INTERNATIONAL ACTUARIAL NOTATION

BY

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The Council of the Society announces that the revised International Actuarial Notation, particulars of which are given below, will be adopted in the *Proceedings* of the Society as from the present Number. The revised Notation will also be used by the Examination Committee for all examinations subsequent to those in May, 1950. The revised Notation has been adopted by the Society of Actuaries and also by the Institute of Actuaries in England, the Faculty of Actuaries in Scotland and by actuarial societies in various other foreign countries.

Before giving an account of the circumstances leading up to the present revision and a complete account of the Notation it is desirable to state here the principal changes which are now being introducd:

" (1) Interest. In place of the symbol $j_{(m)}$ = the nominal rate of interest convertible m times in a year when the effective rate is i, the symbol $i^{(m)}$ has been substituted; the former definitions of $i^{(m)}$ and $\bar{i}$ as the effective rate of interest when interest is convertible m times a year and momently respectively have been abolished.

" (2) Annuities-due. In place of the symbol a for an annuity under which the first payment is to be made at once the symbol ä has been adopted. This adoption of the trema ¨ as a symbol of acceleration has permitted the introduction of s̈ for the accumulated amount of an annuity-due at the end of the term, for which no symbol previously existed.

" (3) Mortality. The use of Q for the probability of dying within a longer term than 1 year has been omitted.

" (4) Compound symbols. The use of D in a compound symbol to indicate decreasing benefits has been introduced.

" (5) Commutation columns. The old definition of $N_x = D_{x+1} + D_{x+2} + D_{x+3}$ etc. has been abolished in favour of $N_x = D_x + D_{x+1} + D_{x+2}$ etc.
This renders the use of N_x and $\mathbf{N}_x$ for the latter series unnecessary."

The complete statement of the revised Notation and its history follows:

The existing international actuarial notation was founded on the 'Key to the Notation' given in the *Institute of Actuaries Text-Book*, Part II, *Life Contingencies*, by George King, and is embodied in an explanatory statement adopted unanimously by the Second International Actuarial Congress held in London in May 1898 and printed on pp. 618–40 of the *Transactions* of that Congress. At the Third International Congress held in Paris in June 1900 a further statement by Dr Sprague was submitted which rearranged the symbols in different orders and grouped them on different principles but did not introduce any changes in the symbols themselves beyond two slight additions. This statement is printed on pp. 622–51 of the *Transactions* of that Congress. At the Eleventh International Congress held in Paris in June 1937 an International Committee was appointed to consider the question of notation and given the duty of submitting definite proposals to the next Congress. This Committee met in Brussels in July 1938 and again in July 1939, and certain changes would have been proposed for adoption at Lucerne in 1940. Owing to the war this could not be done, but in order not to lose the benefit of these changes for an indefinite period the Institute of Actuaries and the Faculty of Actuaries have decided to adopt at a date to be mutually arranged those recommendations which had received unanimous or substantial support at the meetings of the Committee.

As many students and members of the profession have not ready access to the above-mentioned *Transactions* it has been thought desirable to prepare and issue the following statement of the International Actuarial Notation after embodying in it the changes arising from the recommendations mentioned above.

The general principles on which the system is based are as follows:

To each fundamental symbolic letter are attached signs and letters each having its own signification.

The lower space to the left is reserved for signs indicating the conditions relative to the duration of the operations and to their position with regard to time.

The lower space to the right is reserved for signs indicating the conditions relative to ages and the order of succession of the events.

The upper space to the right is reserved for signs indicating the periodicity of the events.

The upper space to the left is free, and in it can be placed signs corresponding to other notions.

In what follows these two conventions are used:

A letter enclosed in brackets, thus (x), denotes 'a person aged x'. A letter or number enclosed in a right angle, thus $\overline{n}|$ or $\overline{15}|$, denotes a term-certain of years.

FUNDAMENTAL SYMBOLIC LETTERS

Interest

$i =$ the effective rate of interest, namely, the total interest earned on 1 in a year on the assumption that the actual interest (if receivable otherwise than yearly) is invested forthwith as it becomes due on the same terms as the original principal.

$v = (1+i)^{-1}$ = the present value of 1 due a year hence.

$d = 1-v$ = the discount on 1 due a year hence.

