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PROCEEDINGS

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FOREWORD

The Casualty Actuarial Society was organized in 1914 as the Casualty Actuarial and Statistical Society of America, with 97 charter members of the grade of Fellow; the Society adopted its present name on May 14, 1921.

Actuarial science originated in England in 1792, in the early days of life insurance. Due to the technical nature of the business, the first actuaries were mathematicians; eventually their numerical growth resulted in the formation of the Institute of Actuaries in England in 1848. The Faculty of Actuaries was founded in Scotland in 1856, followed in the United States by the Actuarial Society of America in 1889 and the American Institute of Actuaries in 1909. In 1949 the two American organizations were merged into the Society of Actuaries.

In the beginning of the twentieth century in the United States, problems requiring actuarial treatment were emerging in sickness, disability, and casualty insurance—particularly in workers' compensation—which was introduced in 1911. The differences between the new problems and those of traditional life insurance led to the organization of the Society. Dr. I. M. Rubinow, who was responsible for the Society's formation, became its first president. The object of the Society was, and is, the promotion of actuarial and statistical science as applied to insurance other than life insurance. Such promotion is accomplished by communication with those affected by insurance, presentation and discussion of papers, attendance at seminars and workshops, collection of a library, research, and other means.

Since the problems of workers' compensation were the most urgent, many of the Society's original members played a leading part in developing the scientific basis for that line of insurance. From the beginning, however, the Society has grown constantly, not only in membership, but also in range of interest and in scientific and related contributions to all lines of insurance other than life, including automobile, liability other than automobile, fire, homeowners and commercial multiple peril, and others. These contributions are found principally in original papers prepared by members of the Society and published in the annual *Proceedings*. The presidential addresses, also published in the *Proceedings*, have called attention to the most pressing actuarial problems, some of them still unsolved, that have faced the insurance industry over the years.

The membership of the Society includes actuaries employed by insurance companies, ratemaking organizations, national brokers, accounting firms, educational institutions, state insurance departments, and the federal government; it also includes independent consultants. The Society has two classes of members, Fellows and Associates. Both classes are achieved by successful completion of examinations, which are held in May and November in various cities of the United States and Canada.

The publications of the Society and their respective prices are listed in the *Yearbook* which is published annually. The *Syllabus of Examinations* outlines the course of study recommended for the examinations. Both the *Yearbook*, at a $20 charge, and the *Syllabus of Examinations*, without charge, may be obtained upon request to the Casualty Actuarial Society, 1100 North Glebe Road, Suite 600, Arlington, Virginia 22201.

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NOTICE

Papers submitted to the *Proceedings of the Casualty Actuarial Society* are subject to review by the members of the Committee on Review of Papers and, where appropriate, additional individuals with expertise in the relevant topics. In order to qualify for publication, a paper must be relevant to casualty actuarial science, include original research ideas and/or techniques or have special educational value, and must not have been previously published or be concurrently considered for publication elsewhere. Specific instructions for preparation and submission of papers are included in the *Yearbook* of the Casualty Actuarial Society.

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PROCEEDINGS

May 13, 14, 15, 16, 1990

EVALUATING THE EFFECT OF REINSURANCE CONTRACT TERMS

JAMES N. STANARD
RUSSELL T. JOHN

Abstract

In many reinsurance pricing situations it is not possible to determine a "correct" absolute price without making a large number of tenuous assumptions. Even so, in order to maximize a company's profitability, it is important for the reinsurance actuary and underwriter to be able to choose the best contract terms among the achievable alternatives. Furthermore, being able to offer different but equivalent terms that better serve the needs of the cedant may help close an important deal.

This paper measures the efficiency of contract terms by estimating the distribution of the present value of cash flows. To do this, the paper examines paid and incurred aggregate distributions as a function of time over the life of a contract. Sensitivity of the results to changes in the parameters of the underlying loss model is investigated.

The authors wish to thank Todd J. Hess for his patience in reading many drafts of this paper and suggesting numerous improvements. He also programmed the analytical model, verified the many cash flow formulas, and produced the accompanying exhibits and graphs.

1. INTRODUCTION

In many reinsurance pricing situations it is not possible to determine a "correct" *absolute* price without making a large number of tenuous assumptions. However, it is often advantageous to make some general statements about *relative* price adequacy. By *relative* price adequacy we mean statements (about a particular layer of subject business), such as:

1. Deal #1 is better than deal #2.
2. Deal #1 is equivalent to deal #2.
3. A deal is better than it was last year.
4. The reinsurer's side of a deal is better than the company's side.

Even if the underwriter cannot accurately estimate an adequate absolute price, consistently choosing the best contract terms among achievable alternatives is important to a company's profitability. Also, being able to offer different but equivalent terms that may better serve the needs of the cedant can help close a deal.

This paper will explore a method to compare *relative* prices for many types of reinsurance contracts, and look at how sensitive the results are to the parameters of the underlying model of losses.

Commonly used methods that utilize ultimate aggregate loss distributions can give some view of the relative price. However, this alone can sometimes lead to incorrect conclusions with regard to maximizing profitability. Additional insight into the relative prices can be seen by examining the distribution of cash flows and the accompanying investment income. To do this, the paper examines paid and incurred aggregate distributions as a function of time over the life of a contract.

Few papers in the casualty actuarial literature have dealt with the cash flow of a contract. For example, Meyers [6] includes investment income to determine the parameters of a primary retrospective rating plan which yields a desired operating profit. Lee [4] uses graphical techniques to lend insight into excess of loss coverages and retrospective rating. Bühlman and Jewell [1], Gerber [2], and Lemaire and Quairiere [5] consider optimal reinsurance and risk exchanges. However, these papers do not consider investment income and only deal with simplified reinsurance contract types (e.g., quota share contracts).

The procedure described herein uses a stochastic model to estimate the distribution of the present value of cash flows. The paper's emphasis will be to derive results that are applicable to real-life pricing decisions. The approach will be to summarize key information rather than to find the single "optimal" solution.

2. AN EXAMPLE

Imagine that it is December 28 and you are a Lloyds underwriter with a long queue of brokers waiting at your box. You are discussing a treaty reinsurance proposal for losses $250,000 excess of $250,000 per loss on a portfolio of long haul trucking liability business that generates a total premium of $5,000,000 (net of commissions). You are very familiar with this account; you have estimated the expected losses to the reinsurance layer as being $1,500,000 (30% of the total subject premium). You are the lead underwriter, so it is up to you to quote terms. After several days of back and forth discussions among you, the broker, and the company, the broker has summarized three types of proposals that he thinks will be acceptable to the company. He wants to know on which one(s) you will give a firm quotation. The alternatives are[1]:

1. Reinsurance premium = 10% of subject premium (sp).
   Aggregate deductible = 20% of sp.
   Aggregate limit = 400% of reinsurance premium.
2. Retrospectively rated contract.
   Provisional premium = 8% of sp.
   Maximum premium = 30% of sp.
   Premium adjusted monthly to 110% of paid losses plus 8% of sp.
   Aggregate limit = 200% of reinsurance premium.
3. Reinsurance premium = 27% of sp.
   Profit sharing after four years of 60% of reinsurer's profit after 10% deduction of reinsurance premium (i.e., 2.7% of sp), on a paid loss basis.
   Aggregate limit = 150% of reinsurance premium.

[1] The alternative contracts will be explained more fully in section 5. Further discussion of reinsurance contracts and terminology may be found in Lee [4], Patrik and John [7], and Reinarz [10].

3. NOTATION

The following notation will be used with respect to the reinsurance layer[2]:

1. N, random number of excess losses,
2. P_t, random variable denoting aggregate paid losses at time t,
3. K_t, random variable denoting aggregate known loss reserves at time t. Note that P_t and K_t can be viewed as the sum of a random number of individual paid or known reserved losses.
4. R_t, random variable denoting reinsurance premium at time t. This may be a function of paid or incurred losses.
5. C_t, random variable denoting the cumulative cash flow (positive and negative) for the reinsurance contract at time t. This is a function of the contract terms, R_t, P_t, and K_t.
6. V, random variable denoting the present value of the net cash flow to the reinsurer defined as:

$$V = \sum_t [C_t - C_{t-1}]v^{t-1}; \; v = \frac{1}{1+i}$$

In addition, it is assumed that losses occur mid-year; premium and loss transactions are made at mid-year; and, production and overhead expenses are ignored.

With this information, one can investigate properties of V in order to judge what set of contract terms is most efficient over a broad range of reasonable assumptions.

4. CRITERIA FOR JUDGING THE EFFECT OF CONTRACT TERMS

There are three ways that a reinsurance contract affects a reinsurer:

Economic Impact: Present value of cash flows, V, from the transaction (pre-tax). The interest rate is assumed to be non-random and known in advance.

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[2] Random variables are denoted by capital letters and non-random quantities are denoted with small letters.

Accounting Impact: An income statement and balance sheet are determined by the contract terms and R_t, P_t, and K_t. Two different reinsurance contracts can produce the same C_t's and therefore have the same economic value, but have very different accounting effects.[3]

Tax Impact: The tax impact is determined from the accounting impact and affects the after-tax economic impact.

This paper considers only the economic impact.

5. DESCRIPTION OF COMMON CONTRACT TYPES

For the purposes of measuring their economic impact, many different types of reinsurance contracts (such as sliding scale commissions, retrospective rating plans, funded programs, aggregate caps, etc.) reduce to a few basic features.

The simplest types of contracts are those where C_t is a function of only P_t, and the function does not vary over different ranges of t. For these, a useful first step in analyzing the economic effect is to graph C as a function of P.

In other words, we are graphing the cumulative cash flow (prior to interest) to the reinsurer (through t) as a function of the underlying paid losses to the contract. The reinsurer prefers larger C's and prefers C's which are less than zero at P's that have a low probability. We would normally expect C to be a declining function of P (as losses increase, the reinsurer's result deteriorates), but this is not always the case.

The following graphs illustrate the functioning of various contract terms; first for simple, then for more complicated types.

[3] As an example, assume that you are choosing among the following three plans described in section 5, to cover the same underlying risks: 1.b. paid loss retro, no minimum; 1.c. funded plan, with interest credit; and 1.d. aggregate deductible. Parameters can easily be chosen such that C_t is the identical function of P_t for all three plans. For those parameters, all three plans have the same economic impact. However, the definition of premium is different in each case. The profit or loss effect of each plan is the same, but the accounting entries producing that result differ.

1. Contracts of the form $C_t = \min(aP_t + b, r - P_t)$

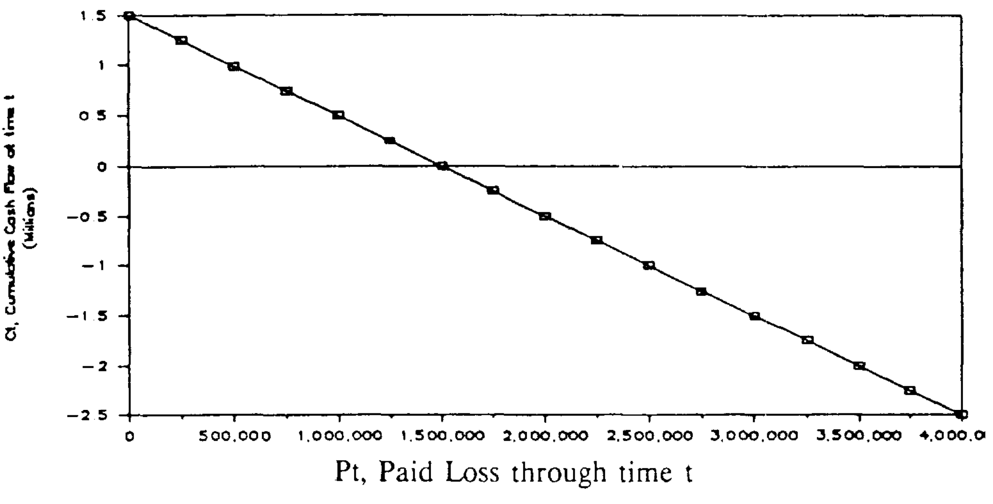
   a. Flat rated: The premium charged by the reinsurer is known in advance of the effective date and is fixed for the life of the contract. The premium is usually expressed as a percentage of the premiums charged by the ceding company on the business subject to the treaty (called subject premium).

$$C_t = r - P_t,$$

   where r = premium.

   For example, let r = \$1,500,000.

FLAT RATED
Cash Flow as a Function of Paid Loss

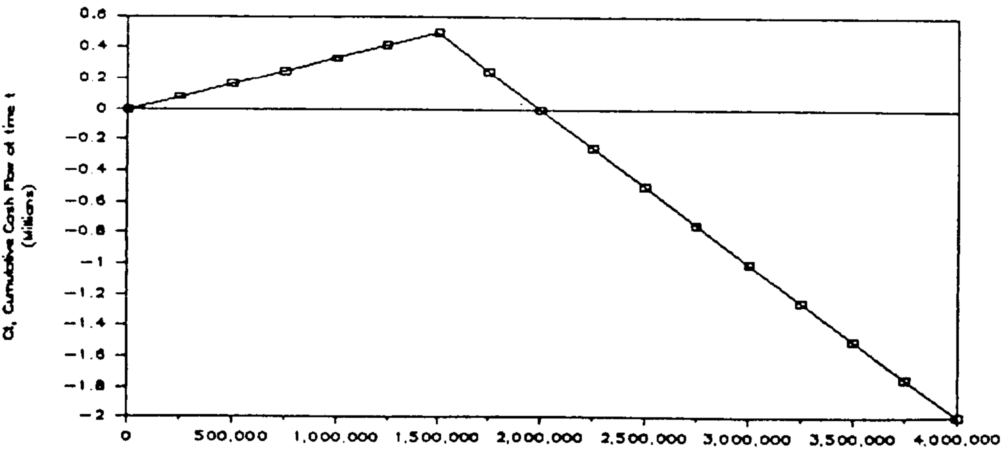


b. Paid loss retro, no minimum (sometimes called cash flow plans): The premium charged by the reinsurer is a function of the actual aggregate paid loss experience. In this case, the developed premium can increase to a maximum of M.

$$C_t = \min (aP_t + b, M - P_t).$$

For example, let $a = .333$, $b = 0$ and $M = \$2,000,000$.

PAID LOSS RETRO, NO MINIMUM
Cash Flow as a Function of Paid Loss



Pt, Paid Loss through time t

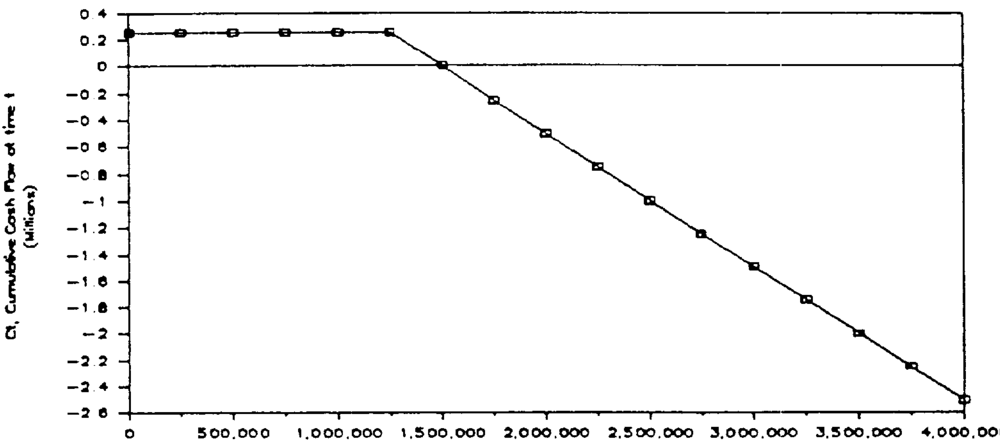
c. Funded plan, with interest credit: The premium less the reinsurer's margin is placed in a fund which accumulates interest at the credited amount and from which losses are paid. When the contract is commuted, the fund balance, if any, is returned to the cedant. The fund would normally be set at an amount sufficiently higher than expected losses to pay for actual losses in most years.

$$C_t = \min(r - f_0, r - P_t),$$

where $r = f_0 +$ margin,
f_0 = fund at time.

For example, let f_0 = \$1,250,000 and margin = \$250,000.

FUNDED PLAN, WITH INTEREST CREDIT
Cash Flow as a Function of Paid Loss



Pt, Paid Loss through time t

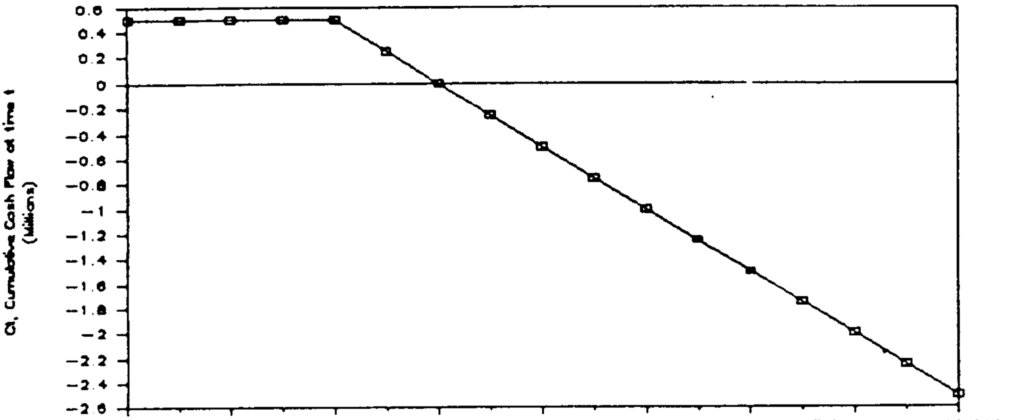
d. Aggregate deductible: For an aggregate deductible, the reinsurer pays no losses until the total losses to the excess layer exceed the deductible. Typically, the aggregate deductible is set lower than the total losses expected for the layer. The graph shows that the economic effect of an aggregate deductible is the same as a funded plan with interest (but the accounting effects are quite different).

$$C_t = \min(p - d, p - P_t),$$

where $p = r + d$,
$d =$ deductible,
$r =$ premium.

For example, let $r = \$500{,}000$ and $d = \$1{,}000{,}000$.

AGGREGATE DEDUCTIBLE
Cash Flow as a Function of Paid Loss



Pt, Paid Loss through time t

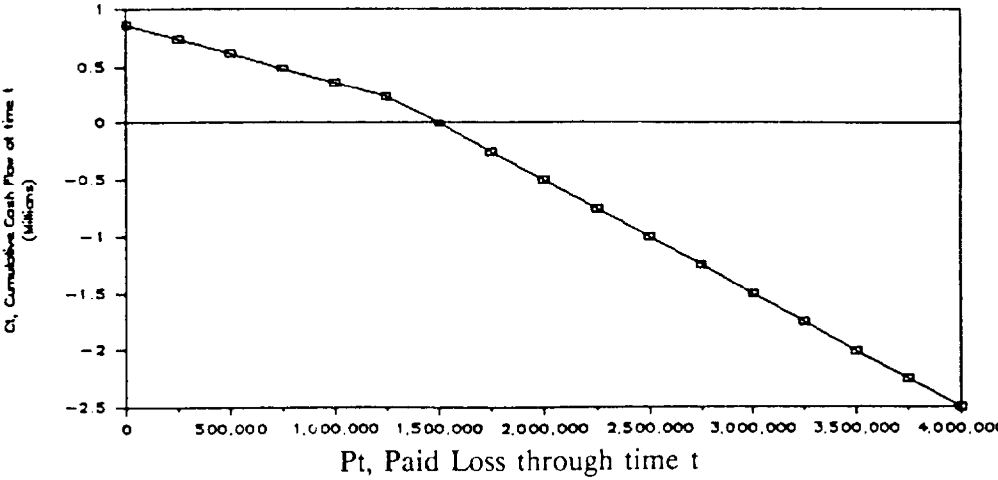
e. Profit Commission: In this plan the reinsurer returns a share of his profits to the cedant. Profit is defined to be premiums less losses and reinsurer's margin. Because actual profit will not be known for many years, profit commission could increase or decrease thereby requiring additional payments by the reinsurer or a return of profit commission by the cedant. However, the profit commission is never less than zero.

$$C_t = \min\left(-(1-h)P_t + r(1-h)(1-e),\ r - P_t\right),$$

where h = profit sharing percent,
e = reinsurer's percent margin,
r = premium.

For example, let $h = .50$, $e = .15$ and $r = \$1{,}500{,}000$.

PROFIT COMMISSION
Cash Flow as a Function of Paid Loss



2. Contracts of the form $C_t = \max(aP_t + b, m - P_t)$

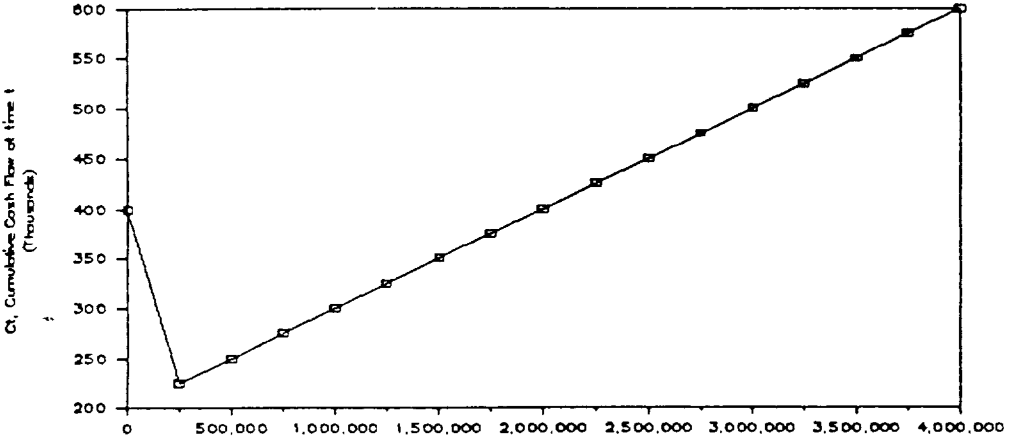
   a. Paid loss retro, no maximum: This is a pure cash flow plan which allows the cedant to spread his incurred loss experience and thereby smooth underwriting results. The cedant usually pays the reinsurer a provisional premium greater than or equal to the minimum, with the final premium based on actual paid losses plus loadings.

$$C_t = \max(aP_t + b, m - P_t),$$

where a = multiplicative loading,
b = additive loading,
m = minimum premium.

For example, let $a = .10$, $b = \$200{,}000$ and $m = \$400{,}000$.

PAID LOSS RETRO, NO MAXIMUM
Cash Flow as a Function of Paid Loss



Pt, Paid Loss through time t

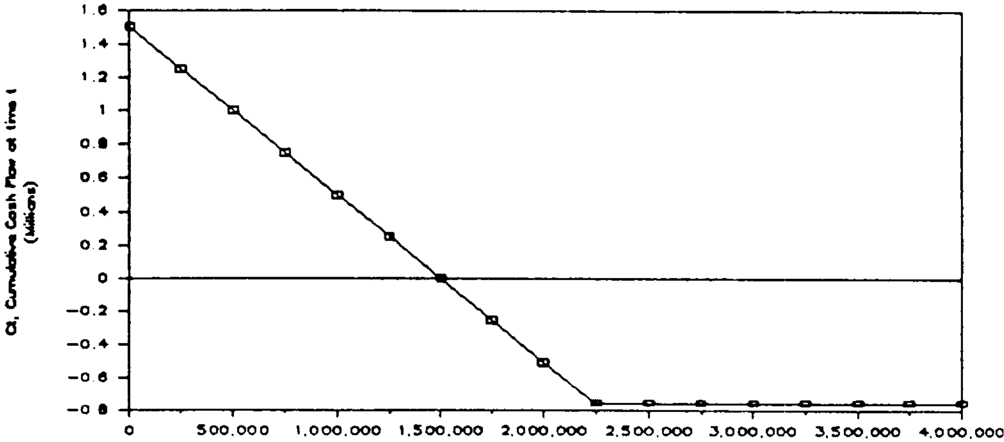
b. Loss ratio aggregate limit or "cap": The reinsurer's aggregate liability for losses is capped at a specific dollar amount expressed as a loss ratio or dollar limit. The loss ratio is usually against the reinsurer's net premiums.

$$C_t = \max\,(r - P_t,\ r - f\,),$$

where r = premium,
f = cap (in dollars).

For example, let r = \$1,500,000 and f = \$2,250,000.

LOSS RATIO CAP
Cash Flow as a Function of Paid Loss



Pt, Paid Loss though time t

3. Plans with both minimums and maximums.

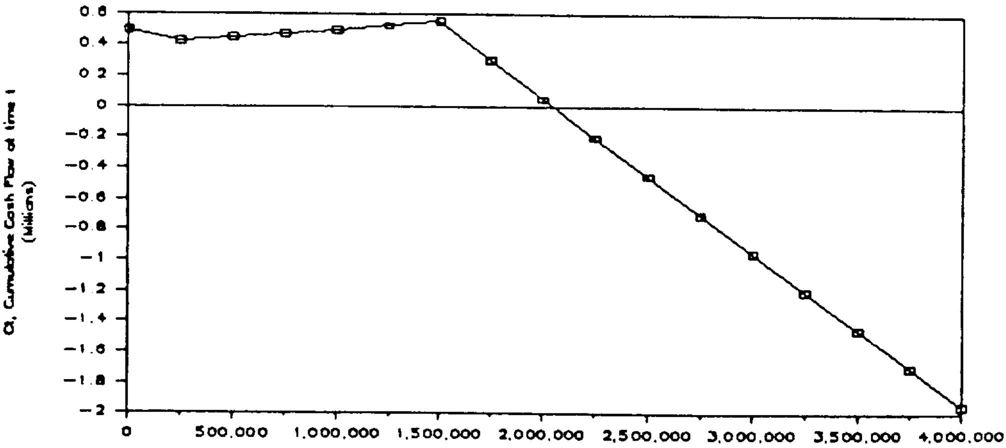
   a. Paid loss retro: This plan is a combination of plans 1b and 2a.

$$C_t = \min(\max(aP_t + b,\ m - P_t),\ M - P_t),$$

where m = minimum premium,
a = multiplicative loading,
b = additive loading,
M = maximum premium.

For example, let $a = .10$, $m = \$500{,}000$, $b = \$400{,}000$, and $M = \$2{,}050{,}000$.

PAID LOSS RETRO
Cash Flow as a Function of Paid Loss



Pt, Paid Loss through time t

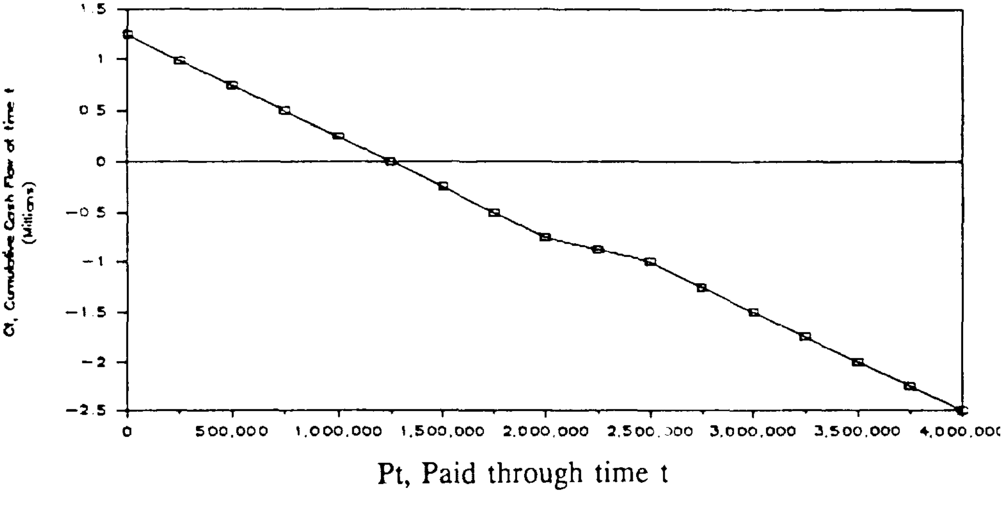
b. Loss corridor: In most loss corridor plans, the reinsurer pays 100% of the losses up to the beginning of the corridor, some share or fraction of the losses in the corridor, and 100% of the losses above the corridor. The corridor is usually expressed in terms of loss ratio points.

$$C_t = \min(\max(r - P_t, r - P_t + h(P_t - u)), r - P_t + h(v - u)),$$

where h = fraction of corridor retained by reinsured,
r = premium,
u = beginning of corridor,
v = end of corridor.

For example, let $h = .50$, $r = \$1,250,000$, $u = \$2,000,000$, and $v = \$2,500,000$.

LOSS CORRIDOR
Cash Flow as a Function of Paid Loss



4. C_t depends on t.

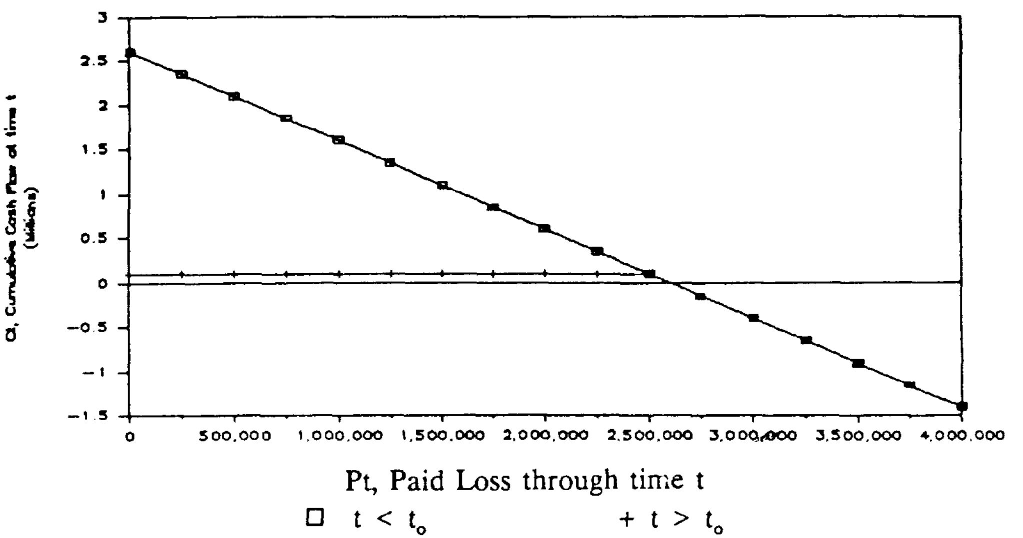
   a. Funded plan with no interest credit to cedant: Under such plans, the fund balance does not accumulate with interest; that is, the reinsurer keeps all interest earned for his own account. At time t_0, the fund, less paid losses and reinsurer's margin, is returned to the cedant provided this balance is positive. The cumulative cash flow at time t_0 is never greater than the margin, though the reinsurer does receive the benefit of full cash flow until the fund is returned.

$$C_t = \begin{cases} r - P_t & t < t_0, \\ \min(r - f, r - P_t) & t \geq t_0, \end{cases}$$

where r = fund + margin,
f = fund,
t_0 = date on which the fund is returned.

For example, let f = \$2,500,000 and margin = \$100,000.

FUNDED PLAN WITH NO INTEREST CREDIT
Cash Flow as a Function of Paid Loss



5. C_t is a function of K_t in addition to P_t.

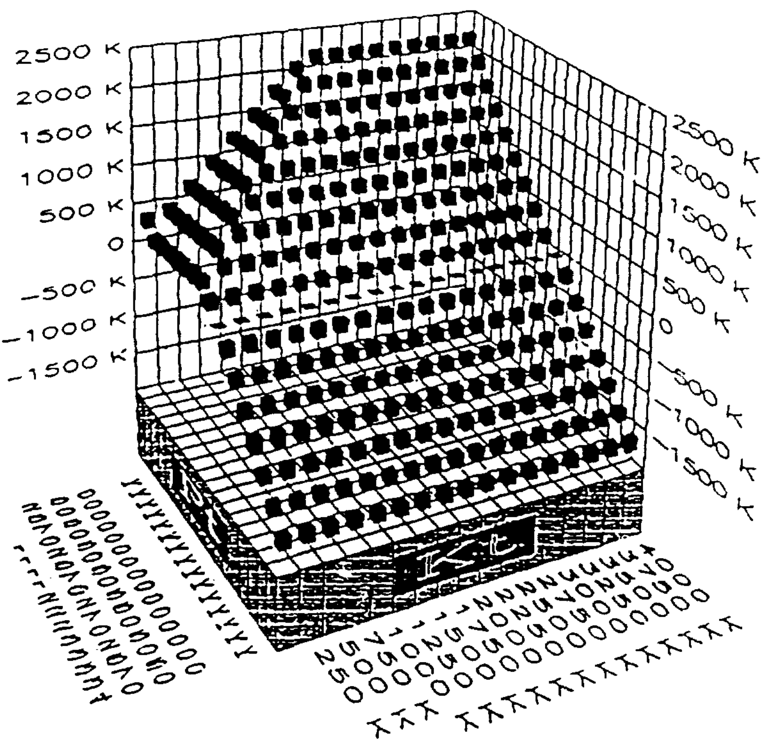
   a. Incurred loss retro: This is similar to a paid loss retro except that the reinsurer's premium, R_t, is a function of known incurred losses $(P_t + K_t)$, multiplied by a loading. The additive load, b_t, may include an IBNR provision that is a function of t.

$$C_t = \min(\max(aP_t + (a + 1)K_t + b_t,\ m - P_t),\ M - P_t),$$

   where a = multiplicative loading,
   b = additive loading,
   M = maximum premium,
   m = minimum premium.

   For example, let $a = .10$, $b = \$400{,}000$, $M = \$2{,}250{,}000$, and $m = \$400{,}000$. Note that this graph is three-dimensional because C_t is a function of two variables, P_t and K_t. In the prior examples C_t was dependent upon one variable, P_t.

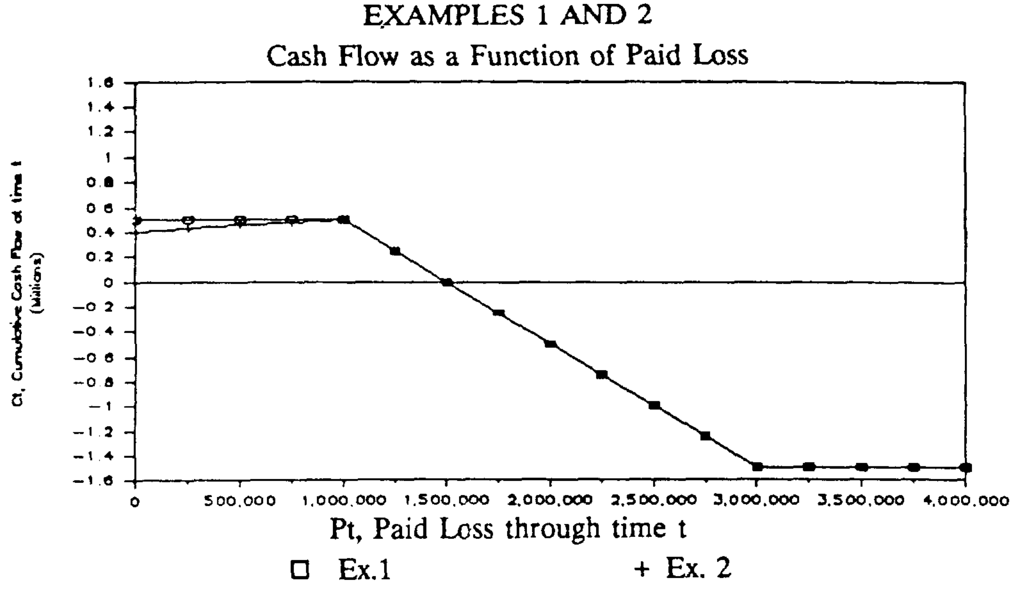
Incurred Loss Retro



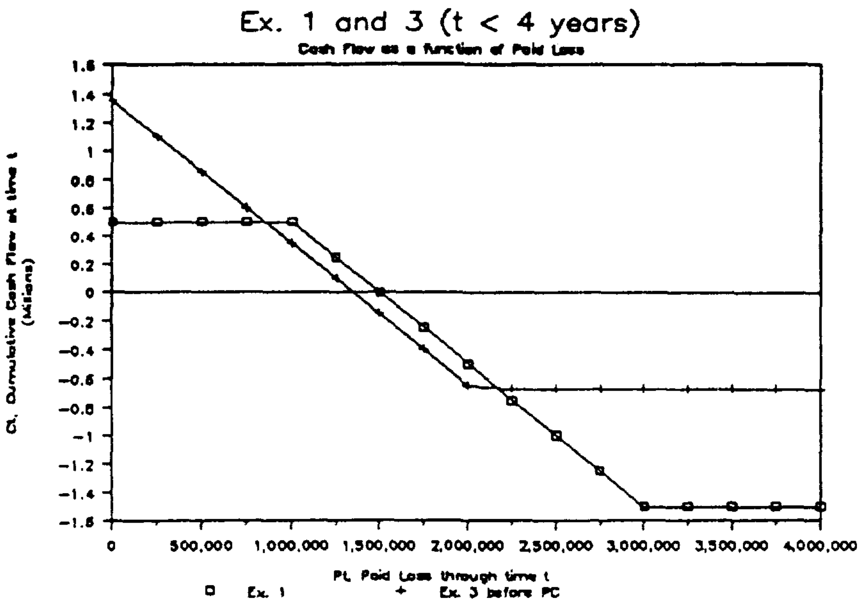
Actual contract terms are often a variation or mixture of the above types, such as the alternatives or the example presented in Section 2.

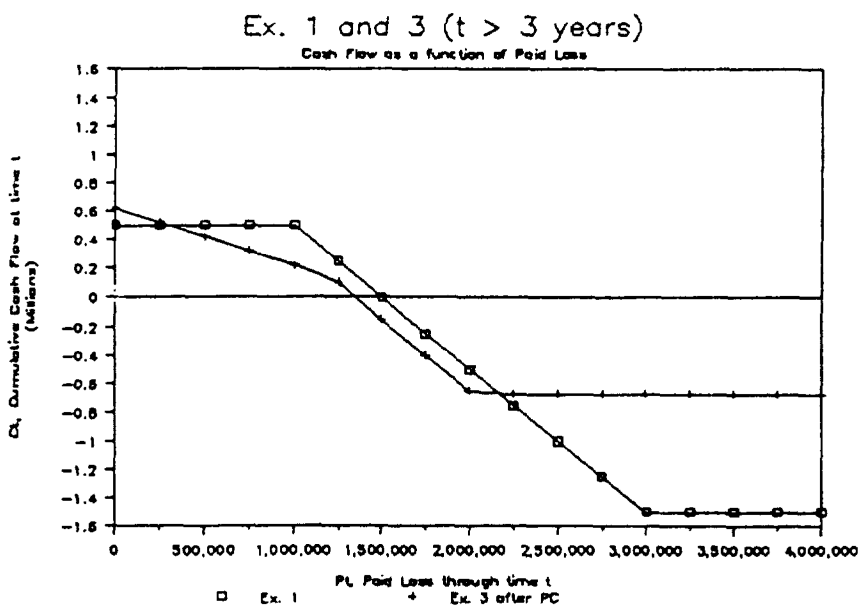
6. COMPARING GRAPHS FOR THE EXAMPLE

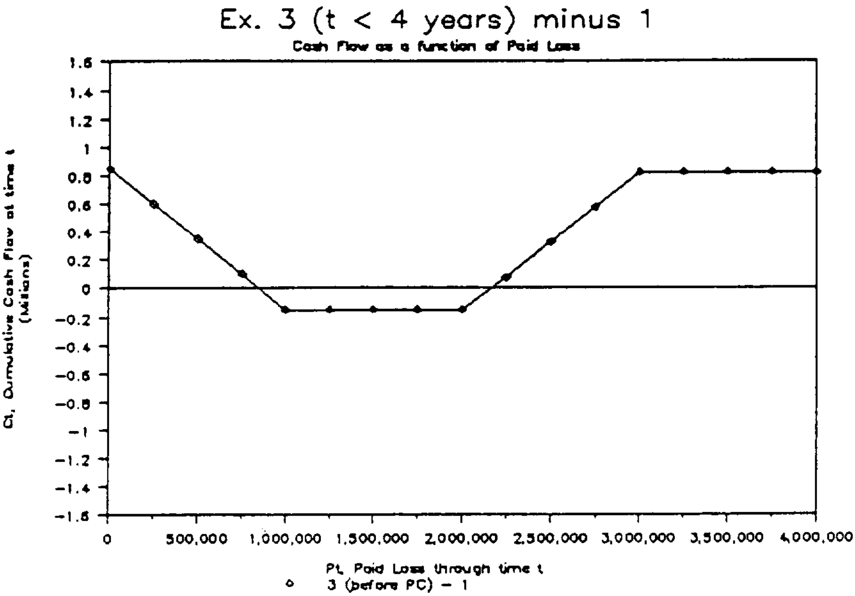
A first step in evaluating relative price adequacy is to examine the graphs of the various alternatives and to examine the graph of a hypothetical contract constructed as the difference between two deals. For the example in section 2, the graph below shows option #1, option #2, and option #1 minus option #2 (the "difference deal" is represented by the triangular region). The obvious conclusions are that the two options are very similar, but that #1 is better than or equal to #2 at all points. Therefore, reject option #2.

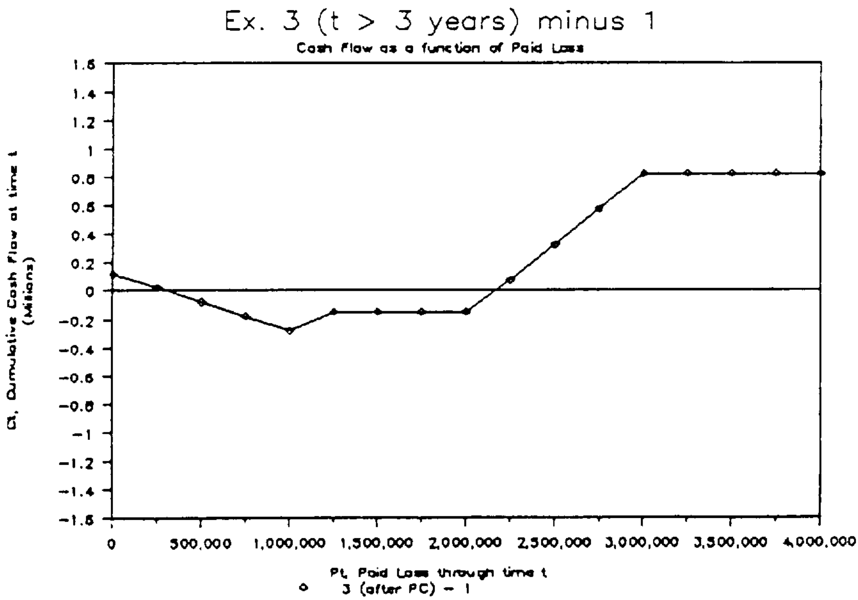


Comparing #1 and #3 is more complex because neither one dominated the other in all cases, and #3 varies with t. The graph of #3 and of #3 minus #1 (referred to as the "difference deal") are shown on the following pages.


Ex. 1 and 3 (t < 4 years)
Cash Flow as a function of Paid Loss
Ct, Cumulative Cash Flow at time t (Millions)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
0
500,000
1,000,000
1,500,000
2,000,000
2,500,000
3,000,000
3,500,000
4,000,000
Pt, Paid Loss through time t
Ex. 1
Ex. 3 before PC


Ex. 1 and 3 (t > 3 years)
Cash Flow as a function of Paid Loss
Ct, Cumulative Cash Flow at time t (Millions)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
0
500,000
1,000,000
1,500,000
2,000,000
2,500,000
3,000,000
3,500,000
4,000,000
Pt, Paid Loss through time t
Ex. 1
Ex. 3 after PC


Ex. 3 (t < 4 years) minus 1
Cash Flow as a function of Paid Loss
Ct, Cumulative Cash Flow at time t (Millions)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
−1.4
−1.6
0
500,000
1,000,000
1,500,000
2,000,000
2,500,000
3,000,000
3,500,000
4,000,000
Pt, Paid Loss through time t
3 (before PC) − 1


Ex. 3 (t > 3 years) minus 1
Cash Flow as a function of Paid Loss
Ct, Cumulative Cash Flow at time t (Millions)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
−1.4
−1.6
0
500,000
1,000,000
1,500,000
2,000,000
2,500,000
3,000,000
3,500,000
4,000,000
Pt, Paid Loss through time t
3 (after PC) − 1

7. DISTRIBUTION OF V

As a next step, it is helpful to compare the contracts over reasonable ranges of parameters of an underlying loss generation model. This will help to focus on the underlying conditions that must be true for one option to be superior to the other. The value V of the difference deal is a random variable. How does the distribution of that variable change as the underlying loss model changes? One clear way to present this information is to look at a matrix (or 3-D graph) of the expected value of V as two important parameters are varied.[4]

To do this one needs to estimate the aggregate distribution of incurred or paid losses. This can be accomplished using simulation or by calculating them directly from the frequency and severity distributions. Using a transformation discussed below, aggregate distributions for excess contracts that reflect the age of the contract can be determined. From this series of distributions, one can calculate the distribution of cash flows to the contract. The specific model of the loss process is based on distributions that are commonly used in casualty actuarial literature.

Consider the aggregate distribution for excess claims:

$$G(x) = \sum_{n=0}^{\infty} \text{Prob}[N = n]\, F(x)^{*N},$$

where $F(x)$ is the individual loss amount distribution. This represents the distribution of P_t at ultimate. The Single Parameter Pareto (see Philbrick [9]) is used to model severity for its ease in estimating excess losses.

The model assumes a negative binomial frequency distribution defined as:

$$\text{Prob}[M = m] = \binom{m + \alpha - 1}{\alpha - 1} p^{\alpha} (1 - p)^{m},$$

where M denotes the number of ground-up claims (i.e., claims from first dollar of loss).

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[4] Although E[V] is probably the most important thing to look at, other information about the distribution of V, such as the Variance [V] and Probability [$V > 0$], can be examined in this format. Also, if you wish to postulate a utility function U (on V), we can look at E[$U(V)$] as the parameters are varied.

It is interesting to see that if ground-up claims are negative binomial, NB(α, p), then the number of excess claims, N, excess of any retention r, is also negative binomial NB (α', p')

$$\text{where } \alpha' = \alpha \text{ and } p' = \frac{p}{1 - F(r) + pF(r)}.$$

In addition, if one assumes that individual claim reporting (or payment) is independent of size[5], then the number of reported (or paid) claims is negative binomial NB (α_t', p_t')

$$\text{where } \alpha_t' = \alpha', \; p_t' = \frac{p'}{w(t) + p'(1 - w(t))}$$

and $w(t)$ is the percent of claims reported (or paid) as of t months from the average accident date. See Appendix D for a general proof of these relationships. Similar relationships hold for other common frequency distributions.

Some of the parameters in this model are "unimportant." That is, the conclusions drawn are insensitive to changes in these parameters. This is because alternative deals covering the same occurrence layer are being compared. (If one tried to compare a \$500,000 excess \$500,000 contract with a \$250,000 excess \$250,000 contract, the result would be much more sensitive to the choice of those "unimportant" parameters.)

The "important" parameters that significantly affect the distribution of V are:

1. Average payment lag. The payment lag is the random delay between loss occurrence and loss payment.

2. Expected total losses to the occurrence layer.

[5] The authors do not believe that the independence assumption of claim reporting (or payment) from size is too restrictive because the claims being considered are already large on a ground up basis and their individual size is bound by the layer limit.

Two different applications of the method, producing equivalent results, were used to calculate E[V]. One was a Monte Carlo simulation described in Appendix B; the other was a calculation of the distribution of cash flows at each t by Panjer's method described in Appendix C.[6]

The E[V] and difference matrices for each application are shown in Exhibits 1 and 2. As can be seen, the results are quite similar and would lead to the same conclusions. Displayed on Exhibits 3–7 are expected cash flow and underlying distributions for the case where E[total loss] equals $1,500,000 and E[lag] equals 36 months.

The E[V] of the difference deal is shown on the graph on the next page for each pair of parameters E[total losses] and E[lag]. Examining this graph shows that #1 is slightly superior if we are quite confident in our estimate of expected losses at $1,500,000 and if the average payment lag is short (less than 24 months). However, for a longer payout or for a misestimate of expected losses (either over or under), #3 is superior. The decision maker can use his subjective assessment of his own risk preferences in choosing between the deals. (The authors prefer deal #3.)

8. CONCLUSION

The methods outlined have the advantage of summarizing the many factors affecting the economic value of a reinsurance contract, first by graphing the contract terms, then by graphing E[V], to allow consistent choices among the alternatives. One can also construct contracts with equivalent terms in the sense that their E[V]'s are approximately equal. The model is general enough to handle most realistic contract types. Applying subjective probabilities to the range of the V distributions corresponding to various lags and expected total losses could further summarize the results. Finally, the model can be made more general by using a random interest rate and applying utility functions to the cash flows. However, such extensions probably do not add much practical value.

[6] The simulation has the advantage of producing the entire distribution of V; the Panjer method only gives E[V] easily. However, when E[N] is large, the Panjer method can be run much more quickly than the simulation.

EXAMPLE 3 MINUS
EXAMPLE 1

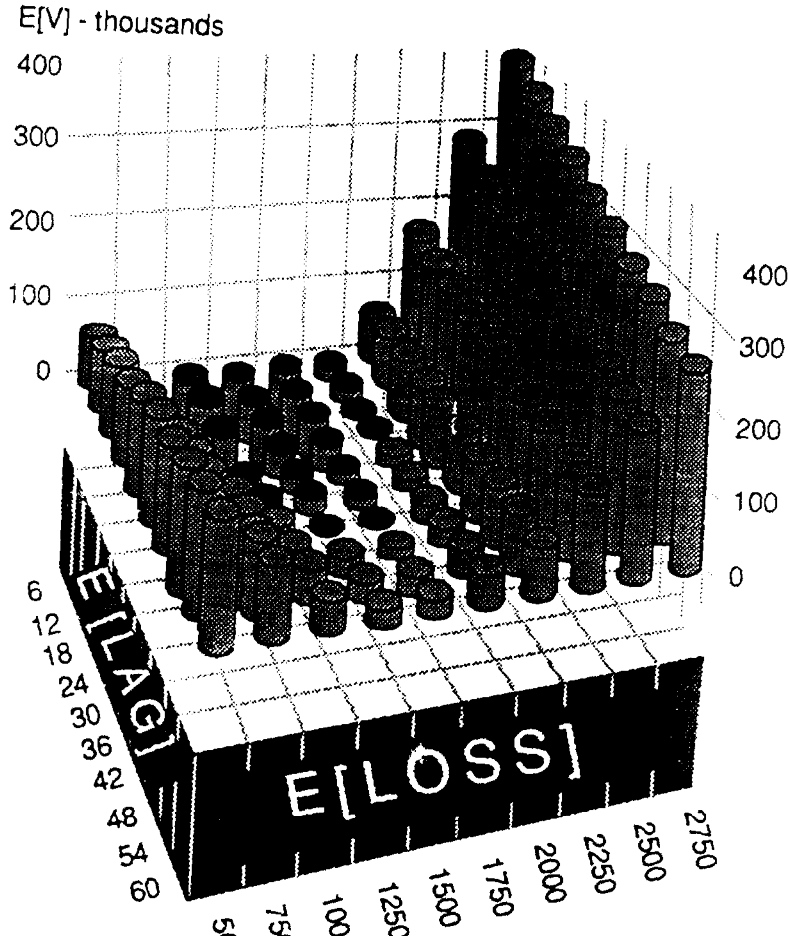

EXHIBIT 1

TABLE OF E[V], EXPECTED PRESENT VALUE OF THE CASH FLOWS
PANJER'S METHOD

Example	E[LOSS]	E[LAG] in months 6	12	18	24	30	36	42	48	54	60
1	500	487	488	489	490	490	491	492	492	493	493
1	750	437	442	446	450	453	456	459	462	464	466
1	1,000	335	346	356	366	374	382	389	396	402	407
1	1,250	181	202	221	238	254	269	282	295	306	316
1	1,500	(8)	24	52	79	103	125	146	165	182	198
1	1,750	(218)	(174)	(136)	(100)	(68)	(37)	(9)	16	40	62
1	2,000	(433)	(378)	(331)	(287)	(246)	(208)	(173)	(141)	(111)	(84)
1	2,250	(640)	(575)	(521)	(470)	(422)	(378)	(337)	(299)	(264)	(231)
1	2,500	(829)	(757)	(696)	(640)	(588)	(538)	(493)	(450)	(410)	(373)
1	2,750	(992)	(915)	(851)	(791)	(735)	(683)	(634)	(588)	(545)	(505)
3	500	548	567	582	595	606	615	624	631	638	644
3	750	406	433	456	476	493	508	521	533	543	553
3	1,000	248	284	315	342	366	386	405	422	437	450
3	1,250	77	121	160	194	224	251	275	297	317	336
3	1,500	(94)	(43)	1	41	77	110	140	167	191	214
3	1,750	(249)	(194)	(146)	(102)	(62)	(25)	9	40	68	95
3	2,000	(378)	(321)	(271)	(226)	(184)	(144)	(108)	(75)	(43)	(15)
3	2,250	(476)	(419)	(370)	(325)	(283)	(243)	(206)	(172)	(139)	(109)
3	2,500	(546)	(491)	(444)	(400)	(359)	(321)	(284)	(250)	(218)	(187)
3	2,750	(593)	(541)	(496)	(455)	(417)	(380)	(345)	(312)	(280)	(250)
3-1	500	61	79	93	105	115	124	132	139	145	151
3-1	750	(32)	(9)	10	26	40	52	62	71	79	87
3-1	1,000	(87)	(62)	(41)	(24)	(9)	4	16	26	35	43
3-1	1,250	(104)	(81)	(61)	(45)	(30)	(18)	(7)	3	12	20
3-1	1,500	(86)	(67)	(51)	(37)	(26)	(15)	(6)	2	9	16
3-1	1,750	(31)	(20)	(10)	(1)	6	12	18	23	28	33
3-1	2,000	55	57	59	61	63	64	65	66	68	69
3-1	2,250	164	156	150	145	140	135	131	127	124	122
3-1	2,500	283	266	252	240	228	218	208	200	192	186
3-1	2,750	399	374	354	336	319	303	289	276	265	254

EXHIBIT 2

TABLE OF E[V], EXPECTED PRESENT VALUE OF THE CASH FLOWS
MONTE CARLO SIMULATION

		E[LAG] in months									
Example	E[LOSS]	6	12	18	24	30	36	42	48	54	60
1	500	467	469	469	470	470	472	472	473	473	474
1	750	418	422	427	429	432	435	438	441	444	447
1	1,000	320	329	338	348	355	364	373	380	383	390
1	1,250	170	192	205	225	243	260	269	282	285	303
1	1,500	(17)	17	47	77	95	115	134	155	173	186
1	1,750	(208)	(162)	(134)	(93)	(70)	(32)	(12)	11	36	57
1	2,000	(415)	(355)	(312)	(272)	(236)	(197)	(167)	(133)	(109)	(81)
1	2,250	(604)	(549)	(494)	(449)	(404)	(365)	(320)	(284)	(249)	(216)
1	2,500	(781)	(720)	(667)	(608)	(560)	(509)	(467)	(419)	(396)	(358)
1	2,750	(937)	(868)	(807)	(753)	(705)	(650)	(599)	(560)	(517)	(481)
3	500	529	546	560	574	583	593	601	607	614	621
3	750	392	419	441	458	474	487	501	513	524	533
3	1,000	241	274	303	330	350	374	393	409	421	436
3	1,250	75	120	152	188	220	248	267	289	300	327
3	1,500	(91)	(40)	7	46	77	107	135	162	188	206
3	1,750	(231)	(175)	(134)	(88)	(56)	14	13	41	69	94
3	2,000	(355)	(296)	(247)	(207)	(169)	(129)	(99)	(62)	(35)	(6)
3	2,250	(446)	(393)	(343)	(302)	(263)	(225)	(189)	(155)	(125)	(94)
3	2,500	(514)	(461)	(418)	(375)	(335)	(299)	(264)	(225)	(205)	(177)
3	2,750	(560)	(509)	(467)	(428)	(393)	(357)	(320)	(292)	(260)	(232)
3-1	500	62	77	91	103	112	121	129	135	140	147
3-1	750	(26)	(3)	14	29	43	52	63	71	80	86
3-1	1,000	(78)	(55)	(35)	(18)	(6)	10	20	29	38	46
3-1	1,250	(95)	(72)	(53)	(37)	(23)	(12)	(2)	7	15	24
3-1	1,500	(74)	(57)	(40)	(31)	(18)	(9)	1	7	15	20
3-1	1,750	(23)	(13)	(0)	5	13	18	24	30	33	37
3-1	2,000	60	58	65	65	67	68	69	70	73	75
3-1	2,250	158	156	150	147	142	139	131	129	124	122
3-1	2,500	267	259	249	233	225	210	203	194	191	181
3-1	2,750	377	359	340	324	312	293	279	267	257	248

EXHIBIT 3

EXPECTED CASH FLOW EXHIBIT: PANJER'S METHOD
EXAMPLE NUMBER 1

THE 100% EXPECTED LAYER LOSSES = $1,500,000
THE AVERAGE PAYMENT LAG IS 36 MONTHS

Yrs	Expected Incurred	Expected Paid	Exp. Earned Premium	Expected Profit Commission	Incurred Loss Ratio	Expected Cash Flow	Present Value of E[CF]
1	4,616	411	500,000	0	0.9%	499,589	499,589
2	121,630	26,837	500,000	0	24.3%	−26,426	−24,469
3	280,569	103,237	500,000	0	56.1%	−76,399	−65,500
4	391,523	196,185	500,000	0	78.3%	−92,949	−73,786
5	457,712	280,569	500,000	0	91.5%	−84,383	−62,024
6	495,311	348,673	500,000	0	99.1%	−68,105	−46,351
7	516,272	400,790	500,000	0	103.3%	−52,116	−32,842
8	527,874	439,578	500,000	0	105.6%	−38,789	−22,633
9	534,263	468,035	500,000	0	106.9%	−28,457	−15,374
10	537,785	488,717	500,000	0	107.6%	−20,682	−10,346
11	539,709	503,708	500,000	0	107.9%	−14,991	−6,944
12	540,780	514,507	500,000	0	108.2%	10,799	−4,631
13	541,351	522,283	500,000	0	108.3%	−7,776	−3,088
14	541,681	527,874	500,000	0	108.3%	−5,591	−2,056
15	541,840	531,885	500,000	0	108.4%	4,011	−1,366
16	541,945	534,763	500,000	0	108.4%	−2,878	−907
17	542,002	536,842	500,000	0	108.4%	−2,079	−607
18	542,030	538,301	500,000	0	108.4%	−1,460	−395
19	542,049	539,382	500,000	0	108.4%	1,081	−270
20	542,068	542,068	500,000	0	108.4%	−2,685	−622
TOTALS						−42,068	125,378

Swing program minimum, provisional, and maximum are: .100, .100, .100.
The aggregate deductible and loss ratio cap percentage of subject premium are: .200, .400.
The aggregate deductible and loss ratio cap dollars are: $0 $0.

EXHIBIT 4

EXPECTED CASH FLOW EXHIBIT: PANJER'S METHOD
EXAMPLE NUMBER 3

THE 100% EXPECTED LAYER LOSSES = $1,500,000
THE AVERAGE PAYMENT LAG IS 36 MONTHS

Yrs	Expected Incurred	Expected Paid	Exp. Earned Premium	Expected Profit Commission	Incurred Loss Ratio	Expected Cash Flow	Present Value of E[CF]
1	388,774	230,278	1,350,000	0	28.8%	1,119,723	1,119,723
2	888,210	590,129	1,349,999	0	65.8%	−359,852	−333,196
3	1,152,781	846,750	1,350,000	0	85.4%	−256,621	−220,011
4	1,289,347	1,027,316	1,350,000	0	95.5%	−180,566	−143,339
5	1,360,387	1,152,781	1,350,000	131,518	100.8%	−256,983	−188,891
6	1,397,986	1,239,555	1,350,001	105,907	103.6%	−61,162	−41,626
7	1,418,170	1,299,695	1,350,000	90,098	105.0%	−44,332	−27,937
8	1,429,110	1,341,569	1,350,001	79,989	105.9%	−31,764	−18,534
9	1,435,067	1,370,895	1,350,001	73,335	106.3%	−22,671	−12,249
10	1,438,328	1,391,533	1,350,000	68,855	106.5%	−16,159	−8,083
11	1,440,110	1,406,136	1,350,000	65,787	106.7%	−11,535	−5,343
12	1,441,091	1,416,491	1,350,000	63,661	106.7%	−8,229	−3,529
13	1,441,622	1,423,858	1,350,000	62,173	106.8%	−5,879	−2,334
14	1,441,921	1,429,110	1,350,000	61,125	106.8%	−4,205	−1,546
15	1,442,076	1,432,855	1,349,999	60,383	106.8%	−3,004	−1,023
16	1,442,168	1,435,533	1,350,000	59,857	106.8%	−2,151	−678
17	1,442,219	1,437,453	1,350,000	59,481	106.8%	−1,543	−450
18	1,442,244	1,438,815	1,350,000	59,214	106.8%	−1,095	−296
19	1,442,260	1,439,803	1,350,001	59,023	106.8%	−797	−199
20	1,442,278	1,442,278	1,350,000	58,542	106.8%	−1,995	−462
TOTALS						−150,820	109,996

Swing program minimum, provisional, and maximum are: .270, .270, .270.
The aggregate deductible and loss ratio cap percentage of subject premium are: .000, .405.
The aggregate deductible and loss ratio cap dollars are: $0, $0.
The profit commission equals .600 after .100 reinsurer's margin, but none until year 5.
Number of intervals tested = 191; largest calculated aggregate loss = 4,517,796

EXHIBIT 5

DISTRIBUTION OF C_t USING PANJER'S METHOD

	Example #										
	1	3 (before PC)	3 (after PC)	t (years)							
P_t	C_t	C_t	C_t	1	2	3	4	5	6	8	10
0	500,000	1,350,000	621,000	0.056	0.007	0.002	0.001	0.000	0.000	0.000	0.000
108,808	500,000	1,241,192	577,477	0.147	0.030	0.009	0.004	0.002	0.001	0.001	0.001
217,616	500,000	1,132,384	533,953	0.293	0.076	0.026	0.012	0.007	0.005	0.003	0.002
326,424	500,000	1,023,576	490,430	0.447	0.150	0.061	0.031	0.018	0.013	0.008	0.006
435,233	500,000	914,768	446,907	0.602	0.248	0.114	0.062	0.039	0.028	0.018	0.015
544,041	500,000	805,959	403,384	0.728	0.363	0.187	0.110	0.073	0.054	0.037	0.030
652,849	500,000	697,151	359,860	0.827	0.483	0.277	0.175	0.122	0.093	0.066	0.055
761,657	500,000	588,343	316,337	0.895	0.599	0.379	0.255	0.187	0.147	0.108	0.092
870,465	500,000	479,535	272,814	0.940	0.701	0.484	0.346	0.264	0.214	0.163	0.141
979,273	500,000	370,727	229,291	0.967	0.787	0.587	0.443	0.351	0.293	0.231	0.203
1,088,081	411,919	261,919	185,768	0.983	0.853	0.679	0.540	0.443	0.379	0.308	0.275
1,196,889	303,111	153,111	142,244	0.991	0.903	0.760	0.631	0.535	0.469	0.391	0.355
1,305,698	194,302	44,302	44,302	0.996	0.938	0.825	0.712	0.622	0.557	0.478	0.439
1,414,506	85,494	(64,506)	(64,506)	0.998	0.962	0.877	0.782	0.702	0.640	0.562	0.523
1,523,314	(23,314)	(173,314)	(173,314)	0.999	0.977	0.916	0.840	0.770	0.715	0.642	0.604
1,632,122	(132,122)	(282,122)	(282,122)	1.000	0.987	0.944	0.885	0.828	0.780	0.714	0.679
1,740,930	(240,930)	(390,930)	(390,930)	1.000	0.992	0.964	0.920	0.874	0.834	0.778	0.746
1,849,738	(349,738)	(499,738)	(499,738)	1.000	0.996	0.977	0.946	0.910	0.878	0.831	0.803
1,958,546	(458,546)	(608,546)	(608,546)	1.000	0.998	0.986	0.964	0.938	0.912	0.874	0.851
2,067,354	(567,354)	(675,000)	(675,000)	1.000	0.999	0.992	0.977	0.958	0.939	0.908	0.890
2,176,163	(676,163)	(675,000)	(675,000)	1.000	0.999	0.995	0.985	0.972	0.958	0.935	0.920
2,284,971	(784,971)	(675,000)	(675,000)	1.000	1.000	0.997	0.991	0.982	0.972	0.954	0.943
2,393,779	(893,779)	(675,000)	(675,000)	1.000	1.000	0.998	0.994	0.988	0.981	0.969	0.961
2,502,587	(1,002,587)	(675,000)	(675,000)	1.000	1.000	0.999	0.997	0.993	0.988	0.979	0.973
2,611,395	(1,111,395)	(675,000)	(675,000)	1.000	1.000	1.000	0.998	0.996	0.992	0.986	0.982
2,720,203	(1,220,203)	(675,000)	(675,000)	1.000	1.000	1.000	0.999	0.997	0.995	0.991	0.988
2,829,011	(1,329,011)	(675,000)	(675,000)	1.000	1.000	1.000	0.999	0.998	0.997	0.994	0.992
2,937,820	(1,437,820)	(675,000)	(675,000)	1.000	1.000	1.000	1.000	0.999	0.998	0.997	0.995
3,046,628	(1,500,000)	(675,000)	(675,000)	1.000	1.000	1.000	1.000	0.999	0.999	0.998	0.997
3,155,436	(1,500,000)	(675,000)	(675,000)	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.998
3,264,244	(1,500,000)	(675,000)	(675,000)	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999
3,373,052	(1,500,000)	(675,000)	(675,000)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
3,481,860	(1,500,000)	(675,000)	(675,000)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3,590,668	(1,500,000)	(675,000)	(675,000)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
			Expected Cash Flow								
			example 1	499,589	473,163	396,764	303,815	219,432	151,327	60,422	11,283
			example 3	1,119,723	759,871	503,250	322,684	65,701	4,539	(71,557)	(110,387)

Note: PC denotes Profit Commission

EXHIBIT 6

DISTRIBUTION OF C_t USING PANJER'S METHOD

	Example #			t (years)				
	1	3 (before PC)	3 (after PC)					
P_t	C_t	C_t	C_t	12	14	16	18	20
0	500,000	1,350,000	621,000	0.000	0.000	0.000	0.000	0.000
108,808	500,000	1,241,192	577,477	0.000	0.000	0.000	0.000	0.000
217,616	500,000	1,132,384	533,953	0.002	0.002	0.002	0.002	0.002
326,424	500,000	1,023,576	490,430	0.005	0.005	0.005	0.005	0.005
435,233	500,000	914,768	446,907	0.013	0.012	0.012	0.012	0.012
544,041	500,000	805,959	403,384	0.027	0.026	0.025	0.025	0.025
652,849	500,000	697,151	359,860	0.050	0.048	0.047	0.046	0.046
761,657	500,000	588,343	316,337	0.085	0.081	0.079	0.078	0.077
870,465	500,000	479,535	272,814	0.131	0.126	0.123	0.122	0.121
979,273	500,000	370,727	229,291	0.189	0.183	0.180	0.178	0.176
1,088,081	411,919	261,919	185,768	0.259	0.251	0.247	0.245	0.243
1,196,889	303,111	153,111	142,244	0.336	0.327	0.323	0.320	0.318
1,305,698	194,302	44,302	44,302	0.419	0.409	0.404	0.402	0.399
1,414,506	85,494	(64,506)	(64,506)	0.503	0.493	0.488	0.485	0.482
1,523,314	(23,314)	(173,314)	(173,314)	0.584	0.574	0.569	0.566	0.564
1,632,122	(132,122)	(282,122)	(282,122)	0.660	0.651	0.646	0.643	0.641
1,740,930	(240,930)	(390,930)	(390,930)	0.729	0.720	0.716	0.713	0.711
1,849,738	(349,738)	(499,738)	(499,738)	0.788	0.781	0.777	0.775	0.772
1,958,546	(458,546)	(608,546)	(608,546)	0.839	0.832	0.828	0.827	0.825
2,067,354	(567,354)	(675,000)	(675,000)	0.879	0.874	0.871	0.869	0.868
2,176,163	(676,163)	(675,000)	(675,000)	0.912	0.907	0.905	0.904	0.903
2,284,971	(784,971)	(675,000)	(675,000)	0.937	0.933	0.932	0.931	0.930
2,393,779	(893,779)	(675,000)	(675,000)	0.956	9.953	0.952	0.951	0.950
2,502,587	(1,002,587)	(675,000)	(675,000)	0.969	0.967	0.966	0.966	0.965
2,611,395	(1,111,395)	(675,000)	(675,000)	0.979	0.978	0.977	0.977	0.976
2,720,203	(1,220,203)	(675,000)	(675,000)	0.986	0.985	0.985	0.984	0.984
2,829,011	(1,329,011)	(675,000)	(675,000)	0.991	0.990	0.990	0.990	0.990
2,937,820	(1,437,820)	(675,000)	(675,000)	0.994	0.994	0.994	0.993	0.993
3,046,628	(1,500,000)	(675,000)	(675,000)	0.996	0.996	0.996	0.996	0.996
3,155,436	(1,500,000)	(675,000)	(675,000)	0.998	0.998	0.997	0.997	0.997
3,264,244	(1,500,000)	(675,000)	(675,000)	0.999	0.999	0.998	0.998	0.998
3,373,052	(1,500,000)	(675,000)	(675,000)	0.999	0.999	0.999	0.999	0.999
3,481,860	(1,500,000)	(675,000)	(675,000)	1.000	0.999	0.999	0.999	0.999
3,590,668	(1,500,000)	(675,000)	(675,000)	1.000	1.000	1.000	1.000	1.000
			Expected Cash Flow					
			example 1	(14,507)	(27,874)	(34,763)	(38,302)	(42,068)
			example 3	(130,151)	(140,235)	(145,390)	(148,028)	(150,820)

Note: PC denotes Profit Commission

EXHIBIT 7

Distribution of Aggregate Paid Losses to Total Layer: Panjer's Method

Limit: 250,000
Retention: 250,000
Layer Expected Losses: 1,500,000
Single Parameter Pareto q: 1.5

The Mean of the Exponential Payment Lag: 36
Var[N]/E[N] for the Excess Layer at Ultimate: 1.032
Alpha = 320.083
Layer Severity = 146.447

	t (years)												
	1	2	3	4	5	6	8	10	12	14	16	18	20
	Summary Statistics												
p'	0.991	0.985	0.980	0.977	0.975	0.973	0.971	0.970	0.970	0.969	0.969	0.969	0.969
E[# Paid]	2.90	4.98	6.47	7.54	8.31	8.86	9.53	9.88	10.06	10.15	10.19	10.22	10.24
E[$ Paid]	425	730	948	1105	1217	1297	1396	1446	1473	1486	1493	1496	1500
CV	0.65	0.50	0.44	0.41	0.39	0.38	0.36	0.36	0.35	0.35	0.35	0.35	0.35
SKW	0.72	0.55	0.49	0.46	0.44	0.42	0.41	0.40	0.40	0.40	0.40	0.40	0.39
Paid Dollars	Cumulative Distribution Function $P(L)(=$ Paid Dollars)												
0	0.056	0.007	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
108,808	0.147	0.030	0.009	0.004	0.002	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
217,616	0.293	0.076	0.026	0.012	0.007	0.005	0.003	0.002	0.002	0.002	0.002	0.002	0.002
326,424	0.447	0.150	0.061	0.031	0.018	0.013	0.008	0.006	0.005	0.005	0.005	0.005	0.005
435,233	0.602	0.248	0.114	0.062	0.039	0.028	0.018	0.015	0.013	0.012	0.012	0.012	0.012
544,041	0.728	0.363	0.187	0.110	0.073	0.054	0.037	0.030	0.027	0.026	0.025	0.025	0.025
652,849	0.827	0.483	0.277	0.175	0.122	0.093	0.066	0.055	0.050	0.048	0.047	0.046	0.046

SKW = Coefficient of Skewness

EXHIBIT 7 (continued)

Paid Dollars	t (years): 1	2	3	4	5	6	8	10	12	14	16	18	20
	Cumulative Distribution Function $P(L)$(= Paid Dollars)												
761,657	0.895	0.599	0.379	0.255	0.187	0.147	0.108	0.092	0.085	0.081	0.079	0.078	0.077
870,465	0.940	0.701	0.484	0.346	0.264	0.214	0.163	0.141	0.131	0.126	0.123	0.122	0.121
979,273	0.967	0.787	0.587	0.443	0.351	0.293	0.231	0.203	0.189	0.183	0.180	0.178	0.176
1,088,081	0.983	0.853	0.679	0.540	0.443	0.379	0.308	0.275	0.259	0.251	0.247	0.245	0.243
1,196,889	0.991	0.903	0.760	0.631	0.535	0.469	0.391	0.355	0.336	0.327	0.323	0.320	0.318
1,305,698	0.996	0.938	0.825	0.712	0.622	0.557	0.478	0.439	0.419	0.409	0.404	0.402	0.399
1,414,506	0.998	0.962	0.877	0.782	0.702	0.640	0.562	0.523	0.503	0.493	0.488	0.485	0.482
1,523,314	0.999	0.977	0.916	0.840	0.770	0.715	0.642	0.604	0.584	0.574	0.569	0.566	0.564
1,632,122	1.000	0.987	0.944	0.885	0.828	0.780	0.714	0.679	0.660	0.651	0.646	0.643	0.641
1,740,930	1.000	0.992	0.964	0.920	0.874	0.834	0.778	0.746	0.729	0.720	0.716	0.713	0.711
1,849,738	1.000	0.996	0.977	0.946	0.910	0.878	0.831	0.803	0.788	0.781	0.777	0.775	0.772
1,958,546	1.000	0.998	0.986	0.964	0.938	0.912	0.874	0.851	0.839	0.832	0.828	0.827	0.825
2,067,354	1.000	0.999	0.992	0.977	0.958	0.939	0.908	0.890	0.879	0.874	0.871	0.869	0.868
2,176,163	1.000	0.999	0.995	0.985	0.972	0.958	0.935	0.920	0.912	0.907	0.905	0.904	0.903
2,284,971	1.000	1.000	0.997	0.991	0.982	0.972	0.954	0.943	0.937	0.933	0.932	0.931	0.930
2,393,779	1.000	1.000	0.998	0.994	0.988	0.981	0.969	0.961	0.956	0.953	0.952	0.951	0.950
2,502,587	1.000	1.000	0.999	0.997	0.993	0.988	0.979	0.973	0.969	0.967	0.966	0.966	0.965
2,611,395	1.000	1.000	1.000	0.998	0.996	0.992	0.986	0.982	0.979	0.978	0.977	0.977	0.976
2,720,203	1.000	1.000	1.000	0.999	0.997	0.995	0.991	0.988	0.986	0.985	0.985	0.984	0.984
2,829,011	1.000	1.000	1.000	0.999	0.998	0.997	0.994	0.992	0.991	0.990	0.990	0.990	0.990
2,937,820	1.000	1.000	1.000	1.000	0.999	0.998	0.997	0.995	0.994	0.994	0.994	0.993	0.993
3,046,628	1.000	1.000	1.000	1.000	0.999	0.999	0.998	0.997	0.996	0.996	0.996	0.996	0.996
3,155,436	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.998	0.998	0.998	0.997	0.997	0.997
3,264,244	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.999	0.999	0.998	0.998	0.998
3,373,052	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.999	0.999	0.999	0.999
3,481,860	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.999	0.999
3,590,668	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

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APPENDIX A
PARAMETERIZING THE MODEL

The two applications of the model described in Section 7 and Appendices B and C (simulation and Panjer's method) were parameterized using the following steps:

1. Select expected losses, E[L], where $L = P_t$ at t equal to ultimate, to the occurrence layer. Several estimates using experience and exposure rating techniques should be applied. The consistency of the results will affect how extensive a range of E[L] estimates should be tested in the model.

2. Estimate the negative binomial (NB) parameter p'. The authors used the fact that the variance/mean, Var/E = $1/p$ for NB. Starting with an ISO ground up Var/E in the 2.0 to 3.0 range, and ISO increased limits severity distributions, one can translate p (½ to ⅓) into p' using the transformation $p' = p/[1 - F(r) + pF(r)]$, r = retention. The examples herein use $p' = .969$ for excess of \$250,000 long haul trucking. As mentioned in Appendix D, E[V] is not sensitive to changes in p'.

3. Estimate a severity distribution $F(x)$. The authors used a Single Parameter Pareto (SPP) with $q = 1.5$. This was suggested by Philbrick [9] as being appropriate for casualty lines. With an SPP severity, changes in the q parameter do not cause significant changes in E[V]. The average severity for the layer (\$250,000 excess of \$250,000) is then calculated.

4. Dividing E[L] by the average severity produces an estimated number of excess claims, E[N]. One can then back into the NB α parameter as:

$$\frac{\mathrm{E}[N]p'}{1 - p'}$$

5. Select payment and/or report lag distributions. While this paper utilizes exponential lags, the results are equally valid for other distributions, although the parameter selection process would change.[7] The cash flows are fundamentally dependent on the payment lag distribution, so care should be taken in the selection of the distribution and its parameters. The extent of sensitivity testing is a function of one's confidence in the payment lag distribution. For a detailed discussion on estimating lag distributions, see Weissner [12]. John [3] discusses report lag distributions by reinsurance line of business.

6. Choose an interest rate to discount the cash flows. The examples used 8%.

[7] For example, if one were testing the sensitivity of results using a two parameter payment lag distribution, the coefficient of variation (CV) would be fixed, and scale parameter selected corresponding to several expected payment lags. This process would then be repeated for several different CV values.

APPENDIX B
MONTE CARLO SIMULATION

The simulation was programmed as follows: Single Parameter Pareto (SPP) q, expected layer losses, and exponential payment lag with lambda equal to 1/mean lag are selected and used to calculate Negative Binomial p' and α parameters as shown in Appendix A. Then, for one iteration:

1. N is drawn from a negative binomial NB (α, p').

2. For each of the N claims, a paid loss amount is drawn from SPP and a payment lag is drawn from the exponential. It was assumed that claims occur mid-year and premium and loss transactions are made at mid-year.

3. The P_t values are calculated by summing total payments in the appropriate time periods using the simulated lags.

4. The reinsurance contract terms were applied to the P_t's to obtain the C_t's.

5. V is calculated $= \sum_{t=1}^{n} (C_t - C_{t-1})\, v^{t-1}$, then V is stored.

The above was repeated for 20,000 iterations, then E[V], Variance [V] and Probability [$V > 0$] are calculated.

The program was written in HSFORTH and run on an IBM PS/2 with a 25 MH 80386 and 80387. For E[N] = 10, 20,000 iterations take 80 seconds, so a 10 × 10 matrix of parameters can be run in 2.3 hours.

APPENDIX C
PANJER'S METHOD

Recall that the aggregate loss distribution is a compound process formed by the infinite sum of n-fold convolutions of the frequency and severity distributions. Panjer [8] showed that this distribution can be estimated recursively provided that the frequency distribution satisfies the recursive relationship

$$p(n) = p(n - 1)(a + b/n) \qquad n = 1,2,3, \ldots ,$$

where $p(n)$ denotes the probability of exactly n claims occurring in a fixed time interval.

Sundt and Jewell [11] showed that the only distributions satisfying this condition are: Poisson, Negative Binomial, Binomial, and Geometric. In the case of the Negative Binomial,

$$p(n) = \binom{n + \alpha - 1}{\alpha - 1} p^{\alpha}(1 - p)^{n}, \; n = 0,1,2, \ldots ;$$

and

$$a = 1 - p, \; b = (1 - \alpha)(1 - p), \; p(0) = p^{\alpha}.$$

Furthermore, if the severity distribution can be represented discretely then the recursive formula for the aggregate distribution $G(\cdot)$ is quite simple:

$$g_i = \sum_{j=1}^{i} (a + bj/i) f_j g_{i-j} \qquad i = 1,2,3, \ldots ;$$

$$g_0 = p(0).$$

Section 7 and Appendix D show that if the number of ground-up claims is negative binomial NB (α, p), then the number of claims excess of a retention r, reported or paid at any time t, is also Negative Binomial with the appropriate transformations of the ground-up parameters.

