AN ACTUARIAL ANALYSIS OF RETROSPECTIVE RATING

BY

THOMAS O. CARLSON

Probably the most universal hallmark of the scientific mind is a persistent penchant for pigeonholing. This trait, which to the layman places the ultimate seal of futile dessication upon research and which has been the butt of a multitude of jokes down the centuries, is nevertheless one of the most potent instruments of investigation; for in the broad it leads to a proper perspective in viewing a given field, and in the narrow it leads to the delineation of distinctive relationships between the constituents of the field. The course often seems wayward and the ports of arrival startling even to the investigator, so that to maintain a balanced keel it is helpful for him to bear in mind Hosea Biglow's observation that

"Facs are contrary 'z mules."

This paper is, in epitome, the pigeonholing technique applied to the gamut of retrospective rating plans, both so-called and not so-called, that fall within the domain of practicability. And in no field that I know of is the Yankee apothegm just quoted better illustrated. The phrase "within the domain of practicability" affords a wide latitude to the author in the establishment of boundaries for his discussion. The term "Analysis" is used in a spirit which is the antithesis of pretentiousness, and has been resorted to only because the ostensibly humbler term "Note" has been so often applied to papers more unassailably definitive than I dare claim this one to be.

Any presentation of such an analysis is simplified in proportion to the succinctness of the symbolism used, but there is a degree of succinctness beyond which one's fellow-workers will not bother to follow. It is hoped that the Table of Symbols set forth in Appendix A falls short of that degree. Properly speaking, an appendix should be reserved for notes of elaboration not essential to the continuity of the paper, but a glance at the length of Appendix A should produce immediate forgiveness for any breach of etiquette in this instance. The symbols conform to the accepted standard notation for common concepts, and have all been selected so as to permit typing as easily as possible. A few comments upon the loss and loss ratio symbols bearing subscripts may be helpful, and have been included in Appendix B.

Classification of Plans

Retrospective rating includes within its scope any rating procedure which determines the premium for a risk after the expiration of the policy period for which the premium is being calculated, and in such a manner as to reflect the loss experience incurred during that period. Since the loss experience of the risk is to be reflected in the rating, it follows that certain items in the final premium will be functions of the risk's losses. Other items will be functions of the standard premium or of the final premium in the general case. Consequently, the formula

$$R = B + CL$$

is implicit in the definition of retrospective rating; in this formula R is the final premium, B is a function of the standard premium, L represents the risk's losses, and the multiplier C in addition to including a credibility factor reflects those elements in the final rate which vary with the losses. Any item which is a function of the final premium, such as taxes, is included as a factor common to both of the terms B and CL. The restriction of R by the imposition of a minimum premium limitation H and a maximum premium limitation G will be expressed by writing the formula in the condensed form,

$$H \leqslant R = B + CL \leqslant G$$

By way of illustration, it may be noted that this is the exact formula for the application of the standard plan for workmen's compensation risks,[1] B being termed the basic premium and C being termed the loss conversion factor. The factor C is constant for a given state, but the values of B, H and G vary by premium size and are set forth in a table of rating values. This plan is referred to hereafter as the standard plan.

This fundamental formula has a most deceptively innocent appearance, for upon resolution into its elements an intricate mutability is discovered which is productive of widely varying particular formulas. It may be well to distinguish at once between what I shall call a particular formula and a particular plan: one formula may embrace a great variety of particular plans, each with its own definite schedule of rating values. A formula completely defined specifies as respects expenses no more than the mode of allocating to the two terms the provisions for the respective items; the aggregate provision for each such item is a characteristic of a particular plan, not of a particular formula. A formula may or may not specify that a limit per claim or per accident is imposed upon the risk's losses reflected in the rating, but a particular plan would have to define such a limit. Therefore, in the determination of a formula, the mode of distributing one expense item may constitute one condition to be imposed arbitrarily, or what we may call one degree of freedom, but the amount of the aggregate provision for that item is not pertinent to the analysis of the formula. In view of the fact that there may be more than a dozen degrees of freedom in a retrospective rating formula, a brief discussion of each element in the standard premium dollar

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[1] Described in detail in "The Retrospective Rating Plan for Workmen's Compensation Risks," by Sydney D. Pinney, P.C.A.S., Vol. XXIV, p. 291.

is in order, and it will be found that for practical consideration the problem may be greatly simplified in advance.

Taxes must be paid upon the final premium. The provision for taxes may therefore be included through the factor $\frac{1}{1-T}$ in each term of the formula, as in the standard plan, or it may be obtained by applying the same factor as a multiplier to the final premium, as in the Comprehensive Rating Plan for National Defense Projects. In the latter event, the basic premium and the loss conversion factor will include no provision for taxes; the minimum and maximum premiums may exclude or may include the provision for taxes according as they are imposed as limitations before or after the application of the tax multiplier to the formula result. It is not the purpose of this paper to discuss the advantages or disadvantages of these different methods. It is clear that the analysis of the respective formulas will not be essentially affected by the inclusion or the exclusion of the tax element.

The production cost allowance and the provisions for general administration, exposure audit and inspection expenses may be variously included in the formula. In the standard plan these four items are fundamentally included in the basic premium and thus vary with the standard premium: the administration, audit and inspection items are determined directly as functions of the standard premium, and the production cost allowance is formally determined as a function of the minimum premium which in turn depends upon the standard premium. Each of the four items could be independently apportioned to the two terms. To my knowledge such a differentiation in their treatment has never been proposed in principle, and since an interpretably simple analysis is dependent upon a reduction in the available number of conditions to be imposed, the four items have been handled herein as one, with the symbol V to designate them jointly.

A reduction in the expense provisions from the amounts contemplated by the full manual rates is not inherent in the concept of retrospective rating but is customarily reflected therein. Any such reduction will involve one or more of these four items. The production cost allowance in the standard plan is equivalent to the full percentage provision in the manual rates applied to the minimum premium, thus effecting a reduction from the allowance under the guaranteed cost basis. It is almost universally accepted that actual administration and audit expenses decrease percentagewise in terms of the standard premium as the premium size increases: the introduction of expense constants recognized this situation to some extent, and a further gradation has been reflected in some of the approved retrospective rating plans. Inspection expense is not in general considered susceptible to reduction from the provisions in the standard rates. In some instances such costs on the larger risks may even amount to a greater per cent of the standard premium than on the smaller risks; as an example may be cited the automobile liability line in which the small risks represent insurance on single automobiles and the larger risks represent the fleet business. The symbol Vr is used in this paper to designate the aggregate provision for these expenses in any retrospective rating formula, so that $(V - Vr)$ represents the total reduction from the standard provisions for these four items.

The expense of investigating and adjusting claims is more closely related to the losses than any of the other expense items, and is commonly reflected in the loss conversion factor. It is primarily because of this fact that claim expense is given consideration apart from other company expenses. It would not appear logical to reflect any of the other expense items, except taxes, in the loss multiplier unless claim expense were also reflected therein. The full provision for claim expense is represented in the following discussion by F. This provision may, like the other company expense items, be variously distributed between the two terms in the fundamental formula. If the entire amount of F is provided through the loss conversion factor C, the component of C reflecting such provision has the form F/E, where E is the permissible loss ratio or the expected losses.

If there is any limitation upon the extent to which the risk's losses may affect the rating, as for example through the use of a credibility factor less than 100% or through the imposition of a minimum or a maximum limit upon the final premium, the "rated" losses included in the final premium through the term CL will be less than the total expected losses in the aggregate, and in order to produce a technical balance within the plan the basic premium must be increased by an amount equal to the "non-rated" losses thus eliminated from the contribution made by the second term to the final premium. Any limitation of the losses entering the second term of the formula will result in a corresponding curtailment of those expense provisions which are functions of the losses; and the balance of such curtailed expense items must be included in the basic premium. The value of the basic premium is unaffected by the manner of making this adjustment and it is most feasibly accomplished by applying the expense component of the loss conversion factor to the loss provisions in the basic premium. The symbol J will be used to designate the loss multiplier which reflects any expense items other than taxes. Then, as already noted, if all claim expense and no other expense is provided through such a factor,

$$J = \frac{F}{E}$$

If we assume as a norm a procedure under which claim expense is a function of losses and under which production cost, administration, audit and inspection expenses are functions of the standard premium, any variation from such a distribution of expenses may be handled by the introduction of a single symbol W to designate the amount by which any expenses provided for by the loss multiplier J exceed the full claim expense provision F. Thus in the general case

$$J = \frac{F + W}{E}$$

and a flat amount equal to W must be deducted from the basic premium, for on the average the expense provision produced by the term $\frac{W}{E}$ in J will be equal to $\frac{W}{E} \cdot E$, or W. The symbol W as thus used is unique among the symbols so far introduced in not being essentially positive. If W is negative, then J produces less than the equivalent of the full provision for claim expense, as in the case, for example, wherein an amount equal to W is included flat (i.e., as a direct percentage of the standard premium) in the basic premium for claim expense. If an amount equal to M is included flat in the basic premium for claim expense and an amount equal to N is provided for other expenses through the medium of J, then by setting $W = N - M$ this interchange of functions is reflected in a simple manner. Thus W operates as a clearing house for any departures from what we have assumed to be the normal procedure in treating expense items other than taxes, a sort of factotum capable of handling two-way traffic if need be.

If any expenses other than taxes are provided through the loss conversion factor, that is, if J is greater than zero, then J may be applied to the losses either modified or unmodified by credibility: the more logical of these two procedures would seem to be that which includes such expense provisions in direct proportion to the "rated" losses, i.e., which applies the multiplier to the credibility-modified losses. If we represent the credibility by the familiar Z, then in the case wherein J is applied to the credibility-modified losses, disregarding the tax multiplier,

$$C = JZ + Z = (1 + J)\, Z,$$

and in the case wherein J is applied to the losses without modification by credibility,

$$C = J + Z$$

The two classes of formulas thus produced do not comprehend the entire field, for it is evident at a glance that each of the formulas given for C is a special case of a more general formula,

$$C = Ja + Jb \cdot Z + Z,$$

wherein

$$Ja + Jb = J,$$

$$Ja = \frac{Fa + Wa}{E},$$

and

$$Jb = \frac{Fb + Wb}{E}$$

The first value given for C represents the case where $Ja = 0$, the second where $Jb = 0$. This general formula, which does not appear to be of much if any practical importance, is treated briefly in Appendix C.