$\delta = \log_e(1+i) = -\log_e(1-d)$ = the force of interest or the force of discount.

Mortality Tables

l = number living.

d = number dying.

p = probability of living.

q = probability of dying.

μ = force of mortality.

m = central death rate.

a = present value of an annuity.

s = amount of an annuity.

e = expectation of life.

A = present value of an assurance.

E = present value of an endowment.

$\left.\begin{array}{l}P\\\pi\end{array}\right\}$ = premium per annum. P generally refers to net premiums, π to special premiums.

V = policy value.

W = paid-up policy.

The methods of using the foregoing principal letters and their precise meaning when added to by suffixes, etc., follow.

Interest

$i^{(m)} = m\{(1+i)^{1/m} - 1\}$ = the nominal rate of interest, convertible m times a year.

$a_{\overline{n}|} = v + v^2 + \ldots + v^n$ = the value of an annuity-certain of 1 per annum for n years, the payments being made at the end of each year.

$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1}$ = the value of a similar annuity, the payments being made at the beginning of each year.

$s_{\overline{n}|} = 1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1}$ = the amount of an annuity-certain of 1 per annum for n years, the payments being made at the end of each year.

$\ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + \ldots + (1+i)^n$ = the amount of a similar annuity, the payments being made at the beginning of each year.

The diaeresis or trema above the letters a and s is used as a symbol of acceleration of payments.

Mortality Tables

The ages of the lives involved are denoted by letters placed as suffixes in the lower space to the right. Thus:

l_x = the number of persons who attain age x according to the mortality table.

$d_x = l_x - l_{x+1}$ = the number of persons who die between ages x and $x+1$ according to the mortality table.

p_x = the probability that (x) will live 1 year.

q_x = the probability that (x) will die within 1 year.

$\mu_x = -\dfrac{1}{l_x}\dfrac{dl_x}{dx} = $ the force of mortality at age x.

$m_x = $ the central death-rate for the year of age x to $x+1 = d_x \Big/ \displaystyle\int_0^1 l_{x+t}\,dt.$

$e_x = $ the curtate 'expectation of life' (or average after-lifetime) of (x).

In the following it is always to be understood (unless otherwise expressed) that the annual payment of an annuity is 1, that the sum assured in any case is 1, and that the symbols indicate the present values:

$a_x = $ an annuity, first payment at the end of a year, to continue during the life of (x).

$\ddot{a}_x = 1 + a_x = $ an 'annuity-due' to continue during the life of (x), the first payment to be made at once.

$A_x = $ an assurance payable at the end of the year of death of (x).

Note. $e_x = a_x$ at rate of interest $i = 0$.

A letter or number at the lower left corner of the principal symbol denotes the number of years involved in the probability or benefit in question. Thus:

$_n p_x = $ the probability that (x) will live n years.

$_n q_x = $ the probability that (x) will die within n years.

Note. When $n = 1$ it is customary to omit it, as shown on page 2, provided no ambiguity is introduced.

$_n E_x = v^n {}_n p_x = $ the value of an endowment on (x) payable at the end of n years if (x) be then alive.

If the letter or number comes before a perpendicular bar it shows that a period of deferment is meant. Thus:

$_n|q_x = $ the probability that (x) will die in a year, deferred n years; that is, that he will die in the $(n+1)$th year.

$_n|a_x = $ an annuity on (x) deferred n years; that is, that the first payment is to be made at the end of $(n+1)$ years.

$_n|_t a_x = $ an intercepted or deferred temporary annuity on (x) deferred n years and, after that, to run for t years.

A letter or number in brackets at the upper right corner of the principal symbol shows the number of intervals into which the year is to be divided. Thus:

$a_x^{(m)} = $ an annuity on (x) payable by m instalments of $1/m$ each throughout the year, the first payment being one of $1/m$ at the end of the first $1/m$th of a year.

$\ddot{a}_x^{(m)} = $ a similar annuity but the first payment of $1/m$ is to be made at once, so that

$\ddot{a}_x^{(m)} = 1/m + a_x^{(m)}.$

$A_x^{(m)} = $ an assurance payable at the end of that fraction $1/m$ of a year in which (x) dies.

