ON THE CREDIBILITY OF THE PURE PREMIUM

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With two exceptions, the many papers on credibility which have appeared in the *Proceedings* of the Casualty Actuarial Society have been concerned only with the credibility of the number of claims. From Whitney[1] to Mayerson[2] the theory has been based on the distribution of the number of claims alone, ignoring the distribution of claim amounts.

By assuming that the number of claims has a Poisson distribution, and approximating probabilities by use of a normal distribution, Whitney[3] and Perryman[4] developed a criterion for full credibility in terms of the expected number of claims. Bailey[5] and Mayerson[6] showed that the partial credibility formula $z = \frac{n}{n+k}$ holds for several distributions in addition to the normal. Buhlmann[7] derived this formula on a distribution-free basis for the claim amounts and the claim frequency.

Two papers which deal specifically with the credibility of the pure premium are Perryman[8] and Longley-Cook.[9] Perryman[10] states: "the volume of exposure required for full credibility of the pure premium requires the multiplication by the factor $1 + \frac{S^2}{M^2}$ of the number of claims required for credibility of the accident frequency." (M and S are the mean and standard

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[1] Whitney, A. W., "The Theory of Experience Rating," *PCAS* Vol. IV, p. 274 (1918).

[2] Mayerson, Allen L., "A Bayesian View of Credibility," *PCAS* Vol. LI, p. 85 (1964).

[3] Whitney, A. W., *op. cit.*

[4] Perryman, F. S., "Some Notes on Credibility," *PCAS* Vol. XIX, p. 65 (1932).

[5] Bailey, A. L., "Credibility Procedures, " PCAS Vol. XXXVII, p. 7 (1950).

[6] Mayerson, Allen L., *op. cit.*

[7] Buhlmann, Hans, "Experience Rating and Credibility," *ASTIN Bulletin* Vol. IV, p. 199 (1967).

[8] Perryman, F. S., *op. cit.*

[9] Longley-Cook, L. H., "An Introduction to Credibility Theory," *PCAS* Vol. XLIX, p. 194 (1962).

[10] Perryman, F. S., *op. cit.*, p. 72.

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deviation of the claim amount distribution.) Longley-Cook[11] derives Perryman's result in a slightly different form.

The credibility tables in general use are based on the distribution of the number of claims, but they are applied, in practice, to the pure premium. The standard for full credibility generally used in automobile liability insurance is 1,084 claims, corresponding to a normal distribution probability of 90% that the actual number of claims will not deviate from the expected number by more than 5% of the expected number (on the assumption that the mean is equal to the variance). For general liability insurance, the standard is usually 683 claims; this corresponds, on the assumption that the mean equals the variance, to a normal curve probability of 95% that the actual number of claims will not deviate from the expected number by more than 7½%. In neither case is the distribution of claim amounts taken into consideration, although these credibility tables are routinely used for ratemaking, where the pure premium, rather than the expected number of claims, is being determined. This procedure has recently been subject to criticism (Braverman[12]).

This paper will present a criterion for full credibility of the pure premium which does not depend on a specific distribution assumption for either the claim frequency or the claim severity.

PERRYMAN'S DERIVATION

Perryman assumed that the expected number of claims "is large so that the frequency distribution of the average claim cost is fairly normal."[13] He also followed earlier authors in assuming that claim frequency is approximately normally distributed and, in working with the pure premium, the product of the claim frequency and the average claim cost, he assumed that it, too, is approximately normally distributed. The approximate normality of the pure premium is an implication of the Central Limit theorem, rather than a consequence of the approximate normality of the claim frequency and the average claim cost. It should be noted that the density of the product of two random variables, both of which are normal, is not normal.

[11] Longley-Cook, L. H., *op. cit.*, Appendix C.

[12] Braverman, Jerome, D., "A Critique of Credibility Tables," *JRI* Vol. XXXIV, p. 409 (1967).

[13] Perryman, F. S., *op. cit.*, p. 72.

We will show that it is not necessary to assume that the claim frequency and the average claim cost are normally distributed. Perryman's results are valid if we assume that the number of claims has any distribution with equal mean and variance, and that the pure premium is normally distributed. We will also derive a criterion for full credibility based only on the moments of the distribution of the number of claims and on the moments of the claim amount distribution—without making an assumption about the specific form of either distribution. This derivation will permit the abandonment of the usual assumption, which underlies credibility tables in use today, that the number of claims, the size of a single claim, or the pure premium is normally distributed. Credibility tables can be based on moments derived from actual data, rather than on a hypothesis about the form of the distribution of the number of claims or of the pure premium.