Other modes of providing expenses may suggest themselves, such for example as having the provision V included by means of a multiplier applied to B rather than a term contained within B. These in my opinion are more of academic than practical interest. The formulas developed later in this paper can be readily adapted to reflect such variations, however, and in fact many variations reduce to the forms discussed in full. In the example just cited, for instance, since B is a function of the standard premium, a multiplier of B is similarly a function of the standard premium, and the variation is reducible for analytic purposes to what I have chosen as a standard form.

No differentiation between allocated and unallocated claim expense has been made in this discussion. Such a division is not important for analytical purposes. The natural procedure would be to treat allocated claim expense as it is treated in the determination of manual rates. For workmen's compensation, all claim expense is treated as a single item in calculating manual rates, and consequently in the standard plan no differentiation is made between the two types of claim expense. For liability lines, on the other hand, allocated claim expense is reported with losses and is treated as a loss in the determination of manual rates: the most natural procedure in developing a retrospective rating plan for such lines would therefore be to include allocated claim expense with the losses; in this case F would represent unallocated claim expense only, and E would represent expected losses plus allocated claim expense. In the Comprehensive Rating Plan for National Defense Projects the allocated claim expense is added to the losses after the latter have been increased by the unallocated claim expense multiplier, the multiplier in this case being applicable to the losses only.

The provision for profit and contingencies in the manual rates will be designated by D, and in the retrospective premium by Dr. These symbols correspond to the symbols V and Vr introduced to designate expenses other than claim or taxes. The rates for workmen's compensation insurance do not include a profit factor except in one state; the contingency factor in the manual rates varies according to the accumulated past experience and is designed to produce neither an underwriting profit nor an underwriting loss over a period of years. On lines other than workmen's compensation the manual rates include a definite provision for profit and contingencies.

There are many ways in which this item may be included in a retrospective rating formula. Any of the modes which have already been discussed for reflecting the respective expense items could be applied to this item as well. In the workmen's compensation line in order to avoid a sharp break in the provision at the eligibility point for application of the rating plan, the introduction could be graded in a variety of ways; for example, the provision for any particular size of risk could be included as some per cent of that portion of the standard premium in excess of the premium required to qualify for rating; or it could be included as D per cent modified either by the credibility or by the ratio of rated losses to expected losses. Although these and other methods produce a slight variation in the individual formulas, we may in the interest of simplicity consider that the provision Dr is included as a flat item in the basic premium, with any reflection of a split in the aggregate provision between the two terms of the fundamental formula taken care of through the "happy" medium of the symbol W, corresponding to the treatment of the expense items. Under any practicable means for introducing this provision, it can ultimately be reduced to the sum thus indicated or it can be reflected by extending the tax multipliers applicable either to the first or to both terms of the formula to reflect both T and D, i.e., by making the multiplier equal to $\frac{1}{1-T-D}$. Any changes in the formulas to reflect the latter mode of including the item will be apparent and should require no elaboration.

The actual losses L reflected in the final premium are limited in the aggregate to those losses lying between the loss allowances in the minimum and in the maximum premiums. These losses may be limited by the application of a credibility factor, and they may be further constrained by a limit per claim or per accident; as examples may be cited the $10,000 limitation per claim in the New York plan for workmen's compensation, and the limitation to the experience rating normal loss amount per case which was a feature of a proposal given extensive consideration some years ago.[2]

The credibility factor, represented by Z, if explicitly expressed is a component of the loss conversion factor C; in such a case some function of Z is also involved in the basic premium. As an independent variable in theory, Z could follow any specified law or no law. In the light of practical considerations, however, we may again considerably reduce the scope of our investigations, since we are interested in only two cases, (1) that in which Z increases between the limits of 0 and 1.00 as the premium size increases, and (2) that in which Z is constant. The standard plan for workmen's compensation is a special case of the latter category, with Z equal to 1.00 for every size of risk.

The minimum and maximum premiums H and G may be subjected to particular conditions. The loss provision in the basic premium, and consequently each term in the fundamental formula, is affected by the variation of H and G. The basic premium may be adapted to a particular progression of values, with corresponding adjustments in H or G or both. The loss conversion factor is determined fundamentally when the credibility and the distribution of the expense items in the formula are known; as will be seen,

[2] P.C.A.S., Vol. XXIV, p. 330.

however, a rounding of the loss conversion factor may produce an effect upon the basic premium.

Of the foregoing elements of variability, all but one in general represent separate degrees of freedom. It should be noted, however, that the determination of C is equivalent to the determination of Z and vice versa.

What started out as a field of several dimensions for classification has by lopping and squeezing now been reduced to an order sufficiently low to permit a reasonably simple treatment, and a design can be developed which is somewhat less like a crazy quilt than an ordered pattern.

The order of handling the separate elements of variability in the development of a formula is immaterial for our purposes. To make a start, however, let us review the expense elements, the limitation of losses, the credibility and the rating factors in succession.

Expense Elements

In the general case the provisions for expenses included flat in the basic premium are equal to

$$Vr - W$$

and the remaining expense provisions other than taxes are produced by a loss multiplier

$$J = \frac{F + W}{E}$$

If $W = 0$, we may consider that claim expense is provided wholly through a loss multiplier and that other expenses except taxes are provided wholly as a direct function of the standard premium. In case any other distribution is contemplated, it can be indicated by using the symbols N and M already explained, setting $N - M = W$. For analysis, such a distinction is immaterial, and the symbols N and M have therefore been omitted from the table in Appendix A. No further mention of such a differentiation will be made.

If $W = Vr$, we may consider that all expenses are provided through a loss multiplier, a case which would be practicable only for a very large risk.

If $W = -F$, it follows that $J = 0$ and no expenses other than taxes vary with the losses.

W, in its representation of expense provisions, may then vary between the limits $-F$ and Vr, or

$$-F \leqslant W \leqslant Vr$$

All these variations are reflected by including W in the formula, and consequently need not be studied further as producing formula types.

The method of including the loss multiplier J in the loss conversion factor C is, however, productive of two types requiring individual consideration:

$$\text{Type I.}\quad C = \frac{JZ + Z}{1 - T}$$

$$\text{Type II.}\quad C = \frac{J + Z}{1 - T}$$

where T is the percentage provision for taxes.

Each of these two types is deducible from a more general formula wherein

$$C = \frac{Ja + Jb \cdot Z + Z}{1 - T}$$

$$Ja + Jb = J$$

Type I is deduced by setting $Ja = 0$, Type II by setting $Jb = 0$. The general formula is discussed in Appendix C.

The provision for profit and contingencies may be included in several ways, none of which would seem to affect our analysis fundamentally. This provision has throughout been represented by Dr as an element in the basic premium, with the understanding that the symbol W would absorb any portion of D reflected in J.

As already indicated, the mode of providing for taxes, although dividing all formulas into two groups according as the loading is applied to each term separately or to the final sum of the two terms, is not important for our present purposes.

Limitation of Losses

In the general case, the risk's losses reflected in a rating are restricted by a limit per claim or per accident in addition to the limitation imposed by the specification of minimum and maximum premiums. In the discussion, unless otherwise noted, the insurance charge for losses in excess of the loss allowance in the maximum premium will be considered as including losses above a specified limit per claim or per accident. Excess pure premium ratio tables which reflect such a limitation may be constructed and in fact have been constructed in at least one state for workmen's compensation risks.[3] One particular plan in which the limit per case is the normal loss amount under the experience rating plan will be accorded separate consideration.

Credibility

Two cases are of particular importance:

Class A. Credibility increases with premium size,

$$0 \leqslant Z \leqslant 1.00$$

Class B. Credibility is constant.

An important case under Class B is that for which the credibility equals 1.00 throughout, as illustrated by the standard plan.

(3) "On Graduating Excess Pure Premium Ratios," by Paul Dorweiler, Vol. XXVIII, p. 132.

Rating Factors

In the general formula no particular conditions are imposed upon the minimum or maximum premiums. The only relationships specified are that the sum of B and CL, equal to R, shall be not less than H nor more than G. The predetermination of certain conditions to be satisfied by H or G or both gives rise to four cases of sufficient practical importance to be worthy of special attention.

Case (a). Minimum premium is greater than basic premium, maximum premium equals standard premium:

$$H > B, G = P$$

Case (b). Minimum premium equals basic premium, maximum premium is greater than standard premium:

$$H = B, G > P$$

Case (c). Minimum premium equals basic premium, maximum premium equals standard premium:

$$H = B, G = P$$

Case (d). Same as (c), but in addition the loss allowance in the maximum (i.e., standard) premium equals the expected losses:

$$H = B, G = P, G' = P' = E$$

The imposition of these conditions cannot of course be made unless the requisite degrees of freedom are available. For example, if the mode of providing for all expense items and the limitation of losses per claim or per accident have been specified, three conditions remain to be determined: the establishment of a constant credibility factor would remove one of these, leaving as possibilities among the foregoing special cases only (a), (b) and (c); the three conditions under (d) could not under such circumstances be satisfied except by coincidence.

For our purposes, therefore, there are two important cases of the general formula, giving rise to two broad categories which will be referred to as Types I and II according as the loss conversion factor excluding taxes takes the form $(JZ + Z)$ or $(J + Z)$. Each of these categories has two important subdivisions designated as Classes A and B according as the credibility increases with the premium size or is constant. And within each subdivision consideration will be given to certain conditions which may be imposed on the rating values, denoted as Cases (a), (b), (c) and (d). The mode of providing for taxes does not affect our analysis of other variables because in one way or another the tax multiplier is common to all terms. The mode of distributing all other expense provisions is conveniently removed as an issue by the versatile symbol W. And any loss limitation per claim or per accident is offset by an increase in the charge for excess losses included in the basic premium. We are then ready to proceed with our study.

It might be noted before embarking upon the formula analysis that more than one formula may be represented in a specific retrospective rating plan. For example, consider a plan under which the credibility increases from 0 to a maximum value of 1.00 as the premium size increases: the formulas involved for the premium range wherein the credibility is variable are different, as will be seen, from those involved for the upper premium range over which the credibility is constant at its maximum value.

