EXPERIENCE RATING PLAN CREDIBILITIES

BY

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For some time past certain criticisms have been made of the Compensation Experience Rating Plan. These have touched on various aspects of the Plan; some of them have been directed to the way in which the Plan works in particular instances. Other criticisms of the Plan have been in respect of some of the more debatable questions such as the period of experience to be used and the swing of the plan. This is the old question of Stability *vs.* Responsiveness and some of the critics have shown a surprising tendency to ignore the essential conflict between these two qualities. With these criticisms, those responsible for setting up and administering the Plan can doubtless deal. It is not in any way my intention to do more than mention them here as leading up to the subject of this paper. The Experience Rating Plan has recently been the subject of intensive studies by the responsible committees with the objects of seeing what there is of merit in the criticisms and of endeavoring to revise the Plan to make it better adapted to present-day conditions. The lessons gained from the, on the whole, successful working of the Plan over a large number of years are, of course, the principal guides in such studies.

One of the ideas being thus investigated is to see whether the Plan could not be simplified, particularly in the actual day-to-day process of rating, which is largely done by clerical help not particularly well trained in actuarial science, and scrutinized by agents, brokers, field men and assureds who, again, are not generally experts in casualty rate-making. One specific suggestion is that considerable simplicity would be obtained if, in respect of the small and medium-sized risks which are a great majority of the total number of rated risks, the large or excess loss experience were not rated. This idea has a lot of merit and the main purpose of this paper is to help it along by working out, systematically, the way in which the credibilities should be handled under such a plan. In effect under it the excess credibility will be zero unless the size of the risk is large, and considerable research and testing has to be done to be sure that such a plan will give consistent results and that the excess experience can be worked in satisfactorily for large risks.

In order to present a logical account of this investigation it is necessary first to give a fairly full account of the treatment of credibility under the present form of the Plan and this is done in the first two parts of the paper. The remaining parts are devoted, first, (since it seemed desirable to discuss some definite plan) to a brief description of a concrete plan, the multi-split plan,* which gives no excess credibility except for large risks. The balance of the paper is given up to a full discussion, with examples, of the determination of credibilities under this Plan.

While the paper discusses a particular Compensation Experience Rating Plan, I have tried to treat the question in such a way as to bring out the principles that should be used with the thought that these principles will be applicable to any similar experience rating plan, whether for Compensation or for any other kind of insurance, for which experience rating is suitable.

Part I

Credibilities in No Split Plans

1. *Analysis of Modification for Simplest Case—No Split Plan.*

First of all we will deal with the case of an experience rating plan with no splits, that is, where all losses (loss costs) are used with equal weight. In this case the ordinary formula for the modification (that is, the multiplier to be applied to manual rates) is

$$\frac{ZA + (1 - Z)E}{E} \tag{1}$$

where A denotes the actual losses
E denotes the expected losses
and Z is the credibility assigned to the risk.

In this paper I will not deal with questions of loss or payroll modification factors, or the number of years experience used, and

* I want to make it clear that no implication is intended that I was the originator of the multi-split plan. I wish I had been.

will assume that these are all incorporated in the "actual" and "expected" losses.

This modification can be put in the form (which I shall often have occasion to use later)

$$1 - Z + Z\frac{A}{E} \tag{2}$$

Note that this expression is in three parts:—

(i) unity, corresponding to no change from manual rates, as, for instance, if $Z = 0$

(ii) $-Z$, being the credit for clear experience, that is, if $A = 0$

and (iii) $+Z\frac{A}{E}$ being the charge for the actual losses of A.

2. *K Formula for the Credibility.*

The values to be given to Z in this modification are usually determined from the formula

$$Z = \frac{E}{E + K} \tag{3}$$

where K is a constant, i.e., does not vary with E.

Substituting this in (1) we get

$$\frac{A + K}{E + K} \tag{4}$$

In practice we can obtain the modifications either from (1) or from (4). If we use (1) we must have a reference table of Z from which to get the value to be substituted in (1). If we use (4) we need only to know the value of K. It is therefore somewhat easier to use (4) in this simple case but, as we shall see, when we come to use a split plan with provision for self rating for large risks, it is then easier to use a formula analogous to (1).

The value of K is determined from consideration of the "swing" it is desired to give the plan. K is usually fixed so as to give for a certain sized risk a definite credit (e.g., 10%) for clear experience or a definite charge (say 25%) for a single maximum loss.

The expression (3) gives for Z a value between 0 and 1, continually increasing as E increases but never quite reaching unity. In fact if Z is plotted as a function of E, Z moves along a branch of a hyperbola which has $Z = 1$ as an asymptote. (See Fig. I).

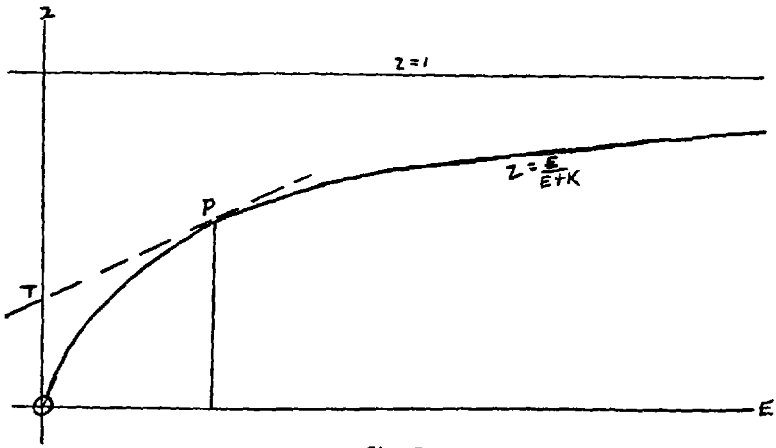

Fig. I.

3. *Conditions to which Z must be Subject.*

At this point it is advantageous to set down some conditions that the credibility Z should satisfy. These are general conditions derived from *a priori* considerations, and are applicable to the more complicated rating formulas we shall consider later.

(i) The credibility should be not less than zero and not greater than unity.

(ii) The credibility should increase (or more strictly speaking not decrease) as the size of the risk increases.

(iii) As the size of the risk increases the percentage charge for any loss of given size should decrease.

(i) and (ii) are obvious requirements; (iii) is perhaps not quite as evident at first, but a little thought will show it is desirable that, given two risks with differing expected losses, then if both have a single actual loss of the same amount the addition to the modification on account of the single loss should be less for the larger risk.

For instance, if we have two risks, the first with expected losses of 1,000 and the second with expected losses of 10,000: if each have a loss of 5,000, then on account of this loss

(a) by (i) above the addition to the premium in each case is positive and not greater than the equivalent of the 5,000 loss (that is if the expected loss ratio is 60%, the addition is not more than 8,333);

(b) by (ii) the addition is greater for the second risk than for the first; and

(c) by (iii) the addition is a *smaller* percentage of the (manual) premium for the second risk than for the first.

If we consider large self rated risks the reasons for (iii) becomes perhaps clearer: For these risks the addition to the premium is the same for a given loss of say 5,000, whatever the size of risk (for example the addition is 8,333 if the expected loss ratio is 60%) but the percentage addition gets smaller as the risk gets bigger.

The conditions mentioned can be expressed mathematically as

$$\left.\begin{array}{ll} \text{(i)} & 0 \leq Z \leq 1 \\ \text{(ii)} & Z' \text{ is not negative} \\ \text{(iii)} & (Z/E)' \text{ is negative} \end{array}\right\} \qquad (5)$$

where to economize space and to facilitate printing we have employed the common notation of Z' for $\frac{dZ}{dE}$: similarly we write W' for $\frac{dW}{dE}$, M' for $\frac{dM}{dE}$ and so on where W, M, etc. are functions of E. All differentiations are to be understood to be with respect to E. We have also written above Z/E for the constantly occurring expression $\frac{Z}{E}$ and we shall often employ this notation. $(Z/E)'$ means of course $\frac{d}{dE}\frac{Z}{E}$. We shall also often say "Z increases" or "Z/E decreases" meaning "Z increases as E increases" or "Z/E decreases as E increases" as will be clear from the context.

It is easily seen that Z as determined by (3) fulfills these conditions: for as E is positive (and K also) Z is > 0 and < 1: also $Z' = K/(E+K)^2$ and is positive, while $(Z/E)' = -1/(E+K)^2$ which is negative.

A useful geometrical interpretation of the conditions is as follows:

Plotting Z as a function of E (as in Fig. I which shows the curve $Z = E/(E + K)$)

(i) means the curve must be bounded by the E axis $Z = 0$ and by the straight line $Z = 1$ parallel to it;

(ii) means that as E increases the curve must always rise from $Z = 0$ towards $Z = 1$ or at most be parallel to the E axis or in other words the tangent at P must slope upwards from left to right or at most be parallel to the E axis;

(iii) means that the tangent must pass above the origin O and cut the Z axis above O; for the tangent at P cuts the Z axis at T where $OT = Z - EZ'$, (where E, Z are the co-ordinates of P), and the condition $(Z/E)' = (EZ' - Z)/E^2$ is negative means that $Z - EZ'$ is positive.

4. *Self Rating.*

In paragraph 2 we have seen that formula (3) for Z gives values that continually approach unity as E increases but never reach that value.

For practical reasons it is often desirable that for risks over a certain size the credibility Z be exactly unity. This certain size is called the self rating point and risks with credibilities equal to unity are called self-rated risks. We will denote the value of E at the self rating point by S. So for $E \geq S$, Z must be unity.

The question now arises as to the proper way to modify formula (3) so as to reach unity at S. Originally all that was done was to draw a straight line from some arbitrary point $(Q_1, Q_1/(Q_1 + K))$ to the self rating point $(S, 1)$ (see Fig. II) and use for Z between Q_1 and S the values given by this line. This however gives discontinuity to the values of Z at Q_1 and at S. So instead of using an arbitrary point Q_1, a tangent was drawn from the point $(S, 1)$ touching the curve $Z = E/(E + K)$ at $E = Q_2$. This is the present practice and does away with the discontinuity at Q_2 but leaves that at S. It would have been better, while making the change to have drawn a curve (e.g., a second degree parabola) touching the line $Z = 1$ at $E = S$ and also touching the curve $Z = E/(E + K)$ at $E = Q$. (See Fig. II).

(*Note*: We shall use Q generally to denote the value of E at the point of departure from the original credibility curve.)

Let us work out the equations of the tangent $s\, q_2$ and the touching parabola $s\, q$.

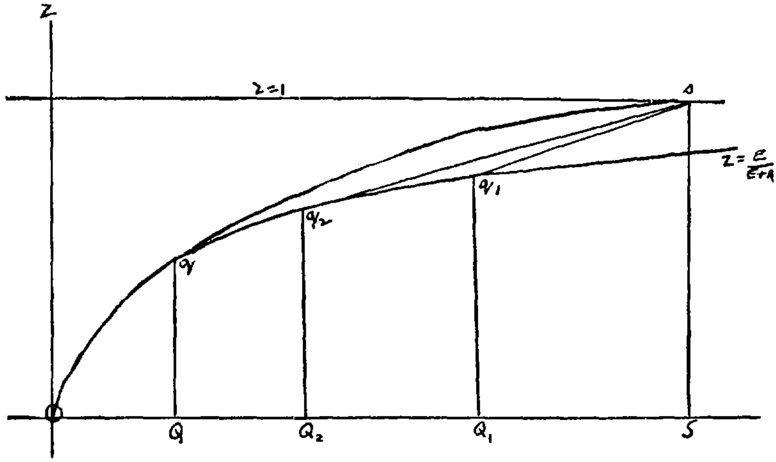

Fig. II.

The tangent to the (hyperbola) $Z = E/(E+K)$ at the point $E = Q_2$, $Z = Q_2\,(Q_2 + K)$ is

$$Z = \frac{E\,K + Q_2^2}{(Q_2 + K)^2}$$

and this passes through $E = S$, $Z = 1$ if

$$\left.\begin{aligned} Q_2 &= \frac{S-K}{2} \\ \text{The tangent is then } Z &= 1 - \frac{4\,K}{(S+K)^2}\,(S-E) \end{aligned}\right\} \qquad (6)$$

A simple parabola of the m-th degree, $Z = 1 - H\,(S-E)^m$, where H is a constant and m is $\nless 1$ will touch $Z = 1$ at $E = S$. It will also touch $Z = E/(E+K)$ at $E = Q$ if

$$\frac{K}{Q+K} = H\,(S-Q)^m \text{ for each must equal } 1 - Z$$

$$\frac{K}{(Q+K)^2} = H\,m\,(S-Q)^{m-1} \text{ for each must equal } Z'$$

from which

$$\left.\begin{aligned} m &= \frac{S-Q}{Q+K} \\ H &= \frac{K\,m}{(S-Q)^{m-1}} \end{aligned}\right\} \qquad (7a)$$

Thus we can either (i) choose m (greater than one) and

$$\left.\begin{aligned} \text{then } Q &= \frac{S - mK}{m+1} \\ \text{and } H &= \frac{K\,(m+1)^{m+1}}{m^m\,(S+K)^{m+1}} \end{aligned}\right\} \qquad (7b)$$

or (ii) choose Q (which must be less than $Q_2 = (S-K)/2$) then m and H can be calculated from equations (7a). If Q is taken as zero $m = S/K$, $H = S^{-\frac{S}{K}}$.

Thus by taking $m > 1$ and $\leq S/K$, or $Q \geq 0$ and $< (S-K)/2$ we can obtain the equation of a simple parabola (not usually a second degree parabola) which touches the credibility curve $Z = E/(E+K)$ at Q and touches the line $Z = 1$ at S. The credibility to be used will be that given by $Z = E/(E+K)$ from 0 to Q, that given by $Z = 1 - H\,(S-E)^m$ from Q to S, and $Z = 1$ for $E > S$.

To determine which parabola (or which value of Q to use, which is the same thing) other considerations (such as the credibilities to be given for various values of E) have to be invoked. Probably for most purposes the second degree parabola obtained by putting $m = 2$ will be satisfactory. For this

$$\left.\begin{aligned} m = 2 \qquad\qquad Q &= \frac{S - 2K}{3} \\ Z &= 1 - \frac{27\,K\,(S-E)^2}{4\,(S+K)^3} \end{aligned}\right\} \qquad (8)$$

This is (in an unfamiliar guise or disguise) the familiar "square root" formula used elsewhere in casualty actuarial science as a credibility formula.

Note that the case of the tangent can be deduced by putting $m = 1$.

Note also that if Q is made equal to zero we use the parabola all the way from 0 to S and the original credibility curve has apparently been dropped entirely. Its influence, however, is still present in determining the slope of the parabola at $E = 0$. This case can of course be treated separately as the use of a family of curves:—

$$Z = 1 - \left(\frac{S-E}{S}\right)^m \qquad (9)$$

where the parameter m has to be settled from other considerations such as the swing to be given to the plan. It will probably be found in many cases that a credibility curve of this type will rise too fast, or in other words if it gives satisfactory values for small values of E it will give too large values for intermediate values. For example this would usually be so if we took $m = 2$ to get the "square root" formula.

It is important to note that as all the parabolas suggested are concave to the E axis the conditions (5) of paragraph 3 are complied with. Z is between 0 and 1, Z' is positive and so is $Z - EZ'$. This is also true of the straight line tangent.

In applying credibilities as thus adjusted to rise to unity at the self rating point it would be very complicated to use the formula in each case, as suggested for the second alternative method in paragraph 2. It is apparently better to use the first alternative there mentioned and have a table of Z values to which reference may be made to get the proper value for a given E; in other words to use as a working formula (1) as opposed to a modified (4).

5. *Another Method of Reaching Self Rating.*

The last sentence represents the general view in the past. However, we can retain most of the advantages of using a formula like (4) by proceeding as follows:—For values of E greater than Q calculate K_E from

$$Z = \frac{E}{E + K_E}$$

where Z is the credibility value from the parabola: thus $K_E = E\,(1 - Z)/Z$. Construct a table for K_E for all values of E, putting $K_E = K$ for $E < Q$. Then apply formula (4) thus

$$\text{modification} = \frac{A + K_E}{E + K_E}. \tag{10}$$

By this method the great majority of risks will be rated by the simple formula (4) with a constant K and for large risks all that is necessary is to ascertain the value of K_E and use the same simple formula. In practice, however, the complications introduced by the present method of splitting into normal and excess would preclude the adoption of this scheme.

