RATE REVISION ADJUSTMENT FACTORS

BY

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INTRODUCTION

Any line of insurance which uses the loss ratio method in rate making relies very heavily on an accurate premium base. If exposure data were available, a pure premium method would most likely be used but in the absence of proper exposure data, the rate revision adjustment factor is vital to the determination of the premium base. Without it, this valuable rate making method based upon loss ratios would be impractical. Rate revision adjustment factors are also useful for individual companies in evaluating their loss experience, projecting premium volumes, establishing comparative statistics under varying rate levels and in budgeting problems where the available amount of expense loading is desired. With so many uses, one would expect to find some literature on the subject, but our Proceedings has never had such a paper presented. Of course, it would be unnecessary to devote much space to a subject if no problems presented themselves or if the solutions to the problems were obvious. Neither is true in this instance, since problems do exist in this area and the solutions are at times difficult and the results surprising.

A rate revision adjustment factor is defined as a number which, when multiplied by a set of collected premiums, will revise or correct these premiums to reflect a new or current set of rates. The definition of a rate revision adjustment factor implies: (a) the existence of a set of rates which are applied to exposures over a period of time; (b) this set of rates is changed; and (c) the new rates are applied to other exposures for a second period of time. The sum of the two sets of premiums produces the collected premium for the entire period. As an example, between January 1 and May 1, five risks are written at $100. each and between May 1 and December 31, seven similar risks are written at the revised rate of $110. each. The collected premium of $1270. can be corrected to a premium at current rates by a rate revision adjustment factor of 1.0394 (i.e., $\frac{1320}{1270}$) to produce the revised premium of $1320. In actual practice we will be given the $100. rate, the $110. rate, the May 1 date of change, and the collected premium of $1270. In some lines of insurance the full year's written exposure of 12 risks will also be known, but in other lines it will not. In either event, it will be our task to determine the rate revision adjustment factor by the appropriate mathematical means, apply it to the collected premium and thus obtain the premium adjusted to current rates.

The object of this paper is to develop a sound approach to obtaining rate revision adjustment factors (hereafter called F) and to compare and discuss various phases of the problem. The paper will (a) treat the most restrictive and simplest case, (b) discuss at length the problem of installment payment of term policies under the annual reporting method of recording installments, (c) relax the restriction requiring a constant volume of business and study its effect, and (d) as a corollary, treat the comparison of two different rate levels to find an "average difference factor" or more familiarly an average deviation. The paper will be confined to consideration of the rate revision adjustment factor necessitated by a single rate change. When it is desired in actual practice to modify premiums to reflect a number of rate changes, a combination factor may be developed by multiplication. For example, a 10% increase followed by a second 10% increase would be equivalent to a 21% increase when adjusting premiums prior to the first increase up to the current level. Finally, it should also be noted that the scope of the paper will be confined to these factors as they apply to a set of written premiums. Results might be quite different if proper factors for application to earned premiums were developed.

The conclusions at the end of the paper are supported by the mathematical development in the next section. For the reader who wants to examine the conclusions immediately, the numbers in parentheses refer to formulas in the next section; the definitions of symbols are presented in Appendix A. Let us now proceed with the development of the formulas.

MATHEMATICAL DEVELOPMENT

Case A is that of a number of exposure units or sum insured of S which are written during the course of a year. Part of these S units are written at a premium rate of r per unit during the first part of the year (1-a). A new rate r′ becomes effective and applies to that part of the S units written during the remaining portion of the year (a). Define d as the rate change expressed as a decimal number from which it follows that

$$d = \frac{r'}{r} - 1 \quad \text{(1)}$$

For future use this may be rewritten as

$$r = \frac{r'}{1 + d} \quad \text{(2)}$$

P will be the premium collected during the year, P′ is the premium P corrected by the rate revision adjustment factor F to the amount which would have been collected if the r′ rates had been in effect for the full year. From this definition we have

$$P' = FP \quad \text{(3)}$$

$$\text{and } P' = Sr' \quad \text{(4)}$$

Under the assumption that S is evenly distributed throughout the year, the collected premium may be expressed as follows:

$$P = [S(1\text{-}a)]\, r + [Sa]\, r' \quad \ldots\ldots (5)$$

By substituting (2), rearranging terms and substituting (4)

$$P = S\left[(1\text{-}a)\frac{r'}{1+d} + ar'\right]$$

$$= Sr'\left[\frac{1+ad}{1+d}\right]$$

$$= P'\left[\frac{1+ad}{1+d}\right]$$

From (3) we thus conclude that

$$F = \frac{P'}{P} = \frac{1+d}{1+ad} \quad \ldots\ldots (6)$$

This is a very general and useful form in that the period under study can be of any length* as long as "a" is the portion on the new rate level, the factor can be used equally well on policy year or calendar year data, and the rate change d may be for a very small subdivision of a line or may be an average change covering a large number of classes or territories. The formula is also applicable in fire where annual renewal business and where prepaid term business is involved. When term business paid on an installment plan is recorded on the company books as a single entry at the inception of the policy (called the full term reporting method) this formula applies equally well. As will be discussed under Case B, this formula is not applicable when installment payment business is recorded on the books only as each installment becomes due—the so-called annual reporting method for installment payment of term business.