Not being qualified to discuss the interaction of policyholders' dividends and retrospective rating, I have omitted all consideration of this matter, in the hope that it might receive attention in a written discussion of this paper.

Rating Formula

The general formula, as we have already seen, is

$$H \leqslant R = B + CL \leqslant G \tag{1}$$

In this formula, for Type I,

$$C = \frac{(1 + J)Z}{1 - T} \tag{2-I}$$

and for Type II,

$$C = \frac{J + Z}{1 - T} \tag{2-II}$$

with

$$J = \frac{F + W}{E} \tag{3}$$

for both types.

We know further that the provision included in the basic premium as a direct function of the standard premium for expenses other than taxes and for profit and contingencies is

$$Vr + Dr - W.$$

In order to determine the loss provision in the basic premium it is most convenient to determine first the average premium collected for all risks of a given size. The assumption underlying the development of any plan is that it is technically in balance, that is, that the provisions for expenses and losses produced in the aggregate for all risks of a given premium size are the same as would be produced if all the risks were written on a guaranteed cost basis reflecting the same aggregate provisions for expenses. The satisfaction of this condition is not only the touchstone for testing the technical validity of any plan, but it also serves as the stepping stone to the establishment of most of the relationships in which we are interested.

Clearly the amount B will be collected on every risk, regardless of loss ratio. Losses up to but not in excess of the loss ratio G' on every risk will be reflected in the second term, G' being the loss allowance in the maximum premium G and determined from the equation

$$G = B + CG' \tag{4}$$

In the aggregate therefore the contribution to the average premium made by the second term, using the summation symbols as defined in Appendix B, will amount to

$$C \cdot \frac{1}{n} \sum_{L=0}^{G'} L = CG'q,$$

n being the total number of risks of the given premium size. Further, by reason of the minimum premium limitation, an additional amount will be collected equal to

$$C \cdot \frac{1}{n} \sum_{L=0}^{H'} (H' - L) = CH's$$

wherein H', the loss allowance in the minimum premium H, is defined by the equation

$$H = B + CH' \tag{5}$$

It should be noted that $G'q$ is a function of a given standard premium P and the loss ratio G', and similarly as respects $H's$, so that for given values of G' and H' these amounts will vary with the premium size. The nature of this variation is indicated in Table I.

The average premium has thus been determined as

$$Rv = B + CG'q + CH's \tag{6}$$

The loss provisions and the expenses other than taxes dependent upon the loss provisions in Rv amount to $(1 + J)\,E$, so that the corresponding provisions to be included in B for Type I must equal

$$\begin{aligned} &(1 + J)\,E - (1 + J)\,Z\,(G'q + H's) \\ &= (1 + J)\,E\,(1 - Z) + (1 + J)\,Z\,(G'p - H's) \\ &= (1 + J)\,E\,(1 - Z) + (1 + J)\,ZI \end{aligned}$$

where $(1 + J)\,ZI$, which has been set equal to $(1 + J)\,Z\,(G'p - H's)$, is commonly termed the "net insurance charge for excess losses." The term $(1 + J)\,ZG'p$ is known as the "gross insurance charge for excess losses," and $(1 + J)\,ZH's$ as the "average loss and expense saving in minimum premium risks."

For Type II, the corresponding provision for losses and expenses dependent thereon to be included in the basic premium is equal to

$$\begin{aligned} &(1 + J)\,E - (J + Z)\,(G'q + H's) \\ &= E\,(1 - Z) + (J + Z)\,(G'p - H's) \\ &= E\,(1 - Z) + (J + Z)\,I, \end{aligned}$$

with interpretations of $(J+Z)\,I$, $(J+Z)\,G'p$ and $(J+Z)\,H's$ analogous to those given in the preceding paragraph for the corresponding terms under Type I.

The resulting expressions for B in (1) are then, for Type I,

$$B = \frac{Vr + Dr - W + (1+J)\,E\,(1-Z) + (1+J)\,ZI}{1-T}, \tag{7-I}$$

and for Type II,

$$B = \frac{Vr + Dr - W + E\,(1-Z) + (J+Z)\,I}{1-T} \tag{7-II}$$

The expanded formula (1) becomes, for Type I,

$$H \leqslant R = \frac{Vr+Dr-W+(1+J)E(1-Z)+(1+J)ZI}{1-T} + \frac{(1+J)ZL}{1-T} \leqslant G, \tag{8-I}$$

and for Type II,

$$H \leqslant R = \frac{Vr+Dr-W+E(1-Z)+(J+Z)I}{1-T} + \frac{(J+Z)L}{1-T} \leqslant G, \tag{8-II}$$

with J defined by (3) and with

$$I = G'p - H's, \tag{9}$$

or

$$I = E\,(1 + G'x - H'x) - H'$$

if expressed directly in terms of excess pure premium ratios.

As we might expect, there is a close analogy between the credibility-weighting process in this formula and under experience rating, so that if we introduce the familiar concept of adjusted losses defined as

$$A = E\,(1-Z) + LZ$$

we may write (8-I) in the form

$$H \leqslant R = \frac{Vr + Dr - W + (1+J)\,(A + ZI)}{1-T} \leqslant G, \tag{10-I}$$

with the loss provision ZI necessary in order to reflect the restrictions imposed by H and G.

The corresponding expression for Type II is somewhat less simple:

$$H \leqslant R = \frac{Vr + Dr - W + A + J\,(I + L) + ZI}{1-T} \leqslant G \tag{10-II}$$

Since

$$P = \frac{V + D - W + (1+J)\,E}{1-T} \tag{11}$$

and the reduction from the standard provisions for expenses other than claim or taxes and for profit and contingencies, all expressed in terms of the standard premium as base, is designated by

$$S = V - Vr + D - Dr,$$

the equations (7) defining B may be written

$$B = P - \frac{S}{1-T} - C\,(E - I), \tag{12}$$

a form which will be found extremely useful.

In the standard plan, the credibility is 100% throughout, so that the expected loss term disappears from both equations (7). Further, by reason of the rounding of the loss conversion factor and its use to absorb differences in the tax loadings from state to state, $W \neq 0$, so that

$$B = \frac{Vr + Dr - W + (1 + J)\,(G'p - H's)}{1 - T} \tag{13}$$

The gross insurance charge for excess losses in the standard plan is then equal to the average losses in excess of the loss allowance in the maximum premium loaded for those expenses dependent upon losses; the offsetting saving on minimum premium risks is the difference between the loss allowance in the minimum premium and the average losses on minimum premium risks, loaded for those expenses dependent upon losses.

Under either Class A or Class B, the only variation in (7) for the special Cases (a) to (d) will be in the expressions for the insurance charge. Case (d) is not possible under Class B if the distributions of the expense provisions have been independently established. The values of the component I in the insurance charge for these special cases are shown below:

$$\text{Case (a):} \quad I = P'p - H's \tag{9a}$$

where

$$P' = \frac{P - B}{C}$$

$$\text{Case (b):} \quad I = G'p \tag{9b}$$

$$\text{Case (c):} \quad I = P'p \tag{9c}$$

$$\text{Case (d):} \quad I = Ep \tag{9d}$$

The basic, minimum and maximum premiums and the loss conversion factor must all be specified if formula (1) is to be definite. These are commonly referred to as the rating values for the particular plan. Their determination and their mutually dependent variation will be discussed in a later section of the paper. Since the determination of the credibility is equivalent to the determination of the loss conversion factor, if the expense apportionment is known, this matter as well will be deferred to the same section.

If a limitation per claim or per accident is imposed upon the risk's losses, modifying the value of L in the second term of the fundamental formula, such a limitation is reflected by an increase in the insurance charge included in the basic premium. In the case of the $10,000 limitation per claim for the workmen's compensation line in New York State, this is accomplished by reflecting the loss limitation in the construction of the excess pure premium ratio tables upon which the insurance charge is based. The Supplementary Rating Plan, given extensive consideration some years ago but never formally made effective, avoided the use of excess pure premium ratio tables by utilizing the split in the adjusted losses produced by application of the prospective experience rating plan to the risk. The losses in the term CL were limited by application of the normal loss amount per case provided for under the prospective plan. The excess loss charge, exclusive of a flat percentage of the standard premium added to reflect the maximum premium limitation, was equal to $\frac{Ae}{A} \cdot E$, where Ae represents the excess adjusted and A the total adjusted losses for the individual risk. Thus the best prospective estimate of the charge for losses above the normal on the particular risk was substituted for an average charge. The use of a flat percentage to reflect the maximum premium limitation could not, however, be justified in the light of our present-day knowledge of the variation of such an item by premium size for a fixed maximum percentage surcharge.

Many variations in the presentation of the retrospective rating formula are possible, some of which possess definite psychological advantages. One of these, the so-called Premium Return Plan, will be discussed in detail in a later section. One other will be briefly analyzed here, to illustrate the procedure of verifying the technical validity of a formula.

Consider the formula,

$$R = B + (1 - Z)\ (E - L) + CL \leqslant P$$

By comparison with formulas (8) it appears that the middle term has borrowed $E\,(1 - Z)$ from the first term in (8) and $-(1 - Z)\,L$ from the second term. Taxes on $-(1 - Z)\,L$, equal in amount to $-\frac{T}{1 - T}\,(1 - Z)$, must be included in C, but C otherwise reflects J and not Z, so that the formula would seem to fall under Type II, Class A, Case (c). In fact, assuming that

$$C = \frac{1 + J}{1 - T} - \frac{T}{1 - T}\,(1 - Z),$$

the complete multiplier of L which is divided between the second and third terms in the rating formula turns out to be

$$\frac{J+Z}{1-T}.$$

This is identical with the loss conversion factor under Type II. Then the B in the rating formula must be equal to

$$\frac{Vr+Dr-W+TE\,(1-Z)+(J+Z)\,P'p}{1-T}$$

in order to balance (6) properly. If B and C have the values thus determined in the course of the analysis, the formula is properly balanced, and the classification indicated above is correct.