This means that we can estimate the aggregate distribution of losses paid, P_t, or reported incurred, $L_t = P_t + K_t$, using Panjer's recursive formula.

For the layer being considered, the Single Parameter Pareto (SPP) severity distribution was discretized into equal intervals, with the number of intervals determined by the following formula subject to a maximum of 20:

$$\#\text{ intervals} = \frac{4}{\mathrm{E}[N]} + 10.$$

Increasing the number of intervals beyond these levels adds significantly to the run time without appreciable improvement in the results.

The above procedure can easily be used to estimate the expected value of V, the present value of net cash flows C_t. The variance of V, $\mathrm{Var}[V]$, on the other hand, is difficult to estimate. Obviously, the sequence of random variables $\{C_t\}$ is not independent. Not so obvious is the fact that, for even simple contract forms, the C_t's do not have independent increments; that is, the sequence $\{C_t - C_{t-1}\}$ is not independent. This means that the $\mathrm{Var}[V]$ contains non-zero covariance terms. This fact is demonstrated by the example in this appendix.

If the decision maker would like to consider other properties besides $\mathrm{E}[V]$ (e.g., $\mathrm{Var}[V]$), then a covariance matrix can be produced, though simulation may be simpler. Aside from that, the Panjer analytical solution is relatively easy to implement.

Example: C_t's Do Not Have Independent Increments

Consider the paid loss retro with no maximum,
$C_t = C(P_t) = \max(aP_t + b, m - P_t)$, where a = multiplicative loading, b = additive loading and m = minimum premium.

For convenience write $C(P_t)$ as:

$$C(P_t) = \begin{cases} m - P_t, & aP_t + b < m \\ aP_t + b, & aP_t + b > m. \end{cases}$$

Recall that the P_t's are non-decreasing and consider the case $aP_1 + b < aP_2 + b < m < aP_3 + b$.

$$
\begin{aligned}
\mathrm{COV}[&C(P_2) - C(P_1), C(P_3) - C(P_2)] \\
&= \mathrm{COV}[-(P_2 - P_1), aP_3 + b - (m - P_2)] \\
&= -a\,\mathrm{COV}[P_2 - P_1, P_3] + \mathrm{COV}[P_2 - P_1, m - P_2] \\
&= -a\,(\mathrm{COV}[P_2, P_3] - \mathrm{COV}[P_1, P_3]) - (\mathrm{COV}[P_2, P_2] \\
&\quad - \mathrm{COV}[P_1, P_2]) \\
&= (1 + a)\,(\mathrm{Var}[P_1] - \mathrm{Var}[P_2]).
\end{aligned}
$$

Note:
$$
\begin{aligned}
\mathrm{COV}[P_{t-1}, P_t] &= \mathrm{COV}[P_{t-1}, (P_t - P_{t-1}) + P_{t-1}] \\
&= \mathrm{COV}[P_{t-1}, P_t - P_{t-1}] + \mathrm{Var}[P_{t-1}] \\
&= \mathrm{Var}[P_{t-1}],
\end{aligned}
$$

because P_t's have independent increments.

APPENDIX D

The following proof derives the transformations shown in Section 7.

Theorem: If Y is negative binomial (α,p) and $X|y$ is binomial (y,w), then X is negative binomial (α',p') where $\alpha' = \alpha$,

and $p' = \dfrac{p}{w + p\,(1 - w)}$

Proof:

$$\Pr[X = x] = \sum_y \Pr[X = x|Y = y]\,\Pr[Y = y]$$

$$= \sum_y \binom{y}{x} w^x (1 - w)^{y-x} \binom{y + \alpha - 1}{\alpha - 1} p^\alpha (1 - p)^y$$

$$= \frac{w^x p^\alpha}{x!\,(\alpha - 1)!} (1 - p)^x \sum_y (1 - w)^{y-x} \frac{(y + \alpha - 1)!}{(y - x)!} (1 - p).^{y-x}$$

Multiplying numerator and denominator by $(x + \alpha - 1)!$ gives:

$$\binom{x + \alpha - 1}{\alpha - 1} [w(1 - p)]^x p^\alpha \sum_y \binom{y + \alpha - 1}{x + \alpha - 1} [(1 - p)\,(1 - w)].^{y-x}$$

Substituting $z = y - x$ and $1 - h = (1 - p)(1 - w) = 1 - (w + p - wp)$:

$$\binom{x + \alpha - 1}{\alpha - 1} \frac{[w\,(1 - p)]^x p^\alpha}{h^{x+\alpha}} \sum_z \binom{z + x + \alpha - 1}{x + \alpha - 1} h^{x+\alpha} (1 - h).^z$$

Notice that the summation over z equals 1.

$$\binom{x + \alpha - 1}{\alpha - 1} \left(\frac{p}{w + p\,(1 - w)}\right)^\alpha \left(\frac{w\,(1 - p)}{w + p\,(1 - w)}\right)^x$$

$$= \binom{x + \alpha - 1}{\alpha - 1} p'^\alpha (1 - p')^x$$

$$= \mathrm{NB}(\alpha',p').$$

Comments:

(i) For claims excess of a retention, r, $w = 1 - F(r)$, where $F(x)$ is the ground-up severity distribution.

(ii) For reported or paid claims $w = w(t)$, where $w(t)$ is the percent of claims reported or paid as of t months from the average accident date.

(iii) $$\frac{\text{Var}[X]}{\text{E}[X]} = \frac{1}{p'}$$
$$= \frac{w + p\,(1 - w)}{p}\,.$$

Note that as w approaches zero (which is the case for excess claims as the retention, r, gets large), the variance/mean approaches 1.0. In fact, the variance/mean approaches 1.0 quite quickly as the following table shows:

VARIANCE/MEAN FOR EXCESS CLAIMS

Ground-Up Var/E	Excess Claim Probability .25	.10	.05	.01
2.0	1.25	1.10	1.05	1.01
3.0	1.50	1.20	1.10	1.02
4.0	1.75	1.30	1.15	1.03

This result is consistent with sensitivity tests of $p' = (\text{Var}[X]/\text{E}[X])^{-1}$ which showed that E[V], the expected present value cash flow, did not change significantly for large changes in p'.

Also, as w approaches 1.0, p'_t approaches p', as expected.

(iv) The coefficient of variation, $CV[X]$, is given as follows:

$$\begin{aligned} CV[X]^2 &= \frac{1}{\alpha(1 - p')} \\ &= CV[Y]^2 \, \frac{w + p\,(1 - w)}{w} \, . \end{aligned}$$

While the variance/mean approaches 1.0 as w approaches zero, the coefficient of variation gets increasingly large as the retention increases.

DISCUSSION OF PAPER PUBLISHED IN VOLUME LXXIV

A NOTE ON THE GAP BETWEEN TARGET AND EXPECTED UNDERWRITING PROFIT MARGINS

EMILIO C. VENEZIAN

DISCUSSION BY SHOLOM FELDBLUM

This paper argues that if forecast and actual insurance costs are random variables, then the traditional actuarial ratemaking procedure produces an average underwriting profit margin lower than the target underwriting profit margin. The argument is correct in that the average profit margin, per policy or per book of business, will indeed be lower than the target profit margin. However, the expected total profit margin, for the insurer or for the industry as a whole, will not differ from the target profit margin. Observed differences between target and actual profit margins are due to marketplace competition, random forecasting errors, or unsustainable target margins, not to biases in the ratemaking procedures. The total profit margin for the insurer is the important figure, not the average margin per policy.

An illustration should clarify these comments. Suppose that:

1. The forecast insurance costs average \$95 per policy, but they vary with equal probability among \$80, \$95, and \$110 per policy.
2. The actual insurance costs have the same probability distribution.
3. The insurer uses a target underwriting profit margin of 5%.

With these probability distributions, the profit margins are as shown below.

TABLE 1

TARGET AND EXPECTED PROFIT MARGINS

	Forecast Cost	Premium	Actual Cost	Profit Margin	Profit Dollars
	\$ 80	\$ 84.21	\$ 80	+5.00%	\$ +4.21
	80	84.21	95	−12.81	−10.79
	80	84.21	110	−30.63	−25.79
	95	100.00	80	+20.00	+20.00
	95	100.00	95	+5.00	+5.00
	95	100.00	110	−10.00	−10.00
	110	115.79	80	+30.91	+35.79
	110	115.79	95	+17.95	+20.79
	110	115.79	110	+5.00	+5.79
Average:	\$ 95	\$100.00	\$ 95	+3.38%	\$ +5.00

The "average profit margin per policy," at 3.38%, is lower than the target profit margin. The total profit margin for the book of business, at 5.00%, is exactly equal to the target profit margin. Moreover, the *weighted average* of the profit margins per policy, where the weights are the premiums per policy, also equals 5.00%.

These observations are not restricted to the particular illustration used here. Equation (2.8) of the paper shows that the profit margin per policy is

$$A = \mathrm{E}(m) = 1 - (1 - T) \cdot \mathrm{E}\,((1 + y)/(1 + x)),$$

where

A is the achieved underwriting profit margin;
T is the target underwriting profit margin;

y and x are independent random variables with means of zero, measuring prediction errors and random cost fluctuations (see Equations (2.7) and (2.4));
E is an expected value operator; and,
m is the underwriting profit margin.
C is the cost per policy (see the following equations).

In other words, the profit margin for an individual policy i is

$$A_i = 1 - (1 - T) \cdot ((1 + y_i)/(1 + x_i)).$$

The dollar profit for this policy is

$$\text{Dollars}_i = C_i \cdot ((1 + x_i) - (1 - T) \cdot (1 + y_i)).$$

The total underwriting profit margin is the ratio of total dollars of profit to total dollars of premium. Since the x and y random variables appear as numerators of separate terms, they both average to zero and the ratio simplifies to

$$(\text{Total Cost} \cdot [1 - (1 - T)]) \,/\, \text{Total Cost} = T.$$

The weighted average of the profit margins is also equal to T. The weighted average is

$$\Sigma\, (\text{premium}_i \cdot \text{profit margin}_i) \,/\, \Sigma\, (\text{premium}_i).$$

This is the total dollars of profit divided by the total premium, and so equals T, as shown above.

When he read his paper before the CAS convention in San Antonio, Professor Venezian indicated that a weighted average by premium is inappropriate, since we are concerned with returns on equity, not returns on premium. This does not alter the situation. Suppose first that the surplus (equity) supporting each policy is a fixed amount that does not vary with the premium, or that it is related to the average cost, not the "forecast" cost. The relevant profit margin is the dollars of profit divided by the surplus amount. The x random variable never enters this ratio, and no "gap" is ever produced. Alternatively, suppose that the surplus supporting each policy varies with the premium charged—say, required surplus equals 50% of the premium. The x random variable does enter the denominator of the profit margin per policy. However, the weighted average profit margin uses surplus as the weights, so the x random variable once more cancels out of the ratio.

The paper's thesis is that the average of ratios is not the ratio of the average. This is unrelated to actuarial pricing procedures. Consider any firm: suppose the costs of two products are \$50 and \$150, and the corresponding revenues for each are \$100. The profit margins are +100% and −33%, for an average of +33%. Yet the firm's total revenues are \$200 and total costs are \$200, for a profit margin of 0%. The latter figure is the important one, since it shows the true profitability of the firm. The former figure varies with the allocation of costs and revenues to products or product lines. Similarly, in the insurance example, if total profit margins are considered, then expected underwriting profits should equal target underwriting profits.

TABLE 2

AVERAGE PROFIT MARGINS

Product	Quantity	Cost	Revenues	Profit
A	1	50	100	+100%
B	1	150	100	−33
Total	2	200	200	0%

Since the paper focuses on average profit margins per policy, instead of the total profit margin, it does not address the original problem:

> *"Over extended periods of time the average underwriting profit margins achieved by the industry as a whole, or by individual firms, in most jurisdictions differ substantially from the targets ostensibly built into the rates."*

This "gap" is the total profit margin achieved by the insurer or by the industry minus the target profit margin. The argument advanced in the paper has no relation to this gap.

ADDRESS TO NEW MEMBERS—MAY 14, 1990

STRAIGHT TALK

STEVEN H. NEWMAN

First of all, I'd like to add my personal congratulations to those already so enthusiastically expressed. Few professions require as much commitment and dedication as a requisite to entry. My remarks today are mainly from the perspective of a shameless and unrepentant realist, and a member of senior corporate management. As such, the thoughts expressed are my own, and may not conform to those held by the CAS.

You've entered a very different professional practice than I did in the 1960s. The job function then was principally advisory. We worked mostly on automobile and workers compensation insurance pricing, and liability loss reserve analysis. Most actuarial departments consisted of just a few people, and many had only one CAS member. The top position, i.e., the top rung of the career ladder, and one achieved at that time by just a handful of actuaries, was "Vice President and Actuary."

Today's mature actuary is far more likely to be "accountable," not advisory; accountable as the decision maker, the policy setter, the strategist. Today there are numerous CAS members heading underwriting departments for their companies. They are also the senior claims, information systems, and even field management officers. Many of the most prominent insurance and reinsurance companies are also led by actuaries.

What does it mean when I say that actuaries are now "accountable"? It means that your decisions will be measured against results. In the 1960s our opinion was sought, and it was generally advisory and academic. In the 1990s, however, your judgements are required, and they will be critical to the success of your business or client. Do your forecasts hold up? Does your pricing yield the intended result? And did *you* impact overall performance favorably? Accountability is unforgiving! Only achievements matter; explanations and excuses are largely irrelevant.

Is the present domination of many company functions by actuaries, particularly in senior management, a trend; or is it just a snapshot of a fleeting moment in the industry's history? While you ponder that question, here are a few related ones.

Have companies headed by actuaries outperformed the others? Would some of the most admired companies in the business like AIG, Chubb, Safeco, Cincinnati Financial, USAA, and State Farm be more successful if they were led by actuaries? Was AIG's remarkable success in the 1970s largely the result of my having established an actuarial department there; and is their even greater success in the 1980s because Bob Sandler is a better chief actuary than I was?

Until now, the thrust of your formal studies and your office work have been directed mostly toward developing fundamental tools for use in analyzing insurance experience. Although there is more to be gained by following that pattern and sharpening further these tools, I urge you to emphasize a different dimension in your continuing studies. That applies particularly to the new Fellows who are now in a position to change course from exam completion to professional and career development.

Believe it or not, it is often overlooked that the object of analysis is to reach a conclusion and to make a decision. Admittedly, more comprehensive and sophisticated analysis ought to ultimately yield a better decision, but does all the input needed for such a decision come in the form of data? Doesn't the quality, the training, the continuity, and even the often-changing mission of the people manning the underwriting and claims functions have a tremendous impact on results? What about understanding the product or exposure? Don't we at times need to understand the ultimate insured and his motives as well? My experience is that evaluating these issues correctly has a critical impact on the quality of the conclusions drawn from analyzing information, and on the overall success of the enterprise as well.

That is why I urge you to begin to broaden your business vision. Try to get as close as you can to the business of the business: the sale, the risk assessment and acceptance, and policyholder service, particularly claims. Take every opportunity to improve your understanding of how what you're doing fits into the whole. In the words of that modern philosopher, Yogi Berra . . . "You can see a lot by just looking!"

Lastly, as general guidance for you at this stage in your career, I suggest that you try to develop a sense of humility as you go about your work. The Bible says, "He that shall humble himself shall be exalted." Recognize the limitations of your art. Can your data be taken seriously, even though it is the best you can compile? What can alter future expected patterns? Management action? Economic conditions and interest rates? Society's values and court decisions? A better appreciation of the significance of these factors, together with a better understanding of your company or client, will inevitably lead to more mature and informed judgements. Whether or not your aspirations push beyond classical actuarial work, you *do* have a great deal more to learn about the business!

Through your efforts on CAS committees, your views will soon begin to impact the profession and in time you will inherit the leadership of the CAS. My personal hope is that through your enlightened professionalism and broader involvement in the business, you will also inherit the leadership of the industry.

Good luck, and Godspeed!

MINUTES OF THE 1990 SPRING MEETING

MAY 13–16, 1990

THE BROADMOOR, COLORADO SPRINGS, COLORADO

Sunday, May 13, 1990

The Board of Directors held their regular quarterly meeting from 1:00 p.m. to 4:00 p.m.

Registration was held from 4:00 p.m. to 6:30 p.m.

A presentation to new Associates and guests was held from 5:30 p.m. to 6:30 p.m.

A welcome reception for all members and guests was held from 6:30 p.m. to 7:30 p.m.

Monday, May 14, 1990

Registration continued from 7:00 a.m. to 8:00 a.m.

A Business Session was held from 8:00 a.m. to 9:30 a.m.

Welcoming comments were made by President Michael Fusco.

A ceremony for new members was held. There were 86 new Associates and 16 new Fellows. The names of these individuals are as follows:

FELLOWS

J. Scott Bradley
Malcolm E. Brathwaite
Scott H. Dodge
Vincent T. Donnelly
Thomas J. Ellefson
William G. Fanning
Kirk G. Fleming
Teresa J. Herderick
Richard J. Hertling
Jane E. Jasper
John J. Joyce
Christine E. Radau
David C. Scholl
Debbie Schwab
G. Clinton Sornberger
Elisabeth Stadler

ASSOCIATES

Gregory S. Beaulieu
Douglas L. Beck
Allan R. Becker
Gavin C. Blair
Jean-Francois Blais
Roberto G. Blanco
Scott K. Charbonneau
Donald L. Closter
Jean Cloutier
Michael A. Coca
Charles Cossette
Jean Cote
Robert J. Curry
Daniel J. Czabaj
Jeffrey F. Deigl
Edward D. Dew
David A. Doe
Bob D. Effinger, Jr.
John W. Ellingrod
Catherine E. Eska
Deborah C. Finnerty
Kerry L. Fitzpatrick
Ross C. Fonticella
France Fortin
Scott F. Galiardo
James M. Gevlin
Linda M. Goss
Edward M. Grab
Marian R. Gross
Michele P. Gust
Thomas L. Hayes
Kathleen A. Hinds
Joanne K. Ikeda
Terrell A. Jones
James W. Jonske
Edward M. Jovinelly
Kevin J. Kelley
Allan A. Kerin
Daniel F. Kligman
Ronald T. Kozlowski
Jean-Marc Leveille
Siu K. Li
Andre Loisel
Brian E. MacMahon
Laura Manley
Burton F. Marlowe
Liam M. McFarlane
Richard B. Moncher
Kelly L. Moore
Kevin J. Moynihan
Daniel M. Murphy
Anthony J. Nerone
William A. Niemczyk
Keith R. Nystrom
Jacqueline E. Pasley
Andre Perez
Marvin Pestcoe
Jill Petker
Michael Petrocik
Jennifer A. Polson
Timothy P. Quinn
Jeffrey C. Raguse
Karin M. Rhoads
A. Scott Romito
Kevin D. Rosenstein
Scott J. Roth
Stuart G. Sadwin
Melissa A. Salton
Pierre A. Samson
Edmund S. Scanlon
Lisa Pouloit Schmidt
Theodore R. Shalack
Rial R. Simons
Christopher M. Smerald
Karen F. Steinberg
Ting-Shih Teng
Richard D. Thomas
Mary L. Turner
Melanie A. Turvill
Christopher P. Walker
Patrick M. Walton
Monty J. Washburn
Scott P. Weinstein
L. Nicholas Weltmann, Jr.
Leigh F. Wickenden
Roger A. Wilk

An address to new members was given by Steven Newman from 8:35 a.m. to 8:45 a.m.

Highlights of the program were presented by Richard Fein.

A summary of call papers was given by David Oakden.

A summary of new *Proceedings* papers was given by Martin Lewis.

Presentation was made of the Harold W. Schloss Memorial Scholarship Fund to Robert J. Moser.

The featured speaker, Dr. Leon Lederman, spoke from 9:30 a.m. to 10:30 a.m.

A general session panel was held from 11:00 a.m. to 12:15 p.m. Introductions were made by Stephen Philbrick, Consulting Actuary, Tillinghast/Towers Perrin. The subject was "Understanding the New Science of Chaos." Mr. Philbrick was the moderator and the panel consisted of David J. Grady, Vice President and Actuary, Prudential Reinsurance Company, and Dr. Leon Lederman.

This was followed by a luncheon from 12:15 to 1:45 p.m.

The afternoon was devoted to concurrent sessions from 1:45 p.m. to 5:00 p.m. with a break from 3:15 to 3:30 p.m.

The concurrent sessions consisted of the following topics:

Long Range Planning Committee.

"Evaluating the Effect of Reinsurance Contract Terms," by James N. Stanard and Russell T. John.

"An Exposure Rating Approach to Pricing Property Excess of Loss Reinsurance," by Stephen J. Ludwig.

"Pricing the Catastrophe Exposure in Homeowners Ratemaking," by David H. Hays.

"An Actuarial Analysis of Servicing Carrier Profit Margins," by Mark W. Littmann.

"Implications of the Mandatory Elimination of a Rating Variable," by Frank J. Karlinski, III.

"Commutation Pricing in the Post Tax Reform Era," by Vincent P. Connor and Richard A. Olsen.

"Individual Risk Loss Development," by Joseph P. Theisen.

"Credibility for Treaty Reinsurance Excess Pricing," by Isaac Mashitz and Gary Patrik.

"The Challenge of Pricing Extended Warranties," by Timothy L. Schilling.

"On the Representation of Loss and Indemnity Distributions," by Yoong-Sin Lee.

"Sources of Distortion in Classification Ratemaking Data and Their Treatment," by John A. Stenmark.

"An Iterative Approach to Classification Analysis," by Joyce Fish and Gary Patrik.

"Property-Liability Insurance Pricing Models: An Empirical Evaluation," by Stephen P. D'Arcy and James R. Garven.

"The Econometric Method of Mixed Estimation—An Application to the Credibility of Trend," by Paul J. Brehm and Denis G. Guenthner.

"Evaluating Workers Compensation Trends Using Data by Type of Disability," by Allan Kaufman.

"Commercial General Liability Ratemaking for Premises and Operations," by Nancy C. Graves and Richard Castillo.

"Pricing the Impact of Adjustable Features and Loss Sharing Provision of Excess-of-Loss Treaties," by Robert Bear and Kenneth Nemlick.

"Basic and Increased Limits Ratemaking: An Integrated Approach," by D. Lee Barclay and Dick Marquardt.

"On Pricing Multiple-Claimant Occurrences for Workers Compensation Per-Occurrence Excess of Loss Reinsurance Contracts," by Gregory T. Graves.

"Overlap Revisited—The Insurance Charge Reflecting Loss Limitation Procedure," by Dr. I. Robbin.

There was an officers' reception from 5:00 p.m. to 6:00 p.m. for new Fellows and guests.

There was a general reception from 6:00 p.m. to 7:00 p.m., which was followed by a special musical presentation, "Cut My Rate," from 7:00 p.m. to 8:30 p.m.

Tuesday, May 15, 1990

Concurrent sessions were held from 8:30 a.m. to 10:00 a.m., covering the following topics:

"Parametrizing the Workers Compensation Experience Rating Plan," by William R. Gillam.

"Evaluating Workers Compensation Trends Using Data by Type of Disability," by Allan Kaufman.

"Commercial General Liability Ratemaking for Premises and Operations," by Nancy C. Graves and Richard Castillo.

"An Exposure Rating Approach to Pricing Property Excess of Loss Reinsurance," by Stephen J. Ludwig.

"Pricing the Catastrophe Exposure in Homeowners Ratemaking," by David H. Hays.

"An Actuarial Analysis of Servicing Carrier Profit Margins," by Mark W. Littman.

"Implications of the Mandatory Elimination of a Rating Variable," by Frank J. Karlinski III.

"Commutation Pricing in the Post Tax Reform Era," by Vincent P. Connor and Richard A. Olsen.

"Individual Risk Loss Development," by Joseph P. Theisen.

"Credibility for Treaty Reinsurance Excess Pricing," by Isaac Mashitz and Gary Patrik.

"The Challenge of Pricing Extended Warranties," by Timothy L. Schilling.

A General Session, "Monitoring for Solvency" was held from 10:30 a.m. to 12:00 noon, with Kevin M. Ryan, Consulting Actuary for Milliman & Robertson, Inc., as Moderator; and John B. Chesson, Majority Counsel, Oversight and Investigations Subcommittee of the House Energy and Commerce Committee; William McCartney, Commissioner, Nebraska Insurance Department, Chairman, NAIC Financial Conditions (Ex4) Committee; and Walter J. Fitzgibbon, Jr., Vice President and Corporate Actuary, Aetna Life & Casualty, as panelists.

The afternoon was free.

There was a Western Barbecue from 6:30 p.m. to 9:30 p.m.

Wednesday, May 16, 1990

Concurrent sessions were held from 8:30 a.m. to 10:00 a.m., on the following topics:

Discussion of "A Note on the Gap Between Target and Expected Underwriting Profit Margins," by Sholom Feldblum.

"Expense Allocation and Policyholder Persistency, Persistency and Profits," by Sholom Feldblum.

"Homeowners Insurance Pricing," by Mark J. Homan.

"Homeowners Ratemaking," by Stacy J. Weinman.

"Property-Liability Insurance Pricing Models: An Empirical Evaluation," by Stephen P. D'Arcy and James R. Garven.

"The Econometric Method of Mixed Estimation—An Application to the Credibility of Trend," by Paul J. Brehm and Denis G. Guenthner.

"Sources of Distortion in Classification Ratemaking Data and Their Treatment," by John A. Stenmark.

"An Iterative Approach to Classification Analysis," by Joyce Fish and Gary Patrik.

"Pricing the Impact of Adjustable Features and Loss Sharing Provisions of Excess-of-Loss Treaties," by Robert Bear and Kenneth Nemlick.

"Basic and Increased Limits Ratemaking: An Integrated Approach," by D. Lee Barclay and Dick Marquardt.

"On Pricing Multiple-Claimant Occurrences for Workers Compensation Per-Occurrence Excess of Loss Reinsurance Contracts," by Gregory T. Graves.

"Overlap Revisited—The Insurance Charge Reflecting Loss Limitation Procedure," by Dr. I. Robbin.

The general session resumed at 10:30 a.m., with presentation of the Michelbacher Award to Joyce Fish for the call paper, "An Iterative Approach to Classification Analysis" by Cecily Gallagher, Joyce Fish, and Howard Monroe. A general session on the new "Course on Professionalism" was given. Janet L. Fagan, Vice President and Senior Actuary, CIGNA Property and Casualty Group was the moderator. Gustave A. Krause, Consulting Actuary, Tillinghast/Towers Perrin and W. James MacGinnitie, Consulting Actuary, Tillinghast/Towers Perrin, were panelists.

Closing remarks were given from 11:45 a.m. to 12:00 noon.

May, 1990 Attendees

In attendance, as indicated by the registration records, were 287 Fellows, and 165 Associates. The list of their names follows:

FELLOWS

Terry Alfuth	Francois Boulanger	Greg Ciezadlo
Mark Allaben	Wallis Boyd	John Coffin
Charlie Angell	Scott Bradley	Robert Conger
Lee Barclay	John Bradshaw	Charles Cook
Bill Bartlett	Jim Brannigan	Frank Corr
Irene Bass	Malcolm Brathwaite	Michael Covney
Edward Baum	Paul Brehm	Richard Cundy
Bob Bear	Dale Brooks	Jim Curley
Al Beer	Brian Brown	Kathleen Curran
Linda Bell	Charles Bryan	Ross Currie
Lenny Bellafiore	John Buchanan	Bob Daino
Janice Berry	George Burger	Steve D'Arcy
Richard Beverage	Claudette Cantin	Gary Dean
William Biegaj	John Captain	Gary Demarlie
Richard Bill	Ruy Cardoso	Howard Dempster
Terry Biscoglia	Jeff Carlson	Bob Deutsch
Bonnie Boccitto	Bill Carpenter	Scott Dodge
Leroy Boison	Andrew Cartmell	Mark Doepke
Parker Boone	Sandy Cathcart	Michael Dolan
Joseph Boor	David Chernick	Vin Donnelly
Ronald Bornhuetter	Diana Childs	Jim Dornfeld

FELLOWS

Lester Dropkin
Brian Duffy
Paul Dyck
Myron Dye
Bruce Earwaker
Richard Easton
Tom Ellefson
Jeffery Englander
David Engles
Janet Ericson
Glenn Evans
Bob Eyers
Doreen Faga
Janet Fagan
Bill Fanning
Richard Fein
Sholom Feldblum
Mark Fiebrink
Russell Fisher
Wayne Fisher
Walter Fitzgibbon
Kirk Fleming
Jim Foote
Louise Francis
Glenn Fresch
Michael Fusco
Cecily Gallagher
Tom Gallagher
Steven Gapp
Chris Garand
Bob Gardner
Boob Giambo
Robin Gillam
Bryan Gillespie
Greg Girard
Spencer Gluck
Daniel Goddard
David Grady
Tim Graham
Patrick Grannan
Gregory Graves
Nancy Graves
Ron Greco
Ann Griffith
Denis Guenthner
Charlie Hachemeister
Larry Haefner
Dave Hafling
Jim Hall
Malcolm Handte
Walter Haner
Jeff Hanson
All Hapke
Dave Hartman
David Hays
Leroy Heer
Agnes Heersink
Paul Henzler
John Herder
Teresa Herderick
Richard Hertling
William Hibberd
Jerry Hillhouse
Mark Homan
Ruth Howald
Doug Hoylman
Heidi Hutter
Jim Inkrott
Richard Jaeger
Steve Jameson
Jane Jasper
Gerry Jerabek
Dick Johe
Russ John
Marvin Johnson
Wendy Johnson
Thomas Johnston
Alan Jones
Bruce Jones
John Joyce
Roy Kallop
Frank Karlinski
Allan Kaufman
Glenn Keatts
Eric Keen
Stan Khury
Fred Kilbourne
Fred Kist
Kyleen Knilans
Lee Koch
John Kollar
Mikhael Koski
Gary Koupf
Israel Krakowski
Gus Krause
Rodney Kreps
Andy Kudera
Gaetane Lafontaine
Michael Lamb
Mike Lamonica
Gary Larose
Michael Larsen
Steve Lehmann
Winsome Leong
Joseph Levin
Martin Lewis
Orin Linden
Rich Lino
Mark Littmann
Roy Livingston
Kevin Lonergan

FELLOWS

Dennis Loper
Robert Lowe
Stephen Ludwig
Aileen Lyle
Jim MacGinnitie
Howard Mahler
Steve Makgill
Joe Marker
Paul Martin
Isaac Mashitz
Bob Matthews
Chuck McClenahan
Betsy McCoy
Gary McDonald
Dennis Mealy
Glenn Meyers
Dave Miller
Mary Frances Miller
Philip Miller
Susan Miller
William Miller
Neil Miner
William Morgan
Jay Morrow
John Muetterties
Donna Munt
John Murad
Thomas Murrin
Jim Muza
Thomas Myers
Kenneth Nemlick
Rick Newell
Patrick Newlin
Steve Newman
Gary Nickerson
Jim Nikstad
Charlie Niles, Jr.
Dave Oakden
Dick Pagnozzi
Robert Palm
Gary Patrik
Marc Pearl
Bruce Petersen
Chuck Petit
Steven Petlick
Roberta Pflum
Steve Philbrick
Emanuel Pinto
Jeff Post
Phil Presley
Virginia Prevosto
Richard Quintano
Andre Racine
Christine Radau
Kurt Reichle
Dan Reppert
Deborah Rosenberg
Gail Ross
Richard Roth, Jr.
Kevin Ryan
Robert Sanders
Donald Sandman
Jerry Scheibl
Jeffrey Scheuing
Tim Schilling
Neal Schmidt
David Scholl
Roger Schultz
Joseph Schumi
Debbie Schwab
Joy Schwartzman
Kim Scott
Rick Sherman
Roy Shrum
Peter Siczewicz
Jerry Siewert
Mel Silver
Leroy Simon
David Skurnick
John Slusarski
Clinton Sornberger
Joanne Spalla
Dan Splitt
Elisabeth Stadler
Jim Stanard
Lee Steeneck
Walt Stewart
Emil Strug
Stuart Suchoff
Bob Tatge
Jane Taylor
Kathleen Terrill
Mike Toothman
Frank Tresco
Michel Trudeau
Lee Van Slyke
Jerry Visintine
Bill Von Seggern
Glenn Walker
Mike Walsh
Mavis Walters
Mike Walters
Stacy Weinman
David Westerholm
Jonathan White
Robin Williams
Richard Woll
Roy Woomer
Paul Wulterkens
Jim Young
James Yow
Richard Zatorski
John Zicarelli

ASSOCIATES

Jean-Luc Allard
Jim Andler
Bernie Battaglin
Greg Beaulieu
Karin Beaulieu
Douglas Beck
Allan Becker
Gavin Blair
Jean-Francois Blais
Robert Blanco
Jack Brahmer
Patrick Burns
Rick Campbell
Christopher Carlson
Martin Cauchon
Mike Caulfield
Scott Charbonneau
David Clark
Don Closter
Jean Cloutier
Michael Coca
Vin Connor
Charles Cossette
Jean Cote
Dan Crifo
Robert Curry
Daniel Czabaj
Rod Davis
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Davis Doe
Frank Douglas
Bob Effinger, Jr.
John Ellingrod
Catherine Eska
Phil Evensen
Debbie Finnerty
Bill Fitzpatrick
Kerry Fitzpatrick
Ross Fonticella
France Fortin
Barry Franklin
Mary Gaillard
Scott Galiardo
Felix Gerard
Scott Gerlach
Terry Goldberg
Linda Goss
Sue Gozzo
Edward Grab
Gray Granoff
Carleton Grose
Marian Gross
Michele Gust
Ewa Gutman
Aaron Halpert
Tom Hayes
Philip Heckman
Joe Herbers
Kathleen Hinds
Alan Hines
Brian Hughes
Joanne Ikeda
Bob Jaso
James Jensen
Dan Johnston
Terry Jones
Jim Jonske
Ed Jovinelly
Kevin Kelley
Marty Kelly
Allan Kerin
James Kleinberg
George Klingman
Tim Kolojay
Ronald Kozlowski
Ken Krissinger
David Lacefield
Claude Lafrenaye
David Lalonde
William Leiner
Jean-Marc Leveille
George Levine
Andre Loisel
Dave Macesic
Brett MacKinnon
Brian MacMahon
Laura Manley
R Marks-Samuelson
Liam McFarlane
Robert Meyer
Madan Mittal
Rich Moncher
Andrew Moody
Kelly Moore
Kevin Moynihan
Daniel Murphy
Rade Musulin
John Nelson
Bill Niemczyk
Chris Nyce
Keith Nystrom
Dale Ogden
Bruce Ollodart
Tim Paddock
Jaci Pasley
Andre Perez
Jill Petker
Michael Petrocik
Jen Polson
Debbie Price
Ron Pridgeon

ASSOCIATES

Timothy Quinn	Michael Scruggs	Rob Waldman
Jeff Raguse	Margaret Seiter	Christopher Walker
Karin Rhoads	Theodore Shalack	David Walker
James Rice	Arlyn Shapiro	Patrick Walton
Rich Robinson	Rial Simons	Monty Washburn
Scott Romito	Chris Smerald	Nancy Watkins
Kevin Rosenstein	Barbara Stahley	Scott Weinstein
Scott Roth	Edward Stanco	Nick Weltmann
George Rudduck	Karen Steinberg	Russell Wenitsky
Stu Sadwin	John Stenmark	Debra Werland
Melissa Salton	Craig Taylor	Leigh Wickenden
Pierre Samson	Glenn Taylor	Peter Wildman
Michael Sansevero	Karen Terry	Roger Wilk
Sandra Santomenno	Joe Theisen	Brad Williams
Edmund Scanlon	Richard Thomas	William Wilson
V Schmid-Sadwin	Ben Tucker	Rich Yocius
Jeffrey Schmidt	Marcie Turner	Heather Yow
Lisa Schmidt	Melanie Turvill	Ron Zaleski

Volume LXXVII, Part 2 No. 147

PROCEEDINGS

November 11, 12, 13, 14, 1990

PRICING THE IMPACT OF ADJUSTABLE FEATURES AND LOSS SHARING PROVISIONS OF REINSURANCE TREATIES

ROBERT A. BEAR
KENNETH J. NEMLICK

Abstract

Excess-of-loss reinsurance contracts often contain loss sharing provisions, such as aggregate deductibles, loss ratio caps or limited reinstatements, and loss corridor provisions. They also frequently contain adjustable premium or commission features, such as retrospective rating plans, profit commission plans, and sliding scale commission plans. Pro rata treaties frequently contain adjustable commission features.

This paper presents an overview of two approaches to pricing aggregate loss distribution problems: the lognormal model and the Heckman-Meyers Collective Risk Model. Applications to reinsurance pricing are then presented. Finally, the paper compares results of applying these approaches to representative working excess-of-loss treaties.

These comparisons suggest that the lognormal model can provide satisfactory approximations to the theoretically more appropriate Collective Risk Model when use of the latter more sophisticated procedure is not necessary due either to data *limitations or to the influence of market conditions and negotiations. The increased efficiency of the lognormal model can lead to greater accuracy by making judgmental estimates unnecessary in many situations.*

The basic lognormal model is generally applicable to pro rata treaties and working excess-of-loss treaties. A mixture of lognormal and discrete distributions is presented that may be applicable in many low mean frequency situations. Cash flow modelling is also discussed.

1. INTRODUCTION

Working excess-of-loss reinsurance contracts, where significant loss frequency is expected, often contain nonproportional coinsurance clauses. These involve provisions where the ceding company is to pay a nonproportional share of losses without receiving a commensurate share of the reinsurance premium. Such clauses include aggregate deductibles, aggregate limits such as loss ratio caps or limited reinstatements, and loss corridor provisions. Quite frequently in the broker market, and less frequently in the direct market, working excess-of-loss treaties contain adjustable premium or commission features. These adjustable features include retrospective rating plans, profit commission or profit sharing plans, and sliding scale commission plans. A relatively small number of excess-of-loss treaties contain both adjustable premium or commission features and nonproportional coinsurance clauses. Pro rata treaties also frequently contain adjustable commission features.

This paper will first present an overview of two approaches to pricing aggregate loss distribution problems: the lognormal model and the Heckman-Meyers Collective Risk Model. Six examples are then presented of how aggregate loss distributions are used in reinsurance pricing. Results of applying these two aggregate loss distribution approaches are compared. Finally, several enhancements of the basic model are discussed. The focus in the paper is on concepts, with technical details and proofs presented in the appendices. Appendices summarize important excess-of-loss pricing methodologies and provide an expanded lognormal table.

The list of references presented at the end of the paper contains several sources for those wishing to delve into reinsurance and excess pricing concepts in greater depth.

The authors' overall purpose is to determine if the lognormal model provides a suitable approximation for reinsurance price monitoring purposes and for pricing situations where limited information is available or a highly precise answer is not required. If the lognormal model provides a satisfactory approximation to the Collective Risk Model results, significant efficiency gains are achievable. A more sophisticated three-parameter alternative to the lognormal is not tested. The reason for this is that the Collective Risk Model or an equivalent approach would be employed if the data and other resources would permit a more sophisticated approach.

2. AGGREGATE LOSS DISTRIBUTIONS

In order to price the impact of adjustable features and nonproportional coinsurance clauses, it is necessary to estimate the aggregate loss distribution. Two methods of estimating this distribution are employed:

(a) *The Lognormal Model*
If the aggregate loss random variable is viewed as the product of a large number of independent, identically distributed random variables, then the logarithm is approximately normally distributed by the Central Limit Theorem. (The stringent condition that the factors be identically distributed may be relaxed [1].) By definition, the aggregate loss random variable is lognormally distributed. In Appendix A, standard formulas based on the Patrik-John Collective Risk Model are employed to estimate the aggregate mean and coefficient of variation from the assumed frequency and severity distributions [2]. An expanded lognormal table with excess pure premium ratios for coefficients of variation between 0.1 and 5 was programmed based on the formulas in Mr. Finger's paper "Estimating Pure Premiums by Layer" [3]. Mr. Finger developed the lognormal model for severity applications, although it is being tested here as an aggregate loss model. Appendix B summarizes the lognormal model and presents the

expanded lognormal table. Parameter uncertainty can be modelled by subjectively weighting indications based on alternative parameter values. (The subjective weights reflect degrees of belief in alternative scenarios, each of which yields a mean and a coefficient of variation of a particular lognormal model.)

(b) *The Collective Risk Model*
This model involves the estimation of parameters for the frequency and severity distributions, along with the judgmental selection of parameters to reflect the degree of uncertainty in the estimated frequency and severity means. If the shape of these distributions is also uncertain, one could assign subjective probabilities reflecting degrees of belief to several scenarios and compute a weighted average of the resulting cumulative probabilities and excess pure premium ratios. These quantities are computed using the Heckman-Meyers algorithm [4], which uses piecewise linear approximations of the cumulative severity distributions together with the assumed frequency distributions to generate the characteristic functions of the severity and aggregate loss distributions. As the characteristic function uniquely determines a probability distribution, numerical methods are employed to evaluate the rather complicated formulas that accomplish this inverse transformation, yielding the aggregate loss cumulative probability distribution function and excess pure premium ratios needed to price the reinsurance conditions that are the focus of this paper. Technical details are summarized in Appendix C.

Appendix D shows that if the ground-up occurrence count distribution for an insured selected at random is negative binomial, then the excess occurrence count distribution for a randomly selected insured is also negative binomial. Based on this result, the formula is derived for calculating the excess occurrence count variance-to-mean ratio for an individual insured selected at random, and it is shown that this formula also applies to the class as a whole. This latter result is then used to demonstrate that, if the proportion of occurrences exceeding the retention is small and the excess frequency mean is known, then the excess occurrence count distribution for the class as a whole is approximately Poisson.[1]

In particular, it is established that

$$VMR_E = (1 - p) + p(VMR_G),$$

where VMR_G and VMR_E are the variance-to-mean ratios for the ground-up and excess occurrence count distributions, respectively, and where p is the probability that a claim will exceed the retention. If VMR_G is two or three (as in the ISO increased limits reviews), and p (a value that may also be calculated via ISO increased limits parameters) is less than .02, then VMR_E is close to unity. This implies that the excess occurrence count distribution for an insured selected at random and for the class as a whole will be approximately Poisson under conditions of parameter certainty. In the Collective Risk Model, uncertainty in the mean frequencies for the various classes of business is reflected through selection of the contagion parameters. This results in negative binomial frequency distributions for the classes under consideration.

The Single Parameter Pareto (SPP) distribution is used to model occurrence severity. Mr. Philbrick's paper "A Practical Guide to the Single Parameter Pareto Distribution" and the discussion by Messrs. Reichle and Yonkunas [6] provide an excellent discussion of this distribution, which is widely used in excess pricing. Ms. Rytgaard recently presented a paper [7] that compares alternative estimates of the SPP parameter and applies credibility theory to obtain more stable estimators of this parameter for portfolios of excess-of-loss treaties with similar characteristics. Appendix E summarizes some of the key properties of the SPP distribution. In particular, it is shown that if ground-up loss occurrences in excess of a particular truncation point are distributed according to the SPP distribution with parameter q, then the excess portions of these occurrences are distributed according to the shifted Pareto distribution (used by Insurance Services Office in increased limits

[1] The proof given in Appendix D is a direct application of the Gamma-Poisson model frequently encountered in the actuarial literature. The authors acknowledge that these results have previously been established elsewhere, and note that Joseph Schumi has established these results using recursive relationships [5].

pricing), where the scale parameter is equal to the truncation point and the shape parameter is equal to q. In the Collective Risk Model, uncertainty in the mean severities is reflected through the selection of the mixing parameter.

Theoretically, if the SPP is appropriate for loss occurrences in excess of a particular attachment, it should be appropriate above all higher attachments, and the parameter should remain constant. Fits to industry data have led the authors to conclude that the SPP parameter varies with the truncation point used in the fitting procedure. Moreover, if the truncation point used in the fitting procedure is less than 50% of the attachment for a particular pricing analysis, the errors become unacceptably large. In order to calculate these parameters, development triangles of SPP parameter estimates for various truncation points were constructed, from which projections were made of the ultimate values of this parameter by class of business and truncation point. In the examples discussed in this paper, the class of business is not identified, because the intent is only to discuss actuarial methodology. Although alternative two- and three-parameter distributions should be tested when data permits, the SPP distribution with these qualifications can be a satisfactory severity model for reinsurance price monitoring work and in pricing situations where limited information is available.

3. EXAMPLES OF TREATIES WITH ADJUSTABLE FEATURES AND LOSS SHARING PROVISIONS

This section discusses the pricing of excess-of-loss treaties containing common types of nonproportional coinsurance clauses and adjustable premium or commission plans. This is accomplished through the examination of six hypothetical treaties, the key provisions of which are summarized at the start of Appendices F through K, respectively. The analysis of each example involves two major steps. First, various parameters (such as the expected claim count, mean severity, and aggregate coefficient of variation) which underlie the distribution of losses in the reinsured layer are calculated. This allows one to obtain an appropriate set of excess pure premium ratios, either by reference to an appropriate lognormal table (via coefficient of variation matching) or by direct generation via the Heckman-Meyers Collective Risk Model. The second step involves the use of the set of excess pure premium ratios derived in the first step in order to determine the expected impact of the particular nonproportional coinsurance clause or adjustable feature being evaluated. For the sake of clarity, excess pure premium ratios (which are called insurance charges in the examples) based on the lognormal assumption are initially used to analyze the six treaty examples. In Section 4, a comparison is made to the results obtained when excess pure premium ratios generated by direct applications of the Collective Risk Model are employed.

Aggregate Deductible Example

Treaty I is an example of a contract containing an annual aggregate deductible provision. The calculation of the treaty's aggregate loss coefficient of variation (CV), which is displayed on Appendix F Exhibit 1, is based on the theory and formulas presented in the second section of this paper as well as in Appendices A through E. The computation of the impact of the aggregate deductible is shown in Appendix F Exhibit 2. The deductible amount is compared to the expected losses in the reinsured layer in order to obtain a corresponding entry ratio, which allows one to look up the appropriate insurance charge from the lognormal tables in Appendix B. (Linear interpolation is used to calculate excess pure premium ratios for CV and entry ratio combinations not explicitly listed in these tables.) Since the insurance charge (29.33% in this case) represents the expected proportion of aggregate losses above the deductible amount, it is easy to see that the complement of this value (70.67%) is the expected percentage of treaty losses eliminated by the aggregate deductible. Thus, if a burning cost or similar study shows that the expected loss cost for the entire layer is 3.75% of subject premium, then the introduction of an aggregate deductible provision reduces this loss cost to 3.75% × [100% − 70.67%], or about 1.10% of subject premium. As shown in Appendix F, this loss cost can easily be converted to an indicated treaty rate through the application of an appropriate expense, profit, and risk loading factor. It should be noted that the factor selected for this purpose should include a provision for risk commensurate with the degree of variability in layer losses after application of the deductible. The degree of variability in this case, and hence the risk load, is higher than that for losses prior to the reflection of this provision.

Aggregate Limit Example

Treaty II contains a limited reinstatement clause. The contract allows three free reinstatements of coverage during the treaty year, which means that the ceding company is covered for losses in the specified layer until those losses exceed four times the width of that layer. After that point, no coverage is provided. (This type of reinstatement clause should be contrasted with the kind that reinstates coverage after a certain number of losses have occurred only if an additional premium is paid. This latter type is really a separate cover, rather than a form of coinsurance on the original treaty.)

The pricing of this treaty is summarized in Appendix G. As was done in the previous example, an entry ratio is calculated by dividing the dollar value of the limited reinstatement provision ($2,800,000 in this case) by the expected losses in the layer prior to all forms of coinsurance. The insurance charge corresponding to this entry ratio (2.37% in this example) is equivalent to the expected percentage of losses eliminated by the limited reinstatement clause. Combining this quantity with the treaty's 20% proportional coinsurance provision yields a 21.89% overall coinsurance percentage. This latter coinsurance percentage is then applied to the expected layer loss cost prior to all coinsurance in order to obtain an expected loss cost and an indicated rate for the treaty. As in the previous example, the loading to convert the expected loss cost to a rate includes a provision for risk that reflects the potential volatility in treaty losses after the limited reinstatement is taken into account. Note that this risk provision should be somewhat lower than that for a similar treaty with no limited reinstatement clause. This is due to the fact that limited reinstatements, along with most other kinds of mechanisms that place a cap on losses, tend to reduce loss variability.

Loss Corridor Example

Treaty III is an example containing a loss corridor provision. Under a loss corridor provision, the reinsurer pays all losses falling in the reinsured layer up to a certain aggregate amount (called the lower bound of the loss corridor interval). Once this amount is reached, the reinsurer stops paying all losses until the total losses in the layer exceed a second threshold amount (the upper bound of the loss corridor interval). After this, the reinsurer resumes payment for all losses in the reinsured layer. The bounds of the loss corridor interval may be expressed in terms of dollar amounts, percentages of expected layer losses, or ratios to treaty premium.

In the example presented in Appendix H, the loss corridor bounds are stated as percentages of expected losses in the layer. This makes the analysis extremely straightforward, since these percentages are directly equivalent to the corresponding entry ratios. The difference between the insurance charges at the lower and upper bounds, respectively, results in the expected percentage of layer losses eliminated by the loss corridor provision. The computation of the expected layer loss cost after coinsurance and the indicated treaty rate is analogous to the calculations presented in the first two examples. Unlike the previous examples, however, there is no definite rule concerning the proper risk load to be included in the factor used to convert the loss cost into a rate. This is due to the fact that the loss corridor provision may either reduce or increase the variability of layer losses, depending on both the location and the size of the eliminated loss interval.

While the simplicity of the loss corridor analysis is not altered very much when the interval bounds are expressed in terms of dollars, the analysis does get complicated when the bounds are stated as ratios to treaty premium. This is due to the fact that the treaty premium is dependent on the treaty rate, which should already reflect the effect of the loss corridor. It is clear that the solution to this problem requires an iterative procedure in which the algorithm presented in Appendix H is repeated until the rate used to compute the loss corridor bounds (expressed as percentages of expected losses) equals the rate indication for the treaty with the loss corridor provision.

Having covered three common types of nonproportional loss sharing plans, the remainder of this section will discuss the analysis of accounts containing adjustable premium or commission plans.

Retrospective Rating Plan Example

Treaty IV is an example of an account with a one-year retrospective rating plan. Similar to the plans encountered in primary insurance, the adjusted treaty rate (and hence the adjusted premium) is based on the account's actual loss experience during the period subject to the plan. This rate is determined by loading the ratio of the treaty's actual losses to subject premium by a multiplicative loss conversion factor and/or an additive flat margin. The computed rate is further subject to a maximum and a minimum as specified in the treaty. (The loss conversion factor or flat margin accounts for the reinsurer's expenses, risk, and profit, and may also contain some provision, subjective or otherwise, to reflect the effect of the plan's maximum and minimum rates.) The main goal of this analysis is to determine the expected rate to be received on this treaty after all retrospective adjustments have been completed. This will enable one to assess the adequacy of the retro plan.

The calculation of the expected treaty rate for this example is outlined in Appendix I. As in the analysis of primary plans, the major step in this calculation is the determination of the true effect of the retro plan's maximum and minimum rates on the expected layer loss cost to be charged to the reinsured (which may differ from any subjective estimates of this effect included in the plan's loss loading factors). This is accomplished by dividing the loss costs that are consistent with the maximum and minimum rates, respectively, by the expected layer loss cost, in order to obtain entry ratios at these two points. These entry ratios enable one to look up the associated excess pure premium ratios, so that the insurance charge at the maximum and the insurance savings at the minimum may be computed. The difference between these latter two quantities is the net insurance charge. Applying the complement of the net insurance charge to the expected layer cost yields the adjusted expected layer cost, which is the loss expected to be charged to the reinsured. This latter quantity is loaded with the retro plan's loss conversion factor and any flat margin in order to obtain the expected treaty rate after retro adjustments. Note that the net insurance charge in this example is negative, indicating that the premium the reinsurer expects to lose because of the maximum rate provision is more than offset by the additional premium expected to be received due to the minimum provision.

The degree of adequacy of the retro plan can be measured by calculating the ratio of the guaranteed cost rate (which is the equivalent treaty rate if the contract were flat rated) to the expected treaty rate after retro adjustments. (To be comparable, the guaranteed cost rate contains the same amount of risk load as that contained in the retro plan parameters but with any insurance charge removed.) As shown on the bottom of the Appendix I Exhibit, the resulting ratio of 0.996 indicates a very slight redundancy in the retro plan.

Profit Sharing Example

Treaty V contains a three-year profit commission plan, in which the profit commission ratio (to treaty premium) is computed via the following formula:

$$\text{Profit Commission Ratio} = 25\% \times [100\% - (\text{Actual 3-Year Treaty Loss Ratio}) - (20\% \text{ Reinsurer's Overhead Provision})].$$

Note that the same formula could be used to compute a profit sharing adjustment that is treated as return premium.

On the surface, the calculation of the expected profit commission ratio for the three-year period (1/1/90–12/31/92 in this case) may seem trivial (i.e., simply plug the three-year expected loss ratio into the formula). It is really not, however, since a three-year loss ratio above 80% (the breakeven point) is implicitly capped at 80% to yield a 0% profit commission for the period. Hence, one must determine the effect of this capping on the expected loss ratio used in the profit sharing formula in order to estimate the expected commission. As in the previous examples, this involves the use of excess pure premium ratios for a lognormal distribution with an appropriate CV.

Appendix J Exhibit 1 displays the calculation of the CV for the distribution of one year's worth of aggregate losses in the reinsured layer. Since this is a three-year profit commission plan, the CV appropriate for aggregate treaty losses for three years combined needs to be determined. This is accomplished on Appendix J Exhibit 2, using the formulas discussed in the second section of the paper and in the related appendices. In reviewing this exhibit, it should be assumed that the subject premium and expected layer cost given for 1990 are values based on ceding company projections and rating analyses, respectively. The numbers shown for 1991 and 1992 are simply copied from 1990, since the information needed to make independent projections for these years is not presently available.

The calculation of the expected profit commission is shown on Appendix J Exhibits 3A and 3B. The expected treaty loss ratio of 48% is computed by reducing the expected loss cost for the entire layer by the 20% proportional coinsurance provision and then dividing the result by the treaty rate. By relating the 80% breakeven loss ratio to the expected loss ratio, an entry ratio is obtained from which the corresponding net insurance charge (*NIC*) is determined. Since the net insurance charge represents the percentage of expected losses eliminated from the profit commission formula by the implicit cap at the breakeven loss ratio, the expected profit commission ratio can be calculated via the following formula:

Expected Profit Commission Ratio =

$$P \times [100\% - ELR \times (100\% - NIC) - EXP],$$

where P = The proportion of profits to be paid to the reinsured;
ELR = Expected treaty loss ratio;
NIC = Net insurance charge;
EXP = Reinsurer's overhead provision.

In the Appendix J exhibits, the expected profit commission based on the formula above is called the "actuarial view," while that obtained by simply plugging the expected loss ratio into the profit commission formula is labelled the "simplistic view." Based on these definitions, it is clear that the expected profit commission based on the actuarial view should generally exceed that based on the simplistic view, as it does in this example.

Sliding Scale Commission Example

Treaty VI contains another kind of adjustable commission provision known as a sliding scale plan. Like the profit commission in the previous example, the commission that is ultimately paid by this plan depends directly on the reinsured's actual experience as measured by the treaty loss ratio. The major difference between these two plans lies in the structure of the formula used to compute the adjustable commission. Whereas the profit commission formula is essentially a straight linear function of the treaty loss ratio (at least up to the breakeven point), the typical sliding scale plan is best described as a piecewise linear function of the loss ratio.

Under a typical sliding scale plan, a minimum commission ratio C_{min} is paid if the treaty loss ratio exceeds a certain fixed value (call it L_1). If the actual loss ratio is less than L_1 but greater than a second fixed value L_2, b_2 points of commission are added to C_{min} for each point by which the actual loss ratio falls short of L_1. Similarly, if the actual loss ratio is below L_2 but greater than some third value L_3, the commission ratio corresponding to L_2 is increased by b_3 points for each point of difference between L_2 and the actual treaty loss ratio. The commissions corresponding to actual loss ratios falling into successively lower intervals (i.e., $[L_i, L_{i-1}]$, where $i = 4,\ldots,n$) are calculated in a manner similar to those for loss ratios falling in the previous two intervals. A maximum commission C_{max} is paid when the loss ratio is zero. It should be noted that the b_i's, which represent the commission slides on the various intervals, are generally less than unity, and some may be equal to zero. The sliding scale plan for Treaty VI (see the bottom of Appendix K Exhibit 1) is expressed in the format described above.

Since the typical sliding scale plan involves both a minimum and maximum commission as well as different commission slide percentages for the various loss ratio intervals, it is clear that the calculation of the expected commission ratio under such a plan requires more than simply looking up the commission that corresponds to the expected loss ratio. Appendix L outlines the derivation of a concise formula for computing this expected commission, which can be expressed as follows:

Expected Sliding Scale Commission Ratio =

$$C_{\max} - \sum_{i=1}^{n} b_i \,\{\text{Expected loss ratio points in the interval } L_i \text{ to } L_{i-1}\},$$

where: $C_{\max}$ is the maximum commission ratio;
b_i is the commission slide on the i^{th} loss ratio interval (b_1 is defined to be 0 and L_0 is infinity);
and Expected loss ratio points in the interval $[L_i, L_{i-1}]$ = (Expected loss ratio) $\times$ $[P_2(L_i) - P_2(L_{i-1})]$.
($P_2(L)$ is the excess pure premium ratio at loss ratio L.)

Appendix L also shows that the above formula is equivalent to saying that the expected commission ratio equals the maximum commission ratio minus the expected points of commission lost over the entire range of possible loss ratios. This interpretation provides a good intuitive justification for the formula stated above.

The above formula is used to calculate the expected commission ratio for the one-year plan given in Treaty VI, the details of which are provided in the Appendix K exhibit. As this exhibit shows, in order to determine the expected number of loss ratio points falling in each interval specified in the plan, it is necessary to multiply the treaty expected loss ratio by the difference between the insurance charges corresponding to both end points of the interval.

On the bottom of the Appendix K exhibit the expected sliding scale commission based on the above formula (the "actuarial view") is compared to the commission that corresponds to the expected loss ratio (the "simplistic view"). Although the actuarial estimate of the expected commission exceeds the simplistic estimate in this example, this is not a general rule. Both the magnitude and the direction of the difference between these two quantities depend on the minimum and maximum commission ratios as well as on the commission slides on the various loss ratio intervals.

4. MODEL COMPARISONS

For the examples presented above, Table 1 compares the key item of interest (either the adjusted rate or expected commissions) under the alternative models. The unadjusted rate is the loaded loss cost before all forms of coinsurance using the same expense and profit loading factor as that used to compute the adjusted rate. (In practice, the loadings for a treaty without coinsurance provisions or premium adjustments would generally not be considered appropriate for a treaty with such provisions.)

The alternative indications for the Heckman-Meyers version of the Collective Risk Model reflect varying levels of parameter uncertainty. The contagion parameter is represented by c and represents the level of parameter uncertainty in the estimated frequency mean. The mixing parameter is represented by b and represents the level of parameter uncertainty in the estimated severity mean.

Values of zero represent no parameter uncertainty, values of .05 represent a moderate level of parameter uncertainty, while values of .10 represent a higher level of parameter uncertainty. Please refer to Appendix C for further technical details. The lognormal model was run under the same assumptions that were used to generate the Collective Risk Model results without parameter uncertainty. Parameter uncertainty was not reflected here for the lognormal model (as it should be in practice using the methods presented in Appendices A and B) in an effort to simplify the presentation.

The comparisons above suggest that the lognormal model provides a satisfactory approximation to the theoretically more appropriate Collective Risk Model results, when use of the latter more sophisticated procedure is not necessary due either to data limitations or to the influence of market conditions and negotiations on final pricing. Application of the lognormal model can lead to significant efficiency gains in reinsurance price monitoring work and in many pricing situations, because it can easily be programmed in spreadsheets and applied efficiently by those with good quantitative skills. The increased efficiency achieved by

TABLE I

COMPARISON OF KEY ITEMS

Example	Unadjusted Rate	Item Compared	Lognormal Model	Collective Risk Model $c = 0$ $b = 0$	Collective Risk Model $c = .05$ $b = .05$	Collective Risk Model $c = .05$ $b = .10$	Collective Risk Model $c = .10$ $b = .10$
I) Aggregate Deductible	5.00%	Adjusted Rate	1.47%	1.58%	1.68%	1.73%	1.77%
II) Limited Reinstatement	25.00	Adjusted Rate	19.53	19.89	19.72	19.55	19.52
III) Loss Corridor	5.00	Adjusted Rate	4.02	3.67	3.71	3.74	3.73
IV) Retro Rating Plan	5.00	Expected Rate After Retro Adjustments	5.02	5.20	5.18	5.14	5.14
V) Profit Commission		Expected Profit Commission	8.37	8.24	8.50	8.69	8.75
VI) Sliding Scale Commission		Expected Sliding Scale Commission	31.04	30.31	30.90	31.22	31.33

this model permits one to apply it much more frequently than more theoretically appropriate methods. Use of a satisfactory quantitative method is usually superior to judgment.

5. MODEL ENHANCEMENTS

The six treaty examples discussed in this paper illustrate methods for pricing common types of nonproportional coinsurance and adjustable features provisions in reinsurance contracts. Although the examples themselves were kept reasonably simple to allow the reader to focus on the basic pricing techniques, the authors recognize that a number of enhancements can be made to the models in order to make them more applicable to specific situations. Unless otherwise stated, the following potential enhancements apply to both the lognormal and Collective Risk Model approaches.

1. Layer Retention and Limits

All the multiline contracts presented in the paper assume that the same layer retention and limit apply to all the classes of business covered by the treaty. In practice, however, some excess-of-loss contracts have retentions and/or limits that vary by line of business (e.g., auto liability losses may be subject to a \$200,000 per occurrence retention, while workers compensation losses have a \$300,000 retention). In these situations, the excess claim severity mean and standard deviation would be calculated for each class of business based on the retention and limit applicable to that class. (The formulas given in Appendix E would be used if a Single Parameter Pareto severity distribution is assumed). Similarly, the expected number of claims in the layer and the excess frequency variance-to-mean ratio would be calculated for each class based on the applicable retention. Once these quantities are computed, the calculation of the aggregate loss distribution for all classes combined would follow the same sequence of steps as if the contract had a single retention and limit applicable to all lines.

A similar procedure could be employed to derive the aggregate loss distribution for a multi-year rating block on an adjustable features contract, if the covered layer of reinsurance varies between the years comprising the block. The method can even be used to reflect underlying primary policy limits. In this case, one would treat each group of insureds purchasing the same policy limit above the layer retention as a separate class of business. The layer limit applicable to each class would be the lesser of the primary policy limit and the reinsurance gross limit (i.e., the retention amount plus the width of the layer). This layer limit, together with the layer retention, would then be used to calculate the expected layer loss cost, as well as the severity mean and standard deviation, for the particular class of policyholders.

2. Severity Distribution Assumptions

The examples in Section 3 use the Single Parameter Pareto severity (SPP) distribution to model occurrence severities. As mentioned in Section 2, the SPP parameter applicable to a particular line of insurance and truncation point can be derived by fitting this curve to empirical claims data. When performing this procedure directly or when utilizing published parameters, it is important to note whether the underlying claims data include allocated loss adjustment expenses (ALAE). If the empirical data used to compute the SPP parameter include ALAE, and if the reinsurance contract handles ALAE as a part of loss, then the formulas presented in the paper for calculating the aggregate loss (and ALAE) distribution can be applied without any modifications. The same is true if the SPP distribution is based on pure losses only and if the reinsurance contract does not cover ALAE.

In most cases, however, the reinsurance contract covers, to some extent, the ALAE associated with layer losses, but the assumed severity distribution describes pure losses only. These situations require one to make minor modifications to the methods presented earlier in the paper.

For the lognormal model, one should use expected layer loss costs excluding ALAE in order to determine the aggregate coefficient of variation (CV). If the ALAE covered by the contract is a constant multiple of layer losses (or nearly so), then the same CV describes the distribution of aggregate losses and ALAE in the layer. One instance when this would be true is where ALAE is a fixed percentage of ground-up losses and where ALAE is shared pro rata between the reinsurer and the ceding company. The particular loss sharing or adjustable feature provision would then be priced, using the expected layer loss cost including ALAE to compute the entry ratios needed to determine the appropriate excess pure premium ratios from the lognormal table. The adjustments to the Collective Risk Model approach for determining the aggregate loss distribution entail adjusting the parameters of the underlying frequency and severity distributions appropriately to reflect the relationship of ALAE to loss and the particular contractual provision concerning the manner by which ALAE will be shared.

Although the SPP distribution was chosen to model claim severities in the treaty examples, it is important to note that other severity distributions could have been used to derive the aggregate loss distribution under either the lognormal or Collective Risk Model approaches. The relaxation of this restriction allows one to use these models to determine the aggregate loss distributions for pro rata reinsurance contracts. (Recall that the SPP distribution is appropriate only above a sufficiently large truncation point, and hence it cannot be used to price pro rata treaties.) Once the aggregate loss distribution has been determined, the particular coinsurance clause or adjustable feature can be priced using the methods presented in the treaty examples.

3. Treaties with Both Coinsurance Provisions and Adjustable Features

The first three treaty examples presented in Section 3 illustrate methods for pricing common types of nonproportional coinsurance provisions, while the latter three examples involve the analysis of treaties with adjustable premium or commission plans. The case in which a treaty contains both a nonproportional coinsurance clause and an adjustable feature has not been considered. In such a situation, one needs to determine not only the effect that the nonproportional coinsurance clause has on expected treaty losses (which can be accomplished using the techniques discussed above) but also the distribution of aggregate losses after the effect of the nonproportional coinsurance has been taken into account. The latter item is necessary in order to compute the expected impact of the adjustable premium or commission plan, since these plans generally operate on actual treaty experience after all coinsurance.

The calculation of the aggregate distribution after nonproportional coinsurance can be accomplished by making direct modifications to the aggregate loss distribution prior to coinsurance (e.g., truncate it at the aggregate deductible amount or censor it at the aggregate limit). The Collective Risk Model would be run again to compute the needed insurance charges, assuming that there will be one claim with a severity distribution equal to the aggregate loss distribution after all forms of nonproportional coinsurance. Another approach is to determine the effects that the nonproportional coinsurance feature has on both the occurrence count and the occurrence severity distributions that underlie the aggregate distribution. The adjusted count and severity distributions can then be combined (using either method discussed in this paper or the alternative recursive or simulation techniques) in order to obtain an aggregate loss distribution that reflects the effects of the nonproportional coinsurance provision.

4. *Aggregate Losses of Zero*

When working with the models presented in the paper, one must consider the probability that no treaty losses will occur during a particular year. Although the chance of this occurring on pro rata or working layer excess-of-loss treaties may be sufficiently small that it can be ignored, treaties reinsuring rare events or high layers could have many loss-free years. One needs to estimate the probability of a loss-free year occurring on the treaty, either subjectively or by examining past treaty experience (if credible), in order to properly estimate the aggregate loss distribution.

If the Collective Risk Model is used to generate the aggregate distribution, the probability of a loss-free year could be reflected directly through the choice of the claim count distribution. A problem arises, however, when one attempts to use the lognormal aggregate loss distribution assumption to price a treaty with a positive probability of having no losses during a particular year. This is due to the fact that the lognormal distribution is not defined at the value zero. One solution to this problem involves the use of a mixture of a lognormal and a discrete distribution to model aggregate losses. This enhanced model may be applicable in many low mean frequency situations. Technical details are summarized in Appendix M.

5. *Investment Income*

The time value of money also has not been considered in the examples presented above, even though it is a legitimate underwriting consideration in evaluating alternative proposals. One way of handling this item would be to develop aggregate loss distributions for the lines of business subject to the treaty prior to all forms of nonproportional coinsurance. (Either the lognormal or Collective Risk Model may be used for this purpose.)

The analysis then becomes a simulation problem. One would simulate annual losses before coinsurance for each line, apply payout patterns to estimate future loss payments by line, apply the nonproportional coinsurance provisions, and finally discount the future treaty losses. (To accomplish this, one might develop and apply stochastic loss reporting, loss payout and interest rate models. Alternatively, one could develop a range of scenarios concerning these parameters and subjectively weight the final results derived from these alternative scenarios.) One would also need to estimate when future premium or commission adjustments would be made and when brokerage and other reinsurance expenses (including taxes [8]) would be paid. The economic value of the proposed treaty would be the difference between discounted reinsurance premium and the sum of the discounted values of all expense items. This economic value should be adjusted for risk considerations, possibly through the selection of the interest rates used in the discounting procedure [9].

A second approach is to estimate the ultimate loss ratio after all coinsurance as a percentage of provisional treaty premium using the methods of Section 3. Payout and loss reporting scenarios that approximately reflect the impact of the coinsurance provisions could then be selected. The loss reporting pattern would be used to estimate both IBNR reserves and the emergence of reported losses. Contractual formulas would be applied to estimate the magnitude of premium or commission adjustments to occur at specified points in time. The remainder of the analysis would proceed as in the first approach.

In this second approach, one item that needs to be considered in calculating premium or commission adjustments at various points in time is the impact of the insurance charges. For a retrospective rating or profit sharing formula, expected reported losses at various stages of development should be multiplied by the complement of the net insurance charge to approximate expected losses subject to the adjustment formula. In sliding scale commission plans, the commission ratio computed by plugging the expected loss ratio into the formula should be adjusted by the difference between the expected commission ratio (the actuarial view) and this formula estimate (the simplistic view), using the methods presented in Section 3. Sliding scale commission adjustments at various points in time would be computed by applying the contractual formula to the expected reported loss ratio and reflecting this commission adjustment gradually.

APPENDIX A
COMPUTATION OF AGGREGATE MEAN AND COEFFICIENT OF VARIATION (PATRIK-JOHN [2] VERSION OF COLLECTIVE RISK MODEL)

Let L represent the random variable of aggregate loss to be paid on a given contract for a particular coverage period.

$$L = L_1 + L_2 + \ldots + L_k,$$

where L_i represents the aggregate loss random variable for group i, $i = 1,2,\ldots,k$.

The groupings may represent distinct groups of classes of insureds or coverages, similar insureds grouped by distinct policy limits, or the overall coverage time period split into sub-periods.

$$L_i = X_{i1} + X_{i2} + \ldots + X_{iN_i},$$

where N_i is the random variable of the number of loss occurrences for group i and X_{ij} is the random variable of loss size of the j^{th} loss for group i.

Let v represent the parameter vector containing all parameters necessary to specify the particular cumulative probability distribution functions (c.d.f.'s) for the L_i's, N_i's, and X_{ij}'s.

The following three assumptions guarantee that the total coverage has been split into independent, homogeneous coverage groups:

Assumption 1: Given v, the L_i's are stochastically independent.

Assumption 2: Given v, the X_{ij}'s are stochastically independent of the N_i's.

Assumption 3: Given v and fixed i (i.e., a particular group), the X_{ij}'s are stochastically independent and identically distributed.

Let $F(x|v)$ represent the c.d.f. of L and let $F_i(x|v)$ represent the c.d.f. of L_i, $i = 1,2,\ldots,k$.

Properties of Model with Known Parameters

(1) The c.d.f. of the aggregate loss L is the convolution of the aggregate loss c.d.f.'s for the individual groups:

$$F(x|v) = P(L \leq x|v) = F_1(x|v) * F_2(x|v) * \ldots * F_k(x|v),$$

where $F_i(x|v) = P(L_i \leq x|v)$ and * denotes the convolution operation.

(2) The cumulants of L given v are sums of the corresponding cumulants of the L_i's given v. This implies that

(a) $E(L|v) = \sum_i E(L_i|v)$ (the means are additive).

(b) $\mathrm{Var}(L|v) = \sum_i \mathrm{Var}(L_i|v)$ (the variances are additive).

(3) The aggregate loss c.d.f. of the i^{th} group, $F_i(x|v)$, can be expressed in the form

$$F_i(x|v) = \sum_n P(N_i = n|v) \cdot G_i^{*n}(x|v),$$

where $G_i(x|v) = P(X_i \leq x|v)$ is the loss amount c.d.f. for the i^{th} group, and G_i^{*n} is the convolution of the n G_i's and represents the c.d.f. of the total amount of exactly n loss occurrences.

(4) The above properties imply that

(a) $E(L_i|v) = E(N_i|v) \cdot E(X_i|v)$.

The mean aggregate loss for the i^{th} group is the product of the mean frequency and mean severity.

(b) $\mathrm{Var}(L_i|v) = E(N_i|v) \cdot \mathrm{Var}(X_i|v) + \mathrm{Var}(N_i|v) \cdot E(X_i|v)^2$.

The variance of the i^{th} group's aggregate loss is the sum of the mean frequency times the variance of severity and the variance of frequency times the square of the mean severity. Substitution into the formulas in (2) above yields the mean and variance of the aggregate loss distribution.

Collective Risk Model

Now delete the restriction that the parameter vector v is known. Assume that the set V of possible parameters is finite and known and that one can specify the subjective likelihood of each element v of V. The structure function $U(v)$ is a discrete probability function that specifies the observer's uncertainty regarding the "best" parameter.

The unconditional c.d.f. $F(x)$ of the aggregate loss L has the following properties:

(1) $F(x) = \sum\limits_{v} F(x|v) \cdot U(v).$

The c.d.f. $F_i(x)$ of L_i is computed similarly.

(2) $\mathrm{E}(L^m) = \sum\limits_{v} \mathrm{E}(L^m|v) \cdot U(v).$

The m^{th} moment of L_i about the origin is computed similarly.

(3) With v unknown, assumptions (1)–(3) above may no longer hold, for the uncertainty regarding v may simultaneously affect the model at all levels. With v unknown, only the first cumulant is additive:

$\mathrm{E}(L) = \sum\limits_{i} \mathrm{E}(L_i),$

but $\mathrm{Var}(L) \neq \sum\limits_{i} \mathrm{Var}(L_i).$

However, $\mathrm{Var}(L) = \mathrm{E}(L^2) - \mathrm{E}(L)^2,$

and $\mathrm{E}(L^2) = \sum\limits_{v} \mathrm{E}(L^2|v) \cdot U(v) = \sum\limits_{v} \{\mathrm{Var}(L|v) + \mathrm{E}(L|v)^2\} \cdot U(v).$

$\mathrm{Var}(L|v)$ and $\mathrm{E}(L|v)$ are evaluated using the formulas above for the model with known parameters.

APPENDIX B
THE LOGNORMAL MODEL [3]

If the aggregate loss random variable is viewed as the product of a large number of independent, identically distributed random variables, the logarithm is then approximately normally distributed by the Central Limit Theorem. (The stringent condition that the factors be identically distributed may be relaxed [1].) This implies that the aggregate loss random variable is lognormally distributed.

The formulas in Appendix A for the model with known parameters are used to estimate the mean and variance of the aggregate loss distribution. It is assumed that the mean aggregate loss for each coverage of the excess-of-loss reinsurance contract has been estimated accurately using standard burning cost and/or exposure rating methods. A Single Parameter Pareto severity distribution is assumed for each coverage and is used to compute the mean and variance of the severity distribution (see Appendix E). The ratio of the mean aggregate loss to the mean severity is the mean number of loss occurrences for a given coverage. The variance of the excess frequency distribution is computed based on the assumptions and the formula developed in Appendix D. Thus, the mean and variance of the frequency and severity distributions for each coverage are specified and used to compute the variance of the aggregate loss distribution for each coverage. The sum of these variances for all of the coverages is the variance of the aggregate loss distribution for all coverages combined, since independence of aggregate losses for the individual coverages is assumed.

The Coefficient of Variation (CV) of the aggregate loss distribution is the ratio of the standard deviation to the mean of L, based on the frequency and severity distributions specified by the vector of parameters v or based on empirical methods applied to burning cost analyses:

$$\mathrm{CV}(L|v) = \frac{\{\mathrm{Var}(L|v)\}^{1/2}}{\mathrm{E}(L|v)} .$$

For simplicity, let $M = \mathrm{E}(L|v)$.

The Entry Ratio r is the ratio of the attachment A to the mean aggregate loss:

$$r = A/M.$$

The Excess Pure Premium (XSP) for a particular attachment A is the expected aggregate losses excess of A:

$$XSP(A|v) = \int_A^\infty (L - A)dP(L|v),$$

where P is the c.d.f. of L, given the vector of parameters v. The Excess Pure Premium Ratio P_2 at entry ratio r is the ratio of the corresponding Excess Pure Premium to the mean aggregate loss:

$$P_2(r|v) = XSP(A|v)/M.$$

Assume that the distribution of L is lognormal, given frequency and severity distributions specified by the vector of parameters v. If the parameters of this lognormal distribution are μ and σ^2, then

$$(1)\ M = \mathrm{E}(L|v) = \exp\{\mu + \frac{\sigma^2}{2}\}, \text{ and}$$

$$(2)\ \mathrm{CV} = \mathrm{CV}(L|v) = \{\exp(\sigma^2) - 1\}^{1/2}.$$

The first moment distribution P_1 is also lognormally distributed, but with parameters $\mu + \sigma^2$ and σ^2. P_1 is defined by

$$P_1(r|v) = \frac{1}{M}\int_0^A L \cdot dP(L|v).$$

The first moment distribution represents the percentage of total aggregate losses from coverage periods where the aggregate loss is less than the attachment. The Excess Pure Premium Ratio can be computed using

$$P_2(r|v) = \{1 - P_1(r|v)\} - r\{1 - P(r|v)\}.$$

Given that M and CV have been established as described above, the parameters of the assumed lognormal aggregate loss distribution can be estimated from formulas (1) and (2) above:

$$\sigma^2 = \ln(1 + CV^2), \text{ and}$$

$$\mu = \ln(M) - \frac{\sigma^2}{2}.$$

As noted above, P_1 is also lognormally distributed with parameters $\mu' = \mu + \sigma^2$ and σ^2. The vector of parameters v determines M and CV through the formulas previously presented. While the Excess Pure Premium is a function of both M and CV, the Excess Pure Premium Ratio is solely a function of the CV. Thus, the Excess Pure Premium Ratios are computed using

$$P_2(r|CV) = \{1 - P_1(r|CV)\} - r\{1 - P(r|CV)\}.$$

This formula was used to compute values for the expanded version of Mr. Finger's famous table which is displayed in Tables 1–3 of this Appendix.

The Excess Pure Premium for attachment A is given by

$$XSP(A|M,CV) = M \cdot P_2(r|CV), \text{ where } r = A/M.$$

Parameter uncertainty may be reflected using the method described under the Collective Risk Model section of Appendix A. For each element v of V, compute M and CV. Since $U(v) = U(M,CV)$, the unconditional Excess Pure Premium for attachment A may be computed using

$$XSP(A) = \sum_M \sum_{CV} XSP(A|M,CV)\ U(M,CV).$$

For the sake of simplicity, a probability of one is assigned to our most likely scenario for the examples in this paper.

TABLE B1

EXCESS PURE PREMIUM RATIOS
LOGNORMAL MODEL

Entry Ratio	Coefficient of Variation .1	.2	.3	.4	.5
0	1.000	1.000	1.000	1.000	1.000
.1	.900	.900	.900	.900	.900
.2	.800	.800	.800	.800	.800
.3	.700	.700	.700	.700	.700
.4	.600	.600	.600	.601	.603
.5	.500	.500	.501	.504	.510
.6	.400	.400	.404	.413	.426
.7	.300	.302	.313	.331	.351
.8	.200	.211	.234	.260	.286
.9	.107	.135	.168	.200	.232
1.0	.040	.079	.117	.153	.187
1.1	.009	.042	.079	.115	.150
1.2	.001	.021	.052	.086	.120
1.3	.000	.010	.034	.064	.096
1.4		.004	.022	.048	.077
1.5		.002	.014	.035	.062
1.6		.001	.009	.026	.049
1.7		.000	.005	.019	.040
1.8			.003	.014	.032
1.9			.002	.010	.026
2.0			.001	.008	.021
2.2			.000	.004	.014
2.4				.002	.009
2.6				.001	.006
2.8				.001	.004
3.0				.000	.003

The Coefficient of Variation is the ratio of the standard deviation to the mean of the assumed lognormal distribution. The Entry Ratio is the ratio of the attachment to the mean. The Excess Pure Premium Ratios are ratios of excess pure premiums to the mean (i.e., ratios of expected excess losses to the total expected loss).

