RISKINESS LEVERAGE MODELS

RODNEY KREPS

Abstract

A general formulation of risk load for total cash flows is presented. It allows completely additive co-measures[1] *at any level of detail for any dependency structure between random variables constituting the total. It is founded on the intuition that some total outcomes are more risky per dollar than others, and the measure of that is a "riskiness leverage ratio." This riskiness leverage function is an essentially arbitrary choice, enabling an infinite variety of management attitudes toward risk to be expressed.*

The complete additivity makes these models useful. What makes them interesting is that attention can be turned toward asking "What is a plausible risk measure for the whole, while being prepared to use the indicated allocation technique for the pieces?" The usual measures are special cases of this form, as shown in some examples.

While the author does not particularly advocate allocating capital to do pricing, this class of models does allow pricing at the individual policy clause level, if so desired.

Further, the desirability of reinsurance or other hedges can be quantitatively evaluated from the cedant's point of view by comparing the increase in the mean cost of underwriting with the decrease in capital cost from reduction of capital required.

[1]Gary Venter coined this term, in parallel with variance and covariance.

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1. INTRODUCTION

The generic problem is that there are a number of random liabilities and assets for a company and a single pool of shared capital to support them. Their mean is usually meant to be supported by the reserves and their variability supported by the surplus, with the total assets of the company being the sum. Frequently, it is desired that the supporting capital be allocated in considerable detail—for example, to underwriter within line of business within state. This is not an end in itself, but is usually meant to help to understand profitability (or lack of it) in a business unit by associating a target rate of return with the allocated surplus and comparing to the actual profit return distribution. Sometimes the allocation is meant to be used for creating a pricing risk load as the allocated surplus times a target rate of return. Really, it is the cost of capital that is being allocated.[2]

One would like to have a methodology that would allow allocation of an essentially arbitrary form for the total capital required, and would also like to have an interpretation of the form in terms of statistical decision theory. The total capital including surplus will usually be represented as the sum of a risk load and a mean outcome. These can be calculated for a given distribution of total results. No attempt to connect risk load to a theory of pricing will be made here, although given the shape of the distribution in the context of a given theory such a connection could be made. It is simply assumed that some appropriate mean return is needed to attract and retain capital for the total risk.

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[2]Gary Venter, private communication.

There are several desirable qualities for an allocatable risk load formulation: (1) it should be able to be allocated to any desired level of definition; (2) the risk load allocated for any sum of random variables should be the sum of the risk load amounts allocated individually; (3) the same additive formula is used to calculate the risk load for any subgroup or group of groups.

This means that senior management can allocate capital to regions, and then regional management can allocate their capital to lines of business, and the allocations will add back up to the original. Further, it also means that the lines of business will add to the allocations for total lines of business as seen at the senior management level.

Ultimately, the choice of the riskiness leverage function will reflect management attitudes toward risk. The intention of this paper is to provide an interpretable framework for infinitely many choices, all of which can be appropriately allocated. It will be argued that the risk load must be considered in the context of the capital to support the risk.

Once management has experimented with various riskiness leverage functions and found a formulation with which they are comfortable, then it can be used to evaluate potential management decisions quantitatively. For example, buying reinsurance or choosing between reinsurance programs can be framed by including the variables representing the reinsurance cash flows. The general effects from a well-designed program will be to increase the mean cost—because the reinsurer needs to make a profit, on average—and to decrease the risk load and its associated cost—because the reinsurance is a good hedge against severe outcomes. If there is a net reduction in total cost, then there is an advantage to the alternative. It is worth noting that no financial information except the price is needed from the reinsurer. In particular, whatever return the reinsurer may think he will get from the contract is irrelevant to the cedant's decision to buy or not.

Section 2 introduces the framework and some practical notes; Section 3 is the development of the form and some of its properties; Section 4 is various examples, including some of the usual suspects for risk measures; Section 5 talks about what general properties might be desirable; and Section 6 is a numerical example with an accompanying spreadsheet.

