UTILITY WITH DECREASING RISK AVERSION

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Abstract

Utility theory is discussed as a basis for premium calculation. Desirable features of utility functions are enumerated, including decreasing absolute risk aversion. Examples are given of functions meeting this requirement. Calculating premiums for simplified risk situations is advanced as a step towards selecting a specific utility function. An example of a more typical portfolio pricing problem is included.

> "The large rattling dice exhilarate me as torrents borne on a precipice flowing in a desert. To the winning player they are tipped with honey, slaying him in return by taking away the gambler's all. Giving serious attention to my advice, play not with dice: pursue agriculture: delight in wealth so acquired."
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> KAVASHA Rig Veda X.3:5

Avoidance of risk situations has been regarded as prudent throughout history, but individuals with a preference for risk are also known. For many decision makers, the value of different potential levels of wealth is apparently not strictly proportional to the wealth level itself. A mathematical device to treat this is the utility function, which assigns a value to each wealth level. Thus, a 50-50 chance at double or nothing on your wealth level may or may not be felt equivalent to maintaining your present level; however, a 50-50 chance at nothing or the value of wealth that would double your utility (if such a value existed) would be equivalent to maintaining the present level, assuming that the utility of zero wealth is zero. This is more or less by definition, as the utility function is set up to make such comparisons possible.

For an individual with a steeply ascending utility function, the value of potential wealth needed to risk losing everything on a 50-50 bet may be less than twice the current level; if the function rises slowly it may be considerably greater than twice the current level; and if the utility increases asymptotically to a value not greater than twice the utility of current wealth, such a bet would not be acceptable for any amount.

Through the device of the utility function, diverse risk situations can be compared. For each situation the expected value of the utility of the possible outcomes can be computed, and the situation with the highest expected utility is preferred.

The pioneering work in the modern application of utility theory was done by von Neumann and Morgenstern[6]. They showed that if a preference ordering for a set of risk situations follows certain consistency requirements, then there is a utility function that will give the same preference ordering on those situations. That in effect removes utility theory from the pleasure-pain area; it can be set up as an essential element of consistent management decision making without addressing questions like "Can a corporation feel pain?"

The consistency requirements can be boiled down to three (e.g., see [4]):

i) any two risk situations can be compared; i.e., one is preferable to the other or they are both equivalent;
ii) if A is preferred to B and B to C, then A is preferred to C;
iii) if A is preferred to B and B to C, then there is a unique r between 0 and 1 such that B is equivalent to $rA + (1 - r)C$.

If risk situations are evaluated consistently, under this definition of consistency, they can be ordered by some utility function.

Thus, utility theory seems to be a potentially valuable tool for choosing between alternative risk situations, such as transferring or accepting the risk of loss for a fixed price, in a consistent way.

The practical actuary, however, finds utility theory somewhat of a dilemma: on one hand it provides the basic theoretical foundation for the worth of the insurance product; on the other hand, no examples of its useful application to insurance are available. Dismissing the whole theory risks throwing out the baby with the bath, but until it can be made to work in practice it will not have much appeal.

The training of most actuaries includes an introduction to utility theory and its general relation to risk situations. There is a gap between this and actual application, however. How to choose a specific utility function is part of this gap; applying this function in realistic situations is another.

The present paper aims to narrow this gap somewhat, but is not so ambitious as to try to close it entirely. Several criteria that a utility function should meet are discussed, and examples are given of functions that meet these requirements. The possible application of utility theory to pricing is also addressed. Since an insurer is taking an uncertain situation for a fixed price, a utility approach may help evaluate the attractiveness of the deal.

Finally, simplified risk situations are evaluated using some of the utility functions discussed. The prices implied by different utility functions for these simplified situations can be used by the analyst to close in on a specific function that most closely reflects the insurer's own risk preferences.

As mentioned above, two risk situations can be compared by computing the expected utility of each, with the higher value being preferred. To apply this to premium calculation, the situation of having the risk and the premium is compared to the situation of having neither; i.e., the expected utility for these two situations is compared. For an insurer with surplus of a and no other policies, the indifference premium g for a random loss variable L is defined as the amount that results in the same expected utility both with and without this premium and potential loss. Thus, assuming the utility function u, a, and L are known, g is the solution of

$$E(u(a)) = E(u(a + g - L))$$
$$(E(u(a)) = u(a) \text{ if } a \text{ is constant}).$$

The calculation of this g in a reinsurance context is illustrated in [5].

