year)
Adjustment Formula—
Adjusted Treaty Premium = 100/75 × (Incurred Losses and ALAE in Layer), subject to a maximum of 10.00% of subject premium and a minimum of 3.00% of subject premium.

APPENDIX I
EXHIBIT I

Adjustable Premiums (Retrospective Rating)

(1) Actual or estimated subject premium for the retrospective rating period	12,000,000
(2) Expected layer loss cost for entire layer prior to the effects of the retro plan (expressed as a percentage of subject premium)	3.7500%
(3) Coinsurance percentage (cedant's participation in layer losses not corresponding to an explicit share of the reinsurance premium, excluding the effects of nonproportional loss sharing plans).	0.00%
(4) Percentage reduction in layer losses due to nonproportional loss sharing provisions only	0.00%
(5) Expected loss cost for entire reinsured portion of layer prior to the effects of the retro plan (expressed as a percentage of subject premium) (2) × [100% − (3)] × [100% − (4)]	3.7500%
(6) Maximum rate (expressed as a percentage of subject premium)	10.000%
(7) Minimum rate (expressed as a percentage of subject premium)	3.000%
(8) Multiplicative loss load (loss conversion factor)	133.333% = 100/75
(9) Additive loss load (flat margin)	0.0000%
(10) Loss cost corresponding to the maximum rate [(6) − (9)]/(8)	7.5000%
(11) Entry ratio corresponding to the maximum rate [(10)/(5)]	2.000
(12) Insurance charge at maximum (excess loss percentage corresponding to maximum entry ratio)*	2.60%
(13) Loss cost corresponding to the minimum rate [(7) − (9)]/(8)	2.2500%
(14) Entry ratio corresponding to the minimum rate [(13)/(5)]	0.600
(15) Insurance charge at minimum (excess loss percentage corresponding to minimum entry ratio)*	43.02%
(16) Insurance savings at minimum [100% × (14)] + (15) − 100%	3.02%
(17) Net insurance charge [(12) − (16)]	−0.42%
(18) Adjusted expected layer loss cost (expected value of losses limited by the retro plan maximum and minimum) (5) × [100% − (17)]	3.7656%
(19) (A) Guaranteed cost treaty rate (equivalent treaty rate if contract were flat rated) (expressed as a percentage of subject premium)	5.0000%
(B) Expected treaty rate after retro adjustments (expressed as a percentage of subject premium) [(8) × (18)] + (9)	5.0208%
(C) Retro plan off-balance factor [(19A)/(19B)] (a factor greater than 1.000 indicates a plan inadequacy; while a factor less than 1.000 indicates a plan redundancy)	0.996

*The insurance charges appearing in items (12) and (15) above are based on a lognormal distribution with a 0.528 coefficient of variation. The insurance charges are obtained via linear interpolation of the table of insurance charges given on Tables B1–B3 of Appendix B. A procedure similar to that employed in the Treaty I example (see Appendix F Exhibit 1) is used to derive the 0.528 coefficient of variation for aggregate losses in the reinsured layer on this treaty.

APPENDIX J

TREATY V

Summary of Key Contract Provisions

Treaty Period: 1/1/90–12/31/90

Layer Reinsured: $700,000 in excess of $300,000 per occurrence

Estimated Treaty Subject Premium: $6,000,000 for 1990,
distributed as follows:
Class 1—$2,000,000
Class 2—$2,000,000
Class 3—$2,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1—10.0%
Class 2—14.0%
Class 3—21.0%
All Classes Combined—15.0%

Treaty Rate (Expressed as a Percentage of Subject Premium): 25.0%

Proportional Coinsurance: 20%

Nonproportional Coinsurance: None

Profit Commission Plan: Adjustment Period—1/1/90 through 12/31/92 (3 years). Reinsurer to pay cedant 25% of the amount by which treaty premiums during the Adjustment Period exceed incurred losses, ALAE, and a 20% provision for the reinsurer's overhead expense.

