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FOREWORD

The Casualty Actuarial Society was organized in 1914 as the Casualty Actuarial and Statistical Society of America, with 97 charter members of the grade of Fellow; the Society adopted its present name on May 14, 1921.

Actuarial science originated in England in 1792, in the early days of life insurance. Due to the technical nature of the business, the first actuaries were mathematicians; eventually their numerical growth resulted in the formation of the Institute of Actuaries in England in 1848. The Faculty of Actuaries was founded in Scotland in 1856, followed in the United States by the Actuarial Society of America in 1889 and the American Institute of Actuaries in 1909. In 1949 the two American organizations were merged into the Society of Actuaries.

In the beginning of the twentieth century in the United States, problems requiring actuarial treatment were emerging in sickness, disability, and casualty insurance—particularly in workers' compensation—which was introduced in 1911. The differences between the new problems and those of traditional life insurance led to the organization of the Society. Dr. I. M. Rubinow, who was responsible for the Society's formation, became its first president. The object of the Society was, and is, the promotion of actuarial and statistical science as applied to insurance other than life insurance. Such promotion is accomplished by communication with those affected by insurance, presentation and discussion of papers, attendance at seminars and workshops, collection of a library, research, and other means.

Since the problems of workers' compensation were the most urgent, many of the Society's original members played a leading part in developing the scientific basis for that line of insurance. From the beginning, however, the Society has grown constantly, not only in membership, but also in range of interest and in scientific and related contributions to all lines of insurance other than life, including automobile, liability other than automobile, fire, homeowners and commercial multiple peril, and others. These contributions are found principally in original papers prepared by members of the Society and published in the annual *Proceedings*. The presidential addresses, also published in the *Proceedings*, have called attention to the most pressing actuarial problems, some of them still unsolved, that have faced the insurance industry over the years.

The membership of the Society includes actuaries employed by insurance companies, ratemaking organizations, national brokers, accounting firms, educational institutions, state insurance departments, and the federal government; it also includes independent consultants. The Society has two classes of members, Fellows and Associates. Both classes are achieved by successful completion of examinations, which are held in May and November in various cities of the United States and Canada.

The publications of the Society and their respective prices are listed in the *Yearbook* which is published annually. The *Syllabus of Examinations* outlines the course of study recommended for the examinations. Both the *Yearbook*, at a $10 charge, and the *Syllabus of Examinations*, without charge, may be obtained upon request to the Secretary, Casualty Actuarial Society, One Penn Plaza, 250 West 34th Street, New York, New York 10119.

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1983 PROCEEDINGS
CONTENTS OF VOLUME LXX

Page

1983 PROCEEDINGS
CONTENTS OF VOLUME LXX — CONTINUED

Page

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PROCEEDINGS

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LOSS RESERVING FOR SOLVENCY

DAVID A. ARATA

Abstract

Loss reserving plays an important role in safeguarding a casualty insurance company's solvency. The specific role, however, depends upon the size of the carrier.

For example, the primary threat to the surplus of most large, multiline insurers is the sudden and unanticipated development of losses from prior accident years. For such a carrier, loss reserving promotes solvency in the most direct manner possible—by attempting to maintain adequate reserves for each unresolved accident year.

Of course, small casualty insurers share this concern over loss reserve adequacy. For this second type of carrier, however, adverse loss development represents only one of several ongoing threats to its existence. A small insurer's surplus can also fall victim to less controllable hazards, such as a year or two of poor underwriting results or undetected rate level deficiencies.

To combat these added dangers, the capital structures of new, monoline casualty companies often incorporate features seldom seen in larger carriers. Small insurers, for instance, sometimes employ policyholder assessments and expensive, low level reinsurance as primary defenses against surplus impairment.

Unfortunately, most small insurance companies do not efficiently utilize these potentially powerful capital structures. A principal cause of this under-utilization is the failure of these companies to choose loss reserving policies appropriate to their specific type of capitalization.

This report demonstrates, by example, how a small insurance company can energize its capital structure by selecting appropriate loss reserving policies. The dramatic impact of such a choice on the company's profit and survival prospects is also quantified.

I. USING COMPUTER SIMULATION TO SELECT LOSS RESERVES THAT COMPLEMENT A SMALL, PRIMARY INSURER'S CAPITAL STRUCTURE

Most loss reserve reviews for large, established casualty insurers focus on the adequacy of the tested reserves. In this type of situation, an actuary's attention centers on factors which directly affect the accuracy of his loss development calculations, such as the quality, availability, and form of underlying data.[1] Once he understands these elements, the actuary typically recommends loss reserves which equal his estimates of each unresolved year's expected loss development.

Usually, recommending appropriate loss reserves for new or small casualty carriers is less straightforward. This complexity is not the result of the obvious, inevitable, and surmountable problem of data unavailability. Rather, loss reserving for a small insurer is more complicated than reserving for an insurance monolith because a smaller company's capital structure is, of necessity, more complex.

These complex capital structures complicate the loss reserving process since a company's loss reserves interact with its capital structure. In a typical captive insurance company, for example, higher than expected loss reserves may trigger additional capital contributions from the insurer's parent, or assessments from its policyholders. Since these contingent capitalizations often form the carrier's first line of defense against insolvency, this interplay between capital and reserves directly impacts the company's chances of survival.

With its survival at stake, a small casualty insurance company should use loss reserve procedures which consider this interaction between loss reserving and its capital structure. Until now, however, an affordable technique for measuring this interplay has not been available.

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[1] A fairly complete list of elements to be considered is found in J. R. Berquist and R. E. Sherman [2].

Thanks to recent advances in microcomputer technology, it is now possible to inexpensively simulate the impact of specific loss reserving programs on a small insurer's capital structure. This report illustrates how Monte Carlo computer simulation can be used to quantify the effect of several different loss reserving methods on a hypothetical captive insurance company's policyholder assessment mechanism. As will be demonstrated, observing this interaction between reserves and capital enables the actuary to recommend a loss reserving procedure which unleashes the full potential of the carrier's capital structure, thereby improving both the company's solvency prospects and its expected profitability.

Outline of Section I

Section I demonstrates, by example, how a small primary casualty company can energize its capital structure by selecting appropriate loss reserving policies. Specifically, the following passages describe how a mythical, monoline captive insurer selects and applies loss reserving policies which enhance the effectiveness of its policyholder assessment provisions.

This example is presented in four parts:

1. The two subsections immediately following this outline introduce the Consulting Actuaries' Reciprocal Exchange (CARE), a hypothetical captive insurance company. These sections also review the circumstances prompting CARE to reserve for solvency.
2. The third subsection following discusses the mechanics of establishing "solvency reserves."
3. The fourth and fifth subsections describe the computer simulation model used to compare the effectiveness of different loss reserving methods.
4. Finally, results and conclusions are summarized in the last three subsections of Section I.

The Company

A recent article in these *Proceedings* [1] discusses a computer model for establishing the appropriate operational requirements of a hypothetical professional liability insurance carrier. This paper extends these earlier findings.

The company analyzed in that article, the Consulting Actuaries' Reciprocal Exchange, is an offshore, mutual insurer with the following features:

1. CARE provides $1 million per occurrence of casualty actuaries' errors & omissions insurance.
2. CARE begins operation with 1,000 members.

3. CARE charges uniform, $1,750 per actuary annual premiums.
4. CARE policyholders pay a flat membership fee.
5. CARE policies are assessable.
6. CARE quota shares a portion of its exposure.

The paper suggested twelve possible CARE membership fee/reinsurance/policyholder assessment combinations. A simulation model then estimated the company's profit and survival expectations under each scenario.

As a result of this analysis, CARE elected to:

- Quota share 25% of its exposures;
- Charge prospective policyholders a $500 membership fee; and,
- Include a policy provision empowering management to assess each member up to 100% of the annual premium whenever operating losses threaten to exhaust the company's surplus.

Given these decisions and the assumptions presented in that paper, CARE management can anticipate average annual surplus growth in excess of 20% with a 4% probability of insolvency over a ten year period.

CARE and the Real World

Unfortunately, the previous paper's idyllic income tax provisions and surplus requirements seldom apply to real captive insurance companies.

- Generally, an insurer's operating income is taxed on a carryback/carryforward basis—that is, operating losses can only be used to offset taxes already paid, or which become due in future years.
- In most cases, a captive insurer's surplus must be maintained at a specified level, sometimes a percentage of net written premium.

Assuming more realistic income tax and surplus requirements dramatically changes the results of the previous analysis. A 250 trial simulation carried out under appropriate tax and surplus assumptions,[2] for example, indicates an 11% chance that CARE's policyholder surplus will be exhausted during the carrier's first ten years of operation. Clearly, this result is in marked contrast to the corresponding 4% probability obtained under the original assumptions.

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[2] Specifically, the model assumes that:
- income is taxed subject to a three year carryback and seven year carryforward of operating losses; and,
- policyholder surplus, after the deduction of all carried reserves, must be maintained at not less than 20% of premium.

Of course, CARE could reduce this probability of insolvency by directly strengthening its capital structure, e.g., by increasing policyholder assessments or purchasing more reinsurance. However, increases in capitalization invariably cost the company money or place a greater risk burden on individual CARE policyholders.

Fortunately, increasing capital or buying more reinsurance are not the company's only options. The remainder of Section I demonstrates that:

1. CARE's existing capital structure operates inefficiently under traditional loss reserving approaches; and
2. CARE can substantially improve its profit and survival prospects by selecting a loss reserving policy which interacts more effectively with its capital structure.

The following paragraphs describe a procedure for choosing such a loss reserving program.

The Mechanics of Establishing Solvency Reserves

The next few subsections investigate the effect on CARE's capital structure, and hence its solvency, of establishing loss liabilities over and above its required reserves.[3] In particular, these subsections examine the profitability and survivability of a carrier that bases its carried loss liabilities on selected upper percentiles of the company's aggregate incurred loss distribution.

This concept of a solvency reserve, then, is quite simple.[4] However, reserving for solvency in a real world situation requires some not so simple decisions regarding methodology.

The following paragraphs present two legitimate techniques for computing solvency reserves. The implications of using each method are also examined; as a result of this examination, one approach is selected for use in this paper.

[3] Required reserves are amounts necessary to fund anticipated loss development.

[4] Some readers may feel that this use of the term "reserve" is inappropriate, since it is not readily apparent that solvency reserves meet generally applicable standards for legal reserves (quantifiability, relationship to specific events, etc.).

Solvency reserves as presented and defined in this paper do meet these traditional standards. For example, Sections I and II establish their quantifiability, and Section III offers a reasonable argument for their foreseeability. In any event, whether they are reserves in the traditional sense or mere accounting nuances, solvency reserves are clearly an effective and necessary management tool for small, risky insurers.

Method 1—Description: The first reviewed calculation sets each accident year's incurred losses equal to a designated percentile of the line's aggregate incurred loss distribution. Thus, the company's total loss reserve equals

Estimated Percentile of Aggregate Incurred Loss Distribution
less
Losses Paid to Date

This total reserve is then broken into its "required" and "solvency" pieces.

For example, suppose that an insurer decides to base its current accident year's loss reserve, at the twelve month valuation, on its estimate of the 95th percentile of the coverage's total loss distribution. Further assume that the company's actuary estimates that this percentile equals 178% of expected losses. Finally, suppose that this coverage's underlying rate levels assume expected losses of $1,000,000.

Given these assumptions, Method 1 sets accident year incurred losses equal to $1,780,000, or 178% of $1,000,000. This $1,780,000 includes both required and solvency reserves, as well as loss payments to date. Required reserves, of course, are merely the difference between developed accident year losses and amounts paid to date. For instance, if developed incurred losses total $1,200,000, of which $250,000 were paid during the year's first twelve months, then the following breakdown applies:

Paid Losses Through 12 Months	$250,000
Reserve for Expected Losses [$1.2 MM − $250,000]	$950,000
Solvency Reserve [$1.78 MM − $1.2 MM]	$580,000
Total Accident Year Incurred Loss:	$1,780,000

Note that Method 1's logic self-destructs whenever a year's developed incurred losses exceed the specified percentile level. In such cases, reserves must be based on the higher estimate; no solvency reserve is established.

Method 1—Analysis: Method 1 and the procedure for developing Schedule P statutory reserves share a number of similarities. First, unless developed incurred losses exceed the target loss level, the total reserve is established without consideration of the year's actual incurred losses. Second, it follows from the previous observation that a given year's total incurred losses, at least through early valuations, can be projected with reasonable precision. Finally, for new, small insurers, this approach usually results in incurred loss development factors less than unity.

Method 2—Description: A second technique for calculating solvency reserves utilizes traditional loss reserving procedures to establish the reserve for expected loss development. A separate calculation then sets the solvency reserve equal to the expected difference between the year's specified percentile and its mean loss levels.[5]

Refer again to our earlier example. Given the previous assumptions, Method 2 always produces a solvency reserve of $780,000, or 78% of expected losses. Of course, the required reserve ($950,000) and loss payments to date ($250,000) again apply. Thus, Method 2's incurred losses of $1,980,000, the sum of these three components, are $200,000 higher than the Method 1 estimate. Reflection reveals the reason for this difference—under Method 2, solvency reserves are not reduced as a result of higher than expected reported losses.

Method 2—Analysis: This second approach has the important advantage of directly incorporating the actuary's best estimate of anticipated loss development into the established reserve. Also, Method 2 avoids the illogic of solvency reserves which vary inversely with a given accident year's incurred losses.

For the above reasons, the solvency reserves examined in the following sections are developed using a Method 2 calculation.

Illustrations of typical solvency reserve calculations used in this paper are provided in Appendix C.

The Model: Assumptions

The next few sections describe the computer simulation model used to compare the effectiveness of alternative loss reserving programs for the Consulting Actuaries' Reciprocal Exchange. This model requires assumptions regarding:

- CARE's expected claim frequency and distribution of the number of claims;
- CARE's average claim size, and the corresponding distributions of claim amounts;
- The parameter error in estimating claim frequency and claim size averages;
- The number of participating actuaries;

[5] The reader may recognize this technique of segregating actual and expected losses in the process of establishing loss reserves. See R. L. Bornhuetter and R. E. Ferguson [3].

- Frequency and severity trends;
- Collectibility of assessments;
- Overhead and other administrative costs;
- Policy terms and the distribution of effective dates;
- Anticipated rate level changes;
- Commissions CARE earns on its ceded reinsurance;
- Payout of incurred losses;
- Interest earned on investable funds;
- Taxation of CARE operating income;
- Rate at which the accuracy of a given accident year's pure premium estimate improves;
- Tax treatment of reserves for losses greater than expected amounts; and,
- Statutory policyholder surplus requirements.

A detailed discussion of each of the above sixteen assumptions is provided in Appendix A.

Four Loss Reserving Alternatives

The Monte Carlo model compares simulated CARE operating results under four loss reserving programs:

1. *Standard loss reserving,* in which all accident year reserves equal the actuary's best estimate of expected loss development.
2. A *90-A Program.* Under this second approach, accident year reserves include, in addition to amounts for anticipated loss development, a solvency reserve equal to the expected difference between the company's 90th percentile and mean loss levels.[6]
3. An *80-A Program,* identical to Program 2, but with solvency reserves based upon 80th percentile loss levels.
4. A *95/90/85/80 Program*—i.e., loss reserving in which:

   (a) The current accident year's reserve contains both funding for anticipated loss development and a Method 2 solvency reserve. This solvency reserve equals the expected difference between the company's 95th percentile and mean loss levels.

   (b) The immediately preceding accident year's reserve equals the reserve required for expected loss development, plus a solvency reserve based on the 90th percentile of the line's total loss distribution.

[6] In instances where establishing reserves greater than the actuary's best estimate of expected loss development impairs CARE's surplus, solvency reserves are reduced to the extent necessary to continue operating the company.

(c) Similarly, solvency reserves for the second and third prior accident years are based upon 85th and 80th percentiles of the respective loss distributions.[7]

For the interested reader, Appendix B provides a table displaying simulated percentiles of CARE's go-in pure premium distribution.

The Model: Results

For each of these four loss reserving programs, our computer model simulates 250 trials of CARE operating experience. Each trial incorporates ten years of randomly generated CARE profits and losses.

To illustrate how our model translates earlier assumptions into program output, Table 1 displays results of the third trial of a 95/90/85/80 simulation.

From each simulation, our computer extracts the following information:

- Whether CARE remains solvent;
- CARE's average annual surplus growth[8] over the simulated ten year period;
- The number of times a call for a policyholder assessment is needed during the ten year period.

Table 2 summarizes these results.

[7] The subtle but significant advantages of a 95/90/85/80 program over a 90A or an 80A approach are discussed in the final passage in Section I: "Selecting the Best Loss Reserving Program."

[8] Average Annual Surplus Growth is determined by the tenth root of the ratio:

$$\frac{\text{Ending Surplus} + \text{Year 10 Solvency Reserve} - \text{Policyholder Assessments}}{\text{Surplus at Start (\$500,000)}}$$

Note that this computation purposely includes investment income earned on policyholder assessments. These funds, once received, become a legitimate and indivisible part of CARE's policyholder surplus. For the sake of the comparisons presented in this paper, however, average surplus growth calculations are adjusted to exclude the assessments themselves. Fairer comparisons result if call funds are removed from all calculations.

Of course, other surplus growth calculations are possible.

TABLE 1

RESULTS OF THE THIRD TRIAL OF 95/90/85/80 SCENARIO
(All Dollar Figures are in Thousands)

	YEAR 1	YEAR 2	YEAR 3	YEAR 4	YEAR 5
Net Earned Premium	$1,313	$1,313	$1,313	$1,444	$1,588
Reinsurance Commission	33	33	33	36	40
Investment Income	106	287	358	460	529
Calendar Year Inc'd Loss*	$1,322	$762	$205	$1,141	$773
Change in Solvency Reserve**	818	536	376	250	191
Expenses Incurred	263	210	175	193	212
Income Taxes Paid	0	0	56	164	451
Surplus at Start***	$500	$863	$987	$1,878	$2,070
Policyholder Assessments	1,313	0	0	0	0
Surplus at End***	863	987	1,878	2,070	2,600
Claim Cost Inflation	12.0%	13.7%	11.7%	12.3%	12.6%
Number of Members	1,000	1,000	1.000	1.000	1.000

	YEAR 6	YEAR 7	YEAR 8	YEAR 9	YEAR 10
Net Earned Premium	$1,922	$2,325	$2,813	$3,404	$4,120
Reinsurance Commission	48	58	70	85	103
Investment Income	629	755	871	991	1,154
Calendar Year Inc'd Loss*	$2,537	$2,831	$2,697	$1,607	$2,329
Change in Solvency Reserve**	347	492	663	804	919
Expenses Incurred	256	310	375	454	549
Income Taxes Paid	−249	−228	9	743	727
Surplus at Start***	$2,650	$2,413	$2,206	$2,283	$3,229
Policyholder Assessments	0	0	0	0	0
Surplus at End***	2,358	2,146	2,217	3,156	4,083
Claim Cost Inflation	12.7%	14.1%	12.0%	9.8%	9.1%
Number of Members	1,100	1,210	1,331	1,464	1,611

Avg. Annual Surplus Growth (includes full solvency reserves):	32.2%
Avg. Annual Surplus Growth (solvency reserves taxed at 46%):	27.5%

* Before inclusion of solvency reserves.
** See Appendix C for underlying calculations.
*** Surplus reflects full deduction of solvency reserves.

The Model: Conclusions

As indicated in Table 2, each nonstandard loss reserving program improves CARE's profit and solvency expectations when compared with corresponding results achieved under standard reserving.

- Column 2 quantifies the dramatic improvement in CARE's solvency prospects which occurs under each alternative loss reserving program. Particular improvement is observed under 90A and 95/90/85/80 reserving.
- Column 3 illustrates that the company's profitability, as measured by its average annual surplus growth over a ten year period, also improves under nonstandard reserving. Reflection (or a glance ahead to Appendix D)

TABLE 2

COMPARISON OF CARE'S SIMULATED SOLVENCY AND PROFITABILITY UNDER ALTERNATIVE LOSS RESERVING PROGRAMS

Reserving Policy (1)	Percent of Trials In Which CARE Becomes Insolvent (2)	Average Annual Surplus Growth[9,10] 50th %ile (3a)	10th %ile (3b)	90th %ile (3c)	# Trials* Requiring: 0 or 1 Call (4a)	2 or More Calls (4b)
Standard Reserving	11.2% (28 Times)	19.1%	CARE Fails	29.1%	193	29
90A	5.6% (14 Times)	29.6%	0	34.4%	96	140
80A	8.8% (22 Times)	25.5%	0	32.1%	132	96
95/90/85/80	5.2% (13 Times)	29.4%	0	34.4%	97	140

* Excluding trials in which CARE becomes insolvent.

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[9] Median surplus growth figures are used instead of mean results due to the extreme skew of the average surplus growth distribution.

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[10] Column 3 assumes that solvency reserves are never taxed. A case can be made that these reserves must ultimately be repatriated and, therefore, taxed at 46%. Taxing year 10's solvency reserve changes average annual growth figures as follows:

Program	50th %ile	10th %ile	90th %ile
90A	24.5%	0	31.0%
80A	22.0%	0	30.0%
95/90/85/80	23.4%	0	30.9%

reveals the two sources of this increased profitability:

Income tax savings, and accrued investment income thereon.

Increased use of policyholder assessments, particularly during CARE's early years. These policyholder assessments generate additional investment income.

Most importantly, column 4 demonstrates the reason for the column 2 and 3 improvements. Specifically, this final column details the increased usage of CARE's principal source of contingent capitalization—policyholder assessments—under all three nonstandard programs.

In summary, Table 2 reveals that each of the three tested loss reserving alternatives energizes CARE's capital structures and thus improve both the company's profit and survival prospects. In so doing, prudent solvency oriented loss reserving policies enable CARE to avoid excessive use of more expensive capitalization or reinsurance options.

The Final Step: Selecting the Best Loss Reserving Program

Finally, CARE's actuary must choose a reserving program to recommend to the company's management. A review of Table 2 narrows his choices to the second and fourth programs tested. Moreover, given the obvious analytical parity of these two alternatives, selecting the more appropriate program—90A or 95/90/85/80—becomes a matter of the actuary's preference.

For both aesthetic and practical reasons, a 95/90/85/80 approach should be favored.

Aesthetically, 95/90/85/80 loss reserving allocates a larger proportion of the company's solvency reserves to recent, unsettled accident years. In so doing, this program places a heavier share of the financial burden of solvency protection on those accounts whose riskiness poses the gravest threat against continued solvency.

More importantly, despite the minimal difference suggested in the second column of Table 2, 95/90/85/80 loss reserving is often more effective than a 90A approach. First, since a 95/90/85/80 program places greatest emphasis on the most recent accident years, this approach better protects solvency during periods of rapid premium growth. Furthermore, the following section demonstrates that this difference in effectiveness increases in proportion to the riskiness of the underlying exposure.

II. LOSS RESERVING FOR SOLVENCY: ANOTHER SITUATION

Intuitively, the solvency reserving program described in Section I might seem less appropriate for a reinsurer dealing in volatile, excess layers of coverage. The following paragraphs test this conjecture by simulating the impact of a solvency oriented reserving policy on an insurance company that provides excess errors & omissions insurance.

The Consulting Actuaries' Reciprocal Exchange (Revisited)

Again consider CARE, our hypothetical captive insurance company. In this case, however, suppose that the company provides $900,000 of insurance in excess of the first $100,000 of loss sustained in any covered occurrence.

Assumptions (Revisited)

As in Section I, we use computer simulation to test CARE's relative solidity under four loss reserving policies. Also, most of the assumptions utilized in the earlier analysis apply again in this second situation. However, note the following three differences:

1. First Year Premium: Given the layer of coverage insured and earlier assumptions regarding applicable claim size distributions, a more realistic per actuary premium is $875 ($1,750 in Section I).
2. Loss Payout: To reflect the slower loss payout anticipated at this higher level of coverage, a uniform five year (i.e., 20/20/20/20/20) payout pattern is assumed.
3. Member Assessability: Due to the smaller premium base and added riskiness of this insurance, CARE empowers its management to assess each member as much as 200% of his annual premium during a given calendar year.

Results and Conclusions (Revisited)

Given these revised assumptions, we used the same model to compare the relative effectiveness of the four reserving programs discussed earlier. Table 3 shows the observed results.[11]

[11] Results for Section I were based upon 250 simulations for each tested reserving policy. The results presented in Table 3, on the other hand, are generated from only 100 trials. Allowances must be made for the larger sampling error in these results.

TABLE 3

EFFECTIVENESS OF ALTERNATIVE RESERVING PROGRAMS FOR EXCESS E&O INSURER (Coverage: $900K excess of $100K)

Reserving Policy (1)	Percent of Trials In Which CARE Becomes Insolvent (2)	Average Annual Surplus Growth			# Trials* Requiring:	
		50th %ile (3a)	10th %ile (3b)	90th %ile (3c)	0 or 1 Call (4a)	2 or More Calls (4b)
Standard Reserving	16% (16 Times)	16.4%	CARE Fails	24.1%	74	10
90A	10% (10 Times)	24.2%	Fails	30.2%	41	49
80A	12% (12 Times)	20.6%	Fails	28.3%	49	39
95/90/85/80	8% (8 Times)	26.1%	0	31.0%	32	60

* Excluding trials in which CARE becomes insolvent.

Columns 2 and 3 confirm that nonstandard loss reserving techniques improve this excess insurer's profit and solvency expectations less dramatically than they improve the corresponding prospects of the primary company analyzed in Section I. Just as clearly, however, some improvements in both profitability and survivability occur.

Again, reasons for these improvements are suggested by Column 4.

III. ON CHOOSING LEGITIMATE LOSS RESERVING POLICIES

The preceding analysis draws specific conclusions regarding the effectiveness of solvency oriented loss reserving as a means of energizing the potentially powerful capital structures of new, small insurance companies. Nothing said thus far, however, addresses the equally important question of what constitutes an acceptable loss reserving policy for such an insurer.

Any legitimate reserving policy must meet at least two standards:

1. Reserves must be based upon reasonably foreseeable estimates of incurred loss.

   Applying this standard to a large, multiline carrier severely restricts the range of legitimate reserving policies, possibly to those based on traditional estimates of developed losses. For this reason, the analysis presented on the preceding pages has little relevance for many domestic U.S. insurance companies.

For small, risky insurers like the ones discussed in this paper, however, estimates of ultimate losses are subject to substantial error, even when made at a 24- or 36-month development. For example, Appendix B illustrates that a given year's losses for the primary insurance company described in Section I can be expected to differ from go-in estimates of expected losses by more than 30% in half of all instances.

Thus, for carriers like the ones considered in this paper, this writer believes that loss reserving programs based on loss percentiles greater than expected levels meet this standard of foreseeability.

2. The principal purpose of such a policy must be to decrease the company's chances of insolvency, and not to avoid paying income taxes.

   In this regard, the following section demonstrates how solvency oriented loss reserving may actually increase the expected present value of the insurer's ultimate tax payments.

IV. ON THE INCOME TAX IMPLICATIONS OF LOSS RESERVING FOR SOLVENCY

The results presented in Sections I and II assume favorable tax treatment of all established loss reserves, including solvency reserves.[12] That is, the preceding analysis assumes that taxing authorities allow a carrier to deduct from its taxable income an amount equal to its solvency reserve.

Of course, the degree to which income tax authorities accept solvency reserves as legitimate deductions will depend upon several external factors, not the least of which is the incorporation of a solvency reserve calculation into the N.A.I.C. Convention Statement. Thus, speculating on the likelihood or timing of solvency reserves becoming deductible is probably premature, and certainly beyond the scope of this report. However, two general observations can be made:

1. For reasons already presented, reserving for solvency works with or without favorable tax treatment of solvency reserves.
2. Properly applied, reserving for solvency increases long-term tax expense. This increase occurs for three reasons:

[12] See assumption 15, Appendix A.

a. As described earlier, a new or risky insurer carrying solvency reserves can be expected to generate more investment income (on additional assessments and income tax savings) than a company not opting for these reserves.

b. Eventually, a surviving insurer becomes large or mature enough to eliminate its dependence on solvency reserves. At such a time, the carrier's solvency reserves would be taken down, resulting in a sizable flow of taxable income.

c. A carrier which reserves for solvency is more likely to survive long enough to pay its income taxes.

A Net Present Value Comparison of Income Tax Liabilities

Straightforward actuarial analysis illustrates that reserving for solvency, when properly applied, actually increases the expected present value of an insurer's income tax payments.

Consider, for example, Appendix D's development of the expected present value of CARE's taxable income, discounted for both inflation and survivability, under traditional and solvency reserving. In particular, a comparison of lines 11.c and 14.c demonstrates that the expected present value of the company's income tax payments under 95/90/85/80 reserving is greater than the corresponding figure developed under standard loss reserving. In addition, comparing lines 12 and 15 establishes that the company can expect to have more investable assets at the end of ten years—a result of additional collected assessments and the investment income accruing thereon—under nonstandard loss reserving. This latter observation, of course, implies that CARE's future taxable earnings will continue to be greater under the nonstandard reserving scenario, despite the just completed takedown of all solvency reserves.

Clearly, loss reserving for solvency is anything but an effective means of avoiding income taxes.

V. LOSS RESERVING FOR SOLVENCY: A POST-MORTEM

This paper demonstrates the importance of a small casualty insurer selecting loss reserving policies which complement its capital structure.

In fact, the previous sections illustrate how a well thought out loss reserving program can energize a dormant but potentially powerful capital structure, thereby improving a carrier's chances of profitably surviving. Moreover, this increased capital effectiveness enables the carrier to reduce its reliance on other, expensive forms of capitalization, such as stop loss or quota share reinsurance.

On a more subjective level, reserving for solvency injects an element of discipline into the financial management of new, risky insurers. In particular, an actuarially sound, solvency oriented loss reserving program provides a carrier with a ready-made philosophy of maintaining specific levels of operating capital within the company. This discipline, in turn, may profoundly affect the insurer's ability to withstand the temptation to distribute to its members premature and potentially unwarranted dividends. Furthermore, for exchanges like the ones discussed in this paper, loss reserving for solvency provides the most legitimate possible justification for delaying repatriation (hence taxation) of questionable captive income.

I hope that this report will encourage a fuller examination of present and possible solvency oriented loss reserving procedures, and thereby promote the development of other applications of the concepts presented in this paper.

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APPENDIX A

SIXTEEN ASSUMPTIONS UNDERLYING CARE SOLVENCY SIMULATION (SECTION I)

1. Expected Claim Frequency / Underlying Frequency Distribution: CARE anticipates 2.5 claims per 100 insured actuaries. Also, a Poisson frequency process is assumed.

2. Expected Average Claim Size / Assumed Severity Distributions: An average claim size (limited to $1,000,000/occurrence) of approximately $56,500 is assumed. CARE also estimates that 98% of all claims will be lognormally distributed with a 3.5 coefficient of variation; the remaining 2% of losses, each above $500,000, will follow a Pareto distribution (Pareto parameter = 1.30).

3. Parameter Error: The average frequency and severity noted in the previous assumptions are subject to standard errors of 0.2 claims and $6,000, respectively.

4. Number Of Insureds: CARE anticipates 1,000 participating actuaries in each of its first five years, with 10% annual membership growth thereafter.

5. Frequency And Severity Trends: A 12% increase in CARE claim costs is assumed for the first year. Thereafter, the annual change in the claim inflation rate is assumed to be normally distributed with an average change of 0 and a 1 point standard deviation. No claim frequency trend is anticipated.

6. Assessment Collectibility: Three fourths (75%) of assessments are assumed to be collectible when due.

7. Expenses: CARE's administrative expenses and unallocated adjustment costs total 15% of premium during year 1, 12% in year 2, and 10% thereafter.

8. Policy Term And Policy Effective Date: All policies are written for one year, effective January 1.

9. Rate Level Changes: Annual premium increases of 10% occur at the end of each of years 3 through 9.

10. Ceded Reinsurance Commission: CARE receives a 7.5% commission on all reinsurance it cedes.

11. Loss Payout: A given policy year's losses are paid over five years, in 30/25/20/15/10 proportions.

12. Investment Return: Funds invested by CARE earn an average 10% return.

13. Taxation Of CARE Income: Operating income is taxed at 46%, subject to standard (3 year/7 year) carryback/carryforward provisions.

14. Improvement In Policy Year Pure Premium Estimates: At its outset, a policy year's pure premium estimate is subject to the variability described in Appendix B. These estimates improve linearly through the end of year five, at which point the pure premium is assumed to be fully known.

15. Favorable Tax Treatment Of Solvency Reserves: All reserves are treated as an offset to CARE income in the year in which they are established.

16. Surplus Requirements: Per the standards of certain offshore jurisdictions, CARE is required to maintain policyholder surplus at not less than 20% of the current year's net written premium.

APPENDIX B

APPROXIMATE PERCENTILES OF A TYPICAL FIRST YEAR CARE TOTAL LOSS DISTRIBUTION

(Based on 1,000 Simulations)

	Insurer Providing First $1 MM of E&O Insurance (Section I)	Insurer Providing $900K x/s $100K of E&O Coverage (Section II)
Percentile	Percentage of Expected Loss	Percentage of Expected Loss
25th	59%	22%
40th	79%	58%
50th	93%	79%
60th	107%	102%
70th	122%	133%
75th	131%	153%
80th	141%	175%
85th	157%	198%
90th	173%	222%
95th	197%	277%
97.5th	219%	314%

APPENDIX C

CALCULATION OF 95/90/85/80 SOLVENCY RESERVES

FIRST FOUR CALENDAR YEARS PRESENTED IN TABLE 1

(Dollar Figures in Thousands)

Valuation at End of Calendar Year	Acc. Year	Accident Yr Net Expected Losses	Selected Percentile: Per 95/90/85/80 Reserving Program	(4A) As a Percentage of Expected Losses (Per App B)	% of Accident Year Variability Remaining (App A, # 14)	A/Y Solvency Reserve As a Percent of Expected Loss {[(4B)-100]X(4C)}	Accident Year Contribution To Solvency Reserve [(3) X (4D)]
(1)	(2)	(3)	(4A)	(4B)	(4C)	(4D)	(5)
1	1	$1,058	95th	196.6%	80%	77.3%	$818
				Total Solvency Reserve for Calendar Year:			$818
				Calendar Year Change in Solvency Reserve:			$818
2	2	$1,151	95th	196.6%	80%	77.3%	$890
	1	$1,058	90th	173.2%	60%	43.9%	$464
				Total Solvency Reserve for Calendar Year:			$1,354
				Calendar Year Change in Solvency Reserve:			$536
3	3	$1,270	95th	196.6%	80%	77.3%	$982
	2	$1,151	90th	173.2%	60%	43.9%	$505
	1	$1,058	85th	157.4%	40%	23.0%	$243
				Total Solvency Reserve for Calendar Year:			$1,730
				Calendar Year Change in Solvency Reserve:			$376
4	4	$1,386	95th	196.6%	80%	77.3%	$1,071
	3	$1,270	90th	173.2%	60%	43.9%	$557
	2	$1,151	85th	157.4%	40%	23.0%	$265
	1	$1,058	80th	140.8%	20%	8.2%	$87
				Total Solvency Reserve for Calendar Year:			$1,980
				Calendar Year Change in Solvency Reserve:			$250

APPENDIX D

COMPARISON OF EXPECTED PRESENT VALUE OF CARE'S INCOME TAX PAYMENTS UNDER STANDARD AND SOLVENCY LOSS RESERVING PROCEDURES

(Dollar Figures are in Thousands)

	Standard Loss Reserving				95/90/85/80 Reserving With Solvency Reserves Taken Down at End of Year 10				
Year	Expected Net Revenues	Expected Losses & Expenses	Taxable Income [(2)−(3)]	Income Taxes Paid [46% of (4)]	Expected Net Revenues‡	Change in Solvency Reserve	Taxable Income [(6)−(7)−(3)]	Tax Loss Carry-Forward From Prior Years	Income Taxes Paid [46% of (8) +(9): 0 if neg.]
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	$1,431	$1,320	$111	$51	$1,434	$818	−$704	N/A	0
2	$1,498	$1,361	$137	$63	$1,638	$536	−$259	N/A	0
3	$1,549	$1,431	$118	$54	$1,709	$365	−$87	N/A	0
4	$1,721	$1,565	$156	$72	$1,903	$244	$94	−$43	0
5	$1,898	$1,715	$183	$84	$2,106	$184	$207	−$95	0
6	$2,279	$2,072	$207	$95	$2,516	$334	$110	−$51	0
7	$2,746	$2,506	$240	$110	$3,017	$471	$40	−$19	0
8	$3,315	$3,036	$279	$128	$3,625	$608	−$19	N/A	0
9	$4,005	$3,682	$323	$148	$4,360	$756	−$78	N/A	0
10	$4,841	$4,474	$368	$169	$5,248	−$4,316	$5,090	−$84	$2,258

SUMMARY—STANDARD LOSS RESERVING:

11. (a) Present Value (at 10%) of Expected Income Tax Payments, Years 1–10	$565
(b) Probability of CARE Surviving Through Year 10 [Table 2]	88.8%
(c) Present Value of Taxes, Discounted for Survivability [(11a) × (11b)]	$502
12. Investable Assets Available to CARE at the End of its Tenth Year	$7,036
13. Policyholder Surplus at the End of CARE's Tenth Year	$1,951

SUMMARY—95/90/85/80 RESERVING WITH RECOVERY:

14. (a) Present Value (at 10%) of Expected Income Tax Payments, Years 1–10	$871
(b) Probability of CARE Surviving Through Year 10 [Table 2]	94.8%
(c) Present Value of Taxes, Discounted for Survivability [(14a) × (14b)]	$825
15. Investable Assets Available to CARE at the End of its Tenth Year	$9,341‡‡
16. Policyholder Surplus at the End of CARE's Tenth Year	$4,257‡‡

‡ Column (6) revenues are higher than column (2) figures because investment income is higher under a 95/90/85/80 program. Investment income is higher due to reduced income taxes and interest on policyholder assessments.
‡‡ Includes $1,312,500 of collected first year assessments, plus accumulated interest thereon.

THE CALCULATION OF AGGREGATE LOSS DISTRIBUTIONS FROM CLAIM SEVERITY AND CLAIM COUNT DISTRIBUTIONS

PHILIP E. HECKMAN
GLENN G. MEYERS

Abstract

This paper discusses aggregate loss distributions from the perspective of collective risk theory. An accurate, efficient and practical algorithm is given for calculating cumulative probabilities and excess pure premiums. The input required is the claim severity and claim count distributions.

One of the main drawbacks of the collective risk model is the uncertainty of the parameters of the claim severity and claim count distributions. Modifications of the collective risk model are proposed to deal with these problems. These modifications are incorporated into the algorithm.

Examples are given illustrating the use of this algorithm. They include (1) calculating the pure premium for a policy with an aggregate limit; (2) calculating the pure premium of an aggregate stop-loss policy for group life insurance; and (3) calculating the insurance charge for a multi-line retrospective rating plan, including a line which is itself subject to an aggregate limit.

1. INTRODUCTION

This paper discusses aggregate loss distributions from the perspective of collective risk theory. Our objective is to provide an efficient algorithm for calculating the cumulative probabilities and excess pure premium ratios for aggregate loss distributions in terms of the claim severity and claim count distributions. Examples illustrating the use of this algorithm will be given.

Aggregate loss distributions are playing an increasingly important role in the pricing of insurance coverages. The insurance buying public is becoming more sophisticated and is recognizing that it is to their advantage to absorb as much of their losses as they possibly can and to purchase excess insurance to cover the catastrophic losses. With the degree of competition that exists in the insurance marketplace, it is extremely important to obtain accurate estimates of the losses that could arise from such an insurance contract.

Aggregate loss distributions have been widely discussed in the insurance literature. Members of the Casualty Actuarial Society are familiar with the papers of Dorweiler [1], Valerius [2], Simon [3] and Hewitt [4]. The aggregate loss distributions in these papers are based on observed aggregate loss data of individual insureds. A problem with this approach is that to get a sufficient volume of data, one must combine the experience of insureds for which one would expect different aggregate loss distributions.

The use of collective risk theory provides an alternative to the above approach. Aggregate loss distributions are calculated in terms of the underlying claim severity and claim count distributions. Empirical data on claim severity and claim count distributions are, in many cases, readily available. Many feel that this approach is superior to observing actual aggregate losses because it makes more efficient use of available data. Much relevant detail is buried when one observes only aggregate loss data.

However, the collective risk model does have some drawbacks. There are problems involved in fitting a distribution to the claim count. For a given insured we get one measurement of the claim count per year. During the years that we get the measurements, the exposure of the insured is most likely changing. In addition, observations are clouded by the fact that we must estimate the number of claims which have been incurred but not reported. Because of these problems it is difficult to fit a distribution to the claim count. Often, we must assume a distribution (usually Binomial, Poisson or Negative Binomial) with the parameters selected by judgment.

While empirical claim severity distributions are readily available, there are still some formidable problems that must be solved. There is no consensus as to how claim severity distributions should be adjusted for inflation. If we try to minimize this problem by choosing a relatively recent claim severity distribution, we will understate the variance of the ultimate claim severity distribution. To see this, consider the following equation.

$$\text{Var}(Z) = E_R(\text{Var}(Z|R)) + \text{Var}_R(E(Z|R))$$

Z = Ultimate Loss

R = Case Reserve

When case estimates are set at the expected value of the ultimate payment, the variance of the immature distribution will be $\text{Var}_R(E(Z|R))$. The variance of the ultimate claim severity distribution will be greater! Great care must be exercised in selecting the ultimate loss distribution. Methods of solving this problem can be found in the literature. [5][6].

Another problem with the collective risk model is that the calculation of the aggregate loss distribution has been very difficult. A great deal has been written about the various methods of solving this problem. We shall attempt to summarize these methods.

One general approach has been to calculate the moments of the aggregate loss distribution in terms of the moments of the claim severity distribution and the claim count distribution. One can then match the moments of the aggregate loss distribution with an assumed distribution. Probably the best known example of this approach is the Normal Power approximation [7]. However, the conditions required to insure the accuracy of this method can be very restrictive. Gary Venter uses the transformed Gamma distribution and obtains better results [8]. While it is easy to compute the results using these methods, one runs the risk of inaccuracies because the assumed distribution is not the same as that implied by the collective risk model.

A very popular method of calculating the aggregate loss distribution is by Monte Carlo simulation. Glenn Meyers has written an article illustrating this approach [9]. This method is easy to understand and can be very accurate. However, it currently requires a great deal of computer time.

A third method of calculating the aggregate loss distribution involves inverting its characteristic function. A recent article illustrating this approach was written by Dr. Shaw Mong [10]. This method requires that we have an explicit representation of the characteristic function of the claim severity distribution. Mong uses a shifted Gamma distribution to describe the claim severity distribution. Mong gives formulas for approximating other claim severity distributions with the shifted Gamma by matching the first three moments. The accuracy of this method depends upon how well the shifted Gamma distribution approximates the desired claim severity distribution.

A fourth method is the so-called "recursive method." This method assumes a discrete claim severity distribution. By choosing a large enough number of points for the claim severity distribution, one can obtain any desired degree of accuracy. For this reason, it has been called an "exact" method. This method requires far less computer time than Monte Carlo simulation. The recursive method is derived in papers by Ethan Stroh [11] and James Tilley [12] by inverting the Laplace transform of the aggregate loss distribution. Much of the mathematics involved is similar to that used in the characteristic function inversion method. Harry Panjer gives a derivation of the recursive method which does not involve inverting the Laplace transform [13].

The method described in this paper inverts the characteristic function of the aggregate loss distribution. Like the recursive method, it is an exact method. Its application goes beyond the recursive method in the following ways.

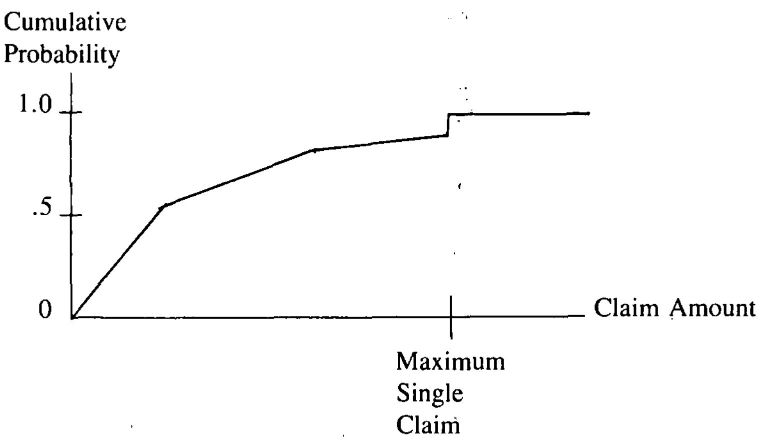
1. This method allows one to combine the aggregate loss distributions of several different lines into a composite aggregate loss distribution. This is necessary if one is to apply the results of the collective risk model to multi-line retrospective rating plans.
2. This method allows for parameter uncertainty in both the claim severity distribution and the claim count distribution. Glenn Meyers and Nathaniel Schenker have shown that allowing for parameter uncertainty significantly improves the fit of the collective risk model to empirical data [14]. It should be noted that Gary Venter's method of Reference [8] also allows for parameter uncertainty.
3. Philip Heckman and Phillip Norton have used the results of this paper to derive a method of selecting the specific and aggregate policy limits that minimize the variance of the retained losses while holding the cost of coverage constant [15].

In short, this method is applicable to a wide variety of insurance pricing problems. We include several examples which illustrate this.

The input required for this algorithm will be the claim count distribution and the claim severity distribution for each exposure class covered by the insurance contract. The claim count distribution can be either Binomial, Poisson or Negative Binomial. The cumulative claim severity distribution is assumed to be piecewise linear. We also allow the highest possible claim amount to occur with some non-zero probability. Figure 1 shows a cumulative distribution function that might typically be considered. Since most claim severity distributions applicable to the insurance business can be approximated to any desired degree of accuracy by a piecewise linear cumulative distribution, we feel we have a completely general method of performing these calculations.

It should be noted that these calculations will require a computer. With the nearly universal availability of computers, we do not consider this a drawback. We will warn the reader that the calculations are very complex, but, at the risk of being repetitious, we will stress the underlying concepts at every opportunity. This method is far more efficient than the more easily understood process of Monte Carlo simulation. Having fulfilled our duty to warn the reader, let us now proceed.

FIGURE 1



2. SUMMARY

We begin by giving a full description of the aggregate loss model. We will show how this distribution can be expressed empirically in terms of Monte Carlo simulation or analytically in terms of convolutions.

After reviewing some basic properties of complex numbers, we will introduce the characteristic function of a probability distribution. One of the remarkable properties of this complex-valued function is that the characteristic function of the convolution of two probability distributions is equal to the product of the characteristic functions of the two individual probability distributions. It is this property of characteristic functions that makes this method work. It is easier to multiply characteristic functions than it is to calculate convolutions by Monte Carlo simulation.

The next section will express the characteristic function of the aggregate loss distribution in terms of the claim count distribution and the characteristic function of the claim severity distribution. We will then derive formulas for the cumulative probabilities and the excess pure premiums for the aggregate loss distribution in terms of its characteristic function. These formulas involve improper integrals which can be evaluated using a Gaussian quadrature formula.

We then provide an analysis of the errors made in numerically evaluating the improper integrals. In some cases, the aggregate loss distribution is known. We test this algorithm by comparing the calculated results with known results. We also provide a comparison of the calculated results with results produced by Monte Carlo simulation.

Four examples illustrating the use of this algorithm will be given: (1) calculating the pure premium of a policy with aggregate limits; (2) calculating the pure premium of an aggregate stop-loss policy for group life insurance; (3) çalculating the insurance charge for a retrospective rating plan involving two policies, one of which is subject to an aggregate limit; and (4) an example similar to (3) except that there is parameter uncertainty for the claim severity distribution.

3. THE COLLECTIVE RISK MODEL

Collective risk theory started by considering the generalized Poisson distribution. However, it soon became apparent that the assumptions of this distribution are violated for many applications. In this section we will discuss the assumptions of the generalized Poisson distribution and indicate some common violations of these assumptions. We will then state a version of the collective risk model that can deal with certain violations of these assumptions.

We start by considering the Poisson distribution. The assumptions underlying this distribution are as follows [16].

1. Claims occurring in two disjoint time intervals are independent.
2. The expected number of claims in a time interval (t_1, t_2) is dependent only on the length of the interval and not on the initial value t_1.
3. No more than one claim can occur at a given time.

There are situations when these assumptions are violated. We give three examples.

1. *Positive Contagion*
   A manufacturer can be held liable for defects in its products which, in many cases, are mass produced. A successful claim against the manufacturer may result in several other claims against the manufacturer. The notion that a higher than expected number of claims in an earlier period can increase the expected number of claims in a future period is what is called positive contagion.

2. *Negative Contagion*
   Consider a group life insurance policy. A death in an earlier period will reduce the expected number of deaths in a later period. One does not die twice. The notion that a higher than expected number of claims in an earlier period can decrease the expected number of claims in a future period is called negative contagion.

3. *Parameter Uncertainty*
   There are many cases when one feels that a Poisson distribution is appropriate, but one does not know the expected number of claims. Two options are available under these circumstances. The first option is to estimate the expected number of claims using historical experience. Parameter uncertainty can come from sample error. A second option is to use the average number of claims for a group of insureds that are similar to the insured under consideration. Parameter uncertainty arises if each member of the group expects to have a different number of claims.

   The effect of parameter uncertainty is similar to that of positive contagion. We give a heuristic argument for this which appeals to modern credibility theory. Suppose one observes n claims during a certain time period. Then one can estimate the number of claims, x, in a future period of equal length using the following formula.

   $$x = Z \cdot n + (1 - Z) \cdot E$$

   where E = Prior estimate
   Z = Credibility factor.

   Note that if the estimate of the expected number of claims is precise or the group of insureds is homogeneous, the credibility factor will be 0.

   If n is greater than expected, the number of claims expected in the future will be greater than the prior estimate for non-zero values of credibility.

   It should be emphasized that we are not arguing that claims in an earlier period will cause claims in a later period, as in the positive contagion case. We are stating only that the claim count distributions observed under the conditions of parameter uncertainty and positive contagion should be similar.

We now turn to specifying the claim count distributions we shall use for each of the above situations. We shall adopt the following notation.

n - A random variable denoting the number of claims
λ - The expected number of claims ($\lambda = E(n)$)
χ - A random variable with $E(\chi) = 1$ and Var $(\chi) = c$

Parameter uncertainty can be modeled by the following algorithm.

Algorithm 3.1

1. Select χ at random from the assumed distribution.
2. Select the number of claims, n, at random from a Poisson distribution with parameter $\chi\lambda$.

We have the following relationships.

$$E(n) = E(n|\chi) \cdot E(\chi) = \lambda \tag{3.1}$$

$$\begin{aligned} \text{Var}\,(n) &= E_\chi(\text{Var}\,(n|\chi)) + \text{Var}_\chi\,(E(n|\chi)) \\ &= E_\chi(\chi\lambda) + \text{Var}_\chi\,(\chi\lambda) \\ &= \lambda + c\lambda^2. \end{aligned} \tag{3.2}$$

If χ has a Gamma distribution, the claim count distribution described by Algorithm 3.1 is the Negative Binomial distribution [17]. We shall use the Negative Binomial distribution to model both the positive contagion and the parameter uncertainty cases.

We shall call the paramter c the contagion parameter for the claim count distribution. We should note that c has also been called the contamination parameter by some authors [18]. It should be noted that if $c = 0$, Algorithm 3.1 yields the Poisson distribution.

We shall use the Binomial distribution to model the negative contagion case. If m is the number of trials and p is the probability of success, we can formally define the contagion parameter to be equal to $-1/m$. Substituting this into Equation 3.2 yields the correct Binomial variance.

$$\text{Var}(n) = \lambda - \lambda^2/m = mp - m^2p^2/m = mp(1 - p)$$

While a negative contagion parameter makes no sense in terms of Algorithm 3.1, we shall see below that this is a very appropriate definition.

We now adopt the following notation.

z—A random variable denoting the amount of a claim
$S(z)$—The cumulative distribution function of z
x—A random variable denoting the aggregate loss for an insured

Aggregate losses can be generated by the following algorithm.

Algorithm 3.2

1. Select the number of claims, n, at random from the assumed claim count distribution.
2. Do the following n times.
   2.1 Select the claim amount, z, at random from the assumed claim severity distribution.
3. The aggregate loss amount, x, is the sum of all claim amounts, z, selected in step 2.1.

Let $F(x)$ denote the cumulative distribution function for the aggregate losses generated by Algorithm 3.2. We now give expressions for the mean and the variance of this distribution.

$$E(x) = E(n) \cdot E(z) = \lambda \cdot E(z) \tag{3.3}$$

$$\begin{aligned} \text{Var}(x) &= E_n(\text{Var}(x|n)) + \text{Var}_n(E(x|n)) \\ &= E_n(n \cdot \text{Var}(z)) + \text{Var}_n(n \cdot E(z)) \\ &= \lambda \cdot \text{Var}(z) + (\lambda + c\lambda^2) \cdot E^2(z) \\ &= \lambda \cdot E(z^2) + c\lambda^2 \cdot E^2(z) \end{aligned} \tag{3.4}$$

Implicit in the use of Algorithm 3.2 is the assumption that the claim severity distribution, $S(z)$, is known. In practice this distribution must be estimated from historical observations, or it must be simply assumed. Parameter uncertainty of the claim severity distribution can significantly affect the predictions of the collective risk model, and it should not be ignored. Our response to this problem is to make the simplifying (and we think reasonable) assumption that the shape of the distribution is known but there is uncertainty in the scale of the distribution.

More precisely, we specify parameter uncertainty of the claim severity distribution in the following manner. Let β be a random variable satisfying the conditions $E(1/\beta) = 1$ and $\text{Var}(1/\beta) = b$. We then model aggregate losses by the following algorithm.

Algorithm 3.3

1. Select the number of claims, n, at random from the assumed claim count distribution.
2. Select the scaling parameter, β, at random from the assumed distribution.

3. Do the following n times.
   3.1 Select the claim amount, z, at random from the assumed claim severity distribution.
4. The aggregate loss amount, x, is the sum of all claim amounts, z, divided by the scaling parameter, β.

Let $\mathcal{F}(x)$ denote the cumulative distribution function for the aggregate losses generated by Algorithm 3.3. Let $U(\beta)$ be the cumulative distribution function for the scaling parameter, β. Then the relationship between $\mathcal{F}(x)$ and $F(x)$ is given by the following equation.

$$\mathcal{F}(x) = \int_0^\infty F(\beta x) dU(\beta) \tag{3.5}$$

We now give formulas for the mean and the variance for the aggregate losses generated by Algorithm 3.3.

$$\begin{aligned} E(x) &= E_\beta(E(x|\beta)) \\ &= E_\beta(\lambda \cdot E(z)/\beta) \\ &= \lambda \cdot E(z) \cdot E(1/\beta) \\ &= \lambda \cdot E(z) \end{aligned} \tag{3.6}$$

$$\begin{aligned} \text{Var}(x) &= E_\beta(\text{Var}(x|\beta)) + \text{Var}_\beta(E(x|\beta)) \\ &= E_\beta[(\lambda \cdot E(z^2) + c\lambda^2 \cdot E^2(z))/\beta^2] + \text{Var}_\beta(\lambda \cdot E(z)/\beta) \\ &= (\lambda \cdot E(z^2) + c\lambda^2 \cdot E^2(z)) \cdot E(1/\beta^2) + \lambda^2 \cdot E^2(z) \cdot \text{Var}(1/\beta) \\ &= (\lambda \cdot E(z^2) + c\lambda^2 \cdot E^2(z)) \cdot (1+b) + \lambda^2 \cdot E^2(z) \cdot b \\ &= \lambda \cdot E(z^2)(1+b) + \lambda^2 \cdot E^2(z) \cdot (b+c+bc) \end{aligned} \tag{3.7}$$

In this paper, we shall assume that β has a Gamma distribution. We shall call b the mixing parameter. The mixing parameter is a measure of parameter uncertainty for the claim severity distribution.

It should be noted that we have chosen mathematically convenient distributions to model contagion and parameter uncertainty. We do not want to imply that these distributions are in any way the "correct" ones. Since parameter uncertainty is not directly observable, it is difficult to discover what the proper distribution should be. It should be noted that it is possible to infer the variance of the parameter uncertainty through the use of Equations 3.4 and 3.7 [14]. But until statistical methodology has advanced to the point where the proper distribution can be determined, it should be acceptable to use ones which are mathematically convenient.

4. CONVOLUTIONS

The above discussion provides a complete description of the aggregate loss model we use in this paper. Algorithms 3.2 and 3.3 provide the means to calculate the cumulative distribution by Monte Carlo simulation. Unfortunately this is a long and expensive process. We now begin to develop the mathematical tools necessary to derive a more efficient process.

Initially we will be concerned with the cumulative distribution function $F(x)$ which is described by Algorithm 3.2. We will then make use of Equation 3.5 to derive the cumulative distribution function $\mathcal{F}(x)$ described by Algorithm 3.3.

Let x be a random variable which has a distribution function $F(x)$. Similarly, let y be a random variable which has distribution function $G(y)$. Let $z = x + y$. Then the convolution of F and G, denoted by $F * G$ is the distribution function for z. We can express this analytically by the equation

$$(F * G)(z) = \int_0^z F(z-y)dG(y).$$

Let $S(z)$ be a claim severity distribution. Define

$$S^{0^*}(z) = \begin{cases} 0 \text{ if } z = 0 \\ 1 \text{ if } z > 0 \end{cases}$$

$$S^{n^*}(z) = (S^{(n-1)^*} * S)(z).$$

One can see that $S^{n^*}(z)$ is the distribution of the total amount of exactly n claims.

Algorithm 3.2 can be expressed in the following manner.

Algorithm 4.1

1. Select the claim count, n, at random.
2. Select the aggregate loss amount, x, from the distribution S^{n^*}.

We now give an analytical expression for this process. Let $F(x)$ denote the distribution function for the aggregate loss distribution. Let $P(n)$ denote the probability of exactly n claims. We then have

$$F(x) = \sum_{n=0}^{\infty} P(n) \cdot S^{n^*}(x). \tag{4.1}$$

5. CHARACTERISTIC FUNCTIONS

It may be helpful at this point to review some properties of complex numbers. A complex number, z, is one which can be written in the form

$$z = a + bi, \tag{5.1}$$

where a and b are real numbers and $i = \sqrt{-1}$. The number a is called the real part of z and b is called the imaginary part of z. Alternatively, z can be written in the form

$$z = re^{i\theta}, \tag{5.2}$$

where r is a nonnegative real number and θ is any real number; r is called the modulus of z, and θ is called the argument of z.

The equivalence of Equations 5.1 and 5.2 can be seen by using Euler's formula.

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta) \tag{5.3}$$

Using this formula it is not difficult to show that the following relationships hold.

$$r = \sqrt{a^2 + b^2} \tag{5.4}$$

$$\theta = \begin{cases} \arctan(b/a) \text{ if } a > 0 \\ \pi + \arctan(b/a) \text{ if } a < 0 \text{ and } b \geq 0 \\ \arctan(b/a) - \pi \text{ if } a < 0 \text{ and } b \leq 0 \\ \pi/2 \text{ if } a = 0 \text{ and } b > 0 \\ -\pi/2 \text{ if } a = 0 \text{ and } b < 0 \end{cases} \tag{5.5}$$

$$a = r\cos(\theta) \tag{5.6}$$

$$b = r\sin(\theta) \tag{5.7}$$

Having given a brief discussion of complex numbers, we define the characteristic function (or Fourier transform) of a cumulative distribution function F.

$$\phi_F(t) = E(e^{itx}) = \int_0^\infty e^{itx} dF(x) \tag{5.8}$$

Let F and G be two cumulative distribution functions.

$$\phi_{(F*G)}(t) = E(e^{itz}) = \int_0^\infty e^{itz} d(F * G) .$$

Since z is the sum of x and y, and x and y are independent, we have

$$\phi_{(F*G)}(t) = E_z(e^{itz}) = E_{x,y}(e^{it(x+y)}) = E_x(e^{itx})\, E_y(e^{ity}) = \phi_F(t)\phi_G(t).$$

Thus we have proved the following well known result.

$$\phi_{(F*G)}(t) = \phi_F(t)\, \phi_G(t) \tag{5.9}$$

As a consequence of this equation we have the following.

$$\phi_{S^{n*}}(t) = (\phi_S(t))^n \tag{5.10}$$

Combining Equations 4.1 and 5.10 we get the following expression for the characteristic function of an aggregate loss distribution, F.

$$\phi_F(t) = \sum_{n=0}^{\infty} P(n)(\phi_S(t))^n \tag{5.11}$$

As stated above, we assume that the claim severity distribution is piecewise linear. We now specify the mathematical form of the claim severity distribution, $S(z)$.

1. Let n be a nonnegative integer.
2. Let $0 \leq a_1 < \ldots < a_n < a_{n+1}$.
3. Let p_k denote the probability that an individual loss is between a_k and a_{k+1}.
4. For $a_k < z < a_{k+1}$, the probability density of z is given by $d_k = p_k/(a_{k+1} - a_k)$.
5. The probability that a claim is equal to a_{n+1} is given by

   $$1 - \sum_{k=1}^{n} p_k.$$

   This allows us to describe the accumulation of claim values at the policy limit (a_{n+1}).

We now calculate the characteristic function of $S(z)$.

$$\phi_S(t) = \int_0^{\infty} e^{itz} dS(z)$$

$$\phi_S(t) = \sum_{k=1}^{n} \int_{a_k}^{a_{k-1}} d_k \cdot e^{itz} dz + \left(1 - \sum_{k=1}^{n} p_k\right) e^{ita_{n-1}}$$

Using Euler's formula (Equation 5.3) we continue.

$$\phi_S(t) = \frac{1}{t}\sum_{k=1}^{n} d_k(\sin(ta_{k+1}) - \sin(ta_k)) + \left(1 - \sum_{k=1}^{n} p_k\right)\cos(ta_{n+1})$$

$$+\, i\,\frac{1}{t}\sum_{k=1}^{n} d_k(\cos(ta_k) - \cos(ta_{k+1})) + i\left(1 - \sum_{k=1}^{n} p_k\right)\sin(ta_{n+1})$$

Let $\tilde{h}(t)$ and $\tilde{k}(t)$ denote the real and imaginary parts of $\phi_S(t)$ respectively.

$$\tilde{h}(t) = \frac{1}{t}\sum_{k=1}^{n} d_k(\sin(ta_{k+1}) - \sin(ta_k)) + \left(1 - \sum_{k=1}^{n} p_k\right)\cos(ta_{n+1}) \quad (5.12)$$

$$\tilde{k}(t) = \frac{1}{t}\sum_{k=1}^{n} d_k(\cos(ta_k) - \cos(ta_{k+1})) + \left(1 - \sum_{k=1}^{n} p_k\right)\sin(ta_{n+1}) \quad (5.13)$$

We now turn to the problem of calculating the characteristic function of the aggregate loss distribution. Our main tool will be Equation 5.11.

Case 1 Binomial Distribution $\mathrm{P(n)} = \binom{\mathrm{m}}{\mathrm{n}}\mathrm{p^n}\,(1-\mathrm{p})^{\mathrm{m-n}}$

$$\phi_F(t) = \sum_{n=0}^{m}\binom{m}{n} p^n(1-p)^{m-n}(\phi_S(t))^n$$

$$\phi_F(t) = \sum_{n=0}^{m}\binom{m}{n}(p\phi_S(t))^n \cdot (1-p)^{m-n}$$

$$\phi_F(t) = (p\phi_S(t)+1-p)^m$$

$$\phi_F(t) = (1 + p(\phi_S(t)-1))^m$$

If we make a change of notation and let $\lambda = mp$ and $c = -1/m$, we get

$$\phi_F(t) = (1 - c\lambda(\phi_S(t) - 1))^{-1/c}. \quad (5.14)$$

Case 2 Poisson Distribution $P(n) = \frac{e^{-\lambda}\lambda^n}{n!}$

$$\phi_F(t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} (\phi_S(t))^n$$

$$\phi_F(t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}(\lambda \cdot \phi_S(t))^n}{n!}$$

$$\phi_F(t) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda \cdot \phi_S(t))^n}{n!}$$

$$\phi_F(t) = e^{-\lambda} e^{\lambda \cdot \phi_S(t)}$$

$$\phi_F(t) = e^{\lambda \cdot (\phi_S(t) - 1)} \tag{5.15}$$

Case 3 Negative Binomial Distribution

$$P(n) = \binom{n + 1/c - 1}{n} (1 + c\lambda)^{-1/c} \left(\frac{c\lambda}{1 + c\lambda}\right)^n$$

$$\phi_F(t) = \sum_{n=0}^{\infty} \binom{n + 1/c - 1}{n} (1 + c\lambda)^{-1/c} \left(\frac{c\lambda}{1 + c\lambda}\right)^n (\phi_S(t))^n$$

$$\phi_F(t) = \sum_{n=0}^{\infty} \binom{n + 1/c - 1}{n} (1 + c\lambda)^{-1/c} \left(\frac{c\lambda\phi_S(t)}{1 + c\lambda}\right)^n$$

$$\phi_F(t) = \sum_{n=0}^{\infty} \binom{n + 1/c - 1}{n} (1 + c\lambda)^{-1/c} \left(\frac{cx}{1 + cx}\right)^n$$

where $x = \frac{\phi_S(t)}{1 - c\lambda(\phi_S(t) - 1)}$

$$\phi_F(t) = (1 + c\lambda)^{-1/c} (1 + cx)^{1/c}$$

$$\phi_F(t) = (1 - c\lambda(\phi_S(t) - 1))^{-1/c} \tag{5.16}$$

Note that Equations 5.14 and 5.16 are identical except for the different interpretation of the contagion parameter c. It should also be noted that the expression in Equations 5.14 and 5.16 approaches the expression in Equation 5.15 as c approaches 0.

In the computer program described below, we set $c = 10^{-7}$ whenever $|c| < 10^{-7}$. Thus the same computer code handles all three cases.

6. THE AGGREGATE LOSS DISTRIBUTION

In the preceding section we derived the characteristic function for the aggregate loss distribution for a single coverage or exposure class. In this section we use the above results to derive formulas for the cumulative probabilities and the excess pure premiums for multiple coverages or exposure classes.

For the sake of convenience, we make the following definitions.

$F(x)$ = Cumulative distribution function of the aggregate losses for all coverages combined

μ = Mean of aggregate loss distribution

σ = Standard deviation of aggregate loss distribution

$f(t)$ = modulus $(\phi_F(t/\sigma))$

$g(t)$ = argument $(\phi_F(t/\sigma))$

For each coverage, j, we define the following.

$$h_j(t) = \bar{h}_j(t/\sigma) - 1 \tag{6.1}$$

$$k_j(t) = \bar{k}_j(t/\sigma) \tag{6.2}$$

where $\bar{h}_j$ and $\bar{k}_j$ are given in Equations 5.12 and 5.13.

Note the $F(x)$ is the convolution of the aggregate loss distributions for each individual coverage. Using Equations 5.4, 5.5, 5.9 and 5.12–5.16 we have the following.

$$f(t) = \prod_j \text{modulus} \quad (1 - c_j\lambda_j(\phi_{S_j}(t/\sigma) - 1))^{-1/c_j}$$

$$f(t) = \prod_j \text{modulus} \quad (1 - c_j\lambda_j(h_j(t) + ik_j(t)))^{-1/c_j}$$

$$f(t) = \prod_j ((1 - c_j\lambda_j h_j(t))^2 + (c_j\lambda_j k_j(t))^2)^{-1/2c_j} \tag{6.3}$$

$$g(t) = \sum_j \text{argument} \quad (1 - c_j\lambda_j(\phi_{S_j}(t/\sigma) - 1))^{-1/c_j}$$

$$g(t) = \sum_j \text{argument} \quad (1 - c_j\lambda_j(h_j(t) + ik_j(t)))^{-1/c_j} \tag{6.4}$$

Once the modulus and the argument of the aggregate characteristic have been determined, it is possible to calculate the cumulative probabilities by use of the following formula.

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{f(t)}{t} \sin\,(tx/\sigma - g(t))\,dt \tag{6.5}$$

The excess pure premium can be obtained from the cumulative distribution function by the following formula.

$$EP(x) = \int_x^\infty (u - x)dF(u)$$

Applying this to Equation 6.5 we get the following formula.

$$EP(x) = \mu - \frac{x}{2} + \frac{\sigma}{\pi}\int_0^\infty \frac{f(t)}{t^2}\,(\cos\,(g(t)) - \cos\,(tx/\sigma - g(t)))dt \tag{6.6}$$

The excess pure premium ratio is defined by the following formula.

$$ER(x) = EP(x)/\mu$$

We now introduce parameter uncertainty of the severity distributions.

$$\mathcal{F}(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{f(t)}{t}\left(1 + \left(\frac{xt}{r\sigma}\right)^2\right)^{-(1+r)/2} \sin\left((1 + r)\arctan\left(\frac{xt}{r\sigma}\right) - g(t)\right) dt \tag{6.7}$$

$$\mathcal{EP}(x) = \mu - \frac{x}{2} + \frac{\sigma}{\pi}\int_0^\infty \frac{f(t)}{t^2}\left[\cos\,(g(t)) - \left(1 + \left(\frac{xt}{r\sigma}\right)^2\right)^{-r/2} \cos\left(r \cdot \arctan\left(\frac{xt}{r\sigma}\right) - g(t)\right)\right] dt \tag{6.8}$$

In the above two formulas, $r = 1 + 1/b$.