Consider for a moment the effect of adopting the intuitive approach to F. This might lead to the use of an erroneous adjusted premium, P'_e, by use of the following formula:

$$P'_e = P \times (1\text{-}a)\,(1+d) + P \times a \times 1.00$$

Or perhaps the reasoning runs

$$P'_e = P + P \times (1\text{-}a) \times d$$

In either event, the equation simplifies to:

$$P'_e = P\,(1 + d - ad) \quad \ldots\ldots (7$$

*Ordinarily, it would be one year.

If we define the erroneous rate revision adjustment factor as F_e, then from (7),

$$F_e = \frac{P'_e}{P} = (1 + d - ad) \qquad (8)$$

To compare the factor F from (6) with F_e from (8), define

$$C = \frac{F}{F_e} \qquad (9)$$

That is, C is a correction factor necessary to correct F_e to the proper factor, F. Substituting (6) and (8) in (9) we have

$$C = \frac{1 + d}{(1 + ad)(1 + d - ad)} = \frac{1 + d}{1 + d + ad^2(1 - a)}$$

$$\text{or } C = \frac{1}{1 + \frac{ad^2(1 - a)}{1 + d}} \qquad (10)$$

The most interesting fact about this equation is that the fraction in the denominator is always positive, thus making $C < 1$ under all circumstances (except $d = 0$ which is trivial). This, of course, means that P'_e is too large a number and rates made by the loss ratio method will consistently include an element of inadequacy. Fortunately, the error is small, ranging up to about 1¼% under a 20% rate reduction, but when we are only dealing with a 5% profit margin, even small errors become important and especially so when they are always in one direction.

Appendix B has been calculated to illustrate the magnitude of the various factors under selected rate revisions when they are made effective in midyear ($a = \frac{1}{2}$). The first section is designated $w = o$ and relates to the equations currently being considered. For example, if a 20% rate increase is made at midyear, the proper rate revision adjustment factor is 1.0909; the one commonly used is 1.1000; the error in using the wrong factor is 0.83%. These interpretations are obtained from the first three entries in the first column of figures in Appendix B. The inadequacy of formula (7) is clearly shown by values of C which reach an inadequacy of 1.23% for a 20% rate reduction.

Case B will be that of a five-year installment payment policy using the annual reporting method of recording the business. Under this system, the policy is written for a five-year term, but the premium is recorded on the company books each year for five years as it is collected. If the year in which the rate revision is made is designated year 0, then the premiums collected on five-year installment business during year 0, denoted ${}_5P_0$, will be made up of premiums from policies written during years 0, -1, -2, -3 and -4.

Define ${}_5S_i$ as the sum insured under such policies written during year i. When a rate revision is made we will collect r ${}_5S_{-4}$ from installments on policies written in year −4 plus similar elements of r ${}_5S_{-3}$, r ${}_5S_{-2}$ and r ${}_5S_{-1}$. The premium collected on policies written in year 0 will be r ${}_5S_o$ (1-a) + r′ ${}_5S_o$ a. Adding up the five segments we have

$${}_5P_o = r({}_5S_{-4} + {}_5S_{-3} + {}_5S_{-2} + {}_5S_{-1} + {}_5S_o - {}_5S_o a + {}_5S_o\, a\, \frac{r'}{r}) \ldots (11)$$

To simplify the evaluation of this equation, two key assumptions are made: (a) ${}_5S_i$ is constant and equal to $(\Sigma_5 S_i)/5$ for each year during the period (this is equivalent to saying that the total exposure insured under five-year installment policies is $\Sigma\, {}_5S_i$ and it is evenly spread over the period) and (b) installments are recorded under the annual reporting method in equal amounts of .20 in each of the five years instead of the actual .22 the first year and .195 for each of the next four years.* This latter assumption will, in fact, be exactly fulfilled under the formula introduced in certain states which sets the installment premium at 35% of the three-year term premium for each of the five years.