Presumably, something in addition to g would be needed to make the transfer worthwhile to the insurer. The excess of the premium offered over the indifference premium can be called the risk adjusted value, or RAV, of the proposal. Applications of the RAV concept can be found in [3] and [8]. However, in this context, any premium above the indifference premium would lead to the acceptance of the contract.

In order to apply this pricing principle, a specific utility function is needed. Several criteria for the selection of a utility function have evolved over time. Among them are:

(1) $u(x)$ is an increasing function on $(0,\infty)$; i.e., $u'(x) > 0$. That is, more is always better. The variable u' is referred to as marginal utility, so this criterion says that marginal utility is always positive.
(2) $u(x)$ is concave downwards; i.e., $u''(x) < 0$. This property is referred to as risk aversion in that it implies that the certainty of the expected value of the outcomes is preferred to an uncertain situation. Concave downward utility also means that marginal utility ($u'(x)$) is a decreasing function of wealth; i.e., as more wealth is accumulated less value is placed on an additional dollar. A gambler might have a utility function

that violates this principle; i.e., a price higher than the expected value might be paid for the chance of a large gain.

(3) Absolute risk aversion decreases as wealth increases. Absolute risk aversion is measured by $ra(x) = -u''(x)/u'(x)$. The $ra(x)$ function so defined can be seen to be the percentage change in the marginal utility $u'(x)$. Decreasing absolute risk aversion means that the percentage decrease in marginal utility is itself decreasing. This property can be shown to equate to greater acceptance of risky situations with greater wealth (see [4], p. 35), which seems intuitively appropriate. This concept is illustrated in Appendix 1.

(4) $u(x)$ is bounded above; i.e., there is a number b such that $u(x) < b$ no matter how large x is. This criterion is necessary to keep very rare large value situations from dominating preferences.

As an example, consider a hypothetical national lottery in which Joe, the winner, receives a choice of either $10 million certain or a risk situation in which he gets a very fabulous sum if he can pick the ace of spades at random from a deck of cards and zero otherwise. If the utility of $10 million is above 1/52 of Joe's maximum possible utility, he will take the $10 million no matter how fabulous the sum may be. On the other hand, if Joe's utility function is not bounded, the choice will depend on what the sum is: for a large enough sum he will choose to draw for the ace.

The bounded utility situation seems more reasonable, but this criterion is somewhat controversial. For instance, it could be argued that Joe would indeed choose to draw if the sum were large enough, but that such a sum would be greater than the current wealth of the world. Since we know that world wealth is finite, we judge Joe's decision to keep the $10 million as reasonable; however, if greater wealth were available the decision to draw would eventually become reasonable and would become compelling as the prize continued to increase.

This argument does not seem persuasive, because the finite wealth of the world does not appear all that relevant to the decision to keep the $10 million and be content with the lifestyle it can support. However, to recognize the degree of subjectivity in this judgment, the opposite opinion has been allowed some consideration. Nonetheless, the boundary criterion will be maintained herein. See [1], p. 35, for a complete discussion of this standard.

A fifth criterion is occasionally advanced.

(5) $u'(x) = 0$ for $x < 0$; i.e., utility is constant for negative values of wealth. This is designed to reflect bankruptcy laws and the corporate form of organization, which presumably make financial entities indifferent as to how bankrupt they become. In a regulated insurance industry, an insurer would not be completely free to act in accordance with this principle, and it probably exaggerates the effect limited liability has on decisions. However, the behavior of the utility function for negative values of wealth is important and must be considered explicitly when choosing a utility function. A more reasonable approach to negative wealth may be to take $u(x) = -u(-x)$. While raising this issue, the current paper does not attempt to settle it. In the examples below, negative values of wealth will not be possible. The following minimal condition will, however, be imposed:

$u(x)$ is defined, continuous, and non-decreasing for $x \leq 0$.

Since preference orderings are not altered by linear transformations of the utility function, by suitable normalization any utility function meeting the criteria 1, 4, and 5 could be transformed to take values between 0 and 1, for $x \geq 0$, without altering the preference orderings. Such utility functions and increasing probability distribution functions for positive variables are, therefore, the same class of mathematical mappings from the positive real numbers to the unit interval. Thus, the literature on probability distributions provides a rich source of functional forms for utility functions. Some distribution functions will not satisfy criteria 2 and 3, however, so these properties must be checked individually.