Equations 6.5–6.8 are derived in Appendix A. It should be noted that Equation 6.5 is the limit of Equation 6.7 as b approaches 0. Similarly Equation 6.6 is the limit of Equation 6.8 as b approaches 0. In our program we set $b = 10^{-7}$ whenever $b < 10^{-7}$ and thus the same computer code handles both situations.

Equations 6.7 and 6.8 are set up so that the parameter uncertainty of claim severity affects all coverages in the same way. This may be realistic if one believes that uncertainty in claim severity is due to inflation and that inflation affects all coverages in the same way. If one wants parameter uncertainty of claim severity for each coverage to be independent, several runs of the program will be required. An example showing how to do this will be given below.

7. NUMERICAL INTEGRATION

We now turn to the problem of evaluating the integrals given in Equations 6.7 and 6.8. It should also be noted that our program is written in FORTRAN to run on a large (IBM 370) computer. In this environment, it gets nearly instantaneous response at the computer terminal. The same algorithm has also been coded in BASIC to run on a TRS80 Model III microcomputer where it reproduces the mainframe results though with substantially greater running time. The actual FORTRAN code is included as Exhibit IX.

We now outline our algorithm. Explanation for the steps will be given below.

Step

1. Enter the parameters for the claim severity and the claim count distributions.
2. Calculate the aggregate mean, μ, and standard deviation, σ.
3. Enter the loss amounts, x.
4. Calculate basic interval length, h.
   $h = 2\pi\sigma$ /(maximum loss amount)
5. In order to apply the Gaussian quadrature formulas, we must evaluate the integrands at specified points. We evaluate the functions $f(t)$ and $g(t)$ at the appropriate points in each of the following intervals.

Interval Number	Interval
1	$(0, h/16)$
2	$(h/16, h/8)$
3	$(h/8, h/4)$
4	$(h/4, h/2)$
5	$(h/2, h)$
6	$(h, 2h)$
.	
.	
.	
$j + 4$	$((j - 1)h, jh)$

j is determined so that $f(t)t < .00002$ for all values of t evaluated in the interval $((j - 1)h, jh)$.

6. For each loss amount, x, evaluate $\mathcal{F}(x)$ and $\mathcal{EP}(x)$ by summing the results of the Gaussian quadrature formulas over each of the intervals given in Step 5.

We now give a more detailed explanation of the above steps.

Step 1

The parameters for each claim severity distribution are the claim severities $a_1, \ldots, a_{n+1}$ and the associated probabilities $p_1, \ldots, p_n$.

The parameters for each claim count distribution are the expected number of claims and the contagion parameter, c. Note that if $|c| < 10^{-7}$, we substitute $c = 10^{-7}$.

We must also enter the mixing parameter, b. If $b < 10^{-7}$ we substitute $b = 10^{-7}$.

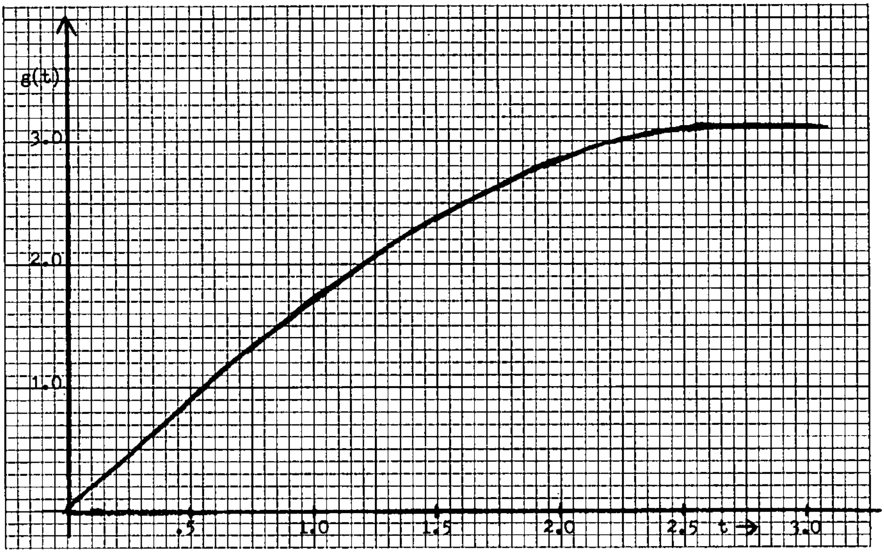
Step 2

For each coverage we calculate the aggregate mean and variance according to Equations 3.6 and 3.7. The aggregate mean and variance are the sums of the individual means and variances for each coverage.

Step 4

Evaluating a typical $g(t)$ showed that $g(t)$ changes slowly. See Figure 2. Also, $r \cdot \arctan(xt/r\sigma)$ is an increasing function of t which is bounded by xt/σ. Thus by choosing $h = 2\pi\sigma/(\text{maximum loss amount})$ we assure that the interval of integration will contain no more than one oscillation of the integrand.

FIGURE 2



Step 5

The evaluation of $f(t)$ and $g(t)$ is the most time consuming operation of this entire algorithm. Thus $f(t)$ and $g(t)$ should only be evaluated once for any given value of t, and the number of points, t, at which these functions are evaluated should be as few as possible. Inspection of the integrands of Equations 6.7 and 6.8 revealed that they changed most rapidly in the interval $(0, h)$. See Figures 3 and 4. Thus it was felt that the intervals used in the numerical integration should be relatively short in the interval $(0, h)$.

By a change of variables, each interval of integration was transformed from the given interval to the interval $(-1, 1)$. The Gaussian 5-point formula is then applied. The points, t_j, where $f(t)$ and $g(t)$ must be evaluated are as follows.

$$t_1 = (-0.90617985\ (b - a) + (b + a))\ /2$$
$$t_2 = (-0.53846931\ (b - a) + (b + a))\ /2$$
$$t_3 = (b + a)\ /2$$
$$t_4 = (0.53846931\ (b - a) + (b + a))\ /2$$
$$t_5 = (0.90617985\ (b - a) + (b + a))\ /2$$

Here a is the left endpoint of the interval, and b is the right endpoint of the interval under consideration. If $f(t_j)\ /\ t_j < .00002$ for $j = 1, \ldots, 5$ or the number of intervals equals 256, no more intervals are used.

FIGURE 3

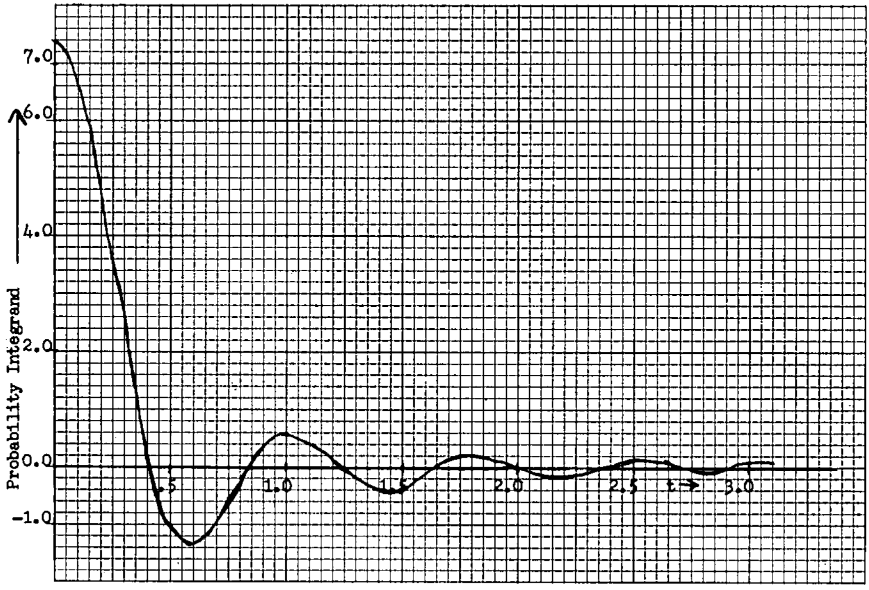
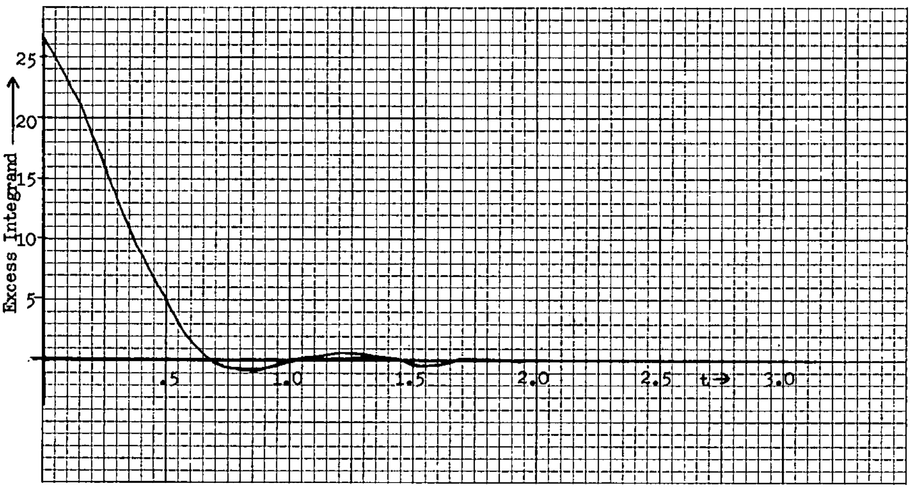


FIGURE 4



Step 6

Now that $f(t)$ and $g(t)$ are evaluated and stored in an array, it becomes an easy task to evaluate $\mathcal{F}(x)$ and $\mathcal{EP}(x)$. For each interval of integration we use the following rule to evaluate the integral.

$$I_p = \left(\sum_{j=1}^{5} W_j P\,(t_j) \right) \cdot \text{(interval length/2)}$$

$$I_E = \left(\sum_{j=1}^{5} W_j Q\,(t_j) \right) \cdot \text{(interval length/2)}$$

$$\text{where } P(t) = \frac{f(t)}{t} \left(1 + \left(\frac{xt}{r\sigma} \right)^2 \right)^{-(1+r)/2} \sin \left((1 + r) \arctan \left(\frac{xt}{r\sigma} \right) - g(t) \right)$$

$$Q(t) = \frac{f(t)}{t^2} \left[(\cos\,(g(t)) - \left(1 + \left(\frac{xt}{r\sigma} \right)^2 \right)^{-r/2} \cos \left(r \cdot \arctan \left(\frac{xt}{r\sigma} \right) - g(t) \right) \right]$$

$W_1 = W_5 = 0.23692689$
$W_2 = W_4 = 0.47862867$
$W_3 = 0.56888889.$

Then $\mathcal{F}(x) = .5 + (\text{Sum of all the } I_p\text{'s}) / \pi$ and
$\mathcal{EP}(x) = \mu - x/2 + (\text{Sum of all the } I_E\text{'s}) \, \sigma/\pi.$

8. ERROR ANALYSIS

There are three sources of error in the above calculations.

Roundoff Error

We use double precision arithmetic at every stage of our calculation. Double precision numbers are accurate to 16 significant digits on IBM equipment. Even though the calculations leading to a particular output value could number in the hundreds, it is doubtful that accumulated roundoff error could be an important factor in our calculations.

Discretization Error

The discretization error for the Gaussian 5-point formula is given by the expression

$$\frac{f^{(10)}(\xi)}{11 \cdot 7^3 \cdot 5^2 \cdot 3^8 \cdot 2} = 8.08 \times 10^{-10} \cdot f^{(10)}(\xi),\ \xi \text{ in } (-1,1).$$

Since the integrands are reasonably smooth (see Figures 3 and 4) the bound on $f^{(10)}$ should be reasonable. Thus the discretization error should not be significant.

Truncation Error

The most significant source of error in these calculations is the truncation error, or the error made by substituting an integral with finite limits of integration for an integral with infinite limits of integration. We now turn to analyzing this truncation error.

The truncation error, E_T, for the excess pure premium is given by

$$E_T = \frac{\sigma}{\pi} \int_a^\infty \frac{f(t)}{t^2} \Bigg[\cos(g(t)) - \left(1 + \left(\frac{xt}{r\sigma}\right)^2\right)^{-r/2} \cos\left(r \cdot \arctan\left(\frac{xt}{r\sigma}\right) - g(t)\right)\Bigg] dt$$

where a is the limit of the finite integral.

$$\text{Now } |E_T| \le \frac{\sigma}{\pi} \int_a^\infty \frac{f(t)}{t^2}(1 + 1)dt$$

$$\le \frac{2\sigma}{\pi} \cdot \max_{t \ge a}(f(t)) \int_a^\infty \frac{dt}{t^2}$$

$$= \frac{2\sigma}{\pi} \cdot \max_{t \ge a}(f(t)) \cdot \frac{1}{a}. \tag{8.1}$$

$$\text{Now } f(t) = \left| \int_0^\infty e^{itx} dF(x) \right| \le \int_0^\infty |e^{itx}|\, dF(x) = 1. \tag{8.2}$$

$$\text{Thus } E_T \le \frac{2\sigma}{\pi} \cdot \frac{1}{a}.$$

The bound on the truncation error given by Equation 8.2 is extremely conservative because, as we show in Appendix B, $\max_{t \geq a} f(t)$ will be significantly less than one for most cases of interest. In fact, if each (piecewise linear) claim severity distribution function is continuous, $f(t)$ approaches the probability of zero claims as t approaches infinity. For example, when the claim count distribution is Poisson with a mean of 10 claims, $f(t)$ will be close to e^{-10} or 0.0000454 for large t.

The bound on the truncation error given by Equation 8.1 is also conservative because the integrand repeatedly changes sign.

In our program, a is usually chosen so that $\max_{t \geq a} (f(t)) \cdot 1/a < .00002$. Thus we would expect the truncation error for the excess pure premium ratio to be bounded by $.000013 \cdot \sigma/\mu$.

The truncation error for the cumulative probabilities does not permit an analysis similar to the above because the denominator of the integrand contains t instead of t^2. The examples in the next section will show that cumulative probabilities calculated by this algorithm seem to be accurate. But they are somewhat less accurate than the excess pure premium.

9. NUMERICAL TESTS OF THE ALGORITHM

There are cases when the algorithm can be compared with known results. We consider two such cases.

If the contagion parameter, c, is equal to -1, then $\phi_F(x) = \phi_S(x)$. The choice of $c = -1$ corresponds to the Binomial distribution with $m = p = 1$.

For our first example, consider the following.

$$F(x) = S(x) = x \text{ for } 0 \leq x \leq 1$$

$$ER(x) = \frac{1}{\mu} \int_x^1 (1 - F(u))du = (1 - x)^2$$

Table 9.1 compares computed to actual results.

TABLE 9.1

x	$F(x)$ Actual	$F(x)$ Computed	$ER(x)$ Actual	$ER(x)$ Computed
.10	.1000	.1000	.8100	.8100
.20	.2000	.2000	.6400	.6400
.30	.3000	.3000	.4900	.4900
.40	.4000	.4000	.3600	.3600
.50	.5000	.5000	.2500	.2500
.60	.6000	.6000	.1600	.1600
.70	.7000	.7000	.0900	.0900
.80	.8000	.8000	.0400	.0400
.90	.9000	.9000	.0100	.0100
1.00	1.0000	.9995	.0000	.0000

For our next example, consider the following.

$F(x) = S(x) = x/2$ for $0 \leq x < 1$
$F(1) = S(1) = 1$

$$ER(x) = \frac{1}{\mu} \int_x^1 (1 - F(u))du = (3 - x)(1 - x)/3$$

For reasons described in Section 7 above, the program required 256 intervals for the numerical integration. The value of $f(t)/t$ for the largest value of t was equal to .001. Using Equation 8.2 we obtained an estimate of .00027 as a bound on the truncation error for $ER(x)$. Table 9.2 compares computed to actual results.

These examples would seem to indicate that the calculation of $ER(x)$ is more accurate than that of $F(x)$. If $F(x)$ is continuous, the error appears to be small, but, if $F(x)$ is not continuous, the errors may not be so small near the points of discontinuity.

We now turn to a more realistic example. Exhibit II shows an actual run of our program. Details concerning the input will be given in the discussion of aggregate increased limits factors which follows. Here we provide a comparison between the results of our program and a Monte Carlo simulation. One should not expect exact agreement between expected and observed results due to

TABLE 9.2

x	$F(x)$ Actual	$F(x)$ Computed	$ER(x)$ Actual	$ER(x)$ Computed
.10	.0500	.0501	.8700	.8700
.20	.1000	.1001	.7467	.7467
.30	.1500	.1502	.6300	.6300
.40	.2000	.2002	.5200	.5200
.50	.2500	.2502	.4167	.4167
.60	.3000	.3003	.3200	.3200
.70	.3500	.3504	.2300	.2300
.80	.4000	.4005	.1467	.1467
.90	.4500	.4511	.0700	.0700
.99	.4950	.4869	.0067	.0067
1.00	1.0000	.7499	.0000	.0001
1.01	1.0000	1.0081	.0000	.0000
1.05	1.0000	.9979	.0000	.0000

simulation error. For this reason we performed a Chi-Square test on the results to see if the difference could be explained by random fluctuations. The results are in Table 9.3.

The expected number of claims in each cell was obtained from Exhibit II. The observed number of claims in each cell was obtained by a Monte Carlo simulation using exactly the same input parameters as those in Exhibit II. Ten thousand trials were used.

If the differences between observed and expected values are due solely to random fluctuations, one should expect a Chi-Square value of 25. In this case we get a slightly higher value of Chi-Square. We have performed similar tests on many occasions and have gotten similar results. The algorithm works.

10. AGGREGATE LIMITS

We now consider how this algorithm can be used to calculate the premium for a policy that is subject to an aggregate limit.

Underwriters have long felt that lines of insurance such as Products Liability and Medical Malpractice present a severe catastrophe potential. For example,

TABLE 9.3

CHI-SQUARE TEST FOR AGGREGATE LOSS DISTRIBUTIONS

Upper Cell Boundary	Observed	Expected
50,000	51	52
100,000	273	268
150,000	432	435
200,000	546	540
250,000	589	587
300,000	632	628
350,000	736	737
400,000	782	782
450,000	789	769
500,000	721	720
550,000	641	662
600,000	625	622
650,000	597	561
700,000	506	491
750,000	402	416
800,000	353	349
850,000	269	294
900,000	227	241
950,000	201	195
1,000,000	135	154
1,050,000	93	121
1,100,000	102	94
1,150,000	73	73
1,200,000	46	55
1,250,000	39	42
Over 1,250,000	140	112

Chi-Square = 26.0
Degrees of Freedom = 25

the publicity given a Products Liability lawsuit may well provoke several additional lawsuits by others who have purchased the same product. Thus underwriters have justifiably sought to limit the total amount of losses that can be paid out under a single policy.

The price for a policy with an aggregate limit (ignoring expense considerations) will be the price of a similar policy with no aggregate limit less the excess pure premium for the aggregate limit. Below, we will give several examples of such calculations using Exhibits II to V. But, before we do this, let us discuss the input parameters.

The claim severity distribution chosen is typical for Products Liability coverages. We will not discuss selection of the claim severity distribution here. Instead we will refer the interested reader to the literature [19] [20].

The claim severity distribution will be subject to a $250,000 occurrence limit.

The mean of the claim count distribution was calculated by dividing total expected losses by the severity mean ($18,198). In Exhibits II, IV and V a contagion parameter of zero was chosen. This choice gives the Poisson distribution. In Exhibit III we chose a contagion parameter of .25. In light of the catastrophe potential for Products Liability that was discussed above, a more highly skewed claim count distribution would indeed seem justified.

A mixing parameter of 0 is used in this example.

Tables 10.1 and 10.2 show the discounts expressed as a proportion of the total expected loss.

While a more highly skewed claim count distribution may be justified for Products Liability, it does not give a conservative price for a policy with an aggregate limit. Thus we would recommend using a Poisson distribution for the claim count unless one has definite evidence that a more skewed distribution is appropriate.

Notice that the discounts depend upon the expected loss. Present tables of increased limits factors do not reflect this dependence. We admit that there is a practical problem involved in publishing increased limits factors that vary by expected loss. However, the "practical" solution of not considering the expected loss can produce embarrassing examples such as the following. This method is identical to that given in I.S.O. rating manuals.

TABLE 10.1

DISCOUNTS FOR AGGREGATE LIMITS

Expected Loss = $500,000

Aggregate Limit	Contagion Parameter 0.00	Contagion Parameter 0.25
$ 600,000	.1394	.2132
800,000	.0516	.1125
1,000,000	.0165	.0570
1,200,000	.0046	.0279
1,400,000	.0012	.0133

TABLE 10.2

DISCOUNTS FOR AGGREGATE LIMITS

Contagion Parameter = 0.0

Aggregate Limit	Expected Loss $250,000	Expected Loss $500,000	Expected Loss $1,000,000
$ 600,000	.0296	.1394	.4202
$ 800,000	.0060	.0516	.2665
$1,000,000	.0010	.0165	.1528
$1,200,000	.0002	.0046	.0791
$1,400,000	—	.0012	.0371

Basic Limits — $25,000 per occurrence and $75,000 aggregate
Base Rate — $1.00 per unit of exposure
Exposure — 1,000,000 units

If an insured bought a policy for the basic limits, he would pay $1,000,000 and the most he could recover in losses is $75,000! While it is unlikely that such a policy has ever been sold, significant errors could be quite common.

We propose the following as a remedy to this situation.

1. Publish increased limits tables for occurrence limits only.
2. Do not give discounts for aggregate limits. Instead, publish a table of aggregate limits which are appropriate for a given expected loss. The aggregate limits should be sufficiently high so that the indicated discount is less than a nominal amount, say 0.5%.

Using Exhibits II, IV and V we can derive the appropriate aggregate limits.

Expected Loss	Aggregate Limit
$ 250,000	$ 825,000
500,000	1,200,000
1,000,000	1,900,000

11. GROUP LIFE AGGREGATE EXCESS INSURANCE

We now give the solution to a problem that was proposed to us by a life actuary of our company.

A large employer wanted to self insure his group life insurance. To protect against a catastrophe, he wanted to purchase aggregate excess insurance to cover losses in excess of 1.25 times the expected loss. The following data were provided to us.

Group	Age Range	Number of Lives	Expected Loss
1	29 and Under	2,073	47,086
2	30–34	1,135	36,342
3	35–39	1,044	35,380
4	40–44	822	54,938
5	45–49	1,004	136,126
6	50–54	1,193	270,050
7	55–59	975	395,471
8	60–64	546	258,525
9	65 and Over	25	13,247

The expected loss was computed using a mortality table and the average amount of insurance in each group.

It was felt that the claim count distribution should be binomial. Thus we chose a contagion parameter of $-1/(\text{number of lives})$ for each group. We were not given a distribution of insurance amounts for each group. Assuming that all insureds had the average amount of insurance in each group would understate the excess pure premium. For this reason we requested rough estimates for those distributions.

The mixing parameter selected was 0.0.

It should be noted that the assumptions of the collective risk model are violated in this example. When a person dies, the amount of his insurance policy is removed from the claim severity distribution. However the turnover of group members should keep the claim severity distribution approximately the same. Thus we feel that the collective risk model will be a good approximation of the true situation.

Exhibit VI gives the computer run for the problem. The pure premium for this coverage was calculated to be 1.53% of the expected loss.

12. RETROSPECTIVE RATING; NESTED AGGREGATES

A retrospective rating plan is a rating plan in which the final premium is determined after the policy period has expired [21]. While these plans have many features, we will limit this discussion to plans where the insurer is liable for all losses above an agreed upon amount.

Retrospective rating plans can cover several different policies under a single plan. Here we provide a simple example showing how to calculate the pure premium, or insurance charge, for such a rating plan. Our example will consist of two coverages, Workers' Compensation and Products Liability.

The Workers' Compensation policy has an expected loss of $500,000. The claim severity distribution is given in Exhibit I. The contagion parameter is .05. The mixing parameter is 0.0.

The Products Liability policy has an expected loss of $500,000 before application of the aggregate limit. The claim severity distribution is given in Exhibit I. The contagion parameter is .25. The policy that is written under the retrospective rating plan is subject to a $1,000,000 aggregate limit. The mixing parameter is 0.0.

The presence of a policy subject to an aggregate limit in the retrospective rating plan makes it necessary to run the program twice to determine the insurance charges. Exhibit III will serve as the first run of the program. For the second run we treat the Workers' Compensation parameters in the usual manner. For the Products Liability, we substitute the aggregate loss distribution in Exhibit

III for the claim severity distribution. We, of course, limit the aggregate losses to $1,000,000. The contagion parameter is -1. This corresponds to a binomial claim count distribution with $m = p = 1$. The results of the second run are shown in Exhibit VII. It can be seen, for example, that the insurance charge for a plan which covers losses in excess of $1,500,000 is $21,894.

We now consider parameter uncertainty for the scale of the claim severity distribution.

Misestimation of the claim severity distribution can occur because of limited information on the individual coverage. In this case one would expect the scale uncertainty for each coverage to be independent. Misestimation of future inflation can also cause scale uncertainty. In this case one could expect the scale uncertainty to affect each coverage in the same way. The following example shows how to handle both of these cases. It will be necessary to run the program once for each individual coverage. A final run is then required to combine the individual coverages.

The Workers' Compensation policy has the same parameters that were specified in the above example with the exception that the mixing parameter is set equal to .05. This reflects uncertainty in the scale of the claim severity distribution for Workers' Compensation. The aggregate loss distribution for this coverage is given in Exhibit VIIIA.

The Products Liability policy has the same parameters that were specified in the above example with the exception that the mixing parameter is set equal to .05. This reflects uncertainty in the scale of the claim severity distribution for Products Liability. The aggregate loss distribution is given in Exhibit VIIIB. It should be noted that this aggregate loss model adjusts the policy limit with the scaling parameter, while in the real world the policy limit remains fixed. However this should not significantly affect the final results.

The aggregate loss distributions for Workers' Compensation and Products Liability are then combined to get the aggregate loss distribution for the total losses of the two coverages. Here the aggregate loss distribution for each coverage is treated as the claim severity distribution for the final run of the program. The Products Liability loss is limited to $1,000,000. The contagion parameter for each coverage is set equal to -1. The mixing parameter is set equal to .05. This reflects uncertainty in the scale of the aggregate loss distribution. For a given year the scale parameter is identical for both coverages. It should be noted that, as we do above, this aggregate loss model adjusts the aggregate limit with the scaling factor.

The results of the third run are shown in Exhibit VIIIC. It can be seen, for example, that the insurance charge for a plan which covers losses in excess of $1,500,000 is $46,424. Here one can see that that parameter uncertainty can significantly affect the required insurance charge.

13. CONCLUSION

We have described an efficient and accurate algorithm which calculates the cumulative probabilities and excess pure premiums for the collective risk model.

The program and related programs have been used at our company in applications described above and many others. These include the analysis of profit sharing plans, large account pricing, aggregate deductibles and sliding scale dividend plans. Also, we are currently exploring applications involving the optimization of reinsurance retentions [15] and designing a retrospective rating plan which properly accounts for the "overlap" problem [22]. In short, this is a very useful program.

Our exposure to these applications has led us to believe that further work needs to be done with the collective risk model. In particular, we need to test the predictions of the collective risk model against actual aggregate loss data. We also need to test the sensitivity of the collective risk model to violations of the assumptions underlying it.

14. ACKNOWLEDGEMENTS

This paper had its origins in an anlysis of Shaw Mong's paper [10] which was done by Glenn Meyers and Nathaniel Schenker. Comparisons of Mong's results with Monte Carlo simulations suggested that Mong's technique worked well when the claim severity distribution was a Gamma distribution, but otherwise it worked poorly. It was also noted that Mong's technique could be modified to work for any claim severity distribution provided one could calculate its characteristic function.

The classic book on risk theory by Beard, Pentikäinen and Pesonen [7] had a very strong influence in our thinking, as can be noted by our several references to it. We regard this paper, in part, as a synthesis of the ideas in Mong's paper and Chapter 8 of the book.

Another strong influence has been observing the various ways this algorithm, and prior Monte Carlo simulations, have been used. We received excellent feedback from the following individuals: Burton Covitz, Michael Larsen, John Meeks, Arthur Placek, and Philip Wolf. The following individuals made several helpful comments while we were preparing this paper: Bradley Alpert, Yakov Avichai, Sam Gutterman, Phillip Norton, Nathaniel Schenker, and Edward Seligman. We offer our sincere thanks.

APPENDIX A
DERIVATION OF EQUATIONS 6.5–6.8

The purpose of this appendix is to derive Equations 6.5–6.8. We will first derive expressions for the cumulative probability and the excess pure premium in terms of $\phi_F(t)$ and $U(\beta)$. Equations 6.5–6.8 will then be special cases of these expressions.

The following formula is given by Kendall and Stuart [23].

$$F(\beta x) = \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{i\beta xt}\cdot\phi_F(-t) - e^{-i\beta xt}\cdot\phi_F(t)}{it}\,dt$$

From Equation 3.5 we have the following.

$$\begin{aligned}\mathcal{F}(x) &= \int_0^\infty F(\beta x)dU(\beta)\\ &= \int_0^\infty \left[\frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{i\beta xt}\,\phi_F(-t) - e^{-i\beta xt}\,\phi_F(t)}{it}dt\right] dU(\beta)\\ &= \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{1}{it}\left[\phi_F(-t)\int_0^\infty e^{i\beta xt}dU(\beta) - \phi_F(t)\int_0^\infty e^{-i\beta xt}dU(\beta)\right] dt\\ &= \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{1}{it}\left[\phi_F(-t)\cdot\phi_U(xt) - \phi_F(t)\cdot\phi_U(-xt)\right] dt \qquad \text{(A.1)}\end{aligned}$$

Thus we have the following.

$$\begin{aligned}\mathcal{E}\mathcal{P}(x) &= \int_x^\infty (v - x)d\mathcal{F}(v) = \int_x^\infty \left[\int_x^v du\right] d\mathcal{F}(v)\\ &= \int_x^\infty \left[\int_u^\infty d\mathcal{F}(v)\right] du = \int_x^\infty (1 - \mathcal{F}(u))du\\ &= \int_0^\infty (1 - \mathcal{F}(v))dv - \int_0^x (1 - \mathcal{F}(v))dv\end{aligned}$$

$$= \mu - \int_0^x (1 - \mathcal{F}(v))dv$$

$$= \mu - \frac{x}{2} - \int_0^\infty \frac{1}{2\pi}\left\{\int_0^\infty \frac{\phi_F(-t)\cdot\phi_U(vt) - \phi_F(t)\cdot\phi_U(-vt)}{it}\,dt\right\} dv$$

$$= \mu - \frac{x}{2} + \frac{1}{2\pi}\int_0^\infty \frac{1}{it}\left[\phi_F(-t)\int_0^x \phi_U(vt)dv - \right.$$

$$\left.\phi_F(t)\int_0^x \phi_U(-vt)dv\right] dt \tag{A.2}$$

Note: $\phi(t) = f(t)e^{ig(t)}$; $f(t) = f(-t)$ and $g(-t) = -g(t)$.

Case 1 $U(\beta) = 0$ for $\beta < 1$ and $U(\beta) = 1$ for $\beta \geq 1$.

$$\phi_U(t) = e^{it}$$
$$\phi_U(xt) = e^{ixt} \tag{A.3}$$
$$\phi_U(-xt) = e^{-ixt} \tag{A.4}$$

$$\int_0^x \phi_U(vt)dv = \frac{e^{ixt} - 1}{it} \tag{A.5}$$

$$\int_0^x \phi_U(-vt)dv = \frac{1 - e^{-ixt}}{it} \tag{A.6}$$

Equation 6.5 is obtained by substituting Equations A.3 and A.4 into Equation A.1. and replacing t with t/σ. Equation 6.6 is obtained by substituting Equations A.5 and A.6 into A.2 and replacing t with t/σ.

Case 2 $dU(\beta) = \dfrac{r}{\Gamma(r + 1)}(r\beta)^r e^{-r\beta}d\beta$

We first show that $U(\beta)$ satisfies the conditions stated for Algorithm 3.3.

$$E(1/\beta) = \frac{r}{\Gamma(r + 1)}\int_0^\infty \frac{1}{\beta}(r\beta)^r e^{-r\beta}d\beta$$

$$= \frac{r}{\Gamma(r)}\int_0^\infty (r\beta)^{r-1}e^{-r\beta}d\beta$$

$$= 1$$

$$E(1/\beta^2) = \frac{r}{\Gamma(r+1)} \int_0^\infty \frac{1}{\beta^2} (r\beta)^r e^{-r\beta} d\beta$$

$$= \frac{r}{r-1} \frac{r}{\Gamma(r-1)} \int_0^\infty (r\beta)^{r-2} e^{-r\beta} d\beta$$

$$= \frac{r}{r-1}$$

If $r = 1 + 1/b$ we have that $\text{Var}(1/\beta) = (r/(r-1)) - 1 = b$.

$$\phi_U(t) = \left(1 - \frac{it}{r}\right)^{-(1+r)}$$

$$\phi_U(xt) = \left(1 - \frac{ixt}{r}\right)^{-(1+r)} = \left(1 + \left(\frac{xt}{r}\right)^2\right)^{-(1+r)/2} e^{i(1+r)\arctan(xt/r)} \tag{A.7}$$

$$\phi_U(-xt) = \left(1 + \frac{ixt}{r}\right)^{-(1+r)} = \left(1 + \left(\frac{xt}{r}\right)^2\right)^{-(1+r)/2} e^{-i(1+r)\arctan(xt/r)} \tag{A.8}$$

$$\int_0^x \phi_U(vt)dv = \frac{1}{it}\left[\left(1 - \frac{ixt}{r}\right)^{-r} - 1\right]$$

$$= \frac{1}{it}\left[\left(1 - \left(\frac{xt}{r}\right)^2\right)^{-r/2} e^{ir\arctan(xt/r)} - 1\right] \tag{A.9}$$

$$\int_0^x \phi_U(-vt)dv = \frac{1}{it}\left[1 - \left(1 + \frac{ixt}{r}\right)^{-r}\right]$$

$$= \frac{1}{it}\left[1 - \left(1 + \left(\frac{xt}{r}\right)^2\right)^{-r/2} e^{-ir\arctan(xt/r)}\right] \tag{A.10}$$

Equation 6.7 is obtained by substituting Equations A.7 and A.8 into Equation A.1 and replacing t with t/σ. Equation 6.8 is obtained by substituting Equations A.9 and A.10 into A.2 and replacing t with t/σ.

APPENDIX B
ASYMPTOTIC BEHAVIOR OF $f(t)$

In the error analysis of Section 8 we indicated that $\max_{t \geq a} f(t)$ could be significantly less than one for large a. We now give a demonstration of this fact. It will be sufficient to consider a single coverage or class of business.

We will adopt the following notation for use in this appendix.

$$D = 1 - \sum_{k-1}^{n} P_k$$

$$A = a_{n+1}$$

As $t \to \infty$, we have the following.

$$h(t) \to D \cos (At)$$
$$k(t) \to D \sin (At)$$
$$\phi_S(t) \to D(\cos (At) + i \sin (At))$$

Case 1 Binomial Distribution

$$\phi_F(t) = [1 + p\,(\phi_S(t) - 1)]^m$$

As $t \to \infty$, we have the following.

$$f(t) \to [(1 - p + D\,p \cos (At))^2 + (D\,p \sin (At))^2]^{m/2}$$
$$f(t) \to [(1 - p)^2 + 2\,D\,p \cos (At) + (D\,p)^2]^{m/2}$$

If $D = 0$, $f(t) \to (1 - p)^m$ which is equal to the probability of having no claims. If $D > 0$, $f(t)$ does not approach a limit, but the asymptotic upper bound of $f(t)$ can be obtained by setting $\cos (At) = 1$.

$$\max_{t \geq a} f(t) \to [(1 - p)^2 + 2\,D\,p + (D\,p)^2]^{m/2}$$

As an example, consider the case when $m = 100$, $p = .1$ and $D = .1$:

$$\max_{t \geq a} f(t) \to .0000905.$$

Case 2 Poisson Distribution

$$\phi_F(t) = e^{\lambda(\phi_S(t)-1)}$$

As $t \to \infty$, $f(t) \to e^{-\lambda} \cdot e^{\lambda D \cos(At)}$. If $D = 0$, $f(t) \to e^{-\lambda}$, which is equal to the probability of having no claims. If $D > 0$, $f(t)$ does not approach a limit, but the asymptotic upper bound of $f(t)$ can be obtained by setting $\cos(At) = 1$.

$$\max_{t \geq a} f(t) \to e^{-\lambda(1-D)}$$

As an example, consider the case when $\lambda = 10$ and $D = .1$:

$$\max_{t \geq a} f(t) \to e^{-9} = .0001234.$$

Case 3 Negative Binomial Distribution

$$\phi_F(t) = [1 - c\lambda(\phi_S(t) - 1)]^{-1/c}$$

As $t \to \infty$, we have the following.

$$f(t) \to [(1 + c\lambda + c\lambda\, D \cos(At))^2 + (c\lambda\, D \sin(At))^2]^{-1/2c}$$
$$f(t) \to [(1 + c\lambda)^2 + 2(1 + c\lambda) \cdot c\lambda\, D \cos(At) + (c\lambda\, D)^2]^{-1/2c}$$

If $D = 0$, $f(t) \to (1 + c\lambda)^{-1/c}$, which is the probability of having no claims. If $D > 0$, $f(t)$ does not approach a limit, but the asymptotic upper bound of $f(t)$ can be obtained by setting $\cos(At) = 1$.

$$\max_{t \geq a} f(t) \to [(1 + c\lambda)^2 - 2(1 + c\lambda)(c\lambda D) + (c\lambda D)^2]^{-1/2c}$$

As an example, consider the case when $\lambda = 10$, $D = .1$ and $c = .1$:

$$\max_{t \geq a} f(t) \to .001631.$$

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NOTE

The exhibits associated with the paper "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions" by Philip E. Heckman and Glenn G. Meyers (*PCAS* LXX, 1983) appear in the subsequent volume of the *Proceedings* (*PCAS* LXXI, 1984).

DISCUSSION BY GARY VENTER

Background

Aggregate losses are easily defined as the sum of individual claims, but the distribution of aggregate losses has not been easy to calculate. In fact, this has been a central, and perhaps *the* central, problem of collective risk theory. The mean of the aggregate loss distribution can be calculated as the product of the means of the underlying frequency and severity distributions; similarly, there are well known formulas for the higher moments of the aggregate distribution in terms of the corresponding frequency and severity moments (e.g., see [5] Appendix C). However the aggregate distribution function, and thus the all important excess pure premium ratio, has been awkward to calculate from the distribution functions of frequency and severity. It is this calculation problem that is addressed and solved in this important paper. The result is generalized somewhat to the case where the severity distribution is known only up to a scale multiplicative factor, which itself follows a specific distribution (inverse gamma). In this review the approach in the paper is abstracted somewhat in an attempt to focus on the areas where the specific assumptions come into play.

Principal Idea

The derivation of the results involves complex mathematics, but the results themselves and the ideas behind the derivation can be easily understood. It is not necessary to know what a characteristic function or a convolution or a complex number is to understand these basic ideas and to use the results. The following properties of the characteristic function are germane to this understanding.

1) It is a transformation of the distribution function.

2) It has an inverse transformation; i.e., the distribution function can be calculated from the characteristic function.

3) The characteristic function of aggregate losses can be calculated from the moment generating function of frequency and the characteristic function of severity.

The basic idea, then, is to calculate the characteristic function of severity and the moment generating function of frequency; use them to compute the characteristic function of aggregate losses; and, use that to calculate the distribution function of aggregate losses and the excess pure premium ratios.

None of the above is actually new to risk theory or even to North American casualty actuaries. What is new and is the heart of this paper's contribution centers around a snag in the above method: the characteristic function of severity is not directly calculable from the distribution function in most cases. The gamma severity is an exception and Mong presented its use to the CAS in this context in the 1980 call paper program. The authors point out that the characteristic function is also calculable when severity is piecewise linear, and the solution they present is for this case. They then assert that any severity distribution needed in property-casualty practice can be closely approximated by a piecewise linear form, which seems reasonable, and thus that this method is completely general. This summarizes the basic ideas of the derivation.

Results

The results can be expressed fairly simply without reference to complex numbers. The formulas below are essentially those derived in the paper, although generalized slightly in that they hold for any severity random variable S, not just one that is piecewise linear, and for binomial or negative binomial frequency with parameters c and λ, defined below. Mong's paper and others have also presented very similar general formulas. As usual, E denotes the expected value.

Result 1: F, the aggregate distribution function, can be expressed as

$$F(x) = \tfrac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin\,(g(t) + tx)\,dt}{t\,f(t)}$$

where
$$f(t) = [(1 + c\lambda - c\lambda E(\cos tS))^2 + (c\lambda E(\sin tS))^2][1/2c]$$
$$g(t) = (-1/c)\arctan\,[E(\sin tS)/((1/c\,\lambda) - E(\cos tS))].$$

Result 2: The expected losses excess of retention x, $EP(x)$, can be calculated as

$$EP(x) = \mu - (x/2) + (1/\pi)\int_0^\infty (1/f(t)t^2)\,(\cos\,(g(t)) - \cos\,(tx + g(t)))dt$$

where μ is the expected aggregate losses.

This is a very nice formula in that 1) aggregate excess losses can be computed without computing the loss probabilities; 2) the integral converges well before infinity because of the t^2 term; and 3) its error structure can be analyzed.

Following Mong, the authors transform the integrals by a change of variables $t \rightarrow t/\sigma$. It is not clear that this is necessary or even useful.

Note that the authors use the negative binomial in the form

$$\Pr(Y = y) = \binom{y + 1/c - 1}{y} (1 + c\lambda)^{-1/c} \left(\frac{c\lambda}{1 + c\lambda}\right)^y.$$

This has mean λ and ratio of the variance to the mean of $1 + c\lambda$. Taking $p = 1/(1 + c\lambda)$ and $\alpha = 1/c$ gives the more usual form

$$\Pr(Y = y) = \binom{\alpha + y - 1}{y} p^{\alpha} (1 - p)^y.$$

Formulas for $E(\cos tS)$ and $E(\sin tS)$ (denoted by the authors as $\tilde{h}(t)$ and $\tilde{k}(t)$ respectively) for piecewise linear S are found as formulas 5.12 and 5.13 of the paper. This is where the piecewise linear assumption is used. Mong's results can be obtained by substituting the corresponding formulas for the gamma severity, namely $E(\cos tS) = (\cos(r \arctan t/a))/(1 + t^2/a^2)^{r/2}$ and $E(\sin tS) = (\sin(r \arctan t/a))/(1 + t^2/a^2)^{r/2}$, where r and a are gamma parameters defined by $E(S) = r/a$ and $\mathrm{Var}(S) = r/a^2$.

It would also be possible to evaluate $E(\cos tS)$ and $E(\sin tS)$ for a discrete severity distribution function S and apply the above formula. Another possibility, which might turn out to be a useful alternative, would be to approximate the severity probability density function by a piecewise linear form, rather than doing so for the cumulative distribution function.

To develop the formulas for the needed trigonometric expectations in this case, suppose the severity density $g(s)$ between two points a_i and a_{i+1} is given by $g(s) = c_i + sd_i$, and there is a probability p of a claim of the largest size a_{n+1}. Then the following formulas can be readily derived using integration by parts.

$$E(\cos tS) = \frac{1}{t} \sum_{i=1}^{n} \left((c_i + sd_i) \sin ts + (d_i/t) \cos ts\right) \Big|_{a_i}^{a_{i+1}} + p \cos ta_{n+1}$$

$$E(\sin tS) = \frac{1}{t} \sum_{i=1}^{n} \left((c_i + sd_i) \cos ts - (d_i/t) \sin ts\right) \Big|_{a_{i+1}}^{a_i} + p \sin ta_{n+1}$$

Note also that for the probabilities to total 1.0,

$$p = 1 - \sum_{i=1}^{n} (a_{i+1} - a_i)(c_i + d_i (a_{i+1} + a_i)/2).$$

For discrete severity distributions, $E(\cos tS)$ and $E(\sin tS)$ can also be directly calculated. For most severity distributions, these expected values can be calculated numerically. In fact, approximating the severity distribution by a piecewise linear function can be regarded as a numerical approximation of $E(\cos tS)$ and $E(\sin tS)$. Other approximation methods are also possible. As this is the only use made of the piecewise linear severity assumption, it can be seen that this assumption is not an essential constraint of the method but rather a convenient numerical device.

In other words, the above formulas for $F(x)$ and $EP(x)$ hold for any severity distribution, S, not just piecewise linear. Since $E(\sin tS)$ and $E(\cos tS)$ need to be calculated for many t's in order to evaluate the integrals, a method is needed to calculate these trigonometric expectations. Any number of numerical integration techniques could be used for the purpose. The point of view of this paper is that approximating the density function of S by a step function provides a simple method for the calculation of $E(\sin tS)$ and $E(\cos tS)$ which is of sufficient accuracy for the end results.

Subsequent discussion with the authors uncovered that this has been supported by further empirical tests which began by approximating a smooth density (e.g. Weibull) by a step function, calculating $F(x)$ and $EP(x)$, and then refining the approximation. It was found that 20 to 25 approximating intervals provided a high degree of accuracy in this process. Thus the characteristic function method can be applied readily to any severity distribution.

Although the formulas above use functions that have not been commonly employed in casualty actuarial practice, their calculation is straightforward. The integrands themselves can be computed on many hand calculators. Carrying out the integration requires numerical methods. The authors adopt a brute force approach, and it gets the job done. More efficient methods may be possible, but a fair amount of expertise in numerical integration would be needed to determine if this were so.

Details of the Method

The formula for the characteristic function of aggregate losses in terms of the frequency moment generating function and the severity characteristic function is $\phi_x(t) = M_n(\ln \phi_s(t))$. This is readily derived from formula 5.11 of the paper. Formulas 5.14 to 5.16 follow directly from this result and the formulas for the moment generating functions of the binomial, Poisson, and negative binomial distributions. In fact, the proofs of those formulas given are essentially derivations of the corresponding moment generating functions.

The derivations of the above general formulas for $F(x)$ and $EP(x)$ are straightforward applications of the inversion formula to the aggregate characteristic function. The inversion formula is the standard procedure for getting the distribution function from the characteristic function and can be found in advanced statistical texts.

Also, the issue of discontinuities in the distribution function deserves further attention. This inversion formula for calculating the distribution function from the characteristic function is not exact at points of discontinuity. This is easy to miss in Kendall and Stuart, which is cited as the source of the inversion formula. Because this has not been taken into account, the above formula for $F(x)$ as well as the paper's formula are incorrect at the discontinuity points. The error is an understatement of the distribution function equal to one half of the jump at those points. This would be an important issue, for example, if a discrete severity were used with the formulas above. In that case the aggregate distribution would also be discrete, and thus its distribution function would be a step function. To evaluate this function at a discontinuity point, then, it would suffice to evaluate it just above the discontinuity, in fact at any point before the next discontinuity.

These errors can also be computed from the underlying distributions. In the case the authors treat most often, namely a severity distribution with a censorship point (e.g., per occurrence limit), the aggregate distribution function is discontinuous, with jumps at n times the censorship point ($n = 0,1,2,\ldots$) equal to the probability of having exactly n claims all of which are total losses (i.e., equal the censorship point). These probabilities can be computed from the frequency and severity distribution function and then the aggregate can be adjusted by half the jump at those points. As an alternative, evaluating at slightly above the discontinuity should give a reasonable approximation. The example in Table 9.2 of the paper illustrates this at $x = 1.00$, where the error is 25%.

In examples given in Exhibits II–VIII, these adjustments would probably not be significant. If, however, the expected number of claims is small (e.g., 5,1,.02) and/or the probability at the censorship point is large, the error at the discontinuity may be significant. In excess insurance/reinsurance applications both these conditions often hold. However, as discussed below under recursive computation, the characteristic function method may not be the most efficient in such applications in any case.

Parameter Uncertainty

The parameter uncertainty issue is an important one and is well considered in the paper. For large individual risks or for insurance companies, this uncertainty can far outweigh the variation that can occur from randomness within known frequency and severity distributions. For example, parameter uncertainty can arise from severity trend and development. Although these may also affect the shape of the severity distribution, they have definite effects on its scale. The authors treat the situation in which the severity distribution is known up to a scale multiplier which is itself inverse gamma distributed. (Actually, they present this as a divisor which is gamma distributed.) The gamma is selected because it leads to tractable results. Note that applying a scale multiplier to severity is equivalent to applying the same multiplier to aggregate losses. This is not true for frequency, as increasing the number of claims changes the shape of the aggregate distribution. This is reflected in the standard formulas for the coefficients of variation and skewness of aggregate losses (e.g., [5], Appendix C).

The derivation in Appendix A of the paper shows that the gamma assumption for a scale is not absolutely required. What is required is a method of calculating the characteristic function of this divisor. This characteristic function can then be plugged into the formulas A1 and A2 to yield expressions for the aggregate distribution function and the excess pure premium, respectively. In fact, the derivations labelled "case 1" and "case 2" do exactly that for the degenerate and gamma divisors, respectively.

Estimating the parameters for the mixing distribution is a problem. The mean can be selected to give the proper severity mean. The variance is more difficult to arrive at. A study of historical errors in trend and development projections could be useful in this regard. The variance of accident year or policy year loss ratios for a large segment of the industry, where process variance can be assumed minimal, should also be a viable approach. The authors seem to suggest comparing the observed variance in loss ratios with the theoretical variance that would occur without parameter risk in order to estimate the degree of parameter risk. This also seems to be a potentially useful approach.

The inverse gamma distribution, i.e., the distribution of X where $1/X$ is gamma distributed, has density $f(x) = \beta\, e^{-1/x\beta} \div \Gamma(r)\, (\beta x)^{1+r}$. This is a fairly dangerous probability distribution, more so than the gamma, in that only finitely many moments exist. In fact $E(X^n) = \Gamma(r - n) \div \beta^n \Gamma(r)$ exists if and only if $n < r$. It is an open question whether or not this will prove appropriate for a mixing distribution.

Besides trend and development factors, parameter uncertainty also arises from risk classification. For computing the aggregate loss distribution of a large and diverse portfolio of risks, this may not be an important factor. However, for a single risk or a carrier specializing in a few classes, this could be an essential consideration. If the risk is not typical of the classification or the class rate is based on insufficient data, the dispersion of possible results will be greater than frequency and severity considerations might suggest. Historical errors in trend and development will also understate the parameter risk in this case.

The parameter uncertainty approach discussed by Bühlmann [2] and developed further by Patrik and John [4] can also be used with the characteristic function method. Bühlmann allowed all parameters of the distributions to have uncertainty and introduced a probability function, called the structure function, to describe the relative weights given to different parameter sets. If the structure function is approximated by a finite number of points, the distribution function of aggregate losses can be calculated for each parameter set by the authors' method and then weighted together by the structure function. This gives a quite general method of dealing with parameter uncertainty.

Recursive Computation of Aggregate Functions

Another method of computing the aggregate distribution function was recently developed by Panjer [3] generalizing Adelson [1]. It is interesting to compare this to the current paper.

Panjer's method involves a recursive formula for $F(x)$ based on discrete severity distributions. For his formula the severity probability function must be given at every multiple of some unit value up to the largest possible loss size, for example $g(1) = .5$, $g(2) = .3$, $g(3) = .1$, $g(4) = .05$, $g(5) = .05$, where g is the severity probability function, 10,000 is the unit, and 50,000 is thus the largest possible loss. In this case the aggregate losses will also come in multiples of the unit. If we now let f denote the aggregate probability, Panjer's formula is

$$f(x) = \sum_{i=1}^{x} (a + b\, i/x)\, g(i)\, f(x - i),$$

where a and b come from the frequency distribution. This formula is valid for binomial, negative binomial, and Poisson frequencies. For the negative binomial

$$\Pr(Y = y) = \binom{\alpha + y - 1}{y} p^{\alpha} (1 - p)^y,$$

$a = 1 - p$ and $b = (\alpha - 1)(1 - p)$. For the Poisson $a = 0$, $b = \lambda$, and for the binomial $\Pr(Y = y) = \binom{m}{y} P^y (1 - p)^{m-y}$, $a = p/p - 1$ and $b = (m + 1)p/1 - p$.

As an example take the above g in units of 10,000 with Poisson $\lambda = 1$. Then $f(x) = \Sigma_{i=1}^{x}\, i\, g(i)\, f(x - 1)/x$.

Now $f(0) = \Pr(N = 0) = e^{-1}$. Thus $f(1) = .5e^{-1}$, $f(2) = .5\, f(1)/2 + .3\, f(0) = .425e^{-1}$, $f(3) = .5\, f(2)/3 + .2\, f(1) + .1\, f(0) = .8125\, e^{-1}/3$, etc.

Thus the aggregate distribution function can be built up by quite simple arithmetic operations using this method.

The excess pure premium can be derived from the aggregate probabilities. The definition in discrete terms is $EP(x) = \Sigma_{i=x}^{\infty}\, (i = x)\, f(i)$. Calculating this requires $f(i)$ for the largest possible i's whereas the recursive procedure builds up from the smallest. But since $\mu = \Sigma_{i=0}^{\infty}\, i\, f(i)$ is known from frequency and severity, if it were possible to calculate $\mu - EP(x)$ then $EP(x)$ would fall out.

$$\text{Now } \mu - EP(x = \sum_{i=0}^{\infty} i\, f(i) - \sum_{i=x}^{\infty} i\, f(i) + x \sum_{i=x}^{\infty} f(i)$$

$$= \sum_{i=0}^{x-1} i\, f(i) + x(1 - \sum_{i=0}^{x-1} f(i)).$$

Thus let $v(x) = \sum_{i=0}^{x-1} i\, f(i)$, $v(0) = 0$, and $w(x) = 1 - \sum_{i=0}^{x-1} f(i)$, $w(0) = 1$.

Then the excess pure premium can be calculated by

$$EP(x) = \mu - v(x) - x\, w(x)$$

where v and w can be calculated recursively by

$$v(x + 1) = v(x) + xf(x) \text{ and } w(x + 1) = w(x) - f(x).$$

By approximating the severity distribution with discrete probabilities the aggregate distribution and excess pure premium functions can thus be estimated recursively. Exhibits 1 and 2 compare this with the characteristic function method. Exhibit 1 shows the piecewise linear severity assumed and the approximating discrete probabilities. A unit of 500 was taken. The largest possible claim is taken as 250,000. The discrete approximation was constructed by matching cumulative probabilities and average severities at 250 + 500 i points, to the extent possible.

Exhibit 2 shows the cumulative probabilities and excess ratios for the two methods. (The excess ratio at x is $EP(x) \div \mu$.) The excess ratio columns are practically identical, suggesting that very little is lost by the discrete approximation. The cumulative probabilities are also rather close. In fact, since the characteristic function method does not provide error estimates for cumulative probabilities, it is not clear which method is closer to the exact probabilities for the piecewise linear severity.

Although the recursive formulas are simpler than those of the characteristic function method, they do not always take less computation, especially when only one or two limits are to be evaluated. On a ground up coverage with a high occurrence limit, a large number of points would be needed to approximate the severity distribution because a small unit would be needed to represent small claims. If, in addition, there are a large number of expected claims, the recursive method can be time consuming. If, on the other hand, an aggregate distribution is being estimated for an excess occurrence layer where there are few expected claims and a large unit can be chosen, this method may be quite efficient.

The recursive method does not provide a mathematically elegant way of accounting for the crucial element of parameter risk. However, this can be handled by enumerating a list of possible scenarios (frequency and severity functions), calculating the aggregate distribution function for each scenario, and then weighting these aggregate functions together by the relative probability attached to each scenario. As discussed above, this is more general than a gamma distributed divisor approach in that it allows for more types of parameter variation.

In conclusion, the authors have produced a practical, efficient method for calculating aggregate probabilities and excess pure premiums. This is not an obscure exercise in complex mathematics but a powerful competitive tool for those who will use it.

Ackowledgment

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EXHIBIT 1

AGGREGATE LOSS DISTRIBUTIONS
COMPARATIVE ASSUMPTIONS

Frequency: Poisson $\lambda = 13.7376$

Piecewise Linear CDF

Limit (000):	1	5	6	7	8	9
Cumulative Prob.:	.38935	.77870	.78438	.78981	.79498	.79993

10	12.5	15	17.5	20	25	35	50
.80466	.81564	.82553	.83449	.84264	.85690	.87927	.90280

75	100	125	150	175	200	225	250
.92739	.94256	.95277	.96009	.96556	.96979	.97316	.97590

Discrete PDF

Amount:	500	1000	1500 to 4000
Probability:	.38326640625	.03041796875	.04866875 each 500

4500	5000	5500 to 249,000 at each $N = 500k$
.054731628	.019691497	Piecewise linear probability from $N - 250$ to $N + 250$

249,500	250,000
.0000685	.0241137

Moments

	Mean	Coefficient of Variation	Coefficient of Skewness
Severity	18,198	2.6600	3.6746
Aggregate	250,000	.7667	1.0744

EXHIBIT 2

AGGREGATE LOSS DISTRIBUTIONS
COMPARATIVE SUMMARY

Aggregate Loss (000)	Characteristic Function Method		Recursive Method	
	Cum. Prob.	Excess Ratio	Cum. Prob.	Excess Ratio
25	.0508	.9016	.0516	.9016
50	.1291	.8107	.1298	.8107
75	.2009	.7273	.2015	.7272
100	.2676	.6507	.2683	.6507
125	.3289	.5806	.3295	.5806
150	.3843	.5163	.3848	.5163
175	.4341	.4573	.4346	.4573
200	.4788	.4030	.4793	.4029
225	.5189	.3529	.5193	.3529
250	.5548	.3066	.5552	.3066
275	.6034	.2642	.6040	.2642
300	.6556	.2273	.6561	.2273
325	.7008	.1951	.7013	.1951
350	.7405	.1672	.7408	.1672
375	.7749	.1431	.7752	.1431
400	.8047	.1221	.8049	.1221
425	.8303	.1039	.8305	.1039
450	.8524	.0880	.8526	.0880
475	.8714	.0742	.8716	.0742
500	.8878	.0622	.8879	.0622
525	.9045	.0518	.9047	.0518
550	.9201	.0430	.9203	.0430
575	.9332	.0357	.9333	.0357
600	.9442	.0296	.9443	.0296
625	.9534	.0245	.9535	.0245
650	.9611	.0202	.9611	.0202
675	.9675	.0167	.9675	.0167
700	.9728	.0137	.9729	.0137
725	.9773	.0112	.9773	.0112
750	.9810	.0091	.9810	.0091
775	.9844	.0074	.9844	.0074
800	.9873	.0060	.9873	.0060
825	.9897	.0048	.9897	.0048
850	.9916	.0039	.9916	.0039

THE PRICING OF MEDI GAP COVERAGE

EMIL J. STRUG

Abstract

This paper presents an abbreviated history of the events leading up to Medicare and the impact of Medicare since its inception upon the health care system and health care costs. Because of the social pressures brought about by ever increasing health care costs, especially to the elderly, Medi Gap policies, their benefits and pricing, have come under close scrutiny of the regulator.

The main body of the text deals with those benefits which not only present particular pricing considerations, but which also are closely evaluated by the regulator. In many instances, the historic data collected and produced by the insurer to develop utilization and pricing trends must be supported by data from external sources.

Hopefully, this presentation provides the reader with not only some new insights into the techniques of pricing programs of this type, but also of the social pressures which influence the regulator in assuring the availability of insurance at reasonable rates. In some states, this has led to subsidization of the Medi Gap policies by those people under age 65. One can see the pressure expanding to use insurance as a means to address the problem of economic and social inequities not only in the area of personal property and casualty coverage, but also in the area of health insurance.

I. BACKGROUND

To set the stage for the current interest of the regulatory authorities in the pricing and benefit content of Medi Gap policies, some analysis of the advent of Medicare and its subsequent impact on the economy might be helpful.

The seeds for providing health care to the aged were planted in 1935 in some of the initial versions of the Social Security Act. Under the study provisions of the Act, the Social Security Board was empowered to conduct research and investigations relative to national health insurance. From 1935 to 1965, when Medicare was enacted, a series of bills dealing with national health insurance were presented to the Congress: 1939, the Wagner Bill; 1943, the Wagner, Murray, Dingell Bill; and, 1946, the Taft Bill. In the 1951 to 1964 era, most of the bills dealt with social insurance proposals for persons aged 65 and over. In 1960, the Kerr-Mills Act was passed establishing a program of medical assistance for the aged. Beginning in 1960, efforts to enact a social insurance program for hospital benefits were stepped up with a series of attempts to enact a sound insurance program known as the King-Anderson Bills. Sufficient momentum was gained so that in 1964 the Senate passed an amendment providing hospital insurance benefits for people aged 65 and over. The House, however, would not agree on a compromise position and the legislation died in conference. In 1965, in addition to a King-Anderson Bill, other proposals were presented such as the Byrnes Bill (named after its author Representative Byrnes), and the Eldicare Bill (sponsored by the American Medical Association and introduced by Representatives Herlong and Curtis). Early in 1965, under the leadership of Chairman Mills, the House Ways & Means Committee put together the Medicare program which was to become effective on July 1, 1966.

The social pressures brought about by the cost to the aged for medical care were a major factor influencing the passage of Medicare. The aged were caught in the bind of fixed incomes, with rising medical care costs continually consuming more of their available income. An examination of Medicare benefits is in order to assess their impact on the covered individual as well as their impact upon the health care system and the group benefit package for people under 65.

II. SUMMARY OF MEDICARE BENEFITS

The Medicare program provides a most comprehensive package of benefits. Regarding hospital benefits, inpatient room and board for a semiprivate accommodation (and where medically necessary, private room) and all special services (general nursing, drugs, operating room, diagnostic services, etc.) were paid in full for the first 60 days, after payment of a deductible. From the 61st to the 90th day, the same benefit provisions prevailed but with a daily copayment equal to 25% of the initial deductible. In addition, there was coverage for care provided in a skilled nursing facility (SNF), plus home health services. Full outpatient diagnostic benefits were also provided to minimize use of inpatient facilities for such services. Benefits in a skilled nursing facility were covered in full for the first 20 days, with the next 80 days of benefits having a daily copayment equal to ⅛th of the initial inpatient deductible. All of these benefits were provided under the hospital insurance portion of Medicare, commonly referred to as Part A.

Physicians benefits, in addition to home health services, were provided under the Supplementary Medical Insurance (SMI) portion of Medicare, generally referred to as Part B. The SMI portion has an annual deductible (as contrasted to spell of illness deductible under Part A), plus a copayment feature with the patient paying 20%. Physicians were reimbursed on a reasonable charge basis.

With the passage of Medicare, persons aged 65 and over had comprehensive benefits available to them which equalled and in many cases exceeded those held by the under age 65 population. Removal of the financial need caused the Medicare population to make full use of the program. Medicare's impact upon the medical care system for the entire population has been well documented by health economists and is summarized in Tables 1–4 which follow.

TABLE 1

PORTION OF HEALTH CARE COSTS PAID BY INDIVIDUALS VERSUS THIRD PARTY PAYORS

	Under 65			65 And Over		
Fiscal Years Ending June 30	Total	Out of Pocket	Third Party	Total	Out of Pocket	Third Party
1966	100%	51%	49%	100%	53%	47%
1967	100%	48%	52%	100%	37%	63%
1970	100%	43%	57%	100%	33%	67%
1973	100%	38%	62%	100%	33%	67%
1976	100%	35%	65%	100%	27%	73%
1977 (Sept)	100%	32%	68%	100%	27%	73%

TABLE 2

Health Care Expenditures As % of Gross National Product

Fiscal Years Ending	Percentage
1966	5.8%
1967	6.2%
1970	7.2%
1973	7.7%
1976	8.7%
1977 (Sept)	8.8%

TABLE 3

Ratio of Personal Expenditures For Medical Care To Disposable Personal Income

Calendar Year	Ratio
1966	6.2%
1967	6.3%
1970	7.1%
1973	7.4%
1976	8.6%
1977	9.1%

TABLE 4

ANNUAL CHANGES IN CONSUMER PRICE INDEX AND IN MEDICAL COMPONENTS OF THE INDEX

Calendar Year	All Items	All Medical Care Items	Physician Fees	Hospital Room	Prescriptions & Drugs
1966	2.9%	4.4%	5.8%	10.0%	1.3%
1967	2.9%	7.1%	7.1%	19.8%	−0.5%
1970	5.9%	6.3%	7.5%	12.9%	2.3%
1973	6.2%	3.9%	3.3%	4.7%	0.3%
1976	5.8%	9.5%	11.3%	13.8%	6.1%
1977	6.5%	9.6%	9.3%	11.5%	6.4%

The results speak for themselves as to the rapid rise in medical care costs. Considering the limited and relatively fixed income of the age 65 and over population, one can see how the social pressures to provide relief in the form of medical care arose in the early 1960's.

A history of the movement of the Medicare deductibles and the cost to purchase Part B (medical) benefits (Table 5) will also show how the increase in these elements has further affected the standard of living of the aged.

It should be noted that in 1972 Medicare benefits were extended to the disabled under Social Security and those receiving treatment for chronic kidney disease. As was mentioned earlier in this paper, deductibles were introduced to keep down the cost of the program to the government. The initial hospital deductible was set equal to the daily cost of care in a semiprivate room. The Part B deductible was set at $50 per calendar year with 20% of the remaining balance coinsured by the recipient, with the first period limited to 6 months to minimize the cost of the program to the government.

III. THE PRICING OF MEDI GAP BENEFITS

To meet the insurance needs of the age 65 and over population for the uncovered portions of the Medicare program, policies were designed which tended to duplicate, when combined with Medicare, comprehensive programs offered by the insurance industry.

TABLE 5

MEDICARE DEDUCTIBLES, COPAYS & COINSURANCE AND PREMIUM

	Part A		
	Benefit Period Deductible	Daily Copay 61st to 90th Hospital Days	21st to 100th SNF Days
7/66	$ 40	$10	$ 5.00
1/69	$ 44	$11	$ 5.50
1/70	$ 52	$13	$ 6.50
1/71	$ 60	$15	$ 7.50
1/72	$ 68	$17	$ 8.50
1/73	$ 72	$18	$ 9.00
1/74	$ 84	$21	$10.50
1/75	$ 92	$23	$11.00
1/76	$104	$26	$13.00
1/77	$124	$31	$15.50
1/78	$144	$36	$18.00
1/79	$160	$40	$20.00

	Part B		
	Premium	Annual Deductible	Coinsurance
7/66	$3.00	$50	20%
4/68	$4.00	$50	20%
7/70	$5.30	$50	20%
7/71	$5.60	$50	20%
7/73	$5.80	$60	20%
7/74	$6.30	$60	20%
7/76	$7.20	$60	20%
7/77	$7.70	$60	20%
7/78	$8.20	$60	20%
7/79	$8.70	$60	20%

The major elements of cost to be met were:

(1) The initial Part A inpatient hospital deductible for each spell of illness.
(2) The Part A inhospital copayment days from the 61st to the 90th day.
(3) Full inhospital coverage from the 91st day on.
(4) The Part A copayment days in a skilled nursing facility from the 21st to the 100th day.
(5) The Part B deductible (currently $60) and coinsurance (20%) for services provided by physicians and the outpatient department of a hospital which were routinely provided under a typical health insurance policy.
(6) Prescription drugs not provided by the hospital.

More than a decade has passed since the Medi Gap program began and there is now ample cost and utilization data particular to the insured Medicare population. Data pertaining to the complementary Part A deductible and copayments are relatively clean as the benefits are for a spell of illness or benefit period. On the other hand, the Part B data present some problems due to the status of the deductible being maintained by Social Security and not by the carrier, plus the difficulty (if not the inability) to maintain appropriate service counts and distribution of losses by size. This constrains the ability to properly measure the impact of inflation upon the deductible and the truncation of service counts under the deductible.

For analytical purposes, we have shown the calculation underlying the rate determination for policies renewed and issued April 1 thru June 30, 1979, for a duration of 12 months. It should be noted that rates are evaluated for each calendar quarter of the year. Rates are calculated to be adequate for all policies with inception dates within that quarter. After the program was introduced in July of 1966, rates were generally changed annually, primarily to reflect the change in the Part A deductible. After an analysis of the distribution of business by effective dates, coupled with the ever increasing unpredictability of Medicare changes, it was decided to evaluate and implement changes in rates on a quarterly basis.

At the outset it should be stated that the methodology employed to produce pure premium generally follows the traditional method of multiplying projected incidence and cost.

As one reviews the various techniques used to develop the underlying data to produce rates, he will note similarities in the methodology used with other lines of insurance. The use of time series regression analysis is found in the development of private passenger automobile rates for both claim incidence and claim cost projections. The time series approach is also used in many property and physical damage lines, as well as for bodily injury coverages and workers compensation to estimate future claim costs.

The adjustment to compensate for the deductible in the calculation of the full claim cost component of drugs is not too dissimilar to that used in collision and property damage coverages where the use of deductibles is common. The one missing element is the loss elimination ratio calculation to adjust for claims below the deductible level. This variance in approach was and is due to the lack of any distribution of losses by size on a full coverage basis.

With the exception of the Part A deductible and copayments (61st to the 90th day and SNF) and Part B Physicians coinsurance, the use of time series regression analysis was for the most part employed in the development of the projected values.