TABLE B2

Excess Pure Premium Ratios
Lognormal Model

Entry Ratio	Coefficient of Variation .6	.7	.8	.9	1.0
0	1.000	1.000	1.000	1.000	1.000
.1	.900	.900	.900	.900	.900
.2	.800	.800	.801	.802	.804
.3	.702	.704	.707	.710	.714
.4	.607	.613	.619	.627	.634
.5	.519	.530	.541	.552	.563
.6	.441	.456	.472	.487	.502
.7	.371	.392	.412	.430	.447
.8	.312	.336	.359	.381	.400
.9	.261	.289	.314	.337	.359
1.0	.218	.248	.275	.300	.323
1.1	.183	.213	.241	.267	.291
1.2	.153	.183	.212	.239	.263
1.3	.128	.158	.187	.213	.238
1.4	.107	.137	.165	.192	.216
1.5	.090	.118	.146	.172	.197
1.6	.076	.103	.130	.155	.180
1.7	.064	.089	.115	.140	.164
1.8	.054	.078	.103	.127	.150
1.9	.046	.068	.092	.115	.138
2.0	.039	.060	.082	.105	.127
2.2	.028	.046	.066	.087	.108
2.4	.020	.036	.053	.073	.092
2.6	.015	.028	.044	.061	.079
2.8	.011	.022	.036	.052	.068
3.0	.008	.017	.030	.044	.059
3.2	.006	.014	.025	.037	.052
3.4	.005	.011	.021	.032	.045
3.6	.004	.009	.017	.028	.040
3.8	.003	.007	.015	.024	.035
4.0	.002	.006	.012	.021	.031
5.0	.001	.001	.006	.011	.018
10.0	.000	.000	.000	.001	.002

The Coefficient of Variation is the ratio of the standard deviation to the mean of the assumed lognormal distribution. The Entry Ratio is the ratio of the attachment to the mean. The Excess Pure Premium Ratios are ratios of excess pure premiums to the mean (i.e., ratios of expected excess losses to the total expected loss).

TABLE B3

EXCESS PURE PREMIUM RATIOS
LOGNORMAL MODEL

Entry Ratio	Coefficient of Variation 1.5	2.0	2.5	3.0	4.0	5.0
0	1.000	1.000	1.000	1.000	1.000	1.000
.1	.902	.905	.908	.911	.916	.921
.2	.813	.824	.834	.842	.855	.864
.3	.736	.756	.772	.786	.805	.820
.4	.670	.699	.721	.738	.764	.782
.5	.612	.649	.676	.697	.728	.750
.6	.562	.605	.637	.661	.697	.721
.7	.518	.567	.602	.630	.669	.696
.8	.479	.532	.571	.601	.644	.673
.9	.444	.502	.544	.575	.621	.652
1.0	.413	.474	.518	.552	.600	.633
1.2	.360	.426	.474	.511	.563	.599
1.4	.316	.386	.437	.476	.531	.570
1.6	.280	.352	.405	.445	.503	.544
1.8	.250	.323	.377	.418	.478	.521
2.0	.224	.297	.352	.394	.456	.500
2.2	.202	.275	.330	.373	.436	.481
2.4	.183	.255	.311	.354	.418	.464
2.6	.167	.238	.293	.337	.402	.448
2.8	.152	.222	.277	.321	.386	.433
3.0	.139	.208	.263	.307	.372	.419
3.5	.114	.179	.232	.275	.341	.389
4.0	.094	.155	.207	.250	.315	.364
4.5	.079	.136	.186	.228	.293	.342
5.0	.067	.120	.168	.209	.274	.322
5.5	.057	.107	.153	.193	.257	.305
6.0	.049	.096	.140	.179	.242	.290
6.5	.043	.087	.129	.167	.229	.276
7.0	.037	.079	.119	.156	.216	.264
7.5	.033	.072	.111	.146	.206	.252
8.0	.029	.066	.103	.137	.196	.242
10.0	.019	.047	.079	.110	.164	.208
20.0	.004	.015	.031	.049	.087	.120
30.0	.001	.007	.016	.029	.056	.083
50.0	.000	.002	.007	.013	.030	.049
100.0	.000	.000	.000	.004	.012	.022

The Coefficient of Variation is the ratio of the standard deviation to the mean of the assumed lognormal distribution. The Entry Ratio is the ratio of the attachment to the mean. The Excess Pure Premium Ratios are ratios of excess pure premiums to the mean (i.e., ratios of expected excess losses to the total expected loss).

APPENDIX C
HECKMAN-MEYERS VERSION OF COLLECTIVE RISK MODEL [4]

This appendix uses the same notation as presented in Appendix A. Let N_i represent the number of loss occurrences for group i and let m_i represent the unconditional mean number of occurrences,

$$m_i = \mathrm{E}(N_i).$$

Let C represent a random variable with $\mathrm{E}(C) = 1$ and $\mathrm{Var}(C) = c$. In this paper, C is assumed to be Gamma distributed. The parameter c is used to model parameter uncertainty in the frequency mean and is called the contagion parameter. Let X_{ij} represent the loss size of the j^{th} loss for group i. L_i is the aggregate loss of the i^{th} group:

$$L_i = X_{i1} + X_{i2} + \ldots + X_{iN_i}.$$

Parameter uncertainty in the severity mean is modelled through a random variable B with $\mathrm{E}(1/B) = 1$ and $\mathrm{Var}(1/B) = b$. B is assumed to be Gamma distributed so $1/B$ is Inverse Gamma distributed. The parameter b is called the mixing parameter.

The Algorithm

(1) Select C at random from the assumed distribution.

(2) Select the number of loss occurrences N_i at random from a Poisson distribution with mean $C \cdot m_i$.

(3) Select B at random from the assumed distribution.

(4) Select the loss occurrence amounts $X_{i1}, X_{i2}, \ldots, X_{iN_i}$ at random from the assumed occurrence severity distribution.

(5) Compute the aggregate loss L_i as the sum of all loss occurrence amounts divided by the scaling parameter B.

Since C is assumed to be Gamma distributed, the frequency distribution generated by the above algorithm will be negative binomial. If the conditions in Appendix D are satisfied, the excess frequency distribution for each group will be approximately Poisson under conditions of parameter certainty, and the excess frequency distribution for all groups combined will also be approximately Poisson due to the independence assumptions. The negative binomial frequency distribution is used to model uncertainty in the mean frequencies.

It is assumed that the shape of the severity distribution is known, and so the mixing parameter b models uncertainty in the severity means for the various groups. If uncertainty exists concerning the shape of the severity distribution, the approach to parameter uncertainty discussed in Appendix A may be applied through assignment of subjective probabilities to alternative scenarios concerning the shape parameter. In this paper, a Single Parameter Pareto severity distribution, as discussed in Appendix E, is assumed. The examples in this paper are evaluated for the following combinations of b and c: $b = c = 0$, $b = c = .05$, $b = .10$ and $c = .05$, and $b = c = .10$. These combinations represent no parameter uncertainty, moderate parameter uncertainty, higher uncertainty concerning the mean severity but moderate uncertainty concerning the mean frequency, and higher parameter uncertainty. Although many other combinations may be appropriate for particular circumstances, these values are used in this paper to illustrate the impact of modelling parameter uncertainty.

The reader may presume that a simulation is performed by running the above algorithm a sufficiently large number of times for each group to generate an accurate estimate of its aggregate loss distribution. Once aggregate loss distributions for each group are obtained in this manner, the aggregate loss distribution for all groups combined can be estimated by conducting a second simulation as follows:

(1) For group i, select L_i at random from the aggregate loss distribution already estimated.

(2) Compute the aggregate loss L for all groups combined by summing the L_i's, $i = 1, 2, \ldots, k$.

This second simulation is performed a sufficiently large number of times to generate an accurate estimate of the aggregate loss distribution for all groups combined. (Note that aggregate limits or deductibles may be applied to individual groups before the second simulation is performed.)

Instead of performing the above simulations, the Heckman-Meyers algorithm computes the aggregate loss distribution directly through application of the characteristic function method briefly summarized in

TREATY IV
COLLECTIVE RISK MODEL

Line	Expected Loss	Claim Severity Distribution	Contagion Parameter	Claim Count Mean	Claim Count Std. Dev.
1	359,995	class1.sev	0.0500 = c_1	5.154	2.546
2	90,033	class2.sev	0.0500 = c_2	1.343	1.197

Mixing Parameter	0.1000 = b
Aggregate Mean	450,028
Aggregate Std. Dev.	297,472

Aggregate Loss Amount	Entry Ratio	Cumulative Probability	Excess Pure Premium	Excess Pure Premium Ratio
0.00	0.0000	0.0015	450,028.21	1.0000
90,005.64	0.2000	0.0572	362,454.04	0.8054
180,011.28	0.4000	0.1577	281,663.70	0.6259
270,016.93	0.6000	0.2988	212,038.72	0.4712
360,022.57	0.8000	0.4477	155,661.12	0.3459
450,028.21	1.0000	0.5832	112,206.06	0.2493
540,033.85	1.2000	0.6949	79,912.14	0.1776
630,039.49	1.4000	0.7811	56,513.45	0.1256
720,045.14	1.6000	0.8450	39,840.27	0.0885
810,050.78	1.8000	0.8911	28,079.72	0.0624
900,056.42	2.0000	0.9237	19,828.73	0.0441

Section 2. The reader is referred to the paper and to the excellent review by Gary Venter for technical details [4]. The alternative recursive method, which is discussed in Mr. Venter's review and in his recent CAS Forum contribution [10], is simpler and in some circumstances more accurate [5], but in other circumstances it is less efficient than the characteristic function method and requires the structure function method discussed in Appendix A to model parameter uncertainty. A sample run of the model is presented in the charts below.

TREATY IV
COLLECTIVE RISK MODEL

Line	Expected Loss	Claim Severity Distribution	Contagion Parameter	Claim Count Mean	Claim Count Std. Dev.
1	359,995	class1.sev	0.1000 = c_1	5.154	2.795
2	90,033	class2.sev	0.1000 = c_2	1.343	1.234

Mixing Parameter	0.1000 = b
Aggregate Mean	450,028
Aggregate Std. Dev.	309,940

Aggregate Loss Amount	Entry Ratio	Cumulative Probability	Excess Pure Premium	Excess Pure Premium Ratio
0.00	0.0000	0.0023	450,028.21	1.0000
90,005.64	0.2000	0.0667	363,013.42	0.8066
180,011.28	0.4000	0.1716	283,301.51	0.6295
270,016.93	0.6000	0.3120	214,944.81	0.4776
360,022.57	0.8000	0.4562	159,564.03	0.3546
450,028.21	1.0000	0.5861	116,620.91	0.2591
540,033.85	1.2000	0.6933	84,374.46	0.1875
630,039.49	1.4000	0.7767	60,690.76	0.1349
720,045.14	1.6000	0.8392	43,547.09	0.0968
810,050.78	1.8000	0.8850	31,246.92	0.0694
900,056.42	2.0000	0.9180	22,462.93	0.0499

APPENDIX D

DERIVATION OF EXCESS OCCURRENCE COUNT VARIANCE-TO-MEAN RATIO

This appendix shows that if the ground-up occurrence count distribution for an insured selected at random is negative binomial, then the excess occurrence count distribution for a randomly selected insured is also negative binomial. Based on this result, the formula for calculating the excess occurrence count variance-to-mean ratio for an individual insured selected at random is derived, and it is shown that this formula also applies to the class as a whole. This latter result is then used to demonstrate that, if the proportion of occurrences exceeding the retention is small and the excess frequency mean is known, then the excess occurrence count distribution for the class as a whole is approximately Poisson.

Assume

(1) An individual policy's distribution of ground-up occurrence counts over a given period of time is Poisson with parameter λ_i.

(2) The policies in the given class are independent and of identical size.

(3) The distribution of the individual policy expected occurrence counts (i.e., the λ_i's) over the class is Gamma with parameters a,r.

(4) The probability of a given occurrence being an excess occurrence (i.e., the probability that it exceeds a fixed retention R) is p. This probability is applicable to all policies and may be calculated from the parameters given in the ISO increased limits reviews.

Given (1) and (3) above, it follows [11] that the distribution of the observed ground-up occurrence counts for an individual policy selected at random is negative binomial with a mean $\mu_G = r/a$ and variance $\sigma_G^2 = (r/a)((a + 1)/a)$. This implies a variance-to-mean ratio $VMR_G = \sigma_G^2/\mu_G = (a + 1)/a = 1 + (1/a)$. Assuming that VMR_G is known (from the ISO increased limits reviews or elsewhere), one can easily solve for a.

It follows [1,2] from the assumptions of a Poisson process that if an individual policy's distribution of ground-up occurrence counts is Poisson with parameter λ_i, then the distribution of excess occurrence counts (claims above R) for the individual policy is also Poisson but with parameter $p\lambda_i$.

The Gamma Distribution has the property [12] that if λ has the distribution $\Gamma\ (a,r)$, then $p\lambda$ has a $\Gamma\ (a/p,r)$ distribution. Hence, the distribution of the individual policy expected excess occurrence counts over the class is $\Gamma\ (a/p,r)$.

Thus, the distribution of observed excess occurrence counts for an individual policy selected at random from the class of policies is negative binomial with a mean $\mu_E = r/[a/p] = pr/a$ and variance

$$\sigma_E^2 = \{r/[a/p]\}\,\{[a/p + 1]/[a/p]\} = [pr/a][1 + p/a].$$

This implies a variance-to-mean ratio $VMR_E = \sigma_E^2/\mu_E = 1 + p/a$. Note that since $p < 1$, $VMR_E < VMR_G$.

One can think of the group of policies covered by a particular excess reinsurance treaty as a statistical sample taken from the theoretically infinite population of all insureds belonging to the particular class [13]. Assuming that the sample is taken at random, the policies selected are independent of each other. From the above, each policy's excess occurrence count distribution has mean μ_E and variance σ_E^2. Given that n policies from the particular class are covered by the reinsurance treaty, the expected number of occurrences subject to the excess treaty is $n\mu_E$ and the variance of the number of occurrences subject to the treaty is $n\sigma_E^2$. This implies a variance-to-mean ratio of $(n\sigma_E^2)/(n\mu_E) = \sigma_E^2/\mu_E = VMR_E$ for the total number of occurrences subject to the treaty. Thus, the excess occurrence count variance-to-mean ratio for the entire group of policies covered by the reinsurance treaty equals the excess occurrence count variance-to-mean ratio for an individual policy selected at random from the class.

If VMR_G is known, a simple formula for calculating VMR_E can be easily derived using the following two relationships (which were proven above):

$$(1)\ VMR_G = 1 + \frac{1}{a}\text{, and}$$

$$(2)\ VMR_E = 1 + \frac{p}{a}\,.$$

Solving equation (1) for a, we get

$$(3)\ a = \frac{1}{VMR_G - 1}\,.$$

Substituting expression (3) into (2), we get

$$\begin{aligned} VMR_E &= 1 + \frac{p}{\left[\dfrac{1}{VMR_G - 1}\right]} \\ &= 1 + p(VMR_G - 1) \\ &= (1 - p) + p(VMR_G). \end{aligned}$$

Based on the above formula, if VMR_G is two or three, as in the ISO increased limits reviews, and p is small (say less than .02), VMR_E will be close to unity. This implies that the excess occurrence count distribution for an insured selected at random and for the class as a whole will be approximately Poisson, provided that the excess frequency mean is known. (Recall that the sum of independent Poisson random variables is also Poisson.)

APPENDIX E
SINGLE PARAMETER PARETO SEVERITY DISTRIBUTION [6]

General Properties of Model

Assume ground-up loss occurrences above the truncation point k are distributed according to the following cumulative distribution function (c.d.f):

$$F(w) = 1 - \left(\frac{k}{w}\right)^q, \text{ where } k > 0,\ q > 0,\ w \geqslant k.$$

Note that

$$F(w) = 1 - \left(\frac{k}{k + (w - k)}\right)^q.$$

Let $y = w - k$ represent the occurrence size excess of k. Then

$$F(y) = 1 - \left(\frac{k}{k + y}\right)^q, \text{ where } y \geqslant 0.$$

Thus, occurrence losses excess of the truncation point k are distributed according to the two-parameter shifted Pareto distribution with scale parameter equal to k and shape parameter equal to q.

Assume ground-up occurrences are censored at limit $k \cdot b$. Then

$$F(y) = 1 - \left(\frac{k}{k + y}\right)^q \text{ if } 0 \leqslant y < k\,(b - 1),$$

$$\text{and } F(y) = 1 \qquad \text{if } y \geqslant k(b - 1).$$

The mean censored excess occurrence is given by

$$\mathrm{E}(y) = \frac{k\,(b^{1-q} - 1)}{1 - q} \quad \text{if } q \neq 1,$$

$$\text{and } \mathrm{E}(y) = k \cdot \ln(b) \quad \text{if } q = 1.$$

The variance of the censored excess occurrences is given by

$$\mathrm{Var}(y) = k^2\left[\frac{q - 2b^{2-q}}{q-2} - \left(\frac{q - b^{1-q}}{q-1}\right)^2\right] \quad \text{if } q \neq 1,\ q \neq 2,$$

$$\mathrm{Var}(y) = k^2\left[2b - 1 - (1 + \ln(b))^2\right] \quad \text{if } q = 1, \text{ and}$$

$$\mathrm{Var}(y) = k^2\left[1 + 2\cdot\ln(b) - \left(\frac{2b-1}{b}\right)^2\right] \quad \text{if } q = 2.$$

Maximum Likelihood Estimation of q

Assume one wishes to compute the Maximum Likelihood Estimator (MLE) of q by fitting n loss occurrences ($W_1, W_2, \ldots, W_n$) above the truncation point k. Let X_i (for $i = 1, 2, \ldots, n$) represent the normalized losses, $X_i = W_i/k$. The c.d.f. of the normalized losses is $F(x) = 1 - x^{-q}$, which is the customary representation of the Single Parameter Pareto (SPP) distribution. Assume m_j occurrences have been censored at limit C_j and let $b_j = C_j/k$, $j = 1, 2, \ldots s$.

Let $u = n - \sum_{j=1}^{s} m_j$ represent the number of uncensored occurrences. Then the MLE of q is given by

$$\hat{q} = \frac{u}{\sum_{i=1}^{u}\ln(X_i) + \sum_{j=1}^{s} m_j\cdot\ln(b_j)},$$

where the X_i's are the uncensored normalized occurrences. If no occurrences have been censored, the MLE of q is

$$\hat{q} = \frac{n}{\sum_{i=1}^{n}\ln(X_i)}.$$

If cases are developing, q should be estimated for each accident year or policy year at each evaluation, and a triangulation approach should be used to project the ultimate estimate of q for losses in excess of the particular truncation point. If cases are not developing and q is to be estimated by pooling the losses from several years, they first need to be adjusted for trend if some of the losses have been censored.

Leveraged Impact of Inflation

Let n represent the number of loss occurrences above truncation k at time 0, and assume the annual loss inflation factor between time 0 and time t is $1 + i$. Based on the SPP distribution with parameter q, the projected number of loss occurrences excess of truncation k at time t is $n(1 + i)^{tq}$.

As long as inflation does not erode the real value of a retention to the point that the SPP distribution is no longer a satisfactory model above the retention, the parameter q and the average occurrence size in the layer of interest will theoretically remain constant over time. The leveraged impact of inflation over a fixed retention is quantified through the application of the adjustment factor $(1 + i)^{tq}$ to excess occurrence frequency.

Change in Layer

Assume that one has credibly estimated losses in the layer from a to b and wishes to estimate expected losses in the layer from c to d, where the SPP distribution with parameter q is appropriate above the lower of the two retentions. The change in expected losses due to the change in reinsurance layer is theoretically given by

$$\text{Change in Layer} = \frac{c^{1-q} - d^{1-q}}{a^{1-q} - b^{1-q}} \text{ if } q \neq 1, \text{ and}$$

$$\text{Change in Layer} = \frac{\ln (d/c)}{\ln (b/a)} \qquad \text{if } q = 1.$$

(The layer limits need not be normalized in the above formulas.) The Change in Layer factor is applied to expected losses in the layer from a to b to estimate expected losses in the layer from c to d.

APPENDIX F

TREATY I

SUMMARY OF KEY CONTRACT PROVISIONS

Treaty Period: 1/1/90–12/31/90

Layer Reinsured: $160,000 in excess of $40,000 per occurrence

Estimated Treaty Subject Premium: $12,000,000 for 1990,
distributed as follows:
Class 1—$9,000,000
Class 2—$3,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1—4.00%
Class 2—3.00%
Both Classes Combined—3.75%

Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Rate: 100/75

Proportional Coinsurance: None

Nonproportional Coinsurance: Aggregate Deductible equal to 3% of Subject Premium

APPENDIX F

EXHIBIT 1

DETERMINATION OF AGGREGATE LOSS DISTRIBUTION SPECIFICATION PARAMETERS FOR NONPROPORTIONAL LOSS SHARING PLANS

	Class of Business		
	Class 1	Class 2	All Classes Combined
(1) Actual or estimated subject premium for treaty period	9,000,000	3,000,000	12,000,000
(2) Expected layer loss cost for entire layer prior to the application of all forms of coinsurance (layer burning cost) [expressed as a percentage of subject premium]	4.0000%	3.0000%	3.7500%
(3) Expected losses for the entire reinsured layer for the treaty period [(1) × (2)]	360,000	90,000	450,000
(4) Single Parameter Pareto q values for severity distributions	0.900	0.950	
(5) Mean excess claim size in layer (μ_s)	69,848	67,039	69,267
(6) Standard deviation of excess claim sizes in layer (σ_s)	60,908	60,084	60,749
(7) Expected number of claims in layer prior to the application of nonproportional loss sharing provisions (μ_c) [(3)/(5)]	5.154	1.343	6.497
(8) Excess claim count variance-to-mean ratio (VMR_c)	1.032	1.067	1.039
(9) Standard deviation of distribution of aggregate losses in layer $[\sigma_s^2 \cdot \mu_c + (\mu_c \cdot VMR_c) \cdot \mu_s^2]^{1/2}$	212,298	106,228	237,391
(10) Coefficient of variation of distribution of aggregate losses in layer [(9)/(3)]	0.590	1.180	0.528
(11) Selected coefficient of variation of aggregate loss distribution for all classes combined			0.528

NOTES:

Lines (5) and (6): The mean excess claim size and the standard deviation of the excess claim sizes are based on a Single Parameter Pareto distribution assumption with the parameter (q value) given in item (4). (See Appendix E for formulas.) The all classes combined mean excess claim size is an average of the individual class mean claim sizes, weighted on the expected excess claim counts on line (7). The all classes combined claim size standard deviation is computed as follows:

(A) For each class of business, calculate the sum of the squares of items (5) and (6), respectively.
(B) Take a weighted average of the sums in (A), using the expected excess claim counts on line (7) as weights.
(C) Subtract the square of the all classes combined mean excess claim size from the result in (B).
(D) Take the square root of the result in (C) to obtain the all classes combined excess claim size standard deviation.

Line (8): The individual class excess claim count variance-to-mean ratios are calculated using the ISO increased limits parameters and the formulas in Appendix D. The all classes combined excess claim count variance-to-mean ratio is an average of the individual class variance-to-mean ratios, weighted on the expected claim counts on line (7).

Line (9): The standard deviation of the aggregate loss distribution for all classes combined is obtained by summing the squares of the aggregate loss distribution standard deviations for the individual classes and then taking the square root of the result.

APPENDIX F
EXHIBIT 2

Aggregate Deductibles

(1) Actual or estimated subject premium for treaty period	12,000,000
(2) Expected layer loss cost for entire layer prior to the application of all coinsurance (layer burning cost) [expressed as a percentage of subject premium]	3.7500%
(3) Coinsurance percentage (cedant's participation in layer losses not corresponding to an explicit share of the reinsurance premium, excluding the presumed effect of the aggregate deductible).	0.00%
(4) Expected dollars of loss for the entire layer prior to the application of all coinsurance [(1) × (2)]	450,000
(5) Loading to convert expected layer loss cost after all forms of coinsurance into a loaded rate (expressed as a multiplicative factor to be applied to the expected layer loss cost)	1.333 = 100/75
(6) Aggregate deductible amount in dollars applicable to the entire layer [3% × $12,000,000]	360,000
(7) Entry ratio corresponding to the aggregate deductible amount [(6)/(4)]	0.800
(8) Insurance charge at entry ratio corresponding to the aggregate deductible amount*	29.33%
(9) Expected percentage of treaty losses eliminated by the aggregate deductible [100% − (8)]	70.67%
(10) Composite coinsurance percentage 100%-{[100% − (3)] × [100% − (9)]}	70.67%
(11) Expected layer loss cost for the entire reinsured portion of layer, after the application of the aggregate deductible (expressed as a percentage of subject premium) (2) × [100% − (10)]	1.0998%
(12) Indicated treaty rate after the application of the aggregate deductible and any proportional coinsurance (expressed as a percentage of subject premium) [(5) × (11)]	1.4664%

*The insurance charge appearing in item (8) above is based on a lognormal distribution with a 0.528 coefficient of variation. The insurance charge is obtained via linear interpolation of the table of insurance charges given on Tables B1–B3 of Appendix B. See Appendix F Exhibit 1 for the derivation of the 0.528 coefficient of variation.

APPENDIX G

TREATY II

SUMMARY OF KEY CONTRACT PROVISIONS

Treaty Period: 1/1/90 – 12/31/90

Layer Reinsured: $700,000 in excess of $300,000 per occurrence

Estimated Treaty Subject Premium: $6,000,000 for 1990,
distributed as follows:
Class 1 – $2,000,000
Class 2 – $2,000,000
Class 3 – $2,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1 – 10.0%
Class 2 – 14.0%
Class 3 – 21.0%
All Classes Combined – 15.0%

Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Rate: 100/60

Proportional Coinsurance: 20%

Nonproportional Coinsurance: Three (3) full free reinstatements permitted under treaty.

APPENDIX G
EXHIBIT 1

AGGREGATE LIMITS

(1) Actual or Estimated Subject Premium for Treaty Period	6,000,000
(2) Expected Layer Loss Cost for Entire Layer Prior to the Application of All Coinsurance (Layer Burning Cost) [Expressed as a Percentage of Subject Premium]	15.000%
(3) Coinsurance Percentage (Cedant's Participation in Layer Losses Not Corresponding to an Explicit Share of the Reinsurance Premium, Excluding the Presumed Effect of the Aggregate Limit Provision)	20.00%
(4) Expected Dollars of Loss for the Entire Layer Prior to the Application of All Coinsurance [(1) × (2)]	900,000
(5) Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Loaded Rate (Expressed as a Multiplicative Factor to be Applied to the Expected Layer Loss Cost)	1.667 = 100/60

Complete Item (6) if the aggregate limit is expressed as a percentage of treaty losses, or Item (7) if the aggregate limit is expressed in terms of limited reinstatements.

(6) Aggregate Limit Amount (Expressed as a Percentage of the Expected Losses for the Treaty Prior to the Application of this Provision)	N/A
(7) (A) Number of Free Reinstatements Allowed Under Treaty	3
(B) Layer Retention	300,000
(C) Layer Gross Limit	1,000,000
(D) Layer Width [(7C) − (7B)]	700,000
(E) Effective Aggregate Limit for the Entire Layer Prior to All Coinsurance (Expressed in Dollars) [1 + (7A)] × (7D)	2,800,000
(F) Effective Treaty Aggregate Limit (Expressed as a Percentage of Treaty Expected Losses) [(7E)/(4)]	311.11%
(8) *Entry Ratio Corresponding to the Aggregate Limit [(6) or (7F). Expressed as a Decimal]*	3.111
(9) Insurance Charge at the Entry Ratio Corresponding to the Aggregate Limit* (This Percentage Is Equivalent to the Expected Percentage of Treaty Losses Eliminated by the Aggregate Limit Provision)	2.37%
(10) Composite Coinsurance Percentage 100% − {[(100% − (3)] × [100% − (9)]}	21.89%
(11) Expected Layer Cost for the Entire Reinsured Portion of Layer, After the Application of the Aggregate Limit Provision (Expressed as a Percentage of Subject Premium) (2) × [100% − (10)]	11.7161%
(12) Indicated Treaty Rate After the Application of Aggregate Limits and Any Proportional Coinsurance (Expressed as a Percentage of Subject Premium) [(5) × (11)]	19.5268%

*The insurance charge appearing in Item (9) above is based on a lognormal distribution with a 0.770 coefficient of variation. The insurance charge is obtained via linear interpolation of the table of insurance charges given on Tables B1–B3 of Appendix B. A procedure similar to that employed in the Treaty I example (see Appendix F Exhibit 1) is used to derive the 0.770 coefficient of variation for aggregate losses in the reinsured layer on this treaty.

APPENDIX H
TREATY III
SUMMARY OF KEY CONTRACT PROVISIONS

Treaty Period: 1/1/90 – 12/31/90

Layer Reinsured: $400,000 in excess of $100,000 per occurrence

Estimated Treaty Subject Premium: $10,000,000 for 1990,
distributed as follows:
Class 1 – $4,500,000
Class 2 – $4,500,000
Class 3 – $1,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1 – 3.20%
Class 2 – 3.80%
Class 3 – 3.50%
All Classes Combined – 3.50%

Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Rate: 100/70

Proportional Coinsurance: None

Nonproportional Coinsurance: Loss Corridor – Reinsurer stops paying losses that fall in the reinsured layer when the ratio of actual losses in the layer to expected losses in the layer reaches 100%, but he resumes full payment of losses in the layer if this ratio goes above 200%.

APPENDIX H
EXHIBIT 1

LOSS CORRIDORS

(1) Actual or Estimated Subject Premium for Treaty Period	10,000,000
(2) Expected Layer Loss Cost for Entire Layer Prior to the Application of All Coinsurance (Layer Burning Cost) (Expressed as a Percentage of Subject Premium)	3.500%
(3) Coinsurance Percentage (Cedant's Participation in Layer Losses Not Corresponding to an Explicit Share of the Reinsurance Premium, Excluding the Presumed Effect of the Loss Corridor Provision)	0.00%
(4) Expected Dollars of Loss for the Entire Layer Prior to the Application of All Coinsurance [(1) × (2)]	350,000
(5) Loading to Convert Expected Layer Loss Cost After All Forms of Coinsurance into a Loaded Rate (Expressed as a Multiplicative Factor to be Applied to the Expected Layer Loss Cost)	1.429 = 100/70
(6) Lower Bound of Loss Corridor Interval (Expressed as a Percentage of Expected Losses for the Treaty Prior to the Application of the Loss Corridor Provision)	100.00%
(7) Upper Bound of Loss Corridor Interval (Expressed as a Percentage of Expected Losses for the Treaty Prior to the Application of the Loss Corridor Provision)	200.00%
(8) Reinsurer's Participation Percentage in Loss Corridor Interval (If Any)	0.00%
(9) Entry Ratio Corresponding to Lower Bound of Interval [(6) Expressed as a Decimal]	1.000
(10) Insurance Charge at Entry Ratio Corresponding to Lower Bound of Interval*	30.11%
(11) Entry Ratio Corresponding to Upper Bound of Interval [(7) Expressed as a Decimal]	2.000
(12) Insurance Charge at Entry Ratio Corresponding to Upper Bound of Interval*	10.61%
(13) Percentage of Expected Treaty Losses Eliminated by the Loss Corridor Provision [(10) − (12)] × [100% − (8)]	19.51%
(14) Composite Coinsurance Percentage 100% − {[(100% − (3)] × [100% − (13)]}	19.51%
(15) Expected Loss Cost for the Entire Reinsured Portion of Layer, After the Application of the Loss Corridor Provision (Expressed as a Percentage of Subject Premium) (2) × [100% − (14)]	2.8173%
(16) Indicated Treaty Rate After the Application of the Loss Corridor Provision and Any Proportional Coinsurance (Expressed as a Percentage of Subject Premium) [(5) × (15)]	4.0248%

*The insurance charges appearing in Items (10) and (12) above are based on a lognormal distribution with a 0.905 coefficient of variation. The insurance charges are obtained via linear interpolation of the table of insurance charges given on Tables B1–B3 of Appendix B. A procedure similar to that employed in the Treaty I example (see Appendix F Exhibit 1) is used to derive the 0.905 coefficient of variation for aggregate losses in the reinsured layer on this treaty.

APPENDIX I
TREATY IV
Summary of Key Contract Provisions

Treaty Period: 1/1/90–12/31/90

Layer Reinsured: \$160,000 in excess of \$40,000 per occurrence

Estimated Treaty Subject Premium: \$12,000,000 for 1990,
distributed as follows:
Class 1—\$9,000,000
Class 2—\$3,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1—4.00%
Class 2—3.00%
Both Classes Combined—3.75%

Indicated Flat Treaty Rate After the Application of All Forms of Coinsurance (Expressed as a Percentage of Subject Premium): 5.00%

Proportional Coinsurance: None

Nonproportional Coinsurance: None

Retrospective Rating Plan:
Adjustment Period—1/1/90 through 12/31/90 (1 year)
Adjustment Formula—
Adjusted Treaty Premium = 100/75 × (Incurred Losses and ALAE in Layer), subject to a maximum of 10.00% of subject premium and a minimum of 3.00% of subject premium.

APPENDIX I
EXHIBIT I

ADJUSTABLE PREMIUMS (RETROSPECTIVE RATING)

(1) Actual or estimated subject premium for the retrospective rating period	12,000,000
(2) Expected layer loss cost for entire layer prior to the effects of the retro plan (expressed as a percentage of subject premium)	3.7500%
(3) Coinsurance percentage (cedant's participation in layer losses not corresponding to an explicit share of the reinsurance premium, excluding the effects of nonproportional loss sharing plans).	0.00%
(4) Percentage reduction in layer losses due to nonproportional loss sharing provisions only	0.00%
(5) Expected loss cost for entire reinsured portion of layer prior to the effects of the retro plan (expressed as a percentage of subject premium) (2) × [100% − (3)] × [100% − (4)]	3.7500%
(6) Maximum rate (expressed as a percentage of subject premium)	10.000%
(7) Minimum rate (expressed as a percentage of subject premium)	3.000%
(8) Multiplicative loss load (loss conversion factor)	133.333% = 100/75
(9) Additive loss load (flat margin)	0.0000%
(10) Loss cost corresponding to the maximum rate [(6) − (9)]/(8)	7.5000%
(11) Entry ratio corresponding to the maximum rate [(10)/(5)]	2.000
(12) Insurance charge at maximum (excess loss percentage corresponding to maximum entry ratio)*	2.60%
(13) Loss cost corresponding to the minimum rate [(7) − (9)]/(8)	2.2500%
(14) Entry ratio corresponding to the minimum rate [(13)/(5)]	0.600
(15) Insurance charge at minimum (excess loss percentage corresponding to minimum entry ratio)*	43.02%
(16) Insurance savings at minimum [100% × (14)] + (15) − 100%	3.02%
(17) Net insurance charge [(12) − (16)]	−0.42%
(18) Adjusted expected layer loss cost (expected value of losses limited by the retro plan maximum and minimum) (5) × [100% − (17)]	3.7656%
(19) (A) Guaranteed cost treaty rate (equivalent treaty rate if contract were flat rated) (expressed as a percentage of subject premium)	5.0000%
(B) Expected treaty rate after retro adjustments (expressed as a percentage of subject premium) [(8) × (18)] + (9)	5.0208%
(C) Retro plan off-balance factor [(19A)/(19B)] (a factor greater than 1.000 indicates a plan inadequacy; while a factor less than 1.000 indicates a plan redundancy)	0.996

*The insurance charges appearing in items (12) and (15) above are based on a lognormal distribution with a 0.528 coefficient of variation. The insurance charges are obtained via linear interpolation of the table of insurance charges given on Tables B1–B3 of Appendix B. A procedure similar to that employed in the Treaty I example (see Appendix F Exhibit 1) is used to derive the 0.528 coefficient of variation for aggregate losses in the reinsured layer on this treaty.

APPENDIX J

TREATY V

Summary of Key Contract Provisions

Treaty Period: 1/1/90–12/31/90

Layer Reinsured: $700,000 in excess of $300,000 per occurrence

Estimated Treaty Subject Premium: $6,000,000 for 1990, distributed as follows:
Class 1—$2,000,000
Class 2—$2,000,000
Class 3—$2,000,000

Expected Layer Loss Costs for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium):
Class 1—10.0%
Class 2—14.0%
Class 3—21.0%
All Classes Combined—15.0%

Treaty Rate (Expressed as a Percentage of Subject Premium): 25.0%

Proportional Coinsurance: 20%

Nonproportional Coinsurance: None

Profit Commission Plan: Adjustment Period—1/1/90 through 12/31/92 (3 years). Reinsurer to pay cedant 25% of the amount by which treaty premiums during the Adjustment Period exceed incurred losses, ALAE, and a 20% provision for the reinsurer's overhead expense.

APPENDIX J

EXHIBIT 1

DETERMINATION OF AGGREGATE LOSS DISTRIBUTION SPECIFICATION PARAMETERS FOR A SINGLE TREATY YEAR

	Class of Business			
	Class 1	Class 2	Class 3	All Classes Combined
(1) Actual or estimated subject premium for treaty period	2,000,000	2,000,000	2,000,000	6,000,000
(2) Expected layer loss cost for entire layer prior to the application of all forms of coinsurance (layer burning cost) [expressed as a percentage of subject premium]	10.0000%	14.0000%	21.0000%	15.0000%
(3) Expected losses for the entire reinsured layer for the treaty period [(1) × (2)]	200,000	280,000	420,000	900,000
(4) Single Parameter Pareto q values for severity distributions	1.500	1.300	1.100	
(5) Mean excess claim size in layer (μ_S)	271,366	303,155	340,296	310,897
(6) Standard deviation of excess claim sizes in layer (σ_S)	246,592	257,600	266,584	260,265
(7) Expected number of claims in layer prior to the application of nonproportional loss sharing provisions (μ_C) [(3)/(5)]	0.737	0.924	1.234	2.895
(8) Excess claim count variance-to-mean ratio (VMR_C)	1.006	1.009	1.019	1.012
(9) Standard deviation of distribution of aggregate losses in layer $[\sigma_S^2 \cdot \mu_C + (\mu_C \cdot VMR_C) \cdot \mu_S^2]^{1/2}$	315,301	383,323	483,065	692,606
(10) Coefficient of variation of distribution of aggregate losses in layer [(9)/(3)]	1.577	1.369	1.150	0.770

NOTES:

Lines (5) and (6): The mean excess claim size and the standard deviation of the excess claim sizes are based on a Single Parameter Pareto distribution assumption with the parameter (q value) given in item (4). (See Appendix E for formulas.) The all classes combined mean excess claim size is an average of the individual class mean claim sizes, weighted on the expected excess claim counts on line (7). The all classes combined claim size standard deviation is computed as follows:

(A) For each class of business, calculate the sum of the squares of items (5) and (6), respectively.
(B) Take a weighted average of the sums in (A), using the expected excess claim counts on line (7) as weights.
(C) Subtract the square of the all classes combined mean excess claim size from the result in (B).
(D) Take the square root of the result in (C) to obtain the all classes combined excess claim size standard deviation.

Line (8): The individual class excess claim count variance-to-mean ratios are calculated using the ISO increased limits parameters and the formulas in Appendix D. The all classes combined excess claim count variance-to-mean ratio is an average of the individual class variance-to-mean ratios, weighted on the expected claim counts on line (7).

Line (9): The standard deviation of the aggregate loss distribution for all classes combined is obtained by summing the squares of the aggregate loss distribution standard deviations for the individual classes and then taking the square root of the result.

APPENDIX J

EXHIBIT 2

DETERMINATION OF AGGREGATE LOSS DISTRIBUTION SPECIFICATION PARAMETERS for ADJUSTABLE PREMIUM OR COMMISSION PLANS

	Adjustment Period			
Dates of Individual Contract Years in Adjustment Period →	Year 1 1/90–12/90	Year 2 1/91–12/91	Year 3 1/92–12/92	Total Adjustment Period
(1) Actual or estimated subject premiums for all classes combined	6,000,000	6,000,000	6,000,000	18,000,000
(2) (A) Expected layer loss cost for entire layer prior to the application of all forms of coinsurance (layer burning cost) [expressed as a percentage of subject premium]	15.0000%	15.0000%	15.0000%	15.0000%
(B) Percentage reduction in layer losses due to nonproportional loss sharing provisions. (Ignore all proportional forms of coinsurance.)	0.00%	0.00%	0.00%	0.00%
(C) Expected layer loss cost for entire layer after the application of nonproportional loss sharing provisions only (expressed as a percentage of subject premium) (2A) × [100% − (2B)]	15.0000%	15.0000%	15.0000%	15.0000%
(3) Expected losses for entire reinsured layer after the effect of all nonproportional coinsurance provisions (1) × (2C)	900,000	900,000	900,000	2,700,000
(4) Mean excess claim size in layer (μ_s) [copied from Appendix J Exhibit 1]	310,897	310,897	310,897	310,897
(5) Standard deviation of excess claim sizes in layer (σ_s) [copied from Appendix J Exhibit 1]	260,265	260,265	260,265	260,265
(6) Expected number of claims in layer [(3)/(4)] (μ_c)	2.895	2.895	2.895	8.685
(7) Excess claim count variance-to-mean ratio (VMR_c) [copied from Appendix J Exhibit 1]	1.012	1.012	1.012	1.012
(8) Standard deviation of distribution of aggregate losses in layer $[\sigma_s^2 \cdot \mu_c + (\mu_c \cdot VMR_c) \cdot \mu_s^2]^{1/2}$	692,606	692,606	692,606	1,199,629
(9) Coefficient of variation of distribution of aggregate losses [(8)/(3)]	0.770	0.770	0.770	0.444
(10) Selected coefficient of variation of aggregate loss distribution for all years in the adjustment block combined				0.444

NOTE: The values of the various items in the total adjustment period column above are calculated using formulas identical to those used to compute the values of similar items shown in the all classes combined column on Appendix J Exhibit 1. (Simply substitute the word "year" for "class.") See the footnotes on the bottom of Appendix J Exhibit 1 for a description of these formulas.

APPENDIX J
EXHIBIT 3A

PROFIT COMMISSIONS

(1) Actual or estimated subject premium for commission adjustment period	18,000,000
(2) Expected layer loss cost for entire layer prior to the application of all coinsurance (layer burning cost) [expressed as a percentage of subject premium]	15.0000%
(3) Coinsurance percentage (cedant's participation in layer losses not corresponding to an explicit share of the reinsurance premium, excluding the effects of nonproportional loss sharing plans.)	20.00%
(4) Percentage reduction in layer losses due to nonproportional loss sharing provisions only	0.00%
(5) Treaty rate [expressed as a percentage of subject premium]	25.000%
(6) Expected treaty loss & ALAE ratio (ELR) $\{(2) \times [100\% - (3)] \times [100\% - (4)]\}/(5)$	48.00%

Profit commission formula is in the form:

Profit commission ratio = $(P) * [100\%$—treaty loss & ALAE ratio—$EXP]$,
subject to a maximum commission ratio

Where: (P) = proportion of profits to be paid to cedant:
EXP = reinsurer's overhead expense provision

(7) Proportion of profits to be paid to the cedant (P)	25.00%
(8) Reinsurer's overhead provision (EXP) [expressed as a percentage of treaty premium]	20.00%
(9) Maximum profit commission ratio (if different from that obtained when a zero loss & ALAE ratio is plugged into the formula above) [expressed as a percentage of treaty premium]	N/A
(10) Simplistic estimate of the expected profit commission ratio [expressed as a percentage of treaty premium] $(7) \times [100\% - (6) - (8)]$, subject to a maximum of (9)	8.00%

APPENDIX J
EXHIBIT 3B

PROFIT COMMISSIONS (CONTINUED)

(11) Breakeven loss & ALAE ratio for profit commission purposes [100% − (8)]	80.00%
(12) Entry ratio corresponding to breakeven point [(11)/(6)]	1.667
(13) Insurance charge at breakeven point* (excess loss percentage corresponding to breakeven entry ratio)	3.09%
(14) Loss & ALAE ratio corresponding to the maximum profit commission ratio 100% − (8) − [(9)/(7)]	0.00%
(15) Entry ratio corresponding to the maximum profit loss & ALAE ratio [(14)/(6)]	0.000
(16) Insurance charge at the maximum profit loss & ALAE ratio* (excess loss percentage corresponding to the maximum profit loss & ALAE ratio)	N/A
(17) Insurance savings at the maximum profit loss & ALAE ratio [100% × (15)] + (16) − 100%	0.00%
(18) Net insurance charge (*NIC*) [(13) − (17)]	3.09%
(19) Actuarial estimate of the expected profit commission ratio (expressed as a percentage of treaty premium) (7) × {100% − (6) × [100% − (18)] − (8)}, subject to a maximum of (9); or $P \times \{100\% - ELR \times [100\% - NIC] - EXP\}$, subject to the maximum.	8.37%
(20) Amount by which the actuarial estimate of the expected profit commission ratio exceeds the simplistic estimate [(19) − (10)]	0.37%

*The insurance charges appearing in items (13) and (16) above are based on a lognormal distribution with a 0.444 coefficient of variation. The insurance charge is obtained via linear interpolation of the table of insurance charges given in Tables B1–B3 of Appendix B. See Exhibits 1 and 2 of Appendix J for the derivation of the 0.444 coefficient of variation.

APPENDIX K
TREATY VI
Summary of Key Contract Provisions

Treaty Period: 1/1/90–12/31/90

Layer Reinsured: $900,000 in excess of $100,000 per occurrence

Estimated Treaty Subject Premium: $25,000,000

Expected Layer Loss Cost for the Entire Layer, Prior to All Forms of Coinsurance (Expressed as a Percentage of Subject Premium): 10.0%

Treaty Rate (Expressed as a Percentage of Subject Premium): 20.0%

Proportional Coinsurance: None

Nonproportional Coinsurance: None

Sliding Scale Commission Plan:
Adjustment Period—1/1/90 through 12/31/90 (1 year)
Plan—Minimum Commission of 20% at a 65% loss ratio.
Commission increases by 0.5% for each 1% decline in loss ratio for loss ratios between 55% and 65%.
Commission increases by 0.75% for each 1% decline in loss ratio for loss ratios between 35% and 55%.
Maximum Commission of 40% at a 35% loss ratio.

APPENDIX K

EXHIBIT 1

SLIDING SCALE COMMISSIONS

(1) Actual or estimated subject premium for commission adjustment period	25,000,000
(2) Expected layer loss cost for the entire layer (expressed as a percentage of subject premium)	10.0000%
(3) Coinsurance percentage (cedant's participation in layer losses not corresponding to an explicit share of the reinsurance premium, excluding the effects of nonproportional loss sharing plans.)	0.00%
(4) Percentage reduction in layer losses due to nonproportional loss sharing provisions only	0.00%
(5) Treaty rate [expressed as a percentage of subject premium]	20.000%
(6) Expected treaty loss & ALAE ratio (*ELR*) {(2) × [100% − (3)] × [100% − (4)]}/(5)	50.00%

(7) Minimum commission ratio 20.00%;
corresponding loss & ALAE ratio 65.00%

(8) The details of the sliding scale commission plan are summarized in columns (A) through (E). Values used in the calculation of the expected sliding scale commission are given in columns (F) through (I).

Loss & ALAE Ratio Interval		(C) Percentage Increase in Commission Ratio Per 1% Decrease in Loss & ALAE Ratio	Corresponding Commission Ratio Interval		(F) Entry Ratio Corresponding to Lower Bound Loss Ratio in Column (A)	(G) Insurance Charge Corresponding to Lower Bound Entry Ratio in Column (F)	(H) Expected Loss Ratio Points in Interval	(I) Expected Reductions from Maximum Commission Rate (C) × (H)
(A) Lower Bound	(B) Upper Bound		(D) Lower Bound	(E) Upper Bound				
65.00% and above		0.00%	20.00%	20.00%	1.300	9.12%	4.56%	0.00%
55.00%	65.00%	0.50%	25.00%	20.00%	1.100	14.47%	2.68%	1.34%
35.00%	55.00%	0.75%	40.00%	25.00%	0.700	34.80%	10.16%	7.62%
0.00%	35.00%	0.00%	40.00%	40.00%	0.000	100.00%	32.60%	0.00%
						Total	50.00%	8.96%

(9) Expected ceding commission ratio from a simplistic point of view [commission ratio corresponding to the treaty *ELR* (item 6), given the plan above.]	28.75%
(10) Expected commission ratio from an actuarial point of view [maximum commission ratio—(total 8I)]	31.04%
(11) Amount by which the actuarial estimate of the expected commission ratio exceeds the simplistic estimate [(10) − (9)]	2.29%

NOTES:

On this exhibit, all commission and loss & ALAE ratios are expressed as percentages of treaty premium.

Column (8G): The insurance charges appearing in this column are based on a lognormal distribution with a 0.485 coefficient of variation. These insurance charges are obtained via linear interpolation of the table of insurance charges given in Tables B1–B3 of Appendix B. A procedure similar to that employed in the Treaty I example (see Appendix F Exhibit 1) is used to derive the 0.485 coefficient of variation for aggregate losses in the reinsured layer on this treaty.

Column (8H): The values in this column are obtained by multiplying the differences between the insurance charges corresponding to consecutive loss & ALAE ratio interval end points in column (8G) by the expected treaty loss & ALAE ratio (item 6).

APPENDIX L

DERIVATION OF A FORMULA FOR CALCULATING THE EXPECTED CEDING COMMISSION UNDER A PIECEWISE LINEAR SLIDING SCALE COMMISSION PLAN

This appendix outlines the derivation of a concise formula for computing the expected ceding commission under a typical sliding scale commission plan. The derivation involves three major steps, as summarized below.

Step 1: Let $L_1, L_2, L_3, \ldots, L_n$ be a series of loss ratios such that $L_1 > L_2 > \ldots > L_n = 0$. This sequence divides the range of possible loss ratios into n consecutive intervals, starting with the first interval $[L_1,\infty)$, followed by the intervals $[L_i, L_{i-1}]$, where $i = 2,3,\ldots,n$. Define $f(L_i)$ to be the ceding commission ratio corresponding to an L_i loss ratio, $i = 1,2,\ldots,n$. Using this notation, $f(L_1)$ represents the minimum commission ratio C_{min}, while $f(L_n)$ equals the maximum commission ratio C_{max}. Furthermore, let b_i represent the commission slide (i.e., the percentage point increase in commission ratio per 1% decline in loss ratio) on the interval $[L_i, L_{i-1}]$, $i = 2,3,\ldots,n$. Also define b_1 to be zero, since the commission ratio is constant (at $C_{\min}$) on the interval $[L_1,\infty)$.

Using the notation defined above, the typical sliding scale commission plan may be expressed as a piecewise linear function of the loss ratio L in the following form:

$$(1) \qquad C = f(L) = \begin{cases} f(L_1) = C_{\min} & \text{if } L \geq L_1 \\ f(L_1) + b_2(L_1 - L) & \text{if } L_2 \leq L < L_1 \\ f(L_2) + b_3(L_2 - L) & \text{if } L_3 \leq L < L_2 \\ \quad\vdots & \quad\vdots \\ f(L_{n-1}) + b_n(L_{n-1} - L) & \text{if } 0 = L_n \leq L < L_{n-1} \end{cases}$$

Step 2: Let $p(L)$ be the probability density function of L. Then the expected ceding commission ratio E(C) is the following:

$$\text{(2)} \qquad \mathrm{E}(C) = \int_L f(L)p(L)dL$$

$$= \int_{L_1}^{\infty} f(L_1)p(L)dL$$

$$+ \sum_{i=2}^{n} \int_{L_i}^{L_{i-1}} [f(L_{i-1}) + b_i(L_{i-1} - L)]p(L)dL.$$

Let $M = \mathrm{E}(L)$ = Expected treaty loss ratio,

$P(L)$ be the cumulative distribution function of L, and

$P_1(L)$ be the first moment distribution function of L.

By definition,

$$P(L_i) = \int_0^{L_i} p(L)dL \text{ and } P_1(L_i) = \frac{1}{M}\int_0^{L_i} Lp(L)dL$$

for any value L_i.

The above definitions allow one to simplify equation (2), since the integral expressions appearing in this equation can easily be stated in terms of $P(L_i)$ and $P_1(L_i)$. Now define $P_2(L)$ to be the excess pure premium ratio at loss ratio L. The reader may recall that the excess pure premium ratio is expressible in terms of $P(L)$ and $P_1(L)$ as follows:

$$\text{(3)} \qquad P_2(L) = [1 - P_1(L)] - \frac{L}{M}[1 - P(L)].$$

The relationship given in (3) is used to eliminate all the $P_1(L_i)$ terms in the simplified version of equation (2) discussed above. The result is an expression for E(C) stated in terms of cumulative distribution function values and excess pure premium ratios.

Step 3: The remainder of the proof involves further algebraic simplification of the expression for E(C). In particular, the facts that

$$f(L_i) = f(L_{i-1}) + b_i(L_{i-1} - L_i) \text{ and that}$$

$$C_{\max} = f(L_1) + \sum_{i=2}^{n} b_i(L_{i-1} - L_i)$$

are employed. The final result is the following expression for the expected sliding scale commission. Note that all the cumulative distribution function terms have cancelled out. (We define L_0 to be infinity, so that $P_2(L_0) = 0$.)

$$\text{(4)} \qquad \mathrm{E}(C) = C_{\max} - M \sum_{i=1}^{n} b_i[P_2(L_i) - P_2(L_{i-1})].$$

Equation (4) provides a convenient formula for calculating the expected ceding commission ratio under a piecewise linear sliding scale plan, since one needs only a description of the plan, the expected treaty loss ratio M, and the appropriate table of excess pure premium ratios in order to use it.

Based on the definitions given above for M and P_2, it follows that the expression $M[P_2(L_i) - P_2(L_{i-1})]$ represents the expected number of loss ratio points falling in the interval from L_i to L_{i-1}. Hence equation (4) may be expressed verbally as follows:

$$\text{(5)} \qquad \mathrm{E}(C) = C_{\max} - \sum_{i=1}^{n} b_i \left\{ \begin{array}{c} \text{Expected loss ratio points in the} \\ \text{interval from } L_i \text{ to } L_{i-1} \end{array} \right\},$$

where: E(C) is the expected commission ratio,

$C_{\max}$ is the maximum commission ratio, and

b_i is the commission slide on the i^{th} loss ratio interval.

Since the product of b_i and the expected number of loss ratio points in the i^{th} interval represents the expected number of commission points lost in that interval, it follows from (5) that the expected ceding commission equals the maximum commission ratio minus the expected points of commission lost over the entire range of possible loss ratios. This provides an intuitive justification of the formula given in (4) above.

APPENDIX M
USE OF A MIXTURE OF A LOGNORMAL AND A DISCRETE DISTRIBUTION TO MODEL AGGREGATE LOSSES

If there is a positive probability that a particular reinsurance treaty will have a loss-free year, then the lognormal model cannot be used to specify the aggregate loss distribution for the treaty. This is due to the fact that the lognormal distribution is not defined at the value zero.

One solution to this problem involves the use of a mixture of a lognormal and a discrete distribution (hereafter referred to as the mixed lognormal distribution) to model aggregate losses. This distribution is defined as follows:

$$(1) \qquad f(r) = \begin{cases} p & \text{if } r = 0 \\ (1 - p) \cdot h(r) & \text{if } r > 0 \end{cases}$$

where r is the entry ratio;
- $f(r)$ is the mixed lognormal probability density function (p.d.f.);
- p is the probability of a loss-free year;
- $h(r)$ is the p.d.f. for a lognormal distribution with parameters μ and σ^2 (the values for these are given below).

Intuitively, the reader may think of the mixed lognormal distribution f as a weighted average of a discrete distribution of unity (which is defined only at the zero entry ratio) and a lognormal distribution h (which is defined at positive entry ratios), using the loss-free probability p and its complement, respectively, as weights. The value for p is determined either subjectively or by analyzing past treaty experience, if the latter is credible. Notice that $f(r)$ becomes a lognormal p.d.f. when p is zero.

It can be shown that for a mixed lognormal distribution, the excess pure premium ratio at a particular entry ratio r is given by

$$(2) \qquad P_2(r) = [1 - H_1(r)] - r(1 - p)[1 - H(r)],$$

where H and H_1 are the cumulative density function (c.d.f.) and first moment distribution function, respectively, corresponding to the lognormal p.d.f. h. (If $p = 0$, this formula reduces to that given for the lognormal distribution in Appendix B.)

To evaluate the above expression, one needs to determine the values of H and H_1 at the particular entry ratio r. This is accomplished by noting that the lognormal distribution h has parameters

(3) $$\mu = -\frac{3}{2}\ln(1-p) - \frac{1}{2}\ln(1+\mathrm{CV}^2) \text{ and}$$

(4) $$\sigma^2 = \ln(1-p) + \ln(1+\mathrm{CV}^2),$$

where CV is the coefficient of variation of the treaty's aggregate loss distribution.

It is important to note that the quantity CV measures the variability inherent among all possible loss amounts on the treaty, including loss amounts of zero, even though the lognormal p.d.f. h is defined only at positive loss amounts.

A value for the CV can be calculated from expected aggregate loss cost estimates, together with assumptions on the treaty's frequency and severity distributions, using the same algorithm as used in the lognormal model. Note again that the expressions for μ and σ^2 reduce to the lognormal model formulas in Appendix B when $p = 0$. (The fact that the quantity CV used in the development of the lognormal model measures the variability inherent only among positive treaty loss amounts, as opposed to that among all possible loss amounts, is the reason the μ and σ^2 expressions given above differ from those in Appendix B when $p > 0$.)

The calculation of $H(r)$ and $H_1(r)$ can be achieved via a transformation from a lognormal to a standard normal distribution. Recall that if a distribution is lognormal with parameters μ and σ^2, then its first moment distribution is also lognormal but with parameters $\mu + \sigma^2$ and σ^2. Hence,

(5) $\quad H(r) = \Phi(z)$ and

(6) $\quad H_1(r) = \Phi(z_1)$,

where Φ is the cumulative standard normal distribution;

$$z = \frac{\ln(r) - \mu}{\sigma};$$

$$z_1 = \frac{\ln(r) - \mu}{\sigma} - \sigma.$$

(μ and σ are defined in (3) and (4) above.)

The reader should be aware that the mixed lognormal distribution model is valid only when

(7) $$p < \frac{CV^2}{1 + CV^2}$$

If the above condition does not hold, then the expression for the lognormal parameter σ^2 in (4) becomes negative, which is impossible. In this case, the aggregate loss distribution must be determined by another approach, such as the Collective Risk Model.

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DISCOUNTED RETURN—MEASURING PROFITABILITY AND SETTING TARGETS

RUSS BINGHAM

Abstract

The Hartford Insurance Group employs a total return approach in ratemaking and performance measurement. This article describes the discounted return methodology used by the Hartford in measuring profit and setting prices based on a target return.

Determination of total income involves the consideration of the time value of money in conjunction with the investment period related to key cash flows. The paramount importance of meeting policyholder liabilities precipitates certain investment principles aimed at reducing risk. Liabilities are fully funded with fixed income assets invested at a "risk-free" treasury market rate where maturities match the average duration of liabilities.

Benchmark surplus, dictated by the consideration of funding and solvency, is introduced as a base for measuring return. The benchmark surplus will differ from the actual surplus of a company depending on past results, dividend payout policy, and debt/equity capital management policy.

A methodology is suggested for determining benchmark rates of return for state regulatory purposes, consistent with the management of solvency risk. The benchmark return will differ from actual total return, which is based on reported income and surplus. In this context, the risks and rewards of investment and capital management policies are borne entirely by the owners of the company and reflected in the total company return.

The benchmark return is suggested for use in ratemaking and regulation since 1) it includes income from all sources, 2) it incorporates investment principles which enhance the protection of policyholder funds, and 3) it measures return against a surplus "standard" not influenced by noninsurance driven capital management practices.

1. SUMMARY

This article presents a practical explanation of the discounted return methodology and its use in measuring profit and setting targets. The approach, applied by line of business, essentially looks at the time value of money in conjunction with the investment period related to key cash flows. The key cash flow aspects are premiums receivable, losses and expenses payable, and the new tax law timing impacts due to the unearned premium reserve offset and loss discounting.

The concept of insurance/benchmark surplus is introduced. Reported surplus is viewed as the sum of benchmark and residual surplus. Benchmark surplus is that which is dictated by the consideration of funding and solvency. The remaining residual surplus depends on past results, dividend payout policy, and debt/equity capital management policy.

An important feature incorporated into this approach is the determination of separate investment yields for operating cash flows, benchmark surplus, and residual surplus. The paramount importance of meeting policyholder liabilities precipitates certain investment principles aimed at reducing risk. Liabilities are fully funded with fixed income assets invested at a "risk-free" treasury market rate where maturities match the average duration of liabilities. This rate applies to all operating cash flows. Liability funding requirements are determined on a discounted basis, based on actual business levels and mix.

The following table summarizes the relationship between industry estimated liability funding requirements and invested assets from 1983 to 1988. Pretax investment yields applicable to operating cash flows and benchmark surplus are also shown.

TABLE 1

HISTORICAL INDUSTRY ASSETS, FUNDING REQUIREMENTS, AND YIELDS

	$Billion			Investment Yields (Before Tax)		Investment Return (Before Tax) On Average Invested Assets
Year	Average Cash & Invested Assets	Estimated Liability Funding Required	% of Invested Assets	Operating Cash Flow	Benchmark Surplus	
1988	381.3	269.5	70.7	8.0	7.2	9.0
1987	337.5	251.1	74.4	7.5	6.8	10.3
1986	286.5	253.3	88.4	7.0	6.3	14.0
1985	239.8	239.4	99.8	9.5	8.6	14.6
1984	216.6	199.4	92.1	11.5	10.4	13.8
1983	205.4	178.1	86.7	10.0	9.0	13.0

It is estimated that $269.5 billion in funding (on a discounted basis) in 1988 was required to meet the expected ultimate liabilities of accident year 1988. This represents 71% of average invested assets in 1988. It is noted that industry invested assets in 1985 were barely sufficient to meet current estimates of ultimate liabilities for that year.

The magnitude of assumed reserve weakening from 1983 to 1985 and reserve strengthening from 1986 to 1988 is substantial and greater than previous cycles due primarily to reinsurance recoverables. Actual investment return on average assets exceeds the investment yields on cash flow due to favorable stock market performance and capital gains realized during this period.

Benchmark surplus is assigned a yield that is tied to the risk-free rate with a reduction to reflect normal overhead (e.g., plant and equipment). The residual surplus yield is that which remains from all other sources of investment and other income and will be affected by items such as investment in common stocks, lower quality bonds, and tax-free investments. Total investment income from cash flow, benchmark surplus, and residual surplus will equal total portfolio yield.

A methodology is suggested for determining benchmark rates of return for state regulatory purposes, consistent with the management of solvency risk. Risk-free investments and controlled benchmark leverage standards are utilized in order to safeguard policyholder funds. The resultant benchmark return differs from actual total return, which is based on reported income and surplus. In this context, the risks and rewards of investment and capital management policies are borne entirely by the owners of the company and reflected in the total company return.

Section 2 introduces the concepts beginning with a simplified example which considers the timing effect of paid losses only. That is, premium and expense are assumed to be settled at the beginning without delay. Discounted return is defined and this example is used to demonstrate the calculations of return on premium and return on surplus. The concept of an underwriting target is also introduced. The example is then expanded to include a discussion of other time related sources of investment income. The establishment of benchmark surplus and the determination of investment yield on operating cash flow versus benchmark and actual surplus is discussed. Finally, the application of this methodology to the prospective rating process is reviewed.

Section 3 applies these concepts and the detailed methodology to historical industry statutory data from 1983 to 1988. Funding requirements are established and the development of investment yields for cash flow and surplus are presented. Actual line of business rates of return and implied target combined ratios are discussed.

The following brief summary, extracted from Table 5, displays the historical pattern of returns and combined industry all lines statutory results.

TABLE 2

HISTORICAL STATUTORY INDUSTRY EXPERIENCE

Accident Year	Combined Ratio	Benchmark Return	Total Return	Calendar Year Return
1988	104.2%	13.4%	15.3%	13.1%
1987	101.7	13.2	18.4	15.1
1986	104.2	8.6	19.3	15.1
1985	119.2	−0.2	5.0	8.0
1984	122.8	1.5	2.2	6.7
1983	118.5	2.2	5.2	11.1
1983 to 1988	119.9%	7.7%	12.1%	12.1%

This analysis, while based on approximations, demonstrates the large swing in returns over the period from 1983 to 1988. The accident year returns are more extreme than those of the calendar years. The benchmark return methodology, due to the "risk free" investment policies, provides some added stability in the measurement of accident year returns. Over the composite six-year period, it is estimated that the accident year return on benchmark statutory surplus was 7.7%, 4.4 points lower than the estimated 12.1% return on total statutory surplus with actual investment results. Returns by any measure over the past six years have been below other industry standards.

Target combined ratios based on 1988 investment yields and expense levels, extracted from Table 6, are shown below. Personal Automobile and Workers Compensation, for example, are more than 7 and 9 points, respectively, above the levels which would produce a 15% benchmark return.

TABLE 3

	Industry Combined Ratios		
	Target for 15% Benchmark Return	Accident Year 1988	Composite 1983 to 1988
Personal Automobile	99.9%	107.1%	108.4%
Total Personal Lines	99.9	105.7	106.8
Commercial Automobile	100.8	100.3	110.0
Workers Compensation	106.5	115.9	119.9
Total Commercial Lines	105.8	103.1	112.4
Total All Lines	103.2	104.2	109.9

Appendix A presents additional discussion and formulae. The treatment of cash flows, specifically the tax law timing items, and the issue of "surplus" surplus (i.e., surplus greater than might be deemed necessary to support premium writings) is discussed. Two specific technical issues are discussed as well: the discounting of reserves and other balance sheet items, and the relationship of the presented methodology to internal rate of return (IRR).

An overview of the Hartford's discounted return methodology and its key features is presented in outline form below.

Outline of Hartford Insurance Group's Approach to Total Return

Accident Year Basis

Components of Total Return
- Underwriting
- Investment income on underwriting cash flow (float)
- Investment income on insurance/benchmark surplus
- Investment income on shareholder/residual surplus

Discounted Operating Return (excluding surplus)
- Underwriting income
- Present value of investment income from five components
  - Premium collection
  - Loss payment
  - Expense payment
  - Tax prepayment due to unearned premium reserve offset
  - Tax prepayment due to loss reserve discounting
- Fixed assumptions for given accident year

Management of Solvency Risk and Protecting Policyholder Funds
- Asset/Liability duration matching
- Discounting at risk-free treasury rate; independent of asset yields, and at tax rate of 34%
- Investment yields variable by line of business

Insurance/Benchmark Surplus Versus Shareholder/Residual Surplus (Equity)
- Setting benchmark surplus
- Funding basis
- Volatility adjustment
- Benchmark surplus investment yield
- Residual surplus and investment yield

Return on Benchmark Surplus Versus Return on Total Surplus
- Rate of return regulation on benchmark basis
- Shareholder risk, investment and capital management policies, and total return

Alternative Minimum Tax and Changing Tax Rates

2. DISCOUNTED RETURN

What is Discounted Return?

The term statutory underwriting results generally means the current estimate of the ultimate cash values of premiums, losses, and expenses for a given year. Because the magnitude and timing of these cash flows are quantifiable, the investment income to be achieved can be estimated. This investment income, together with underwriting income, produces total operating income.

The concept of discounting involves the recognition of the time value of money. For example, if we had to pay someone $105 next year, we would invest only $100 at a 5% interest rate today to satisfy that obligation. There is $5 worth of value (from investment income) in the one year delay of payment.

This principle of time value applied to insurance says that premiums are worth less to us, due to delay in collection, while expenses and losses cost us less, since payments are paid at some future time. We lose investment income on premiums while we gain investment income on expenses and losses.

Discounted return joins the two concepts by attempting to determine the ultimate financial performance of a year, resulting from both underwriting and investment, valued at the time of the year's exposure. For instance, if the discounted profitability of a book of business is determined to be $1,000, then this means the value of this business is $1,000 at the time it was written.

How It Works: Nominal (Ultimate Cash Amounts) vs. Discounted Return

It is important to understand the relationship between nominal and discounted results. To illustrate this relationship and demonstrate the concepts, the following simplified example will be used:

- Premium of $1,000,
- Expense of $300,
- Loss of $800, to be paid at the end of 2 years,
- Interest rate of 10% before tax, 6.6% after tax for both years.

To simplify the example, it is assumed that no delay in collecting premium or paying expenses exists. The premium offset and loss discounting provisions of the tax law are also ignored.

In this example, underwriting losses are $100 before tax. The after-tax underwriting losses of $66.00 and investment income of $48.44 produce a first-year nominal loss of $17.56. Investment income in the second year is $51.64, resulting in a cumulative nominal income of $34.08 after two years. The present value of the $34.08 is $30.00 on a basis discounted to the beginning of the first year. This is determined by dividing $34.08 by 1.066 squared, that is, discounting for two years.

The discounted profitability of $30.00 is equivalent to the $34.08 that is expected in retained earnings at the end of two years. For example, if $30.00 were in the bank, it would grow to $34.08 in two years, compounded at 6.6%.

Important smoothing of income is achieved by using the discounted calendar year as compared to the nominal calendar year. The nominal calendar income, which begins with a loss of ($66.00) reaches ($17.56) at the end of year 1 and achieves a gain of $34.08 at the end of year 2. The discounted calendar income begins at $30.00 and reaches $31.98 and $34.08 at the end of years 1 and 2, respectively. While the ending retained earnings value is identical, the yearly calendar flows do not provide as clear a picture of profitability.

The $30 discounted calendar year profit from below is the operating income for this business. The underwriting income and investment income is limited to only cash flow sources related to the business; investment income from surplus is treated separately. This discounted operating income then, by convention, consists of nominal underwriting income plus the investment income credit, both after-tax:

$$\text{Discounted Operating Income} = \text{Underwriting Income} + \text{Investment Credit}. \qquad (2.1)$$

ILLUSTRATION 1

Nominal and Discounted Income Example

	At Beginning of 1st Year	At End of 1st Year	At End of 2nd Year
Nominal Income Before Tax			
Underwriting	($100.00)	$ —	$ —
Investment from Loss Reserves	—	80.00	80.00
from Retained Earnings	—	(6.60)	(1.76)
Nominal Income After Tax			
Underwriting	(66.00)	—	—
Investment from Loss Reserves	—	52.80	52.80
from Retained Earnings	—	(4.36)	(1.16)
Nominal Balance Sheet			
Loss Reserves	800.00	800.00	0
Retained Earnings	(66.00)	(17.56)	34.08
Net Income			
Nominal	(66.00)	(17.56)	34.08
Discounted	30.00	31.98	34.08

The investment income credit is the present value of the investment income derived from the investment of loss reserves. This is normally determined directly by formula to avoid the need of calculating yearly cash flows as we did above. Since the average payment (in this example, the only payment) is delayed two years, the following formula is used:

$$\begin{aligned}\text{Investment Credit} &= (1 - 1/(1 + r^N)) \times \text{Loss} \\ &= (1 - 1/(1.066^2)) \times \$800 \\ &= \$96\end{aligned} \tag{2.2}$$

The present value of the investment income from loss reserves is \$96. The discounted operating income is Underwriting Income + Investment Credit = (\$66) + \$96 = \$30. This answer is identical to the figure produced from the more detailed calculations in Illustration 1.

Calculating ROP and ROS

Once the dollars of income have been determined, it is desirable to express them as a measure of return. The first of these is Discounted Operating Return, defined as discounted operating income as a percentage of premium. In our example, ROP is \$30 ÷ \$1,000, or 3%.

$$\text{Discounted Operating Return } (ROP) = \text{Discounted Operating Income/Premium.} \qquad (2.3)$$

Total income includes investment income on surplus-related assets in addition to operating income. Surplus-related assets are the residual invested assets that remain after all operating liabilities are funded. That is, invested assets must first be "put aside" to pay out loss reserves and other liabilities. Only the remaining, uncommitted invested assets produce investment income for surplus purposes.

Surplus is considered in total at this time, with benchmark surplus to be discussed later. Also, the determination of funding requirements plays a key role in the actual process of performance measurement and target setting (also to be discussed later).

Although total income can be related to premium in order to produce a return on premium figure, normally total income is expressed as a percentage of surplus (equity). If investment income on uncommitted assets equates to 6.0% of surplus, when operating at a 2 to 1 premium to surplus ratio, the *ROS* is calculated as follows:

$$\begin{aligned}&\text{Return on Surplus } (ROS) = \\ &\text{Operating Return} \times \text{Premium/Surplus Ratio} + \text{Yield on Surplus} \\ &\qquad = 3\% \times 2 + 6.0\% = 12.0\% \qquad (2.4)\end{aligned}$$

Setting Targets

Selection of an appropriate *ROS* target requires an assessment of the risk/return relationship for a line of business. After this has been evaluated, the process of setting targets involves determining what combined ratio will produce the desired *ROS*. In the simplest terms, the loss figure that will result in the target *ROS*, when fed through the calculations described above, is "backed into."

Suppose the goal is to achieve an *ROS* of 15%. Then it is necessary to determine what level of loss will result in this return. Continuing with the earlier example, it is known that the $800 loss figure (loss ratio of 80% and combined ratio of 110%) is too high since the *ROS* is 12.0%. With a little bit of math, it is determined that a loss of $772.22 (loss ratio of 77.2% and combined ratio of 107.2%) will produce an *ROS* of 15%.

Current Applications at the Hartford

The discounted return methodology is being used today to both measure returns and set targets. Historical returns are measured by line of business throughout the year and over each planning time horizon using the formula approach described. Targets are being determined by this approach as well. In addition, the more detailed approach using specific multi-year cash flows is being applied to specific business programs and large accounts.

The methodology is also being used in the development of prospective rates. Most of the required components are developed as part of the traditional rating process. The prospective underwriting view, which includes a contingency margin, together with the additional assumptions on interest rates and benchmark leverage, lead directly to the determination of an expected rate of return.

Several other factors are considered in this process beyond the simple case illustrated and will be discussed. They do not change the basics discussed so far.

In Practice

The illustration presented above is a simplification. Several other factors are considered when this methodology is applied to actual experience or in prospective rating.

Other Cash Flow Sources of Investment Credit

Five basic cash flow components are considered to impact total investment income. In addition to payment of loss revenues, they are: premium collection, expense payment, and prepayment of tax due to both loss discounting and the 20% unearned premium offset. These latter two are products of the new tax law and result in a loss of investment income. Premium collection delays also result in a loss of investment income while expense payment delays result in a gain of investment income.

Benchmark Leverage

Three sources of insurance company risk are: insurance, investment and solvency. Insurance risk, derived from both the activities of underwriting and the investing of underwriting cash flows, will principally be a function of underwriting if the investment of underwriting cash flows is at a "risk-free" rate where maturities match liabilities (i.e., loss payouts). This investment approach essentially isolates the total operating income from the effects of investment policy and market volatility.

Investment risk is a function of company investment policy concerning types of investments and maturities, which gives rise to yield and default risks and related volatility.

Solvency risk is the exposure of surplus to both insurance (underwriting) and investment risk. The magnitude and volatility of underwriting losses, along with fluctuating investment results with their associated probabilities, are key determinants of this risk.

An important aspect of total return and management of solvency risk lies in determining the proper level of surplus. Surplus should be a function of two factors:

1. The degree and magnitude of financial exposure. This is essentially the amount and length of time over which funds are committed to the insurance operation of a respective line of business, and
2. The degree of volatility in the loss/combined ratio which will establish the prudent amount of surplus required to guard against the probability of ruin.

The approach developed determines funding requirements by line of business to address the first aspect of financial exposure. Surplus is set initially in direct proportion to funding requirements (i.e., money at risk). Judgemental evaluation of the relative volatility of a line of business, to reflect characteristics such as catastrophes, is then incorporated to arrive at a final benchmark leverage.

Statutory policyholder surplus should be the reference basis upon which the benchmark standards are established, since this is the most readily available and is consistent across all companies.

Yields on Operating Cash Flow vs. Surplus

It is important to distinguish among investment yields on operating cash flow, benchmark surplus, and residual surplus. In order to protect policyholder funds and permit market pricing, operating cash flows are assumed to be invested in current fixed income-producing treasury assets, with yields varying by line of business and maturities related to length of payout. Benchmark surplus is assumed to be invested at approximately 90% of the average operating cash flow rate, a reduction for normal noninvested overhead. Other aspects which affect overall yield are reflected in the residual surplus yield so that the total income from both cash flow and all surplus assets when combined will equal the total portfolio yield.

Items which are part of the residual surplus yield determination include realized capital gains, other income, investment in lower-yielding common stock and real estate, adjustments for nonearning assets, and allowance for tax-oriented investment policies aimed at maximizing after-tax income.

GAAP equity versus statutory surplus issues would not affect the benchmark return previously calculated. Total return is calculated after combining both benchmark and residual surplus investment income and is stated as a percentage of total surplus. This total would be the equity as defined by a given company.

Application to Rating

The discounted return methodology is integral to the rating process since it estimates the expected ultimate total return from both underwriting and investment income to be realized from the business under review, measured in today's dollars. The approach presented provides for the calculation of a rate of return coincident with any prospective rate, consistent with the objective of managing solvency risk, and protecting policyholder funds.

Tax Law

The approach utilizes the corporate tax rate effective at the time the business was written (now at 34%). This applies to underwriting and investment income on cash flow and benchmark surplus, since these are assumed to be invested in risk-free Treasurys, which are taxable. Non-taxable investments will affect only the residual surplus yield and the total return, but not benchmark return.

Although we don't know what future revisions may be made in the tax law, existing law should be reflected in the measurement of historical returns. Accident year 1986, for example, would now have a discounted return that is different than originally planned even if all underwriting and investment assumptions held true. The after-tax return would differ due to the tax rate change and its effect on subsequent calendar year investment income derived from the 1986 accident year loss reserves.

The alternative minimum tax (AMT) must also be considered with allowances made for different tax rates and a redefinition of taxable income when this occurs. In the methodology presented, 20%, rather than the 34%, would be used to determine the benchmark return if the AMT were applicable.

3. APPLICATION TO HISTORICAL INDUSTRY DATA

The objective of this section is to demonstrate the application of this methodology by utilizing 1988 industry annual statement data. Only approximate estimates of reserve strengthening and average delay in collection of premium and payment of loss and expense have been made since the focus here is on the presentation of the methodology. The magnitude of reserve weakening from 1983 to 1985, and strengthening from 1986 to 1988, is fairly substantial and greater than previous cycles due primarily to reinsurance recoverables.

Full realization of the investment income credit (i.e., present value of investment income on all cash flows) requires a specific level of funding based on the volume and mix of business written. Funding is the present value equivalent that is invested to produce the investment income credit. This is determined first using the formulas shown in Appendix D.

In order to determine benchmark and total return, investment income in excess of that produced by operating cash flows must also be considered. It is therefore necessary to look at total investment income and reported total surplus to determine this residual after funding is established. This is also shown in Appendix D.

The results of this approach are summarized for personal and commercial lines in the following Table 4. The industry's 1988 statutory data and the Insurance Expense Exhibit as published by A. M. Best are the basis for this analysis.

It should be noted that the application is required by line of business. Funding and discounting, for example, are affected by loss payout. Surplus allocation is based on relativities derived from a simultaneous evaluation of all lines of business using the volatility-adjusted funding approach, with the total benchmark leverage set at 2 to 1. These benchmark leverage standards are not discussed in detail here since their development involves some subjective judgement. They are sufficiently reasonable to serve in the demonstration of the methodology.