Examples of functions that do not meet the above criteria are:

$u(x)$	Fails
x	2
$x - a/2x^2$ $(x \leq a)$	3
$a + b\ln(x + c)$	4
$1 - \exp(-bx)$	3
x^a $(0 < a < 1)$	4
$1 - 1/x$	5

The Weibull and Pareto distribution functions do meet all criteria for proper parameters; e.g., $u(x) = 1 - \exp(-bx^c)$, $c < 1$ and $u(x) = 1 - (bx + 1)^{-c}$.

Other special cases of the transformed beta and gamma distribution functions [7] will also suffice.

Although exponential utility fails criterion 3, it leads to relatively simple computations. Advocates of exponential utility argue that decreasing absolute risk aversion is important but its effects can be provided by changing the parameter of the function as wealth increases. While feasible, it seems undesirable to do this, as the example in Appendix 1 illustrates. A utility function should capture the preferences of the decision maker, including the relationship of preferences to wealth. Questions such as, "if you had $50 million and were offered . . ." should be used to help determine these preferences. In other words, the utility function should be able to get at fundamental attitudes towards risk including how reactions will change with wealth.

Looking at absolute risk aversion as the percentage change in marginal utility provides another approach to this issue. The marginal utility of wealth should decrease as wealth increases, but decreasing absolute risk aversion means that the percentage decrease in marginal utility should itself be declining. If a utility function does not reflect this decline, it is not properly valuing various wealth potentials. In other words, decreasing absolute risk aversion is not simply a matter of having different attitudes towards risk at different wealth levels. It is rather an aspect of the shape of the utility function at every point and reflects the relative desirability of the different levels of wealth themselves.

Exponential utility has other aspects that make it unrealistic in insurance situations. One of these is additivity. Of course, the individual risk premiums must add up to the portfolio premium; this does not mean, however, that the indifference premium for a single risk should be .001 of the premium for 1000 such risks. Under exponential and linear utility, and only with these forms, the indifference premium for a number of independent risks will be the sum of the indifference premiums for the risks separately [1], [4]. This is contrary to usual practice. For instance, there is generally thought to be a benefit to pooling, since the probability of being a large percentage away from expected results becomes less as individual risks are pooled. In other cases, adding independent risks might jeopardize surplus enough that a higher charge would be needed for the last one. Neither such effect is captured by exponential or linear utility.

Risk decisions under exponential utility do not reflect the other risks that may be in the portfolio [8], which again appears unrealistic. All these problems essentially derive from the constant absolute risk aversion of the exponential, which renders decision making independent of wealth. Although calculations are more difficult when decreasing risk aversion is required, this seems unavoidable in realistic insurance situations.

Using utility concepts can help bring consistency to risk decisions. However, the selection of a utility function requires some time and attention. Examining many simplified risk situations to determine which functions best reflect the preferences of the decision maker is one approach.

For example, the indifference premium g is calculated below for some very simple loss distributions, using the utility functions $u(x) = 1 - \exp(-.01x^{.25})$ and $v(x) = 1 - (1 + 10^{-7}x)^{-1}$. These functions are primarily illustrative and are not necessarily advocated. Companies with surplus, a, of \$20,000,000 and \$50,000,000 will be considered. A risk with a .001 probability of a total loss of \$10,000,000 and a .999 probability of no losses will be used.

The indifference premium for v is the solution of:

$$v(a) = .001\ v(a + g - 10{,}000{,}000) + .999\ v(a + g)$$

or $1/(1 + a10^{-7}) = .001/((a + g)10^{-7}) + .999/(1 + (a + g)10^{-7})$.

For selected values of a the equation can be solved for g algebraically. In fact, $g = -a + (10^7/2c)(1 - c + ((1 + c)^2 + .004\ c)^{1/2})$ where $c = (1 + 10^{-7}a)^{-1}$. The similar equation for u may be solved iteratively. The indifference premiums are shown below.