For ease of reference the exhibits and their content are summarized below:

Exhibit	Content
1	Part A Inpatient Hospital Deductible Annual Claims Incidence.
2	Estimate of 1980 Medicare Part A Inpatient Hospital Deductible and Copayment Amounts.
3	Part A Inpatient Hospital Copayment Annual Day Incidence 61st to 90th Day.
4	Part A Skilled Nursing Facility Copayment Annual Day Incidence 21st to 100th Day.
5	Inpatient Hospital Benefits from the 91st Day on-Monthly Pure Premiums.
6	Part B Physicians and Outpatient Services Annual Deductible Monthly Pure Premium.
7	Part B Physicians Coinsurance Annual Service Incidence.
8	Physicians Coinsurance Average Cost Per Service.
9	Part B Outpatient Hospital Service Coinsurance Annual Incidence.
10	Part B Outpatient Hospital Service Coinsurance Average Cost Per Service.
11	Prescription Drugs Annual Claim Incidence.
12	Prescription Drugs Average Number of Prescriptions Per Prescription Drug Claim.
13	Prescription Drugs Calculation of the Average Charge per Prescription Drug Claim.

Exhibit	Content
14	Prescription Drugs Average Charge Per Prescription.
15	Calculation of Expected Monthly Pure Premium By Benefit Category.

We will only address those calculations which presented some particular problems or modifications before being introduced into the pure premium calculations.

To develop the cost elements of the Part A deductible and copayments we made use of data issued by Medicare. To estimate the value to be introduced into the Medicare formula the technique used was to raise the incomplete data from the base period (that would be used to calculate the deductible) to its ultimate value, much as one would do in developing the average claim cost for claim reserve calculations. In this case calendar year 1978 serves as the base year for the 1980 deductible calculation. The development of the value is shown in Exhibit 2. Once having developed the deductible, the calculation of the copayment values becomes an arithmetic exercise, as they are a proportion of the deductible amount.

For two benefit categories, inhospital coverage from the 91st day on, and Part B physician and outpatient service deductible, the results were projected by using pure premiums rather than incidence and cost.

The development of expected pure premiums for in hospital benefits beyond the 90th day is contained in Exhibit 5. Benefits for days beyond 90 days are paid in full by the insurance carrier. As previously noted, one would normally calculate this value by estimating the day utilization and the average daily costs. An analysis of these elements indicated erratic behavior in terms of utilization, length of stay, and per diem costs, whereas the pure premiums produced stable as well as reasonable results.

The most difficult element of pure premium to calculate is that related to the Part B annual deductible, in whole or in part, for physician and outpatient hospital services. As was previously mentioned, there are no available statistics by size of loss to determine the impact of inflation and utilization upon the deductible value because the status of the Part B deductible and the benefits applicable to satisfy the deductible are maintained by Medicare.

To solve the problem, the choice of the regression curve was paramount. It not only had to show a high degree of correlation to historical data, but also demonstrate a pattern of future development that was logical. With an increasing unit cost one would expect that in successive years the average deductible would increase at a decreasing rate and become asymptotic as it approached the deductible limit.

The most recent observation would indicate that the values have become asymptotic; therefore, the last observed value was chosen as the expected pure premium for the rating period. The historic values and the projected pure premiums are shown in Exhibit 6.

For the coinsurance benefits that complement the Part B 80% coinsurance payments, a return to the more traditional technique of using utilization (frequency) and average cost per service for calculating pure premiums was adopted. The physicians and hospital elements are handled separately as each is influenced by the inflation factors particular to each of the segments. The increase in physicians' prevailing fees is controlled by the Department of Health, Education and Welfare. For 1979, this value was calculated to be 5.08% above 1978 values and the same rate of increase was assumed to continue in 1980. The increase in hospital charges reflects the inflationary pressures of the local hospital area.

Exhibit 7 develops the expected service utilization for physicians coinsurance benefits. The average service cost associated with this benefit is developed in Exhibit 8. Projections are based upon values as issued by Health, Education and Welfare to Part B intermediaries. The companion piece to the physicians coinsurance is the outpatient hospital coinsurance benefit. The utilization and cost considerations are displayed in Exhibits 9 and 10.

To assess the reasonableness of the cost trends for physicians and outpatient hospital services, a comparison is made of the estimates for these services made by the Medicare actuaries in developing Part B rates. These values can be found in the Part B rate promulgation as published in the Federal Register.

The next and final benefit to be analyzed is prescription drugs. Prescription drugs, outside of those provided in a hospital setting, are not covered by Medicare. The benefit to be priced provides payment for prescription drugs subject to a $25 quarterly deductible and 20% coinsurance payment by the insured. Pure premiums are developed by estimating the number of claims, the average number of prescriptions per claim, and the average cost per prescription. The estimations of the number of claims and the average number of prescriptions present no unusual or unique considerations. Generally, the number of claimants has increased over time with the number of prescriptions showing a continuing decline. The underlying data and projections for these two elements are shown in Exhibits 11 and 12. In order to develop the full prescription charge, the average prescription claim payment has to be adjusted to reflect the removal of the 20% coinsurance and the $25 deductible. Projecting the average prescription charge without modification would obviously produce erroneous results. The conversion of the average prescription cost from a partial to a full basis is developed in Exhibit 13. The resultant values are then transferred to Exhibit 14 where the projected value is developed. To evaluate the reasonableness of this value, the inherent annual trend from the last observed value to the projected value is compared to the trends observed for the most recent annual values in the Consumer Price Index for drugs and for those shown in the Lilly Drug Digest. At the time of preparation of the filing, the Consumer Price Index trend, as of October 1978, was 7.5%, while the Lilly Digest (1977) showed 9.4%. The 5.5% trend used in the pure premium projections was therefore considered to be reasonable. The estimated pure premium for the benefit was calculated by developing the estimated full charge per claim and then reducing this value by the deductible amount and 20% coinsurance.

The pure premium for each of the benefit categories previously described and its detailed calculations are contained in Exhibit 15.

In reviewing the data contained in the calculation of the expected incidence and costs for the various benefits shown, the reader should be aware of the characteristics of the population, health care providers, and the manner in which the business is underwritten.

In terms of the population, it is for all practical purposes totally resident in one state. By being essentially a single state program, the health care practices of the providers have a definite impact and influence upon the cost of the covered services.

The physicians serving this population have almost universally accepted payment on an assignment basis from Medicare. This means that the physician accepts the level of fees established by Medicare as being payment in full, thereby limiting the patient's liability to the coinsurance amount after satisfaction of the Part B deductible. This removes the problem of the patient being assessed an additional charge which could, depending upon the policy design, impact the pricing process.

Hospital benefits, with the exception of those which are fill-ins of Part A deductibles or copayments, are subject to local inflationary pressures rather than that being experienced on a countrywide basis. Local statistics are therefore more appropriate for determining this movement of hospital costs than those developed from regional or national data.

As regards the manner in which this program is underwitten, there are two major considerations. The first is that the rate is uniform regardless of the age or sex of the member. The morbidity characteristics of this population are that the utilization and cost, and therefore the pure premium, increase by age. Additionally, the pure premiums by sex would require a higher rate for males versus females.

The second and probably most liberal consideration is the provision of benefits for any pre-existing condition, no requirement of any symptom-free period before benefits become effective, and no waiting period before benefits are available. Because of the general health condition of the aged, the introduction of any of the previously mentioned limitations into the policy would require a reduction in rate.

EXHIBIT 1

Part A Inpatient Hospital Deductible Annual Claims Incidence

Per 100 Contracts For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
24.932	24.966	25.001	25.025	25.346	25.513	25.750	25.910	25.771	25.917	25.889	26.215	26.968*

* The projected values resulting from the three projection methods indicated below were initially considered. Despite the significantly high indexes of determination and the reasonability of the values, it was determined to be appropriate to calculate the projected claim incidence value using the most recently observed annual rate of increase (1.2%) which is somewhat lower than the annual trends underlying the aforementioned projected values. [(26.215)(1.012**(28.5/12)) = 26.968].

Projection Method	Form of Equation	Index of Determination	Projected Value
Linear	$Y = A + BX$	.928	27.329
Hyperbolic	$Y = 1/(A + BX)$	.927	27.462
Exponential	$Y = A(\mathrm{Exp}(BX))$	.926	27.392

The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

EXHIBIT 1 (CONT.)

Form of Equation	Type of Function	Equation Number	Index of Determination	A	B	Proj. Value	Ann. Trend
1. $Y = A + (B * X)$	Linear	1	.928	24.735696	.120598	27.329	1.8%
2. $Y = 1/(A + B * X)$	Hyperbolic	5	.927	.040403	−.000186	27.462	2.0%
3. $Y = A * \mathrm{Exp}(B * X)$	Exponential	2	.926	24.743391	.004730	27.392	1.9%
4. $Y = A * (X ** (B))$	Power	3	.827	24.625612	.021323	26.290	.1%
5. $Y = A + B * \mathrm{Log}(X)$	Logarithmic	7	.826	24.615411	.542850	26.281	.1%
6. $Y = X/(A + B * X)$	Hyperbolic	6	.540	.001940	.038695	25.783	−.7%
7. $Y = A * \mathrm{Exp}(B/X)$	Exponential	8	.537	25.843334	−.049305	25.784	−.7%
8. $Y = A + (B/X)$	Hyperbolic	4	.534	25.843736	−1.253488	25.785	−.7%

EXHIBIT 2

Estimate of 1980 Medicare Inpatient Hospital Deductible and Copayment Amounts*

Item	Amount	Source
A. Average hospital charge per day for the period January 1, 1977 to December 31, 1977	$197.07	Appendix B
B. Average per diem rate for the period January 1, 1977 to December 31, 1977	$160.69	Page 44891, Federal Register, Vol. 43 No. 190, dated 9/29/78. Appendix A
C. Ratio of per diem rate to average hospital charge per day for the period January 1, 1977 to December 31, 1977	.815	Item B ÷ Item A
D. Average hospital charge per day for the period January 1, 1977 to June 30, 1977	$190.77	Appendix B
E. Average hospital charges per day for the period January 1, 1978 to June 30, 1978	$217.21	Appendix B
F. Estimated average hospital charge per day for the period January 1, 1978 to December 31, 1978	$224.38	(Item E ÷ Item D) (Item A)

* The law provides that for spells of illness beginning in calendar years after 1968 the inpatient hospital deductible shall be equal to $40 multiplied by the ratio of (1) the current average per diem rate for inpatient hospital services for the calendar year preceding the year in which the promulgation is made to (2) the current average per diem rate for such services for 1966. Changes in the amount of the inpatient hospital deductible also affect certain other cost-sharing provisions under the Medicare hospital insurance program, the patient co-payment for the 61st to 90th inpatient day which equals 25 percent of the inpatient hospital deductible, and the skilled nursing home daily co-payment which is equal to 12.5 percent of the inpatient hospital deductible.

EXHIBIT 2 (CONT.)

ESTIMATE OF 1980 MEDICARE INPATIENT HOSPITAL DEDUCTIBLE AND COPAYMENT AMOUNTS

Item	Amount	Source
G. Estimated ratio of per diem rate to average hospital charge per day for the period January 1, 1978 to December 31, 1978	.815	Based on 1977 experience. Item C.
H. Estimated average per diem rate for the period January 1, 1978 to December 31, 1978	$183.68	(Item F) (Item G)
I. Average per diem rate for the period January 1, 1966 to December 31, 1966	$ 40.01	Page 44891, Federal Register, Vol. 43 No. 190, dated 9/29/78. Attachment I
J. Estimated 1980 inpatient hospital deductible	$184.00	(Item H/Item I) ($40) rounded to the nearest multiple of $4.00
K. 1979 Medicare inpatient hospital deductible	$160.00	Page 44891, Federal Register, Vol. 43 No. 190, dated 9/29/78. Attachment I
L. Estimated 1980 Medicare inpatient hospital deductible	$184.00	Item J
M. Medicare inpatient hospital deductible for the period 5/15/79 to 5/14/80	$169.00	[(7.5/12) (Item J) + (4.5/12) (Item J)]
N. Co-payment for the 61st to the 90th inpatient hospital day for the period 5/15/79 to 5/14/80	$ 42.25	(Item M) (.250)
O. Skilled nursing facility daily co-payment for the period 5/15/79 to 5/14/80	$ 21.13	(Item M) (.125)

EXHIBIT 3

PART A INPATIENT HOSPITAL COPAYMENT ANNUAL DAY INCIDENCE
61ST TO 90TH DAY

Per 100 Contracts For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
15.732	16.504	16.633	17.384	17.995	18.137	18.420	18.407	18.453	18.484	18.443	18.644	19.225*

* The projected values resulting from the two projection methods indicated below were initially considered. Despite the significantly high indexes of determination and the reasonability of the values, it was determined to be appropriate to calculate the projected day incidence value using the most recently observed annual rate of increase (1.3%) which is somewhat lower than the annual trends underlying the aforementioned projected values [(18.644)(1.013**(28.5/12)) = 19.225].

Projection Method	Form of Equation	Index of Determination	Projected Value
Logarithmic	$Y = A + B(\ln X)$	.951	19.529
Power	$Y = AX^B$	.951	19.648

The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

EXHIBIT 3 (CONT.)

Form of Equation	Type of Function	Equation Number	Index of Determination	A	B	Proj. Value	Ann. Trend
1. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.951	15.679929	1.254644	19.529	2.0%
2. $Y = A * (X ** (B))$	Power	3	.951	15.721701	.072661	19.648	2.2%
3. $Y = X/(A + B * X)$	Hyperbolic	6	.867	.011565	.053449	18.523	−.3%
4. $Y = A * \text{Exp}(B/X)$	Exponential	8	.855	18.675771	−.197837	18.505	−.3%
5. $Y = A + (B/X)$	Hyperbolic	4	.843	18.646558	−3.390909	18.489	−.4%
6. $Y = A + (B * X)$	Linear	1	.807	16.194574	.242322	21.404	6.0%
7. $Y = A * \text{Exp}(B * X)$	Exponential	2	.797	16.206697	.013945	21.873	7.0%
8. $Y = 1/(A + B * X)$	Hyperbolic	5	.786	.061665	−.000804	22.532	8.3%

EXHIBIT 4

PART A SKILLED NURSING FACILITY COPAYMENT ANNUAL DAY INCIDENCE 21ST TO 100TH DAY

Per 100 Contracts For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
38.222	37.110	36.874	36.101	34.642	34.094	32.028	29.945	27.113	23.493	20.563	18.111	11.257*

* The projected value is the result of an exponential projection [$Y = A(\text{Exp}(BX))$] which has an index of determination of .879. This value is considered to be appropriate for inclusion in the rate calculation in view of the acceptable index of determination as well as the fact that the annual trend underlying the projected value is consistent with the expectation that day incidence for Skilled Nursing Facilities will continue to decrease, but at a somewhat lesser rate than has been historically observed. A linear projection [$Y = A + BX$] has a higher index of determination (i.e., .926); however the resulting projected value of 3.161 was considered to be clearly inadequate and therefore rejected. The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

Form of Equation	Type of Function	Equation Number	Index of Determination	*A*	*B*	Proj. Value	Ann. Trend
1. $Y = A + (B * X)$	Linear	1	.926	42.621060	−1.835343	3.161	−52.0%
2. $Y = A * \text{Exp}(B * X)$	Exponential	2	.879	45.623749	−.065091	11.257	−18.1%
3. $Y = 1/(A + B * X)$	Hyperbolic	5	.821	.018952	.002392	14.207	−9.7%
4. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.697	43.344607	−7.596821	20.037	4.3%
5. $Y = A * (X ** (B))$	Power	3	.631	46.313503	−.263028	20.665	5.7%
6. $Y = A + (B/X)$	Hyperbolic	4	.395	26.444535	16.422209	27.208	18.7%
7. $Y = A * \text{Exp}(B/X)$	Exponential	8	.340	25.892833	.554400	26.569	17.5%
8. $Y = X/(A + B * X)$	Hyperbolic	6	.288	−.019405	.039521	25.894	16.2%

EXHIBIT 5

INPATIENT HOSPITAL BENEFITS FROM THE 91ST DAY ON-MONTHLY PURE PREMIUM

Per Contract For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
$1.210	$1.324	$1.433	$1.525	$1.606	$1.601	$1.631	$1.643	$1.653	$1.643	$1.661	$1.650	$1.678*

* The projected value is the result of a hyperbolic projection [$Y = X/(A + BX)$] which has an index of determination of .944, the highest index of determination of the projection methods employed. A logarithmic projection [$Y = A + B(\ln X)$] has virtually the same index of determination (i.e., .943); however the resulting projected value of $1.816 was considered to be excessive in view of the relative stability of the recent actual experience. The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

Form of Equation	Type of Function	Equation Number	Index of Determination	*A*	*B*	Proj. Value	Ann. Trend
1. $Y = X/(A + B * X)$	Hyperbolic	6	.944	.265492	.583515	1.670	.7%
2. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.943	1.229987	.191130	1.816	4.1%
3. $Y = A * (X ** (B))$	Power	3	.936	1.237339	.131828	1.854	5.0%
4. $Y = A * \text{Exp}(B/X)$	Exponential	8	.928	1.698944	−.376911	1.669	.5%
5. $Y = A + (B/X)$	Hyperbolic	4	.910	1.687731	−.539045	1.663	.3%
6. $Y = A + (B * X)$	Linear	1	.736	1.318242	.035399	2.079	10.2%
7. $Y = A * \text{Exp}(B * X)$	Exponential	2	.713	1.317486	.024125	2.213	13.2%
8. $Y = 1/(A + B * X)$	Hyperbolic	5	.689	.759790	−.016557	2.476	18.6%

EXHIBIT 6

PART B PHYSICIANS AND OUTPATIENT SERVICES ANNUAL DEDUCTIBLE MONTHLY PURE PREMIUM

Per Contract For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
$1.822	$1.851	$1.852	$1.837	$1.975	$2.065	$2.109	$2.134	$2.238	$2.235	$2.242	$2.234	$2.234*

* The most recent observation (i.e., the year ending 12/31/77) has been carried forward to the period of the rates. The three projection methods indicated below have significantly high indexes of determination; however due to the relative stability of the four most recent observations, the projected values were judged to be excessive and therefore rejected.

Projection Method	Form of Equation	Index of Determination	Projected Value
Linear	$Y = A + BX$	.923	$2.745
Exponential	$Y = A(\text{Exp}(BX))$	.919	$2.876
Hyperbolic	$Y = 1/(A + BX)$	.914	$3.102

The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

EXHIBIT 6 (CONT.)

Form of Equation	Type of Function	Equation Number	Index of Determination	*A*	*B*	Proj. Value	Ann. Trend
1. $Y = A + (B * X)$	Linear	1	.923	1.748182	.046357	2.745	9.1%
2. $Y = A * \text{Exp}(B * X)$	Exponential	2	.919	1.761175	.022810	2.876	11.2%
3. $Y = 1/(A + B * X)$	Hyperbolic	5	.914	.564407	−.011259	3.102	14.8%
4. $Y = A * (X ** (B))$	Power	3	.828	1.719763	.103302	2.361	2.4%
5. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.822	1.701871	.208711	2.342	2.0%
6. $Y = X/(A + B * X)$	Hyperbolic	6	.550	.119619	.460290	2.147	−1.7%
7. $Y = A * \text{Exp}(B/X)$	Exponential	8	.540	2.173125	−.239441	2.149	−1.6%
8. $Y = A + (B/X)$	Hyperbolic	4	.530	2.173832	−.480788	2.151	−1.6%

EXHIBIT 7

PART B PHYSICIANS COINSURANCE
ANNUAL SERVICE INCIDENCE

Per 100 Contracts For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
349.034	361.880	379.235	397.626	405.828	419.269	434.288	447.282	448.633	451.196	445.098	444.293	444.293*

* The most recent observation (i.e., the year ending 12/31/77) has been carried forward to the period of the rates. The two projection methods indicated below have significantly high indexes of determination; however, due to the relative stability of the five most recent observations, the projected values, which represent upward trends, were judged to be inappropriate and therefore rejected.

Projection Method	Form of Equation	Index of Determination	Projected Value
Power	$Y = AX^B$	.955	487.301
Logarithmic	$Y = A + B(\ln X)$	.947	480.797

The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

EXHIBIT 7 (CONT.)

Form of Equation	Type of Function	Equation Number	Index of Determination	*A*	*B*	Proj. Value	Ann. Trend
1. $Y = A * (X ** (B))$	Power	3	.955	340.755047	.116595	487.301	4.0%
2. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.947	337.525093	46.697891	480.797	3.4%
3. $Y = A + (B * X)$	Linear	1	.877	354.056190	9.422917	556.649	10.0%
4. $Y = A * \text{Exp}(B * X)$	Exponential	2	.868	355.636978	.023301	586.911	12.4%
5. $Y = 1/(A + B * X)$	Hyperbolic	5	.856	.002802	−.000058	641.861	16.8%
6. $Y = X/(A + B * X)$	Hyperbolic	6	.806	.000769	.002227	441.929	−.2%
7. $Y = A * \text{Exp}(B/X)$	Exponential	8	.783	447.533321	−.303118	441.268	−.3%
8. $Y = A + (B/X)$	Hyperbolic	4	.759	446.350307	−120.050450	440.767	−.3%

EXHIBIT 8

Physicians Coinsurance Average Cost Per Service

Item	Amount	Source
A. Calculation of the cost trend factor to project the average cost per service for physicians' coinsurance benefit category from the year ending 12/31/77 to the year ending 5/14/80.		
1. The economic index applicable to physicians' services announced by the Social Security Administration for the period July 1, 1976 through June 30, 1977.	1.276	Part B Intermediary Letter No. 76-34 from Department of Health, Education and Welfare, dated August 1976. Appendix D1
2. The economic index applicable to physicians' services announced by the Social Security Administration for the period July 1, 1977 through June 30, 1978.	1.357	Part B Intermediary Letter No. 77-24 from Department of Health, Education and Welfare, dated June 1977. Appendix D2
3. Percent of increase for fiscal year 1978 over fiscal year 1977	6.35%	Item A.2. ÷ Item A.1.
4. The economic index applicable to physicians' services announced by the Social Security Administration for the period July 1, 1978 through June 30, 1979	1.426	Part B Intermediary Letter No. 78-23 from Department of Health, Education and Welfare, dated June 1978. Appendix D3
5. Percent of increase for fiscal year 1979 over fiscal year 1978	5.08%	Item A.4. ÷ Item A.2.
6. Expected percent of increase for fiscal year 1980 over fiscal year 1979	5.08%	Judgment

EXHIBIT 8 (CONT.)

PHYSICIANS COINSURANCE AVERAGE COST PER SERVICE

Item	Amount	Source
B. Cost trend factor to project the year ending 12/31/77 to the year ending 5/14/80.	1.132	$(1.0635)^{6/12}(1.0508)$ $(1.0508)^{10.5/12}$
C. Cost per service for the physicians' coinsurance benefit category for the year ending 12/31/77.	$7.85	Corporate Statistics
D. Expected average cost per service for physicians' coinsurance benefit category for the year ending 5/14/80	$8.89	(Item B)(Item C)

EXHIBIT 9

PART B OUTPATIENT HOSPITAL SERVICE COINSURANCE ANNUAL INCIDENCE

Per 100 Contracts For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
70.307	74.164	78.924	83.151	85.813	90.751	95.921	99.602	102.056	105.553	108.745	113.426	150.742*

* The projected value is the result of a linear projection [$Y = A + BX$] which has an index of determination of .996, the highest index of determination of the projection methods employed. This value is considered to be appropriate for inclusion in the rate calculation in view of the extremely high index of determination as well as the fact that the annual trend underlying the projected value is consistent with the decelerating annual rates of increase observed in the recent historical experience. An exponential projection [$Y = A(\text{Exp}(BX))$] and a hyperbolic projection [$Y = 1/(A + BX)$] also have extremely high indexes of determination (i.e., .987 and .970, respectively); however the resulting projected values (i.e., 173.859 and 257.553, respectively) were considered to be excessive and therefore rejected. The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

Form of Equation	Type of Function	Equation Number	Index of Determination	A	B	Proj. Value	Ann. Trend
1. $Y = A + (B * X)$	Linear	1	.996	67.072222	3.891619	150.742	12.7%
2. $Y = A * \text{Exp}(B * X)$	Exponential	2	.987	69.133700	.042893	173.859	19.7%
3. $Y = 1/(A + B * X)$	Hyperbolic	5	.970	.014183	−.000479	257.553	41.2%
4. $Y = A * (X ** (B))$	Power	3	.942	65.492350	.199877	120.924	2.7%
5. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.912	62.790400	17.757761	117.272	1.4%
6. $Y = X/(A + B * X)$	Hyperbolic	6	.739	.005724	.009589	101.474	−4.6%
7. $Y = A * \text{Exp}(B/X)$	Exponential	8	.608	103.716866	−.490394	101.378	−4.6%
8. $Y = A + (B/X)$	Hyperbolic	4	.636	103.382814	−42.594857	101.402	−4.6%

EXHIBIT 10

PART B OUTPATIENT HOSPITAL SERVICE COINSURANCE AVERAGE COST PER SERVICE

For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
$6.66	$6.98	$7.23	$7.36	$7.56	$7.72	$7.92	$8.16	$8.32	$8.47	$8.68	$8.89	$11.03*

* The three projection methods indicated below result in extremely high and nearly equal indexes of determination. The projected value produced by the hyperbolic projection was rejected as being clearly excessive. It was determined to be appropriate to use the mean of the linear projection and the exponential projection [($10.73 + $11.33)/2 = $11.03] in the rate calculation in consideration of the nearly equal validity of the linear and exponential projection methods, as well as the fact that the annual trend underlying the mean value is consistent with both recent historical experience and reasonable expectations of future hospital cost increases for outpatient services.

Projection Method	Form of Equation	Index of Determination	Projected Value
Linear	$Y = A + BX$	.996	$10.73
Exponential	$Y = A\ (\mathrm{Exp}(BX))$	.991	$11.33
Hyperbolic	$Y = 1/(A + BX)$	.983	$12.43

The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

EXHIBIT 10 (CONT.)

Form of Equation	Type of Function	Equation Number	Index of Determination	*A*	*B*	Proj. Value	Ann. Trend
1. $Y = A + (B * X)$	Linear	1	.996	6.572121	.193392	10.730	8.2%
2. $Y = A * \text{Exp}(B * X)$	Exponential	2	.991	6.635739	.024875	11.328	10.7%
3. $Y = 1/(A + B * X)$	Hyperbolic	5	.983	.149577	−.003215	12.429	15.2%
4. $Y = A * (X ** (B))$	Power	3	.940	6.434750	.115542	9.172	1.3%
5. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.919	6.353128	.886190	9.072	.9%
6. $Y = X/(A + B * X)$	Hyperbolic	6	.730	.037955	.118866	8.290	−2.9%
7. $Y = A * \text{Exp}(B/X)$	Exponential	8	.695	8.397684	−.285364	8.287	−2.9%
8. $Y = A + (B/X)$	Hyperbolic	4	.660	8.386769	−2.156228	8.286	−2.9%

EXHIBIT 11

PRESCRIPTION DRUGS
ANNUAL CLAIM INCIDENCE

Per 100 Contracts For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
45.596	46.638	47.320	48.467	49.514	51.017	53.018	54.695	56.173	57.436	58.618	59.663	72.772*

* The projected value is the result of a linear projection [$Y = A + BX$] which has an index of determination of .991. This value is considered to be appropriate for inclusion in the rate calculation in view of the extremely high index of determination as well as the fact that the annual trend underlying the projected value is consistent with the decelerating annual rates of increase observed in the recent historical experience. An exponential projection [$Y = A(\text{Exp}(BX))$] and a hyperbolic projection [$Y = 1/(A + BX)$] have slightly higher indexes of determination (i.e., .992), however the resulting projected values (i.e., 77.042 and 85.039, respectively) were considered to be excessive and therefore rejected. The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

Form of Equation	Type of Function	Equation Number	Index of Determination	A	B	Proj. Value	Ann. Trend
1. $Y = A * \text{Exp}(B * X)$	Exponential	2	.992	44.016928	.026036	77.042	11.4%
2. $Y = 1/(A + B * X)$	Hyperbolic	5	.992	.022510	−.000500	85.039	16.1%
3. $Y = A + (B * X)$	Linear	1	.991	43.495270	1.361689	72.772	8.7%
4. $Y = A * (X ** (B))$	Power	3	.866	42.970726	.116049	61.348	1.2%
5. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.847	42.342538	6.006066	60.769	.8%
6. $Y = X/(A + B * X)$	Hyperbolic	6	.595	.005300	.017889	55.141	−3.3%
7. $Y = A * \text{Exp}(B/X)$	Exponential	8	.569	55.903593	−.269983	55.206	−3.2%
8. $Y = A + (B/X)$	Hyperbolic	4	.544	55.918910	−13.815348	55.276	−3.2%

EXHIBIT 12

Prescription Drugs
Average Number of Prescriptions per Prescription Drug Claim

For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
9.875	9.750	9.542	9.402	9.277	9.149	9.081	9.011	8.925	8.866	8.788	8.712	8.054*

* The three projection methods indicated below result in extremely high and nearly equal indexes of determination. It was determined to be appropriate to use a mean of the logarithmic, exponential, and hyperbolic projections [(7.877 + 7.778 + 8.058)/3 = 8.054] in the rate calculation in consideration of nearly equal validity of these three projection methods as well as the fact that the annual trend underlying the mean value is equal to the most recently observed annual rate of decrease (−3.3%).

Projection Method	Form of Equation	Index of Determination	Projected Value
Hyperbolic	$Y = 1/(A + BX)$	.976	7.877
Exponential	$Y = A(\text{Exp}(BX))$	.971	7.778
Logarithmic	$Y = A + B(\ln N)$	.970	8.508

The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

EXHIBIT 12 (CONT.)

Form of Equation	Type of Function	Equation Number	Index of Determination	A	B	Proj. Value	Ann. Trend
1. Y = 1/(A + B * X)	Hyperbolic	5	.976	.101054	.001204	7.877	−4.2%
2. Y = A * Exp(B * X)	Exponential	2	.971	9.880607	−.011128	7.778	−4.7%
3. Y = A + B * Log(X)	Logarithmic	7	.970	10.017760	−.492071	8.508	−1.0%
4. Y = A + (B * X)	Linear	1	.965	9.867120	−.102916	7.654	−5.3%
5. Y = A * (X ** (B))	Power	3	.964	10.037782	−.052904	8.534	−.9%
6. Y = A + (B/X)	Hyperbolic	4	.745	8.877872	1.238568	8.935	1.1%
7. Y = A * Exp(B/X)	Exponential	8	.733	8.881842	.132370	8.937	1.1%
8. Y = X/(A + B * X)	Hyperbolic	6	.720	−.014160	.112545	8.938	1.1%

EXHIBIT 13

Prescription Drugs
Calculation of the Average Charge per Prescription Drug Claim

For Fiscal Years Ending

	3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77
1. Average cost per claim	$32.00	$32.47	$32.71	$32.97	$32.93	$32.86	$32.86	$33.16	$33.35	$33.68	$33.92	$34.37
2. Average charge per claim	$65.00	$65.59	$65.89	$66.21	$66.16	$66.08	$66.08	$66.45	$66.69	$67.10	$67.40	$67.96
3. Average number of prescriptions per claim	9.875	9.750	9.542	9.402	9.277	9.149	9.081	9.011	8.925	8.866	8.788	8.712
4. Average charge per prescription (Item 2 ÷ Item 3)	$ 6.58	$ 6.73	$ 6.91	$ 7.04	$ 7.13	$ 7.22	$ 7.28	$ 7.37	$ 7.47	$ 7.57	$ 7.67	$ 7.80

* Drug benefit covers 80% of cost after the satisfaction of a $25 deductible.

EXHIBIT 14

PRESCRIPTION DRUGS AVERAGE CHARGE PER PRESCRIPTION

For Fiscal Years Ending

Actual												Projected
3/31/75	6/30/75	9/30/75	12/31/75	3/31/76	6/30/76	9/30/76	12/31/76	3/31/77	6/30/77	9/30/77	12/31/77	5/14/80
$6.58	$6.73	$6.91	$7.04	$7.13	$7.22	$7.28	$7.37	$7.47	$7.57	$7.67	$7.80	$8.86*

* The three projection methods indicated below have extremely high and nearly equal indexes of determination. The value produced by the hyperbolic projection was rejected as being excessive in view of the historical rates of increase. It was determined to be appropriate to use the mean of the linear projection and the exponential projection [($8.77 + $8.95)/2 = $8.86] in the rate calculation in consideration of the nearly equal validity of the linear and exponential projection methods, as well as the fact that the annual trend underlying the mean value is consistent with recent historical experience.

Projection Method	Form of Equation	Index of Determination	Projected Value
Linear	$Y = A + BX$	.987	$8.77
Exponential	$Y = A(\text{Exp}(BX))$	.982	$8.95
Hyperbolic	$Y = 1/(A + BX)$	.976	$9.19

The remaining projection methods employed produce values and/or indexes of determination that were judged to be inappropriate for consideration.

EXHIBIT 14 (CONT.)

Form of Equation	Type of Function	Equation Number	Index of Determination	*A*	*B*	Proj. Value	Ann. Trend
1. $Y = A + (B * X)$	Linear	1	.987	6.561969	.102902	8.774	5.1%
2. $Y = A * \text{Exp}(B * X)$	Exponential	2	.982	6.580818	.014302	8.950	6.0%
3. $Y = 1/(A + B * X)$	Hyperbolic	5	.976	.151582	−.001991	9.193	7.2%
4. $Y = A * (X ** (B))$	Power	3	.952	6.457253	.067192	7.936	.7%
5. $Y = A + B * \text{Log}(X)$	Logarithmic	7	.941	6.432230	.479468	7.903	.6%
6. $Y = X/(A + B * X)$	Hyperbolic	6	.742	.023777	.132492	7.485	−1.7%
7. $Y = A * \text{Exp}(B/X)$	Exponential	8	.722	7.542393	−.167905	7.484	−1.7%
8. $Y = A + (B/X)$	Hyperbolic	4	.701	7.537976	−1.187710	7.483	−1.7%

EXHIBIT 15

Calculation of Expected Monthly Pure Premium By Benefit Category for the Period 5/15/79 to 5/14/80

Item	Amount	Source
A. Inpatient hospital deductible per admission		
1. Annual claim incidence per 100 contracts	26.968	Exhibit 1
2. Average payment per inpatient hospital deductible	$169.00	Exhibit 2, Item M
3. Expected monthly pure premium	$ 3.798	[(Item A1) (Item A2) ÷ 1200]
B. Co-payment for the 61st to the 90th inpatient hospital day		
1. Annual day incidence per 100 contracts	19.225	Exhibit 3
2. Average payment per day	$ 42.25	Schedule 2, Item M
3. Expected monthly pure premium	$ .677	[(Item B1) (Item B2) ÷ 1200]
C. Expected monthly pure premium for the 91st to the 120th inpatient hospital day	$ 1.678	Exhibit 5
D. Expected monthly pure premium for the joint physicians' services and outpatient services annual deductible	$ 2.234	Exhibit 6
E. Physicians' services coinsurance		
1. Annual services incidence per 100 contracts	444.293	Exhibit 7
2. Average payment per service	$ 8.89	Exhibit 8, Item D
3. Expected monthly pure premium	$ 3.291	[(Item E1) (Item E2) ÷ 1200]

EXHIBIT 15 (CONT.)

CALCULATION OF EXPECTED MONTHLY PURE PREMIUM BY BENEFIT CATEGORY FOR THE PERIOD 5/15/79 TO 5/14/80

Item	Amount	Source
F. Outpatient hospital service coinsurance		
1. Annual service incidence per 100 contracts	150.742	Exhibit 9
2. Average payment per service	$ 11.03	Exhibit 10
3. Expected monthly pure premium	$ 1.386	[(Item F1) (Item F2) ÷ 1200]
G. Skilled Nursing Facility		
1. Annual day incidence per 100 contracts	11.257	Exhibit 4
2. Average payment per day	$ 21.13	Exhibit 2, Item 0
3. Expected monthly pure premium	$ .198	[(Item G1) (Item G2) ÷ 1200)]
H. Prescription Drugs		
1. Average number of prescriptions per claim	8.054	Exhibit 12
2. Average charge per prescription	$ 8.86	Exhibit 14
3. Average charge per claim	$ 71.36	(Item H1) (Item H2)
4. Expected average payment per claim	$ 37.09	[$71.36 − $25.00][.80] = $37.09
5. Annual claim incidence per 100 contracts	72.772	Exhibit 11
6. Expected monthly pure premium	$ 2.249	[(Item H4) (Item H5) ÷ 1200]

PARAMETER UNCERTAINTY IN THE COLLECTIVE RISK MODEL

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Abstract

This paper proposes a new version of the collective risk model that allows for uncertainty in selecting the expected number of claims and the claim severity distribution. We provide two different methods of estimating the parameters of this model. It is demonstrated by computer simulation that one must combine the experience of several insureds in order to accurately quantify parameter uncertainty. Tests on a very large sample of individual insured data show a significant improvement in the accuracy of the collective risk model when parameter uncertainty is taken into account. The tests do not show perfect agreement between the model and the empirical data, but the agreement is close enough to be useful in many applications.

1. INTRODUCTION

This paper discusses the role of collective risk theory in making insurance pricing decisions. Collective risk theory provides a way of calculating the probability that a loss arising out of an insurance contract will exceed a given amount. The calculation is done in terms of the underlying claim severity and claim count distributions. Related to this is the excess pure premium, which is the cost of insuring all losses above a given amount. Also of interest is the excess pure premium ratio, which is the excess pure premium divided by the expected loss.

Of principal concern is the relationship between the variance of the loss ratio and the size of the insured. A common assumption was stated by Simon [10, p. 44] as follows: "As the risk size increases, we expect the variance (of the loss ratio) to approach zero." Large insureds are typically written on a retrospective rating plan or an aggregate excess contract. The effect of this assumption would be that for all loss amounts greater than the expected loss, the excess pure premium ratio would approach zero as the size of the insured becomes large.

The practical underwriter would feel very uncomfortable with an agreement to provide coverage for all losses above the expected loss for a zero or nominal premium for even a very large insured. His complaint would be that the expected loss cannot be estimated with the necessary precision.

Of interest is the distribution of actual losses about an unbiased estimate of the expected losses. Estimates of the expected losses vary because of many things such as future economic conditions, changes in loss development patterns, changes in the insured's operations, and changes in loss control procedures. Many of these changes are independent of the size of the insured. Thus, one should not expect the variance of the loss ratio to approach zero as the size of the insured becomes large.

The traditional models used in collective risk theory, such as the generalized Poisson distribution, do not allow for uncertainty in estimating the expected loss. This may be acceptable for the small insured, since the variance of the losses due to the random nature of the loss process is large compared to the variance due to the misestimation of the expected loss. As the insured increases in size, however, the variance due to the misestimation of the expected loss will dominate.

Below we will propose a version of the collective risk model that allows for uncertainty in estimating the expected loss. Most excess pure premium ratios used in the United States are calculated from the well known "Table M" [7]. This table is based on empirically derived excess pure premium ratios. Because of the predominant use of this table we must first address the following question: why use a model?

2. THE ACTUARY'S DILEMMA

It has been a long standing debate among actuaries as to whether one should use empirical data or theoretical models to derive aggregate loss distributions. After completing the mammoth task of tabulating aggregate loss distributions, Simon [10, p. 14] wrote: "To avoid the difficulties and pitfalls of empiricism we should borrow from the theory of risk, from Monte Carlo techniques, and from operations research. Let's begin pushing some frontiers today, so that we'll be ready to solve tomorrow's problems."

Officially, it appears that those who favor the empirical approach have prevailed and Mr. Simon's advice has gone unheeded. Skurnick [11] constructed a table for the state of California based on empirical observations. In 1980, a National Council subcommittee constructed another table based on empirical observations. Mr. Simon's table was in effect for seventeen years.

While the use of empirical distributions does not require one to make the assumptions that are necessary with the theoretical approach, there are some fundamental problems with the empirical approach. It is generally agreed that the variance of the loss ratio distribution decreases as the size of the insured increases. It is also agreed that the variance of the loss ratio distribution increases as the average claim severity increases. But it is necessary to combine the experience of insureds of different sizes and average claim severities in order to get a sufficiently large sample. For example, the tables constructed by the National Council on Compensation Insurance in 1980 combined all insureds on a countrywide basis into expected loss ranges that include $25,000 to $50,000, $50,000 to $100,000 and $100,000 to $200,000.

Thus, the actuary is faced with the dilemma of choosing between two undesirable alternatives. If the empirical approach is chosen, a sample from a heterogeneous population is required. If the theoretical model is used, a number of simplifying assumptions must be made.

By proposing a mathematical model, we do not advocate abandoning the use of empirical data. Once a model has been constructed, one should form hypotheses which can be tested on live data. If statistical tests demonstrate that the model is consistent with the data, the dilemma will be resolved.

It will become apparent that this is easier said than done, but this is the goal toward which we all must strive. If this goal is reached, there are many advantages to the theoretical approach. Since the size of the insured and the claim severity distribution are input variables, it is possible to adjust the parameters of the model to account for situations when there is little or no data available. For example, it would be a simple matter to find the aggregate loss distribution that results when all claims are subject to an accident limitation.

3. THE COLLECTIVE RISK MODEL

In this section, we propose a version of the collective risk model that allows for uncertainty in estimating the expected loss. Heckman and Meyers [2] discuss this model in great detail, and so a full mathematical description will not be given here. Much of what follows is taken from their paper and is included here for the sake of completeness.

We start by considering the Poisson distribution. In their classic book on risk theory, Beard, Pentikainen, and Pesonen [1, p. 18] give the assumptions underlying this distribution as follows:

1. Claims occurring in two disjoint time intervals are independent.
2. The expected number of claims in a time interval (t_1, t_2) depends only on the length of the time interval and not the initial value of t_1.
3. No more than one claim can occur at a given time.

There are many cases when one feels that a Poisson distribution is appropriate, but one does not know the expected number of claims. Two options are available under these circumstances. The first option is to estimate the expected number of claims from historical experience. Parameter uncertainty can arise from sampling variability and changes in claim frequency over time. A second option is to use the average number of claims for a group of insureds that are similar to the insured under consideration. Parameter uncertainty arises when some members of the group have different expected numbers of claims.

We now turn to specifying the claim count distributions that we shall use when parameter uncertainty is present. We shall adopt the following notation.

Let N be a random variable denoting the claim count,
λ be the expected number of claims, and
χ be a random variable with $E[\chi] = 1$ and $\text{Var}[\chi] = c$.

The claim count distribution can be modeled by the following algorithm.

Algorithm 3.1

1. Select χ at random from the assumed distribution.
2. Select the number of claims, N, at random from a Poisson distribution with parameter $\chi \cdot \lambda$.

We have the following relationships.

$$E[N] = E_{\chi}[E(N|\chi)] = E_{\chi}[\chi \cdot \lambda] = \lambda \tag{3.1}$$

$$\begin{aligned} \text{Var}[N] &= E_{\chi}[\text{Var}(N|\chi)] + \text{Var}_{\chi}[E(N|\chi)] \\ &= E_{\chi}[\chi \cdot \lambda] + \text{Var}_{\chi}[\chi \cdot \lambda] \\ &= \lambda + c \cdot \lambda^2 \end{aligned} \tag{3.2}$$

If χ has a Gamma distribution, the claim count distribution described by Algorithm 3.1 is the negative binomial distribution (Beard et al., [1, p. 110]). We shall use the negative binomial distribution to model the claim count distribution when parameter uncertainty is present.

We shall call the parameter c the contagion parameter for the claim count distribution. If $c = 0$, Algorithm 3.1 yields the Poisson distribution.

We now adopt the following notation.

Let Z be a random variable denoting claim severity,
$S(z)$ be the cumulative distribution function for the claim severity, z, and
X be a random variable denoting the aggregate loss for an insured.

Aggregate losses can then be generated by the following algorithm.

Algorithm 3.2

1. Select the number of claims, N, at random from the assumed claim count distribution.
2. Do the following N times
   2.1 Select the claim amount, Z, at random from the assumed claim severity distribution.
3. The aggregate loss amount, X, is the sum of all claim amounts, Z, selected in step 2.1.

We now give expressions for the mean and variance of the aggregate loss distribution generated by Algorithm 3.2.

$$E[X] = E[N] \cdot E[Z] = \lambda \cdot E[Z] \tag{3.3}$$

$$\begin{aligned}\text{Var}[X] &= E_N[\text{Var}(X|N)] + \text{Var}_N[E(X|N)] \\ &= \lambda \cdot \text{Var}[Z] + (\lambda + c \cdot \lambda^2) \cdot E^2[Z] \\ &= \lambda \cdot E[Z^2] + c \cdot \lambda^2 \cdot E^2[Z] \end{aligned} \tag{3.4}$$

Implicit in the use of Algorithm 3.2 is the assumption that the claim severity distribution, $S(z)$, is known. In practice, this distribution must be estimated from historical observations, or it must be simply assumed. Under such conditions, errors in selecting the parameters of the claim severity are inevitable. To model parameter uncertainty in the claim severity distribution, we make the simplifying assumption that the shape of the distribution is known, but there is uncertainty in the scale of the distribution. Venter [12] makes the same assumption in his treatment of parameter uncertainty.

More precisely, we specify parameter uncertainty of the claim severity distribution in the following manner.

Let β be a random variable satisfying the conditions $E[1/\beta] = 1$ and $\text{Var}[1/\beta] = b$. We then model aggregate losses by the following algorithm.

Algorithm 3.3

1. Select the number of claims, N, at random from the assumed claim count distribution.
2. Select the scaling parameter, β, at random from the assumed distribution.
3. Do the following N times.
   3.1 Select the claim amount, Z, at random from the assumed claim severity distribution.
4. The aggregate loss amount, X, is the sum of all claim amounts, Z, divided by the scaling parameter, β.

We now give formulas for the mean and variance for the aggregate loss distribution generated by Algorithm 3.3.

$$\begin{aligned}E[X] &= E_\beta[E(X|\beta)] \\ &= E_\beta[\lambda \cdot E(Z)/\beta] \\ &= \lambda \cdot E[Z] \cdot E[1/\beta] \\ &= \lambda \cdot E[Z] \end{aligned} \tag{3.5}$$

$$\begin{aligned}\text{Var}[X] &= E_\beta[\text{Var}(X|\beta)] + \text{Var}_\beta[E(X|\beta)] \\ &= E_\beta[(\lambda \cdot E(Z^2) + c \cdot \lambda^2 \cdot E^2(Z))/\beta^2] + \text{Var}_\beta[\lambda \cdot E(Z)/\beta] \\ &= (\lambda \cdot E[Z^2] + c \cdot \lambda^2 \cdot E^2[Z]) \cdot E[1/\beta^2] \\ &\quad + \lambda^2 \cdot E^2[Z] \cdot \text{Var}[1/\beta] \end{aligned}$$

$$= (\lambda \cdot E[Z^2] + c \cdot \lambda^2 \cdot E^2[Z]) \cdot (1 + b) + \lambda^2 \cdot E^2[Z] \cdot b$$
$$= \lambda \cdot E[Z^2] \cdot (1 + b) + \lambda^2 \cdot E^2[Z] \cdot (b + c + b \cdot c) \quad (3.6)$$

In this paper, we shall assume that β has a Gamma distribution. We shall call b the mixing parameter. The mixing parameter is a measure of parameter uncertainty for the claim severity distribution.

Let R denote the ratio $X/E[X]$. From Equations 3.5 and 3.6, we get the following result:

$$\text{Var}[R] = (1 + b) \cdot E[Z^2]/(\lambda \cdot E^2[Z]) + b + c + b \cdot c \quad (3.7)$$

Under the above assumptions on parameter uncertainty, it is possible to calculate the cumulative probabilities and excess pure premium ratios in an efficient manner (Heckman and Meyers [2]). We have chosen mathematically convenient distributions to model parameter uncertainty. We do not want to imply that these distributions are in any way the "correct" ones. Since parameter uncertainty is not directly observable, it is difficult to discover what the correct distribution should be. As we shall show, it is possible to infer the values of b and c through the use of Equations 3.6 and 3.7. But until statistical methodology has advanced to the point where the proper distributions can be determined, it should be acceptable to use ones which are mathematically convenient.

Interpreting the Model

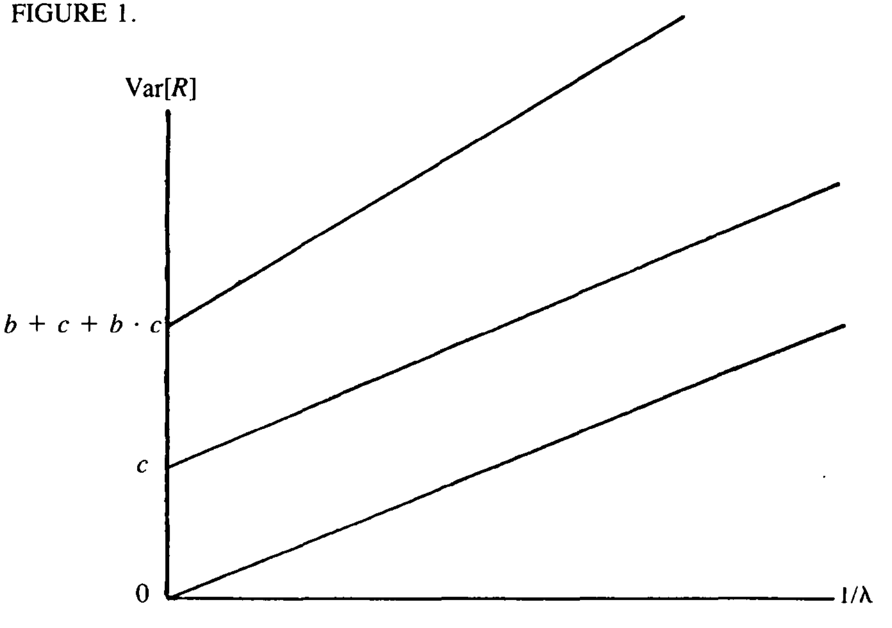
As mentioned in the introduction, we are concerned with the relationship between the variance of the loss ratio and the size of the insured. Parameter uncertainty can perhaps best be understood in terms of how it affects this relationship.

In what follows, it will be helpful to recognize that Var[R] is equal to the squared coefficient of variation of the loss ratio.

It can be seen from Equation 3.7 that this model implies a linear relationship between Var[R] and $1/\lambda$.

Figure 1 illustrates the effect of parameter uncertainty on Var[R]. If $b = c = 0$, the aggregate loss distribution is the generalized Poisson distribution. In this case Var[R] approaches zero as $1/\lambda$ approaches zero (or as the insured becomes larger). If we introduce parameter uncertainty in the claim count distribution, Var[R] lies on a line parallel to that implied by the generalized Poisson distribution, with the variance approaching c as $1/\lambda$ approaches zero. If we add parameter uncertainty in the claim severity distribution, the slope of the line increases by a factor of $1 + b$. The Var[R] approaches $b + c + b \cdot c$ as $1/\lambda$ approaches zero.

FIGURE 1.



4. ESTIMATING THE PARAMETERS OF THE MODEL

In the previous section, we proposed a version of the collective risk model that accounts for parameter uncertainty. This model depends upon the expected number of claims, λ, the claim severity distribution, $S(z)$, the contagion parameter, c, and the mixing parameter, b.

The expected number of claims can be estimated from historical claim frequencies and estimates of current exposure.

A complete discussion of estimating claim severity distributions is beyond the scope of this paper. In our work, we typically obtain claim severity distributions from bureau circulars, or we estimate them from company data using methods similar to those described by Patrik [8]. These claim severity distributions are often derived from experience other than that of the insured under consideration. For this reason, we adjust the scale of the distribution to match the average claim size that we project for the insured.

Before discussing our methods for estimating b and c, we should mention the work of Patrik and John [9]. They deal with parameter uncertainty by picking a finite set of claim severity distributions and claim count distributions for the collective risk model. They then combine the various outputs of the model by taking a weighted average. The weights are probabilities which they assign subjectively.

The use of subjective probabilities has always been controversial. Many consider the word "guess" to be more appropriate. It is unfortunate that in many situations an answer is demanded, but no data is available. Under these circumstances, the use of subjective probabilities may be acceptable.

Regardless of how one feels toward the use of subjective probabilities, one should always consider the possibility of estimating b and c from observations of aggregate loss data. The remainder of this section will develop ways of doing this.

We will describe two approaches for estimating b and c. The choice of estimators will depend on the kind of data available. The general idea underlying both of these approaches will be to evaluate an expression which "resembles" $\text{Var}[N]$, $\text{Var}[X]$ or $\text{Var}[R]$. Using Equations 3.2, 3.6 or 3.7, we can then set the expression equal to its expected value, which depends upon b and c. The estimates are then obtained by solving for b and c. The details are given in the appendices.

We first consider the case of a single insured for which we have r years of experience. We assume that all systematic adjustments of the data, such as trend and loss development, have been made.

The following estimators for b and c are derived in Appendix A.

For $j = 1,\ldots,r$, let N_j be the claim count for year j. Let e_j be a number such that $e_j = K \cdot E[N_j]$ for each year j. K is a constant of proportionality. The number e_j represents either exposure or premium.

Let $\hat{\lambda}_1 = (1/r) \sum_{j=1}^{r} N_j \cdot (e_1/e_j)$ and

$$V = \sum_{j=1}^{r} ((e_1/e_j) \cdot N_j - \hat{\lambda}_1)^2.$$

Then an estimator for c is given by

$$\hat{c} = \frac{V - (r-1)/r \sum_{j=1}^{r} (e_1/e_j) \cdot \hat{\lambda}_1}{(r-1) \cdot \hat{\lambda}_1^2}. \tag{4.1}$$

Let $X_1,\ldots,X_r$ be r independent aggregate loss amounts associated with $N_1,\ldots,N_r$, respectively, and let $A_j = X_j/N_j$ be average claim costs. Furthermore let

$$N = \sum_{j=1}^{r} N_j,$$

$\hat{\mu} = \sum_{j=1}^{r} X_j/N$ be an estimate of $E[Z]$,

$\hat{\sigma}^2$ be an estimate of Var$[Z]$, and

$$W = \sum_{j=1}^{r} N_j \cdot (A_j - \hat{\mu})^2.$$

Then an estimator for b is given by

$$\hat{b} = \frac{W - (r-1) \cdot \hat{\sigma}^2}{(r-1) \cdot \hat{\sigma}^2 + \hat{\mu}^2 \cdot (N - (1/N) \sum_{j=1}^{r} N_j^2)}. \tag{4.2}$$

We used Monte Carlo simulation to test the accuracy of these estimators. Specifically, we selected a claim severity distribution (given in Exhibit I) along with $c = 0.1$ and $b = 0.1$ We then simulated aggregate losses and claim counts using Algorithm 3.3, and tested how well the estimates $\hat{b}$ and $\hat{c}$ compared with the selected b and c. We obtained $\hat{\sigma}^2$ by multiplying the coefficient of variation of our claim severity distribution by the estimate of $E[Z]$ obtained from the simulation. The results of these tests are given in Table 1.

TABLE 1

SIMULATED ESTIMATES OF b AND c

Expected Losses	r	Average $\hat{c}$	Std Dev $\hat{c}$	Average $\hat{b}$	Std Dev $\hat{b}$
250,000	5	.1004	.0720	.0364	.1746
"	25	.1002	.0326	.0759	.1203
"	100	.1005	.0153	.1017	.0798
"	400	.1013	.0086	.0999	.0441
1,000,000	5	.0957	.0672	.0787	.1080
"	25	.0973	.0305	.0956	.0624
"	100	.1014	.0155	.0961	.0273
"	400	.0986	.0077	.1006	.0145
5,000,000	5	.0973	.0670	.0873	.0847
"	25	.1001	.0308	.0956	.0390
"	100	.0992	.0218	.0995	.0252
"	400	.0988	.0084	.0993	.0098

The averages and standard deviations of the estimates for r = 5, 25, 100, and 400 were based on 2000, 400, 100, and 25 trials, respectively. While the estimates of the errors are subject to simulation error, the above table suggests that one may need several hundred observations to accurately estimate b and c. Clearly, this is impossible for a single insured. We have repeated this experiment for different values of b and c and have gotten similar results.

Table 1 does not tell how much accuracy is necessary. To answer this, one must first ask how the collective risk model will be used. An almost certain use is the calculation of excess pure premium ratios. How much error one may tolerate for b and c will then depend on how much error one may tolerate for excess pure premium ratios.

Table 2 gives excess pure premium ratios for various sizes of insureds and various b's and c's. The method of calculating the excess pure premium ratios is that of Heckman and Meyers [2].

TABLE 2

EXCESS PURE PREMIUM RATIOS
Expected Loss = 1,000,000

Entry Ratio	$b = c = 0.0$	$b = c = .01$	$b = c = .05$	$b = c = .10$
0.50	0.500	0.500	0.504	0.513
1.00	0.083	0.100	0.149	0.191
1.50	0.005	0.009	0.032	0.064
2.00	0.000	0.001	0.006	0.022
2.50	0.000	0.000	0.001	0.007

Expected Loss = 5,000,000

Entry Ratio	$b = c = 0.0$	$b = c = .01$	$b = c = .05$	$b = c = .10$
0.50	0.500	0.500	0.502	0.509
1.00	0.038	0.068	0.130	0.176
1.50	0.000	0.001	0.020	0.053
2.00	0.000	0.000	0.003	0.016
2.50	0.000	0.000	0.000	0.005

Table 2 shows that significant differences in excess pure premium ratios result from different values of b and c. Taking this result along with the results indicated by Table 1, we are forced to the rather unpleasant conclusion that parameter uncertainty cannot be adequately quantified on the basis of the experience of a single insured.

If we are to estimate b and c from empirical data, it would appear that our only alternative is to combine the experience of several insureds, and assume that the same b and c are appropriate for all of them. It is this question that we now address.

5. ESTIMATING THE PARAMETERS OF THE MODEL—CONTINUED

If we were to combine the experience of several insureds to estimate b and c, we might consider using the above estimators and treating the combined observations as annual observations of a single insured. We feel, however, that this would be inappropriate for the following reasons.

First, a key assumption in the estimation procedure for c is that the expected number of claims is directly proportional to the measurement of exposure. While this assumption may be appropriate for a single insured, different insureds may have different exposure bases or different inherent claim frequencies.

Second, a key assumption in the estimation procedure for b is that the same claim severity distribution is appropriate for all observations of a single insured. Different insureds can expect to have different claim severity distributions.

Below we will give estimators for b and c. The data requirements for these estimators are as follows.

For each insured and each year we require three items:

1. exposure,
2. incurred losses, and
3. incurred claim count.

Some remarks concerning the data requirements are in order.

First, it will be assumed that the exposure is directly proportional to the expected claim count for each insured. The constant of proportionality may vary with the insured. Many exposure bases, such as payroll, are inflation sensitive. Thus, trends in the exposure base that do not reflect expected claim count should be removed.

Second, it will be assumed that the expected claim severity is the same for all observations of a single insured. The expected claim size need not be the same for all insureds. Incurred losses must be adjusted for trends in claim severity. The trend factors must be derived from external sources so as not to introduce bias in the estimates.

Third, every effort should be made to get the maximum number of observations. A minimum of two observations per insured will be required. We should strive for the maximum number of observations per insured and the maximum number of insureds.

The following estimators for b and c are derived in Appendix A:

T = number of insureds
r_i = number of observations for insured i
N_{ij} = number of claims for observation j of insured i
e_{ij} = exposure for observation j of insured i
X_{ij} = incurred loss for observation j of insured i

$A_{ij} = X_{ij}/N_{ij}$
Z_i = a random variable denoting claim severity for insured i.

For each i, let

$$\hat{\lambda}_{i1} = (1/r_i) \sum_{j=1}^{r_i} N_{ij} \cdot (e_{i1}/e_{ij}) \quad \text{and}$$

$$V = \sum_{i=1}^{T} \sum_{j=1}^{r_i} (N_{ij} \cdot e_{i1}/e_{ij} - \hat{\lambda}_{i1})^2.$$

Then, an estimator for c is given by

$$\hat{c} = \frac{V - \sum_{i=1}^{T} (r_i - 1)/r_i \sum_{j=1}^{r_i} (e_{i1}/e_{ij}) \cdot \hat{\lambda}_{i1}}{\sum_{i=1}^{T} (r_i - 1) \cdot \hat{\lambda}_{i1}^2}. \tag{5.1}$$

For each i, let

$$N_{i.} = \sum_{j=1}^{r_i} N_{ij},$$

$$\hat{\mu}_i = \sum_{j=1}^{r_i} X_{ij}/N_{i.} \text{ be an estimate for } E[Z_i],$$

$\hat{\sigma}_i^2$ be an estimate for Var[Z_i], and

$$W = \sum_{i=1}^{T} \sum_{j=1}^{r_i} N_{ij} \cdot (A_{ij} - \hat{\mu}_i)^2.$$

Then, an estimator for b is given by

$$\hat{b} = \frac{W - \sum_{i=1}^{T} (r_i - 1) \cdot \hat{\sigma}_i^2}{\sum_{i=1}^{T} ((r_i - 1) \cdot \hat{\sigma}_i^2 + \hat{\mu}_i^2 \cdot (N_{i.} - (1/N_{i.}) \sum_{j=1}^{r_i} N_{ij}^2))}. \tag{5.2}$$

Note that Equations 5.1 and 5.2 reduce to Equations 4.1 and 4.2, respectively, when $T = 1$.

Many companies and rating bureaus do not have the data required for the above estimators. However, all is not lost. We shall now show that it is possible to get rough estimates of b and c with quite a bit less data.

One of the predictions of this version of the collective risk model is the linear relationship of the squared coefficient of variation of the loss ratio with $1/\lambda$ (or equivalently, 1/expected losses). See Equation 3.7 and Figure 1. The following estimators of b and c will exploit this relationship.

The data requirements for these estimators will be loss ratios and premiums for each insured, and a single claim severity distribution that represents all the insureds. Divide the insureds into T groups of size r_i. For reasons stated above, we would prefer that the groups consist of multiple observations of the same insured. We shall say something about this in the next section. For observation j of group i, let

e_{ij} = exposure (premium),
X_{ij} = incurred loss, and
$R_{ij} = X_{ij}/e_{ij}$.

For each group i, let

$$\hat{\mu}_i = (1/r_i) \sum_{j=1}^{r_i} R_{ij},$$

$$E_i = (1/r_i) \sum_{j=1}^{r_i} 1/e_{ij}, \text{ and}$$

$$W_i = \sum_{j=1}^{r_i} (R_{ij} - \hat{\mu}_i)^2.$$

An estimate of the squared coefficient of variation for the ith group is given by the expression

$$\frac{W_i}{(r_i - 1) \cdot \hat{\mu}_i^2}.$$

An estimate of 1/expected losses for the ith group is given by the expression

$$\frac{E_i}{\hat{\mu}_i}.$$

Using linear regression, we can find an approximate relationship of the following form:

$$\frac{W_i}{(r_i - 1) \cdot \hat{\mu}_i^2} = \hat{A} \cdot \frac{E_i}{\hat{\mu}_i} + \hat{B}. \tag{5.3}$$

Estimators for b and c are then given by

$$\hat{b} = \hat{A} \cdot E[Z]/E[Z^2] - 1 \text{ and} \tag{5.4}$$
$$\hat{c} = (\hat{B} - \hat{b})/(1 + \hat{b}). \tag{5.5}$$

These estimators for b and c are derived in Appendix B. As mentioned above, we must select a single claim severity distribution that represents all insureds. In practice, it is questionable that this can be done. If estimates are obtained in this manner, then the variance of the aggregate loss distribution will be overstated for the low severity insured and understated for the high severity insured. Meyers [4] discusses several problems associated with this. It should be noted that the linear relationship between the squared coefficient of variation and 1/expected losses derived from Equation 5.3 will be preserved in the model if any reasonable claim severity distribution is selected. However, one should not put undue faith in the particular estimates of b and c.

6. TESTING THE MODEL

Thus far we have proposed a version of the collective risk model which allows for parameter uncertainty, and we have given ways to estimate the parameters for this model. We now turn to the crucial question, how well does it fit empirical data?

In 1980, a National Council committee assembled a large sample of individual insured data for the purpose of constructing a new table of excess pure premium ratios, otherwise known as Table M. This sample contained the standard premium and the incurred losses for all insureds during the policy year beginning July 1, 1973 for all states in which the National Council had jurisdiction.

The data was grouped by premium size, and the empirical loss ratio distributions were used to calculate the excess pure premium ratios for the smaller premium sizes. For those insureds with premium of $200,000 or more, it was felt that the empirical excess pure premiums were not credible, and so a combination of modeling and empirical data was used.

After the table was completed, we requested and received from the National Council a tape containing this experience. This data forms the basis of our analysis.

Estimating the Parameters of the Model

Since we had only premium and loss data, we used Equations 5.4 and 5.5 with the e_{ij}'s representing standard premiums. Since we had only one observation for each insured, we chose our groups on the basis of premium size. Thus there are two main sources of parameter uncertainty. The first source is heterogeneity of insureds, and the second is differences in premium adequacy. This is consistent with the construction of the new Table M.

It should be noted that if we choose our groups on the basis of multiple observations on a single insured with the e_{ij}'s representing exposure, the sources of parameter uncertainty are quite different. Heterogeneity of insureds is not involved. Changes in the insured's operations, changes in economic conditions, changes in loss development patterns and other changes over time are the main sources of parameter uncertainty. Thus, estimates of b and c under these conditions could be quite different.

At the time this study was done, the National Council did not have a claim severity distribution available. The closest thing we had to a comparable claim severity distribution was estimated from our own company data for accident year 1975 developed to 42 months. We chose 42 months because it matched the average maturity of the NCCI data. We then changed the scale of the distribution to match the average claim size which was reported by the National Council for the policy year 1973–74. The resulting claim severity distribution is given in Exhibit I.

In choosing the groups, we put those observations with the lowest r_1 premiums in the first group, those observations with the next lowest r_2 premiums in the second group, and so on. The problem remained of choosing the r_i's, $i = 1, \ldots, n$ for the n groups. We observed that when the r_i's were equal for all i the variance of the residuals of the regression decreased as the premium increased. In statistical terminology, this is known as heteroscedasticity. We dealt with this problem in two ways. One way was to have r_i decrease as the premium increases. The other way was to use a weighted regression.

The weighted regression can be described as follows. If the model $Y = AX + B + \epsilon$ is to be fitted, but if it appears that the standard deviation of ϵ is proportional to X, then let $Y' = Y/X$ and let $X' = 1/X$. In the new model $Y' = A' + B'X' + \epsilon'$, ϵ' will have approximately constant variance. A' will be an estimate of A and B' will be an estimate of B.

Exhibit II gives the various sets of r_i's that we considered. Table 3 gives the resulting estimates of b and c.

TABLE 3

ESTIMATES OF b AND c

Set of r_i's	$\hat{A}$	Std Err $\hat{A}$	$\hat{B}$	Std Err $\hat{B}$	R^2	$\hat{b}$	$\hat{c}$
1	62831	5746	.275	.020	.820	.340	−.048
2	85383	1609	.443	.038	.870	.181	.222
2(1–11)	55378	3786	1.054	1.269	.071	.181	.740
2(12–22)	52831	2728	.469	.046	.921	.127	.304
3	58997	2967	.305	.013	.976	.258	.037
3(1–8)	55418	2226	.495	.176	.570	.182	.265
3(9–15)	64572	8328	.291	.025	.964	.377	−.062
4	55539	4430	.445	.164	.401	.184	.220
5*	59311	13939	.350	.103	.721	.265	.068

* Used unweighted regression

There are several points that should be made about these estimates. First, negative estimates of b and c are possible; $\hat{b}$ will be negative whenever

$$\hat{A} \cdot E[Z]/E[Z^2] < 1;$$

$\hat{c}$ will be negative whenever

$$(\hat{B} + 1) \cdot E[Z^2]/E[Z] < \hat{A}.$$

This can happen if the assumed mean and variance of the claim severity distribution are not appropriate for the given observations. Negative estimates of b and c can also occur because of random variation of the regression coefficients. Examination of the standard errors of $\hat{A}$ suggests that random variation could explain the two negative estimates of c.

If a negative estimate of b is obtained, we suggest setting $\hat{b}$ equal to zero and setting $\hat{c} = \hat{B}$. If a negative estimate of c is obtained, we suggest setting $\hat{c} = 0$ and $\hat{b} = \hat{B}$. If $\hat{B}$ is negative, we suggest setting $\hat{b}$ and $\hat{c}$ equal to zero.

Second, the estimates of A and B vary by the set of r_i's chosen. Examination of the standard errors of the coefficient $\hat{A}$ indicates that this variation could be random. However, the variation in the estimates of B cannot be explained by random variation. It would appear that the estimate of B is decreasing as the size of the insured increases. This can be seen by comparing the pairs of estimates #2(1–11) with #2(12–22), #3(1–8) with #3(9–15), and #4 with #5. In all three comparisons, the estimate of B corresponding to higher premium observations is lower than the estimate of B corresponding to lower premium observations.

This means that the sum, $b + c + b \cdot c$, is decreasing as the premium increases. As mentioned above, the division of parameter uncertainty between b and c is suspect for the estimators used. It seems that parameter uncertainty decreases as the size of the insured increases.

This seems to be a reasonable conclusion. Because of experience rating, one would expect the standard premium to be more accurate for large insureds than for small insureds.