2. THE FRAMEWORK

Assume n random financial variables X_k, $k = 1$ to n; and let $X = \sum_{k=1}^{n} X_k$ be their sum, the net result to the company. These variables may be from assets and/or liabilities but we will think of them for the initial exposition as liabilities. The convention used here is the actuarial view that liabilities are positive and assets are negative. This is an odd point of view for financial reports, and so in the accompanying exemplar spreadsheet, to be discussed at length in Section 6, the formulas are rephrased with the variables being net income streams and positive income being positive numbers.

Denote by μ the mean of X, C the total capital to support X, and R the risk load for X. Their relationship is

$$C = \mu + R \tag{2.1}$$

In more familiar terms, for balance sheet variables the capital would be the total assets, the mean the booked net liabilities, and the risk load the surplus.

Correspondingly, let μ_k be the mean of X_k, C_k be the capital allocated to X_k and R_k be the risk load for X_k. These satisfy the equation analogous to Equation (2.1):

$$C_k = \mu_k + R_k. \tag{2.2}$$

Using the abbreviation

$$\overline{dF} \equiv f(x_1,x_2,\ldots,x_n)dx_1dx_2\ldots dx_n, \tag{2.3}$$

where $f(x_1,x_2,\ldots,x_n)$ is the joint probability density function of all the variables, the individual means are defined by

$$\mu_k \equiv \int x_k \overline{dF}, \tag{2.4}$$

and the overall mean is

$$\mu \equiv \int \left[\sum_{k=1}^{n} x_k\right] \overline{dF} = \sum_{k=1}^{n} \mu_k. \tag{2.5}$$

Riskiness leverage models have the form

$$R_k \equiv \int \overline{dF}(x_k - \mu_k)L(x) \qquad \text{with} \quad x \equiv \sum_{k=1}^{n} x_k. \tag{2.6}$$

Then

$$R = \int \overline{dF}(x-\mu)L(x) = \int f(x)(x-\mu)L(x)dx. \tag{2.7}$$

The essential key to this formulation is that *the riskiness leverage L depends only on the sum of the individual variables*. In the second form of Equation (2.7), $f(x)$ is the density function for X, the sum of random variables.

It follows directly from their definitions that $R = \sum_{k=1}^{n} R_k$ and $C = \sum_{k=1}^{n} C_k$, no matter what the joint dependence of the variables may be.

In analogy with the relation of covariance to variance, the R_k will be referred to as co-measures of risk for the measure R. On occasion, the C_k will also be referred to as co-measures when the context is clear. Since additivity is automatic with these co-measures, what remains is to *find appropriate forms for the riskiness leverage $L(x)$*.

The form can be thought of as the risk load being a probability-weighted average of risk loads over outcomes of the total net loss:

$$R = \int dx\, f(x)r(x) \qquad \text{where} \quad r(x) = (x-\mu)L(x). \tag{2.8}$$

Again, the riskiness leverage reflects that not all dollars are equally risky, especially dollars that trigger analyst or regulatory tests.

Equation (2.8) is a standard decision-theoretic formulation for R. It could have been written down immediately, except that the special form for the risk load for outcomes is needed so that the co-measures have good properties. Another version of Equation (2.8) is to represent the risk load as an integral over risk load density:

$$R = \int rld(x)dx \qquad \text{where} \quad rld(x) = f(x)(x-\mu)L(x). \tag{2.9}$$

This has the advantage of showing which outcomes most contribute to the risk load. Another formulation, of note to theorists, is to say that the riskiness leverage modifies the joint density function and that the allocations are statistical expectations on a risk-adjusted density function. However, the support of L needs to be the same as the support of f to make this really work.

$$R = \int dx\, f^*(x)(x-\mu) \qquad \text{with} \quad f^*(x) = f(x)L(x). \tag{2.10}$$

A closely related useful form for thinking about the risk loads is that they are conditional expectations of a variable less its mean on the risk-adjusted measure, and that the conditions refer to the overall total variable. A typical condition might be that the total loss is greater than some specified value.