A CRITERION FOR FULL CREDIBILITY

In a given classification (which may be, for instance, a territory), let N be the number of claims and let $X_1, X_2, X_3 \ldots\ldots$ be the individual claim amounts, in order of occurrence. Define a random variable T to be $X_1 + X_2 + X_3 + \ldots\ldots + X_N$, the sum of a random number of random variables. T, of course, can be interpreted as the total amount of claims. We will assume that: (a) the random variables $X_1, X_2, X_3, \ldots$ are indentically distributed, (b) that the random variables $N, X_1, X_2, \ldots\ldots$ are independent, and (c) that the random variables $N, X_1, X_2, \ldots\ldots$ have fourth moments.

We now define the concept of full credibility for the pure premium derived from the experience of a given classification during a given time period. This definition will depend upon two parameters, k and P, and is identical with Perryman's criterion (the observed pure premium should be within 100 $k\%$ of the expected pure premium with probability P). We will express our criterion in terms of T, the total amount of claims. (The pure premium can be derived from T by dividing by the exposure, which is a constant.)

Definition: A classification is said to be *fully credible (k, P)* if

$$\text{(A)}\quad Pr\,[(1-k)\,E(T) \leq T \leq (1+k)\,E(T)] \geq P$$

or equivalently, in terms of the standardized linear translate of T

$$\text{(B)}\quad Pr\left[\frac{-k\,E(T)}{\sigma_T} \leq \frac{T-E(T)}{\sigma_T} \leq \frac{k\,E(T)}{\sigma_T}\right] \geq P.$$

In practice, the exact distribution of T, the total amount of claims, is not available. Some approximation must be used to determine whether, for a given k and P, inequality (B) is satisfied. Perryman used the standard normal distribution to approximate the distribution of $\frac{T - E(T)}{\sigma_T}$. Since the normal distribution is symmetric, inequality (B) is satisfied if and only if $t_{(1+P)/2} \leq k \frac{E(T)}{\sigma_T}$ where $t_{(1+P)/2}$ is the $100\left(\frac{1+P}{2}\right)$ percentile point of the approximating distribution for $\frac{T - E(T)}{\sigma_T}$, in this case the standard normal distribution. Thus, for $P = .90$ we use the value $t_{.95} = 1.645$, as given in any table of the normal distribution function.

In this paper, the percentiles of the distribution of $\frac{T - E(T)}{\sigma_T}$ will be approximated by the first few terms of an expansion due to E. A. Cornish and R. A. Fisher (see Bowers[14]). The Cornish-Fisher expansion expresses a percentile of the distribution of $\frac{T - E(T)}{\sigma_T}$ as a percentile of the standardized normal distribution and certain correction terms, which adjust for the departure from normality of the distribution of T. The expansion requires a knowledge only of the moments of T. If z_e denotes the $100e$ percentile of the standard normal distribution, and t_e denotes the $100e$ percentile of $\frac{T - E(T)}{\sigma_T}$, the sum of the first few terms of the Cornish-Fisher expansion is:

$$\text{(C)} \quad t_e = z_e + \frac{\gamma_1}{6}(z_e^2 - 1) + \left[\frac{\gamma_2}{24}(z_e^3 - 3z_e) - \frac{\gamma_1^2}{36}(2z_e^3 - 5z_e)\right],$$

where $\gamma_1 = \frac{E[T - E(T)]^3}{\sigma_T^3}$ and $\gamma_2 = \frac{E[T - E(T)]^4}{\sigma_T^4} - 3$. Perryman's result can be obtained by omitting all terms after the first (thereby assuming that T is normally distributed) and solving the inequality $t_{(1+P)/2} \leq k \frac{E(T)}{\sigma_T}$, for a given P and k, under the assumption that $E(T) = \sigma_T^2$.

We now express the moments of T in terms of the moments of N and of the X_i's. These moments are needed in inequality (B) and to compute

[14] Bowers, Newton L., Jr., "An Approximation to the Distribution of Annuity Costs," *TSA* Vol. XIX, p. 295 (1967).

the γ's used in the Cornish-Fisher expansion (C). We will use a property of conditional expectations (see Brunk[15]):

$$E\,[g(T,N)] = E_N\,[E\{g(T,N) \mid N\}]\,.$$

In the formulas which follow:

$$E(N) = \lambda \qquad E[(N-\lambda)^i] = \lambda_i \qquad \text{for } i = 2, 3, \ldots\ldots$$

$$E(X) = \mu \qquad E[(X-\mu)^i] = \mu_i \qquad \text{for } i = 2, 3, \ldots\ldots$$

Since $E(T \mid N) = E(X_1 + X_2 + X_3 + \ldots\ldots\ldots + X_N \mid N) = N\mu$, it follows by setting $g(T,N) = T$ that $E(T) = E_N\,[E(T \mid N)] = E(N\mu) = \mu E(N) = \mu\lambda$.