APPENDIX J

EXHIBIT 1

DETERMINATION OF AGGREGATE LOSS DISTRIBUTION SPECIFICATION PARAMETERS FOR A SINGLE TREATY YEAR

	Class of Business			
	Class 1	Class 2	Class 3	All Classes Combined
(1) Actual or estimated subject premium for treaty period	2,000,000	2,000,000	2,000,000	6,000,000
(2) Expected layer loss cost for entire layer prior to the application of all forms of coinsurance (layer burning cost) [expressed as a percentage of subject premium]	10.0000%	14.0000%	21.0000%	15.0000%
(3) Expected losses for the entire reinsured layer for the treaty period [(1) × (2)]	200,000	280,000	420,000	900,000
(4) Single Parameter Pareto q values for severity distributions	1.500	1.300	1.100	
(5) Mean excess claim size in layer (μ_S)	271,366	303,155	340,296	310,897
(6) Standard deviation of excess claim sizes in layer (σ_S)	246,592	257,600	266,584	260,265
(7) Expected number of claims in layer prior to the application of nonproportional loss sharing provisions (μ_C) [(3)/(5)]	0.737	0.924	1.234	2.895
(8) Excess claim count variance-to-mean ratio (VMR_C)	1.006	1.009	1.019	1.012
(9) Standard deviation of distribution of aggregate losses in layer $[\sigma_S^2 \cdot \mu_C + (\mu_C \cdot VMR_C) \cdot \mu_S^2]^{1/2}$	315,301	383,323	483,065	692,606
(10) Coefficient of variation of distribution of aggregate losses in layer [(9)/(3)]	1.577	1.369	1.150	0.770

NOTES:

Lines (5) and (6): The mean excess claim size and the standard deviation of the excess claim sizes are based on a Single Parameter Pareto distribution assumption with the parameter (q value) given in item (4). (See Appendix E for formulas.) The all classes combined mean excess claim size is an average of the individual class mean claim sizes, weighted on the expected excess claim counts on line (7). The all classes combined claim size standard deviation is computed as follows:

(A) For each class of business, calculate the sum of the squares of items (5) and (6), respectively.

(B) Take a weighted average of the sums in (A), using the expected excess claim counts on line (7) as weights.

(C) Subtract the square of the all classes combined mean excess claim size from the result in (B).

(D) Take the square root of the result in (C) to obtain the all classes combined excess claim size standard deviation.

Line (8): The individual class excess claim count variance-to-mean ratios are calculated using the ISO increased limits parameters and the formulas in Appendix D. The all classes combined excess claim count variance-to-mean ratio is an average of the individual class variance-to-mean ratios, weighted on the expected claim counts on line (7).

Line (9): The standard deviation of the aggregate loss distribution for all classes combined is obtained by summing the squares of the aggregate loss distribution standard deviations for the individual classes and then taking the square root of the result.

APPENDIX J

EXHIBIT 2

DETERMINATION OF AGGREGATE LOSS DISTRIBUTION SPECIFICATION PARAMETERS for ADJUSTABLE PREMIUM OR COMMISSION PLANS

	Adjustment Period			
Dates of Individual Contract Years in Adjustment Period →	Year 1 1/90–12/90	Year 2 1/91–12/91	Year 3 1/92–12/92	Total Adjustment Period
(1) Actual or estimated subject premiums for all classes combined	6,000,000	6,000,000	6,000,000	18,000,000
(2) (A) Expected layer loss cost for entire layer prior to the application of all forms of coinsurance (layer burning cost) [expressed as a percentage of subject premium]	15.0000%	15.0000%	15.0000%	15.0000%
(B) Percentage reduction in layer losses due to nonproportional loss sharing provisions. (Ignore all proportional forms of coinsurance.)	0.00%	0.00%	0.00%	0.00%
(C) Expected layer loss cost for entire layer after the application of nonproportional loss sharing provisions only (expressed as a percentage of subject premium) (2A) × [100% − (2B)]	15.0000%	15.0000%	15.0000%	15.0000%
(3) Expected losses for entire reinsured layer after the effect of all nonproportional coinsurance provisions (1) × (2C)	900,000	900,000	900,000	2,700,000
(4) Mean excess claim size in layer (μ_s) [copied from Appendix J Exhibit 1]	310,897	310,897	310,897	310,897
(5) Standard deviation of excess claim sizes in layer (σ_s) [copied from Appendix J Exhibit 1]	260,265	260,265	260,265	260,265
(6) Expected number of claims in layer [(3)/(4)] (μ_c)	2.895	2.895	2.895	8.685
(7) Excess claim count variance-to-mean ratio (VMR_c) [copied from Appendix J Exhibit 1]	1.012	1.012	1.012	1.012
(8) Standard deviation of distribution of aggregate losses in layer $[\sigma_s^2 \cdot \mu_c + (\mu_c \cdot VMR_c) \cdot \mu_s^2]^{1/2}$	692,606	692,606	692,606	1,199,629
(9) Coefficient of variation of distribution of aggregate losses [(8)/(3)]	0.770	0.770	0.770	0.444
(10) Selected coefficient of variation of aggregate loss distribution for all years in the adjustment block combined				0.444

NOTE: The values of the various items in the total adjusment period column above are calculated using formulas identical to those used to compute the values of similar items shown in the all classes combined column on Appendix J Exhibit 1. (Simply substitute the word "year" for "class.") See the footnotes on the bottom of Appendix J Exhibit 1 for a description of these formulas.