TABLE 4

1988 ACCIDENT YEAR ALL LINES FUNDING AND INVESTMENT CREDIT (ASSUMED AVERAGE 8% PRETAX INTEREST RATE)

($BILLION)

Committed Assets for Liability Funding	Nominal Value	Years Avg Lag	Nominal Balance Sheet	Discounted Balance Sheet	Investment Credit
Personal Lines					
Premium	85.7	.17	−14.3	−13.9	−.7
Expense	22.1	.17	3.7	3.6	.2
Loss	68.6	1.38	94.5	89.0	4.3
Tax Law:					
Unearned Prem. Res. Offset			−2.9	−2.8	−.1
Loss Discounting			−2.8	−2.7	−.1
Commercial Lines					
Premium	113.9	.25	−28.5	−27.6	−1.5
Expense	32.6	.25	8.1	7.9	.4
Loss	84.8	3.11	264.1	232.5	12.6
Tax Law:					
Unearned Prem. Res. Offset			−3.9	−3.7	−.2
Loss Discounting			−14.4	−12.8	−.7
Total All Lines					
Premium	199.6	.21	−42.8	−41.5	−2.1
Expense	54.7	.22	11.8	11.5	.6
Loss	153.4	2.34	358.6	321.5	16.9
Tax Law:					
Unearned Prem. Res. Offset			−6.8	−6.5	−.3
Loss Discounting			−17.2	−15.5	−.8
Net Funded Liabilities and Investment Credit			303.6	269.5	14.2
Yield (after tax)					5.3%
Funding Equity				34.1	1.9
Benchmark Surplus			99.8	99.8	4.7
Yield (after tax)					4.8%
Residual Surplus			11.3	11.3	1.8
Net Other Assets/Liabilities			−33.4	−33.4	0
Total Statutory Surplus			111.1	111.1	8.4
Yield (after tax)					7.5%
Cash & Invested Assets			381.3	381.3	22.6
Yield (after tax)					5.9%

It is assumed that all insurance liabilities are fully funded and that only the remaining balance of invested assets is available for surplus-related investment income. In this case, the total (average) statutory cash and invested assets is $381.3 billion, which produced investment income of $22.6 billion. Assets are committed for insurance liability funding for premium (agents' balances viewed as a negative liability), expense payable, loss reserves, and tax law prepayment due to the unearned premium offset and loss reserve discounting. These latter two are also viewed as negative liabilities.

Fixed income treasury market yields have been used to discount the accident year and to determine investment income on funding. These yields average 4.9% and 5.4% after tax, respectively, for personal and commercial lines. Yield curves are used to produce various yields by line assuming an approximate match of investment period with liability payout.

The investment of funds committed to support policyholder liabilities in risk-free Treasurys with maturities matching liability payout essentially eliminates investment and interest rate risk as a factor affecting operating return.

Balance sheet loss reserves (in billions of dollars) are estimated at $358.6 nominal, and $321.5 discounted. Total insurance liabilities to be funded are $303.6 nominal and $269.5 discounted. Invested assets of $269.5 produce an investment income credit of $14.2, for an overall 5.3% average yield. This is based on an average 2-year treasury bill yield of 8% pretax for 1988.

Restated, we need to set aside $269.5 to fund ultimate liabilities of $303.6, which will produce $14.2 of investment income credit on a present value basis. This calculation recognizes the timing of premium, loss, expense, and tax flows and related investment income, on an after-tax basis.

All remaining assets are viewed as uncommitted. Benchmark surplus is established first at an overall 2 to 1 basis, with its yield set at 90% of the cash flow yield, or 4.8%, to adjust for normal noninvested overhead. All investment income on residual assets (after deduction of the benchmark surplus), as well as other income, is related to the residual surplus.

Since surplus is considered to be invested uniformly for all lines, investment income produced on these uncommitted assets is assumed to be at a fixed rate and not variable by line of business.

For convenience, the combined benchmark and residual surplus and related investment income are used in the calculation of total return. The equity due to discounted funding of \$34.1 (\$303.6 less \$269.5) is also considered to be shareholder equity on a discounted basis and income on this is credited to residual surplus. The resultant discount-based total surplus yield is 7.5%.

This yield will fluctuate in comparison to the yield applicable to funding and reserve discounting since it is affected by investment policy. Stock investment is particularly influential in this regard since it brings lower dividend yields, but potentially significant, although erratic, capital gains. The level of invested assets in relation to required funding, especially impacted during periods of substantial business growth, is also a key factor in determining this yield. Increased business writings require the commitment of more funds, leaving relatively fewer residual invested assets to produce investment income for surplus.

It is important to note that all investment income is included in this approach. This methodology has simply determined the allocation of investment income among operating cash flows, benchmark surplus, and residual surplus.

The industry's estimated 1988 statutory discounted operating income is:

$$\begin{aligned}\text{Discounted Operating Income} &= \text{Underwriting Income} + \\ &\quad\ \text{Investment Credit} \\ &= (\$5.6) + \$14.2 = \$8.6 \end{aligned} \tag{3.1}$$

Estimated discounted operating return is:

$$\begin{aligned}\text{Discounted Operating Return} &= \text{Discounted Operating Income} / \\ &\quad\ \text{Premium} \\ ROP &= \$8.6 / \$199.6 = 4.3\% \end{aligned} \tag{3.2}$$

Estimated return on benchmark surplus is:

$$\text{Return on Benchmark Surplus} = ROP \times \text{Premium} / \text{Benchmark Surplus} + \text{Benchmark Surplus Yield}$$

$$BROS = 4.3\% \times 2.0 + 4.8\% = 13.4\% \quad (3.3)$$

Estimated return on total surplus is:

$$\text{Return on Total Surplus} = ROP \times \text{Premium} / \text{Surplus} + \text{Surplus Yield}$$

$$TROS = 4.3\% \times 1.8 + 7.5\% = 15.3\% \quad (3.4)$$

Table 5 summarizes six years of historical industry returns. Approximate assumptions were made as to the degree of reserve strengthening in each year to estimate accident year results. The changing tax law and effective tax rate differences in 1986 and 1987 have been considered.

The accident year returns exhibit more extreme swings over the period as compared to published calendar year returns. The benchmark returns provide some stability, since they are not influenced by capital gains, as are the total returns.

It is worth noting that the returns by any of the measures have been poor over the six-year period. The composite benchmark return has been 7.7% over this period, compared to the total return of 12.1%. The calendar year reported composite is also 12.1%.

Line of Business Targets

Target combined ratios, based on the achievement of a 15% return on benchmark surplus, are shown in Table 6 by line of business. These targets have been developed utilizing the investment yields and expense levels from 1988. Personal lines accident year combined ratios would have to improve (from the 1988 105.7 ratio) by nearly 6 points to 99.9 in order to achieve a return of 15% on benchmark surplus. Commercial lines returns in total, on the other hand, have improved dramatically from the unprofitable experience of earlier years and are at reasonable levels, although cumulatively still below a long run 15% average. Some specific lines still lag substantially behind.

TABLE 5

INDUSTRY STATUTORY RETURNS

Year	Line of Business	Cash Flow Yield (Before Tax)	Acc. Year Comb. Ratio	*ROP*	Benchmark Prem/ Surplus	Benchmark *BROS*	Total Prem/ Surplus	Total *TROS*	Cal Year *TROS*
1988	Personal	7.4	105.7	0.4	2.8	5.8	2.5	8.4	
	Commercial	8.2	103.1	7.3	1.7	16.8	1.5	18.4	
	Combined	8.0	104.2	4.3	2.0	13.4	1.8	15.3	13.1
1987	Personal	7.0	103.9	0.6	2.8	5.7	2.7	11.3	
	Commercial	7.7	100.1	7.5	1.7	16.5	1.6	21.5	
	Combined	7.5	101.7	4.6	2.0	13.2	1.9	18.4	15.1
1986	Personal	6.5	107.0	−1.0	2.8	0.5	2.7	11.4	
	Commercial	7.2	102.0	5.3	1.7	12.2	1.6	22.8	
	Combined	7.0	104.2	2.6	2.0	8.6	1.9	19.3	15.1
1985	Personal	8.7	111.8	−2.5	2.7	−2.0	2.5	3.3	
	Commercial	9.7	125.7	2.4	1.6	0.7	1.6	5.8	
	Combined	9.5	119.2	−2.4	2.0	−0.2	1.9	5.0	8.0
1984	Personal	10.5	108.2	0.4	2.6	6.6	2.3	6.7	
	Commercial	11.8	136.8	−4.4	1.6	−1.6	1.4	−0.6	
	Combined	11.5	122.8	−2.0	2.0	1.5	1.8	2.2	6.7
1983	Personal	9.2	105.6	0.9	2.6	7.3	2.2	9.5	
	Commercial	10.2	130.9	3.5	1.6	−0.9	1.4	2.7	
	Combined	10.0	118.5	−1.3	2.0	2.2	1.7	5.2	11.1
1983 to 1988 Composite	Personal		106.8	0.2	2.7	4.0	2.5	8.6	
	Commercial		112.4	3.1	1.7	9.5	1.5	13.9	
	Combined		109.9	1.6	2.0	7.7	1.8	12.1	12.1

TABLE 6

TARGET COMBINED RATIOS FOR 15% BENCHMARK RETURN
(BASED ON 1988 INVESTMENT AND EXPENSE LEVELS)

Line of Business	Cash Flow Yield BT	Payment Lag			Expense Ratio	Target Comb. Ratio	*ROP*	Benchmark	
		Prem	Exp	Loss				Prem/ Surp	*BROS*
Homeowners	7.3	.17	.17	1.25	32.0	99.7	3.2	3.2	15.0
Per Auto Liab.	7.5	.17	.17	1.75	24.1	101.0	4.5	2.3	15.0
Per Auto PhyD.	7.3	.17	.17	.75	24.2	98.4	2.8	3.6	15.0
Total Per Auto		.17	.17	1.34	24.2	99.9	3.8	2.7	15.0
Com Auto Liab.	7.8	.25	.25	2.50	28.3	102.5	5.7	1.8	15.0
Com Auto PhyD.	7.3	.25	.25	.75	30.0	97.0	3.2	3.2	15.0
Total Com Auto		.25	.25	1.97	28.8	100.8	4.9	2.1	15.0
Workers Comp.	8.0	.25	.25	3.00	24.5	106.5	5.7	1.8	15.0
Other Liab.	8.8	.25	.25	5.50	23.6	119.4	9.5	1.1	15.0
Medical Liab.	9.0	.25	.25	6.50	13.9	136.7	9.5	1.1	15.0
Com F & Allied	7.3	.25	.25	1.25	36.5	99.6	2.8	3.6	15.0
Com Multi Peril	7.6	.25	.25	2.00	36.1	98.5	5.7	1.8	15.0
Other	7.6	.25	.25	2.00	31.7	99.1	5.7	1.8	15.0
Personal	7.4	.17	.17	1.32	25.8	99.9	3.7	2.8	15.0
Commercial	8.2	.25	.25	2.94	28.6	105.8	6.2	1.7	15.0
All Combined	8.0	.21	.21	2.24	27.4	103.2	5.1	2.0	15.0

It is important to note that many of the assumptions used in determining these targets are rough approximations for the purpose of demonstrating the methodology. In particular, personal lines and commercial lines premium collections and expense payments are assumed to lag two months and three months, respectively.

Individual company application of this methodology would require determination of the applicable cash flow patterns and modification of the benchmark leverage to reflect unique characteristics of that company's business.

In summary, determination of this target combined ratio requires simply the following input assumptions:

- Expense ratio
- Investment yield (Treasurys)
- Average premium, expense, and loss payment dates
- Benchmark leverage
- Target benchmark return

These should, of course, reflect unique company and state characteristics.

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APPENDIX A

SUPPORTING DETAIL AND FORMULAS

Analysis of financial performance is best accomplished on a discounted accident period basis. This presents the truest picture of business being written since the present values of all cash flows are considered, regardless of which calendar period they are reported in. The results may appear quite different than calendar year results viewed on a nominal basis, which are affected by reserve adjustments, underwriting, and investment results attributable to prior accident years.

A detailed model can be used to portray the exact cash flows for a given accident year, and would include items such as the loss payout in subsequent calendar years. Formulas are shown in Appendices B, C, and D which closely approximate such exact model results but provide a much simpler means of application in practice. In these formulas, it is assumed that premium, loss and expense are paid on a single payment date.

In practice, not much accuracy is sacrificed by using average payment date; for that reason, the formulas are presented using average dates. Furthermore, it is assumed that taxes are incurred and paid on underwriting at the beginning of the year without delay. Tax on investment income is paid without delay also. All discounting is as of the beginning of the year.

The same approach used for losses is applied to premium and expense. It should be noted, however, that premiums received late (positive payment lag) or expenses paid early (negative payment lag) will result in a reduction of funding. The tax law timing items—loss discounting and unearned premium offset—are considered to be reductions to funding since credits are to be received at a date in the future.

Industry or company specific cash flow patterns, if credible, can be utilized. This is similar, for example, to the loss discounting approach under the present tax law with regard to loss payout.

Tax Law Timing Items Viewed as a Reserve

The two tax law timing items, the loss discounting and 20% unearned premium offset, are viewed as analogous to losses in terms of their time value effect. When considered as a "loss discount tax reserve" and an "unearned premium tax reserve," parallels to loss reserves are drawn.

Loss reserves are the reserve liabilities which are invested until payments are made. The loss discount tax reserve is the prepaid tax which is paid due to loss discounting and is reduced as recoveries are achieved through successive period discounting. Loss discounting is viewed in this manner as a negative loss reserve with recovery payments resulting in an ending reserve of zero. This negative reserve is invested and produces negative investment income. The unearned premium tax reserve is similar except that the recovery occurs in the following year and the negative investment income is for one year only. All taxes are assumed paid when incurred.

Reconciliation of Nominal and Discounted Income for a Single Year

When following the flows for a single accident year until settlement of all losses, it is possible to provide an exact reconciliation of nominal and discounted results. The nominal ending retained earnings after the settlement of all cash flows for a given year is the total return from underwriting and all investment income. This value, discounted back to the present, will equal the accident year discounted operating return. This is calculated by adding underwriting income and the investment credits from premium, expense, loss, loss discount tax reserve, and unearned premium tax reserve.

Funding

Funding plays a key role in the methodology. Funding is the amount of assets/liabilities that are needed to support a particular level of business. Specifically, it is defined as the present value equivalent in assets that are required to produce the present value of investment income from all future cash flows. Funding is based on the magnitude of the cash flows and the length of time that it takes to settle them, summed across all flows after discounting to present value.

Funding is a beginning point in the establishment of leverage as it provides a measure of the amount of funds committed when writing a line of business—a basis for assigning surplus (before adjusting for volatility). Total funding across all lines of business determines the total invested assets that must be committed by a company to support all writings. The remaining investment assets determine the residual surplus yield.

Surplus Runoff

The need for benchmark surplus remains beyond the initial year that the business is written. It is suggested that surplus committed to support the business be allowed to run off in proportion to the reduction in funding over time. Since loss reserves are the primary long-term user of funds, then consequently surplus should run off as loss reserves decline to zero.

Surplus Surplus and Related Investment Income

In practice, a company operates each year with a beginning reported surplus that results from several factors including past results, dividend payout policy, and debt/equity capital management policies. Also, some of the assets are not income-producing investments. The approach recognizes actual invested assets and accepts the surplus as given.

By introducing benchmark leverage and the associated policyholder risk minimization investment principles, however, the state rate of return calculations can be divorced from reported actual returns. Both the investment income and surplus (or equity) bases differ between the benchmark and reported returns. In this sense, the concept of surplus surplus is irrelevant to the determination of rate of return for state regulatory purposes.

The relationship of invested assets, required funding, and actual surplus is an important one. During periods of rapid growth, funding demands can "use up" invested assets. The result is a reduced yield on residual surplus since relatively fewer invested assets remain to produce residual income.

Technical Issues

Discounting

Discounting (primarily of loss reserves) has gained prominence due to the increasing recognition that much of insurance profitability involves investment income derived from the delay in paying losses subsequent to the collection of premium. Certainly investment income has become increasingly important to insurance industry total returns.

Discounting is viewed as a means of determining the value inherent in balance sheet assets. A discounted loss reserve, for example, is felt to represent the true ultimate liability of losses in present dollars, that is, the amount of money that needs to be set aside to pay ultimate losses. The difference between this discounted reserve and the nominal (i.e., reported) reserve is thus considered company, or shareholder, equity.

Unfortunately, several other items having significant cash flow (and investment income) impact are usually ignored. Agents' balances and reinsurance recoverables, for example, are two important ones. A complete valuation solution requires that a discounted balance sheet be determined in parallel to the present nominal balance sheet in which all items are considered. Focusing on loss reserves only and attempting to introduce this discount into the nominal balance sheet is not sufficient, and perhaps misleading. Also, market valuation of assets is equally important.

The treatment of taxes with respect to loss discounting is a particular problem. The difficulty of handling taxes is often glossed over, if discussed at all. A pretax discount factor is often used and a vague statement made as to the need to "tax effect" the result. Unfortunately, a pretax discount factor will produce an overstated discount equity.

Consider a loss payable in one year of $1,000. If the pretax discount rate is 10%, then the $1,000 reserve discounted for one year will be $909.09, with an apparent discount equity of $90.91. If $909.09 is invested for one year at 10%, however, the $90.91 in investment income must be taxed, leaving an insurance company short of the $1,000 required to pay the claim.

The correct amount is determined by discounting the reserve at the after-tax rate of 6.6%, assuming a 34% tax rate, and also applying the loss discount provision of the tax law. The discounted reserve then will be $938.09, which invested at 10% for one year and taxed at 34% will produce exactly the $1,000 needed to pay the claim, excluding the effect of the discount provision. The reserve discount equity is $1,000 minus $938.09, or $61.91—the investment credit defined earlier. With recognition of the tax law discounting provision, this is reduced to $60, the net investment credit attributable to loss reserves. This is the true equity in the reserve.

RESERVE DISCOUNT EQUITY
$1,000 LOSS PAYABLE IN GIVEN NUMBER OF YEARS
WITH 10% PRETAX DISCOUNT, 34% TAX RATE

	Number of Years		
	1	2	3
After-Tax Based Discounted Reserve	938	880	825
Discount Equity			
—Before Tax Loss Discounting	62	120	175
—Including Tax Loss Discounting	60	114	164
Pre-Tax Based Discounted Reserve	909	826	751
Discount Equity	91	174	249
After Tax at 34%	60	114	164

It should be noted that use of a pretax discount with subsequent taxing of the discount will produce the identical discount equity, as long as the tax law discount rate and payout pattern equals the actual investment rate and payout. This was not true prior to introduction of the loss discount provision of the tax law.

Internal Rate of Return (IRR)

Internal rate of return is sometimes used to determine the "value" of insurance from the shareholder perspective. It essentially provides a single measurement, via the IRR, which indicates the rate of return on shareholder capital over a period of time produced from the net capital flows.

The IRR has inherent limitations, due to its simplicity, that reduce its value in comparison to the discounted return, or net present value, approach. The IRR approach is unable to distinguish between different accident years and/or different types of cash flows and yields. Each accident year should be viewed as a unique entity (much as an individual project in manufacturing), in which each year's characteristics dictate initial benchmark surplus needs and subsequent runoff as this need diminishes. The release of benchmark capital is thus related to the respective accident year operating flows. In addition, operating cash flow and surplus should be viewed as fundamentally different, in terms of cash flow pattern, related invested assets, and yields produced from these invested assets.

By its very definition, the IRR is a single number which represents an overall return from net calendar period capital flows, without recognition of different accident year or cash flow components. The IRR, furthermore, can be altered by arbitrary capital withdrawal (i.e., dividend payment) policies. The discounted return, a specific measurement of profit, is not subject to such alteration.

If the IRR methodology is applied by accident year and capital is withdrawn as liabilities are settled, then the results will be essentially the same as the discounted return methodology. If a single accident year is analyzed and capital is released only after all operating cash flows are settled, the following formula demonstrates the relationship between the IRR and yields applicable to operating cash flow and surplus as used in the discounted return methodology.

$$\text{Ending Capital} = S(1 + \text{IRR})^N = S(1 + R_S)^N + OR(1 + R)^N,$$

where

S = Beginning Surplus,
R_S = Surplus Yield,
OR = Discounted Operating Return,
R = Cash flow yield,
N = Year in which last flows are settled.

Risk Adjustments and Differing Cash Flow Yields

The yield applicable to operating cash flows discussed in this paper has not differentiated between the components of this flow nor have risk adjustments been made. In certain situations the yields applicable to the flows should be different and adjusted for added risk. Yields applicable to retrospective loss rated premiums, for example, should reflect the added credit risk associated with the substantial collection delays. Another example is in the treatment of reinsurance where credit risk adjustments are appropriate both in the premium and loss areas.

Appendices B, C, and D

Appendix B presents further detail on general definitions and formulas, and Appendix C presents the formulas for the loss discount investment income credit when actual payouts and yields differ from the tax law.

It should be noted that the difference between actual yield and that assumed under the tax law has a greater effect than do payout differences. Even though a loss may be modeled on a single payment date, the loss discount is not recovered on a single date but rather as dictated by successive year discounting.

Appendix D summarizes all formulas in tabular form, on both a nominal and a discounted basis. Although this paper has focused on a discounted basis for valuation, a further understanding of the methodology is achieved by considering the parallel nominal balance sheet and income statements.

APPENDIX B

GENERAL DEFINITIONS AND FORMULAS

Underwriting Income = $(P - E - L)(1 - T)$,

where P = Premium, E = Expense, L = Loss, T = Tax Rate

Nominal Basis

Operating Return = Underwriting Income
+ Investment Income on Insurance Liabilities

Total Return = Operating Return
+ Investment Income on Surplus

Discounted Basis

Operating Return = Underwriting Income
+ Investment Income Credit on Insurance Float

Investment Income Credit (ICC) = Present Value of Investment Income on All Cash Flows Related to the Accident Period

Premium $ICC = -(1 - D_p)P$
Expense $ICC = (1 - D_e)E$
Loss $ICC = (1 - D_l)L$
UPR Tax $ICC = (1 - D_u)(.2T)PU$
Disc Tax ICC: See Appendix C for formula

where $D = 1/(1 + R)^N$, i.e., Discount Factor
R = rate for calculating discount, *after tax*
R_b = tax law discount rate before tax
N = average payment date for Premium, Expense, or Loss, respectively
for D_u, $N = 1$, UPR tax recovery payment date
U = Annual Premium Year End Unearned factor (i.e., Unearned Premium/Premium)

Nominal/Discounted Reconciliation

Discounted Operating Return = Nominal Ending Total Return / $(1 + R)^N$,

where N is the ending period when all insurance cash flows have been settled.

All dollar figures and discount factors are after-tax except discount factor for loss discounting using R_b, the tax law discount rate.

APPENDIX C

LOSS DISCOUNTING INVESTMENT INCOME CREDIT FACTOR (FACTOR TIMES LOSS FOR $ IMPACT)

1) Actual and Law Rates and Payouts Same

$-\{(D_b - D_a) + T(1 - D_b)\}$,

where
$D = 1/(1 + R)^N$, i.e., Discount Factor
R = rate for calculating discount
N = payment date
b = before tax
a = after tax
T = tax rate
$D_a = 1/(1 + R_a)^N$
$R_a = (1 - T)R_b$

2) Actual and Law Rates Different, Payouts Same

$- \{(Dr'_b - D_a) + T(1 - Dr'_b)\}$
$+ (Dr'_b - D_a)(R_a - R'_a)/(R_a - R'_b)$, (rate adjustment).

where $'$ signifies using law rate

3) Actual and Law Rates and Payouts Different

$- \{(Dn'r'_b - Dn'_a) + T(1 - Dn'r'_b)\}$
$+ (Dn'r'_b - Dn'_a)(R_a - R'_a)/(R_a - R'_b)$ (rate adjustment)
$+ TD_a \{(1 - Dn''r'_b) - (Dn''r'_b - Dn''_a)R'_b/(R_a - R'_b)\}$ (date adjustment),

where $'$ signifies using law rate or payment date
$n'' = n' - n$, i.e., difference in payment date

Effect of different rates is greater than payout differences and formula 2 is sufficiently accurate for most applications.

An approximate formula to the above is

$- T\{(1 - Dmr_a) \times (1 - Dn'r'_b)\}$, where $m = (n + 1)/2$
$= - T\{(1 - 1/(1 + R_a)^m) \times (1 - 1/(1 + R'_b)^{n'})\}$

APPENDIX D

BALANCE SHEET AND INVESTMENT INCOME FORMULAS

Committed Assets for Liability Funding (By Line)	Nominal Value	Years Pay Lag	Nominal Balance Sheet	Investment Income	Discounted Balance Sheet	Investment Credit
Premium	P	N_p	$-N_pP$	$-RN_pP$	$-PD(N_p)/R$	$-PD(N_p)$
Expense	E	N_e	N_eE	RN_eE	$ED(N_e)/R$	$ED(N_e)$
Loss	L	N_l	N_lL	RN_lL	$LD(N_l)/R$	$LD(N_l)$
Tax Law:						
UPR Offset			$-.2TPU$	$-.2RTPU$	$-.2TPUD(1)/R$	$-.2TPUD(1)$
Loss Discounting			ZL/R	ZL	KL/R	KL
Total Funding for all Lines of Business			Sum of Above (F_n)	Sum of Above (RF_n)	Sum of Above (F_d)	Sum of Above (RF_d)
Funding Equity			0	0	F_n-F_d	$R(F_n-F_d)$
Benchmark Surplus			MB	RMB	MB	RMB
Residual Surplus			$S-B$	$I-RF_n-RMB$ (I_r)	$S-B$	$I-RF_n-RMB$ (I_r)
Net Other Assets/Liabilities			$A-S-F_n$ $+(1-M)B$	0	$A-S-F_n$ $+(1-M)B$	0
Total Invested Assets and Investment Income			A	I	A	I

APPENDIX D

BALANCE SHEET AND INVESTMENT INCOME FORMULAS

(CONTINUED)

Discount amount factor, $D(N) = \{1-1/(1+R)^N\}$
R = Risk free treasury interest rate, applicable to cash flows, after tax
T = Corporate tax rate, presently 34% (20% if AMT applies)
Loss discount investment income factor approximately equal to $Z = -RT\{(N_t+1)/2\}\ \{1-1/(1+R_t)^N\}$, where R_t = tax law discount rate
K = Loss discount investment credit factor from Appendix C
M = Benchmark surplus yield overhead adjustment factor
S = Total Surplus
A = Total Invested Assets
I = Total Investment Income

The discounted yield on combined total surplus (benchmark & residual) including reserve discount equity, $R_s = (I-F_d)/S$
Alternatively, $R_s = R_I(A/S) - R(F_d/S)$, where $R_I = I/A$, the total invested asset yield
Return on Benchmark $BROS = U/B + R(F_d/B) + RMB$, where U = Underwriting income after tax
Total Return on Surplus $TROS = U/S + R(F_d/S) + R_s$
The nominal yield on residual surplus, $R_m = I_r/(S-B)$
The discounted yield on residual surplus including reserve discount equity, $R_a = \{I_r+R(F_n-F_d)\}/(S-B)$

RISK LOADS FOR INSURERS

SHOLOM FELDBLUM

Abstract

Insurance companies are risk averse, even as individuals are. Casualty actuaries have suggested several methods of calculating risk loads to compensate the insurer for the risk it accepts. Methods currently in use, and reviewed in this paper, consider (a) the standard deviation and variance of the loss distribution, (b) utility functions, (c) the probability of ruin, and (d) reinsurance costs.

These methods are theoretically unsound. They consider the wrong type of risk; they arbitrarily equate risk with a mathematically more tractable variable; and, they require equally arbitrary assumptions about an insurer's aversion to risk. More importantly, they concentrate on the size of loss distribution, though the true risk to the insurance company resides in profit fluctuations.

Modern portfolio theory measures the risk assumed by investors in securities. Systematic risk, the overall risk faced by a diversified stock portfolio, requires an additional premium. Firm-specific risk, or the fluctuations in an individual stock's price, can be eliminated by diversification and is not compensated for in security returns. Insurance equivalents to modern portfolio theory can be applied to insurance portfolios to determine risk premiums by line of business. Such analysis reveals the Commercial Liability lines to be highly risky and the Personal Property lines to be less risky. In sum, this method allows insurers to measure the true risk they face in each line of business.

I am indebted to Richard Woll and Benjamin Lefkowitz, who made numerous corrections to an earlier version of this paper. The remaining errors, of course, are my own.

1. INTRODUCTION

Most persons are risk averse: they prefer a stable income to a fluctuating one, even if the two have equal expected values. Risk aversion is one of the foundations of insurance, for the insured trades the chance of a fortuitous but large loss for the payment of a fixed annual premium.

Insurers also are risk averse, although their large size masks their preference for a stable income. When faced with a large risk, an insurer may decline the application, seek reinsurance, or charge an additional premium, a "risk load." The third option is the most desirable, since declining the application reduces business volume, and buying reinsurance gives up potential profit on the ceded business.

Yet calculating risk loads is a complex task. On the one hand, insurers often incorporate "contingency" provisions in premium rates, whether for conflagration hazards in turn of the century fire rates or unanticipated liabilities in current General Liability rates. On the other hand, there is no established procedure for determining the size of the risk load.

So actuaries have devised numerous methods, which are grouped below into four categories:

(1) The risk load may vary with the random loss fluctuations of the individual risk; e.g., "standard deviation" and "variance" methods.

(2) The risk load may vary with the characteristics of the overall portfolio of risks; e.g., "utility function" and "probability of ruin" methods.

(3) The risk load may vary with the empirical costs of reducing risk; e.g., "reinsurance" method.

(4) The risk load may vary with fluctuations in profitability; e.g., "modern portfolio theory" methods.

Some methods are simple to implement but lack theoretical justification; others are mathematically elegant but difficult to apply. The advantages and deficiencies of each method are examined below. Only the last method, however, measures the true risk faced by insurers.

2. STANDARD DEVIATION AND VARIANCE METHODS

The simplest approach is to conceive of the insurer's risk in the same fashion as the insured's risk. Suppose an insurer sells a General Liability policy to a contractor, who has a 1% chance of being liable for a \$100,000 loss, and a 99% chance of no loss. The expected value of the loss is \$1,000, but the contractor may be willing to pay \$2,000 to entirely avoid the risk of loss. Similarly, the insurer may require a pure premium of *more* than \$1,000, to compensate it for the risk it assumes.

Suppose a second contractor also purchases an insurance policy. This insured has a 0.1% chance of a \$1,000,000 loss, and a 99.9% chance of no loss. The expected value of the loss is again \$1,000, but both the standard deviation and the variance of the loss are higher, as shown below.[1]

TABLE 1

STANDARD DEVIATION AND VARIANCE OF LOSS

Amount of Loss	Probability of Loss	Expected Value of Loss	Standard Deviation of Loss	Variance of Loss
\$ 100,000	1.0%	\$1,000	\$ 9,950	\$ 99,000,000
1,000,000	0.1	1,000	31,607	999,000,000

The loss distribution on the second policy has a standard deviation about three times as large and a variance about ten times as large as that for the first policy, though their expected losses are the same. If the risk load is proportional to the standard deviation or the variance of the losses, then the risk load for the second policy should be either three times or ten times as large as that for the first policy. The standard

[1] Consider the first contractor, with a 1% chance of a \$100,000 loss. The variance of loss is $(0.01)(100{,}000^2) + (0.99)(0^2) - (1{,}000^2) = 99{,}000{,}000$ "dollars squared." The standard deviation is the square root of this, or \$9,950.

deviation method, as currently applied by the Insurance Services Office,[2] would determine the pure premium as

Pure premium = expected loss + (constant · standard deviation).

If the constant is 0.5%, the pure premiums are \$1,050 for the first policy and \$1,158 for the second policy.[3] This method is now in vogue, as it requires information only about the loss distribution, not about other insurer characteristics. Therefore, it can be applied to all carriers equally.[4] The Insurance Services Office (ISO), the major U.S. rate making bureau for the non-Compensation lines of business, presently uses the standard deviation of the loss distribution to calculate risk loads for General Liability, Products Liability, and Commercial Automobile increased limits factors.[5] Until the mid-1980's, ISO used the variance of the loss distribution for this purpose, a method proposed by Robert S. Miccolis [38] in 1977.

Loss frequencies and severities vary by policy, and no insurer could estimate all the needed figures. As an approximation, one can determine the standard deviation or variance of the loss distribution for policies with a specified limit of liability. A General Liability policy with a limit of \$25,000 truncates all loss indemnification at that amount. The expected value, standard deviation, and variance of the loss distribution are all lower than those for a similar policy with a \$1,000,000 limit. Using the standard deviation method and a loss distribution modeled by a Pareto curve, ISO calculated the following risk loads and increased limits factors for one group of Premises/Operations risks:

[2] The probability of loss for any particular policy is indeterminate. Rather, ISO estimates the loss distribution for policies of a given limit of liability, and applies the resultant risk loads to the increased limits factors.

[3] For the first policy, \$1,000 + (0.005)(\$9,950) = \$1,050. For the second policy, \$1,000 + (0.005)(\$31,607) = \$1,158.

[4] This is particularly important for rating bureaus, which have information only about the size of loss distribution for the block of business.

[5] For details, see the memoranda of ISO's Actuarial Research Committee.

TABLE 2

RISK LOADS AND INCREASED LIMITS FACTORS FOR PREMISES/OPERATIONS RISKS (MEDIUM TABLE)

Policy Limit	Average Severity	ALAE per Claim	ULAE per Claim	ILF without RL	Risk Load	ILF with RL
$ 25,000	$ 4,039	$2,325	$ 477	1.00	$ 521	1.00
50,000	5,314	2,325	573	1.20	797	1.22
100,000	6,698	2,325	677	1.42	1,179	1.48
200,000	8,135	2,325	784	1.64	1,706	1.86
⋮	⋮	⋮	⋮	⋮	⋮	⋮
50,000,000	14,828	2,325	1,287	2.70	8,503	3.66
100,000,000	16,227	2,325	1,391	2.92	11,943	4.33

ALAE: Allocated loss adjustment expense (ISO uses a constant dollar amount for each policy limit; although unrealistic, this simplifies the calculations).
ULAE: Unallocated loss adjustment expense (ISO determines the ULAE as 7.5% of expected loss plus ALAE for this line of business).
ILF: Increased limits factor.
RL: Risk load.

Unfortunately, this method has no theoretical justification, for several reasons. First, the *insurer's* risk is different from the *insured's* risk. The insured is more concerned about random loss fluctuations—which could ruin him financially—than about the accuracy of the expected loss estimate. But the insurer may have thousands of policies in each line of business. It is less concerned about random loss fluctuations, which even out over a large volume of risks, than about the accuracy of its expected loss estimate.

To illustrate this, suppose 10,000 insureds buy General Liability policies. Each insured has the same probability of a $100,000 loss. This probability is not known exactly, but is estimated to be between 0.5% and 1.5%. The expected value of the loss on each policy may be as low as $500 or as high as $1,500, but these figures are not the major concern of the insured. He seeks relief from worry, from the risk of possible bankruptcy. For him, the range of probable losses—for which actuaries use standard deviation and variance statistics—is the primary concern.

Suppose the insurer charges a $2,000 premium for each policy. Its expected loss ratio lies between 25% and 75%, depending upon the true probability of loss. For example, if the probability of loss is actually 1%, then the expected loss for each policy is $1,000 and the expected loss ratio is 50%. Random loss fluctuations will not cause the actual loss ratio to deviate much from the expected, since many homogeneous risks are covered. But the actual loss ratio *will* differ greatly from the forecasted loss ratio if the probability of loss is incorrectly estimated. A 0.5% chance of loss will bring large profits, while a 1.5% chance of loss will have the opposite effect. This is the "risk" that the insurer must guard against.

Actuaries use the terms "process risk" and "parameter risk" to denote these two causes of fluctuation in insurance losses. Process risk refers to random loss fluctuations about a stable mean; this is the major risk for the insured. Parameter risk refers to uncertainty in estimating the expected loss; this is the major risk for the insurer.[6]

If the standard deviation and variance methods capture process risk, not parameter risk, why are they used to calculate insurer risk loads for liability policies? After all, process risk and parameter risk are independent, so estimating one is of no help for the other. The usual explanation is that: "There is no easy method of estimating parameter risk. To satisfy their member companies, rating bureaus must somehow calculate risk loads. Basing the factors on Pareto curves and process risk is sophisticated enough that no further questions will be asked." Sophisticated it may be, but a satisfying explanation it is not.[7]

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[6] The actuarial use of the terms "process risk" and "parameter risk" is due to Robert L. Freifelder [25]. Freifelder speaks of the probability distribution function of the loss process (whence *process risk*) and of an *a priori* distribution of the unknown parameters of the loss distribution function (whence *parameter risk*).

[7] There are two problems in estimating parameter risk. One is to quantify the magnitude of this risk—e.g., the expected fluctuation in forecasted average pure premiums due to estimation errors. The second is to use these estimates of parameter risk to evaluate needed actuarial figures, such as Workers Compensation excess loss factors. This second part is a mathematical exercise, albeit a complex one. Philip Heckman and Glenn Meyers [30] outline a sophisticated method of solving this problem. Moreover, actuaries often *assume* an *a priori* distribution for the parameters of the loss function, and thereby "quantify" the parameter risk. However, they have yet to address the crucial first question noted above: *How does one estimate the true parameter risk*?

Even if one seeks to calculate process risk, one must measure the standard deviation of the insurance portfolio as a whole, not that of individual risks. The ratio of the standard deviation to the expected value decreases as additional homogeneous risks are added to the portfolio. Consider the first policy in Table 1. If the insurer issues a single policy, the ratio of standard deviation to expected loss is 9.950 (standard deviation of $9,950 divided by expected loss of $1,000). If the insurer issues two such policies, the expected loss is $2,000 and the standard deviation is $14,071, for a ratio of 7.036.[8] If the insurer issues one hundred such policies, the ratio is less than one. In other words, the standard deviation of the individual policy's loss distribution is no guide even to the process risk faced by the insurer.[9]

On first reflection, it might seem that using the variance of the loss distribution avoids this problem. After all, the ratio of the variance to the expected value does not change when similar risks are added to the portfolio. In truth, using the variance simply aggravates the problem. The process risk faced by the insurer does in fact decrease as additional

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[8] The probability of loss is 1% for each policy. Thus, the probability of a loss on both policies, for a total loss of $200,000, is 0.01^2, or 0.0001. The probability of one loss of $100,000 is (2)(0.99)(0.01), or 0.0198. The probability of no loss is $(0.99)^2$, or 0.9801. The expected loss is (2)(0.01)($100,000), or $2,000. These figures, as well as the calculation of the variance and standard deviation, are shown below.

Number of Losses	Total Loss (1)	Probability (2)	Expected Loss (3)	Variance Calculation: $(3) \cdot (2)^2$
Two losses	200,000	0.0001	20	4,000,000
One loss	100,000	0.0198	1,980	198,000,000
No losses	0	0.9801	0	0
Total		1.0000	2,000	202,000,000

Variance = $202{,}000{,}000 - 2{,}000^2 = 198{,}000{,}000$ (dollars squared).
Standard deviation = $198{,}000{,}000^{0.5}$ = $14,071

The ratio of standard deviation to expected loss is $14,071 ÷ $2,000, or 7.036.

[9] David B. Houston [33] makes a similar distinction between an individual's and an insurer's risk. The individual is concerned with variations in outcomes of a particular action. The insurer is concerned with sampling error that affects the estimated mean pure premium. This distinction is similar to that in the text, except that Houston ascribes all parameter risk to sampling error.

policies are issued, but the ratio of the variance to the expected loss does not show this. The variance method ignores the problem; it does not solve it.[10]

The second theoretical failure of the standard deviation and variance methods is that they determine only relative risk, not absolute risk. The ISO exhibit for Premises/Operations risk loads (see Table 2 above) says that the risk load for a policy with a $50,000 limit should be about one and one half times that for a policy with a $25,000 limit. But how are the dollar amounts of the risk loads determined—or the ratio of risk load to expected loss?

The mathematics provide no answer. ISO simply chooses an overall risk load for the line of business, and then spreads this risk load by size of policy limit using the standard deviation or variance method. But determining the overall risk load is our primary concern, and an arbitrary choice is no solution.

The third theoretical failure is that these methods determine relative standard deviation, or relative variance, not relative risk. The simplified illustration of two General Liability risks in Table 1 provides different "risk loads" depending upon whether the standard deviation or variance method is used. The risk load for the second policy is either three times or ten times that for the first policy. There is no *a priori* reason to equate risk with either the standard deviation or the variance. These statistics are used because they are mathematically tractable. But the goal is to measure actual risk, not to equate risk with an appealing mathematical concept and then to measure the latter.

To sum up the standard deviation and variance methods: Parameter risk, the real concern, is too hard to measure, so process risk is substituted for it. The standard deviation is a tractable mathematical construct, so it replaces "risk." Then an overall portfolio risk load is chosen arbitrarily, and the standard deviation method spreads it over policies according to the size of the policy limit. Somehow, this hardly sounds like proper actuarial practice.

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[10] Advocates of exponential utility functions often cite the invariance of exponential utility to the wealth of the insurer as an advantage; see the quotation from Freifelder in footnote 16 below. Again, just the opposite is true. The risk does vary with the wealth of the insurer. A method which ignores this is defective.

3. UTILITY FUNCTIONS

Microeconomists have long used utility functions in consumer demand theory, and casualty actuaries have recently suggested using them to calculate risk loads. Utility functions allow the rate maker to vary the risk load on a policy with the composition of the entire insurance portfolio and with the insurer's attitude toward risk. Unfortunately, the mathematics required are complex and needed assumptions can only be guessed at, so this method is not popular.[11]

A utility function expresses the value of a given basket of assets to its owner. Utility functions provide an ordinal, not a cardinal, sequence of values. In other words, it is meaningless to speak of the absolute utility of a loaf of bread or a quart of milk to an individual. We can say only that the individual prefers a loaf of bread to a quart of milk, or vice versa.[12] Similarly, we cannot determine the absolute utility of a $2,000 premium for the insured, but we can say that he or she prefers paying this premium to suffering a 1% chance of a $100,000 loss.

The discussion below seems to imply cardinal values for utility. For instance, an exponential utility function assigns a cardinal value to a given basket of goods. This is not the intention, however. The implication is only that the utility is proportional to the value of the exponential function, not that it is equal to it. The same comment applies to all the utility functions discussed below.

Utility functions are an ideal tool for calculating risk loads, since they are the mathematical equivalent of the "attitude toward risk." Utility functions depend upon the insurer's degree of risk aversion, the composition of its insurance portfolio, and its corporate wealth.

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[11] On the use of utility functions in demand theory, see James M. Henderson and Richard E. Quandt [31] or Angus Deaton and John Muellbauer [20], pages 25–26.

[12] See, for example, Paul A. Samuelson [44], page 91: " . . . a cardinal measure of utility is in any case unnecessary; . . . only an *ordinal* preference, involving 'more' or 'less' but not 'how much,' is required for the analysis of consumer's behavior"; or Armen A. Alchian [1], page 39: "Any numbering sequence which gives the most preferred sure prospect the highest number, the second preferred sure prospect the second highest number, etc., will predict his choices according to 'utility maximization.' But any other sequence of numbers could be used so long as it is a *monotone transformation* of the first sequence. And this is exactly the meaning of the statement that utility is *ordinal* and not cardinal."

As a simple illustration, suppose the utility of an asset is proportional to the square root of its price: $U_x = Kx^{0.5}$.[13] If an insured has \$10,000 of assets, with a 1% chance of losing it all, and a 99% chance of no loss, his present utility is $(0.01)(0)K + (0.99)(10{,}000^{0.5})K = 99K$. This equals the utility of \$9,801 of assets. In other words, the insured would be willing to pay \$199 to avoid the risk of loss.

Suppose the insurer begins with \$1,000,000 of assets. If it accepts the risk of loss from the insured, its total utility is

$$(0.01)(990{,}000^{0.5})K + (0.99)(1{,}000{,}000^{0.5})K = 999.95K,$$

which is equal to the utility provided by assets of \$999,899.74. That is, it needs a pure premium of \$100.26, or a risk load of 0.26%. The larger the insurance portfolio, and the greater the surplus of the insurer, the smaller is the risk load needed.

But what is an appropriate utility function? Theoretical economists do not have this problem, since they use utility functions to prove mathematical theorems, not to solve practical problems. But actuaries desirous of using utility functions to calculate risk loads must first determine what utility functions are most realistic.

There are two considerations in determining an appropriate utility function.

First, the function should satisfy the mathematical properties needed for utility theory. Gary Venter [48] lists several such properties:[14]

1. *Utility is an increasing function of wealth*; that is, as wealth increases, the utility of that wealth increases.

2. *Actors are risk averse*; that is, each incremental increase in wealth yields progressively less incremental utility for the actor.

3. *Risk aversion decreases as wealth increases*; that is, the poor individual has greater absolute risk aversion than the wealthy individual has (on average).

[13] The constant "K" is a proportionality factor that transforms the utility function from a cardinal to an ordinal measure. This is not as general as Alchian's *monotone transformation* (see preceding footnote). A monotone transformation is appropriate for the theory, but it cannot generate the absolute risk loads needed in practice.

[14] Only the first three of Gary Venter's criteria are listed in the text. His last two, that the utility function be bounded from above and that the utility be equal to zero for negative amounts of wealth, are less commonly accepted by economists.

Gary Venter's criteria mirror reality, and many persons would agree with them.[15] But casualty actuaries have found that one of the simplest and most tractable utility functions, the exponential function, has a risk aversion level that is invariant with the wealth of the actor. With an exponential utility function, the utility of a portfolio of insurance contracts equals the sum of the utilities of each individual contract.

This attribute of exponential utility functions simplifies the mathematics of calculating risk loads, and it has made the exponential function the utility function of choice for calculating risk loads. But it does not accord with reality. The essence of insurance is that the insurance company, due to its large size, is less risk averse than each individual insured.[16]

ISO has noted that even if one posits a given family of curves for the utility function, such as the exponential family, varying the parameters of the family provides different risk loads for each size of risk. One can determine whatever risk load one wants, as well as various relationships among risk loads for different policies, simply by varying the parameters of the utility function.[17] To avoid this problem, John Cozzolino and Naomi Kleinman [15] have suggested using the reciprocal of the insurer's surplus as the parameter of the exponential utility function. This does indeed provide a simple formula for the parameter of the utility function. But what evidence is there that it accurately reflects differences in risk aversion among insurers of different sizes? Presum-

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[15] Venter's first criterion simply says that people prefer more wealth to less wealth. His second criterion seems realistic. It is not universally true, but it seems to hold for most persons in most situations. His third criterion has been formulated rigorously by Ken Arrow [2], though it is hardly amenable to a simple proof.

[16] See, for example, Robert Freifelder [25], *op. cit.*, as well as his shorter article [26]. Note his theorem 1 on page 75 of this article: "If premium rates are based on an exponential utility function, the total premium required for a class of independent contracts is equal to the sum of the premiums required for each of the contracts individually." His justification for the exponential utility function strikes the practical businessman as strange, but it fits well with a desire for elegant and tractable procedures: "There are no 'portfolio' or 'wealth' effects with an exponential utility function. What this means is that with an exponential utility theory ratemaking model, the decision maker does not have to know the exact characteristics of the company's portfolio or its wealth. In practical situations the above information is not generally available" (p. 74).

[17] Note the comment by J. David Cummins and David J. Nye [17], page 429: "Risk loadings and hence solvency are very sensitive to the choice of the risk aversion parameter when the expected utility approach is used."

ably, large insurers are less risk averse than small insurers are. But what evidence is there that the risk aversion varies directly with the reciprocal of the insurer's surplus?

The second problem in determining an appropriate utility function is equally serious. The utility function must model reality, or the risk load procedure becomes a sterile mathematical exercise. Is the risk aversion demonstrated by insurers indeed similar to that implied by an exponential utility function, or a square root utility function, or some other function? This question is difficult to answer, and no one has yet proposed a method of doing so.[18]

Utility function analysis translates the vague "attitude toward risk" into concrete mathematical expressions. But it provides no practical guidance towards measuring either risk aversion or utility. In other words, it restates the problem of determining risk loads; it does not solve it.

The earlier comments regarding process risk and parameter risk apply to utility function analysis as well. In the example above, parameter risk refers to the uncertainty regarding the probability of loss. It might be 1%; it might be 2%; it might be some other probability. If we knew the distribution of the probability of loss, we could incorporate this into utility function analysis. But utility function analysis provides no aid for measuring this distribution, so we are no better off than when we began.[19]

[18] The Society of Actuaries life contingencies textbook, *Actuarial Mathematics*, models insurance transactions between a risk averse insured and a risk neutral insurer. This is a standard economic model. Since insureds are more risk averse than insurers are, it also reflects reality (though imperfectly). Nevertheless, it leaves unanswered our question: "What is the appropriate risk load for insurers?" See Newton L. Bowers, Jr., et al., [8], pages 7–16.

[19] Freifelder, following Bühlmann, proposes one means of empirically measuring parameter risk. Using automobile accident data, he assumes a Poisson loss distribution for each driver, and an underlying Gamma distribution of the Poisson means in the population of drivers. Thus, one driver may have a 10% chance of an accident, so his loss distribution is Poisson with a mean of 10%. A second driver may have a 20% chance of an accident, so his loss distribution is Poisson with a mean of 20%. The Gamma distribution may be estimated by examining the moments of the empirical loss distribution. See Robert L. Freifelder [25], pages 83–84, Hans Bühlmann [11], and Lester B. Dropkin [21].

This procedure masks the true parameter risk; namely, that *the underlying distribution of means changes over time*. Richard Woll has pointed out that if the underlying distribution of means remained constant, then average loss frequencies for a large insurer would not vary from year to year. Yet they do vary. That is, the parameter risk is not just that the Poisson means are unknown, but that they change over time. For further discussion, see Richard Woll's review [50] of Cozzolino and Kleinman's paper [15], especially pages 21–22.

In sum, utility theory is no more promising than the "standard deviation" and "variance" methods discussed previously. For the theoretical economist, utility theory produces mathematical theorems. But no one has yet even suggested how to model an insurer's risk aversion. Instead, the theoreticians say: "Let us choose a simple and tractable utility function, regardless of its accuracy or applicability, and determine risk loads accordingly." This is hardly a suitable actuarial procedure.

4. PROBABILITY OF RUIN

European actuaries developed probability of ruin analysis to determine surplus requirements for insurers of different sizes and with different insurance portfolios. "The probability of ruin" is the probability that the insurer will become technically insolvent during a specified time period, such as the coming year. In other words, it is the probability that required reserves will exceed available assets sometime during the period.[20]

The analysis may concentrate on any of three variables: the probability, the assets, or the liabilities (required reserves). That is, one may formulate the problem in three ways: (1) What is the probability that an insurer with given assets and a given portfolio of risks will become technically insolvent? (2) For an insurer with a given portfolio of risks, how much assets (or surplus) are needed such that the probability of ruin is less than a given amount?[21] (3) For an insurer with given assets (or surplus) and a given insurance portfolio, what risk loading must be added to the premium such that the probability of ruin is less than a given value?

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[20] See, for example, R. E. Beard, T. Pentikainen, and E. Pesonen [5], especially pages 132–159. For an American exposition, see Alfred E. Hofflander [32]. A stochastic cash flow probability of ruin model, which considers the availability of assets to pay claims instead of insurance regulatory requirements, is presented in C. D. Daykin, et al., [19], as well as in earlier papers by these authors.

[21] For example, Robert Cooper [14], pages 22–43, uses probability of ruin analysis to determine the *necessary invested capital* for an insurance company, though his "high degree of confidence" seems low.

At first glance, probability of ruin analysis seems to solve some of the problems associated with utility function analysis. Absolute risk loads are still not provided, since they require an assumption about an appropriate probability of ruin—one in a thousand? one in ten thousand? But one may calculate the relative risk load for any risk in a given insurance portfolio: it is the extra premium such that the addition of that risk does not change the overall probability of ruin.

An illustration should clarify this—and show the problems with this procedure as well. Suppose an insurer sells General Liability policies, and all its insureds have a 1% chance of a loss equal to the policy limit. The insurer has $50,000 of assets, and it may issue either two $100,000 policies to two independent insureds or one $200,000 policy to a single insured. Finally, the insurer demands that the probability of ruin be no more than one in one thousand.

The expected loss of either portfolio is $2,000. A pure premium of $2,000 brings total assets to $52,000. This leaves a chance of ruin of 1%, as any loss would exceed available assets. For the portfolio of two risks, the insurer needs $100,000, or $50,000 in addition to its original assets, to lower the probability of ruin to one in a thousand. Note that the chance of total loss on *both* policies is one in ten thousand, less than the probability of ruin set by the insurer. A pure premium of $25,000 is therefore needed for each policy, of which $1,000 is the expected loss and $24,000 is the risk load.

For the portfolio of one $200,000 risk, the insurer needs $200,000 to lower the probability of ruin to one in a thousand. Since it has original assets of $50,000, it requires a pure premium of $150,000. Of this amount, $2,000 is the expected loss, and $148,000 is the risk load. The single large risk needs a greater risk load than do the two small risks if the probability of ruin is to be equal.[22]

Unfortunately, probability of ruin analysis concentrates on the chance of technical insolvency. It does not balance this against the income from the additional premium. In practice, one must choose an extremely low probability of ruin (say, one in ten thousand) so that risk loads are needed to prevent insolvency. Suppose one determines that, to ensure a proba-

[22] This illustration is extreme; no insurer writes only one or two policies. The oversimplification is for heuristic purposes only. The same analysis may be applied to an insurer writing a thousand policies.

bility of ruin less than one in ten million, the needed risk loads are 10% of premium on a $100,000 premium policy and 50% of premium on a $1,000,000 premium policy.

Even if the marketplace allowed only a 20% risk load on the latter policy, almost all insurers would prefer the second policy to ten of the first. After all, the probability of ruin is low, and the additional risk load is extra income. In truth, the needed risk load for the second policy is between 1 time and 5 times that for the first policy. Somewhere between these two numbers, the additional profit makes up for the additional risk.

Probability of ruin analysis helps define the boundaries, or endpoints, for the needed risk load. It does not determine where within that interval the appropriate risk load lies. It is useful for solvency regulation, since only the endpoint is desired. It is useless for risk loads, since the actual load is needed.[23]

5. REINSURANCE METHOD

The risk for insurers is the possibility of unexpected losses either on an individual policy or on a book of business. To stabilize loss fluctuations, a primary insurer may enter into an excess of loss reinsurance treaty. Such protection is not costless. The reinsurance premium must cover not only costs but also the reinsurer's administrative expenses and profit margin. The primary insurer must balance the additional cost of reinsurance protection against the reduction in risk afforded by the treaty.[24]

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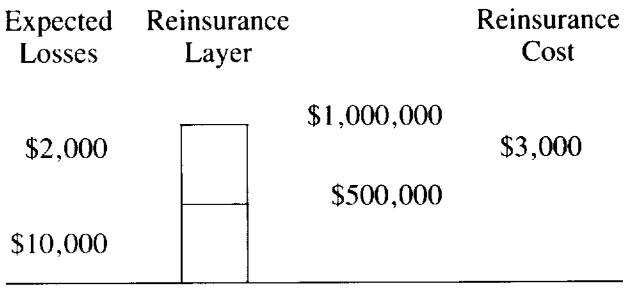
[23] Stephen P. D'Arcy and Neil A. Doherty [18], page 3, present a similar argument in another context: "The ruin probability, no doubt, forms an important constraint on managerial decisions if only because insurers operate in a regulatory environment that focuses attention on solvency. However, constraints are not objectives. Additionally, as an objective, the probability of ruin is quite incomplete since no account is taken of the value of the equityholders', policyholders', and other parties' claims in the respective states of solvency and ruin. There is indeed a world of difference between surviving and prospering that is ignored by the probability of ruin objective." In other words, two policies may both pass the probability of ruin test set by the insurer. Nevertheless, the insurer may judge one of the policies to be more "risky" and require a higher risk load.

[24] Reinsurance involves various costs, such as underwriting profits and investment income received by the reinsurer and administrative and processing costs of the primary carrier. The *former* costs would be used to estimate the risk load. For a clear description of these costs, see Daniel A. Bailey [3].

The reinsurance treaty is the real-world counterpart of the theoretical risk load. Suppose the expected losses and expenses (i.e., not including a risk load) for a General Liability policy are \$10,000 for a \$500,000 limit and \$12,000 for a \$1,000,000 limit (that is, \$2,000 for the second \$500,000 layer). Suppose also that the charge for facultative per risk excess of loss reinsurance protection of \$500,000 over \$500,000 is \$3,000, as shown below.



The "empirical" risk load for the second \$500,000 of coverage is \$1,000: the reinsurer's charge minus the expected losses. The "empirical" risk load for the lower layer would be determined in the same manner (e.g., by examining the cost of facultative reinsurance of \$250,000 over \$250,000, then \$150,000 over \$100,000, and so forth). Since reinsurance underwriters vary their premium rates by the characteristics of the primary insurer, such as its financial stability, insurance portfolio, and underwriting stringency, the complete risk faced by the insurer is considered, not just the process risk on individual policies.[25]

Unfortunately, this method places the cart before the horse. Reinsurers need actuarial guidance as much as other insurers do. Risk theory is as much for their benefit as it is for that of primary insurers. Reinsurance underwriters evaluate risk as best they can: some succeed and some go bankrupt. Actuaries can help both primary insurers and reinsurers by recommending appropriate risk loads.

[25] Robert Butsic [12], analyzing the economic value of a loss reserve portfolio, compares the risk adjusted discount rate to a hypothetical loss reserve transfer to a reinsurer. The profit margin required by the reinsurer should equal the difference between present values of the loss reserves using a risk free versus a risk adjusted discount rate.

Moreover, reinsurance premiums are based on more than just evaluations of risk. If there is strong competition for a certain type of business, reinsurers cut rates. If some reinsurers leave a line of business, others raise rates. Marketplace pressures influence prices as much as risk characteristics do, and their independent influences cannot be easily distinguished.

The risk load may be subsumed under "profit and contingencies." In practice, the profit margin depends more on competitive pressures and marketplace constraints than it does on actuarial cost considerations. But insurers need the cost analysis as much as they need the marketing analysis, for they must continually decide whether to match competitors' prices. The question here is, "What is the appropriate cost of the additional risk to the insurer?"

6. LOSSES AND PROFITS

The risk load methods discussed above concentrate on insurance losses. But insurers do not just pay losses. They collect premiums as well, and they try to match premium rates to anticipated expenditures. Risk is a function of profitability, or net income, not just of loss payments.

Three examples should clarify this. Each illustration is idealized, but their combination provides a realistic portrayal of insurance operations.

(1) Suppose an insurer issues a retrospective rating plan, with no maximum or minimum premium, and no loss limit. In other words, the final premium is equal to the actual losses, with a loading for expenses and profit.

The variability of loss payments has no effect on the insurer's profit. The profit is set by the retrospective rating plan. It is not dependent upon random loss fluctuations or even "parameter risk."[26]

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[26] The major risk for the insurer stems from the potential uncollectability of additional premiums. See Roy P. Livingston [36].

(2) Suppose two lines of business have the same size of loss distributions but different loss payout patterns. In one line, the average loss is paid out six months after the accident date; and, in the other line, the average loss is paid out four years after the accident date. Inflation and investment returns affect the second line much more than they do the first, so insurance profitability, or net after-tax operating income, will vary more for the second line. The size of loss distributions, however, do not show this.

Similarly, competitive pressures affect insurance profitability. Again, suppose two lines of business have the same size of loss distributions. One line earns a constant 10% return on equity. In the second line, however, fluctuating market conditions cause profitability to vary substantially from year to year. Clearly, there is more risk for the insurer in the second line of business.

(3) Size of loss distributions are only meaningful for determining risk loads when the risks insured are homogeneous and the premiums are the same for each of them. When the risks insured are *heterogeneous*, and the insurer, by its underwriting and pricing expertise, charges different premiums based upon the anticipated hazards, then size of loss distributions give no clue to the insurer's risk.

Gary Koupf illustrated this with a simple Commercial Liability example.[27] Suppose an insurer sells Commercial Automobile Bodily Injury and Property Damage coverages to a group of homogeneous insureds. Each insured incurs one Bodily Injury claim for $10,000 and one Property Damage claim for $1,000. The insurer charges $15,000 for the BI coverage and $1,500 for the PD coverage. Clearly, there is no risk for the insurer. Profitability is stable, and the size of loss distributions are degenerate for each coverage.

If one combines the Bodily Injury and Property Damage coverages, however, the size of loss distribution becomes highly variable. For a single insured, the average expected loss is $5,500, but the variance of the loss distribution is $20,250,000. The variance of the loss distribution depends upon the degree of heterogeneity of the coverages or of the risks insured. Yet the insurer's profitability remains stable, as long as appropriate premiums are charged for each coverage.

[27] Gary Koupf, comments at the ISO Actuarial Research Commmittee meeting, June 15, 1988.

The combination of these three examples portrays reality well:

(1) Underwriters vary premium rates with the anticipated hazards. Most policies are not retrospectively rated, but they are not purely random contracts either. Much of the variance in the size of loss distribution is reflected in premium rate differences.

(2) Many factors besides size of loss distributions affect insurer profitability: investment income, competitive pressures, and regulatory decisions. Insurers in many lines of business are comfortable with the statistical loss distributions. They are concerned, however, whether regulators will allow needed rate revisions, whether investment returns will match loss cost inflation, and whether competitive pressures will force them to cut rates in order to retain market share.

(3) Most Commercial Liability insureds are heterogeneous. Each has different loss characteristics, and each has its own hazards. Insurance underwriters adjust policy conditions, vary premiums, and select insureds to obtain a profitable book of business. If one ignores the insurance operations, and one examines only the size of loss distributions, one finds great variability. But much of this variability is neither "process risk" nor "parameter risk." It is the anticipated variability reflected by the different coverages and risks.

In sum, the size of loss distribution is but one influence on the insurer's profitability and risk—and not even the most important one. To appropriately determine the risk faced by insurers, one must examine overall profitability, not individual losses.

7. MODERN PORTFOLIO THEORY METHODS

Investors face risks similar to those of insurers. *Process risk* in insurance refers to the random fluctuations of actual losses about their expected values; *firm-specific risk* in financial theory refers to the random fluctuations of a specific stock's price that are unrelated to market movements. *Parameter risk* in insurance refers to the uncertainty of expected losses; *systematic risk* in financial theory refers to the unexpected movements of the stock market as a whole.[28] Diversifying an insurance portfolio smooths process risk but does not affect parameter risk. Diversifying a financial portfolio eliminates specific risk but has little effect on systematic risk.

Modern portfolio theory rests on two assumptions. First, the risk premium varies with systematic risk, not specific risk. Portfolio diversification eliminates specific risk, so the investor should receive no additional return for voluntarily assuming such risk. Second, the original formulation of modern portfolio theory (Markowitz) assumed that systematic risk varies as the standard deviation of returns on a diversified portfolio. Historical returns, on a weekly or monthly basis, can be used to measure the standard deviation. More recent approaches (Capital Asset Pricing Model) assume that systematic risk varies as the regression coefficient (termed "beta") of the diversified portfolio's return on the total market return.[29,30]

One can apply this method to determine insurance risk loads as well.

The risk load should depend upon fluctuations in overall insurance portfolio returns. It should not vary with the loss fluctuations of individual risks, since these can be reduced and often eliminated by proper diversification.

Fluctuations in insurance portfolio returns can be measured by the standard deviation of historical operating returns by line of business. Alternatively, they can be measured by the regression coefficient of the return from a particular line of business on the return of all lines combined.

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[28] Systematic risk is often termed *diversifiable* or *market* risk. Specific risk is also termed *unsystematic*, *residual*, *unique*, or *undiversifiable* risk. See Richard A. Brealey and Stewart C. Myers [9], page 132.

[29] A good introduction to modern portfolio theory is J. Fred Weston and Thomas E. Copeland [49], chapters 16 and 17. The development of the theory is due to William F. Sharpe [45] and John V. Lintner [35].

[30] Several technical assumptions used in the Capital Asset Pricing Model are more relevant to securities than to insurance products, such as costless financial transactions and the availability of various quantities of securities at a given market price. See below in the text for further discussion of these issues.

Of course, insurance policies do differ from financial investments, and modern portfolio theory is more applicable to the latter than to the former.

Financial investments can be broken down into small pieces. Even a small investor can diversify his portfolio by purchasing shares in a mutual fund. A portfolio of thirty or more unrelated stocks is well diversified, and most investors can afford such purchases. In contrast, insurance policies are discrete units. Distinct policies, if written by the same agency or branch office, may not be unrelated—just as one does not diversify a financial portfolio by purchasing a dozen oil stocks.

The price of a stock reflects not only current earnings but also investors' expectations for future earnings. A well-established but cyclical industry may show severe fluctuations in year to year profitability, but milder changes in stock prices. The insurance industry shows consistent "underwriting" or "profitability" cycles. The standard deviation of insurance returns may not accurately reflect investors' expectations of long term profitability.[31]

Neither of these problems is insurmountable. A small General Liability insurer faces not only systematic risk borne by the industry as a whole, but also some specific risk due to its particular book of business. This implies that the insurer must examine the standard deviation of historical returns on a book of the same size and quality, not on a fully diversified book, such as the industry book. The small- or moderate-size insurer needs a slightly larger risk load than that indicated by industry-wide experience.

The second difference mentioned above has the opposite effect. Since insurance profitability is cyclical, yearly operating ratios show greater fluctuation than investors' expectations do. In other words, the insurer

[31] Nahum Biger and Yehuda Kahane [7] state this as follows: ". . . underwriting profits reported by insurers are not necessarily equal to the way market participants assess those profits, their variability, and the systematic portion of the risk. It follows that evaluation of the systematic risk of underwriting, which is not based on market returns but on reported profits, may result in biased estimates of the coefficients."

needs a slightly smaller risk load than that indicated by the standard deviation of insurance operating ratios.[32]

Best's *Aggregates and Averages* shows industry-wide operating returns by line of business. There are two problems with these figures: (1) there is no adjustment for reserve deficiencies and redundancies and (2) operating income is determined by spreading net investment income to line of business, not by discounting all cash flows to a common date. Nevertheless, Best's figures are carefully compiled and widely available, and they are sufficient for the illustrative purposes of this paper.[33]

Best's determines operating ratios by line of business as:

(Losses + loss adjustment expenses incurred) / net premiums earned

+

(commissions, brokerage, and other underwriting expenses) / net premiums written

+

(policyholder dividends − net investment income) / net premiums earned.

[32] The mathematical derivation of the Capital Asset Pricing Model relies on the opportunity of borrowing or lending at the risk-free interest rate. This is not true for insurers, but it is not true for investors either. Both investors and insurers must pay a premium to borrow money.

The major difference between the financial and insurance markets is that investors can quickly modify their portfolios, whereas insurers are constrained by competitive pressures, high new business production costs, and higher pure premiums among new policyholders. (See Conning & Co. [13] and Sholom Feldblum [23] for further discussion of these costs.) Modern portfolio theory presumes that optimal portfolios are determined by risk and return. In truth, numerous other factors are also relevant.

Risk and return considerations are important, but they cannot—in isolation—argue for restructuring an insurance portfolio. J. D. Hammond and N. Shilling [29] note that the "efficient" insurance portfolios determined by their analysis consist mostly of minor lines of business. J. R. Ferrari [24] finds that an "efficient" insurance portfolio would require separation of automobile bodily injury from property damage, and separation of fire from extended coverage. As Ferrari notes, these other factors must be considered when structuring an insurance portfolio. See also the discussion of Ferrari's paper by Matthew Rodermund [42].

Thus, we do not determine "efficient" insurance portfolios, or recommend restructuring an insurer's writings to "optimize" risk-return relationships. Rather, we simply analyze the variance in insurance profitability by line of business to suggest the risk loading appropriate to each.

[33] A comprehensive model for determining discounted insurance profits by line of business is provided by Richard G. Woll [51]. For methods of examining insurance reserve adequacy, see Ruth E. Salzmann [43].

For instance, the 1988 operating ratios for the Fire (profitable) and Private Passenger Automobile Liability (unprofitable) lines of business are as follows:[34]

TABLE 3

INSURANCE OPERATING RATIOS

	Loss Ratio	Expense Ratio	Dividends	Investment Income	Operating Ratio
Fire	53.9%	37.8%	0.9%	4.9%	87.7%
Pers Auto Liab	93.2	22.7	0.8	9.7	107.1

Instead of operating ratios, we use profit margins. That is, an 87.7% operating ratio is a profit margin of 12.3%, and a 107.1% operating ratio is a profit margin of -7.1%. Profit margins, and the standard deviations of profit margins by line of business over the past 10 years, are shown in Table 4.

The Commercial Liability lines of business—Commercial Multiple Peril, Other Liability, Medical Malpractice, and Commercial Auto Liability—are highly risky: the standard deviations of their profit margins average 12.5. The Personal Property lines of business—Homeowners and Private Passenger Auto Physical Damage—are less risky: the standard deviations of their profit margins average 3.8.[35]

[34] Data from Best's *Aggregates and Averages*, Property-Casualty, 1989 Edition (Oldwick, NJ: A. M. Best Company, 1989), pages 96 and 98.

[35] Natural catastrophes, such as hurricanes and earthquakes, present the greatest risks in Homeowners insurance. During most years, Homeowners experience is favorable, but a major hurricane may cause enormous industry losses. U.S. catastrophe experience was mild in the late 1970s and in the 1980s, so operating ratios have been relatively stable. Hurricane Hugo and the California earthquake of 1989, which are not yet included in the data presented in the text, demonstrate the catastrophe potential in this line of business. Many climatologists believe that the experience of the early and mid-1980s has been exceptional, and we may expect more severe catastrophes in the future. If so, the Homeowners stability is deceptive. The risk may be hidden, but it is still there. See also footnote 37.

TABLE 4

PROFIT MARGINS AND THEIR STANDARD DEVIATIONS BY LINE OF BUSINESS (1979–1988)

	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988	Std. Dev. (79–88)
Fire	8.2	5.1	5.6	5.6	−0.6	−6.6	8.8	12.7	7.2	12.3	6.48
Allied	1.5	−1.4	7.8	3.2	−5.6	0.1	2.0	22.7	24.0	22.5	10.58
Farmowners	9.5	−9.1	−7.9	−16.3	−7.3	−14.8	−7.4	−4.3	8.6	4.9	8.66
Homeowners	4.0	−2.0	1.8	−0.1	0.5	−1.8	−6.6	1.6	8.0	4.8	3.87
CMP	11.4	6.6	−0.5	−9.1	−14.8	−25.1	−12.2	11.1	6.5	3.1	13.49
Ocean Marine	−1.1	−6.9	−1.6	−3.3	−2.4	−2.4	5.5	8.5	9.4	2.6	5.14
Inl. Marine	8.2	1.2	1.5	1.5	0.9	−1.8	7.4	21.8	22.1	18.0	8.76
Group A&H	4.1	2.5	0.6	1.9	1.8	7.4	2.3	−10.6	−6.8	−2.5	5.05
Other A&H	4.1	1.8	1.3	3.3	1.9	8.1	9.4	6.8	−0.1	1.9	3.03
Work Comp	6.3	9.3	10.2	11.1	3.7	−5.3	−3.8	−7.4	−4.8	−5.7	7.08
Other Liab	14.0	7.3	3.5	−6.4	−13.8	−25.1	−25.8	−2.5	3.7	8.5	13.22
Med Mal	8.0	0.2	−1.4	−9.8	−8.9	−18.3	−29.5	−8.7	8.1	16.0	12.84
Aircraft	−3.1	4.6	7.6	10.0	5.6	4.6	6.7	16.4	17.8	9.2	5.71
PPA Liab	5.2	3.9	−1.4	−2.1	−2.9	−3.9	−9.5	−9.2	−7.7	−7.1	4.86
CA Liab	2.4	−1.1	−8.4	−15.3	−21.3	−30.8	−16.4	−3.6	1.0	2.3	10.82
PPA Phy Dam	1.4	4.9	3.0	0.2	5.1	1.1	2.9	8.2	11.7	9.5	3.67
CA Phy Dam	8.2	5.9	2.1	−4.2	−4.0	−8.5	5.5	19.9	22.7	22.4	10.80
Fidelity	21.1	15.4	10.6	−3.2	−11.9	−6.2	16.1	32.0	36.1	35.9	16.53
Surety	13.2	−1.2	18.3	15.4	18.4	12.3	−2.2	−7.6	−0.6	17.2	9.51
Burglary	24.0	13.5	9.6	8.2	15.9	23.5	30.5	34.8	41.3	39.5	11.54
Boiler & M	15.6	13.1	11.2	3.5	−3.7	−10.6	24.1	11.8	18.1	4.8	9.88
Reinsurance	6.6	5.9	4.5	−1.3	−6.4	−22.2	−7.0	0.4	−2.6	10.0	8.81
Other Lines	−12.7	−1.5	−1.4	−16.3	2.9	6.7	15.7	−14.3	−8.8	−12.3	9.94
Total	5.9	4.1	2.4	−0.5	−2.3	−7.4	−6.3	0.9	4.4	4.3	4.39

These standard deviations reflect fluctuations in returns, not dispersion of the loss distribution. For instance, Ocean Marine has great random loss fluctuations on individual policies. But most Ocean Marine claims are small partial losses: the standard deviation of the profit margin is low (5.1), since there is not much uncertainty in the expected loss values.[36] Workers Compensation also has high variation in the size of loss distribution, since there is no limit on medical payments in the

[36] See Klaus Gerathewohl, et al. [28].

policy. But the statutory benefits and bureau rate making reduce the fluctuation in overall portfolio returns to a manageable level.[37]

Commercial Liability insurers must continually adjust their expected loss values as social conditions change. The proliferation of new causes of action hampers General Liability expected loss forecasts, while the increasing claims consciousness of the public frustrates Medical Malpractice loss forecasts. This is the risk which insurers face, and for which they need additional "risk loads."[38]

How stable are these results over time? Are the high standard deviations noted for the Commercial Liability lines characteristic of these types of risks or are they peculiar to the time period used?

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[37] David Appel, a research economist formerly with the National Council on Compensation Insurance, has pointed out to me an important difference in pricing strategies between Workers' Compensation and other Commercial lines of business. Dr. Appel's insights are correct, and they modify the conclusion in the text. Many carriers write "accounts," providing Commercial Automobile, General Liability, Commercial Property, and Workers Compensation coverages for the insured. During downturns of the underwriting cycle, insurers reduce their Commercial Auto and GL rates, or they provide large schedule modifications, to retain the business. Conversely, during upturns of the cycle, Commercial Auto and GL rates increase rapidly and schedule modifications diminish.

Workers Compensation rates, however, show less variation from year to year. Thus, the high variability in Commercial Automobile and General Liability profits may reflect on all the coverages marketed together, and does not necessarily indicate that these lines are more risky than Workers Compensation.

Similar business and competitive considerations apply to all the figures in this paper. Financial theory is abstract: it provides directions, but it does not offer decisions for concrete cases. The pricing actuary must temper the abstract theory with practical judgment to arrive at an equitable risk load for any line of business.

[38] Fluctuations in reported operating returns by line of business depend primarily on insurance risk, not on investment risk. The more stable investment returns, such as interest, dividends, rents, and realized capital gains are carried to the income statement. Unrealized capital gains and losses, which vary widely from year to year, are a direct charge to surplus. Thus, CMP, with a short average settlement lag but great insurance risk, has a high standard deviation and a high β in Table 5, as well as in the studies by Hammond and Shilling and by Cummins and Nye (see following footnote). Workers Compensation, with a long settlement lag but less insurance risk, has lower standard deviations and βs in this paper and in the previous studies.

An alternative possibility for fluctuating insurance returns, that they are caused more by stock value variations than by insurance risk, is considered by Yehuda Kahane [34].

Hammond and Shilling [29] analyzed the standard deviations of *underwriting profits* by line of business for 1956–1970.[39] Among the major lines of business, they found high standard deviations for Commercial Multiple Peril and General Liability BI, and low standard deviations for Workers Compensation, similar to the results of the present analysis. However, they found a somewhat higher standard deviation for the Personal Property lines than for automobile liability.[40]

Modern portfolio theory considers the historical variance of returns of a single segment of a portfolio an incomplete approximation for risk. Equally important is the covariance of returns among securities.[41] Unfortunately, estimating covariances among securities or lines of insurance is an arduous task.[42]

The Capital Asset Pricing Model provides an elegant means of determining the risk on an individual security, composed of both the variance of its own returns and the covariances with the returns on other securities.[43] Returns from each security are regressed against the returns of the total market portfolio, thereby quantifying price fluctuations that cannot be reduced by diversification.

A prudent investor diversifies his financial holdings. Variances of return that can be eliminated by diversification should receive no reward for the additional risk undertaken. Variances of return that are correlated with total market fluctuations, however, cannot be eliminated by diversification. The CAPM posits that this "risk" is rewarded by a higher expected return.

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[39] Investment income by line was not readily available in the 1970s, so Hammond and Shilling [29] used the complement of the combined ratio. Interest rates were relatively stable from 1956 through 1970, so the standard deviations of underwriting income and operating profits should be similar.

Cummins and Nye [17] examined the variability of returns by line of business for one insurance company from 1958 to 1975 and found the same results for the major lines of business as in this paper: high variability for CMP and General Liability, low variability for Auto Physical Damage and Fire (which in the 1960s accounted for most of Personal Property insurance), low variability for Workers Compensation, and low to moderate variability for Automobile Liability.

[40] This accords with the more rigorous estimation method discussed below; see Table 5.

[41] See Harry Markowitz [37].

[42] Ferrari [24], Brubaker [10], and Cooper [14] emphasize the importance of covariance among lines of business. The development of the Capital Asset Pricing Model has obviated the need for quantifying covariances, so there has been little subsequent work on Ferrari's or Brubaker's methods.

[43] See William F. Sharpe [46].

Formally, the following regression model quantifies the undiversifiable or systematic risk (β):

security's return = $a + \beta$ (market return).

The β is determined from historical returns. The current expected return, R, is

$$R = R_f + \beta(R_m - R_f),$$

where R_f is the risk free rate, such as the rate on Treasury bills, and R_m is the overall market return.[44]

Several writers have applied modern portfolio theory to the investment of stockholders in insurance firms.[45] The "risk" associated with insuring a given block of business is related to the covariance of return from that business with the diversified financial portfolios held by the investors in the insurance firm. These covariances, or the *underwriting betas* associated with writing a line of insurance, have generally been low and unstable.[46]

A stockholder chooses between investing his money in an insurance firm and investing it in other securities. The insurance firm itself does not have this option. Were it to invest part of its equity in securities, instead of using it to "support" insurance writings, it would subject its stockholders to double income taxation: the insurer pays taxes on its investment earnings and its stockholders pay taxes on dividends and

[44] This equation relies on the assumed availability of borrowing and lending at the risk free rate; see J. Fred Weston and Thomas E. Copeland [49] for a good summary. This assumption is unrealistic, but the agreement of the CAPM with empirical returns is the major justification of its use. Sharpe and Alexander [47] use a quotation from Milton Friedman [27] to clarify this issue: "The relevant question to ask about the 'assumptions' of a theory is not whether they are descriptively 'realistic,' for they never are, but whether they are sufficiently good approximations for the purpose at hand. And this question can be answered only by seeing whether the theory works, which means whether it yields sufficiently accurate predictions." On empirical testing of the CAPM, see D. W. Mullins, Jr. [39] and the references cited therein.

[45] See particularly William Fairley [22].

[46] Fairley [22], *op. cit.*, estimates an underwriting β of -0.21. Biger and Kahane [7], *op. cit.*, conclude that "preliminary empirical evidence presented shows that the 'systematic risk' of underwriting profits approaches zero in most lines." J. David Cummins and Scott Harrington [16], using quarterly accounting data, find a highly unstable underwriting β through the 1970s, averaging to -0.03 for 1970–1981 (or -0.01 for an annual data value).

capital gains. The insurer's stockholders would prefer to invest their monies directly in securities and pay income taxes only once.[47]

Thus, the traditional use of the Capital Asset Pricing Model for estimating underwriting β's quantifies the risk faced by the investor in insurance stocks, not the risk of the insurer. Commenting on CAPM based pricing models, D'Arcy and Doherty say, "Notice that nowhere is there a direct relationship between the competitive underwriting profit and risk. The riskiness of the insurance operations *per se* is not at issue. Much of the risk can be diversified by the insurance company's own equityholders in the management of their personal portfolios. Only that component of risk that is not so diversifiable, the systematic risk, is reflected in the competitive underwriting profit. Thus the competitive price is related only to the beta, which picks up this systematic risk."[48]

An insurer chooses lines of insurance (or blocks of business) to maximize its expected return while minimizing its "risk." The market return R_m in the CAPM model should be replaced by the return on a fully diversified insurance portfolio. The appropriate equation is

$$R = R_f + \beta(R_p - R_f),$$

where R_p is the return on the all lines combined insurance portfolio.[49]

Operating returns from Best's *Aggregates and Averages* (see Table 3) are used to determine the β's by line shown below. Note carefully: these do *not* reflect the risk to the investor in insurance stocks. Rather, they reflect the risk to the insurer of writing different lines of business.