This suggests, nevertheless, another method of attaining self rating, namely, by using (4) and gradually reducing the constant

K as E goes from Q to S. Thus if we were to construct values of K_E so that, at Q, $K_E = K$ and $K'_E = 0$ and, at S, $K_E = 0$ and $K'_E = 0$ we would get credibility values which would join smoothly with those given by $Z = E/(E+K)$ at Q and with $Z = 1$ at S.

We will not at present pursue this further, but as will be seen later this idea is used in the more complicated questions of split plans and multi-split plans.

6. *Justification for Departing from Usual Credibility Formula.*

At this point it would seem desirable to see what theoretical objections there may be to departing from the usual or standard credibility formula $Z = E/(E+K)$ or, to put it the other way, whether we can justify departures such as dealt with above. The first thing to be remembered here is that the standard credibility formula itself does not give an exact measure of the proper credibility that shall be given to the risk experience. It is an approximation to an approximation of an expression for the credibility that was based on some necessarily rather arbitrary assumptions as will be seen from the classic papers of Messrs. Whitney and Michelbacher, (P.C.A.S., Vol. IV), describing the genesis of the present form of experience rating. I do not mean to be understood to be attacking the general validity of the usual formula or to be advocating its abandonment. The formula is a very satisfactory, practical instrument that gives credibility values conforming in a reasonable manner to what we would expect and it is because of this that it has stood the test of time. I do mean to state, however, that any not too violent departures from the formula arising out of the self-rating adjustments given in the preceding paragraph cannot be condemned merely for the reason that they are departures. If—as they do—these departures give values that also are reasonable in the light of our a priori judgment and that conform to the criteria of paragraph 3, then our system of credibility values is just as defensible as those given by the unadulterated standard credibility formula.

To anticipate a little so as to collect together all the remarks on departure from the standard formula, similar considerations apply to the usual form of split plan dealt with in Part II. As for the multi-split plan dealt with in the remainder of the paper, the question there arises as to the validity of the method used of handling the excess credibility. This is kept at zero for small and medium-sized risks and for large risks is brought up to unity at the self-rating point. If the excess portion is considered by itself there is little theoretical justification for this procedure but excess experience *is* excess and always arises in connection with the corresponding normal experience and never by itself, so we must consider the normal and excess parts together. Then whether we look at the risk's average or over-all credibility or whether we look at the effect of any reasonable combination of normal and excess experience we will find that the credibilities by the multi-split plan are not unreasonable.

PART II

CREDIBILITIES IN SPLIT PLANS

7. *Application to "Split" Plans.*

So far we have dealt with a no-split plan as explained in paragraph 1. We now shall consider the necessary modifications of the preceding theory so as to apply it to a split plan. It is not my intention to deal with the history of experience rating (for which see Mr. Kormes' recent papers, P.C.A.S., Vols. XXI and XXII) and so I will merely state here that almost invariably losses (both Actual and Expected) are divided into "normal" and "excess," that is to say the risk is considered in two parts; first, the experience on losses limited to a certain amount per case (say $1,000 indemnity and $100 medical), this being the "normal" part; and second, the experience on the loss cost in excess of this certain amount, this being the "excess" part. The expected losses are divided in the same way (from the available statistics) and the final rate for the risk is the sum of the adjusted rates for each of the two parts.

Less credibility is given to the excess losses since they are more unusual. The reason for making the split is fairly obvious. Without a split a single loss of, say, 3,000 gets as much weight as six losses of 500 each and it is both theoretically and practically desirable to give the six losses much more weight.

The rating formula is as follows where E_n, A_n, Z_n denote the normal expected losses, actual losses and credibility respectively and E_e, A_e, Z_e are the same for the excess part, (note that $E_n + E_e = E$ and $A_n + A_e = A$).

$$\text{Modification} = \frac{E_n}{E} \frac{Z_n A_n + (1 - Z_n) E_n}{E_n} + \frac{E_e}{E} \frac{Z_e A_e + (1 - Z_e) E_e}{E_e}$$

$$= \frac{Z_n A_n + (1 - Z_n) E_n + Z_e A_e + (1 - Z_e) E_e}{E} \qquad (11)$$

If as usual we use

$$\frac{E_n}{E_n + K_n} \text{ for } Z_n \text{ and } \frac{E_e}{E_e + K_e} \text{ for } Z_e$$

(where by making K_e much larger than K_n we give much less credibility to the excess losses) we get for the modification

$$\frac{E_n}{E} \frac{A_n + K_n}{E_n + K_n} + \frac{E_e}{E} \frac{A_e + K_e}{E_e + K_e}$$

which is not subject to much simplification for working purposes. In fact, it is easier to read Z_n and Z_e out of a prepared table and apply (11) particularly as (i) the normal and excess ratios E_n/E and E_e/E vary for risks according to the classifications involved and (ii) by using (11) it is easy to modify Z_n and Z_e (in accordance with the principles set out in Part I) to attain self-rating at S_n and S_e respectively (these self-rating points usually differ). Z_n and Z_e are usually brought to self-rating by means of tangents as shown in paragraph 4, equations (6), although I think it would be better to use a second degree parabola as per equations (8).

It is to be noted that since both Z_n and Z_e comply with the conditions (5) of paragraph 3, so does also the combination of the two in (11) whatever be the proportions of the normal and the excess portions.

8. *Analysis of Split Plan Modification.*

It is useful to note (for it will be needed later) the following analysis of (11).

$$\left.\begin{array}{l} 1 - Z_n \dfrac{E_n}{E} + Z_n \dfrac{A_n}{E} - Z_e \dfrac{E_e}{E} + Z_n \dfrac{A_e}{E} \\ \text{or } 1 + \dfrac{E_n}{E}\left\{ - Z_n + Z_n \dfrac{A_n}{E_n} \right\} + \dfrac{E_e}{E}\left\{ - Z_e + Z_e \dfrac{A_e}{E_e} \right\} \end{array}\right\} \qquad (12)$$

This is analogous to the analysis in paragraph 1 of expression (1) into (2): here the parts are:

(i) unity $\left(\text{equal to } \frac{E_n + E_e}{E}\right)$.

(ii) (a) $-Z_n E_n/E$ the credit for clear normal experience.

(b) $-Z_e E_e/E$ the credit for clear excess experience.

(iii) (a) $+Z_n \frac{A_n}{E}$ or $Z_n \frac{A_n}{E_n} \cdot \frac{E_n}{E}$ the charge for the actual normal losses of A_n.

(b) $+Z_e \frac{A_e}{E}$ or $Z_e \frac{A_e}{E_e} \cdot \frac{E_e}{E}$ the charge for the actual excess losses of A_e.

Part III

The Multi-Split Plan—Derivation of Formulas

9. *The Multi-Split Plan.*

The present state of the experience rating plan (as far as the scope of this paper is concerned) is practically as described in Part II. Recently, however, studies have been made with a view to improve the plan and the remainder of this paper arose out of considering some aspects of suggestions which took the form of (i) advocating the so-called multi-split plan and (ii) endeavoring to reduce the working formula to as simple a form as possible, the aim being something like (4).

The so-called multi-split plan consists of a different way of dividing the total losses into "normal" and "excess", or rather as originally proposed, it reduced all losses to normal losses leaving out of account the remainder (or excess) losses, which are not so great as under the ordinary plan. The principle invoked is to take the first (say) 500 of each loss at its face value, the next 500 at (say) two-thirds of its actual value or at a reduction of one-third, the next 500 at another one-third reduction, namely, four-ninths of its actual value, and so on. Thus a very large loss could not be taken at more than 1,500 (using the above values which are illustrative only). The reduction is achieved by means of a table of discounted values showing the discounted value to be used for each size of loss exceeding 500. For losses not greater than 500 the full value is to be used. Thus a loss of 1,000 would have a discounted value of 833 (equal to 500 plus two-thirds of 500), a loss of 1,500 a discounted value of 1,055 (equal to 833 plus two-thirds of two-thirds of 500) and so on. Intermediate values (e.g. for a loss of 800) would be shown in the tables, calculated from the formula:—

$$\text{Discounted value for loss of } x \ (x > 500) = 1{,}500\left\{1 - (2/3)^{\frac{x}{500}}\right\}$$

or if a is the starting point (corresponding to the 500 above) and ρ (< 1) is the discounting ratio (corresponding to the $2/3$ above)

$$\text{Discounted value for loss of } x \ (x > a) = a\,\frac{1 - \rho^{\frac{x}{a}}}{1 - \rho} \qquad (13)$$

The maximum discounted value is obviously $a/(1 - \rho)$.

From the risk's experience the discounted losses A_n would be determined (it being necessary to enter the table of discounted values only for losses $> a$) and from collective statistics the corresponding expected discounted losses E_n would be determined.

From A_n and E_n by a simple credibility formula (several suggestions as to this are given below) the risk's modification would be calculated. For the great majority of risks, no attention would be paid to the "remainder" losses $A - A_n$ (or excess losses) the experience on these being brought in only above a certain size of risk (i.e., after a certain Q point) to attain ultimate self-rating (at a certain S point).

It is not my purpose here to go into the details or to discuss the soundness or otherwise, or the merits and demerits of the multi-split plan except to say that I believe the idea to be a good one (better than the current split-plan) and that the discounted values given by the exponential curve (13) seem, from tests and from theoretical considerations, to give a good approximation to the relative weight that should be given to losses of various sizes. I hope to give a fuller account of these tests, theoretical and practical, at another time. In this paragraph I have given the above brief account of the plan so as to render intelligible the ideas of the remainder of this paper which is concerned with the credibility formulas to be used in connection with the multi-split plan *or any other plan* where the excess credibility used is zero up to a certain (Q) point and then is gradually brought up to unity at a self-rating (S) point as is in effect done in the multi-split plan. In any case it is not desirable to pass judgment on the multi-split plan until an exploration has been made of how to manage the credibilities this plan is to grant. It is the main purpose of this paper to do some of this exploring.

10. *First Formula for the Modification.*

The first formula we shall consider for the modification to be used in the multi-split plan is arrived at in this way.

If in (11) we put $Z_e = 0$ we get

$$\frac{Z_n A_n + (1 - Z_n) E_n + E_e}{E}$$

and now if, for simplicity, we put $Z_n = E/(E + K)$ (instead of the usual $E_n/(E_n + K_n)$) we get

$$\frac{A_n + E_e + K}{E + K}$$

and we take this for the modification when $E \leq Q$, when $Z_e = 0$.

Now we can get self-rating by adding $A - (A_n + E_e + K)$ or $A_e - E_e - K$ to the numerator of this expression and subtracting $(E + K) - E$ or K from the denominator: we accordingly use for the modification for $E > Q$

$$\frac{A_n + E_e + K + W\,(A_e - E_e - K)}{E + K - W K}$$

where W is to be zero for $E \leq Q$ and unity for $E \geq S$, and in between zero and unity for E between Q and S.

Thus:

$$\left.\begin{aligned} \text{Modification} &= \frac{A_n + E_e + K}{E + K} \\ &\qquad \text{for } E \leq Q \\ \text{and} &= \frac{A_n + E_e + K + W\,(A_e - E_e - K)}{E + K\,(1 - W)} \\ &\qquad \text{for } E > Q \text{ and } \leq S \end{aligned}\right\} \quad (14)$$

where W is a function of E (to be determined), equal to zero for $E = Q$ and rising from 0 to 1 as E goes from Q to S.

This is perhaps not quite as simple as a formula (see (31)) to be considered later but I deal with it first because of the greater ease of handling the theoretical work.

It will be observed that if $A_n = E_n$ (and $A_e = E_e$ if $E > Q$) the modification equals unity as it should.

Now (14) can be analyzed into:

$$\left.\begin{array}{ll} 1 - \dfrac{E_n}{E+K} + \dfrac{A_n}{E+K} & \text{for } E < Q \\ \text{and } 1 - \dfrac{E_n}{E+K\,(1-W)} + \dfrac{A_n}{E+K\,(1-W)} & \\ \quad - \dfrac{W\,E_e}{E+K\,(1-W)} + \dfrac{W\,A_e}{E+K\,(1-W)} & \text{for } E > Q \end{array}\right\} \quad (15)$$

whence by a comparison with (12)

$$\left.\begin{array}{lll} Z_n = \dfrac{E}{E+K},\ Z_e = 0 & & \text{for } E < Q \\ Z_n = \dfrac{E}{E+K\,(1-W)},\ Z_e = \dfrac{W\,E}{E+K\,(1-W)} & & \text{for } E > Q \end{array}\right\} \quad (16)$$

We see that $Z_n = 0$ for $E = 0$
and $Z_n = Z_e = 1$ for $E = S$ where $W = 1$
also $Z_n > Z_e$ for $E < S$ (except for $E = 0$)

It will be noted that here, and this is true generally of the multi-split plan as we shall discuss it, that there is only one self-rating point, not one for normal losses and one for excess as in the case of the present plan. This is deliberately done as one means of simplification, and is justifiable if the self-rating point is not too low.

11. *Conditions for W to fulfill.*

Before proceeding to the determination of W, it is necessary to consider how this function must behave. We see at once that as well as $W = 0$ for $E = Q$ and $W = 1$ for $E = S$, we must have $W' = 0$ for $E = Q$ and for $E = S$ in order that we have smooth junctions with $Z_n = E/(E+K)$ and $Z_e = 0$ at $E = Q$ and with $Z_n = Z_e = 1$ at $E = S$.

Furthermore we must also determine W in such a manner that the credibilities comply with the conditions (5), paragraph 3. For $E < Q$, Z_n obviously complies with these (as has been shown above) and so does Z_e and therefore so does any combination of Z_n and Z_e.

For $E > Q$ both Z_n and Z_e comply with (5i) but, on the other hand, Z_e *cannot* comply with (5iii) as will readily be seen from the geometrical interpretations of this condition given in paragraph 3. As Z_e has to rise from zero at $E = Q$ to unity at $E = S$ the tangent to the curve $Z_e =$ function of E must, at any rate for the first part of the range $E = Q$ to $E = S$ cut the Z_e axis below the origin (see Fig. III). This of course applies to all varieties of plan where $Z_e = 0$ up to a point $E = Q$ and then rises to unity at a point $E = S$, in such a manner that there is a smooth junction at Q.

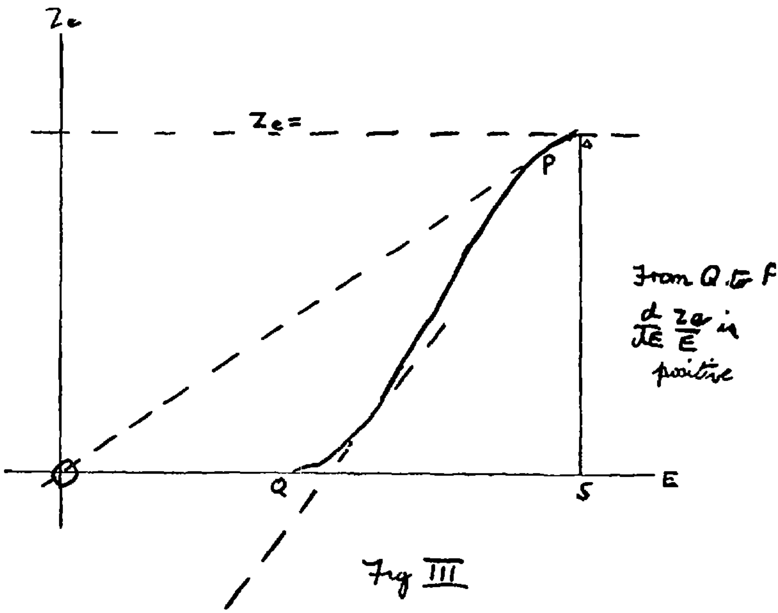

Fig III

Let us consider, however, any single loss and let the ratio of the excess portion of this to the normal be θ.