Define ${}_5P'_i$ as the collected premium in year i under five-year installment policies and ${}_5F_i = {}_5P'_i/{}_5P_i$. Then (11) may be simplified by use of (2), (4), and the foregoing assumptions and definitions:

$${}_5P_o = \frac{r'}{1+d}\left[\frac{\Sigma\, {}_5S_i}{5}(5) - \frac{\Sigma\, {}_5S_i}{5}\, a + \frac{\Sigma\, {}_5S_i}{5}(a + ad)\right]$$

$${}_5P_o = {}_5P'_o \left[\frac{1 + \frac{a}{5} d}{1 + d}\right] \ldots\ldots (12)$$

$${}_5F_o = \frac{1 + d}{1 + \frac{a}{5} d} \ldots\ldots (13)$$

Similar reasoning can be applied to each of the years 1, 2, 3 and 4 which result in successively dropping off r ${}_5S_{-4}$, r ${}_5S_{-3}$, etc. while successively adding r′ ${}_5S_1$, r′ ${}_5S_2$, etc. The resulting solutions form a pattern which may be generalized:

$${}_5F_i = \frac{1 + d}{1 + \frac{a + i}{5} \cdot d} \quad (i = 0, 1, 2, 3, 4) \ldots\ldots (14)$$

*This latter system of annual recording introduces a further distortion in the rate making process. Since the premium is earned too fast because of the .22 element being used the first year, we again have an overstatement of the premium base and, hence, an inadequacy in the rates made on this basis. See also Proceedings of the National Association of Insurance Commissioners, Eighty-third Session, 1952, pp. 45–46.

We see from (14) that a rate change should be reflected in each of the five years following its effective date if business can be written under an installment plan and recorded on the annual reporting method. Under any system that ignores the consequences of five-year business we would only get the effect of applying (6) to year 0. This formula makes it necessary to investigate the rate levels over nine years if a rate change is to be based on five years of experience. This is necessitated because the earliest one of the five years has its income affected by installments collected on policies written four years earlier —hence, if there were a rate change during this fourth previous year, strict accuracy would require that part of its effect be reflected in the earliest year. With high speed electronic equipment containing large storage capacity, such a program could possibly be carried out. Some simplification would be desirable under present conditions which usually employ desk calculators and this leads us to the next case.

Case C will "telescope" the five-year effect of a rate change on installment business into the initial year 0. The reasoning here is that the full effect of a rate change will be reflected immediately in the premium and it is hoped the distortion produced by not using (14) will be small enough to be offset by the computational savings. To accomplish this "telescoping" we add to ${}_5P_o$ only the increment of change from each of the years 1 through 4. Define ${}_5P''_o$ as the premium of year 0 under installment policies recorded on the annual reporting method which has been adjusted to reflect the changes in premium over each of five years due to a rate change made in 0.

$$\begin{aligned} {}_5P''_o &= {}_5P_o + ({}_5P'_o - {}_5P_o) + ({}_5P'_1 - {}_5P_1) + ({}_5P'_2 - {}_5P_2) \\ &\quad + ({}_5P'_3 - {}_5P_3) + ({}_5P'_4 - {}_5P_4) \\ &= {}_5P_o + \sum_{i=o}^{4} ({}_5P'_i - {}_5P_i) \\ &= {}_5P_o + \sum_{i=o}^{4} \left(1 - \frac{{}_5P_i}{{}_5P'_i}\right) {}_5P'_i \end{aligned}$$

Under our assumption of an even distribution of exposure over the five-year period, all the ${}_5P'_i$ will be equal, so we substitute ${}_5P'_o$ for the term outside the parenthesis and then substitute (12). Simultaneously, (14) will be substituted inside the parenthesis.

$${}_5P''_o = {}_5P_o + {}_5P_o \left[\frac{1 + d}{1 + \frac{a}{5} \cdot d}\right] \sum_{i=o}^{4} \left[1 - \frac{1 + \frac{a + i}{5} \cdot d}{1 + d}\right]$$

Upon simplification, this becomes

$$ {}_5P''_o = {}_5P_o\left[1 + \frac{15d - 5ad}{5 + ad}\right] \qquad (15) $$

Then if ${}_5F''_0$ is defined as $\frac{{}_5P''_o}{{}_5P_o}$ we have

$$ {}_5F''_o = 1 + \frac{15d - 5ad}{5 + ad} \qquad (16) $$

Now let us study the effect of using (6) on the year 0 premium for five-year installment business when we should use (16). Define a correction factor

$$ {}_5C'' = \frac{{}_5F''_o}{F} $$

$$ {}_5C'' = \frac{(5 - 4ad + 15d)\,(1 + ad)}{(5 + ad)\,(1 + d)} $$

or

$$ {}_5C'' = 1 + \frac{d(10 - 4a^2d + 14ad)}{(5 + ad)\,(1 + d)} \qquad (17) $$

The second term has its sign controlled by the sign of d. So, if $d > 0$, ${}_5C'' > 1$ which means that (6) will produce too small a premium (and would need a correction factor in excess of 1 to rectify it). This means that if the rate trend has been generally upward, (6) would tend to continue this trend beyond the time true experience would call for a downturn. Conversely, if a rate trend has been downward, (6) tends to perpetuate the trend even after the true experience would call for an upward revision. Rate increases are often hard to come by—it would be unfortunate if we continued a practice that gives us more rate decreases than the truth warrants.

Appendix B illustrates the values taken by the various formulae.