The highest β's occur in the Commercial Liability lines of business: Commercial Multiple Peril, General Liability, Medical Malpractice, and Commercial Auto Liability.[50] In other words, when the insurance industry as a whole does well, these lines show excellent returns; when the

[47] Myers and Cohn [40] therefore argue that policyholders should compensate the insurer for federal income taxes on the investment income from surplus.

[48] Stephen P. D'Arcy and Neil A. Doherty [18], page 37.

[49] β may be calculated either by a least squares regression or as $COV(R,R_p) \div VAR(R_p)$; see Simon Benninga [6]. I am indebted to Gabriel Baracat for aid in estimating the risk loads by line of business.

[50] The high β for fidelity is due to the strong profits in this line during the most recent years. This is presumably a random event, due to the low premiums in this line and the U.S. economic prosperity, which reduces fidelity losses.

industry is less profitable, these lines fare even worse. The Personal Property lines of business—Homeowners and Private Passenger Auto Physical Damage—have low β's. These lines have smoother underwriting cycles than does the experience of all lines combined.

TABLE 5

β's BY LINE OF INSURANCE
(BASED ON 1979–1988 EXPERIENCE)

Line of Ins.	Beta	Line of Ins.	Beta	Line of Ins.	Beta
Fire	0.92	Work Comp	0.46	Fidelity	2.32
Allied Lines	1.04	General Liab	2.98	Surety	0.04
Farmowners	1.35	Med Mal	2.65	Burglary	0.33
Homeowners	0.65	Aircraft	0.07	Boiler & Mach	0.87
CMP	2.78	Pers Auto Liab	0.45	Reinsurance	1.74
Ocean Marine	0.04	Comm Auto Liab	2.27	Other Lines	−1.62
Inland Marine	0.88	PPA Phy Dam	0.37		
Group A&H	−0.46	CA Phy Dam	1.52	Total	1.00
Other A&H	−0.51				

Most accident and health insurance is sold by life companies. The profitability of these lines is unrelated to the Property/Casualty underwriting cycle; the historical correlation is negative. Much reinsurance is bought for Commercial Property and General Liability risks, and its profitability follows the returns of these primary lines.

Workers Compensation and Private Passenger Auto Liability have not been profitable lines in recent years, but their returns have been relatively stable. In Auto Liability, the large number of small risks smooths the fluctuations in insurance returns, though consumer complaints about high premium rates keep profits low. Administered rating and account pricing smooth the fluctuations in Workers Compensation returns. The divergence between state legislators, who mandate WC benefits (often in response to labor desires) and state regulators, who oversee rates (sometimes in response to employer needs), depresses profits.[51]

[51] See William Bailey [4].

To determine risk loads by line of business, one needs the risk free rate and the expected return for all lines combined. Actuaries and financial analysts regularly forecast returns for the Property/Liability insurance industry.[52] The risk free rate may be derived from returns on Treasury bills and bonds. Thus, if the risk free rate is 7% per annum, and the expected return for the industry as a whole is 14% per annum, the risk premium for Reinsurance is 12.2% per annum [$= 1.74 \cdot (14 - 7)$].

This is a return on equity; it must be converted into a return on premium for the rate making calculation. In other words, one needs appropriate premium-surplus ratios by line of business.

Inasmuch as surplus is needed to support *insurance* risk, represented by fluctuations in reported operating ratios, the loadings discussed here compensate the insurer for the risk it undertakes. The return on equity can be directly converted to a return on premiums, and a relationship such as the Kenney rule is appropriate.[53]

If surplus is also needed to support asset value fluctuations, such as unrealized capital gains and losses, which do not flow through to the income statement, then additional surplus is needed for long-tailed lines of business. The ratemaking loadings would be slightly different from those shown here. In particular, CMP would have a somewhat lower load and Workers Compensation would have a higher load.[54]

But is modern portfolio theory correct even for financial investments? Financial analysts note two problems with the theory: First, different β's result when different experience periods or different statistical methods are used. Second, the "security market line,"—the empirical relationship of returns afforded by stocks and their historical β's—is less steeply sloped than the theoretical Capital Asset Pricing Model line predicts

[52] Stock analysts estimate the average CAPM β for property/casualty insurers to be approximately unity.

[53] The β's determined here are based on operating ratios, which relate profits to premiums. The magnitude of the profit fluctuations is viewed relative to annual premium, not to loss reserves. Surplus allocation should vary with premium if these βs are used. If one allocates surplus relative to loss reserves, one should relate profit fluctuations to reserves. If so, the βs for the Commercial Liability lines of business would be lower, since their reserves are larger.

[54] On the functions of supporting surplus, see Alfred E. Hofflander [32]. See also the National Association of Insurance Commissioners [41], page 8: "In addition to providing protection against unusually large losses, surplus also provides a cushion against declines in the value of equity investments, such as common and preferred stocks."

(though the difference is not great). That is, the actual returns demanded by investors increase somewhat less rapidly with the increase in β than is predicted by modern portfolio theory.

The actuary must be aware of these problems. Refinements of modern portfolio theory may lead to improvements in estimating risk loads by line of business. But this method at least quantifies the true risk faced by insurers, not some substitute that has no relationship to the insurer's risk.

How might this analysis be improved? First, cash flow discounting should be used instead of spreading investment income to line of business. Different growth rates by line of business cause the Insurance Expense Exhibit allocation of investment income by line to distort the true expected present values of insurance operations. Divergences between embedded yields and expected new money rates are also a problem. Second, if an insurer's writings are large enough, the historical returns on its own book of business should be used instead of industry totals, since one insurer's book may have different characteristics from that of another insurer. Third, quarterly returns by line of business should be examined over different time periods. Reinsurance had stable and favorable returns for 1979–1981, but highly variable profits in subsequent years. Quarterly returns for the most recent seven years may better reflect the risks in this line of business. Fourth, the expected return for the industry as a whole should be estimated by various methods and by type of insurer. For instance, the expected returns in Personal Automobile Liability insurance differ between agency companies and direct writers.

These are refinements, additional bells and whistles. Even without these enhancements, this method is superior to current "risk load" estimation procedures. It quantifies the true risk faced by insurers, not the "process risk" faced by insureds. It rests on the relationship found in financial investments of greater expected returns for portfolios with greater variance. It relies on the empirical risk aversion demonstrated by institutional investors, which is presumably similar to the risk aversion characteristic of insurers. In sum, it provides insurers with a measure of the true cost of insuring "risky" lines of business.

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REINSURER RISK LOADS FROM MARGINAL SURPLUS REQUIREMENTS

RODNEY KREPS

Abstract

The return on the marginal surplus committed to support the variability of a proposed reinsurance contract is used to derive an appropriate risk load for reinsurers. The risk load is a linear combination of the standard deviation and variance of the return on the contract, and depends upon the covariance of the contract with the existing book, the standard deviation of the contract, the standard deviation of the existing book, the acceptable probability of "ruin" of the company, and the yield required on marginal surplus (the additional surplus required for this contract). A new term is defined, the reluctance to write risk, and relatively simple formulas result for it and the premium, which satisfy intuitive reasonableness criteria. Extensions to include expenses and an existing "bank" are discussed, and application is made to the interesting case of excess layer pricing. Empirical comparison suggests that the market pricing is consistent with this approach.

1. MICROECONOMICS

The underlying economic point of view taken is that of a reinsurer considering a new contract. The reinsurer has committed surplus to support the variability of his existing book; the new contract will require additional surplus to support its variability.[1] The return on this marginal surplus required must be at least as much as is available in the capital markets; otherwise the reinsurer might just as well invest directly. It is assumed, with Brubaker [1], that the company expresses the part of its surplus required to support the variability of a book of business with

[1] The remarks here apply equally well to insurance contracts, but the ratemaking procedures for primary insurers typically do not allow explicit risk loads. An implicit load is present from whatever provisions are present for profit, which is economically the reward for bearing risk.

expected return R and standard deviation S as[2]

$$V = zS - R, \tag{1.1}$$

where z is a distribution percentage point corresponding to the acceptable probability that the actual result will require even more surplus than allocated.[3] For example, if the distribution is Normal, then a z of 3.1 is a 1/1000 probability, and an amount of surplus given as above will cover the actual losses 999 years out of 1000 years, on average. The choice of the appropriate value of z is an upper management decision, reflecting the overall conservatism of the company and explicit and implicit regulatory requirements.

Consider a potential new contract with an expected return (premium less losses and expenses) r and standard deviation σ, and indicate the resulting new book values with a prime ($'$). The new values are given by

$$R' = R + r, \tag{1.2}$$

and

$$V' = zS' - R'. \tag{1.3}$$

It is assumed that the nature of the total book distribution has not changed significantly, so that the same value of z is appropriate. The marginal surplus required by the contract is then given as

$$V' - V = z(S' - S) - r. \tag{1.4}$$

Now, the return from the contract and the amount of the marginal surplus required to support the contract imply a yield rate y on this surplus. The value of y must be (at least) equal to the rate in the capital markets, otherwise management might as well simply invest this surplus.[4] Setting

[2] We take all values as present values. A desirable property possessed by this form of surplus allocation is that it is invariant with respect to change in currency value.

[3] This is very similar in spirit and calculation to the "stability constraint [2]." The total surplus need of a company will consist of this contribution, plus that needed to support expenses and equity in any unearned premium reserve for new writings, plus any other contributions required by regulators and/or management.

[4] There are reasons, such as a desire to maintain market presence, which could allow y to fall below the capital market rate temporarily.

the yield rate equal to the management target gives the required return on the contract[5]

$$r = y(V' - V), \tag{1.5}$$

which leads to

$$r = [yz/(1 + y)](S' - S). \tag{1.6}$$

Denoting by C the correlation of the contract with the existing book,

$$(S')^2 = S^2 + \sigma^2 + 2\sigma SC. \tag{1.7}$$

The value of C will be between -1 and $+1$, and

$$S' - S = \sigma(2SC + \sigma)/(S' + S). \tag{1.8}$$

Finally, combining the above and taking σ as the measure of risk, say that r, the risk load, is equal to reluctance times risk:

$$r = \mathcal{R}\sigma, \tag{1.9}$$

where $\mathcal{R}$, the reinsurer's reluctance to take on risk, is defined by

$$\mathcal{R} = [yz/(1 + y)](2SC + \sigma)/(S' + S). \tag{1.10}$$

2. INSURANCE

If the expected mean losses on the contract are μ and the expenses are E, then the appropriate premium P is given by

$$P = \mu + \mathcal{R}\sigma + E. \tag{2.1}$$

In the overwhelmingly typical case, $\sigma \propto \Sigma$, the reluctance has an excellent approximation as

$$\mathcal{R} = [yz/(1 + y)](C + \sigma/2S). \tag{2.2}$$

[5] This approach is actually an extension of the discussion on page 453 of Patrik and John [3]. We adopt this for its simplicity, while acknowledging that there are interesting questions with respect to the surplus flow needed to support the expected return of the book and of the contract, and the consequent internal rate of return.

These two equations form the heart of the paper, and both should and do make sense intuitively. In a competitive market, the yield rate y required will decrease, and so will the reluctance. A more conservative company will have a higher value of z, and hence a higher reluctance. A reinsurer whose book is regional will have a larger reluctance to take on a contract from a national carrier than from a carrier from a different region, and a still higher reluctance for a carrier in his region, because of the increasing values of the covariance.

In the very pessimistic case where $C = 1$, the exact form for $\mathcal{R}$ becomes

$$\mathcal{R} = [yz/(1 + y)], \tag{2.3}$$

which depends only on factors external to the contract. The premium still depends, of course, on μ and σ. Back in the general case, if there is a "bank" B built up,[6] then the marginal surplus required is reduced by B, and the premium becomes

$$P = \mu + \mathcal{R}\sigma + E - yB/(1 + y). \tag{2.4}$$

3. EXCESS LAYERING APPLICATION

In the case of high excess layers, generally speaking the mean loss μ will be a small part of the premium, and the contribution from the risk load will be the most significant. This is intuitive and also mathematically demonstrable.

The layer payout function $P(x;A,L)$ for loss in the layer with attachment point A and limit L from an unlayered loss of x is defined by, as usual,

$$P(x;A,L) = \begin{cases} 0, & x \leq A \\ (x - A), & a \leq x \leq (A + L) \\ \mathrm{L}, & (A + L) \leq x. \end{cases} \tag{3.1}$$

[6] That is, on a long-term treaty the premiums have exceeded the losses enough for some years that the reinsurer feels that the reassured has some measure of moral, if not legal, equity. Conversely, the losses may have exceeded the premiums enough that the reinsurer wants to add to the premium "to be made whole," which corresponds here to a negative B.

Denote by $E\{\text{any function of } x\}$ the expected value of that function over the distribution. That is, if $f(x)$ is the probability density function defined on the interval $(0,\infty)$ and $h(x)$ is any function of x, then

$$E\{h\} = \int_0^{\infty} h(x)f(x)dx. \tag{3.2}$$

Of particular interest are $E\{P\}$ and $E\{P^2\}$. For convenience define $G(x)$ as the probability that a loss is greater than x. That is,

$$G(x) = \int_x^{\infty} f(x)dx. \tag{3.3}$$

Then, a direct substitution of P in the expectation formula and an integration by parts yields the mean $\mu = E\{P\}$ as

$$\mu = \int_0^{L} G(A + x)dx; \tag{3.4}$$

and, similarly,

$$E\{P^2\} = \int_0^{L} 2xG(A + x)dx. \tag{3.5}$$

By definition,

$$\sigma^2 = E\{P^2\} - \mu^2. \tag{3.6}$$

Now keep L fixed and increase A; that is, examine higher and higher layers. Since G goes to zero as its argument becomes large, both μ and σ do also. In the cases of much practical interest (e.g., varieties of Pareto and Burr) where $G(x)$ has a power law behavior for large x,

$$G(x) \sim g/x^{\alpha}. \tag{3.7}$$

The integrals to lowest order in L/A may be approximated as

$$\mu \sim gL/A^{\alpha} \text{ and } E\{P^2\} \sim gL^2/A^{\alpha}. \tag{3.8}$$

Since G is essentially constant across the layer, it should come as no surprise that the result is that of a binomial distribution (with "success" probability μ/L) and that

$$\sigma^2 = \mu(L - \mu). \tag{3.9}$$

Thus, μ goes to zero faster than σ (which goes as $\sqrt{\mu}$); and the risk load dominates the expected loss, as intuition would suggest.

4. A USEFUL LEMMA

Further, one often has many layers stacked to create a program of protection; and a reinsurer may want to be on, for example, the first, fourth, and seventh layers. Clearly, there are correlations between layers, since to reach the fourth layer the loss must have exceeded the limit of the first. Fortunately, it is not necessary to do simulations or calculations for all possible combinations of layers. If one knows L, μ, and σ for each layer, then the appropriate mean and standard deviation can be calculated easily for any contract.

Suppose there is a set of layers P_i, where i runs over the set of values 1 to N; these layers do not overlap; and they are in increasing sequence ($A_i + L_i \leq A_j$ for $i < j$). This is the usual case. The layers need not actually be contiguous, although they generally are. Now, using Σ_i to mean summation over values of the layer index appropriate to the contemplated contract (one, four, and seven in the example above),

$$\mu = E\left\{\sum_i P_i\right\} = \sum_i E\{P_i\} = \sum_i \mu_i. \tag{4.1}$$

This is the obvious, but useful, result; and

$$E\left\{\left(\sum_i P_i\right)^2\right\} = E\left\{\sum_i (P_i)^2\right\} + 2\sum_{i<j} E\{P_iP_j\}, \tag{4.2}$$

where the $\Sigma_{i<j}$ means summation is restricted to values of i and j such that $i < j$. The essential point is that when $i < j$, for values of x where P_j is non-zero, P_i is constant at L_i. Thus,

$$E\{P_iP_j\} = L_iE\{P_j\} = L_i\mu_j \tag{4.3}$$

and

$$E\left\{\left(\sum_i P_i\right)^2\right\} = \sum_i [(\sigma_i)^2 + (\mu_i)^2] + 2\sum_{i<j} L_i\mu_j. \tag{4.4}$$

Hence

$$\sigma^2 = \sum_i (\sigma_i)^2 + 2\sum_{i<j} (L_i - \mu_i)\mu_j. \tag{4.5}$$

The second term represents the covariance between the layers. This formula is a great convenience in actual simulation modeling, since it means one only has to do the layers separately, and then any combination of layers may be easily derived.

5. PRACTICAL CONSIDERATIONS

Where does one obtain μ and σ for the layers? Typically, from doing simulation modeling on the underlying data. There, one has to make explicit the assumptions on trend, development, exposure, curve family, and so on. However, once done, the statistics are obtained fairly easily in these days of powerful personal computers. In principle, σ should contain the uncertainty from the underlying assumptions (parameter variability) as well as the process variance from the distributions.

Expenses of the reinsurer can be modeled as a flat piece, for handling the contract per se, plus a piece proportional to the number of losses, representing the loss handling cost. The expected number of losses is also available from the simulation runs.

One would surmise that the market pricing would be relatively efficient, in the sense of producing rates appropriate to the risk. Reinsurers have, after all, been in the business a long time. Of course, in the golden years of the past, the expectation was that relationships would be long-term, and that rates each year would be adjusted for past results so that in the not too long run reinsurers would make a profit.[7] In such circumstances, precise pricing was not as necessary, nor was competition perhaps as fierce as in today's environment.

Where does "rate on line" pricing fit in? For those not in the reinsurance field, this is the inverse of "payback period:" the number of years that premium would have to be collected to equal one total loss. For example, for a limit of one million dollars, a 10% rate on line gives

[7] Hence the notion of a "bank." See the preceding footnote.

a premium of $100,000, or equivalently a payback period of ten years. Underwriters seem to have definite notions of what a maximum payback period should be, more or less independent of the nature of the cover. Assuming a continuing relationship, what seems to underlie this kind of thinking is the notion that every individual program should make a profit in a time frame during which the reassured is likely to remain solvent. In the present context, this translates into an additional contribution to σ coming from credit risk. This contribution would not go to zero as the layer gets higher.

Returning to the reluctance, it is expected to be relatively constant across layers as long as $\sigma/\Sigma \propto C$, for example when a reinsurer is considering a piece of a layer of a large multi-line primary. Further, as remarked earlier, to the extent that the covariance is large, we would expect the reluctance to be a product of only the reinsurance market conditions and the reinsurer's conservatism measure.

In actual practical use of this work, for any given reinsurance program reluctance has been taken as constant across layers, and σ reflects only the process variability. On a relatively small sample, the reluctance has values varying typically from 30% to 70% or more. Note that with a z of 3.1 and a pessimistic $C = 1$, a 12% return is a reluctance of 33%, and a 20% return is a reluctance of 52%, so this type of range might have been expected.

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ON THE REPRESENTATION OF LOSS AND INDEMNITY DISTRIBUTIONS

YOONG-SIN LEE

Abstract

In this paper many relations and equations relating pure premium or expected value quantities are presented in terms of random variables. This is made possible by the use of the indicator function so that awkward representations of functions of loss are simplified. Relations and formulas on such topics as basic limits losses, excess of loss coverages and retrospective rating are presented in stronger, more primitive forms. The related mathematics is often simplified and, in particular, an effective technique for handling trend is presented.

1. INTRODUCTION

The loss distribution is an essential component in actuarial work but because of the various limitations of payment of loss in an insurance contract, the indemnity is not always identical to the loss. Hence the indemnity often has a rather complicated representation in terms of the original loss. Actuarial formulas and expressions become less tractable and more difficult to understand. For this reason the treatment of basic limits losses, excess of loss coverages and retrospective rating, for example, are replete with complicated mathematical relations, and the formulas and equations presented in the literature provide little insight into their meanings. This paper uses the indicator function to give the indemnity a single representation as a random variable. Many of the mathematical relations connecting expected value pure premiums are now turned into more primitive, stronger relations between random variables. The algebra in manipulating the mathematical relations is often reduced, and the relations themselves become more transparent when viewed this way. Mathematical notations are known to have revolutionized mathematics and science in the long history of these disciplines;

witness the invention of zero, the use of Arabic numerals in place of the Roman numerals, and the introduction of vectors and matrices in modern mathematics. While it is not pretended that the use of the indicator function will have such portentous effects in actuarial science, it does simplify actuarial mathematics, add new insight in many areas, and lead more easily to some new results.

2. DEFINITIONS

It will clarify matters if we distinguish between the original loss incurred by the insured and the indemnity paid by the insurer. Let us represent the original loss by the random variable X, which follows the loss distribution. We also assume that the indemnity depends solely on the loss so that, being a function of the random variable X, it is itself a random variable with distribution called the indemnity distribution. The indemnity relates to the loss X in many ways, depending on the nature of the insurance contract. Typically, the indemnity as a function of the loss assumes different functional forms over different ranges of the size of loss. This contributes to the unwieldiness in the mathematics of the indemnity distribution. For example, if the original loss is X and the indemnity is the basic limits loss with limit k, then the indemnity $g(X;k)$ may be described as

$$g(X;k) = \begin{cases} X & 0 < X \leq k \\ k & X > k. \end{cases}$$

Note that $g(X;k)$ is a random variable, but is represented above and elsewhere in the *Proceedings* by two separate expressions. This representation has obstructed the view of the users of this random variable, making it awkward to work with. It can be represented in a single expression using the indicator function:

$$g(X;k) = XI_{(0,k]}(X) + kI_{(k,\infty)}(X),$$

where $I_S(x)$ is an indicator function defined as follows:

$$I_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

with S representing a set of possible values of x. The inclusion of $X = k$

in the lower line segment instead of in the upper agrees with the convention in defining the distribution function $F(x) = \text{Prob}\ (X \leq x)$. Although in the above definition of $I(.)$ the argument is a numerical variable, the definition extends easily to the case of a random variable argument in the usual way. When expressed in this form many expressions involving the random variable $g(X;k)$ can be manipulated more easily. The following simple properties of the indicator function contribute much to its power:

$$I_{S_1}(x)\, I_{S_2}(x) = I_{S_1 \cap S_2}(x),$$

$$I_{S_1}(x) + I_{S_2}(x) = I_{S_1 \cup S_2}(x) \quad \text{if } S_1 \cap S_2 = \emptyset.$$

It can be easily deduced that

$$I_{S_1}(x) - I_{S_2}(x) = I_{S_1 \cap \bar{S}_2}(x) \quad \text{if } S_2 \subseteq S_1$$

and

$$I_{S_1}(x)\, I_{S_2}(x) = 0 \quad \text{if } S_1 \cap S_2 = \emptyset.$$

The prescribed statistical text, Mood, Graybill and Boes [4], uses the indicator function quite freely. However, examples in casualty actuarial science where the indicator function is applicable are much more interesting and richer, and its use could also be more sophisticated. LaRose [2] presents the following notations for expected values of certain functions of the loss. In his notation

$$X1(k) = \frac{1}{\alpha} \int_0^k t\,\mathrm{dF}(t),$$

$$\text{where} \quad \alpha = \int_0^\infty t\,\mathrm{dF}(t) \quad \text{is the mean loss,}$$

$$X2(k) = \frac{1}{\alpha} \int_0^k t\,\mathrm{dF}(t) + \frac{k}{\alpha} \int_k^\infty \mathrm{dF}(t),$$

and

$$X3(k) = \frac{1}{\alpha} \int_k^\infty (t - k)\,\mathrm{dF}(t).$$

He shows that many actuarial functions expressed in a variety of forms can be represented in terms of these three quantities. It is clear that $X1(k)$, $X2(k)$ and $X3(k)$ are respectively the expected values of the following functions of loss, measured in units of the mean loss:

$$XI_{(0,k]}(X)$$

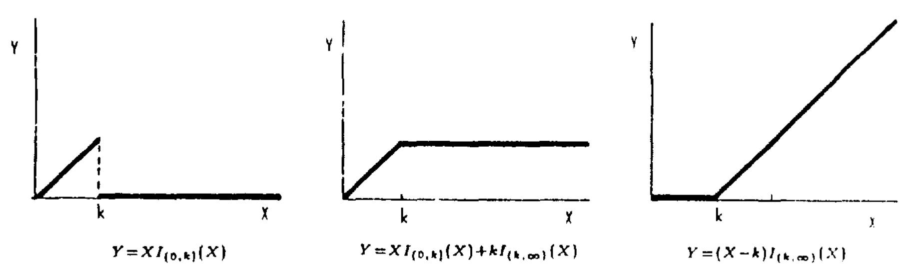
$$XI_{(0,k]}(X) + kI_{(k,\infty)}(X)$$

and

$$(X - k)I_{(k,\infty)}(X).$$

As functions of X, they are shown graphically in Figure 1. These quantities are more closely related to the loss and are more easily understood in their random variable forms. Most of the relations treated by LaRose [2] can be generalized to random variable versions; some of them are presented in the rest of this paper. A glance through Figures 1–4 shows visually the underlying similarity of many quantities derived from the loss function. Some quantities known by different names are the same function of the loss, and some bear simple relationships to others.

FIGURE 1
SOME FUNCTIONS OF THE LOSS



3. EXCESS OF LOSS COVERAGE

An excess of loss coverage pays the amount of loss in excess of r for losses exceeding r but not greater than s, and the amount $j = s - r$ for losses exceeding s:

$$h(X;r,j) = \begin{cases} X - r & \text{if } r < X \leq s, \\ j & \text{if } s < X. \end{cases}$$

In terms of the indicator function

$$h(X;r,j) = (X - r)I_{(r,\,s]}(X) + jI_{(s,\,\infty)}(X).$$

Miccolis [3] shows that

$$h(X;r,j) = g(X;s) - g(X;r),$$

where $r + j = s$. As an example of algebraic manipulation with the indicator function representation, we derive this result as follows:

$$\begin{aligned} g(X;s) - g(X;r) &= XI_{(0,\,s]}(X) + sI_{(s,\,\infty)}(X) - \{XI_{(0,\,r]}(X) + rI_{(r,\,\infty)}(X)\} \\ &= X\{I_{(0,\,s)]}(X) - I_{(0,\,r]}(X)\} + sI_{(s,\,\infty)}(X) \\ &\quad - rI_{(r,\,s]}(X) - rI_{(s,\,\infty)}(X) \\ &= XI_{(r,\,s]}(X) - rI_{(r,\,s]}(X) + sI_{(s,\,\infty)}(X) - rI_{(s,\,\infty)}(X) \\ &= (X - r)I_{(r,\,s]}(X) + (s - r)I_{(s,\,\infty)}(X). \end{aligned}$$

Hence

$$g(X;s) - g\,(X;r) = h\,(X;r,j).$$

(See equations 10–11 of Miccolis [3]). Miccolis derives a result on the expectation of $h^2(X;r,j)$ (see his equation 13), which can also be conveniently derived as follows:

$$\begin{aligned} g^2(X;s) &= \{g(X;r) + h(X;r,j)\}^2 \\ &= g^2(X;r) + h^2(X;r,j) + 2g(X;r)h(X;r,j). \end{aligned}$$

But

$$
\begin{aligned}
& g(X;r)h(X;r,j) \\
& = \{XI_{(0,r]}(X) + rI_{(r,\infty)}(X)\}\{(X - r)I_{(r,s]}(X) + (s - r)I_{(s,\infty)}(X)\} \\
& = r\{(X - r)I_{(r,s]}(X) + (s - r)I_{(s,\infty)}(X)\} \\
& = rh(X;r,j).
\end{aligned}
$$

(This result, in terms of random variables, is also given by Miccolis [3] in his equation 39.) We have the random variable version of his equation 13:

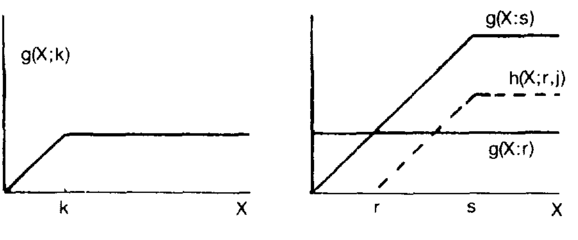
$$h^2(X;r,j) = g^2(X;s) - g^2(X;r) - 2rh(X;r,j).$$

In statistics the calculus of expectations is made easier by manipulating the random variables or their functions, rather than dealing with integrals directly. For example,

$$
\begin{aligned}
& \operatorname{Var}[XI_{(0,k]}(X) + kI_{(k,\infty)}(X)] \\
& = \operatorname{Var}[XI_{(0,k]}(X)] + \operatorname{Var}[kI_{(k,\infty)}(X)] \\
& \quad + 2\operatorname{Cov}[XI_{(0,k]}(X), kI_{(k,\infty)}(X)] \\
& = \operatorname{Var}[XI_{(0,k]}(X)] + \operatorname{Var}[kI_{(k,\infty)}(X)] \\
& \quad - 2k\mathrm{E}\{(XI_{(0,k]}(X)\}\,\mathrm{E}\{I_{(k,\infty)}(X)]\},
\end{aligned}
$$

since in the covariance the cross-product is zero. It is easier to perform manipulations such as this than to work with integrals. Figure 2 shows some relations between these quantities, which are treated as functions of the loss X.

FIGURE 2
BASIC LIMITS AND EXCESS OF LOSS COVERAGES



An Example on Moments

As an example using the expectation operator, consider first the h^{th} moment of $g(X;k)$. We have

$$\mathrm{E}\,\{g^h(X;k)\} = \mathrm{E}\,\{[XI_{(0,k]}(X) + kI_{(k,\infty)}(X)]^h\}.$$

Expanding the power on the left hand side we see easily that all the cross terms are zero and so the h^{th} moment of $g(X;k)$ is

$$\mathrm{E}\,\{g^h(X;k)\} = \mathrm{E}\,\{X^h I_{(0,k]}(X)\} + k^h[1 - F(k)].$$

Now consider the h^{th} central moment of $g(X;k)$. We have the general result

$$\mu_h[g(X;k)] = \sum_{j=0}^{h-2} \binom{h}{j} (-1)^j \mu'_{h-j}\mu^j - (-1)^h(h-1)\mu^h,$$

where μ'_j is the h^{th} moment of $g(X;k)$ and $\mu = \mu'_1$ is its mean. We may then make use of our result for the moments of $g(X;k)$ to obtain the h^{th} central moment of $g(X;k)$

$$\mu_h[g(X;k)] = \sum_{j=0}^{h-2} \binom{h}{j} (-1)^j[\alpha_{h-j} + \beta_{h-j}]\mu^j - (-1)^h(h-1)\mu^h,$$

where

$$\alpha_j = \mathrm{E}\,\{[X^j I_{(0,k]}(X)]\},\ \beta_j = k^j[1 - F(k)] \text{ and}$$

$$\mu = \mathrm{E}\,\{XI_{(0,k]}(X)\} + k\,[1 - F(k)].$$

4. TREND

First we derive a result which will considerably simplify and clarify the treatment of trend effect. Let y be a monotone function of x. To fix the idea, we assume the function to be increasing in x:

$$y = \alpha(x),$$

so that the transformation is invertible:

$$x = \alpha^{-1}(y).$$

Then

$$I_{(a,b]}(y) = I_{(a,b]}(\alpha(x))$$
$$= I_{(\alpha^{-1}(a),\,\alpha^{-1}(b)]}(x) \qquad (1)$$

because the following statements are equivalent:

$I_{(a,b]}(y) = 1,$

$y \in (a,b],$

$\alpha^{-1}(y) \in (\alpha^{-1}(a),\, \alpha^{-1}(b)],$

$x \in (\alpha^{-1}(a),\, \alpha^{-1}(b)],$

$I_{(\alpha^{-1}(a),\,\alpha^{-1}(b)]}(x) = 1;$

and similarly whenever one of the indicator functions assumes the value 0, the other does also. The result is also true for a decreasing function, in which case the terminal points of the intervals are reversed.

Consider a loss X being subject to inflation. Suppose that at a future time point the loss becomes

$$Y = \alpha(X)$$

with $\alpha(.)$ increasing. Then the basic limits loss becomes

$$g(\alpha(X);k) = \alpha(X)I_{(0,k]}(\alpha(X)) + kI_{(k,\infty)}(\alpha(X)).$$

Using (1) we have

$$g(\alpha(X);k) = \alpha(X)I_{(0,\alpha^{-1}(k)]}(X) + kI_{(\alpha^{-1}(k),\infty)}(X) \qquad (2)$$

This can be rewritten in the form of a rescaled g function:

$$g(a(X);k) = \frac{k}{\alpha^{-1}(k)}\left\{\frac{\alpha^{-1}(k)\alpha(X)}{k} I_{(0,\alpha^{-1}(k)]}(X) + \alpha^{-1}(k)I_{(\alpha^{-1}(k),\infty)}(X)\right\}$$
$$= \{k/a^{-1}(k)\}\, g\,\{\alpha^{-1}(k)\alpha(X)/k;\alpha^{-1}(k)\}.$$

In this representation we have a rescaled g function of the form $g(w(X);b)$ which takes the value $w(X) = \alpha^{-1}(k)\alpha(X)/k$ over the interval $(o,b]$ and the value $b = \alpha^{-1}(k)$ over the interval (b,∞), with $w(b) = b$.

It is easily verified that when X takes the value $\alpha^{-1}(k)$, the first argument of the g function above also takes the value $\alpha^{-1}(k)$. The manipulation of random variables in this manner is a more general method of treating the trend effect and could be of practical use if the assumption of uniform inflation rate over the range of the size of loss is too serious a deviation from reality. Similarly, the effect of inflation on an excess of loss coverage can be represented as

$$h(\alpha(X);r,j) = [(\alpha(X) - r]\, I_{(\alpha^{-1}(r),\alpha^{-1}(s)]}(X) + (s - r)I_{(\alpha^{-1}(s),\infty)}(X);$$

or alternatively described in the form of a rescaled h function:

$$\begin{aligned}
h(\alpha(X);r,j) &= c'h(X';r',j') \\
&= \frac{s - r}{\alpha^{-1}(s) - \alpha^{-1}(r)} \left\{ \frac{\alpha^{-1}(s) - \alpha^{-1}(r)}{s - r} [(\alpha(X) - r]I_{(\alpha^{-1}(r),\alpha^{-1}(s)]}(X) \right. \\
&\quad \left. + [\alpha^{-1}(s) - a^{-1}(r)]I_{(\alpha^{-1}(s),\infty)}(X) \right\} \\
&= c'\{(X' - r')I_{(r',s']}(X') + (s' - r')I_{(s',\infty)}(X')\},
\end{aligned}$$

where

$$c' = \frac{s - r}{\alpha^{-1}(s) - \alpha^{-1}(r)},$$

$$X' = \alpha(X)/c',$$

$$r' = r/c', \quad s' = s/c',$$

$$j' = s' - r' = (s - r)/c'$$

In the rescaled form the function $h(X';r',j')$, where X' is a function of X, takes the value $X' - r'$ in the interval $(X' = r', X' = s']$ and the value j' in the interval $(X' = s', \infty)$, with $X' - r' = 0$ when $X' = r'$ and $X' - r' = j'$ when $X' = s'$. When $\alpha(.)$ is the identity function, the original definition of h is recovered. The quantity of interest is the expectation of the h function. In general the functional form would not be easy to obtain, but the numerical computation should not be much more difficult than the computation of the untransformed h function.

If the trend function $\alpha(.)$ is not monotone, then it can be broken up into pieces each of which forms a monotone function. The above analysis can then be carried out piecemeal. It is more usual to assume that the inflation rate is uniform over the loss size:

$$\alpha(X) = aX.$$

Venter [8] gives an extensive treatment for this case. It can be shown by straightforward substitution in the rescaled versions of the formulas above that

$$g(aX;k) = ag(X;k/a),$$

and

$$h(aX;r,j) = ah(X;r/a,j/a).$$

Formulas involving trend could cause difficulties because of the lack of suitable tools for handling the transformed loss payment resulting from inflation. Bickerstaff [1] describes a model for automobile physical damage loss in which there is a deductible D representable by the random variable

$$XI_{(0,D]}(X) + DI_{(D,\infty)}(X)$$

and an upper limitation of loss payment L such that the reduction in loss payment can be represented by the random variable

$$(X - L)I_{(L,\infty)}(X) = XI_{(L,\infty)}(X) - LI_{(L,\infty)}(X).$$

The total reduction due to deductible and limitation in year 1 is then

$$XI_{(0,D]}(X) + DI_{(D,\infty)}(X) + XI_{(L,\infty)}(X) - LI_{(L,\infty)}(X),$$

with $D < L$. The loss payment limitation L is subject to a discount at an annual rate of $1 - d$ and the loss incurred is subject to inflation at an annual rate of r, simultaneously, so that in the n^{th} year X becomes $X(1 + r)^{n-1}$ and L becomes Ld^{n-1}. Consequently the total reduction becomes

$$(1 + r)^{n-1}XI_{(0,D]}((1 + r)^{n-1}X) + DI_{(D,\infty)}((1 + r)^{n-1}X)$$
$$+ (1 + r)^{n-1}XI_{(Ld^{n-1},\infty)}((1 + r)^{n-1}X) - Ld^{n-1}I_{(Ld^{n-1},\infty)}((1 + r)^{n-1}X)$$

which is obtained by simply applying the appropriate factors to X and D. Using equation (1), we can reduce this to

$$
\begin{aligned}
&(1 + r)^{n-1} X I_{(0,\, D(1+r)^{1-n}]}(X) + D I_{(D(1+r)^{1-n},\, \infty)}(X) \\
&+ (1 + r)^{n-1} X I_{(Ld^{n-1}(1+r)^{1-n},\, \infty)}(X) - Ld^{n-1} I_{(Ld^{n-1}(1+r)^{1-n},\, \infty)}(X)
\end{aligned}
$$

Thus with the results developed earlier in this section the effect of trend can be handled in a fairly formal way. Taking expectations leads to the pure premium version of Bickerstaff's [1] formula. The original formula given by Bickerstaff is incorrect; Philbrick [5] corrects the error.

A Numerical Example

We now give an example in the calculation of trend when the inflation rate varies with the amount of loss. There does not seem to be any theory on the specific functional form for rates of inflation which vary over the range of the amount of loss. In any case such rates cannot be precisely determined in practice. One way is to break up the range of loss value into sub-intervals and for each of the sub-intervals to approximate the rate of inflation by a linear function. This would often lead to mathematically tractable solutions and is quite satisfactory for handling practical problems.

We assume that the inflation rate i of the loss X increases with the value of X. Specifically, we assume that i starts at 10%, increases linearly until it reaches 20% at $X = 20{,}000$, and thereafter remains at 20%. Thus $i = i\,(X)$ can be described by the formula

$$
i(X) = (0.10 + 0.10X/20000) I_{(0,\, 20000]}(X) + 0.20 I_{(20000,\, \infty)}(X).
$$

The loss after inflation is then described by the formula

$$
\begin{aligned}
\alpha(X) &= [1 + i(X)]X \\
&= (1.10X + 0.10X^2/20000) I_{(0,\, 20000]}(X) \\
&\quad + 1.20X I_{(20000,\, \infty)}(X). \qquad (3)
\end{aligned}
$$

The overall effect of inflation over the full range of loss can be described by the average rate of inflation $E\{\alpha(X)\}/E\{X\} - 1$, where

$$E\{\alpha(X)\} = E\{(1.10X + 0.10X^2/20000)I_{(0,\,20000]}(X)\} + 1.20E\{XI_{(20000,\,\infty)}(X)\}. \tag{4}$$

In this example we have for the purpose of illustration divided the range of loss into only two sub-intervals; there is actually no difficulty in dividing the range into any finite number of sub-intervals. In this approach the main task in the calculation of the effect of inflation is the evaluation of the incomplete first and second moments of the distribution of loss, as is clear from the preceding formula. In fact, formula (4) can be rewritten as follows:

$$E\{\alpha(X)\} = 1.10E_{(s)}\{X\} + (0.10/s)E_{(s)}\{X^2\} + 1.20[E\{X\} - E_{(s)}\{X\}]$$

where we have written s for 20,000 and where

$$E_{(s)}\{X^k\} = \int_0^s x^k dF(x)$$

is the incomplete k^{th} moment of X up to s. For most distributions commonly used to model loss data, explicit formulas for such moments are available.

Suppose we are interested in the effect of inflation on the basic limits loss $g(X;k)$ with $k = 10{,}000$. From (2) the indemnity after inflation would be

$$\begin{aligned} g(\alpha(X);k) &= \alpha(X)I_{(0,\,\alpha^{-1}(k)]}(X) + kI_{(\alpha^{-1}(k),\,\infty)}(X) \\ &= [1.10X + (0.10/20{,}000)X^2]I_{(0,\,\alpha^{-1}(k)]}(X) \\ &\quad + 10{,}000I_{(\alpha^{-1}(k),\,\infty)}(X). \end{aligned}$$

The value of $\alpha^{-1}(k)$ is determined by the quadratic equation

$$1.10x + (0.10/20{,}000)x^2 = k = 10{,}000,$$

and the solution is easily found to be $x = \alpha^{-1}(10{,}000) = 8{,}743$.

Thus the expected value of the basic limits indemnity would be

$$\begin{aligned}\mathrm{E}\{g(\alpha(X);k)\} &= \mathrm{E}\{1.10X + (0.10/20{,}000)X^2]I_{(0,\,8743]}(X)\} \\ &\quad + 10{,}000[1 - F(8{,}743)]. \qquad (5)\end{aligned}$$

Suppose the limit is $k = 25{,}000$; the solution would be somewhat different. A close look at equation (3) reveals that the inverse $x = \alpha^{-1}(k)$ is given by $k/1.20 = 20{,}833$. The expected value of the indemnity under this contract would then be

$$\begin{aligned}\mathrm{E}\{g(\alpha(X;k)\} &= \mathrm{E}(1.10X + (0.10/20{,}000)X^2]I_{(0,\,20000]}(X)\} \\ &\quad + 1.2\mathrm{E}\{XI_{(20000,\,20833]}(X)\} \\ &\quad + 25{,}000[1 - F(20{,}833)], \qquad (6)\end{aligned}$$

remembering the change in functional form at $X = 20{,}000$. The actual calculation can then be carried through by evaluation of the distribution function and the appropriate incomplete moments.

Now let us take the specific functional form for the loss distribution to be lognormal with parameters

$$\mu = 7.6 \text{ and } \sigma = 1.8$$

so that the mean loss is

$$\mathrm{E}\{X\} = 10{,}097 \text{ with CV} = 4.953,$$

where CV stands for the coefficient of variation (standard deviation / mean). For the lognormal the k^{th} moment is

$$\mathrm{E}\{X^k\} = \exp[k\mu + (1/2)k^2\sigma^2]$$

and the incomplete k^{th} moment is

$$\mathrm{E}_{(x)}\{X^k\} = \mathrm{E}\{X^k\}\Phi[(\ln x - \mu)/\sigma - k\sigma]$$

where $\Phi(.)$ denotes the distribution function of the standard normal variable. Table 1 shows the incomplete moments with orders shown in column (1) up to the various values shown in the first row. Order zero means the value of the distribution function.

TABLE 1

INCOMPLETE MOMENTS

(1) k	(2) 20,000	(3) 8743	(4) 10,000	(5) 20,833	(6) 25,000
0	0.8997	0.7939	0.8145	0.9036	0.9198
1	3,044	1,651	1,844	3,124	3,493
2	26,455,454	7,075,407	8,880,632	28,093,176	36,534,773

First we will calculate the overall inflation; formula (4) and the numbers in column (2) of Table 1 give

$$1.10 \times 3044 + .10 \times 26{,}455{,}454\ /20{,}000$$
$$+\ 1.20 \times 20{,}000 \times (10{,}097 - 3{,}044) = 11{,}945,$$

corresponding to an overall rate of $11{,}945/10{,}097 - 1 = 18.3\%$. Next consider the basic limits indemnity with a limit at 10,000. Noting that $\alpha^{-1}(k)$ in formula (5) has the value 8743 and using the numbers in column (3), we have the expected basic limits indemnity

$$1.10 \times 1651 + .10 \times 7{,}075{,}407/20{,}000$$
$$+\ 10{,}000 \times (1 - .7939) = 3{,}912.$$

From column (4) we can easily obtain the limited indemnity without inflation:

$$1{,}844 + 10{,}000(1 - .8145) = 3{,}699$$

so that the effective inflation rate is $3{,}912/3{,}699 - 1 = 5.76\%$.

Similarly, we can calculate the inflation rate for the basic limits indemnity with a limit of 25,000. From formula (6) and the numbers in columns (2) and (5) we find this to be

$$1.10 \times 3044 + .10 \times 26{,}455{,}454/20{,}000$$
$$+\ 1.2 \times (3{,}124 - 3{,}044) + 25{,}000 \times (1 - .9036) = 5{,}987.$$

Again, we can calculate the expected indemnity without inflation for this case with the numbers in column (6):

$$3{,}493 + 2{,}500 \times (1 - .9198) = 5{,}498$$

so that the effect of inflation is to increase the expected indemnity by $5{,}987/5{,}498 - 1 = 8.89\%$.

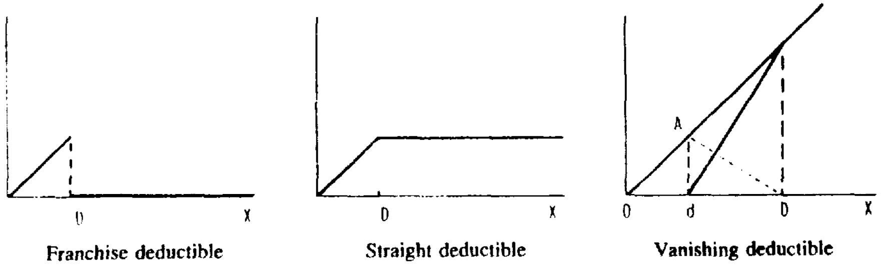
5. DEDUCTIBLES

The common type of deductible, called straight deductible, has the simple representation

$$XI_{(0,D]}(X) + DI_{(D,\infty)}(X).$$

A franchise deductible is represented as $XI_{(0,D]}(X)$. These are depicted as functions of the loss X in Figure 3. A more complicated form of deductible is the vanishing deductible, which equals the loss up to the amount d, but thereafter reducing linearly to 0 when the loss becomes $D > d$ (Snader [7]); see Figure 3 for a pictorial description.

FIGURE 3
DEDUCTIBLES



It is easier to describe the indemnity after the deductible. The geometry in Figure 3 shows easily that the equation for the indemnity is

$$y = \frac{D}{D - d}(x - d)$$

over the range $(d, D]$. Thus the indemnity can be written as

$$Y = \frac{D}{D - d}(X - d)I_{(d, D]}(X) + XI_{(D, \infty)}(X).$$

The deductible itself can be found by taking the difference:

$$X - Y = XI_{(0, d]}(X) + \frac{d}{D - d}(D - X)I_{(d, D]}(X).$$

The expectation of the deductible can then be obtained as

$$\begin{aligned} \mathrm{E}\{X - Y\} &= \mathrm{E}\{XI_{(0, d]}(X)\} + \frac{d}{D - d}\mathrm{E}\{(D - X)I_{(d, D]}(X)\} \\ &= \int_0^d x\mathrm{d}F(x) + \frac{d}{D - d}\int_d^D (D - x)\mathrm{d}F(x). \end{aligned}$$

6. RETROSPECTIVE RATING

In retrospective rating, the charge over rE, where E stands for the expected loss, can be represented as a random variable:

$$\Phi(r)E = (X - rE)I_{(rE, \infty)}(X).$$

$\Phi(r)$ may be interpreted as the excess of the loss X over rE, measured in units of E, and

$$E\{\Phi(r)\} = \phi(r)$$

where $E\{\ \}$ means expectation and $\phi(r)$ has the usual meaning of charge over rE as an average. Similarly, we represent the random variable savings as

$$\Psi(r)E = (rE - X)I_{(0, rE]}(X)$$

with

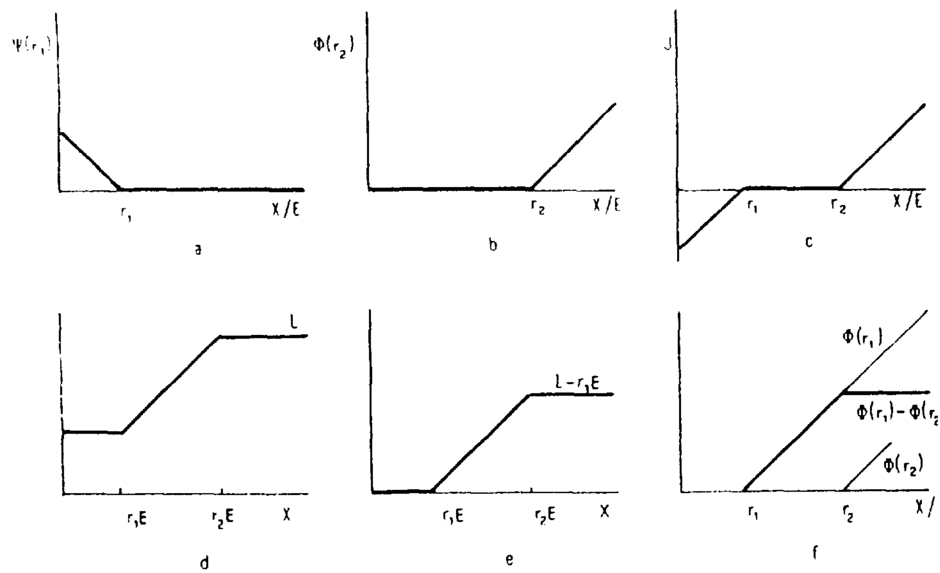
$$E\{\Psi(r)\} = \psi(r).$$

The graphs of $\Phi(r)$ and $\Psi(r)$ as functions of X/E are shown in Figures 4a and 4b. The loss portion of the limited pure premium paid by the insured, L, is a random variable under the insurance contract:

$$L = r_1 E I_{(0,\, r_1 E]}(X) + X I_{(r_1 E,\, r_2 E]}(X) + r_2 E I_{(r_2 E,\, \infty)}(X). \tag{7}$$

The insured pays the minimum premium $_{r_1}E$ if the actual loss incurred is not more than $r_1 E$, the actual loss if it is greater than $r_1 E$ but not greater than $r_2 E$, and the maximum premium $r_2 E$ if the actual loss exceeds $r_2 E$, as far as the loss portion of the premium is concerned. The graph of L as a function of X is shown in Figure 4d.

FIGURE 4
RETROSPECTIVE RATING



By adding and subtracting $XI_{(0,\, r_1E]}(X)$ and $XI_{(r_2E,\, \infty]}(X)$ in the formula (7) for L we have

$$L = r_1EI_{(0,\, r_1E]}(X) - XI_{(0,r_1E]}(X) + XI_{(0,\, r_1E]}(X) + XI_{(r_1E,\, r_2E]}(X)$$
$$+ XI_{(r_2E,\, \infty]}(X) - XI_{(r_2E,\, \infty]}(X) + r_2EI_{(r_2E,\, \infty)}(X)$$
$$= (r_1E - X)I_{(0,\, r_1E]}(X) + X - (X - r_2E)I_{(r_2E,\, \infty)}(X)$$

Thus

$$L = X - \{\Phi(r_2)E - \Psi(r_1)E\}$$
$$= X - J$$

where

$$J = \Phi(r_2)E - \Psi(r_1)E$$

is the random variable version of the net charge I as defined, for example, in Skurnick [6] and, of course, $E\{J\} = I$. Note that J is equal to $-(r_1E - X)$ if the loss X is less than r_1E, zero if the loss X is between r_1E and r_2E, and $X - r_2E$ if the loss X exceeds r_2E. Its graph as a function of X is shown in Figure 4c.

The following identity is interesting:

$$\Phi(r)E - \Psi(r)E = (X - rE)I_{(rE,\, \infty)}(X) - (rE - X)I_{(0,\, rE]}(X)$$
$$= X - rE,$$

being the random variable version of the well-known identity $\phi(r) - \psi(r) = 1 - r$. The graph of $\Phi(r)E - \Psi(r)E$ as a function of X is simply a 45-degree line through the point $(rE,0)$.

The following equality is easily derived from equation (7):

$$L - r_1E = r_1EI_{(0,\, r_1E]}(X) + XI_{(r_1E,\, r_2E]}(X) + r_2EI_{(r_2E,\, \infty)}(X) - r_1E$$
$$= (X - r_1E)I_{(r_1E,\, r_2E]}(X) + (r_2E - r_1E)I_{(r_2E,\, \infty)}(X). \qquad (8)$$

This equality is illustrated in Figure 4e, and its relationship to L is clearly visualized by comparison to Figure 4d. Whereas

$$\begin{aligned}\Phi(r_1)E - \Phi(r_2)E &= (X - r_1E)I_{(r_1E,\infty)}(X) - (X - r_2E)I_{(r_2E,\infty)}(X)\\ &= (X - r_1E)I_{(r_1E,r_2E]}(X) + (X - r_1E)I_{(r_2E,\infty)}(X)\\ &\quad - (X - r_2E)I_{(r_2E,\infty)}(X)\\ &= (X - r_1E)I_{(r_1E,r_2E]}(X)\\ &\quad + (r_2E - r_1E)I_{(r_2E,\infty)}(X). \end{aligned} \tag{9}$$

See Figure 4f for an illustration. Equations (8) and (9) show that

$$\Phi(r_1)E - \phi(r_2)E = L - r_1E, \tag{10}$$

and both are the excess of loss function $h(X;r_1E,r_2E - r_1E)$ as described in Section 2. Let BP be the basic premium of the retrospective plan. Adding and subtracting this on the right-hand side of (10) yields

$$\Phi(r_1) - \Phi(r_2) = [\{BP + CL\} - \{BP + Cr_1E\}]/CE, \tag{11}$$

where C is the loss conversion factor to be applied to the loss to obtain the premium. Equation (11) is useful in determining the exact entry ratios as well as the minimum and maximum premiums in a retrospective rating plan. Noting that

$$E\{BP + CL\} = P(1 - D),$$

which is the premium after adjustment for expense gradation D, and

$$E\{BP + r_1CE\} = H,$$

where H is the minimum premium, we have by taking expectations on both sides of (11) the familiar identity (Skurnick [6])

$$\phi(r_1)E - \phi(r_2)E = (P - PD - H)/CE.$$

7. CONCLUSION

An alternative method of representing a function assuming different functional forms over the range of its argument is by the Heaviside or delta function, which is defined as

$$H(x) = \begin{cases} 1 & \text{if } 0 \leq X \\ 0 & \text{otherwise.} \end{cases}$$

Thus the function

$$g(X;k) = \begin{cases} X & 0 < X \leq K \\ k & X > k. \end{cases}$$

can be represented in terms of the Heaviside function as

$$\begin{aligned} g(X;k) &= XH(X) - XH(X - k) + kH(X - k) \\ &= XH(X) - (X - k)H(X - k). \end{aligned} \tag{12}$$

The Heaviside function obviates the explicit use of the set and so is more parsimonious in notation. Most people, however, would take a relatively long time to picture the shape of the function represented by (12). While the indicator function representation visually shows the sets of points where the g function assumes different forms, it is not so with the Heaviside function. Thus, although at times clumsy in form, the indicator function representation is preferred here.

In mathematics and, more generally, scientific work, given relations are to be made as general as possible. Relations between random variables are certainly more general than those derivable from these relations but pertaining to their expectations only. In actuarial work, the most important quantity related to a loss is the expected value. It is natural that much of the work on the topics described in this paper has focused on expected values. This paper has shown that it is often possible to express the relations in terms of the random variables, thus strengthening the existing mathematical results. The results are stronger in the sense that when a relation holds for random variables, it is true for each realization, whereas a relation for expected values holds only on the average.

Such an approach allows us to look at the results in another way. The stronger results give not only the expectation relationship, but also relationships pertaining to other characteristics of the indemnity distribution, such as higher order moments. Also, quite often the mathematics become simpler and easier to understand. In particular, the treatment of trend in this fashion is more effective than techniques hitherto available.

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AN EXAMPLE OF CREDIBILITY AND SHIFTING RISK PARAMETERS

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Abstract

In this paper, the won-lost record of baseball teams will be used to examine and illustrate credibility concepts. This illustrative example is analogous to the use of experience rating in insurance. It provides supplementary reading material for students who are studying credibility theory.

This example illustrates a situation where the phenomenon of shifting parameters over time has a very significant impact. The effects of this phenomenon are examined.

Three different criteria that can be used to select the optimal credibility are examined: least squares, limited fluctuation and Meyers/Dorweiler. In applications, one or more of these three criteria should be useful.

It is shown that the mean squared error can be written as a second order polynomial in the credibilities with the coefficients of this polynomial written in terms of the covariance structure of the data. It is then shown that linear equation(s) can be solved for the least squares credibilities in terms of the covariance structure.

The author wishes to thank Julie Jannuzzi and Gina Brewer for typing this paper.

1. INTRODUCTION

In this paper, the won-lost record of baseball teams will be used to examine and illustrate credibility concepts. This illustrative example is analogous to the use of experience rating in insurance. The mathematical details are contained in the appendices.

One purpose of this paper is to provide supplementary reading material for students who are studying credibility theory. However, this paper also contains a number of points which should prove of interest to those who are already familiar with credibility theory.

Of particular interest is the effect of shifting risk parameters over time on credibilities and experience rating. This example illustrates a situation where the phenomenon of shifting parameters over time has a very significant impact.

The general structure of the paper is to go from the simplest case to the more general. The mathematical derivations are confined to the appendices.

Section 2 briefly reviews the use of credibility in experience rating.

Section 3 describes the data sets from baseball that are used in this paper in order to illustrate the concepts of the use of credibility in experience rating.

Section 4 is an analysis of the general structure of the data. It is demonstrated that the different insureds (baseball teams) have significantly different underlying loss potentials. It is also shown that for this example a given insured's relative loss potential does shift significantly over time.

Section 5 states the problem whose solution will be illustrated. One wishes to estimate the future loss potential using a linear combination of different estimates.

Section 6 discusses simple solutions to the problem presented in Section 5.

Section 7 discusses three criteria that can be used to distinguish between solutions to the problem in Section 5.

Section 8 applies the three criteria of Section 7 to the forms of solution presented in Section 6. The results of applying the three different criteria are compared. The reduction in squared error and the impact of the delay in receiving data are both discussed.

Section 9 discusses more general solutions to the problem than those presented in Section 6.

Section 10 applies the three criteria of Section 7 to the forms of the solution presented in Section 9.

Section 11 shows equations for Least Squares Credibility that result from the covariance structure assumed.

Section 12 discusses miscellaneous subjects.

Section 13 states the author's conclusions.

2. CREDIBILITY AND EXPERIENCE RATING

Experience rating and merit rating modify an individual insured's rate above or below average. From an actuarial standpoint, the experience rating plan is using the observed loss experience of an individual insured in order to help predict the future loss experience of that insured. Usually this can be written in the form:

$$\begin{aligned}\text{New Estimate} &= (\text{Data}) \times (\text{Credibility}) \\ &\quad + (\text{Prior Estimate}) \times (\text{Complement of Credibility})\end{aligned}$$

For most experience rating plans, the prior estimate is the class average. However, in theory the prior estimate could be a previous estimate of the loss potential of this insured relative to the class average. This paper will treat both possibilities.

2.1 Shifting Parameters Over Time

There are many features of experience rating plans that are worthy of study by actuaries. Meyers [1], Venter [2], Gillam [3], and Mahler [4] present examples of recent work. The example in this paper will deal with only one aspect, that is, how to best combine the different years of past data.

The author, in a previous paper [5], came to the following conclusion concerning this point:

"When there are shifting parameters over time, older years of data should be given substantially less credibility than more recent years of data. There may be only a minimal gain in efficiency[1] from using additional years of data."

3. THE DATA SETS

This paper will examine two very similar sets of data in order to illustrate certain features of credibility. Each set of data is the won-lost record for a league of baseball teams.[2] One set is for the so-called National League while the other is for the American League.[3] Each set of data covers the sixty years from 1901 to 1960. During this period of time each league had eight teams.

For each year, called a season in baseball, for each team, we have the losing percentage, i.e., the percentage of its games that the individual team lost.

3.1 Advantages of this Data

This example has a number of advantages not to be found using actual insurance data. First, over a very extended period of time there is a constant set of risks (teams). In insurance there are generally insureds who leave the data base and new ones that enter.

Second, the loss data over this extended period of time are readily available, accurate and final. In insurance the loss data are sometimes hard to compile or obtain and are subject to possible reporting errors and loss development.

Third, each of the teams in each year plays roughly the same number of games.[4] Thus the loss experience is generated by risks of roughly equal "size." Thus, in this example, one need not consider the dependence of credibility on size of risk.

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[1] Meyers [11] defines the efficiency of an experience rating plan as the reduction in expected squared error accomplished by the use of the plan. The higher the efficiency the smaller the expected squared error.

[2] Appendix A gives some relevant features of the sport of baseball.

[3] These two leagues are referred to as the major leagues. They generally contain the best players in North America. The data for the two leagues are independent of each other, since no inter-league games are included in the data.

[4] Over the 60 years in question, teams usually played about 150 games per year.

4. ANALYSIS OF THE GENERAL STRUCTURE OF DATA

The loss experience[5] by risk (team) by year are given in Table 1 for the National League and Table 2 for the American League.[6]

4.1 Is There an Inherent Difference Between Teams?

The first question to be answered is whether there is any real difference between the experience of the different teams, or is the apparent difference just due to random fluctuations. This is the fundamental question when considering the application of experience rating.

It requires only an elementary analysis in order to show that there is a non-random difference between the teams. The average experience for each team over the whole period of time differs significantly from that of the other teams. If the experience for each team were drawn from the same probability distribution, the results for each team would be much more similar. The standard deviation in losing percentage over a sample of about 9000 games[7] would be .5%.[8] Thus if all the teams' results were drawn from the same distribution, approximately 95% of the teams would have an average losing percentage between 49% and 51%.[9]

The actual results are shown on Table 3. In fact, only 3 of 16 teams have losing percentages in that range. The largest deviation from the grand mean is 15 times the expected standard deviation if the teams all had the same underlying probability distribution.

---

[5] For each of 60 years, the percentage of games lost is given for each team. The data are from *The Sports Encyclopedia* [6].

[6] For the National League the teams are in order: Brooklyn, Boston, Chicago, Cincinnati, New York, Philadelphia, Pittsburgh and St. Louis. For the American League the teams are in order: Boston, Chicago, Cleveland, Detroit, New York, Philadelphia, St. Louis and Washington. In both cases, the city given is that in which the team spent the majority of the data period.

[7] About 150 games for a team each year times 60 years.

[8] A binomial distribution with a 50% chance of losing, for 9000 games, has a variance of $9000(1/2)(1 - 1/2) = 2250$. This is a standard deviation of 47 games lost, or $47 \div 9000 = .5\%$ in losing percentage.

[9] Using the standard normal approximation, 95% of the probability is within two standard deviatioins of the mean which in this case is 50%.

TABLE 1

National League Losing Percentages

	NL1	NL2	NL3	NL4	NL5	NL6	NL7	NL8
1901	.500	.419	.619	.626	.620	.407	.353	.457
1902	.467	.457	.504	.500	.647	.591	.259	.582
1903	.580	.485	.406	.468	.396	.637	.350	.686
1904	.641	.634	.392	.425	.307	.658	.431	.513
1905	.669	.684	.399	.484	.314	.454	.373	.623
1906	.675	.566	.237	.576	.368	.536	.392	.653
1907	.608	.561	.296	.569	.464	.435	.409	.660
1908	.591	.656	.357	.526	.364	.461	.364	.682
1909	.706	.641	.320	.497	.399	.516	.276	.645
1910	.654	.584	.325	.513	.409	.490	.438	.588
1911	.709	.573	.403	.542	.353	.480	.448	.497
1912	.660	.621	.393	.510	.318	.520	.384	.588
1913	.543	.564	.425	.582	.336	.417	.477	.660
1914	.386	.513	.494	.610	.455	.519	.552	.471
1915	.454	.474	.523	.539	.546	.408	.526	.529
1916	.414	.390	.562	.608	.434	.405	.578	.608
1917	.529	.536	.519	.494	.364	.428	.669	.461
1918	.573	.548	.349	.469	.427	.553	.480	.605
1919	.590	.507	.464	.314	.379	.657	.489	.606
1920	.592	.396	.513	.464	.442	.595	.487	.513
1921	.484	.493	.582	.542	.386	.669	.412	.431
1922	.654	.506	.481	.442	.396	.627	.448	.448
1923	.649	.506	.461	.409	.379	.675	.435	.484
1924	.654	.403	.471	.458	.392	.636	.412	.578
1925	.542	.556	.558	.477	.434	.556	.379	.497
1926	.566	.536	.468	.435	.510	.616	.451	.422
1927	.610	.575	.444	.510	.403	.669	.390	.399
1928	.673	.497	.409	.487	.396	.717	.441	.383
1929	.636	.542	.355	.571	.444	.536	.425	.487
1930	.545	.442	.416	.617	.435	.662	.481	.403

TABLE 1

(CONTINUED)

	NL1	NL2	NL3	NL4	NL5	NL6	NL7	NL8
1931	.584	.480	.455	.623	.428	.571	.513	.344
1932	.500	.474	.416	.610	.532	.494	.442	.532
1933	.461	.575	.442	.618	.401	.605	.435	.464
1934	.483	.533	.430	.656	.392	.624	.507	.379
1935	.752	.542	.351	.556	.405	.582	.438	.377
1936	.539	.565	.435	.519	.403	.649	.455	.435
1937	.480	.595	.396	.636	.375	.601	.442	.474
1938	.493	.537	.414	.453	.447	.700	.427	.530
1939	.583	.451	.455	.370	.490	.702	.556	.399
1940	.572	.425	.513	.346	.526	.673	.494	.451
1941	.597	.351	.545	.429	.516	.721	.474	.366
1942	.601	.325	.558	.500	.441	.722	.551	.312
1943	.556	.471	.516	.435	.641	.584	.481	.318
1944	.578	.591	.513	.422	.565	.601	.412	.318
1945	.559	.435	.364	.604	.487	.701	.468	.383
1946	.385	.471	.464	.565	.604	.552	.591	.372
1947	.390	.442	.552	.526	.474	.597	.597	.422
1948	.455	.405	.584	.582	.494	.571	.461	.448
1949	.370	.513	.604	.597	.526	.474	.539	.377
1950	.422	.461	.582	.569	.442	.409	.627	.490
1951	.382	.506	.597	.558	.376	.526	.584	.474
1952	.373	.582	.500	.552	.403	.435	.727	.429
1953	.318	.403	.578	.558	.545	.461	.675	.461
1954	.403	.422	.584	.519	.370	.513	.656	.532
1955	.359	.448	.529	.513	.481	.500	.610	.558
1956	.396	.403	.610	.409	.565	.539	.571	.506
1957	.455	.383	.597	.481	.552	.500	.597	.435
1958	.539	.403	.532	.506	.481	.552	.455	.532
1959	.436	.449	.519	.519	.461	.584	.494	.539
1960	.468	.429	.610	.565	.487	.617	.383	.442

TABLE 2

American League Losing Percentages

	AL1	AL2	AL3	AL4	AL5	AL6	AL7	AL8
1901	.419	.390	.599	.452	.489	.456	.650	.545
1902	.438	.448	.493	.615	.638	.390	.426	.551
1903	.341	.562	.450	.522	.463	.444	.532	.686
1904	.383	.422	.430	.592	.391	.464	.572	.748
1905	.487	.395	.506	.484	.523	.378	.647	.576
1906	.682	.384	.418	.523	.404	.462	.490	.633
1907	.604	.424	.441	.387	.527	.393	.546	.675
1908	.513	.421	.416	.412	.669	.556	.454	.559
1909	.417	.487	.536	.355	.510	.379	.593	.724
1910	.471	.556	.533	.442	.417	.320	.695	.563
1911	.490	.490	.477	.422	.500	.331	.704	.584
1912	.309	.494	.510	.549	.671	.408	.656	.401
1913	.473	.487	.434	.569	.623	.373	.627	.416
1914	.405	.545	.667	.477	.545	.349	.536	.474
1915	.331	.396	.625	.351	.546	.717	.591	.444
1916	.409	.422	.500	.435	.481	.765	.487	.503
1917	.408	.351	.429	.490	.536	.641	.630	.516
1918	.405	.540	.425	.563	.512	.594	.525	.437
1919	.518	.371	.396	.429	.424	.743	.518	.600
1920	.529	.377	.364	.604	.383	.688	.503	.553
1921	.513	.597	.390	.536	.359	.654	.474	.477
1922	.604	.500	.494	.487	.390	.578	.396	.552
1923	.599	.552	.464	.461	.355	.546	.513	.510
1924	.565	.569	.562	.442	.414	.533	.513	.403
1925	.691	.487	.545	.474	.552	.421	.464	.364
1926	.699	.471	.429	.487	.409	.447	.597	.460
1927	.669	.542	.569	.464	.286	.409	.614	.448
1928	.627	.532	.597	.558	.344	.359	.468	.513
1929	.623	.612	.467	.545	.429	.307	.480	.533
1930	.662	.597	.474	.513	.442	.338	.584	.390

TABLE 2

(CONTINUED)

	AL1	AL2	AL3	AL4	AL5	AL6	AL7	AL8
1931	.592	.634	.494	.604	.386	.296	.591	.403
1932	.721	.675	.428	.497	.305	.390	.591	.396
1933	.577	.553	.503	.513	.393	.477	.636	.349
1934	.500	.651	.448	.344	.390	.547	.559	.566
1935	.490	.513	.464	.384	.403	.611	.572	.562
1936	.519	.464	.481	.461	.333	.654	.625	.464
1937	.474	.442	.461	.422	.338	.642	.701	.523
1938	.409	.561	.434	.455	.349	.651	.638	.503
1939	.411	.448	.435	.474	.298	.638	.721	.572
1940	.468	.468	.422	.416	.429	.649	.565	.584
1941	.455	.500	.513	.513	.344	.584	.545	.545
1942	.388	.554	.513	.526	.331	.643	.457	.589
1943	.553	.468	.464	.494	.364	.682	.526	.451
1944	.500	.539	.532	.429	.461	.532	.422	.584
1945	.539	.523	.497	.425	.467	.653	.464	.435
1946	.325	.519	.558	.403	.435	.682	.571	.506
1947	.461	.545	.481	.448	.370	.494	.617	.584
1948	.381	.664	.374	.494	.390	.455	.614	.634
1949	.377	.591	.422	.435	.370	.474	.656	.675
1950	.390	.610	.403	.383	.364	.662	.623	.565
1951	.435	.474	.396	.526	.364	.545	.662	.597
1952	.506	.474	.396	.675	.383	.487	.584	.494
1953	.451	.422	.403	.610	.344	.617	.649	.500
1954	.552	.390	.279	.558	.331	.669	.649	.571
1955	.455	.409	.396	.487	.377	.591	.630	.656
1956	.455	.448	.429	.468	.370	.662	.552	.617
1957	.468	.416	.503	.494	.364	.614	.500	.643
1958	.487	.468	.497	.500	.403	.526	.516	.604
1959	.513	.390	.422	.506	.487	.571	.519	.591
1960	.578	.435	.506	.539	.370	.623	.422	.526

TABLE 3

AVERAGE LOSING PERCENTAGES (1901-1960)

Risk (Team)	NL1	NL2	NL3	NL4	NL5	NL6	NL7	NL8
National League	53.4	49.9	47.3	51.8	44.7	56.5	47.8	48.8

Risk (Team)	AL1	AL2	AL3	AL4	AL5	AL6	AL7	AL8
American League	49.5	49.4	47.0	48.5	42.6	52.9	56.4	53.5

Thus there can be no doubt that the teams actually differ.[10] It is therefore a meaningful question to ask whether a given team is better or worse than average.

A team that has been worse than average over one period of time is more likely to be worse than average over another period of time. If this were not true, we would not have found the statistically significant difference in the means of the teams.

Thus if we wish to predict the future experience of a team, there is useful information contained in the past experience of that team. In other words, there is an advantage to experience rating.

4.2 Shifting Parameters Over Time

A similar, but somewhat different question of interest is whether for a given team the results for the different years are from the same distribution (or nearly the same distribution). In other words, are the observed different results over time due to more than random fluctuation? The answer is yes. This is a situation where the underlying parameters of the risk process shift over time.

[10] The situation here is somewhat complicated by the fact that one team's loss is another team's win. Thus the won-loss records of seven teams determine that of the remaining team. However, the author confirmed with a straightforward simulation that in this case this phenomenon would not affect the conclusion. For 8 teams each with the 50% loss rate playing 9000 games each, in 32 out of 600 cases (5%) a team had a winning percentage lower than 49% or more than 51%. In none of the 600 cases did a team have a winning percentage as low as 48% or as high as 52%.

As discussed in Section 2.1, the extent to which risk parameters shift over time has an important impact on the use of past insurance data to predict the future.

Whether the risk parameters shift over time can be tested in many ways. Two methods will be demonstrated here. These methods can be applied to insurance data as well as the data presented here.

The first method of testing whether parameters shift over time uses the standard chi-squared test. For each risk, one averages the results over separate 5 year periods.[11] Then one compares the number of games lost during the various 5 year periods. One can then determine by applying the chi-squared test that the risk process could not have the same mean over this entire period of time. The results shown in Table 4 are conclusive for every single risk. Even the most consistent risk had significant shifts over time.

In the second method of testing whether parameters shift over time, one computes the correlation between the results for all of the risks for pairs of years. Then one computes the average correlation for those pairs of years with a given difference in time. Finally, one examines how the average correlation depends on this difference. The results in our case are displayed in Table 5.

Observed values of the correlation different from zero are not necessarily statistically significant. For this example, a 95% confidence interval around zero for the correlation is approximately plus or minus .10.[12] Thus, for this example, the correlation decreases as the difference in time increases until about ten years when there is no longer a significant correlation between results.[13]

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[11] The data were grouped in five year intervals for convenience. Other intervals could also have been used.

[12] For larger distances between the years, we have fewer observations to average, so the confidence interval expands to approximately plus or minus .12. The confidence intervals were determined via repeated simulation in which the actual data for each year were separately assigned to the individual risks at random; thus for the simulated data any observed correlation is illusory.

[13] For a difference of between 15 and 20 years there is again a small but significant positive correlation. The author has no explanation for this long term cycle.

TABLE 4

Results of Chi-Squared Test of Shifting Parameters Over Time

For each risk (team) its experience over the 60 year period was averaged into 12 five-year segments. (The simplifying assumption was made of 150 games each year; this did not affect the results.) Then for each risk separately, the chi-square statistic was computed in order to test the hypothesis that each of the five year segments was drawn from a distribution with the same mean. The resulting chi-square values are:

NL1	NL2	NL3	NL4	NL5	NL6	NL7	NL8
107	45	98	35	39	73	114	119

AL1	AL2	AL3	AL4	AL5	AL6	AL7	AL8
114	69	34	30	97	162	53	65

For example, for the risk (team) NL2 the data by five-year segments are as follows:

	'01 –'05	'06 –'10	'11 –'15	'16 –'20	'21 –'25	'26 –'30	'31 –'35	'36 –'40	'41 –'45	'46 –'50	'51 –'55	'56 –'60
(1) Games Lost*	402	451	412	357	370	389	391	386	326	344	354	310
(2) Expected Games Lost**	374	374	374	374	374	374	374	374	374	374	374	374
$(3)=[(1)-(2)]^2/(2)$	2.1	15.9	3.9	.8	0	.6	.8	.4	6.2	2.4	1.1	11.0

The sum of row (3) is 45, which is the chi-square value for this risk.

For each risk there is less than a .2% chance that the different five-year segments were drawn from distributions with the same mean.*** Thus we reject the hypothesis that the means are the same over time; we accept the hypothesis of shifting risk parameters over time.

*Assuming 150 games per year, and the observed losing percentage for the five year segment.

**Assuming 150 games per year, and the observed losing percentage for the whole 60 years.

***For 11 degrees of freedom, there is a .16% chance of having a chi-square value of 30 or more. There is a .004% chance of having a chi-square value of 40 or more.

TABLE 5

AVERAGE CORRELATIONS OF RISKS EXPERIENCE OVER TIME (1901-1960)

Difference Between Pairs of Years of Experience	Correlation NL	Correlation AL
1	.651	.633
2	.498	.513
3	.448	.438
4	.386	.360
5	.312	.265
6	.269	.228
7	.221	.157
8	.190	.124
9	.135	.078
10	.100	.090
11	.083	.058
12	.103	.063
13	.154	.101
14	.176	.104
15	.180	.141
16	.246	.178
17	.278	.166
18	.219	.198
19	.176	.219
20	.136	.225
21	.090	.159
22	.065	.125
23	.055	.093
24	.004	.048
25	−.024	.006

TABLE 5

(CONTINUED)

Difference Between Pairs of Years of Experience	Correlation NL	Correlation AL
26	−.028	.010
27	−.095	−.002
28	−.128	−.013
29	−.107	−.032
30	−.062	.006
31	−.061	−.019
32	−.028	.027
33	−.015	.002
34	.017	.088
35	.038	.143
36	−.014	.156
37	−.024	.214
38	−.012	.238
39	−.017	.138
40	−.095	.093
41	−.174	.055
42	−.216	.028
43	−.332	−.043
44	−.423	−.018
45	−.363	−.035
46	−.332	.066
47	−.324	.069
48	−.373	.136
49	−.423	.075
50	−.475	.145

The correlation between years that are close together is significantly greater than those further apart. This implies that the parameters of the risk process are shifting significantly over time. If the parameters were reasonably constant over time, the correlations would not depend on the length of time between the pair of years.

On the other hand, there is a significant correlation between the results of years close in time. Thus recent years can be usefully employed to predict the future.

5. STATEMENT OF THE PROBLEM

Let X be the quantity we wish to estimate. In this case, X is the expected losing percentage for a risk.

Let Y_1, Y_2, Y_3, etc., be various estimates for X. Then one might estimate X by taking a weighted average of the different estimates Y_i.

$$X = \sum_{i=1}^{n} Z_i Y_i,$$

where X = quantity to be estimated,
Y_i = one of the estimates of X,
Z_i = weight assigned to estimate Y_i of X.

Here only linear combinations of estimators are being considered. In addition, the estimators themselves will be restricted to a single year of past data for the given risk or to the grand mean (which is 50% in this case).[14] No subjective information or additional data beyond the past annual losing percentages will be used.[15] In other words, this is a situation analogous to (prospective) experience rating. This is not a situation analogous to schedule rating.

[14] In other words, in this case, Y either equals the observed losing percentage for the risk in one year or equals the grand mean of 50%. Credibility methods can be applied to more general estimators.

[15] The use of information on the retirement of players or acquisition of new players might enable a significant increase in the accuracy of the estimate. The breakdown of the data into smaller units than an entire year might enable a significant increase in the accuracy of the estimate.

The problem to be considered here is what weights Z_i produce the "best" estimate of future losing percentage. In order to answer that question, criteria will have to be developed that allow one to compare the performance of the different methods to determine which is better. In the example being dealt with in this paper, it is easy to get unbiased estimators. Since all of the estimators being compared will be unbiased, the question of which method is better will focus on other features of the estimators.

Usually the weights Z_i are restricted to the closed interval between 0 and 1. In the most common situation we have two estimates, i.e., $i = 2$. In that case we usually write:

$$X = Z \cdot Y_1 + (1 - Z) \cdot Y_2$$

where Z is called the credibility and $(1 - Z)$ is called the complement of credibility. However, it is important to note that the usual terminology tempts us into making the mistake of thinking of the two weights and two estimates differently. The actual mathematical situation is symmetric.

6. SIMPLE SOLUTIONS TO THE PROBLEM

In this section, various relatively simple solutions to the problem will be presented.

6.1 Every Risk is Average

The first method is to predict that the future losing percentage for each risk will be equal to the overall mean of 50%. This method ignores all the past data; i.e., the past data are given zero credibility. While this is not a serious candidate for an estimation method in the particular example examined in this paper, it is a useful base case in general.

6.2 The Most Recent Year Repeats

The second method is to predict that the most recent past year's losing percentage for each risk will repeat. This is what is meant by giving the most recent year of data 100% credibility.

6.3 *Credibility Weight the Most Recent Year and the Grand Mean*

In the third method, one gives the most recent year of data for each risk weight Z, and gives the grand mean, which in this case is 50%, weight $1 - Z$.

When $Z = 0$, one gets the first method; when $Z = 1$, one gets the second method. Since each of these is a special case of this more general method, by the proper choice of Z one can do better than or equal to either of the two previous methods. This is an important and completely general result. It does not depend on either the criterion that is used to compare methods or the means of deciding which value of Z to use.