Now let $g(T,N) = (T-\mu\lambda)^2 = [T - E(T)]^2$.

Since $E\,[(T-\mu\lambda)^2 \mid N] = E\,[\{(T - N\mu) + (N\mu - \lambda\mu)\}^2 \mid N]$

$= N\mu_2 + \mu^2(N-\lambda)^2$, it follows that

$$E\,[\{T - E(T)\}^2] = E_N\,[N\mu_2 + \mu^2(N-\lambda)^2]$$

$$= \mu_2\lambda + \mu^2\lambda_2.$$

We can also show, by a similar algebraic development, that:

$$E[\{T - E(T)\}^3] = \mu_3\lambda + 3\mu_2\mu\lambda_2 + \mu^3\lambda_3,$$

$$E[\{T - E(T)\}^4] = \mu_4\lambda + 3\mu_2^2(\lambda_2 - \lambda + \lambda^2) + 4\mu\mu_3\lambda_2 + 6\mu^2\mu_2(\lambda_3 + \lambda\lambda_2) + \mu^4\lambda_4.$$

We now apply the Cornish-Fisher expansion, using the moments just developed. In applications involving risk theory, $E[\{T - E(T)\}^3]$ is greater than zero, since all terms in the formula, except μ_3 and λ_3, are positive. In the usual models for the number of claims, the Poisson distribution or the negative binomial, λ_3 is positive; claim amounts, too, have positive skewness in most lines of property and casualty insurance, so μ_3 is also greater than zero. Thus the third central moment of T, hence λ_1, is positive, hence $t_{(1-P)/2}$ and $t_{(1+P)/2}$ are not equal, as they are in the case of a symmetric distribution like the normal. Because positive skewness implies that the longer "tail" of the distribution is to the right of the mean, $t_{(1+P)/2}$ is greater than $\mid t_{(1-P)/2} \mid$ for values of P of interest in credibility theory. To satisfy the inequality (B) we will set $\dfrac{k\,E(T)}{\sigma_T} = t_{(1+P)/2}$ which produces an

[15] Brunk, H. D., *An Introduction to Mathematical Statistics*, Blaisdell Publishing Co., 1965.

interval with probability greater than P and a slightly conservative full credibility standard.

For a given k and P we can let $t_e = t_{(1+P)/2} = \dfrac{k\,E(T)}{\sigma_T}$ in the Cornish-Fisher expansion. For algebraic simplicity, we will use only two terms of the expansion (C):

$$\frac{k\,E(T)}{\sigma_T} = z_e + \frac{E[T - E(T)]^3}{6\sigma_T^3}(z_e^2 - 1).$$

Substituting the moments of T just developed, we obtain:

$$\frac{k\mu\lambda}{\sqrt{\mu_2\lambda + \mu^2\lambda_2}} = z_e + \frac{\mu_3\lambda + 3\mu_2\mu\lambda_2 + \mu^3\lambda_3}{6(\mu_2\lambda + \mu^2\lambda_2)^{3/2}}(z_e^2 - 1).$$

We then have the following equation which must be satisfied by λ, the expected number of claims:

$$\text{(D)}\quad k\lambda = z_e\sqrt{\lambda}\sqrt{\frac{\lambda_2}{\lambda} + \frac{\mu_2}{\mu^2}} + \frac{z_e^2 - 1}{6}\,\frac{\dfrac{\lambda_3}{\lambda} + \dfrac{3\lambda_2}{\lambda}\dfrac{\mu_2}{\mu^2} + \dfrac{\mu_3}{\mu^3}}{\dfrac{\lambda_2}{\lambda} + \dfrac{\mu_2}{\mu^2}}.$$

If we ignore the term involving the third moment, we obtain the following simple equation for λ:

$$\text{(E)}\quad \lambda = \frac{z_e^2}{k^2}\left(\frac{\lambda_2}{\lambda} + \frac{\mu_2}{\mu^2}\right).$$

If we assume that the mean and the variance of the number of claims are equal, and remember that z_e is the $100\,\dfrac{1+P}{2}$ percentile of the standard normal distribution [$\sqrt{2}$ times Perryman's $f(P)$, since Perryman used a normal distribution with variance ½] and that μ and μ_2 are the mean and variance of the claim amount distribution (Perryman's M and S^2), formula (E) becomes

$$\lambda = \frac{2f(P)^2\left(1 + \dfrac{S^2}{M^2}\right).}{k^2}$$[16]

[16] Perryman, F. S., *op. cit.*, p. 72.