If we just want to think about co-measures without the explicit breakout into mean and risk load, we can use the generalization

$$R_k \equiv \int \overline{dF}(x_k - a\mu_k)L(x) \qquad \text{with} \quad x \equiv \sum_{k=1}^{n} x_k, \tag{2.11}$$

where any constant value can be used for a. A prime candidate is $a = 0$, and in the exemplar spreadsheet in Section 6 this is done because the variables considered there are net income variables.

It is also clear from Equation (2.6) that some variables may have negative risk loads, if they happen to be below their mean when the riskiness leverage on the total is large. This is a desirable feature, not a bug, as software developers say. Hedges in general and reinsurance variables in particular should exhibit this behavior, since when losses are large they have negative values (ceded loss) greater than their mean costs.

Practical Notes

Actual calculation of Equations (2.6) and (2.7) cannot be done analytically, except in relatively simple cases. However, in a true Monte Carlo simulation environment they are trivially evaluated. All one has to do is to accumulate the values of X_k, $L(X)$, and $X_k L(X)$ at each simulation. At the end, divide by the number of simulations and you have the building blocks[3] for a numerical evaluation of the integrals. As usual, the more simulations that are done the more accurate the evaluation will be. For companies that are already modeling with some DFA model it is easy to try out various forms for the riskiness leverage.

This numerical procedure is followed in the spreadsheet of Section 6, which has assets and two correlated lines of business. All the formulas are lognormal so that the exact calculations for moments could be done. However, the spreadsheet is set up to do simulation in parallel with the treatment on a much more complex model. It is also easy to expand the scope. If one starts at a very high level and does allocations, these allocations will not change if one later expands one variable (e.g., countrywide results) into many (results by state) so long as the total does not change.

[3]The mean for X_k is just the average over simulations, and it might be advantageous to calculate this first. The risk load is just the average over simulations of $X_k L(X)$ minus the mean of X_k times the average over simulations of $L(X)$.

Fundamentally, a risk measure should arise from economic requirements and management attitudes toward risk as part of the management business model. In this paper's class of models the risk attitude information is in the riskiness leverage function.

Gedanken[4] experiments indicate that to get the riskiness leverage it is probably desirable to start with plausible relativities between outcomes. After that is done, set the overall scale by some criterion such as probability of ruin (Value At Risk), mean policyholder deficit, Tail Value At Risk (TVAR)[5] or anything else that references the total capital and suits management's predilections. It is best if the overall level can be framed in the same terms as the relativities. In the Section 6 spreadsheet, TVAR is used.

In general, it might be good to start with simple representations, say with two parameters, and then see what consequences emerge during the course of testing. More remarks will be made later on specific forms. It will also be shown that the usual forms of risk measure can be easily framed and the differences between them interpreted in terms of different riskiness leverages.

A warning: there is no sign of time dependence in this formulation so far. Presumably the variables refer to the present or future value of future stochastic cash flows, but there is considerable work to be done to flesh this out.[6]

3. FORM DEVELOPMENT

Here we will start from a covariance formulation and proceed to the framework above by a detailed mathematical derivation.

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[4]That is, thought experiments, as contrasted with the real thing. The term is from the early days of relativity.

[5]TVAR is the average value of a variable, given that it is past some defined point in the tail. For example, one could ask for the average loss size given that the loss is excess of $10M.

[6]The work of Leigh Halliwell "The Valuation of Stochastic Cash Flows" may provide a way of looking at this problem.