Throughout the discussion thus far we have always assumed the exposure to be written evenly over the period. Let us instead now define ${}_o\phi$ as the exposure in force at the beginning of the year and ϕ_1 as exposure in force at the end of the year. In Case D we treat annual policies as we did in Case A but now they will have a continuous rate of growth of w (corresponding to the investment concept of interest convertible continuously). Define $\overline{P}$ and $\overline{P'}$ as before but now a continuous rate of growth is involved in our assumptions. The premium at revised rates will be

$$ \overline{P'} = \int_0^1 \phi_o\, r'\, (1 + w)^t dt $$

where t is an increment of time between the beginning and the end of the year. This reduces to

$$\overline{P'} = \frac{\phi_o r' w}{\log(1 + w)} \quad \text{.......................(18)}$$

where the abbreviation "log" is the base e logarithm.

The collected premium may be expressed as

$$\overline{P} = \phi_o r \int_o^{1-a} (1 + w)^t dt + \phi_o r' \int_{1-a}^{1} (1 + w)^t dt$$

Integrating, evaluating and substituting (2) we have

$$\overline{P} = \frac{\phi_o r'}{\log(1 + w)} \left[(1 + w)^{1-a} \left(\frac{-d}{1 + d}\right) + \frac{d + w(1 + d)}{1 + d} \right] \quad \text{...(19)}$$

By (18), substitute $\frac{\overline{P'}}{w}$ for the term outside the brackets and at the same time define $\overline{F} = \frac{\overline{P'}}{\overline{P}}$. This results in

$$\overline{F} = \frac{(1 + d)w}{(1 + d)w + d - d(1 + w)^{1-a}} \quad \text{...........(20)*}$$

or

$$\overline{F} = \frac{1}{1 + \frac{d[1 - (1 + w)^{1-a}]}{(1 + d)w}} \quad \text{....................(21)}$$

As shown in Appendix C, w can be calculated from observed data as

$$w = \log \frac{\phi_1}{\phi_o} \quad \text{.........................(22)}$$

To compare (21) with our assumption of a constant volume in (6), define $\overline{C}$ as the correction factor necessary to change F (which is based on an otherwise correct calculation) to $\overline{F}$. That is,

$$\overline{C} = \frac{\overline{F}}{F} = \frac{1 + ad}{1 + d + \frac{d}{w}[1 - (1+w)^{1-a}]} \quad \text{..............(23)}$$

*If $w = o$, $\overline{F}$ becomes the indeterminate form $\frac{o}{o}$. Upon differentiating both numerator and denominator, $\overline{F}_{w=o} = \frac{1+d}{1+ad}$. This is the same as (6), which it should be.

In search of an approximation,

$$1-(1+w)^{1-a} = 1 - \left[1 + (1-a)w + \frac{(1-a)\,(-a)}{2!}w^2 + \frac{(1-a)\,(-a)\,(-a-1)}{3!}w^3 + \cdots \right]$$

$$= - \left[(1-a)w - \frac{(1-a)\,(a)}{2}w^2 + \frac{(1-a)\,(a)\,(a+1)}{6}w^3 - \cdots\cdots \right]$$

Thus

$$\overline{C} = \frac{1 + ad}{(1+d) - (1-a)d + \frac{d(1-a)\,(a)w}{2} - \frac{d(1-a)\,(a)\,(a+1)}{6}w^2 + \cdots} \quad (24)$$

While w can theoretically reach values in excess of 1.00, it seems that a practical working limit would be between + .20 and — .20. A reasonable figure for d might be ± .15 and a is selected at ½ as a typical figure. Under these conditions, the maximum error in the $\overline{C}$ caused by omitting the last term and all subsequent terms in the denominator of (24) is given by

$$\frac{d(1-a)\,(a)\,(a+1)}{6(1+ad)}w^2$$

Under the conditions outlined, this is on the order of .0004. This is sufficiently small that (24) may be written as

$$\overline{C}_x = \frac{1 + ad}{1+ad + \frac{adw(1-a)}{2}}$$

where $\overline{C}_x$ indicates an approximation of $\overline{C}$.

$$\overline{C}_x = \frac{1}{1 + \frac{adw(1-a)}{2(1+ad)}} \quad (25)$$

In the light of (25) we can better judge whether the effect of increasing volume is sufficient to warrant the use of the more complicated (21) in lieu of (6). Equation (21) can be simplified by using the series expansion employed in arriving at (25) if the user is willing to waive the possible effect of a maximum error in $\overline{F}$ of