Surplus	u	v
20,000,000	13,422.56	14,988.78
50,000,000	11,101.62	11,997.13

Two such independent risks would have a .000001 probability of \$20,000,000 in losses, a .001998 probability of a single \$10,000,000 loss, and a .998001 probability of no losses. Thus for u, the indifference premium g is the solution of:

$$u(a) = .000001\ u(a + g - 20{,}000{,}000) + .001998\ u(a + g - 10{,}000{,}000) + .998001\ u(a + g).$$

This and the corresponding equation for v can be solved iteratively to yield:

Surplus	u	v
20,000,000	26,889.03	29,985.23
50,000,000	22,203.42	23,994.49

As would be anticipated, the less wealthy company needs a higher premium to take on these risks. Also, contrary to what might be expected from pooling considerations, the premium for two independent risks is somewhat more than twice the premium for a single risk in these cases, especially for the smaller company. This may be realistic in this case because the risk loss is a substantial proportion of surplus and two losses will nearly bankrupt the company. Since the coefficient of variation of the two independent risks is lower than that of a single risk, premium calculation principles based on the standard deviation (or variance) of the aggregate loss distributions would not capture this effect. It would be nice to have an example of a case where a pooling benefit were shown. This would probably require the specification of utilities of negative wealth. The benefit of pooling might be to reduce the surplus needed per risk at a given price; i.e., for a fixed ratio $g/E(L)$, the ratio of needed surplus to number of risks may decrease with the addition of independent risks.

The consideration of simplified situations such as those above can help determine a utility function. The indifference premium for a portfolio of real business can be calculated by these same principles from the utility function and the probability distribution function of aggregate losses, although the calculation will be more intricate for a continuous loss distribution function. An example is given in Appendix 2.

Spreading a portfolio premium to the individual risks is unfortunately a somewhat arbitrary process in this context. Possibilities include spreading in proportion to expected losses or finding the exponential utility function that gives the same overall portfolio premium, and using that to determine the individual insured's premium.

The expected value method does not differentiate contracts by hazard, and thus is probably most appropriate when the riskiness is fairly homogeneous. Exponential utility will give such a differentiation, but this may be somewhat artificial. In the typical situation, where individual insureds are not independent, due to common parameter risk, even the exponential utility premiums will not add up to the portfolio premium. Spreading premium to individual insureds in a realistic way is a problem that merits further research.

An elegant suggestion has been presented by Borch [2]. He recommends calculating the premium for the random loss variable X by the formula $(1 + i)\ E(X) + j\ \mathrm{cov}(X,L)$, where L is the portfolio aggregate loss random variable. This formula gives premiums that add up to the portfolio premium even when risks are not independent. An example is discussed in Appendix 2.

The selection of a realistic utility function requires careful consideration of the implications of this choice in comparison with the judgments the function aims to model. Starting with functions that meet certain general criteria and then examining how they perform in simplified situations can help in this process. The Weibull and Pareto distribution functions provide forms that meet all the criteria discussed herein, although the extension to negative wealth deserves further attention. A practitioner would need to consider specific parameter values and decide which, if any, are appropriate for a specific application. The rewards of this effort would be a procedure for evaluating diverse risk situations from a consistent perspective.

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APPENDIX 1

RISK AVERSION EXAMPLE

Consider two utility functions $u(x) = 1 - \exp(-x/b)$ and $v(x) = 1 - \exp(-(x/b)^{.5})$, which have the $ra(x)$ functions $1/b^2$ and $(1/2b)(b/x)^{.5}(1 + (b/x)^{.5})$ respectively. Thus, u has constant risk aversion and v has decreasing risk aversion.

Now it is easy to show for u that decisions do not depend on the current wealth: some algebraic manipulation yields $E(u(a + L)) = 1 - \exp(-a/b)E(\exp(-L/b))$ for any wealth level a and risk situation L. This is greater than $u(a)$ if and only if $E(\exp(-L/b))$ is less than unity; thus, preferences are independent of wealth.

However, for v this is not true. The acceptance of risk will in fact increase as wealth increases. Consider a risk which will yield a profit of \$11,750 with 90% probability and a loss of \$100,000 with 10% probability. This is examined at two levels of wealth, a = \$1,000,000 and a = \$5,000,000, below. A value of 1,000,000 is selected for b.

a:	1,000,000	5,000,000
$u(a)$:	.632121	.99326205
$u(a + 11{,}750)$:	.636418	.99334076
$u(a - 100{,}000)$:	.593430	.99255342
$E(u(a + L))$:	.632119	.99326203
$v(a)$:	.632121	.893122
$v(a + 11{,}750)$:	.634269	.893402
$v(a - 100{,}000)$:	.612749	.890693
$E(v(a + L))$:	.632117	.893131

Thus for u, the risk is rejected at every wealth level, while for v it is rejected at a = 1,000,000 and accepted at a = 5,000,000.