Various proposed schemes[7] have utilized the fact that an allocation formula of the form

$$C_k = \alpha\mu_k + \beta\,\mathrm{Cov}(X_k, X) \tag{3.1}$$

will always be additive no matter what the dependency between the X_k may be. That is,

$$\begin{aligned} C &\equiv \alpha\mu + \beta\,\mathrm{Var}(X) \\ &= \alpha\mathrm{E}(X) + \beta\,\mathrm{Cov}(X, X) \\ &= \alpha\sum_{k=1}^{n}\mu_k + \beta\sum_{k=1}^{n}\mathrm{Cov}(X_k, X) \\ &= \sum_{k=1}^{n} C_k. \end{aligned} \tag{3.2}$$

A similar result will hold for the sum of any subset of the variables, thus ensuring the desired properties of the allocation. The sum of covariances of the individual variables with the total is the covariance of the total with itself. This paper generalizes this notion.

This form can be pushed further by imposing the reasonable requirement[8] that if a variable has no variation, then the capital to support it is simply its mean value with no additional capital requirement. This requires $\alpha = 1$. Then, with capital being the sum of the mean and the risk load,

$$R_k = \beta\,\mathrm{Cov}(X_k, X) \tag{3.3}$$

and

$$R = \beta\,\mathrm{Var}(X) \tag{3.4}$$

and so finally

$$R_k = R\frac{\mathrm{Cov}(X_k, X)}{\mathrm{Var}(X)}. \tag{3.5}$$

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[7]For a sampling, try [6], [2], and [4]. There are no doubt others.

[8]In [6], since the company can default, a constant value carries a negative risk load. We are assuming an ongoing company.

This form is familiar from CAPM.

However, it is clear that there are many independent linearly additive statistics. Back up a little to the definitions of mean and covariance, expressed as integrals over the joint density function:

$$\begin{aligned}\mu_k \equiv \mathrm{E}(X_k) &= \int x_k f(x_1,\ldots,x_n)dx_1\ldots dx_n \\ &\equiv \int x_k \overline{dF}. \end{aligned} \tag{3.6}$$

The additivity of the mean then comes from

$$\mu \equiv \mathrm{E}(X) = \int \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \int x_k = \sum_{k=1}^{n} \mu_k. \tag{3.7}$$

The covariance of one variable with the total is defined as

$$\mathrm{Cov}(X_k, X) \equiv \int \overline{dF}(x_k - \mu_k)(x - \mu), \tag{3.8}$$

where $x \equiv \sum_{k=1}^{n} x_k$. The additivity of the covariance is from

$$\begin{aligned}\mathrm{Cov}(X,X) &= \int \overline{dF}(x-\mu)^2 \\ &= \int \overline{dF}\left[\sum_{k=1}^{n}(x_k - \mu_k)\right](x-\mu) \\ &= \sum_{k=1}^{n}\int \overline{dF}(x_k-\mu_k)(x-\mu) \\ &= \sum_{k=1}^{n}\mathrm{Cov}(X_k, X). \end{aligned} \tag{3.9}$$

We want to generalize this result, and to do so we need more independent statistics that are linear functionals in X_k. Define the moment expectations

$$\mathrm{E}_m(X_k) \equiv \int \overline{dF}[(x_k - \mu_k)(x-\mu)^m]. \tag{3.10}$$

Then, following the same argument as in Equation (3.9), for any m

$$\mathrm{E}_m(X) = \sum_{k=1}^{n} \mathrm{E}_m(X_k). \tag{3.11}$$

Notice that the moment expectation for $m = 1$ is just the covariance of X_k with the total.

The individual risk load may now be formulated as

$$R_k = \sum_{m=1}^{\infty} \beta_m \mathrm{E}_m(X_k), \tag{3.12}$$

and there are now an infinite number of arbitrary constants to play with. Since there are so many independent constants, essentially any form can be approximated arbitrarily well.