$$\frac{d(1-a)\,(a)\,(a+1)}{6(1+ad)}w^2$$

This approximation of $\overline{F}$, called $\overline{F}_x$ is

$$\overline{F}_x = \frac{1}{1 - \frac{d(1-a)\,(2-aw)}{2(1+d)}} \qquad (26)$$

As we look at (25) the effect can clearly be seen of assuming a constant volume of business when it is in fact changing over the year. If d and w are both positive or both negative, then assuming a constant volume will produce too high a revised premium and, hence, too low a rate. Thus, in an expanding economy and in a time of generally rising rates, a constant volume assumption will put an element of inadequacy in the rates. When combined with the element of inadequacy from equation (10), we may be reaching serious proportions. If d and w are of opposite sign, rates produced on the constant volume assumption would contain an element of excessiveness which would be somewhat counterbalanced by the inadequacy from (10). When installment business is involved, (17) introduces another element which will sometimes increase and sometimes decrease the rates. Appendix B contains a section for $w = +.10$ and one for $w = -.10$. It can be seen that the approximations are very good for the selected values. Another interesting observation is that for a given value of d, the values for $w = +.10$ and for $w = -.10$ multiply to 1.000. This is a case then where an increase followed by a decrease of the same percentage are offsetting. Finally, in the opinion of the author, $\overline{C}$ is sufficiently close to 1.000 that for most practical purposes it can be ignored up to values of $w = \pm .10$ if computational simplicity is desired. This will then permit the use of (6).

Case E (corresponding to Case B) will study five-year installment business under an assumption of a continuous rate of growth W. Define ${}_5\phi_i$ as the five-year installment exposure in force at the beginning of the year "i" which will be rewritten during the year at the rates then in effect. (Note: The total exposure in force for all the policies would be roughly five times this amount, but only one-fifth of all policies will be up for rewriting during any year. This definition corresponds to the definition of ${}_5S_i$). Corresponding to equation (11) we may now write:

$$
\begin{aligned}
{}_5\overline{P}_o &= \int_{-4}^{-3} r\,{}_5\phi_o\,(1+W)^t\,dt + \int_{-3}^{-2} r\,{}_5\phi_o\,(1+W)^t\,dt \\
&+ \int_{-2}^{-1} r\,{}_5\phi_o\,(1+W)^t\,dt + \int_{-1}^{0} r\,{}_5\phi_o\,(1+W)^t\,dt \\
&+ \int_{0}^{1-a} r\,{}_5\phi_o\,(1+W)^t\,dt + \int_{1-a}^{1} r'\,{}_5\phi_o\,(1+W)^t\,dt
\end{aligned}
$$

This may be compressed into one integral involving r and one integral involving r′ and generalized to

$$ {}_5\overline{P}_i = \int_{-4+i}^{1-a} r\ {}_5\phi_o\,(1+W)^t\,dt + \int_{1-a}^{1+i} r'\ {}_5\phi_o\,(1+W)^t\,dt \ldots\ldots (27) $$

Evaluating and putting in terms of r′:

$$ {}_5\overline{P}_i = \frac{{}_5\phi_o\ r'}{(1+d)\log(1+W)} \left\{ (1+W)^{1+i}\left[1+d-(1+W)^{-5}\right] - d\,(1+W)^{1-a} \right\} \quad (28) $$

Using similar reasoning

$$ {}_5\overline{P}'_i = \int_{-4+1}^{1+i} {}_5\phi_o\ r'\,(1+W)^t\,dt $$

or:

$$ {}_5\overline{P}'_i = \frac{{}_5\phi_o\ r'}{\log(1+W)}\left[1-(1+W)^{-5}\right](1+W)^{i+1} \ldots\ldots\ldots\ldots (29) $$

Define:

$$ {}_5\overline{F}_i = \frac{{}_5\overline{P}'_i}{{}_5\overline{P}_i} $$

Substituting (28) and (29) and simplifying

$$ {}_5\overline{F}_i = \frac{1+d}{1+\left[\dfrac{1-(1+W)^{-a-i}}{1-(1+W)^{-5}}\right]\cdot d} \quad \ldots\ldots\ldots\ldots (30) $$

Although (1+W) could be obtained from the observation of ${}_5\phi_o$ and ${}_5\phi_1$, it would be more practical to measure it as a function of (a) ${}_5\phi_{-4}$ and ${}_5\phi_1$ thus covering the most recently expired five-year period, (b) ${}_5\phi_{-4}$ and ${}_5\phi_5$ thus covering the entire period of time involved in (27) or (c) ${}_5\phi_{-2}$ to ${}_5\phi_3$ thus covering the centermost five years. The author's preference is for (a) since it will be always available whereas (b) and (c) may reach into the future. Then, by analogy with (22),

$$ (1+W) = \left[1+\log\frac{{}_5\phi_1}{{}_5\phi_{-4}}\right]^{1/5} \ldots\ldots\ldots\ldots\ldots\ldots (31) $$

Following a process similar to that that produced (15), we may "telescope" the effect of the five-years under (30) by writing the telescoped premium as

$$ {}_5\overline{P}''_o = {}_5\overline{P}_o + \sum_{i=o}^{4} ({}_5\overline{P}_i{}' - {}_5\overline{P}_i) $$