Since we can have a "normal" loss with no excess portion but cannot have an "excess" loss without a corresponding "normal" portion, it follows that θ can range from 0 to some maximum value which we will call α. To take the illustration given in paragraph 9 where $\alpha = 500$ and $\rho = 2/3$, if the maximum possible actual loss is 7,500, it is easily seen that α will be very nearly equal to 4 for the maximum normal loss is 1,500. (The actual value of α in this case is 4.01).

Then it is $Z_n + \theta Z_e$ which must comply with the conditions (ii) and (iii) of (5) and for all possible values of θ. Since θ can be zero, Z_n must certainly comply with these conditions; and then $Z_n + \theta Z_e$ will also comply for all values of θ if it complies for the maximum value of θ regardless of whether Z_e complies or not, for the conditions in question are linear in Z_n and Z_e. Thus we must have Z_n and $Z_n + \alpha Z_e$ (which we will call ζ) both complying with (ii) and (iii). As regards condition (ii) it is desirable (but not necessary) that Z_e also comply (and this can be arranged.)

We observe that at Q $\zeta/E = 1/(Q+K)$ and at S it equals $(1+\alpha)/S$ so that for ζ/E to decrease from Q to S as required by conditions (iii) we must have

$$S > (1+\alpha)(Q+K) \tag{17}$$

This is of course a condition limiting the choice of S when Q has been chosen and vice versa.

12. *Examination of Conditions*

We see from (16) and from

$$\zeta = Z_n + \alpha Z_e = \frac{E(1+\alpha W)}{E + K(1-W)} \tag{18}$$

that we can either determine W directly or first settle on ζ from which we can get W and the other functions. Before deciding which we will do we shall first collect together and "boil down" the requirements that must be fulfilled.

A. Terminal Conditions

(i) W must be 0 at $E = Q$ and 1 at $E = S$
W' must be 0 at Q and at S

(ii) Z_n must be $Q/(Q+K)$ at $E = Q$ and 1 at $E = S$
Z'_n must be $K/(Q+K)^2$ at $E = Q$ and 0 at $E = S$

(iii) ζ must be $= Z_n$ at $E = Q$ and $= 1 + \alpha$ at $E = S$
ζ' must be $= Z'_n$ at $E = Q$ and $= 0$ at $E = S$

(iv) Z_e must be 0 at $E = Q$ and $= 1$ at $E = S$
Z'_e must be 0 at $E = Q$ and at $E = S$

It is easily seen that any one of the sets of conditions (i) to (iv) is equivalent to the other three, e.g., if (iii) holds then (i), (ii) and (iv) must.

B. Conditions for $E > Q$ and $< S$

As E increases

(i) Z_n should increase
(ii) Z_n/E should decrease
(iii) ζ should increase
(iv) ζ/E should decrease

It is also desirable but not mandatory that in addition

(v) Z_e should increase
(vi) W should increase

(The solutions given will comply with (v) and (vi))

Let us see if all the B conditions are independent and if not let us reduce them to the fewest possible.

First expressing Z_n in terms of ζ by eliminating W from (18) and the expression for Z_n in (16) we get

$$Z_n = \frac{a E + \zeta K}{a E + (a+1) K} \tag{19}$$

Differentiating* this

$$\{a E + (a+1) K\}^2 Z'_n = a K (a+1-\zeta) + \{a E + (a+1) K\} K \zeta'$$

and as $a + 1 - \zeta$ is positive, we find that Z'_n is if ζ' is. So B (iii) includes B (i).

Also

$$\frac{Z_n}{E} = \frac{a + K \zeta/E}{a E + (a+1) K}$$

and it is obvious, without differentiating, that if ζ/E decreases as E increases, so does Z_n/E. Thus B (iv) includes B (ii).

Further, differentiating (18) we get

$$\{E + K(1-W)\}^2 \zeta' = K (1-W)(1+aW) + \{a E + (a+1) K\} E W'$$

* We shall frequently have occasion to differentiate an expression of the form $Z = \frac{X}{Y}$ where X, Y and Z are functions of E. To save space we will usually not write the result in the form $Z' = \frac{Y X' - X Y'}{Y^2}$ but instead will put it in the form

$$Y^2 Z' = Y X' - X Y'$$

which shows that if W' is positive so is ζ'. Thus B (iv) includes B (iii) and therefore also B (i).

Also, as $Z_e = W Z_n$, if W' is positive and therefore Z'_n is, so is Z'_e. Thus B (vi) includes B (v).

The B conditions therefore can be reduced to:

B (iii) ζ should increase
B (iv) ζ/E should decrease

which are mandatory, or to the following which comprises all the mandatory and desirable conditions:

B (iv) ζ/E should decrease
B (v) W should increase

We could now proceed for example to make Z_n go from its value $Q/(Q+K)$ at $E=Q$, to 1 at $E=S$ (using the methods of paragraph 4) and see whether the resulting Z_n values gave W and ζ values which complied with B (iv) and B (iii) or B (v), but this is an indirect way of working. It is better to determine one of the functions so that the conditions are directly complied with. It appears that the most suitable function to operate on is either ζ or W for these are the functions appearing in the conditions B (iv), B (iii) and B (v).

I have found that ζ is somewhat preferable. I construct a formula for it so as to satisfy B (iii) and B (iv) and then find it also satisfies B (v).

The alternative of constructing W itself so as to comply with B (v) and B (iv) is a little more complicated but (as shown in Appendix III) leads to identically the same results as by the method I have used, namely, constructing ζ first.

13. *Construction of ζ.*

We have then to construct ζ so that (i) at $E=Q$, ζ equals $Q/(Q+K)$ and $\zeta' = K/(Q+K)^2$; (ii) at $E=S$, ζ equals $1+\alpha$ and $\zeta'=0$; (iii) ζ' must be always positive, and (iv) $(\zeta/E)'$ must be always negative. It is understood (17) that $S > (1+\alpha)(Q+K)$.

We could try drawing a simple parabola of the m-th degree as in paragraph 4 from $(S, 1+\alpha)$ touching the curve $E/(E+K)$ at $E=Q$, but this is possible only if the tangent at $E=Q$ to the curve $E/(E+K)$ cuts the line $\zeta = \alpha + 1$ at $E = S_1$ where $S_1 < S$. It is easily found that

$$K\,S_1 = (Q+K)^2\,(\alpha+1) - Q^2$$

while S_2 the minimum value of S from (17) is given by

$$K\,S_2 = K\,(Q+K)\,(\alpha+1)$$

and therefore

$$K\,(S_1 - S_2) = Q\,\{\alpha\,Q + (\alpha+1)\,K\} \text{ and so } S_1 > S_2.$$

So if S lies between S_1 and S_2, no such parabola can be drawn. (What the above proves is that if S is between S_1 and S_2, the curve for ζ must contain a point of inflexion between Q and S which is evident if a diagram is drawn.)

We could use in some cases a non-simple cubic parabola of the form

$$\zeta = a_1\,(S-E)^3 + a_2\,(S-E)^2 + a_3\,(S-E) + (1+\alpha)$$

but this again would not work for all combinations of Q, K and S and in any event if we used such a parabola we would have to investigate to see that the necessary requirements for ζ and W were met, and this would lead to many restrictions. As we are looking for a universal construction we must try something else.

14. *Construction of ζ by Method Finally Used.*

I have accordingly devised a method of constructing an expression for ζ which will give the required values to ζ and its first differential coefficient at both $E = Q$ and $E = S$ and for which ζ continually increases and ζ/E continually decreases as E increases. In order not to burden the body of the paper unduly with mathematics, I have relegated the details of this construction to Appendix I. However, in order to preserve continuity I have numbered the equations in that appendix just as though the appendix were placed here; thus equations (20) to (27i) inclusive are to be found in Appendix I.

The construction is given in detail but it will be seen that all the calculation of the constants is contained in the equations (27b) to (27g). Then from (27h) and (27i) ζ is readily obtainable for all required values of E from Q to S.

15. *This Construction Fulfills Required Conditions.*

From ζ as thus determined W is found from (18) which gives

$$W = \frac{(\zeta - 1)\,E + \zeta\,K}{\alpha\,E + \zeta\,K} \qquad (28)$$

from which W is readily calculated for values of E.

If our object is to calculate W as quickly as possible, we can eliminate the step of calculating ζ from Y—see equation (27i)—and use instead

$$W = \frac{E + K - Y}{\alpha Y + K} \tag{28a}$$

We also have for E from Q to S,

$$\left.\begin{aligned} Z_n &= \frac{\alpha E + \zeta K}{\alpha E + (\alpha + 1) K} \\ Z_e &= \frac{(\zeta - 1) E + \zeta K}{\alpha E + (\alpha + 1) K} \end{aligned}\right\} \tag{29}$$

These of course give the proper values to Z_n, Z_e, Z'_n and Z'_e at Q and at S. Also of course W, Z_n, Z_e are all between 0 and 1 and $Z_n > Z_e$ (because $\zeta < \alpha + 1$).

We also know from paragraph 12, that as ζ' is positive and $(\zeta/E)'$ is negative Z'_n is also positive and $(Z_n/E)'$ is negative.

We can prove that W (and therefore also Z_e) increases with E for our construction. The proof will be found in Appendix II.

This completes, for the moment, the discussion of formula (14) for the modification. Let us note, however, that the construction for W does not depend upon the value of the excess ratio E_e/E or r.

16. *Second Formula for the Modification.*

We will now consider another formula that has been suggested for the modification for the multi-split plan on the ground that is rather simpler than (14) in practical application.

This formula was derived as follows: For $E < Q$ use the normal modification as the modification for the risk: For $E > Q$ amplify the formula so as to equal A/E at $E = S$ just as was done for the previous formula (14). The result is

$$\left.\begin{aligned} &\frac{A_n + K}{E_n + K} && \text{for } E < Q \\ \text{and} \\ &\frac{A_n + K + W(A_e - K)}{E_n + K + W(E_e - K)} && \text{for } E > Q \end{aligned}\right\} \tag{30}$$

but if we analyze this as per (12) we get

$$Z_n = \frac{E}{E_n + K + W(E_e - K)}, \; Z_e = W Z_n.$$

Now if $E_e > K$, Z_n is greater than unity, contravening condition (5) (i) of paragraph 3. This means that if $E_e > K$ (whether E is less or greater than Q and whatever W is—except unity) the charge for a normal loss will be greater than the premium equivalent. However, we can adjust (14) so as to overcome this, as follows:—First of all we must lay down the condition that K must be greater than E_e for $E = Q$; then instead of the constant, K, in (30) we put a function of E, which we will call K_E, such that this is equal to the constant K for $E \leq Q$ but increases as E increases above Q so that K_E is always greater than E_e and also so that $K'_E = 0$ for $E = Q$ (this insures a continuous join of K and K_E at Q.)

We thus have for the modification

$$\left.\begin{array}{ll} \dfrac{A_n + K}{E_n + K} & \text{for } E < Q \\[2ex] \dfrac{A_n + K_E + W\,(A_e - K_E)}{E_n + K_E + W\,(E_e - K_E)} & \text{for } E > Q \text{ and } \leq S \end{array}\right\} \quad (31)$$

Leaving the determination of K_E aside for the moment and putting $M = K_E - E_e$ where M is of course a function of E we have

$$\left.\begin{array}{ll} \text{for } E < Q & Z_n = \dfrac{E}{E_n + K} = \dfrac{E}{E + M} \\[2ex] \text{for } E > Q & Z_n = \dfrac{E}{E_n + K_E - W\,(K_E - E_e)} = \dfrac{E}{E + M\,(1 - W)} \\[2ex] & Z_e = W\,Z_n = \dfrac{W\,E}{E + M\,(1 - W)} \end{array}\right\} (32)$$

Now M is positive and so Z_n is > 0 and < 1 until $W = 1$ when $Z_n = 1$: $Z_e = 0$ while $W = 0$ and then as W rises from 0 to 1, Z_e is > 0 and < 1 until $W = 1$ when $Z_e = 1$. Also $Z_n > Z_e$.

17. *Construction of W for Formula (31).*

We now determine $\zeta = Z_n + \alpha Z_e$ in a manner similar to that used for formula (14).

Put M_Q for the value of M at Q. We have

$M' = K'_E - E'_e = K'_E - E_e/E$, and $M - E\,M' = K_E - E\,K'_E$

and

$$\frac{d}{dE}\,\frac{E}{E + M} = \frac{(E+M) - E\,(1+M')}{(E + M)^2} = \frac{M - E\,M'}{(E+M)^2} = \frac{K_E - E\,K'_E}{(E + M)^2}$$

Now at $E = Q$, $K'_E = 0$ and so at that point

$$\frac{d}{dE}\frac{E}{E+M} = \frac{K}{(Q+M_Q)^2}$$

So we must have

$$\text{at } Q \qquad \zeta = \frac{Q}{Q+M_Q} \quad \zeta' = \frac{K}{(Q+M_Q)^2}$$

$$\text{at } S \qquad \zeta = \alpha + 1 \qquad \zeta' = 0.$$

Now if we denote E_e/E, the excess ratio, by r and put

$$\zeta = \frac{E}{Y(1-r)} \tag{33}$$

we must have (compare with the method used in Appendix I)

$$\text{at } Q \qquad Y = (Q+M_Q)\,/(1-r) = Q + K/(1-r),\ Y' = 1$$

$$\text{at } S \qquad Y = \frac{S}{(\alpha+1)(1-r)} \qquad\qquad Y' = \frac{1}{(\alpha+1)(1-r)}$$

Now if (i) $(\alpha+1)(1-r)$ is greater than unity, which it will be for r is small, say less than ½, while α is greater than one,

$$\text{and if (ii)} \qquad S > (\alpha+1)\{Q(1-r)+K\} \tag{34}$$

(this corresponds to the condition (17) and means that ζ/E must be less at S than at Q), we can proceed to determine Y just as previously (see after equation (20)—Appendix I)

u will in this case be $1/(1+\alpha)(1-r)$ and w will be

$$\frac{S-(\alpha+1)\{Q(1-r)+K\}}{(S-Q)(1+\alpha)(1-r)}$$

Note that w is positive and $u-w$ is positive, by (34).
Thus $0 < w < u < 1$.
Thus we determine λ, p, k, h, j, t as before and we get:

$$Y = \left\{\frac{h}{k} - \frac{t}{j}\right\}\left(S-Q\right) + \left(Q + \frac{K}{1-r}\right) - \frac{h(S-Q)^2}{E-Q+(S-Q)k} + \frac{t(S-Q)^2}{j(S-Q)+Q-E} \tag{35a}$$

$$\zeta = \frac{E}{Y(1-r)} \tag{35b}$$

Note that because K_E increases with E, $E+M$, which is the same as E_n+K_E, increases as E does and faster than E_n+K so that $(1-r)\,Y$, which is less than E_n+K, is *a fortiori* less than E_n+K_E: thus ζ is greater than $E/(E_n+K_E)$ or $E/(E+M)$.

18. *This Construction Fulfills Requirements.*

Now as for formula (14) we have

$$W=\frac{(\zeta-1)\,E+\zeta\,M}{a\,E+\zeta\,M} \tag{36}$$

As before we can express W in terms of Y namely

$$W=\frac{E+M-Y\,(1-r)}{a\,Y\,(1-r)+M} \tag{36a}$$

Also

$$\left.\begin{aligned} Z_n&=\frac{a\,E+\zeta\,M}{a\,E+(a+1)\,M}\\ Z_e&=\frac{(\zeta-1)\,E+\zeta\,M}{a\,E+(a+1)\,M}\end{aligned}\right\} \tag{37}$$

These of course give the proper values to Z_n, Z_e, W and their first derivatives at $E=Q$ and at $E=S$.

Also, since $\zeta>E/(E+M)$, W, Z_n and Z_e are all between 0 and 1, and Z_n is greater than Z_e (except at S).