THREE EXAMPLES

To illustrate the use of the Cornish-Fisher expansion, we give three examples of the calculation of λ, the expected number of claims necessary for full credibility. We will, in each case, use for k and P the values commonly used in automobile insurance credibility tables, $k = .05$ and $P = .90$.

Poisson Distribution — Automobile Insurance

If we assume that N, the number of claims, has a Poisson distribution, then $\lambda_3 = \lambda_2 = \lambda$. This is the assumption underlying most credibility work done to date.

To use the Cornish-Fisher expansion, we need the moments of X, the size of an individual claim. These may be obtained from a study of claims by size of loss, data which are not readily available. A study of one large company's 1952 experience on 2,116 automobile property damage claims yielded the following moments:

$$\mu = 89.82$$

$$\mu_2 = 26{,}060$$

$$\mu_3 = 28{,}740{,}000$$

Thus the values of the ratios $\frac{\mu_2}{\mu^2}$ and $\frac{\mu_3}{\mu^3}$ are 3.230 and 39.658 respectively.

Because of the Poisson assumption, formula (D) becomes:

$$\text{(F)} \quad k\lambda = z_e \sqrt{\lambda} \sqrt{1 + \frac{\mu_2}{\mu^2}} + \frac{z_e^2 - 1}{6} \, \frac{1 + 3\frac{\mu_2}{\mu^2} + \frac{\mu_3}{\mu^3}}{1 + \frac{\mu_2}{\mu^2}}$$

and, substituting $k = .05$, $z_e = z_5 = 1.645$ and the moments of X given above, we have:

$$.05\lambda = 1.645 \sqrt{\lambda} \sqrt{4.230} + (.2843) \frac{50.348}{4.230}, \text{ and}$$

$$\lambda = 4{,}713.$$

If, instead, we solve equation (E), we obtain $\lambda = 4{,}577$. The similarity of these two results indicates the modest effect of including the third moment of the claim distribution in the calculation, but emphasizes the importance of recognizing the effect of the variation in size of claim in credibility calculations. Assuming that claim frequency follows a Poisson distribution, it takes 4,713 claims, not 1,084, to achieve a 90% probability that actual claims will be within 5% of expected. If we solve equation (D) for k, using $\lambda = 1084$, we find that a full credibility criterion of 1,084 claims produces a probability of 90% that actual claims will be within 10.6%, not 5%, of expected claims.

Negative Binomial Distribution

The negative binomial distribtuion has now replaced the Poisson distribution in the affections of casualty actuaries. In his paper, Dropkin[17] used the negative binomial to represent the distribution of the number of auto claims per policy. His data show a mean of .163 and variance of .193, which are inconsistent with the moments of the Poisson distribution.

In Dropkin's notation, the probability function for the number of claims can be written as:

$$Pr(N = x) = \binom{x + r - 1}{x} \left(\frac{a}{1 + a}\right)^r \left(\frac{1}{1 + a}\right)^x \quad \text{for } x = 0, 1, 2, \ldots .$$

The first three moments are:

$$\lambda = E(N) = \frac{r}{a} \qquad \lambda_2 = E[\{N - E(N)\}^2] = \frac{r}{a}\,\frac{a+1}{a}$$

$$\lambda_3 = E[\{N - E(N)\}^3] = \frac{r}{a}\,\frac{a+1}{a}\,\frac{a+2}{a}.$$

From this, it is easy to verify that $\lambda_3 = \frac{\lambda_2}{\lambda}(2\lambda_2 - \lambda)$ and that

$$\frac{\lambda_3}{\lambda} = \frac{\lambda_2}{\lambda}\left(2\frac{\lambda_2}{\lambda} - 1\right).$$

If we assume that we have E independent exposure units and use Dropkin's data, then $\lambda = .163E$ and $\lambda_2 = .193E$, $\frac{\lambda_2}{\lambda} = 1.184$ and $\frac{\lambda_3}{\lambda} = 1.620$. We will use the same moment ratios for the claim amount distribution as in the preceding example, namely $\frac{\mu_2}{\mu^2} = 3.230$ and $\frac{\mu_3}{\mu^3} = 39.658$. We can now write equation (D) for λ, the number of claims required for full credibility, $k = .05$ and $P = .90$:

$$.05\lambda = 1.645\sqrt{\lambda}\sqrt{4.414} + (.2843)\,\frac{52.751}{4.414}.$$

$$\lambda = 4{,}913.$$

[17] Dropkin, L. B., "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records," *PCAS* Vol. XLVI, p. 165 (1959).