Substituting (28) and (29) and simplifying:

$$ {}_5\overline{P}''_o = {}_5\overline{P}_o \left\{ 1 + \frac{5(1+W)^{-a} - \sum_{i=0}^{4} (1+W)^{i-5}}{1 - (1+W)^{-a} + \frac{1}{d}[1 - (1+W)^{-5}]} \right\} \qquad W \neq 0 $$

The quantity following the summation sign may be further simplified since it is a geometric progression and becomes:

$$ {}_5\overline{P}''_o = {}_5\overline{P}_o \left\{ 1 + \frac{5(1+W)^{-a} - \frac{1}{W}[1 - (1+W)^{-5}]}{1 - (1+W)^{-a} + \frac{1}{d}[1 - (1+W)^{-5}]} \right\} \quad W \neq 0 \quad \ldots(32) $$

Define:

$$ {}_5\overline{F}''_o = \frac{{}_5\overline{P}''_o}{{}_5\overline{P}_o} $$

Then

$$ {}_5F''_o = 1 + \frac{5(1+W)^{-a} - \frac{1}{W}[1 - (1+W)^{-5}]}{1 - (1+W)^{-a} + \frac{1}{d}[1 - (1+W)^{-5}]} \quad W \neq 0 \quad \ldots(33) $$

Finally, define

$$ {}_5\overline{C}'' = \frac{{}_5\overline{F}''_o}{\overline{F}} \ldots\ldots\ldots\ldots(34) $$

and

$$ {}_5\overline{C} = \frac{{}_5\overline{F}''_o}{{}_5F''_o} \ldots\ldots\ldots\ldots(34a) $$

Appendix B gives numerical examples of equations (30), (33), (34) and (34a). In the author's opinion ${}_5\overline{C}$ does not come close enough to 1.000 to permit an assumption of $W = 0$ unless W in itself is quite small (say, $\pm$ .02). The error caused by ignoring the effect of five-year installment business if it is recorded under the annual reporting system is quite large, even under small values of d as shown by ${}_5\overline{C}''$.

The next natural development which suggests itself is that of more than one rate change within the one year period. Since this rarely happens and since the formulae will follow from the general pattern laid down, their development will be left to those forced to use them. If the changes are small, the repeated application of the formulae developed will not introduce much error.

As a corollary to the main subject, it has also been observed that certain intuitive reactions can lead to erroneous results in the matter of comparing rate levels between two organizations. This is most commonly done in comparing a company rate per unit of exposure, K, with a bureau rate per unit of exposure, B, where S is the exposure as before. Also, let $p = SK$; that is, the company premium, and let j be used as a subscript to identify the finest breakdown of the data with which we are working. R_j is the ratio of the company rate to the bureau rate; i.e., $R_j = \frac{K_j}{B_j}$ and μ is the composite or average ratio of rate levels which we are seeking. Finally, V_j is the proportion of volume in the j^{th} classification and equals $\frac{p_j}{\Sigma p_j}$. (Since all summations will be over j, this will be omitted from Σ). Intuition seems to lead to an erroneous μ, called μ_e by the following reasoning: To get a weighted average deviation, apply the weights to the individual deviations. This sounds innocent enough and leads to the following:

$$\mu_e - 1 = \Sigma[V_j(R_j - 1)]$$

Of course, $\Sigma V_j = 1.00$ which leads to

$$\mu_e = \Sigma V_j R_j = \Sigma \frac{V_j K_j}{B_j} \quad \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

The true comparison of composite rate levels is arrived at by extending exposures, in their finest breakdown, first at one set of rates and then at the other set of rates; thus obtaining the total premium for the entire group of business at each rate level. Then the ratio of the two totals would give the composite ratio of rate levels. In terms of our definitions:

$$\mu = \frac{\Sigma S_j K_j}{\Sigma S_j B_j} = \frac{\Sigma p_j}{\Sigma S_j B_j} \quad \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

This is a perfectly good form for the equation, provided the statistical breakdown of S is fine enough to identify unique manual rates. If this is not the case, or if S is not a coded item (as in fire insurance), other means of getting at the results must be obtained. From the definitions $S = \frac{p}{K}$, so substituting this in (36) and rearranging,

$$\mu = \frac{\Sigma p_j}{\Sigma \frac{p_j}{K_j} \cdot B_j}$$

Therefore

$$\mu = \frac{1}{\Sigma \frac{V_j}{R_j}} = \frac{1}{\Sigma V_j \frac{B_j}{K_j}} \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

Thus, it is the harmonic mean that is correct to use instead of the more usual arithmetic mean. It can be shown that $\mu_e > \mu$ under all cases where the formula would be used.* Care must be exercised in ascertaining V_j which is a weighting system based on the company's premium volume and not on its exposure units.