For any choice of the constants β_m, the total risk load is the sum of the individual risk loads:

$$R = \sum_{m=1}^{\infty} \beta_m \mathrm{E}_m(X) = \sum_{m=1}^{\infty} \beta_m \sum_{k=1}^{n} \mathrm{E}_m(X_k) = \sum_{k=1}^{n} R_k. \tag{3.13}$$

This risk load can be put into a more transparent form by writing it as

$$R_k = \sum_{m=1}^{\infty} \beta_m \mathrm{E}_m(X_k) = \int \overline{dF}(x_k - \mu_k) \sum_{m=1}^{\infty} \beta_m (x - \mu)^m. \tag{3.14}$$

Since the term with $m = 0$ integrates to 0 (that being the definition of the mean), what is present is a Taylor series expansion of a function of the total losses about μ. Thus, Equation (3.14) may be written as

$$R_k = \int \overline{dF}(x_k - \mu_k) L(x). \tag{3.15}$$

This is the framework described earlier.

Properties

Clearly, the allocation properties are all satisfied for any choice of $L(x)$. The risk load has no risk for constant variable

$$R(c) = 0.$$

It also will scale with a currency change

$$R(\lambda X) = \lambda R(X),$$

provided $L(x)$ is homogeneous of order zero:

$$L(\lambda x) = L(x).$$

The reason this is required is that there is already a currency dimension in the term multiplying L. This can be made to happen, for example, by making L a function of ratios of currencies such as x/μ or x/σ, where σ is the standard deviation of X.

However, a more interesting possibility is to make L also be a function of x/S, where again S is the total surplus of the company. Since asset variability is in principle included in the random variables, S should be a guaranteed-to-be-available, easily liquefiable capital. This could come, for example, by having it in risk-free instruments or by buying a put option on investments with a strike price equal to what a risk-free investment would bring, or any other means with a sure result.

It is intuitively clear that S must come into the picture. Consider the case where loss is normally distributed with mean 100 and standard deviation 5. Is this risky for ruin, from a business point of view? If the surplus is 105, it is—but if it is 200 it is not. The natural interpretation is that the riskiness leverage should be a function of the ratio of the difference of the outcome from the mean to the surplus. Since the riskiness leverage could be used (with a pre-determined leverage) to give the surplus, there is a certain recursive quality present.

This formulation of risk load may or may not produce a coherent risk measure.[9] The major reason is that subadditivity[10] $[R(X+Y) \leq R(X) + R(Y)]$ depends on the form of $L(x)$. It might be remarked that superadditivity $[R(X+Y) > R(X) + R(Y)]$ is well known in drug response interactions, where two drugs taken separately are harmless but taken together are dangerous. While axiomatic treatments may prefer one form or another, *it would seem plausible that the risk measure should emerge from the fundamental economics of the business* and the mathematical properties should emerge from the risk measure, rather than vice versa.

A riskiness leverage formulation clearly allows the entire distribution to influence the risk load, and does not prescribe any particular functional form for the risk measure. In addition, many familiar measures of risk can be obtained from simple forms for the riskiness leverage ratio.

4. EXAMPLES

Risk-Neutral

Take the riskiness leverage to be a constant; the risk load is zero.

The positive risk load balances the negative risk load. This would be appropriate for risk of ruin if the range of x where $f(x)$ is significant is small compared to the available capital, or if the capital is infinite. It would be appropriate for risk of not meeting plan if you don't care whether you meet it or not.

Variance

Take

$$L(x) = \frac{\beta}{S}(x - \mu). \tag{4.1}$$

This riskiness leverage says that the whole distribution is relevant; that there is risk associated with good outcomes as much as

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[9]In the sense of [1] the actual risk measure is mean $+R$.

[10]A requirement for coherence. See [5] or [1].

bad; and that the outcome risk load increases quadratically out to infinity.

This gives the usual

$$R = \frac{\beta}{S}\int_0^\infty dx\, f(x)(x-\mu)^2 \tag{4.2}$$

and

$$R_k = \frac{\beta}{S}\int \overline{dF}(x_k - \mu_k)\left(\sum_{j=1}^{n} x_j - \mu\right). \tag{4.3}$$

Note that Equation (4.1) is suggestively framed so that β is a dimensionless constant available for overall scaling. The total capital then satisfies

$$C = \mu + S, \tag{4.4}$$

and the solution for $S = R$ is proportional to the standard deviation of the total:

$$S = \sqrt{\beta\,\mathrm{Var}(X)}. \tag{4.5}$$

It is perfectly possible, of course, to use some other formulation of the constant, say β/μ, which would then give a different measure. Such a measure would imply that the riskiness leverage does not depend on the amount of surplus available unless it was hidden in the scaling factor β.