Average Reduction from Standard Premium

Since the same aggregate amount must be provided for losses regardless of what the rating plan may be, the average premium for all risks of a given standard premium size is equal to the standard premium less the net reduction in the standard provisions for expenses, profit and contingencies including taxes thereon. This produces directly the first of the formulas (14) shown below for Rv. The second is the same as (6) in the foregoing section, the third is derived from (12), the fourth from (6) and (5), and the fifth from (6) and (4). Thus,

$$\left.\begin{aligned} Rv &= P-\frac{S}{1-T} \\ &= B+C\,(G'q+H's) \\ &= B+C\,(E-I) \\ &= H+C\,(H'p-G'p) \\ &= G-C\,(G's-H's) \end{aligned}\right\} \qquad (14)$$

It follows from these equations that the average reduction from the standard premium may take the following forms, each of interest:

$$\left.\begin{aligned} P-Rv &= \frac{S}{1-T} \\ &= C\,(P'-G'q-H's) \\ &= C\,(P'-E+I) \\ &= C\,(P'-H')-C\,(H'p-G'p) \\ &= C\,(G's-H's)-C\,(G'-P') \end{aligned}\right\} \qquad (15)$$

In the two last equations, $(P-H)$ may be substituted for $C\,(P'-H')$ and $(G-P)$ for $C\,(G'-P')$.

The average reduction, or the net expense reduction (as the expression for $\frac{S}{1-T}$ will be abbreviated henceforth), is thus equivalent to the following loss provisions all modified by application of the loss conversion factor:

(1) The loss allowance in the standard premium, less the average losses up to the loss allowance in the maximum premium, less the average loss savings on minimum premium risks.

(2) The excess of the loss allowance in the standard premium over the total expected losses for that size of risk, plus the loss portion of the insurance charge without modification by credibility.

(3) The allowance for losses between the minimum and standard premiums, less the sum of the average losses between the loss allowances in the minimum and maximum premiums.

(4) The excess of the allowance for losses over the average losses between the loss allowances in the minimum and maximum premiums, less the allowance for losses between the standard and maximum premiums.

These formulas apply for the general case regardless of the value of the credibility. The corresponding formulas for Cases (a) to (d) will be set forth, without verbal interpretation; the first and third are applicable to all four cases and will not be repeated here.

$$\text{Case (a)}: \quad \left.\begin{aligned} P - Rv &= C\,(P's - H's) \\ &= C\,(P' - H') - C\,(H'p - P'p) \end{aligned}\right\} \tag{15a}$$

$$\text{Case (b)}: \quad \left.\begin{aligned} P - Rv &= C\,(P' - G'q) \\ &= CG's - C\,(G' - P') \end{aligned}\right\} \tag{15b}$$

$$\text{Case (c)}: \quad P - Rv = CP's \tag{15c}$$

$$\text{Case (d)}: \quad P - Rv = CEs = CEp \tag{15d}$$

Rate for Individual Risk

A thorough understanding of the plan is aided by analysis of the credits or surcharges on individual risks as the loss ratio varies throughout its possible range. In the following, the contributions of the loss and expense portions respectively to the premium reductions are shown, and the surcharges are broken down into the loss contribution and the offsetting reduction produced by the expense gradation. The constitution of the loss conversion factor C as defined in (2) and (3) should be born in mind.

All of these formulas may be deduced from the formulas (15) for $(P - Rv)$, since the net expense reduction which we wish to segregate from the loss portion of the individual risk's credit or surcharge is identically equal to $(P - Rv)$. Many other expressions for the credit or surcharge could of course be derived, but only the more significant ones have been set forth.

Group 1: $L \leqslant H'$

These are the minimum premium risks, i.e.,

$$R = H$$

$$P - H = \frac{S}{1-T} + C\,(H'p - G'p) \tag{16}$$

The maximum credit a risk may earn is equal to the net expense reduction, plus the average converted losses between the loss allowances in the minimum and maximum premiums.

$$\text{Case (a):}\quad P - H = \frac{S}{1-T} + C\,(H'p - P'p) \tag{16a}$$

$$\text{Case (b):}\quad P - H = \frac{S}{1-T} + CG'q \tag{16b}$$

$$\text{Case (c):}\quad P - H = \frac{S}{1-T} + CP'q \tag{16c}$$

$$\text{Case (d):}\quad P - H = \frac{S}{1-T} + CEq \tag{16d}$$

The last term on the right-hand side of each of the formulas (16) to (16d) is equal to $(Rv - H)$, or the additional reduction on minimum premium risks beyond the average reduction on all risks.

Group 2: $H' < L \leqslant P' \leqslant G'$

These risks earn premiums between the minimum and standard premiums or equal to the latter.

$$\left.\begin{aligned} R &= B + CL \\ P - R &= C\,(P' - L) \\ &= (P - H) - C\,(L - H') \\ &= \frac{S}{1-T} + C\,(H'p - G'p) - C\,(L - H') \\ &= \frac{S}{1-T} + C\,(G'q + H's) - CL \end{aligned}\right\} \tag{17}$$

From the second (or third) of these equations, it is seen that the premium reduction on such risks is equal to that on minimum premium risks less the converted losses in excess of the loss allowance in the minimum premium, an obvious result. From the fourth, the loss portion of the reduction from the standard premium is equal to the converted losses collected on the average from other than the basic premium (refer to (6)), less the converted losses on the individual risk. This last form is the one chosen for presenting the results for Cases (a) to (d).

$$\text{Case (a)}: \quad P - R = \frac{S}{1-T} + C\,(P'q + H's) - CL \tag{17a}$$

$$\text{Case (b)}: \quad P - R = \frac{S}{1-T} + CG'q - CL \tag{17b}$$

$$\text{Case (c)}: \quad P - R = \frac{S}{1-T} + CP'q - CL \tag{17c}$$

$$\text{Case (d)}: \quad P - R = \frac{S}{1-T} + CEq - CL \tag{17d}$$

Group 2a: $H' < L < G' < P'$

It is possible to establish a maximum premium limitation below the standard premium, so long as the relationship

$$Rv < G < P$$

is observed. G cannot equal Rv, because in that event the plan would degenerate (in the mathematical sense) to a guaranteed cost plan. In this special case the formulas (17) to (17d) still are true, and to (17) may be added the formula,

$$P - R = (P - G) + C\,(G' - L)$$

Group 3: $P' < L \leqslant G'$

These risks earn premiums between the standard and maximum premiums or equal to the latter, and thus develop surcharges.

$$\left.\begin{aligned} R &= B + CL \\ R - P &= C\,(L - P') \\ &= (G - P) - C\,(G' - L) \\ &= C\,(G's - H's) - C\,(G' - L) - \frac{S}{1-T} \\ &= CL - C\,(G'q + H's) - \frac{S}{1-T} \end{aligned}\right\} \tag{18}$$

From the first and fourth of these equations, it is evident that the formula for the surcharge on risks in this group is the exact negative of the formula for the credit on risks in Group 2. The negatives of the second and third of the formulas (17) could have been included, but the corresponding formulas involving the maximum premium have been substituted. The surcharge is equal to the maximum surcharge less the amount by which the risk's converted losses fall short of the provision for such losses in the maximum premium. The interpretation of the third equation will be apparent from the interpretation of the maximum surcharge (obtained by setting $G' = L$ and thus eliminating the middle term) under Group 4 below. From the fourth equation, the following formula for Case (b) is deduced:

$$\text{Case (b)}: \quad R - P = CL - CG'q - \frac{S}{1-T} \tag{18b}$$

Cases (a), (c) and (d) are not possible in this group.

Group 4: $L > G' \geqslant P'$

These risks all earn the maximum premium, i.e.,

$$R = G$$

$$G - P = C\,(G's - H's) - \frac{S}{1-T} \tag{19}$$

Consequently, the maximum surcharge is equal to the excess of the allowance for converted losses over the average actual converted losses between the loss allowances in the minimum and maximum premiums, less the net expense reduction under the plan. This should be compared with the interpretation for risks in Group 1.

$$\text{Case (b)}: \quad G - P = CG's - \frac{S}{1-T} \tag{19b}$$

For Cases (a), (c) and (d),

$$R = P$$

$$\text{Case (a)}: \quad R - P = 0 = C\,(P's - H's) - \frac{S}{1-T} \tag{19a}$$

$$\text{Case (c)}: \quad R - P = 0 = CP's - \frac{S}{1-T} \tag{19c}$$

$$\text{Case (d)}: \quad R - P = 0 = CEs - \frac{S}{1-T} = CEp - \frac{S}{1-T} \tag{19d}$$

Group 4a: $L \geqslant G', G' < P'$

This special group corresponds to Group 2a already discussed, and is the group of risks earning the maximum premium under a plan wherein the maximum is less than the standard premium. In this special case,

$$R = G$$

$$P - G = \frac{S}{1 - T} - C\,(G's - H's), \tag{20}$$

or if $H = B$,

$$P - G = \frac{S}{1 - T} - CG's \tag{20b}$$

The credit on such maximum premium risks is equal to the average credit on all risks diminished by the amount by which the average converted losses between the loss allowances in the minimum and maximum premiums fall short of the allowance for such losses.

Premium Return Plan

The first of the formulas (17),

$$P - R = C\,(P' - L),$$

suggests an approach to the application of retrospective rating which has obvious psychological advantages. The plan as thus far discussed provides for the determination of the individual risk's premium by formula (1), that is, the final retrospective premium is built up from the basic premium. The formula just given, however, with a rearrangement of terms determines the final premium by deduction of a discount from the standard premium, the discount being a specified percentage C of the loss saving on the individual risk as measured from the loss allowance in the standard premium. For the case in which L exceeds P', the first of the formulas (18) could be similarly converted into a formula setting forth the final premium as equal to the standard premium plus a surcharge which is expressed as the product of C times the excess of L over P'. The psychological advantages of this presentation of a surcharge are not so apparent as in the corresponding case of a discount, however, so that this approach which determines the final premium through a departure from the standard premium is of particular importance in cases under which the maximum does not exceed the standard premium, such as Cases (a), (c) and (d). In such cases, this formula is the basis for what is commonly termed a "Premium Return Plan," the rating procedure under which may be expressed as follows:

> Final retrospective premium = standard premium minus [per cent of loss saving to be returned × (loss allowance in standard premium minus actual losses)]; the standard premium is charged if the actual losses exceed the loss allowance in the standard premium.