Examining now Z'_n we find

$$\{a\,E+(a+1)\,M\}^2\,Z'_n=a\,(a+1-\zeta)\,(M-E\,M') \\ +\{a\,E+(a+1)\,M\}\,M\,\zeta'$$

and Z_n will certainly be positive if $M-E\,M'$ is. Now, as shown above, this last expression is the same as $K_E-E\,K'_E$: this means Z'_n will certainly be positive if $(K_E/E)'$ is negative and we will so construct K_E.

Now to examine $(Z_n/E)'$

$$\frac{Z_n}{E}=\frac{a+M\,\zeta/E}{a\,E+(a+1)\,M}$$

Now the denominator of this equals

$$E\,\{(1+a)(1-r)-1\}+(a+1)\,K_E$$

which, as $(1+a)(1-r)-1$ is positive, increases with E. As for the numerator, ζ/E decreases as E increases and if M does also then the whole numerator does, and so if M decreases, Z_n/E will also unquestionably decrease. On the other hand if M increases, we find by differentiation that

$$\{a\,E+(a+1)\,M\}^2\,(Z_n/E)'=\{a\,E+(a+1)\,M\}\,M\,(\zeta/E)' \\ -a\,(a+M\,\zeta/E)-a\,(a+1-\zeta)\,M'$$

and the right hand side is certainly negative if M' is positive for $(\zeta/E)'$ is negative. Thus whether M increases or decreases, Z_n/E decreases. (Note, the construction we adopt, in paragraph 19, makes M' negative for the first part of the range Q to S and positive for the latter part).

We can also show that W (and therefore Z_e) increases with E, for our construction. As in the case of the corresponding proof for the formula (14) construction we have put this proof in Appendix II.

19. *Determination of K_E.*

We now come to the determination of K_E. We must have

(a) $K_E = K$ for $E = Q$.
(b) $K'_E = 0$ at Q and positive for $E > Q$.
(c) $(K_E/E)'$ negative.
(d) $K_E > E_e$.

We first note that (d) is the only condition involving E_e (or in other words r) and if $K_E > E_e$ for the maximum value of r it will be so for all values of r: so we will make $K_E > E_e$ for the maximum value of r and then we can use the same series of values of K_E for all values of r. Let this maximum value of r be g; note that as K must be greater than E_e for $E = Q$ we must have $K > Q\,g$.

For $E > Q$ we will let K_E be given by the hyperbola

$$(K_E - g\,E)(E + a_1) = a_2$$

which is asymptotic to $K_E = g\,E$ (see Fig. IV). We will determine the constants a_1 and a_2 so that the curve touches $K_E = K$ at $E = Q$. We have

$$K_E = \frac{a_2}{E + a_1} + g\,E \qquad \text{so that } K = \frac{a_2}{Q + a_1} + E\,Q$$

$$K'_E = g - \frac{a_2}{(E + a_1)^2} \qquad \text{so that } g = \frac{a_2}{(Q + a_1)^2}$$

whence $a_1 = \dfrac{K - 2\,Q\,g}{g}$ $\qquad a_2 = \dfrac{(K - Q\,g)^2}{g}$

and thus:

$$K_E = \frac{(K - Q\,g)^2}{g\,E + (K - 2\,Q\,g)} + g\,E \tag{38}$$

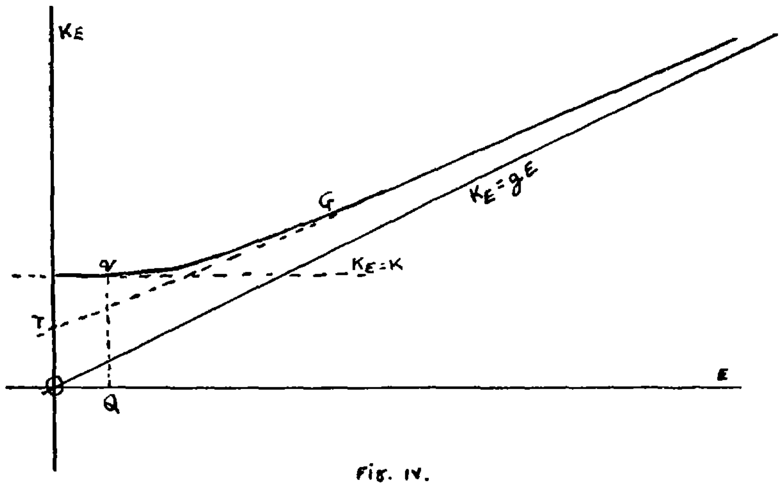

Fig. IV.

The curve is $q\,G$ in Fig. IV. The tangent at any point G cuts the K_E axis at T above 0 showing that $(K_E/E)'$ is negative. Thus all the conditions (a) to (d) are complied with.

For Appendix II it is necessary to note that the maximum value of $O\,T$ occurs for $E = Q$, that is, the maximum value of $K_E - E\,K'_E$, which equals $M - E\,M'$, is K.

We have now completed for the moment the discussion of formula (31). We will return later to consider how to deal with the different values of r that arise. The construction given above for W depends on the value of r used; note, however, that the formula (38) for K_E is useable for all values of r.

20. *Third Formula for the Modification.*

Let us now see what we must do if we apply the ordinary modification formula (11) to the multi-split plan. Making $Z_e = 0$ we have for the modification for $E < Q$

$$\frac{E_n}{E}\ \frac{A_n + K_n}{E_n + K_r} + \frac{E_e}{E}$$

which we can write as

$$\frac{K + \frac{E_n}{E}(A_n + E_e)}{E_n + K} \quad \text{for } E < Q \tag{39a}$$

For $E > Q$ and $\leq S$ we can put $E_n + K(1 - W)$ for the denominator and we must add to the numerator

$$W\{(A - A_n - E_e)\,E_n/E - K\}$$

and we get the rather cumbersome formula

$$\frac{K + (A_n + E_e)\frac{E_n}{E} + W\left\{(A_e - E_e)\frac{E_n}{E} - K\right\}}{E_n + K(1 - W)} \quad \text{for } E > Q \tag{39b}$$

for which $Z_n = \dfrac{E_n}{E_n + K(1 - W)}$, $Z_e = W Z_n$

It will be seen that

$$Z_n = \frac{E}{E + \frac{(1 - W)K}{1 - r}}$$

which is of the same form as Z_n in (16) with $K/(1 - r)$ for the K there; and indeed if we multiply the top and bottom of (39b) by E/E_n and put ${}^rK$ for $K/(1 - r)$ we get

$$\frac{{}^rK + A_n + E_e + W(A_e - E_e - {}^rK)}{E + {}^rK(1 - W)} \tag{40}$$

which is of the same form as (14) with ${}^rK$ for K.

So we can determine W just as for (14) but using ${}^rK$ for K. We note, however, that as for formula (31) the values of W depend on the value of r.

21. *Value of Excess Ratio to be Used.*

Now let us consider this question of the value of r that enters into the determination of W. We have discussed three formulas for modifications, namely, (14), (31) and (39). For the first W does *not* depend on r but for the last two it does. It is obviously impractical to calculate a series of values of W for each separate possible value of r and we will therefore see if we cannot use, for all values of r, the values of W calculated for one particular r, say the average value or the maximum or the minimum value. Let us take (39) first, and suppose we have calculated values of W for a certain excess ratio r and use them for risks with a different excess ratio x. Then, since $W = 0$ at Q and $= 1$ at S and $W' = 0$ at both Q and S, Z_n will join smoothly at Q with the values below Q, and will be tangential to $Z_n = 1$ at S; also Z_e will $= 0$ at Q and 1 at S and Z'_e will $= 0$ at both Q and S.

Now since $\zeta = E\,(1 + \alpha W)/\{E + {}^xK\,(1 - W)\}$

$$\{E + {}^xK\,(1 - W)\}^2\,\zeta' = {}^xK\,[(1 - W)(1 + \alpha W) + (1 + \alpha)\,E\,W'] + \alpha\,E^2\,W'$$

which will be positive for all values of x since W' is positive. We also see that Z'_n is positive by putting $\alpha = 0$ in the above, when ζ becomes Z_n. Also Z'_e is positive, for $Z_e = W\,Z_n$.

Now to consider Z_n/E and ζ/E. We easily find by differentiation that

$$\{E+{}^xK\,(1-W)\}^2\,(\zeta/E)'={}^xK\,(1+\alpha)\,W'-\{(1+\alpha\,W)-\alpha\,E\,W'\}$$

and by considering that this expression is negative if r is put for x, we see it remains negative if ${}^rK > {}^xK$: we see similarly $(Z_n/E)'$ is certainly negative if ${}^rK > {}^xK$ (put $\alpha = 0$ in the above expression).

So if ${}^rK > {}^xK$ or $r > x$ we can certainly use with safety for the case of an excess ratio x the W's derived for the ratio r. On the other hand there is some margin in the fulfillment of the conditions by the W's derived for ratio r (except perhaps in a borderline case where S is only a little greater than $(1 + \alpha)(Q + {}^rK)$—see (17)) and if x is not much greater than r we probably will still have Z_n/E and ζ/E decreasing.

We note that the condition $r > x$ is what we would expect; for if $r > x$, then ${}^rK > {}^xK$ and Z_n for $E = Q$ will be greater for x than for r. Thus at Q, Z_n/E and ζ/E, which are equal at Q, will be greater for x than for r. On the other hand at S, Z_n/E and ζ/E are equal for all values of excess ratio being equal to $1/S$ and $(1 + \alpha)/S$ respectively. So for x the ratio Z_n/E or ζ/E has further to decrease as E goes from Q to S than it has for r and we should not be surprised therefore that the W values calculated for r will work satisfactorily for a smaller ratio x.

To come now to formula (31) we first note that we have taken care of K_E by using the maximum excess ratio in fixing it. As far as Z_n and ζ are concerned, we easily find that if the excess ratio is x

$$\{E + M\ (1-W)\}^2\, \zeta' = (M - E\, M')(1-W)(1+\alpha\, W) + \{\alpha\, E + (\alpha+1)\ M\}\ E\, W'$$

Now $M - E\, M' = K_E - E\, K'_E$ which is positive and so the right hand side is positive whatever the value of x. If we put $\alpha = 0$ in the above equation, ζ becomes Z_n and the right hand side is of course still positive. Thus Z'_n and ζ' are positive for all values of x. The question, however, is not so simple when we come to consider Z_n/E and ζ/E.

We have

$$\{E + M\ (1-W)\}^2\ (\zeta/E)' = M\ (1+\alpha)\ W' - \{(1+\alpha\, W) - \alpha\, E\, W'\} - M'\ (1-W)(1+\alpha\, W)$$

Now in this M refers to an excess ratio x and if we write, temporarily, $\overline{M}$ for the M for the ratio r, we have

$$M = \overline{M} - (x - r)\ E \qquad\qquad M' = \overline{M}' - (x - r)$$

and the right hand side of the above equation becomes

$$\overline{M}\ (1+\alpha)\ W' - \{(1+\alpha\, W) - \alpha\, E\, W'\} - \overline{M}'\ (1-W)(1+\alpha\, W) + (x-r)\ \{(1-W)(1+\alpha\, W) - (1+\alpha)\ E\, W'\}$$

which we will call $X + (x - r)\ \mu$.

Now X we know is negative for it is what the above right hand side becomes if $x = r$. As for μ, this $= 1$ for $E = Q$ and $= 0$ for $E = S$, but as we shall see as E goes from Q to S μ rapidly becomes negative and remains negative till E reaches S. If we write, for the moment, V for $W - E\, W'$, V is the distance above the origin that the tangent to the curve for W (as a function of E) cuts the W axis $E = 0$. μ becomes $(1-2\, W - \alpha\, W^2) + (1 + \alpha)\ V$. The first term in this equals 1 for $W = 0$ $(E = Q)$, equals 0 for $W = \{-1 + \sqrt{1 + \alpha}\}/\alpha$, equals $-(1 + \alpha)$ for $W = 1$ $(E = S)$ and decreases continually from $W = 0$ to $W = 1$. As for the second term, V equals 0 at $E = Q$ and equals 1 at $E = S$. As will be seen from the examples given below V is negative from $E = Q$ until E is well advanced towards S. Thus we find μ starting from

1 at Q rapidly becomes negative, reaches a minimum and then rises to 0 at S. Now if $x > r$ and μ is negative $(\zeta/E)'$ will be negative, but if μ is positive $(\zeta/E)'$ will be negative only if $(x - r)\,\mu$ is not greater than $-X$. Thus if $x > r$, $(\zeta/E)'$ will be certainly negative over the greater part of the range from Q to S and the only region it can be positive is in the earlier part of the range and then only if there is not much "margin," i.e., only if the relationship of Q and S is such that there is not much drop in ζ/E from Q to S. Further, if in any particular case where there is not much margin and where, therefore, ζ/E does not decrease continuously in the earlier part of the range Q to S, we can improve the situation by using a higher value of η in calculating the W values. It will readily be seen on examination of the construction of Y in Appendix I that a higher value of η will give higher values of Y and lower values of W and ζ/E. Thus increasing η should tend to eliminate the up and down behaviour of ζ/E in the early part of Q to S in borderline cases.

On the other hand, if $r > x$, (ζ/E) will certainly decrease in the first part of Q to S but in the latter part there is danger of an increase and the only thing to prevent this is the "margin" (in the sense used above): but here we must note that in the case of formula (31) if $r > x$, Z_n for Q is *less* for x than for r and therefore ζ/E for Q is less for x than for r and so (as at S ζ/E is the same for x and for r) there is less drop in ζ/E from Q to S for x than for r so it will be easier for ζ/E to increase. The opposite is, of course, the case if $r < x$: there will be a bigger drop in ζ/E from Q to S for x than for r.

The conclusion is that x should be greater than r for formula (31). This is borne out by the examples given below—where it will be seen that $x < r$ gives quite unsatisfactory results, while $x > r$ gives usually quite good ones though not in all borderline cases. An example is given of how increasing the value of η improves a borderline case.

In the above discussion we have dealt with ζ/E. A similar analysis can be made of Z_n/E but it is fairly plain that if we get proper results for ζ/E we will also get them for Z_n/E.

Thus in the case of formula (39), to calculate the W values we should use a value of r at or nearly at the maximum of its range while for formula (31) we should use r near the minimum.

A word about the minimum value for S. In respect of formula (39) we must have

$$S > (1+\alpha)\{Q + K/(1-r)\} \tag{41a}$$

and we should see that S complies with this for the maximum value of r. (Some margin of compliance is desirable.)

In respect of formula (31) we must have

$$S > (1+\alpha)\{Q(1-r) + K\} \tag{41b}$$

and in this case we should see that S complies for the minimum value of r. (The values of K will, of course, probably be quite different for the two cases). We see that the necessity here of using, for r, the maximum value for formula (39) and the minimum for formula (31) agrees with the requirements for the W values.

In respect of formula (14) no question of r arises and we must simply have

$$S > (1+\alpha)(Q+K) \tag{41c}$$

22. *Other Formulas for the Modification.*

I have now given three different formulas, (14), (31) and (39), for the multi-split plan modification and it is clear that many more could be devised, but the three given are sufficient to illustrate the principles involved. It will be observed that the procedure consists of

(a) Choosing a formula for the modification for $E < Q$. This is the most important step since the greater number of risks fall in this range, and in addition the credibilities for risks where $E > Q$ are settled, to a large extent, by the "swing" below Q.

(b) Adjusting the modification formula for $E > Q$ by the addition of terms involving a parameter W so that the credibilities join smoothly at Q to those below Q and reach unity tangentially at S.

(c) Calculating the values of W so as to fulfill these conditions and the conditions set out in paragraph 3. The technique developed above consists in calculating ζ so that it and its first derivative ζ' take the required values at Q and at S and so that ζ increases and ζ/E decreases. Then it is necessary to check that these values when used in conjunction with the modification formula give values of Z_n, Z_e and W that increase and values of Z_n/E that decrease.