Comparison of Expected with Actual Results

Using the estimates of b and c obtained above, it is possible to calculate the cumulative probabilities and the excess pure premium ratios implied by the model. We now compare the results predicted by the model with the actual results in the National Council data base. This comparison will take two forms. We will first perform chi-square goodness of fit tests. A description of the chi-square goodness of fit test can be found in Hoel [3, p. 226]. We will then compare excess pure premium ratios predicted by the model with those of the new Table M.

We chose three sets of parameters for our testing. In the first test, we set $b = 0$ and $c = 0$ because it represents the case with no parameter uncertainty. For the second test, we chose the estimates $\hat{b} = .258$ and $\hat{c} = .037$ from regression #3 since it produced the highest R^2 over all the points. For the third test, we chose the estimates $\hat{b} = .184$ and $\hat{c} = .220$ when the premium was less than \$125,000 (regression #4), and $\hat{b} = .263$ and $\hat{c} = .068$ when the premium was greater than \$125,000 (regression #5). This enabled us to test if B decreases as the premium increases.

Since the variance of the loss ratio distribution changes with the size of the insured, we decided to estimate the distribution implied by the model and perform the chi-square test on each of several groups of insureds. Each group was to have a fairly narrow range of premium sizes. The results are given in Exhibit III.

No set of parameter values performed well when the premium was less than \$15,000. While the second and third sets of parameters performed better than the first, all sets severely underestimated the number of zero loss ratios. It appears that higher values of c are needed for small premium sizes.

It is difficult to note a pattern in the results of the chi-square tests on individual groups. The chi-square test is simply not powerful enough to distinugish between the various sets of parameter values on individual groups. However, the chi-square test permits combining the results of independent tests. (Actually, the tests are not independent since the parameters b and c were estimated from all the observations. Since the number of observations used in estimating the parameters was far greater than the number of observations in each chi-square test, however, the tests are very nearly independent.) When the results are combined, a clear pattern emerges.

The results predicted by the second and third sets of parameters are better than the results predicted by the parameters $b = 0$ and $c = 0$. Allowing for parameter uncertainty significantly improves the performance of the collective risk model. The results predicted by the third set of parameters are better than the results predicted by the second set for the smaller premium sizes. This is consistent with our hypothesis that B decreases as the size of the insured increases.

Comparisons with the new Table M are given in Exhibit IV. Again we see that allowing for parameter uncertainty significantly improves the performance of the collective risk model. While the model does not fit the new Table M perfectly, it does come reasonably close.

Interpretation of the Results

The combined chi-square statistic calculated in Exhibit III indicates that we should reject the hypothesis that aggregate losses have the distribution predicted by the model. This shows that we have indeed made a number of simplifying assumptions.

This brings us back to the "Actuary's Dilemma." As noted above, the construction of an empirical Table M is suspect because of the necessity of using heterogeneous groups of insureds. It is extremely difficult to tell which is the more accurate. One must look to the applications in order to determine which to use.

Through the end of 1982, Table M was used to determine insurance charges in retrospective rating plans. The same insurance charges were used regardless of what claim severity distributions were appropriate for the insured and what accident limit was selected. Meyers [4] demonstrated that the claim severity distribution and the accident limit have a significant effect on the insurance charge. By examining Meyers' tables, one can see that these differences are much larger than the differences between the collective risk model with parameter uncertainty and the new Table M.

A revision of the retrospective rating plan is currently being considered by the National Council. This revision contains an adjustment for the "overlap" between the insurance charge and the excess loss premium factor. This adjustment was derived using the collective risk model with parameter uncertainty.

While the "Actuary's Dilemma" is not resolved, we see that the collective risk model with parameter uncertainty can make a significant contribution to the solution of today's problems.

7. LARGE INSUREDS

As demonstrated above, it is necessary to combine the experience of several insureds to get stable estimates of b and c. The methods given for estimating b and c assume that these parameters are the same for all insureds. It seems unlikely that b and c are the same for all insureds. For example, a stable company that has been working in the same line of business for many years should have a lower b and c than a company that has recently made material changes to its operations. A detailed examination of a company's operations may reveal additional sources of parameter uncertainty.

For small insureds, it may not be cost effective to conduct such an examination. Thus, it should be acceptable to assume that b and c are the same for all small insureds.

For large insureds, close examinations are routine. It seems quite possible that an underwriter could more accurately quantify parameter uncertainty on the basis of judgmental factors. However, skeptical actuaries respond that while underwriters are very sensitive to both the natural desire to sell insurance and aversion to risk, their quantitative estimates depend very much on what the competition offers. We regard it as an open question as to which method performs the best.

What is an actuary to do under these circumstances? First, we should provide estimates of b and c based on the combined experience of several large insureds. As Morell [6] remarked in his review of the first version of this paper, "We owe them at least that much." Furthermore, the data should contain several years of experience for each insured and the appropriate estimators for b and c should be used. Parameter uncertainty arising from heterogeneity between members of a group of insureds is not applicable for large account pricing.

If a close examination reveals additional sources of parameter uncertainty, sensitivity testing should be done to determine the effect of this uncertainty.

Quite often, the results of such testing can aid in designing a contract that is agreeable to all parties.

The remarks in this section are quite speculative. But they point out the need for extreme caution in using the collective risk model with large accounts.

8. CONCLUSION

This paper proposes a new version of the collective risk model that allows for uncertainty in selecting the expected number of claims and the claim severity distribution. We provide two different methods of estimating the parameters of this model. It is demonstrated by computer simulation that one must combine the experience of several insureds in order to accurately quantify parameter uncertainty. Tests on a very large sample of individual insured data show a significant improvement in the accuracy of the collective risk model when parameter uncertainty is taken into account. The tests do not show perfect agreement between the model and the empirical data, but the agreement is close enough to be useful in many applications.

9. ACKNOWLEDGEMENTS

We would like to thank the National Council for providing us with the data that was used in the paper. An earlier version of this paper was presented in the 1982 CAS call for papers. While preparing a review, Roy Morell made several comments which aided us in preparing this version. We urge the reader to read his review [6]. We would like to thank Bradley Alpert, Yakov Avichai, Philip Heckman and Edward Seligman for their helpful comments.

APPENDIX A — DERIVATION OF EQUATIONS 4.1, 4.2, 5.1 AND 5.2

The equations derived in this appendix require the following data.

T = number of insureds
r_i = number of observations for insured i, $(r_i > 1)$
N_{ij} = number of claims for observation j of insured i
e_{ij} = exposure for observation j of insured i
X_{ij} = incurred loss for observation j of insured i
$\hat{\sigma}_i^2$ = an estimate of $\mathrm{Var}[Z_i]$, where Z_i is a random variable denoting claim severity for insured i.

Estimating c

Let λ_{ij} be the expected number of claims for insured i and observation j. Assume $\lambda_{ij} = K_i \cdot e_{ij}$. Then

$$\lambda_{ij} = \lambda_{i1} \cdot e_{ij}/e_{i1}. \tag{A.1}$$

It follows from Equations 3.2 and A.1 that

$$\mathrm{Var}[N_{ij}] = \lambda_{i1} \cdot e_{ij}/e_{i1} + c \cdot (\lambda_{i1} \cdot e_{ij}/e_{i1})^2. \tag{A.2}$$

Let

$$\hat{\lambda}_{i1} = (1/r_i) \sum_{j=1}^{r_i} N_{ij} \cdot e_{i1}/e_{ij} \text{ and}$$

$$V = \sum_{i=1}^{T} \sum_{j=1}^{r_i} (N_{ij} \cdot e_{i1}/e_{ij} - \hat{\lambda}_{i1})^2.$$

Adding and subtracting λ_{i1} inside the parentheses gives us

$$V = \sum_{i=1}^{T} \left(\sum_{j=1}^{r_i} (N_{ij} \cdot e_{i1}/e_{ij} - \lambda_{i1})^2 - r_i \cdot (\hat{\lambda}_{i1} - \lambda_{i1})^2 \right).$$

Thus,

$$E[V] = \sum_{i=1}^{T} (r_i - 1)/r_i \sum_{j=1}^{r_i} (e_{i1}/e_{ij})^2 \cdot \mathrm{Var}[N_{ij}].$$

Using Equation A.2 we get

$$E[V] = \sum_{i=1}^{T} (r_i - 1)/r_i \sum_{j=1}^{r_i} (e_{i1}/e_{ij}) \cdot \lambda_{i1} + c \cdot \sum_{i=1}^{T} (r_i - 1) \cdot \lambda_{i1}^2.$$

Solving for c we get

$$c = \frac{E[V] - \sum_{i=1}^{T} (r_i - 1)/r_i \sum_{j=1}^{r_i} (e_{i1}/e_{ij}) \cdot \lambda_{i1}}{\sum_{i=1}^{T} (r_i - 1) \cdot \lambda_{i1}^2}.$$

Equation 5.1 is obtained by substituting V for $E[V]$ and $\hat{\lambda}_{i1}$ for λ_{i1} ($i = 1, \ldots, T$). Equation 4.1 is simply Equation 5.1 with $T = 1$.

Estimating b

Let $\mu_i = E[Z_i]$,

$\sigma_i^2 = \text{Var}[Z_i]$,

$N_{i.} = \sum_{j=1}^{r_i} N_{ij}$, and

$A_{ij} = X_{ij}/N_{ij}$.

Let β denote the severity scaling factor.

Then $E[A_{ij}|N_{ij}, \beta] = (1/\beta) \cdot \mu_i$ and

$\text{Var}[A_{ij}|N_{ij}, \beta] = (1/\beta^2) \cdot \sigma_i^2/N_{ij}$.

Thus $E[A_{ij}|N_{ij}] = \mu_i$ and

$$\text{Var}[A_{ij}|N_{ij}] = (1 + b) \cdot \sigma_i^2/N_{ij} + b \cdot \mu_i^2. \qquad \text{(A.3)}$$

Let

$$\hat{\mu}_i = \sum_{j=1}^{r_i} X_{ij}/N_{i.}, \text{ and}$$

$$W = \sum_{i=1}^{T} \sum_{j=1}^{r_i} N_{ij} \cdot (A_{ij} - \hat{\mu}_i)^2.$$

Adding and subtracting μ_i inside the parentheses gives us

$$W = \sum_{i=1}^{T} \left(\sum_{j=1}^{r_i} N_{ij} \cdot (A_{ij} - \mu_i)^2 - N_{i.} \cdot (\hat{\mu}_i - \mu_i)^2 \right).$$

Thus,

$$E[W|N_{ij}\text{'s}] = \sum_{i=1}^{T} \sum_{j=1}^{r_i} (N_{ij} \cdot \text{Var}[A_{ij}|N_{ij}] - (1/N_{i.}) \cdot N_{ij}^2 \cdot \text{Var}[A_{ij}|N_{ij}]).$$

Using Equation A.3 we get

$$E[W|N_{ij}\text{'s}] = (1 + b) \sum_{i=1}^{T} (r_i - 1) \cdot \sigma_i^2 + b \sum_{i=1}^{T} (N_{i.} - (1/N_{i.}) \sum_{j=1}^{r_i} N_{ij}^2) \cdot \mu_i^2.$$

Solving for b we get

$$b = \frac{E[W|N_{ij}\text{'s}] - \sum_{i=1}^{T} (r_i - 1) \cdot \sigma_i^2}{\sum_{i=1}^{T} ((r_i - 1) \cdot \sigma_i^2 + \mu_i^2 \cdot (N_{i.} - (1/N_{i.}) \sum_{j=1}^{r_i} N_{ij}^2))}.$$

Equation 5.2 is obtained by substituting W for $E[W|N_{ij}\text{'s}]$, $\hat{\mu}_i$ for μ_i and $\hat{\sigma}_i^2$ for σ_i^2. Equation 4.2 is simply Equation 5.2 with $T = 1$.

APPENDIX B—DERIVATION OF EQUATIONS 5.4 AND 5.5

The equations derived in this appendix require that individual observations be divided into T groups of size r_i. They also require the first moment, $E[Z]$, and the second moment, $E[Z^2]$, of an assumed claim severity distribution.

For observation j of group i, let

e_{ij} = exposure (premium),
X_{ij} = incurred loss, and
$R_{ij} = X_{ij}/e_{ij}$.

For each group i, let

$\mu_i = E[R_{ij}]$, and

$$\hat{\mu}_i = (1/r_i) \sum_{j=1}^{r_i} R_{ij}.$$

It follows from Equation 3.6 that

$$\text{Var}[R_{ij}] = \frac{\mu_i \cdot (1 + b) \cdot E[Z^2]}{e_{ij} \cdot E[Z]} + \mu_i^2 \cdot (b + c + b \cdot c). \tag{B.1}$$

For each group i, let

$$E_i = (1/r_i) \sum_{j=1}^{r_i} (1/e_{ij}), \text{ and}$$

$$W_i = \sum_{j=1}^{r_i} (R_{ij} - \hat{\mu}_i)^2.$$

Adding and subtracting μ_i inside the parentheses gives us

$$W_i = \sum_{j=1}^{r_i} (R_{ij} - \mu_i)^2 - r_i \cdot (\hat{\mu}_i - \mu_i)^2.$$

It then follows that

$$E[W_i] = (r_i - 1)/r_i \sum_{j=1}^{r_i} \mathrm{Var}[R_{ij}]. \tag{B.2}$$

Combining Equations B.1 and B.2 we get

$$\frac{E[W_i]}{(r_i - 1) \cdot \mu_i^2} = \frac{(1 + b) \cdot E[Z^2]}{E[Z]} \cdot \frac{E_i}{\mu_i} + b + c + b \cdot c. \tag{B.3}$$

If one finds an approximate relationship of the form

$$\frac{W_i}{(r_i - 1) \cdot \hat{\mu}_i^2} = \hat{A} \cdot \frac{E_i}{\hat{\mu}_i} + \hat{B},$$

Equation 5.4 follows by equating $\hat{A}$ with the coefficient of E_i/μ_i in Equation B.3. Equation 5.5 follows by equating $\hat{B}$ with the constant term in Equation B.3 and using Equation 5.4.

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EXHIBIT I

THE CLAIM SEVERITY DISTRIBUTION

Loss Amount	Cumulative Probability
0.00	0.00000
19.79	0.21384
39.57	0.51025
79.15	0.74056
118.72	0.79959
158.29	0.82665
197.86	0.84450
277.01	0.86657
395.73	0.88626
593.59	0.90606
791.45	0.91797
1187.18	0.93388
1582.91	0.94464
1978.63	0.95223
2770.09	0.96242
3957.27	0.97156
5935.90	0.97998
7914.54	0.98476
9893.17	0.98785
11871.80	0.99001
15829.07	0.99281
19786.34	0.99452
27700.87	0.99649
39572.68	0.99790
59359.02	0.99890
79145.31	0.99934
98931.69	0.99956
118718.00	0.99970
158290.69	0.99983
197863.37	0.99990
277708.75	0.99996
395726.56	0.99998
593590.00	0.99999
791453.44	1.00000

Summary Statistics:

Severity Mean = 632.56

Severity Standard Deviation = 5704.69

EXHIBIT IIa

GROUPINGS USED FOR THE REGRESSIONS

	Set of r_i's				
i	#1	#2	#3	#4	#5
1	48600	45000	87922	13632	350
2	34221	39000	60000	11500	325
3	26730	34000	50000	9500	300
4	21008	29000	40000	8000	275
5	31690	25000	30000	7000	250
6	22575	21000	20000	6000	225
7	29624	17500	12800	5000	200
8	18702	15000	6400	4500	175
9	12955	13000	3200	4000	150
10	9478	11000	1600	3500	
11	7304	10000	800	3000	
12	5477	9000	400	2500	
13	4592	8000	200	2000	
14	8573	7000	100		
15	5684	6000	50		
16	7318	5000			
17	4461	4500			
18	2933	4000			
19	3638	3500			
20	2265	3000			
21	1516	2500			
22	1057	1472			
23	724				
24	610				
25	453				
26	827				
27	541				
28	351				
29	231				
30	351				
31	188				
32	222				
33	129				
34	65				
35	41				
36	29				
37	33				
38	9				
39	23				
40	12				
41	8				
42	7				
43	10				

Grouping	Range of Premium Sizes
#1	Premium ≥ 1000
#2	Premium ≥ 1000
#3	Premium ≥ 1000
#4	5000 ≤ Premium ≤ 125000
#5	Premium > 125000

EXHIBIT IIb

Groupings Used for the Regressions

The following table should provide one with an indication of how the premium sizes were spread among the various r_i's.

Premium Lower Boundary	Insured Count	Average Loss Ratio	Squared C. V. of Loss Ratio
1000	20444	.706	105.78
1100	26275	.689	45.16
1250	34277	.698	73.15
1500	26756	.724	45.29
1750	21024	.710	45.67
2000	31724	.711	37.34
2500	22588	.730	23.94
3000	29273	.745	20.28
4000	18709	.728	15.64
5000	26329	.800	13.99
7500	13759	.743	9.03
10000	14261	.764	5.38
15000	11779	.757	4.57
25000	4994	.733	2.69
35000	3843	.760	2.53
50000	2946	.722	1.49
75000	1414	.696	1.48
100000	1368	.655	1.02
150000	582	.684	.96
200000	539	.645	.76
300000	221	.614	.68
400000	130	.654	.49
500000	122	.538	.47
750000	55	.503	.43
1000000	60	.432	.39
Total	313472		

EXHIBIT III

CHI-SQUARE TESTS

Premium Range	Sample Size	$b = 0$ and $c = 0$			$b = .258$ and $c = .037$			$b = .184$ and $c = .220$ for Prem $\leq$ 125000 $b = .263$ and $c = .058$ for Prem $>$ 125000		
		χ^2	DF	P*	χ^2	DF	P*	χ^2	DF	P*
200001-205000	35	1.54	4	.82	5.20	3	.33	5.44	4	.16
175001-178000	41	4.02	5	.55	9.57	4	.05	7.18	4	.13
150001-153000	50	2.21	5	.82	5.68	5	.34	8.16	6	.23
125001-127500	58	17.61	7	.01	7.57	6	.27	2.82	7	.90
100001-102500	129	34.76	10	.00	15.76	10	.11	16.57	10	.08
90001-91500	64	14.27	7	.05	4.22	7	.75	2.38	8	.97
80001-81000	70	13.69	8	.09	11.54	10	.17	17.05	8	.03
70001-70700	48	4.18	5	.52	2.82	5	.73	5.44	5	.36
65001-65550	75	12.61	7	.08	13.34	8	.10	18.17	8	.02
60001-60600	73	8.76	8	.36	10.94	8	.20	14.17	8	.08
55001-55550	94	36.77	8	.00	15.22	8	.05	9.03	8	.34
50001-50500	103	25.82	8	.00	22.45	8	.00	19.63	8	.01
Subtotal		176.24	80	.00	124.32	82	.00	126.04	84	.00
45001-45450	99	34.28	8	.00	13.12	8	.11	7.82	8	.45
40001-40400	99	6.81	8	.44	6.68	8	.57	7.24	8	.51
35001-35350	119	76.36	10	.00	38.25	10	.00	15.01	.10	.11
30001-30300	146	55.40	10	.00	31.45	10	.00	7.22	10	.71
25001-25250	195	118.57	10	.00	56.35	10	.00	12.03	10	.28
20001-20200	222	60.48	10	.00	31.94	10	.00	18.70	10	.04
15001-15075	148	46.82	10	.00	24.36	9	.01	14.96	10	.13
Subtotal		391.91	66	.00	202.15	65	.00	82.98	66	.07
Total		568.15	146	.00	326.47	147	.00	209.02	150	.00

* P = probability that χ^2 is greater than the observed χ^2 if the hypothesis is true.

EXHIBIT IV

Empirical and Model Excess Pure Premium Ratios

Empirical—NCCI 1980 Table M

	Expected Loss					
Entry Ratio	25000	50000	75000	100000	150000	200000
0.25	.789	.771	.764	.760	.756	.753
0.50	.631	.592	.574	.562	.545	.537
0.75	.516	.458	.431	.414	.388	.373
1.00	.430	.360	.330	.310	.280	.260
1.25	.365	.290	.258	.238	.208	.184
1.50	.316	.239	.208	.187	.159	.133
1.75	.277	.200	.171	.154	.128	.098
2.00	.246	.171	.144	.128	.104	.074
2.25	.220	.148	.123	.110	.087	.057
2.50	.198	.129	.106	.094	.073	.044
2.75	.180	.114	.091	.080	.061	.033
3.00	.163	.101	.080	.069	.051	.026

Model—b = .184 and c = .220 for premium $\leq$ 125000
b = .263 and c = .058 for premium $>$ 125000

	Expected Loss					
Entry Ratio	25000	50000	75000	100000	150000	200000
0.25	.785	.771	.765	.762	.753	.752
0.50	.633	.597	.581	.572	.542	.536
0.75	.522	.470	.445	.430	.288	.377
1.00	.438	.376	.346	.328	.281	.267
1.25	.373	.305	.272	.252	.207	.193
1.50	.322	.251	.218	.197	.156	.142
1.75	.281	.209	.176	.156	.120	.106
2.00	.247	.176	.144	.125	.093	.081
2.25	.219	.150	.119	.101	.074	.063
2.50	.195	.129	.100	.083	.059	.049
2.75	.175	.111	.084	.069	.048	.039
3.00	.158	.097	.071	.057	.040	.032

EXHIBIT IV (CONT.)

EMPIRICAL AND MODEL EXCESS PURE PREMIUM RATIOS

Empirical—NCCI 1980 Table M

Model—$b = 0$ and $c = 0$

	Expected Loss					
Entry Ratio	25000	50000	75000	100000	150000	200000
0.25	.764	.753	.751	.750	.750	.750
0.50	.588	.546	.528	.518	.509	.505
0.75	.465	.398	.364	.342	.317	.301
1.00	.377	.296	.254	.227	.192	.170
1.25	.313	.226	.182	.154	.119	.097
1.50	.263	.176	.133	.107	.076	.057
1.75	.224	.140	.101	.077	.050	.036
2.00	.193	.113	.078	.057	.035	.023
2.25	.168	.093	.061	.043	.025	.015
2.50	.148	.078	.049	.034	.018	.011
2.75	.130	.066	.040	.027	.013	.008
3.00	.116	.056	.033	.021	.010	.005

UTILITY WITH DECREASING RISK AVERSION

GARY G. VENTER

Abstract

Utility theory is discussed as a basis for premium calculation. Desirable features of utility functions are enumerated, including decreasing absolute risk aversion. Examples are given of functions meeting this requirement. Calculating premiums for simplified risk situations is advanced as a step towards selecting a specific utility function. An example of a more typical portfolio pricing problem is included.

> "The large rattling dice exhilarate me as torrents borne on a precipice flowing in a desert. To the winning player they are tipped with honey, slaying him in return by taking away the gambler's all. Giving serious attention to my advice, play not with dice: pursue agriculture: delight in wealth so acquired."
>
> KAVASHA Rig Veda X.3:5

Avoidance of risk situations has been regarded as prudent throughout history, but individuals with a preference for risk are also known. For many decision makers, the value of different potential levels of wealth is apparently not strictly proportional to the wealth level itself. A mathematical device to treat this is the utility function, which assigns a value to each wealth level. Thus, a 50-50 chance at double or nothing on your wealth level may or may not be felt equivalent to maintaining your present level; however, a 50-50 chance at nothing or the value of wealth that would double your utility (if such a value existed) would be equivalent to maintaining the present level, assuming that the utility of zero wealth is zero. This is more or less by definition, as the utility function is set up to make such comparisons possible.

For an individual with a steeply ascending utility function, the value of potential wealth needed to risk losing everything on a 50-50 bet may be less than twice the current level; if the function rises slowly it may be considerably greater than twice the current level; and if the utility increases asymptotically to a value not greater than twice the utility of current wealth, such a bet would not be acceptable for any amount.

Through the device of the utility function, diverse risk situations can be compared. For each situation the expected value of the utility of the possible outcomes can be computed, and the situation with the highest expected utility is preferred.

The pioneering work in the modern application of utility theory was done by von Neumann and Morgenstern[6]. They showed that if a preference ordering for a set of risk situations follows certain consistency requirements, then there is a utility function that will give the same preference ordering on those situations. That in effect removes utility theory from the pleasure-pain area; it can be set up as an essential element of consistent management decision making without addressing questions like "Can a corporation feel pain?"

The consistency requirements can be boiled down to three (e.g., see [4]):

i) any two risk situations can be compared; i.e., one is preferable to the other or they are both equivalent;
ii) if A is preferred to B and B to C, then A is preferred to C;
iii) if A is preferred to B and B to C, then there is a unique r between 0 and 1 such that B is equivalent to $rA + (1 - r)C$.

If risk situations are evaluated consistently, under this definition of consistency, they can be ordered by some utility function.

Thus, utility theory seems to be a potentially valuable tool for choosing between alternative risk situations, such as transferring or accepting the risk of loss for a fixed price, in a consistent way.

The practical actuary, however, finds utility theory somewhat of a dilemma: on one hand it provides the basic theoretical foundation for the worth of the insurance product; on the other hand, no examples of its useful application to insurance are available. Dismissing the whole theory risks throwing out the baby with the bath, but until it can be made to work in practice it will not have much appeal.

The training of most actuaries includes an introduction to utility theory and its general relation to risk situations. There is a gap between this and actual application, however. How to choose a specific utility function is part of this gap; applying this function in realistic situations is another.

The present paper aims to narrow this gap somewhat, but is not so ambitious as to try to close it entirely. Several criteria that a utility function should meet are discussed, and examples are given of functions that meet these requirements. The possible application of utility theory to pricing is also addressed. Since an insurer is taking an uncertain situation for a fixed price, a utility approach may help evaluate the attractiveness of the deal.

Finally, simplified risk situations are evaluated using some of the utility functions discussed. The prices implied by different utility functions for these simplified situations can be used by the analyst to close in on a specific function that most closely reflects the insurer's own risk preferences.

As mentioned above, two risk situations can be compared by computing the expected utility of each, with the higher value being preferred. To apply this to premium calculation, the situation of having the risk and the premium is compared to the situation of having neither; i.e., the expected utility for these two situations is compared. For an insurer with surplus of a and no other policies, the indifference premium g for a random loss variable L is defined as the amount that results in the same expected utility both with and without this premium and potential loss. Thus, assuming the utility function u, a, and L are known, g is the solution of

$$E(u(a)) = E(u(a + g - L))$$
$$(E(u(a)) = u(a) \text{ if } a \text{ is constant}).$$

The calculation of this g in a reinsurance context is illustrated in [5].

Presumably, something in addition to g would be needed to make the transfer worthwhile to the insurer. The excess of the premium offered over the indifference premium can be called the risk adjusted value, or RAV, of the proposal. Applications of the RAV concept can be found in [3] and [8]. However, in this context, any premium above the indifference premium would lead to the acceptance of the contract.

In order to apply this pricing principle, a specific utility function is needed. Several criteria for the selection of a utility function have evolved over time. Among them are:

(1) $u(x)$ is an increasing function on $(0,\infty)$; i.e., $u'(x) > 0$. That is, more is always better. The variable u' is referred to as marginal utility, so this criterion says that marginal utility is always positive.
(2) $u(x)$ is concave downwards; i.e., $u''(x) < 0$. This property is referred to as risk aversion in that it implies that the certainty of the expected value of the outcomes is preferred to an uncertain situation. Concave downward utility also means that marginal utility $(u'(x))$ is a decreasing function of wealth; i.e., as more wealth is accumulated less value is placed on an additional dollar. A gambler might have a utility function

that violates this principle; i.e., a price higher than the expected value might be paid for the chance of a large gain.

(3) Absolute risk aversion decreases as wealth increases. Absolute risk aversion is measured by $ra(x) = -u''(x)/u'(x)$. The $ra(x)$ function so defined can be seen to be the percentage change in the marginal utility $u'(x)$. Decreasing absolute risk aversion means that the percentage decrease in marginal utility is itself decreasing. This property can be shown to equate to greater acceptance of risky situations with greater wealth (see [4], p. 35), which seems intuitively appropriate. This concept is illustrated in Appendix 1.

(4) $u(x)$ is bounded above; i.e., there is a number b such that $u(x) < b$ no matter how large x is. This criterion is necessary to keep very rare large value situations from dominating preferences.

As an example, consider a hypothetical national lottery in which Joe, the winner, receives a choice of either $10 million certain or a risk situation in which he gets a very fabulous sum if he can pick the ace of spades at random from a deck of cards and zero otherwise. If the utility of $10 million is above 1/52 of Joe's maximum possible utility, he will take the $10 million no matter how fabulous the sum may be. On the other hand, if Joe's utility function is not bounded, the choice will depend on what the sum is: for a large enough sum he will choose to draw for the ace.

The bounded utility situation seems more reasonable, but this criterion is somewhat controversial. For instance, it could be argued that Joe would indeed choose to draw if the sum were large enough, but that such a sum would be greater than the current wealth of the world. Since we know that world wealth is finite, we judge Joe's decision to keep the $10 million as reasonable; however, if greater wealth were available the decision to draw would eventually become reasonable and would become compelling as the prize continued to increase.

This argument does not seem persuasive, because the finite wealth of the world does not appear all that relevant to the decision to keep the $10 million and be content with the lifestyle it can support. However, to recognize the degree of subjectivity in this judgment, the opposite opinion has been allowed some consideration. Nonetheless, the boundary criterion will be maintained herein. See [1], p. 35, for a complete discussion of this standard.

A fifth criterion is occasionally advanced.

(5) $u'(x) = 0$ for $x < 0$; i.e., utility is constant for negative values of wealth. This is designed to reflect bankruptcy laws and the corporate form of organization, which presumably make financial entities indifferent as to how bankrupt they become. In a regulated insurance industry, an insurer would not be completely free to act in accordance with this principle, and it probably exaggerates the effect limited liability has on decisions. However, the behavior of the utility function for negative values of wealth is important and must be considered explicitly when choosing a utility function. A more reasonable approach to negative wealth may be to take $u(x) = -u(-x)$. While raising this issue, the current paper does not attempt to settle it. In the examples below, negative values of wealth will not be possible. The following minimal condition will, however, be imposed:

$u(x)$ is defined, continuous, and non-decreasing for $x \le 0$.

Since preference orderings are not altered by linear transformations of the utility function, by suitable normalization any utility function meeting the criteria 1, 4, and 5 could be transformed to take values between 0 and 1, for $x \ge 0$, without altering the preference orderings. Such utility functions and increasing probability distribution functions for positive variables are, therefore, the same class of mathematical mappings from the positive real numbers to the unit interval. Thus, the literature on probability distributions provides a rich source of functional forms for utility functions. Some distribution functions will not satisfy criteria 2 and 3, however, so these properties must be checked individually.

Examples of functions that do not meet the above criteria are:

$u(x)$	Fails
x	2
$x - a/2x^2$ $(x \le a)$	3
$a + b \ln(x + c)$	4
$1 - \exp(-bx)$	3
x^a $(0 < a < 1)$	4
$1 - 1/x$	5

The Weibull and Pareto distribution functions do meet all criteria for proper parameters; e.g., $u(x) = 1 - \exp(-bx^c)$, $c < 1$ and $u(x) = 1 - (bx + 1)^{-c}$.

Other special cases of the transformed beta and gamma distribution functions [7] will also suffice.

Although exponential utility fails criterion 3, it leads to relatively simple computations. Advocates of exponential utility argue that decreasing absolute risk aversion is important but its effects can be provided by changing the parameter of the function as wealth increases. While feasible, it seems undesirable to do this, as the example in Appendix 1 illustrates. A utility function should capture the preferences of the decision maker, including the relationship of preferences to wealth. Questions such as, "if you had $50 million and were offered . . ." should be used to help determine these preferences. In other words, the utility function should be able to get at fundamental attitudes towards risk including how reactions will change with wealth.

Looking at absolute risk aversion as the percentage change in marginal utility provides another approach to this issue. The marginal utility of wealth should decrease as wealth increases, but decreasing absolute risk aversion means that the percentage decrease in marginal utility should itself be declining. If a utility function does not reflect this decline, it is not properly valuing various wealth potentials. In other words, decreasing absolute risk aversion is not simply a matter of having different attitudes towards risk at different wealth levels. It is rather an aspect of the shape of the utility function at every point and reflects the relative desirability of the different levels of wealth themselves.

Exponential utility has other aspects that make it unrealistic in insurance situations. One of these is additivity. Of course, the individual risk premiums must add up to the portfolio premium; this does not mean, however, that the indifference premium for a single risk should be .001 of the premium for 1000 such risks. Under exponential and linear utility, and only with these forms, the indifference premium for a number of independent risks will be the sum of the indifference premiums for the risks separately [1], [4]. This is contrary to usual practice. For instance, there is generally thought to be a benefit to pooling, since the probability of being a large percentage away from expected results becomes less as individual risks are pooled. In other cases, adding independent risks might jeopardize surplus enough that a higher charge would be needed for the last one. Neither such effect is captured by exponential or linear utility.

Risk decisions under exponential utility do not reflect the other risks that may be in the portfolio [8], which again appears unrealistic. All these problems essentially derive from the constant absolute risk aversion of the exponential, which renders decision making independent of wealth. Although calculations are more difficult when decreasing risk aversion is required, this seems unavoidable in realistic insurance situations.

Using utility concepts can help bring consistency to risk decisions. However, the selection of a utility function requires some time and attention. Examining many simplified risk situations to determine which functions best reflect the preferences of the decision maker is one approach.

For example, the indifference premium g is calculated below for some very simple loss distributions, using the utility functions $u(x) = 1 - \exp(-.01x^{.25})$ and $v(x) = 1 - (1 + 10^{-7}x)^{-1}$. These functions are primarily illustrative and are not necessarily advocated. Companies with surplus, a, of \$20,000,000 and \$50,000,000 will be considered. A risk with a .001 probability of a total loss of \$10,000,000 and a .999 probability of no losses will be used.

The indifference premium for v is the solution of:

$$v(a) = .001\ v(a + g - 10{,}000{,}000) + .999\ v(a + g)$$

or $1/(1 + a10^{-7}) = .001/((a + g)10^{-7}) + .999/(1 + (a + g)10^{-7})$.

For selected values of a the equation can be solved for g algebraically. In fact, $g = -a + (10^7/2c)(1 - c + ((1 + c)^2 + .004\ c)^{1/2})$ where $c = (1 + 10^{-7}a)^{-1}$. The similar equation for u may be solved iteratively. The indifference premiums are shown below.

Surplus	u	v
20,000,000	13,422.56	14,988.78
50,000,000	11,101.62	11,997.13

Two such independent risks would have a .000001 probability of \$20,000,000 in losses, a .001998 probability of a single \$10,000,000 loss, and a .998001 probability of no losses. Thus for u, the indifference premium g is the solution of:

$$u(a) = .000001\ u(a + g - 20{,}000{,}000) + .001998\ u(a + g - 10{,}000{,}000) + .998001\ u(a + g).$$

This and the corresponding equation for v can be solved iteratively to yield:

Surplus	u	v
20,000,000	26,889.03	29,985.23
50,000,000	22,203.42	23,994.49

As would be anticipated, the less wealthy company needs a higher premium to take on these risks. Also, contrary to what might be expected from pooling considerations, the premium for two independent risks is somewhat more than twice the premium for a single risk in these cases, especially for the smaller company. This may be realistic in this case because the risk loss is a substantial proportion of surplus and two losses will nearly bankrupt the company. Since the coefficient of variation of the two independent risks is lower than that of a single risk, premium calculation principles based on the standard deviation (or variance) of the aggregate loss distributions would not capture this effect. It would be nice to have an example of a case where a pooling benefit were shown. This would probably require the specification of utilities of negative wealth. The benefit of pooling might be to reduce the surplus needed per risk at a given price; i.e., for a fixed ratio $g/E(L)$, the ratio of needed surplus to number of risks may decrease with the addition of independent risks.

The consideration of simplified situations such as those above can help determine a utility function. The indifference premium for a portfolio of real business can be calculated by these same principles from the utility function and the probability distribution function of aggregate losses, although the calculation will be more intricate for a continuous loss distribution function. An example is given in Appendix 2.

Spreading a portfolio premium to the individual risks is unfortunately a somewhat arbitrary process in this context. Possibilities include spreading in proportion to expected losses or finding the exponential utility function that gives the same overall portfolio premium, and using that to determine the individual insured's premium.

The expected value method does not differentiate contracts by hazard, and thus is probably most appropriate when the riskiness is fairly homogeneous. Exponential utility will give such a differentiation, but this may be somewhat artificial. In the typical situation, where individual insureds are not independent, due to common parameter risk, even the exponential utility premiums will not add up to the portfolio premium. Spreading premium to individual insureds in a realistic way is a problem that merits further research.

An elegant suggestion has been presented by Borch [2]. He recommends calculating the premium for the random loss variable X by the formula $(1 + i)\ E(X) + j\ \mathrm{cov}(X,L)$, where L is the portfolio aggregate loss random variable. This formula gives premiums that add up to the portfolio premium even when risks are not independent. An example is discussed in Appendix 2.

The selection of a realistic utility function requires careful consideration of the implications of this choice in comparison with the judgments the function aims to model. Starting with functions that meet certain general criteria and then examining how they perform in simplified situations can help in this process. The Weibull and Pareto distribution functions provide forms that meet all the criteria discussed herein, although the extension to negative wealth deserves further attention. A practitioner would need to consider specific parameter values and decide which, if any, are appropriate for a specific application. The rewards of this effort would be a procedure for evaluating diverse risk situations from a consistent perspective.

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APPENDIX 1

RISK AVERSION EXAMPLE

Consider two utility functions $u(x) = 1 - \exp(-x/b)$ and $v(x) = 1 - \exp(-(x/b)^{.5})$, which have the $ra(x)$ functions $1/b^2$ and $(1/2b)(b/x)^{.5}(1 + (b/x)^{.5})$ respectively. Thus, u has constant risk aversion and v has decreasing risk aversion.

Now it is easy to show for u that decisions do not depend on the current wealth: some algebraic manipulation yields $E(u(a + L)) = 1 - \exp(-a/b)E(\exp(-L/b))$ for any wealth level a and risk situation L. This is greater than $u(a)$ if and only if $E(\exp(-L/b))$ is less than unity; thus, preferences are independent of wealth.

However, for v this is not true. The acceptance of risk will in fact increase as wealth increases. Consider a risk which will yield a profit of $11,750 with 90% probability and a loss of $100,000 with 10% probability. This is examined at two levels of wealth, a = $1,000,000 and a = $5,000,000, below. A value of 1,000,000 is selected for b.

a:	1,000,000	5,000,000
$u(a)$:	.632121	.99326205
$u(a + 11{,}750)$:	.636418	.99334076
$u(a - 100{,}000)$:	.593430	.99255342
$E(u(a + L))$:	.632119	.99326203
$v(a)$:	.632121	.893122
$v(a + 11{,}750)$:	.634269	.893402
$v(a - 100{,}000)$:	.612749	.890693
$E(v(a + L))$:	.632117	.893131

Thus for u, the risk is rejected at every wealth level, while for v it is rejected at $a = 1{,}000{,}000$ and accepted at $a = 5{,}000{,}000$.

Now, if one wanted to stay with exponential utility because of its easier calculations, one could change b to 5,000,000 when a changed. This would lead to the acceptance of the risk at the higher wealth level. However, the situation at this level is similar to a choice between L + $4,000,000 and $4,000,000 certain at the lower wealth level. In fact, with the fixed value $b = 1{,}000{,}000$, v will evaluate this choice at $a = 1{,}000{,}000$ the same as L versus zero at $a = 5{,}000{,}000$. However, u with the changed parameter will evaluate them differently.

APPENDIX 2

PREMIUM CALCULATIONS

In this Appendix the indifference premium is calculated for a continuous aggregate loss distribution with a stop loss cover. The insurer has \$50 million of surplus and the utility function $v(x) = 1 - (1 + 10^{-7}x)^{-1}$. The aggregate losses, L, to be insured are transformed gamma distributed [7] with mean, coefficient of variation, and coefficient of skewness of \$50 million, .363, and .406 respectively. The density function is taken in the form $f(x) = (ab/(r - 1)!)(bx)^{ar-1}\exp(-(bx)^a)$, where $(r - 1)!$ denotes the gamma function evaluated at r. This gives $E(X^n) = (r - 1 + n/a)!/b^n(r - 1)!$. The moments given arise when $a = r = 2$ and $b^{-1} = 37{,}612{,}639$. This distribution is a bit more dangerous than would arise in many property-casualty insurance lines with that much volume, but not exceptionally so. It could represent one of the more risky liability lines.

Stop loss insurance with a \$100 million retention is proposed, so negative surplus would not be possible if at least \$50 million is charged. The indifference premium for the retained business is desired. This will be the solution g of:

$$v(a) = E(v(a + g - L)) \text{ or}$$
$$1 - (1 + 10^{-7}a)^{-1} = 1 - E((1 + 10^{-7}(a + g - L))^{-1}), \text{ or}$$
$$(1 + 10^{-7}a)^{-1} = E((1 + 10^{-7}(a + g - L))^{-1}), \text{ or}$$
$$1/6 = E((6 + h - 10^{-7}L)^{-1}), \text{ where } h = 10^{-7}g.$$

Because of the stop loss, any loss greater than \$100 million will be cut off at \$100 million in computing this expectation. Thus the equation for g becomes:

$$1/6 = \int_0^{100M} \frac{f(x)\,dx}{6 + h - 10^{-7}x} + \int_{100M}^{\infty} \frac{f(x)\,dx}{6 + h - 10}$$

where $f(x) = 2x^3b^4\exp(-(bx)^2)$.

Now $Pr\,(L \leq 100M) = .00687$ can be calculated via the incomplete gamma function [7], and so we seek h, the solution of:

$$1/6 - \frac{.00687}{h - 4} = \int_0^{100M} \frac{f(x)dx}{6 + h - 10^{-7}x}.$$

By numerical integration and iteration, $h = 5.6568$ can be found, yielding the indifference premium $g = \$56{,}568{,}000$. This calculation can be done by computer or a good programmable calculator.

To apply Borch's formula to distribute this premium to individual insureds we must first choose constants i and j so that $(1 + i)E(L) + j\ \mathrm{var}(L) = g$. Then the premium for an insured with random loss variable X will be $(1 + i)E(X) + j\ \mathrm{cov}(X,L)$. These will add up to g since $\mathrm{cov}(X + Y,L) = \mathrm{cov}(X,L) + \mathrm{cov}(Y,L)$ and $\mathrm{cov}(L,L) = \mathrm{var}(L)$.

One way to select i and j might be to first select i as a desired profit load for a hypothetical insured that does not contribute to the overall portfolio variance, i.e., for which $\mathrm{cov}(X,L) = 0$. Then j can be solved for from g and the moments of L. Thus suppose $i = .02$ is selected. Then $j = (g - 1.02\ E(L))/\mathrm{var}(L)$.

For instance, suppose that in the above example g = \$60,000,000 were calculated for the case where the stop loss is removed. (This calculation would require specification of $v(x)$ for $x < 0$.) Since $E(L) = 50{,}000{,}000$ and $\mathrm{var}(L) = 3.3 \times 10^{14}$, $j = 9{,}000{,}000/3.3 \times 10^{14} = 2.7 \times 10^{-8}$ can be computed for this case.

TRANSFORMED BETA AND GAMMA DISTRIBUTIONS AND AGGREGATE LOSSES

GARY VENTER

Abstract

Distribution functions are introduced based on power transformations of beta and gamma distributions, and properties of these distributions are discussed. The gamma, beta, *F*, Pareto, Burr, Weibull and loglogistic distributions are special cases. The transformed gamma mixed with a gamma yields a transformed beta.

The transformed gamma is used to model aggregate distributions by matching moments. The transformed beta is used to account for parameter uncertainty in this model. Calculation procedures are discussed and *APL* program listings are included.

The transformed gamma is compared to exact methods of computing the aggregate distribution function based on the entire frequency and severity distributions.

INTRODUCTION

For pricing aggregate covers it is useful on occasion to have a way to estimate the distribution function for aggregate losses from the moments of this distribution. The usual approximation methods are designed primarily to calculate percentiles of the far right tail for mildly skewed distributions (e.g., see Pentikainen [9]). The gamma distribution has been suggested for this purpose (e.g., Hewitt [7]). However, the skewness of the gamma is always twice the coefficient of variation (see Hastings & Peacock [6]). Adding a third parameter to the gamma has been suggested by Seal [10], but the added parameter shifts the origin, sometimes resulting in the possibility of negative losses, which is often unsatisfactory. The transformed gamma distribution offers an alternative third parameter that affects the shape of the distribution but not its location.

The transformed beta and its special cases could be tried in this regard, also. However, its principal application herein is to deal with one kind of parameter uncertainty in the transformed gamma. The distributions are introduced below and then applications are discussed for each.

TRANSFORMED GAMMA AND TRANSFORMED BETA DISTRIBUTIONS

Transformed Gamma

The gamma function at r is defined as $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt$. The percentage of this integral reached by integrating up to some point x defines a probability distribution, i.e., the probability of being less than or equal to x. The gamma distribution usually is given by adding a scalar transformation of the variable; i.e., the probability of being less than or equal to x is given by the percentage of the integral that occurs up to λx for some positive number λ. The transformed gamma distribution adds a power transformation; i.e., the cumulative probability is given by:

$$G(x;r,\alpha,\lambda) = \frac{\int_0^{(\lambda x)^\alpha} t^{r-1}e^{-t}dt}{\Gamma(r)}$$

This distribution will be considered below as a model for aggregate losses although it may be a reasonable candidate for severity distributions as well. As it has three parameters it can match three moments of the distribution being modeled.

The gamma and exponential distributions are special cases given by $\alpha = 1$ and $\alpha = r = 1$ respectively. The Weibull distribution is also reached by taking $r = 1$. Thus the transformed gamma distribution provides a common generalization of the gamma and Weibull distributions and offers the possibility of improved fits whenever either have been found approximately suitable.

The moments are given by

$$E(X^n) = \frac{\Gamma(r + (n/\alpha))}{\lambda^n\,\Gamma(r)}$$

and the moment distributions

$$\frac{\int_0^a x^n dG_x}{E(X^n)}$$

are given by $G(a; (r + (n/\alpha),\alpha,\lambda)$. The probability density function is

$$g(x,r,\alpha,\lambda) = \frac{\alpha\lambda}{\Gamma(r)}(\lambda x)^{\alpha r-1}e^{-(\lambda x)^\alpha}.$$

These formulas require $n > -\alpha r$ but not necessarily an integer.

Finding parameters r, α, and λ from data involves the solution of non-linear equations whether matching moments or maximum likelihood is used. These equations can be quite readily solved by numerical means, e.g., Newton-Raphson iteration, as discussed more fully in Appendices A and B.

To match moments it has proven quite practical to solve for α and r using the known (e.g., known from sampling or calculated from frequency and severity) coefficients of variation and skewness, which do not depend on λ, in a system of two equations in two unknowns, and then to solve for λ using the mean. Handy equations are:

$$CV^2 + 1 = \Gamma(r + 2/\alpha)\,\Gamma(r) \div \Gamma(r + 1/\alpha)^2, \quad \text{and}$$
$$(SK \times CV^3) + 3CV^2 + 1 = \Gamma(r + 3/\alpha)\,\Gamma(r)^2 \div \Gamma(r + 1/\alpha)^3,$$

where CV is the coefficient of variation and SK skewness. See Appendix A for a discussion of how to solve this system.

Maximum likelihood techniques are discussed in Appendix B.

Once the parameters r,α, and λ have been estimated, the expected losses, higher moments, and percentiles of the aggregate layer from a to b can be read from the distribution. For example, expected losses for the layer are expected losses excess of a less expected losses excess of b. Define $R(a)$ to be the ratio of expected losses excess of a to all expected losses, i.e.,

$$R(a) = \frac{\int_a^\infty (x - a)dG_x}{E(X)}.$$

It is not difficult to show that

$$R(a) = 1 - \frac{\int_0^a x\, dG_x}{E(X)} - \frac{a}{E(X)}(1 - G(a)).$$

So far this is valid for any positive distribution G. Now using the moment ratio property of the transformed gamma:

$$R(a) = 1 - G(a;(r + (1/\alpha)), \alpha,\lambda) - \frac{a\lambda\Gamma(r)}{\Gamma(r + (1/\alpha))}(1 - G(a;r,\alpha,\lambda)).$$

Thus, if we knew how to compute the probability distribution function G, the aggregate layer expected losses would follow immediately. G can be calculated using numerical integration, but there is a series expansion for the incomplete gamma function that is also fairly quick to use. The incomplete gamma function is defined as

$$IG(x;r) = \int_0^x t^{r-1}e^{-t}\,dt \div \Gamma(r).$$

Then $G(x;r,\alpha,\lambda) = IG((\lambda x)^\alpha;r)$. From Abramowitz and Stegun [1] formula 6.5.29 (page 262) the expansion

$$IG(x;r) = \frac{e^{-x}x^{r-1}}{\Gamma(r)} \sum_{i=0}^{\infty} \prod_{k=0}^{i} \frac{x}{r+k}$$

can be derived. From 30 to 200 terms of this sum generally give acceptable accuracy. Exhibit 1 lists an *APL* program for *IG*.

For cases where the expected number of losses is low, there is a non-negligible probability that no losses will occur. The transformed gamma can not account for this because it is an entirely positive distribution. An alternative is a point mass at zero with the conditional probability on losses greater than zero being modeled by a transformed gamma. The probability of no losses can be computed from the frequency distribution. Formulas for computing the moments of the positive (conditional) distribution from the moments of the entire loss distribution and the probability of having a loss are given in Appendix C, along with standard formulas for computing aggregate moments from those for frequency and severity.

Example

Professional liability losses limited to $1 million per occurrence for a small group of hospitals are believed to have expected losses of $219,316 with coefficients of variation and skewness of 1.550 and 2.510 respectively and a probability of .123 of no losses. The aggregate expected losses excess of $1 million will be calculated by the above method.

By the formulas in Appendix C the positive portion of the aggregate distribution has expected losses of 250,000 and coefficients of variation and skewness of 1.409 and 2.344. Using the method in Appendix A gives parameters $r = .2478$, $\alpha = 1.470$, and $\lambda = 1.144 \times 10^{-6}$ for the positive portion. Thus the entire distribution has the cumulative probability function $\Pr(L < x) = .123 + .877\, G(x;\, .2478,\, 1.470,\, 1.144 \times 10^{-6})$. The excess ratio at a = $1,000,000 can be calculated by the methods above to be .0728 for the conditional positive distribution, so the excess expected losses are $18,200 = .0728 × $250,000 for this piece and .877 × 18,200 = $16,000 for the entire distribution.

Transformed Beta

The beta function $B(r,s)$ may be defined as

$$B(r,s) = \frac{\int_0^\infty t^{r-1}\,dt}{(t+1)^{r+s}}\,.$$

This is a transformation of the more usual definition

$$B(r,s) = \int_0^1 u^{r-1}(1-u)^{s-1}\,du$$

accomplished by taking $t = u \div (1 - u)$ or $u = t \div (t + 1)$. The beta is related to the gamma by

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}\,.$$

As in the gamma case a distribution function F may be defined by the partial integral, i.e.,

$$F(x;r,s,\alpha,\beta) = \int_0^{(x/\beta)^\alpha} \frac{t^{r-1}\,dt}{(t+1)^{r+s}} \div B(r,s).$$

This will be called the transformed beta distribution. Its density is

$$f(x;r,s,\alpha,\beta) = \frac{(\alpha/\beta)(x/\beta)^{\alpha r-1}}{B(r,s)(1+(x/\beta)^\alpha)^{r+s}}\,.$$

For $r = 1$ the closed form

$$F(x;1,s,\alpha,\beta) = 1 - ((x/\beta)^\alpha + 1)^{-s}$$

results. This is coming to be known as the Burr distribution, and in turn has two special cases, namely $\alpha = 1$ which is the Pareto, and $s = 1$ which gives the log transform of the logistic. As the logistic is like a heavy-tailed normal the loglogistic can be thought of as being like a lognormal with heavier right and left tails. Its distribution function

$$F(x;\,1,1,\alpha,\beta) = 1 - \frac{\beta^\alpha}{x^\alpha + \beta^\alpha}$$

is of particularly simple form.

The case $\alpha = 1$, i.e., $F(x;r,s,1,\beta)$ is a version of the transformed beta that has been investigated for severity applications. This will be called the generalized-F as its special case $\alpha = 1$, $\beta = s/r$ gives the F distribution where $2r$ and $2s$ are integers. The Pareto is also a special case of the generalized-F given by $r = 1$.

There is an interesting mixture property of the transformed gamma that generates a transformed beta, namely that with a population of transformed gamma random variables with fixed r and α, and with the transformed scale parameter λ^{α} itself gamma-distributed across the population, the compound process of picking a variable from the population, then taking a realization of that variable, is a transformed beta process. This is proved in Appendix D. Several corollary statements follow by taking the special cases of the transformed gamma (i.e., Weibull, gamma, and exponential) and mixing by a gamma, viz.,

(a) Weibull mixed by gamma yields Burr;
(b) Gamma mixed by gamma yields generalized-F;
(c) Exponential mixed by gamma yields Pareto;
(d) Weibull mixed by exponential yields loglogistic.

Exhibit 2 diagrams this situation.

Robert Hogg proved (a), (b), and (c) separately and Gary Patrik independently proved (c). The transformed beta and gamma distributions originally were developed in order to unify these results. Robert Miccolis pointed out that the generalized-F is a ratio of two gamma variates. This suggested the result, proved in Appendix E, that if X is transformed beta with parameters r, s, α, β, then $1/X$ is also, with parameters s, r, α, β^{-1}.

If X is transformed beta in r, s, α, β then

$$E(X^n) = \beta^n B(r + n/\alpha, s - n/\alpha) \div B(r,s)$$

if $-\alpha r < n < \alpha s$ and non-existent otherwise. This is an example of a distribution with unbounded moments for $n \geq \alpha s$ which arises in a natural way as a combination of distributions with all moments finite. For $\alpha = 1$ (generalized-F, Pareto) the moments simplify to

$$\beta^{-n}E(X^n) = (r)\,\frac{(r+1) \times \cdots (r+n-1)}{(s-1) \times (s-2) \cdots (s-n)} = \prod_{i=1}^{n} \frac{r+i-1}{s-i}\,.$$

This makes methods of moments parameter estimation quite simple for this special case. Maximum likelihood parameter estimation for the transformed beta is similar to that for the transformed gamma as covered in Appendix H. Loss severity distributions also have been fit by the transformed beta and gamma distributions by matching sample and formula values of the excess ratio $R(a)$ in a manner similar to that in Harwayne [5].

As with the transformed gamma, the moment distributions are of the same form as the original distribution, in fact

$$\int_0^a x^n \, dF_x \div E(X^n) = F(a; r + n/\alpha, s - n/\alpha, \alpha, \beta).$$

Thus, as with the transformed gamma, a calculation of excess losses can be made if the cumulative distribution can be calculated. This has proven to be most practically accomplished through numerical integration. Appendix F discusses one method. The moment distribution formulas for the transformed beta and gamma show that the Burr and Weibull moment distributions do not maintain the original form, i.e., $r = 1$.

The mixture derivation of the transformed beta provides an interesting way to deal with so called "parameter risk." It is fairly plausible that aggregate losses for a given company (insured or insurer) are distributed transformed gamma and that the shape parameters r and α are fairly well known and stable but because of uncertain trend (or other factors) there is substantial uncertainty about the scale parameter λ, which relates to the overall level of expected results. If λ^α is gamma distributed in s and γ then the overall aggregate distribution is transformed beta in r, s, α, β where $\beta = \gamma^{1/\alpha}$. It also is not difficult to show that λ^α is gamma in s, α means that λ is transformed gamma in s, α, β (see Appendix G). Thus it can be concluded that if aggregate losses are transformed gamma in r, α, Λ where Λ is unknown but is itself transformed gamma in s, α, β (same α) then the aggregate losses are transformed beta in r, s, α, β.

In theory it would be a great coincidence if the uncertainty about λ had the same parameter α as did the aggregate losses themselves. As a practical technique for quantifying this uncertainty, however, it should not be too burdensome to use the α already in hand for aggregate losses. There will still be two parameters, s and β, available to match to the uncertainty the analyst feels is inherent.

There are several ways in which s and β could be determined. Different values could be tried and the 25th, 50th, and 75th percentile λ calculated for each, with the corresponding percentile of aggregate expected losses $\Gamma(r + 1/\alpha) \div \lambda\Gamma(r)$ following. These can be compared with the uncertainty that seems inherent in the overall level of losses. The latter uncertainty can be estimated by trying to combine the uncertainties in the trend, development, and other factors used to estimate the overall level. The regression statistics used in developing these factors may be helpful if regression was used.

Another approach to measuring the distribution of λ is using industry loss ratios. Expected losses for an aggregate loss distribution with cdf $G(x; r, \alpha, \lambda)$

are $\Gamma(r + (1/\alpha)) \div \lambda\Gamma(r)$. Thus, for fixed r,α, the reciprocal of the aggregate losses, and thus the reciprocal of the loss ratio, is proportional to λ. Therefore if λ is unknown but is a realization of a random variable Λ which is transformed gamma in s, α, β, where α is fixed, the shape parameter s can be estimated by looking at the historical distribution of loss ratio reciprocals. This would measure some of the variation that would occur even if Λ were known, however. An alternative is to look at some broader base of comparable experience, such as the line for the industry or state or class in question where the process variance is minimal and hence the principal source of variation is the parameter uncertainty. Depending on the similarity between the company in question and the broader base as to projection methods for trend and loss development, the stability of the historical data base, and so forth, this approach may give a reasonable estimate of the parameter uncertainty.

Estimating β then could proceed by matching the formula $E(1/\Lambda)$ for the transformed gamma distribution to the expected value of $1/\Lambda$ calculated for the year and company in question. For Λ with cdf $G(\lambda;s,\alpha,\beta)$ the $E(1/\Lambda)$ is $\beta\,\Gamma(s - 1/\alpha) \div \Gamma(s)$ from the transformed gamma moment formula.

Borrowing loosely from our earlier example, suppose a malpractice risk has aggregate losses distributed according to the transformed gamma with $r = .2478$, $\alpha = 1.470$ and $E(1/\Lambda) = 1 \div (1.144 \times 10^{-6})$, where Λ is transformed gamma in s, 1.470, β. Suppose the previous four years of industry malpractice experience produced loss ratios of .505, .750, 1.001, and 1.357, which have reciprocals 1.980, 1.333, .999, and .737. The reciprocals average 1.262 and have an unbiased sample standard deviation estimate of .5370 for an estimated CV of .4255. The formula

$$1 + CV^2 = \Gamma(s + 2/\alpha)\,\Gamma(s) \div \Gamma(s + 1/\alpha)^2 \text{ then becomes}$$

$$1.181 = \Gamma(s + 1.36)\,\Gamma(s) \div \Gamma(s + .68)^2,$$

which can be solved numerically to find $s = 2.597$. Then

$$\begin{aligned} 1 \div 1.144 \times 10^{-6} = E(1/\Lambda) &= \beta\,\Gamma(s - 1/\alpha) \div \Gamma(s) \\ &= \beta\,\Gamma(2.597 - .68) \div \Gamma(2.579) \end{aligned}$$

can be solved directly to yield $\beta = 1{,}288{,}500$. From the transformed beta in $r = .2478$, $s = 2.597$, $\alpha = 1.470$, $\beta = 1{,}288{,}500$ expected losses of

$$\frac{\beta\,\Gamma(r + 1/\alpha)\,\Gamma(s - 1/\alpha)}{\Gamma(r)\,\Gamma(s)} = 250{,}000$$

can be calculated, confirming the calculation of β.

The expected losses excess of \$1 million in the aggregate increase substantially when this additional uncertainty is included. For this transformed beta an excess ratio of .1348 can be computed at \$1,000,000 which yields excess expected losses of \$33,700 compared to .0728 and \$18,200 for the transformed gamma.

The great disparity between these figures comes from the wide divergence in loss ratios in the period studied. If the uncertainty in Λ really is so great that next year's loss ratio for the whole industry can come out anywhere in the range .50 to 1.35, then there is a much greater chance that total losses for a small segment of the industry will exceed the target \$1 million.

For other more stable lines a similar analysis would show a much smaller difference. In those cases there is a danger that the potential variation in level would be understated by looking at industry loss ratios. Swings in calendar year ratios may be dampened by reserve changes. Also, a particular sector of the industry would probably have wider variation than the total industry in the degree to which the proper level could be projected. This would be important if the company under study were concentrated in one area. The selection of the parameter s probably should be made with a good deal of judgement because of these considerations.

SUMMARY AND EXTENSIONS

The above gives a method of approximating the distribution function of aggregate losses from the moments of that distribution, based on the transformed beta and gamma distributions. Since a distributional assumption is involved, the method is likely to be less precise than the exact methods of Adelson [11], Panjer [12] and Heckman and Meyers [13]. Those methods do, however, require more input information, namely the underlying frequency and severity distribution functions, and they also require substantially more computation. As computing becomes faster and less expensive and as good parameterized frequency and severity distributions become available those methods become increasingly viable, and the assumption of a distributional form for aggregate losses becomes more avoidable. Methods based on moments only are nonetheless of definite value at present.

The transformed beta distribution is a good candidate for casualty loss severity distributions, because it generalizes the Pareto and Burr which have been used with moderate success. The problems of trend and development by layer of loss have yet to be settled entirely in casualty lines, however, especially with regard to having factors that are independent of distributional assumptions. Thus, there currently is a fair amount of uncertainty as to casualty severity distributions.

The transformed gamma may be useful in loss severity, for example, in workers' compensation. Also, the inverse transformed gamma, i.e., the distribution of Y when $X = 1 \div Y$ is transformed gamma, is a heavy-tailed distribution which may have application to casualty loss severity. This distribution function is:

$$G'(y) = \int_0^{(y/\lambda)^\alpha} \frac{t^{-r-1} e^{-1 \div t}}{\Gamma(r)}\, dt$$

and $E(X^n) = \lambda^n\, \Gamma(r - n/\alpha) \div \Gamma(r)$ for $n < r\alpha$.

A problem that sometimes arises with maximum likelihood estimation with these distributions is that no maximum exists. Usually this happens because the maximum likelihood, given α, increases as α decreases. After some point the increase becomes negligible however. One alternative in this case is to pick a "low enough" value of α and maximize the likelihood fixing that value. This usually gives much better fits than the Weibull, gamma, Burr, etc., in these cases.

Another alternative is that there may be other functions that are limiting values of these distributions. For instance, in the Burr case, $F(x) = 1 - ((x/\beta)^\alpha + 1)^{-s}$, small α often leads to large β but with $(x/\beta)^\alpha$ near zero for the range of interest, so $1 + (x/\beta)^\alpha$ is close to $e^{(x/\beta)^\alpha}$ and $F(x)$ is approximately $1 - e^{-s(x/\beta)^\alpha}$ which is a Weibull. Conversely, small β and large α make (x/β) very close to $(x/\beta)^\alpha + 1$, relatively speaking, so $F(x)$ is approximately $1 - (x/\beta)^{-\alpha s}$, which is a non-shifted Pareto. Similar relationships may occur for the general cases.

A limitation of the above methods is that the transformed gamma does not seem able to take on any combination of moments. For example, it appears that the coefficient of skewness must be greater than the coefficient of variation (CV) if $CV > 1.25$. In the gamma case the coefficient of skewness is always twice the CV. Thus, the transformed gamma allows a fair amount of departure from gamma-ness but not complete latitude. Appendix J discusses this problem and suggests alternate approaches.

Much of the interest in the gamma stems from a 1940 theorem of Lundberg [14] which shows that under certain conditions the negative binomial frequency leads to an approximately gamma aggregate distribution. Since aggregate distributions seem to be positively skewed for the most part, but do not always have the skewness double the *CV*, gamma-like distributions allowing some deviation from the gamma are thus appealing candidates for this purpose.

Exhibit 3 gives the results of a test of the transformed gamma against an exact calculation of an aggregate distribution using the characteristic function method. The severity distribution is piecewise linear. Approximating the severity by a discrete distribution also permits a comparison to the recursive method of Adelson and Panjer. Intervals of $500 were chosen for this discrete approximation. Details are provided in Exhibit 3. The results show that the two exact methods are extremely similar, indicating that not much is lost by the discrete approximation to severity. The transformed gamma also is reasonably close over a wide range of loss sizes, confirming, at least in this one case, the usefulness of this simplifying approximation.

EXHIBIT 1

```
      ∇IG[⎕]∇
       ∇ E←V IG I;R;X;D
[1]   ⍝INCOMPLETE GAMMA FCT 0 TO X, PARAM R; I IS PRECISION SUGGEST~35 TO 350
[2]    X←V[1]
[3]    R←V[2]
[4]    →((R>55)∨(175<X)∨X>7E75*÷R+1)/BIG
[5]    D←((X*(R-1))×*(-X))÷!(R-1)
[6]    →END
[7]   ⍝BIG:R←1E¯12×⌊0.5+R×1000000000000
[8]   ⍝SOMETIMES ABOVE LINE NEEDED TO AVOID TRUNCATION PROBLEMS
[9]   BIG:D←(X*1|R)×(×/X÷(*X÷⌊R-1)×R-⍳⌊R-1)÷!1|R
[10]  END:E←D×+/×\X÷R+¯1+⍳I
       ∇
```

EXHIBIT 2

TRANSFORMED GAMMA MIXED BY GAMMA WITH SPECIAL CASES

$$
\begin{array}{ccc}
\dfrac{1}{\Gamma(r)} \displaystyle\int_0^{\theta x^{\alpha}} t^{r-1} e^{-t}\, dt & & \dfrac{1}{\Gamma(r)} \displaystyle\int_0^{\theta x} t^{r-1} e^{-t}\, dt \\
\text{(Transformed Gamma)} & \xRightarrow[\alpha=1]{} & \text{(Gamma)} \\
\Big\downarrow r = 1 & & \Big\downarrow r = 1 \\
1 - e^{-\theta x^{\alpha}} & & 1 - e^{-\theta x} \\
\text{(Weibull)} & \xRightarrow{\alpha=1} & \text{(Exponential)}
\end{array}
$$

If θ is distributed Gamma in s, γ:

$$
\begin{array}{ccc}
\dfrac{1}{B(r,s)} \displaystyle\int_0^{(x/\beta)^{\alpha}} \dfrac{t^{r-1}\, dt}{(t + 1)^{r+s}} & & \dfrac{1}{B(r,s)} \displaystyle\int_0^{(x/\beta)} \dfrac{t^{r-1}\, dt}{(t + 1)^{r+s}} \\
\text{(Transformed Beta)} & \xRightarrow{\alpha=1} & \text{(Generalized-}F\text{)} \\
\Big\downarrow r = 1 & & \Big\downarrow r = 1 \\
1 - ((x/\beta)^{\alpha} + 1)^{-s} & \xRightarrow[\alpha=1]{} & 1 - (x/\beta + 1)^{-s} \\
\text{(Burr)} & & \text{(Pareto)}
\end{array}
$$

where $\beta = \gamma^{1/\alpha}$

EXHIBIT 3
PART 1

AGGREGATE LOSS DISTRIBUTIONS
COMPARATIVE SUMMARY

Aggregate Loss ($000)	Characteristic Function Method: Cumulative Probability	Characteristic Function Method: Excess Ratio	Recursive Method: Cumulative Probability	Recursive Method: Excess Ratio	Transformed Gamma: Cumulative Probability	Transformed Gamma: Excess Ratio
25	.0508	.9016	.0516	.9016	.0621	.9031
50	.1291	.8107	.1298	.8107	.1260	.8125
75	.2009	.7273	.2015	.7272	.1895	.7283
100	.2676	.6507	.2683	.6507	.2520	.6503
125	.3289	.5806	.3295	.5806	.3129	.5786
150	.3843	.5163	.3848	.5163	.3717	.5129
175	.4341	.4573	.4346	.4573	.4280	.4529
200	.4788	.4030	.4793	.4029	.4817	.3984
225	.5189	.3529	.5193	.3529	.5324	.3491
250	.5548	.3066	.5552	.3066	.5801	.3047
275	.6034	.2642	.6040	.2642	.6245	.2650
300	.6556	.2273	.6561	.2273	.6658	.2295
325	.7008	.1951	.7013	.1951	.7039	.1981
350	.7405	.1672	.7408	.1672	.7388	.1702
375	.7749	.1431	.7752	.1431	.7707	.1457
400	.8047	.1221	.8049	.1221	.7995	.1243
425	.8303	.1039	.8305	.1039	.8255	.1055
450	.8524	.0880	.8526	.0880	.8488	.0893
475	.8714	.0742	.8716	.0742	.8696	.0752
500	.8878	.0622	.8879	.0622	.8881	.0631
525	.9045	.0518	.9047	.0518	.9043	.0528
550	.9201	.0430	.9203	.0430	.9186	.0439
575	.9332	.0357	.9333	.0357	.9310	.0364
600	.9442	.0296	.9443	.0296	.9418	.0301
625	.9534	.0245	.9535	.0245	.9511	.0247
650	.9611	.0202	.9611	.0202	.9592	.0203
675	.9675	.0167	.9675	.0167	.9660	.0165
700	.9728	.0137	.9729	.0137	.9718	.0134
725	.9773	.0112	.9773	.0112	.9768	.0109
750	.9810	.0091	.9810	.0091	.9809	.0088
775	.9844	.0074	.9844	.0074	.9844	.0070
800	.9873	.0060	.9873	.0060	.9873	.0056
825	.9897	.0048	.9897	.0048	.9897	.0045
850	.9916	.0039	.9916	.0039	.9917	.0035

EXHIBIT 3
PART 2

AGGREGATE LOSS DISTRIBUTIONS
COMPARATIVE ASSUMPTIONS

Frequency: Poisson $\lambda = 13.7376$

Piecewise Linear *CDF*

Limit (000)	Cumulative Probability	Limit (000)	Cumulative Probability
1	.38935	25	.85690
5	.77870	35	.87927
6	.78438	50	.90280
7	.78981	75	.92739
8	.79498	100	.94256
9	.79993	125	.95277
10	.80466	150	.96009
12.5	.81564	175	.96556
15	.82553	200	.96979
17.5	.83449	225	.97316
20	.84264	250	.97590

Discrete *PDF*

Amount	Probability
500	.38326640625
1000	.03041796875
1500 to 4000	.04866875 each 500
4500	.054731628
5000	.019691497
5500 to 249,000 at each N = 500k	Piecewise linear probability from $N - 250$ to $N + 250$
249,500	.0000685
250,000	.0241137

Moments

	Mean	Coefficient of Variation	Coefficient of Skewness
Severity	18,198	2.6600	3.6746
Aggregate	250,000	.7667	1.0744

Transformed Gamma Parameters

r : .5613125

α : 1.8300318

λ : $1 \div 417896.414$

APPENDIX A

SOLVING TWO EQUATIONS

Many systems of two equations in two unknowns, including the transformed gamma moment system in the text, can be solved by Newton-Raphson iteration, with the partial derivatives taken numerically. The numerical partial derivative of $f(x,y)$ with respect to y, for example, is $(f(x,y(1 + \Delta)) - f(x,y)) \div y\Delta$, where Δ is a small number; e.g., 10^{-7}. Because of limits to computer accuracy, Δ should not be too small, e.g., $\Delta = 10^{-50}$ would be too small for most computer installations. This method is quite useful when the partials are not available in closed form or are excessively intricate.