TVAR (Tail Value At Risk)

Take the riskiness leverage

$$L(x) = \frac{\theta(x - x_q)}{1-q}. \tag{4.6}$$

The value q is a management-chosen percentage; for example, $q = 99\%$. The quantile x_q is the value of x where the cumulative distribution of X, the total, is equal to q. That is, $F(x_q) = q$. $\theta(x)$ is the step function: zero for negative argument and 1 for positive. See Appendix A for mathematical asides on this function.

This riskiness leverage ratio is zero up to a point, and then constant. Here the constant is chosen so as to exactly recreate TVAR, but clearly any constant will give a similar result. In fact, a riskiness leverage ratio that is constant up to a point and then jumps to another constant will give a similar result.

$$\begin{aligned} C &= \mu + \int dx\, f(x)(x-\mu)\frac{\theta(x-x_q)}{1-q} \\ &= \mu + \int_{x_q}^{\infty} dx\, f(x)\frac{x-\mu}{1-q} \\ &= \mu - \frac{\mu}{1-q}(1-q) + \frac{1}{1-q}\int_{x_q}^{\infty} dx\, f(x)x \\ &= \frac{1}{1-q}\int_{x_q}^{\infty} dx\, f(x)x. \end{aligned} \tag{4.7}$$

This is the definition of TVAR, well known to be coherent.[11]

We see shortly that the allocated capital is just the average value of the variable of interest in the situations where the total is greater than x_q. This is one example of the conditional expectation referred to earlier.

$$\begin{aligned} C_k &= \mu_k + \int \overline{dF}(x_k-\mu_k)\frac{\theta(x-x_q)}{1-q} \\ &= \mu_k - \frac{\mu_k}{1-q}\int \overline{dF}\theta(x-x_q) + \frac{\int \overline{dF} x_k \theta(x-x_q)}{1-q} \\ &= \frac{\int \overline{dF} x_k \theta(x-x_q)}{1-q}. \end{aligned} \tag{4.8}$$

This measure says that only the part of the distribution at the high end is relevant.

[11] [5], Op. cit.

VAR (Value At Risk)

Take the riskiness leverage

$$L(x) = \frac{\delta(x - x_q)}{f(x_q)}. \tag{4.9}$$

In Equation (4.9) $\delta(x)$ is the Dirac delta function.[12] Its salient features are that it is zero everywhere except at (well, arbitrarily close to) zero and integrates to one.[13] See Appendix A for remarks about this very useful function. Here the riskiness leverage ratio is all concentrated at one point. The constant factor has been chosen to reproduce VAR exactly, but clearly could have been anything.

$$\begin{aligned} C &= \mu + \int dx\, f(x)(x - \mu)\frac{\delta(x - x_q)}{f(x_q)} \\ &= \mu + x_q - \mu \\ &= x_q. \end{aligned} \tag{4.10}$$

This gives value at risk, known not to be coherent.[14] This measure says that only the value x_q is relevant; the shape of the loss distribution does not matter except to determine that value.

The capital co-measure is the mean of the variable over the hyperplane where the total is constant at x_q:

$$\begin{aligned} C_k &= \mu_i + \int \overline{dF}(x_k - \mu_k)\frac{\delta(x - x_q)}{f(x_q)} \\ &= \frac{1}{f(x_q)} \int \overline{dF} x_k \delta\left(\sum_{j=1}^{n} x_j - x_q\right). \end{aligned} \tag{4.11}$$

In a simulation environment one would have to take a small region rather than a plane. This could most easily be done as the

[12]Introduced in 1926.