It is of interest to note that when the modification formula for $E < Q$ is settled, it is possible to choose more than one formula for $E > Q$ and that the calculation of the ζ values is independent of the choice of the modification formula for E greater than Q. For instance, instead of formula (14) for E greater than Q we could have

$$\frac{A_n + E_e + K\,(1 - W)}{E + K\,(1 - W)} + W\,\frac{A_e - E_e}{E} \tag{14A}$$

which gives $Z_n = \dfrac{E}{E + K\,(1 - W)},\ Z_e = W.$

The same ζ's as determined for (14) are applicable here and it will be found that the resulting values for W, Z_n, and Z_e are satisfactory. However, to calculate W from ζ requires the solution of a quadratic equation and all-in-all (14A) is not as simple to work with as is (14).

Another, and easily worked, variation of 14 is

$$\frac{A_n + E_e + K}{E + K}\,(1 - W) + \frac{A}{E}\,W \tag{14B}$$

which gives

$$Z_n = \frac{E + W\,K}{E + K},\ Z_e = W.$$

Here again the ζ's are the same as for (14) and it will be found that

$$Z_n = \frac{\alpha E + \zeta K}{\alpha E + (\alpha + 1)\,K} \qquad Z_e = \frac{(\zeta - 1)\,E + \zeta K}{\alpha E + (\alpha + 1)\,K}$$

These are the same as for (14) showing that (14B) gives the *same* values of Z_n and Z_e as does (14). (The W values are different, of course.) Thus (14B) could be used in place of (14) if it gives a better "working formula" and if it is felt that it is easier of explanation, to the layman, than is (14).

However, I will not pursue further this discussion of alternative formulas but will proceed to consider some practical aspects of the three original formulas.

PART IV

MULTI-SPLIT PLAN—PRACTICAL CONSIDERATIONS

23. *Comparison of the Three Formulas.*

We will now examine some of the characteristics of the three formulas (14), (31) and (39), we are discussing. We will pay particular attention to the credibilities given for low values of E, that is those below Q.

For $E \not> Q$ Z_e is zero and Z_n is equal to:—

$$\frac{E}{E+K} \text{ by formula (14)}$$

$$\frac{E}{E_n+K} \text{ by formula (31)}$$

$$\frac{E_n}{E_n+K} \text{ by formula (39)}$$

(The K's will not necessarily be the same).

Therefore (a) for a fixed value of E, i.e. for a fixed total premium the (normal) credibility for varying normal ratios E_n/E, i.e. for varying amounts of normal premiums contained in the fixed total premiums, will

(i) not vary, for formula (14)
(ii) increase as the amount of normal premiums *decreases*, and vice versa, for formula (31)
(iii) increase as the amount of normal premium *increases*, and vice versa, for formula (39)

and (b) for a fixed value of E_n, i.e. for a fixed normal premium, the (normal) credibility for varying normal ratios, i.e. for varying amounts of total premium, will

(i) increase as the amount of total premium *increases*, and vice versa, for formula (14)
(ii) increase as the amount of total premium *increases*, and vice versa, for formula (31)
(iii) not vary for formula (39)

For formula (39) this behavior is, of course, in accordance with our accepted notions (as the formula is, of course, the ordinary one) but for formula (31) the behavior in particular in respect of (a) (ii) is rather strange.

Formula (14) comes in between the other two and its characteristics are quite defensible. Nevertheless, as the excess ratios are low for the multi-split plan, the disadvantages of (31) are not as serious as they otherwise would be and the working scheme for this formula is very simple.

Now let us look at another aspect of the three credibilities. If as is customary we fix K by its effect for a low or minimum value of E (either by way of the charge for a maximum loss or the credit for clear experience) we find the formulas give different results for larger values of E say in the neighborhood of Q. Since in thus fixing K it is customary to use an average value of the excess ratio, formulas (14) and (39) will give the same credibilities (for the average value of r) at higher values of E if the K's are chosen so as to give the same effect at a low value of E. (The K's will differ—if r is the average excess ratio used, K by formula (39) will be $(1 - r)$ times the K by formula (14)). On the other hand the credibilities at higher values of E given by (31) will be considerably greater than those given by formula (14) or (39) with the same effect at a low value of E. This will be an advantage of formula (31) if we desire to give a wider swing to the plan for medium values of E without opening up the swing too much for small sizes of E, and it has been suggested that there would be considerable merit in doing this since no credibility is given to the excess experience as long as E is less than Q.

24. *Working formulas.*

We come now to the question of the form in which the "working formula" should be put.

First we call attention to the point that both for formulas (14) and (31) if in either the numerator or the denominator we take the sum of the coefficient of W and of the remaining terms we get A in the case of the numerator and E in the case of the denominator. For formula (39) we get $A\ (1 - r)$ and E_n respectively but if we put this formula in the alternative form (40) we again get A and E respectively. This, of course, is the same as saying that we get self-rating for $W = 1$.

Thus we can arrange our working formulas as follows:

I. Formula (14)

We give two alternatives

(i) We require a table of W for values of $E > Q$ and $< S$. We arrange our work sheet to give (a) ballasted actual discounted (normal) losses plus unrated expected excess losses, namely, $A_n + E_e + K$ where K is the "ballast" (b) ballasted expected losses, $E + K$. Then if $E \not> Q$ the modification is $\frac{(a)}{(b)} = \frac{A_n + E_e + K}{E + K}$

but if $E > Q$ we subtract from the top (c) the proportionate surplus of ballasted actual losses being W times the difference between (a) and the total actual losses, namely, $W\{(A_n + E_e + K) - A\}$, and we subtract from the bottom (d) the proportionate surplus of ballasted expected losses, being W times the difference between (b) and the actual expected losses or $W\{(E + K) - E\}$ and the modification is $\frac{(a) - (c)}{(b) - (d)}$

or (ii) We require a table of W as before and also a table of ballasts B equal to $K(1 - W)$. For $E < Q$, $B = K$. We arrange our work sheets to give (a) actual discounted (normal) losses plus unrated expected excess losses $A_n + E_e$ (b) the total expected losses. Then if $E < Q$ the modification is $\frac{(a) + \text{ballast}}{(b) + \text{ballast}} = \frac{A_n + E_e + K}{E + K}$

If $E > Q$ to the top we add (c) the proportionate remainder losses being W times the difference between the total actual losses and (a) or $W\{A - (A_n + E_e)\}$. Then the modification is

$$\frac{(a) + (c) + \text{ballast}}{(b) + \text{ballast}}$$

where the ballast is B from the table.

The second alternative seems to me to be the preferable.

II. Formula (31)

As before we give alternatives

(i) We require a table of W for $E > Q$ and of K_E the ballast ($= K$ for $E < Q$). Then we get (a) ballasted actual discounted losses, $A_n + K_E$ and (b) ballasted expected discounted (normal) losses. If $E < Q$ the modification is

$$\frac{(a)}{(b)} = \frac{A_n + K}{E_n + K}$$

but if $E > Q$ we subtract from the top (c) the proportionate surplus ballasted discounted losses being W times the difference between (a) and the total actual losses or $W\{(A_n + K_e) - A\}$, and from the bottom we subtract (d) the proportionate surplus expected discounted losses being W times the difference between (b) and the total expected losses; then the modification is

$$\frac{(a) - (c)}{(b) - (d)}$$

or (ii) We require a table of W as before and also a table of ballasts B equal to $K_E (1 - W)$. For $E < Q, B = K$. We get (a) actual discounted (normal) losses (b) expected discounted losses and if $E < Q$ the modification is

$$\frac{(a) + \text{ballast}}{(b) + \text{ballast}} = \frac{A_n + K}{E_n + K}$$

but if $E > Q$ we add to the top (c) the proportionate remainder actual losses being W times the difference between the total actual losses and (a), and to the bottom we add (d) the proportionate remainder expected losses being W times the difference between the total expected losses and (b). Then the modification is

$$\frac{(a) + (c) + \text{ballast}}{(b) + (d) + \text{ballast}}$$

where the ballast is B from the table.

Again the second alternative seems to be the preferable.

III. Formula (39)

In the form (39) this formula is not very suitable for easy working. It would be best to put it in the form (40) and then proceed as for formula (14) but in all cases dividing the ballast—whether K or B—by $(1 - r)$ before using so as to give ${}^rK$ or ${}^rB$ as the case may be. This makes the application of this formula a little more complicated than (14) which again, at any rate for $E < Q$, is neither quite as simple as (31) nor perhaps as attractive when explained to the layman. For (31) the layman is told, we get the modification by dividing the ballasted discounted actual losses by the ballasted (discounted) expected losses, while for (14) he is told we get the modification by dividing the ballasted discounted actual loss plus the (unrated) expected excess losses by the ballasted (undiscounted) expected losses.

25. *The Basic Constants.*

The fundamental quantities entering into all the calculation in connection with the multi-split plan credibilities as set out above are S, Q, and K and the auxiliary quantities are r (except in the case of (14)) and α. A few observations on these are offered.

Taking α first, we see that no particular harm is done by choosing it on the high side and therefore it seems possible and desirable to choose a value for it which can be the same for all states and need not be changed for every rate revision. This will simplify our calculations by eliminating one source of variation. As for the value to be assigned, if we use actual values in respect of death and more particularly permanent total cases, we shall obtain very high values but if as seems desirable we use, as at present, average values for these types of losses α will come out at a moderate value. In the examples given below I have used the value 4. This is possibly on the small side for universal use.

As for the excess ratio r, this does not enter into (14) at all (except incidentally into the determination of K). It enters into the calculations for (39) (apart from its use in fixing K) so that theoretically we should have different sets of W values for each r. If we use a fixed value of r, preferably near the maximum value we should get satisfactory results (see paragraph 21). There is not yet much information available as to the range of r except that it seems probable it will be fairly small (e.g. with a maximum of perhaps 40% and an average of 15% to 20%) for the values of a and ρ likely to be used in practice for discounting (see paragraph 7). In formula (31) the ratio r enters first into the determination of K_E and as shown in paragraph 20, a maximum value g should be used here. In the examples given below, I have used $g = .333$ which is possibly too low. As for the value of r to be used for formula (31) in determining the W values, the investigation in paragraph 21 shows that a low value should be used but it is not certain in respect of this formula (31) that a single value of r will work satisfactorily in all cases—particularly if the inequality (41b) is complied with by only a small margin. As in the case of α it would be a great simplification in practice if a universal value could be adopted for the fixed value of r to be used in determining the W's but until more is known about the actual values r can take, it cannot be decided if this is possible for formula (31).

Coming now to K, we have mentioned above the usual procedure for the fixing of this constant. As for Q and S these also must be settled on in some more or less arbitrary manner. Suggestions have been made to take S as a certain multiple (say twenty) of the average D. and P. T. value and Q as a fixed percentage of S. (Care must be taken, of course, that S and Q together with the K value chosen satisfy the condition (41) (a), (b) or (c) as the case may be). The taking of Q as a fixed proportion of S would greatly simplify the calculation of the W's.

If α (and the value of r if any to be used) are fixed then the determination of y depends solely on one parameter, namely, the value of w, which can vary, in accordance with the choice of K in relation to S and Q, from 0 to u. This assumes we take η equal to a fixed value say ½ in (27a). So it would be easy to compile a standard table of y. Now if in addition Q/S is a fixed ratio q then Y/S (which equals $y(1-q)+(q+K/S)$ for (14) for example) will also depend solely on a single parameter fixed by the relationship of K and S and therefore so will ζ/S and there also W expressed in terms of E/S. Thus if q is fixed W depends only on the relationship of K and S (and if this were fixed one table of W would do!)

The task of preparing a table of W for any state can thus be made much easier by deciding on fixed values for α, r, g and q, although as a matter of fact it is not burdensome to calculate W *ab initio*. We first calculate u and w: the expressions for these quantities are in Appendix I for formula (14) and in paragraph 17 for formula (31); for formula (39) use the same expressions as for formula (14) but with ${}^rK$ in place of K.

Then by equations (27b) to (27h) we get the expressions for Y (for formula (31) use equation (35a) instead of (27h)). From Y we get W by using equation (28a) for formula (14), (36a) for formula (31) and (28a) with ${}^rK$ for K for formula (39). For formula (31) we must in addition calculate K_E and M.

26. *Which Formula should be used?*

As to which of the three formulas should be used, the final determination of this question will rest on practical grounds, regard being had principally to the ease of explanation and facility of operation of the plan. This seems to rule out the rather more complicated (39) and give a slight preference to (31), or in other words the order of preference is likely to be (31), (14) and (39), the exact reverse of the order of theoretical desirability. However, if theoretical soundness is given enough weight then the "middle of the road" (14) might be chosen—and the mathematics of derivation and calculation will be considerably simplified. Of course (see paragraph 22) many other formulas are possible and it may well be that one far better may be devised.

My personal preference so far is with (14) but I have tried to present the alternatives impartially.

Part V

Illustrations of Multi-Split Plan Credibilities

27. At the end of the paper will be found some tables giving examples of W values and credibilities for the multi-split plan. These have been calculated in accordance with the foregoing and with basic values similar to those that might be expected to be used in practice.

The examples are chosen so as to be applicable to

I. New York State—with high benefits
II. Massachusetts—with medium benefits
III. Georgia with low benefits

In all cases the S values has been taken as approximately twenty times the average D. and P. T. value and the Q value is 10% of the S value (so that the q of paragraph 25 is 0.1). The actual S and Q values used were

New York	$S = 140000$	$Q = 14000$
Massachusetts	90000	9000
Georgia	42000	4200

(Note that as everywhere else in this paper these are in terms of *expected losses* so that the subject premiums would be about two-thirds greater).

In all cases the value of a used is 4, and the value of η is ½.

In all the tables the various values are given for specimen values of E/S so as to facilitate comparisons from one state and one table to another. The at first sight odd percentages between Q and S were chosen as to give round percentages of the interval between Q and S: thus $E/S = 55\%$ represents a point half way between Q and S. This scheme of specimen values is possible because Q/S is constant.

28. In table I are given values worked out on the assumption that formula (14) is used for the modification.

The value of r is accordingly immaterial, except in fixing K where an average value of one-sixth was used. The values used for K are New York 6900, Massachusetts 5520, Georgia 4140; these were chosen so as to give a charge of 20% for a maximum loss and a credit of 6⅔% for clear experience for expected losses of 600 for New York, 480 for Massachusetts, and 360 for Georgia, the maximum losses used being 1500 for New York, 1200 for Massachusetts, and 900 for Georgia. (These are discounted values of course).

In table II are given values on the assumption that formula (31) is to be used. The excess ratio used in calculating K_E (that is the g of the paragraph 20) is in all cases one-third. For each of the three states three sets of values are given—with $r = 333$, $r = .167$ and $r = 0$ respectively. (Of course the value $r = 0$ cannot arise in practice but the values are given for this to show how the formulas behave when r is very small). The values of K used are New York 7000, Massachusetts 5000, Georgia 4200, which as before, were chosen so as to give the same charge for a maximum loss and the same credit for clear experience for the same expected losses (with the same average value of one-sixth for r) as for Table I.

The values shown in Tables I and II, for each selected value of E/S are E, K_E (Table II only), W, B, Z_n, Z_e, $S\,Z_n/E$ and $S\,\zeta/E$. The last two functions are given to show the way in which they decrease with E, or in other words to illustrate the negativeness of $(Z_n/E)'$ and $(\zeta/E)'$.

The values of u and w involved in the example in Tables I and II are

	u	w		
	All States	New York	Massachusetts	Georgia
Formula (14) Table I	.2	.0563	.0430	.00159
Formula (31) Table II				
$r = .333$	.3	.1388	.1184	.0555
$r = .167$	.24	.0890	.0727	.0223
$r = 0$	.2	.0550	.0420	0

The fact that $w = 0$ for Table II, Georgia, $r = 0$, shows that for this example S is equal to instead of being greater than $(1 + \alpha) \{Q(1 - r) + K\}$. So in this case, $y = 0$ for all values of x and therefore V is also constant. Thus W is linear and equal to $(E - Q)/(S - Q)$ and there is no smooth junction for any of W, Z_n and Z_e at Q or at S. This is, of course, the limiting case and as observed above $r = 0$ does not arise in practice. If w were equal to (or less than) zero for a possible value of r, then S, or Q or K would have to be changed.