If $m \to \infty$ then instead of writing (∞) a bar is placed over the principal symbol. Thus:

$\bar{a}$ = a continuous or momently annuity.

$\bar{A}$ = an assurance payable at the moment of death.

A small circle placed over the principal symbol shows that the benefit is to be complete. Thus:

$\overset{\circ}{a}$ = a complete annuity.

$\overset{\circ}{e}$ = the complete expectation of life.

> *Note.* Some consider that $\bar{e}$ would be as appropriate as $\overset{\circ}{e}$. As $e_x = a_x$ at rate of interest $i = 0$, so also the complete expectation of life $= \bar{a}_x$ at rate of interest $i = 0$.

When more than one life is involved the following rules are observed:

If there are two or more letters or numbers in a suffix without any distinguishing mark, joint lives are intended. Thus:

$$l_{xy} = l_x \times l_y, \quad d_{xy} = l_{xy} - l_{x+1:y+1}.$$

> *Note.* When, for the sake of distinctness, it is desired to separate the letters or numbers in a suffix, a colon is placed between them. A colon is used instead of a point or comma to avoid confusion with decimals when numbers are involved.

a_{xyz} = an annuity, first payment at the end of a year, to continue during the joint lives of (x), (y) and (z).

A_{xyz} = an assurance payable at the end of the year of the failure of the joint lives (x), (y) and (z).

In place of a life a term-certain may be involved. Thus:

$a_{x:\overline{n}|}$ = an annuity to continue during the joint duration of the life of (x) and a term of n years certain; that is, a temporary annuity for n years on the life of (x).

$A_{x:\overline{n}|}$ = an assurance payable at the end of the year of death of (x) if he die within n years, or at the end of n years if (x) be then alive; that is, an endowment assurance for n years.

If a perpendicular bar separates the letters in the suffix, then the status after the bar is to follow the status before the bar. Thus:

$a_{y|x}$ = a reversionary annuity, that is, an annuity on the life of (x) after the death of (y).

$A_{z|xy}$ = an assurance payable on the failure of the joint lives (x) and (y) provided both these lives survive (z).

If a horizontal bar appears above the suffix then survivors of the lives, and not joint lives, are intended. The number of survivors can be denoted by a letter or number over the right end of the bar. If that letter, say r, is not distinguished by any mark, then the meaning is *at least* r survivors; but if it is enclosed in square brackets, [r], then the meaning is *exactly* r survivors. If no letter or

number appears over the bar, then unity is supposed and the meaning is *at least one* survivor. Thus:

$a_{\overline{xyz}}$ = an annuity payable so long as at least one of the three lives (x), (y) and (z) is alive.

$a_{\overline{xyz}}^{2}$ = an annuity payable so long as at least two of the three lives (x), (y) and (z) are alive.

$p_{\overline{xyz}}^{[2]}$ = probability that exactly two of the three lives (x), (y) and (z) will survive a year.

$_nq_{\overline{xy}}$ = probability that the survivor of the two lives (x) and (y) will die within n years = $_nq_x \times {_nq_y}$.

$_nA_{\overline{xy}}$ = an assurance payable at the end of the year of death of the survivor of the lives (x) and (y) provided the death occurs within n years.

When numerals are placed above or below the letters of the suffix, they designate the order in which the lives are to fail. The numeral placed *over* the suffix points out the life whose failure will finally determine the event; and the numerals placed *under* the suffix indicate the order in which the other lives involved are to fail. Thus:

A_{xy}^{1} = an assurance payable at the end of the year of death of (x) if he dies first of the two lives (x) and (y).

A_{xyz}^{2} = an assurance payable at the end of the year of death of (x) if he dies second of the three lives (x), (y) and (z).

$A_{xyz}^{2}{}_{1}$ = an assurance payable at the end of the year of death of (x) if he dies second of the three lives, (y) having died first.

$A_{\overline{xy}:z}{}_{3}$ = an assurance payable at the end of the year of death of the survivor of (x) and (y) if he dies before (z).

$A_{x:\overline{n}|}^{1}$ = an assurance payable at the end of the year of death of (x) if he dies within a term of n years.

$\left. \begin{array}{l} a_{\overline{yz}|x}^{1} \\ \text{or} \\ a_{\overline{yz}|x}^{2} \end{array} \right\}$ = an annuity to (x) after the failure of the survivor of (y) and (z), provided (z) fails before (y).