[13]This implies that $\int dx\, f(x)\delta(x - a) = f(a)$. See Appendix 1.

[14][5], Op. cit.

difference of two closely neighboring TVAR regions. This was done using the formulation of the exemplar spreadsheet and a 1% width of the region.

SVAR (Semi-Variance)

Take the riskiness leverage

$$L(x) = \frac{\beta}{S}(x-\mu)\theta(x-\mu). \tag{4.12}$$

The risk load is the semi-variance—the "downside" of the variance:

$$R = \frac{\beta}{S}\int_{\mu}^{\infty} dx\, f(x)(x-\mu)^2, \tag{4.13}$$

and

$$R_k = \frac{\beta}{S}\int \overline{dF}(x_k-\mu_k)(x-\mu)\theta(x-\mu). \tag{4.14}$$

This measure says that risk loads are only non-zero for results worse (greater) than the mean. This accords with the usual accountant's view that risk is only relevant for bad results, not for good ones. Further, this says the load should be quadratic to infinity.

Mean Downside Deviation

Take the riskiness leverage

$$L(x) = \beta\frac{\theta(x-\mu)}{1-F(\mu)}. \tag{4.15}$$

$F(x)$ is the cumulative distribution function for X, the total. This risk load is a multiple of the mean downside deviation, which is also TVAR with $x_q = \mu$. This riskiness leverage ratio is zero below the mean, and constant above it. Then

$$R(X) = \frac{\beta}{1-F(\mu)}\int_{\mu}^{\infty} dx\, f(x)(x-\mu), \tag{4.16}$$

and

$$R_k = \frac{\beta}{1-F(\mu)} \int \overline{dF}(x_k - \mu_k)\theta(x-\mu). \tag{4.17}$$

In some sense this may be the most natural naive measure, as it simply assigns capital for bad outcomes in proportion to how bad they are. Both this measure and the preceding one could be used for risks such as not achieving plan, even though ruin is not in question.

In fact, there is a heuristic argument suggesting that $\beta \approx 2$. It runs as follows: suppose the underlying distribution is uniform in the interval $\mu - \Delta \leq x \leq \mu + \Delta$. Then in the cases where the half-width Δ is small compared to μ, the natural risk load is Δ. For example, if the liability is \$95M to \$105M, then the natural risk load is \$5M. So from Equation (4.17)

$$\Delta = R(X) = \frac{\beta}{0.5} \int_{\mu}^{\mu+\Delta} \frac{dx}{2\Delta}(x-\mu) = \frac{\beta\Delta}{2}. \tag{4.18}$$

However, for a distribution that is not uniform or tightly gathered around the mean, if one decided to use this measure, the multiplier would probably be chosen by some other test such as the probability of seriously weakening surplus.

Proportional Excess[15]

Take the riskiness leverage

$$L(x) = \frac{h(x)\theta[x-(\mu+\Delta)]}{x-\mu}, \tag{4.19}$$

where to maintain the integrability of R_k either $h(\mu) = 0$ or $\Delta > 0$. Then

$$R = \int f(x)h(x)\theta[x-(\mu+\Delta)]dx, \tag{4.20}$$

and

$$R_k = \int \overline{dF} \frac{x_k - \mu_k}{x-\mu} h(x)\theta[x-(\mu+\Delta)]. \tag{4.21}$$

[15] Another contribution from Gary Venter.

The last form has the simple interpretation that the individual allocation for any given outcome is pro-rata on its contribution to the excess over the mean.

5. GENERIC MANAGEMENT RISK LOAD

Most of the world lives in a situation of finite capital. Frame the question as "given the characteristics of the business, what is an appropriate measure of risk to the business, which generates a needed surplus S?" In the spreadsheet example this is done with a simplistic riskiness leverage function.

Clearly, the question at the heart of the matter is what an appropriate measure of riskiness might be. There are many sources of risk among which are the risk of not making plan, the risk of serious deviation from plan, the risk of not meeting investor analysts' expectations, the risk of a downgrade from the rating agencies, the risk of triggering regulatory notice, the risk of going into receivership, the risk of not getting a bonus, etc.