Given $f(x,y)$ and $g(x,y)$, initial estimates x_0 and y_0 and derivatives f_x, f_y, g_x, g_y the iteration proceeds by setting

$$x_{i+1} = x_i - (fg_y - gf_y) \div (f_xg_y - g_xf_y)$$
$$y_{i+1} = y_i - (gf_x - fg_x) \div (f_xg_y - g_xf_y)$$

where the functions and derivatives are evaluated at (x_i,y_i). See Conte and de Boor [3] page 86 for details.

Exhibit A1 gives an *APL* system for this procedure. The user interactively defines the equations to be solved. Any user-defined functions may be called in this process. A sample run of the system is shown in Exhibit A2.

EXHIBIT A1
PAGE 1

```
∇DELUXENR[⎕]∇
      ∇ DELUXENR;ΔA;ΔB;LOOPTOL;DELTOL;MODEFLAG;PFQA;PFQB;PGQA;PGQB
     ⍝WRITTEN BY STAN STIEFEL
      'SPECIFY ONE FUNCTIONAL RELATION. . .'
      'USE THE VARIABLE NAMES A AND B FOR THE UNKNOWNS.'
      'FQ' MAKEFX ⍞
      'SPECIFY THE OTHER RELATION'
      'GQ' MAKEFX ⍞
      'ENTER INITIAL VALUE FOR A'
      A←⎕
      'ENTER INITIAL VALUE FOR B'
      B←⎕
      MODEFLAG←1↑,DELTOL-DELTOL←LOOPTOL←0.00001
      'WOULD YOU LIKE TO USE DEFAULT CONDITIONS (0)'
      'OR SEE A MENU OF OPTIONS (1). . .0 OR 1'
      ⍎⎕/'MENU'
     LP:PARTIALS DELTOL
      A←A-ΔA←(DET(2 2 ρ(A FQ B),PFQB,(A GQ B),PGQB))÷DET(2 2 ρPFQA,PFQB,PGQA,PGQB)
      ⍎MODEFLAG/'PARTIALS DELTOL'
      B←B-ΔB←(DET(2 2 ρPFQA,(A FQ B),PGQA,(A GQ B)))÷DET(2 2 ρPFQA,PFQB,PGQA,PGQB)
      →(∨/LOOPTOL<|(ΔA,ΔB)÷(A,B)+0=A,B)/LP
      'A:  ';A;'      B:     ';B
      ⎕WA←⎕EX 2 2 ρ'FQGQ'
      ∇
```

```
      ∇MAKEFX[⎕]∇
      ∇ NAME MAKEFX RELAT;X;TITLE
      →(0='='∊RELAT)/DID
      RELAT[RELAT⍳'=']←'-'
     DID:TITLE←'RSLT',NAME,'←A ',NAME,' B'
      RELAT←'RSLT',NAME,'←',RELAT
      RELAT←RELAT,(0.5×X+|X←(ρTITLE)-ρRELAT)ρ' '
      TITLE←TITLE,((ρRELAT)-(ρTITLE))ρ' '
      ⎕WA←⎕FX TITLE,[0.5] RELAT
      ∇
```

EXHIBIT A1
PAGE 2

```
∇MENU[⎕]∇
      ∇ MENU
[1]     'FOR PURPOSES OF TAKING NUMERICAL DERIVATIVES, FUNCTIONS WILL BE EVALUATED AT A,  A-ΔA, B, B-ΔB.'
[2]     'ΔA AND ΔB ARE SPECIFIED AS FRACTIONS OF A AND B. . .1E¯5 IS THE DEFAULT.    PLEASE SPECIFY THE FRACTION.'
[3]     DELTOL←⎕
[4]     'ITERATION WILL BE CONSIDERED COMPLETE WHEN BOTH A AND B HAVE CHANGED BY LESS THAN SOME FRACTION OF THEMSELVES'
[5]     'DEFAULT IS 1E¯5.   PLEASE SPECIFY THE FRACTION.'
[6]     LOOPTOL←⎕
[7]     'SEQUENCE OF CALCULATION CAN BE EITHER OF TWO OPTIONS'
[8]     '(0) GET PARTIALS, GET NEW A, GET NEW B.'
[9]     '(1) GET PARTIALS, GET NEW A, GET PARTIALS, GET NEW B.'
[10]    'DEFAULT IS 0.  PLEASE SPECIFY 0 OR 1.'
[11]    MODEFLAG←⎕
      ∇


      ∇PARTIALS[⎕]∇
      ∇ PARTIALS XXXX;Z
[1]     PFQA←((A FQ B)-((A-Z) FQ B))÷Z←1E¯10⌈|Z←XXXX×A
[2]     PGQA←((A GQ B)-((A-Z) GQ B))÷Z
[3]     PFQB←((A FQ B)-(A FQ(B-Z)))÷Z←1E¯10⌈|Z←XXXX×B
[4]     PGQB←((A GQ B)-(A GQ(B-Z)))÷Z
      ∇


      ∇DET[⎕]∇
      ∇ Y←DET X
[1]     Y←(X[1;1]×X[2;2])-X[1;2]×X[2;1]
      ∇
```

EXHIBIT A2

```
       ∇CV[⎕]∇
      ∇ Y←A CV R
[1]    Y←(!¯1+R)×(!¯1+R+2÷A)+(!¯1+R+÷A)*2
[2]    Y←(Y-1)*0.5
      ∇



      ∇SKW[⎕]∇
      ∇ Y←A SKW R
[1]    Y←((!¯1+R)*2)×(!¯1+R+3÷A)÷(!¯1+R+÷A)*3
[2]    Y←Y+2-3×(!¯1+R)×(!¯1+R+2÷A)÷(!¯1+R+÷A)*2
[3]    Y←Y÷(A CV R)*3
      ∇


      DELUXENR
SPECIFY ONE FUNCTIONAL RELATION. . .
USE THE VARIABLE NAMES A AND B FOR THE UNKNOWNS.
(A CV B)=1.409
SPECIFY THE OTHER RELATION
(A SKW B)=2.344
ENTER INITIAL VALUE FOR A
⎕:
      1.2
ENTER INITIAL VALUE FOR B
⎕:
      .3

WOULD YOU LIKE TO USE DEFAULT CONDITIONS (0)
OR SEE A MENU OF OPTIONS (1). . .0OR 1
⎕:
      0
A:  1.47     B:  0.2478

      A CV B
1.409
      A SKW B
2.344



      ∇SKEW2[⎕]∇
      ∇ Y←A SKEW2 R
[1]    N←⌊¯1+R+÷A
[2]    M←(-⍳N)+R+÷A
[3]    O←!(-N+1)+R+÷A
[4]    S←(-⍳N)+R+3÷A
[5]    T←!(-N+1)+R+3÷A
[6]    U←(×/S+M)×(T+O)
[7]    Y←((!¯1+R)*2)×U÷(!¯1+R+÷A)*2
[8]    Y←Y+2-3×(!¯1+R)×(!¯1+R+2÷A)÷(!¯1+R+÷A)*2
[9]    Y←Y÷(A CV R)*3
      ∇
```

APPENDIX B

MAXIMUM LIKELIHOOD FOR THE TRANSFORMED GAMMA

Maximum likelihood in the case where there are no problems of truncation or censorship of the sample reduces to one non-linear equation to solve for α, then linear equations for r and λ. The α equation is somewhat intricate but is solved easily numerically. Given a sample y_i, $i = 1$ to n, the likelihood function is

$$L(r,\alpha,\lambda) = \prod_{i=1}^{n} \alpha\lambda^{\alpha r} y_i^{\alpha r-1} e^{-(\lambda y_i)^\alpha} \div \Gamma(r) \text{ and}$$

$$\ln L(r,\alpha,\lambda) = n \ln \alpha + n\, \alpha\, r \ln \lambda - n \ln \Gamma(r)$$
$$+ (\alpha r - 1) \Sigma \ln y_i - \lambda^\alpha \sum_{i=1}^{n} y_i^\alpha .$$

Setting the partial derivatives of this to zero, and denoting the derivative of $\ln \Gamma(r)$ by $\psi(r)$ yields the likelihood equations:

(a) $\psi(r) - \ln r = \alpha\, \overline{\ln y} - \ln \overline{y^\alpha}$
(b) $r = \overline{y^\alpha} \div \alpha(\overline{y^\alpha \ln y} - \overline{y^\alpha}\ \overline{\ln y})$
(c) $\lambda = (\overline{y^\alpha} \div r)^{1/\alpha}$

Substituting for r in (a) via (b) gives a single equation for α which when solved allows r and λ to be calculated from (b) and (c). This is a generalization of the method found in Hachemeister [4] for the gamma distribution. Note that to solve (a),

$$\overline{y^\alpha} = \frac{1}{n} \sum_{i=1}^{n} y_i^\alpha, \quad \overline{\ln y} = \frac{1}{n} \sum_{i=1}^{n} \ln y_i,$$

$$\text{and } \overline{y^\alpha \ln y} = \frac{1}{n} \sum_{i=1}^{n} y_i^\alpha \ln y_i,$$

must be calculated from the sample at each iteration.

As suggested on page 152 of Aquino [2], differentiating Abramowitz and Stegun's [1] formula 6.1.34 (page 256) gives the series approximation

$$\psi(z) = \Gamma(z) \sum_{k=1}^{26} k c_k\, z^{k-1},$$

where c_1 to c_{26} are as shown in Exhibit B1. This expansion gives more than 13 place accuracy on [1,2] and the recursive relation $\psi(1 + z) = \psi(z) + 1/z$ can be used outside of this interval.

To solve equation (a) with (b) substituted for r we have an equation $f(\alpha) = 0$ where f is calculable by computer or calculator. This can be solved iteratively by numerical Newton-Raphson:

Start with a guess α_0. Then let

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{\dfrac{f(\alpha_i\,(1 + \Delta)) - f(\alpha_i)}{\alpha_i}}$$

$$\text{i.e. } \alpha_{i+1} = \alpha_i \left(1 - \frac{\Delta}{\dfrac{f(\alpha_i\,(1 + \Delta))}{f(\alpha_i)} - 1} \right)$$

where Δ is small, e.g. 10^{-7}.

A reasonable starting value α_0 usually is given by calculating the sample ratio of the coefficient of variation over half the coefficient of skewness, as this is greater, less than, or equal to 1 when α is.

As an alternative, the secant method

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)\,(\alpha_i - \alpha_{i-1})}{f(\alpha_i) - f(\alpha_{i-1})}$$

can be used to solve for α. This involves only one computation of f each iteration, so it may be faster than Newton-Raphson iteration.

EXHIBIT B1

SERIES EXPANSION FOR $\psi(z)$

$$\psi(z) = \Gamma(z) \sum_{k=1}^{26} k c_k z^{k-1}$$

k	c_k		
1	−1.00000	00000	000000
2	−0.57721	56649	015329
3	0.65587	80715	202538
4	0.04200	26350	340952
5	−0.16653	86113	822915
6	0.04219	77345	555443
7	0.00962	19715	278770
8	−0.00721	89432	466630
9	0.00116	51675	918591
10	0.00021	52416	741149
11	−0.00012	80502	823882
12	0.00002	01348	547807
13	0.00000	12504	934821
14	−0.00000	11330	272320
15	0.00000	02056	338417
16	−0.00000	00061	160950
17	−0.00000	00050	020075
18	0.00000	00011	812746
19	−0.00000	00001	043427
20	−0.00000	00000	077823
21	0.00000	00000	036968
22	−0.00000	00000	005100
23	0.00000	00000	000206
24	0.00000	00000	000054
25	−0.00000	00000	000014
26	−0.00000	00000	000001

APPENDIX C

AGGREGATE MOMENTS

A. In terms of frequency and severity moments, assume individual claim sizes are independent, identically distributed, and independent of the number of claims.

Let N denote number of claims, X claim size, L aggregate losses, μ the mean, σ the standard deviation, γ the coefficient of skewness, c the coefficient of variation, and

$$N_i = \frac{E\,(N - \mu_N)^i}{\mu_N}.$$

Then

$$\mu_L = \mu_X \mu_N$$

$$\sigma_L^2 = \mu_N \sigma_X^2 + (\mu_X \sigma_N)^2$$

$$\gamma_L \sigma_L^3 = \mu_N \gamma_X \sigma_X^3 + 3\mu_X \sigma_X^2 \sigma_N^2 + \mu_X^3 \gamma_N \sigma_N^3$$

$$\sigma_L^2 = \mu_X^2 \mu_N (c_X^2 + N_2)$$

$$\gamma_L = (\gamma_X c_X^3 + 3c_X^2 N_2 + N_3) \div \sqrt{\mu_N (c_X^2 + N_2)^3}$$

$$c_L^2 = (c_X^2 + N_2) \div \mu_N$$

B. Moments of conditional (positive) distribution in terms of moments of entire distribution and probability of losses being non-zero

$$F(a) = \Pr\,(L \le a) = \begin{cases} 1 - p \text{ when } a = 0 \\ 0 \quad\quad \text{when } a < 0 \\ (1 - p) + pG(a) \text{ when } a > 0 \end{cases}$$

Then

$$\mu_F = p\mu_G$$

$$\mu_G = \mu_F \div p$$

$$c_G^2 = p\, c_F^2 + p - 1$$

$$\gamma_G = \frac{p^2 \gamma_F\, c_F^3 + (p - 1)(3pc_F^2 + p - 2)}{c_G^3}$$

EXHIBIT C1

```
      ∇CONDITMO[⎕]∇
      ∇ X←P CONDITMO Q;TERM1;TERM2;COEFVAR;GAMMA
[1]   ⍝WRITTEN BY VICTOR PUGLISI
[2]   ⍝  THIS PROGRAM CALCULATES CONDITIONAL MOMENTS IN THE FORM OF THE COEFFICIENT OF VARIATION (CV) AND THE SKEWNESS
[3]   ⍝  (GAMMA) BASED UPON RISKMODEL OUTPUT FOR THE PART OF THE DISTRIBUTION GREATER THAN 0.
[4]   ⍝  IT TAKES AS LEFT-HAND ARGUMENT THE PROBABILITY OF CLAIMS BEING LARGER THAN 0, CURRENTLY FOUND AT THE TOP OF THE
[5]   ⍝  RISKMODEL OUTPUT FOR EACH LAYER DENOTED BY 'PROBABILITY OF LOSS' AND FOR RIGHT-HAND ARGUMENT REQUIRES A TWO
[6]   ⍝  ELEMENT VECTOR CONSISTING OF THE COEFFICIENT OF VARIATION AND THE COEFFICIENT OF SKEWNESS FOR EACH MAJOR GROUP.
[7]   ⍝  THESE ARE FOUND IN COLUMNS 8 AND 9 RESPECTIVELY OF THE RISKMODEL OUTPUT.
[8]    COEFVAR←((P×Q[1]*2)+P-1]*0.5
[9]    TERM1←(P*2)×Q[2]×Q[1]*3
[10]   TERM2←(P-1)×(3×P×Q[1]*2)+P-2
[11]   GAMMA←(TERM1+TERM2)÷COEFVAR*3
[12]   X←COEFVAR,GAMMA
      ∇
```

APPENDIX D

TRANSFORMED BETA IS TRANSFORMED GAMMA MIXED BY A GAMMA

The transformed gamma density function

$$g(x;r,\alpha,\lambda) = \frac{\alpha\lambda^{\alpha r}\, x^{\alpha r-1}\, e^{-\lambda^\alpha x^\alpha}}{\Gamma(r)}$$

can also be parameterized as $\alpha\, \theta^r x^{\alpha r-1} e^{-\theta x^\alpha} \div \Gamma(r)$, taking $\theta = \lambda^\alpha$. Given a family of such random variables with α and r fixed and θ itself gamma distributed with parameters s and γ, i.e., having density $\gamma^s\, \theta^{s-1}\, e^{-\gamma\theta} \div \Gamma(s)$, then the compound process is transformed beta.

To demonstrate this the density for the compound distribution will be calculated. This is the probability-weighted average of the densities of the family, that at x equals:

$$\int_0^\infty \frac{\alpha\, \theta^r x^{\alpha r-1}\, e^{-\theta x^\alpha}}{\Gamma(r)} \frac{\gamma^s \theta^{s-1}\, e^{-\gamma\theta}\, d\theta}{\Gamma(s)}$$

$$= \frac{\alpha\gamma^s\, x^{\alpha r-1}}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \theta^{r+s-1}\, e^{-\theta(x^\alpha+\gamma)}\, d\theta$$

which, after the change of variable $\phi = \theta(x^\alpha + \gamma)$, becomes

$$\frac{\alpha\gamma^s\, x^{\alpha r-1}}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left(\frac{\phi}{x^\alpha + \gamma}\right)^{r+s-1} e^{-\phi}\, \frac{d\phi}{x^\alpha + \gamma}$$

$$= \frac{\alpha\gamma^s\, x^{\alpha r-1}}{\Gamma(r)\,\Gamma(s)(x^\alpha + \gamma)^{r+s}} \int_0^\infty \phi^{r+s-1}\, e^{-\phi}\, d\phi$$

$$= \frac{\alpha\gamma^s\, x^{\alpha r-1}}{\Gamma(r)\,\Gamma(s)(x^\alpha + \gamma)^{r+s}}\, \Gamma(r + s)$$

$$= \frac{\alpha\gamma^s\, x^{\alpha r-1}}{B(r,s)(x^\alpha + \gamma)^{r+s}} .$$

Now defining β by $\gamma = \beta^\alpha$ gives for the compound density

$$\frac{\alpha\beta^{\alpha s}\, x^{\alpha r-1}}{B(r,s)(x^\alpha + \beta^\alpha)^{r+s}} = \frac{\beta^{-\alpha(r+s)}\,\alpha\beta^{\alpha s}\, x^{\alpha r-1}}{B(r,s)((x/\beta)^\alpha + 1)^{r+s}}$$

$$= (\alpha/\beta)(x/\beta)^{\alpha r-1} \div B(r,s)((x/\beta)^\alpha + 1)^{r+s}$$

which is the transformed beta density.

APPENDIX E

RECIPROCAL OF TRANSFORMED BETA VARIATE IS TRANSFORMED BETA

Let $Y = \dfrac{1}{X}$ where X has cdf $F(x;r,s,\alpha,\beta)$.

Now $Y \leq a \rightleftharpoons X \geq (1/a)$ so $\Pr\,(Y \leq a) = 1 - \Pr\,(X < (1/a)$

$$= 1 - \frac{1}{B(r,s)} \int_0^{(a\beta)^{-\alpha}} \left[\frac{t^{r-1}}{(1 + t)^{r+s}}\right] dt.$$

Let $u = (1/t)$; $t = (1/u)$; $dt = -du/u^2$.

$$\text{Then } \Pr(Y \leq a) = 1 + \frac{1}{B(r,s)} \int_\infty^{(a\beta)^\alpha} \frac{u^{1-r}\, du}{(1 + (1/u))^{r+s}\, u^2}$$

$$= 1 - \frac{1}{B(r,s)} \int_{(a\beta)^\alpha}^{\infty} \left[\frac{u^{1+s-2}}{(u + 1)^{r+s}}\right] du$$

$$= \frac{1}{B(r,s)} \int_0^{(a\beta)^\alpha} \left[\frac{u^{s-1}}{(u + 1)^{r+s}}\right] du.$$

Therefore Y has cdf $f(y;s,r,\alpha,1/\beta)$.

APPENDIX F

NUMERICAL INTEGRATION BY GAUSSIAN QUADRATURE

Gaussian quadrature is a method of numerical integration that estimates the integral by taking a weighted sum of the value of the function being integrated at several points. In general

$$\int_a^b f(y)dy \approx \frac{b-a}{2} \sum_{i=1}^{n} W_i f(y_i),$$

where $2y_i = (b - a)x_i + b + a$ and W_i and x_i are somewhat complex to calculate. Exhibits F1 and F2 give W_i and x_i for a few values of n. See Abramowitz [1] pages 916–919 for others. Hildebrand [8] discusses the mathematical background.

This approach works best for functions that can be closely approximated by polynomials of degree n.

The integration of the transformed beta distribution function is more accurate if two transformations are made. First the mapping $u = t/(t + 1)$ transforms the integral to

$$F(x;r,s,\alpha,\beta) = \int_0^{x^\alpha/x^\alpha+\beta^\alpha} [u^{r-1}(1-u)^{s-1}]\, du \div B(r,s)$$

$$= IB\left(\frac{x^\alpha}{x^\alpha + \beta^\alpha}\,;\, r,s\right),$$

which can be taken as the definition of the function IB. However, the approximation of this integral by the above quadrature formula is not close for small values of r and s, e.g., below 1. A recurrence relation was derived to express $IB(x;r,s)$ as a function of $IB(x;r + 1, s + 1)$, putting the integral to be solved in a more satisfactory area. This relationship is $rsIB(x;r,s) = x^r(1 - x)^s(s - (r + s)x) + (r + s + 1)(r + s)\, IB(x;r + 1, s + 1)$, and was derived by George Phillips from Abramowitz's [1] formulas 26.5.2 and 26.5.16 on page 944. In practice this formula is applied thrice to get to the $r + 3$, $s + 3$ level. Exhibit F3 gives a series of *APL* programs which performs the calculation of $F(x; r,s,\alpha,\beta)$.

EXHIBIT F1

ABSCISSAS AND WEIGHTS FOR n POINT GAUSSIAN QUADRATURE

$n=6$

x_i			W_i		
±0.23861	91860	83197	0.46791	39345	72691
±0.66120	93864	66265	0.36076	15730	48139
±0.93246	95142	03152	0.17132	44923	79170

$n=10$

±0.14887	43389	81631	0.29552	42247	14753
±0.43339	53941	29247	0.26926	67193	09996
±0.67940	95682	99024	0.21908	63625	15982
±0.86506	33666	88985	0.14945	13491	50581
±0.97390	65285	17172	0.06667	13443	08688

$n=24$

±0.06405	68928	62606	0.12793	81953	46752
±0.19111	88674	73616	0.12583	74563	46828
±0.31504	26796	90163	0.12167	04729	27803
±0.43379	35076	26045	0.11550	56680	53726
±0.54542	14713	88840	0.10744	42701	15966
±0.64809	36519	36976	0.09761	86521	04114
±0.74012	41915	78554	0.08619	01615	31953
±0.82000	19859	73903	0.07334	64814	11080
±0.88641	55270	04401	0.05929	85849	15437
±0.93827	45520	02733	0.04427	74388	17420
±0.97472	85550	71309	0.02853	13886	28934
±0.99518	72199	97021	0.01234	12297	99987

EXHIBIT F2

$n = 96$

	X_i	W_i		X_i	W_i
1	−.999689503883231	.000796792065552	49	.016276744849603	.032550614492363
2	−.998364375863182	.001853960788947	50	.048812985136050	.032516118713869
3	−.995981842987209	.002910731817935	51	.081297495464426	.032447163714064
4	−.992543900323763	.003964554338445	52	.113695850110666	.032343822568576
5	−.988054126329624	.005014202742928	53	.145973714654897	.032206204794030
6	−.982517263563015	.006058545504236	54	.178096882367619	.032034456231993
7	−.975939174585136	.007096470791154	55	.210031310460567	.031828758894411
8	−.968326828463264	.008126876925698	56	.241743156163840	.031589330770727
9	−.959688291448743	.009148671230783	57	.273198812591049	.031316425596861
10	−.950032717784438	.010160770535008	58	.304364944354496	.031010332586314
11	−.939370339752755	.011162102099838	59	.335208522892625	.030671376123669
12	−.927712456722309	.012151604671088	60	.365696861472314	.030299915420828
13	−.915071423120898	.013128229566962	61	.395797649828909	.029896344136328
14	−.901460635315852	.014090941772315	62	.425478988407301	.029461089958168
15	−.886894517402420	.015038721026995	63	.454709422167743	.028994614150555
16	−.871388505909297	.015970562902562	64	.483457973920596	.028497411065085
17	−.854959033434601	.016885479864245	65	.511694177154668	.027970007616848
18	−.837623511228187	.017782502316045	66	.539388108324357	.027412962726029
19	−.819400310737932	.018660679627411	67	.566510418561397	.026826866725592
20	−.800308744139141	.019519081140145	68	.593032364777572	.026212340735672
21	−.780369043867433	.020356797154333	69	.618925840125469	.025570036005349
22	−.759602341176647	.021172939892191	70	.644163403784967	.024900633222484
23	−.738030643744400	.021966644438744	71	.668718310043916	.024204841792365
24	−.715676812348968	.022737069658329	72	.692564536642172	.023483399085926
25	−.692564536642172	.023483399085926	73	.715676812348968	.022737069658329
26	−.668718310043916	.024204841792365	74	.738030643744400	.021966644438744
27	−.644163403784967	.024900633222484	75	.759602341176647	.021172939892191
28	−.618925840125469	.025570036005349	76	.780369043867433	.020356797154333
29	−.593032364777572	.026212340735672	77	.800308744139141	.019519081140145
30	−.566510418561397	.026826866725592	78	.819400310737932	.018660679627411
31	−.539388108324357	.027412962726029	79	.837623511228187	.017782502316045
32	−.511694177154668	.027970007616848	80	.854959033434601	.016885479864245
33	−.483457973920596	.028497411065085	81	.871388505909297	.015970562902562
34	−.454709422167743	.028994614150555	82	.886894517402420	.015038721026995
35	−.425478988407301	.029461089958168	83	.901460635315852	.014090941772315
36	−.395797649828909	.029896344136328	84	.915071423120898	.013128229566962
37	−.365696861472314	.030299915420828	85	.927712456722309	.012151604671088
38	−.335208522892625	.030671376123669	86	.939370339752755	.011162102099838
39	−.304364944354496	.031010332586314	87	.950032717784438	.010160770535008
40	−.273198812591049	.031316425596861	88	.959688291448743	.009148671230783
41	−.241743156163840	.031589330770727	89	.968326828463264	.008126876925698
42	−.210031310460567	.031828758894411	90	.975939174585136	.007096470791154
43	−.178096882367619	.032034456231993	91	.982517263563015	.006058545504236
44	−.145973714654897	.032206204794030	92	.988054126329624	.005014202742928
45	−.113695850110666	.032343822568576	93	.992543900323763	.003964554338445
46	−.081297495464426	.032447163714064	94	.995981842987209	.002910731817935
47	−.048812985136050	.032516118713869	95	.998364375863182	.001853960788947
48	−.016276744849603	.032550614492363	96	.999689503883231	.000796792065552

EXHIBIT F3

```
∇TBXR[⎕]∇
      ∇ Y←X TBXR AGD;B;A;G;D
[1]   ⍝TRANSFORMED BETA XS RATIO AT X TIMES MEAN
[2]    A←AGD[1]
[3]    G←AGD[2]
[4]    D←AGD[3]
[5]    B←(G CBETA D)÷(G+÷A) CBETA D-÷A
[6]    Y←X TBETAXR A,B,G,D
       ∇

      ∇TBETAXR[⎕]∇
      ∇ Y←X TBETAXR ABGD;A;B;G;D;H;L
[1]   ⍝TRANSFORMED BETA XS RATIO PARAMS A B G D
[2]    A←ABGD[1]
[3]    B←ABGD[2]
[4]    G←ABGD[3]
[5]    D←ABGD[4]
[6]    L←L÷1+L←(X÷B)*A
[7]    Y←(H←(G+÷A) CBETA D-÷A)-L IB(G+÷A),D-÷A
[8]    Y←Y-X×(÷B)×(G CBETA D)-L IB G,D
[9]    Y←Y÷H
       ∇

      ∇CBETA[⎕]∇
      ∇ Y←V CBETA W
[1]   ⍝COMPLETE BETA OF V AND W
[2]    Y←÷(V!V+W)×V×W÷V+W
       ∇

      ∇IB[⎕]∇
      ∇ R←X IB AB;Y1;Y2;Y3;Y4;Y5;A;B
[1]   ⍝ WRITTEN BY GEORGE PHILLIPS
[2]    A←AB[1]×1 ⎕CC 0
[3]    B←AB[2]
[4]    Y1←¯1↑×\(X,1-X)*AB
[5]    Y2←((B-1)+⍳3)-X×(A+B-2)+2×⍳3
[6]    Y3←(X×1-X)* 0 1 2
[7]    Y4←1,×\(A+B-1)+⍳6
[8]    Y5←×\(1,(A+1),A+2)×1,(B+1),B+2
[9]    R←(÷A×B)×(Y1×+/Y2×Y3×Y4[1 3 5]÷Y5)+(Y4[7]÷Y5[3])×(X INCBETA(A+3),B+3)
       ∇

      ∇INCBETA[⎕]∇
      ∇ RST←A INCBETA VW
[1]   ⍝WRITTEN GREGG EVANS
[2]    RST←1 GSQD '(X*VW[1]-1)×(1-X)*VW[2]-1/DX'
       ∇

      ∇GSQD[⎕]∇
      ∇ RST←X GSQD Y;A;B;C;D;E;F
[1]   ⍝WRITTEN BY GREGG EVANS
[2]    F←(E←⍎,(22⍴ 1 0)\(0 11)↓(11 0)⍕0,[1.5] D÷(10*10)×~D←C≠¯1↑Y)\C←(∧\Y≠'/')/Y
[3]    C←'(X÷2)×',F,0↑E[D]←(⍴D←,(¯1+D[;1]),D←(B,(⍴D)÷B←(+/~E)÷10)⍴D←(0=E)/⍳⍴,E)⍴'(0.5×X×A+1)'
[4]    RST←+/(2÷+/GSQDVAR[2;])×GSQDVAR[2;]×⍎C,0↑A←GSQDVAR[1;]
       ∇
```

APPENDIX G

RELATIONSHIP BETWEEN GAMMA AND TRANSFORMED GAMMA

To show: Λ^α is gamma in s, γ if and only if Λ is transformed gamma in s,α,β where $\beta = \gamma^{1/\alpha}$.

Note that

$$\begin{aligned}\Pr(\Lambda \le \lambda) &= \Pr(\Lambda^\alpha \le \lambda^\alpha)\\ &= G(\lambda^\alpha;s,1,\gamma)\\ &= \int_0^{\gamma\lambda^\alpha} t^{s-1}\, e^{-t}\, dt\\ &= \int_0^{(\beta\lambda)^\alpha} t^{s-1}\, e^{-t}\, dt = G(\lambda;s,\alpha,\beta).\end{aligned}$$

APPENDIX H

MAXIMUM LIKELIHOOD ESTIMATORS FOR TRANSFORMED BETA PARAMETERS

Given a sample $x_1, \ldots, x_n$, fitting the parameters r, s, α, and β of the transformed beta by maximum likelihood involves finding the maximum of the log-likelihood function

$$\ln L(r, s, \alpha, \beta) = n \ln \Gamma(r + s) + n \ln \alpha + (\alpha r - 1) \sum_{i=1}^{n} \ln x_i$$

$$- (n\alpha r \ln \beta + n \ln \Gamma(r) + n \ln \Gamma(s) + (r + s) \sum_{i=1}^{n} \ln(1 + x_i/\beta)^{\alpha}.$$

As with the transformed gamma let the derivative of $\ln \Gamma(x)$ be denoted $\psi(x)$. Dividing the partials of $\ln L$ by n and setting to zero gives the following 4 equations:

$$(r)\text{: } \psi(r + s) = \psi(r) + \overline{\ln(1 + \beta/x_i)^{\alpha})}$$

$$(s)\text{: } \psi(r + s) = \psi(s) + \overline{\ln(1 + x_i/\beta)^{\alpha})}$$

$$(\alpha)\text{: } 1/\alpha + r\,\overline{\ln(x_i/\beta)} = (r + s)\overline{(\ln(x_i/\beta))(\beta/x_i)^{\alpha} + 1)^{-1}}$$

$$(\beta)\text{: } r = (r + s)\overline{(1 + (\beta/x_i)^{\alpha})^{-1}}$$

where the bar denotes the average over the sample of the barred function.

The (α) and (β) equations are linear in r and s, so they can be solved to yield r and s as functions of α and β. These can be substituted into the (r) and (s) equations to give two non-linear equations in two unknowns (α,β) which can be solved by the methods of Appendix A.

An *APL* system for solving these equations is shown in Exhibit H1 and a run with sample data in Exhibit H2.

EXHIBIT H1

APL PROGRAMS FOR
TRANSFORMED BETA MLE

```
    ∇NRFN[⎕]∇
      ∇ AB1←V NRFN AB;YA;YB;J;Z
[1]   ⍝WRITTEN BY GARY VENTER
[2]   ⍝NEWTON RAPHSON ITERATION FOR TRBET PARAMS, SAMPLE IN V
[3]    AB1←AB
[4]    Z←1E¯7
[5]   TOP:AB←AB1
[6]    Y←V FN AB
[7]    YA←V FN(AB[1]×1+Z),AB[2]
[8]    YB←V FN AB[1],AB[2]×1+Z
[9]    YA←(YA-Y)÷Z×AB[1]
[10]   YB←(YB-Y)÷Z×AB[2]
[11]   J←(YA[1]×YB[2])-YA[2]×YB[1]
[12]   AB1←AB[1]-((Y[1]×YB[2])-Y[2]×YB[1])÷J
[13]   AB1←AB1,AB[2]-((Y[2]×YA[1])-Y[1]×YA[2])÷J
[14]   '2 OLD         TOLERANCES         2 NEW'
[15]   AB,Y,AB1
[16]   'R.S:';R,S
[17]   →(2E¯7<+/|¯1+AB1÷AB)/TOP
[18]   ⎕←Y←V FN AB1
[19]   'R,S,ALPHA,BETA'
[20]   R,S,AB1
       ∇


       ∇FN[⎕]∇
       ∇ Y←V FN AB;D;F;G;H;N;PS;PR;PRS;DL;LL
[1]   ⍝R AND S ARE GLOBALS
[2]   ⍝V A VECTOR OF OBSERVATIONS, AB IS ALPHA,BETA
[3]   ⍝Y IS A 2 VECTOR TRYING TO GET TO 0,0 FOR TBET MLE
[4]    N←÷⍴V
[5]    G←V÷AB[2]
[6]    H←⍟G
[7]    D←1+G*-AB[1]
[8]    F←N×+/H÷D
[9]    H←AB[1]×N×+/H
[10]   D←N×+/÷D
[11]   R←-÷H-AB[1]×F÷D
[12]   S←R×-1-÷D
[13]   G←N×+/⍟1+G*AB[1]
[14]   PS←SI S
[15]   Y←H+PS-PR←SI R
[16]   Y←Y,G+PS-PRS←SI R+S
       ∇
```

EXHIBIT H1 PAGE 2

```
  ∇SI[⎕]∇
      ∇ PSIX←SI X;Z;PSIZ;Y;M;N
[1]   ⍝PSI FUNCTION IE DERIVATIVE LOG GAMMA FUNCTION
[2]   ⍝WRITTEN HARRY SOUL
[3]    Z←X-⌊X
[4]    →(L1,L2)[1+Z=0]
[5]   L1:PSIZ←-(!-1-Z)×+/(⍳26)×CEE×Z*¯1+⍳26
[6]    Y←1000|⌊X
[7]    N←0
[8]    M←⌊X÷1000
[9]    PSIX←PSIZ++/÷Z+¯1+⍳Y
[10]   →(M=0)/0
[11]  LT:N←N+1
[12]   PSIX←PSIX++/÷Z+(1000×N-1)+Y+¯1)+Y+-1+⍳1000
[13]   →(N<M)/LT
[14]   →0
[15]  L2:PSIZ←-(!Z)×+/(⍳26)×CEE×(Z+1)*¯1+⍳26
[16]   PSIX←PSIZ++/÷⍳X-1
      ∇
```

60	CEE
61	1
62	0.5772156649015329
63	−0.6558780715202538
64	−0.0420026350340952
65	0.1665386113822915
66	−0.0421977345555443
67	−0.009621971527877
68	0.007218943246663
69	−0.0011651675918591
70	−0.0002152416741149
71	0.0001280502823882
72	−2.01348547807E−5
73	−1.2504934821E−6
74	1.133027232E−6
75	−2.056338417E−7
76	6.116095E−9
77	5.0020075E−9
78	−1.1812746E−9
79	1.043427E−10
80	7.7823E−12
81	−3.6968E−12
82	5.1E−13
83	−2.06E−14
84	−5.4E−15
85	1.4E−15
86	1E−16

EXHIBIT H2

SAMPLE RUN OF TRANSFORMED BETA MLE WITH GOOD STARTING ESTIMATES

V

2.201825487277711	1.747798995989603	1.555619456471727	1.434261861491408
1.345898293955564	1.276532762732432	1.219472497925706	1.171009053335359
1.128878212884788	1.091598560297855	1.058149169964544	1.027797375655266
0.9999999999999999	0.9743434501286376	0.9505056924135983	0.9282311847924588
0.9073138148639067	0.8875849650558165	0.8689049641059015	0.8511568358547295
0.8342416436253031	0.81807496609125	0.8025841905289312	0.7877064064383325
0.7733867467937893	0.7595770676095318	0.7462348863847357	0.733322520898723
0.7208063846790024	0.7086564061646645	0.6968455463959924	0.6853493958275812
0.674145835167939	0.6632147483965766	0.6525377785832587	0.6420981190348407
0.6318803337678372	0.6218702024548765	0.6120545858977834	0.6024213087964627
0.5929590571538267	0.5836572881149439	0.5745061504078917	0.5654964138531555
0.5566194066522674	0.5478669593658831	0.5392313546553542	0.53070528199689
0.5222817966889851	0.5139542825661741	0.5057164179087162	0.4975621441012036
0.4894856366454348	0.4814812781759202	0.4735436331613866	0.4656674240036885
0.4578475082673738	0.4500788567894164	0.4423565324296311	0.434675669228307
0.4270314517386136	0.4194190942972299	0.4118338199870745	0.404270839030415
0.3967253263281841	0.3891923978309076	0.3816670853866948	0.3741443096602311
0.3666188506509329	0.3590853152547595	0.3515381012078956	0.3439713566152265
0.3363789340936847	0.328754338338611	0.3210906656344104	0.3133805334572376
0.305615997826835	0.2977884554141212	0.2898885265394574	0.2819059140149433
0.273829231162068	0.2656457900782352	0.2573413380345932	0.248899725301121
0.2403024809791267	0.2315282633825993	0.2225521361535894	0.2133445971882582
0.2038702484700424	0.1940859297219838	0.1839380254588229	0.1733584487947235
0.1622584092416313	0.1505182593963702	0.1379699089521555	0.1243638396796979
0.1093001477080087	0.09205965646857106	0.07106750819518526	0.04089307909136584

V NRFN 1.521 1.553

```
2 OLD                                  TOLERANCES                                    2 NEW
1.521 1.553                            1.456996026050206E-6 1.088012693967189E-7     1.520915599542439 1.553092179774157
R, S: 1.441569975759713 6.476705211863293
2 OLD                                  TOLERANCES                                    2 NEW
1.520915599542439 1.553092179774157    2.314481939436064E-11 2.418509836843441E-12   1.520915600822739 1.553092175281865
R, S: 1.441699580189243 6.477401387277938
4.440892098500626E-16 2.775557561562891E-16
R, S, ALPHA, BETA
1.441699614500499 6.477400647693872 1.520915600822739 1.553092175281865
```

APPENDIX J

TRANSFORMED GAMMA
RELATION BETWEEN COEFFICIENTS OF VARIATION AND SKEWNESS

Empirical investigations suggest that not all pairs of positive real numbers can be realized as the coefficients of variation (*CV*) and skewness (*SKW*) of a transformed gamma distribution. For example, as mentioned in the text, for *CV*s of 1.25 and greater the *SKW* always seems to exceed the *CV*.

While not proven analytically, observation suggests the following:

(1) For fixed *r* the ratio *SKW/CV* is a decreasing function of alpha.
(2) If the ratio *SKW/CV* is held constant (by increasing alpha), then the *CV* and *SKW* increase as *r* decreases.
(3) These increases are asymptotic to some finite value as *r* goes to zero.

Thus for a fixed *SKW/CV* ratio, the *CV* and *SKW* can not exceed a maximum. The following table gives these approximate maximum values for selected ratios.

SKW/CV	*Maximum CV*	*Maximum SKW*
1.4	11.1	15.5
1.3	3.9	5.1
1.2	2.0	2.4
1.1	1.51	1.66
1.0	1.25	1.25
.9	1.09	.98
.83	1.00	.83
.8	.97	.78
.7	.88	.62
.6	.81	.49
.5	.76	.38
.4	.71	.28
.3	.67	.20
.2	.64	.13
0.00	.58	0.00

This relationship thus restricts the values which the *CV, SKW* pairs can take on. As the maxima seem to be increasing functions of the ratio *SKW/CV*, each maximum is an upper bound over all lower values of that ratio. For example, if the *SKW* is less than or equal to .83*CV*, then the *CV* does not exceed 1.0. Conversely, if the *CV* is above 1.0 the *SKW* is .83 or greater.

It is interesting to note that the skewness can be negative. This seems possible for any value of r. For small r, SKW reaches zero at about an alpha of $1/r$. In the Weibull case ($r = 1$) zero skewness occurs for α just above 3.6.

The use of empirical studies in mathematical investigations is of course subject to pitfalls. The findings in this appendix should thus be regarded as hypotheses until more rigorous demonstrations can be provided.

Further investigation has also revealed that matching transformed gamma moments is not possible if the CV is very small and the SKW is large. In this case, it has been possible to match transformed beta moments. The case $\alpha = 1$ often suffices, and this yields closed form solutions for the parameters as follows:

Define $M_j = E(X^j)/E(X)^j$ for any random variable X. Then the transformed beta parameters r and s are:

$$r = 2\,\frac{M_3 - M_2^2}{M_2^2 + M_2M_3 - 2M_3}$$

$$s = \frac{r + 1 - 2M_2r}{r + 1 - M_2r}$$

Unfortunately, those equations sometimes yield negative parameters. In that case the transformed beta with $r = 20$ ($\alpha \neq 1$) has seemed to give satisfactory fits.

Using the transformed beta to match moments in this way would seem to give up the parameter uncertainty. This is not necessary, however, as the moments of the combined process-parameter system can be found by combining the process and parameter moments. In fact,

$$M_j(\text{combined}) = M_j(\text{process})\, M_j(\text{parameter}).$$

Thus the combined moments can be used to calculate the transformed beta or gamma parameters. This, in fact, allows for greater freedom in selecting the parameter distribution moments, in that the skewness need not be strictly determined by the CV.

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GOOD AND BAD DRIVERS—A MARKOV MODEL OF ACCIDENT PRONENESS

EMILIO VENEZIAN

VOLUME LXVIII

DISCUSSION BY SANFORD R. SQUIRES

A series of studies by the Department of Transportation in the 1960's and early 1970's first popularized analysis of automobile insurance experience through accident proneness models. Unlike many actuarial approaches which tend to be empirical and emphasize practical interpretations, accident proneness models tend to be heavily theoretical and often prove difficult to interpret on a non-technical level. If one is not already familiar with the various Poisson process models found in actuarial literature, Dr. Venezian's paper might be extremely slow going.

Dr. Venezian's summary of other authors' models in the introduction is remarkable in its terseness without omitting the salient points. In the same style, the author points out many strengths and implications of his own proposed model. This is a paper which requires careful reading, working out details derived by the author, for full appreciation and understanding.

Dr. Venezian analyzes some of the weaknesses in his own model. In particular, the crucial role of mileage in determining the fitting parameters is noted. According to his data, males have almost twice the accident frequency of females in the younger age groups. The paper attributes all this difference to mileage alone, since mileage differences conveniently follow the same pattern of accident likelihood differences between the sexes. The author notes that detailed data of mileage by age and sex would be helpful. Such data was analyzed in the early seventies[1]—but the mileage patterns would seem to disrupt rather than enhance Dr. Venezian's model. In particular, average annual mileage for male drivers rises steeply in the younger ages. This is directly counter to the exponential decay of accidents with age shown in the model. Female mileages show much less variation by age than male mileages. Mileage does contribute to explaining

[1] The *National Transportation Survey,* 1969–1970, conducted by the Bureau of Census for the Federal Highway Administration developed average annual mileage data for the general driver population by age and sex of drivers. This data has recently been published and used in analysis of automobile classification plans by the Rate Regulation Division of the California Department of Insurance in Phase II of their *Study of California Driving Performance* (November 1979), p. 17.

the variation in accident frequencies by age and sex, but not in as simple a way as used in the author's paper.

Dr. Venezian has created a "Markov process" model. That is, drivers bounce between "good" and "bad" states with a predetermined probability of transition back and forth. To someone familiar with Markov processes, it is obvious that the model will produce an exponential decay of accident likelihood over time. If driver data shows accidents decreasing with age in a somewhat exponential fashion, this model has to fit. It is disappointing that the way it fits is so philosophically unappealing—at least from this reviewer's viewpoint. It is difficult to give a layman's explanation for the Markov process other than "it fits the data."

Only three drivers in a hundred go from the good to the bad state every year while seventeen in a hundred go the other direction. Intuitively one would like to see more frequent transitions of some sort reflecting heavy use of autos during the day, but not at night, occasional bad driving conditions and occasional non-use of the automobile for weeks at a time while on vacation—or any of the hundreds of other everyday real life situations where accident likelihood for an individual varies considerably over a short period of time. The long time between transitions in the model stems from the transition probability parameters fitted to the data. These parameters are a direct result of the overall rate of exponential decay in accident likelihood with age. These low transition probabilities fit best, but seem to defy intuitive interpretation other than to produce the steady, long term decrease in accident likelihood with age. High frequency transitions are more intuitive but are practically useless to produce the desired data fit from the model.

The ad-hoc assumption made by the author that males and females start their transitions at different ages seems non-intuitive. In the author's model, female transitions start before male transitions. This seems intuitively counter to the fact that females tend to start driving at later ages than males in our society. It would seem more plausible to either allow different accident likelihood ratios between "bad" and "good" states for men and women or allow different transition probabilities or starting ages. These possibilities are somewhat ad-hoc, too, but could fit the data. They are more in keeping with some particular data published by the state of California in Phase II of its recent study.[2] A mileage-adjusted surrogate for accident likelihood shown graphically

[2] *Study of California Driving Performance Phase II*, Rate Regulation Division—California Department of Insurance (November 1979), p. 22.

in that study suggests different "bad" to "good" ratios for males and females. Better data from strictly insurance sources with loss costs and mileage would be helpful, but is not generally available.

The data fit in the author's paper does not strike this reviewer as particularly great. However, the calculated statistical significance test on the excess variance test shows that other models, like a simple Poisson, produce much worse fits to the data! This is the best fit offered so far.

Dr. Venezian observes that one might be able to distinguish different models by looking at the distribution of time between accidents. For his own model, he implies that one would most likely see two clustered distributions for the number of accidents for an individual over subintervals of time. Both statements would seem to be only of theoretical interest and difficult to measure in the real world because of the low frequencies of accidents and the likelihood of changing driving environments during the period of years between accidents. Although data in this regard might support assertions about excess variance in total, it seems unlikely that one could use this test to distinguish between different competing models.

Models on the order of complexity of the author's will quickly approach the point where their mathematical robustness will necessitate more detailed data and more intricately structured statistical tests in order that further progress be made. It is this reviewer's opinion that intuitive interpretation of models will become important in selecting the best among competing models.

Several extensions of the author's model are possible.

One could possibly add more discrete states. In addition to "good" and "bad," I'm sure someone will suggest a third "ugly" state. Surely, this would produce much improved data fits—but new insights are unlikely to result.

One could create a continuous spectrum of states and a corresponding set of transition probabilities—a task to be tackled by only the purest of pure actuaries, but not an impossible task.

Finally, one might follow up on the author's speculations on the further utilization of mileage data. One might create a model using cumulative lifetime mileage and its first derivative (or first difference), annual mileage, as the entire basis for accident likelihood. (The mathematical treatment of accident likelihoods as separate from mileage in the author's paper is cosmetic, resulting from a two variable notation, rather than any intrinsic functional independence in the equations.)

Dr. Venezian has written a very professional and accurate paper proposing a new model for accident proneness. Although this review has made general criticisms on non-intuitive aspects of the author's model, the paper is of professional calibre and few technical flaws can be found. Dr. Venezian has taken us a step in complexity beyond previous models. The long journey yet to go should provide significant challenge for future contributors to the *Proceedings*.

ACTUARIAL VALUATION OF PROPERTY/CASUALTY INSURANCE COMPANIES

ROBERT W. STURGIS

VOLUME LXVIII

DISCUSSION BY ROBERT ROTHMAN AND ROBERT V. DEUTSCH

VOLUME LXIX

AUTHOR'S REPLY TO DISCUSSION

My paper points out that, in spite of voluminous readings in the Society of Actuaries literature, our *Proceedings* do not deal with this subject at all. The reviews by Mr. Lowe and by Messrs. Rothman and Deutsch have added significantly to the discussion, and subsequently to my deeper understanding of the underlying interrelationships affecting company valuations.

My paper is largely a synthesis of classical life literature. Thus, I was somewhat taken aback by the Rothman-Deutsch review suggesting that the present value of future cash flows was a better method than present value of future earnings. Nowhere in my review of the life insurance literature had this been suggested. Further, I had no intuitive understanding of what the real difference was between the two methods. So, I set about to reconcile the two approaches.

The reconciliation, presented below, is an algebraic representation of the cash and earnings process for a theoretical insurance enterprise.

First some definitions:

V = Value of company
S_t = Net Worth at time t
CF_t = Cash flow at time t
E_t = Statutory earnings at time t
R_t = Reserves at time t (not just loss reserves; all reserves)
i = Assumed interest rate
j = Risk-adjusted interest rate
v = $1/(1 + i)$ = Interest discount rate
u = $1/(1 + j)$ = Risk-adjusted discount rate

It immediately struck me that one major difference between the earnings and the cash flow methods is the interest versus the risk-adjusted discount rate. In the cash flow method, they are the same ($i = j$). In the classical earnings method, the use of a higher risk rate of return is stressed.[1]

The two methods can be formularized as follows for a theoretical enterprise.

CASH FLOW:

$$V = S_0 + R_0 + \sum_{t=1}^{\infty} u^t CF_t$$

or value is present cash (reserves and surplus), plus the present value of future cash.

EARNINGS:

$$V = S_0 + \sum_{t=1}^{\infty} u^t E_t$$

or value is present surplus (net worth) plus the present value of future earnings. (One thing that may not be clear from the text of my paper, but should be clear from the example, is that beginning surplus and future earned surplus contributions are not retained and compounded, but rather, present valued to the owner(s).)

Keeping in mind that earnings are equal to cash flow plus interest on reserves less reserve changes, the earnings formula can be restated as follows:

$$V = S_0 + \sum_{t=1}^{\infty} u^t [CF_t - (R_t - R_{t-1}) + iR_{t-1}].$$

The appended exhibit algebraically restates this formula as

$$V = [S_0 + R_0 + \sum_{t=1}^{\infty} u^t CF_t] - \sum_{t=1}^{\infty} (j - i) u^t R_{t-1}.$$

Thus, the earnings formula can be restated as equal to the cash flow formula less an interest penalty $(j - i)$ on funds held in reserve. Therefore:

(1) If $j = i$, the two methods are equivalent, and

(2) If $j > i$, the earnings method introduces an interest penalty on reserves based on the assumption that such funds need to be invested conservatively. This penalty is analogous to the interest penalty on required

[1] James C. H. Anderson, "Gross Premium Calculations and Profit Measurement for Non-Participating Insurance," *Transactions Society of Actuaries*, Volume XI (1959), p. 378.

surplus, which is cited in the paper but not included in the above formulation.

In summary, the cash flow method can be seen as a special case of the classical earnings formula. When the discount rate equals the assumed interest earnings rate, the timing of the booking of earnings no longer affects present values. That is, the present values of statutory earnings, GAAP earnings, and cash flows are all the same.

Algebraic Representation of Earnings Formula

$$(1)\ V = S_0 + \sum_{t=1}^{\infty} u^t[CF_t - (R_t - R_{t-1}) + iR_{t-1}]$$

This is algebraically equivalent to:

$$(2)\ V = S_0 + \sum_{t=1}^{\infty} u^t CF_t + \sum_{t=1}^{\infty} u^t(1 + i)R_{t-1} - \sum_{t=1}^{\infty} u^t R_t$$

Removing the first term from the second summation:

$$(3)\ V = S_0 + \sum_{t=1}^{\infty} u^t CF_t + u(1 + i)R_0 + \sum_{t=2}^{\infty} u^t(1 + i)R_{t-1} - \sum_{t=1}^{\infty} u^t R_t$$

$$(4)\ V = S_0 + \sum_{t=1}^{\infty} u^t CF_t + \left[\frac{1 + i}{1 + j}\right] R_0 + \sum_{t=2}^{\infty} u^t(1 + i)R_{t-1} - \sum_{t=1}^{\infty} u^t R_t$$

$$(5)\ V = S_0 + \sum_{t=1}^{\infty} u^t CF_t + R_0 - \left[\frac{j - i}{1 + j}\right] R_0 + \sum_{t=2}^{\infty} u^t(1 + i)R_{t-1} - \sum_{t=1}^{\infty} u^t R_t$$

Rearranging terms, and executing a change of variables on t in the second summation:*

$$(6)\ V = [S_0 + R_0 + \sum_{t=1}^{\infty} u^t CF_t] - u(j - i)R_0 + \sum_{t=1}^{\infty} u^{t+1}(1 + i)R_t - \sum_{t=1}^{\infty} u^t R_t$$

Then, combining the second and third summations:

$$(7)\ V = [S_0 + R_0 + \sum_{t=1}^{\infty} u^t CF_t] - u(j - i)R_0 - \sum_{t=1}^{\infty} u^t R_t \left[1 - \frac{1 + i}{1 + j}\right]$$

* This transformation is possible because the series converges to zero.

$$(8)\quad V = [S_0 + R_0 + \sum_{t=1}^{\infty} u^t CF_t] - u(j - i)R_0 - \sum_{t=1}^{\infty} u^t R_t \left[\frac{j - i}{1 + j}\right]$$

$$(9)\quad V = [S_0 + R_0 + \sum_{t=1}^{\infty} u^t CF_t] - u(j - i)R_0 - \sum_{t=1}^{\infty} u^{t+1} R_t (j - i)$$

Finally, bringing the R_0 term back inside the summation and executing a second transformation of the variable t:

$$(10)\quad V = [S_0 + R_0 + \sum_{t=1}^{\infty} u^t CF_t] - \sum_{t=1}^{\infty} (j - i) u^t R_{t-1}$$

A NOTE ON LOSS DISTRIBUTIONS

J. GARY LAROSE

VOLUME LXIX

DISCUSSION BY CHARLES C. HEWITT, JR.

This is a remarkable piece of synthesis! It finds a thread running through many significant papers in our *Proceedings* and weaves an entirely new fabric which, when reviewed on an overall or on a modular basis, should make life much simpler for both the student and the practicing actuary. The author recognizes that mathematicians constantly strive for generalizations which provide solutions for *superficially* different problems.

He has achieved that goal in three basic elements found in this paper:

(1) Simplified (standardized?) notation,
(2) Recognition of a commonalty of approach, and
(3) Application of simplified notation to a gallimaufry of actuarial problems.

Standardized Notation

There is an opportunity here for the Casualty Actuarial Society to intervene on behalf of present and future generations of actuaries in the matter of actuarial notation. The textbook, currently in preparation under the auspices of the Actuarial Education and Research Fund (AERF), on distributions of a (single) loss will suggest a more mathematically oriented notation. Gary LaRose, with attribution to Robert Finger, suggests another notation.

Taking advantage of symbols readily available on computers and word processors, I have proposed still another:

(1) $F\#(x) = \int_0^x f(t)dt$

(2) $F\$(x) = \frac{1}{E(t)} \int_0^x t\, f(t)dt$ and

(3) $F\&(x) = F\$(x) + \frac{x}{E(t)}\,[1 - F\#(x)]$

where all three *cumulative* functions have a range from *zero* to *one*.

It is my feeling that this latter notation has a mnemonic value not possessed by the other forms. A comparison of the LaRose/Finger and Hewitt notations appears in the Appendix to this review.

Loss Distribution Table

A useful tool in valuing coverage limitation is a loss distribution table*.

LOSS DISTRIBUTION TABLE

Loss Amount	Number	Amount	Loss Elimination Ratio (LER)
(x)	$F\#(x)$	$F\$(x)$	$F\&(x)$
x_1	$F\#(x_1)$	$F\$(x_1)$	$F\&(x_1)$
x_2	$F\#(x_2)$	$F\$(x_2)$	$F\&(x_2)$
x_i	$F\#(x_i)$	$F\$(x_i)$	$F\&(x_i)$
∞	1	1	1

where $x_1 < x_2 < \ldots < x_i < \ldots < \infty$

The values in the "Number," "Amount," and "LER" columns are cumulative. Values of x are selected so as to make for easy calculation of frequently used deductibles (retentions) and limits. Formulas for commonly used actuarial expressions in both LaRose/Finger and Hewitt notations are contained in the Appendix.

Gary LaRose has earned the gratitude of many for the thought, research and clarity of expression which has gone into his effort.

* See Charles C. Hewitt, Jr. and Benjamin Lefkowitz, "Methods for Fitting Distributions to Insurance Loss Data," *PCAS* LXVI (1979), p. 147.

APPENDIX

Item	LaRose/Finger	Hewitt
Cumulative Distribution Function (c.d.f.)	$F(x)$	$F\#(x)$
Proportion of total losses on those losses whose amount is less than or equal to x	$X1(x)$	$F\$(x)$
Loss Elimination Ratios Deductible (Retention)	$X2(x)$	$F\&(x)$
Limit(s)	$X3(x)$	$1 - F\&(x)$
Interval Mean $\alpha = E(t)$	$\frac{X1(x_2) - X1(x_1)}{F(x_2) - F(x_1)} \alpha$	$\frac{F\$(x_2) - F\$(x_1)}{F\#(x_2) - F\#(x_1)} E(t)$
Value of a Layer ($x_2 > x_1$) Proportional	$X2(x_2) - X2(x_1)$	$F\&(x_2) - F\&(x_1)$
Absolute	$[X2(x_2) - X2(x_1)]\alpha$	$[F\&(x_2) - F\&(x_1)]\ E(t)$
Mean Value of Coverage Deductible (Retention) = x_1 Limit = x_2	$\frac{X2(x_2) - X2(x_1)}{1 - F(x_1)} \alpha$	$\frac{F\&(x_2) - F\&(x_1)}{1 - F\#(x_1)} E(t)$

Effect of changes in deductible (retention) from x_0 to x_1 and limit from x_2 to x_3	Modifiers LaRose/Finger	Modifiers Hewitt
On Frequency	$\frac{1 - F(x_1)}{1 - F(x_0)}$	$\frac{1 - F\#(x_1)}{1 - F\#(x_0)}$
On Severity	$\frac{[1 - F(x_0)][X2(x_3) - X2(x_1)]}{[1 - F(x_1)][X2(x_2) - X2(x_0)]}$	$\frac{[1 - F\#(x_0)][F\&(x_3) - F\&(x_1)]}{[1 - F\#(x_1)][F\&(x_2) - F\&(x_0)]}$
On Pure Premium	$\frac{X2(x_3) - X2(x_1)}{X2(x_2) - X2(x_0)}$	$\frac{F\&(x_3) - F\&(x_1)}{F\&(x_2) - F\&(x_0)}$

ESTIMATING PROBABLE MAXIMUM LOSS WITH ORDER STATISTICS

MARGARET E. WILKINSON

VOLUME LXIX

DISCUSSION BY JOHN S. MCGUINNESS

It is always refreshing to see new thinking. This paper introduces to our *Proceedings* an approach—order statistics—that has not before been mentioned there as far as I can find. It would be most welcome if all our papers, as this and many others do, contained clear examples of how to apply practically the new ideas or techniques they propose. Miss Wilkinson is to be complimented for giving us this example of how to use examples.

The paper also has a number of other major values. It points up several important needs and raises several important questions from which we all can learn. The essentials of the paper can be summarized as:

1. It introduces and explains an unfamiliar method which seems to be a generalization of rank correlation and use of dummy variables in correlation problems into a broader realm in which quantitative values of a variable do not exist, or are not for some reason handy to use.
2. It shows the need for, and benefits of, thorough presentation and thorough research.
3. It shows the need by American actuaries to expand their horizons to foreign actuarial work and references.
4. It raises several important questions, the answers to which can be very instructive:
   a. What does "probable maximum loss" mean, most particularly to the author?
   b. What shape does a curve of percentage losses really have?
   c. What work has been done in investigating partial losses and their frequency distribution?
   d. How well do order statistics work with a U-shaped curve and with others that depart materially in shape from that of the normal curve?

The first point needs no further comment, but it will be useful to look at each of the other points in turn, the better to appreciate the paper. Prior to this, however, basics such as the purpose and nature of PML require consideration.

The Purpose of PML

Noting the purpose of computing a PML can be helpful in keeping our eyes on the essentials. As the author mentions, the concept originated in property insurance (more specifically, in connection with fire insurance on fixed-location properties). It was observed that the preponderance of losses were only partial, due in large measure to public fire protection. Total losses were so rare in categories such as protected, fire-resistive structures that, instead of limiting the amount of liability retained on such structures according to the rules followed by ordinary risks, it was safe to anticipate a partial loss and base the retention on that amount instead. A real world PML is thus always relative: to a limit of insurance; to the value of a property; to the value of another insurable interest.

Few underwriters are actuaries,[1] so the concept of probable maximum loss among underwriters has usually been a matter of feel and completely unaided by the concept of a confidence interval. Errors such as a 10 percent estimate of PML on Chicago's late McCormick Place and a 25 percent estimate of PML on the totally destroyed Lake Charles, Louisiana refinery have been the not infrequent result.

The term "PML" appeared only after World War II, although a rough "theory of lines" and elaborate retention schedules or "line sheets" existed in fire insurers prior to 1900. Some time ago the Reinsurance Offices Association in London, after lengthy discussion, recognized the very imprecise nature of PML estimates that are actually used. It has instead standardized on EML (estimated maximum loss), a far more accurate name that is reflective of the judgmental nature of customary practice. The use of EML does not indicate that accurate, fact-based PML's cannot be calculated, but only that they are not being calculated in common practice.

Thorough Presentation and Thorough Research

Frequency distributions of losses by percentage of the limit of insurance are not easily come by. This reviewer first thought that the author had the distinction of obtaining a new one. Nowhere are the data in the paper's Exhibit I labelled as synthetic or hypothetical. The exhibit calls them "Sample Data." The third

[1] One leading underwriter who merited the designation was Benjamin Rush, whose researches substituted facts for underwriting feelings, and thereby changed the marine business of the Insurance Company of North America from a disaster to a profitable operation, and revolutionized ocean marine insurance ratemaking in the process. Another was Francis C. Moore of the Continental Insurance Company, whose *Fire Insurance and How to Build* (New York: Bacon and Taylor Company, 1903) was for fifty years the leading work on how to set underwriting retentions.

paragraph in the section on application of order statistics states "Exhibit I contains a list of 100 claims that are representative of a particular problem in which a PML estimate is needed." But the distribution in the paper was seen to be quite different from any that the reviewer had ever encountered, so the author was queried as to the source of the "data." The author was completely forthcoming in response and acknowledged that they are not real data but synthetic numbers derived by assuming a particular (humped) frequency distribution. Clear and complete labelling should be a *sine qua non*. All evidence the reviewer has seen indicates that the author's assumption is wrong, thus throwing into question the results and conclusions of the paper. Checking easily available references could supply real data that can test these conclusions.

Insurance to Value by Dr. George Head has been in our syllabus of required reading for examinations for several years. Dr. Head cites a paper[2] by one of our charter members, Professor A. W. Whitney, and quotes from it some actual data of the type needed.[3] A second ready source is Ruth Salzmann's "Rating by Layer of Insurance" in our 1963 *Proceedings*.[4] A third source that was not available to Miss Wilkinson when she wrote her paper is one by Gunnar Benktander, presented to the 1982 ASTIN meeting.[5] Although residing in Europe, Dr. Benktander had been able to find Miss Salzmann's paper and referred to it with admiration.

Foreign Actuarial Work and Reference

All of the references in the CAS Syllabus are to works published in America. None of the American or foreign contributions to the *Transactions* of the International Congresses of Actuaries, the ASTIN *Bulletin*, the proceedings of foreign national actuarial bodies, or foreign books, are being used. There is much material in these other publications that could be of real value to our students, even as much of the work in our *Proceedings* is being read and used by foreigners. Miss Wilkinson's paper shows that use by this Society of foreign references as study materials would acquaint our newer members with sources

[2] A. W. Whitney, "The Actuarial Theory of Fire Insurance Rates as Depending on the Ratio of Insurance to Sound Value, Hence the Determination of the Rates for Use With the Coinsurance Clause," *Transactions*, VI International Congress of Actuaries, Vol. 2, 1909, pp. 395–403.

[3] G. Head, *Insurance to Value*, (Homewood, Illinois: Richard D. Irwin, Inc., 1971) pp. 83–88.

[4] R. Salzmann, "Rating by Layer of Insurance," *PCAS* L (1963), pp. 15–26.

[5] G. Benktander, "First and Second Risks," 1982 ASTIN Colloquium, Liège.

they do not know about, and would lead ultimately to a considerably broadened perspective within our membership.

Defining Probable Maximum Loss

The author starts her paper by saying: "In the past there has been much discussion about the definition of probable maximum loss (PML), but little attention has been given to its quantification." She quotes a little later an integrated set of three definitions that were designed to quantify the term quite precisely and thus to permit derivation of PML estimates directly from measured facts; she then quotes a contrasting definition that ignores facts and the concept of a confidence interval and makes a PML estimate a pure judgment or feeling; she then declines to adopt a definition for her paper. This leaves the reader puzzled as to what she is writing about.

By her later use of confidence intervals, she implicitly seems to adopt the first set, but gives no hint of (1) why she mentioned the underwriter's definition (which cannot serve as a base for her statistical analysis), (2) why in view of this limitation she treats it as of equal importance or validity with the former, (3) why she does not state the definition on which she is basing her paper, and (4) why she is *not* explicitly adopting a definition. This source of confusion for the reader of the paper seems to have adversely affected the paper itself.

Failure to make another fine but important distinction also causes trouble. The statement is made:

> The PML depends upon (i) estimates of the likelihood that losses of various sizes will occur, (ii) the amount of losses and associated probabilities that the insured is willing to accept, and (iii) the amount of losses and associated probabilities that the underwriter is not willing to accept. Thus, the insured and the underwriter can have different estimates of the PML for the same loss exposure.

Correctly stated, this would read:

> The PML depends upon the probabilities that losses of various proportions of the relevant limit of loss will occur, and upon the confidence level selected.

The author's statement ignores the fact that a PML is always relative to some limit of loss; the term cannot have meaning otherwise. It also fails to embrace the fact that a PML is always fact-based and fact-related. *Estimates* of the PML—not the actual PML—are what the author describes in the quote above. Naturally, by making different assumptions and guesses, insured and underwriter will make different *estimates* of PML, but that does not change the *actual* PML. The quotation also fails to allow for the fact that the insured and underwriter (or any two persons) can also make different PML estimates simply by having different confidence levels, assuming that both know what a confidence level is and think in such terms, or implicitly without knowing it by using different confidence levels.

It is hoped that the author will in her reply set down precisely what she is writing her paper about: what she is measuring or estimating and calling "PML."

Frequency Distribution of Proportional Losses

Ignoring the essential proportional character of PML has led the author to some questionable conclusions. This proportional character is evidenced by more than the data on losses to fixed-location property that were previously cited. For further example, the existence of "total loss only" insurance on waterborne hulls would make no sense but for the U-shaped curve of proportional partial losses. The land-based fire insurance underwriter's customary assumption that properties not under public fire protection are "total-loss" risks is an exact parallel, based on the observation that if a fire gets beyond a minor stage in such property it is generally extinguished only by the burning of all the combustible material that is present.