Note. Sometimes to make quite clear that a joint-life status is involved a symbol ⌐ is placed above the lives included. Thus $A_{\overline{xy}:\overline{n}|}^{1}$ = a joint life temporary assurance on (x) and (y).

In the case of reversionary annuities, distinction has sometimes to be made between those where the times of year at which payments are to take place are determined at the outset and those where the times depend on the failure of the preceding status. Thus:

$a_{y|x}$ = annuity to (x), first payment at the end of the year of the death of (y) or, on the average, about 6 months after his death.

$\overset{.}{a}_{y|x}$ = annuity to (x), first payment 1 year after the death of (y).

$\overset{\circ}{\overset{.}{a}}_{y|x}$ = complete annuity to (x), first payment 1 year after the death of (y).

ANNUAL PREMIUMS

The symbol P with the appropriate suffix or suffixes is used in simple cases, where no misunderstanding can occur, to denote the annual premium for a benefit. Thus:

P_x = the annual premium for an assurance payable at the end of the year of death of (x).

$P_{x:\overline{n}|}$ = the annual premium for an endowment assurance on (x) payable after n years or at the end of the year of death of (x) if he die within n years.

P^1_{xy} = the annual premium for a contingent assurance payable at the end of the year of death of (x) if he die before (y).

In all these cases it is optional to use the symbol P in conjunction with the principal symbol denoting the benefit. Thus instead of $P_{x:\overline{n}|}$ we may write $P(A_{x:\overline{n}|})$. In the more complicated cases it is necessary to use the two symbols in this way. Suffixes, etc., showing the conditions of the benefit are to be attached to the principal letter, and those showing the conditions of payment of the premium are to be attached to the subsidiary symbol P. Thus:

$_nP(\overline{A}_x)$ = the annual premium payable for n years only for an assurance payable at the moment of the death of (x).

$P_{xy}(A_x)$ = the annual premium payable during the joint lives of (x) and (y) for an assurance payable at the end of the year of death of (x).

$_nP(_{n|}a_x)$ = the annual premium payable for n years only for an annuity on (x) deferred n years.

$_tP^{(m)}(A_{x:\overline{n}|})$ = the annual premium payable for t years only, by m instalments throughout the year, for an endowment assurance for n years on (x) (see below as to $P^{(m)}$).

Notes. (1) As a general rule the symbol P could be used without the principal symbol in the case of assurances where the sum assured is payable at the end of the year of death, but if it is payable at other times, or if the benefit is an annuity, then the principal symbol should be used.

(2) $P^{(m)}_x$. A point which was not brought out when the international system was adopted is that there are two kinds of premiums payable *m* times a year, viz. those which cease on payment of the instalment immediately preceding death and those which continue to be payable to the end of the year of death. To distinguish the latter the *m* is sometimes enclosed in square brackets, thus $P^{[m]}$.

POLICY VALUES AND PAID-UP POLICIES

$_tV_x$ = the value of an ordinary whole-life assurance on (x) which has been *t* years in force, the premium then just due being unpaid.

$_tW_x$ = the paid-up policy the present value of which is $_tV_x$.

The symbols V and W may, in simple cases, be used alone, but in the more complicated cases it is necessary to insert the full symbol for the benefit thus:

$$_tV^{(m)}(\overline{A}_{x:\overline{n}|}) \text{ (corresponding to } P^{(m)}(\overline{A}_{x:\overline{n}|})), \quad _tV(_{n|}a_x).$$

Note. As a general rule V or W can be used as the main symbol if the sum assured is payable at the end of the year of death and the premium is payable periodically throughout the duration of the assurance. If the premium is payable for a limited number of years, say n, the policy value after t years could be written $_tV[_nP(A)]$, or, if desired $_{\overline{n}|}^tV(A)$.

In investigations where modified premiums and policy values are in question such modification may be denoted by adding accents to the symbols. Thus, when a premium other than the net premium (a valuation premium) is used in a valuation it may be denoted by P′ and the corresponding policy value by V′. Similarly, the office (or commercial) premium may be denoted by P″ and the corresponding paid-up policy by W″.