APPENDIX J
EXHIBIT 3A

PROFIT COMMISSIONS

(1) Actual or estimated subject premium for commission adjustment period	18,000,000
(2) Expected layer loss cost for entire layer prior to the application of all coinsurance (layer burning cost) [expressed as a percentage of subject premium]	15.0000%
(3) Coinsurance percentage (cedant's participation in layer losses not corresponding to an explicit share of the reinsurance premium, excluding the effects of nonproportional loss sharing plans.)	20.00%
(4) Percentage reduction in layer losses due to nonproportional loss sharing provisions only	0.00%
(5) Treaty rate [expressed as a percentage of subject premium]	25.000%
(6) Expected treaty loss & ALAE ratio (*ELR*) {(2) × [100% − (3)] × [100% − (4)]}/(5)	48.00%

Profit commission formula is in the form:

Profit commission ratio = (*P*) * [100%—treaty loss & ALAE ratio—*EXP*],
subject to a maximum commission ratio

Where: (*P*) = proportion of profits to be paid to cedant:
EXP = reinsurer's overhead expense provision

(7) Proportion of profits to be paid to the cedant (*P*)	25.00%
(8) Reinsurer's overhead provision (*EXP*) [expressed as a percentage of treaty premium]	20.00%
(9) Maximum profit commission ratio (if different from that obtained when a zero loss & ALAE ratio is plugged into the formula above) [expressed as a percentage of treaty premium]	N/A
(10) Simplistic estimate of the expected profit commission ratio [expressed as a percentage of treaty premium] (7) × [100% − (6) − (8)], subject to a maximum of (9)	8.00%

APPENDIX J
EXHIBIT 3B

PROFIT COMMISSIONS (CONTINUED)

(11) Breakeven loss & ALAE ratio for profit commission purposes [100% − (8)]	80.00%
(12) Entry ratio corresponding to breakeven point [(11)/(6)]	1.667
(13) Insurance charge at breakeven point* (excess loss percentage corresponding to breakeven entry ratio)	3.09%
(14) Loss & ALAE ratio corresponding to the maximum profit commission ratio 100% − (8) − [(9)/(7)]	0.00%
(15) Entry ratio corresponding to the maximum profit loss & ALAE ratio [(14)/(6)]	0.000
(16) Insurance charge at the maximum profit loss & ALAE ratio* (excess loss percentage corresponding to the maximum profit loss & ALAE ratio)	N/A
(17) Insurance savings at the maximum profit loss & ALAE ratio [100% × (15)] + (16) − 100%	0.00%
(18) Net insurance charge (*NIC*) [(13) − (17)]	3.09%
(19) Actuarial estimate of the expected profit commission ratio (expressed as a percentage of treaty premium) (7) × {100% − (6) × [100% − (18)] − (8)}, subject to a maximum of (9); or $P \times \{100\% - ELR \times [100\% - NIC] - EXP\}$, subject to the maximum.	8.37%
(20) Amount by which the actuarial estimate of the expected profit commission ratio exceeds the simplistic estimate [(19) − (10)]	0.37%

*The insurance charges appearing in items (13) and (16) above are based on a lognormal distribution with a 0.444 coefficient of variation. The insurance charge is obtained via linear interpolation of the table of insurance charges given in Tables B1–B3 of Appendix B. See Exhibits 1 and 2 of Appendix J for the derivation of the 0.444 coefficient of variation.

APPENDIX K

TREATY VI

Summary of Key Contract Provisions

Treaty Period: 1/1/90–12/31/90

Layer Reinsured: $900,000 in excess of $100,000 per occurrence

Estimated Treaty Subject Premium: $25,000,000

Expected Layer Loss Cost for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium): 10.0%

Treaty Rate (Expressed as a Percentage of Subject Premium): 20.0%

Proportional Coinsurance: None

Nonproportional Coinsurance: None

Sliding Scale Commission Plan:
- Adjustment Period—1/1/90 through 12/31/90 (1 year)
- Plan—Minimum Commission of 20% at a 65% loss ratio.
  - Commission increases by 0.5% for each 1% decline in loss ratio for loss ratios between 55% and 65%.
  - Commission increases by 0.75% for each 1% decline in loss ratio for loss ratios between 35% and 55%.
  - Maximum Commission of 40% at a 35% loss ratio.