6.4 *Determining the Credibility*

When employing the third method, the obvious question is how does one determine the value of credibility to use. Ideally one would desire a theory or method that would be generally applicable, rather than one that only worked for a single example. There have been many fine papers on this subject in the actuarial literature.

Generally, the credibility considered "best" is determined by some objective criterion. This will be discussed later.

Using either Bühlmann/Bayesian credibility methods or classical/limited fluctuation credibility methods, one determines which credibility will be expected to optimize the selected criterion in the future. One can also empirically investigate which credibility would have optimized the selected criterion if it had been used in the past; i.e., one can perform retrospective tests. This will be discussed in more detail later.

6.5 *Equal Weight to the* N *Most Recent Years of Data*

In the fourth method, one gives equal weight to the N most recent years of data for each risk, and gives the grand mean, which in this case is 50%, weight $1 - Z$. This method gives each of the N most recent years weight of Z/N.[16] When $N = 1$ this reduces to the previous method. Thus this method will perform at least as well as the previous method, with the proper choices of N and Z.

7. CRITERIA TO DECIDE BETWEEN SOLUTIONS

In this section, we will discuss *three* criteria that can be used to distinguish between solutions. These criteria can be applied in general and not just to this example.

7.1 Least Squared Error

The first criterion involves calculating the mean squared error of the prediction produced by a given solution compared to the actual observed result. The smaller the mean squared error, the better the solution.

The Bühlmann/Bayesian credibility methods attempt to minimize the squared error; i.e., they are least squares methods. Minimizing the squared error is the same as minimizing the mean squared error.

7.2 Small Chance of Large Errors

The second criterion deals with the probability that the observed result will be more than a certain percent different than the predicted result. The less this probability, the better the solution.

This is related to the basic concept behind "classical" credibility which has also been called "limited fluctuation" credibility [7]. In classical credibility, the full credibility criterion is chosen so that there is a probability, P, of meeting the test that the maximum departure from expected is no more than k percent.

The reason the criterion is stated in this way rather than the way it is in classical credibility is that, unlike the actual observations, one cannot observe directly the inherent loss potential.[17] However, the two concepts are closely related, as discussed in Appendix G.

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[16] In later methods, the weights given to the different years of data will be allowed to differ from each other.

[17] It has been shown that the loss potential varies for a risk over time. Thus it cannot be estimated as the average of many observations over time.

7.3 Meyers/Dorweiler

The third criterion has been taken from Glenn Meyers' paper [1]. Meyers in turn based his criterion upon the ideas of Paul Dorweiler [8].

This criterion involves calculating the correlation between two quantities. The first quantity is the ratio of actual losing percentage to the predicted losing percentage. The second quantity is the ratio of the predicted losing percentage to the overall average losing percentage. The smaller the correlation between these two quantities, i.e., the closer the correlation is to zero, the better the solution.

To compute the correlation, the Kendall τ statistic is used.[18] This is explained in detail in Appendix B. The relation of this criterion as used here and as it is used by Meyers to examine experience rating is also discussed in that appendix.

8. THE CRITERIA APPLIED TO THE SIMPLE SOLUTIONS

In this section the three criteria in Section 7 will be applied to the simple solutions given in Section 6. More knowledgeable readers may wish to skip to Section 8.4 which compares the results of applying the three different criteria. Section 8.5 discusses the reduction in squared error. Section 8.6 examines the impact of a delay in receiving data.

8.1 The Two Base Cases

The two simplest solutions either always use as the estimate the overall mean ($Z = 0$), or always use as the estimate the most recent observation ($Z = 1$). While neither of these solutions is expected to be chosen, they serve as the base cases for testing the other solutions.

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[18] Meyers in [1] used the Kendall τ statistic. In the example here, any other reasonable measure of the correlation could be substituted.

The first criterion is the smallest mean squared error. For the two data sets the results are:

	Mean Squared Error	
	NL	AL
$Z = 0$	.0091	.0095
$Z = 1$	.0059	.0068

The second criterion is to produce a small probability of being wrong by more than k percent. For the two data sets the results are as follows:

	Percent of time that the estimate is in error by more than 5%		Percent of time that the estimate is in error by more than 10%		Percent of time that the estimate is in error by more than 20%	
	NL	AL	NL	AL	NL	AL
$Z = 0$	82.2%	80.3%	64.8%	63.8%	29.0%	31.4%
$Z = 1$	75.8%	72.9%	52.3%	55.7%	19.1%	22.0%

The third criterion is to have a correlation as close to zero as possible between the ratio of the actual to estimated and the ratio of estimated to the overall mean. For the two data sets the results are as follows:

	Correlation (Kendall τ)	
	NL	AL
$Z = 0$*	.48	.46
$Z = 1$	$-.24$	$-.27$

* Limit as Z approaches zero.

8.2 Applying Credibility to the Latest Year of Data

The third prediction method, explained in Section 6.3, uses credibility to combine the latest year of data and the grand mean. The mean squared error depends on the credibility. As shown in Table 6, the mean squared error is a minimum for Z between 60% and 70%.[19] The probability of having errors of 20% or more is displayed in Table 7. Based on this second criterion, the optimal Z is between 50% and 80%.[20] This criterion does not distinguish very sharply between the different values of credibility.

The correlations used in the third criterion are displayed in Table 8. Based on the third criterion the optimal Z is approximately 70%.[21]

8.3 Applying Credibility to the Latest N *Years of Data*

The fourth method, explained in Section 6.4, uses credibility to combine the grand mean with the latest N years of data (giving each year of data the same weight.)

The results of applying the first criterion are shown in Table 6. Based on most actuarial uses of credibility, an actuary would expect the optimal credibilities to increase as more years of data are used. In this example they do not. In fact, using more than one or two years of data does an inferior job according to this criterion.

This result is to be expected, since the parameters shift substantially over time. Thus the use of older data (with equal weight) eventually leads to a worse estimate.[22]

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[19] For the NL data set, the minimum occurs when $Z = 68\%$. For the AL data set, the minimum occurs when $Z = 66\%$. Also, it should be noted that the squared errors for $Z = 0$ vary somewhat with the number of years of data used, solely due to the differing periods of time over which the test can be performed.

[20] For the NL data set, the optimal Z is 75%. For the AL data set, the optimal Z is 55%. It should be noted that, given the limited number of observations, two values of Z can produce identical results for this criterion.

[21] For the NL data set, the correlation is closest to zero for $Z = 71\%$. For the AL data set, the correlation is closest to zero for $Z = 66\%$.

[22] The number of years of data to use to get the best estimate will depend on the particular example. This general subject was explored in Mahler [5].

TABLE 6

Mean Squared Error (.0001)

NL

Z	N = 1	N = 2	N = 3	N = 4	N = 5	N = 7	N = 10	N = 15	N = 20	N = 25
0	91	90	90	89	87	84	80	80	80	80
.10	80	80	80	80	79	77	74	75	76	77
.20	70	72	72	72	71	71	70	72	72	74
.30	62	65	65	65	65	66	67	69	69	71
.40	56	59	60	60	60	62	64	67	67	69
.50	52	55	56	56	57	59	63	66	65	68
.60	50	53	53	53	55	57	62	65	64	68
.70	49	52	52	52	53	56	63	65	63	68
.80	51	53	52	52	54	57	64	66	64	69
.90	54	55	53	53	55	58	66	68	65	70
1.00	59	59	56	56	57	61	70	70	66	72

AL

Z	N = 1	N = 2	N = 3	N = 4	N = 5	N = 7	N = 10	N = 15	N = 20	N = 25
0	95	96	96	95	95	95	95	92	91	95
.10	84	85	86	86	88	89	90	88	87	91
.20	75	77	78	79	81	83	86	85	83	87
.30	67	69	71	73	75	79	82	82	80	83
.40	61	64	66	68	71	75	80	81	78	80
.50	58	60	62	64	68	73	78	79	76	78
.60	56	57	60	62	66	71	78	79	74	76
.70	56	56	59	61	66	71	78	79	73	74
.80	58	57	59	62	66	72	79	79	73	73
.90	62	59	61	64	68	74	81	81	73	73
1.00	68	63	64	67	71	77	84	83	74	73

TABLE 7

PERCENT OF TIME THAT THE ESTIMATE IS IN ERROR BY MORE THAN 20%

NL

Z	N = 1	N = 2	N = 3	N = 4	N = 5	N = 7	N = 10	N = 15	N = 20	N = 25
0	29	29	29	28	28	27	25	25	25	26
.10	25	26	25	25	25	25	25	24	24	25
.20	23	23	23	23	24	24	23	24	23	26
.30	19	21	22	22	21	22	23	23	23	24
.40	18	19	20	20	22	23	22	22	23	25
.50	17	19	18	19	21	21	21	22	24	25
.60	17	18	18	18	18	21	21	22	24	25
.70	17	18	18	19	19	21	21	22	26	26
.80	17	17	18	19	20	22	23	24	25	27
.90	18	19	17	18	21	22	24	25	25	27
1.00	19	20	18	20	21	23	25	26	25	28

AL

Z	N = 1	N = 2	N = 3	N = 4	N = 5	N = 7	N = 10	N = 15	N = 20	N = 25
0	31	31	32	32	32	32	33	32	32	34
.10	27	27	27	27	27	28	28	28	29	30
.20	23	24	25	25	25	27	27	27	28	30
.30	21	21	22	23	24	25	27	27	27	29
.40	19	19	21	23	24	26	27	26	26	27
.50	18	19	21	22	23	24	28	26	26	27
.60	18	18	21	21	22	25	27	25	26	26
.70	20	18	19	20	22	25	26	25	25	27
.80	19	19	18	20	21	26	27	26	25	26
.90	20	21	19	22	23	27	26	27	25	27
1.00	22	23	22	24	26	28	29	28	27	26

TABLE 8

CORRELATION (KENDALL τ)

NL

Z	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 7$	$N = 10$	$N = 15$	$N = 20$	$N = 25$
0*	.48	.45	.46	.45	.43	.39	.31	.28	.29	.25
.10	.44	.41	.42	.41	.40	.35	.27	.24	.26	.22
.20	.38	.36	.37	.36	.35	.30	.23	.20	.22	.18
.30	.32	.30	.31	.31	.30	.25	.18	.16	.18	.14
.40	.25	.24	.25	.25	.24	.20	.12	.11	.14	.10
.50	.17	.17	.19	.19	.18	.14	.07	.06	.10	.06
.60	.09	.09	.12	.12	.12	.08	.02	.02	.05	.02
.70	.01	.02	.04	.05	.05	.02	−.04	−.03	.01	−.02
.80	−.08	−.06	−.03	−.02	−.02	−.04	.09	−.07	−.03	−.06
.90	−.16	−.13	−.01	−.09	−.08	−.10	.14	−.12	−.08	−.10
1.00	−.24	−.21	−.17	−.16	−.15	−.16	−.19	−.16	−.12	−.14

*Limit as Z approaches zero

CORRELATION (KENDALL τ)

AL

Z	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 7$	$N = 10$	$N = 15$	$N = 20$	$N = 25$
0*	.46	.45	.44	.41	.38	.34	.28	.24	.27	.30
.10	.42	.41	.40	.38	.35	.30	.25	.21	.25	.28
.20	.36	.36	.35	.33	.30	.26	.21	.17	.21	.25
.30	.29	.30	.30	.27	.25	.21	.16	.13	.18	.22
.40	.22	.24	.23	.22	.19	.16	.12	.09	.15	.19
.50	.14	.16	.17	.15	.13	.11	.07	.05	.11	.16
.60	.05	.08	.10	.08	.07	.05	.02	.01	.07	.12
.70	−.03	.00	.02	.02	.00	−.01	−.03	−.03	.03	.09
.80	−.11	−.07	−.05	−.05	−.06	−.07	−.07	−.07	−.01	.05
.90	−.19	−.15	−.12	−.12	−.12	−.12	−.12	−.11	−.05	.01
1.00	−.27	−.22	−.19	−.18	−.18	−.17	−.16	−.15	−.09	−.02

*Limit as Z approaches zero

The results of applying the second criterion are displayed in Table 7. This criterion does not sharply distinguish between the different values of credibility. There is a broad range of credibilities all of which do reasonably well.[23] This is particularly true for larger values of N. Again the use of more years of data eventually leads to an inferior estimate.

The results of applying the third criterion are displayed in Table 8. Again the optimal credibility does not increase as N increases. Unlike the other criteria, the third criterion cannot be used to distinguish between values of N. For each N, there is a Z, such that the correlation is zero. Thus each value of N performs as well as all the others.

Meyers points out that the distribution of Kendall's τ can be used to obtain a confidence interval for the credibility. As explained in Appendix B, for this example a 95% confidence interval for τ around zero has a radius of about .07.

For example, using 10 years of data, the optimal credibility using the Meyers/Dorweiler criterion for the NL set of data is 63%. However, this point estimate for the credibility is actually an estimate of an interval of credibilities that correspond to τ between plus and minus .07. The optimal credibility is 63% ± 13%.[24]

8.4 *Comparison of the Results of the Three Criteria*

In Table 9 the optimal credibilities are displayed as determined by the three criteria for various values of N. Note that the listed values of credibility are those that happened to work best over the period of time observed. Values close to these values would also work well over this period of time.

One should think of the point estimates listed in Table 9 as the centers of interval estimates. This is illustrated when one compares the different estimates obtained by analyzing the NL and AL data sets. There is no inherent difference in the two data sets. Thus one would expect the credibilities from the two analyses to be the same. They are similar,

[23] This is true to a lesser extent for the first criterion. This subject is explored in Mahler [9].

[24] For $Z = 63.3\%$, $\tau = 0$. For $Z = 50.1\%$, $\tau = .07$. For $Z = 76.4\%$, $\tau = -.07$.

but far from identical. This indicates that the peculiarities of the specific observed values are sufficient to affect the answers somewhat. There is some lack of precision in the estimates in Table 9.

TABLE 9

OPTIMAL CREDIBILITY

Number of Years of Data Used	NL Criterion #1	NL Criterion #2	NL Criterion #3	AL Criterion #1	AL Criterion #2	AL Criterion #3
1	68%	75%	71%	65%	55%	66%
2	71	80	72	70	56	70
3	74	87	76	72	77	73
4	76	57	77	72	69	72
5	74	61	77	70	70	71
7	71	64	73	67	51	68
10	60	49	63	62	70	64
15	63	43	64	65	69	62
20	71	40	73	81	82	77
25	64	30	64	97	61	94

Criterion #1: Least Squares (Section 7.1)
Criterion #2: Small Chance of Large Errors (Section 7.2)
Criterion #3: Meyers/Dorweiler (Section 7.3)

This can be illustrated further by reversing the time arrow and analyzing the data sets going backwards in time rather than forwards. For example, one could use data from years 1902 to 1911 to "predict" 1901. This analysis is equally valid for determining optimal credibilities in this example as was the original anlysis.

For $N = 10$, one gets the following optimal credibilities for the different data sets, where NLR and ALR represent respectively the NL and AL data sets reversed in time.

Optimal Credibilities (N = 10)

	NLR	NL	ALR	AL	Average
Criterion #1	72%	60%	57%	62%	63%
Criterion #2	58	49	42	70	55
Criterion #3	77	63	58	64	65

The optimal credibilities differ between the four data sets. The amount of variation provides some idea of the imprecision of the different estimates. While the optimal credibilities differ between the three criteria, the differences do not appear to be sufficiently large to allow one to draw any definitive conclusions.

In this case, the use of any value of credibility between 50% and 70% would perform reasonably well according to all three criteria for all four data sets. As a practical matter, the difference in the predictions will not vary that much depending on which value of credibility is chosen in that range.[25]

In most applications of credibility, values for the credibility that differ somewhat from optimal perform reasonably well and the choice between these values has a relatively small practical impact.

8.5 Putting the Reduction in Squared Error in Context

The first criterion used to determine the optimal credibility is to minimize the squared error. Using the optimal credibility based on this criterion will reduce the squared error between the observed and predicted result. What should be considered a significant reduction in squared error?

[25] The maximum difference in any prediction for N = 10 between using 50% and 70% credibility is 3.3% in the losing percentage. In most cases it is much smaller. On average it would make about a 1% difference.

Let us examine an example. For the NL data, using one year of data, the optimal credibility is 68% as shown in Table 9. As shown in Table 6 the mean squared errors are:

Z	Mean Squared Error
0	.0091
68%	.0049
100%	.0059

In this case, by the use of credibility, the squared error has been reduced from .0059 if the data were relied upon totally, or .0091 if the data were totally ignored, to .0049. In this case, the squared error has been reduced to 83% (.0049/.0059) of its previous value.[26]

As discussed in Appendix E, in the current case, the best that can be done using credibility to combine two estimates is to reduce the mean squared error between the estimated and observed values to 75% of the minimum of the squared errors from either relying solely on the data or ignoring the data.[27]

The reduction of the squared error to 83% of its previous value appears significant in light of the maximum possible reduction to 75%.[28]

8.6 Effect of Delay in Receiving Data

It has been shown previously for the data set examined in this paper that the further apart in time two years are, the lower the correlation between them. Thus if there is a delay before the data are available for use in experience rating, the resulting estimate of the future will be less accurate.

[26] The "previous" value of the squared error is considered to be the minimum of the squared errors that result from either ignoring the data entirely or relying on the data entirely.

[27] When using more than two or more years of data, the reduction in squared error depends on the impact of shifting parameters over time. However, in the absence of shifting parameters over time, for N years with the same weight applied to each year, the maximum possible reduction is $1 \div (2(N + 1))$.

[28] The maximum reduction is possible when the squared errors for $Z = 0$ and $Z = 1$ are equal.

As is shown in Table 10, as the delay increases, the squared error increases significantly. The increase in squared error is particularly significant as one goes from a situation of having the data from the most recent year available to predict the coming year to a situation of having

TABLE 10

MINIMUM SQUARED ERROR (.0001)

Time Between Latest Data Point and Future Prediction	NL $N = 1$	NL $N = 2$	NL $N = 3$	NL $N = 4$	NL $N = 5$
1	49	52	51	51	53
2	66	62	60	60	60
3	69	66	65	64	65
4	73	71	69	69	70
5	77	73	73	72	72
6	76	75	75	73	74
7	78	77	75	75	75
8	79	77	77	76	75
9	78	78	77	76	75
10	78	78	76	75	75

Time Between Latest Data Point and Future Prediction	AL $N = 1$	AL $N = 2$	AL $N = 3$	AL $N = 4$	AL $N = 5$
1	56	56	59	61	66
2	71	70	71	74	76
3	78	77	80	81	83
4	83	85	85	87	88
5	89	89	90	91	91
6	91	91	92	93	93
7	93	93	94	93	94
8	95	94	94	94	93
9	95	94	94	93	93
10	94	94	93	93	94

only the next most recent year available. Unfortunately, the latter situation is more common in insurance than is the former.

As is shown in Table 11, the optimal credibility (as determined using the least squares criterion) decreases as the delay increases. Less current information is less valuable for estimating the future.

TABLE 11

OPTIMAL CREDIBILITY (CRITERION #1, LEAST SQUARES)

Time Between Latest Data Point and Future Prediction	NL				
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
1	68	71	74	76	74
2	51	59	64	64	63
3	47	53	55	56	55
4	40	45	47	47	45
5	33	38	40	39	36
6	30	33	34	32	30
7	24	26	26	25	24
8	19	20	21	21	20
9	14	16	17	18	20
10	11	13	15	18	21

Time Between Latest Data Point and Future Prediction	AL				
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
1	65	70	72	72	70
2	51	57	58	57	56
3	42	47	46	46	45
4	35	36	37	36	36
5	25	28	28	28	25
6	21	22	22	19	18
7	15	16	14	14	13
8	11	9	10	10	9
9	6	7	8	7	9
10	7	7	7	9	10

9. MORE GENERAL SOLUTIONS

In Section 6, four relatively simple forms of a solution were given. In this section, more general forms of a solution will be given.

9.1 Combine Previous Estimate and Most Recent Data

In the fifth method, one gives the latest year of *data* weight Z, and gives the previous *estimate* weight $1 - Z$. Of course, one has to choose an initial estimate.[29] In this case, for each risk the initial estimate will be taken as the grand mean of 50%.[30] Once this estimation method has been used for several years, the initial estimate has very little weight.

For example, let us assume $Z = 60\%$. Then the weights assigned to the given years of data used in estimating the result for the year 1911 would be as follows:

Year of Data	Weight in Estimate of 1911
1910	$Z = 60\%$
1909	$Z(1 - Z) = 40\% \times 60\% = 24\%$
1908	$Z(1 - Z)^2 = 40\% \times 40\% \times 60\% = 9.6\%$
1907	$Z(1 - Z)^3 = 9.6\% \times 40\% = 3.84\%$
1906	$Z(1 - Z)^4 = 3.84\% \times 40\% = 1.54\%$
1905	$Z(1 - Z)^5 = 1.54\% \times 40\% = .61\%$
1904 and Prior	$(1 - Z)^6 = .41\%$

The above assumes that the latest year of data is always given 60% weight, while the current estimate is given 40% weight.

Thus in this case, one gets a geometrically decreasing weight. This procedure is called (single) exponential smoothing [10]. It is an example of what mathematicians call a "filter."[31] Once the process of exponential

[29] This is precisely analogous to choosing a "seed" value in exponential smoothing.

[30] One could use subjective judgement to choose the initial estimate for each risk. Also one could use data from the period prior to that displayed in this paper; this has been avoided for the sake of simplicity.

[31] Morrison [11] gives this as an example of a "fading-memory polynomial filter."

smoothing gets "up to speed," it is equivalent to a weighted least squares regression, where the fitted line is horizontal,[32] and where the weights are geometrically decreasing as the data get less recent.

9.2 *More General Varying Weights*

In Section 9.1, one gave geometrically decreasing weight to years of data further in the past. More generally one can make the estimate:

$$F = \Sigma Z_i X_i + (1 - \Sigma Z_i)M$$

where the weights Z_i depend on how far in the past are the data X_i. For years for which data are not available (presumably because they are too far in the past) one uses the grand mean M instead of the data. This method is a generalization of the methods in Sections 6 and 9.1.

Unfortunately, calculating or empirically determining the optimal values of the weights Z_i becomes difficult as more years of data are used. Also, there are many vectors of Z_i that are very close to optimal; i.e., the n-dimensional volume of values $Z_1, \ldots, Z_n$ that produce close to optimal results is relatively large.

10. THE CRITERIA APPLIED TO THE MORE GENERAL SOLUTIONS

In this section the three criteria in Section 7 will be applied to the more general solutions to the problem given in Section 9. For simplicity, the results will be shown for the situation where there is no delay in obtaining the data for use in making the next estimate. In Section 8.6, an example was given of the results of such a delay in receiving data. The same general pattern would apply here.

10.1 *Geometrically Decreasing Weights*

In Section 9.1, weight Z is applied to the latest available year of data, while weight $1 - Z$ is applied to the previous estimate.

---

[32] Double exponential smoothing, sometimes called linear exponential smoothing, would be equivalent to a weighted linear least squares regression, with geometrically decreasing weights as the data got less recent.

Table 12 gives the mean squared errors for various values of Z. The optimal values of Z, using criterion #1 (least squares), are all close to 55%.[33] This results in weights to the various years of data very similar to those in the example in Section 9.1.

TABLE 12

MEAN SQUARED ERRORS* (.0001) THAT RESULT FROM APPLYING Z TO LATEST YEAR OF DATA AND $1 - Z$ TO PREVIOUS ESTIMATE

Z	NL	NLR**	AL	ALR**
0	79	97	95	96
.1	61	70	72	78
.2	56	63	65	71
.3	52	60	60	67
.4	50	57	57	64
.5	49	56	55	63
.6	50	56	55	63
.7	50	57	55	64
.8	52	58	56	66
.9	54	59	58	69
1.0	57	62	61	73

* First 10 years are not included in the computation of the squared errors in order to eliminate the calibration period.

** Data reversed in time.

In this case there is no significant reduction in squared error beyond what was previously obtained by applying credibility to the latest available year.[34]

Table 13 displays the results of applying criterion #2, limited fluctuation. Values of the credibility between 40% and 80% generally perform well.

[33] For the NL data set the optimal credibility is 53%. For the NLR data set, it is 58%. For the AL data set it is 60%. For the ALR data set it is 54%.

[34] Compare the results in Table 6 for $N = 1$ with those in Table 12.

Table 14 displays the results of applying criterion #3, Meyers/Dorweiler.[35] Unlike the previous two cases, the optimal credibilities are close to zero; 5% to 10% credibility produces correlations close to zero. The use of such small credibilities is approximately the same as using 10 to 20 years of data as the basis for the estimate, since the geometrically decreasing weights decline only slowly.

TABLE 13

PERCENT OF TIME* THAT THE ESTIMATE IS IN ERROR BY MORE THAN 20%
APPLYING Z TO LATEST YEAR OF DATA
AND $1 - Z$ TO PREVIOUS ESTIMATE

Z	NL	NLR**	AL	ALR**
0	25	31	33	31
.1	23	23	25	26
.2	21	22	24	25
.3	18	21	21	23
.4	16	21	20	21
.5	16	20	19	22
.6	16	20	19	21
.7	17	19	20	22
.8	18	19	19	22
.9	18	20	18	23
1.0	19	21	19	26

* First 10 years are not included in the computation in order to eliminate the calibration period.

** Data reversed in time.

[35] In this case, the results of the first 20 years were excluded from the computation, in order to eliminate the calibration period. Twenty years were used, rather than ten years as in the previous two tables, since in this case smaller credibilities are optimal and smaller credibilities require a longer calibration period.

TABLE 14

CORRELATIONS* (KENDALL TAU) THAT RESULT FROM APPLYING Z TO LATEST YEAR OF DATA AND $1 - Z$ TO PREVIOUS ESTIMATE

Z	NL	NLR**	AL	ALR**
0***	.11	.16	.28	.14
.1	−.03	.01	.00	−.09
.2	−.05	−.05	−.04	−.10
.3	−.08	−.09	−.07	−.12
.4	−.10	−.12	−.10	−.13
.5	−.13	−.15	−.12	−.15
.6	−.15	−.18	−.14	−.17
.7	−.18	−.20	−.16	−.20
.8	−.20	−.23	−.18	−.22
.9	−.23	−.25	−.21	−.24
1.0	−.26	−.27	−.23	−.28

* First 20 years are not included in the computation of the correlations in order to eliminate the calibration period.

** Data reversed in time.

*** Limit as Z approaches zero.

10.2 More General Varying Weights

In Section 9.2, varying weights Z_i are applied to the most recent N years, while the remaining weight is given to the grand mean. This method will only be examined using criterion #1, least squares. One can solve numerically for the set of weights which produce the least squared error, using a given number of years of data.[36] The results are as follows:

[36] Unfortunately, as the number of years increases, the amount of computer time required also increases.

Using Most Recent Two Years of Data ($N = 2$, $\Delta = 1$)

	Credibility		
	Second Most Recent Year	Most Recent Year	Mean Squared Error (.0001)
NL	9.6%	61.1%	48
AL	13.1%	56.9%	54

Using Most Recent Three Years of Data ($N = 3$, $\Delta = 1$)

	Credibility			
	Third Most Recent Year	Second Most Recent Year	Most Recent Year	Mean Squared Error (.0001)
NL	16.4%	1.1%	59.0%	45
AL	8.1%	9.1%	55.7%	53

Most of the credibility is assigned to the most recent year. The complement of credibility, which is assigned to the grand mean, is about 25 to 35 percent, decreasing as N increases.

Complement of Credibility

	$N = 1$*	$N = 2$	$N = 3$
NL	32%	29%	24%
AL	35%	30%	27%

* One minus the optimal credibility from Table 9.

The mean squared error is reduced from that using only the latest year of data.[37]

[37] Since the use of fewer years of data is just a special case, the least squared error using more years of data must be less than or equal that using fewer years of data.

Mean Squared Error (.0001)

	$N = 1$*	$N = 2$	$N = 3$
NL	49	48	45
AL	56	54	53

* From Table 6.

11. EQUATIONS FOR LEAST SQUARES CREDIBILITIES

In Section 11.2 are equations to solve for the least squares credibility. These equations follow from the assumed covariance structure discussed in Section 11.1. In Section 11.3 the equations in Section 11.2 are modified to constrain them to place no weight on the grand mean. Section 11.4 compares the mean squared errors that result from different credibilities. Section 11.5 briefly discusses the validity of the results derived in this paper.

11.1 The Covariance Structure

By analyzing the covariance structure, one can set up matrix equations to solve for the credibilities that minimize the squared error. These matrix equations are discussed in the next section.

As shown in Appendix D, the variance of the data can be broken down into two pieces. There is the variance between the risks.[38] There is also the variance within the risks.[39] These two variances add up to the total variance.

	Between Variance	Within Variance	Total Variance[40]
NL	.001230	.007892	.009121
AL	.001619	.007875	.009494

[38] This has been denoted as τ^2.

[39] This has been denoted as $\delta^2 + \zeta^2$. δ^2 is what is usually termed process variance, while ζ^2 is the variance due to shifting parameters over time.

[40] May differ slightly from the sum of the other two variances due to rounding.

Also of interest is the covariance between the years of data. It is assumed that this is a function of the number of years separating the data. The observed values are given in Table 15. As was seen in Table 5, the covariance decreases as the years of data are further apart. After about 6 years the covariances are relatively close to zero.

TABLE 15

COVARIANCE (.0001)

Years Separating Data	NL	AL
0*	7892	7875
1	4919	4527
2	3416	3175
3	3128	2411
4	2541	1766
5	1810	780
6	1566	383
7	955	−99
8	387	−561
9	−74	−1068
10	−394	−878
11	−558	−980
12	−389	−1092
13	3	−737
14	59	−814
15	212	−453
16	603	−39
17	786	−139
18	302	214
19	47	279
20	−268	415

*Equal by definition to the within variance.

It is possible to divide the within variance into two parts. The first part is the process variance excluding the effect of shifting parameters over time.[41] The second part is that portion of the within variance due to shifting parameters over time.[42] While this division may aid our understanding, it is not necessary for the calculation of the least squares credibilities. Not coincidentally, this division cannot be performed based solely on the reported data contained in Tables 1 and 2. This subject is discussed in more detail in Appendix D.

11.2 Matrix Equations for Least Squares Credibilities

Using the estimation method described in Section 9.2:

$$F = \sum_{i=1}^{N} Z_i X_i + \left(1 - \sum_{i=1}^{N} Z_i\right) M \tag{11.1}$$

As derived in Appendix C, one gets the following expression for the expected squared error between the observation and prediction:

$$\begin{aligned} V(Z) = & \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i Z_j \left(\tau^2 + C(|i-j|)\right) \\ & - 2 \sum_{i=1}^{N} Z_i \left(\tau^2 + C(N + \Delta - i)\right) \\ & + \tau^2 + C(0) \end{aligned} \tag{11.2}$$

In equation (11.2) we have used the following quantities defined in Appendix D.

τ^2	=	between variance
$C(k)$	=	covariance for data for the same risk, k years apart
	=	"within covariance"
$C(0)$	=	within variance
Δ	=	the length of time between the latest year of data used and the year being estimated

[41] This has been denoted as δ^2.
[42] This has been denoted as ζ^2.

Equation 11.2 shows that the squared error is a second order polynomial in the Z_i.[43] This equation is the fundamental result for analyzing least squares credibility.

One can differentiate equation 11.2 in order to get N linear equations in N unknowns, which can be solved for the optimal credibilities.

$$\sum_{j=1}^{N} Z_j(\tau^2 + C(|i - j|)) = \tau^2 + C(N + \Delta - i) \quad i = 1, 2, \dots N \qquad (11.3)$$

The set of equations 11.3 can be solved on a computer relatively easily using the usual methods from matrix theory. The results of doing so for $\Delta = 1$, using the average of the variances and covariance determined from the NL and AL data separately,[44] are shown in Table 16.

TABLE 16

LEAST SQUARES CREDIBILITIES, SOLUTIONS OF MATRIX EQUATIONS 11.3 ($\Delta = 1$)

Number of Years of Data Used (N)	Years Between Data and Estimate									
	1	2	3	4	5	6	7	8	9	10
1	66.0%	—	—	—	—	—	—	—	—	—
2	57.7	12.6	—	—	—	—	—	—	—	—
3	56.1	4.8	13.5	—	—	—	—	—	—	—
4	55.6	4.6	11.5	3.5	—	—	—	—	—	—
5	55.7	5.1	11.7	6.0	−4.4	—	—	—	—	—
6	55.9	4.9	11.3	5.8	−6.6	3.9	—	—	—	—
7	56.0	4.7	11.5	6.2	−6.5	5.9	−3.5	—	—	—
8	56.0	4.7	11.4	6.2	−6.3	5.9	−2.8	−1.2	—	—
9	56.1	4.9	11.0	6.6	−6.7	5.3	−3.1	−4.3	5.6	—
10	55.9	5.0	11.2	6.4	−6.4	5.1	−3.4	−4.5	3.6	3.5

The complement of credibility is applied to the grand mean.

First column is the credibility applied to the most recent year, second column is the credibility applied to the next most recent year, etc.

Note: Based on the average of the variances and covariances determined from the NL and AL data separately; however, assumes that for a separation of eight years or more, the covariance is zero.

[43] When $N = 1$, the squared error is a parabola as a function of the credibility. This has been noted before, for example in Appendix B of Meyers [12].

[44] It is assumed that for a separation of eight years or more, the covariance is zero.

The results conform reasonably well to those determined in Section 10.2.

The credibilities applied to the most recent year quickly converge to about 56%. The credibilities for the less recent years are much smaller. However, these credibilities do not monotonically decline as the years get less recent. There is a complicated pattern of weights determined by the covariance matrix. Some of the weights are even less than zero.[45]

The optimal credibilities are uniquely determined given the covariance structure. However, there are many other sets of credibilities which produce expected squared errors very close to minimal. The precise values of the credibilities are not particularly important, although the general range of credibilities that perform well might be instructive.

One can apply equation (11.2) to the method discussed in Sections 6.5 and 8.3 of applying equal weight Z_i to the latest N years of data, where

$Z_i = Z/N$ for $i = 1, \ldots, N$

As shown in Appendix C, the least squares credibility in this case is given by:

$$Z = N \frac{N\tau^2 + \sum_{i=1}^{N} C(N + \Delta - i)}{N^2\tau^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} C(|i - j|)} \tag{11.4}$$

The results of applying this equation for $\Delta = 1$, using the average of the variances and covariances determined from the NL and AL data separately,[46] are shown in Table 17.

Table 17 can be compared to Table 11.

The results in Table 11 conform reasonably well to those determined empirically for each data set (for $\Delta = 1$).

[45] Giving negative weight to some years allows a larger weight to be given to other years. The net effect is to reduce the expected squared error.

[46] It is assumed that for a separation of eight years or more, the covariance is zero.

TABLE 17

LEAST SQUARES CREDIBILITY, SOLUTION TO EQUATION 11.4 ($\Delta = 1$)

Number of Years of Data Used (N)	Z	$Z \div N$
1	66.0%	66.0%
2	70.3	35.2
3	72.9	24.3
4	73.6	18.4
5	72.2	14.4
6	71.3	11.9
7	69.9	10.0
8	68.2	8.5
9	67.3	7.5
10	66.9	6.7

Equal weight Z/N is applied to each of the N most recent years of data. The complement of credibility, $1 - Z$, is applied to the grand mean.

Note: Based on the average of the variances and covariances determined from the NL and AL data separately; however, assumes that for a separation of eight years or more, the covariance is zero.

11.3 Placing No Weight on the Grand Mean

Once the estimation method described in Sections 9.1 and 10.1 gets "up to speed," the initial estimate, which was taken as the grand mean, has very little weight. For all intents and purposes each risk is estimated based on its own past data, without relying on the data of other risks, in particular the grand mean.[47]

[47] The covariance structure is herein estimated using the data for all risks. This in turn is used to estimate the optimal credibilities. However, the credibilities are applied to the data for the particular risk we are estimating.

One can constrain the credibilities used in equation 11.1, so that they add to unity, thus giving no weight to the grand mean. Equation 11.1 then becomes

$$F = \sum_{i=1}^{N} Z_i X_i \tag{11.5}$$

with the constraint

$$\sum_{i=1}^{N} Z_i = 1. \tag{11.6}$$

The least squres credibilities for equations 11.5 and 11.6 are derived in Appendix C using the method of Lagrange Multipliers. The result is a set of $N + 1$ linear equations in $N + 1$ unknowns, the Z_i for $i = 1, \ldots, N$, and λ, the Lagrange Multiplier. There is the single constraint equation 11.6, plus the N equations 11.7.

$$\sum_{j=1}^{N} Z_j \, C(|i - j|) = C(N + \Delta - i) + \frac{\lambda}{2}, \quad i = 1, 2, \ldots, N \tag{11.7}$$

The set of equations 11.6 and 11.7 can be solved on a computer relatively easily using the usual methods from matrix theory. The results of doing so for $\Delta = 1$, using the average of the variances and covariances determined from the NL and AL data separately,[48] are shown in Table 18.

11.4 Mean Squared Errors

The mean squared errors that result from using the credibilities in Tables 16, 17, and 18 are displayed in Table 19.

When applying general weights to the latest N years of data, giving the most remote year of data no weight is equivalent to the case of using the latest $N - 1$ years of data. Since using the latest $N - 1$ years of data is a special case of using the latest N years of data, we expect the squared errors to decline, or remain constant.

This is what we observe for the credibilities from Table 16. They decline until $N = 6$, where the point of diminishing returns is reached.

[48] It is assumed that for a separation of eight years or more, the covariance is zero.

TABLE 18

LEAST SQUARES CREDIBILITIES, SOLUTIONS OF EQUATIONS 11.6 AND 11.7 ($\Delta = 1$)

Number of Years of Data Used (N)	Years Between Data and Estimate									
	1	2	3	4	5	6	7	8	9	10
1	100.0%	—	—	—	—	—	—	—	—	—
2	72.6	27.4	—	—	—	—	—	—	—	—
3	66.1	10.3	23.6	—	—	—	—	—	—	—
4	63.5	9.1	16.0	11.4	—	—	—	—	—	—
5	63.1	8.7	15.8	9.5	2.9	—	—	—	—	—
6	62.8	7.6	14.1	8.6	−3.9	10.8	—	—	—	—
7	62.5	7.7	13.8	8.2	−4.1	9.0	2.9	—	—	—
8	62.3	7.3	14.0	7.7	−4.8	8.6	−0.2	5.1	—	—
9	61.8	7.3	13.0	8.3	−5.7	7.0	−1.1	−1.9	11.2	—
10	60.8	7.5	13.1	7.7	−5.2	6.3	−2.2	−2.5	6.1	8.4

The credibilities are constrained to sum to unity.

First column is the credibility applied to the most recent year, second column is the credibility applied to the next most recent year, etc.

Note: Based on the average of the variances and covariances determined from the NL and AL data separately; however, assumes that for a separation of eight years or more, the covariance is zero.

Applying the same weight to each year is a special case of using the general weights. Thus the squared errors that result from using the credibilities from Table 17 should be greater than or equal to those that result from the credibilities from Table 16. This is the case, as shown in Table 19. Also, as was observed in Section 8.3, using more years of data leads in this case to larger squared errors.

Applying no weight to the grand mean is a special case of using the general weights. Thus the squared errors that result from using the credibilities from Table 18 should be greater than or equal to those that result from the credibilities from Table 16. As is shown in Table 19, the squared errors are substantially greater, with the gap narrowing as the number of years increases.

TABLE 19

	Mean Squared Errors (.0001)*		
Number of Years of Data Used (N)	Using the Credibilities From Table 16**	Using the Credibilities From Table 17***	Using the Credibilities From Table 18****
1	52	52	63
2	51	54	58
3	49	55	54
4	48	57	52
5	48	60	52
6	47	61	51
7	47	64	51
8	47	66	51
9	47	68	51
10	47	70	50

* Mean squared error using the stated credibilities to predict for the NL and AL data sets.

** The complement of credibility is given to the grand mean.

*** Equal weight to N years, with the complement of credibility given to the grand mean.

**** The credibilities add up to one, and thus no weight is given to the grand mean.

11.5 Validity of Results

The credibilities determined in Sections 10 and prior are all determined empirically by directly working with the data. In this section equations for the least squares credibilities have been introduced along with an assumed covariance structure.

The credibilities resulting from the use of the equations in this section are comparable to those determined in the previous sections empirically. As is shown in Appendix F, the observed pattern of squared errors is comparable to that derived from the assumed covariance structure.

Therefore, the results of this section are an appropriate means of estimating least squares credibilities for this example. How well these results would apply to another situation would depend on the covariance structure that underlies the particular data set.

12. MISCELLANEOUS

Section 12.1 contrasts the Meyers/Dorweiler Criterion vs. the other criteria. Section 12.2 discusses a somewhat artificial ratemaking example. It is intended to point the way towards applying these or similar methods to practical situations. Section 12.3 compares the baseball example to typical insurance applications. Section 12.4 shows that the estimates that result herein from the use of the credibilities are in balance. Section 12.5 discusses the question of what estimation method to select for predicting the future loss record of baseball teams. It is included in order to complete the illustrative example used throughout this paper.

12.1 Contrasting the Meyers/Dorweiler Criterion vs. the Other Criteria

Section 10.1 provides a good example of how criterion #3, Meyers/Dorweiler, differs on a basic conceptual level from the first two criteria. Both of the other criteria are concerned with eliminating large errors.[49] Criterion #1, least squares, does this since even a few large errors will

[49] Mahler [7] compares the credibilities that result from the application of the Bühlmann/least squares criterion and the credibilities that result from the application of the classical/limited fluctuation criterion.

greatly increase the sum of squared errors. Criterion #2, limited fluctuation, does this directly by minimizing the number of errors larger than the selected size.

In contrast, criterion #3, Meyers/Dorweiler, is concerned with the pattern of the errors. Large errors are not a problem, as long as there is no pattern relating the errors to the experience rating modifications. For example, consider the following two situations. In each case, for simplicity, only four risks are assumed.

Situation #1

Modification	Error
1.20	+30%
1.20	−30%
.80	+40%
.80	−40%

Situation #2

Modification	Error
1.30	+2%
1.10	+1%
.90	−1%
.70	−2%

Situation #2 with its small errors is preferable under the first two criteria. Situation #1 with its lack of a pattern of errors is preferable under the Meyers/Dorweiler criterion. Most actuaries would prefer Situation #2.

This example is not meant to discourage use of the Meyers/Dorweiler criterion. Rather it is meant to point out the potential hazards of relying solely on any single criterion, as well as the importance of understanding exactly what is being tested by any criterion that is being used.

12.2 A Ratemaking Example

Assume for a given line of insurance that the five most recent annual loss ratios are being combined to calculate a rate level indication.[50] Assume that it is three years from the latest year of data to the average date of loss under the proposed new rates.[51] A weighted average of the annual loss ratios will be used to estimate the future loss ratio.

If we assume a given covariance structure, equations 11.6 and 11.7 can be used to calculate the optimal least squares set of weights, Z_i, such that

$$\sum_{i=1}^{5} Z_i = 1.$$

Assume the covariance of the loss ratios separated by a given number of years is as follows:[52]

Separation in Years	Covariance in Loss Ratios (.00001)
0	130
1	60
2	55
3	50
4	45
5	40
6	35
7	30

Then the optimal weights are: 11.6%, 13.4%, 17.3%, 23.8%, 33.9%, with the more recent data receiving more weight. It is interesting to note that these weights can be reasonably approximated by the weights used in Walters [13], i.e., 10%, 15%, 20%, 25%, and 30%.

This example is for illustrative purposes only. It should not be taken as a derivation of the correct weights to use in any real world application. Unfortunately, in order to apply this idea to real world applications one

[50] The loss ratios for the separate years are presumed to have been adjusted for trend, development, and any other factors such as law changes.

[51] This period will vary, but $\Delta = 3$ is not uncommon.

[52] This would be produced by $\delta^2 = .0004$, $\zeta^2 = .0009$, $\ell(1) = .667$, $\ell(2) = .611$, $\ell(3) = .556$, $\ell(4) = .500$, $\ell(5) = .494$, $\ell(6) = .389$, $\ell(7) = .333$, where the quantities are defined as in Appendix D.

has to estimate the covariance matrix. This will be affected by shifting parameters over time. It will also be affected by the varying quantity of data available in each year.[53] It will be affected by the uncertainty in the trend estimates and loss development estimates applied to each year. These complications are beyond the scope of this paper.

12.3 Baseball Example vs. Typical Insurance Applications

In many typical insurance applications, credibility is used in the process of determining relativities. For example, credibility is used to determine the rate for a class or territory relative to the overall rate level. Credibility is also used in experience rating, where the rate for an individual risk is adjusted relative to an average.

In these situations, where a class, territory, or individual risk is compared relative to an average, the result depends on the other classes, territories, or risks which make up the average. An automobile territory with a low relativity in Massachusetts could have a higher loss potential than a automobile territory with a high relativity in Vermont. A workers compensation insured could have a credit experience modification simply because of the bad loss experience of several other employers in the same business in the same state. An insured with a .9 experience modification could have a higher loss potential than another risk with 1.1 modification in a different business or in a different state. The baseball example has this same feature. A team is being compared relative to the average in the league. The losing percentage only has relevance to rank teams in a single league relative to the average for that league. The difference in this example is that the average is a known constant. The grand mean is always .500.

In baseball if somebody loses, then somebody else wins. Thus the win-loss records of seven teams determine that of the remaining team in an eight team league.[54]

[53] In this paper, each year of baseball data represented a comparable number of games, so this aspect was not important.

[54] The win-loss record of teams in the same league should be negatively correlated by an amount proportional to the number of games the two teams have played.

This could have had a major impact on the analysis of this example. However, each team played each other team in the league approximately the same number of times each year and each team played approximately the same number of games in total.[55] Thus no team had its results distorted by playing a weaker or stronger schedule.

12.4 Estimates in Balance

The estimation methods used herein are always in balance.

The most general estimation method considered herein was given by equation 9.1, where the subscript j has been added to identify team j:

$$F_j = \sum_{i=1}^{N} Z_i X_{ij} + \left(1 - \sum_{i=1}^{N} Z_i\right) M$$

Then the average of the estimates F_j for all the teams in the league is given by:

$$\frac{1}{8} \sum_{j=1}^{8} F_j = \sum_{i=1}^{N} Z_i \left(\frac{1}{8} \sum_{j=1}^{8} X_{ij} \right) + \left(1 - \sum_{i=1}^{N} Z_i\right) M$$

$$= \left(\sum_{i=1}^{N} Z_i \right) M + \left(1 - \sum_{i=1}^{N} Z_i\right) M$$

$$= M$$

Note that for a given year i, the credibility Z_i assigned to each team's experience X_{ij} for that year is the same for all teams. Also note the fact that the grand mean is the same for all years.

That the estimates are in balance can be verified directly for the example given in Table 20. The predicted losing percentages for each year average to .500, subject to rounding.

12.5 Choice of a Prediction Method

The example in this paper is for illustrative purposes only; the purpose of this paper was not to predict baseball teams' win-loss records. Nevertheless, it may be of interest to choose a reasonable prediction method

[55] The schedule was exactly balanced, but a few scheduled games are sometimes not played.

TABLE 20

National League, Predictions of Losing Percentages

	NL1	NL2	NL3	NL4	NL5	NL6	NL7	NL8
1904	.541	.479	.461	.495	.469	.575	.379	.606
1905	.582	.568	.432	.456	.398	.610	.423	.534
1906	.615	.613	.424	.480	.368	.504	.408	.588
1907	.627	.568	.334	.533	.389	.531	.421	.598
1908	.594	.559	.351	.544	.448	.463	.426	.616
1909	.578	.598	.375	.529	.408	.476	.405	.631
1910	.633	.599	.366	.508	.427	.498	.354	.614
1911	.614	.576	.371	.509	.426	.492	.430	.581
1912	.651	.563	.411	.524	.400	.490	.443	.522
1913	.624	.582	.414	.511	.376	.508	.425	.557
1914	.561	.555	.438	.550	.377	.454	.471	.596
1915	.458	.526	.478	.570	.441	.504	.515	.509
1916	.468	.493	.505	.541	.504	.443	.517	.529
1917	.437	.438	.536	.574	.464	.440	.551	.559
1918	.503	.506	.519	.511	.423	.442	.603	.492
1919	.534	.519	.425	.493	.440	.512	.514	.565
1920	.560	.512	.467	.394	.413	.584	.509	.565
1921	.567	.448	.488	.458	.449	.573	.490	.528
1922	.509	.486	.543	.501	.419	.618	.449	.474
1923	.592	.492	.499	.469	.426	.596	.461	.466
1924	.596	.503	.485	.448	.412	.626	.450	.479
1925	.615	.448	.478	.462	.418	.605	.440	.536
1926	.553	.522	.525	.474	.441	.562	.418	.505
1927	.556	.516	.485	.458	.488	.583	.452	.465
1928	.571	.550	.472	.497	.441	.610	.422	.436
1929	.613	.509	.441	.487	.434	.648	.452	.418
1930	.603	.530	.406	.539	.449	.558	.442	.471
1931	.556	.472	.430	.570	.448	.614	.476	.434
1932	.564	.487	.452	.586	.448	.559	.498	.403
1933	.513	.478	.441	.584	.504	.520	.467	.492

TABLE 20

(CONTINUED)

	NL1	NL2	NL3	NL4	NL5	NL6	NL7	NL8
1934	.487	.537	.455	.588	.442	.564	.460	.468
1935	.487	.523	.447	.609	.434	.578	.492	.433
1936	.633	.534	.405	.558	.427	.568	.460	.417
1937	.545	.543	.442	.532	.426	.603	.470	.440
1938	.518	.563	.421	.582	.412	.579	.457	.467
1939	.498	.536	.436	.490	.449	.635	.450	.507
1940	.543	.486	.456	.437	.477	.641	.518	.445
1941	.547	.458	.494	.398	.508	.635	.495	.466
1942	.569	.406	.522	.433	.510	.659	.491	.411
1943	.572	.381	.538	.477	.472	.661	.525	.378
1944	.551	.452	.519	.457	.573	.590	.492	.368
1945	.559	.530	.515	.451	.544	.586	.455	.363
1946	.546	.470	.428	.543	.513	.629	.472	.399
1947	.450	.487	.468	.538	.562	.559	.538	.400
1948	.434	.459	.511	.531	.495	.579	.559	.433
1949	.453	.439	.548	.554	.504	.554	.497	.451
1950	.413	.492	.571	.564	.511	.502	.527	.419
1951	.440	.470	.564	.556	.470	.454	.570	.477
1952	.414	.501	.572	.548	.429	.503	.563	.472
1953	.411	.542	.518	.541	.428	.458	.646	.457
1954	.375	.455	.553	.543	.503	.475	.627	.469
1955	.416	.456	.554	.521	.423	.497	.626	.507
1956	.395	.454	.532	.515	.481	.497	.594	.531
1957	.419	.434	.572	.453	.521	.523	.566	.512
1958	.451	.421	.567	.482	.533	.504	.571	.471
1959	.507	.425	.538	.492	.501	.532	.492	.512
1960	.464	.451	.523	.509	.482	.551	.502	.518

Note: Using latest three years of data, with weights of 10%, 10%, 55% (55% weight to the most recent year; 25% weight to grand mean).

for this particular problem. Assume that $\Delta = 1$, i.e., 1910 data are available to predict 1911, etc.

Based on Table 19, the credibilities in Table 16 work well.

The author would recommend avoiding using many years of data unless it substantially improved the accuracy. It is better to keep things simple. For this particular problem, based on Table 19, there seems little advantage to using more than 3 years of data. For example, giving 55% weight to the most recent year, 10% weight to the next most recent year, 10% weight to the third most recent year, and the remaining 25% weight to the grand mean works reasonably well.[56]

The predictions that result from this method of estimation applied to the National League data are shown in Table 20.[57] The errors are shown in Table 21.

The mean squared error is .0046.[58] There is a 14% chance of an error of more than 20%. The correlation used in the Meyers/Dorweiler criterion is .02, not significantly different from zero. Thus according to all three criteria this prediction method works well.

13. CONCLUSIONS

The data from baseball used in this paper provide a useful way to examine and illustrate credibility concepts.

The methods and concepts illustrated here can be applied to problems actuaries deal with in insurance. However, this paper is only a first step; there is further work required to apply these general concepts to any specific practical situation.

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[56] Many other choices would also work reasonably well. This illustrates the typical situation where once the general form of the weights is determined, there is a range of weights that work well. Usually, the specific choice of weights within that range has relatively little impact on the final result.

[57] For example, the 1904 entry under NL2: .479 = (.10)(.419) + (.10)(.457) + (.55)(.485) + (.25)(.500), where the first three values are from Table 1, and .500 is the grand mean.

[58] The mean squared error is .0049 when the method is applied to both the AL and NL data sets. This is a standard deviation of 10½ losses out of a season of 150 games; the process standard deviation is about 6 losses out of a season of 150 games.

TABLE 21

National League, Errors of Predictions in Table 20

	NL1	NL2	NL3	NL4	NL5	NL6	NL7	NL8
1904	−.100	−.155	.069	.070	.162	−.083	−.052	.093
1905	−.087	−.116	.033	−.028	.084	.156	.050	−.089
1906	−.060	.047	.187	−.096	.000	−.032	.016	−.065
1907	.019	.007	.038	−.036	−.075	.096	.012	−.062
1908	.003	−.097	−.006	.018	.084	.002	.062	−.066
1909	−.128	−.043	.055	.032	.009	−.040	.129	−.014
1910	−.021	.015	.041	−.005	.018	.008	−.084	.026
1911	−.095	.003	−.032	−.033	.073	.012	−.018	.084
1912	−.009	−.058	.018	.014	.082	−.030	.059	−.066
1913	.081	.018	−.011	−.071	.040	.091	−.052	−.103
1914	.175	.042	−.056	−.060	−.078	−.065	−.081	.125
1915	.004	.052	−.045	.031	−.105	.096	−.011	−.020
1916	.054	.103	−.057	−.067	.070	.038	−.061	−.079
1917	−.092	−.098	.017	.080	.100	.012	−.118	.098
1918	−.070	−.042	.170	.042	−.004	−.111	.123	−.113
1919	−.056	.012	−.039	.179	.061	−.145	.025	−.041
1920	−.032	.116	−.046	−.070	−.029	−.011	.022	.052
1921	.083	−.045	−.094	−.084	.063	−.096	.078	.097
1922	−.145	−.020	.062	.059	.023	−.009	.001	.026
1923	−.057	−.014	.038	.060	.047	−.079	.026	−.018
1924	−.058	.100	.014	−.010	.020	−.010	.038	−.099
1925	.073	−.108	−.080	−.015	−.016	.049	.061	.039
1926	−.013	−.014	.057	.039	−.069	−.054	−.033	.083
1927	−.054	−.059	.041	−.052	.085	−.086	.062	.066
1928	−.102	.053	.063	.010	.045	−.107	−.019	.053
1929	−.023	−.033	.086	−.084	−.010	.112	.027	−.069
1930	.058	.088	−.010	−.078	.014	−.104	−.039	.068
1931	−.028	−.008	−.025	−.053	.020	.043	−.037	.090
1932	.064	.013	.036	−.024	−.084	.065	.056	−.129
1933	.052	−.097	−.001	−.034	.103	−.085	.032	.028

TABLE 21

(CONTINUED)

	NL1	NL2	NL3	NL4	NL5	NL6	NL7	NL8
1934	.004	.004	.025	−.068	.050	−.060	−.047	.089
1935	−.265	−.019	.096	.053	.029	−.004	.054	.056
1936	.094	−.031	−.030	.039	.024	−.081	.005	−.018
1937	.065	−.052	.046	−.104	.051	.002	.028	−.034
1938	.025	.026	.007	.129	−.035	−.121	.030	−.063
1939	−.085	.085	−.019	.120	−.041	−.067	−.106	.108
1940	−.029	.061	−.057	.091	−.049	−.032	.024	−.006
1941	−.050	.107	−.051	−.031	−.008	−.086	.021	.100
1942	−.032	.081	−.036	−.067	.069	−.063	−.060	.099
1943	.016	−.090	.022	.042	−.169	.077	.044	.060
1944	−.027	−.139	.006	.035	.008	−.011	.080	.050
1945	.000	.095	.151	−.153	.057	−.115	−.013	−.020
1946	.161	−.001	−.036	−.022	−.091	.077	−.119	.027
1947	.060	.045	−.084	.012	.088	−.038	−.059	−.022
1948	−.021	.054	−.073	−.051	.001	.008	.098	−.015
1949	.083	−.074	−.056	−.043	−.022	.080	−.042	.074
1950	−.009	.031	−.011	−.005	.069	.093	−.100	−.071
1951	.058	−.036	−.033	−.002	.094	−.072	−.014	.003
1952	.041	−.081	.072	−.004	.026	.068	−.164	.043
1953	.093	.139	−.060	−.017	−.117	−.003	−.029	−.004
1954	−.028	.033	−.031	.024	.133	−.038	−.029	−.063
1955	.057	.008	.025	.008	−.058	−.003	.016	−.051
1956	−.001	.051	−.078	.106	−.084	−.042	.023	.025
1957	−.036	.051	−.025	−.028	−.031	.023	−.031	.077
1958	−.088	.018	.035	−.024	.052	−.048	.116	−.061
1959	.071	−.024	.019	−.027	.040	−.052	−.002	−.027
1960	−.004	.022	−.087	−.056	−.005	−.066	.119	.076

Note: Predicted Losing Percentage minus Actual Losing Percentage

When shifting parameters over time is an important phenomenon, older years of data should be given substantially less credibility than more recent years of data. The more significant this phenomenon, the more important it is to minimize the delay in receiving the data that is to be used to make the prediction.

In this paper three different criteria were examined that can be used to select the optimal credibility: least squares, limited fluctuation, and Meyers/Dorweiler. In applications, one or more of these three criteria should be useful. While the first two criteria are closely related, the third criterion can give substantially different results than the others.

Generally the mean squared error can be written as a second order polynomial in the credibilities. The coefficients of this polynomial can be written in terms of the covariance structure of the data. This in turn allows one to obtain linear equation(s) which can be solved for the least squares credibilities in terms of the covariance structure.

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APPENDIX A

SOME RELEVANT FEATURES OF BASEBALL

Baseball is a competitive sport involving a combination of luck and skill. Two teams play against each other in a game; the team that scores the most "runs" wins the game, the other team loses.[1]

Each team has nine players in the game at a time.[2] Players may be substituted for, but once they leave the game they cannot return. Over this period of time each team had 20 to 25 players on its roster.[3] The individual skills of the players, as well as how their skills complement each other, has a direct impact on the quality of the team.

In addition to the players, a team has coaches and a field manager. By supervising the players' training and conditioning, providing advice, deciding who plays, and by various decisions throughout the game, these people have some effect on the percentage of games lost or won by the team.

Each team has an owner(s) and other office personnel.[4] By developing new players, trading for players with other teams, etc., management has some effect on the percentage of games lost or won by the team.

All of these elements that affect the quality of the team shift over time. A team's roster of players typically changes a little during the course of a single year; over the course of several years the changes are substantial. It is unusual for a player to be with a single team for more than 10 years, although on very rare occasions a player has played for a single team for 20 years.

Even if the identity of the players were to stay the same, the skill level of individual players changes over time. The most important effect is aging; as a player gets older he generally improves until he reaches a

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[1] While it is possible for a baseball game to end in a tie, such games are ignored in major league standings.

[2] Currently the American League has added a tenth player, the designated hitter.

[3] Of the players on the roster, about half get most of the playing time, while the remainder see much less playing time.

[4] During the latter half of this period a team had a general manager.

peak and then declines. Injuries can have a profound impact on a player's skill; sometimes that impact is temporary while sometimes it is permanent.

The field managers and coaching staff also change over time.[5] In addition, the owner(s) and upper management change, but much less frequently.

Finally, a team occasionally relocates to another city.

It might be useful to think of the following analogy to a workers compensation risk. The baseball players correspond to the workers in the factory. The field manager corresponds to the plant manager. The baseball upper management corresponds to the corporation's upper management.

[5] Quite often the departure of the field manager will be related to the poor record of the team.

APPENDIX B

MEYERS/DORWEILER CRITERION AND KENDALL'S TAU

If an experience rating plan works properly, then after the application of experience rating, an insurer should be equally willing to write debit and credit risks. In other words, the modified loss ratio of expected losses to modified premiums should be the same for debit and credit risks.

Mathematically, we desire that the correlation between the experience modification and the modified loss ratio be zero.[1]

In the example in this paper, the experience modification corresponds to the ratio of predicted losing percentage to the grand mean losing percentage.[2] For example, a predicted losing percentage of 60% is equivalent to an experience modification of 60% ÷ 50% = 1.2. The modified loss ratio corresponds to the ratio of the actual losing percentage and the predicted losing percentage.[3] The third criterion used in this paper is that the correlation between these two ratios be zero. This corresponds to the criterion used by Meyers.

Meyers [1] uses the Kendall τ to measure correlation.

Let X and Y be two vectors of length n.[4] Kendall's τ can be calculated as follows [14]. Suppose Y is arranged in its natural order. Assume that the corresponding ranks of X are $X_1, X_2, \ldots X_n$, a permutation of 1, 2, ... n. Let Q be the number of inversions in $X_1, X_2, \ldots X_n$.[5] Then let

$$\tau = 1 - \frac{4Q}{n(n-1)}$$

[1] If the correlation is positive, then insurers would prefer to write credit risks. The credits and debits given are on average too small, i.e., the credibility assigned to the experience is too small. The situation is reversed for a negative correlation.

[2] The predicted losses are equal to the experience modification times the expected losses for an average risk in the class. In a more general situation one would have classifications of risks; in this example we have only one such classification and thus use the grand mean rather than the class mean.

[3] In general the modified loss ratio is equal to the expected loss ratio times the actual losses over the predicted losses. In this example, the expected loss ratio can be thought of as unity.

[4] In our case, X would be the experience modifications and Y would be the corresponding modified loss ratios.

[5] For example, in the X-ranking 3214 for $n = 4$, there are 3 inversions of order.

τ is symmetrically distributed on the range $[-1, +1]$. As is usual for measures of correlation, $+1$ signifies complete agreement and -1 signifies complete disagreement.

As shown in Kendall and Stuart [14],

$$\text{Var}\ \tau = \frac{2(2n + 5)}{9n(n - 1)}$$

As n approaches infinity the distribution of τ approaches the normal distribution.

In the examples in this paper, the variance of τ varies from .0009 to .0016.[6] The standard deviation of τ goes from .031 to .040. Thus an approximate 95% confidence interval around zero for τ has a radius of approximately .07, about two standard deviations.

[6] $n = 8$ teams times 59 years $= 472$ to $n = 8$ times $35 = 280$.

APPENDIX C

MATRIX EQUATIONS FOR LEAST SQUARES CREDIBILITY

In this appendix, equations 11.2, 11.3, 11.4, and 11.7 in the main text are derived. The squared error is written as a second order polynomial in the credibilities, with the coefficients depending on the covariance structure discussed in Appendix D. This squared error is minimized by setting the partial derivative(s) with respect to the credibilities equal to zero.

Assume an estimate for year $N + \Delta$, using N years of data, is given by:

$$F = \sum_{i=1}^{N} Z_i X_i + (1 - \Sigma Z_i) M$$

where X_i is the data for year i, and M is the grand mean.[1] Let $Z_0 = 1 - \Sigma Z_i$. Write Z for the vector $Z_0, Z_1, \ldots, Z_N$.

Then the mean squared error between the prediction and the observation is given by the expected value of the squared difference between F and $X_{N+\Delta}$.

$$\begin{aligned} V(Z) &= \mathrm{E}[(F - X_{N+\Delta})^2] \\ &= \mathrm{E}\left[\left\{ \sum_{i=1}^{N} Z_i \left(X_i - X_{N+\Delta}\right) + Z_0 \left(M - X_{N+\Delta}\right)\right\}^2\right] \\ &= \sum_{i=1}^{N} Z_i^2 \mathrm{E}[(X_i - X_{N+\Delta})^2] \\ &\quad + \sum_{i=1}^{N} \sum_{j \neq i} Z_i Z_j \, \mathrm{E}[(X_i - X_{N+\Delta})(X_j - X_{N+\Delta})] \\ &\quad + 2 \sum_{i=1}^{N} Z_0 Z_i \, \mathrm{E}[(X_i - X_{N+\Delta})(M - X_{N+\Delta})] \\ &\quad + Z_0^2 \, \mathrm{E}[(M - X_{N+\Delta})^2] \end{aligned}$$

[1] It is assumed that the grand mean is known. This is the case in this paper. It is the case whenever one is only concerned with relativities compared to the overall average.

From Appendix D we have,[2]

$$E[(X_i - X_{N+\Delta})^2] = 2\delta^2 + 2\zeta^2(1 - \ell(N + \Delta - i))$$

$$E[(X_i - X_{N+\Delta})(X_j - X_{N+\Delta})] = \delta^2 + \zeta^2(1 + \ell(|i - j|) - \ell(N + \Delta - i) - \ell(N + \Delta - j))$$

$$E[(X_i - X_{N+\Delta})(M - X_{N+\Delta})] = \delta^2 + \zeta^2(1 - \ell(N + \Delta - i))$$

$$E[(M - X_{N+\Delta})^2] = \delta^2 + \zeta^2 + \tau^2$$

Therefore

$$\begin{aligned} V(Z) = {} & \sum_{i=1}^{N} Z_i^2(2\delta^2 + 2\zeta^2(1 - \ell(N + \Delta - i))) \\ & + \sum_{i=1}^{N} \sum_{j \neq i} Z_i Z_j \Big[\delta^2 + \zeta^2(1 + \ell(|i - j|) - \ell(N + \Delta - i) \\ & - \ell(N + \Delta - j)\Big] \\ & + 2Z_0 \sum_{i=1}^{N} Z_i(\delta^2 + \zeta^2(1 - \ell(N + \Delta - i)) \\ & + Z_0^2(\delta^2 + \tau^2 + \zeta^2) \end{aligned}$$

$$\begin{aligned} V(Z) = {} & \delta^2 \left[\sum_{i=0}^{N} \sum_{j=0}^{N} Z_i Z_j + \sum_{i=1}^{N} Z_i^2 \right] + Z_0^2 \tau^2 \\ & + \zeta^2 \Big[\sum_{i=0}^{N} \sum_{j=0}^{N} Z_i Z_j + \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i Z_j(\ell(|i - j|) - \ell(N + \Delta - i) \\ & - \ell(N + \Delta - j)) - 2Z_0 \sum_{i=1}^{N} Z_i \ell(N + \Delta - i) \Big] \end{aligned}$$

but

$$\sum_{i=0}^{N} Z_i = Z_0 + \sum_{i=1}^{N} Z_i = (1 - \sum_{i=1}^{N} Z_i) + \sum_{i=1}^{N} Z_i = 1$$

[2] In Appendix D, $X(\theta,t)$ = the observation for risk θ at time t. Since in this appendix none of the calculations are performed for individual risks, the θ has been suppressed in order to simplify the notation.

Therefore

$$V(Z) = \delta^2 + Z_0^2\tau^2 + \zeta^2 + \delta^2 \sum_{i=1}^{N} Z_i^2$$

$$+ \zeta^2 \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i Z_j(\ell(|i - j|) - \ell(N + \Delta - i) - \ell(N + \Delta - j))$$

$$- \zeta^2 Z_0 2 \sum_{i=1}^{N} Z_i \ell(N + \Delta - i)$$

$$V(Z) = \delta^2 + \zeta^2 + \tau^2 + \tau^2 \left[\sum_{i=1}^{N} Z_i \right]^2 - 2\tau^2 \sum_{i=1}^{N} Z_i$$

$$+ \delta^2 \sum_{i=1}^{N} Z_i^2 + \zeta^2 \sum_{i=1}^{N} \sum_{j=i}^{N} Z_i Z_j(\ell(|i - j|) - \ell(N + \Delta - i)$$

$$- \ell(N + \Delta - j)) - 2\zeta^2 \sum_{i=1}^{N} Z_i \ell(N + \Delta - i)$$

$$+ 2\zeta^2 \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i Z_j \ell(N + \Delta - i)$$

$$V(Z) = \delta^2 + \zeta^2 + \tau^2 + \tau^2 \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i Z_j - 2\tau^2 \sum_{i=1}^{N} Z_i$$

$$+ \delta^2 \sum_{i=1}^{N} Z_i^2 + \zeta^2 \sum_{i=1}^{N} \sum_{j=i}^{N} Z_i Z_j(\ell(|i - j|) - \ell(N + \Delta - j))$$

$$- 2\zeta^2 \sum_{i=1}^{N} Z_i \ell(N + \Delta - i)$$

$$V(Z) = \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i Z_j(\delta^2 \delta_{ij} + \tau^2 + \zeta^2 \ell(|i - j|))$$

$$- 2 \sum_{i=1}^{N} Z_i(\tau^2 + \zeta^2 \ell(N + \Delta - i)) + \delta^2 + \zeta^2 + \tau^2$$

This is equation 11.2 in the main text, with $\delta^2\delta_{ij} + \zeta^2\ell(|i - j|) = C(|i - j|)$ the covariance between data for a given risk $|i - j|$ years apart. It is left as an exercise to the reader to verify that the formula for the mean squared error compared to the underlying mean rather than the observed value would be exactly δ^2 less.

In order to minimize this squared error, one sets the partial derivatives with respect to Z_i equal to zero. This yields the following set of N equations.

$$\begin{aligned} 2Z_i(\delta^2 + \tau^2 + \zeta^2) + \sum_{j \neq i} 2Z_j(\tau^2 + \zeta^2\ell(|i - j|)) \\ - 2(\tau^2 + \zeta^2\ell(N + \Delta - i)) \\ = 0, \quad i = 1, \ldots, N \end{aligned}$$

$$\sum_{j=1}^{N} Z_j(\delta^2\delta_{ij} + \tau^2 + \zeta^2\ell(|i - j|)) = \tau^2 + \zeta^2\ell(N + \Delta - i),$$
$$i = 1, \ldots, N$$

This is equation 11.3 in the main text, again with

$\delta^2\delta_{ij} + \zeta^2\ell(|i - j|) = C(|i - j|)$.

It is worth noting that equation 11.3 is very similar to the usual general matrix equation for optimal least squares credibilities:

$$\vec{Z} = \frac{\mathrm{COV}[\vec{X},Y]}{\mathrm{COV}[\vec{X},\vec{X}]}$$

where $\vec{X}$ is the vector of observations, and Y is the quantity to be estimated.[3] Here in equation 11.3, there is an additional term of τ^2, the between variance, added to the covariances. This is due to the application of the complement of credibility to the grand mean.

In the absence of shifting parameters over time ($\zeta^2 = 0$), the squared error is given by:

$$V(Z) = \delta^2\left(1 + \sum_{i=1}^{N} Z_i^2\right) + \tau^2\left(1 - \sum_{i=1}^{N} Z_i\right)^2$$

[3] See, for example, Theorem 3.3 in Chapter III of De Vylder [15].

The optimal credibilities are given by the solution to the equations:

$$\sum_{j=1}^{N} Z_j(\delta^2\delta_{ij} + \tau^2) = \tau^2, \quad i = 1, \ldots, N$$

The solution has all the credibilities equal:

$$Z_i = \frac{\tau^2}{N\tau^2 + \delta^2}, \quad i = 1, \ldots, N$$

$$\sum_{i=1}^{N} Z_i = \frac{N\tau^2}{N\tau^2 + \delta^2} = \frac{N}{N + \delta^2/\tau^2}$$

This is the familiar expression for the least squares credibility in the absence of shifting parameters over time.

If we set $Z_i = Z/N$ for $i = 1, \ldots, N$ then equation 11.2 becomes:

$$V(Z) = \frac{Z^2}{N^2}\left\{N\delta^2 + N^2\tau^2 + \zeta^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \ell(|i-j|)\right\}$$

$$- 2\frac{Z}{N}\left\{N\tau^2 + \zeta^2 \sum_{i=1}^{N} \ell(N + \Delta - i)\right\} + \delta^2 + \zeta^2 + \tau^2$$

Setting the derivative of $V(Z)$ equal to zero gives the least squares credibility:

$$Z = N\frac{N\tau^2 + \zeta^2 \sum_{i=1}^{N} \ell(N + \Delta - i)}{N^2\tau^2 + N\delta^2 + \zeta^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \ell(|i-j|)}$$

This is equation 11.4 in the main text, with $C(|i-j|) = \delta^2\delta_{ij} + \zeta^2\ell(|i-j|)$.

We can minimize $V(Z)$ in equation 11.2, given the constraint $\Sigma_{i=1}^{N} Z_i = 1$, by using Lagrange Multipliers.

We set the partial derivatives with respect to Z_i of

$V(Z) - \lambda\left(\sum_{i=1}^{N} Z_i - 1\right)$ equal to zero.

This produces the following N equations:

$$\sum_{j=1}^{N} Z_j(\delta^2\delta_{ij} + \zeta^2\ell(|i - j|)) = \zeta^2\ell(N + \Delta - 1) + \frac{\lambda}{2} \quad i = 1, \ldots, N$$

This is equation 11.7 in the main text. It is worth noting the absence from the above equation of τ^2, the between variance. This follows logically from the fact that the grand mean is given no weight and each risk is estimated solely from its own data.

APPENDIX D

COVARIANCE STRUCTURE

In this appendix, the covariance structure for the data sets in Tables 1 and 2 will be analyzed. As discussed in Section 11.1, the variance is the sum of three pieces, the between variance, the variance due to shifting parameters over time, and the process variance excluding the effect of shifting parameters over time. The analysis herein will define these three pieces.

Let $X(\theta,t)$ be the observation for risk θ at time t.

Let $\mu(\theta,t)$ be the expected value for risk θ at time t.

$\mu(\theta,t) = \mathrm{E}[X(\theta,t)]$.

Let $\mu(\theta) = \mathrm{E}_t[X(\theta,t)]$.

Let M be the grand mean.

$M = \mathrm{E}[\mu(\theta,t)] = \mathrm{E}_\theta[\mu(\theta)]$.

In our case, θ and t are both discrete rather than continuous variables. We can observe X. M is known since we are dealing with relativities compared to the overall average. On the other hand $\mu(\theta,t)$ is unknown and can never be observed directly.

We can observe the squared error that results from using different estimations. This squared error can be usefully expressed in another form. To do so, we split the variance of X into various pieces. Define

$$\delta^2 = \mathrm{E}_\theta[\mathrm{E}_t[\mathrm{E}[(X(\theta,t) - \mu(\theta,t))^2|\theta,t]]]$$

$$\zeta^2 = \mathrm{E}_\theta[\mathrm{E}_t[(\mu(\theta,t) - \mu(\theta))^2|\theta]]$$

$$\begin{aligned}\zeta^2\ell(s) &= \mathrm{E}_\theta[\mathrm{E}_t[\mathrm{COV}[X(\theta,t),X(\theta,t+s)]|\theta]] \\ &= \mathrm{E}_\theta[\mathrm{E}_t[\mathrm{COV}[\mu(\theta,t),\mu(\theta,t+s)]|\theta]]\end{aligned}$$

$$\tau^2 = \mathrm{VAR}_\theta[\mathrm{E}_t[\mu(\theta,t)]] = \mathrm{VAR}_\theta[\mu(\theta)]$$

Then δ^2 is the process variance excluding any impact of shifting risk parameters over time. ζ^2 is the variance due to shifting parameters over time. $\ell(s)$ is a correlation measuring how much the risk parameters shift over time. $\ell(0) = 1$. $\ell(s) \leq 1$ for $s > 0$.[1] τ^2 is the parameter variance, the variance between the different risks.

For later convenience of notation define

$$\delta^2(\theta,t) = \mathrm{E}[(X(\theta,t) - \mu(\theta,t))^2|\theta,t]$$

$$\delta^2(\theta) = \mathrm{E}_t[\delta^2(\theta,t)]$$

$$\zeta^2(\theta) = \mathrm{E}_t[(\mu(\theta,t) - \mu(\theta))^2|\theta]$$

$$\ell(s,\theta) = \mathrm{E}_t[\mathrm{COV}[\mu(\theta,t), \mu(\theta,t + s)]] + \zeta^2(\theta)$$

then

$$\delta^2 = \mathrm{E}_\theta[\delta^2(\theta)] = \mathrm{E}_{\theta,t}[\delta^2(\theta,t)]$$

$$\zeta^2 = \mathrm{E}_\theta[\zeta^2(\theta)]$$

$$\ell(s)\zeta^2 = \mathrm{E}_\theta[\ell(s,\theta)\zeta^2(\theta)]$$

It is useful to rearrange the definitions of the variances in the usual manner so as to express the expected value of a quantity squared as the sum of a squared mean and a variance.

$$\mathrm{E}[X^2(t,\theta)|t,\theta] = \mu^2(t,\theta) + \delta^2(\theta,t)$$

$$\mathrm{E}_t[\mu^2(t,\theta)] = \mu^2(\theta) + \zeta^2(\theta)$$

$$\mathrm{E}_\theta[\mu^2(\theta)] = M^2 + \tau^2$$

A similar expression can be derived from the definition of the covariance.

$$\mathrm{E}_t[\mu(t,\theta)\mu(t + s,\theta)] = \mu^2(\theta) + \ell(s,\theta)\zeta^2(\theta)$$

For the formula for the expected value of the squared error of the estimate from the observation, one needs to express various expected values in terms of the variances and correlations defined above.

[1] One should note that it is an assumption that this correlation depends only upon the separation of the two years in question. Whether or not this is a reasonable approximation to reality is an empirical question which depends on the particular application.

$$\begin{aligned}
\mathrm{E}_{t,\theta}[X^2(t,\theta)] &= \mathrm{E}_\theta[\mathrm{E}_t[\mathrm{E}[X^2(t,\theta)|t,\theta]]] \\
&= \mathrm{E}_\theta[\mathrm{E}_t[\mu^2(t,\theta) + \delta^2(\theta,t)]] \\
&= \mathrm{E}_\theta[\mu^2(\theta) + \zeta^2(\theta) + \delta^2(\theta)] \\
&= M^2 + \tau^2 + \zeta^2 + \delta^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}_{t,\theta}[X(t,\theta)X(t + s,\theta)] &= \mathrm{E}_\theta[\mathrm{E}_t[\mathrm{E}[X(t,\theta)X(t + s,\theta)|t,\theta]]] \\
&= \mathrm{E}_\theta[\mathrm{E}_t[\mu(t,\theta)\mu(t + s,\theta)]] \\
&= \mathrm{E}_\theta[\mu^2(\theta) + \ell(s,\theta)\zeta^2(\theta)] \\
&= M^2 + \tau^2 + \ell(s)\zeta^2
\end{aligned}$$

$$\mathrm{E}_{t,\theta}[MX(t,\theta)] = M\mathrm{E}[X(t,\theta)] = M^2$$

Then it follows that:

$$\begin{aligned}
\mathrm{E}_{t,\theta}[(X(t,\theta) - X(t + s,\theta))^2] &= \mathrm{E}_{t,\theta}[X^2(t,\theta)] + \mathrm{E}_{t,\theta}[X^2(t + s,\theta)] \\
&\quad - 2\mathrm{E}_{t,\theta}[X(t,\theta)X(t + s,\theta)] \\
&= M^2 + \tau^2 + \zeta^2 + \delta^2 + M^2 + \tau^2 \\
&\quad + \zeta^2 + \delta^2 - 2(M^2 + \tau^2 + \ell(s)\zeta^2) \\
&= 2\delta^2 + 2\zeta^2(1 - \ell(s))
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}_{t,\theta}[(X(t,\theta) - X(t + s,\theta))(X(t + u,\theta) - X(t + s,\theta))] \\
&\quad = \mathrm{E}_{t,\theta}[X(t,\theta)X(t + u,\theta)] + \mathrm{E}_{t,\theta}[X^2(t + s,\theta)] \\
&\qquad - \mathrm{E}_{t,\theta}[X(t + s,\theta)X(t + u,\theta)] - \mathrm{E}_{t,\theta}[X(t,\theta)X(t + s,\theta)] \\
&\quad = M^2 + \tau^2 + \ell(u)\zeta^2 + M^2 + \tau^2 + \zeta^2 + \delta^2 - (M^2 + \tau^2 \\
&\qquad + \ell(s - u)\zeta^2) - (M^2 + \tau^2 + \ell(s)\zeta^2) \\
&\quad = \delta^2 + \zeta^2(1 + \ell(u) - \ell(s - u) - \ell(s))
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}_{t,\theta}[(X(t,\theta) - X(t + s,\theta))(M - X(t + s,\theta))] \\
&\quad = M^2 - M^2 + (M^2 + \tau^2 + \zeta^2 + \delta^2) \\
&\qquad - (M^2 + \tau^2 + \ell(s)\zeta^2) \\
&\quad = \delta^2 + \zeta^2(1 - \ell(s))
\end{aligned}$$

$$\begin{aligned} \mathrm{E}_{t,\theta}[(M - X(t,\theta))^2] &= M^2 - 2M^2 + (M^2 + \tau^2 + \zeta^2 + \delta^2) \\ &= \delta^2 + \zeta^2 + \tau^2 \end{aligned}$$

These results are used in Appendix C.