COMPOUND SYMBOLS

(Ia) = an annuity $\Big\}$ commencing at 1 and increasing 1 per annum.
(IA) = an assurance

If the whole benefit is to be temporary the symbol of limitation is placed outside the brackets. Thus:

$(Ia)_{x:\overline{n}|}$ = a temporary increasing annuity.

$(IA)^1_{x:\overline{n}|}$ = a temporary increasing assurance.,

If only the increase is to be temporary but the benefit is to continue thereafter, then the symbol of limitation is placed immediately after the symbol I. Thus:

$(I_{\overline{n}|}a)_x$ = a whole-life annuity $\Big\}$ increasing for n years and thereafter stationary.
$(I_{\overline{n}|}A)_x$ = a whole-life assurance

If the benefit is a decreasing one, the corresponding symbol is D. From the nature of the case this decrease must have a limit, as otherwise negative values might be implied. Thus:

$(D_{\overline{n}|}A)^1_{x:\overline{n}|}$ = a temporary assurance commencing at n and decreasing by 1 in each successive year.

If the benefit is a varying one the corresponding symbol is v. Thus:

(va) = a varying annuity.

COMMUTATION COLUMNS

Single lives

$$D_x = v^x l_x,$$
$$N_x = D_x + D_{x+1} + D_{x+2} + \text{etc.,}$$
$$S_x = N_x + N_{x+1} + N_{x+2} + \text{etc.,}$$
$$C_x = v^{x+1} d_x,$$
$$M_x = C_x + C_{x+1} + C_{x+2} + \text{etc.,}$$
$$R_x = M_x + M_{x+1} + M_{x+2} + \text{etc.}$$

When it is desired to construct the assurance columns so as to give directly assurances payable at the moment of death the symbols are distinguished by a bar placed over them. Thus:

$$\overline{C}_x = v^{x+1}d_x \text{ which is an approximation to } \int_0^1 v^{x+t}\mu_{x+t}l_{x+t}\,dt.$$
$$\overline{M}_x = \overline{C}_x + \overline{C}_{x+1} + \overline{C}_{x+2} + \text{ etc.}$$
$$\overline{R}_x = \overline{M}_x + \overline{M}_{x+1} + \overline{M}_{x+2} + \text{ etc.}$$

Joint lives

$$D_{xy} = v^{\frac{1}{2}(x+y)}l_{xy},$$
$$N_{xy} = D_{xy} + D_{x+1:y+1} + D_{x+2:y+2} + \text{ etc.,}$$
$$C_{xy} = v^{\frac{1}{2}(x+y)+1}d_{xy},$$
$$M_{xy} = C_{xy} + C_{x+1:y+1} + C_{x+2:y+2} + \text{ etc.,}$$
$$C_{xy}^1 = v^{\frac{1}{2}(x+y)+1}d_x l_{y+\frac{1}{2}},$$
$$M_{xy}^1 = C_{xy}^1 + C_{x+1:y+1}^{\ 1} + C_{x+2:y+2}^{\ 1} + \text{ etc.}$$

SELECTION

If the suffix to a symbol which denotes the age is enclosed in a square bracket it indicates the age at which the life was selected. To this may be added, outside the bracket, the number of years which have elapsed since selection, so that the total suffix denotes the present age. Thus:

$l_{[x]+t}$ = the number in the select life table who were selected at age x and have attained age $x + t$.

$d_{[x]+t} = l_{[x]+t} - l_{[x]+t+1}.$

$a_{[x]}$ = value of an annuity on a life now aged x and now select.

$a_{[x-n]+n}$ = value of an annuity on a life now aged x and select n years ago at age $x - n$.

$N_{[x]} = D_{[x]} + D_{[x]+1} + D_{[x]+2} + \ldots.$

$\ddot{a}_{[x]} = N_{[x]} \div D_{[x]} = 1 + a_{[x]},$

and similarly for other functions.

When Dr Sprague presented his statement, he mentioned that an objection had been raised that the notation in some cases offers the choice of two symbols for the same benefit. For instance, a temporary annuity may be denoted either by $_na_x$ or by $a_{x:\overline{n}|}$. This is, he says, a necessary consequence of the principles underlying the system, and neither of the alternative forms could have been suppressed without injury to the symmetry of the system.