CONCLUSIONS

From the definition of the rate revision adjustment factor and from a cursory examination of it, there does not seem to be anything too complex or mysterious about what it is, how it should be calculated or how it should be applied. Intuition would lead us to calculate the rate revision adjustment factor as based on pro rata of the number of months involved at each rate level. This results in (8) which is not correct and the error caused by such reasoning consistently produces inadequate rates. If the assumptions are met of a level volume of business evenly distributed over the period and the recording of all premiums (both term and installment) is made at the time the contract is entered into, then equation (6) is the only correct one to use. This formula is sufficiently accurate if the volume is rising or falling slightly (say, 10% or less per year), but when the rate of growth (or decline) is very large, such as in the early years of a new line of business, equation (21) would have to be used despite its calculating complexity. Equation (26) is an approximation to (21) which may be used when the rate of growth is moderate and judgment indicates its appropriateness. When installment payment term business is recorded annually as each installment falls due, the proper evaluation of the rate revision adjustment factor becomes quite tedious as shown by both equation (14) which assumes a level volume of business and equation (30) which recognizes a rate of growth in the volume. Short cut equations (16) and (33) "telescope" the effect of a rate change into the original year it becomes effective and save a great deal of difficulty when compared to (14) and (30).

In applying these formulae to specific cases, the full ingenuity of the actuary must be used to adapt them to the prevailing conditions. For example, if both the annual reporting method and the full term reporting method are permitted, it may be necessary to use some form of a composite formula which takes this into consideration. It may also be a problem to ascertain the true date on which rates were revised. For example, if rates on policies written to be effective 45 days after the effective date of a rate change are allowed to remain on the old basis, then the true effective date of the change from the viewpoint of the actuary may have to be modified. Care must also be exercised if substantial rate decreases are made at any one time in such a manner that it is advantageous to cancel short rate and rewrite the policy.

*This is the usual proof that the arithmetic mean is larger than the harmonic mean and is not shown here.

This would not likely occur on small personal lines but is a definite possibility in any class generating a large premium per risk. Here the rate change could introduce other considerations not reflected in the formulas.

The final section of the paper established (37) as the proper means of obtaining the average deviation of a company's rates from those of a bureau (or other similar comparisons) when detailed exposure data is not available. If the erroneous formula (35) were used, the ratio of rate levels would be stated too high and thus the deviation of the company would be understated.

Perhaps the outstanding lesson to be learned from the analyses presented is that intuitive reasoning can often lead to seriously defective results. Sound conclusions can be reached only by solid reasoning from the firm foundation of fundamental principles. In this way, the limitations as well as the area of application will be known.

APPENDIX A

SYMBOL DEFINITIONS

In general, P represents premium, r rate, F factor, S and ϕ are amounts insured or exposures in force and C is a correction or comparison factor.

S	Exposure units or sum insured.
1-a	Portion of the period prior to the rate change.
a	Portion of the period after the rate change.
r	Rate per unit of exposure prior to the rate change.
r′	Rate per unit of exposure after the rate change.
d	Rate change expressed as a decimal number; positive sign indicates a rate increase; negative sign indicates a rate decrease.
P	Premium actually collected or recorded on the company books during the year.
P′	Premium which would have been collected if all business during the year had been written at the r′ rates.
F	Rate revision adjustment factor to adjust P to P′.
P_e	An erroneously calculated value of P′.
F_e	An erroneously calculated value of F.
C	A factor to compare P'_e with P′, or to compare F with F_e.
i	Used as a subscript to identify various years with 0 designating the year in which the rate change is made; negative numbers designate prior years; positive numbers designate subsequent years.

5 Used as a subscript preceding symbols such as P and F to indicate they deal with 5-year term business written on an installment basis and recorded on the company books as each installment is collected.

$''$ Double primes indicate a quantity based on "telescoping" the five-year effect of a rate change on installment business into one year.

${}_5C''$ A factor to compare ${}_5F''_0$ with F; that is, a measurement of the error introduced if five year installment payment term business recorded annually is treated the same as annual business.

ϕ_i The exposure in force at the beginning of year i.

w The continuous rate of growth at which policies are being written.

— A bar over a symbol indicates that a continuous rate of growth is involved in the assumptions.

t An increment of time between the beginning and end of the year.

log Natural or base e logarithms.

$\overline{C}$ A factor to compare $\overline{F}$ with F; that is, a measurement of the error introduced by assuming business is written evenly throughout a year when, in fact, it is written at a changing rate w.

$\overline{C}_x$ An approximation to $\overline{C}$.

$\overline{F}_x$ An approximation to $\overline{F}$.

W The continuous rate of growth at which policies are being written under five-year installment payment plans, subject to annual recording on the company books. This symbol is used in lieu of ${}_5w$ for simplicity of notation.

${}_5\overline{C}''$ A factor to compare ${}_5\overline{F}''_0$ with $\overline{F}$; that is the same as ${}_5C''$ except it involves a continuous rate of growth.

${}_5\overline{C}$ A factor to compare ${}_5\overline{F}''_0$ with ${}_5F''_0$; that is, the same as $\overline{C}$ except involving five-year installment business recorded annually.

K A company rate per unit of exposure.