I have given no examples of the application of formula (39) for this is a simple modification of (14). In fact, Table I gives the values for formula (39) for K values equal to the K's of that table multiplied by $(1 - r)$ whatever r may be. There is little to comment on in these Tables I and II. The functions behave of course as they should in the light of the foregoing theory.

29. To illustrate the discussion in paragraph 21, in respect of formula (31), of the effect of using values of W, derived from a fixed value of the excess ratio, for the case of a different, varying, value of the ratio, I show in Table III values of Z_n, Z_e, $S\,Z_n/E$ and $S\,\zeta/E$ that occur with a variable excess ratio x if W values are used calculated for a fixed value r. These are shown for the same values of E/S as before, for each of the three States, for all combinations of r and x equal to .333, .167 and 0. The values for $r = x$ are not given as they are in Table II. (Here again I must mention that the results shown for r or $x = 0$ are merely illustrative of the limit of the effect of a low excess ratio.)

Chart I (shown at the end of the Tables) has been included to show graphically and a little more fully the behavior of ζ/E if r does not equal x. It shows for each of the nine combinations of the three States and the three x values how ζ/E behaves in going from Q to S when r equals each of the three values we have selected (including the case of $r = x$).

It will be seen that in accordance with the theory given in paragraph 21

(a) if $r = x$ the function ζ/E decreases satisfactorily (for Georgia, $r = x = 0$, ζ/E follows a horizontal straight line which at Q and S is not tangential to the curves for $E < Q$ and $> S$—but this is a limiting case);

(b) if $r > x$ ζ/E does not behave satisfactorily: it decreases, then rises and then falls again.

(c) if $r < x$ ζ/E decreases satisfactorily, except in the case of the Georgia values: there, for $r = .167$, $x = .333$, the behavior is bad for the early part of the interval Q to S (but not bad as, say, for $r = .333$ $x = .167$). In any case this is quite close to a borderline case. For $r = 0$ Georgia, the values of ζ/E are of course even worse.

In paragraph 21 it was suggested that in a borderline case such as Georgia $r = .167$, $x = .333$ where ζ/E, instead of continually decreasing, first decreases then increases and then decreases again, improvement would result if we increased the value of η used to calculate the W's. To show how this works out in this particular case I give on Chart II a graph of ζ/E for Georgia $r = .167$ $x = .333$ both for $\eta = 1/2$ (the value used in Chart I and Table III) and for $\eta = 1$, the highest possible value. It will be seen that the up and down behavior of ζ/E is eliminated when $\eta = 1$.

30. Finally, I give Table IV to illustrate the remarks in paragraph 23 regarding the different effects of the three formulas with respect to the credibilities given at higher value of E if the K values are chosen so as to give the same effect at a certain low value of E. In the table IV the K values used for formulas (14) and (31) are the same as in the previous tables and the K values used for formula (39) were chosen so as to give the same effects as the other formulas at minimum values of E. In Table IV are shown for selected E values the Z_n values and also the average credibilities (i.e. the credit for clear experience) taking into account the (zero) excess credibility.

. . . .

Appendix I

Construction of ζ for formula (14).

The construction referred to in paragraph 14 is as follows:

$$\text{Put } \zeta = \frac{E}{Y} \tag{20}$$

We will construct Y and derive ζ from it.

Y must be such that

(i) at $E = Q$, Y must equal $Q + K$ and be tangent to the line $Y = E + K$ i.e. Y' must equal 1;

(ii) at $E = S$, Y must equal $S/(1 + \alpha)$ and be tangent to the line $Y = E/(1 + \alpha)$ i.e. Y' must equal $1/(1 + \alpha)$;

(iii) $Y' = (E/\zeta)'$ must be always positive;

(iv) $(Y/E)' = (1/\zeta)'$ must be always negative.

Thus (see Fig. V) we must make Y go from q to s and be tangent at q to $L\,q$ and at s to O_s, so that Y continually rises and its tangent cuts $0\,Y$ above 0.

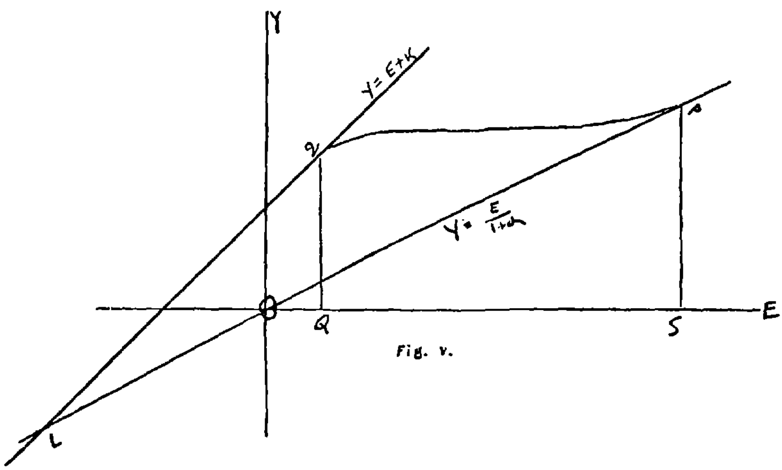

Fig. V.

q must be lower than s which is, of course, the same as the necessary condition (17).

We now put Y equal to the sum of the ordinates of two hyperbolas

$$Y = A_1 - \frac{B_1}{C_1 + E} \quad \text{and } Y = \frac{B_2}{C_2 - E} - A_2$$

where the A's, B's and C's are constants that will be determined so that the sum of these partial curves will meet the necessary conditions, namely that the combined curve touches $L\,q$ at q and O_s at s. B_1 and B_2 are to be positive and $C_1 > -Q, C_2 > S$: then the vertical asymptotes of the two hyperbolas are to the left of Q and the right of S respectively. In both hyperbolas Y' is positive (between Q and S) for in both Y increases from $E = Q$ to $E = S$, therefore, for the combined curve Y' is positive. Again the first partial curve is continually concave to the E axis from Q to S and so Y'' is always negative but it increases continually (that is, gets less negative) from Q to S; also the second curve is continuously convex to the E axis from Q to S, and so Y'' is always positive and it increases continually from E to S: so the sum of the two Y'''s which commences by being negative at q and ends by being positive at S can change sign only once between Q and S: in other words there is one and only one point of inflexion between Q and S and the tangent to the combined curve, starting from $L\,q$ at $E = Q$ and ending at $L\,s$ at $E = S$ can never cut $O\,Y$ below O as an examination of Fig. V will show. In other words, for the combined curve $(Y/E)'$ will always be negative, as required. (The tangent not only always cuts $O\,Y$ above O but also always cuts $L\,q$ above L: this fact will be needed in Appendix II).

To determine the constants we will simplify the calculations by transferring the origin to q and making $S - Q$ the unit i.e. we put

$$y = \frac{Y - Q - K}{S - Q} \qquad x = \frac{E - Q}{S - Q} \tag{21}$$

then the required curve will be

$$y = \frac{h}{k} - \frac{h}{x + k} + \frac{t}{j - x} - \frac{t}{j} \tag{22}$$

where h, k, t must be > 0 and $j > 1$.

In addition we must have

(i) for $x = 0$ $y = 0$ (this is taken care of the form of (22))

(ii) for $x = 0$ $y' = 1$

(iii) for $x = 1$ $y = \{S/(1+\alpha) - (Q+K)\}/(S-Q)$ or w (say)

(iv) for $x = 1$ $y' = 1/(1 + \alpha)$ or u (say) (ii), (iii) and (iv) give us

$$\left.\begin{aligned} \frac{h}{k^2} + \frac{t}{j^2} &= 1 \\ \frac{h}{k\,(k+1)} + \frac{t}{j\,(j-1)} &= w \\ \frac{h}{(k+1)^2} + \frac{t}{(j-1)^2} &= u \end{aligned}\right\} \qquad (23)$$

Note that $w = \dfrac{S - (1+\alpha)(Q+K)}{(S-Q)(1+\alpha)}$ which is positive and that $u - w = \dfrac{\alpha Q + (\alpha+1)K}{(S-Q)(1+\alpha)}$ which is also positive and so

$$0 < w < u < 1 \qquad (24)$$

To solve (23) I put

$$p = \frac{h}{k\,(k+1)} \qquad \lambda = \frac{h}{k^2} \qquad (25)$$

Then

$$w - p = \frac{t}{j\,(j-1)} \qquad 1 - \lambda = \frac{t}{j^2}$$

from which we get

$$\frac{p^2}{\lambda} = \frac{h}{(k+1)^2} \qquad \frac{(w-p)^2}{1-\lambda} = \frac{t}{(j-1)^2}$$

so we must have

$$\frac{p^2}{\lambda} + \frac{(w-p)^2}{1-\lambda} = u \qquad (26)$$

and if we can find values of p and λ that satisfy this and such that $w - p > 1 - \lambda > 0$ and $\lambda > p > 0$ then these values will give a solution of (23).

Now (26) can be written

$$(p - w\lambda)^2 = \lambda\,(1-\lambda)(u - w^2)$$

and as $w < u < 1$ therefore $u > w^2$ so put $u - w^2 = \sigma$ which is positive and we have

$$(p - w\lambda)^2 + (\lambda - \tfrac{1}{2})^2\,\sigma = \sigma/4$$

which is an ellipse in p and λ (see Fig. VI) with center $\lambda = \tfrac{1}{2}$, $p = w/2$, passing through the origin $(0, 0)$ and touching the p axis there, also passing through $(w, 1)$ and touching $\lambda = 1$ there. It cuts the λ axis at $\lambda = 0$ and $\lambda = \sigma/u$ and also cuts $p = w$ at $\lambda = 1$ and $\lambda = w^2/u$.

Also the line $w - p = 1 - \lambda$ passes through $(w, 1)$ and cuts the λ axis at $\lambda = 1 - w$. Also since w is less than unity the line $p = \lambda$ which is parallel to $w - p = 1 - \lambda$ passes through the origin and lies to the left of $w - p = 1 - \lambda$.

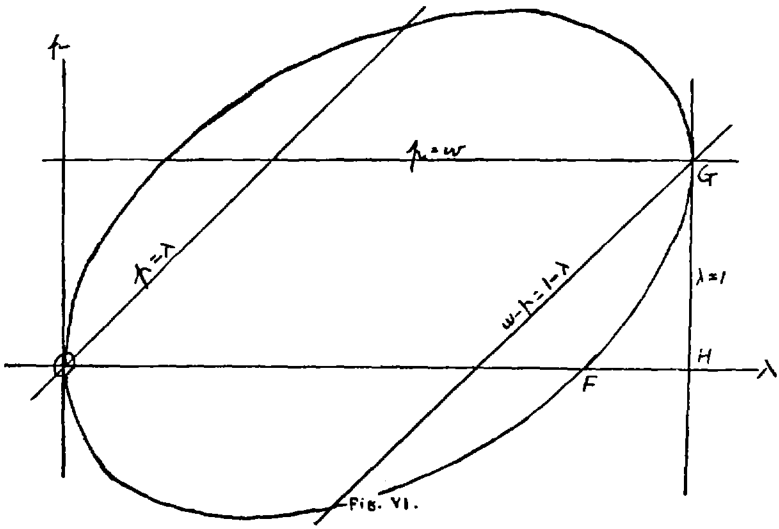

FIG. VI.

Thus all the solutions are given by the arc of the ellipse from $p = 0$, $\lambda = \sigma/u$ to $p = w$, $\lambda = 1$ or (in Fig. VI) from F to G.

There is one "degree of freedom" in this solution as there is one more constant in (22) than there are conditions to be fulfilled.

This is expressed by the possibility of choosing any point on the arc $F\,G$ to give values of p and λ. As $F\,H = w^2/u$ and is usually small compared with $O\,H$ which equals one, a good set of values for p and λ is usually obtained by putting $\eta = \frac{1}{2}$ in $p/\lambda = (1 - \eta)\,w$, the equation which gives all the solutions by varying η from 0 to 1.

The solution is thus:

$$\text{Put } \frac{p}{\lambda} = (1 - \eta)\,w \qquad 0 < \eta < 1 \qquad (27a)$$

Then solving (26) for λ

$$\lambda = \frac{u - w^2}{u - w^2\,(1 - \eta^2)} \qquad (27b)$$

$$p = (1 - \eta)\,w\,\lambda. \qquad (27c)$$

Then from (25)

$$k = \frac{p}{\lambda - p} \tag{27d}$$

$$h = \lambda k^2 \tag{27e}$$

$$j = \frac{w - p}{(w - p) - (1 - \lambda)} \tag{27f}$$

$$t = (1 - \lambda) j^2 \tag{27g}$$

Then from $y = \frac{h}{k} - \frac{t}{j} - \frac{h}{k + x} + \frac{t}{j - x}$

$$Y = \left(\frac{h}{k} - \frac{t}{j}\right)(S-Q)+(Q+K) - \frac{h\,(S-Q)^2}{E-Q+(S-Q)\,k} + \frac{t\,(S-Q)^2}{j\,(S-Q)+Q-E} \tag{27h}$$

$$\zeta = \frac{E}{Y} \tag{27i}$$

If η is taken as 0, $p = 0$ and $\lambda = \sigma/u$, the partial curve $y = \frac{h}{k} - \frac{h}{k + x}$ degenerates to $y = 0$ and the curve for Y is not a proper tangent at q: similarly if $\eta = 1$, $p = w$ and $\lambda = 1$, the partial curve $y = \frac{t}{j - x} - \frac{t}{j}$ degenerates to $y = 0$ and the curve for Y is not a proper tangent at S. η should therefore be taken between 0 and 1 say at ½ as suggested above.

The equation for Y is of the form

$$Y = A_1 - \frac{B_1}{C_1 + E} + \frac{B_2}{C_2 - E} = \frac{C_3 + B_3 E - A_1 E^2}{(C_1 + E)(C_2 - E)}$$

and so the equation for ζ is of the form

$$\zeta = \frac{E\,(C_1 + E)(C_2 - E)}{C_3 + B_3 E - A_1 E^2}$$

a cubic equation. (All the A's, B's and C's are constants).

Appendix II

Proof that W (and therefore Z_e) increases with E.

We wish to show that for our construction of W for formula (14)—and for formula (31)—W increases with E. An algebraical proof is given below but first it is constructive to examine the question geometrically and in terms of Y and E as shown in Fig. V.

Taking equation (28a)

$$(a\, Y + K)\, W = E - Y + K$$

we can regard this as the equation of a family of curves in Y and E with W as the parameter. The equation can be written as

$$(a\, W + 1)\; Y = E + K\,(1 - W)$$

showing this represents a family of straight lines. Each one passes through the point L of Figure V, the intersection of $Y = E + K$ and $Y = E/(1 + a)$, the coordinates of which are

$$E_L = -\frac{1 + a}{a}\,K,\quad Y_L = -\,\frac{K}{a}$$

For $W = 0$ the line is $Y = E + K$ or the line $L\,q$, and for $W = 1$ the line is $(a + 1)\;Y = E$ or the line $L\,O\,s$: and as W goes from 0 to 1 the line rotates round L from $L\,q$ to $L\,s$. Now drawing

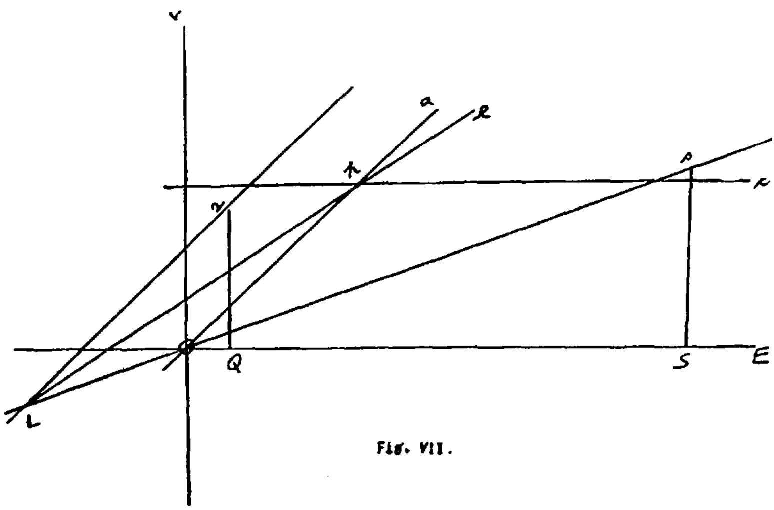

Fig. VII.