It is of interest to note that variance of $X = \mathrm{E}_{t,\theta}[(M - X(t,\theta))^2] = \delta^2 + \zeta^2 + \tau^2$. This is the split of the variance of X into three pieces that was discussed above.

Let $\mathrm{C}(s)$ = Covariance for data for the same risk, Δ years apart. Then for $s > 0$

$$\mathrm{C}(s) = \mathrm{E}[(X(t,\theta) - \mu(\theta))(X(t + s,\theta) - \mu(\theta))]$$

$$\begin{aligned} \mathrm{C}(s) &= \mathrm{E}[X(t,\theta)X(t + s,\theta)] - \mathrm{E}[X(t,\theta)\mu(\theta)] - \mathrm{E}[X(t + s)\mu(\theta)] \\ &\quad + \mathrm{E}[\mu^2(\theta)] \\ &= M^2 + \tau^2 + \ell(s)\zeta^2 - (M^2 + \tau^2) - (M^2 + \tau^2) + M^2 + \tau^2 \\ &= \ell(s)\zeta^2 \end{aligned}$$

$$\begin{aligned} \mathrm{C}(0) &= \mathrm{E}[(X(t,\theta) - \mu(\theta))^2] = \mathrm{E}(X^2(t,\theta)] - 2\mathrm{E}[X(t,\theta)\mu(\theta)] \\ &\quad + \mathrm{E}[\mu^2(\theta)] \\ &= M^2 + \tau^2 + \zeta^2 + \delta^2 - 2(M^2 + \tau^2) + M^2 + \tau^2 \\ &= \zeta^2 + \delta^2 \end{aligned}$$

It is worth noting that the covariance structure assumed herein differs from that in Gerber and Jones [16]. The covariance structure which in Gerber and Jones is shown to give credibility formulas of the updating variety[2] can be written as:

$$\mathrm{COV}[X_i,X_j] = \begin{bmatrix} W_i & i < j \\ W_i + V_i & i = j \end{bmatrix}$$

That covariance structure would assume for example that the covariance of the 1940 data with the data for each of the years earlier than

[2] Credibilities of the updating variety are such that new estimate = (prior estimate × complement of credibility) + (new data × credibility). This is the form of the estimate discussed in Section 9.1.

1940 is the same. In fact we observe that the distance between the years has an extremely significant impact on the covariance between the years.

The covariance structure assumed here can be written as:

$$\mathrm{COV}[X_i,X_j] = \left[\begin{array}{ll} \ell(j-i)\zeta^2 & i<j \\ \zeta^2+\delta^2 & i=j \end{array} \right.$$

Thus the optimal least squares credibilities that result from the matrix equations that are given in Appendix C will generally not be of the updating variety.[3]

We can directly estimate only the following quantities from the data: τ^2, C(0), C(1), C(2), etc. Not coincidentally, these are the quantities that enter into the formula in Appendix C for the squared error. Thus, these are also the quantities that enter into the calculation of the optimal credibilities.

Thus, it is not necessary to estimate δ^2 by itself. However, if one does so, the values for ζ^2 and $\ell(i)$ follow. We will estimate δ^2 here solely in order to aid our understanding; it does not affect any of the calculated values of the credibilities.[4]

For a binomial process, with a success rate of .4 or .6, the variance is $.24n$.[5] This is approximately the variance for the average risk in this example, with $n = 150$.[6] The resulting variance of games lost is (150)(.24). The variance in losing percentage is $(150)(.24)/(150)^2 = .0016$.

Thus a reasonable approximate value for δ^2 is .0016. The values for the variances and correlations are shown in Table D1. It should be noted that as the difference in years increases, the correlations get close to zero.

For example, the observed value for the NL data for $\delta^2 + \zeta^2 = .007892$. Thus since we assume $\delta^2 = .001600$, we estimate $\zeta^2 = .006292$. The observed value of $\zeta^2\ell(1) = .004919$. Thus we estimate $\ell(1) = .004919/.006292 = .782$. For this example, the observed value of $\tau^2 = .001230$.

[3] They will be of the updating variety when $\ell(s) = 1$ for all s.

[4] In general if something cannot be observed in the squared errors, then it is not needed to calculate the optimal least squares credibilities.

[5] The variance is $p(1-p)n$.

[6] Teams played about 150 games per year over this period of time.

It is important to note that the total variance of the observations is equal to $\delta^2 + \zeta^2 + \tau^2 = .009122$. Thus, what has been done here is just an analysis of variance, breaking the variance into its various sources. For this example, about 13.5% of the variance of the observation is due to the differences between the risks, about 17.5% is due to the process variance, and about 69.0% is due to shifting parameters over time.

One can verify that the observed pattern in the covariance structure in Table D1 is not due solely to random chance. One can rearrange the data in random fashion, and observe the covariances.

TABLE D1

COVARIANCE STRUCTURE

	NL	AL
τ^2	.001230	.001619
δ^2*	.001600	.001600
ζ^2**	.006292	.006275
$\ell(0)$***	1.000	1.000
$\ell(1)$	.782	.721
$\ell(2)$	.543	.506
$\ell(3)$	.497	.384
$\ell(4)$	.404	.283
$\ell(5)$	.288	.124
$\ell(6)$	.249	.061
$\ell(7)$	.158	−.016
$\ell(8)$	.062	−.089
$\ell(9)$	−.012	−.170
$\ell(10)$	−.063	−.140

* δ^2 estimated as .001600 based on an assumed binomial process.
** ζ^2 is based on the assumed value of δ^2 and the observed value of $\zeta^2 + \delta^2$.
*** $\ell(0)$ is unity by definition.

First one can rearrange the entries in each row of Table 1; for each row separately, assign each entry in that row to a randomly selected column. Similarly one can rearrange the entries in each column of Table 1; for each column separately, assign each entry in that column to a randomly selected row. The resulting covariances that are computed for these two "scrambled" data sets are shown in Table D2. All of the covariances $\ell(i)$, $i > 0$ are close to zero. Therefore, one can conclude that there is a significant pattern being displayed in Table D1.

TABLE D2

COVARIANCE STRUCTURE, SCRAMBLED DATA

	NL Entries in Each Row Rearranged	NL Entries in Each Column Rearranged
τ^2	.000191	.001230
δ^2*	.001600	.001600
ζ^2**	.007330	.006292
$\ell(0)$***	1.000	1.000
$\ell(1)$	.010	−.117
$\ell(2)$	−.009	.021
$\ell(3)$	.008	−.070
$\ell(4)$	−.084	−.035
$\ell(5)$	−.025	−.039
$\ell(6)$	−.020	−.006
$\ell(7)$	−.030	−.053
$\ell(8)$	−.058	.082
$\ell(9)$	.049	.091
$\ell(10)$	.042	−.019

* δ^2 estimated at .001600 based on an assumed binomial process.

** ζ^2 is based on the assumed value of δ^2 and the observed value of $\zeta^2 + \delta^2$.

*** $\ell(0)$ is unity by definition.

APPENDIX E

PUTTING THE REDUCTION IN SQUARED ERROR IN CONTEXT

The first criterion used to determine the optimal credibility is to minimize the squared error. Using the optimal credibility based on this criterion will reduce the squared error between the observed and predicted result. What should be considered a significant reduction in squared error?

Let us examine an example. For the NL data set, using one year of data, the optimal credibility is 68% as shown in Table 9. As shown in Table 6 the mean squared errors are:

Z	Mean Squared Error
0	.0091
68%	.0049
100%	.0059

In this case, by the use of credibility, the squared error has been reduced from .0059 if the data were relied upon totally, or .0091 if the data were totally ignored, to .0049. In this case, the squared error has been reduced to 83% (.0049/.0059) of its previous value.[1]

All of these squared errors include the variation of the observed results around the expected value.[2] The use of credibility does not affect this source of variation. Thus credibility methods cannot reduce the squared error between the observed value and the estimated/predicted value to as great an extent as they reduce the squared error between the true mean and the estimated/predicted mean.[3]

It is shown in Mahler [9] that the best that can be done using credibility to combine two estimates is to halve the mean squared error between the estimated and theoretical true underlying mean. However,

[1] The "previous" value of the squared error is considered to be the minimum of the squared errors that result from either ignoring the data entirely or relying on the data entirely.

[2] This random variation is usually referred to as process risk.

[3] It should be noted that the former squared error is concrete and easily observed, while the latter squared error is theoretical and difficult if not impossible to observe.

in this paper the squared error being examined is between the estimated/predicted and the observed result, rather than the true underlying mean. This squared error is inherently larger due to the random variation in the observed result. Also the result derived in Mahler [9] was derived in the absence of shifting parameters over time.

It turns out that, in the current case, the best that can be done using credibility to combine two estimates is to reduce the mean squared error between the estimated and observed values to 75% of the minimum of the squared errors from either relying solely on the data or ignoring the data.[4] One can think of half[5] of the squared error as being due to two sources: the inherent process variance associated with comparing to observed results, and the presence of shifting parameters over time. This portion of the squared error is independent of the value chosen for the credibility. The remainder of the squared error can be thought of as that which is affected by the choice of the value of credibility; as stated above this can be at most cut in half by the use of credibility methods. If half of the squared error is cut in half, this reduces the total squared error to 75% of what it was.

Assume one is estimating the future by credibility weighting together a single year of data and the grand mean.[6] Let $V(0)$ be the squared error between the predicted and observed results for $Z = 0$. Let $V(1)$ be the squared error between the predicted and observed results for $Z = 1$. Then as is shown in Appendix F:

Z	Squared Error Between Predicted and Observed
0	$V(0)$
Optimal	$V(1)\left(1 - \dfrac{V(1)}{4V(0)}\right)$
100%	$V(1)$

with the optimal credibility given by: Z optimal $= 1 - V(1)/2V(0)$.

[4] When using more than two or more years of data, the reduction in squared error depends on the impact of shifting parameters over time. However, in the absence of shifting parameters over time, for N years with the same weight applied to each year, the maximum possible reduction is $1/(2(N + 1))$.

[5] This is only a half for the case when the squared errors for $Z = 0$ and $Z = 1$ are equal. However, this is the case when one gets the maximum reduction in squared error.

[6] The formula given below does not hold when using several years of data.

In the example above, we had $V(0) = .0091$, $V(1) = .0059$. Using these values in the above formula gives Z optimal $= 68\%$, equal to the empirically determined 68%. The formula for the minimum squared error gives a value of .0049, which is equal to the empirical minimum squared error. The reduction of the squared error to 83% of its previous value appears significant in light of the maximum possible reduction to 75%.[7]

[7] The maximum reduction is possible when the squared errors for $Z = 0$ and $Z = 1$ are equal.

APPENDIX F

SQUARED ERRORS

In Appendix C, the fundamental formula for the squared error was derived:

$$V(Z) = \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i Z_j(\delta^2\delta_{ij} + \tau^2 + \zeta^2\ell(|i - j|))$$

$$- 2 \sum_{i=1}^{N} Z_i(\tau^2 + \zeta^2\ell(N + \Delta - i)) + \delta^2 + \zeta^2 + \tau^2.$$

One can actually check this result against the observed squared errors.[1] For example, let $N = 2$ and $\Delta = 3$. Then

$$V(Z_1,Z_2) = Z_1^2(\delta^2 + \tau^2 + \zeta^2) + 2Z_1Z_2(\tau^2 + \zeta^2\ell(1))$$
$$+ Z_2^2(\delta^2 + \tau^2 + \zeta^2) - 2Z_1(\tau^2 + \zeta^2\ell(4))$$
$$- 2Z_2(\tau^2 + \zeta^2\ell(3)) + \delta^2 + \zeta^2 + \tau^2$$

Using the average of the NL and AL values in Table D1 for the covariance structure:

$$\tau^2 = .001425 \qquad \delta^2 + \zeta^2 = .007884$$

$$\zeta^2\ell(1) = .004723 \qquad \zeta^2\ell(3) = .002770 \qquad \zeta^2\ell(4) = .002158$$

$$V(Z_1, Z_2) = Z_1^2(.009309) + Z_1Z_2(.012296) + Z_2^2(.009309)$$
$$- Z_1(.007166) - Z_2(.008390) + .009309$$

Table F1 contains the results of the test for various values of Z_1 and Z_2. (Z_1 is the credibility applied to the less recent year of the two.) The mean squared errors are a close match to those given by the equation.[2]

[1] The covariances were estimated from the same data as is being used to test the equation for the squared error. Thus, the magnitude of the covariances is not being tested. However, the validity of the assumed form of the covariance structure as well as the validity of the derivation of the equation for $V(Z)$ are being tested.

[2] The differences are largely due to the fact that at the two ends of the data period there are either no predictions or no actual observation to enter into the computation of an error.

When $N = 1$, one gets the following parabola for $V(Z)$:

$$V(Z) = Z^2(\delta^2 + \tau^2 + \zeta^2) - 2Z(\tau^2 + \zeta^2\ell(\Delta)) + \delta^2 + \zeta^2 + \tau^2$$

$V(0) = \delta^2 + \tau^2 + \zeta^2 =$ squared error ignoring the data

$V(1) = 2\delta^2 + 2\zeta^2(1 - \ell(\Delta)) =$ squared error relying solely on the data

$$Z \text{ optimal} = \frac{\tau^2 + \zeta^2\ell(\Delta)}{\tau^2 + \delta^2 + \zeta^2} = \frac{V(0) - V(1)/2}{V(0)} = 1 - \frac{V(1)}{2V(0)}$$

$$V(Z \text{ optimal}) = -\frac{(\tau^2 + \zeta^2\ell(\Delta))^2}{\tau^2 + \delta^2 + \zeta^2} + \delta^2 + \zeta^2 + \tau^2$$

$$= -\frac{(V(0) - V(1)2)^2}{V(0)} + V(0)$$

$$= -V(0) + V(1) - \frac{V(1)^2}{4V(0)} + V(0)$$

$$= V(1)\left(1 - \frac{V(1)}{4V(0)}\right)$$

This is the result referred to in Appendix E. The reduction in mean squared error is greatest when $V(1) = V(0)$; then the squared error is reduced to 75% of the minimum of the squared errors that result from relying solely on the data or ignoring the data.

In the absence of shifting parameters over time,[3] the estimate improves as one uses more and more years of data. For large N, relying solely on the data produces a very good estimate; this is reflected in the fact that the optimal credibility approaches 1 as N gets large. Thus for large N, one cannot reduce the squared error significantly by using credibility.

[3] In the presence of shifting parameters over time the situation is much more complicated.

TABLE F1

MEAN SQUARED ERRORS (.0001)

Z_1	Z_2	Observed	Estimated by 2nd Order Polynomial
0	0	9,182	9,309
0	.25	7,592	7,793
.25	0	7,963	8,099
0	.5	7,202	7,441
.15	.35	7,087	7,293
.25	.25	7,172	7,352
.5	0	7,949	8,053
0	.75	8,011	8,253
.25	.5	7,581	7,769
.5	.25	7,957	8,057
.75	0	9,140	9,171
0	1	10,020	10,228
.25	.75	9,189	9,349
.5	.5	9,165	9,260
.75	.25	9,947	9,961
1	0	11,536	11,452
.75	.75	15,162	15,031
1	1	25,162	24,667

Note: Mean Squared Errors in estimating NL and AL data. $N = 2$, $\Delta = 3$. Estimate uses data from the fourth and third years prior to the estimation period with weights Z_1 and Z_2, respectively, and the complement of credibility applied to the grand mean. $Z_1 = 15\%$ and $Z_2 = 35\%$ is the solution to equation 11.3 for the least squares credibility.

The exact behavior can be derived using the results of Appendix C. In the absence of shifting parameters over time ($\zeta^2 = 0$), and applying equal weight Z/N to each of N years, based on the result in Appendix C, the squared error is given by:

$$V(Z) = Z^2 \left(\frac{\delta^2}{N} + \tau^2\right) - 2Z\tau^2 + \delta^2 + \tau^2$$

$$V(0) = \delta^2 + \tau^2$$

$$V(1) = \delta^2 \left(\frac{N+1}{N}\right)$$

$$Z \text{ optimal} = \frac{N\tau^2}{N\tau^2 + \delta^2} = \frac{(N+1)V(0) - NV(1)}{(N+1)V(0) - (N-1)V(1)}$$

$$V(Z \text{ optimal}) = \delta^2 + \tau^2 - \frac{\tau^4 N}{N\tau^2 + \delta^2}$$

$$= V(1)\left(1 - \frac{V(1)}{(N+1)^2 V(0) - (N^2 - 1)V(1)}\right)$$

The maximum reduction in squared error compared to the minimum of $V(0)$ and $V(1)$ occurs when $V(0) = V(1)$. For this case

$$Z \text{ optimal} = 1/2$$

$$V(Z \text{ optimal}) = V(1)\left(1 - \frac{1}{2(N+1)}\right)$$

As N gets large, there is no significant reduction in squared error due to using credibility (in the absence of shifting parameters over time).

APPENDIX G

THE SECOND CRITERION AND LIMITED FLUCTUATION CREDIBILITY

The second criterion in Section 7 deals with the probability that the observed result will be more than a certain percent different than the predicted result. The less this probability, the better the solution.

This is related to the basic concept behind "classical" credibility which has also been called "limited fluctuation" credibility [7]. In classical credibilty, the full credibility criterion is chosen so that there is a probability, P, of meeting the test, that the maximum departure from expected is no more than k percent.

The reason the criterion is stated in this way rather than the way it is in classical credibility is that, unlike the actual observations, one cannot observe directly the inherent loss potential.[1]

However, the two concepts are closely related. If there is a small chance of the estimate differing by a large amount from the true value of the inherent loss potential, then, since the observed values are distributed about the true value, the chance of the estimate differing by a large amount from the observed value will be smaller than it would otherwise be.

For example, assume the inherent loss potential is .550 and that the observed values are distributed approximately normally with a standard deviation of .050. Then there is approximately a 95% probability that the observed value will be between .452 and .648.[2]

Assume the estimated values are also approximately normally distributed about the inherent loss potential.[3] Assume a standard deviation of .028. Then there is a 95% chance that the estimate will be between .495 and .605, i.e., within 10% of the true inherent loss potential.

[1] It has been shown that the loss potential varies for a risk over time. Thus, it cannot be estimated as the average of many observations over time.

[2] The mean plus or minus 1.96 standard deviations.

[3] An unbiased estimator has the same expected value as the inherent loss potential.

The difference between the estimated value and the observed value will also be approximately normally distributed about zero.[4] The standard deviation is .057.[5] Thus, there would be a 95% chance that the absolute difference between the estimated and observed value will be less than .112. This corresponds to about a 95% chance that the estimated value will be within ±20% of the observed value.[6]

In a particular example, the result would depend on the relative size of the variances of the observations and the estimates. However, the smaller the variance in the estimates, the smaller the variance in the difference between the estimates and the observations. Thus the smaller the probability that the estimate and the true mean differ by a large amount, the smaller the probability that the estimate and the observation differ by a large amount.

[4] The sum or difference of two normal distributions is also a normal distribution. The new mean is the difference of the two means.

[5] The new variance is the sum of the two variances.

[6] $.112 \div .550 = .204$.

THE DISTRIBUTION OF AUTOMOBILE ACCIDENTS—ARE RELATIVITIES STABLE OVER TIME?

EMILIO C. VENEZIAN

Abstract

Data on the distribution of automobile accidents typically reject the hypothesis that accident rates are the same for all members of a group. Given these findings, policy analysis is usually based on models that assume that accident proneness differs among individuals of a group and that the differences are stable over time. The analysis presented in this paper is aimed at assessing the validity of these assumptions.

A simple model that allows for variability in both proneness and exposure level is used to estimate the potential contribution of variability in exposure levels to total variability in accident rates. Data indicate that variability of exposures may have a substantial bearing on the variability of accident rates.

Data from several groups of drivers from California and North Carolina are used for direct tests of the stability of relative accident rates. When the California data are used, the tests do not lead to a rejection of the hypothesis that relative accident rates are stable. When the North Carolina data, based on a larger number of observations, are used, the tests clearly reject the hypothesis.

The implications of these findings for economic and policy analysis are discussed.

1. INTRODUCTION

For most types of accidental events, the number of accidents over a period does not exhibit a Poisson distribution, even when data are restricted to a group of individuals who are presumed homogeneous. This finding has been reported in a variety of settings such as automobile accidents [11, 13, 16], health insurance claims [6], and professional liability incidents [9, 15]. This empirical finding is often rationalized by appealing to the notion of differences in "accident proneness" among the individuals in the groups under analysis. The typical model assumes that each individual within the group has an inherent accident rate, and that, for each individual, the number of accidents in a given period has a Poisson distribution with the appropriate parameter. Each individual in the group under study is viewed as having an inherent accident rate, and this rate is assumed to differ among individuals according to some probability distribution. This model, often called the "compound Poisson model"[1] has reportedly been successful in fitting the distribution of the number of accidents or claims observed in a given time interval.

The apparent success of these attempts, especially those based on the assumption that accident proneness has a Gamma distribution, does not provide a sound basis for the formulation of public policy.[2] In the first place, the usual interpretations of the compound Poisson require that the accident rate of a given individual be stable over time, a characteristic for which tests have seldom been performed. Another shortcoming of these methods is that the distribution inferred from this model is identical to the distributions inferred from other models [2, 10]. Moreover, models exist that provide results which are as good as, or better than, those obtained under the assumption that accident proneness differs among individuals; however, these models have very different implications for public policy [10, 14, 17]. It is therefore of interest to examine more closely the relationship between data and hypothesis on one hand and the relationship between hypothesis and policy on the other. This paper examines the issue in the context of automobile insurance.

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[1] See, for example, Feller [2], pages 288–293; Seal [10], page 31.

[2] Public policy generally refers to policies adopted by governmental or quasi-governmental entities. In the present context, it includes such diverse areas as licensing, limitation of privileges, and the imposition of premium penalties for past events.

The paper first discusses briefly, in Section 2, the data that will be used in exploring the theoretical issues. Section 3 discusses the compound Poisson model that is often used to justify both private and public initiatives in accident prevention. The paper then consists of three main sections. Section 4 discusses an alternate source of differences in the inferred proneness of individuals. This alternate model leads to a different valuation of the benefits of any policy that restricts driving by individuals who have had relatively large numbers of accidents in the past. There is relatively little that can be done with existing data to discriminate between the models. Since the models lead to different conclusions, data to permit an assessment of the alternatives should be collected if at all possible. Section 5 considers a test of the hypothesis that claim propensities (or more accurately, indices of claim propensity) are constant over time when taken over reasonably long durations. Such constancy is essential if we are to use historical data to implement a policy whose benefits can be asserted to exist only to the extent that past accidents predict future accident propensity for the individual. The available data on automobile accident involvement indicate that constancy is not a reasonable assumption. Section 6 provides a discussion of the findings in the context of economic and policy analysis of insurance issues.

2. DATA FOR ANALYSIS

In order to examine these issues, this article uses two sets of data from available literature. Both of these sets related to the accident records of groups of drivers whose records were followed over a long period of time.

The first body of data relates to a sample of drivers in California. The data used in the present analysis were derived by the author from available tabulations [5]. The data relate to accidents experienced in the years 1969 to 1974 by a sample of California drivers who had licenses active for the period 1961 to 1974. The original paper gives extensive tabulations by sex and by pattern of accidents. The basic data used in this paper were derived from the original data and are presented in three tables in Appendix A. The analysis will be performed separately on the three sets of data: (1) for female drivers, (2) for male drivers, and (3) female and male drivers combined.

The second set of data is available in the form used directly for computations [12]. It relates to accidents experienced in the periods[3] 1967–1968 and 1969–1970 by all North Carolina drivers who were at least 22 years old in November of 1970, and in the twelve-month periods 1969 and 1970 for North Carolina drivers who were 21 years old in November of 1970. For drivers whose age at the end of the study was 22 years or more, the data are available separately for ages 22–25, 26–39, 40–59, and 60 and over. In all cases, the drivers are classified by their age at the beginning of the study period.

3. THE COMPOUND POISSON MODEL AND ITS INTERPRETATION

The compound Poisson model pictures each individual as having an inherent propensity to be involved in an accident. Most models picture that propensity as a fixed number that does not vary over time. Strict constancy from day to day is not necessary as long as it holds over periods of time comparable with those for which data are available. Moreover, the mathematical and statistical analysis would not be affected substantially if that element which is constant were an index of proneness which modifies the average rate for the group as a whole. What is of major importance to the arguments surrounding the compound Poisson model is that this index is immutable for a given individual.

The principal statistical implication of the compound Poisson hypothesis is that individuals with large numbers of accidents are relatively more common than would be predicted by the simple Poisson model. In statistical terms, the consequence of having a compound Poisson distribution is that the variance of the number of claims will be larger than the mean number of claims. In contrast, for the simple Poisson, the mean and the variance of the number of claims are identical.

The usual interpretation of the compound Poisson hypothesis is that individuals with large accident propensities affect the group adversely, leading to a higher average number of claims. This affects the insurability of those members of the group who have low propensity indices. In

[3] The periods do not cover the calendar years. As explained in the original reference, the nominal year 1970, for example, covers the twelve-month period beginning in December 1969.

automobile insurance, two streams of rhetoric have arisen from this interpretation. Some argue that failure to reflect the differences among group members in insurance rates amounts to "guilt by association." Others bemoan the fact that insurance premiums are made to depend on factors that are not controllable by the individual. In the context of medical professional liability, the model elicits the picture that a few "bad apples" are responsible for most of the problems and has resulted in calls to revoke the licenses of these "bad apples" and thus reduce the number of claims.

If this picture is true, an economic analysis of restricting the privilege of driving, either through license restrictions or through the provision of insurance only at high rates, would be useful. The best level of restrictions would be determined by balancing the costs and the benefits of such a decision. The costs arise primarily from curtailing the freedom of some individuals to drive automobiles; they have a monetary component related to the difference in price between driving one's own car and relying on alternate modes of transportation, and a nonmonetary component related to loss of freedom. The benefits arise from the reduced number of accidents, and these also have monetary and nonmonetary elements. For society as a whole, monetary benefits[4] arise from avoiding costs to rectify the consequences of accidents, while the nonmonetary component stems from the reduction in pain and suffering associated with the avoided mishaps. The calculation of the benefits depends very strongly on the exact hypothesis which motivates the compound Poisson model. To explore the extent to which this might affect our thoughts about policy, it is worthwhile to contrast the usual assumption, that the differences in accident experience are due to differences in inherent ability, with a specific alternate hypothesis.

The key parameter in the Poisson distribution is not an "accident propensity" that measures inherent ability, but a weighted measure that recognizes both the ability to perform a dangerous task and how often the task is performed. The expected number of automobile accidents which one individual might have in a year may not be a fixed quantity; it might, for example, depend on the number of miles that the individual

[4] Distributional costs and benefits will also result. Individuals with low accident rates will not have to subsidize individuals in the same group that have higher accident rates.

chooses to drive under various sets of conditions. Similarly, the expected number of claims against an engineer might depend on the number of plants she designs, and the number of claims that a physician might expect could depend on the number of patients that are treated by that physician. Thus variation in the Poisson parameter among drivers does not require that the rate of accidents differs among individuals when the level of activity is identical. It could be explained equally well, from a statisical point of view, if all drivers had exactly the same accident proneness under every given set of driving conditions but they differed in the miles they drove under various conditions.

This alternate hypothesis as to sources of variability suggests a different interpretation of the compound Poisson process. In this picture, all drivers in a group have exactly the same probability of having an accident in each mile they drive; but, they differ from each other inherently in the mileage driven. To keep an exact correspondence to the previous model, it is important that the distance and nature of driving in this version be as immutable as the inherent accident probability in the previous one; these measures of exposure may change only in ways that are strictly coupled with the average for the group as a whole.

Neither of these simple models is likely to be strictly valid. In all likelihood, drivers differ with respect to the probability that they will be involved in an accident under a given set of conditions. In all likelihood, they also differ with respect to the exposure level they chose. Thus a model that recognizes differences in both propensity and activity levels is likely to provide a better explanation of actual experience.[5]

4. INTERPRETATION OF THE EXCESS VARIANCE

When the number of accidents observed for each of many members of the group is analyzed, we expect to see a variance that is approximately equal to the mean if all members have the same probability of having

[5] The combined effect of individual propensity and exposure is particularly important in economic contexts in which the driver has control of the exposure level, at least within broad limits. Unfortunately, this dual determination has seldom been considered. The literature on moral hazard, for example, appeals to a "level of care" which might be selected by the driver, but does not take into account the possible direct choice over the level of exposure by restricting or expanding the mileage driven.

an accident in a unit of time, and a variance greater than the mean if individual members differ in this respect. A positive difference between the variance and the mean, or "excess variance," results from differences in the Poisson parameters of the members of the group.

If we are observing individuals over a period of time T, the Poisson parameter for individual i will be:

$$M_i = k_i p_i T, \tag{1}$$

where k_i is the number of opportunities for individual i to have an accident and

p_i is the individual's probability of an accident on any given opportunity.

Usually there is a measure of T, but there are no measures of k_i or p_i; so this model does not have an operational meaning.[6] The model adopted in arguing that probability of an accident varies across individuals in the group is equivalent to arguing that k_i is the same for all individuals. The alternate model discussed earlier assumes that k_i varies across individuals but p_i does not. Equation 1 provides a more general formulation and can be made operational if there are measures of the level of activity; for example, the number of miles driven per year for automobile accidents, the number of takeoffs for small aircraft accidents, or the number of specific surgical procedures for medical professional liability. Even in the absence of such measures, formulation is worth considering because it may yield some insight into the process.

If the Poisson parameter, M_i, varies across individuals, it can be proved that the average number of accidents for individuals in a group is given by:

$$\mathrm{E}_i(N) = \mathrm{E}_i(M_i) = T\mathrm{E}_i(k_i p_i), \tag{2}$$

and

$$\mathrm{Var}_i(N) - \mathrm{E}_i(N) = T^2\mathrm{Var}_i(k_i p_i). \tag{3}$$

[6] In some contexts it would be possible to obtain information about the level of exposure, even though imperfect. In relation to automobile accidents, the mileage driven per year might serve as a measure of k_i.

In these equations, $E_i(Z)$ denotes the expected value of Z and $Var_i(Z)$ denotes the variance of Z, both measured over the population in the group.

If we denote the excess variance, $Var_i(N) - E_i(N)$, as $X_i(N)$, the variance of this statistic under the null hypothesis that the Poisson parameter is identical for all members of the population is given by:

$$Var_i(X_i(N)) = \frac{2}{I} E_i^2(k_i p_i) = \frac{2}{I} E_i^2(N) \quad (4)$$

where I is the total number of individuals observed [14].

It is worth noting that if k_i is the same for all individuals, the excess variance is proportional to the variance of p_i, whereas if p_i is the same for all individuals, then the excess variance is proportional to the variance of k_i. If both k_i and p_i vary, then the excess variance will depend on the joint distribution of p_i and k_i.

Assuming for now that a stable compound Poisson is the proper model, it is of interest to determine whether the data indicate that there is significant heterogeneity in a given group and to interpret the excess variance, if it can indeed be said to be positive. The equations given above can be used for this purpose. A test requires simply computing the observed excess variance and the variance of that quantity under the null hypothesis; this statistic can be estimated by using Equation 4. The sample estimate of the excess variance is the sample estimate of the variance minus the sample estimate of the mean. If the number of observations is large, both these sample estimates are asymptotically normal [1]; it follows that the difference is asymptotically normal, so the ratio of its sample value to the standard deviation should, under the null hypothesis, be distributed as a standard normal deviate. When interest is centered on determining whether there is significant heterogeneity among members of the group, the null hypothesis is that there is no heterogeneity; under those conditions, the distribution of claims would follow a simple Poisson distribution. Table 1 summarizes the data used in assessing the significance of the excess variance. Table 2 presents the main results. It is clear that the excess variance is positive and highly significant for all the groups under consideration, since the ratio of the estimate to its standard deviation is always greater than 15.

TABLE 1

NUMBER OF DRIVERS BY GROUP AND NUMBER OF ACCIDENTS

State and Group	Number of Accidents								
	0	1	2	3	4	5	6	7+	Total
CA Females	19,634	3,573	558	83	19	4	1	0	23,872
CA Males	21,800	6,589	1,476	335	69	16	4	4	30,293
CA All	41,434	10,162	2,034	418	88	20	5	4	54,165
NC 22–25	276,081	69,811	16,770	4,060	967	236	74	26	121,221
NC 26–39	709,649	143,601	29,401	6,658	1,725	447	124	51	891,656
NC 40–59	762,592	138,955	23,580	4,492	1,054	254	74	33	931,044
NC 60+	254,255	47,095	8,159	1,511	344	97	28	24	311,513
NC 21	144,803	25,302	4,007	637	105	15	5	1	174,875

TABLE 2

ANALYSIS OF EXCESS VARIANCE BY GROUP

State and Group	Number of Accidents		Sample Excess Variance		
	Mean[a]	Variance[b]	Value[c]	Std.Dev.[d]	Value/Std.Dev.
CA Females	0.2111	0.2483	0.0372	0.0014	27.22
CA Males	0.3617	0.4435	0.0818	0.0042	19.68
CA All	0.2953	0.3631	0.0678	0.0025	26.72
NC 22–25	0.3294	0.4322	0.1028	0.0011	94.66
NC 26–39	0.2609	0.3439	0.0830	0.0006	155.63
NC 40–59	0.2211	0.2752	0.0541	0.0005	118.05
NC 60+	0.2252	0.2822	0.0570	0.0008	70.63
NC 21	0.2167	0.2353	0.1867	0.0010	180.14

a. Mean of the number of accidents
b. Variance of the number of accidents
c. Variance minus mean of the number of accidents
d. Calculated as the square root of the variance given by Equation 4

The statistical significance of the excess variance is of importance in examining private policy issues such as merit rating and freedom to underwrite. From this perspective, it is important to know whether the data suggest that the Poisson parameter differs among individual members of the group. The existence of variability among individuals suggests that differential pricing based on experience may be useful in achieving an equitable allocation of future costs. From the point of view of public policy issues such as restricting the ability of individuals to drive, however, this information is not sufficient because the variability may be due to differences in the level of activity of individuals rather than to differences in claim propensity.

While this distinction is not important in dealing with private mechanisms such as classification by individual companies in a market with open competition, it is important in dealing with public mechanisms such as classifications mandated by the state or licensing restrictions. As discussed earlier, if the difference in Poisson parameters arises predominantly from differences in the level of activity, restrictions placed on the privilege of driving by individuals with large numbers of accidents will either restrict their mobility or force them to use alternate drivers who have comparable or higher propensities to have accidents for corresponding exposures. Thus social benefits might not be experienced, but substantial social costs would be incurred.

It is not possible to draw firm conclusions about the relative importance of level of exposure and accident propensity from the available data. The information is sufficient, however, to permit drawing tentative conclusions.[7] The line of inference begins by noting that the excess variance measures the variance of Poisson parameters, as is shown by Equation 3. The ratio of this quantity to the square of the Poisson parameter represents the coefficient of variation of the parameter.

TABLE 3

VARIATION OF POISSON PARAMETER BY GROUP

State and Group	Poisson Parameter[a]	Excess Variance[b]	Coefficient of Variation[c]
CA Females	0.2111	0.0372	0.83
CA Males	0.3617	0.0818	0.63
CA All	0.2953	0.0678	0.78
NC 22–25	0.3294	0.1028	0.95
NC 26–39	0.2609	0.0830	1.22
NC 40–59	0.2211	0.0541	1.11
NC 60+	0.2252	0.0570	1.12
NC 21	0.2167	0.1867	3.98

a. From Table 2, column 2
b. From Table 2, column 4
c. Coefficient of variation of the Poisson parameter

[7] Another possibility that deserves consideration is that the classification system is inadequate.

Table 3 shows the results for the various groups. In most cases, the coefficient of variation of the Poisson parameter among members of a group is very close to one. In the case of North Carolina drivers of 21 years of age, it is almost four.

The interpretation of this number must, unfortunately, rely on the context of the problem because firm data are not available.[8] At one extreme, if the level of exposure is the same for all individuals, the coefficient of variation of the Poisson parameter would approximately equal that of accident proneness. At the other extreme, if the accident proneness were the same for all individuals, this number would equal the coefficient of variation in the exposure level. Any measure of the coefficient of variation of the exposure level will therefore serve to help to place the results in context. In the present case, exposure might be measured by mileage driven in a unit of time [8] and might well exhibit a large coefficient of variation. Rough estimates are discussed in Appendix B; they range from 0.3 to 0.9.

The observed coefficients of variation of the Poisson parameter are generally higher than the corresponding estimates for mileage driven. However, even with the lower estimates for the latter, variation in mileage driven would account for about 25 percent of the variance of Poisson parameters. Thus exposure may play a substantial role in determining the accident rates of individuals. Data relating accident experience and mileage driven by individuals in different time periods could provide better measures of the relative contribution of exposure; even accurate data on the distribution of mileage driven would be useful in assessing the relative effects of exposure and propensity on the Poisson parameter of individuals.

5. A TEST FOR GENERAL COMPOUND POISSON MODELS

The discussion presented above indicates that caution must be exercised in using results from a simple static analysis to guide policy. The usual analyses do not pinpoint the reason for variation in Poisson param-

[8] Even if data were available, it should be remembered that the model used here assumes that individuals select their exposure level without regard to their accident proneness. This may be appropriate when individuals are insured but may be a poor assumption in the absence of insurance.

eters and these reasons may have a bearing on policy issues. For example, even people who would accept the hypothesis that the accident propensity of an individual, p_i, does not change over time might question the hypothesis that the exposure level of the individual, k_i, does not change. Yet the predictability of the Poisson parameter plays a key role in the ideology of classification and merit rating [7]. The relevance of statistical analysis to policy requires analysis of models that are realistic and address the key issues. This is more likely to happen if the public policy issues are examined and statistical tools are developed to analyze the key issues.

One of the important issues in automobile liability is the measurement of the benefits to be derived from restricting the mobility of drivers with several claims.[9] The costs that would be incurred by such restrictions would depend primarily on the number of people on whom restrictions would be placed, not on the model assumed. The benefits, however, may be estimated only in relation to a model. In this context, statistical methods can provide assistance only if they are designed to provide relevant information and if they are valid. A common feature of the two models discussed earlier is the assumption that the likelihood that an individual driver will have an accident is an inherent characteristic of the individual. Statistics are useful in establishing whether this is a valid conclusion.

For the most part, compound Poisson models have been tested by assuming a specific form for the distribution of accident propensities, inferring a theoretical distribution to the number of accidents and performing a goodness of fit test to establish that the fit is adequate in a statistical sense. If they result in a good fit, these statistical procedures can, at best, establish that it is plausible that during a given time period, individuals in a group differ with respect to accident propensity and that the propensities can be characterized as having a distribution similar to the one assumed. The procedures do not test the assumption that the claim propensity of an individual is the same in two different time

[9] The statement is valid whether the restriction occurs by exercise of the power of the state to limit the privilege of driving, or by exercise of economic power to increase the cost faced by certain individuals in order to drive. The discussion will be limited to the former, since analysis of the latter requires knowledge of tradeoffs whose value cannot be estimated readily.

intervals.[10] That assumption is important in most arguments related to either pricing of insurance or to restrictive public policy, since these arguments assume that past experience is a good predictor of future performance for an individual.

The analysis presented above still retains the untested assumption that the Poisson parameter corresponding to i^{th} individual, M_i, is constant over time. Direct tests of this assumption are not feasible since we cannot observe the same individual repeatedly during the same time interval. The literature does provide a method for determining whether this key assumption is correct. The method uses data from a single population studied in successive time periods. It was first suggested by Lundberg [6], who showed that, for a general compound Poisson with the Poisson parameter of each individual being equal to an individual parameter times the average rate for all individuals in any given subinterval of time, the probability that an individual will have m claims in the subinterval t_2, and n claims in subinterval t_1, given that he had $m + n$ claims in the interval $t_1 + t_2$, is given by:

$$P_r(m,t_2|n,t_1) = \frac{(m + n)!}{m!\, n!}\, \Theta_1^m(1 - \Theta_1)^n \tag{5}$$

where $\Theta_1 = r_1t_1/(r_1t_1 + r_2t_2)$,

r_j is the average accident rate in period j, and

t_j is the duration of period j.

The operational time intervals, r_jt_j contain the average accident rates, r_j, which are not known, along with the calendar time, t_j. In order to provide a valid test, it is necessary to develop measures of the ratios of operational time intervals. Lundberg argues that the ratios may be estimated by the ratio of the number of accidents or claims in each subperiod to the total number of accidents or claims. Once the parameters are known, the conditional distributions for all relevant values of $m + n$ can be computed and compared to the observed data by using a chi-squared test. Lundberg recommends grouping cells so that the expected number of claims is five or more. The degrees of freedom for each value of m +

[10] A notable exception is the analysis of Weber [16], who used methods attributed to Greenwood and Yule [3] and Kerrich [4] for the case in which the compounding distribution is the Gamma distribution. The method of Greenwood and Yule actually turns out to be valid for general compounding distributions and is equivalent to the method used here.

n is one less than the number of cells used in the test; the additivity of chi-squared may be used to construct an overall test by adding the contributions to chi-squared and adding the degrees of freedom. Lundberg used this test with data on health insurance claims in Sweden and found the results did not reject the hypothesis of a compound Poisson distribution with stable parameters.

It is worth noting that the method recommended by Lundberg for the estimation of the parameters relies on ratios of the average accident rates. It follows that, if the average accident rates change and the individual accident rates change proportionately, the test will not be affected.[11] Thus the test will be valid if the accident relativities are constant, even though the actual accident rates change. From this perspective, the null hypothesis could be characterized as the assertion that rate relativities are constant over time.

To illustrate the procedure, the data for all drivers in California, given in Table A3 of Appendix A, will be used. The data there were 7,967 accidents for the period 1969–71 and 8,030 accidents for the period 1972–1974. The total number of accidents was 15,997. The best estimate of the needed parameter is $\Theta_1 = 7{,}967/15{,}997 = 0.4980$.

This parameter is used in Equation 5 to estimate the expected fraction of drivers with m accidents in the first period and n accidents in the second period given a total number of $m + n$ accidents in the whole period. These probabilities are multiplied by the observed number of drivers with $m + n$ accidents in the whole period to obtain the expected value of the number of drivers with m accidents in the first period and n accidents in the second period. The observed and predicted numbers of drivers with one, two, three, and four accidents in the period 1969–1974 are shown in Table 4. Tables 5 to 11 provide analogous information for the other groups. Note that in Lundgren's scheme, the data for individuals that had no accidents in either the first or second interval contribute only to the estimation of the frequency of accidents in the two periods, but do not contribute to the value of the test statistic. Accordingly the tables do not show this group.

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[11] Lundgren's original application to data on health insurance claims actually involved substantially different claim rates in the first and second period.

TABLE 4

Observed and Predicted Distribution of Female Drivers in California, by Number of Accidents

Number of Accidents in 1969–74		Number of Accidents in 1969–1971						Chi-Squared
		0	1	2	3	4	5	
1	Observed	1,816	1,757					
	Predicted	1,838.5	1,734.8					0.3
2	Observed	164	266	128				
	Predicted	147.7	278.8	131.5				2.5
3	Observed	16	24	33	10			
	Predicted	11.3	32.0	30.2	9.5			4.2
4	Observed		5[a]	8	6[b]			
	Predicted		6.4[a]	7.1	5.5[b]			2.0
Total (8 degrees of freedom)								9.0

a. Drivers with either zero or one accidents in 1969–71
b. Drivers with either three or four accidents in 1969–71

TABLE 5

Observed and Predicted Distribution of Male Drivers in California, by Number of Accidents

Number of Accidents in 1969–74		Number of Accidents in 1969–1971						Chi-Squared
		0	1	2	3	4	5	
1	Observed	3,226	3,363					
	Predicted	3,269.5	3,319.5					1.2
2	Observed	403	692	381				
	Predicted	363.4	738.0	374.6				7.3
3	Observed	44	126	124	41			
	Predicted	40.9	124.7	126.6	42.8			0.4
4	Observed	7	15	26	21[a]			
	Predicted	4.2	17.0	25.9	22.0[a]			2.2
Total (9 degrees of freedom)								11.1

a. Drivers with either three or four accidents in 1969–71

TABLE 6

OBSERVED AND PREDICTED DISTRIBUTION OF ALL DRIVERS IN CALIFORNIA, BY NUMBER OF ACCIDENTS

Number of Accidents in 1969–74		Number of Accidents in 1969–1971: 0	1	2	3	4	5	Chi-Squared
1	Observed	5,042	5,120					
	Predicted	5,101.0	5,061.0					1.4
2	Observed	567	958	509				
	Predicted	512.5	1,014.0	504.5				9.2
3	Observed	60	150	157	51			
	Predicted	52.9	157.4	156.1	51.6			1.3
4	Observed	9	18	34	23	4		
	Predicted	5.6	22.2	33.0	21.8	5.4		3.3
Total (10 degrees of freedom)								15.3

TABLE 7

OBSERVED AND PREDICTED DISTRIBUTION OF DRIVERS OF AGE 22–25 IN NORTH CAROLINA, BY NUMBER OF CLAIMS

Number of Accidents in 1967–70		Number of Accidents in 1967–1968: 0	1	2	3	4	5	Chi-Squared
1	Observed	33,615	36,196					
	Predicted	33,847.3	35,963.0					3.1
2	Observed	4,639	7,164	4,967				
	Predicted	3,942.1	8,377.3	4,450.6				358.8
3	Observed	648	1,339	1,358	715			
	Predicted	462.7	1,475.0	1,567.2	555.1			160.7
4	Observed	73	239	313	237	105		
	Predicted	53.4	227.1	362.0	256.4	68.1		35.9
5	Observed	14	41	54	63	36	28	
	Predicted	6.3	40.3	71.4	75.8	40.3	8.6	60.3
6	Observed		9[a]	18	17	18	12[b]	
	Predicted		7.1[a]	16.3	23.1	18.4	9.2[b]	3.2
Total (19 degrees of freedom)								622.0

a. Drivers with either zero or one accidents in 1967–68

b. Drivers with either five or six accidents in 1967–68

TABLE 8

Observed and Predicted Distribution of Drivers of Age 26–39 in North Carolina, by Number of Claims

Number of Accidents in 1967–70		Number of Accidents in 1967–1968: 0	1	2	3	4	5	Chi-Squared
1	Observed	68,827	74,774					
	Predicted	69,682.9	73,918.1					2.3
2	Observed	7,817	13,273	8,311				
	Predicted	6,923.1	14,687.7	7,790.2				92.3
3	Observed	1,064	2,286	2,291	1,017			
	Predicted	760.8	2,421.0	2,568.2	908.1			44.8
4	Observed	149	434	576	398	168		
	Predicted	95.6	405.8	645.8	456.7	121.1		64.9
5	Observed	28	75	115	128	71	30	
	Predicted	12.0	63.8	135.3	143.6	76.1	16.1	40.1
6	Observed		13[a]	28	38	28	17[b]	
	Predicted		11.9[a]	27.3	38.6	30.7	15.4[b]	0.5
Total (19 degrees of freedom)								244.9

a. Drivers with either zero or one accidents in 1967–68
b. Drivers with either five or six accidents in 1967–68

TABLE 9

Observed and Predicted Distribution of Drivers of Age 40–59 in North Carolina, by Number of Claims

Number of Accidents in 1967–70		Number of Accidents in 1967–1968: 0	1	2	3	4	5	Chi-Squared
1	Observed	66,875	72,080					
	Predicted	67,156.4	71,798.6					2.3
2	Observed	5,928	11,048	6,604				
	Predicted	5,507.7	11,776.8	6,295.5				92.3
3	Observed	629	1,540	1,644	679			
	Predicted	507.1	1,626.4	1,738.8	619.7			44.8
4	Observed	76	265	365	261	87		
	Predicted	57.5	245.9	394.4	281.1	75.1		12.9
5	Observed	10	35	85	80	43	11	
	Predicted	7.0	37.2	79.6	85.1	45.5	9.7	2.4
6	Observed		10[a]	20	19	13	12[b]	
	Predicted		7.0[a]	16.2	23.0	18.5	9.3[b]	5.3
Total (19 degrees of freedom)								160.0

a. Drivers with either zero or one accidents in 1967–68

TABLE 10

Observed and Predicted Distribution of Drivers of Age 60 or more in North Carolina, by Number of Claims

Number of Accidents in 1967–70		Number of Accidents in 1967–1968: 0	1	2	3	4	5	Chi-Squared
1	Observed	23,367	23,728					
	Predicted	23,252.8	23,842.2					1.1
2	Observed	2,083	3,878	2,198				
	Predicted	1,989.0	4,078.9	2,091.1				17.6
3	Observed	205	529	549	228			
	Predicted	181.9	559.4	573.6	196.1			10.9
4	Observed	23	70	115	90	46		
	Predicted	20.4	83.8	129.0	88.2	22.6		17.6
5	Observed		14[a]	30	29	24[b]		
	Predicted		17.4[a]	29.9	30.7	18.6[b]		2.4
Total (13 degrees of freedom)								49.6

a. Drivers with either zero or one accidents in 1967–68
b. Drivers with either four or five accidents in 1967–68

TABLE 11

Observed and Predicted Distribution of Drivers of Age 21 in North Carolina, by Number of Claims

Number of Accidents in 1967–70		Number of Accidents in 1967–1968: 0	1	2	3	4	5	Chi-Squared
1	Observed	13,118	12,184					
	Predicted	13,135.0	12,167.0					0.1
2	Observed	1,204	1,767	1,036				
	Predicted	1,079.9	2,000.6	926.6				54.5
3	Observed	113	215	232	77			
	Predicted	89.1	247.6	229.4	70.8			11.3
4	Observed	13	23	33	26	10		
	Predicted	7.6	28.3	39.3	24.2	5.6		9.3
Total (10 degrees of freedom)								75.2

The results indicate that the data from California do not reject the hypothesis that Poisson parameters for each individual bear the same relationship to the aggregate average level in the two periods under consideration. When the two sexes are combined, the value of chi-squared is 15.3 with 10 degrees of freedom, a value that would occur by chance about ten percent of the time if the null hypothesis were valid. This finding is consistent with that of Weber [16], based on California data for a different and shorter period. These data do not provide strong evidence against the hypothesis that relativities are stable.

The results from the North Carolina data, on the other hand, lead to the clear rejection of the null hypothesis. Given the degrees of freedom, the observed values of chi-squared in any one group would correspond to probabilities much lower than one in one thousand if the null hypothesis were true. It is worth noting that large contributions to chi-squared arise from relatively small values of $m + n$. This indicates that the data leading to the rejection of the null hypothesis are not concentrated in the cells corresponding to individuals with a high aggregate accident propensity or to cells with relatively small numbers of observations.

In view of the results with the data from North Carolina, the fact that the California data do not reject the null hypothesis could be attributed to the fact that different mechanisms are operating in the two environments. However, the number of observations in California is much smaller than that in North Carolina. The total number of drivers observed in California is just over 54,000, compared to over 2.5 million in North Carolina. Thus, an alternate explanation of the results is that the power of the test to reject the null hypothesis is so low that the California data cannot attain the conventional levels of confidence. Unfortunately, there are no other formal tests of the hypothesis of interest.[12]

[12] Thyrion [13], among others, has pointed out that there is an interesting recursive relation for the compound Poisson of arbitrary compounding distribution. Statistical tests based on that relation have not been developed.

6. DISCUSSION

The data analysis suggests that driver accident frequencies may be determined jointly by the driver's ability to drive and the exposure that the driver experiences. This finding is important for both economic analysis and policy formulation.

Economic analysis usually assumes that economic agents act rationally, in the sense that they select the options that provide them the greatest level of satisfaction. Analyses of insurance purchasing and the related issue of moral hazard have not, however, considered seriously the possibility that individuals may, in the absence of insurance, select the level of exposure with due regard to the individual's accident propensity and risk aversion. Given the possibility of such effects, the analysis of the insurance purchasing decision may be misleading unless one explicitly recognizes that the utility function depends on both consumption of goods and ability to travel. It may even be important to take into account the relationship between ability to travel and ability to generate income.

This possibility also creates some interesting problems in the analysis of insurance classifications. Given that the individual selects exposure by considering both accident propensity and risk aversion, the net experience of that individual, measured in terms of the expected number of accidents, will reflect a complex interaction of accident propensity and risk aversion. Moreover, this expected number of accidents is not likely to provide much information regarding what its corresponding value after insurance is likely to be, since the existence of insurance coverage may have large effects on the individual's choice of exposure. Also, the experience of an individual under one classification scheme will serve to predict the experience of the same individual under a different classification scheme only to the extent that the new classification scheme will affect neither the individual's propensity to have an accident, nor his selection of a level of exposure, nor his inclination to purchase coverage. In this respect, particular care should be exercised in drawing inferences about plans based on merit rating or bonus-malus systems from corresponding information gathered under classification plans that do not include experience rating.

From the perspective of public policy, the possible effect of variability of exposure suggests that the Poisson parameter of individuals will not be constant from period to period, even if the accident proneness remains fixed. The possibility that individuals will change their exposure level implies that past experience may not be the best predictor of the future experience for an individual. While prediction of average group performance based on the past may be appropriate and necessary for proper functioning of insurance markets, public policy should reflect concern about distributional equity if the past is not a good predictor of the future for an individual. Variation of the Poisson parameter over time implies that the public policy arguments favoring merit rating may not be properly based on fact. If the individual's accident propensity varies from period to period, a rate based on past exposure is not necessarily a good predictor of future experience for the individual. In fact, if propensity varies over time, the issue of whether merit rating is a better predictor of future performance than classification rating must be examined empirically rather than assumed.

Given the likely effect of exposure levels, it is perhaps not surprising that the data do not support the hypothesis that Poisson parameters are constant over time or bear a constant relationship to the group average. At present, the conclusion that the data reject the hypothesis is based largely on the data from North Carolina. The other body of data that is currently available, that of California, does not reject the hypothesis. The California data for both sexes combined is barely consistent with the hypothesis at the ten percent level.[13] It may well be that the hypothesis would be rejected by a larger sample of drivers from this state and period. Additional data with which to probe this question would be valuable, especially if the data base included estimates of the exposure level.

[13] It may be worth noting that if we were to focus our attention on the group with two accidents in the total observation period, the California data would reject the hypothesis. The more comprehensive data do not reject the hypothesis, since the other groups contribute more to the degrees of freedom than they contribute to the chi-square value.

APPENDIX A

SUMMARY OF THE CALIFORNIA DATA

This appendix records the data from California relevant to this study. The data were derived from Table N, Appendix I, of a report prepared by the Department of Motor Vehicles of the State of California [5]. That table gives the distribution of licensed drivers in the California data base by number of accidents in each of the calendar years 1961–63 and 1969–74. The number of accidents in the period 1961–63 was ignored in the present analysis since interest was focused on the accidents in two subintervals of a common length, and because instability of accident rates over the intervening period 1963–1969 could be due to the long gap in information. The tabulations presented below were obtained by grouping all combinations of accident numbers that gave the same total for the years 1969–71, and those that gave the same totals for the years 1972–74.

TABLE A1

NUMBER OF DRIVERS WITH M CLAIMS IN PERIOD 1969–71 AND N CLAIMS IN THE PERIOD 1972–74

CALIFORNIA, FEMALE DRIVERS

1972–74	1969–71 Claims						
Claims	m = 0	m = 1	m = 2	m = 3	m = 4	m = 5	Total
n = 0	19,634	1,757	128	10	1	0	21,530
n = 1	1,816	266	33	5	1	0	2,121
n = 2	164	24	8	1	0	0	197
n = 3	16	3	0	1	0	0	20
n = 4	2	0	0	0	0	0	2
n = 5	1	1	0	0	0	0	2
Total	21,633	2,051	170	16	2	0	23,872

TABLE A2

NUMBER OF DRIVERS WITH M CLAIMS IN PERIOD 1969–71 AND N CLAIMS IN THE PERIOD 1972–74

CALIFORNIA, MALE DRIVERS

1972–74	1969–71 Claims						
Claims	m = 0	m = 1	m = 2	m = 3	m = 4	m = 5	Total
n = 0	21,800	3,363	381	41	3	1	25,589
n = 1	3,226	692	124	18	5	0	4,065
n = 2	403	126	26	5	0	1	561
n = 3	44	15	3	2	0	0	64
n = 4	7	2	2	0	0	1	12
n = 5	0	0	1	1	0	0	2
Total	25,480	4,198	537	67	8	3	30,293

TABLE A3

NUMBER OF DRIVERS WITH M CLAIMS IN PERIOD 1969–71 AND N CLAIMS IN THE PERIOD 1972–74

CALIFORNIA, ALL DRIVERS

1972–74	1969–71 Claims						
Claims	m = 0	m = 1	m = 2	m = 3	m = 4	m = 5	Total
n = 0	41,434	5,120	509	51	4	1	47,119
n = 1	5,042	958	157	23	6	0	6,186
n = 2	567	150	34	6	0	1	758
n = 3	60	18	4	2	0	0	84
n = 4	9	2	2	0	0	1	14
n = 5	1	1	1	1	0	0	4
Total	47,113	6,249	707	83	10	3	54,165

APPENDIX B

ESTIMATES OF THE COEFFICIENT OF VARIATION OF EXPOSURE LEVEL

A number of studies have used data on mileage driven per unit of time by individual drivers. Unfortunately, none of these studies presents the essential summary statistics, the mean and variance of the mileage. These summary statistics would be sufficient to estimate the coefficient of variation and provide a standard for comparison. Since data are not available for direct estimation, indirect methods of estimation are needed.

The approach taken in this appendix is to assume that the distribution of mileage driven by members of a population is lognormally distributed. Given this assumption, information on fractiles of the distribution would be sufficient to permit an estimate of the coefficient of variation. Even this, however, is not available directly.

A plausible way of inferring fractiles of the distribution is to assume that published statistics will relate to groups that are of reasonable size. The *California Driver Record Book* for 1976 gives accidents per driver and per mile for drivers using their vehicles for specified annual mileages. The lowest category listed is from zero to 2,250 miles; the highest one is over 100,000 miles. We assume that 2,250 and 100,000 are corresponding fractiles on the left and right tails of the distribution. This assumption leads to an estimate of 15,000 for the median annual mileage driver by California drivers, an estimate that appears acceptable. By postulating which fractile corresponds to these numbers, we obtain estimates of the mean annual mileage and the coefficient of variation in this quantity. The estimates are shown in Table B1. Note that the largest and smallest assumed fractiles are not likely to be correct. The highest, one percent, leads to a coefficient of variation for the exposure level which is comparable to that of the aggregate accident rate; this would imply virtually no variability in the accident propensity per mile driven among individuals. The lowest, one per million, would not allow enough drivers in the extreme groups to provide reliable statistics. Between these extremes, the inferred coefficient of variation is fairly stable. Thus, in spite of the lack of direct data, it is plausible that the coefficient of variation in mileage driven is between one quarter and one half.

TABLE B1

ESTIMATES OF THE MEAN AND COEFFICIENT OF VARIATION OF ANNUAL MILEAGE DRIVEN

Assumed Fractile	Inferred Value of: Mean	Inferred Value of: Coefficient of Variation
1/100	20,900	0.94
1/1,000	18,100	0.46
1/10,000	17,100	0.30
1/100,000	16,600	0.22
1/1,000,000	16,200	0.17

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DISCUSSION OF PAPER PUBLISHED IN VOLUME LXXV

MINIMUM BIAS WITH GENERALIZED LINEAR MODELS

ROBERT L. BROWN

DISCUSSION BY GARY G. VENTER

1. INTRODUCTION

This paper is a welcome addition to CAS literature on cross-classification ratemaking. This review considers it in the context of other recent work outside the *PCAS*. Despite the title of the paper, the connection with general linear models does not seem to be the primary emphasis of the paper, and some skepticism about this aspect is voiced below.

In his paper, Robert Brown provides additional insight into minimum bias procedures as well as an introduction to generalized linear models. The cross-classification framework is that provided by Bailey [1]. For data with n rows and p columns, the cell in the ith row and jth column has n_{ij} exposure units, e.g., premium, which generate data, e.g., a loss ratio, of $r_{ij} = L_{ij}/n_{ij}$. This is modeled by n row parameters $x_1 \ldots x_n$ and p column parameters $y_1 \ldots y_p$.

Bailey models the ijth cell as an arithmetic function of x_i and y_j; for example, the multiplicative model uses the function $f(x_i,y_j) = x_iy_j$ to estimate future observations of r_{ij}. He then, in effect, applies the principle of balance; he requires that the row and column totals from the model balance to those from the data. In formulas, for each row i:

$$\sum_j n_{ij}r_{ij} = \sum_j n_{ij}f(x_i,y_j),$$

and for each column j:

$$\sum_i n_{ij}r_{ij} = \sum_i n_{ij}f(x_i,y_j).$$

There are $n + p$ such equations, which are enough to solve for all the x's and y's, and as Bailey notes, the solutions can be obtained iteratively. In fact, usually the equations are of the form

$$x_i = g_i(y_1, \ldots ,y_p) \text{ and } y_j = h_j(x_1, \ldots ,x_n).$$

By starting with reasonable initial values for the x's and y's, the g and h functions can be used to iteratively refine these values until stability is achieved. This is called fixed point iteration, and its convergence properties can be found in numerical analysis texts. Thus an estimation method is specified by giving its system of equations. Brown follows this convention, as does this review. A detail not usually mentioned is that only $n + p - 1$ independent equations are specified in such systems. The result of this is that one of the $n + p$ parameters can be set arbitrarily, e.g., to 1. In a multiplicative model, for example, multiplying the x's by a factor and dividing the y's by the same factor will not affect the cell estimates, so one less parameter is really needed.

As will be discussed more fully below, at least four types of alternatives to Bailey's method have been developed, mostly outside the CAS *Proceedings* or not recognized as relating to the minimum bias procedure. These are: (1) alternatives to the balance principle; (2) more general arithmetic functions; (3) using the arithmetic function as a base, but allowing individual cells to vary from that, based on their own data; and (4) estimating individual cells without postulating an arithmetic relationship between rows and columns. Brown's paper addresses primarily the first area. This review points out some remaining difficulties, and briefly recaps how they have been approached in other studies, using the above alternatives. The connection with general linear models is also discussed.

2. ALTERNATIVES TO BALANCE PRINCIPLE

Brown provides several alternatives to the principle of balance, although he does not give explicit reasons for abandoning it. One such reason may be that it assigns full credibility to each row and column in total, which may not be appropriate. A possible response, however, would be to credibility-adjust the row and column totals before applying the balance principle. Another response might be to find models that automatically quantify the likely deviations from the cell estimates. However, this could probably be done without discarding the balance principle. Perhaps the basic motivation for abandoning balance is that the principle, while appealing, is not self evident, and thus more fundamental principles should be sought.

In any case, the first alternative Brown presents is to model the numerator of r_{ij}, i.e., L_{ij}, as a random draw from a distribution with mean $n_{ij}f(x_i,y_j)$. Given a distribution and an arithmetic function f, maximum likelihood estimation can be used to solve for all the parameters from the observations. Several distributions are illustrated, and for each a system of $n + p$ equations in $n + p$ unknowns is derived.

For instance, assuming a normal distribution with a multiplicative model, i.e., that the L_{ij} are normally distributed with mean $n_{ij}x_iy_j$ and variance σ^2, gives the following equation for each x_i:

$$x_i\sum_j n_{ij}^2y_j^2 = \sum_j n_{ij}^2r_{ij}y_j,$$

and similarly for each y_j. Interestingly, these equations do not involve the σ^2 parameter of the normal distribution.

For the multiplicative model with a Poisson distribution assumption, Brown finds that the system of equations for Bailey's balanced multiplicative model is reproduced. This result was also shown by van Eeghen, Greup, and Nijssen [8]. While it shows that the Poisson distribution satisfies the principle of balance, it does not give much support for using a balanced model, in that the cell data is not usually Poisson distributed. In fact, this might be a reason for dropping the balance requirement, since most distributions will not reproduce it. Thus, the equivalence of the Poisson and Bailey models, rather than supporting their use, suggest that alternatives might be more appropriate.

For the exponential distribution, the following simple equations are produced:

$$x_i = \sum_{j=1}^{p} \frac{r_{ij}}{py_j}, \text{ and } y_j = \sum_{i=1}^{n} \frac{r_{ij}}{nx_i}.$$

This reviewer has found that the same equations hold for the gamma model, which adds a parameter to the exponential. Like the σ^2 of the Normal distribution, this parameter does not enter the equations for the x's and y's.

The most logical distribution for the multiplicative model would probably be the lognormal, because that results when the errors are also due to multiplicative effects. The estimating equations can be derived by using the additive normal model with the logs of the data.

The normal distribution models Brown uses are unusual in that each cell has the same variance σ^2 for L_{ij}. It is hard to see how this could occur from cells with different exposures. For instance, if each exposure unit has the same variance τ^2, then the cell variance would be $n_{ij}\tau^2$ due to the additivity of independent exposures, that is, it would not be constant but would be proportional to n_{ij}. Or, if there are additional gaps between the arithmetic function and the exposure unit means, which average to zero over all cells, i.e., $E(L_{ij}) = x_i y_j + g_{ij}$, with $E(g_{ij}) = 0$, then the variance of this gap, $\mathrm{Var}(g_{ij})$, would be added to $n_{ij}\tau^2$ to give the variance of L_{ij}. Only if the variance of the gap, i.e., error from the arithmetic function assumption, were large compared to the risk variance $n_{ij}\tau^2$ would the constant variance assumption be a reasonable approximation. However, in this case the use of that arithmetic function would be questionable.

It is not difficult to carry out the estimation assuming a variance of $n_{ij}\sigma^2$ rather than σ^2 for L_{ij}. For instance, for the multiplicative model, the x equations become:

$$x_i \sum_j n_{ij} y_j^2 = \sum_j n_{ij} r_{ij} y_j,$$

which are in fact the equations Brown derived for the least squares multiplicative model.

The latter is another alternative Brown presents, namely minimizing the weighted least squares difference between the data and the model. For instance, for the multiplicative model, minimize $\Sigma_{ij} n_{ij}(r_{ij} - x_i y_j)^2$.

This was also advocated by Sant [7], and, under the label "analysis of variance" approach, by Chamberlain [2] and others. The least squares approach has the advantage of not assuming a distributional form, although it does still assume a particular arithmetic function of the parameters x_i and y_j. If the different cell means themselves come from a highly skewed distribution, e.g., display very large percentage differences among the cells, then minimizing the sum of squared errors could allow significant percentage errors for the low mean cells. Thus least squares works reasonably well only for certain types of distributions.

It is generally advisable when doing weighted least squares to use weights which are inversely proportional to the cell variances. The weights Brown uses are thus consistent with r_{ij} having variance inversely proportional to n_{ij}, which seems appropriate. However, the constant variance model for L_{ij} would lead to weights of n_{ij}^2, which, for the least squares model, would produce the system of equations Brown gave for the normal model.

3. GENERALIZATIONS OF ARITHMETIC FUNCTIONS

Although not mentioned in the paper, both of Brown's alternatives, as well as Bailey's original method, can be generalized to use other arithmetic functions of the row and column parameters. For example, the function $x_iy_j + z_j$ has sometimes been used to good effect in class-by-territory ratemaking. This is a combination of additive and multiplicative effects that uses $n + 2p$ parameters. Maximum likelihood estimation with the constant variance normal distribution, for instance, provides a set of $n + 2p$ equations which have the forms:

$$x_i \sum_j n_{ij}^2 y_j^2 = \sum_j n_{ij}^2 (r_{ij} - z_j) y_j,$$

$$y_j \sum_i n_{ij}^2 x_i^2 = \sum_i n_{ij}^2 (r_{ij} - z_j) x_i, \text{ and}$$

$$z_j \sum_i n_{ij}^2 = \sum_i n_{ij}^2 (r_{ij} - x_i y_j).$$

The squares on the exposures would be dropped under the assumption of the variance of L_{ij} proportional to $n_{ij}\sigma^2$. The combined additive-multiplicative function is sometimes appropriate when the high rated classes in the high rated territories, for example, get too much charge from a multiplicative model and not enough from an additive one. Other arithmetic functions are possible, also, such as $x_i^{.8}y_j^{1.2}$, etc., although the term "arithmetic" might be a misnomer for such functions. There is a wide variety of possibilities of this type which have been largely unexplored. An important exception is Harrington [4], who applies an additive model after applying the Box-Cox transformation to the data. This transformation is $r_{ij}^c = (r_{ij}^c - 1)/c$. This is really a common generalization of both the additive ($c = 1$) and multiplicative models, in that the limit of r_{ij}^c as c goes to zero is $\ln(r_{ij})$, giving an additive log model. By searching for the best fitting c parameter, improved fits can be produced.

4. GLIM DISCUSSION

The GLIM section of Brown's paper is somewhat difficult to follow, but he does recommend background material. Even so, it will not be clear to those without experience with linear models how GLIM as defined might apply to the cross-classification problem. The following example illustrates how this can be done for the multiplicative model with Normal constant variance errors.

If L_{ij} denotes the numerator $n_{ij}r_{ij}$ of r_{ij}, and μ_{ij} its expected value, the Normal density can be put in the GLIM form:

$$f(L_{ij}) = \exp\left[\frac{\mu_{ij}L_{ij} - .5\mu_{ij}^2}{\sigma^2} - \frac{L_{ij}^2}{2\sigma^2} - .5\ \ln(2\pi\sigma^2)\right].$$

Since the GLIM definition uses variables x and y, let the row and column effects formerly denoted by x and y now be denoted by w and z instead. The observed vector Y to be modeled is the set of L_{ij} all strung out in a single vector, i.e., if $k = (i - 1)p + j$, then $y_k = L_{ij}$. There are $m = np$ of these y_k's. The coefficients β_h to be estimated will be interpreted as the $\ln(w_i)$'s and $\ln(z_j)$'s listed as a single vector (z's after all the w's), followed by a constant term which should turn out to be 1. Thus, there are $q = n + p + 1$ of these β_h's. The explanatory vector, x_h, for $h < q$ is a list of m elements x_{kh} that are all 0's except for 1's which occur when y_k comes from either a row or a column corresponding to β_h. That is for $k = (i - 1)p + j$, $x_{kh} = 1$ only for $h = i$ and $h = n + j$. The last vector x_q, consists of the logs of all the exposures n_{ij}.

With these definitions, let $n_k = \Sigma_{h=1}^{q} x_{kh}\beta_h$. If we defined the x's right, then $n_k = \ln(w_i) + \ln(z_j) + \ln(n_{ij}) = \ln(\mu_{ij}) = \ln(\mu_k)$. Therefore the link function g is the log function. From the form of the density function, it can be seen, in Brown's notation, that the dispersion parameter ϕ is σ^2, $a(\phi) = \phi$, and $c(y,\phi) = -.5[(y^2/\phi) + \ln(2\pi\phi)]$. Also, $\theta_k = \mu_k$, and $b(\theta) = .5\theta^2$.

Thus, this GLIM model is just the original Normal model with constant variance, assuming that maximum likelihood is used to estimate the GLIM parameters. For some reason, the constant variance assumption seems to be inherent in the GLIM models, although it is not necessary when using regular maximum likelihood methods outside of GLIM. For this application, then, GLIM seems to require a fair amount of work to properly arrange the data, with benefits that are unclear.

From the deviances shown in the paper for 12 models, as well as their apparent reliance on density functions, it would appear that deviances cannot be compared across distributions to determine the best fitting model. They probably can be compared to evaluate link functions for one distribution.

5. ALTERNATIVES TO ARITHMETIC FUNCTIONS

Another criticism of minimum bias methods has been the strict reliance on the arithmetic function. Just because data is organized in rows and columns does not imply that there is such an arithmetic relationship. For instance, if loggers have 20% more injuries than cab drivers nationwide, can we expect this will hold true in New York? If office workers have a 90% lower work related accident frequency than workers in general, will this be the case in lower Manhattan? The multiplicative models assume such relationships will hold, and the additive models are based on similar assumptions. In some lines of insurance, it is felt that any arithmetic function of row and column averages can adequately model individual cell results.

At least two methods have been developed in response to this criticism: allowing individual cells to vary from the arithmetic function, or estimating individual cells without using an arithmetic function, e.g., by credibility methods.

The first method was used in the 1981 Massachusetts auto rate hearings, where the calculated relativity was credibility weighted with the cell data r_{ij}. Thus, cells with enough credibility could be based largely upon their own experience. As described in DuMouchel [3], the arithmetic function f was the combined additive-multiplicative function, and the credibility for cell ij was given by:

$$Z_{ij} = \frac{n_{ij}}{n_{ij} + K_j} .$$

Here K_j is the ratio of two variance components s_j^2/t^2, where s_j^2 is the within-cell variance scalar over time, and t^2 is the average variance of true cell means from their calculated relativities. More precisely, for time period t, r_{ijt} has mean μ_{ij} and variance s_j^2/n_{ij}, and μ_{ij} has mean $f(x_i,y_j,z_j)$ and variance t^2. If there are c time periods in the data, s_j^2 is estimated by:

$$\hat{s}_j^2 = \sum_{i,t} n_{it}(r_{ijt} - r_{ij})^2/n(c - 1).$$

DuMouchel gives a somewhat intricate method of estimating t^2. A Bühlmann-Straub type estimation would also be possible. For this, let

$$W = \sum_{i,j,t} n_{ijt}(r_{ijt} - f(x_i,y_j,z_j))^2.$$

Then it can be shown that

$$E(W) = nc \sum_j s_j^2 + t^2 \sum_{i,j,t} n_{ijt}.$$

This means that W is an unbiased estimator of the right hand side, and can thus be used to estimate t^2. That is,

$$t^2 = \left[W - nc \sum_j \hat{s}_j^2 \right] \div \sum_{i,j,t} n_{ijt}.$$

If the estimate is negative, it should be set to zero, which would give full credibility to the model and none to the cell data. In the Massachusetts case, DuMouchel found that the combined additive-multiplicative model fit the data very well, so that the credibilities given individual cell data were low.

Other approaches to giving credibility to individual cell variation from the arithmetic function can be used. An example is found in Weisberg, Tomberlin, and Chatterjee [10], who use similar model assumptions to those of DuMouchel, but just with pure additive or multiplicative functions f. They use a different, possibly more general, statisical method to estimate the credibilities.

Another alternative is to incorporate so-called interaction effects, which are essentially additional parameters for specific cells. This was suggested by Chamberlain [2], who showed how to measure the significance of such terms. Jee [6], who summarizes and tests many of the above methods, added all individual cell variables that improved the F statistic at a 15% significance level, and found that this improved the predictive accuracy of the additive, multiplicative, and Box-Cox models.

The credibility only method, not using any arithmetic function, is illustrated by the national relativity approach often used in workers compensation, as described by Harwayne [5]. The indicated percentage change in non-serious pure premiums for the ith class in industry group 1 in state j, for example, is calculated by a variant of the following. Let x_i be the indicated change for class i countrywide, and let y_j be the indicated change for industry group 1 in state j, with r_{ij} the indicated change based on the cell data alone. If the expected number of claims for the ijth cell is at least 300, r_{ij} receives full credibility. Otherwise, the credibility it receives, z_{ij}, is the ratio of expected claims to 300, raised to the two-thirds power. The credibility given to x_i is calculated by essentially the same rule, but it is limited to $(1 - z_{ij}) \div 2$. The balance of the credibility goes to y_j. In formulas, the estimate for the ij cell is:

$$\hat{r}_{ij} = z_{ij}r_{ij} + z_i x_i + z_j y_j,$$

where z_{ij} and z_i are calculated by the rule (expected claims/300)$^{2/3}$, where the expected claims are for the class in the state or the class countrywide, as appropriate. Although the estimate uses the row and column averages, there is no mathematical relationship postulated between the cell and the totals for the row and column it is in. The x's and y's in the previous models were parameters to be estimated from the data, presumably with some estimation error, while here they are statistics calculated exactly.

The credibilities above may work well in practice, but they could be criticized as being ad hoc. A least squares credibility type approach is given in Venter [9]. The estimate for the gth row and hth column for a future time period is a linear sum of the observations for all the cells available, i.e.,

$$\hat{r}_{gh} = q + \sum_{i,j} z_{ij} r_{ij},$$

where the z's are the weights in the linear function, and q is the constant term. These are found by minimizing the expected squared error $E(\hat{r}_{gh} - r_{gh0})^2$, where r_{gh0} is a future observation of the cell. Thus the credibility estimator is the linear function of all the cell data that minimizes the expected squared error between the estimate and a future observation. This is the standard least squares credibility, applied to the cross-classification problem. As is often the case with credibility, it will probably work better with indicated changes than with pure premium itself.

To express the resulting weights z_{ij} more compactly, introduce the notation $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. The weights are derived as functions of four variance components: u^2 is the variance between row means, v^2 is the variance between column means, w^2 is the variance of a cell mean from row-column additivity, and s^2 is the average relative variance of the cells from their means over time. Also, m is the overall mean of all cells. More precisely, the assumption is:

$$\text{Cov}(r_{ijt} r_{abk}) = u^2\delta_{ia} + v^2\delta_{jb} + w^2\delta_{ia}\delta_{jb} + s^2\delta_{ia}\delta_{jb}\delta_{tk} \div n_{ijt}.$$

This holds for both additive and multiplicative models, and many others as well. The weights z are expressed in terms of ratios of the variance components. K, J, and L are the ratios of s^2 to u^2, v^2, and w^2, respectively. Using a dot in a subscript to denote summation over that subscript, the weights are:

$$q = m(1 - z_{..}), \text{ and}$$

$$\frac{z_{ij}}{w_{ij}} = \frac{\delta_{gi}}{K} + \frac{\delta_{hj}}{J} + \frac{\delta_{gi}\delta_{hj}}{L} - \frac{z_{i.}}{K} - \frac{z_{.j}}{J}, \text{ where } \frac{1}{w_{ij}} = \frac{1}{n_{ij}} + \frac{1}{L}.$$

This requires the summed row and column weights $z_{i.}$ and $z_{.j}$, which can be found from the system of $n + p$ linear equations below, one for each row a and column b:

$$z_{a.}\left[1 + \frac{w_{a.}}{K}\right] = \frac{1}{J}\left[w_{ah} - \sum_j w_{aj}z_{.j}\right] + \delta_{ga}\left[\frac{w_{a.}}{K} + \frac{w_{ah}}{L}\right] \text{, and}$$

$$z_{.b}\left[1 + \frac{w_{.b}}{J}\right] = \frac{1}{K}\left[w_{gb} - \sum_i w_{ib}z_{i.}\right] + \delta_{hb}\left[\frac{w_{.b}}{J} + \frac{w_{gb}}{L}\right] .$$

As these equations are linear, they can be solved by matrix methods, although iteration may also work well. The resulting weights differ from credibilities in that they are not necessarily between zero and one, although they are derived in the same manner as credibility weights in the single dimension case.