This result may, or may not, be subject to a specified minimum premium. There are thus four essential rating values, as in the case of formula (1) which we may call the accretive formula, three of which, the minimum and maximum premiums and the loss conversion factor, are common to both formulas; the basic premium in the accretive formula is replaced in the premium return formula by the loss allowance in the standard premium. The loss conversion factor, which in the rating procedure above was termed "per cent of loss saving to be returned," is also referred to frequently as the "premium return factor."

The symbolic representation, with the maximum premium equal to the standard premium, is

$$H \leqslant R = P - C\ (P' - L) \leqslant P \tag{21a}$$

This is labeled (21a) because it represents our Case (a) under retrospective rating.

For Cases (c) and (d),

$$R = P - C\ (P' - L) \leqslant P, \tag{21c}$$

and

$$R = P - C\ (E - L) \leqslant P \tag{21d}$$

In Case (c),

$$H = B = P - CP'$$

and in Case (d)

$$H = B = P - CE$$

A plan with the rating formula (21c) has been made effective in Pennsylvania for application to workmen's compensation insurance on National Defense Projects.

Under (21a) there are two degrees of freedom in the determination of the rating values, under (21c) there is one, and under (21d) none, if the modes of expense allocation are known. For analysis it is convenient to revert these formulas to their equivalent accretive forms. Considerable light may be shed upon the interpretation if we anticipate to some degree the section to be devoted to the determination of the rating values. For Case (a), from (19a) it follows that

$$\frac{S}{1-T} = C\ (P's - H's)$$

so that

$$C = \frac{\frac{S}{1-T}}{P's - H's} \tag{22a}$$

For Type I,

$$Z = \frac{S}{(1+J)\ (P's - H's)}, \tag{23a-I}$$

and for Type II,

$$Z = \frac{S}{P's - H's} - J \tag{23a-II}$$

The loss conversion factor, or the per cent of loss saving below the loss allowance in the standard premium which is returned on the individual risk, is thus equal to the net expense reduction including taxes divided by the loss savings which are to be reflected in determining the return premiums on all risks. In other words, the premium return factor distributes the amount obtained through reduced provisions for expenses and contingencies in direct proportion to loss savings as measured from the loss allowance in the standard premium, with due regard to the minimum premium limitation on the result.

For Case (c),

$$C = \frac{\frac{S}{1-T}}{P's} \tag{22c}$$

$$\text{Type I:} \quad Z = \frac{S}{(1+J)\,P's} \tag{23c-I}$$

$$\text{Type II:} \quad Z = \frac{S}{P's} - J \tag{23c-II}$$

For Case (d),

$$C = \frac{\frac{S}{1-T}}{Es} = \frac{\frac{S}{1-T}}{Ep} \tag{22d}$$

$$\text{Type I:} \quad Z = \frac{S}{(1+J)\,Es} = \frac{S}{(1+J)\,Ep} \tag{23d-I}$$

$$\text{Type II:} \quad Z = \frac{S}{Es} - J = \frac{S}{Ep} - J \tag{23d-II}$$

Clearly, it is possible for either Z or C in these formulas to exceed unity. But as a practical matter, it is not feasible to permit Z to exceed unity, so that if Z is the dependent variable in general, it will have to be considered independent when it is limited to 1.00. In such an event the dependent variable in (23a) will be either H or P', while in (23c) it will be P'. In (23d) the credibility cannot be fixed at 1.00 without changing the loss allowance in the standard premium. But that value has been fixed at E by the conditions defining Case (d); consequently, if Case (d) has been used in developing a premium return plan, then for those premium sizes for which Z (or C) is limited to a fixed value, such as 1.00, Case (c) must be used and P' considered as a dependent variable.

If $J = 0$, then it is more practicable to limit C to unity, rather than to limit Z thus, because of the difficulty of explaining a return of more than 100% of the loss saving on a risk; in theory, however, since taxes on such a return are also saved, it is perfectly logical (though not so practicable) to have a 100% credibility, in which case

$$C = \frac{1}{1 - T},$$

or the premium return factor is equal to the credibility of 100% loaded for taxes.

Further examination of (22a) brings a revelation which is startling at first thought, namely, that if the minimum premium is one of the rating values independently determined, and if the credibility is actually variable (has not attained its upper limit), the individual pays the same premium regardless of the value of J, i.e., regardless of the manner in which the provisions for expense items are distributed as functions of the standard premium on the one hand and as functions of the losses or loss provisions on the other. That this is true is shown by the fact that the expression on the right-hand side of (22a),

$$\frac{\dfrac{S}{1 - T}}{P's - H's},$$

is independent of any element relating to the apportionment of expense items to the two terms of the fundamental formula (1). From (23a) it is seen that as J decreases, Z increases, but $(1 - T)\,C$ remains constant if S, P' and H are given.

Once the credibility attains its upper limit, however, so that it can no longer behave as a variable, the value of P' (which as we have seen becomes a dependent variable in such an event) is dependent upon the value of J, and will vary if J varies, but not in proportion to J.

This survey of retrospective rating from the vantage-point of the premium return concept has cast new light upon the nature of the loss conversion factor. In the formulas (21) this factor turned out to be the proportion of the loss saving to be returned on each risk developing a premium less than standard, i.e., a factor governing the distribution of the amount available for return to those risks earning a credit, with due reflection of the minimum premium limiting the final premium.

It is reasonable to hypothesize an analogous function of the loss conversion factor in the general formula (1), to wit, that the loss conversion factor represents the proportion of the loss saving to be returned on each risk developing such a saving, the factor being so determined that the aggregate of all such returns is equal to the aggregate reduction from the standard provisions for expenses and contingencies plus the aggregate of the surcharges above standard premium on those risks developing such surcharges. The effect of the minimum and maximum premiums upon the returns and surcharges must be reflected in the calculation of the loss conversion factor.

In order to test the propriety of this analogical deduction, the value of C may be calculated on the basis of such an assumption, and then compared with its value obtained from a solution of (8) for any given value of R. The use of P for R in the latter solution is convenient and does not affect the result.

The reduction from the standard provisions for expenses and contingencies is equal to

$$\frac{S}{1-T}=\frac{V-Vr+D-Dr}{1-T}$$

The aggregate of the surcharges on risks developing a final premium in excess of the standard is equal to

$$C\cdot\frac{1}{n}\cdot\sum_{L=P'}^{G'}L=C\,(G'q-P'q) \qquad \text{(See Appendix B)}$$

The aggregate of the savings to be returned to risks developing a final premium less than the standard is equal to

$$C\,(P'-H')-C\cdot\frac{1}{n}\cdot\sum_{L=H'}^{P'}L=C\,(P's-H's) \qquad \text{(See Appendix B)}$$

By the hypothesis to be tested,

$$C\,(P's-H's)=\frac{S}{1-T}+C\,(G'q-P'q)$$

so that

$$\left.\begin{aligned} C&=\frac{\dfrac{S}{1-T}}{(P's-H's)-(G'q-P'q)}\\ &=\frac{\dfrac{S}{1-T}}{P'-G'q-H's}\end{aligned}\right\} \qquad (24)$$

On the other hand, from (8) and (9),

$$P=\frac{Vr+Dr-W+(1+J)\,E}{1-T}-C\,(E-G'p+H's)+CP',$$

and by (11),

$$P=\frac{V+D-W+(1+J)\,E}{1-T}$$

Therefore, subtracting the one value of P from the other,

$$\frac{S}{1-T}-C\,(P'-G'q-H's)=0$$

and

$$C = \frac{\frac{S}{1-T}}{P' - G'q - H's}$$

as before, proving the validity of the deduction by analogy which was being tested.

The second of equations (18) indicates that the fundamental formula (1) may be cast into the form,

$$H \leqslant R = G - C\,(G' - L) \leqslant G, \tag{25}$$

starting with the maximum premium and determining a "premium return" proportionate to the loss saving up to the allowance for losses in the maximum premium. The premium return factor in this case has of course the value defined by (24). If $G = P$, the condition determining Case (a), the formula (25) reduces to (21a).

The development of the generalized formula in this form is more of theoretical than of practical interest. There are advantages, however, in generalizing the first formulas in the groups (17) and (18) to reflect minimum and maximum limits, thus setting forth the fundamental formula in two parts in such a fashion as to emphasize the premium returns on risks with favorable loss ratios and the surcharges on risks with unfavorable loss ratios, the loss allowance in the standard premium marking the division line between the two groups, as follows:

$$\left.\begin{aligned} H \leqslant R &= P - C\,(P' - L), & L \leqslant P' \\ G \geqslant R &= P + C\,(L - P'), & L > P' \end{aligned}\right\} \tag{26}$$

The rating procedure could then be phrased:

1. If the risk's losses are less than the loss allowance in the standard premium, the premium reduction below standard is equal to (loss allowance in standard premium less risk losses) $\times$ loss conversion factor, the final premium being subject to a specified minimum premium.
2. If the risk's losses are greater than the allowance for losses in the standard premium, the premium surcharge above standard is equal to (risk losses less loss allowance in standard premium) $\times$ loss conversion factor, the final premium being subject to a specified maximum premium.

The dual formula (26) suggests a more general formula under which the loss conversion factor for credit risks would differ from the loss conversion factor for surcharge risks. If Ca designate the first and Cb the second of these two factors, then Ca and Cb will be connected by the relation,

$$Ca\,(P's - H's) = \frac{S}{1-T} + Cb\,(G'q - P'q) \tag{27}$$

This generalization has not been deemed sufficiently important to receive separate consideration in the establishment of categories of plans. As an example of its application, however, consider a plan which provides for surcharges up to a limit of K times the standard premium, the surcharge on an individual risk being equal to the losses in excess of the loss allowance in the standard premium, with the tax provision thereon. In such a case,

$$Cb = \frac{1}{1-T},$$
$$G = 1 + K,$$

and Ca is determined from (27). The amount obtained from the surcharge augments the amount available for distribution as a premium return to risks with favorable loss ratios.

As a final illustration of the many possible applications of the premium return concept, the standard premium P could be increased on all eligible risks by applying to it a factor $\left(1+\frac{Y}{P}\right)$, producing $(P+Y)$ as a premium from which to make available reductions to risks with favorable experience. The rating formula would be the same as (25) with $(P+Y)$ substituted for G. The premium return factor C would be determined by formula (22a) with $(P+Y)$'s substituted for P's.