B A Bureau rate or base rate per unit of exposure.

j A subscript to designate the finest breakdown of the data with which we are working. Usually the breakdown would be to the point of unique manual rates.

p Company premium.

V_j The proportion of volume in the j^{th} cell.

μ The composite or average ratio of rate levels, $(\mu-1)$ is the average deviation of company rates from Bureau rates.

μ_e An erroneous μ.

APPENDIX B

Evaluation of Formulae When a = ½ and d Assumes Various Values

					d			
Section	Symbol	Equation	.20	.10	.05	−.05	−.10	−.20
	F	(6)	1.0909	1.0476	1.0244	.9744	.9474	.8889
	F_e	(8)	1.1000	1.0500	1.0250	.9750	.9500	.9000
	C	(10)	.9917	.9977	.9994	.9993	.9972	.9877
	${}_5F_0$	(14)	1.1765	1.0891	1.0448	.9548	.9091	.8163
w = 0	${}_5F_1$	(14)	1.1321	1.0680	1.0345	.9645	.9278	.8511
	${}_5F_2$	(14)	1.0909	1.0476	1.0244	.9744	.9474	.8889
	${}_5F_3$	(14)	1.0526	1.0280	1.0145	.9845	.9677	.9302
	${}_5F_4$	(14)	1.0169	1.0092	1.0048	.9948	.9890	.9756
	${}_5F''_0$	(16)	1.4902	1.2475	1.1244	.8744	.7475	.4898
	${}_5C''$	(17)	1.3660	1.1908	1.0976	.8974	.7890	.5510
	$\overline{F}$	(21)	1.0886	1.0464	1.0238	.9750	.9486	.8912
	$\overline{F}_x$	(26)	1.0884	1.0464	1.0238	.9750	.9486	.8914
	$\overline{C}$	(23)	.9978	.9989	.9994	1.0006	1.0013	1.0027
	$\overline{C}_x$	(25)	.9977	.9988	.9994	1.0006	1.0013	1.0028
	${}_5\overline{F}_0$	(30)	1.1712	1.0867	1.0436	.9559	.9112	.8201
w or	${}_5\overline{F}_1$	(30)	1.1212	1.0627	1.0319	.9670	.9328	.8605
W = +.10	${}_5\overline{F}_2$	(30)	1.0793	1.0417	1.0214	.9773	.9533	.9008
	${}_5\overline{F}_3$	(30)	1.0438	1.0234	1.0121	.9869	.9728	.9408
	${}_5\overline{F}_4$	(30)	1.0135	1.0073	1.0038	.9958	.9912	.9804
	${}_5\overline{F}''_0$	(33)	1.5029	1.2545	1.1280	.8704	.7392	.4718
	${}_5\overline{C}''$	(34)	1.3806	1.1989	1.1018	.8927	.7793	.5294
	${}_5\overline{C}$	(34a)	1.0085	1.0056	1.0032	.9954	.9889	.9633
	$\overline{F}$	(21)	1.0935	1.0489	1.0250	.9737	.9461	.8863
	$\overline{F}_x$	(26)	1.0934	1.0489	1.0250	.9737	.9461	.8864
	$\overline{C}$	(23)	1.0024	1.0013	1.0006	.9993	.9986	.9971
	$\overline{C}_x$	(25)	1.0023	1.0012	1.0006	.9994	.9987	.9972
	${}_5\overline{F}_0$	(30)	1.1816	1.0915	1.0459	.9537	.9071	.8127
w or	${}_5\overline{F}_1$	(30)	1.1435	1.0735	1.0372	.9619	.9228	.8416
W = −.10	${}_5\overline{F}_2$	(30)	1.1041	1.0542	1.0277	.9711	.9409	.8761
	${}_5\overline{F}_3$	(30)	1.0633	1.0335	1.0173	.9816	.9618	.9181
	${}_5\overline{F}_4$	(30)	1.0213	1.0115	1.0060	.9934	.9863	.9696
	${}_5\overline{F}''_0$	(33)	1.4727	1.2382	1.1195	.8795	.7581	.5123
	${}_5\overline{C}''$	(34)	1.3468	1.1805	1.0922	.9033	.8013	.5780
	${}_5\overline{C}$	(34a)	.9883	.9925	.9956	1.0058	1.0142	1.0459

APPENDIX C

To evaluate w, the continuous rate of increase, consider the function

$$\left(1 + \frac{w}{t}\right)^t$$

As t increases from 1, we are dividing the interval into more and more subdivisions as we go from ϕ_o to ϕ_1. The continuous rate of growth is when t becomes infinite. So,

$$\frac{\phi_1}{\phi_o} = \lim_{t \to \infty} \left(1 + \frac{w}{t}\right)^t$$

This limit is the very common one involved in the base of natural logarithms and equals e^w.

Hence

$$\frac{\phi_1}{\phi_o} = e^w$$

$$w = \log \frac{\phi_1}{\phi_o}$$