If we solve equation (E), which ignores third moments, we obtain $\lambda = 4{,}776$.

Poisson Distribution — Workmen's Compensation Insurance

Our third example will again use the Poisson distribution for the number of claims, but will use, for the moments of the claim amount distribution, the data in Dropkin[18] on the distribution of California workmen's compensation losses for major permanent partial cases and for temporary total disability cases, policy year 1961 first reports.

For the 4,721 major permanent partial cases included in Dropkin's study, $\mu = 13{,}687.67$, $\mu_2 = 85{,}715 \times 10^3$, $\mu_3 = 461{,}448 \times 10^7$, $\frac{\mu_2}{\mu^2} = .4575$ and $\frac{\mu_3}{\mu^3} = 1.7994$. Substituting these values in formula (F) and again using $k = .05$ and $P = .90$, we have:

$$.05\lambda = 1.645\sqrt{\lambda}\sqrt{1.4575} + (.28434)\frac{4.1719}{1.4575} \text{ and}$$
$$\lambda = 1{,}610.$$

If we solve equation (E), we obtain $\lambda = 1{,}578$.

For the 60,398 temporary total disability claims included in the Dropkin data, $\mu = 513.80$, $\mu_2 = 689{,}244$, $\mu_3 = 345{,}857 \times 10^4$, $\frac{\mu_2}{\mu^2} = 2.6109$ and $\frac{\mu_3}{\mu^3} = 25.4985$.

Substituting these values in equation (F), with $k = .05$ and $P = .90$, we have:

$$.05\lambda = 1.645\sqrt{\lambda}\sqrt{3.6109} + (.28434)\frac{34.3312}{3.6109} \text{ and}$$
$$\lambda = 4{,}016.$$

If we solve equation (E) with these same data, we obtain $\lambda = 3{,}908$.

CONCLUSION

On the basis of the above results, the following conclusions seem to be justified:

(1) The usual criteria for full credibility, 1,084 claims in automobile

[18] Dropkin, L. B., "Size of Loss Distributions in Workmen's Compensation Insurance," *PCAS* Vol. LI, p. 198 (1964).

insurance and 683 claims in general liability insurance, are too low. If we adopt a Poisson distribution for number of claims and the moments of the claim amount distribution derived from the automobile data available to us, 1,084 claims as a standard for full credibility results in a probability of 90% that actual claims will be within 10.6% of expected claims. The use of 683 claims as a standard for full credibility, on the assumption that the shape of the general liability claim amount distribution is not too dissimilar from that for automobile insurance, yields a 90% probability that actual claims will be within 13.4% of expected. In view of the 5% margin for underwriting profit and contingencies built into most liability insurance rates, swings of 10.6% or 13.4% in claims seem to be larger than prudent management ought to be willing to accept.

(2) If the third moment of the claim amount distribution is used, thereby recognizing the positive skewness inherent in most insurance claim patterns, the number of claims needed for full credibility is increased by 3% to 10% (based on the data used in the paper).

(3) If a negative binomial distribution is adopted for the number of claims instead of a Poisson, the number of claims needed for full credibility is increased. Credibility tables currently in use for liability insurance are based on the assumption that the mean equals the variance, as in the Poisson distribution.

(4) The number of claims needed for full credibility of the pure premium varies substantially by coverage. The results shown in the paper, 4,713 claims for automobile liability insurance, 1,610 claims for major permanent partial disability, and 4,016 claims for temporary total disability, indicate the need for separate credibility tables by coverage and, for workmen's compensation, by type of claim. (It should be noted that the automobile insurance data on which the moments of the claim amount distribution were based comprised only 2,116 claims. Thus no particular credence should be given to the particular figure of 4,713 claims until it is substantiated by a calculation based on a larger and more recent block of claims by size of loss.)

By expressing the number of claims required for the pure premium in a given classification to have full credibility in terms of the moments of the distribution of the number of claims, the moments of the distribution of claim amounts, and a selected normal distribution percentile, this paper has attempted to supply a basis for more accurate and scientific credibility tables. The formula, however, requires much more data on losses by size than is currently available. If losses by size data were available for various coverages, both countrywide and by state, it would be possible to calculate the full credibility point for each coverage and state. It should be noted, however, that the number of claims required for full credibility does not depend on the magnitude of the moments of the claim distribution, but only on the relationship between the higher moments and the mean. We suspect, therefore, that the credibility calculation for a given coverage will be relatively stable from state to state and from year to year.