The changes in these respective formulas to reflect Cases (c) and (d) are obvious and the variant formulas need not be given in detail.

Determination of Rating Values

Aside from a loss limitation per claim or per accident, there are four rating values which must be known for the application of a retrospective rating plan to an individual risk. Under formula (1) these are the basic, minimum and maximum premiums and the loss conversion factor; under formula (26) the loss allowance in the standard premium replaces the basic premium. It will be noted that if C is known, the determination of the basic premium is equivalent to the determination of the loss allowance in the standard premium, and vice versa, the two values being related through the equation

$$P = B + CP'$$

We may then confine our attention to a discussion of the determination of B, H, G and C.

If all other elements affecting the rating are known except the four rating values, three degrees of freedom remain and one of the rating values will be dependent upon the choice of the remaining three. These values in the general case vary by premium size, both because of the variation in the excess pure premium ratios by premium size and also because of the variation in the reductions from the standard provisions for expenses.

The case in which C (or Z) is the dependent variable has already received attention under the preceding section, wherein the formulas (24) were developed for the determination of C. It is evident that this value is not directly calculable, since P', G' and H' are dependent upon the choice of C. The loss conversion factor must therefore in the general case be determined by trial-and-error methods. In lieu of (24), a choice of other equations for the testing of trial values of C is available, such as the following, which is expressed so as to facilitate use of the excess pure premium ratio tables:

$$(P-H)-\frac{S}{1-T}=CE(H'x-G'x) \tag{28}$$

The corresponding formulas for the Cases (a) to (d) are given below:

$$\text{Case (a):}\quad (P-H)-\frac{S}{1-T}=CE(H'x-P'x) \tag{28a}$$

$$\text{Case (b):}\quad (P-B)-\frac{S}{1-T}=CE(1-G'x) \tag{28b}$$

$$\text{Case (c):}\quad (P-B)-\frac{S}{1-T}=CE(1-P'x) \tag{28c}$$

$$\text{Case (d):}\quad (P-B)-\frac{S}{1-T}=CE(1-Ex) \tag{28d}$$

Only in Case (d) is the value of C directly calculable. In this case, as could have been deduced immediately from (15d),

$$C=\frac{\frac{S}{1-T}}{Ep}$$

This equation reveals concretely why Case (d) is impossible when the credibility is constant (Class B). For if $Z=K$, a constant, the gradation of expenses will be governed by the following equations derived from the above:

$$\text{Type I:}\quad S=(1+J)\,KEp$$

$$\text{Type II:}\quad S=(J+K)\,Ep$$

Since Ep decreases as the premium size increases, the net expense reduction S would also decrease, with J constant, as the premium increases, a variation which is the contrary of what we know to be proper and which is therefore ruled out of the domain of practicability. The difference between the values of Ep for a small and a large premium size is too great to be offset by any variation in J, for J in practicable application is bounded by the limits

$$0\leqslant J\leqslant\frac{F+Vr+Dr}{E}$$

If the credibility is variable (Class A) and is independent of the other rating values, any law for its determination can be imposed so long as within its limits of variation, 0 and 1.00, it does not decrease as the premium size increases; that is, Z must be a monotonic increasing function of P. As an illustration, the credibility could be determined in accordance with the formula familiar in the experience rating plans,

$$Z = \frac{P}{P + K}$$

where K is an arbitrary constant.

Theoretically the credibility could be introduced on a one-split or multi-split basis, as is done in the prospective rating plans, but because of the imposition of the minimum and maximum premium limitations an accurate determination of the loss provision in the basic premium would be extremely difficult if not indeed impossible. This and other generalizations of the loss conversion factor are discussed briefly in Appendix C.

The practical determination of the basic premium from the other rating values is subject to a trial-and-error procedure because the mathematical forms of the functions

$$Lx = f\ (L, P)$$

and

$$L = g\ (Lx, P)$$

are so complex as to "defy computation" (as the small-town plumber advertised his work to do). This matter is discussed at length in Appendix B. Probably the simplest procedure is to test trial values of B in the following equation until the desired value of Dr is produced:

$$\left.\begin{array}{l} \text{Type I: } Dr = (1-T)\,B - Vr + W - \\ \qquad\qquad (1+J)\,E(1-Z) - (1+J)\,Z\,(G'p - H's) \\ \text{Type II: } Dr = (1-T)\,B - Vr + W - E\,(1-Z) - (J+Z)(G'p - H's) \end{array}\right\} \quad (29)$$

If the loss allowance in the standard premium replaces the basic premium as a rating value, it is feasible in the general case to calculate B from the trial value of P' and test it by the procedure just outlined. In some of the restricted cases, however, a more direct procedure is available. For example, under Case (c), the value of $P's$ can be determined directly from (28c), and the value of P' obtained from $P's$ by trial and error, through the relationship

$$P's = P' - E + EP'x$$

Of course, if any great number of calculations for a given state were being made, a table of values of $P's$ corresponding to given values of P' could be computed and P' obtained by inverse interpolation in such a table. An abbreviated table of this nature is included in Table I. For workmen's compensation, however, more than twenty such tables would be necessary, and their construction is so laborious as to be impractical. By the table-entry procedure explained in Appendix B, one table of excess pure premium ratios can be made to serve for all states (disregarding the effect of experience and benefit levels), but such a simplification is not possible in the case of Lp, Lq or Ls, because the permissible loss ratio enters into the determination of all these functions.

If B, G and C are given, in order to determine H the value of $H's$ may first be calculated by the formula

$$H's = P' - G'q - \frac{\frac{S}{1-T}}{C} \tag{30}$$

derived from (15), and the value of H' corresponding to $H's$ obtained by trial and error, as in the case of $P's$ and P' just discussed. Other formulas are available which may be more convenient for particular instances. But the one given appears to be the simplest for general use. H is then calculated from H' by (5).

If B, H and C are given, G' may be obtained by inverse interpolation in tables of excess pure premium ratios, entering with the value of $G'x$ determined from the formula

$$G'x = 1 + \frac{\frac{S}{1-T}}{CE} - \frac{P' - H's}{E} \tag{31}$$

derived from (15). Then G is determined by (4).

The operation of W as a clearing-house for otherwise untractable odds and ends of expense and contingency elements was described in the introduction to this paper. In the interest of simplicity and economy in rating, the loss conversion factor is rounded as a rule to two decimals; the effect of this rounding is absorbed in the basic premium through the element W, whether the rounding be up or down. In the standard plan, in order to permit one table of basic, minimum and maximum premiums for the facilitation of interstate rating, the loss conversion factor was adjusted in reflection of the differing expense loadings so as to produce no negative contingency balances; this adjustment was balanced by the introduction of the appropriate value of W in the basic premium.

The other rating values are also frequently rounded as a matter of convenience, the effect of the rounding being absorbed by the provision for contingencies.

VARIATION OF RATING VALUES

For practical application, the limits of variation of the respective rating values B, H, G and C are given by the following inequalities,

$$\frac{Vr + Dr - W}{1 - T} \leqslant B < Rv \tag{32}$$

$$B \leqslant H < Rv \tag{33}$$

$$Rv < G \leqslant K, \tag{34}$$

K being so determined that K' is the lowest loss ratio for which $K'x = 0$. The variation of C is governed by the following limits of variation of J and Z which have already been given:

$$0 \leqslant J \leqslant \frac{F + Vr + Dr}{E} \tag{35}$$

$$0 < Z \leqslant 1, \tag{36}$$

so that

$$0 < C \leqslant 1 + \frac{F + Vr + Dr}{E} \tag{37}$$

The value of Rv, as given by the first of equations (14) is

$$Rv = P - \frac{S}{1 - T}$$

This value is the upper limit of the minimum premium and the lower limit of the maximum premium. If either the minimum or the maximum premium is equal to Rv then

$$H = G = B = Rv$$

The plan is thus in such an event reduced to a guaranteed cost plan, with a discount from standard premium equal to $\frac{S}{1 - T}$ made available to every risk without regard to its loss ratio.

The lower limit of B has been given as equal to

$$\frac{Vr + Dr - W}{1 - T}$$

This limit reflects the assumption that negative values of the insurance charge, I, are not admissible. This assumption is debatable, and as a matter of fact the net insurance charge for certain low premium sizes in some states under the standard plan is negative, the loss saving on minimum premium risks being greater than the excess losses on maximum premium risks. Theoretically the value of B could be less than zero. Thus, if the credibility is 100% and

$$C\,(H's - G'p) > \frac{Vr + Dr - W}{1 - T},$$

B will be less than zero. Such values are not practicable, however, and the lower limit given for B would seem to be reasonable for practical application. This will insure retention of the provision for expenses and contingencies deemed necessary as an irreducible minimum, even if the risk develops no losses. On a very large risk, the provision may be small expressed as a percent of standard premium, with the balance for expenses collected through the factor J, but for risks small enough to produce loss ratios close to zero it is not practicable to include a large proportion of the expense provision other than claim as dependent upon the losses.

These limits of variation are all interdependent in a given case, since the rating values must at all times satisfy (24).

If all conditions other than the determination of the four rating values are assumed to be known, the relative variation of the rating values may be examined through the following equation:

$$B + C\,(G'q + H's) = Rv = \text{constant} \tag{38}$$

It is convenient to examine first the limiting case wherein the premium size is so great that there is no divergence of individual risk loss ratios from the permissible loss ratio. In the original charts of excess pure premium ratios for the workmen's compensation line, such a situation was assumed for a \$500,000 risk; for at that time an annual premium of \$500,000 lay in the domain of opiate visions. Now, less than a decade later, even larger risks are being written, and the cautious individuals responsible for the newer New York and Massachusetts charts have wisely refused to put to paper their idea of a premium size so fantastically remote as to warrant the assumption of such a state of perfection.