Contrasting the numbers used by the author with some of Miss Salzmann's real data can be highly informative, so this has been done in Exhibits I and II. The table in Exhibit I sets side by side the number of claims, total monetary amount of claims, and decimal fraction of claim dollars in each class. Each class contains claims of a particular size. Size is shown as a proportion of the limit of insurance ("insured value"), consistent with the purpose of determining a PML. Miss Wilkinson's top loss is, for want of a stated loss limit, used therefor. The statistics from Miss Salzmann's first table, covering protected frame homes, are used, although any of her other three sets or those from the other references could be.

The great difference between the two frequency distributions is apparent. The graph in Exhibit II presents the contrast pictorially. These exhibits show the basis for the curves given in one reference Miss Wilkinson cited[6] and the basis for the cautions appearing in the same source about setting confidence intervals,[7] both of which seem to have escaped her attention.

[6] J. S. McGuinness, "Is Probable Maximum Loss (PML) a Useful Concept?", *PCAS* LVI (1969), pp. 34–35.

[7] *Ibid.*, pp. 32–33.

Order Statistics and Non-Normal Distributions

The paper is based on the assumption of at least asymptotic normality in the underlying data. All available evidence indicates this assumption to be incorrect, so the paper needs to be reworked.

One fact that the Salzmann data reveal is a hump at the left extreme of the data when they are finely enough divided there (in this case, by tenths of one per cent). This would suggest the possibility of a Poisson distribution were it not for the rising right-hand tail.[8] The Salzmann data were, unfortunately, not split by individual percentages between 90 and 100 per cent, but clearly would display the typical rise were this done; only the extent is left unknown.

Since proportions of a single limit are the relevant numbers in determining PML, assuming them to be mutually independent seems at least questionable. Robert Hurley's cautions (in the first four paragraphs of his review of the Salzmann paper)[9] about dealing with this type of data are well taken and to the point here.

In her Exhibit III Miss Wilkinson acknowledges that at least four of her six estimates are not distribution-free. Consequently, at least these are made erroneous by the incorrect assumption of normal data. The 4th estimates, which exceeds by 71 per cent the upper limit of the numbers she presents as data, is thereby also inconsistent with any rational concept of PML, and any actual data so far revealed.

Potential for Order Statistics

The unrealistic results displayed in the paper are due to faulty data and not necessarily to the use of order statistics. Although order statistics hide information in the data that is relevant and important, they possibly can be useful in work on PML estimates. It is sincerely hoped that the author will accept the challenge to apply them to real world data and let us see in her reply whether she has given us a tool that is both new to us and practically useful. I hope she has.

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[8] For a relevant discussion of curve shapes see H. Buehlmann, *Mathematical Methods in Risk Theory* (New York: Springer-Verlag, 1970), pp. 4–12.

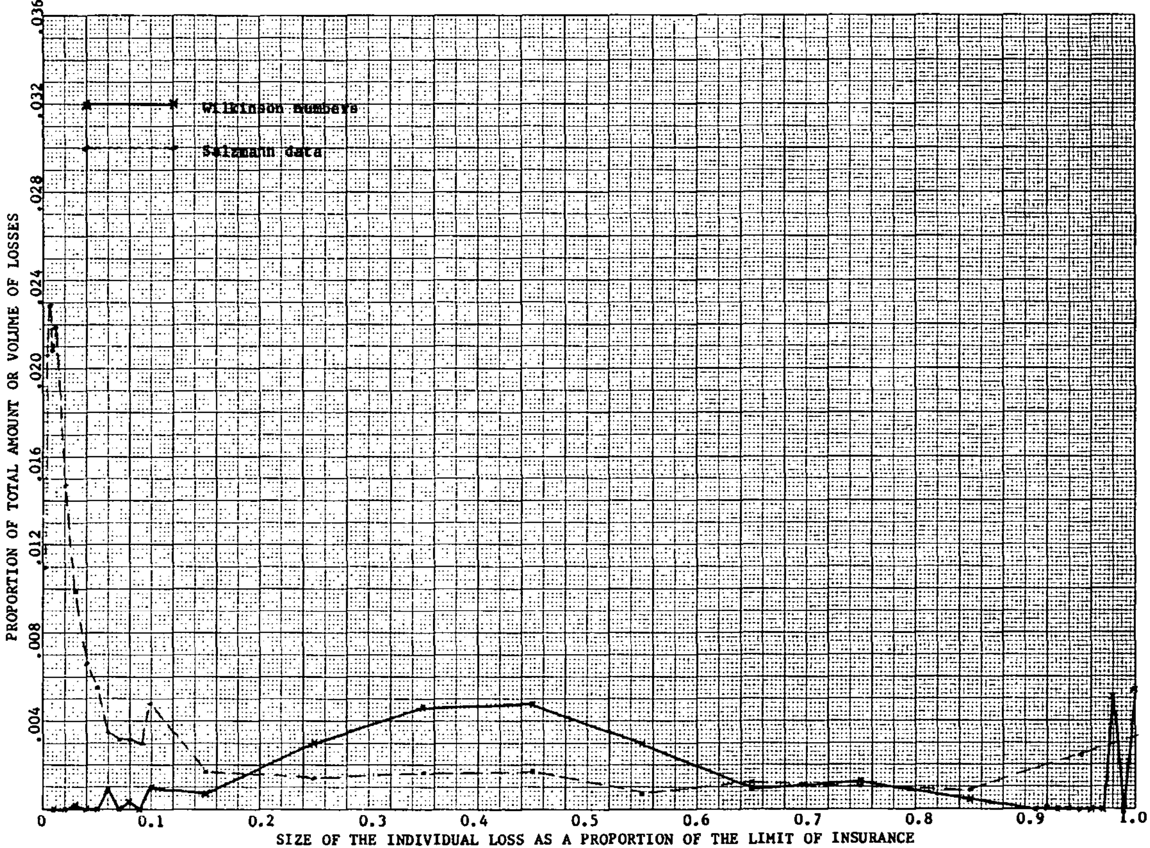
[9] R. Hurley, Discussion of "Rating by Layer of Insurance," *PCAS* L (1963), p. 27.

EXHIBIT I

Comparison of Two Proportional Claim Frequency Distributions

Upper Class Limit as a % of Insurance Amount	Number of Claims		Amount of Claims in Dollars		Proportion of Total Amount of Claims	
	Wilkinson	Salzmann	Wilkinson	Salzmann	Wilkinson	Salzmann
1	0	3,310	0	194,386	0	.097196
2	0	671	0	146,114	0	.073732
3	0	275	0	98,098	0	.049502
4	1	132	19,874	65,746	.000935	.033177
5	0	86	0	54,913	0	.027710
6	3	46	96,884	35,328	.004559	.017827
7	0	34	0	31,578	0	.015935
8	1	31	40,660	31,793	.001913	.016043
9	0	20	0	30,192	0	.015235
10	2	31	110,051	47,294	.005178	.023865
20	9	94	733,184	168,544	.034499	.035050
30	22	37	3,214,792	135,034	.151269	.068140
40	24	27	4,854,234	155,985	.228412	.078713
50	19	16	4,922,466	168,850	.231622	.085204
60	10	8	3,178,831	72,536	.149577	.036603
70	3	10	1,136,200	122,774	.053463	.061954
80	3	9	1,322,259	104,923	.062218	.052946
90	1	6	482,259	78,378	.022692	.039551
91	0	—	0	—	0	—
92	0	—	0	—	0	—
93	0	—	0	—	0	—
94	0	—	0	—	0	—
95	0	19	0	239,237	0	.120723
96	0	—	0	—	0	—
97	0	—	0	—	0	—
98	1	—	563,899	—	.026534	—
99	0	—	0	—	0	—
100	1	—	576,525	—	.027128	—

COMPARISON OF TWO PROPORTIONAL CLAIM FREQUENCY DISTRIBUTIONS



DISCUSSION BY ALBERT J. BEER

With the increased importance of utilizing quantitative analysis in risk management decision-making, Miss Wilkinson's paper should provide our profession with a valuable use of the concept of probable maximum loss (PML), a term that has been a fixture of the insurance vernacular for decades. Previously, underwriters have used the PML, or other related tools, to establish the range for the "working layer of coverage." While it was always acknowledged that a larger loss was possible, the PML estimated the expected maximum loss potential for the risk, with the exposure beyond the PML being treated as a catastrophe. Today, the dramatic increase in the amount of risk retained by insureds has made the pricing of large accounts more complex, since the "buffer" of the working layer is no longer available. It is at these extreme values that the author's work with order statistics may provide a variety of applications.

Before I discuss the results of the paper, I would like to resolve what I perceive to be an ambiguity in the treatment of PML as defined by the author. In my opinion, any discussion of PML is unclear without a quantification of the term "probable." If a pair of dice are rolled, is it reasonable to say the total will "probably" be less than eight ($p = {}^{21}\!/_{36}$); less than ten ($p = {}^{30}\!/_{36}$); or, less than twelve ($p = {}^{35}\!/_{36}$)? How certain of an outcome must one be in order to say it is probable? It is precisely this subjectivity that leads to the potential conflict between the insured and the carrier which is alluded to by the author. This dilemma could easily be resolved by quantifying the term "probable." McGuinness[1] accomplishes this by means of a reference to a "*stated proportion of all cases*" which will be equaled or exceeded by the PML. This concept is similar to the confidence coefficient of a one-sided confidence interval. With these ideas in mind, I would suggest that the PML could be redefined as follows:

Definition: PML_α is that amount (or proportion of total value) which will equal or exceed $100\alpha\%$ of all losses that are incurred.

For example, $PML_{.95}$ would represent that amount which would be expected to equal or exceed 95% of the losses incurred by the risk.

If the PML_α is so defined, an insured and underwriter who agree on the underlying loss distribution would arrive at the same PML_α. It is true that the respective risk aversion and risk acceptance levels would certainly affect the degree of satisfaction each would have at various α levels. However, at any fixed α point, there would be technical agreement on PML_α. The "negotiation"

[1] John S. McGuinness, "Is Probable Maximum Loss (PML) A Useful Concept?" *PCAS* LVI, 1969, p. 31.

on the appropriate price for risk transfer would at least have a common starting point.

Miss Wilkinson's definition of PML as the "worst loss likely to happen" does not include any quantification of the term "likely." Therefore, as is noted in the paper, the PML estimates that appear in Exhibit III are not approximating the same quantities. For example, the nth sample order statistic $X_{(n)}$ is intended to be an estimator for the upper bound of the loss variate X. Therefore, $X_{(n)}$ is more closely related to the maximum *possible* loss. Clearly, this is not the same concept McGuinness had in mind when he discussed the generalized PML. It may be noted that my suggested definition of PML_α allows for this degenerate case by choosing $\alpha = 1.00$. (Of course, it may not be technically possible to derive a $PML_{1.00}$ if the distribution has no finite upper bound.) In contrast, in a situation with 100 losses, using $X_{(95)}$ as an estimate for $k_{.95}$, the 95th percentile, is equivalent to approximating $PML_{.95}$. I will try to demonstrate that the results displayed in Exhibit III are much more consistent than they appear.

Throughout this discussion an attempt will be made to provide more general results derived from the author's excellent foundation. I hope these additional comments help to clarify any imprecision in the PML concept.

General Results Concerning $X_{(n)}$

This section concisely presents the theory upon which most of the remainder of the paper is based. In addition to the results which appear, the corresponding distribution for $X_{(r)}$ could be given by:

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!}(F_x(x))^{r-1} f(x)\,(1 - F_x(x))^{n-r}.$$

The reason for introducing this more general result is to allow for the derivation of properties of $X_{(r)}$ similar to those presented for $X_{(n)}$. In particular, it may be shown that the order statistics from a uniform distribution over (0,1), with $u_{(r)} = F_x(x_{(r)})$, have a beta distribution with parameters $a = r$, $b = n - r + 1$.

Therefore, $E(u_{(r)}) = r/(n+1)$

$\text{Var}(u_{(r)}) = r(n-r+1)/((n+1)^2(n+2))$ for $r = 1,2,\ldots,n$.

Additionally, the first approximations displayed in the paper as (4) and (5) can be extended to:

$E(X_{(r)}) \doteq F_{xx}^{-1}(r/(n+1))$

$\text{Var}(X_{(r)}) \doteq r(n-r+1)/((n+1)^2(n+2))\,(f_x[F_x^{-1}(r/(n+1))])^{-2}.$

These results form the basis of the author's initial three estimates of PML. Using the generalized forms above (with $r = 100\ \alpha$), estimates for our PML_{α} may be computed as follows:

Method	Estimates for : $PML_{.90}$	$PML_{.95}$	$PML_{.99}$	$PML_{1.00}$
1) $X_{(r)}$ from sample data	\$331,179	\$434,449	\$563,899	\$576,525
2) $E(X_{(r)})$	344,158	404,453	516,532	589,468
3) $E(X_{(r)}) + 2(\text{Var}\ (X_{(r)}))^{1/2}$	399,632	482,839	662,380	803,420

Methods 2 and 3 assume an underlying lognormal distribution with μ = \$212,521 and σ = \$110,506.

It may be noted that the $PML_{1.00}$ estimates are those derived in the paper under the author's definition of PML.

Using $X_{(r)}$ *As An Estimate for the PML*

Although this is obviously the most convenient approach, it relates only to the data that are available from reported claims and may not be an accurate indication of the underlying exposure in the future. For example, immature loss history may not show any losses in excess of a few thousand dollars. Should the PML be chosen to be the largest claim paid to date, or the largest reported claim, or some other choice?

From another point of view, suppose $X_{(99)}$ = \$400,000 and the largest claim $X_{(100)}$ = \$2,000,000. Is the \$2,000,000 loss catastrophic and, by definition, not probable? Clearly $X_{(n)}$ alone should not be used in any of these cases and judgment would play a critical role in the choice of an appropriate PML.

I would also add that, technically, this method could have been described as distribution-free in Exhibit III since it requires no assumption regarding the underlying probability distribution.

Distribution-Free Bounds for $E(X_{(n)})$

The advantage of a reliable distribution-free bound for any variable is obvious. Hopefully, some work may be done in the future to test the sensitivity of this bound with regard to accuracy for various distributions.

The clever use of the Schwartz inequality was a novel application to this realm of actuarial science. In fact, this same technique may be used to derive the generalized result:

$$E(X_{(r)}) \leq \mu + \sigma \left[\frac{B(2r-1, 2n-2r+1)}{(B(r,n-r+1))^2} - 1 \right]^{1/2}$$

where $B(a,b,) = \dfrac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$.

Assuming the same lognormal distribution as mentioned above, the following bounds may be computed:

$$\begin{aligned} E(X_{(90)}) &\leq \$531{,}509 \\ E(X_{(95)}) &\leq 590{,}319 \\ E(X_{(99)}) &\leq 756{,}736 \\ E(X_{(100)}) &\leq 988{,}044 \end{aligned}$$

General Results For Quantiles

The introduction of the pth quantile technique is a useful concept for quantifying the meaning of "probable" in PML. Based upon the discussion of order statistics both in the paper and above, it is easily seen that a reasonable estimator of k_p is the rth sample order statistic $x_{(r)}$ where: i) $F_x(k_p) = p$

ii) $r = np$ for p fixed.

It is interesting to note that this sample quantile estimate $X_{(r)}$ is asymptotically distributed as a normal variate; i.e.

$$X_{(r)} \rightarrow N(k_p,\, p(1-p)/n\, f^2(k_p)$$

for $r = np$, as n increases with p fixed.

The author has provided the technique for approximating the appropriate moments of this distribution by differentiating the Taylor series; namely,

$$\begin{aligned} E(X_{(r)}) = E(F_x^{-1}(u_{(r)})) \doteq F_x^{-1}(E(u_{(r)}) &= F_x^{-1}(r/(n+1)) \\ &= F_x^{-1}(np/(n+1)) \\ &\rightarrow F_x^{-1}(p) = k_p \end{aligned}$$

$$\begin{aligned} \text{Var}\,(X_{(r)}) &= \text{Var}\,(F_x^{-1}(u_{(r)})) \doteq \text{Var}\,(u_{(r)})\,(f_x\,(E(F_x^{-1}(u_{(r)}))^{-2} \\ &= \frac{r(n-r+1)}{(n+1)^2(n+2)}\,(f_x\,(F_x^{-1}(E(u_{(r)}))^{-2} \\ &= \frac{np(n-np+1)}{(n+1)^2(n+2)}\,(f_x\,(F_x^{-1}(E(np/(n+1)))^{-2} \\ &\rightarrow \frac{p(1-p)}{n}\,\frac{1}{f_x^{-2}(k_p)} \end{aligned}$$

This analysis demonstrates the theory behind the intuitive appeal of using $X_{(r)}$ as an estimate of k_p, which can be interpreted as PML_p as defined above.

Distribution-Free Confidence Interval For k_p

The clarity of this section is enhanced by the interesting heuristic explanation of the result:

$$P(X_{(r)} < k_p < X_{(s)}) = \sum_{i=r}^{s-1} \binom{n}{i} p^i (1 - p)^{n-i}$$

as a binomial distribution.

As a technical note, the equation which appears in the paper as:

$$P(X_{(r)} < k_p < X_{(s)}) = P(F_x(X_{(s)}) < p) - P(F_x(X_{(r)}) < p)$$

may be expressed as:

$$P(X_{(r)} < k_p < X_{(s)}) = [1/B(s,n - s + 1)] \int_0^p x^{s-1} (1 - x)^{n-s}\, dx$$

$$- [1/B(r,n - r + 1)] \int_0^p x^{r-1} (1 - x)^{n-r}\, dx.$$

These integrals may be evaluated by means of an Incomplete Beta Function Table, a method which appears more efficient than actually calculating the various binomial probabilities.

Since the distributions of $X_{(r)}$, and hence k_p, are severely skewed, similar results for $p = .99$ and $p = 1.00$ are not practical. However, I performed the related calculations for $p = .90$, $\alpha = .10$ with the following results:

$$P(X_{(83)} < k_p < X_{(94)}) = .887349$$
$$P(X_{(85)} < k_p < X_{(95)}) = .902531$$
$$P(X_{(86)} < k_p < X_{(96)}) = .903715$$
$$P(X_{(87)} < k_p < X_{(97)}) = .868286$$

By minimizing $s - r$ and $X_{(s)} - X_{(r)}$, we would choose the upper bound for $k_{.90}$ as $X_{(95)} = \$434,449$.

Summary

The author's results as displayed in Exhibit III are not as disparate as they may appear at first glance if the various methods are recognized for what they are designed to produce. By allowing PML_α to be defined as I have suggested above, the consistency of the techniques proposed in the paper are better demonstrated as follows.

Method	Estimates for : PML$_{.90}$	PML$_{.95}$	PML$_{.99}$	PML$_{1.00}$
1) $X_{(r)}$	$331,179	$434,449	$563,899	$576,525
2) $E(X_{(r)})$	344,158	404,453	516,532	589,468
3) $E(X_{(r)}) + 2(\text{Var}\ (X_{(r)}))^{1/2}$	399,632	482,839	662,380	803,420
4) Upper bound for $E(X_{(r)})$	531,509	590,319	756,736	988,044
5) $X_{(r)}$ as an estimate of k_p	331,179	434,449	563,899	576,525
6) Upper bound for k_p	434,449	563,899	—	—

I believe this type of analysis would be extremely informative to individuals charged with determining proper retention limits on a per occurrence basis as well as to the underwriter and actuary who must set a price for the related excess coverage. In addition, these methods could similarly be used on an aggregate basis to help select appropriate stop-loss thresholds. In one sense this latter approach would imply the existence of a new concept which is the aggregate analog to the PML$_\alpha$. Perhaps this new term could be defined as:

Definition: The Probable Maximum Aggregate Loss at the α level (PMAL$_\alpha$) is that amount (or proportion of total value) which will equal or exceed the accumulation of all losses to the risk during a fixed period of time with probability 100α %.

For example, if PMAL$_{.95}$ = $1,000,000, you would expect the aggregate loss over a particular period to be less than $1,000,000 ninety-five percent of the time. Expressed differently, it could be stated that the actual aggregate loss for the risk is expected to exceed $1,000,000 five percent of the time, or once every twenty similar periods.

Conclusion

Ms. Wilkinson has provided the literature with a number of valuable techniques for analyzing and determining estimates of the Probable Maximum Loss. The clarity of presentation and the numerous intuitive explanations are excellent pedagogical methods to utilize in the discussion of a term (PML) as familiar to the non-technician as it is to the actuary.

My suggested generalizations were introduced only to present further applications of the author's ideas as well as to, hopefully, clarify what I perceived to be an ambiguity in the definition of PML. As mentioned above, these generalized results actually give the results of Exhibit III in the paper a greater semblance of consistency than it may seem to display initially.

With regard to further study of this topic, I would be very interested in seeing more work done analyzing the accuracy of these estimates. In particular, it would be informative to investigate this sensitivity of accuracy with respect to the skewness and kurtosis of the various distributions applicable to the property/casualty lines of business.

AUTHOR'S REPLY TO DISCUSSIONS

Discussion by Albert J. Beer

My paper was originally written for the May, 1982, Casualty Actuarial Society Discussion Paper Program "Pricing, Underwriting and Managing the Large Risk." Mr. Beer reviewed my paper at that time. His discussion so enhanced my paper that I made my submission to the *Proceedings* conditional on his agreement to submit his discussion.

Mr. Beer's definitions of PML_α and $PMAL_\alpha$ easily clarify an area I treated by implication, but did not address directly. These definitions explain how the risk manager and the underwriter can have different PML's when they use the same loss distribution; the different PML's are due to different values for α.

Mr. Beer's generalizations of my results nicely complete the ideas I presented. It should be noted that his generalizations reduce the six methods presented to five, since "$X_{(r)}$" and "$X_{(r)}$ as an estimate of k_p" reduce to the same estimator.

Discussion by John S. McGuinness

Many of Mr. McGuinness's questions, particularly those concerning definitions, are answered if my paper and Mr. Beer's discussion are read as a unit, as intended. Mr. McGuinness has raised other issues, several of which I will address.

1) Sample Data

The sample data presented in Exhibit I of my paper are used to illustrate the calculation of the proposed estimators for probable maximum loss (PML), to ensure that the concepts are clearly understood by the readers. The source of the data is not relevant to the concepts presented in the paper.

The proposed estimators for PML were derived theoretically and tested on several sets of real and synthetic data. While of varying usefulness, I judged the estimators presented in the paper to be of sufficient interest to merit presentation to fellow actuaries.

I purposely did not choose data used by A. W. Whitney[1] or Ruth Salzmann[2]

[1] A. W. Whitney, "The Actuarial Theory of Fire Insurance Rates as Depending on the Ratio of Insurance to Sound Value, Hence the Determination of the Rates for Use with the Coinsurance Clause," *Transactions*, VI International Congress of Actuaries, Vol. 2, 1909, pp. 395–403.

[2] R. Salzmann, "Rating by Layer of Insurance," *PCAS* L, pp. 15–26.

to illustrate calculations because I wanted to get away from the idea that the PML concept applies only to property. Other actual data could not be presented in the paper because releases from clients could not be obtained in a timely fashion.

The data presented in my paper are computer-generated. They are the aggregate distribution of claims, the number of claims having a Poisson distribution, and the claim size having a lognormal distribution. Several papers in the *Proceedings* and other journals have stated the success of the Poisson and lognormal distributions in approximating number of claims and claim size, respectively[3]. Consequently, I feel the data presented in my paper are representative of a "real-life" situation where $PMAL_\alpha$ might be sought.

2) Order Statistics and Non-Normal Distributions

This paper is *not* based on the assumption of asymptotic normality, as Mr. McGuinness states. $E(X_{(r)})$ and $E(X_{(r)}) + 2(\text{Var}\,(X_{(r)}))^{1/2}$ as estimators for PML_α do require assumptions concerning the underlying distribution of the data. However, $X_{(r)}$, the upper bound of $E(X_{(r)})$, and the upper bound for κ_p are distribution-free.

3) Potential for Order Statistics

One of the reasons for writing this paper was to call attention to order statistics as a useful tool in actuarial work. As stated in the original paper:

> Order statistics are particularly useful for studying certain phenomena because quite a few of the results concerning the properties of $X_{(r)}$ and the properties of functions of some subset of the order statistics are distribution-free. If an inference is distribution-free, assumptions regarding the underlying population are not necessary.

If reliable information about the underlying population is available, it should, of course, be used to the greatest reasonable extent. However, it has been my experience and, I suspect, that of many other actuaries, that we usually know very little about the underlying population. Consequently, I find order statistics a frequently useful, and often underutilized, tool, particularly with reference to PML_α and $PMAL_\alpha$.

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[3] For excellent bibliographies, see:

A. L. Mayerson, "A Bayesian View of Credibility," *PCAS* LIII, pp. 85–104.

G. Patrik, "Estimating Casualty Insurance Loss Amount Distributions," *PCAS* LXVII, pp. 57–109.

AN INFORMATION STRATEGY FOR INSURANCE

DOUGLAS M. SWEENY
IBM CORPORATION

Thank you for that kind introduction. I am delighted to be here and honored to be your keynote speaker. It is always a pleasure to meet with some of our best customers—and you and the companies you represent are just that. In fact, as an industry, you're IBM's fastest growing business segment. Perhaps that says something about automation in the insurance industry. It is also a pleasure to return to Miami—to catch some sun—and to check the progress of Christo's "Plastic Islands."

This morning I would like to share with you a perspective on the impact that information systems technology is having, and may have in the future, on your business. The material is very current and represents the collective thoughts of my staff, along with those of some prominent industry analysts, academics, and insurance company executives, who serve as consultants to IBM.

I would like to look briefly at the past, spend more time on current impacts, and then explore the future in greater depth. At the end, I'd like to suggest some basic precepts for an information strategy for insurance and provide some insight as to why we are witnessing the dawning of "the age of the workstation." My premise is that as technology increasingly affects insurance business operations, product design, company base economics, and even the structure of the industry, the role of the "Information Systems Executive" will change dramatically, and the role and involvement of the user executive, which is you, will change as well.

Although as a unit of measure a decade is a crude yardstick, it does help to identify some major trends. Let's begin with the recent past. In the Seventies your companies collectively grew 142% in premiums. That's pretty impressive, and during that period data processing played a key role—although more in a back office or supportive way.

The operational departments of your companies stepped up to this volume upsurge often by relying more and more on the function and capacity of their data processing systems. In general, the systems progressed in the decade of the Seventies from batch overnight . . . to online inquiry . . . and finally to interactive processing.

It was just 7 years ago—at a meeting in Florida—that insurance data processing executives were sharing their successes using exciting new technologies: online interactive claims processing systems with the 3270 Display Station, profitable uses of the 3850 Mass Storage System, and distributed data processing.

Now, most of your companies have installed online interactive systems, and the breadth and sophistication of today's systems far exceeds those of the mid-Seventies. Dozens of your companies now use the mass storage concept, and a number of you count your distributed processors in the hundreds. Perhaps even more revealing, many of the future technologies envisioned a short time ago, including voice store and forward products, image transmission, and color graphics are already here today, from IBM and other companies. Technology is indeed galloping, and the past is truly a prologue to the future.

And so, the focus in the Seventies was on learning new but basic information systems concepts—data base, online inquiry, distributed processing—and applying them to the insurance business.

I suppose in any decade analysis, the current must be viewed as a bridge between the past and future—and it's true here. It seems that the changes—economics, competitive, governmental, and even consumer forces—are all peaking at once in the early Eighties. In fact, more change has occurred in insurance in the last two years than in the past two decades. On a broad basis, we have entered the era of Toffler's "Global Economic Village." Some key indicators are the recent spate of overseas acquisitions by your firms, and the projections that networks may soon interconnect the insurance exchanges around the world. Just last month, executives from Lloyd's visited us in Princeton to see and discuss the new Insurance Value Added Network Service (IVANS) which will go live this July.

Our national economy is in its own state of flux, and within the broad financial community each component is looking enviously at his neighbor's turf. As a recent NBC documentary pointed out, "even the staid, local business of banking has now become global, complex, and aggressively competitive." As traditional industry lines blur, it's become popular to refer to insurance companies as "non-bank financial institutions." Some insurers are now acquiring banks, and banks are circumventing regulations to buy insurance companies. The business of labeling is becoming quite complex. A more fitting term for banks might be "non-insurance financial institutions."

Several other factors are having a profound effect on your companies and in turn on information systems. First of all, consumers have become more sensitive to the value of their assets and their yield. Public perception of money has changed. The "basis point consumer" will move for a quarter of a percent of additional yield, and loyalty to the financial institutions has lessened. Perhaps more importantly, insurance buyers are viewing insurance more as a commodity—one that can easily be replaced or even terminated. Consider that for ordinary life insurance the voluntary policy surrender rate jumped from 5.8% to 8.8% in just the last decade! Plus, increased attacks from the outside, like Walter Kenton's new book, *How Life Insurance Companies Rob You, and What You Can Do About It,* do nothing for a company's public image!

Another issue is overcapacity. A recent university study indicates that the property and liability segment has the financial capacity to grow at twice the recent rate. Add to that capacity the entry of other financial firms into insurance, and we have a significant over-capacity, now and well into the future. As you well know, the result has been cut-throat price competition and "cash-flow" underwriting, which in turn has increased the pressure on information systems to be much more efficient.

A third factor is that of "disintermediation." Literally, this concept connotes the removal of any link in the process that does not add value—that does not pay its own way. For example, as many of you from companies writing life insurance know, continued customer borrowing against their policies removes the companies from the investment process. Applied to an insurance agent, clients may elect other means to secure insurance if the agent is not perceived as adding value to the transaction. To data processing support, this means accelerated change as clients and companies elect to add or remove links from the usual chain.

All of these factors have created a driving force for productivity, expense control, and new applications of technology in data processing.

Let's look at how this technology is being applied today. First, there has certainly been a significant expansion of networks—a controlled explosion if you will—to regions, districts and branches, to claims offices, to agencies, and even to clients and health providers. Over 50% of insurance companies now have distributed processors in their networks, and over 90% have remote online interactive terminals. In the agency area, a recent study by Frost and Sullivan revealed that more than 25,000 agents will automate over the next six years, and most of those will be network-connected to carriers in some way. And there is a recent flurry of activity in connecting health providers to insurers via regional or national networks.

In addition to networking, technology has been used to restructure and integrate business functions, from automated rating and quoting to machine-adjudication of claims; from integrated data and text applications to fully automated correspondence.

A third area, and perhaps the most exciting, is the one that led *Time* magazine to name the computer the "machine of the year." The intelligent workstation has found its way into every segment of the insurance industry: as an executive workstation in New York, as an insurance service center device in Ohio, and as an extension to the information center across the country. It's already being used as a total agency management system, a health insurance provider system, and an actuarial workstation. The fascination with this powerful microcomputer has resulted in it appearing as an executive Christmas present in one company, an award for top insurance producers in another, and as an agent incentive for writing more business. One insurance company has announced its intentions to provide a personal computer to every employee over the next four or five years. They foresee the day when management reports will arrive in diskette form and management meetings are attended with diskette in hand.

Another insurance company has discovered that by attaching a communicating model/85 electronic typewriter to a personal computer, you can have the highest quality printed output, as well as the flexibility to use the typewriter independently when needed. A company in Europe is actually planning two workstations per employee, at least in the data processing area, so that creativity at home will not go unfulfilled. And with so many universities now requiring students to have their own personal computers, like the slide rules and calculators of yesterday, it is clear that the information age and workstation age will continue to gain momentum.

As a result of the proliferation of individual workstations, users are becoming more independent. Their confidence is growing: they are now ready, willing, and able to use technology; and thus, the number of people demanding information processing is exploding, touching all parts of the U.S. workforce. That's an enormous audience, nearly 100 million people. And so, understandably, our focus is increasingly on the end user.

The problem is, users come in all shapes and sizes. That leaves us with the question—how do you understand the needs represented by this many individuals? That's a key area we have been studying over the last 2 or 3 years—categorizing these users in a way that we think contributes to understanding their characteristics as users of information technology. So, I would like to outline this concept of user segmentation, and along the way, call out things we are doing or plan to do as a direct consequence. Some of the results will apply to you—helping you to tailor your own information systems plans. Using federal government job classifications and Bureau of Labor Statistics, all of the occupations in the U.S. workforce were grouped into six broad segments. We did this tentatively—the six seemed to exhibit key differences in support and workstation needs, and we are testing these differences now.

Of course, nothing in user segmentation implies that each segment is strictly homogeneous. Your world does not fit neatly into six cubicles. In time, I am sure we will increase the number of segments we study, but let me touch briefly on five segments that are important to insurance.

Let us start with the engineers/scientists segment. This is a group that has characteristics and requirements that many of you will identify with. The actuarial community has long been a heavy user of computing support, often being the most knowledgeable end user group. In fact, you were among the first in your industry to use timesharing terminals and build sophisticated computer models. Across the insurance industry, actuaries, statisticians, and industrial engineers represent only 2% of the workforce, but their computational requirements are, of course, very high.

In these respects you resemble engineers and scientists in other industries. However, as we examined the variety of functions actuaries and actuarial departments perform, it seemed clear to us that they have more in common with members of the second segment—the business professional. As a professional, this kind of user has substantial discretion in the way problems are addressed and what tools are used. Productivity is very important for this user, but is measured by the *quality* of enterprise decisions, rather than transaction volumes. The business professional understands the productivity benefits of information systems, but does not want to become a data processing professional to obtain them. The business professional has urgent requests—and can easily become impatient at the queue of jobs that arrive at central data processing first.

We have responded to these characteristics as they exist in all industries with a solution we call the information center—a solution that has been so successful I'd like to dwell on it a moment. What a business professional does can be broadly categorized into three application types—Decision Support, Professional Support, and Data Base/Data Communications—each supported by a variety of software and systems. Within these application types, however, are many individual needs, and the larger problem of how to apply the best solution without becoming a data processing professional.

The information center solves this by focusing the power and competence of the central staff on the user through an information center consultant. This consultant has the ability to understand the business professional application, and how the ultimate user can do it himself. In fact, a number of user departments have established their own internal consultants, and I am sure some of you have staff professionals who function in this way. The results: faster, more relevant solutions and support of management's objective to drive the information system resource to new users and uses.

By the way, we regularly ask our customers about the reasons they see for data processing growth. Last year was the first time that the reason I just mentioned—new uses and new users—was the most frequent answer, and it occupied first place by a comfortable margin.

Acceptance of the information center as the vehicle for user expansion was borne out by the same survey, with over 60% of all respondents expecting to have an information center in operation by the end of this year, and 87% of our largest customers.

Overall, the information center, with its extension to include intelligent workstations, appears to be the best approach to satisfy the information systems support needs of the business professional.

Another segment is the commercial or clerical segment. For some insurance companies this may be 50% of their employee population. The commercial workers—tellers, cashiers, clerks, and claims adjudicators, for example—tend not to be trained in data processing. What they do depends heavily on specific industry practices and procedures, and is usually justified by volume-based productivity. It is this segment of the population that has received the bulk of automated support to date.

The fourth segment is the Data Processing Professional—a resource that is scarce both in insurance and in industry generally. Although only ½% of the U.S. workforce, in some insurance companies 5% of the employees are data processing professionals. Evidence exists that over the long haul, the last 25 years, this segment has maintained an astonishing 20% growth rate in productivity, but the question is what can be done now to maintain and then enhance productivity growth. One answer is an approach we call the ideal programming environment. In a nutshell, you trade off less expensive computer power and workstation availability against more expensive programmer time and effort. The payoff is twofold: vastly improved programming productivity, and programmer satisfaction.

The most prominent members of the last segment are secretaries; occupationally characterized by the creation, retrieval, and manipulation of text, but very dependent on others for the specifics of their functions. IBM, as well as literally hundreds of other office systems vendors, offers a variety of devices to address the needs of this segment. IBM also issued a statement of direction in which we promised to interconnect our various solutions for more fully integrated office systems, and to integrate text functions with the data processing applications which demand them.

So let's take a look, briefly, at where the workstation of the Eighties is and where it's going. While much of what I say will apply to all workstation vendors, I am sure you'll understand my tendency to use IBM products as examples!

Now that we understand that there are some specific classifications of end users who need different amounts of either text, computation, information access, special output, or other workstation function, the next step is to select the best family of workstations to support each type of user. First, it is important to understand the key workstation trends. Across our entire industry, performance and function are improving dramatically, and price is dropping. You may read that in this "Age of the Workstation," the computer terminal will become as pervasive as the telephone. While the price curve has not dropped that low yet, the trend is certainly in the right direction.

Let me give you three or four examples—product announcements that IBM made in March of this year—that illustrate these trends.

The IBM 3178 Display Station is a good example of a nonintelligent cathode ray tube, or CRT, whose price was cut by 50% from that of the 3278, the predecessor product, and yet has significant ergonomic, or human engineering, enhancements such as reduced footprint, tilt screen, and removable base. The square base can in fact be suspended vertically in a desk well by a small bracket, to further reduce occupied space. I know in my office, the credenza is only 20 inches wide, and most workstations are just too big to fit on it!

Another product has dramatically enhanced function by the attachment of PC (Personal Computer) intelligence to a nonintelligent 3270 workstation at a small cost. The importance of this announcement is that if you already have a major investment in 3270 Displays that you've purchased over the last 5 years, you can add personal computer power to the display very inexpensively.

Also announced in March was the enhancement of the Personal Computer to include color graphics and substantially more storage and processor power. The new IBM PC/XT, as it's called, is now being adopted as a standard by several insurance companies for use as an agency system, because its increased capacity will help to avoid future constraints. Also, I think this product, which can operate independently or be tied to your central system, is probably the best solution for you, as actuaries, today. By the way, this is one of the products we're demonstrating downstairs.

And finally, the 3290 Information Panel is clearly the most exciting technology we've seen in the recent past. This device uses a gas panel concept which produces a flicker-free sharp image. The screen itself is only 4 inches deep and contains up to 9,920 characters of information, about 4 times the capacity of a standard display. A close-up of the 3290 screen shows that the device can actually display 4 different screens of information from 4 different application programs or data bases. I saw this product being tested for a Blue Cross Plan in Ohio in a claims application. Instead of an operator looking up some information, writing it down, and doing another inquiry, the 3290 saved and displayed multiple requests for that one application. A 27% productivity enhancement was realized in processing health claims.

From a direction standpoint, I think over the next few years you will continue to see us expand upon the alpha/numeric display base and to integrate and enhance facilities for text, image, graphics, and facsimile, and at the same time offering variations in screen size, color, and increasing the levels of user programmability. And, of course, the other major technology to be integrated into the workstation will undoubtedly be voice systems as they develop.

With all the new users and the proliferation of workstations, a variety of industry sources project the ratio of workstations to employees will grow eleven times—from 1:23 in 1980 to 1:2 before the end of the decade. I should add that a number of insurance companies we work with are already at a ratio of 1:4, and at least one firm plans to be at 1:1 by 1987!

Well, let's turn to the future. From the current, where tactics are necessary, to the future, where strategy counts. The key really is planning. Both tactical and strategic planning are needed, and they must be in balance; one for survival, one for growth.

Over the past year, we've noticed a new emphasis in insurance on long-range business planning, spurred in part by the financial services "free for all." In fact, our department has hosted nearly a dozen strategic planning briefings in the past 12 months, mostly for insurance companies, but in a few instances for banks or manufacturers peering over the proverbial pasture fence. In our briefings we provide our assessment of the emerging financial services industry and the role that technology is playing, particularly in electronic delivery of products. We also share some of IBM's planning techniques that may be transferable to insurance.

We explore strategic questions that insurance executives are, or should be, asking . . . such as:

- If we decide to expand into financial services, what value would we bring to the marketplace, and, what would be the impact on our traditional insurance business?
- If we decide not to expand into financial services, but our peers and others do, how would that affect our competitive position?

And finally we assess the comparative technological strengths and weaknesses of the major players: the banks, security brokers, and insurers.

Strategic vision and long-term thinking is beginning to be a familiar theme. It is a megatrend that John Naisbitt recognized, and it's happening in insurance right now.

As a result of more strategic thinking, insurance executives are now beginning to make organizational changes within their companies. From monolithic organizations, they are moving to more free-form or modular ones. Insurance companies will no longer be the tightly coupled businesses of the past. Products, functions, market segments, and distribution channels will be seen as separate entities or individual business units. Each company can then select those to be emphasized and those to be sold or discontinued.

At the same time, top management is beginning to view technology and information systems in a new role, a role as a competitive weapon.

It can be used:

- To strengthen customer/client relationships and reach new markets,
- To create new value-added features for product differentiation,
- To establish barriers to entry or barriers to change between suppliers,
- And, internally to increase productivity and efficiency, to help an insurer be the low cost producer or distributor of its products.

With information systems as both a strategic and tactical weapon, financial executives in the most progressive firms are recognizing their potential as a sound investment. As so many bankers are learning today, the long-term return on investment in technology can be more attractive than alternative investments in brick and mortar, especially so when competitive positioning is considered.

For example, a recent *New York Times* article described Citicorp's technology push that has introduced change so rapidly that both the consumer and bank management can barely cope. Two positive results have occurred, however; they have discovered that 25% of their consumer base actually prefers a mechanical teller to a human one. An automated teller machine, by the way, is ten times more productive than a teller of similar expense. Secondly, the firm has discovered that its market share has grown from 4.4% to 9.6% in New York in just five years! Doubling market share—that's not bad, is it? Perhaps most importantly for Citicorp, technology is redefining what a bank is. I think the same may be said for insurance in the immediate future.

In light of all these shifts in the insurance environment and in top management's thinking, we are beginning to see some fundamental precepts on which to build an information strategy for insurance.

First, as insurance has grown to be critically dependent on technology, the requirement has increased for the information systems plan to be fully integrated with the business plan and for the information systems executive to become a full partner with you and other executive management. Strongly supportive of this is an analysis done by Warren McFarlan in his new book, *Corporate Information Systems Management*. In it he introduces a grid to help position the role of information systems in light of various industry climates. It is his view that the strategic relevance of information systems is different by industry. He points out that unlike most other industries, insurance has moved steadily into the strategic quadrant, in which company success, both short and long term, has become critically dependent upon information systems activities.

Second, as insurance organization structures become modular to respond rapidly to market conditions, information systems must mirror that "loosely coupled" structure. This is indeed a challenge; maintaining sufficient integration so as not to duplicate data, programs, or networks, but modular enough so that component systems can be modified or even relocated without causing major disruption in connected systems.

Third, with top management looking for any technological advantage which might be a competitive weapon, it is becoming increasingly important to scan the environment for new technologies with high payback potential, and to pilot them early before they mature.

A recent *Harvard Business Review* article provides a useful framework for placing emerging technologies in perspective, by phases. Phase I calls for the early identification of potentially useful technologies. This initial phase involves selection and evaluation on a research and development basis. For insurance, we see several new products that promise significant benefits such as the 3290 information panel, which I referenced earlier. Another new technology, videotex (a concept linking Cable TV, a mini-computer, and low cost keyboards), is already in use in European insurance companies as an information center extension and as an agency support function. One U.S. life company identified over 30 potential uses of that technology in their Phase I planning process.

The point is, for new technologies, early experimentation is needed to gain experience and confront technical issues.

In Phase II, the technology can begin to move out into selected user departments. This is still an experimental stage and often the users' views of the new product differ substantially from the first data processing assessment. In this second pilot phase, detailed cost justification is still premature. As an example, the IBM Scanmaster I, a recently announced image storage and retrieval product, is here now. More than two dozen insurance companies are planning Scanmaster pilots to assess the new technology in user departments such as legal, personnel, and claims. I believe audio distribution has also reached this point. In fact, one insurance company has made substantial progress in putting voice store and forward into productive use for their top executives. Their goal is to eliminate telephone tag.

Phase III is the controlled use of a technology. At this point it is ready for large scale, profitable use. Here economics become of prime importance, and each company must determine the extent to which it will invest in the technology. Office systems, such as the IBM Displaywriter, are in Phase III for many companies, as are personal computers.

The transfer of the technology to widespread loosely controlled use represents Phase IV. The technology is mature, understood, and easily installed. Online inquiry, distributed processing, and data base interaction are in this phase for most companies.

I think the recognition of these phases, and the proper placing of each technology within that framework, is vitally important. What is needed is an ongoing technology assessment, a strong research and development function, and the flexibility to conduct user experiments before a new technology is mature.

Fourth, with the explosion of personal workstations among users, it is imperative that both information systems and user department management focus on connectivity, workstation standards, and data control to ensure productive growth and to protect company assets. There are two areas of concern, with the first being the management of data. A potential exposure exists here because the volume of data is growing exponentially as a result of client orientation and the complexity of insurance products. But even more importantly, dispersed users, armed with personal workstations at work or home, may proliferate hundreds of local data bases at a pace outstripping corporate core systems. Standards for data access and use must be established and enforced in order to protect this vital corporate asset. The second concern is network control. As it becomes less and less possible to manage *what* is communicated, it becomes more and more critical for companies to exercise responsible control over the *means* of communications.

Fifth, and finally, all of insurance management must begin to view technology in its emerging new role: the financial executive—as an attractive investment alternative; the marketing executive—as a connection to the customer and as a product differentiator; the functional executive (such as claims, underwriting, and actuarial)—as a productivity enhancer; and, of course, the top executive—as a tool for both survival and growth.

So what's ahead of us is an exciting era—one that's being termed the "information age," more appropriately the "workstation age," or perhaps the "age of the end user." Successful firms, and especially insurance companies (or should I say financial service companies), will be those who apply technology most effectively.

For you, the challenge of the Eighties will be to become and remain comfortable with the new technology, to promote computer literacy throughout your organizations, and to accelerate your already innovative use of information processing. I think the payoff will be an economic one—some estimate a billion dollars per year industrywide. But more important than economics, it may be the key to survival.

I have enjoyed being with you today, and hope you have a great meeting here in Miami.

MINUTES OF THE 1983 SPRING MEETING

May 15–18, 1983

DORAL COUNTRY CLUB, MIAMI, FLORIDA

Sunday May 15, 1983

The Board of Directors held their regular quarterly meeting from 1:00 p.m. to 4:00 p.m.

Registration was held from 4:30 p.m. to 6:30 p.m.

The Officers held a reception for new Fellows and their spouses from 5:30 p.m. to 6:30 p.m.

The general reception for all members and guests was held from 6:30 p.m. to 7:30 p.m.

Monday, May 16, 1983

President Frederick W. Kilbourne opened the meeting at 9:00 a.m. Mr. Gary Granoff, Deputy Commissioner, Florida Insurance Department welcomed our Society to Florida.

Mr. Kilbourne presented the new Associates and then introduced the new Fellows and presented them with their diplomas. As part of the introduction, Mr. Kilbourne emphasized the responsibilities of the new members.

Michael Walters, Chairman of the Organizational Transition Team, reviewed the proposed changes to the Constitution and Bylaws, and answered questions on the new organization.

Mr. Kilbourne concluded the business session at 10:15 a.m.

Mr. Douglas M. Sweeney, Manager, Insurance Industry Marketing, IBM Corporation, delivered the Keynote Address. He emphasized the growing use of data processing within the insurance industry and the important role it will play in the future.

The afternoon session consisted of a series of concurrent presentations. The presentations included; six new papers, five workshops, and four data processing presentations.

The new papers presented were:

1. "The Pricing of Medi Gap Coverage,"
   Emil J. Strug
   Actuary
   Blue Cross & Blue Shield of Massachusetts
2. "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions"
   Philip E. Heckman
   Senior Actuarial Analyst
   CNA Insurance Companies
   Glenn G. Myers
   Assistant Actuary
   CNA Insurance Companies
3. "Loss Reserving for Solvency"
   David A. Arata
   Western Pricing Officer
   Sentry Insurance Company
4. "Transformed Beta and Gamma Distributions and Aggregate Losses"
   Gary G. Venter
   Actuary
   National Council on Compensation Insurance
5. "Utility with Decreasing Risk Aversion"
   Gary G. Venter
   Actuary
   National Council on Compensation Insurance
6. "Parameter Uncertainty in the Collective Risk Model"
   Glenn G. Myers
   Assistant Actuary
   CNA Insurance Companies
   Nathaniel Schenker
   Ph.D. Candidate
   University of Chicago

The workshops covered the following subjects:

1. "Fitting Loss Distributions"
   Dr. Stuart Klugman
   Associate Professor of Statistics
   University of Iowa
2. "Loss Reserve Modeling"
   Paul S. Liscord
   Consulting Actuary
   Liscord, Ward & Roy, Inc.

3. "Risk Exchange"
   Dr. Jean Lemaire
   Institute de Statistique
   Brussels, Belgium
4. "Risk Management and Small Insurance Company Applications"
   James A. Hall, III
   Partner
   Coopers & Lybrand

   Richard T. Zatorski
   Consulting Actuary
   Frank B. Hall & Co.
5. "Anistics"
6. "Visicorp"

The data processing presentations were demonstrations of personal computers and various software by representatives from Apple, GRID, Hewlett Packard, and IBM.

Tuesday, May 17, 1983

All day Tuesday was devoted to a continuation of the concurrent sessions from Monday afternoon.

Wednesday, May 18, 1983

At 8:30 a.m., W. James MacGinnitie moderated a panel on "Data Base Availability in The Future." His panel consisted of:

Joseph T. Brophy
Senior Vice President
Travelers Insurance Companies

Anthony J. Grippa
Senior Vice President
National Council on Compensation Insurance

Harry Schuford, Ph.D.
Manager, Captive Development & Planning Division
Fred S. James Co.

After a short presentation by each panel member, Mr. MacGinnitie moved into the audience to lead a lively question and answer session. The discussion covered a wide range of topics including Mr. Brophy's use of personal computers within his company, the direction of the National Council on Compensation Insurance with regard to data availability, and the importance of data processing in risk management.

At 9:30 a.m. Mr. Kilbourne reconvened the business session and presented Mr. P. Adger Williams, President of the American Academy of Actuaries and Past-President of our Society. He updated the members on the Academy's activities and expressed his thoughts on the proposed CAS reorganization.

Mr. Kilbourne then called for the vote on amending the CAS Constitution and Bylaws with regard to Organization. The proposed changes were approved unanimously by a hand vote.

At 10:45 a.m., Mr. Joseph W. Brown moderated a panel entitled "Agency Automation, Telecommunication Network and Agency Interface." His panel members were:

John W. Folk
President
Insurance Institute of Research

Carol A. Hammes
Executive Vice President
Hales & Associates, Inc.

Chris Scattliff
Sr. Vice President, Marketing
ARC — Automation Services

Kirk T. Foley
Project Manager
Hartford Insurance Co.

The panel presented a variety of views reflecting their individual areas of expertise directed at the question, "Will agency automation, telecommunications network, and agency interface make the american agency system a more efficient competitor?" Although there was a consensus of the panel that the various technology components will result in efficiency, the panelists offered different views as to the magnitude and the timetable for realization of these efficiencies.

Mr. Kilbourne closed the meeting at 12:00 noon.

In attendance, as indicated by registration records, were 150 Fellows, 121 Associates, 18 guests, 13 subscribers, and 5 students. The list follows

FELLOWS

Addie, B. J.
Adler, M.
Aldorisio, R. P.
Anker, R. A.
Barrette, R.
Bartik, R. F.
Bass, I. K.
Beverage, R. M.
Bill, R. A.
Bland, W. H.
Boison, L. A., Jr.
Bornhuetter, R. L.
Carpenter, T. S.
Childs, D. M.
Christie, J. K.
Cohen, H. L.
Conger, R. F.
Connell, E. C.
Conners, J. B.
Corr, F. X.
Covney, M. D.
Curry, A. C.
Davis, L. S.
Dawson, J.
Dean, C. G.
Degerness, J. A.
Demers, D.
Dempster, H. V.
Doepke, M. A.
Donaldson, J. P.
Easton, R. D.
Eland, D. D.
Eliason, E. B.
Fallquist, R. J.
Fiebrink, M. E.
Fisher, R. S.
Fisher, W. H.
Fresch, G. W.
Gallagher, T. L.
Garand, C. P.
Giambo, R. A.
Gillespie, B. C.
Gluck, S. M.
Goldfarb, I. H.
Gottheim, E. F.
Graham, T. L.
Gutterman, S.
Hachemeister, C. A.
Hall, J. A., III
Haner, W. J.
Hartman, D. G.
Harwayne, F.
Heer, E. L.
Heersink, A. H.
Henry, D. R.
Herder, J. M.
Hine, C. A.
Hillhouse, J. A.
Honebein, C. W.
Horowitz, B. A.
Hough, P. E.
Ingco, A. M.
Inkrott, J. G.
Irvan, R. P.
Jameson, S.
Jean, R. W.
Johe, R. L.
John, R. T.
Johnson, W. H.
Johnston, T. S.
Kelly, A. E.
Khury, C. K.
Kilbourne, F. W.
Kollar, J. J.
Krause, G. A.
Kraysler, S. F.
LaRose, J. G.
Ledbetter, A. R.
Lerwick, S. N.
Linden, O. M.
Liscord, P. S.
Lo, R. W.
Lommele, J. A.
Lowe, S. P.
Lyle, A. C.
MacGinnitie, W. J.
MakGill, S. S.
McGovern, W. G.
McGuinness, J. S.
McManus, M. F.
Meyers, G. G.
Miccolis, R. L.
Miller, D. L.
Miller, M. J.
Moore, B. C.
Muetterties, J. H.
Munro, R. E.
Murad, J. A.
Naffziger, J. V.
Nash, R. K.
Newlin, P. R.
Newman, S. H.
Newville, B. S.
Nickerson, G. V.
Oakden, D. J.
Palm, R. G.
Patrick, G.
Pearl, M. B.
Petersen, B.
Phillips, H. J.
Pratt, J. J.
Purple, J. M.
Reynolds, J. J., III
Riddlesworth, W. A.

FELLOWS

Rodermund, M.
Rogers, D. J.
Rosenberg, M.
Scheibl, J. A.
Schumi, J. R.
Schoop, E. C.
Simon, L. J.
Snader, R. H.
Squires, S. R.
Steeneck, L. R.
Stephenson, E. A.
Stewart, C. W.
Streff, J. P.
Strug, E. J.
Taht, V.
Taranto, J. V.
Tatge, R. L.
Truttmann, E. J.
Tuttle, J. E.
Venter, G. G.
Walker, R. D.
Walsh, A. J.
Walters, M. A.
Wasserman, D. L.
Weller, A. O.
Wess, C.
Williams, P. A.
Wilson, J. C.
Wilson, R. L.
Woll, R. G.
Wulterkens, P. E.
Yingling, M. E.
Young, E. W.
Youngerman, H.
Zatorski, R. T.
Zubulake, T. J.

ASSOCIATES

Abramson, G. R.
Andler, J. A.
Andrus, W. R.
Bartlett, J. W.
Baum, E. J.
Belden, S. A.
Bensimon, A. S.
Bhagavatula, R. R.
Bothwell, P. T.
Brooks, D. L.
Bujaucuis, G. S.
Chanzit, L. G.
Chorpita, F. M.
Chou, P. S.
Connor, V. P.
Crifo, D. A.
Currie, R. A.
Davis, R. D.
Deutsch, R. V.
Downer, R. B.
Duffy, B.
Dye, M. L.
Edmondson, A. H.
Ellefson, T. J.
Eramo, R. P.
Fasking, D. D.
Fiebrink, D. C.
Flanagan, T. A.
Forde, C. S.
Friedman, H. H.
Fueston, L. L., Jr.
Gerard, F. R.
Godbold, N. T.
Goldberg, T. L.
Gould, D. E.
Granoff, G.
Gruber, C.
Halpert, A.
Hanson, J. L.
Harwood, C. B.
Head, T. F.
Henkes, J. P.
Henry, T. A.
Henzler, P. J.
Hobart, G. P.
Hofmann, R. A.
Hutter, H. E.
Kadison, J. P.
Kane, A. B.
Kaur, A. F.
Kelly, M. K.
Klawitter, W. A.
Kleinberg, J. J.
Klingman, G. C.
Kolojay, T. M.
Kooken, M. W.
Kozik, T. J.
Lafrenaye, C.
Leong, W.
Livingston, R. P.
Marks, S. D.
Martin, P. C.
Mashitz, I.
McIntosh, K. A.
Merlino, M. P.
Meyer, R. E.
Miller, D. L.
Millman, N. L.
Mittal, M. L.

ASSOCIATES

Morgan, S. T.
Morrow, J. B.
Napierski, J. D.
Narvell, J. C.
Neale, C. L.
Nelson, J. K.
Neuhauser, F., Jr.
Nichols, R. S.
Nichols, R. W.
Nikstad, J. R.
Normandin, A. G.
O'Connell, P. G.
Ogden, D. F.
Petit, C. I.
Piazza, R. N.
Pierson, F. D.
Pilon, A.
Potter, J. A.
Potts, C. M.
Powell, D. S.
Pulis, R. S.
Rapoport, A. J.
Reynolds, J. D.
Sansevero, M., Jr.
Schmidt, N. J.
Schwartzman, J. A.
Sellitti, M.
Sherman, H. A.
Singer, P. E.
Smith, R. A.
Stanco, E. J.
Steinen, P. A.
Stroud, R. A.
Surrago, J.
Swisher, J. W.
Symnowski, D. M.
Toczylowski, D. L.
Townsend, C. J.
Varca, J. J.
Visner, S. M.
Wacek, M. G.
Waldman, R. H.
Watford, J. D.
Weimer, W. F.
Weiner, J. S.
Westerholm, D. C.
White, C. S.
Whiting, D. R.
Windwehr, D. R.
Withers, D. A.
Yau, M. W.
Young, R. G.

GUESTS—STUDENTS—SUBSCRIBERS

Allen, T. C.
Belton, E. F.
Brophy, J. T.
Cassuto, I. A.
Chansky, J. S.
Curran, K. F.
Earls, R.
Fave, L. D.
Folk, J. W.
Gartlan, R.
Gunter, B.
Gutman, E.
Hager, G.
Hammes, C. A.
Johnson, M. B.
Keating, R.
Klugman, S.
Kuhn, W. N.
Lemaire, J.
Libbery, W.
Mong, S.
Neff, R.
Pichler, K. J.
Rapoport B.
Rowe, R.
Roy, T. S.
Schenker, N.
Schmidt, L. D.
Schuford, H.
Spangler, J. L.
Stenmark, J. A.
Sweeny, D. M.
Thompson, R. M., Sr.
Valenti, A. T.
Vandernoth, J. P.
Wootton, G. A.

Respectfully submitted,

BRIAN E. SCOTT
Secretary

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PROCEEDINGS

November 6, 7, 8, 9, 1983

SOCIAL INSURANCE AND THE CASUALTY ACTUARY

PRESIDENTIAL ADDRESS BY FREDERICK W. KILBOURNE

Sixty-nine years ago, some forty actuaries gathered in New York and formed the Casualty Actuarial and Statistical Society of America. Article II of the constitution of this Society proclaimed its object to be ". . . the promotion of actuarial and statistical science as applied to the problems of casualty and social insurance . . ." We have always done well in fulfilling this object concerning workers' compensation, the original line of social insurance. In our early days, we attended fairly well to both sides of the field our forefathers had tilled for us. The first thirty volumes of our *Proceedings* included 34 papers and actuarial notes on social insurance, of which a dozen dealt specifically with unemployment insurance. In more recent years, however, one section of our original field has come to be neglected. The next thirty volumes contained only four papers on social insurance, and the ten most recent volumes are essentially silent on the subject. Yet social insurance, in its various versions and perversions, is one of the great burning issues of the Twentieth Century. The resolution of social insurance problems, for better or for worse, will play a major role in determining the quality of life in the Twenty-First Century. Society needs the Casualty Actuarial Society now more than ever, as it grapples with its problems in our field. We will be derelict in our responsibilities to the public and to our profession if we abandon the field, allowing it to become the exclusive property of the politician. Social insurance and the actuary should never have been divorced, and must be reconciled, if only for the sake of the children.

Consider, first, the actuary. What is this "actuary?" *The Fact Book* of the American Academy of Actuaries says that actuaries, who are "trained in the science of mathematical probabilities and finance," "evaluate the current financial implications of future contingent events." *The Statement of Purpose* of the Society of Actuaries says the actuary is "trained to analyze uncertainty, risk and probabilities," and that actuarial science is "built on the evaluation of the financial, economic and other implications of future contingent events." An upcoming encyclopedia article contends that "if both money and uncertainty are involved in a problem, the actuary can be important to the construction of a solution." Succinctly, if not poetically, I see the actuary as a "future cost analyst." Note that all these definitions encompass territory going well beyond the fields now being cultivated by the actuarial profession. Also note, on the other hand, that they are specific, dealing always with money and with the future. Perhaps the shortest definition of all is that we are "financial futurists." It has been said that a profession must be both unique and necessary, and who but the actuary can claim training which has been directed at producing no more and no less than a financial futurist? The criterion that we be necessary, of course, is quite another matter. Yet the government of this country believes a valuation actuary to be necessary to the proper functioning of a life insurance company. And the U.S. Government requires an enrolled actuary where there are private pension plan liabilities. And stockholders in general need help in evaluating the recommendations of managers whose interests may favor short-term earnings at the expense of long-term survival; with the need existing whether or not those stockholders currently are getting that help. And taxpayers need help in evaluating the promises of politicians whose interests in being elected may overwhelm their skills as financial futurists. Nor are the taxpayers the only losers if these elected officials prove to have been better salesmen than engineers. We all lose if our social institutions come apart at the seams; whether they go bankrupt or whether they merely serve to distort or dampen our economy. So we actuaries are necessary to public institutions such as social insurance, not because we can unerringly predict the future, but because we are experts in evaluating the current and long-term financial implications of programs involving uncertain future events.

Consider, next, social insurance. Ralph Blanchard defined social insurance as "any form of insurance in which the government goes beyond the regulation of practices and the dissemination of information" and does so by "compelling insurance" or by "becoming itself an insurer." This was in his 1942 presidential address entitled "The Casualty Actuary and Social Insurance." Also in the 1942 *Proceedings* was "An Approach to a Philosophy of Social Insurance," by Jarvis Farley and Roger Billings. This paper considered social insurance to be "when the government role is extended beyond the regulatory function, whether by making the purchase of insurance compulsory on certain classes of citizens or by actually underwriting and assuming the risk, or both." So perhaps we may agree that social insurance is insurance that is required or provided by the government. This includes, for example, auto insurance in New York; state disability insurance in California; medical insurance in Canada; and, workers' compensation insurance everywhere.

The distinction between insurance and subsidization should be noted. You are being insured when you pay your expected loss cost, as well as may be determined, into a common pool for the provision of benefits to those who suffer a fortuitous loss. You are being subsidized when you pay less than that cost. The two concepts are distinct, although a given program may have elements of both. A government program of subsidization is welfare, not social insurance. A private insurance company program of subsidization, in a competitive market, leads to bankruptcy.

It is unfortunate that welfare has come to have negative connotations, for it is essential in a humane society with productive capacity beyond minimum subsistence for all. Some people are unable to produce, and they must be subsidized or they will die. Consider all of us at the beginning and end of our life span, for example. Furthermore, even producers may suddenly find themselves unable to produce. Contingent events lurk everywhere, trying to keep you from your job, taking your arm or your youth, bringing depression to you or to your country, and more, and worse. Social insurance as opposed to welfare, is arguably a good solution to such producer problems, particularly where private insurance has failed to fill the need for one reason or another. It is often paid for by the employer, who treats the expense as an element of employee compensation and passes it on to the consumer of his goods or services. This seems appropriate in the case of workers' compensation, at least, for it permits the price of a manufactured item, for example to reflect the true cost of its production. Social insurance is not the same thing as welfare, but together they have the potential of taking the rough edges off a free market economy, while retaining the incentives that are essential to a productive society.

Social insurance can be abused, of course, as can all human institutions. Benefit levels and eligibility criteria can be debated, but most will agree that the public good suffers if incentives to produce are severely eroded. Social insurance, like all insurance, must be sold. The sales prospect here is the voter, however, rather than the would-be policyholder. Government has the responsibility, particularly since it is unregulated except by an inexpert electorate, to present the social insurance plan fairly when it is being sold or revised. If the plan includes a welfare or subsidy element, this fact should be identified. If financing is to be on a "pay-as-you-go" basis, future cost estimates should not be predicted on "pray-as-you-go" assumptions. These simple rules of fair play and survival have not always been followed by North American and other governments. Some social insurance plans have been soundly based at their inception, and in subsequent operation; but others have not, including some of the more important ones. It is misrepresentation or worse to present a welfare system as insurance; to imply that one's taxes are premiums which are being invested on one's behalf when they are not. It is worse to promise benefits which are unsupported by adequate taxation, and which will be deliverable only with the help of divine intervention. Qualified actuaries in government have generally been doing a valiant job against powerful opposition in trying to keep their programs sound, or in struggling to restore them to health. But they are few in number, being only 3% of the Casualty Actuarial Society and less than 2% of the actuarial profession at large. We should be doing more to help them.

It is not my purpose to be critical in general of the government of the United States or of Canada. Many others are far worse. Consider the socialist states. The popular appeal of socialism is contained almost entirely within the issues covered by social insurance. Most of us have sufficient compassion to be moved—even to the point of parting with a considerable portion of our earnings—by the truly needy family with dependent children, or the worker who is unemployed because of injury or recession. Few of us are so moved by the desire of those in government to have a state monopoly in the production of widgets—or bread, for that matter. So the socialist states rely on social insurance issues for the degree of legitimacy they enjoy. Some have even elevated the matter to a religion, with all the attendant horrors that absolutism invariably brings. It is surely a religion speaking when we are asked to have faith in a system because of a promise that it will bring us a bountiful tomorrow, while ignoring its dismal past record of underproductivity relative to systems based on freedom. There are worse governments than those found in North America.

Social insurance was very much on the minds of the founders of our society. Our first president is a case in point. Isaac Rubinow published, in 1913, a book entitled "Social Insurance with Special Reference to American Conditions." Twenty years later he published "Quest for Security" and along the way contributed a number of seminal papers on technical aspects of workers' compensation insurance. Unfortunately, Dr. Rubinow was not with us long as a casualty actuary. An immigrant from Russia (not the Soviet Union) as a teenager, he became a medical doctor before he became an actuary. After only five years in the insurance business he left to become Director of the American Zionist Medical Unit in Palestine (not Israel). His second presidential address, in 1916, was mailed to the CASSA from another distant land, California. Returning to the United States before long, he continued his lifelong devotion to the cause of social insurance, or at least to the issues that underlie the field. Before his death in 1936, he served as a member of the Ohio Commission on Unemployment Insurance and was a consultant to the economic security committee which formulated the U.S. Social Security Act. In that 1916 address, he prophesied that social insurance developments would "swell the membership and importance of our Society beyond the wildest dreams of those who were responsible for the initial steps in its organization," and that we would become "an institution which can and will apply the scientific methods of mathematics and statistics to the elimination of grave social ills, and to the betterment of the world we live in." We have done much, but we can do so much more.

Though I never met Dr. Rubinow, I did have the privilege of talking several times with another charter member, Bill Breiby. The founders of the CAS had a considerable sense of purpose, and social insurance was much on their minds, though they were far from unanimous in their opinions on the subject. The original Fellowship examinations had two parts: the first dealt substantially with rates and reserves, while the second covered the principles and history of social insurance, government statistics, old age and disability insurance, unemployment compensation, and premiums for and valuation of pension funds. Nor was Rubinow the only one exhorting us to play a role in the development of social insurance programs. Blanchard, in his 1942 address, suggested that "the casualty actuary, whether motivated by social consciousness or by self interest, should devote more attention . . . to the social insurance field," for this casualty actuary, "can be of great usefulness in giving technical guidance to government action, and in determining the lines to be drawn between social and private insurance." Farley, in his 1942 paper, clearly had the actuary in mind when he spoke of "the obligation of our national leaders neither to overstate the need for (social) insurance nor to underestimate (its) cost, but to tell the nation candidly and to the best of their ability the actual facts upon which the decisions must be made." An interesting historical note is that Farley's paper was discussed in the 1942 *Proceedings* by two of our Fellows who have been prominent in American social insurance circles, W. Rulon Williamson and Robert J. Myers. Bringing matters up to date, it was just last month that I discussed the foregoing paper with Jarvis Farley, and that I talked with Bob Myers about his upcoming social insurance assignment in Granada. The CAS may be the oldest North American actuarial body, but it remains a young organization. After all, our first meeting took place only a few years before the Bolshevik Revolution.

The question naturally arises as to just what we can do to make the world safe for social insurance—or from it, depending on your point of view. There are many things we can do, both as the CAS and as individual members. We can reintroduce social insurance topics into our syllabus and examinations. We can add topics designed to broaden the world view of the actuary, and to suggest that his or her training is useful in solving problems beyond current narrow applications. We can broaden our own view and scope geographically; this meeting and an escalating relationship with the Canadian Institute of Actuaries are auspicious in this regard. We can tender a public expression of professional opinion when an egregiously unsound social insurance plan is proposed or enacted. We can support the American Academy of Actuaries in its government and public relations programs concerning matters such as unisex legislation. The new structure of the CAS should make it more feasible to undertake ambitious activities, and I know that Carl Honebein, and Stan Khury, and their troops will serve us all well in this regard.