APPENDIX K

EXHIBIT 1

SLIDING SCALE COMMISSIONS

(1) Actual or estimated subject premium for commission adjustment period	25,000,000
(2) Expected layer loss cost for the entire layer (expressed as a percentage of subject premium)	10.0000%
(3) Coinsurance percentage (cedant's participation in layer losses not corresponding to an explicit share of the reinsurance premium, excluding the effects of nonproportional loss sharing plans.)	0.00%
(4) Percentage reduction in layer losses due to nonproportional loss sharing provisions only	0.00%
(5) Treaty rate [expressed as a percentage of subject premium]	20.000%
(6) Expected treaty loss & ALAE ratio (*ELR*) {(2) × [100% − (3)] × [100% − (4)]}/(5)	50.00%

(7) Minimum commission ratio 20.00%;
corresponding loss & ALAE ratio 65.00%

(8) The details of the sliding scale commission plan are summarized in columns (A) through (E). Values used in the calculation of the expected sliding scale commission are given in columns (F) through (I).

Loss & ALAE Ratio Interval		(C) Percentage Increase in Commission Ratio Per 1% Decrease in Loss & ALAE Ratio	Corresponding Commission Ratio Interval		(F) Entry Ratio Corresponding to Lower Bound Loss Ratio in Column (A)	(G) Insurance Charge Corresponding to Lower Bound Entry Ratio in Column (F)	(H) Expected Loss Ratio Points in Interval	(I) Expected Reductions from Maximum Commission Rate (C) × (H)
(A) Lower Bound	(B) Upper Bound		(D) Lower Bound	(E) Upper Bound				
65.00% and above		0.00%	20.00%	20.00%	1.300	9.12%	4.56%	0.00%
55.00%	65.00%	0.50%	25.00%	20.00%	1.100	14.47%	2.68%	1.34%
35.00%	55.00%	0.75%	40.00%	25.00%	0.700	34.80%	10.16%	7.62%
0.00%	35.00%	0.00%	40.00%	40.00%	0.000	100.00%	32.60%	0.00%
						Total	50.00%	8.96%

(9) Expected ceding commission ratio from a simplistic point of view [commission ratio corresponding to the treaty *ELR* (item 6), given the plan above.]	28.75%
(10) Expected commission ratio from an actuarial point of view [maximum commission ratio—(total 8I)]	31.04%
(11) Amount by which the actuarial estimate of the expected commission ratio exceeds the simplistic estimate [(10) − (9)]	2.29%

NOTES:

On this exhibit, all commission and loss & ALAE ratios are expressed as percentages of treaty premium.

Column (8G): The insurance charges appearing in this column are based on a lognormal distribution with a 0.485 coefficient of variation. These insurance charges are obtained via linear interpolation of the table of insurance charges given in Tables B1–B3 of Appendix B. A procedure similar to that employed in the Treaty I example (see Appendix F Exhibit 1) is used to derive the 0.485 coefficient of variation for aggregate losses in the reinsured layer on this treaty.

Column (8H): The values in this column are obtained by multiplying the differences between the insurance charges corresponding to consecutive loss & ALAE ratio interval end points in column (8G) by the expected treaty loss & ALAE ratio (item 6).

APPENDIX L
DERIVATION OF A FORMULA FOR CALCULATING THE EXPECTED CEDING COMMISSION UNDER A PIECEWISE LINEAR SLIDING SCALE COMMISSION PLAN

This appendix outlines the derivation of a concise formula for computing the expected ceding commission under a typical sliding scale commission plan. The derivation involves three major steps, as summarized below.

Step 1: Let $L_1, L_2, L_3, \ldots, L_n$ be a series of loss ratios such that $L_1 > L_2 > \ldots > L_n = 0$. This sequence divides the range of possible loss ratios into n consecutive intervals, starting with the first interval $[L_1,\infty)$, followed by the intervals $[L_i, L_{i-1}]$, where $i = 2,3,\ldots,n$. Define $f(L_i)$ to be the ceding commission ratio corresponding to an L_i loss ratio, $i = 1,2,\ldots,n$. Using this notation, $f(L_1)$ represents the minimum commission ratio C_{min}, while $f(L_n)$ equals the maximum commission ratio C_{max}. Furthermore, let b_i represent the commission slide (i.e., the percentage point increase in commission ratio per 1% decline in loss ratio) on the interval $[L_i, L_{i-1}]$, $i = 2,3,\ldots,n$. Also define b_1 to be zero, since the commission ratio is constant (at $C_{\min}$) on the interval $[L_1,\infty)$.