Now, if one wanted to stay with exponential utility because of its easier calculations, one could change b to 5,000,000 when a changed. This would lead to the acceptance of the risk at the higher wealth level. However, the situation at this level is similar to a choice between L + \$4,000,000 and \$4,000,000 certain at the lower wealth level. In fact, with the fixed value b = 1,000,000, v will evaluate this choice at a = 1,000,000 the same as L versus zero at a = 5,000,000. However, u with the changed parameter will evaluate them differently.

APPENDIX 2

PREMIUM CALCULATIONS

In this Appendix the indifference premium is calculated for a continuous aggregate loss distribution with a stop loss cover. The insurer has \$50 million of surplus and the utility function $v(x) = 1 - (1 + 10^{-7}x)^{-1}$. The aggregate losses, L, to be insured are transformed gamma distributed [7] with mean, coefficient of variation, and coefficient of skewness of \$50 million, .363, and .406 respectively. The density function is taken in the form $f(x) = (ab/(r - 1)!)(bx)^{ar-1}\exp(-(bx)^a)$, where $(r - 1)!$ denotes the gamma function evaluated at r. This gives $E(X^n) = (r - 1 + n/a)!/b^n(r - 1)!$. The moments given arise when $a = r = 2$ and $b^{-1} = 37{,}612{,}639$. This distribution is a bit more dangerous than would arise in many property-casualty insurance lines with that much volume, but not exceptionally so. It could represent one of the more risky liability lines.

Stop loss insurance with a \$100 million retention is proposed, so negative surplus would not be possible if at least \$50 million is charged. The indifference premium for the retained business is desired. This will be the solution g of:

$v(a) = E(v(a + g - L))$ or
$1 - (1 + 10^{-7}a)^{-1} = 1 - E((1 + 10^{-7}(a + g - L))^{-1})$, or
$(1 + 10^{-7}a)^{-1} = E((1 + 10^{-7}(a + g - L))^{-1})$, or
$1/6 = E((6 + h - 10^{-7}L)^{-1})$, where $h = 10^{-7}g$.

Because of the stop loss, any loss greater than \$100 million will be cut off at \$100 million in computing this expectation. Thus the equation for g becomes:

$$1/6 = \int_0^{100M} \frac{f(x)\,dx}{6 + h - 10^{-7}x} + \int_{100M}^{\infty} \frac{f(x)\,dx}{6 + h - 10}$$

where $f(x) = 2x^3b^4\exp(-(bx)^2)$.

Now $Pr\,(L \leq 100M) = .00687$ can be calculated via the incomplete gamma function [7], and so we seek h, the solution of:

$$1/6 - \frac{.00687}{h - 4} = \int_0^{100M} \frac{f(x)dx}{6 + h - 10^{-7}x}.$$

By numerical integration and iteration, $h = 5.6568$ can be found, yielding the indifference premium $g = \$56{,}568{,}000$. This calculation can be done by computer or a good programmable calculator.

To apply Borch's formula to distribute this premium to individual insureds we must first choose constants i and j so that $(1 + i)E(L) + j\ \mathrm{var}(L) = g$. Then the premium for an insured with random loss variable X will be $(1 + i)E(X) + j\ \mathrm{cov}(X,L)$. These will add up to g since $\mathrm{cov}(X + Y,L) = \mathrm{cov}(X,L) + \mathrm{cov}(Y,L)$ and $\mathrm{cov}(L,L) = \mathrm{var}(L)$.

One way to select i and j might be to first select i as a desired profit load for a hypothetical insured that does not contribute to the overall portfolio variance, i.e., for which $\mathrm{cov}(X,L) = 0$. Then j can be solved for from g and the moments of L. Thus suppose $i = .02$ is selected. Then $j = (g - 1.02\ E(L))/\mathrm{var}(L)$.

For instance, suppose that in the above example $g = \$60{,}000{,}000$ were calculated for the case where the stop loss is removed. (This calculation would require specification of $v(x)$ for $x < 0$.) Since $E(L) = 50{,}000{,}000$ and $\mathrm{var}(L) = 3.3 \times 10^{14}$, $j = 9{,}000{,}000/3.3 \times 10^{14} = 2.7 \times 10^{-8}$ can be computed for this case.