Fig. VII we see that if, at any point p of the curve $q\,s$ we are constructing for Y, W is to decrease, the tangent to the curve at p

must fall in the angle $a\,p\,l$ where a is on $O\,p$ extended and l is on $L\,p$ extended.

Now the conditions to which the curve $q\,p\,s$ is subject are that the tangent is in the angle $a\,p\,c$ where $p\,c$ is parallel to the E axis $O\,Q\,S$ and since $a\,p\,l$ falls inside $a\,p\,c$ it is possible to construct the curve so that W decreases: but we observe that our construction does not permit of this: for as noted in Appendix I the tangent always cuts $L\,q$ above L and thus the tangent always falls in the angle $l\,p\,c$. Thus W cannot decrease for our construction.

We can now give an algebraic proof of the increasing of W with E. To do this we obtain the inequality expressing the fact noted above that the tangent to $q\,s$ cuts $L\,q$ above L. If the coordinates of the intersection of $L\,q$ and the tangent are E_T and L_T we have

$$Y_T = Y'\,(E_T - E) + Y = E_T + K$$

whence $E_T = \dfrac{Y - K - E\,Y'}{1 - Y'}$

therefore $\dfrac{Y - K - E\,Y'}{1 - Y'} > E_L > -\dfrac{\alpha + 1}{\alpha}\,K$

$$\text{or } \alpha\,Y + K > Y'\,\{\alpha\,E + (\alpha + 1)\,K\}.$$

Translating this back into terms of ζ we put $Y = E/\zeta$ and $Y' = (\zeta - E\,\zeta')/\zeta^2$ and get

$$E\,\zeta'\,\{\alpha\,E + (\alpha + 1)\,K\} > \zeta\,K\,(\alpha + 1 - \zeta).$$

Now differentiating (28)

$$(\alpha\,E + \zeta\,K)^2\,W' = E\,\zeta'\,\{\alpha\,E + (\alpha + 1)\,K\} - \zeta\,K\,(\alpha + 1 - \zeta)$$

which is positive by the inequality just proved.

Thus W' is positive.

We will now give a proof in the case of the construction given for formula (31): the geometrical proof is considerably complicated by the variability of M and we will not give it. We can, however, readily extend the algebraic proof as follows:

Proceeding as in the proof for formula (14) we have E_L given by

$$\frac{E_L}{(1 + \alpha)(1 - r)} = E_L + \frac{K}{1 - r} \quad \text{or} \quad E_L = -K\,\frac{\alpha + 1}{\alpha\,(1 - r) - r}$$

and E_T is given by $Y'\,(E_T - E) + Y = E_T + \dfrac{K}{1 - r}$

so $E_T = \dfrac{Y - E\,Y' - K/(1 - r)}{1 - Y'}$ which is greater than E_L

Thus

$$\{\alpha\,(1-r)-r\}\;Y+K/(1-r)>Y'\;[\{\alpha\,(1-r)-r\}\;E+(\alpha+1)\;K].$$

Now putting $Y=\dfrac{E}{\zeta\,(1-r)}$ $\qquad Y'=\dfrac{\zeta-E\,\zeta'}{\zeta^2\,(1-r)}$

we get $E\,\zeta'\;[\{\alpha\,(1-r)-r\}\;E+(\alpha+1)\;K]>\zeta\,K\;(\alpha+1-\zeta)$.

Now $K_E \not< K$ so we can put K_E for K in the left hand side of this inequality which then becomes $E\,\zeta'\,\{\alpha\,E+(\alpha+1)\,M\}$.

Also the maximum value of $M-E\,M'$ is, as we have seen in paragraph 19, equal to K: so we can put $M-E\,M'$ for K in the right hand side. So we have

$$\zeta'\,E\,\{\alpha\,E+(\alpha+1)\,M\}>\zeta\,(M-E\,M')(\alpha+1-\zeta).$$

Differentiating (36) we get

$$(\alpha\,E+\zeta\,M)^2\,w'=\zeta\,E\,\{\alpha\,E+(\alpha+1)\;M\}-\zeta\,(M-E\,M')\,(\alpha+1-\zeta)$$

and by the inequality just proved the right hand side is positive and so W (and therefore Z_e) increases with E.

Appendix III

Direct Construction of W for Formula (14).

At the end of paragraph (12) I had to choose between

(a) constructing ζ so that ζ' is positive and $(\zeta/E)'$ negative and then seeing if W' is positive; or

(b) constructing W so that W' is positive and $(\zeta/E)'$ negative.

I chose (a) but stated that (b) would lead to identical values of W.

In this Appendix we will work out (b).

We must first express in terms of W the condition that $(\zeta/E)'$ must be negative. Dividing (18) through by E and differentiating we get

$$\{E+K\,(1-W)\}^2\;(\zeta/E)'=W'\;\{\alpha\,E+(\alpha+1)\;K\}-(1+\alpha\,W)$$

and the right hand side multiplied by α is equal to

$$\{\alpha\,E+(\alpha+1)\;K\}^2$$

times the derivative with respect to E of

$$\frac{1+\alpha\,W}{\alpha\,E+(\alpha+1)\,K}$$

So the condition that ζ/E must decrease is equivalent to the condition that $(1+\alpha\,W)/\{\alpha\,E+(\alpha+1)\;K\}$ must decrease.

Now if we put

$$a\omega = 1 + aW \qquad a\epsilon = aE + (a+1)K$$

or in other words change the origin from $E = 0$, $W = 0$ to

$$E = -\frac{a+1}{a}K \qquad W = -\frac{1}{a}$$

the conditions W' is to be positive and $(\zeta/E)'$ is to be negative become ω' is to be positive and $(\omega/\epsilon)'$ is to be negative (where the differentiations are here with respect to ϵ). These are very similar to the conditions under which we constructed ζ. We have the terminal conditions that

(i) when $\epsilon = Q + \frac{a+1}{a}K \qquad \omega = \frac{1}{a},\ \omega' = 0$

(ii) when $\epsilon = S + \frac{a+1}{a}K \qquad \omega = \frac{1+a}{a},\ \omega' = 0.$

Now if we put $\omega = \frac{\epsilon}{aV}$ we have to go

from $\epsilon = Q + \frac{a+1}{a}K \qquad V = Q + \frac{a+1}{a}K$ with $V' = 1$

to $\epsilon = S + \frac{a+1}{a}K \qquad V = \frac{S}{1+a} + \frac{K}{a}$ with $V' = \frac{1}{1+a}$

so that V' is positive and $(V/\epsilon)'$ is negative.

These conditions are very similar to those for Y in Appendix I. In fact if we refer to Fig. V in Appendix I we see that if we change the origin from O (or $E = 0$, $Y = 0$) to L

$$(E = -K\,(a+1)/a,\ Y = -K/a)$$

by putting

$$a\epsilon_1 = aE + (a+1)K \qquad aV_1 = aY + K$$

the conditions to which V_1 is subject become exactly those to which V is subject—except that the condition Y/E must decrease does not become the condition V_1/ϵ_1 must decrease. In other words the ϵ_1 and V_1 which we get this way, by transfering E and Y are exactly the ϵ and the V we have just derived from E and W: for it is easily seen that the two ϵ's are the same and as for the two V's the V_1 derived from Y equals

$$\frac{aY+K}{a}$$

which is the same as $\dfrac{\alpha E/\zeta + K}{\alpha}$

or $\dfrac{\alpha E\{E + K(1 - W)\} + KE(\alpha W + 1)}{\alpha E(\alpha W + 1)}$ by (18)

or $\dfrac{\alpha E + (\alpha + 1)K}{\alpha(\alpha W + 1)}$

which equals $\epsilon/\alpha\,\omega$ or V derived from E and W.

Thus the only difference between the conditions for V and for Y are that for the former V/ϵ must decrease and for the latter Y/E. These represent the difference between the conditions with which we started. In constructing Y in Appendix I we required that this should make ζ' positive and in setting up V we required that this should make W' positive.

Now if Y/E is to decrease the tangent to the curve $q\,s$ must cut $O\,Y$ above O: and if V/ϵ is to decrease the tangent must pass above L or, as it can be put must cut $L\,q$ above L. It will be recalled that our construction actually fulfills *both* these conditions (or rather as it fulfills the harder condition that the tangent should pass above L it also fulfills the easier condition that it should pass above O) and it was because of this that W' proved to be positive as well as ζ'.

So if we finish the construction of V by

(i) transferring the origin $\epsilon = 0$ $V = 0$ from L to q at the same time making the unit $S - Q$ (just as we did in Appendix I for Y) and denoting the transformed ϵ by x and the transformed V by y; and

(ii) constructing y in terms of x just as in Appendix I

then we get the same values of y as in Appendix I and these give values of V in terms of ϵ that give the same values of W in terms of E as we get from the values of Y as obtained in Appendix I. Thus we see that if we set out to construct W direct so as to make W' positive and $(\zeta/E)'$ negative we arrive at exactly the same W values as we do by constructing ζ first as in Appendix I.

TABLE I

Examples of Results Produced by Formula (14)

$a = 4 \qquad \eta = \frac{1}{2}$

		E/S: .01	.05	.10	.145	.19	.28	.37	.55	.73	.91	1.00
				Q								S
New York	E	1400	7000	14000	20300	26600	39200	51800	77000	102200	127400	140000
	W	..	..	.000	.040	.095	.211	.328	.558	.776	.958	1.000
$S = 140000$	B	6900	6900	6900	6624	6245	5444	4637	3050	1546	290	0
$Q = 14000$	Z_n	.169	.504	.670	.754	.810	.878	.918	.962	.985	.998	1.000
$K = 6900$	Z_e	.000	.000	.000	.030	.077	.185	.301	.536	.763	.956	1.000
	$S\,Z_n/E$	16.90	10.08	6.70	5.20	4.26	3.14	2.48	1.75	1.35	1.10	1.00
	$S\;\zeta/E$	16.90	10.08	6.70	6.03	5.88	5.78	5.74	5.65	5.53	5.30	5.00
Massachusetts	E	900	4500	9000	13050	17100	25200	33300	49500	65700	81900	90000
	W	..	..	.000	.041	.095	.208	.320	.543	.760	.950	1.000
$S = 90000$	B	5520	5520	5520	5294	4996	4372	3754	2523	1325	276	0
$Q = 9000$	Z_n	.140	.449	.620	.711	.774	.852	.899	.952	.980	.997	1.000
$K = 5520$	Z_e	.000	.000	.000	.029	.074	.177	.288	.517	.745	.947	1.000
	$S\,Z_n/E$	14.00	8.98	6.20	4.90	3.07	3.02	2.43	1.73	1.34	1.10	1.00
	$S\;\zeta/E$	14.00	8.98	6.20	5.71	5.62	5.57	5.54	5.49	5.42	5.26	5.00
Georgia	E	420	2100	4200	6090	7980	11760	15540	23100	30660	38220	42000
	W	..	..	.000	.050	.100	.200	.301	.502	.703	.904	1.000
$S = 42000$	B	4140	4140	4140	3933	3726	3312	2894	2062	1230	397	0
$Q = 4200$	Z_n	.092	.337	.504	.607	.682	.780	.843	.918	.961	.990	1.000
$K = 4140$	Z_e	.000	.000	.000	.030	.068	.158	.253	.460	.675	.894	1.000
	$S\,Z_n/E$	9.20	6.74	5.04	4.19	3.59	2.79	2.28	1.67	1.32	1.09	1.00
	$S\;\zeta/E$	9.20	6.74	5.04	5.02	5.02	5.02	5.02	5.02	5.02	5.01	5.00

TABLE II

Examples of Results Produced by Formula (31)—when excess ratio of risk is the same as that for which the W's are calculated

$a = 4$ $\eta = \frac{1}{2}$ $g = .333$

			E/S: .01	.05	.10	.145	.19	.28	.37	.55	.73	.91	1.00
					Q								S
New York		E	1400	7000	14000	20300	26600	39200	51800	77000	102200	127400	140000
		K_B	7000	7000	7000	8000	9700	13580	17630	25900	34240	42610	46790
$S = 140000$	$r = .333$	W	..	..	.000	.034	.092	.218	.346	.594	.813	.971	1.000
$Q = 14000$		B	7000	7000	7000	7728	8808	10620	11530	10515	6403	1236	0
$K = 7000$		Z_n	.176	.600	.857	.946	.972	.990	.995	.999	1.000	1.000	1.000
		Z_e	.000	.000	.000	.032	.089	.216	.344	.593	.813	.971	1.000
		$S\,Z_n/E$	17.60	12.00	8.57	6.52	5.11	3.54	2.69	1.82	1.37	1.09	1.00
		$S\ \zeta/E$	17.60	12.00	8.57	7.41	6.85	6.62	.641	6.13	5.82	5.36	5.00
	$r = .167$	W	..	..	.000	.041	.105	.238	.369	.611	.819	.970	1.000
		B	7000	7000	7000	7672	8682	10348	11125	10075	6197	1278	0
		Z_n	.171	.546	.750	.821	.850	.880	.901	.938	.970	.995	1.000
		Z_e	.000	.000	.000	.034	.089	.210	.333	.573	.794	.965	1.000
		$S\,Z_n/E$	17.10	10.92	7.50	5.66	4.47	3.14	2.44	1.71	1.33	1.09	1.00
		$S\ \zeta/E$	17.10	10.92	7.50	6.60	6.35	6.14	6.04	5.87	5.68	5.34	5.00
	$r = 0$	W	..	..	.000	.049	.118	.258	.389	.623	.820	.968	1.000
		B	7000	7000	7000	7608	8555	10076	10772	9764	6163	1364	0
		Z_n	.167	.500	.667	.727	.757	.795	.828	.888	.943	.989	1.000
		Z_e	.000	.000	.000	.036	.090	.205	.322	.553	.773	.957	1.000
		$S\,Z_n/E$	16.70	10.00	6.66	5.01	3.99	2.84	2.24	1.61	1.29	1.09	1.00
		$S\ \zeta/E$	16.70	10.00	6.67	6.01	5.87	5.77	5.72	5.64	5.53	5.30	5.00

TABLE II — *Continued*

Examples of Results Produced by Formula (31)—when excess ratio of risk is the same as that for which the W's are calculated

$a = 4$ $\eta = \frac{1}{2}$ $g = .333$

			E/S: .01	.05	.10 Q	.145	.19	.28	.37	.55	.73	.91	1.00 S
Massachusetts		E	900	4500	9000	13050	17100	25200	33300	49500	65700	81900	90000
		K_E	5600	5600	5600	6060	6980	9250	11730	16920	22210	27550	30230
$S = 90000$	$r = .333$	W	..	..	.000	.030	.083	.202	.326	.572	.796	.967	1.000
$Q = 9000$		B	5600	5600	5600	5878	6401	7382	7906	7242	4531	909	0
$K = 5600$		Z_n	.145	5.23	.776	.887	.935	.973	.987	.996	.999	1.000	1.000
		Z_e	.000	.000	.000	.027	.077	.197	.322	.569	.795	.967	1.000
		$S\,Z_n/E$	14.50	10.46	7.76	6.12	4.92	3.48	2.67	1.81	1.37	1.10	1.00
		$S\ \zeta/E$	14.50	10.46	7.76	6.86	6.55	6.29	6.15	5.95	5.72	5.35	5.00
	$r = .167$	W	..	..	.000	.038	.098	.225	.352	.591	.802	.965	1.000
		B	5600	5600	5600	5830	6296	7169	7601	6920	4398	964	0
		Z_n	.142	.481	.687	.778	.821	.866	.890	.933	.967	.994	1.000
		Z_e	.000	.000	.000	.030	.080	.195	.314	.551	.775	.959	1.000
		$S\,Z_n/E$	14.20	9.62	6.87	5.37	4.32	3.09	2.41	1.70	1.32	1.09	1.00
		$S\ \zeta/E$	14.20	9.62	6.87	6.18	6.01	5.88	5.81	5.71	5.57	5.31	5.00
	$r = 0$	W	..	..	.000	.048	.113	.247	.374	.605	.804	.961	1.000
		B	5600	5600	5600	5769	6191	6965	7343	6683	4353	1074	0
		Z_n	.138	.446	.616	.693	.734	.783	.819	.881	.938	.987	1.000
		Z_e	.000	.000	.000	.033	.083	.193	.306	.553	.754	.949	1.000
		$S\,Z_n/E$	13.80	8.92	6.16	4.78	3.86	2.80	2.21	1.60	1.28	1.08	1.00
		$S\ \zeta/E$	13.80	8.92	6.16	5.69	5.61	5.56	5.53	5.48	5.42	4.69	5.00