In this limiting case the following values obtain:

Condition	*Lx*	*Lp*	*Lq*	*Ls*
$L < E$	$1 - \frac{L}{E}$	$E - L$	L	0
$L \geqslant E$	0	0	E	$L - E$

Since in any practicable plan for such a premium size,

$$H' < E \leqslant G',$$

it follows that

$$I = 0$$

and

$$Rv = B + CE,$$

wherein for Type I,

$$B = \frac{Vr + Dr - W + (1 + J)\,E\,(1 - Z)}{1 - T},$$

and for Type II,

$$B = \frac{Vr + Dr - W + E(1 - Z)}{1 - T}$$

If the credibility is equal to 100%, as might be expected for such a large risk, B is further simplified to include provisions for expense only:

$$B = \frac{Vr + Dr - W}{1 - T}$$

Variations in H and G do not affect the result so long as the condition given above relating to H' and G' is maintained. An increase in B must be offset by a decrease in C, and vice versa; or B and C vary in opposite directions.

In studying cases other than this limiting case, the following comparative variations of the loss functions for a given premium size are useful. K', as in the discussion of the variation of G, will designate the lowest loss ratio for which $K'x = 0$. Reference to Table I may assist the reader. The expressions for increments may fail of fulfillment by one point in the last decimal place to which the increments are rounded, but are true in principle.

> Lp decreases continuously from E to 0 as L increases from 0 to K', and equals 0 for $L > K'$. (That is, Lp is a monotonic decreasing function of L.)
>
> $Lq = E - Lp$, and therefore increases continuously from 0 to E as L increases from 0 to K', and equals E for $L > K'$.
>
> $Ls = L - Lq$, and increases continuously from 0 to $(L - E)$ as L varies from 0 to K', and equals $(L - E)$ for $L > K'$.
>
> $\frac{dLq}{dL} = -\frac{dLp}{dL}$, and is a monotonic decreasing function of L, equal to 0 for $L \geqslant K'$.
>
> $\frac{dLs}{dL} = 1 - \frac{dLq}{dL}$, and is therefore a monotonic increasing function of L, equal to 1 for $L \geqslant K'$.

For a constant increment ΔL, therefore,

$$\Delta Lp + \Delta Lq = 0$$

and

$$\Delta Lq + \Delta Ls = \Delta L,$$

so that ΔLq and ΔLs have the same sign as ΔL, while ΔLp has the opposite sign. Further, if

$$\Delta H' = \Delta G' > 0,$$

$$\Delta H'q > \Delta G'q$$

so that in this case

$$\Delta G'q + \Delta H's < \Delta H'$$

If B and C are constant, $G'q$ increases as G increases; to balance (38), $H's$ must decrease and hence H' must decrease. Therefore H and G vary in opposite directions.

If C and G are constant, and B is increased by ΔB,

$$\Delta H' = \Delta G' = -\frac{\Delta B}{C}$$

But
$$\Delta H' = \Delta(G'q + H's) - \Delta(G'q - H'q),$$
so that
$$C \cdot \Delta(G'q + H's) = -\Delta B + C \cdot \Delta(G'q - H'q)$$
Further,

$\Delta(G'q - H'q) > 0$, since $\Delta H'q$ and $\Delta G'q$ are negative and $|\Delta H'q| > |\Delta G'q|$.

Therefore,
$$\Delta B + C \cdot \Delta(G'q + H's) = C \cdot \Delta(G'q - H'q) > 0,$$

and the equation (38) must be balanced by a decrease in H.

By analogous reasoning, if C and H are constant, B and G vary in opposite directions.

Thus far, the results are clear-cut. Such definiteness is not possible when C is one of the variables. If C is increased by an increment ΔC,

$$\frac{\Delta H'}{H'} = \frac{\Delta G'}{G'} = -\frac{\Delta C}{C + \Delta C}$$

The total increment on the left-hand side of (38) is equal to

$$\Delta C\ (G'q + H's) + (C + \Delta C) \cdot \Delta(G'q + H's)$$

the first term of which is positive and the second is negative. The total increment is then positive, zero or negative according as

$$\left|\frac{\Delta C}{C + \Delta C}\right| \gtreqless \left|\frac{\Delta(G'q + H's)}{G'q + H's}\right|$$

By reason of the nature of the variation of Lx by loss ratio and premium size, and the restrictions placed by common sense upon the rating values, it is probable that the total increment is always positive, to be offset by a decrease in B, G or H. The author has never encountered a case in which this was not true, and after experimenting with a wide variety of cases is convinced that the total increment will be positive for any practicable combination of rating values. The mathematical proof is not possible without a knowledge of the mathematical expression of Lx as a function of L and P.

The relative variations for premium sizes below the limiting case may then be summarized by stating that any pair of the rating values B, C, H and G must vary in opposite directions to maintain a balance in the equation (38). This can be proved in the cases where C is not one of the two variables and is probably true for all practicable combinations of rating values when C is one of the variables.

APPENDIX A

Table of Symbols

(All Symbols except C, J, K, Z and those with subscript "x" may represent either amounts or ratios to standard premium.)

A = adjusted losses $= E\,(1-Z) + LZ$
B = basic premium
C = loss conversion factor
D = provision for profit and contingencies on manual rate basis
E = expected losses
F = claim expense provision on manual rate basis
G = maximum premium
H = minimum premium
I = net provision for excess losses unmodified by credibility (see p. 295)
J = loss multiplier reflecting those provisions for expenses, profit and contingencies which are functions of the losses, equal to $\dfrac{F+W}{E}$
K = any constant
L = actual losses
P = standard premium
R = final premium
S = reduction in the provisions for commissions, expenses other than claim and taxes, and profit and contingencies under the plan $= V - Vr + D - Dr$
T = provision for taxes
V = provision for expenses other than claim and taxes on manual rate basis
W = provision for items other than claim included in the plan through the multiplier J
Z = credibility

Prime ($'$) attached to a premium symbol indicates the allowance for losses in that premium under the plan. Thus, G' designates the allowance for losses in the maximum premium.

The subscript "r" designates the provision for a particular item in the retrospective rating plan; in this paper it has been found necessary to attach such a subscript only to the two symbols D and V.

The subscript "v" designates the average value of the item bearing the subscript for the size of risk being discussed. Thus, Rv = average final premium for the particular size of risk.

A loss ratio symbol with the subscript "x" attached designates the ratio to total losses of losses in excess of that loss ratio on every risk for a given premium size, i.e., the excess pure premium ratio corresponding to the loss ratio bearing the subscript.

A loss ratio symbol with the subscript "p" attached designates the average amount of losses in excess of that loss ratio on every risk of a given premium size. It will be noted that $Lp = ELx$.

A loss ratio symbol with the subscript "q" attached designates the average amount of losses up to and including that loss ratio on every risk of a given premium size. It will be noted that $Lq = E\,(1 - Lx) = E - Lp$.

A loss ratio symbol with the subscript "s" attached designates the difference between the losses which will produce that loss ratio and the average losses up to and including that loss ratio on every risk of a given premium size. Thus, $Ls = L - Lq = L - E + Lp$.

APPENDIX B

The Loss Functions Lx, Lp, Lq and Ls

The excess pure premium ratio, Lx, corresponding to the loss ratio L is a concept familiar to all students of retrospective rating, and enters into the evaluation of the effect of imposing specified minimum and maximum premiums in the plan. The mathematical form of the function

$$Lx = f\,(L, P)$$

has not been determined. The excess pure premium ratios used in developing the standard plan for workmen's compensation risks were determined by graphical methods. As Mr. Dorweiler in his paper at the last meeting has explained,[4] the tables recently constructed in New York, with results close to those obtained some years previously by graphical methods, were based upon the formula (using the symbols set forth in Appendix A)

$$Lx = \left(1 - \frac{L}{E}\right) + \frac{L}{E}\,c^{-P^n}$$

In this equation c and n are parameters calculated for each of fourteen distinct loss ratios by determining log log c and n so as to produce the straight line of closest fit by the least squares method for the equation

$$n \log P - \log\log \frac{L}{Ls} + \log\log c = 0$$

Fourteen values of P were used in this procedure. It may be noted in explanation of the equation given above for determining c and n that

$$\frac{Lx - \left(1 - \frac{L}{E}\right)}{\frac{L}{E}} = \frac{Ls}{L}$$

The forms of the functions

$$n = f_1\,(L, P)$$

and

$$c = f_2\,(L, P)$$

(4) P.C.A.S., Vol. XXVIII, p. 132.

are extremely complex and are accurately determinable for only those loss ratios marking the limits of the respective loss ratio groups in the experience tabulations. In New York, the formula was used to calculate Lx for these loss ratios within certain size groups, the results were plotted against the adjusted average premium for each size group, and the final table was graduated graphically.

This comment is inserted to explain why no attempt has been made in the foregoing analysis to use any expression for Lx as a mathematical function of L and P.

In the construction of a table of excess pure premium ratios, the actual loss ratios used are adjusted to reproduce on the average a definite permissible loss ratio. In the case of the standard table, this permissible loss ratio was taken to be 60%; in New York 59.8% was used. If this permissible loss ratio underlying the table be designated by Et, and if E be the permissible loss ratio in the state for which the table is being used, the proper procedure in determining Lx corresponding to a given loss ratio L is to enter the table with the adjusted loss ratio

$$L \cdot \frac{Et}{E}$$

The resulting value of Lx is of course multiplied by E to give Lp. The interpolated value is the same as would be produced by interpolation in a table based upon the same experience data but with the actual loss ratios adjusted so as to reproduce E as the average (instead of Et).

The symbol Lp represents the excess pure premium as a ratio to premium, and may be termed an "excess premium ratio" to distinguish it from an excess pure premium ratio. It is also used to represent the product of the premium P by the ratio Lp.

The average losses within the loss ratio L, designated by Lq, are equal to the expected losses less the losses in excess of the loss ratio L.

The average loss saving within the loss ratio L, or Ls, is equal to $L - Lq$ and is therefore the difference between the total possible losses and the average losses within the loss ratio L.