There also is a great deal that we can do as individual actuaries to promote the public interest goals of social insurance, whether by sound social insurance programs or by other means. One thing we can do is to speak out, professionally but emphatically, against unsound social insurance programs. We can evaluate and report on the financial implications of the redistribution of income and wealth. We can point out that the government can redistribute but it does not generate, and that the multitudes will go hungry if all that is available is two loaves and five fish. If we can establish that a social security plan is likely to turn into antisocial insecurity, we can say so, and then search for an audience capable of taking corrective action. We can put actuarial talent and techniques to work in helping to develop solutions to long-range economic problems. For example, an actuarial model of the economy could be developed in order to test the impact of making all government decisions in terms of maximizing productivity. Another model could be designed based on maximizing the long-term flow of income from the haves to the have-nots; such a model might have great promise in terms of bringing together so-called liberals and so-called conservatives. Actuaries are probably as well, or better, equipped than others to measure the long-term implications of radical economic plans, such as a hypothetical economy under which each person would receive net compensation directly proportional to his relative productivity—defined as absolute output relative to his potential. We can also endeavour to show our employers, and through them the public, that long-term profits in a competitive insurance market may not be at all inconsistent with the goals of social insurance. We can seek clients in the public sector, conduct seminars on actuarial matters of interest to the public or their representatives, and write papers and articles on social insurance themes. We can each spend part of our career in government employment. We can do much, and we can make a difference. The children need us.

APPENDIX
PERTINENT QUOTATIONS

This appendix contains quotations selected to enhance and shed light upon the address to which it is appended.

"Socialism is a stage in social development from a society guided by the dictatorship of the proletariat to a society wherein the state will have ceased to exist." Joseph Stalin

"Insurance is an ingenious modern game of chance in which the player is permitted to enjoy the comfortable conviction that he is beating the man who keeps the table." Ambrose Bierce

"Every form of refuge has its price." *Lying Eyes*, The Eagles

"You can see the stars and still not see the light." *Already Gone*, The Eagles

"I could be wrong, but I'm not." *Victim of Love*, The Eagles

REINSURING THE CAPTIVE/SPECIALTY COMPANY

LEE R. STEENECK

Abstract

This paper primarily discusses one quantitative excess of loss reinsurance pricing technique. European actuarial literature of the 1960's explores mathematical utility theory in the context of insurance. Recently, Freifelder and Cozzolino have written about exponential utility's value in pricing. This paper explores the relationship between wealth, reinsurance dollars and retention/cession. It is hoped that actuaries can supplement management judgment on cost effective reinsurance programs with analyses such as described here.

Introduction

Much has been written about reinsurance lately. The topic has scored highly in topics of current interest to actuaries, regulators, and others. The scope of this reinsurance paper is limited to selecting and pricing an excess of loss reinsurance coverage for a captive or specialty company. Many of these are single line insurers, so applying theory is simplified.

I intend to introduce risk theory but concentrate on utility theory concepts and applications. I believe utility theory presents an entire framework for risk-reward evaluation. A contract of reinsurance can be consummated only when an offer and acceptance has occurred. Since both parties to the contract have different and distinct expectations, each must be realistic in evaluating cost versus benefit. Utility theory allows for a mathematical treatment of the problem.

Reinsurance Programs

Virtually every insurance company must concern itself with the various forms of reinsurance that are available and the functions they perform. The establishment of a good reinsurance program is essential (a) to contain to a manageable level claim variance and (b) to reduce adverse effects on company growth and solvency caused by claim variance.

> It has been said that the object of reinsurance is, in the first place, to protect the direct writing company, the cedent, against payments of such claims as would threaten his solvency, and, secondly, to secure the cedent a result of his risk business (*'earnings'*) as even as possible.[1] (Emphasis supplied.)

To purchase reinsurance economically means to select a form suitable to the needs of the company, with a retention high enough to control costs, yet low enough to minimize loss experience fluctuations over the years.

There are basically two types of treaty reinsurance: (a) pro rata or proportional reinsurance, which calls for the equal sharing of premiums and losses, and (b) excess of loss. Much reinsurance sold today is on an excess of loss form. Coverage can apply (a) per occurrence to an individual insured or (b) per event to a group of insureds. Event reinsurance is termed catastrophe reinsurance. Excess of loss can also be time dependent, as opposed to occurrence dependent. For example, aggregate or stop loss reinsurance is used to restrict total claims incurred for typically an annual period either (a) on a per risk basis or (b) for a collection of risks.

Excess of loss per risk or per occurrence reinsurance is very popular today. Coverage usually is divided into several layers. According to Reinarz,[2] layers are either "exposed" or "unexposed." An exposed or working layer is expected to have reasonably predictable frequency/severity characteristics. If a moderate sized hospital company issues $1 million policy limits and its appropriate retention is $250,000, the layer $250,000 xs 250,000 could be a working layer ("xs" means "in excess of a retention of"). This narrow layer with substantial premium per annum should be self-funding over a three-to-five-year time horizon according to reinsurance practice. A layer of $500,000 xs 500,000 also would be exposed since any single loss could attach, but the layer would not work as often. Presumably, there would not be enough premium in the second layer to sustain full layer losses (an unbalanced condition); hence, the reinsurer should have highly variable accident year results. This layer would be expected to be self funding over a much longer time horizon. Since chronological stabilization is more valuable here for the cedent (and riskier to the reinsurer), rates for this layer would include a higher profit and risk charge than for the layer $250,000 xs 250,000.

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[1] S. Bjerreskov, "On the Principles for the Choice of Reinsurance Method and for the Fixing of Net Retention for an Insurance Company," *International Congress of Actuaries*, 1954.

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[2] R. Reinarz, *Reference Book of Property and Liability Reinsurance Management*, Mission Publishing, Fullerton, Cal., 1969.

Of course, an employed physician with separate limits may have attended the claimant negligently while he was hospitalized. Although, in my example, an individual policy would not pay beyond $1 million, the claimant might recover $1.5 million because both policies would be expected to contribute. Reinsurance can protect against these multiple claims through a "clash cover." Two policies with losses from the same occurrence would be subject to one retention (e.g., $250,000).

Stability is enhanced as large losses are truncated, as far as the insurer is concerned, at a cost of modest premium outlay. Modest premium outlay is important in these days of high investment returns on funds withheld. The environment is one of knowledgeable buyer dealing with knowledgeable seller so transactions are free of rate and form regulation. This places great pressure on the negotiators to form an equitable alliance.

Reinsurance Loss Loadings

Reinsurance actuaries believe contracts exhibiting low risk should be priced at low expected reward, and conversely, high risk reinsurance should be priced with a high expected reward. When we divide the variance in a loss portfolio between insurer and reinsurer, we have a two-person game. More determined attempts to minimize variance on the retained portfolio concomitantly bring about more costly reinsurance. European actuarial literature discusses this.

Lambert[3] notes that the reinsurance loading generally increases according to the form of reinsurance—(a) pro rata, (b) excess of loss, and (c) stop loss or aggregate excess respectively. Vajda[4] demonstrates that for a given level of premium, the reinsurer's variance is minimized if the form is pro rata:quota share. Borch[5] notes that stop loss reinsurance minimizes the variance of the portfolio retained by the ceding insurer.

It is no wonder that most reinsurance sold for capacity, stability, and catastrophe protection today is of the excess of loss form. The form functions well and in an era where investment income on retained funds is extremely important, excess of loss reinsurance is in some sense optimal. Pro rata requires a large premium outlay. Stop loss reinsurance is heavily loaded for profit and contin-

[3] H. Lambert, "Contribution to the Study of . . . Collective Risk Theory" (French), *ASTIN Bulletin #2*, 1963.

[4] S. Vajda, "Minimum Variance Reinsurance," *ASTIN Bulletin #2*, 1963.

[5] K. Borch, "An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance," *16th International Congress of Actuaries*, 1960.

gency. Furthermore, it does not return cash quickly.

Utility Theory

Very little has been written in the U.S. about the quantitative study of (a) relative costs of various reinsurance forms and (b) methods of establishing retentions. One text, however, by Reinarz,[6] illustrates several pragmatic approaches that can be taken. If the excess of loss form is chosen, a cost effective retention can be viewed in light of (a) the reinsurer's loss loading, (b) minimizing the variation in retained loss ratio, (c) reinsuring where claims frequency drops off, and others. These are judgmental approaches calling for the actuary or reinsurance purchaser to guess at relative effectiveness. Can the consequences of the decision be measured objectively in advance?

In European literature, beginning in the 1960's, we see risk theory being applied in the insurance context. Retention and reinsurance programs are selected to help constrain the probability of ruin. For large companies more concerned with stable earnings growth, a fraction of surplus can be placed at risk. Stockholder or policyholder (in a mutual company) disappointment will certainly precede financial ruin or insolvency.

For those interested in risk theory, I suggest reading Gerber,[7] Seal,[8] Bühlmann,[9] Philipson,[10] Wilhelmsen,[11] Bjerreskov,[12] Pentikäinen,[13] Wooddy,[14] and

[6] R. Reinarz, *op. cit.*

[7] H. Gerber, *An Introduction to Mathematical Risk Theory,* Huebner Monograph #8, Richard Irwin, Homewood, Ill., 1979.

[8] H. Seal, *Stochastic Theory of a Risk Business,* John Wiley & Sons, New York, N.Y., 1969.

[9] H. Bühlmann, *Mathematical Methods in Risk Theory,* Springer-Verlag, Berlin, 1970.

[10] C. Philipson, "A Review of the Collective Theory of Risk," supplement to *ASTIN Bulletin, Vol. V,* (from *Skandinavisk Aktuarietidskrift,* 1968).

[11] L. Wilhelmsen, "On the Stipulation of Maximum Net Retentions in Insurance Companies," *International Congress of Actuaries,* 1954.

[12] S. Bjerreskov, *op. cit.*

[13] T. Pentikäinen, "On the Reinsurance of an Insurance Company," *International Congress of Actuaries,* 1954; T. Pentikäinen, "Reserves of Motor-Vehicle Insurance in Finland," *ASTIN Bulletin,* 1962; T. Pentikäinen, "On the Reinsurance of an Insurance Company," *op. cit.*

[14] J. Wooddy, "Part 5 Study Notes—Risk Theory," Education and Examination Committee of the Society of Actuaries.

Beard et al.[15] The methods they note generally are complicated in theory, simplified in practice, and may not be as safety oriented as stated.

Game theory, developing at the same time, can be viewed in the insurance context.[16] Various players, employing competing strategies, obtain payoffs which they seek to maximize by some measure. Payoffs depend on each player's strategy but all strategies are interactive. The simplest is the two-person zero sum game where "my gain is your loss." Properly structured reinsurance programs can benefit both the seller and buyer. Reinsurance should be thought of as a partnership arrangement.

Traditional economic theory "at first glance" may not seem to apply to insurance. Businessmen seek to maximize profits. The purchase of insurance at a cost greater than expected losses is, therefore, an irrational business decision. The resulting reduction in profits is contrary to the businessman's primary motive. But Bernoulli[17] stated that a rational man does not seek to maximize gain but instead to maximize the expected *utility* of gain. Uncertainty creates anxiety. Supply and demand forces are altered. Current economic theory embraces utility theory.

Let us explore utility. Briefly, the utility of money, the value an individual places on an amount of money, varies depending on the individual's wealth. Different individuals view $1, $10, and $1000 differently. One thousand dollars to the beggar is worth substantially more than $1000 to the millionaire. To the beggar, it represents food, shelter, and warmth. To the millionaire, it may only cover repairs to his prestigious automobile.

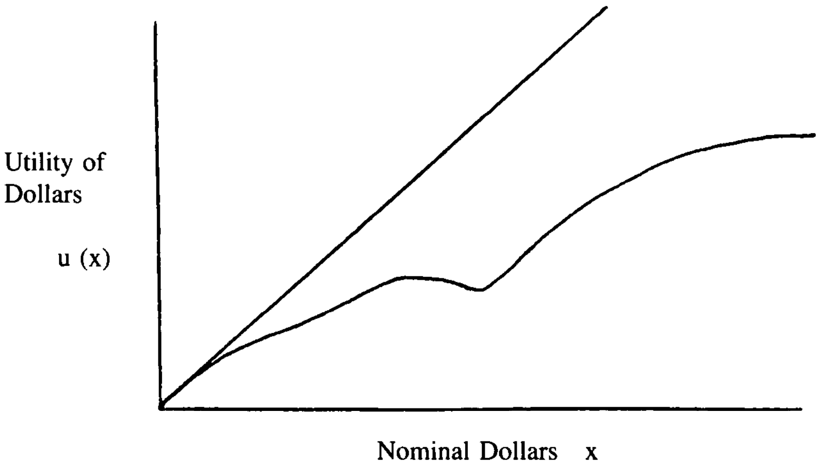
Figure 1 illustrates one utility curve. Along the forty-five degree line each dollar is worth no less and no more than the previous one, an unrealistic situation. Instead, most likely, we should see a convex down curve. The value of additional dollars decreases generally over the length of the curve. There may be risk-taking sections of the curve, however, where we play unfair lotteries because of our aspirations. Siegel[18] writes about levels of aspiration.

[15] R. Beard, T. Pentikäinen and E. Pesonen, *Risk Theory,* Methuen & Co., London, England, 1969.

[16] K. Borch, "Recent Developments in Economic Theory and Their Application to Insurance," *ASTIN Bulletin,* 1964.

[17] D. Bernoulli, "Exposition of a New Theory on the Measurement of Risk," translation of the original 1738 work, *Econometrica,* 1954.

[18] S. Siegel, "Level of Aspiration and Decision Making," *Psychological Review #64,* 1957.



We often see charities offering $10 tickets on a chance to win a new car. Although the game is unfair if ticket sales are brisk (the expected winnings are less than $10 per ticket sold), we might aspire to own that new car so we take a chance. The point is that the ticket price has lower utility than our aspiration to own the car. The equilibrium price or balance of indifference is what utility theory measures. In the case of insurance, how much premium is one willing to pay (the certain result) so as to escape an uncertain loss process? This is the mirror image of the car lottery example. In that example, you pay to gain (utility); for insurance, you pay not to lose (disutility).

Savage[19] gives an interesting history of utility and the papers written about it. Arrow[20] and Pratt[21] give accurate and meaningful interpretations of the concepts of risk aversion and risk preference.

It may appear that some insurers are nearly indifferent to risk. Only recently has the ISO varied profit and contingency loadings from the traditional 5%

[19] L. Savage, *The Foundations of Statistics*, John Wiley & Sons, New York, N.Y., 1954.

[20] K. Arrow, *Essays in the Theory of Risk Bearing*, Markham Publishing Co., Chicago, Ill., 1971.

[21] J. Pratt, "Risk Aversion in the Small and in the Large," *Econometrica #32*, 1964.

generally used. A $10,000 premium ($500 profit and contingencies loading) OL&T large risk at 25/75 limits was priced for profit and contingencies indifferently to a $10,000 neurosurgeon at $1/3 million limits.

Insurer underwriting practices reflect preferences. Certain insureds are desirable, as evidenced in Bailey's paper on "Skimming the Cream."[22] (Automobile classes weren't homogeneous.) Just as this example demonstrates risk preference, we see FAIR plans with loss-free insureds. Insurers obviously prefer not to insure these policyholders at the voluntary market price.

Utility theory is not abstract, incapable of practical use. Insurers can and do specify preferences. Utility theory quantitatively handles preferences.

The Utility Function

Figure 2 illustrates four families of utility functions.

Logarithmic utility was first suggested by Bernoulli.[23] It implies decreasing risk aversion. The family can be particularly useful for insurers if they become more risk prone or daring as they develop more wealth over time.

Quadratic utility also may be useful for insurance companies. Markowitz[24] shows that if a decision maker maximizes expected utility and always prices on a best mean-minimum variance principle, he will develop a Pareto-optimal portfolio. This occurs only if his utility function is quadratic. Borch[25] demonstrates that stop loss reinsurance should be preferred for insurers exhibiting quadratic utility toward risk.

Quadratic utility curves have two drawbacks, however. First, the curves only increase up to a wealth level of $b/2$ (see Figure 2). Second, it can be demonstrated that these curves imply an increasing aversion to risk as wealth increases. So the larger the insurer gets, the more likely he will raise prices and reinsure more of his business. In my experience, insurers do not behave in this manner; thus, quadratic utility curves are not very useful for insurance companies.

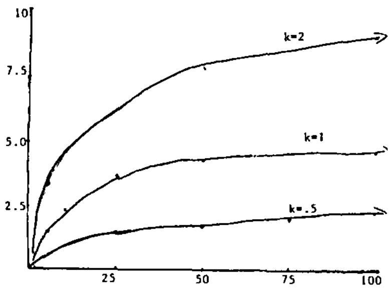
[22] R. Bailey, "Any Room Left for Skimming the Cream?" *PCAS*, XLVII, 1960.

[23] D. Bernoulli, *op. cit.*

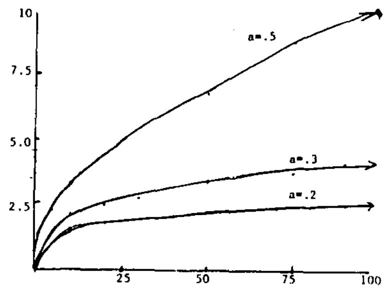
[24] H. Markowitz, *Portfolio Selection*, John Wiley & Sons, New York, N.Y., 1959.

[25] K. Borch, *op. cit.*

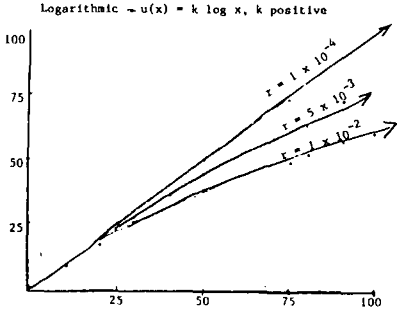
FOUR FAMILIES OF UTILITY FUNCTIONS



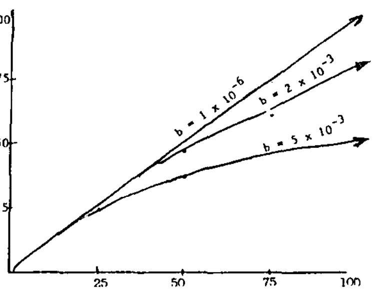
Logarithmic — $u(x) = k \log x$, k positive



Fractional Power — $u(x) = x^a$, x positive, a between--(0,1)



Exponential — $u(x) = 1/r\ (1-e^{-rx})$, r positive



Quadratic — $u(x) = x-bx^2$, b positive, x less than ½b

A third family of utility functions is termed exponential. Gould[26] shows that consumers choose deductibles consistent with those that exponential utility dictates. Shpilberg and DeNeufville[27] note that results are not particularly sensitive to the family of utility functions used; so, since the exponential is easiest to work with, use it. Cozzolino[28] and Freifelder[29] have developed ratemaking models relying exclusively on exponential utility.

The fourth family is termed fractional power.

Until recently insurance was largely priced on expected value (after expenses). Depending on the insurer, a level of underwriting return or total return was targeted. This implied the utility function $u(x) = x$, the forty-five degree line from Figure 1. (This linear function is a special case of the exponential utility family where, as r approaches zero, the quantity $(1/r) \times (1 - \exp(-rx))$ approaches x. An r of 0 would mean no risk aversion, or indifference.)

Utility functions generally are concave down in the first quadrant. "More is better" so the curve is increasing, but the rate of climb slows since added dollars are worth slightly less than prior dollars. Mathematically, the first derivative $u'(x)$ is greater than zero, but the second derivative $u''(x)$ is negative. If we calculate $-u''(x)/u'(x)$ as an index of risk preference, then only for the exponential family does everything cancel, and we are left with r: constant risk aversion. Wealth is immaterial. The reader can verify that logarithmic utility has decreasing risk aversion with wealth.

The constant risk aversion of the exponential family makes pricing a multiplicity of insureds over time easier. Decisions can be made independent of order or time. Other families of functions rely on wealth for pricing purposes, and all decisions must be made in light of others. The exponential function is both clean and aesthetically appealing.

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[26] J. Gould, "The Expected Utility Hypothesis and the Selection of Optimal Deductible for a Given Insurance Policy," *Journal of Business*, April, 1969.

[27] D. Shpilberg and R. DeNeufville, "Use of Decision Analysis for Optimizing Choice of Fire Protection and Insurance: An Airport Study," *Journal of Risk and Insurance*, College of Business, University of Georgia, Athens, March, 1975.

[28] J. Cozzolino, "A Method for the Evaluation of Retained Risk," *Journal of Risk and Insurance XLV #3*, College of Business, University of Georgia, Athens, 1978.

[29] L. Freifelder, *A Decision Theoretic Approach to Insurance Ratemaking*, Huebner Monograph #4, Richard Irwin, Homewood, Ill., 1976.

We could assume that for the long run r can vary; call it a different r each year. Then, exponential utility can displace logarithmic utility's prime appeal. Risk aversion could decline periodically with increasing wealth with no complications in application.

One final and most important item. Assuming exponential utility and given particular reinsurance terms for a book of business, an insurer can determine an indifference price such that the insurer does not care whether it is ceding the business or keeping it. The cedent must be willing to make a fair offer of reinsurance to the reinsurer. The striking price for reinsurance can be determined using exponential utility theory as a guide. The retention can be the most cost effective one of a group tested.

The Mathematics of Utility

Suppose an insured with wealth a is given a choice of self-insuring completely a loss process X or paying a gross premium G for full coverage. Assume the insured has a linear utility attitude so that $u(x) = bx + d$.

To determine G, we solve the general equation $u(a - G) = E(u(a - X))$. The utility of net wealth after insurance must equate to the expectation of the utility of wealth without insurance. From our expression $bx + d$, we substitute $a - G$ and $a - X$ respectively for x, and get:

$$\begin{aligned} b(a - G) + d &= E(b(a - X) + d) \\ &= b(a - E(X)) + d \\ &= b(a - m) + d, \text{ where } E(X) = m, \text{ the mean expected losses} \\ G &= m \end{aligned}$$

Recall I said linear utility implied risk indifference. In this case an insured would pay no more than expected losses to relieve himself of the uncertain loss process.

Now suppose the insured's utility function is exponential so $u(x) = (1/r)(1 - \exp(-rx))$. Let us modify this somewhat. Let us make the process X negative so the function relates to losses. Let us also negate the entire expression and speak of the disutility (Du) of losses (See Cozzolino). In this case, G is given by:

$$\begin{aligned} Du\,(a - G) &= E(u(a - X)) \\ -(1/r)\,(1 - \exp(r(a - G))) &= E(-(1/r)\,(1 - \exp(r(a - X)))) \\ &= -(1/r)\,(1 - E\,(\exp(r(a - X)))) \\ \exp(r(a - G)) &= E(\exp(r(a - X))) \end{aligned}$$

$$\exp(-rG) = E(\exp(-rX))$$
$$G = -(1/r)\ln E(\exp(-rX))$$
$$G = (1/r)\ln E(\exp(rX)) \quad \text{translated back!}$$

To make this arithmetically workable, we can take the claim size distribution and separate it into n partitions, if necessary, each with probability p_i. Then if we assume a uniform distribution over the interval (x_i, x_{i+1}), the risk-adjusted severity is given by the following formula:

$$G = \frac{1}{r}\ln\left[\sum_{i=1}^{n}\frac{p_i}{x_{i+1}-x_i}\cdot\frac{(\exp(r^{x_{i+1}}) - \exp(r^{x_i}))}{r}\right]$$

It is now only necessary to bring in the frequency distribution. Let k represent the number of claims. Then the risk premium adjusted for frequency and severity equals:

$$G' = \frac{1}{r}\ln\left(\sum_{k=0}^{\infty} p(k)\exp(krG)\right)$$

In the case where frequency is Poisson distributed with parameter k, we have $G' = (k/r)(\exp(rG) - 1)$. If frequency is distributed according to the negative binomial with parameters p and b, (mean $b(1-p)/p$, variance $b(1-p)/p^2$) then $G' = (b/r)\ln(p/(1-(1-p)\exp(rG)))$.

At this point an illustration is in order. Suppose a property owner has a utility function $u(x) = \exp(-.005x)$. Further suppose there is a 1 in 10 chance of a property loss whose distribution is $f(x) = .10\,(.01\exp(-.01x))$. Then expected loss is given by:

$$E(X) = (.90)(0) + .10\int_0^\infty x\,(.01\exp(-.01x))\,dx = 10$$

Risk-adjusted premium, G' is given by:

$$u(a-G') = .90\,u(a) + \int_0^\infty u(a-x)\,f(x)dx$$
$$-\exp(-.005(a-G')) = -.90\exp(-.005a) - .10\int_0^\infty \exp(-.005(a-x))\,(.01\exp(-.01x))dx$$
$$\exp .005G' = .90 + (.10)\,(2)$$
$$G' = 200\ln(1.10)$$
$$G' = 19.06$$

The insured is willing to pay almost double expected losses because of the danger in the frequency/severity distributions coupled with his risk averseness.

A Test Case

Assume a hospital company writes only policy limits of \$5 million. According to recent ISO increased limit studies, losses can be modeled by a shifted Pareto distribution.[30]

The following chart provides a representative example of average severities:

Policy Limit	Average Loss	Average Allocated Loss Expense	Sum
$ 250,000	$ 54,402	$15,000	$ 69,402
$5,000,000	$112,227	$15,000	$127,227

Further, assume a claim frequency of .006 against 16,667 occupied beds, producing 100 expected claims. If acquisition; general expenses; taxes, licenses, fees; and profit amount to 25%, premium volume at $5 million limits should be:

$$100\ (127{,}227)/.75 = \$16{,}963{,}600$$

Expected losses in the $4,750,000 xs 250,000 layer (excluding pro rata allocated loss adjustment expenses) equal:

$$100\ (127{,}227 - 69{,}402) = \$5{,}782{,}500 \text{ and}$$

divided by the 10 claims over $250,000 implied by the shifted Pareto, yields an average loss in this layer of $578,250.

Now let us view the reinsurer's loss distribution. If we move the y-axis of the gross loss distribution over to the right to $250,000; we have a decreasing reinsurance loss function defined on the interval (0; $4,750,000). Let us assume it is nearly exponential. (For ease in calculus the tail is included.)

A characteristic of the exponential is that the mean, $578,250 here, is the reciprocal of the value r, so $r = 1.729 \times 10^{-6}$. The loss function is then given by:

$$f(x) = .10\ (.000001729 \exp(-.000001729x));\ x \text{ positive.}$$

Mean losses are given by:

$$E(X) = 100((.90)(0) + .10 \int_0^{4,750,000} .000001729 \exp(-.000001729\, x)dx)$$

$$= 100\ (0 + (.10)\ (578{,}250))$$

$$= \$5.782{,}500$$

[30] Insurance Services Office, "Report of the Increased Limits Subcommittee: A Review of Increased Limits Ratemaking," 1980.

Suppose the insurer has a utility function given by

$$u(x) = -\exp(-.00000025x)$$

Then,

$$-\exp(-.00000025(a - (G'/100))) = .90\,u(a) +$$
$$\int_0^{4,750,000} \exp(-.00000025(a - x))\,(.10)\,(.000001729\exp(-.000001729))dx$$

Dividing through by $u(a)$ gives

$$\exp(.00000025(G'/100))$$
$$= .90 + .10\,(.000001729)\int_0^{4,750,000} \exp((.00000025 - .000001729)x)dx$$
$$= .90 + .10(.000001729)\,(675,000)$$
$$= 1.0167$$

Finally, we have

$$\frac{G'}{100} = \frac{\ln(1.0167)}{.00000025} = 66,248$$

or

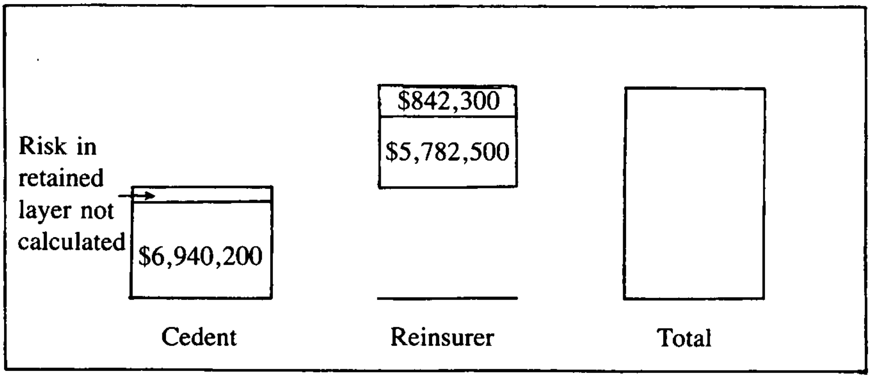
$$G' = \$6,624,800$$

(The appendix gives the framework of a more complete mathematical/statistical analysis.)

In this example, the reinsured should be willing to cede $4,750,000 xs 250,000 for $6,624,800 − 5,782,500 or $842,300 more than expected losses. If the reinsurer has the same utility function or is less risk averse, a deal can be struck. The reinsurer might express this as $16,963,600 (57,825/127,227) = $7,710,000 less a 14% ceding commission or $6,630,600. The $842,300 loading would have to cover all reinsurer operating expenses, including service, and a profit/risk charge.

This retention pricing example probably is not optimal. Other retention levels should be studied. Diagramatically, Figure 3 shows the first attempt loss costs.

An approach would be to minimize the sum of "Total" subject to a restriction on the risk proneness of the ceding company and a reasonably risk averse function for the reinsurer.



Each layer of loss by size will have certain frequencies. The insurer will calculate his risk load for each layer. The reinsurer, viewing the same data, may have similar pure premiums, but in any event, will also calculate risk loads. Depending on the optimism and risk proneness of the reinsurer, the ceding company may find layers cheaper (in terms of utility) to cede than retain. Other considerations will impact the purchased retention level (beyond the scope of this paper). Neither party has to know or attempt to negotiate the other's risk propensity. The bottom line will determine whether reinsurance is purchased, and at what level.

"r" Values

The question always rises, "How can management specify their risk aversion function?" Kalcek and McIntyre[31] begin to explore this. They suggest risk capital can be determined as: (a) 1 to 5% of annual working capital, (b) 1 to 3% of total assets, (c) 3 to 5% of annual earnings, or (d) 0.1 to 0.5% of annual sales. The rules of thumb come from the manufacturing environment; insurers might substitute cash on hand and cash flow for working capital. Other measures could be invented. Suppose we set a value on risk capital of x. We have a desire to

[31] K. Kalcek and W. McIntyre, *Financial Executive*, April, 1977.

bet x in annual adverse claim variability, but only lose it with small probability. Small companies must be aggressive with x as a percent of base capitalization, large companies would tend to select at range minimums. Risk capital may be defined loosely as an amount of money an insurer is prepared to lose in the case of unusual adverse claim variability.

From our exponential disutility (*Du*) function, we can take the first derivative. The slope constantly increases. Suppose we set our risk-reward level at 10:1. One dollar is worth the risk of 10. By analogy, horse race handicappers don't generally bet on "sure things." If a win ticket for \$2 will pay \$2.20, they won't bet \$10 to win \$11. The risk-reward is judgmentally poor.

Suppose, for a small company, x must be \$1 million. We can then calculate the r value.

$$Du(x) = -(1/r)\,(1-\exp(-rx))$$

$$\frac{d}{dx} Du(x) = -\exp\,(-rx)$$

$$10 = -\exp\,(-1{,}000{,}000\,r)$$

$$(\ln 10)/1{,}000{,}000 = r$$

$$r = .000002$$

We can also use a polling technique. By interviewing management, we can determine risk propensity. Ask what premium management would charge for several loss/no-loss situations, then graph expected payoffs (abscissa) against premium (ordinate). For example, "How much would you pay for a lottery ticket with a .001 chance of winning \$1 million?" Although the expected value is \$1000, the risk avoider might pay only \$500. If the question were asked, however, in the disutility context where there is a .001 chance of losing \$1 million, he might say \$1500. By getting premiums for a wide variety of expected payoffs, utility or disutility curves can be constructed.

A third method for determining r is to perform price and resulting earnings studies based on a variety of r values near zero. A company's earnings target, coupled with a business mix, can lead to an implied r value.

Risk Assessment

Once an appropriate excess of loss retention is determined, underwriters and actuaries can meet to discuss pricing techniques. Proposed treaty rates must be assessed both analytically and judgmentally. The pricing method previously described is completely analytic once utility is specified. This price indication can be compared with both empirical and exposure methods. Empirically, the company would have a history of observed losses per exposure unit by layer (after trend and development/IBNR). The gross price for insurance can also be layered by exposure. National Council ELPF's, ISO increased limit factors, and property distributions such as published by Salzmann[32] are useful.

If no credible past data exist, reinsurance collective experience and judgment are used to rate the account. It is of great benefit to reinsure or quote on many state doctor and hospital companies. Each lacks complete credibility, but collective experience fills whatever gaps exist. When a totally new risk presents itself, such as in 1973 New Jersey no-fault excess of loss coverage, reinsurers price by analogy. This no-fault should be similar to a combination of first party long-tail workers' compensation and automobile liability/medical payments.

Only after actuarial, claims, and underwriting personnel have evaluated the company's experience does a responsible quotation emerge.

Conclusion

It is not surprising that reinsurance has received little mathematical attention until lately in the *Proceedings*. Until recently, there have been but a handful of actuarial practitioners in the field. Mathematical and statistical tools, such as utility theory, were not studied in the U.S. for application to reinsurance. Utility theory, I believe, is a key to understanding which reinsurance forms make sense and what retentions are desirable.

Throughout history, reinsurance has operated along traditional lines. Excess of loss reinsurance is very popular today. The burning question is, "What retention is appropriate for my business and how much should reinsurance be worth to me?" This essay primarily attempts to seek an analytical solution to an otherwise judgmental decision. (Two examples were given, an individual property risk and a portfolio of hospital bed exposures.) By setting limits on retained loss variability (as measured by utility) a natural consequence is excess cession, and furthermore excess pure loss cost and risk charge. No attempt has been made to define a corporate utility function but several curves have been noted and insight given in how to interpret and use them.

My thanks go to William Weimer for extending my example. He eliminated the constraint that reinsurance frequency of loss be constant and I am grateful to him for the mathematics expressed in the appendix.

[32] R. Salzmann, "Rating by Layer of Insurance," *PCAS* L, 1963.

APPENDIX—FREQUENCY IN UTILITY CALCULATIONS

We can formalize the mathematical structure of the hospital example stated earlier. Specifically, we can eliminate the assumption of a constant number of excess $250,000 claims. Our choice of a Poisson frequency distribution will provide an elegant path to follow. In a collective risk theory framework, this will be a derivation using a particular frequency distribution and a particular severity distribution. We hope that after reviewing this example, the reader will gain more insight into the general formulas stated in the mathematical section and will be able to apply them with distributions of his or her choice.

We make the following assumptions:

Total losses (each is the excess of $250,000 portion)

$$X = X_1 + X_2 + \cdots + X_N$$

Frequency of claims distribution: Poisson (h) with $h = 10$

$$P(N = n) = \exp(-h)\, h^n/n!;\ n = 0,1,2,\ldots$$

Severity of claims distribution: Exponential with mean = $5,782,500

$$f(x) = s \exp(-sx);\ x > 0 \text{ and } s = .000\,001\,729$$

Utility function:

$$u(x) = -\exp(-rx);\ r = .000\,000\,25$$

Initial Net Worth = a

With these assumptions, the hospital company should be willing to pay an amount G' for a $250,000 excess of loss cover, where G' satisfies the equation:

$$u(a - G') = E(u(a - X)).$$

The "no memory" property of $u(x)$ leaves us with:

$$\begin{aligned}
\exp(rG') &= E(\exp(rX)) \\
&= E(E(\exp(rX)/N)) \\
&= \sum_n P(N = n)\, E(\exp(r(X_1 + X_2 + \cdots + X_n))) \\
&= \sum_n P(N = n)\, (s/(s - r))^n
\end{aligned}$$

$$= \sum_n (\exp(-h)h^n/n!)\ (s/(s - r))^n$$

$$= \exp(-h) \sum_n (hs/(s - r))^n/n!$$

Solving for G' gives $G' = (h/r)\ ((s/(s - r)) - 1)$.
Replacing h, s, and r with their selected values leaves $G' = \$6{,}761{,}325$.

We see that by letting the frequency vary, we are introducing more uncertainty into our problem, and the premium G' has gone up from \$6,624,800. (Actually, some of the increase, \$41,578, is due to the severity distribution no longer being truncated.)

DURATION

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Abstract

This paper begins by highlighting some of the changes in the macroeconomic environment that have affected the way insurers and reinsurers price their products. Attention is next focused on the importance of the time value of money for certain insurance products. The next topic is reinvestment risk and the ways investors try to deal with this problem. Working with fixed income securities, immunization theory and, in particular, duration are discussed. Duration is the word given to the statistic derived by weighting each year (of the bond's life) by the present value of the associated cash flows; all aggregated and divided by the price of the bond. Duration provides a theory or framework that the investor can use to more or less guarantee (as far as the investment or reinvestment risks are concerned) a targeted wealth position.

Today's economic environment has caused all of us to think a lot more about the investment side of the insurance and reinsurance business. While the time value of money has always been at least an implicit factor in insurance and reinsurance pricing, it has now, for better or worse, come out of the closet. Certain insurance products—structured settlements and loss portfolio transfers, for example—put the investment issue on center stage.

The pricing of some products is, in part, dependent upon an accurate determination of the present value of the dollars that will ultimately be paid out. For example, a structured settlement, made in lieu of a lump sum claim payment, guarantees that a specified number of dollars will be paid to the claimant at specified intervals for life or for a determined number of years. A loss portfolio transfer involves the transfer or reinsurance of a defined block or portfolio of known and/or unknown losses from one party to another. A present value is assigned to the portfolio of losses; pricing is largely a function of this estimated present value. Once the ultimate value of the structured settlement or loss portfolio transfer has been determined (estimated), the insurer normally purchases one or more bonds that will fund the claim payments. As we will see, the concept of "duration" is useful in attempting to harmonize the liabilities and the related invested assets.

A factor that makes the time value of money both critical and hard to deal with is that we are living in turbulent financial times. One set of statistics that highlights this is the frequency of prime rate changes. The prime rate changed only twice in the 1940's; 16 times in the 1950's; 17 times in the 1960's; 132 times in the 1970's, and so far, with over two years of the 1980's finished, we've already seen over 70 changes! [1]

Since the time value of money is now an explicit and important part of some of our pricing and the financial environment is unstable or volatile, we will have to pay more attention to the investment side of the business. It is recognized that there are many factors that influence the rate of return of an insurance program. For life insurance, the principal underwriting risks are the mortality, lapse and long term expense assumptions. Property/casualty insurers face critical frequency and severity assumptions, both confounded by price and social inflation.

Once a view is taken on these assumptions, no matter how certain or uncertain they may be, there remain timing (i.e., the actual incidence of payments) and investment risks. This paper does not address underwriting risk (i.e., in the context of this paper, the expected ultimate loss level) or the timing of the related payments. These are important but separate topics. Similarly, this paper does not address the important topic of profit and contingency loadings.

In pricing the financial, as opposed to underwriting, aspect of an insurance or reinsurance arrangement, the first step is usually to calculate a breakeven or internal rate of return. The internal rate of return or yield for an investment is the discount rate that equates the present value of the expected cash outflow with the present value of the expected inflows [2][1]. Next, an attempt is made to design an investment program to produce a return greater than the breakeven or internal rate of return. Obviously there would be little or no incentive to write the business if the internal rate of return cannot be exceeded with an acceptable level of investment risk.

There are, of course, a number of financial risks or reasons why the targeted return might not be achieved. The principal investment related risks are: *timing* (i.e., the timing or incidence of the cash outflows, which might be very predictable for auto physical damage, for example, but quite uncertain for the so-called long tail lines); *credit risk* (i.e., default as to interest and principal) which won't be treated in this paper; and *reinvestment risk* which will be discussed.

Our company recently had a submission that highlighted reinvestment risk in a dramatic way. Stripped of nonessential features and somewhat disguised, the proposal involved a single premium, paid in advance, in return for a commitment to pay \$100,000,000 at the end of 20 years. Assuming the money is invested at 12% per annum and ignoring the credit risk, profit, overhead, taxes, arbitrage opportunities, etc., the price or "pure premium" is $\$100,000,000/(1.12)^{20}$ or \$10,366,677.

How do we get from \$10,366,677 to \$100,000,000? Let's assume we buy a bond for \$10,366,677 (assume cost = par value = redemption value) with a 12% annual coupon. At the end of 20 years we have:

Our \$10,366,677	(redemption)
and	
\$24,880,023	(20 years of interest, at a 12% coupon)

[1] In mathematical terms, r = the internal rate of return where CF_t is the cash flow for period t such that

$$\sum_{t=0}^{n}\left[\frac{CF_t}{(1+r)^t}\right] = 0$$

(A discussion of the possible shortcomings of the Internal Rate of Return technique can be found in *Financial Theory and Corporate Policy*, by Thomas E. Copeland and J. Fred Weston, Reading, Massachusetts: Addison-Wesley Publishing Company, 1979 edition, pages 28–33).

The rest, a staggering $64,753,300, comes from interest on interest or reinvestment! Put another way, 72% of the return comes from the interest earned on the interest.[2] Clearly, reinvestment can make or break an insurance program. Let's say upon reinvestment we get 10%—not 12%—we'll then be short of the goal by $18,383,000. Of course, rates could go the other way, say to 14%, in which case we would have a windfall profit of $23,602,000.

Volatile interest rates and the recognition of the time value of money in insurance and reinsurance pricing make it appropriate for us to put a new word in our vocabulary—immunization. Unfortunately, the word immunization isn't always used precisely and, perhaps, it means different things to different people. The definition I offer is: the investment risks are immunized if the desired wealth level (of the investment portfolio) has been achieved at the end of the investment horizon (i.e., holding period) regardless of interest rate changes during the holding period. This, of course, implies that all intervening cash flows during the holding period have been met.

Immunization, although much talked about today, is not a new concept. Some fairly sophisticated work was done on immunization theory at least as far back as 1938 by Mr. Frederick R. Macauley [3]. The earliest traces in the actuarial literature date to 1952 when a British life insurance actuary, Mr. Frank Reddington [4], suggested that insurance companies really ought to think about synchronizing their investments and underwriting risks. In this country, the cause has been championed, for the last 10 years or so, by Mr. Irwin T. Vanderhoof, FSA, ACAS [5].

To understand immunization techniques one needs to understand the several ways bonds can be characterized. First, there is the simple, but not very useful, notion of years or term to maturity. This is self-explanatory—a bond maturing in 2002 has a 20 year maturity as of 1982. As demonstrated in the above example, buying a 20 year bond to cover a liability maturing in 2002 does not immunize one from the "disease" of changing interest rates. In a period of volatile interest rates, characterizing a bond as a 20 or 30 or whatever years to maturity really isn't very useful.

Recognition of the fact that the years to maturity isn't a useful way to describe a bond has led to another measure known as the weighted term or years to maturity. Under this approach *all* cash flows occurring over the life of

[2] See Appendix I for some other examples.

the bond (i.e., interest coupons and the redemption value) are used as weights for the year involved. Put on a formula basis:[3]

$$\text{Weighted average term to maturity} = \frac{CF_1 \cdot 1}{\Sigma CF} + \frac{CF_2 \cdot 2}{\Sigma CF} + \frac{CF_3 \cdot 3}{\Sigma CF} \ldots ; \frac{CF_n \cdot n}{\Sigma CF} \text{ or,}$$

$$\frac{\sum_{t=1}^{n} CF_t(t)}{\sum_{t=1}^{n} CF_t},$$

where

t = year of cash flow (i.e., year 1, year 2, etc.)
CF_t = cash flow in year t
n = number of years to maturity

Thus, a 10 year bond with a 4% coupon would have a weighted average term to maturity of 8.71 years while the same type of bond with an 8% coupon would have a weighted average term to maturity of 8 years (see Table A below). Since this measure recognizes the cash flow differences between the bonds, it is somewhat more useful than the years to maturity in determining a portfolio's overall sensitivity to changing interest rates. Although a better measure than years to maturity, weighted average years to maturity doesn't have (much) operational significance. The problem with this measure is that each dollar has equal weight; that is, the time value is not considered.

The quest for immunization has led to an even more sophisticated and more useful concept known as "duration" of the bond. Duration is a measure of a bond's price volatility. Thus, duration can be derived using differential calculus (see Appendix II).

In simpler terms, duration is a weighted average term to maturity where the years are weighted by the present value of the related cash flow.

[3] In this and all other formulas in this paper, annual end-of-year interest payments have been assumed. It would be relatively easy to modify the formulas to accommodate the more typical mode of semi-annual interest payments.

TABLE A [6]

Bond A—$1,000 face value with a 4% coupon, maturing in 10 years
Bond B—$1,000 face value with a 8% coupon, maturing in 10 years

Weighted Average Term to Maturity (assuming annual interest payments)

Bond A

(1) Year	(2) Cash Flow	(3) Cash Flow/TCF	(4) (1) × (3)
1	$ 40	0.02857	0.02857
2	40	0.02857	0.05714
3	40	0.02857	0.08571
4	40	0.02857	0.11428
5	40	0.02857	0.14285
6	40	0.02857	0.17142
7	40	0.02857	0.19999
8	40	0.02857	0.22856
9	40	0.02857	0.25713
10	1,040	0.74286	7.42860
Sum	$1,400	1.00000	8.71425

Weighted Average Term to Maturity—8.71 Years

Bond B

(1) Year	(2) Cash Flow	(3) Cash Flow/TCF	(4) (1) × (3)
1	$ 80	0.04444	0.04444
2	80	0.04444	0.08888
3	80	0.04444	0.13332
4	80	0.04444	0.17776
5	80	0.04444	0.22220
6	80	0.04444	0.26664
7	80	0.04444	0.31108
8	80	0.04444	0.35552
9	80	0.04444	0.39996
10	1,080	0.60000	6.00000
Sum	$1,800	1.00000	7.99980

Weighted Average Term to Maturity—8.00 Years

$$\text{Duration} = \frac{\displaystyle\sum_{t=1}^{n} \frac{t \cdot CF_t}{(1+y)^t}}{\displaystyle\sum_{t=1}^{n} \frac{CF_t}{(1+y)^t}}$$

CF_t = cash flow in year t
y = yield to maturity[4] (not the coupon rate)
t = year of cash flow
n = number of years to maturity.

Using the above formulas, for example, the 8% bond (maturing in 10 years) discussed above has a duration of 7.25 years compared with a weighted average term to maturity of 8.0 years. (See Table B.)[5]

There are a couple of other ways to compute a bond's duration (see Appendix II for a discussion of volatility and duration), one of which is a crude, but useful shortcut/approximation. If the coupon rate is fairly close to the yield to maturity, say 70% or more, the duration can be very roughly approximated as $1/y + 1$, the formula for a perpetuity (See Appendix II, Section II, Equation 10).

Duration can do some very interesting and wonderful things for the investor seeking to achieve a certain ultimate return or wealth level. Duration allows the interest rate risk (i.e., reinvestment) to be balanced with the price or capital

[4] There are three different yields associated with a bond

1. *Nominal yield* is the ratio of interest to principal (without regard to compounding). This is also called the coupon rate.
2. *Current yield* is the ratio of interest to the amount actually paid for the bond. The current yield overstates the return on premium bonds and understates the return on discount bonds.
3. *Yield to maturity*—sometimes called the net yield to maturity, takes into account all cash flows associated with the bond, i.e., the amount paid, the interest and redemption amounts to be received if the bond is held to maturity.

   $$\text{Price} = \sum_{t=0}^{n} \frac{CF_t}{(1+r)^t}$$

   r is the yield to maturity

[5] See Appendix V for a replication of the Table B Duration Values and a simple program to calculate a bond's duration.

TABLE B [7]
DURATION OF A BOND

DURATION (ASSUMING EIGHT PER CENT MARKET YIELD)

BOND A—4% COUPON

(1) Year	(2) Cash Flow	(3) PV at 8%	(4) PV of Flow	(5) PV = % of Price	(6) (1) × (3)
1	$ 40	0.9259	$ 37.04	0.0506	0.0506
2	40	0.8573	34.29	0.0469	0.0936
3	40	0.7936	31.75	0.0434	0.1302
4	40	0.7350	29.40	0.0402	0.1606
5	40	0.6806	27.22	0.0372	0.1860
6	40	0.6302	25.21	0.0345	0.2070
7	40	0.5835	23.34	0.0319	0.2233
8	40	0.5403	21.61	0.0295	0.2360
9	40	0.5002	20.01	0.0274	0.2466
10	1,040	0.4632	481.73	0.6585	6.5850
Sum			$731.58	1.0000	8.1193

Duration—8.12 Years

BOND B—8% COUPON

(1) Year	(2) Cash Flow	(3) PV at 8%	(4) PV of Flow	(5) PV = % of Price	(6) (1) × (3)
1	$ 80	0.9259	$ 74.07	0.0741	0.0741
2	80	0.8573	68.59	0.0686	0.1372
3	80	0.7938	63.50	0.0635	0.1906
4	80	0.7350	55.80	0.0588	0.1906
5	80	0.6806	54.44	0.0544	0.2720
6	80	0.6302	50.42	0.0504	0.3024
7	80	0.5835	46.68	0.0467	0.3269
8	80	0.5403	43.22	0.0432	0.3456
9	80	0.5002	40.02	0.0400	0.3600
10	1,080	0.4632	500.26	0.5003	5.0030
Sum			$1000.00	1.000	7.2470

Duration—7.25 Years

risk. This balancing arises out of the *inverse*[6] relationship between interest rate or reinvestment risk and price risk.

The zero coupon bonds that have recently become fashionable may help to illustrate duration.[7] An obligation to fund a liability of known proportion at the end of 10 years would be totally satisfied by a 10 year zero coupon bond; that is, there is no reinvestment risk. It doesn't matter what happens to interest rates if the only goal is to exactly achieve a certain wealth position at the end of the holding period. Put another way, since a zero coupon bond has no interim cash flow, its *term to maturity* is equal to its *weighted term to maturity* which is also equal to its *duration*.

It turns out that, for several reasons, zero coupon bonds are not a panacea.

- Zero coupon bonds are not as yet widely available.
- Zero coupon bonds may not be available at a credit risk level that suits the investor. (Also note that the *entire* credit risk is "stacked" at the redemption date.)
- Zero coupon bonds, other things being equal, carry a slight premium.
- Tax exempt institutions (such as pension funds) are currently the major investors in these bonds. The tax implication of zero coupon bonds can—to certain investors—be onerous (i.e., the bond owner is subject to tax on income which is accrued—not received).[8]

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[6] Other things being equal, as interest rates rise the price or market value of fixed rate bonds decline and as interest rates decline the price or market value of fixed rate bonds rises.

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[7] According to George L. Shinn's article, "Innovative Approaches to Financing" appearing in the Winter 81/82 issue of *Chief Executive*, J. C. Penney Inc. issued the first public zero coupon bond in April, 1981.

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[8] The obverse of this coin is, of course, a great attraction to the bond issuer, but the IRS wants to spoil the game—a little. See page 43 of the May 5, 1982 edition of the *Wall Street Journal*. Currently the issuer takes a deduction on a pro-rata or equal installment accrual of interest. The proposed IRS change, which will require Congressional approval, will reflect the compounding of accrued interest.

Example: 30 year $1,000 bond, purchased for $50, yielding 10.5%

	Deductible Interest	
	Current tax basis	Proposed
Year 1	$950/30 = $31.67	$50.00 × .105 = $5.25
Year 2	$950/30 = $31.67	$55.25 × .105 = $5.80

It seems clear that such a change in the tax law will, other things being equal, reduce the enthusiasm of would-be zero coupon bond issuers.

An interesting recent development strikes at the first two shortcomings: availability and credit risk. In July of 1982, Merrill Lynch brought to market a cleverly designed new product called TIGR's (Treasury Investment Growth Receipts) [8]. Other investment bankers have since followed suit.

TIGR's are treasury bonds repackaged to look and behave like zero coupon bonds. The new tax act, Tax Equity and Fiscal Responsibility Bill of 1982, effective July 1, 1983, added I.R.C. Section 1232B, which prescribes the tax treatment for bonds that have been stripped (i.e., for which the interest coupons have been separated). An investment bank might, for example, buy a 25-year bond, strip out the 50 interest coupons yielding 51 zero coupon bonds (including the bond itself). The "mini" zero coupon bonds are kept in a custodian bank. The investor gets a receipt as evidence of his claim on the securities.

The last shortcoming, taxes, may yet be solved. Some state and local housing authorities have been issuing zero coupon municipal bonds. Most of the issues so far have call features, thus taking away one of the presumed advantages of zero coupon bonds and Original Issue Discount bonds. (According to Woolridge and Gray, Original Issue Discount, "OID", bonds of which zero coupon is the extreme case are priced to yield as much as 100 basis points less than otherwise comparable full coupon bonds. They offer two reasons: non-callability and immunization. [9])

Working with bond durations, one can achieve nearly the same immunized result offered by zero coupon bonds. In other words, if a company has a 10 year obligation and invests in a bond with a duration of 10 years, which may in fact involve a bond with a term to maturity of 18 years (for example), the return/wealth would be immunized. This happy result comes about because of the counterbalancing of interest rate risk and price risk. In other words, if interest rates go down, the investment return is less than anticipated but there is a counterbalancing capital gain in the market price of the bond. Put another way, by buying a bond with a longer than apparently needed term to maturity—but with the right duration—the investor creates an interest sensitive overhang (i.e., the difference between the duration and the term to maturity) on the bond which is engineered to the right proportions. Consider the following examples in Table C:

TABLE C [10]

REALIZED RETURN FROM A 5-YEAR 9% PAR BOND OVER VARIOUS HORIZON PERIODS

Reinvestment Rate and Yield-to-Maturity At Horizon		Horizon Period			
		1 Year	3 Years	4.13 Years	5 Years
	Coupon Income	$90	$270	$372	$450
7%	Capital Gain	$68	$37	$16	$0
	Interest-On-Interest	$2	$25	$51	$78
	Total Dollar Return	$160	$331	$439	$528
	Realized Compound Yield	*15.43%*	*9.77%*	*9.00%*	*8.66%*
9%	Capital Gain	$0	$0	$0	$0
	Interest-On-Interest	$2	$32	$67	$103
	Total Dollar Return	$92	$302	$439	$553
	Realized Compound Yield	*9.00%*	*9.00%*	*9.00%*	*9.00%*
11%	Capital Gain	−$63	−$35	−$16	$0
	Interest-On-Interest	$2	$40	$83	$129
	Total Dollar Return	$29	$275	$439	$579
	Realized Compound Yield	*2.89%*	*8.26%*	*9.00%*	*9.36%*

Table C illustrates a striking compensation effect for investment periods of less than 5 years. For the 3-year period, at the 7% reinvestment rate assumption, the interest-on-interest naturally falls short of the amount required to support a target return of 9%. However, if the bond could be sold at the price corresponding to the assumed 7% yield-to-maturity rate, then a capital gain would be realized which would more than compensate for the lower value of interest-on-interest. Table C illustrates the well-known facts that over the short term, lower interest rates lead to increased returns through price appreciation while, over the longer term, lower interest rates lead to reduced returns through reduced interest-on-interest. For periods lying between the short term and the longer term, it is not surprising to find these two effects providing some compensation for each other.

Duration is particularly useful when pricing a single risk or insurance program. Indeed, it is probably the only way to immunize the investment risks of such individual undertakings. Duration can also be used by the insurance company as it aggregates risks and a corresponding portfolio of invested assets. The required duration of the portfolio could be computed as the weighted average of the various constituent durations or computed on an aggregate basis—by expected payout year (e.g., a certain block of assets with a duration of .5 for payments in the first year, 1.5 for payments in the second year, etc.).

While duration is an elegant and appealing concept, it is not without a few practical problems. First, it's nearly impossible to find bonds with a duration greater than 20 years.[9] Second, a bond's duration changes over time. For example, as one year of a 10 year holding period passes there remain 9 years on the obligation, but the duration has decreased only by perhaps 6/10 of a year. Thus, the liability and corresponding assets are out of synchronization and no longer immunized. Third, when interest rates change the bond duration also changes with the result that the investor is not immunized against *further* interest rate changes. (See Appendix III.) Fourth, sinking funds and call features can make the whole process fairly complicated. These four problems can, however, be overcome by constantly retuning or rebalancing the duration of assets. Fifth, transaction costs and taxes can be a drag on the immunization program.

The duration of the portfolio must be tracked—an ideal computer application—and the portfolio tuned as

1. interest payments become available for investment
2. bonds mature
3. time passes
4. market yields change, and
5. new liabilities are taken on.

The easiest way to envision the retuning is to sell the entire portfolio and reinvest at the new required/computed durations: not the most efficient approach, but easy to understand. Alternatively, retuning can be accomplished by merely shifting funds from longer term bonds to shorter term bonds to shorten the

[9] As noted earlier, duration is a specially weighted average term to maturity—the weights being the present value of the cash flows. A bond with a duration of 20 involves a term to maturity of 45 at 4% yield to maturity with a coupon of 6%, 70 years at 5% YTM. At a 6% YTM, the duration starts to converge on 17.65 at about 120 years. Since bonds are rarely issued with terms succeeding 35 years, it is nearly impossible to achieve a duration of more than 20 years with today's coupon rates and yields to maturity. See also Appendix II, section II and Appendix III Exhibit A2.

duration (the usual requirement as time passes) or vice versa. Simply reinvesting the coupons, as they are paid, in short term bonds may fund and accomplish the needed rebalancing. For a fuller discussion of portfolio rebalancing, see Gushee's article [11].

Another portfolio approach is to strive for a perfect matching of the expected payments and the interest income and redemptions from the investment portfolio. The usual motivation for matching of this type is solvency considerations. The National Association of Insurance Commissioners has, from time to time, expressed an interest in the matter and commissioned Tillinghast, Nelson and Warren, Inc. to study the idea and develop a program or protocol [12].

In theory, an immunized portfolio can be achieved by cash flow matching. This condition will be obtained only if all interest coupons go directly to loss payments so that there is *no* reinvestment exposure. Similarly, all loss payments and maturities have to be precisely matched. All in all, a difficult but not impossible task.

It would, of course, be easy to construct a cash flow matched portfolio using zero coupon bonds. While such a portfolio might be called a cash flow matched portfolio, it is in reality a duration based portfolio.

It is also possible to construct a perfect cash flow matched portfolio using conventional bonds—especially if the liability or payment stream is decreasing over time. If the payment stream is increasing or variable (i.e., up and down from year to year), it may not be possible to achieve perfect matching (i.e., avoid reinvestment risks).

Merrill Lynch and several other investment houses (Salomon Brothers, First Boston and Goldman Sachs, to name a few) have developed cash flow matching models. Under those systems, the customer specifies the "cash flow liability stream". The "system" accesses the firm's bond data base and develops a portfolio consistent with the customer's expected payment profile, the customer's credit risk appetite, and the customer's attitude toward call risk. (The greater the coupon, the greater the call risk. Put the other way around, bonds with lower coupons, other things being equal, will sell at a greater discount and hence afford greater call protection.)

A sample portfolio is set forth in Appendix IV. A study of Appendix IV reveals that all coupon interest goes directly to loss payments leaving no reinvestment risk. There is a very strong similarity or connection between a perfect cash flow matched portfolio and the TIGR's discussed earlier. Indeed, as can be seen from Appendix IV, a perfect cash flow matched portfolio is developed by taking a conventional bond and breaking it into n + 1 (i.e., n coupons plus a redemption value) constituent zero coupon bonds. Thus it can be seen that cash flow matching is actually a subset or type of duration.

In general, a matching program will keep maturities shorter than a duration program. This, of course, will produce a yield penalty in (normal) times of a positive yield curve.[10]

Cash flow matching can be a very useful concept—especially for the regulator. Depending on the objectives and circumstances, matching may or may not be a good investment strategy. Cash flow matching is difficult to achieve at the individual risk or program level (where there's a single payment or where the payments are so small that it would not be feasible to put together a portfolio of bonds.) Duration, on the other hand, can and does work both at the individual risk and portfolio level.

Summary

As Dr. Leibowitz has suggested, "the traditional motivation for bond investment was to secure a fixed cash flow over some appropriate time frame. The typical bond investor was highly risk averse. He was more than willing to sacrifice the excitement of potentially spectacular results in order to achieve a reasonably reliable pattern of return. However, in recent years, the traditional role of bonds as an asset category has been buffeted by a series of dramatic changes in the marketplace. Surging interest rates and an explosion in volatility have characterized recent markets. This environment imposes a harsh dilemma on the bond portfolio manager: how to pursue prudent active strategies and still provide his client with the comfort level that probably served as the primary basis for allocating funds to the fixed-income market in the first place?" [13]

In most insurance and reinsurance pricing, the reinvestment problem is a small, although not unimportant, element in the parcel of risks assumed. For the so-called long tail liability lines and life insurance, and specialty products

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[10] A yield curve is a plot of yields (usually on the ordinate) and maturities (usually on the abscissa) for bonds of comparable quality. The bonds differ as to maturities but need to be identical in creditworthiness, e.g., treasury bonds or bonds issued at the same time by the same issuer. The yield curve tells us what a knowledgeable investor requires, other things being equal, to commit for longer times (i.e., maturities). A yield curve is said to be positive or upward sloping if yields increase as maturities lengthen. Yield curves are normally positive reflecting the greater uncertainty and "risk" premium associated with longer maturities. There have been periods, notably in the early 1980's, when the yield curve has been negative (i.e., short duration money commands higher interest rates than long term money).

such as structured settlements and loss portfolio assumptions, the reinvestment risk is very important. Often, this additional risk element is either underestimated or, perhaps more often, simply ignored.

Sometimes an attempt is made to quantify the reinvestment risk, charge an additional premium for the risk, and use conventional investment techniques. In other words the actuary might price the business using yields derived from an immunized investment portfolio—even though the funds might be invested on another basis.

Generally it would be prudent to immunize or harmonize the liabilities and the related invested assets. As discussed in this paper, immunization can be achieved by (exact) cash flow matching or by tuning the investment portfolio to the appropriate duration.

Cash flow matching and duration are very useful concepts and are but the first steps (in a way, building blocks) in more sophisticated contemporary portfolio management. I believe actuaries as well as others in today's insurance company need to be more familiar with the management or harmonization of assets and liabilities. We also need to have some familiarity with the newer active (versus the traditional passive) portfolio management theories and techniques. It is hoped that this paper is a step, even if a modest one, in that direction.

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[15] John P. Rountree, III, "Duration as a Tool in Bond Valuation," *Fixed Income Research,* Kidder, Peabody and Co., Inc., January 23, 1980, page 1.

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[17] Macauley, op. cit.

[18] Leibowitz, op. cit., page 5.

[19] Rountree, op. cit., page 4.

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APPENDIX I

Magnitude of Interest-On-Interest To Achieve 9% Realized Compound Yield From 9% Par Bonds of Various Maturities [14]

Maturity In Years	Total Dollar Return	Interest-On-Interest At 9% Reinvestment Rate	Interest-On-Interest As Percentage Of Total Return
1	\$92	\$2	2.2%
2	193	13	6.5
3	302	32	10.7
4	422	62	14.7
5	553	103	18.6
7	852	222	26.1
10	1,412	512	36.2
20	4,816	3,016	62.6
30	13,027	10,327	79.3

APPENDIX II

Bond Price Volatility and Duration

An idea suggested by Rountree [15] and others is to get a different perspective on duration by developing a measure of bond price volatility. Consider the financial world's rule of thumb that one basis point change in yield for a long term coupon bond drives a ⅛% bond price change. This implies a duration of .125%/.01% or 12.5.

Studying bond price volatility leads to a more rigorous explanation/derivation of duration.

If the price of the bond is P,

$$P = \sum_{t=1}^{N} \frac{CF_t}{(1 + y)^t} \tag{1}$$

we can, using simple differential calculus, measure the change in the price of the bond related to a change in the yield:

$$\lim_{\Delta y \to 0} \frac{\Delta P}{\Delta y} = \frac{dP}{dy} = \sum_{t=1}^{N} \frac{CF_t}{(1 + y)^{t+1}} \cdot (-t) \tag{2}$$

Relating this to the price of the bond,

$$\lim_{\Delta y \to 0} \frac{\Delta P/\Delta y}{P} = \frac{\Sigma_{t=1}^{N} CF_t(-t)/(1 + y)^{t+1}}{\Sigma_{t=1}^{N} CF_t/(1 + y)^t}$$

$$= \left(\frac{1}{(1 + y)}\right) \frac{\Sigma_{t=1}^{N} CF_t(-t)/(1 + y)^t}{\Sigma_{t=1}^{N} CF_t/(1 + y)^t} \tag{3}$$

Thus, it can be seen that duration is a function of bond price volatility

$$\lim_{\Delta y \to 0} \frac{\Delta P/\Delta y}{P} = \frac{-1}{1 + y} \cdot \text{Duration} \tag{4}$$

Put another way, the relation between the duration of a bond and its price volatility (as set forth by Hopewell and Kaufman [16], is:

$$\lim_{\Delta y \to 0} \frac{\Delta P}{P} = -D^*(\Delta y) \tag{5}$$

where

$\frac{\Delta P}{P}$ = the % change in bond price

D^* = the adjusted duration of the bond in years which is equal to $\left(\frac{1}{1 + y}\right) D$, and

Δy = the change in the market yield.

Rearranging, we get:

$$\lim_{\Delta y \to 0} \frac{\Delta P/\Delta y}{P} = -D^* = \left(\frac{1}{1 + y}\right) \cdot -D = \left(\frac{1}{1 + y}\right) \frac{\Sigma_{t=1}^{N} CF_t(-t)/(1 + y)^t}{\Sigma_{t=1}^{N} CF_t/(1 + y)^t} \tag{6}$$

as in (3).

In practice, the unadjusted duration figure (D) is used when computing the impact of market rate changes. Without the factor $(1/1 + y)$ we have the time-weighted "corporate average-life" formula with each payment period weighted by its present value discount factor as originally proposed by Macauley. [17]

Thus,

$$D^* = \lim_{\Delta y \to 0} \frac{-\Delta P/\Delta y}{P} = \frac{-dP/dy}{P} = \frac{\Sigma_1^N CF_t(t)/(1 + y)^t}{\Sigma_1^N CF_t/(1 + y)^t} \tag{7}$$

Consider a discrete case:

$$D^* = \frac{-\Delta P/\Delta y}{P} = \frac{-\Delta P}{P} \cdot \frac{1}{\Delta y} = \frac{-\text{ price change}}{\text{price}} \cdot \frac{1}{\text{yield change}} \tag{8}$$

(from 5), or

$$D^* = \frac{-(P_2 - P_1)}{(P_1 + P_2)/2} \cdot \frac{1}{y_2 - y_1} = \frac{2(P_1 - P_2)}{P_1 + P_2} \cdot \frac{1}{y_2 - y_1}$$

Example: A 10 year bond with an 8% coupon (at the end of the year)

$P_1 =$ 99.9665 $\quad Y_1 = .08005$

$P_2 =$ 100.0336 $\quad Y_2 = .07995$

The adjusted duration equals 6.71 From (8)

Duration = 7.247 as in Table A, Bond B. From (4)

APPENDIX II—SECTION II

Consider a perpetuity (of \$1)

$$P = \frac{1}{y} \tag{9}$$

$$\frac{dP}{dy} = \frac{-1}{y^2}$$

$$D^* = \frac{-dP/dy}{P} = \frac{1/y^2}{1/y} = \frac{1}{y}$$

$$D = \frac{1}{y} + 1 \tag{10}$$

Thus for very long term bonds, $1/y + 1$ may be a good approximation for duration.

This exercise also sheds some light on a comment in the text that it's not possible with high yields for a bond to have a duration of 20 years or longer. Formula (10) would suggest that durations of 20 or more years can only be achieved when y, the yield, is less than 5.26%.

APPENDIX III

EXHIBIT A1

DURATION OF VARIOUS BONDS ALL PRICED TO YIELD 9% [18]

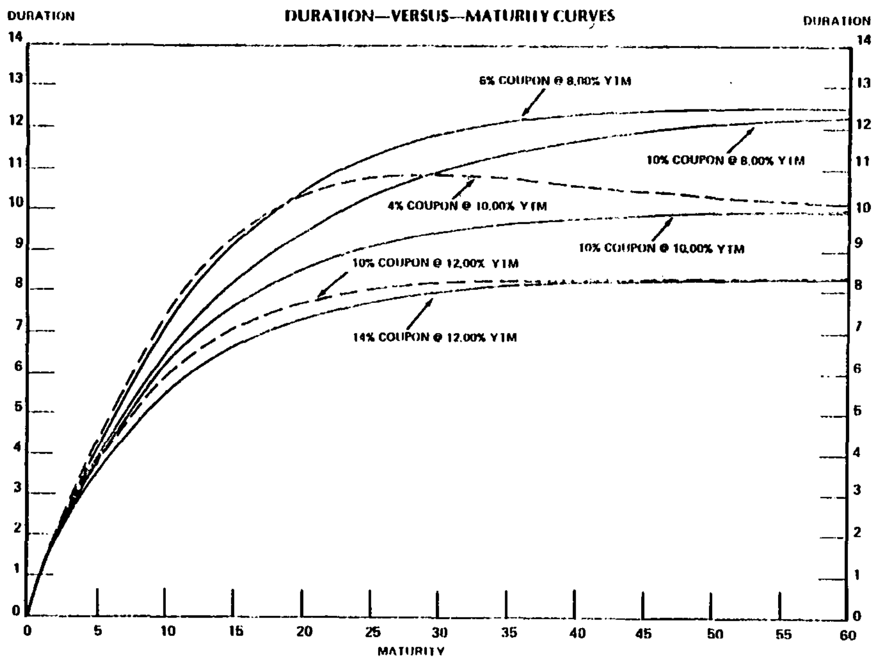
Maturity in Years	Coupon			
	0%	7.5%	9.0%	10.50%
1	1.00	0.98	0.98	0.98
2	2.00	1.89	1.87	1.86
3	3.00	2.74	2.70	2.66
4	4.00	3.51	3.45	3.38
5	5.00	4.23	4.13	4.05
7	7.00	5.50	5.34	5.20
10	10.00	7.04	6.80	6.59
20	20.00	9.96	9.61	9.35
30	30.00	11.05	10.78	10.59
100	100.00	11.61	11.61	11.61

Ex. A1 shows the *Duration* of various bonds. Returning to the original objective of providing an assured 9% target return over a 5-year period, we can see that one should choose a bond having a *Duration* of 5 years (as opposed to a maturity of 5 years!)