TABLE II — *Continued*

Examples of Results Produced by Formula (31)—when excess ratio of risk is the same as that for which the W's are calculated

$a = 4$ $\eta = 1/2$ $g = .333$

			E/S: .01	.05	.10 (Q)	.145	.19	.28	.37	.55	.73	.91	1.00 (S)
Georgia		E	420	2100	4200	6090	7980	11760	15540	23100	30660	38220	42000
		K_E	4200	4200	4200	4320	4590	5390	6370	8560	10890	13290	14510
$S = 42000$	$r = .333$	W	..	..	.000	.028	.072	.174	.283	.510	.737	.942	1.000
$Q = 4200$		B	4200	4200	4200	4199	4260	4452	4567	4194	2973	771	0
$K = 4200$		Z_n	.094	.375	.600	.732	.817	.906	.948	.982	.994	.999	1.000
		Z_e	.000	.000	.000	.021	.059	.158	.168	.500	.733	.941	1.000
		$S\,Z_n/E$	9.40	7.50	6.00	5.05	4.30	3.23	2.56	1.79	1.36	1.10	1.00
		$S\;\zeta/E$	9.40	7.50	6.00	5.63	5.54	5.49	5.46	5.42	5.38	5.23	5.00
	$r = .167$	W	..	..	.000	.040	.091	.201	.314	.535	.745	.936	1.000
		B	4200	4200	4200	4147	4172	4307	4370	3980	2777	850	0
		Z_n	.092	.353	.545	.658	.729	.811	.857	.914	.954	.989	1.000
		Z_e	.000	.000	.000	.026	.067	.163	.269	.489	.711	.925	1.000
		$S\,Z_n/E$	9.20	7.06	5.45	4.54	3.84	2.90	2.31	1.66	1.46	1.13	1.00
		$S\;\zeta/E$	9.20	7.06	5.45	5.25	5.24	5.23	5.22	5.22	5.20	5.15	5.00
	$r = 0$	W	..	..	.000	.053	.109	.224	.338	.552	.745	.919	1.000
		B	4200	4200	4200	4091	4090	4183	4217	3835	2777	1076	0
		Z_n	.091	.333	.500	.598	.661	.738	.787	.858	.917	.973	1.000
		Z_e	.000	.000	.000	.032	.072	.166	.266	.473	.683	.894	1.000
		$S\,Z_n/E$	9.10	6.66	5.00	4.12	3.48	2.64	2.13	1.56	1.26	1.07	1.00
		$S\;\zeta/E$	9.10	6.66	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00

TABLE III

Examples of Results Produced by Formula (31)—when excess ratio of risk (x) is different from that (r) for which the W's are calculated

$a = 4$ $\eta = \frac{1}{2}$ $g = .333$

r	x		E/S: .01	.05	.10	.145	.19	.28	.37	.55	.73	.91	1.00
					Q								S
New York		E	1400	7000	14000	20300	26600	39200	51800	77000	102200	127400	140000
.333	.167	Z_n	.171	.546	.750	.820	.848	.877	.898	.936	.970	.995	1.000
		Z_e	.000	.000	.000	.028	.078	.191	.311	.556	.789	.966	1.000
		$S\,Z_n/E$	17.10	10.92	7.50	5.66	4.47	3.14	2.42	1.71	1.33	1.09	1.00
		$S\ \zeta/E$	17.10	10.92	7.50	6.43	6.10	5.87	5.80	5.74	5.66	5.33	5.00
	0	Z_n	.167	.500	.667	.725	.751	.787	.818	.880	.941	.990	1.000
		Z_e	.000	.000	.000	.025	.069	.172	.283	.523	.765	.961	1.000
		$S\,Z_n/E$	16.70	10.00	6.67	5.00	3.95	2.81	2.21	1.60	1.43	1.09	1.00
		$S\ \zeta/E$	16.70	10.00	6.67	5.68	5.40	5.26	5.26	5.40	5.47	5.31	5.00
.167	.333	Z_n	.177	.600	.857	.945	.973	.990	.996	.999	1.000	1.000	1.000
		Z_e	.000	.000	.000	.039	.102	.236	.368	.610	.819	.970	1.000
		$S\,Z_n/E$	17.70	12.00	8.57	6.52	5.12	3.54	2.70	1.82	1.37	1.09	1.00
		$S\ \zeta/E$	17.70	12.00	8.57	7.59	7.27	6.90	6.66	6.26	5.85	5.36	5.00
	0	Z_n	.167	.500	.667	.726	.754	.792	.823	.884	.943	.990	1.000
		Z_e	.000	.000	.000	.030	.079	.188	.304	.540	.772	.960	1.000
		$S\,Z_n/E$	16.70	10.00	6.67	5.01	3.97	2.83	2.22	1.67	1.29	1.09	1.00
		$S\ \zeta/E$	16.70	10.00	6.67	5.84	5.63	5.52	5.52	5.53	5.52	5.31	5.00

TABLE III — *Continued*

Examples of Results Produced by Formula (31)—when excess ratio of risk (x) is different from that (r) for which the W's are calculated

$a = 4$ $\eta = ½$ $g = .333$

r	x		E/S .01	.05	.10 Q	.145	.19	.28	.37	.55	.73	.91	1.00 S
New York (Cont'd)													
0	.333	Z_n	.176	.600	.857	.947	.973	.990	.996	.999	1.000	1.000	1.000
		Z_e	.000	.000	.000	.046	.115	.255	.387	.622	.820	.961	1.000
		$S\ Z_n/E$	17.60	12.00	8.57	6.54	5.12	3.54	2.69	1.82	1.37	1.09	1.00
		$S\ \zeta/E$	17.60	12.00	8.57	7.80	7.54	7.18	6.87	6.34	5.87	5.35	5.00
	.167	Z_n	.171	.546	.750	.823	.851	.882	.904	.940	.971	.995	1.000
		Z_e	.000	.000	.000	.040	.100	.228	.352	.586	.796	.963	1.000
		$S\ Z_n/E$	17.09	10.92	7.50	5.08	4.48	3.15	2.45	1.71	1.33	1.09	1.000
		$S\ \zeta/E$	17.09	10.92	7.50	6.78	6.58	6.41	6.24	5.96	5.70	5.32	5.00
Massachusetts		E	900	4500	9000	13050	17100	25200	33300	49500	65700	81900	90000
.333	.167	Z_n	.142	.481	.687	.776	.819	.862	.889	.930	.966	.994	1.000
		Z_e	.000	.000	.000	.023	.068	.174	.290	.532	.769	.961	1.000
		$S\ Z_n/E$	14.20	9.62	6.87	5.36	4.31	3.08	2.40	1.69	1.32	1.09	1.00
		$S\ \zeta/E$	14.20	9.62	6.87	5.99	5.74	5.56	5.54	5.56	5.54	5.32	5.00
	0	Z_n	.138	.446	.616	.689	.728	.773	.808	.872	.935	.989	1.000
		Z_e	.000	.000	.000	.021	.060	.156	.263	.499	.744	.956	1.000
		$S\ Z_n/E$	13.80	8.92	6.16	4.75	3.84	2.76	2.19	1.58	1.28	1.09	1.00
		$S\ \zeta/E$	13.80	8.92	6.16	5.33	5.09	4.99	5.03	5.21	5.36	5.29	5.00

TABLE III — *Continued*

Examples of Results Produced by Formula (31)—when excess ratio of risk (x) is different from that (r) for which the W's are calculated

$a = 4$ $\eta = \frac{1}{2}$ $g = .333$

Massachusetts (Cont'd)

r	x		E/S: .01	.05	.10 Q	.145	.19	.28	.37	.55	.73	.91	1.00 S
.167	.333	Z_n	.145	.523	.776	.888	.936	.974	.988	.996	.999	1.000	1.000
		Z_e	.000	.000	.000	.034	.092	.219	.348	.589	.801	.965	1.000
		$S\,Z_n/E$	14.50	10.46	7.76	6.12	4.92	3.48	2.67	1.81	1.38	1.10	1.00
		$S\;\zeta/E$	14.50	10.46	7.76	7.07	6.87	6.61	6.44	6.09	5.76	5.34	5.00
	0	Z_n	.138	.446	.616	.691	.731	.778	.814	.877	.937	.988	1.000
		Z_e	.000	.000	.000	.026	.072	.175	.287	.518	.751	.953	1.000
		$S\,Z_n/E$	13.80	8.92	6.16	4.77	3.84	2.78	2.20	1.59	1.29	1.08	1.00
		$S\;\zeta/E$	13.80	8.92	6.16	5.48	5.36	5.28	5.30	5.36	5.40	5.27	5.00
0	.333	Z_n	.145	.523	.776	.881	.938	.975	.988	.997	.999	1.000	1.000
		Z_e	.000	.000	.000	.042	.106	.241	.370	.603	.803	.961	1.000
		$S\,Z_n/E$	14.50	10.46	7.76	6.07	4.93	3.48	2.67	1.81	1.37	1.10	1.00
		$S\;\zeta/E$	14.50	10.46	7.76	7.24	7.16	6.92	6.67	6.20	5.77	5.32	5.00
	.167	Z_n	.142	.481	.687	.780	.824	.869	.896	.935	.968	.993	1.000
		Z_e	.000	.000	.000	.037	.093	.215	.335	.566	.778	.954	1.000
		$S\,Z_n/E$	14.20	9.62	6.87	5.38	4.34	3.11	2.42	1.70	1.32	1.09	1.00
		$S\;\zeta/E$	14.20	9.62	6.87	6.40	6.29	6.17	6.04	5.81	5.59	5.28	5.00

TABLE III — *Continued*

Examples of Results Produced by Formula (31)—when excess ratio of risk (x) is different from that (r) for which the W's are calculated

$a = 4$ $\eta = 1/2$ $g = .333$

	r	x	E/S	.01	.05	.10	.145	.19	.28	.37	.55	.73	.91	1.00
						Q								S
Georgia			E	420	2100	4200	6090	7980	11760	15540	23100	30660	38220	42000
	.333	.167	Z_n	.092	.353	.545	.655	.725	.806	.852	.909	.953	.990	1.000
			Z_e	.000	.000	.000	.018	.052	.140	.241	.464	.702	.933	1.000
			$S\,Z_n/E$	9.20	7.06	5.45	4.52	3.82	2.88	2.30	1.65	1.31	1.09	1.00
			$S\ \zeta/E$	9.20	7.06	5.45	5.01	4.91	4.88	4.91	5.03	5.15	5.19	5.00
		0	Z_n	.091	.333	.500	.592	.652	.725	.773	.846	.915	.980	1.000
			Z_e	.000	.000	.000	.017	.047	.126	.219	.431	.674	.923	1.000
			$S\,Z_n/E$	9.10	6.66	5.00	4.08	3.43	2.59	2.09	1.54	1.25	1.08	1.00
			$S\ \zeta/E$	9.10	6.66	5.00	4.55	4.42	4.39	4.46	4.67	4.95	5.13	5.00
	.167	.333	Z_n	.094	.375	.600	.735	.820	.910	.950	.983	.994	.999	1.000
			Z_e	.000	.000	.000	.029	.075	.183	.298	.526	.741	.935	1.000
			$S\,Z_n/E$	9.40	7.50	6.00	5.07	4.32	3.25	2.57	1.79	1.36	1.10	1.00
			$S\ \zeta/E$	9.40	7.50	6.00	5.87	5.90	5.86	5.79	5.61	5.42	5.21	5.00
		0	Z_n	.091	.333	.500	.595	.657	.732	.781	.853	.917	.978	1.000
			Z_e	.000	.000	.000	.024	.060	.147	.245	.456	.683	.915	1.000
			$S\,Z_n/E$	9.10	6.66	5.00	4.10	3.46	2.61	2.11	1.55	1.26	1.07	1.00
			$S\ \zeta/E$	9.10	6.66	5.00	4.77	4.72	4.71	4.76	4.87	5.00	5.10	5.00

TABLE III — *Continued*

Examples of Results Produced by Formula (31)—when excess ratio of risk (x) is different from that (r) for which the W's are calculated

$a = 4$ $\eta = \frac{1}{2}$ $g = .333$

Georgia (Cont'd)

r	x		.01	.05	.10	.145	.19	.28	.37	.55	.73	.91	1.00
					Q			E/S					S
0	.333	Z_n	.094	.375	.600	.737	.824	.912	.952	.983	.994	.999	1.000
		Z_e	.000	.000	.000	.039	.090	.204	.322	.543	.741	.918	1.000
		$S\ Z_n/E$	9.40	7.50	6.00	5.08	4.34	3.26	2.57	1.79	1.36	1.10	1.00
		$S\ \zeta/E$	9.40	7.50	6.00	6.16	6.23	6.17	6.05	5.74	5.42	5.13	5.00
	.167	Z_n	.092	.353	.545	.660	.733	.816	.861	.916	.954	.985	1.000
		Z_e	.000	.000	.000	.035	.080	.183	.291	.506	.711	.905	1.000
		$S\ Z_n/E$	9.20	7.06	5.45	4.55	3.86	2.91	2.33	1.67	1.31	1.08	1.00
		$S\ \zeta/E$	9.20	7.06	5.45	5.52	5.54	5.53	5.47	5.35	5.20	5.06	5.00

TABLE IV

Credibilities given for selected value of E if K is chosen to give credit of 6⅔% for clear experience at qualification point

	r	E	E_n	Formula (14) Z_n	Formula (14) Average Credibility	Formula (31) Z_n	Formula (31) Average Credibility	Formula (39) Z_n	Formula (39) Average Credibility
New York									
	.333	14000	9333	.670	.447	.857	.571	.619	.413
Qualification	.167	14000	11667	.670	.558	.750	.625	.670	.558
point $E = 600$	0	14000	14000	.670	.670	.667	.667	.709	.709
	.333	14000	9333	.670	.447	.857	.571	.619	.413
	.167	11200	9333	.619	.516	.686	.571	.619	.516
	0	9333	9333	.575	.575	.571	.571	.619	.619
Massachusetts									
	.333	9000	6000	.620	.413	.776	.517	.566	.377
Qualification	.167	9000	7500	.620	.517	.687	.573	.620	.517
point $E = 480$	0	9000	9000	.620	.620	.616	.616	.662	.662
	.333	9000	6000	.620	.413	.776	.517	.566	.377
	.167	7200	6000	.566	.472	.621	.517	.566	.472
	0	6000	6000	.521	.521	.517	.517	.566	.566
Georgia									
	.333	4200	2800	.504	.336	.600	.400	.448	.299
Qualification	.167	4200	3500	.504	.420	.546	.456	.504	.420
point $E = 360$	0	4200	4200	.504	.504	.500	.500	.549	.549
	.333	4200	2800	.504	.336	.600	.400	.448	.299
	.167	3360	2800	.448	.373	.480	.400	.448	.373
	0	2800	2800	.404	.404	.400	.400	.448	.448

CHART II