Thus,

$$Lp + Lq = E$$
$$Lq + Ls = L$$

All of these symbols are in reality abbreviations for loss summations for a given premium size. To express them as such, let n represent the total number of risks of a given premium size; further, let the summation extend over only those risks on which the total risk losses are at least as great as the lower of the two summation limits, and include on each such risk only that portion of the losses actually lying within the summation limits. Thus, $\sum_{L=H'}^{G'} L$ indicates a summation extending over every risk incurring a total loss ratio not less than H', with only that portion of the losses in excess of H' but not greater than G' included in the summation. If P were equal to \$10,000, H' equal to 20% and G' equal to 80%, no risk would be included unless its total losses were equal to at least \$2000; if its losses were between \$2000 and \$8000 it would contribute to the summation the excess of its risk losses over \$2000; if its losses were greater than \$8000 it would contribute \$6000 (or \$8000 — \$2000) to the summation. It should be noted that since n equals the total number of risks of the given premium size, it is not equal to the number of risks actually contributing to the summation unless the lower limit of summation is zero. With these conventions, we may write:

$$G'p = \frac{1}{n} \sum_{L=G'}^{\infty} L$$

$$G'q = \frac{1}{n} \sum_{L=0}^{G'} L$$

$$H's - \frac{1}{n} \sum_{L=0}^{H'} (H' - L) = H' - \frac{1}{n} \sum_{L=0}^{H'} L$$

$$H'p - G'p = G'q - H'q = \frac{1}{n} \sum_{L=H'}^{G'} L$$

$$G's - H's = G' - H' - \frac{1}{n} \sum_{L=H'}^{G'} L$$

$$G'p + H'q = E - \frac{1}{n} \sum_{L=H'}^{G'} L$$

$$G'q + H's = H' + \frac{1}{n} \sum_{L=H'}^{G'} L$$

APPENDIX C

Generalized Loss Conversion Factor

In the general class which includes both Types I and II as special cases, the expenses which are reflected in J vary in part with the total risk losses L, and in part with the ratable risk losses ZL.

$$C = \frac{Ja + Jb \cdot Z + Z}{1 - T}$$

with

$$J = Ja + Jb,$$

$$Ja = \frac{Fa + Wa}{E},$$

and

$$Jb = \frac{Fb + Wb}{E}.$$

The provision for losses with expenses dependent thereon to be included in the basic premium is then equal to

$$(1+J)\,E-(Ja+Jb\cdot Z+Z)\,(G'q+H's)$$
$$=(1+Jb)\,E\,(1-Z)+(Ja+Jb\cdot Z+Z)\,I$$

where I is defined by (9). The general form of (7) becomes

$$B=\frac{Vr+Dr-W+(1+Jb)\,E\,(1-Z)+(Ja+Jb\cdot Z+Z)\,I}{1-T},$$

and the corresponding change in (8) is apparent.

If $Ja=0$ these equations reduce at once to the formulas derived for Type I; if $Jb=0$, they reduce to the Type II formulas.

Clearly, with due regard to the significance of C, virtually all of the formulas and discussion in the main body of the paper are valid for this general case. The important exception is found in the formulas (23) for the determination of Z:

$$\text{Case (a):}\quad Z=\frac{S}{(1+Jb)\,(P's-H's)}-Ja$$

$$\text{Case (c):}\quad Z=\frac{S}{(1+Jb)\,P's}-Ja$$

$$\text{Case (d):}\quad Z=\frac{S}{(1+Jb)\,Es}-Ja$$
$$=\frac{S}{(1+Jb)\,Ep}-Ja$$

It was noted in the section on "Determination of Rating Values" that theoretically the credibility could be introduced on a one-split or on a multi-split basis, as is done in the prospective rating plans. Because of the imposition of the maximum and minimum premium limitations an accurate determination of the loss provision to be included in the basic premium would be extremely difficult if not indeed impossible. The difficulty is apparent upon consideration of the simplest case, a one-split plan with no minimum premium specified other than the basic premium. If the subscript n designate normal, and the subscript e excess, in accordance with the customary nomenclature and symbolism, we may write

$$G=B+Cn\,G'n+Ce\,G'e$$

wherein Cn and Ce differ only to the extent of the differing credibilities reflected in them. But for given values of G, B, Cn and Ce, this equation is still indeterminate as respects the two unknown quantities $G'n$ and $G'e$, and there are an infinity of possible pairs of values for these two loss allowances to be considered in determining the loss provision in B; the relationship within each pair would vary moreover with variations in the credibilities.

If $Ce = 0$, the problem reduces to the one discussed in the paper in which the losses reflected in the rating are subject to a specified limiting amount per claim or per accident.

If Zn and Ze are both constant, for example if it should be desired to assign 100% credibility to the normal portion and 50% credibility to the excess portion of the losses, a table of excess pure premium ratios could be constructed corresponding to the loss ratio represented by the sum

$$Zn\,Ln + Ze\,Le$$

on each risk. A separate table would have to be constructed, i.e., a separate experience tabulation made, for each pair of values Zn and Ze if more than one pair were to be used, and in general the procedure would involve too much labor to be worthwhile unless it were definitely known in advance that a certain pair of credibilities would in fact be used. The New York table with losses limited to $10,000 per claim for medical and indemnity combined[5] represents such a table with $Ze = 0$. In this case, with $Ze = 0$, the value of Zn in any plan based on the table is not restricted to a constant.

[5] P.C.A.S., Vol. XXVIII, p. 312.

TABLE I

TABLE OF VALUES OF Lp, Lq AND Ls FOR SELECTED LOSS RATIOS AND PREMIUM SIZES
WORKMEN'S COMPENSATION STANDARD TABLE
PERMISSIBLE LOSS RATIO = 60%

L	$1,000			$5,000			$10,000			$25,000			$50,000			$100,000			$250,000		
	Lp	Lq	Ls	Lp	Lq	Ls	Lp	Lq	Ls	Lp	Lq	Ls	Lp	Lq	Ls	Lp	Lq	Ls	Lp	Lq	Ls
.05	.568	.032	.018	.556	.044	.006	.551	.049	.001	.550	.050	—	.550	.050	—	.550	.050	—	.550	.050	—
.10	.537	.063	.037	.512	.088	.012	.503	.097	.003	.501	.099	.001	.500	.100	—	.500	.100	—	.500	.100	—
.15	.510	.090	.060	.470	.130	.020	.457	.143	.007	.453	.147	.003	.452	.148	.002	.451	.149	.001	.451	.149	.001
.20	.486	.114	.086	.433	.168	.032	.414	.186	.014	.409	.191	.009	.405	.195	.005	.403	.197	.003	.402	.198	.002
.25	.466	.134	.116	.398	.202	.048	.373	.227	.023	.367	.233	.017	.361	.239	.011	.355	.245	.005	.353	.247	.003
.30	.448	.152	.148	.368	.232	.068	.337	.263	.037	.329	.271	.029	.322	.278	.022	.311	.289	.011	.307	.293	.007
.35	.432	.168	.182	.343	.257	.093	.307	.293	.057	.295	.305	.045	.284	.316	.034	.268	.332	.018	.262	.338	.012
.40	.417	.183	.217	.318	.282	.118	.280	.320	.080	.263	.337	.063	.249	.351	.049	.227	.373	.027	.218	.382	.018
.45	.404	.196	.254	.295	.305	.145	.253	.347	.103	.233	.367	.083	.215	.385	.065	.189	.411	.039	.178	.422	.028
.50	.391	.209	.291	.275	.325	.175	.230	.370	.130	.205	.395	.105	.182	.418	.082	.154	.446	.054	.142	.458	.042
.55	.379	.221	.329	.257	.343	.207	.209	.391	.159	.179	.421	.129	.153	.447	.103	.123	.477	.073	.109	.491	.059
.60	.368	.232	.368	.241	.359	.241	.189	.411	.189	.156	.444	.156	.127	.473	.127	.096	.504	.096	.082	.518	.082
.65	.358	.242	.408	.226	.374	.276	.172	.428	.222	.136	.464	.186	.105	.495	.155	.075	.525	.125	.059	.541	.109
.70	.348	.252	.448	.212	.388	.312	.155	.445	.255	.119	.481	.219	.088	.512	.188	.058	.542	.158	.041	.559	.141
.75	.339	.261	.489	.200	.400	.350	.142	.458	.292	.106	.494	.256	.074	.526	.224	.044	.556	.194	.027	.573	.177
.80	.331	.269	.531	.189	.411	.389	.130	.470	.330	.094	.506	.294	.062	.538	.262	.032	.568	.232	.016	.584	.216
.85	.323	.277	.573	.179	.421	.429	.119	.481	.369	.085	.515	.335	.053	.547	.303	.024	.576	.274	.009	.591	.259
.90	.316	.284	.616	.169	.431	.469	.111	.489	.411	.076	.524	.376	.046	.554	.346	.017	.583	.317	.004	.596	.304
.95	.309	.291	.659	.158	.442	.508	.103	.497	.453	.069	.531	.419	.039	.561	.389	.013	.587	.363	.001	.599	.351
1.00	.303	.297	.703	.149	.451	.549	.096	.504	.496	.063	.537	.463	.032	.568	.432	.008	.592	.408	—	.600	.400
1.05	.296	.304	.746	.140	.460	.590	.089	.511	.539	.057	.543	.507	.028	.572	.478	.005	.595	.455	—	.600	.450
1.10	.290	.310	.790	.131	.469	.631	.082	.518	.582	.052	.548	.552	.023	.577	.523	.002	.598	.502	—	.600	.500
1.15	.284	.316	.834	.122	.478	.672	.076	.524	.626	.047	.553	.597	.020	.580	.570	.001	.599	.551	—	.600	.550
1.20	.278	.322	.878	.114	.486	.714	.070	.530	.670	.042	.558	.642	.016	.584	.616	—	.600	.600	—	.600	.600
1.25	.272	.328	.922	.106	.494	.756	.065	.535	.715	.038	.562	.688	.013	.587	.663	—	.600	.650	—	.600	.650
1.30	.267	.333	.967	.097	.503	.797	.060	.540	.760	.034	.566	.734	.009	.591	.709	—	.600	.700	—	.600	.700
1.35	.262	.338	1.012	.089	.511	.839	.056	.544	.806	.030	.570	.780	.006	.594	.756	—	.600	.750	—	.600	.750
1.40	.256	.344	1.056	.082	.518	.882	.051	.549	.851	.026	.574	.826	.004	.596	.804	—	.600	.800	—	.600	.800
1.45	.251	.349	1.101	.076	.524	.926	.047	.553	.897	.023	.577	.873	.002	.598	.852	—	.600	.850	—	.600	.850
1.50	.246	.354	1.146	.070	.530	.970	.043	.557	.943	.019	.581	.919	.001	.599	.901	—	.600	.900	—	.600	.900