Using the notation defined above, the typical sliding scale commission plan may be expressed as a piecewise linear function of the loss ratio L in the following form:

$$(1) \qquad C = f(L) = \begin{cases} f(L_1) = C_{\min} & \text{if } L \geq L_1 \\ f(L_1) + b_2(L_1 - L) & \text{if } L_2 \leq L < L_1 \\ f(L_2) + b_3(L_2 - L) & \text{if } L_3 \leq L < L_2 \\ \quad\cdot & \quad\cdot \\ \quad\cdot & \quad\cdot \\ \quad\cdot & \quad\cdot \\ f(L_{n-1}) + b_n(L_{n-1} - L) & \text{if } 0 = L_n \leq L < L_{n-1} \end{cases}$$

Step 2: Let $p(L)$ be the probability density function of L. Then the expected ceding commission ratio $\mathrm{E}(C)$ is the following:

$$(2) \qquad \mathrm{E}(C) = \int_L f(L)p(L)dL$$

$$= \int_{L_1}^{\infty} f(L_1)p(L)dL$$

$$+ \sum_{i=2}^{n} \int_{L_i}^{L_{i-1}} [f(L_{i-1}) + b_i(L_{i-1} - L)]p(L)dL.$$

Let $M = \mathrm{E}(L) =$ Expected treaty loss ratio,
$P(L)$ be the cumulative distribution function of L, and
$P_1(L)$ be the first moment distribution function of L.

By definition,

$$P(L_i) = \int_0^{L_i} p(L)dL \text{ and } P_1(L_i) = \frac{1}{M}\int_0^{L_i} Lp(L)dL$$

for any value L_i.

The above definitions allow one to simplify equation (2), since the integral expressions appearing in this equation can easily be stated in terms of $P(L_i)$ and $P_1(L_i)$. Now define $P_2(L)$ to be the excess pure premium ratio at loss ratio L. The reader may recall that the excess pure premium ratio is expressible in terms of $P(L)$ and $P_1(L)$ as follows:

$$(3) \qquad P_2(L) = [1 - P_1(L)] - \frac{L}{M}[1 - P(L)].$$

The relationship given in (3) is used to eliminate all the $P_1(L_i)$ terms in the simplified version of equation (2) discussed above. The result is an expression for $\mathrm{E}(C)$ stated in terms of cumulative distribution function values and excess pure premium ratios.

Step 3: The remainder of the proof involves further algebraic simplification of the expression for E(C). In particular, the facts that

$$f(L_i) = f(L_{i-1}) + b_i(L_{i-1} - L_i) \text{ and that}$$

$$C_{\max} = f(L_1) + \sum_{i=2}^{n} b_i(L_{i-1} - L_i)$$

are employed. The final result is the following expression for the expected sliding scale commission. Note that all the cumulative distribution function terms have cancelled out. (We define L_0 to be infinity, so that $P_2(L_0) = 0$.)

$$\text{(4)} \qquad \mathrm{E}(C) = C_{\max} - M \sum_{i=1}^{n} b_i[P_2(L_i) - P_2(L_{i-1})].$$

Equation (4) provides a convenient formula for calculating the expected ceding commission ratio under a piecewise linear sliding scale plan, since one needs only a description of the plan, the expected treaty loss ratio M, and the appropriate table of excess pure premium ratios in order to use it.

Based on the definitions given above for M and P_2, it follows that the expression $M[P_2(L_i) - P_2(L_{i-1})]$ represents the expected number of loss ratio points falling in the interval from L_i to L_{i-1}. Hence equation (4) may be expressed verbally as follows:

$$\text{(5)} \qquad \mathrm{E}(C) = C_{\max} - \sum_{i=1}^{n} b_i \begin{Bmatrix} \text{Expected loss ratio points in the} \\ \text{interval from } L_i \text{ to } L_{i-1} \end{Bmatrix},$$

where: E(C) is the expected commission ratio,

$C_{\max}$ is the maximum commission ratio, and

b_i is the commission slide on the i^{th} loss ratio interval.