To obtain a *Duration* of 5-years in a 9% par bond, it turns out that one would need a maturity of around 6.3-years.

EXHIBIT A2

DURATION-VERSUS-MATURITY CURVES [19]



APPENDIX IV

Declining Liability Stream [20]

Year	Benefit Payment Required	Cash Flow Matched Portfolio: Par	Name	Coupon	Maturity	Cost	Cash flow
1981	$1,260,000	$505,000	Mt. States Tel.	8.700	9/1/1981	$485,000	$1,260,662.00
1982	1,219,000	528,000	Norway	7.500	6/15/1982	486,615	1,219,927.00
1983	1,175,000	503,000	Australia	8.125	11/15/1983	447,473	1,175,127.00
1984	1,127,000	522,000	Export Dev. Bank	9.850	1/15/1984	475,542	1,127,549.75
1985	1,078,000	521,000	Manuf. Han. Tr.	8.500	6/1/1985	453,270	1,078,699.25
1986	1,026,000	516,000	Sweden	9.500	4/15/1986	448,941	1,027,047.25
1987	974,000	514,000	European Invest. Bk.	9.875	6/1/1987	445,828	975,158.50
1988	919,000	484,000	Ford Mtr. Credit	8.250	11/1/1988	353,823	919,779.75
1989	864,000	493,000	Trailer Train	10.000	5/15/1989	403,023	864,199.75
1990	807,000	461,000	GMAC	7.125	12/1/1990	328,850	807,549.75
1991	748,000	435,000	Ford Mtr. Credit	7.500	11/15/1991	298,815	748,703.50
1992	688,000	424,000	Commercial Credit	7.750	2/15/1992	281,544	688,648.50
1993	628,000	396,000	Heller	7.750	4/1/1993	265,724	628,873.50
1994	567,000	350,000	GMAC	7.750	10/1/1994	238,763	567,528.50
1995	508,000	318,000	Ohio Edison	8.750	9/1/1995	212,427	508,403.50
1996	451,000	289,000	Phil Electric	8.250	8/1/1996	184,870	451,578.50
1997	396,000	269,000	H. F. C.	8.450	1/15/1997	180,351	396,370.75
1998	344,000	239,000	Canada	8.625	4/1/1998	173,784	344,698.63
1999	296,000	201,000	New England Pwr.	8.375	9/1/1999	131,138	296,391.75
2000	252,000	174,000	Carolina P & L	8.750	8/1/2000	118,673	252,558.00
2001	212,000	156,000	Public Svc. Ele.	8.375	5/15/2001	105,030	212,801.00
2002	176,000	126,000	So. Cal. Edison	8.250	7/1/2002	82,618	176,269.00
2003	144,000	105,000	Long Island Light	8.125	12/1/2003	62,174	144,874.00
2004	117,000	90,000	Florida P & L	8.500	1/1/2004	60,050	117,517.50
2005	93,000	73,000	Commonwealth Edison	8.750	3/1/2005	47,451	93,499.00
2006	73,000	56,000	Duke Power	8.375	10/1/2006	36,047	73,305.25
2007	56,000	46,000	Ontario Prov.	8.400	1/15/2007	31,037	56,683.25
2008	42,000	34,000	Central P & L	8.875	9/1/2008	23,514	42,751.25
2009	31,000	27,000	Ches. Pot. Tel.	8.875	6/1/2009	19,412	31,535.63
2010	23,000	21,000	Ohio Bell Tel.	8.750	1/1/2010	14,822	23,418.50
2011	16,000	16,000	Pacific G & E	9.375	2/1/2011	11,408	16,750.00
	$16,307,000					$6,908,017	$16,324,859.51

APPENDIX V

```
10 REM   ************   DURATION CALCULATION   ************
20 REM  THIS PROGRAM WAS WRITTEN IN "BASICA" ON AN IBM PC.
30 REM THIS PROGRAM WILL CALCULATE DURATIONS FOR A SERIES OF BONDS AT FIVE
40 REM YEAR MATURITY INTERVALS UP TO A SPECIFIED MAXIMUM--NN
50 REM WITH THE INPUT SPECIFIED FOLLOWING THE PROGRAM LISTING THE PROGRAM
60 REM WILL REPLICATE THE DURATION VALUES SHOWN IN TABLE B.
70 REM     VARIABLE      DESCRIPTION
80 REM     --------      -----------
90 REM        Y            YIELD TO MATURITY
100 REM       MAT           REDEMPTION VALUE
110 REM        CF           THE COUPON OR INTEREST RECEIVED EACH YEAR
120 REM                     AS IN THE PAPER IT IS ASSUMED THAT THE COUPON
130 REM                     OR INTEREST IS RECEIVED AT YEAR END.
140 REM        NN           NUMBER OF YEARS TO MATURITY
150 REM
160 REM
170 REM
180 REM
190 REM    PROMPT FOR INPUT OF DATA
200 REM
210 PRINT "INPUT YIELD (EG. 8% AS .08)";
220 INPUT Y
230 PRINT "INPUT REDEMPTION VALUE(EG. 1000)";
240 INPUT MAT
250 PRINT "INPUT COUPON (EG. 80)";
260 INPUT CF
270 PRINT "INPUT # OF YEARS TO MATURITY";
280 INPUT NN
290 REM
300 REM    PRINT REPORT HEADINGS
310 REM
320 FOR P=1 TO 5
330 PRINT
340 NEXT P
350 PRINT"     YIELD(%)  REDEMPTION($)  COUPON($)  MATURITY(YRS)  DURATION"
360 PRINT"     --------  -------------  ---------  -------------  --------"
370 REM
380 REM  DURATION CALCULATION
390 REM
400 FOR N=5 TO NN STEP 5
410 NUM=(MAT*N)/(1+Y)^N
420 DEN =MAT/(1+Y)^N
430 FOR T= 1 TO N
440 NUM =NUM +(CF*T)/(1+Y)^T
450 DEN= DEN + CF/(1+Y)^T
460 NEXT T
470 DUR = NUM/DEN
480 PRINT USING"      #.##          #,###        ###        ###          ###.####
";Y*100,MAT,CF,N,DUR
490 NEXT N
500 FOR P=1 TO 5
510 PRINT
520 NEXT P
530 END
```

```
RUN
INPUT YIELD (EG. 8% AS .08)? .08
INPUT REDEMPTION VALUE(EG. 1000)? 1000
INPUT COUPON (EG. 80)? 40
INPUT # OF YEARS TO MATURITY? 25
```

YIELD(%)	REDEMPTION($)	COUPON($)	MATURITY(YRS)	DURATION
8.00	1,000	40	5	4.5907
8.00	1,000	40	10	8.1184
8.00	1,000	40	15	10.6238
8.00	1,000	40	20	12.2635
8.00	1,000	40	25	13.2452

```
OK
```

```
RUN
INPUT YIELD (EG. 8% AS .08)? .08
INPUT REDEMPTION VALUE(EG. 1000)? 1000
INPUT COUPON (EG. 80)? 80
INPUT # OF YEARS TO MATURITY? 25
```

YIELD(%)	REDEMPTION($)	COUPON($)	MATURITY(YRS)	DURATION
8.00	1,000	80	5	4.3121
8.00	1,000	80	10	7.2469
8.00	1,000	80	15	9.2442
8.00	1,000	80	20	10.6036
8.00	1,000	80	25	11.5288

```
OK
```

INSURANCE REGULATION IN CANADA AND THE ROLE OF THE CASUALTY ACTUARY

ROBERT M. HAMMOND

It is an honour to be invited to speak at the annual meeting of the Casualty Actuarial Society. I am particularly pleased for two reasons. First, being an actuary myself, I have a great interest in all branches of the actuarial profession. I have been active in the Canadian Institute of Actuaries and recently have been elected to the Board of Governors of the Society of Actuaries. Attendance at this meeting gives me the opportunity to meet members of the Casualty Actuarial Society and to hear first hand the major current issues they are facing. I think this is important because I firmly believe that an exchange of views and ideas between the various actuarial organizations in North America is in the best interests of us all.

The second reason for being particularly pleased about being here is that my responsibilities as federal Superintendent of Insurance during the past year or so have clearly demonstrated to me just how important actuarial advice can be to the well-being of the property and casualty insurance industry. I became Superintendent just at the point in time that the Canadian property and casualty industry recorded its largest underwriting losses ever. My experiences since have convinced me that, as a regulator, I should do what I can to encourage a greater involvement of the casualty actuarial profession in the industry in Canada. Consequently, I welcome the opportunity to put some of my ideas in this regard before you.

Most of you are from the United States. To a large degree, the supervisory system in Canada corresponds to the system in the United States. However, there are some significant differences. For this reason, I thought that it would be worthwhile to take a few minutes to summarize for you the main features of the existing Canadian supervisory system. This will put in better perspective my main subject today, namely, the changes to the Canadian supervisory system that the Department is thinking of recommending, including the change that hopefully will work to expand the role of casualty actuaries in the Canadian industry.

In Canada, the supervision of insurance companies is shared by the provincial and federal governments. The federal insurance legislation regulates certain activities of all federally-incorporated companies and all non-resident companies.

The objective of the federal legislation is to try to ensure to the extent possible that federally-supervised companies remain able to meet their obligations to their policyholders. The provisions of the federal insurance legislation aimed at achieving this objective fall into four main groups: control of entry into business, gathering of information, requirements relating to investments and valuation of assets and liabilities, and what we generally describe as the "discipline provisions."

As respects control of entry, all non-resident companies transacting insurance in Canada first must obtain a certificate of registry from the Minister of State for Finance, the elected member of the federal Cabinet to whom I report. For the incorporation of a Canadian company, ministerial consent and prescribed minimum levels of capital and surplus are required.

Gathering of information is achieved by requiring companies to file annual statements in a prescribed form and such other information as may be requested. The annual statements of Canadian companies must be accompanied by a report from an independent auditor. The statutes require the Superintendent to examine a company's affairs at its head office at least once every three years.

As regards assets and liabilities, non-resident companies are required to maintain assets in Canada under the control of the Minister sufficient to cover their Canadian liabilities. The statutes impose both quantitative and qualitative requirements on the investments of Canadian companies and on the assets deposited by non-resident companies, and stipulate that assets must be valued in a prescribed manner.

For life insurance business, a company must appoint a valuation actuary who is a Fellow of the Canadian Institute of Actuaries, and he or she must certify as to the adequacy of the actuarial reserves and the appropriateness of the assumptions. The valuation actuary is free to choose the valuation assumptions subject to their acceptability to the Superintendent. On the presumption that the actuarial reserves would contain adequate margins, no explicit minimum surplus requirements have been prescribed for life insurance business up until this time.

For property and casualty business, liabilities for the purposes of the statement filed with the Department must include unearned premiums and a provision for claims, including a provision for claims incurred but not reported. However, no professional certification as to the adequacy of the reserves is required currently.

As a continuing capital and surplus requirement, property and casualty companies must maintain assets equal to liabilities plus 15% of claims reserves and unearned premiums, with some reduction in the margin of unearned premiums being permitted if a company's experience is favorable.

The statutes impose penalties for failure to comply with the legislation. For example, failure to submit annual statements can result in withdrawal of the certificate of registry. When a company's assets appear to be less than the minimum required under the provisions of the legislation, a number of remedies are available to the Minister, including, where necessary, taking control of the assets of the company and applying to a court for an order to rehabilitate the company or wind it up. However, before taking any action, the Minister must give the company the opportunity to be heard.

The majority of the Canadian insurance business is done by federally-supervised companies. However, it also is possible for companies to be incorporated provincially and all provinces have legislation applicable to provincial companies that corresponds to the federal insurance legislation. In addition, the provinces have exclusive jurisdiction over matters such as licensing of agents and brokers, the terms and conditions of contracts and their interpretation, and filing and approval of premium rates. Provincial legislation on these subjects applies with equal force to both provincial and federal companies. I mentioned that the provincial governments have exclusive jurisdiction over premium rates. However, in the context of what I am going to say later, it is important to know that, in general, the provinces have not acted to regulate premium rates in a very significant way.

The sharing of jurisdiction between the federal and provincial governments probably sounds confusing and may appear to be burdensome for the companies. However, the cooperation and mutual respect existing between the federal and the provincial superintendents are such that there is little, if any, duplication of effort.

The Canadian property and casualty industry has a good record of financial soundness by world standards and we who are in the regulatory business like to think that the supervisory system has played some part in the establishment of this record. Prior to 1981, there had been only one very small federally-incorporated property and casualty insurance company failure in fifty years and that failure was caused by someone disappearing with the securities. However, in 1981, the industry experienced its worst underwriting results in history and two relatively small federally-incorporated companies failed. A third small federal company failed in 1982 and since then, two provincially-incorporated companies have encountered difficulty.

Probably because the Canadian industry has had a good record of soundness, we do not have any government-run compensation schemes in place that guarantee claim payments in the event of company failure. The first federal company to fail had $40 million of claims outstanding. However, because of financial assistance provided by a consortium of interested brokers, insurance companies, and other interested parties, all claimants will be paid in full. The second company to fail had some $23 million of claims outstanding and the ultimate payment to policyholders is expected to be in the range of 70 to 75 cents on the dollar. The third federal company to fail had some $14 million owing to claimants. The amount of the ultimate payment to these claimants will depend very much on the determination by the Courts of the validity of one of the company's major reinsurance treaties. The reinsurer has denied any liability under the contract on various grounds.

I mentioned that there is no guaranty scheme in Canada to protect policyholders. I should clarify this statement by saying that, in a sense, a very limited form of compensation scheme does exist. Under the federal insurance law, if the Superintendent is appointed as liquidator of an insurance company, all the administrative costs, as opposed to the claim costs, involved in liquidating the company are assessed against the industry. As a consequence, the remaining assets of a troubled company are preserved for the benefit of the policyholders. The industry cannot be assessed for policy claims; policy claims must be paid from the company's remaining assets. In the case of the three failed companies, the industry will have been assessed a total of $10 million for administrative costs incurred in the liquidations to the end of this year. So, you can see that, just as is the case in other sectors of the business community, the administrative costs involved in carrying out a liquidation of an insurance company are extremely expensive.

As you can imagine, the failure of the three federal companies caused strong public reaction, particularly from those whose claims were not being paid in full. Understandably, the common perception was that the Department should have acted to prevent the failures.

As a result, the Department was directed to review its administrative practices and the pertinent legislation with a view to recommending changes that would improve protection to policyholders. After analyzing in detail what went wrong with the three failed companies, we put forward a series of proposals for amendments to the legislation for discussion with the industry and other interested parties. When putting our proposals together, we tried not to lose sight of the fact that the Canadian industry has an excellent record of financial soundness and that the vast majority of companies operate in a sound and responsible manner. What we are trying to catch with our proposals are the companies that operate at the so-called fringe. One of the toughest problems a regulator must face is to try to devise rules that will catch the companies operating at the fringe but at the same time will not impose undue restrictions on sound and responsible companies. This balance is not always easy to achieve.

The proposals put forward by the Department for legislative reform focussed on four main areas.

1. Initial and Continuing Capital and Surplus Requirements

The companies that failed were all relatively new and small and appear to have been undercapitalized. At the present time, the legislation requires a minimum of $1.5 million of capital and surplus for incorporation of a property and casualty company. The proposal put forward was to increase the minimum requirement from $1.5 million to $5 million. A strengthening of the existing continuing capital and surplus requirements (the 15% margin requirement) also was proposed in the form of the addition of some new tests. One of the most important features of these proposed new tests is that reinsurance would not be permitted to reduce the capital and surplus margin requirement below 50% of the amount that would be required if there had been no reinsurance. This approach was taken in recognition of the fact that the direct writing insurer is ultimately responsible for the entire loss if the reinsurance fails and should take this into account in its financial planning.

2. The Adequacy of Security Provided by Reinsurance Arrangements and, Directly Related to this Question, the Incentive for Direct Writing Companies to Write Good Quality Business

Problems in collecting on reinsurance ceded to unregistered companies were a significant factor in the failure of one of the companies. The Department recognizes that some access to unregistered reinsurance companies continues to be necessary. However, the proposal put forward was that recourse to unregistered reinsurance should be more restrictive than it is now. Under the proposal, small newly incorporated companies would be prohibited from reinsuring with unregistered companies. Other companies would generally be prohibited from ceding more than 50% of their reinsurance to unregistered companies.

As an incentive for a direct writing company to write good quality business, it was proposed that a company be required to retain a minimum percentage of its total gross premium written. We are currently thinking of an ultimate figure of 35% with more latitude being given in the early years immediately after a company's formation.

3. Quality and Collectability of Assets

In the context of improving the quality and collectability of assets, we suggested a shortening of the standard ninety day period during which premiums due from agents, but unpaid, can be recognized as an asset for solvency purposes. Our current thinking is that a sixty-five day period would be more appropriate. Also suggested was a limit for solvency purposes on the proportion of a company's assets that can take the form of amounts due from associated companies.

4. The Adequacy of Claims and Other Reserves

Given that claims reserves that ultimately proved inadequate were a factor in the failure of at least one of the companies, it was proposed that a property and casualty company be required to provide an annual certificate signed by an appropriately qualified actuary. The final wording of the proposed certificate has not yet been developed. However, it is expected that the person signing the certificate will be required to state that in his or her opinion, the provision for outstanding claims represents a fair and reasonable estimate of the amounts that, together with amounts to be recovered from reinsurers, will be required to settle the claims. The objective of such a requirement is to force more attention to be paid to the adequacy of claims reserves and to encourage more direct involvement in the industry of actuaries qualified in the casualty field.

You will have noted that I made reference to amounts to be recovered from reinsurers in the proposed wording of the certificate. Reinsurance is so important to the financial soundness of most property and casualty insurance companies that we think we must insist that the actuary expand his certificate beyond the net reserves retained by the company. We think that the actuary must review the adequacy and the appropriateness of the reinsurance arrangements and be prepared to express a view on the recoverability of amounts that will be due from reinsurers.

As I mentioned earlier, in general, there is no regulation of premium rates in Canada. All three of the companies that failed appear to have been charging inadequate premium rates. As a means of trying to deal with this situation, we also have proposed that an actuarial report be required with respect to the adequacy of the liability for unearned premiums to cover the liability for claims expected to be incurred in the unexpired portion of the policy. In other words, if in the opinion of the actuary the current premium structure appears inadequate, this view will have to be expressed in the annual report to the regulating authorities.

We acknowledge that these proposals regarding required reports from casualty actuaries need to be refined. Certainly, before anything is put into legislation, we plan to discuss our proposals in detail with the profession. I think the important point is that we see our proposals as being a step towards requiring more actuarial involvement in the property and casualty industry in Canada. Reliance on the professionalism of the actuary has been one of the cornerstones in the development of the existing supervisory system in Canada for life insurance companies and I believe that we should be moving in the same direction for the property and casualty industry.

One of the practical problems we must face in moving in this direction is the lack of actuaries in Canada qualified to practice in the casualty field. For this reason, the proposals as originally published suggested that in certain specified circumstances, and with the approval of the Superintendent, an opinion from someone other than an actuary might be accepted. I view this option as being only part of a transition stage until such time as there are a sufficient number of qualified actuaries practicing in Canada to meet the need.

In simplified form, those were the major proposals put forward by the Department. The objective of the proposals is to improve protection for policyholders. However, in recognition of the fact that no practical system of government supervision can provide a 100% guarantee against loss, the Department has also put forward, for discussion purposes, a form of consumer protection plan to protect policyholders within certain specified limits in the event of company failure. No decision has yet been made as to whether such a plan should be implemented.

As I mentioned earlier, our original proposals were circulated some time ago to interested parties for comment and suggestion. Since many of the proposals are quite technical in nature, it was decided this was the appropriate and prudent course of action to take. A number of very good submissions have been received. Although we continue to believe that the basic thrust of the proposals was sound, a careful review of the submissions indicates that some changes in our proposals are needed. As a consequence, we are revising our proposals and hope to have a revised set to put before the industry in the not-too-distant future. At the present, it is difficult to predict when Parliament might have time to deal with changes in the insurance legislation. Nevertheless, we think we should continue trying to obtain some sort of consensus on the proposals so that we will be ready with our recommendations should Parliament find time to deal with the matter.

I mentioned that we have also been directed to revise our administrative practices in light of recent experiences. We already have made some changes in this regard. To the extent possible, a much greater proportion of the Department's supervisory resources is being allocated to what we perceive to be the weaker companies. More attention than ever is being devoted to the examination of the adequacy of claims reserves. Certainly, reinsurance arrangements are being studied more closely and the Department is taking steps to improve its expertise in the reinsurance field. Most important of all, we are trying to put into practice what we are preaching about the need for more involvement of the casualty actuarial profession in the Canadian property and casualty insurance industry. In fact, we are currently trying to recruit a casualty actuary for the Department's staff.

Thank you for giving me the opportunity to tell you a bit about some of the changes we are thinking of making in regard to the Canadian property and casualty insurance industry. As I have indicated, we think the casualty actuarial profession has an important role to play in these changes and we are looking forward to working with the profession to implement these changes in a manner that is in the best interests of the public, the industry, and the profession.

MINUTES OF THE 1983 FALL MEETING

November 6–9, 1983

TORONTO HILTON HARBOUR CASTLE, TORONTO, ONTARIO

Sunday, November 6, 1983

The Board of Directors held their regular quarterly meeting from 1:00 p.m. to 4:00 p.m.

Registration took place from 4:00 p.m. to 6:30 p.m.

The President's reception for new Fellows and their spouses was held from 5:30 p.m. to 6:30 p.m.

A general reception for all members and guests was held from 6:30 p.m. to 7:30 p.m.

Monday, November 7, 1983

Registration was continued from 7:45 a.m. to 8:30 a.m.

The meeting opened with general remarks from the C.A.S. President, Frederick W. Kilbourne. The President introduced the Honorable Robert G. Elgie, M.D., Minister of Consumer and Commercial Relations for the Province of Ontario. This was followed by the reading of a letter of welcome from Art Eggleton, Mayor of the City of Toronto.

The President then recognized the 28 new Fellows and 7 new Associates and diplomas were awarded. The names of these individuals follow.

FELLOWS

Gregory N. Alff
Stephen A. Belden
Regina M. Berens
James E. Biller
Ralph S. Blanchard, III
François Boulanger
Paul Braithwaite
Jeanne H. Camp
Claudette Cantin
Gregory J. Ciezadlo
Robert B. Downer
James M. Foote
Joseph A. Gilles
Roger M. Hayne
Gaëtane LaFontaine
Michael A. LaMonica
Winsome Leong
Kevin F. Lonergan
Kevin C. McAllister
Robert T. Muleski
Donna S. Munt
James R. Nikstad
Glenn J. Pruiksma
Harold N. Schneider
Frances A. Smith
Harry W. Soul
Daniel L. Splitt
John D. Zicarelli

ASSOCIATES

Victoria L. Bailey	Timothy T. Hein	Gary P. McDonald
David L. Barclay	Jeffrey H. Mayer	Mark A. Ruegg
Michael A. DeConti		

The presentation of diplomas was then followed by a summary of new papers and a review of previous papers.

Mr. Stephen G. Kellison, Executive Director of the American Academy of Actuaries, reported on the activities of the Academy.

The President announced the election results for Officers and Directors.

Officers

C. K. Khury	President-Elect
Herbert J. Phillips	Vice President–Administration
Robert A. Anker	Vice President–Development
Wayne H. Fisher	Vice President–Membership
Michael A. Walters	Vice President–Programs

Directors

Robert A. Bailey	
E. Frederick Fossa	
Robert B. Foster	
George D. Morison	
Ronald E. Ferguson	Appointed Two Years
Hugh G. White	Appointed One Year

From 9:30 a.m. to 10:30 a.m., a keynote address was given by William G. Ouchi, Professor of Management, Graduate School of Management, University of California at Los Angeles. The subject was "The Revitalization of American Industry."

From 11:00 a.m. to 11:55 a.m., a panel, "The Classification Controversy," was conducted by Michael A. Walters, Moderator. The panelists were:

Mavis A. Walters
Vice President
Insurance Services Office

Sanford R. Squires
Vice President and Actuary
Commercial Union Insurance

Hugh G. White
Assistant Vice President and Actuary
Travelers, Canada

A formal luncheon was served from 12:00 noon to 1:25 p.m. and Mr. Robert M. Hammond, Superintendent of Insurance for the Canadian Federal Department of Insurance, was the guest speaker. His subject was "Insurance Regulation in Canada and the Role of the Casualty Actuary."

From 1:30 p.m. to 2:40 p.m., a panel discussion on "Futurism, Economics and Demography—Are They Relevant to the Actuary?" was moderated by Frederick W. Kilbourne. The panelists were:

Roy R. Anderson
Vice President
Allstate Life

Robin B. Leckie
Senior Vice President
Manufacturers Life Insurance

Diane J. Macunovich
President
Plan Technics Consultants

From 3:00 p.m. to 4:55 p.m., eight concurrent workshops were held. They were:

A. Continuation of Panel:
The Classification Controversy

B. Continuation of Panel:
Futurism, Economics and Demography—
Are They Relevant to the Actuary?

C. Structured Settlements

Lee R. Steeneck—Moderator
Second Vice President
General Reinsurance Corp.

Vincent J. Mangano
President
Vincent J. Mangano Associates

Charles R. Meyer
Tax Partner
Coopers & Lybrand

D. No-Fault Auto Insurance Revisited

C. K. Khury—Moderator
Vice President and Actuary
Prudential Property and Casualty Insurance

Gary Granoff
President
Granoff Resources

Anne E. Kelly
Assistant Chief Casualty Actuary
New York State Insurance Department

E. Considerations in the Establishment of Domestic and Off-Shore Captives

Alfred O. Weller—Moderator
Vice President
B.R.I. Coverage Corp.

Frank Dunn
Manager
Citibank

L. Brian Shaull
Partner
Coopers & Lybrand

P. Bruce Wright
Partner
Trubin, Sillcocks, Edelman, Knapp

F. New Paper: Duration

Ronald E. Ferguson
President
General Reinsurance Corp.

G. New Paper: Reinsuring the Captive/Specialty Company

Lee R. Steeneck
Second Vice President
General Reinsurance Corp.

H. Discussion of the Paper:
The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions
Authors: Philip E. Heckman
Glenn G. Meyers

Gary Venter
Actuary
N.C.C.I.

A general reception was held from 6:30 p.m. to 7:30 p.m.

Tuesday, November 8, 1983

The concurrent workshops were continued from 8:30 a.m. to 9:25 a.m.

At 9:30 a.m., the business session was reconvened to include:

Committee Reports
Secretary's Report
Treasurer's Report

The President noted the decease of four members:

Harmon T. Barber
Joseph J. MaGrath
Arthur N. Matthews
Harry F. Richardson

The Woodward-Fondiller Prize was jointly awarded to Philip Heckman and Glenn Meyers.

The Dorweiler Prize was awarded to Stephen P. D'Arcy.

From 10:30 a.m. to 11:00 a.m., President Frederick W. Kilbourne delivered his Presidential Address.

At 11:00 a.m., a panel entitled "Actuarial Advocacy—An Oxymoron?" was moderated by Stephen G. Kellison. The panelists were:

Irene K. Bass
Vice President
Crum and Forster

James A. Hall, III
Partner
Coopers & Lybrand

Frank Harwayne
Vice President and Director
N.C.C.I.

The remainder of the programme, Tuesday afternoon and Wednesday morning until noon, was jointly sponsored with the Canadian Institute of Actuaries, consisting of four separate panels on loss and premium reserving.

From 1:30 p.m. to 2:45 p.m., a panel, "Considerations in the Reserving Process," was given. The panelists were:

Martin Adler
Vice President and Actuary
GEICO

David C. Westerholm
Manager and Senior Actuarial Analyst
CNA Insurance Companies

From 3:15 p.m. to 4:30 p.m., a panel, "Analysis of U.S. and Canadian Reserving Practices," was presented. The participants were:

Herbert J. Phillips—Moderator
Senior Vice President and Chief Actuary
Insurers' Advisory Organization of Canada

David Atkins
Partner
Coopers & Lybrand

David J. Oakden
Actuary
Aetna Casualty Company of Canada

Alain Thibault
Consulting Actuary
Blondeau and Company

A general reception was held from 6:00 p.m. to 7:00 p.m.

Wednesday, November 9, 1983

From 9:00 a.m. to 10:15 a.m., a panel "Survey of Reserving Techniques I," was presented by:

John J. Schultz, III
Vice President and Actuary
California Casualty Group

Richard E. Sherman
Senior Consultant
Coopers & Lybrand

From 10:30 a.m. to 11:45 a.m., a panel, "Survey of Reserving Techniques II," was also presented by:

John J. Schultz, III
Richard E. Sherman

At 12:00 noon the meeting adjourned following closing remarks by Frederick W. Kilbourne and Carlton W. Honebein of the Society and Christopher D. Chapman, President of the Canadian Institute of Actuaries.

In attendance, as indicated by registration records, were 203 Fellows, 71 Associates, 19 guests, 8 subscribers, and 18 students. The list follows.

FELLOWS

Addie, B. J.
Adler, M.
Alexander, L. M.
Alff, G. N.
Anker, R. A.
Bailey, R. A.
Bashline, D. T.
Bass, I. K.
Beer, A. J.
Ben-Zvi, P. N.
Berens, R. M.
Biller, J. E.
Biondi, R. S.
Bishop, E. G.
Blanchard, R. S., III
Bornhuetter, R. L.
Boulanger, F.
Braithwaite, P.
Brannigan, J. F.
Brouillette, Y. J.
Burger, G.
Camp, J. H.
Cantin, C.
Carbaugh, A. B.
Carponter, J. D.
Carter, E. J.
Cheng, J. S.
Cheng, L. W.
Christie, J. K.
Ciezadlo, G. J.
Cis, M. M.
Clinton, R. K.
Conger, R. F.
Conners, J. B.
Cook, C. F.
Covney, M. D.
Crowe, P. J.
Cundy, R. M.
Curry, A. C.

FELLOWS

Daino, R. A.
Dean, C. G.
Donaldson, J. P.
Dorval, B. T.
Downer, R. B.
Drennan, J. P.
Dropkin, L. B.
Drummond-Hay, E. T.
Dussault, C.
Easton, R. D.
Egnasko, G. J.
Eliason, E. B.
Evans, G. A.
Fallquist, R. J.
Ferguson, R. E.
Finger, R. J.
Fisher, W. H.
Flaherty, D. J.
Flynn, D. P.
Foote, J. M.
Ford, E. W.
Forker, D. C.
Fossa, E. F.
Foster, R. B.
Fresch, G. W.
Friedberg, B. F.
Frohlich, K. R.
Fusco, M.
Gilles, J. A.
Goldberg, S. F.
Gottlieb, L. R.
Grannan, P. J.
Graves, J. S.
Hafling, D. N.
Hall, J. A., III
Hallstrom, R. C.
Hartman, D. G.
Harwayne, F.
Hayne, R. M.
Hazam, W. J.
Hennessy, M. E.
Hewitt, C. C., Jr.
Hine, C. A.
Holmberg, R. D.
Honebein, C. W.
Hoylman, D. J.
Hughey, M. S.
Jaeger, R. M.
John, R. T.
Karlinski, F. J., III
Kaufman, A.
Kelly, A. E.
Khury, C. K.
Kilbourne, F. W.
Klaassen, E. J.
Kleinman, J. M.
LaFontaine, G.
LaMonica, M. A.
Lange, D. L.
LaRose, J. G.
Lehmann, S. G.
Leimkuhler, U. E., Jr.
Leong, W.
Levin, J. W.
Linden, O. M.
Lino, R. A.
Liscord, P. S.
Lonergan, K. F.
Lotkowski, E. P.
Lowe, S. F.
MacGinnitie, W. J.
Mahler, H. C.
Makgill, S. S.
Marker, J. O.
Masterson, N. E.
Mathewson, S. B.
McAllister, K. C.
McCarter, M. G.
McClure, R. D.
McConnell, C. W., II
McGuinness, J. S.
McLean, G. E.
McManus, M. F.
McMurray, M. A.
Miller, M. J.
Miller, R. R.
Mohl, F. J.
Moody, R. A.
Morison, G. D.
Muetterties, J. H.
Muleski, R. T.
Munro, R. E.
Munt, D. S.
Murad, J. A.
Murray, E. R.
Myers, N. R.
Neidermyer, J. R.
Newman, S. H.
Newville, B. S.
Nikstad, J. R.
Niswander, R. E.
Oakden, D. J.
O'Brien, T. M.
O'Neil, M. L.
Pagnozzi, R. D.
Patrik, G. S.
Phillips, H. J.
Pollack, R.
Pratt, J. J.
Prevosto, V. R.
Pruiksma, G. J.
Racine, A. R.
Radach, F. R.
Retterath, R. C.

FELLOWS

Richardson, J. F.
Robertson, J. P.
Rodermund, M.
Roland, W. P.
Salzmann, R. E.
Scheibl, J. A.
Schneider, H. N.
Schultz, J. J., III
Sheppard, A. R.
Sherman, R. E.
Shoop, E. C.
Shrum, R. G.
Smith, F. A.
Smith, L. M.
Soul, H. W.
Splitt, D. L.
Squires, S. R.
Stanard, J. N.
Steeneck, L. R.
Steer, G. D.
Strug, E. J.
Sturgis, R. W.
Taht, V.
Taranto, J. V.
Taylor, J. C.
Thibault, A.
Tiller, M. W.
Toothman, M. L.
Tverberg, G. E.
Van Ark, W. R.
Venter, G. G.
Verhage, P. A.
Walker, R. D.
Walters, M. A.
Walters, M. A.
Weiland, W. T.
Weissner, E. W.
Weller, A. O.
Westerholm, D. C.
White, H. G.
Whitman, M.
Wilcken, C. L.
Wilson, J. C.
Wilson, R. L.
Wiser, R. F.
Woll, R. G.
Woods, P. B.
Young, R. J.
Zicarelli, J. D.
Zubulake, T. J.

ASSOCIATES

Bailey, V. M.
Banfield, C. J.
Barclay, D. L.
Bennett, R. S.
Bensimon, A. S.
Bertrand, F.
Boone, J. P.
Briere, R. S.
Brown, R. L.
Bursley, K. H.
Chorpita, F. M.
Clark, D. G.
Cohen, A. I.
Connor, V. P.
Costner, J. E.
DeConti, M. A.
Degarmo, L. W.
Deutsch, R. V.
Driedger, K. H.
Edie, G. M.
Egnasko, V. M.
Einck, N. R.
Gaillard, M. B.
Gannon, A. H.
Gillam, W. R.
Gould, D. E.
Granoff, G.
Halpern, N. S.
Harrison, E. E.
Hein, T. T.
Hutter, H. E.
Inderbitzin, P. H.
Jensen, J. P.
Kolk, S. L.
Koupf, G. I.
Lamb, J. A.
Limpert, J. J.
Mayer, J. H.
McConnell, D. M.
McDonald, G. P.
Meyer, R. E.
Miner, N. B.
Mokros, B. F.
Moody, A. W.
Murphy, F. X., Jr.
Murphy, W. F.
Murray, J. B. M.
Nolan, J. D.
Normandin, A. G.
Ogden, D. F.
Potts, C. M.
Ratnaswamy, R.
Robbins, K. B.
Ross. L. A.

ASSOCIATES

Ruegg, M. A.
Sandler, R. M.
Sansevero, M., Jr.
Schulman, J.
Singer, P. E.
Silverman, M. J.
Skolnik, R. S.
Skrodenis, D. P.
Suchoff, S. B.
Thompson, P. R.
Townsend, C. J.
Wainscott, R. H.
Walker, G. M.
Weimer, W. F.
White, D. C.
Whiting, D. R.
Wilson, W. F.

GUESTS—STUDENTS—SUBSCRIBERS

Allard, J.
Anderson, R. R.
Atkins, D.
Belton, E. F.
Benson, D. W.
Bradley, J. S.
Cartmell, A.
Chapman, C. D.
Dufresne, J.
Dunn, F.
Earles, R. R.
Elgie, R. G.
Elliott, P. L.
Englander, J.
Fontaine, A.
Fromentin, P.
Fung, C.
Graves, G. G.
Hager, G.
Hammond, R. M.
Haughey, T. D.
Homer, B.
Johnson, A. P.
Kartechner, J. W.
Kellison, S. G.
Lautzenheiser, B. J.
Laws, M. H.
Leckie, R. B.
Ludwig, P. A.
Macunovich, D. J.
McSally, M. J.
Meyer, C. R.
Nielsen, L.
Novik, J.
Ouchi, W. G.
Paquette, S.
Roeser, K. G.
Ross, D. P.
Schmidt, L. D.
Shaull, L. B.
Simcock, C. E.
Smith, D. A.
Spangler, J. L.
Wilson, G. S.
Wright, P. B.

Respectfully submitted,
BRIAN E. SCOTT
Secretary

REPORT OF THE SECRETARY

The purpose of this report is to provide the membership with a summary of significant activities of the CAS during the past year.

Our Society is an active one and many of our members contribute to our Society's activities. Time won't allow me to mention all of the contributions but I will try to cover the highlights.

1983 was a year of change for us. In May the membership approved a new organizational structure and a transition program to implement the new structure. The transition is already under way as the Board appointed four vice presidents at the September meeting. They are

Vice President–Administration	Herbert J. Phillips
Vice President–Program	Michael A. Walters
Vice President–Membership	Wayne H. Fisher
Vice President–Development	Robert A. Anker

These appointments, combined with the appointment of two short term directors, and the election of four new directors will establish our new organization. 1983 business will be conducted under this structure.

The Education Policy Committee proposed and the Board of Directors approved a new policy on waiver of examination requirements for membership. It reaffirmed the practice of granting membership only to those who have demonstrated competence through the examination process. It also delineated the process for receiving credits for examinations sponsored by other actuarial societies.

The CAS, in conjunction with the American Academy of Actuaries, conducted the third loss reserve seminar. The seminar was well attended by members as well as non-members.

Our Committee on Career Enhancement developed a program for bringing career opportunities in actuarial science to the attention of handicapped individuals. Published material on our profession will be made available to various organizations that are involved with handicapped persons for dissemination to their members. They also initiated a two-phase program to assist in the recruiting to our profession from among minorities.

A budget for 1984 was approved which included an increase in dues and examination fees. Dues were increased to $100 for Fellows and Associates of more than 5 years and to $75 for Associates of less than 5 years. Exam fees were increased by $10.

Our two membership meetings were well attended. The November, 1982 meeting was in San Francisco and featured Dr. Thomas Sowell as the keynote speaker. Our spring, 1983 meeting was at the Doral Country Club in Miami and featured a number of computer software and hardware dealers who demonstrated and lectured on the use of their wares for actuaries.

Our membership ranks continued to grow with the admittance of 46 Fellows and 66 Associates.

This is my final report as Secretary. I would be remiss if I did not acknowledge the great contribution made to our Society by Edee Morabito. Secretaries have come before me and the Vice President–Administration will handle these duties under the new organization but none of us could operate without Edee's help. We are fortunate to have her managing our New York Office.

Respectfully submitted,

BRIAN E. SCOTT
Secretary

REPORT OF THE TREASURER

This is my first and final Treasurer's Report to the membership since, under the reorganization, the Vice President–Administration includes the duties of both the Secretary and the Treasurer.

The 1983 fiscal year was one of significant activity, starting with the reorganization itself. The Editorial Committee was quite active and the *Proceedings* are now current. Membership has passed the 1,000 mark and by the close of the 1984 fiscal year, could be close to 1,100, depending on the examination results this year.

The Finance Committee audited the assets and accounts maintained by the Treasurer and found them to be correct. The year ended with a small increase in Surplus, far less than was budgeted, but, nevertheless, an increase. Members' Equity now stands at $206,547.33, up $9,304.34 from last year. This Equity is subdivided into $49,367.64 for the Michelbacher Fund, $8,547.66 for the Dorweiler Fund, $1,616.64 for the CAS Trust and $147,015.39 for Surplus.

The 1984 fiscal year budget has been reviewed by both the Finance Committee and the Board of Directors and approved with certain modifications. It became apparent that in order to meet the needs of the Society and cover the anticipated disbursements in fiscal 1984, dues and certain other fees needed to be increased. While the Society has been able to maintain level dues and fees for the past several years, it is impossible to continue at that level of income with today's costs.

Accordingly, for the 1984 fiscal year, dues for Fellows and Associates over five years will be increased to $100 from the present $80; Associates for the first five years will be increased to $75 from the present $60. Examination fees will be increased to $50 per part for Parts 4 through 10, up from the present $40. Finally, the fees for Study Notes and Study Kits have also been increased.

The major reasons for these increases are a combination of lower interest rates resulting in a lower rate of return on investments, increased postage costs, and increased printing costs.

As stated earlier, the Society has been successful in keeping dues and other fees constant for several years; it is just impossible to continue this schedule in 1984.

Respectfully submitted,

HERBERT J. PHILLIPS
Treasurer

TREASURER'S REPORT

FISCAL YEAR ENDED 9/30/83 (ACCRUAL BASIS)

INCOME

Dues	$ 68,550.10
Exam Fees	71,773.00
Meetings	119,533.90
Proceedings	13,240.00
Readings	11,842.56
Invitational Program	6,300.00
Interest	25,191.28
Actuarial Review	230.00
Other	803.17
Total	$315,857.67

DISBURSEMENTS

Printing	$ 80,579.52
Office Expenses	96,807.04
Other Exam Expenses	4,689.98
Meeting Expenses	115,949.25
Library	659.28
Insurance	2,797.17
Math Assoc. of America	2,000.00
Expenses—President	5,000.00
Expenses—Pres.-Elect	2,500.00
Outside Services	0
Miscellaneous	1,245.95
Total	$312,228.19

Income	$315,857.67
Disbursements	312,228.19
Change in CAS Surplus	$ +3,629.48

ACCOUNTING STATEMENT (ACCRUAL BASIS)

ASSETS	9/30/82	9/30/83	CHANGE
Checking Account	$ 586.76	$ 8,553.45	$+ 7,966.69
Money Market Fund	80,582.40	31,883.98	−48,698.42
Bank Certificates of Deposit	97,855.51	100,115.58	+ 2,260.07
U.S. Treasury Notes	99,971.90	99,971.90	0
Accrued Income	5,324.00	14,658.01	+ 9,334.01
Total	$284,320.57	$255,182.92	$−29,137.65

LIABILITIES	9/30/82	9/30/83	CHANGE
Office Services	$ 14,500.00	$ 27,000.00	$+12,500.00
Printing Expenses	41,379.13	0	−41,379.13
Examination Expenses	0	0	0
Meeting Expenses & Prepaid Fees	8,038.63	4,500.00	− 3,538.63
Prepaid Exam Fees	22,319.82	17,136.00	− 5,183.82
Other	840.00	0	− 840.00
Total	$ 87,077.58	$ 48,636.00	$−38,441.58

MEMBERS' EQUITY	9/30/82	9/30/83	CHANGE
Michelbacher Fund	$ 43,678.40	$ 49,367.64	$+ 5,689.24
Dorweiler Fund	8,836.52	8,547.66	− 288.86
CAS Trust	1,342.16	1,616.64	+ 274.48
CAS Surplus	143,385.91	147,015.39	+ 3,629.48
Total	$197,242.99	$206,547.33	$+ 9,304.34

Herbert J. Phillips
Treasurer

This is to certify that the assets and accounts shown in the above financial statement have been audited and found to be correct.

Finance Committee
Glenn W. Fresch, Chairman
James H. Kreuzer
William J. Rowland
Michael A. Walters

1983 EXAMINATIONS—SUCCESSFUL CANDIDATES

Examinations for Parts 4, 6, 8 and 10 of the Casualty Actuarial Society were held on May 4, 5 and 6, 1983. Examinations for Parts 5, 7 and 9 were held on November 9 and 10, 1983.

Examinations for Parts 1, 2 and 3 are jointly sponsored by the Casualty Actuarial Society and the Society of Actuaries. These examinations were given in May and November of 1983. Candidates who passed these examinations were listed in the joint releases of the two societies.

The Casualty Actuarial Society and the Society of Actuaries jointly awarded prizes to the undergraduates ranking the highest on the General Mathematics examination. For the May, 1983, examination, the $200 prize was awarded to Bruce W. Brandt. The additional $100 prize winners were Christine B. Burnley, John H. Kerper, Pedro G. Nebres, and Eduardo M. Reyes. For the November, 1983, examination, the $200 prize was awarded to Howard M. Pollack. The additional $100 prize winners were Alden V. De la Rosa, Mark E. Glickman, Valerie M. Harris, and Eric K. Lossin.

The following candidates were admitted as Fellows and Associates at the November, 1983, meeting as a result of their successful completion of the Society requirements in the May, 1983, examinations.

FELLOWS

Alff, Gregory N.
Belden, Stephen A.
Berens, Regina M.
Biller, James E.
Blanchard, Ralph S., III
Boulanger, Francois
Braithwaite, Paul
Camp, Jeanne H.
Cantin, Claudette
Ciezadlo, Gregory J.
Downer, Robert B.
Foote, James M.
Gilles, Joseph A.
Hayne, Roger M.
Lafontaine, Gaetane
LaMonica, Michael A.
Leong, Winsome
Lonergan, Kevin F.
McAllister, Kevin C.
Muleski, Robert T.
Munt, Donna S.
Nikstad, James R.
Pruiksma, Glenn J.
Schneider, Harold N.
Smith, Frances A.
Soul, Harry W.
Splitt, Daniel L.
Zicarelli, John D.

ASSOCIATES

Bailey, Victoria M.
Barclay, David L.
DeConti, Michael A.
Hein, Timothy T.
Mayer, Jeffrey H.
McDonald, Gary P.
Ruegg, Mark A.

The following is the list of successful candidates in examinations held in May, 1983.

Part 4

Allen, Roger G.
Bauer, Bruno P.
Behrendt, David
Boucek, Charles H.
Bray, Rosemary P.
Carlson, Karyl T.
Cartmell, Andrew R.
Chisholm, Thomas J.
Conley, Kevin J.
Curry, Michael K.
Danielson, Guy R.
Dearr, Ryan L.
DeLiberato, Robert V.
Desjardins, Charles
Dezube, Janet B.
DiDonato, Anthony M.
Fitzgerald, Beth E.
Flannery, Nancy G.
Flemming, Kirk G.
Gaylor, Paul F.
Girard, Gregory S.
Glotzer, Leonard R.
Graves, Gregory T.
Greene, Alex R.
Guenthner, Denis G.
Hall, Alison A.
Harbage, Robin A.
Kneuer, Paul J.
Kreps, Rodney E.
Kufera, Joseph A.
Kulik, John M.
Lacek, Mary Lou
Langhorst, R. David
Lombardi, Paul M.
Maharajh, Bindranath
McCarthy, Michael J.
McCoy, Mary E.
Membrino, Conrad O.
Penso, Michael A.
Peterson, Scott A.
Pitbladdo, Richard B.
Raffa, Guy P.
Roupas, Theodore G.
Rosenberg, David J.
Rosenstein, Kevin D.
Scheuing, Jeffrey R.
Sealand, Pamela J.
Shadman-Valavi, Ahmad
Shepherd, Linda A.
Sterling, Mary E.
Terrill, Kathleen W.
Tremblay, Martin-Eric
Turner, George W., Jr.
Veilleux, Andre
Votta, James
Waldman, Jeffrey M.
Williams, Robin M.
Yit, Bill S. C.
Yow, James W.

Part 6

Aldin, Neil C.
Arvanitis, Robert J.
Bailey, Victoria M.
Barclay, David L.
Batdorff, Rober N.
Battles, John E. N.
Bellafiore, Leonard A.
Boor, Joseph A.
Boyd, Wallis A.
Bradford, David K.
Bradley, J. Scott
Brodie, Pamela E.
Bryan, Susan E.
Buchanan, John W.
Bury, John M.
Captain, John E.
Carpenter, William M.
Cascio, Michael J.
Cellars, Ralph M.
Chansky, Joel S.
Chodnicki, Karen A.
Christhilf, David A.
Cieslak, Walter P.
Clark, Daniel B.
Cohen, Elliot J.
Cutler, Janice Z.
Davis, Dan J.
DeConti, Michael A.
Deede, Martin W.
DeFalco, Thomas J.
Desilets, Claude
Dickinson, Donna R.
Donelson, Norman E.
Dyck, N. Paul
Earwaker, Bruce G.
Easlon, Kenneth
Englander, Jeffrey A.
Farwell, Randall A.
Forbus, Barbara L.
Fung, Charles C. K.
Gardner, Robert W.
Gebhard, James J.
Gevlin, James M.
Gidos, Peter M.
Gorvett, Richard W.
Greco, Ronald E.
Griffin, Dale C.
Guernsey, Anne L.
Guiahi, Farrokh
Gunn, Christy H.
Gutman, Ewa
Hein, Timothy T.
Herbers, Joseph A.
Hollister, Jeanne M.
Homan, Mark J.
Housholder, Timothy J.
Hughes, Brian A.
Huyck, Brenda J.
Johnson, Andrew P.
Kasner, Kenneth R.
Klenow, Jerome F.
Kline, Charles D., Jr.
Lacroix, Marthe
Lewis, Martin A.
Lipton, Barry
Littmann, Mark W.
Maguire, Brian P.
Maier, Linda L.
Mayer, Jeffrey H.
McDonald, Gary P.
McQuilkin, Mary T.
Miller, Brett E.
Miller, John V.
Mohrman, David F.
Montgomery, Warren D.
Mucci, Robert V.
Mueller, Robert A.
Musante, Donald R.
Newman, Henry E.
Noback, Jodee B.
Olson, Carol L.
Paddock, Timothy A.
Peterson, Steven J.
Phillips, George N.
Placek, Arthur C.
Reppert, Daniel A.
Robinson, Richard D.
Roesch, Robert S.
Roth, Randy J.
Ruegg, Mark A.
Sandman, Donald D.
Santomenno, Sandra C.
Schilling, Timothy L.
Siczewicz, Peter J.
Silver, Melvin S.
Slusarski, John
Sornberger, G. Clinton
Steingiser, Russell
Tistan, Ernest S.
Treitel, Nancy R.
Wallace, Thomas A.
Wilk, Roger A.
Zaleski, Ronald J.

Part 8

Alpert, Bradley K.
Bakel, Leo R.
Balchunas, Anthony J.
Basson, Steven D.
Baum, Edward J.
Berry, Janice L.
Bertrand, Francois
Biegaj, William P.
Biscoglia, Terry J.
Boone, James P.
Bothwell, Peter T.
Braithwaite, Paul
Bujaucius, Gary S.
Camp, Jeanne H.
Canetta, John A.
Carlson, Jeffrey R.
Cathcart, Sanders B.
Chiang, Jeanne D.
Cripe, Frederick F.
Curren, Kathleen F.
Dornfeld, James L.
Egnasko, Valere M.
Ellefson, Thomas J.
Faltas, Bill
Forney, John R., Jr.
Hale, Jonathan B.
Hanson, Jeffrey L.
Harrison, Eugene E.
Hayne, Roger M.
Henzler, Paul J.
Hoppe, Kenneth J.
Howald, Ruth A.
Hurley, Paul M.
Hutter, Heidi E.
Kaplan, Robert S.
Keen, Eric R.
Kolk, Stephen L.
Konopa, Milan E.
Loper, Dennis J.
Loucks, William D., Jr.
Mailloux, Patrick
McSally, Michael J.
Mendelssohn, Gail A.
Merlino, Matthew P.
Mill, Ralph A.
Miner, Neil B.
Murdza, Peter J., Jr.
Nester, Karen L.
Nichols, Raymond S.
Nikstad, James R.
Palmer, Donald W.
Pelletier, Bernard A.
Potts, Cynthia M.
Raman, Rajagopalan K.
Rapoport, Andrew J.
Rosenberg, Deborah M.
Silverman, Janet K.
Smith, Michael B.
Somers, Edward C.
Spalla, Joanne S.
Symnoski, Diane M.
Tresco, Frank J.
Vaughan, Richard L.
Vitale, Lawrence A.
Wacek, Michael G.
Weber, Dominic A.
Whiting, David R.
Wickman, Alan E.
Withers, David A.

Part 10

Alff, Gregory N.
Belden, Stephen A.
Berens, Regina M.
Biegaj, William P.
Biller, James E.
Blanchard, Ralph S., III
Boccitto, Bonnie L.
Boulanger, Francois
Bouska, Amy S.
Bowen, David S.
Cantin, Claudette
Chanzit, Lisa G.
Chuck, Allan
Ciezadlo, Gregory J.
Coffin, John D.
Dembiec, Linda A.
Dodd, George T.
Downer, Robert B.
Eagelfeld, Howard M.
Ehrlich, Warren S.
Foote, James M.
Friedman, Howard H.
Gilles, Joseph A.
Hall, Allen A.
Hapke, Alan J.
Holdredge, Wayne D.
Johnson, Larry D.
Kane, Adrienne B.
Knilans, Kyleen
Kooken, Michael W.
Lafontaine, Gaetane
LaMonica, Michael A.
Leong, Winsome
Lonergan, Kevin F.
Marks, Steven D.
McAllister, Kevin C.
Miller, Robert A., III
Muleski, Robert T.
Munt, Donna S.
O'Connell, Paul G.
Paquette, Sylvie L.
Pierson, Frank D.
Pruiksma, Glenn J.
Ross, Lois A.
Schmidt, Neil J.
Schneider, Harold N.
Schwartzman, Joy A.
Siewert, Jerome J.
Smith, Frances A.
Smith, Judith P.
Soul, Harry W.
Splitt, Daniel L.
Tom, Darlene P.
Weimer, William F.
White, David L.
Wiseman, Michael L.
Zicarelli, John D.

The following candidates will be admitted as Fellows and Associates at the May, 1984 meeting as a result of their successful completion of the Society requirements in the November, 1983 examinations.

FELLOWS

Boccitto, Bonnie L.
Bouska, Amy S.
Chanzit, Lisa G.
Coffin, John D.
Dodd, George T.
Duffy, Thomas J.
Kane, Adrienne B.
Knilans, Kyleen
Kooken, Michael W.
Kozik, Thomas J.
Marks, Steven D.
O'Connell, Paul G.
Pinto, Emanuel
Schmidt, Neil J.
Schwartzman, Joy A.
Tom, Darlene P.
Weimer, William F.
Wiseman, Michael L.

ASSOCIATES

Anderson, Bruce C.
Bakel, Leo R.
Balchunas, Anthony J.
Balling, Glenn R.
Basson, Steven D.
Bear, Robert A.
Becraft, Ina M.
Belden, Scott C.
Berry, Janice L.
Biegaj, William P.
Boccitto, Bonnie L.
Bouska, Amy S.
Boyd, Wallis A.
Bryan, Susan E.
Campbell, Kenrick A.
Captain, John E.
Carlson, Jeffrey R.
Chansky, Joel S.
Chiang, Jeanne D.
Deede, Martin W.
Desilets, Calude
Dupuis, Camille
Dyck, N. Paul
Elliott, Paula L.
Forney, John R., Jr.
Grace, Gregory S.
Greco, Ronald E.
Haskell, Gayle E.
Hurley, Paul M.
Huyck, Brenda J.
Johnson, Andrew P.
Keller, Wayne S.
Kelley, Robert J.
Levenglick, Arthur B.
Licht, Peter M.
Loper, Dennis J.
Lyons, Daniel K.
Matthews, Robert W.
McQuilkin, Mary T.
McSally, Michael J.
Mendelssohn, Gail A.
Mozeika, John K.
Nester, Karen L.
Onufer, Layne M.
Palmer, Donald W.
Paquette, Sylvie L.
Peterson, Steven J.
Port, Rhonda D.
Raman, Rajagopalan K.
Rathjen, Ralph L.
Roth, Randy J.
Schultheiss, Peter J.
Silver, Melvin S.
Smith, Byron W.
Smith, Judith P.
Trinh, Minh
Walker, Leigh M.
Walsh, Michael C.
Webster, Patricia J.
Woomer, Roy T., III

The following is a list of successful candidates in examinations held in November, 1983.

Part 5

Aldin, Neil C.
Allaire, Christiane
Anderson, Bruce C.
Apfel, Kenneth
Bellafiore, Leonard A.
Bender, Robert K.
Billings, Holly L.
Blakinger, Jean M.
Boor, Joseph A.
Brissman, Mark D.
Busche, George R.
Campbell, Kenrick A.
Carlson, Karyl T.
Cellars, Ralph M.
Chen, Chyen
Cieslak, Walter P.
Clark, Daniel B.
Cunninghis, Sharon B.
Dekle, James M.
Der, William
Dezube, Janet B.
Dodge, Scott H.
Downing, Jeremiah M.
DuFresne, Jacques
Edlefson, Dale R.
Edwards, Nancy D.
Ericson, Janet M.
Eschenbrenner, Denise
Flannery, Nancy G.
Girard, Gregory S.
Glicksman, Steven A.
Gorvett, Richard W.
Goselin, Craig A.
Grace, Gregory S.
Graves, Gregory T.
Greene, Alex R.
Guenthner, Denis G.
Haidu, James W.
Harbage, Robin A.
Hay, Randolph S.
Hill, Tony D.
Homan, Mark J.
Johnson, Eric J.
Jordan, Jeffrey R.
Kaplan, Robert S.
Kasner, Kenneth R.
Kaster, Donna L.
Kelly, Beverley A.
Kinson, Paul E.
Kline, Charles D., Jr.
Kneuer, Paul J.
Kohan, Richard F.
LaPointe, Susan E.
Levenglick, Arthur B.
Lewis, Stuart A.
Licht, Peter M.
Littmann, Mark W.
Llewellyn, Barry I.
Mailloux, Patrick
Marles, Blaine C.
Membrino, Conrad O.
Menning, David L.
Mohler, Elena D.
Mohrman, David F.
Mozeika, John K.
Mueller, Nancy D.
Muller, Robert G.
Musulin, Rade T.
Myers, Thomas G.
Noyce, James W.
Overgaard, Wade T.
Pace, Michelle M.
Paddock, Timothy A.
Pitbladdo, Richard B., Jr.
Plano, Richard A.
Post, Jeffrey H.
Quintano, Richard A.
Reppert, Daniel A.
Rice, James W.
Salton, Jeffrey C.
Scheuing, Jeffrey R.
Schlissel, Joanne
Schwandt, Jeffory C.
Sclafane, Susanne
Shepherd, Linda A.
Silver, Melvin S.
Slotznick, Lisa A.
Spalding, Keith R.
Svendsgaard, Christian
Swords, Elaine E.
Terrill, Kathleen W.
Theisen, Richard J.
Tingley, Nanette
Tistan, Ernest S.
Treitel, Nancy R.
Visintine, Gerald R.
Von Seggern, William J.
Wachter, Christopher J.
Walder, Lawrence M.
Wilk, Roger A.

Part 7

Almagro, Manuel, Jr.
Bakel, Leo R.
Balchunas, Anthony J.
Balling, Glenn R.
Basson, Steven D.
Bear, Robert A.
Becraft, Ina M.
Belden, Scott C.
Berry, Janice L.
Biegaj, William Pl
Boyd, Wallis A.
Bryan, Susan E.
Captain, John E.
Carlson, Jeffrey R.
Carlton, Kenneth E.
Chansky, Joel S.
Chiang, Jeanne D.
Deede, Martin W.
DeFalco, Thomas J.
Desilets, Claude
Donnelly, Vincent T.
Dupuis, Camille
Dyck, N. Paul
Elliott, Paula L.
Erickson, John A.
Fletcher, James E.
Forney, John R., Jr.
Gardner, Robert W.
Goldberg, Steven L.
Grace, Gregory S.
Greco, Ronald E.
Handte, Malcolm R.
Haskell, Gayle E.
Hertling, Richard J.
Hurley, Paul M.
Huyck, Brenda J.
Johnson, Andrew P.
Keller, Wayne S.
Kelley, Robert J.
Kudera, Andrew E.
Lipton, Barry
Loper, Dennis J.
Lyons, Daniel K.
Lyons, Mark D.
Matthews, Robert W.
McClure, John W., Jr.
McQuilkin, Mary T.
McSally, Michael J.
Mendelssohn, Gail A.
Nester, Karen L.
Onufer, Layne M.
Palmer, Donald W.
Paquette, Sylvie L.
Pence, Clifford A., Jr.
Peterson, Steven J.
Petit, Charles I.
Port, Rhonda D.
Putney, Alan K.
Raman, Rajagopalan K.
Rathjen, Ralph L.
Roth, Randy J.
Schultheiss, Peter J.
Sealand, Pamela J.
Smith, Byron W.
Smith, Judith P.
Trinh, Minh
Walker, Leigh M.
Walsh, Michael C.
Webster, Patricia J.
Weinman, Stacy J.
Woomer, Roy T., III
Yard, Roger A.
Yen, Chung-Ye

Part 9

Amundson, Richard B.
Bothwell, Peter T.
Brooks, Dale L.
Chanzit, Lisa G.
Chernick, David R.
Coffin, John D.
Deutsch, Robert V.
Dodd, George T.
Duffy, Thomas J.
Gillam, William R.
Hale, Jonathan, B.
Hein, Timothy T.
Hoppe, Kenneth J.
Kane, Adrienne B.
Knilans, Kyleen
Kolk, Stephen L.
Kooken, Michael W.
Kozik, Thomas J.
Marks, Steven D.
McDonald, Gary P.
McIntosh, Karol A.
Merlino, Matthew P.
Miner, Neil B.
Murdza, Peter J., Jr.
Neale, Catharine L.
Nichols, Richard W.
Normandin, Andre
O'Connell, Paul G.
Pelletier, Bernard A.
Pierson, Frank D.
Pinto, Emanuel
Plunkett, Richard C.
Ruegg, Mark A.
Sanders, Robert L.
Schmidt, Neil J.
Schwartzman, Joy A.
Sherman, Harvey A.
Siewert, Jerome J.
Thompson, Kevin B.
Tom, Darlene P.
Vaughan, Richard L.
Vitale, Lawrence A.
Wacek, Michael G.
Weimer, William F.
Wiseman, Michael L.

NEW FELLOWS ADMITTED MAY, 1983: The eighteen new Fellows admitted at the Doral are shown.

NEW ASSOCIATES ADMITTED MAY, 1983: Fifty of the fifty-nine new Associates admitted at the Doral are shown.

NEW FELLOWS ADMITTED NOVEMBER, 1983: Twenty-seven of the twenty-eight new Fellows admitted at Toronto are shown with President Kilbourne.

NEW ASSOCIATES ADMITTED NOVEMBER, 1983: Four of the seven new Associates admitted at Toronto are shown with President Kilbourne.

OBITUARIES

HARMON T. BARBER
EDWIN W. KITZROW
ARTHUR N. MATTHEWS
HARRY F. RICHARDSON

HARMON T. BARBER
1897–1983

Harmon T. Barber, a Fellow and past President of the Casualty Actuarial Society died February 19, 1983.

Mr. Barber was a native of Hartford, Connecticut. He was a graduate of Hartford Public High School and received a B.S. degree from Trinity College in 1919. He was a former president of the Trinity Board of Trustees. In 1955, he was the recipient of the highest honor that can be bestowed upon a Trinity graduate—the Eigenbrodt Cup.

Mr. Barber was associated with the Travelers Insurance Conpanies for over 43 years, retiring in 1962 as second vice president and actuary. During his distinguished career at the Travelers, he achieved his Fellowship in 1924.

Ham Barber's contributions to the Casualty Acturial Society and his work for the Society were extensive. He served on many of the standing committees of the CAS, and had two terms as an elected member of the Council, 1930 to 1933, and 1946 to 1947. From 1938 to 1940 he was one of the two Vice Presidents of the Society and in 1947 his second term on the Council was interrupted when he was again elected Vice President, serving from 1947 to 1949. The Society saluted his efforts in 1949 by electing him President and he served what was then the customary two-year term, from 1949 to 1951.

He was the first President of the CAS to deliver a presidential address at both the Spring meetings and the November meetings. During his term Ham Barber delivered four such addresses, all notable and worth reading or rereading: May 1950, "A Mid-Century Look at Casualty Insurance"; November 1950, "The Enigma of the Permissible Loss Ratio"; May 1951, "The Casualty Actuarial Profession"; November 1951, "The Gateway to Membership."

In addition, Mr. Barber was the author of four papers and at least one discussion of a paper authored by someone else. His papers were: "A Suggested Method for Developing Automobile Rates" (1928); "Compensation Expenses per Policy" (1934); "Can We Improve the Compensation Rating Method?" (1936); and "Mechanized Unit Reporting" (1946). Fourteen years later, in 1960, he contributed a discussion on L. H. Longley-Cook's paper, "The Census Method."

Mr. Barber also served with distinction on a number of special committees of the CAS: the Committee on Mortality for Disabled Lives, formed in 1939; the Committee on Unemployment Insurance (which later became the Committee on Social Insurance), formed in 1941; and the Committee on Compensation and Liability Loss and Loss Expense Reserves, formed in 1947.

He served as a second lieutenant with the field artillery during World War I, and was a member of the American Legion from 1933 to 1945 he was a chairman of the Windsor Board of Education. He was also a member of the First Church of Christ in Wethersfield.

He is survived by his wife, Louisa; a son, a daughter, a step-son, and a step-daughter; six grandchildren; and two great grandchildren.

ERWIN W. KITZROW
1900–1983

Erwin W. Kitzrow, an Associate of the Casualty Actuarial Society since 1935, died on October 26, 1983, after a long illness.

Mr. Kitzrow was a native of Milwaukee and attended the University of Wisconsin—Extension Division in Milwaukee. He served in the U.S. Coast Guard and U.S. Navy during World War I.

During the 1920's he was chief rater for the Workmen's Compensation Bureau in Milwaukee and later served as secretary of the state Insurance Commission.

In 1931, he joined Hardware Mutual Casualty Company as actuary and secretary, and eventually became vice president of underwriting. In 1950, he moved to Glendale, California, where he became president of Mid-Century Insurance Company.

Following his retirement in 1965, he lived in Altadena, and subsequently Escondido, California, where he did yacht and boat inspections along the coast of Southern California.

He is survived by his wife, Ruth; three sons, a step-daughter; and eleven grandchildren.

ARTHUR N. MATTHEWS
1898–1983

Arthur N. Matthews, a Fellow of the Casualty Actuarial Society since 1926, died October 16, 1983.

Mr. Matthews' entire insurance career was spent with the Travelers Insurance Companies. In 1943 he was made assistant actuary, in 1950 associate actuary, in 1956 actuary, and, in 1962 second vice president and actuary.

Born in Windsor, he graduated as valedictorian of the first class at Loomis School. He graduated as salutatorian in the Class of 1921 at Trinity, when he received his B.S. degree. He served on the College's endowment committee and had been a class agent since 1960.

Throughout his career, Mr. Matthews performed extensive service on behalf of the Society. He was a chairman of the Examination Committee, the Committee on Papers, and the Committee on Mortality for Disabled Lives. He was elected to the Council for three year terms in 1934 and 1942, the latter of which was extended to 1946, a one year term in 1947, and another three year term in 1952. He was elected vice president of the Society in 1955 and served two years.

In Windsor, he served five years on the Town Council, on the Zoning Board of Appeals and the Historical Society. He was a member of the First Church Congregational in Windsor, where he held numerous offices over the years. A fellow and past vice president of the Casualty Actuarial Association, he also served many years with SCORE (Service Corps of Retired Executives), the volunteer organization of the Small Business Administration.

He is survived by a son, Edgar W., of New Britain; a daughter, Nancy M. Swain, of Avon; five grandchildren and a great-granddaughter.

HARRY F. RICHARDSON
1887–1983

Harry F. Richardson, an associate of the Casualty Actuarial Society since 1932, died on June 5, 1983, after a sudden illness.

A native of the Commonwealth of Massachusetts, Mr. Richardson earned an S.B. degree in electrical engineering from the Massachusetts Institute of Technology in 1908.

Prior to entering the insurance industry, he worked as an electrical engineer. In 1920, he joined the National Council on Workmen's Compensation Insurance, in 1920 as secretary/treasurer with responsibility for underwriting and engineering. He was the third employee hired by the firm, which was the predecessor organization to the National Council on Compensation Insurance. At the end of a 35 year career, he retired in 1955 as the National Council general manager, the chief executive officer's title at that time.

Widely respected by consumer groups, regulatory and other government officials, and insurance carriers, Mr. Richardson was acknowledged as an outstanding authority on workers' compensation. He, in effect, grew up with the business at a time when the fledgling system was beginning to cope with rapid growth and a complex and challenging regulatory environment.

He is survived by his wife, Naomi; a daughter; and several grandchildren and great grandchildren.

INDEX TO VOLUME LXX

INDEX — CONTINUED