A method for estimating the required variance components is to compute the four sums of squared differences below:

$$D1 = \sum_{i,j,t} n_{ijt}(r_{ijt} - r_{ij})^2 ,$$

$$D2 = \sum_{i,j,t} n_{ijt}(r_{ijt} - x_i)^2 ,$$

$$D3 = \sum_{i,j,t} n_{ijt}(r_{ijt} - y_j)^2 \text{, and}$$

$$D4 = \sum_{i,j,t} n_{ijt}(r_{ijt} - \hat{m})^2 .$$

Using their expected values below, these can be used to estimate s^2, J, K, and L.

$$E(D1) = s^2 np(c - 1),$$

$$E(D2) = s^2 \left[n(pc - 1) + \frac{1}{J + L} \left(n_{..} - \sum_{ij} n_{ij}^2 \div n_{i.} \right) \right],$$

$$E(D3) = s^2 \left[p(nc - 1) + \frac{1}{K + L} \left(n_{..} - \sum_{ij} n_{ij}^2 \div n_{.j} \right) \right], \text{ and}$$

$$E(D4) = s^2 \left[npc - 1 + \frac{1}{K} \left(n_{..} - \sum_{i} n_{i.}^2 \div n_{..} \right) \right.$$

$$\left. + \frac{1}{J} \left(n_{..} - \sum_{j} n_{.j}^2 \div n_{..} \right) + \frac{1}{L} \left(n_{..} - \sum_{i,j} n_{ij}^2 \div n_{..} \right) \right].$$

Brown's paper is a valuable addition to the *Proceedings*, particularly the least squares and maximum likelihood methods. Further empirical studies on how well all of the above models work would be a good area for future research. Both the goodness of fit and accuracy of prediction should be tested, and any distributional assumptions should be reviewed through an analysis of the residuals.

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ADDRESS TO NEW MEMBERS—NOVEMBER 12, 1990

FREDERICK W. KILBOURNE

Good morning, and congratulations to you new Fellows and Associates of the Casualty Actuarial Society. My job over the next five minutes or so is to imprint the wisdom of the ages on you; your job is to become inspired to contribute even more to mankind and the actuarial profession over the decades remaining to you.

After accepting this assignment, I wondered why my remarks are to be directed exclusively at new members. I learned that actuaries are like coins: those without the requisite number of exams are unformed, and are too soft for any imprint to take hold and stay; those minted more than a few months ago, on the other hand, have become so hardened and sullied that nothing leaves any impression on them. Only those of you who are freshly struck are suitable candidates for inspirational imprinting. My remarks are for you alone.

There's a great deal of inspirational wisdom out there—even beyond that which you've gleaned from the Syllabus and the exams—but our time is short. I considered offering you nine commandments, or warning you about the six deadly sins. I decided instead to focus on the three R's—and the big F. If each of you will conduct your actuarial practice keeping in mind the three R's—and the big F—the world will be a better place. And you'll get more out of your work.

While I don't mean to discourage you from reading, 'riting, and 'rithmetic, these are not the three R's I have in mind today. To distinguish my three R's, I shall sometimes refer to them as Be-R's. I worried briefly whether you'd be able to absorb wisdom simply by memorizing a confusing and arbitrary list, but I soon realized you are experts in such matters.

First, *be reliable*. If you agree to do a job, do it. If you agree to complete it by Friday, plan to complete it on Tuesday, so you can actually finish it on Wednesday, and deliver it on Thursday. Use the same approach on your budget. If you think or suspect that you can't do the job right, on time, within budget—don't accept it, or try to have it changed. If unanticipated major problems arise along the way, let the person relying on you know immediately, so that together you can take corrective or defensive action. If your work product is important, unreliability will have serious repercussions. If it isn't important, get a new job.

Second, *be rigorous*. While our profession does need more and better communicators and people who can see the forest, as we remind ourselves frequently, we must never forget that, in our heart of hearts, we are essentially tree-people. Among the questions to be asked when undertaking an assignment are: what exactly is your actuarial problem of the hour, what specific question do you need answered, and what precision actuarial tools are available to you in doing the work?

Actuaries have an important advantage due to the fact that we focus on the tree—and often on the branch or the leaf—relative to sometimes competing professionals, such as the forester economist, or the blue-sky futurist. Focus ensures that someone cares enough about our work to pay us. And focus is the key to rigor. The fact that the actuarial territory is the future—and thus that our conclusions necessarily are subject to uncertainty—makes it especially important that we know exactly what it is we're trying to estimate, that our logic and thinking be rigorous rather than fuzzy or wishful or conservative, and that we brainstorm freely, but avoid tangents like the plague. Which reminds me to share with you Joe Brownlee's speech at the Annual Meeting of the Academy regarding the three kinds of actuaries: those who can count, and those who can't. Finally, we must express our actuarial conclusions, and the uncertainty that lurks beneath them, with precision. You might say to your employer or client: "My best estimate of the actuarial number you seek is exactly $15 million—or exactly $13 million under scenario A and $18 million under scenario B—but never that it is $15.376 million, and never that it is in the range of $10 to $20 million," unless you're willing to personally insure against the result coming in at $21 million. You might go on to say, "the uncertainty underlying my exact best estimate, by the way, may be precisely defined as 'uncertain'—but the CAS is working on the problem, and we'll keep you posted."

Third, *be right*. Actually, this is redundant. The best way to be right more often than the next fellow is to be more rigorous. While you're at it, follow up on your actuarial projections, and keep a running score for yourself. I wish I'd done that over the years, and recommend it to those with the time to see their IBNR's converted into paids. But, back to being right, and living up to my promise to have a confusing and arbitrary list, what I really mean to say is *do the right thing*. You know what that covers: don't use your actuarial skills on projects that are contrary to the public interest. By the same token, don't fail to speak out when actuarial principles are trashed, whether or not you are asked for your opinion. The Federal Budget comes to mind, and you can undoubtedly think of other examples closer to home (such as Proposition 103). Contribute to the development of standards for your profession, and then follow them. Tell the truth. Treat other people fairly. Few, if any of you, don't know right from wrong, or don't care.

We may not always agree on the particulars, but if each of us does the right thing as we see it, we all will advance together. This is important for all professions—well, almost all—but it is critical for the actuarial profession, which is totally dependent on public trust. Jim Anderson recently spoke of our need to maintain our integrity and objectivity. If we had an oath, as was proposed some years ago, those terms would certainly be in it. Thank God we don't have an oath, and thank God we have a sufficient supply of integrity and objectivity to go around as we do the right thing in our actuarial work.

BIG F IS FOR FREEDOM. Freedom is our most important possession, the most undervalued, and the hardest to regain once it is lost. Whatever strength our economy may have is due to freedom. Our medical and technological advances would have been retreats without freedom. The joyful act without freedom is sorrow, except for those few who prefer bondage. When freedom goes, the actuary will follow, for both require a strong and open market in risk to survive.

So I close with a plea for you to use your talent and your actuarial skills to defend and promote freedom—while at the same time being reliable and rigorous, and doing the right thing. If you think I may have overdone this freedom thing, ask someone from eastern Europe—or China—or ask an enrolled actuary.

PRESIDENTIAL ADDRESS—NOVEMBER 12, 1990

THE SEVEN DEADLY SINS

MICHAEL FUSCO

Like all presidents before me, I struggled trying to decide what to say. I turned to past presidents and their addresses for guidance, and I heard a lot about the traditions of our Society. While many former presidents provided input and some even lamented that they would like to revise their remarks, none offered to actually help me write mine. To future presidents in the room, please remember that I will also be a traditionalist soon.

I am and have been all year a strong advocate of the hyphenated actuary concept. I call myself an actuary-casualty and think of all of you in the same way. We know that we are part of the broader actuarial profession, but we tend to focus on the casualty specialty of that profession. While that focus may be appropriate at most times, I am fearful that, by thinking of ourselves in such a narrow way, we may be losing sight of the rest of the world.

We do not live in a monastery, but rather in a world cluttered with regulation, financial ramifications and social issues. Yet perhaps we should remember the concept articulated by Cassian in the fifth century and embellished by St. Thomas Aquinas in their advices to monastery monks. Seven deadly sins were enumerated as the chief obstacles to perfection. These deadly sins can be distinguished not so much by their gravity, but by their power to generate even more misdeeds. As actuaries-casualty in the Casualty Actuarial Society, we have been striving for the last 76 years for perfection. Let's examine how we relate to these seven deadly sins.

DEADLY SIN #1—PRIDE

It is best defined as "inordinate self-esteem." Although we are guilty of this sin, we rationalize that it is a virtue as well and is justified. We are proud of our profession and pleased that it was ranked number one by an outside objective source. We take pride in our work and well we should because it is filled with creativity. We are pleased with our Society's growth, its uniqueness, and its independence. I personally swelled with pride as I travelled this year from the Seattle Space Needle, to the Alamo in San Antonio, to the Inn at Sturbridge Village, visiting our regional affiliates and seeing so many students eager to learn and enter this profession, so many competent actuaries seeking to further their education, and so many potential leaders for our Society. My crystal ball clearly displays only a bright future for the CAS.

But we must not be so blinded by our pride that we diminish the value of other specialty groups of actuaries or of other professions; that we erect artificial barriers of entry to our own specialty group; or that we overlook the incompetence of some actuaries. We are spending resources wisely to establish standards of practice. When these standards are violated—and unfortunately that is inevitable—we must take the necessary disciplinary actions, as distasteful as those actions may be. If we fail to, we will have little to be proud of as a professional society.

DEADLY SIN #2—COVETOUSNESS

This is best described as "an inordinate desire for wealth or possessions." While I cannot name too many spendthrifts in our group, I cannot label us as sinners in this regard. We despise the evil associated with waste and therefore attempt to establish insurance rating systems that are efficient and accurate, but we are not greedy. The excess profits laws in several states make certain of that.

Our main wealth is in our knowledge, and we readily share that with others. Our Society membership is open to all who wish to study for exams, our literature is freely available, and our seminars are open to nonmembers, although we do levy a modest surcharge. No, if misers are what we are, then counting would be our main pleasure, and that we know is the domain of another profession, while analysis is reserved for actuaries.

DEADLY SIN #3—LUST

If I were brave, I would use the definition that is fairly common—manly necessity—but I fear that that would disturb the female Fellows (one of my favorite oxymorons) in our Society. Instead, I will define it as "unlawful passion" and again, without taking a straw poll, I will say that we fail as sinners here. What we have done instead, through an extraordinary act of will, has been to transform this "unlawful passion" that is innate to all of us, into "lawful passion" and then display our lust openly.

We lust for our work—not only to stay busy but to succeed at it. We yearn for the right answer or for the best answer possible, given the limitations of data and tools. Look around the room and guess at how often the person seated next to you has stayed up until 3 in the morning working on a personal computer to solve a problem that appeared unsolvable. Oh, if only we used these lustful energies and hours differently, our profession too could have been the subject of a prime time TV series.

DEADLY SIN #4—ANGER

I define it as rage or desire for vengeance or, if anger can be righteous, then maybe justice. But guess what? We fail miserably as sinners here again. Perhaps we have become very good at controlling our anger, as we've often had to when testifying as expert witnesses and being abused on cross examination. But we don't display anger even when we should. I'll use IBNR as my example.

For years we have accepted "qualified loss reserve specialist" (QLRS) as the description of one who can be allowed to determine IBNR reserves, when we knew that "actuary" should be in that definition. Even with the change this year to the term "qualified actuary" (QA), we accepted addons to that definition to make it not too difficult for others to intrude into our professional specialty. We know that only with the proper basic education, continuing education, and experience can one accurately determine the reserves needed to pay ultimate liabilities. And yet, we allow others to practice this purest form of casualty actuarial science without a license. Other professions would not stand for this, and we should not either. I am not recommending that you throw things, as I have been known to do on occasion, but make it clear that you're mad as hell and not going to stand for it anymore!

DEADLY SIN #5—GLUTTONY

"Overindulgence to the point of absurdity." The word "too" is very important here—whether too much or too eagerly or too expensively or with too much ado. Notwithstanding our performance last night at the cocktail reception, I cannot classify us as sinners here either.

We use terms like range of reasonableness and margins for error to prevent us from going to extremes. And while we very much want our Society to grow, we are content to do so from within and not to cannibalize other societies. We must retain respect for other specialty groups or actuaries, yet maintain our distance from them, and should insist on similar treatment. To misquote from a famous Shakespearean character, our motto should be "neither an acquirer, nor an acquiree be!"

But we should seek to expand our horizons in the next decade, without being gluttonous. The CAS name is not well known internationally and it should be. We must improve our visibility overseas in the 1990s.

DEADLY SIN #6—ENVY

It's easily defined as jealousy—petty or otherwise. Let's not kid ourselves—we have all experienced it. The first time probably was as a student when we failed an actuarial exam while watching someone else in our study group pass. And in our careers, I'll bet there have been a few occasions when we've vied for a particular position or for a particular client and finished second best. But we've managed to overcome those disappointments and continue to work together successfully on CAS committees—a tribute to true professionals.

No, we have not let envy overwhelm us—if we had, we would never be creators, only imitators. But, we must learn to deal better with others who envy us. We are well thought of by the insurance industry that we serve and by other actuarial specialty groups and other professionals. We are very much in demand. And why not? What other profession, other than fortune tellers, does as well as we do in predicting the future?

DEADLY SIN #7—SLOTH

This is "disinclination to action or labor" and is not characteristic of us at all. Instead of indolence, we exhibit hyper-activity. We try to stay too busy, not only in our work, but in our professional society responsibilities as well; and yet we never hide behind that simple fact. In truth, our slogan is: "When you want something done, give it to a busy Fellow (or Associate)."

The point is, our Society runs because you make it run. It is only through our volunteer efforts that anything gets done. My advice to the new Fellows and new Associates in the room is to get involved and stay involved. The meritocracy works—keep it going!

That completes the 7 deadly sins. If the sins are deadly to us as individuals, they are no less deadly to our Society. But I hope I have convinced you that, while not perfect, we can hardly be classified as "sinners."

This quest for perfection began 76 years ago. Some say the CAS's basic mission has not changed much. Thomas Jefferson recognized the necessity for institutions to change many years ago in a letter to a friend: ". . . institutions must go hand in hand with the progress of the human mind. As that becomes more developed, more enlightened, as new discoveries are made, new truths discovered and manners and opinions change, with the change of circumstances, institutions must advance also to keep pace with the times."

I maintain that the CAS has kept pace with the times. Our three primary functions—giving exams, sponsoring meetings, and stimulating research—remain, but through the years, we have strived for more, better, and different.

There are more exams today than before—even without partitioning—because the subject matter to be mastered is greater. The Syllabus is different because society needs actuaries to know different things. And the exams are more rigorous—ask anyone who has taken one recently. We have improved in this area and will continue to do so.

Our meetings are more plentiful—seminars are given on many new topics and regional affiliates are providing additional forums. We didn't need an Environmental Issues seminar in the early days of the CAS, but we need one now. And the meetings are much more professionally run with the advances in technology that have been made. I predict a video-conference meeting soon.

Research remains the area of high visibility. Computer technology should make our modeling processes better, and our call papers are on new and different subjects. The Global Issues Discussion Paper Program for next Spring could not have been issued by our founding fathers. And your Board of Directors has recently approved a major funding expenditure for 1991 to stimulate even more research, so good things are in store for us.

In researching our *Proceedings*, I noticed that a former President in his Presidential Address some 20 years ago, predicted that a merger of the various actuarial societies into one society was inevitable. That did not occur and it will not occur. Whether we keep our name as the Casualty Actuarial Society or change it one day to the Society of Actuaries-Casualty, we will retain our uniqueness and our independence.

But preserving our independence requires cohesion among ourselves. On this point, I offer to all of us the following words from Benjamin Franklin: "We must all hang together, or assuredly we shall all hang separately."

And now, please let me close by thanking a few individuals.

Thanks to my family for accepting the demands on my time this past year.

Thanks to my employer, ISO, for also accepting the demands on my time this year.

Thanks to the 1990 Executive Council, not only for their hard work, but also for their patience and guidance. They are symbolic of not just the Executive Council, but also the Board, the Committee Chairmen, and so many of you who served on committees or task forces.

Thanks to Kathy Spicer, not only as a representative of the CAS office, but also for being my tangible legacy to the CAS. I hired her as CAS Meeting Planner 3 years ago; and she has made our meetings much more professional and, as a result, has made my hiring decision look very good.

And thanks to Denise De Angelis for being my partner in all this not just this year, but for the past 14 years. She knows only too well that I could not have done it without her.

And lastly, thanks to all of you—the CAS membership—for giving me the privilege and the pleasure to serve as President of the Society I love.

MINUTES OF THE 1990 ANNUAL MEETING

NOVEMBER 11–14, 1990
LE MERIDIEN, NEW ORLEANS, LOUISIANA

Sunday, November 11, 1990

The Board of Directors held their regular quarterly meeting from 1:00 p.m. to 4:00 p.m.

Registration was held from 4:00 p.m. to 6:30 p.m.

From 5:30 p.m. to 6:30 p.m., there was a special presentation to new Associates and their guests. This session included an introduction to standards of professional conduct and the CAS committee structure.

A general reception for all members and guests was held from 6:30 p.m. to 7:30 p.m.

Monday, November 12, 1990

Registration continued from 7:00 a.m. to 8:00 a.m.

President Michael Fusco opened the meeting at 8:00 a.m. The first order of business was to introduce Robert Conger to give the Secretary's and Treasurer's report. Robert Conger also announced the results of elections of officers and directors.

The members of the 1991 Executive Council will be Vice President-Administration, Robert F. Conger; Vice President-Admissions, Steven G. Lehmann; Vice President-Continuing Education, Irene K. Bass; Vice President-Programs and Communications, Albert J. Beer; and Vice President-Research and Development, Alan M. Kaufman. President-Elect will be Michael L. Toothman. New Board members will be Robert A. Anker, Linda L. Bell, W. James MacGinnitie, James N. Stanard, and David J. Oakden.

Michael Toothman introduced 55 new Associates for 1990. Mr. Bryan introduced 54 new Fellows for 1990. The names of these individuals follow:

FELLOWS

Kenneth Apfel	David R. Heyman	Kai-Jaung Pei
Lawrence J. Artes	Anthony D. Hill	Isabelle Perigny
Robert K. Bender	Alan M. Hines	Steven J. Peterson
Stephen W. Book	John M. Hurley	Kevin B. Robbins
Christopher S. Carlson	Steven J. Johnston	Richard D. Robinson
Lynn R. Carroll	Edward M. Jovinelly	Randy J. Roth
Michael J. Caulfield	Mary Jean King	Jeffrey C. Salton
Danielle Charest	Charles D. Kline, Jr.	Richard D. Schug
Walter P. Cieslak	Kenneth R. Krissinger	Mark E. Schultze
Carol Desbiens	John R. Kryczka	Mark J. Silverman
Timothy B. Duffy	Kay E. Kufera	Lisa A. Slotznick
Denis Dumulon	Paul E. Lacko	David Spiegler
Richard L. Fox	David A. LaLonde	Lawrence J. Steinert
Jacque B. Frank	Jon W. Michelson	Edward C. Stone
Richard N. Gibson	H. Elizabeth Mitchell	Christopher M. Suchar
Bonnie S. Gill	Karl G. Moller, Jr.	Ernest I. Wilson
Steven A. Glicksman	Chris E. Nelson	Martha A. Winslow
David C. Harrison	Jonathan Norton	Heather E. Yow

ASSOCIATES

Richard R. Anderson	Diane K. Hausserman	Laura A. Olszewski
Guy A. Avagliano	Gordon K. Hay	Naomi S. Ondrich
Katharine Barnes	Thomas G. Hess	Margaret O'Brien
David Bechtel	Peter H. James	Teresa K. Paffenback
Jeffrey R. Cole	Tony J. Kellner	Susan R. Pino
Francois Dagneau	Bryan J. Kincaid	Richard W. Prescott
Edgar W. Davenport	France LeBlanc	Alfred Raws III
Kendra M. Felisky-Watson	Eric F. Lemieux	James E. Rech
David N. Fields	Stephen J. McGee	Diane R. Rohn
David A. Foley	Dennis T. McNeese	John M. Ruane, Jr.
Jacque B. Frank	M. Sean McPadden	Timothy J. Rundle
Deborah A. Greenwood	Christopher J. McShea	Gary E. Shook
Dawson T. Grubbs	Robert L. Miller	Christy L. Simon
George M. Hansen	Charles B. Mitzel	Patricia E. Smolen
	Todd B. Munson	Tom A. Smolen
	John Nissenbaum	Elizabeth L. Sogge

ASSOCIATES

Thomas N. Stanford	Susan M. Treskolasky	Beth M. Wolfe
Elissa M. Sturm	Scott D. Vandermyde	Kathy A. Wolter
Jeffrey L. Subeck	Marjorie C. Weinstein	Nancy E. Yost

Fred Kilbourne was introduced next and gave an address to the new members.

Al Beer gave a summary of the program and Irene Bass summarized the *Proceedings* papers being presented. She called for reviews of previous papers from the floor. There were none. The following awards were presented:

Woodwood-Fondiller Prize: Amy S. Bouska

Dorweiler Prize: Glenn G. Meyers

The featured speaker, Heiko H. Thieme, spoke from 9:00 a.m. to 10:00 a.m. A general session panel was held from 10:30 a.m. to 12:00 p.m. Ernest G. Jacob, CFA, Vice President, Barclays deZoete Wedd, moderated a panel on the "Capital Management of Property/Casualty Insurance Companies." The panel consisted of Ronald E. Compton, President, Aetna Life & Casualty; Joseph W. Brown, Jr., President, Fireman's Fund Insurance Companies; and Ronald L. Bornhuetter, President and Chief Executive Officer, NAC Re Corporation.

This panel was followed by a luncheon with the Presidential Address by Michael Fusco. Lunch was from 12:30 p.m. to 1:45 p.m.

The afternoon was devoted to concurrent sessions which consisted of various panels and papers.

The panel presentations covered the following topics:

1. Catastrophe Pricing and Trends

Panelists: David H. Hays, Senior Assistant Actuary
State Farm Fire and Casualty Company

William M. Gray, Professor
Department of Atmospheric Science
Colorado State University

Stuart B. Mathewson, Vice President
E. W. Blanch Company

2. The Anti-Trust Mine Field

Presenter: Roger Moak, Senior Vice President and General Counsel
Insurance Services Office

3. No-Fault Automobile Insurance

Moderator: John B. Conners, Senior Vice President and Manager
Liberty Mutual Insurance Company

Panelists: Gregory L. Hayward, Actuary
State Farm Mutual Automobile Insurance Company

Joseph A. Herbers, Consulting Actuary
Tillinghast/Towers Perrin

Donald Segraves, Executive Director
Insurance Research Council

4. Workers Compensation Issues

Moderator: Richard L. Johe, Executive Consultant
Coopers & Lybrand

Panelists: Ronald C. Retterath, Senior Vice President and Actuary
National Council on Compensation Insurance

Richard A. Hoffman, Vice President
Midwest Employers Casualty Company

John P. Tierney, Consulting Actuary
Tillinghast/Towers Perrin

5. Increased Limits Issues

Moderator: Glenn R. Meyers, Assistant Vice President and Actuary
Insurance Services Office

Panelists: Robert J. Finger, Principal
William M. Mercer, Inc.

Oakley E. Van Slyke, Director
Coopers & Lybrand

Isaac Mashitz, Vice President and Actuary
North American Reinsurance Corporation

6. Questions and Answers with the CAS Board of Directors

Moderator: Albert J. Beer, Senior Vice President
Skandia America Group

Panelists: Lee R. Steeneck, Vice President
General Reinsurance Corporation

Jerome A. Scheibl, Vice President-Industry Affairs
Wausau Insurance Companies

Charles A. Bryan, Partner
Ernst & Young

Janet L. Fagan, Vice President and Senior Actuary
CIGNA Property & Casualty Group

The new *Proceedings* papers were:

1. "On the Representation of Loss and Indemnity Distributions"

   Author: Yoong-Sin Lee
   National University of Singapore

2. "Risk Load for Insurers"

   Author: Sholom Feldblum
   Liberty Mutual Insurance Company

3. "The Distribution of Automobile Accidents—Are Relativities Stable Over Time?"

   Author: Emelio C. Venezian
   Rutgers University

The officers held a reception for new Fellows and their guests from 5:30 p.m. to 6:30 p.m.

There was a general reception for all members from 6:30 p.m. to 7:30 p.m.

Tuesday, November 13, 1990

Following breakfast, there were concurrent sessions. The panel presentations, in addition to some of the subjects covered on Monday, covered the topic of:

1. Risk Margins for Discounted Loss Reserves

Panelists: Michael A. McMurray, Consulting Actuary
Milliman & Robertson, Inc.
Committee on Reserves

Stephen W. Philbrick, Consulting Actuary
Tillinghast/Towers Perrin
Committee on Theory of Risk

The new *Proceedings* papers were:

1. "Discounted Return—Measuring Profitability and Setting Targets"

Author: Russ Bingham
Hartford Insurance Company

2. "An Example of Credibility and Shifting Risk Parameters"

Author: Howard C. Mahler
Workers' Compensation Rating and
Inspection Bureau of Massachusetts

From 10:30 a.m. to 12:00 p.m. a general session was held. The subject was actuarial standards and professional guidelines. This panel consisted of:

Moderator: Mavis A. Walters, Executive Vice President
Insurance Services Office
President–American Academy of Actuaries

Panelists: Harry D. Garber, Vice Chairman
Equitable Life Assurance Society
Chairperson–Joint Task Force on Professionalism
President-Elect–American Academy of Actuaries

Michael A. Walters, Consulting Actuary
Tillinghast/Towers Perrin
Chairperson–Casualty Practice Council

Walter N. Miller, Vice President and Actuary
Prudential Insurance Company
Chairperson–Actuarial Standards Board

Michael J. Miller, Consulting Actuary
Tillinghast/Towers Perrin
Chairperson–Casualty Committees of the
Actuarial Standards Board

The afternoon was free.

Dinner was held from 6:00 p.m. to 8:30 p.m. on the Cajun Queen Riverboat.

Wednesday, November 14, 1990

There were concurrent sessions from 8:00 a.m. to 10:00 a.m.

The panel presentation, in addition to some of the subjects presented Monday and Tuesday, included:

1. Closing the Actuarial Communications Gap

   Presenter: C. A. Miller Associates, Inc.

2. Expert Systems

   Panelists: Joseph J. DeSalvo, Director
   Insurance Industry Decision Support Group
   Coopers & Lybrand

   Eric Marcus, Managing Associate
   Coopers & Lybrand

The new *Proceedings* papers included:

1. "Reinsurer Risk Loads from Marginal Surplus Requirements"

   Author: Rodney E. Kreps
   Sullivan Payne Company

2. A Discussion of "Minimum Bias with Generalized Linear Models" Paper by Robert L. Brown, PCAS LXXV

   Author: Gary G. Venter
   Workers' Compensation Reinsurance Bureau

3. "Pricing the Impact of Adjustable Features and Loss Sharing Provisions of Reinsurance Treaties"

   Authors: Robert A. Bear
   North Star Reinsurance Corporation

   Kenneth J. Nemlick
   North Star Reinsurance Corporation

Following a break, a general session was held on contemporary auto insurance cost containment opportunities.

Moderator: Steven F. Goldberg, Senior Vice President
and Chief Actuary USAA

Panelists: Brian O'Neill, President
Insurance Institute for Highway Safety

Honorable Leo W. Fraser, Jr., Representative
State of New Hampshire House of Representatives
Chairperson–Committee on Commerce, Small Business and Consumers Affairs

Paul Hasse, Consultant
McKinsey and Company

After the transfer of the Presidency, Charles Bryan gave the closing remarks.

November 1990 Attendees

In attendance, as indicated by the registration records, were 287 Fellows, 120 Associates, and 37 guests, subscribers, and students. The list of their names follows.

FELLOWS

Addie, B.
Aldin, N.
Almagro, M., Jr.
Amundson, R.
Angell, C.
Anker, R.
Apfel, K.
Artes, L.
Asch, N.
Atkinson, R.
Bailey, V.
Bashline, D.
Bass, I.
Bassman, B.
Bear, R.
Beer, A.
Ben-Zvi, P.
Bender, R.
Berens, R.
Berquist, J.
Biondi, R.
Blakinger, J.
Blanchard, R.
Blivess, M.
Book, S.
Bornhuetter, R.
Bowen, D.
Braithwaite, P.
Brannigan, J.
Brooks, D.
Brown, J., Jr.
Bryan, C.
Bujaucius, G.
Cardoso, R.
Carlson, C.
Carponter, J.
Carroll, L.
Cascio, M.
Caulfield, M.
Charest, D.
Chuck, A.
Cieslak, W.
Conger, R.
Connell, E.
Conners, J.
Conway, A.
Crawshaw, M.
Curry, A.
Dahlquist, R.
De Falco, T.
Degerness, J.
Dembiec, L.
Desbiens, C.
Di Donato, A.
Diamantoukos, C.
Dodd, G.
Dolan, M.
Donaldson, J.
Dornfeld, J.
Drennan, J.
Drummond-Hay, E.
Duda, D.
Duffy, B.
Duffy, T.
Dumulon, D.
Eagelfeld, H.
Easlon, K.
Edie, G.
Egnasko, G.
Egnasko, V.
Ericson, J.
Evans, G.
Fagan, J.
Fallquist, R.
Feldblum, S.
Finger, R.
Fisher, R.
Fisher, W.
Fitzgerald, B.
Fitzgibbon, W.
Fleming. K.
Flynn, D.
Forbus, B.
Fox, R.
Frank, J.
Friedberg, B.
Frohlich, K.
Fusco, M.
Gannon, A.
Gardner, R.
Gebhard, J.
Ghezzi, T.
Giambo, R.
Gibson, R.
Gill, B.
Gilles, J.
Gillespie, J.
Glicksman, S.
Goldberg, S.
Goldfarb, I.
Gottlieb, L.
Grace, G.
Grady, D.
Graves, G.
Greco, R.
Grippa, A.
Gunn, C.
Hachemeister, C.
Haefner, L.
Hallstrom, R.
Harrison, D.
Hartman, D.

FELLOWS

Haskell, G.
Hayne, R.
Hays, D.
Hayward, G.
Herbert, B.
Herbert, N.
Hermes, T.
Heyman, D.
Hill, A.
Hines, A.
Hough, P.
Hurley, J.
Irvan, R.
Jaeger, R.
Johe, R.
Johnson, A.
Johnson, E.
Johnson, W.
Johnston, S.
Jordan, J.
Josephson, G.
Jovinelly, E.
Kasner, K.
Keatinge, C.
Kelley, R.
Kilbourne, F.
King, M.
Kleinman, J.
Kline, C.
Kollar, J.
Koupf, G.
Kozik, T.
Krause, G.
Kreps, R.
Krissinger, K.
Kryczka, J.
Kucera, J.
Kudera, A.
Kufera, K.
Lacko, P.
Lacroix, M.
Lalonde, D.
Lamb, D.
Lange, D.
Lattanzio, F.
Lee, R.
Lee, Y.
Lehman, M.
Lehmann, S.
Levin, J.
Linden, O.
Lino, R.
Lipton, B.
Lockwood, J.
Lotkowski, E.
Loucks, W.
Lyons, D.
Mac Ginnitie, J.
Mahler, H.
Mashitz, I.
Mathewson, S.
Mc Clenahan, C.
Mc Clenahan, D.
Mc Dermott, S.
Mc Murray, M.
Mc Padden, M. S.
Meyer, R.
Meyers, G.
Michelson, J.
Miller, M.F.
Miller, M.
Miller, P.
Miller, R.
Miller, S.
Mitchell, E.
Mohl, F.
Morell, R.
Moylan, T.
Mueller, N.
Mulder, E.
Muleski, R.
Muller, R.
Murphy, F., Jr.
Murphy, W.
Murrin, T.
Myers, N.
Nelson, C.
Nemlick, K.
Nester, K.
Ng, K.
Niswander, R.
Norton, J.
Noyce, J.
O'Connell, P.
Overgaard, W.
Patrik, G.
Pei, K.
Perigny, I.
Petersen, B.
Peterson, S.
Philbrick, S.
Phillips, G.
Pinto, E.
Potts, C.
Pratt, J.
Pruiksma, G.
Purple, J.
Quintano, R.
Raman, R.
Rapoport, A.
Reale, P.

FELLOWS

Reichle, K.
Retterath, R.
Robbins, K.
Robertson, J.
Robinson, R.
Roland, W. P.
Rosenberg, S.
Ross, G.
Ruegg, M.
Ryan, K.
Salton, J.
Scheibl, J.
Schneider, H.
Schug, R.
Schultze, M.
Schwartzman, J.
Shoop, E.
Silverman, M.
Simon, L.
Skurnick, D.
Slotznick, L.
Smith, R.
Snader, R.
Spiegler, D.
Steeneck, L.
Steer, G.
Steinen, P.
Steinert, L.
Stone, E.
Suchar, C.
Surrago, J.
Svendsgaard, C.
Tatge, R.
Taylor, C.
Terrill, K.
Thompson, K.
Tierney, J.
Toothman, M.
Treitel, N.
Van Ark, W.
Van Slyke, O.
Venter, G.
Volponi, J.
Votta, J.
Wacker, G.
Wainscott, R.
Wallace, T.
Walters, Mavis
Walters, Michael
Webb, B.
Webb, N.
Webster, P.
Whitlock, R.
Wickwire, J.
Wildman, P.
Wilson, E.
Wilson, R.
Winslow, M.
Woll, R.
Woods, P.
Yingling, M.
Yonkunas, J.
Yow, H.
Yow, J.

ASSOCIATES

Allen, D.
Anderson, R.C.
Anderson, R.R.
Avagliano G.
Barnes, K.
Bauer, B.
Cadorine, A.
Caron, L.
Cellars, R.
Chorpita, F.
Christhilf, D.
Clark, D.
Coca, M.
Cole, J.
Connor, V.
Cutler, J.
Dagneau, F.
Danielson, G.
Dashoff, T.
Davenport, E.
Diss, G.
Felisky-Watson, K.
Fields, D.
Fisher, N.
Fletcher, J.
Foley, D.
Gelinne, D.
Goldberg, S.
Greenhill, E.
Greenwood, D.
Gutman, E.
Gwynn, H.
Hansen, G.
Harbage, R.
Hausserman, D.
Hay, G.
Hay, R.
Herbers, J.
Hess, T.

ASSOCIATES

Hess, T.G.
Higgins, J.
Hofmann, R.
James, P.
Jensen, J.
Johnston, D.
Jones, B.
Kellner, T.
Kincaid, B.
Kolk, S.
Kolojay, T.
La Berge, C.
Lacek, M.
Leblanc, F.
Lemieux, E.
Lepage, P.
Letourneau, R.
Licht, P.
Limpert, J.
Llewellyn, B.
Mahon, M.
Malik, S.
Marles, B.
Marlowe, B.
Mc Creesh, J.
Mc Gee, S.
Mc Govern, E.
Mc Neese, D.
Mc Shea, C.
Miller, R.
Mitzel, C.
Miyao, S.
Moody, A.
Mozeika, J.
Munson, T.
Nelson, J.
Nissenbaum, J.
O'Brien, M.
Olszewski, L.
Ondrich, N.
Ottone, J.
Paffenback, T.
Pagliaccio, J.
Pino, S.
Prescott, R.
Pulis, R. S.
Raman, S.
Raws, A.
Rech, J.
Reed, D.
Riff, M.
Rohn, D.
Ruane, J.
Salton, M.
Sandler, R.
Santomenno, S.
Sarosi, J.
Schmitt, K.
Schoenberger, S.
Schwandt, J.
Simon, C.
Smith, B.
Smolen, P.
Smolen, T.
Snow, D.
Sogge, E.
Stanford, T.
Stenmark, J.
Strommen, D.
Sturm, E.
Thorne, J.
Tingley, N.
Torgrimson, D.
Treskolasky, S.
Vandermyde, S.
Weinstein, M.
White, L.
Williams, B.
Wolfe, B.
Wolter, K.
Yen, C.Y

GUESTS-SUBSCRIBERS-STUDENTS

Arnold, D.
Behan, D.
Berry, B.
Bingham, R.
Burnett, L.
Colver, C.
Compton, R.
Cullinan, T.
D'Alto, L.
De Angelis, D.
Desalvo, J.
Folkesson, J.
Fraser, L.
Garber, H.
Gibbs, S.
Gray, B.
Hasse, P.
Jacob, E.
Kaufman, D.
Marcus, E.
Miller, C.
Miller, W.
Moak, R.
Murphy, J.

GUESTS-SUBSCRIBERS-STUDENTS

O'Neill, B.
Radhakrishnan, R.
Reott, J.
Rice, V.
Schneider, C.
Segraves, D.
Spangler, J.
Spicer, K.
Stenson, T.
Thieme, H.
Van Leer, P.
Venezian, E.
Winn, J.

REPORT OF THE VICE PRESIDENT—ADMINISTRATION

The objective of this report is to provide the membership with a brief summary of CAS activities since the last annual meeting.

As stated in Article II of our Constitution, the purposes of the Casualty Actuarial Society are to:

- advance the body of knowledge of actuarial science in applications other than life insurance,
- establish and maintain standards of qualification for membership,
- promote and maintain high standards of conduct and competence for the members, and
- increase the awareness of actuarial science.

There has been significant activity in all of these areas during the past twelve months.

Particularly noteworthy are the number and quality of continuing education opportunities that the CAS has made available to its members—and the enthusiasm with which our members partake of these activities. These opportunities clearly fulfill the purpose of promoting and maintaining high standards of competence for our members. Meetings and seminars have included:

- the Spring meeting in Colorado Springs, attended by 466 members and 233 non-members;
- the Annual meeting in New Orleans, with advance registrations of 408 members and 213 non-members;
- the Casualty Loss Reserve Seminar in Dallas-Fort Worth, of which the CAS is a co-sponsor, attended by nearly 700;
- the Canadian Property and Casualty Insurance Liability Seminar in Toronto, 122 attendees;
- the Ratemaking Seminar, which attracted 621 registrants;
- the Specialty Lines Seminar, attended by 75;
- the Environmental Issues Seminar, which was attended by 111; and
- the cosponsored ASTIN Colloquium, which was held in New York in November, 1989.

With the Environmental Issues Seminar, the CAS initiated a process of producing and selling audio tapes of selected meetings and seminars, so that non-attendees may benefit from these educational opportunities.

Our nine Regional Affiliates and two Special Interest Sections continued to prosper, and also provided rich educational opportunities during this past year. Most of the Regional Affiliates held two meetings during the year, attracting Fellows, Associates and students from their local areas. Our newest Special Interest Section, Casualty Actuaries in Reinsurance, conducted a seminar on reinsurance issues which attracted 190 attendees from all over the country. Two more Regional Affiliates are in the formative process. A specific goal established by the Long Range Planning Committee has been to nurture the Regional Affiliates, and the leaders of all the Regional Affiliates will be meeting with the CAS Executive Council in New Orleans during the Annual Meeting to discuss ways in which the CAS and the Regional Affiliates can work together, among other topics.

Another category of continuing education resources to which the CAS can point with pride is our publications. Most noteworthy among our publications during this past year is the CAS textbook, entitled *Foundations of Casualty Actuarial Science*. The initial print run of 2,000 copies was virtually sold out by the time it was received from the printer, and the second print run is selling steadily. The textbook represents an important addition to the CAS literature, and an important addition to the libraries of students and practitioners alike. Other CAS publications important to the continuing education of our members are the *Proceedings*, the *Actuarial Review*, the *Forum*, and the Discussion Paper Program.

Clearly, as members of the actuarial profession attempt to fulfill the continuing education requirements of the American Academy of Actuaries, CAS members have no shortage of opportunities to partake of continuing education. Indeed, based on the high level of participation, attendance and interest at our various meetings and seminars, it seems likely that most of our members already meet the requirements.

The Education and Examination process of the CAS supports two of the CAS purposes: to establish and maintain standards of qualification for membership, and to promote high standards of conduct and competence for members.

In addition to the annual review and update of examination reading requirements (which now also include several readings from the new CAS textbook), a significant *Syllabus* change was implemented during 1990. Operations Research, formerly Part 3b and jointly sponsored by the CAS and Society of Actuaries, was removed from the CAS *Syllabus* (but not from the Society of Actuaries *Syllabus*). In its place, the CAS incorporated as Part 3B, Introduction to Property and Casualty Insurance. This exam part covers property and casualty coverages, operations, and an introduction to ratemaking and reserving. Formerly, a student did not encounter these topics until Part 5. With the movement of these introductory topics out of Part 5, a new subject, Finance, was added to Part 5. As has been the practice when *Syllabus* changes are introduced, transition rules are provided for students that are in the midst of the examination track.

Another important change introduced in 1990, but not yet implemented, is the addition of a Course on Professionalism to the requirements for becoming an Associate. This course will focus on conduct, ethics, and business practice—not on technical actuarial issues. As currently planned, the one-day seminar-format course will be offered in various locations to students who have completed five or more exams. Implementation is scheduled for 1991, and the new requirement will apply to students who complete their Associateship examinations after May, 1991.

These *Syllabus* changes are the result of recommendations contained in a landmark White Paper on the educational content of the *Syllabus*.

An exam-related matter of considerable interest to our students is the question of whether to partition the examinations into smaller units. Much input has been collected and much study and discussion conducted over the past twelve months, and the Board of Directors took action at its November, 1990 meeting. Specifically: (1) Part 4 will be partitioned beginning in May, 1992, and will be offered in both May and November thereafter; (2) Part 5 will be partitioned beginning in November, 1993, and will be offered in both May and November thereafter; (3) the current Part 5 transition program will be extended through 1992, and students having credit for half of Part 5 will not lose that credit at the end of the transition; (4) Parts 6 and 7 will not be partitioned; and (5) a decision on Parts 8–10 has been deferred at least three years.

New papers published in the *Proceedings*, the Discussion Paper book, and the *Forum* represent very important efforts "to advance the body of knowledge of actuarial science in applications other than life insurance." During this past year, six papers were accepted for publication in the 1990 *Proceedings*; and the May, 1990 Discussion Paper Program, entitled "Pricing," attracted 23 papers.

Another anticipated source of research and papers is a new class of participation in the CAS that was introduced this year: the Academic Correspondent Program. This program is for non-members who are involved in teaching actuarial science, mathematics, business, or related courses, and who have an interest in the CAS. Academic Correspondents receive the publications of the CAS, are invited to attend meetings, and may submit *Proceedings* papers.

Historically, the work of our committees also has been an important source of research, which generally has been conducted on a volunteer basis by the members of the committees. The Board of Directors, at its September meeting, took steps to extend the leverage of the committee efforts by creating a program that will allow the committees to solicit and manage funded research, rather than having to rely solely on the volunteer efforts of the committee members. One piece of funded work, entitled "The Profit Provision in the Ratemaking Formula," is nearly complete, and we can look forward to more work products in the coming years.

Of course, committees play a vital role not just in research efforts, but in all facets of the CAS. Currently, the CAS has approximately two dozen active committees, staffed by over 350 of our members. Beginning with the staffing of the 1990 committees, the CAS instituted a new Participation Survey to identify members who are interested in serving on the various committees. At the same time, committee staffing procedures were changed to invite CAS Associates to participate on committees (other than Board Committees and Admissions Committees).

Continuing a valued tradition, the Committee Chairs met as a group with the Executive Council in April, 1990, to exchange ideas and to provide input to the Executive Council.

Efforts to "increase the awareness of actuarial science" were brought to a focus this year when, following an audit of the CAS public relations needs, the Board of Directors formally established public relations objectives:

- to increase recognition of the casualty actuarial profession at the university student level;
- to increase the stature of the CAS as an organization, particularly internationally;
- to provide adequate communications to the membership, including exam-taking students, on all CAS activities; and
- to increase the awareness of the value added by casualty actuaries in insurance-related fields.

The Executive Council will be establishing several goals for 1991 that are intended to achieve these objectives.

Two external events created publicity for the CAS this year. First, the continuing debate surrounding California Proposition 103 kept rate regulation and casualty actuaries in the public eye. Second, the National Association of Insurance Commissioners approved a change in the 1990 Annual Statement that requires an actuarial statement of opinion on property/casualty loss and loss adjustment expense reserves by a "qualified actuary"; the first definition of "qualified actuary" is "a member in good standing of the Casualty Actuarial Society."

Our relationships with foreign actuarial bodies continue to develop. The past year was marked by the cosponsorship of the ASTIN Colloquium, mentioned previously, and by the establishment of an International Relations Policy by the Board of Directors.

We have actively nurtured our relationships with the other North American actuarial organizations as well. The Executive Council held a joint meeting with the Society of Actuaries Executive Council last December. More recently, the Board of Directors authorized the signing of a "Working Agreement" with the other actuarial organizations. This working agreement defines various areas of responsibility for each organization and areas for cooperation among the organizations.

The Board of Directors, with prime responsibility for setting policy, met four times in 1990. New members elected to the Board for next year include Robert Anker, Linda Bell, James MacGinnitie, James Stanard, and David Oakden. The membership elected Michael Toothman to the position of President-Elect, and Charles Bryan will be President for the 1990-1991 year.

The Executive Council, with primary responsibility for day-to-day operations, met several times during the year. The Board of Directors elected the following Vice Presidents for the coming year:

Vice President-Administration	Robert Conger
Vice President-Admissions	Steven Lehmann
Vice President-Continuing Education	Irene Bass
Vice President-Programs and Communications	Albert Beer
Vice President-Research and Development	Allan Kaufman

In summary, the CAS continues to grow and thrive. During 1990, 141 new members joined our ranks, and 70 new Fellows were named. The CAS remains financially healthy as well. A budget of approximately $1.2 million for the 1990-1991 year was approved by the Board of Directors. Dues for next year will be $190, an increase of $15; examination fees for Parts Four through Ten will be increased $10 to $120. The fee for the Academic Correspondent program was reduced to $75.

Finally, the Audit Committee examined the CAS books for fiscal year 1990 and found the accounts to be properly stated. The year ended with an increase in surplus of $107,989.86. Members' equity now stands at $576,137.46, subdivided as follows:

Michelbacher Fund	$ 76,654.10
Dorweiler Fund	7,531.38
CAS Trust	2,772.12
Scholarship Fund	7,543.86
CLRS Fund	5,000.00
CAS Surplus	476,636.00
TOTAL MEMBERS' EQUITY	$576,137.46

Respectfully submitted,

ROBERT F. CONGER
Vice President-Administration

FINANCIAL REPORT
FISCAL YEAR ENDED 9/30/90

OPERATING RESULTS BY FUNCTION

FUNCTION	INCOME	DISBURSEMENTS	NET RESULTS
Exams	$ 398,213.71	$311,476.52 (a)	$ 86,737.19
Member Services (b)	410,542.00	568,301.14	(157,759.14)
Programs	164,847.52	82,759.32 (c)	82,088.20
Other (d)	96,923.61	0.00	96,923.61
TOTAL	$1,070,526.84	$962,536.98	$107,989.86 (e)

Notes: (a) Does not include exam-related expenses incurred by the development function
(b) Areas under the supervision of VP-Administration & VP-Development
(c) Does not include program-related expenses incurred by the development function
(d) Investment income less Foreign Exchange and Miscellaneous bank debits, and ASTIN Fund
(e) Change in CAS Surplus

BALANCE SHEET

ASSETS	9/30/89	9/30/90	CHANGE
Checking Account	$ 128,887.11	$ 192,268.29	$ 63,381.18
Money Market Fund	104,287.45	0.00	(104,287.45)
Bank Certificates of Deposit	300,000.00	100,000.00	(200,000.00)
U.S. Treasury Notes & Bills	678,127.65	806,526.28	128,398.63
Accrued Interest	9,082.19	22,895.21	13,813.02
CLRS Fund	5,000.00	5,000.00	0.00
TOTAL ASSETS	$1,225,384.40	$1,126,689.78	($ 98,694.62)
LIABILITIES			
Office Expenses	$ 92,591.79	$114,138.60	$ 21,546.81
Printing Expenses	183,407.86	239,306.11	55,898.25
Prepaid Exam Fees	148,005.00	166,698.00	18,693.00
Prepaid Inv. Program 1991	0.00	12,685.00	12,685.00
New Orleans Mtg. Fees	0.00	12,460.00	12,460.00
ASTIN Meeting	59,377.63	0.00	(59,377.63)
Diamond Jubilee	275,713.55	0.00	(275,713.55)
Other	971.86	5,264.61	4,292.75
TOTAL LIABILITIES	$760,067.69	$550,552.32	($209,515.37)
MEMBERS EQUITY			
Michelbacher Fund	$ 73,418.34	$ 76,654.10	$ 3,235.76
Dorweiler Fund	8,048.47	7,531.38	(517.09)
CAS Trust	2,615.21	2,772.12	156.91
Scholarship Fund	7,588.55	7,543.86	(44.69)
CLRS Fund	5,000.00	5,000.00	0.00
CAS Surplus	368,646.14	476,636.00	107,989.86
TOTAL EQUITY	$465,316.71	$576,137.46	$110,820.75

Robert F. Conger, *Vice President-Administration*

This is to certify that the assets and accounts shown in the above financial statement have been audited and found to be correct.

AUDIT COMMITTEE

Lee M. Smith, Chairman
Albert J. Quinn
William J. Rowland
Charles Walter Stewart

1990 EXAMINATIONS—SUCCESSFUL CANDIDATES

Examinations for Parts 4, 6, 8 and 10 of the Casualty Actuarial Society were held on May 8, 9, 10 and 11. Examinations for Parts 3(B), 5, 5(A), 5(B), 7, and 9 were held on November 7, 8, 9, and 13.

Examinations for Parts 1, 2, and 3 (SOA courses 100, 110, 120, 130 and 135) are jointly sponsored by the Casualty Actuarial Society and the Society of Actuaries. Parts 1 and 2 were given in February, May and November of 1990 and Part 3 was given in May and November of 1990. Candidates who were successful on these examinations were listed in the joint releases of the two societies.

The Casualty Actuarial Society and the Society of Actuaries jointly awarded prizes to the undergraduates ranking the highest on the Part 1 examination.

For the February, 1990 examination the $200 first prize was awarded to Raymond Jung. The $100 prize winners were Corey Bilot, Lance Dyrland, Pak-Chuen Li, and Questor Ng.

For the May, 1990 examination the $200 first prize was awarded to Joel Rosenberg. The $100 prize winners were Ken Eliezer, Andrew Erman, Daniel Golberg, Ronald Neath, and Philip Wunderlich.

For the November, 1990 examination the $200 first prize was awarded to Edward Eickenberg. The $100 prize winners were Steven Grondin, Gilbert Pak, Erik Vee, Kai-Yip Wong, and Jeung-Horng Wu.

The following candidates were admitted as Fellows and Associates as a result of their successful completion of the Society requirements in the May, 1990 examinations.

FELLOWS

Kenneth Apfel	Michael J. Caulfield	Richard L. Fox
Lawrence J. Artes	Danielle Charest	Jacque B. Frank
Robert K. Bender	Walter P. Cieslak	Richard N. Gibson
Stephen W. Book	Carol Desbiens	Bonnie S. Gill
Christopher S. Carlson	Timothy B. Duffy	Steven A. Glicksman
Lynn R. Carroll	Denis Dumulon	David C. Harrison

FELLOWS

David R. Heyman
Anthony D. Hill
Alan M. Hines
John M. Hurley
Steven J. Johnston
Edward M. Jovinelly
Mary Jean King
Charles D. Kline, Jr.
Kenneth R. Krissinger
John R. Kryczka
Kay E. Kufera
Paul E. Lacko
David A. Lalonde
Jon W. Michelson
H. Elizabeth Mitchell
Karl G. Moller, Jr.
Chris E. Nelson
Jonathon Norton
Kai-Jaung Pei
Isabelle Perigny
Steven J. Peterson
Kevin B. Robbins
Richard D. Robinson
Randy J. Roth
Jeffrey C. Salton
Richard D. Schug
Mark E. Schultze
Mark J. Silverman
Lisa A. Slotznick
David Spiegler
Lawrence J. Steinert
Edward C. Stone
Christopher M. Suchar
Ernest I. Wilson
Martha A. Winslow
Heather E. Yow

ASSOCIATES

Richard R. Anderson
Guy A. Avagliano
Katharine Barnes
Jeffrey R. Cole
Francois Dagneau
Edgar W. Davenport
Kendra M. Felisky-Watson
David N. Fields
David A. Foley
Jacque B. Frank
Deborah A. Greenwood
Dawson T. Grubbs
George M. Hansen
Diane K. Hausserman
Gordon K. Hay
Thomas G. Hess
Peter H. James
Tony J. Kellner
Bryan J. Kincaid
France LeBlanc
Eric F. Lemieux
Stephen J. McGee
Dennis T. McNeese
M. Sean McPadden
Christopher J. McShea
Robert L. Miller
Charles B. Mitzel
Todd B. Munson
John Nissenbaum
Margaret O'Brien
Laura A. Olszewski
Naomi S. Ondrich
Teresa K. Paffenback
Susan L. Pino
Richard W. Prescott
Alfred Raws III
James E. Rech
Diane R. Rohn
John M. Ruane Jr.
Timothy J. Rundle
Gary E. Shook
Christy L. Simon
Patricia E. Smolen
Tom A. Smolen
Elizabeth L. Sogge
Thomas N. Stanford
Elissa M. Sturm
Jeffrey L. Subeck
Susan M. Treskolasky
Scott D. Vandermyde
Marjorie C. Weinstein
Beth M. Wolfe
Kathy A. Wolter
Nancy E. Yost

The following is the list of successful candidates in examinations held in May, 1990.

Part 4

Kristen M. Albright
Craig A. Allen
Rhonda K. Allison
Ann L. Alnes
Mark D. Ames
Newton E. Amoh
Scott C. Anderson
Michael J. Andring
Michael E. Angelina
Bhim D. Asdhir
Gad Attias
Kim G. Balls
Katharine Barnes
Rose D. Barrett
Philip A. Baum
Brian P. Beckman
Douglas S. Benedict
Lyne Bergeron
Annie Bisaillon
Stephen D. Blaesing
Betsy L. Blue
Alain Boisvert
John T. Bonsignore
Christopher L. Bowen
David B. Bowls
Christopher K. Bozman
George P. Bradley
Mark L. Brannon
James L. Bresnahan
Lisa M. Brieden
Louis M. Brown
Stephen J. Bruce
Russell J. Buckley
Paul A. Bukowski
Michelle M. Bull
Peter V. Burchett
John F. Butcher II
Robert N. Campbell
Michael E. Carpenter
Daniel G. Carr
Martin Carrier
Michael W. Cash
Tania J. Cassell
Jessalyn Chang
Jean-Francois Charest
Wanchin W. Chou
Pin J. Chung
Susan D. Ciardiello
Christopher J. Claus
Thomas D. Coatney
James Paul Cochran
Danielle Comtois
Mary L. Corbett
Gregory L. Cote
Timothy J. Cremin
Richard J. Currie
Thomas V. Daley
Edgar W. Davenport
Karen L. Davies
Michael L. DeMattei
Mike Devine
Jeffrey E. Doffing
Andrew J. Doll
Jeffrey L. Dollinger
Victor G. dos Santos
William F. Dove
Alicia G. Doyle
Peter F. Dragon
Mary Ann Duchna-Savrin
Jean-Sebastien Dumais
Bernard Dupont
Sondra H. Einig
David M. Elkins
Paul E. Ericksen
Jennifer L. Ermisch
Madelyn C. Faggella
Matthew G. Fay
Judith Feldmeier
John D. Ferraro
Audrey M. Ferrier
Stephen A. Finch
Joyce L. Fish
Robert F. Flannery
John B. Folkesson
Douglas E. Franklin
Charles S. Fuhrer
Nathalie Gamache
James E. Gant
Andrea Gardner
Jeffrey N. Garnatz
Susan T. Garnier
Rita M. Geraghty
Bradley J. Gleason
Richard S. Goldfarb
Mathew L. Gossell
Jeffrey S. Goy
Odile Goyer
M. Harlin Grove
Julie K. Halper
Bradley A. Hanson

Diane K. Hausserman
Shlomo O. Haviv
Gordon K. Hay
Paul D. Henning
Keith D. Holler
Beth M. Hostager
Annie Houle
Thomas A. Huberty
Jeffrey R. Hughes
Daniel Hurtubise
Annette Y. Jiang
Mark R. Johnson
Dieter E. Jurkat
Hariharan Kanagalingam
Stephen H. Kantor
Trina C. Kavacky
Stephen G. Kellison
Mary V. Kelly
Susan E. Kent
Robert W. Kirklin
Paul H. Klauke
Joan M. Klucarich
Timothy F. Koester
Gilbert M. Korthals
Karen L. Krainz
Peter A. Kraus
Adam J. Kreuser
Sun J. Kwon
Eurico A. Lacerda
Roch Lacroix
Carl Lambert
W. Keith Landry
Mylene Landry
Christian B. Lapointe
James W. Larkin
Claude Larochelle
Michael D. Larson
Lawrence K. Law
Scott J. Lefkowitz
Marie-Pierre Legault
Elizabeth A. Lemaster
Deanne C. Lenhardt
Chantal Letourneau
Richard S. Light
Kevin E. Litton
Ronald P. Lowe, Jr.
Robert G. Lowery
Richard Maguire
Cathy A. Mahanna
William G. Main
Donald E. Manis
Anthony L. Manzitto
Stephen N. Maratea
Maria Mattioli
Melinda H. Mayerchak
Heidi J. McBride
Robert D. McCarthy
Teresa J. McCorkle
Thomas S. McIntyre
Christian Menard
Stephen V. Merkey
Stephen J. Meyer
Anne C. Meysenburg
Todd M. Miller
Steven L. Minuk
John H. Mize
Annie Monfet
Russell E. Moore
Francois R. Morin
Josee Morin
Thomas M. Mount
Raymond D. Muller
Timothy O. Muzzey
David Y. Na
Denis P. Neumann
Quang C. Nguyen
Keon Nielsen
Douglas K. Nishimura
Victor Njakou
Stephen R. Noonan
Michel Nystrom
Russell R. Oeser
Steven J. Olson
William L. Oostendorp
John E. Pannell
Prabha Pattabiraman
Fanny C. Paz-Prizant
Karen L. Pehrson
Beverly L. Phillips
Mark W. Phillips
Genevieve Pineau
Susan L. Pino
Joseph W. Pitts
Ted Poon
On Cheong Poon
John F. Radwanski
Andrew T. Rippert
Anthony V. Rizzuto
Douglas A. Roemelt
Paul J. Rogness
Diane R. Rohn
Luis Romero
Laura A. Romine
David A. Rosenzweig
Lisa M. Ross
Martin Roussel
Robert A. Rowe
James B. Rowland
Caroline Roy
David Ruhm
Asif Mohammad Sardar
Leigh A. Saunders

Melodee J. Saunders
Marilyn E. Schafer
Peter Senak
Cynthia S. Siemers
Christy L. Simon
Patricia E. Smolen
David B. Sommer
Keith R. Spalding
Angela K. Sparks
William G. Stanfield
Douglas W. Stang
Deborah A. Stobo
Brian M. Stoll
John W. Stonehill
Collin J. Suttie
Scott J. Swanay
Yachung Syu
David M. Terne
Michel Theberge
Georgia A. Theocharides
Edward D. Thomas
Daniel J. Tinter
Michael J. Toth
Stacy L. Trowbridge
Son Trong Tu
David B. VanKoevering
Monique Veilleux
Jennifer A. Violette
Linda M. Waite
Qing Wang
Bryan C. Ware
Stephen D. Warfel
Robert G. Weinbert
John P. Welch
Calvin Wolcott
Lynne M. Woody
Eva M. Woolley
Donald S. Wroe
Floyd M. Yager
Gerald T. Yeung
Shawn M. Young
Joshua A. Zirin
Barry C. Zurbuchen

Part 6

Marc J. Adee
Richard R. Anderson
Guy A. Avagliano
Todd R. Bault
Nathalie Begin
Wayne E. Blackburn
Daniel D. Blau
Alicia E. Bowen
Anthony J. Burke
Julie S. Chadowski
Mario Champagne
John Chittenden
Bryan C. Christman
Cindy C. Chu
Jeffrey R. Cole
Kathleen F. Connor
Martin L. Couture
David A. Cullather
Francois Dagneau
Michael K. Daly
Manon Debigare
Herb Desson
Laura Deterding
Patrick K. Devlin
Stephen R. DiCenso
Kevin G. Dickson
Michel Dionne
Pierre Dionne
Michael C. Dubin
Francois Dumas
Patrick Dussault
Charles C. Emma
Thomas R. Fauerbach
Kendra M. Felisky-Watson
David N. Fields
David A. Foley
Yves Francoeur
Kim B. Garland
Bruce R. Gifford
Michael A. Ginnelly
Richard S. Goldfarb
Charles T. Goldie
Matthew E. Golec
Bradley A. Granger
Jeffrey W. Graver
Deborah A. Greenwood
Dawson T. Grubbs
Paul J. Hancock
George M. Hansen
Steven T. Harr
Thomas G. Hess
David B. Hostetter
Kathleen M. Ireland
Peter H. James
Hou-Wen Jeng

Tony J. Kellner
Deborah E. Kenyon
Joseph P. Kilroy
Changseob Kim
Bryan J. Kincaid
David O. Kirste
Howard A. Kunst
Benoit Laganiere
D. Scott Lamb
Alan E. Lange
Paul W. Lavrey
France LeBlanc
Eric F. Lemieux
Giuseppe F. LePera
Cornwell H. Mah
Blair E. Manktelow
Leslie R. Marlo
Keith A. Mathre
Stephen J. McGee
Dennis T. McNeese
M. Sean McPadden
Christopher J. McShea
Robert L. Miller
Charles B. Mitzel
Todd B. Munson
David A. Murray
Sarah L. Nellis
Richard N. Nevins
John Nissenbaum
Randall S. Nordquist
Margaret O'Brien
Laura A. Olszewski
Naomi S. Ondrich
Teresa K. Paffenback
Chandrakant C. Patel
Timothy B. Perr
Brian D. Poole
C. Stuart Powers
Richard W. Prescott
Lewis R. Pulliam
Mark S. Quigley
Kenneth P. Quintilian
Eric K. Rabenold
Donald K. Rainey
Thomas O. Rau
Alfred Raws III
James E. Rech
Elizabeth M. Riczko
William E. Roche
Diane R. Rohn
Bradley H. Rowe
Michael R. Rozema
John M. Ruane, Jr.
Timothy J. Rundle
Yves Saint-Loup
Stephen P. Sauthoff
David M. Savage
Suzanne E. Schoo
Lisa M. Scorzetti
Vincent M. Senia
Derrick D. Shannon
Gary E. Shook
David A. Smith
Tom A. Smolen
Elizabeth L. Sogge
Thomas N. Stanford
Elissa M. Sturm
Jeffrey L. Subeck
Yuan-Yuan Tang
Susan M. Treskolasky
James F. Tygh
Peter S. Valentine
Scott D. Vandermyde
Michael A. Visintainer
Sebastien Vu
Alice M. Wang
Marjorie C. Weinstein
Robert J. White
Kevin Wick
Gnana K. Wignarajah
Beth M. Wolfe
Kathy A. Wolter
John M. Woosley
Vincent F. Yezzi
Nancy E. Yost
Sheng Hau Yu

Part 8

Gary R. Abramson
Kenneth Apfel
W. Brian Barnes
Gavin C. Blair
Jean-Francois Blais
Roberto G. Blanco
Jack B. Brauner
Lynn R. Carroll
Martin Cauchon
Charles Cossette
David J. Darby
Brian W. Davis
Edward D. Dew
Brad C. Eastwood
Bob D. Effinger, Jr.

John W. Ellingrod
Catherine E. Eska
William G. Fitzpatrick
Louis Gariepy
David B. Gelinne
Richard J. Gergasko
John F. Gibson
Peter M. Gidos
Eric L. Greenhill
Anne G. Greenwalt
Cynthia M. Grim
Anthony D. Hill
George A. Hroziencik
Jeffrey R. Ill
Brian A. Jones
Kevin A. Kesby
Jean-Marc Leveille
Sam F. Licitra
Andre Loisel
Brian E. MacMahon
Robert J. Meyer
Richard B. Moncher
Daniel M. Murphy
Jonathan Norton
G. Christopher Nyce
Gregory V. Ostergren
Timothy A. Paddock
Rudy A. Palenik
Julia L. Perrine
Jill Petker
Deborah Price
Timothy P. Quinn
Kay K. Rahardjo
John F. Rathgeber
Scott E. Reddig
Steven C. Rominske
Pierre A. Samson
Karen E. Schmitt
Gordon L. Scott
Robert F. Scott, Jr.
Christopher M. Smerald
Linda D. Snook
David Spiegler
Stephen D. Stayton
John A. Stenmark
Sharon K. Sublett
Rae M. Taylor
Mary L. Turner
Melanie A. Turvill
Ricardo Verges
Leigh M. Walker
Patrick M. Walton
Nancy P. Watkins
Peter A. Weisenberger
Elizabeth A. Wellington
Gregory S. Wilson
Windrie Wong
Richard P. Yocius

Part 10

Jeffrey Adams
Rebecca C. Amoroso
Lawrence J. Artes
Bruno P. Bauer
Robert K. Bender
Stephen W. Book
Christopher S. Carlson
Kenneth E. Carlton
Michael J. Caulfield
Danielle Charest
Walter P. Cieslak
Carol Desbiens
Timothy B. Duffy
Denis Dumulon
Steven R. Fallon
Nancy G. Flannery
Richard L. Fox
James J. Gebhard
Richard N. Gibson
Bonnie S. Gill
Steven A. Glicksman
David C. Harrison
Todd J. Hess
David R. Heyman
Alan M. Hines
John M. Hurley
Steven J. Johnston
Edward M. Jovinelly
Mary Jean King
Charles D. Kline, Jr.
Constantine G. Koufacos
Kenneth R. Krissinger
John R. Kryczka
Kay E. Kufera
Paul E. Lacko
David A. Lalonde
Peter M. Licht
Jon W. Michelson
H. Elizabeth Mitchell
Karl G. Moller, Jr.
Brian A. Montigney

Chris E. Nelson
Kathleen M. Pechan
Kai-Jaung Pei
Steven J. Peterson
Jennifer A. Polson
Robert Potvin
Kevin B. Robbins
Richard D. Robinson
Randy J. Roth
Jeffrey C. Salton
Melissa A. Salton
Valerie Schmid-Sadwin
Richard D. Schug
Mark E. Schultze
Mark J. Silverman
Lisa A. Slotznick
Lawrence J. Steinert
Russell Steingiser
Edward C. Stone
Christopher M. Suchar
Ting-Shih Teng
Ernest I. Wilson
Martha A. Winslow
Heather E. Yow

The following candidates were admitted as Fellows and Associates as a result of their successful completion of the Society requirements in the November, 1990 examinations.

FELLOWS

James J. Gebhard
Melissa A. Salton
Valerie Schmid-Sadwin
Warren B. Tucker
Nancy P. Watkins

ASSOCIATES

Marc J. Adee
Karen L. Ayres
Nathalie Begin
Thomas S. Boardman
Pierre Bourassa
Alicia E. Bowen
Anthony J. Burke
Janet L. Chaffee
Mario Champagne
Cindy C. M. Chu
Dianne Costello
Martin L. Couture
Kenneth M. Creighton
Patrick K. Devlin
Kevin G. Dickson
Pierre Dionne
Yves Doyon
Patrick Dussault
Brad C. Eastwood
Charles C. Emma
William E. Emmons
Yves Francoeur
Louis Gariepy
Bruce R. Gifford
Michael A. Ginnelly
Richard S. Goldfarb
Charles T. Goldie
Matthew E. Golec
Todd A. Gruenhagen
Ellen M. Hardy
Jeffrey R. Ill
Kathleen M. Ireland
Changseob Kim
Richard O. Kirste
David R. Kunze
Frank O. Kwon
D. Scott Lamb
Mathieu Lamy
Nicholas J. Lannutti
Paul W. Lavrey
Giuseppe F. Lepera
Donald F. Mango
Blair E. Manktelow
Donald R. McKay
Sara L. Nellis
Marlene D. Orr
Donald D. Palmer
Chandrakant C. Patel
Timothy B. Perr
Julie L. Perrine
Brian D. Poole
Kenneth P. Quintilian
Eric K. Rabenold
Donald K. Rainey
Elizabeth M. Riczko
William E. Roche
Bradley H. Rowe
Yves Saint-Loup
Joanne Schlissel
Vincent M. Senia
Derrick D. Shannon
David A. Smith
Keith R. Spalding
Stephen D. Stayton
Frederick M. Strauss
Rae M. Taylor
Peter S. Valentine
Kenneth R. Van Laar
William Vasek
Michael A. Visintainer
Sebastian Vu
Kevin Wick
Gnana K. Wignarajah
John M. Woosley
Vincent F. Yezzi
Sheng Hau Yu

The following is the list of successful candidates in examinations held in November, 1990.

Part 3B

Rimma Abian
Kim M. Abramek
Shawna S. Ackerman
Michael W. Allard
Craig A. Allen
Rhonda K. Allison
K. Athula Alwis
Scott C. Anderson
Michael J. Andring
Richard T. Arnold
Bhim D. Asdhir
Nathalie J. Auger
Lewis V. Augustine
Rita Ann Basile
Dominic Bazin
John A. Beckman
Brian K. Bell
Sheila J. Bertelsen
Donna K. Bever
Gina S. Binder
Bruce E. Binnig
Annie Bisaillon
Laverne J. Biskner III
Annie Blais
Ming Y. Blinn
Betsy L. Blue
Maurice P. Bouffard
Christopher K. Bozman
Lori M. Bradley
Kevin M. Brady
Lisa M. Brieden
Tracy L. Brooks-Szegda
Eric J. Brosius
David Browan
Lisa A. Brown
Stephanie J. Brown
Laura L. Burnaford
John F. Butcher II
Robert N. Campbell
Anthony E. Cappelletti
Kristi I Carpine-Taber
Benoit Carrier
Martin Carrier
Terri L. Cartwright
Tania J. Cassell
Richard J. Castillo
Julia C. Causbie
Kevin J. Cawley
Julie S. Chadowski
Andrea L. S. Chan
Debra S. Charlop
Sigen Chen
John S. Chittenden
Bryan C. Christman
Kuei-Hsia R. Chu
Rita E. Ciccariello
Denise R. Clark
Kay A. Cleary
William B. Cody
Jeffrey R. Coker
Pamela A. Conlin
Sharon L. Cooper
Conni J. Craig
Richard S. Crandall
Malcolm H. Curry
Michael T. Curtis
Charles A. Dalcorobbo
David J. Darby
Karen L. Davies
Kristin D. Defrain
Marie-Julie Demers
Ronald M. Dennis
Dina M. Deschino
Dawn M. Desousa
Marybeth Diffley
Lisa A. Doedtman
Shawn F. Doherty
Bernard Dupont
Colleen L. Eberts
David M. Elkins
Gregg Evans
Joseph G. Evleth
Charles V. Faerber
Arlene M. Fahey
Patrick V. Fasciano
Bruce D. Fell
John R. Ferrara
Audrey M. Ferrier
George Fescos
Brian C. Fischer
David I. Frank
Mark R. Frank
Russell Frank
Cynthia J. Friess

Jeffery N. Garnatz
Carol A. Garney
Lynn A. Gehant
Rita M. Geraghty
Margaret W. Germani
Judith E. Ghirardelli
Thomas J. Ginnelly
Michael F. Glatz
Donna L. Glenn
Annette J. Goodreau
Chris D. Goodwin
Karl Goring
Odile Goyer
Lawrence E. Grabowski
Marc C. Grandisson
Jeffrey W. Graver
Sheri M. Green
Charles R. Grilliot
William A. Guffey
Jean-Francois Guimond
Howard A. Gullbrand
Kristy A. Hadaway
Nasser Hadidi
Elizabeth E. Hansen
Robert L. Harnatkiewicz
Adam D. Hartman
Curtis D. Harvey
Gary M. Harvey
Jonathon B. Hayes
Barton W. Hedges
Rhonda R. Hellman
Shohreh Heshmati
Lisa R. Hilton
Amy J. Himmelberger
Carl F. Hirschman
Brook A. Hoffman
Robert J. Hopper
Geoffrey W. Horton
Melissa K. Houck
Annette Y. Hu
Paul R. Hussian
Jonathon D. Imboden
Cindy Jacobowitz
Patrick C. Jensen
Brian E. Johnson
Anthony N. Katz
James M. Kelly
Rebecca A. Kennedy
Susan E. Kent
Glenda J. Kettelson
Jean-Luc E. Kiehm
Martin T. King
Jean-Raymond Kingsley
Bradley J. Kiscaden
Joan M. Klucarich
Terry A. Knull
Timothy F. Koester
Fred S. Koppenheffer
Karen L. Krainz
Debra K. Kratz
Dean F. Kruger
Jason A. Kundrot
Mylene J. Labelle
Josee Lambert
Mathieu Lamy
John B. Landkamer
Gregory D. Larcher
David L. Larson
Robert J. Larson
Helen P. Leclair
Doris Lee
Jeanne P. Lee
Glen Leibowitz
Elizabeth A. Lemaster
Julie Lemieux-Roy
Deanne C. Lenhardt
Chantal Letourneau
Shu Ching Lin
Steven C. Lin
Andrew M. Lloyd
Ronald P. Lowe, Jr.
Tai-Kuan Ly
Susan A. Lynch
Christine Macisaac
Kenneth W. Macko
Joleen P. Mallon
Donald F. Mango
Richard J. Marcks
Keith M. Marcus
Leslie R. Marlo
Maria Mattioli
Bonnie C. Maxie
Melinda H. Mayerchak
Robert D. McCarthy
Timothy L. McCarthy
Jay E. McClain
Richard J. McElligott
Carole R. McIntyre
Thomas S. McIntyre
Matthew M. McKenzie
William E. McWithey
Anne C. Meysenburg
Brenda D. Miller
John H. Mize
Douglas J. Moeller
Francois R. Morin
Randy J. Murray
Karen E. Myers
Donna M. Nadeau
Kathleen V. Najim
Aaron W. Newhoff

James J. Niemann
Michael A. Nori
Michael Nystrom
Kimberly A. Oaks
Steven J. Olson
William L. Oostendorp
James D. O'Malley
Linda M. O'Shea
Jennifer J. Palo
Brian S. Pauling
Rick S. Pawelski
Charles C. Pearl
Kenneth J. Peschell
William Peter
Anne M. Petrides
Genevieve Pineau
Glen-Roberts Pitruzzelo
Gregory J. Poirier
Ted Poon
Christine A. Porcelli
Alfredo Portillo
Mary E. Potts
Charlene M. Pratt
Walter D. Price
Arlie Proctor
Richard B. Puchalski
Patricia A. Pyle
Robert E. Quane
Karen L. Queen
Kathleen M. Quinn
John F. Radwanski
Darin L. Rasmussen
Michael T. Ray
Yves Raymond
Brenda L. Reddick
Timothy O. Reed
Cynthia L. Rice
Patsy A. Riley
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Marn Rivelle
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Dave H. Rodriguez
Douglas A. Roemelt
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Colleen M. Sullivan
Francis P. Sullivan
Anuradha Sundram
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Pamela A. Thomas
W. Mont Timmins
Mike Toledano
Katherine A. Tollar
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Marie-Claire Turcotte
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Keith A. Walsh
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Perry K. Wooley
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Sondra H. Einig
George Fescos
Sy Foguel
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Shiu-Shiung Lin
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Gerson Smith
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Part 5A

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Part 9

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Edward D. Dew
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France Fortin
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Andrea Gardner
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David B. Gelinne
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Bryan J. Kincaid
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Warren B. Tucker
Mary L. Turner
Melanie A. Turvill
Patrick M. Walton
Nancy P. Watkins
Marjorie C. Weinstein
Kathy A. Wolter
Ronald J. Zaleski

Fellows admitted in May 1990

Front Row: Jane E. Jasper, J. Scott Bradley, John J. Joyce, Michael Fusco, president, Vincent T. Donnelly, Teresa J. Herderick, Thomas J. Ellefson, Christine E. Radau, Elisabeth Stadler.

Back Row: G. Clinton Somberger, Malcolm E. Brathwaite, Richard J. Hertling, Debbie Schwab, William G. Fanning, Kirk G. Fleming, Scott H. Dodge, David C. Scholl.

FELLOWS ADMITTED IN NOVEMBER 1990

Seated in Front: Thomas Duffy, Isabelle Perigny, Danielle Charest, Kay Kufera, Michael Fusco (CAS President), Steven Johnston, Jon Michelson, Lawrence Artes. **Middle Row:** Chris Nelson, Carol Desbiens, Kai-Jaung Pei, Robert Bender, Paul Lacko, Richard Gibson, Walter Cieslak, Richard Schug, Jeffrey Salton. **Top Row:** Steven Book, Mark Schultz, Edward Stone, Michael Caulfield, Christopher Carlson, Charles Kline, Richard Fox, and Kenneth Apfel.

FELLOWS ADMITTED IN NOVEMBER 1990

Seated in Front: Steven Peterson, Edward Jovinelly, David Harrison, Bonnie Gill, Michael Fusco (CAS President), Mark Silverman, Lynn Carroll, Martha Winslow. **Middle Row:** Elizabeth Mitchell, Jacque Frank, Richard Robinson, John Kryczka, David LaLonde, David Heyman, Lisa Slotznick, Ernest Wilson, Kevin Robbins, Mary King, Heather Yow. **Top Row:** Anthony Hill, David Spiegler, Kenneth Krissinger, Alan Hines, Jonathan Norton, Chris Suchar, Lawrence Steinert, and Denis Dumulon, Steven Glicksman, John Hurley, Karl Moller and Randy Roth are not pictured.

OBITUARIES

George B. Elliott
Jarvis Farley
Gerald J. Jerabek

GEORGE B. ELLIOTT
1908–1990

George B. Elliott, a Fellow of the Casualty Actuarial Society since 1940, died on November 5, 1990 at the age of 82.

George was rightly called the patriarch of the Pennsylvania Compensation Rating Bureau. He became General Manager in 1948, and continued in that responsibility until he retired in 1973. He was the central figure in all that went on in workers compensation in Pennsylvania throughout those years. Prior to his arrival at the Pennsylvania Bureau, he had been employed by the Pennsylvania Insurance Department as a rate actuary, his job when he obtained his Fellowship in 1940.

George served two three-year terms on the CAS Council (now the Board), from 1947 to 1950 and from 1951 to 1954. From 1964 to 1966, he was Chairman of the Committee on Development of Papers. And he was the author of two notable papers: "Multiple Injury Accidents and Losses in Excess of any Specific Retention—Pennsylvania Workmen's Compensation" (1946), and "The Making of Workmen's Compensation Rates as Illustrated by the 1951 Pennsylvania Rate Revision" (1951).

JARVIS FARLEY
1910–1991

Jarvis Farley, a Fellow of the Casualty Actuarial Society since 1940, died on July 12, 1991 at the age of 81.

When Mr. Farley became a CAS Fellow, he was actuary and assistant treasurer of the Massachusetts Indemnity Company. He remained with that company (which became the Massachusetts Indemnity and Life Insurance Company) throughout his career, rose in the ranks, and was made Chairman of the Board and Chief Executive Officer in 1970. He retired in 1974 and lived in Needham, Massachusetts.

Mr. Farley was a member of the CAS Council from 1947 to 1950, joining George Elliott for that term, and he was Chairman of the Committee on Social Insurance from 1964 to 1967. He was the author of two important papers in his field: "A 1940 View of Non-Cancellable Disability Insurance" (1940), and "An Approach to a Philosophy of Social Insurance" (1942).

Jarvis Farley, in his retirement years, continued to be busy in the affairs of the American Academy of Actuaries. He was Chairman of the AAA's Specialty Committee of the Actuarial Standards Board, and was an active member of the Joint Task Force on Professionalism, the Committee on Guides to Professional Conduct, and the Committee on Continuing Care Retirement Communities.

GERALD J. JERABEK
1948–1990

Gerald J. Jerabek, a Fellow of the Casualty Actuarial Society and a Member of the American Academy of Actuaries since 1979, died on August 14, 1990, at the age of 42. He was born on July 22, 1948, in Manitowoc, Wisconsin.

Mr. Jerabek received a bachelor of science degree in mathematics from Michigan Technological University in 1970 and a masters degree in statistics from Purdue University in 1972. From 1972 to 1976, he performed actuarial pricing duties at State Farm Insurance Companies. He joined Nationwide Insurance Companies in 1976. There, his responsibilities included personal auto and homeowners ratemaking, state filings, and special projects. He was elected Vice President, Personal Lines Pricing in August, 1989.

Mr. Jerabek served the Casualty Actuarial Society as a member of the Education and Examination Committee, the Examination Committee, the Committee on Principles of Ratemaking, and the Committee on Ratemaking.

Mr. Jerabek had a wide range of interests. He was well known among CAS golfers and tennis players for his proficiency in those sports. He competed in regional and national tournaments and won several awards.

Mr. Jerabek is survived by his wife, Jaclyn, of Columbus, Ohio; sister, Valerie Joens, of Topeka, Kansas; and parents, Mr. and Mrs. Milos Jerabek, of Pembine, Wisconsin.

Mr. Jerabek is deeply missed by his family and by his friends and colleagues.

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