Since the product of b_i and the expected number of loss ratio points in the i^{th} interval represents the expected number of commission points lost in that interval, it follows from (5) that the expected ceding commission equals the maximum commission ratio minus the expected points of commission lost over the entire range of possible loss ratios. This provides an intuitive justification of the formula given in (4) above.

APPENDIX M
USE OF A MIXTURE OF A LOGNORMAL AND A DISCRETE DISTRIBUTION TO MODEL AGGREGATE LOSSES

If there is a positive probability that a particular reinsurance treaty will have a loss-free year, then the lognormal model cannot be used to specify the aggregate loss distribution for the treaty. This is due to the fact that the lognormal distribution is not defined at the value zero.

One solution to this problem involves the use of a mixture of a lognormal and a discrete distribution (hereafter referred to as the mixed lognormal distribution) to model aggregate losses. This distribution is defined as follows:

$$(1) \qquad f(r) = \begin{cases} p & \text{if } r = 0 \\ (1 - p) \cdot h(r) & \text{if } r > 0 \end{cases}$$

where r is the entry ratio;
$f(r)$ is the mixed lognormal probability density function (p.d.f.);
p is the probability of a loss-free year;
$h(r)$ is the p.d.f. for a lognormal distribution with parameters μ and σ^2 (the values for these are given below).

Intuitively, the reader may think of the mixed lognormal distribution f as a weighted average of a discrete distribution of unity (which is defined only at the zero entry ratio) and a lognormal distribution h (which is defined at positive entry ratios), using the loss-free probability p and its complement, respectively, as weights. The value for p is determined either subjectively or by analyzing past treaty experience, if the latter is credible. Notice that $f(r)$ becomes a lognormal p.d.f. when p is zero.

It can be shown that for a mixed lognormal distribution, the excess pure premium ratio at a particular entry ratio r is given by

$$(2) \qquad P_2(r) = [1 - H_1(r)] - r(1 - p)[1 - H(r)],$$

where H and H_1 are the cumulative density function (c.d.f.) and first moment distribution function, respectively, corresponding to the lognormal p.d.f. h. (If $p = 0$, this formula reduces to that given for the lognormal distribution in Appendix B.)

To evaluate the above expression, one needs to determine the values of H and H_1 at the particular entry ratio r. This is accomplished by noting that the lognormal distribution h has parameters

(3) $$\mu = -\frac{3}{2}\ln(1 - p) - \frac{1}{2}\ln(1 + \mathrm{CV}^2) \text{ and}$$

(4) $$\sigma^2 = \ln(1 - p) + \ln(1 + \mathrm{CV}^2),$$

where CV is the coefficient of variation of the treaty's aggregate loss distribution.

It is important to note that the quantity CV measures the variability inherent among all possible loss amounts on the treaty, including loss amounts of zero, even though the lognormal p.d.f. h is defined only at positive loss amounts.

A value for the CV can be calculated from expected aggregate loss cost estimates, together with assumptions on the treaty's frequency and severity distributions, using the same algorithm as used in the lognormal model. Note again that the expressions for μ and σ^2 reduce to the lognormal model formulas in Appendix B when $p = 0$. (The fact that the quantity CV used in the development of the lognormal model measures the variability inherent only among positive treaty loss amounts, as opposed to that among all possible loss amounts, is the reason the μ and σ^2 expressions given above differ from those in Appendix B when $p > 0$.)

The calculation of $H(r)$ and $H_1(r)$ can be achieved via a transformation from a lognormal to a standard normal distribution. Recall that if a distribution is lognormal with parameters μ and σ^2, then its first moment distribution is also lognormal but with parameters $\mu + \sigma^2$ and σ^2. Hence,

(5) $\quad H(r) = \Phi(z)$ and

(6) $\quad H_1(r) = \Phi(z_1)$,

where Φ is the cumulative standard normal distribution;

$$z = \frac{\ln(r) - \mu}{\sigma};$$

$$z_1 = \frac{\ln(r) - \mu}{\sigma} - \sigma.$$

(μ and σ are defined in (3) and (4) above.)

The reader should be aware that the mixed lognormal distribution model is valid only when

(7) $$p < \frac{\mathrm{CV}^2}{1 + \mathrm{CV}^2}$$

If the above condition does not hold, then the expression for the lognormal parameter σ^2 in (4) becomes negative, which is impossible. In this case, the aggregate loss distribution must be determined by another approach, such as the Collective Risk Model.

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