Given the above, it seems plausible that company management's list for the properties of the riskiness leverage ratio should be that it:

1. be a downside measure (the accountant's point of view);
2. be more or less constant for excess that is small compared to capital (risk of not making plan, but also not a disaster);
3. become much larger for excess significantly impacting capital; and
4. go to zero (or at least not increase) for excess significantly exceeding capital—once you are buried, it doesn't matter how much dirt is on top.

With respect to (3), the risk function probably has steps in it, especially as regulatory triggers are hit. For (4), a regulator might want to give more attention to the extreme areas. In fact, a regulator's list of properties for the riskiness leverage might include that it

1. be zero until capital is seriously impacted, and
2. not decrease, because of the risk to the state guaranty fund.

TVAR could be used as such a risk measure if the quantile is chosen to correspond to an appropriate fraction α of surplus. This would be

$$L_{\text{Regulator}}(x) = \frac{\theta(x-\alpha S)}{1-F(\alpha S)}. \tag{5.1}$$

However, everyone recognizes that at some level of probability management will have to bet the whole company. There is always business risk.

Management may more typically formulate its risk appetite in forms such as "For next year, we want not more than a 0.1% chance of losing all our capital, and not more than a 10% chance of losing 20% of capital." This is basically two separate VAR requirements, and can be satisfied by using the larger of the two required capital amounts. Or, as in the spreadsheet, management may choose to say something like, "We want our surplus to be $1\frac{1}{2}$ times the average bad result in the worst 2% of cases."

A (much too) simple example approximately satisfying (1) to (3) on management's list consists of linear downside riskiness leverage:

$$L(x) = \begin{cases} 0 & \text{for} \quad x < \mu \\ \beta\left[1 + \alpha\dfrac{(x-\mu)}{S}\right] & \text{for} \quad x > \mu \end{cases}. \tag{5.2}$$

The value of α is essentially the relative riskiness at the mean and at excess over mean equal to surplus. The value of β is again an overall scale factor. In the spreadsheet the allocations are nearly independent of the value of α, and TVAR is used for the example. The suggested use is to get the riskiness leverage function, and then to evaluate the effects of reinsurance (approximated by an increase in the mean and a decrease in the coefficient of variation) by seeing how the capital requirement changes for the same leverage function.

6. EXEMPLAR SPREADSHEET

The Excel workbook "Mini DFA.xls" has two lines of business with a correlation between the lines and investment income. The example is meant to be oversimplified but plausible, and takes the underwriting result for each line as a fixed premium less random draw on loss and expense. There is investment income on the surplus but no explicit consideration of it within the reserves. On the other hand, the lines of business are priced to a net positive underwriting result, so we could say that we are looking at future values including all investment income.

Cells with a blue background are input cells, and the reader is invited to change them and see how the results change. All the formulas are lognormal so that the exact calculations could be done. However, there is a "Simulate" button on the spreadsheet that will give statistics and cumulative distribution functions for whatever set of cells is selected. Simulation is used to get the overall results and the allocation ratios for different risk measures.

The sheets in the workbook are of two types: the data sheets (e.g., "basics") and the simulations done on them ("Sim basics"). The different sheets are generally different business alternatives. We start with "basics," which gives the basic setup of the business, and continue on: "TVARS" calculates various TVAR measures, "change volume" changes the volumes of the lines, and "reinsurance" and "reinsurance (2)" explore the effects of reinsurance. We will walk through them in detail, with commentary.

In all of them, the layout is the same. The two lines of business and the investment on surplus are laid out in columns, with blue background for user input. The financial variables are the two net underwriting results and the investment result, all of which vary randomly. F9 will recalculate to a new set of results. Below the income variables are the starting and ending surplus, and calculated mean and current (random) return. Interesting simulation results such as allocation percentages are displayed to the right of the surplus calculation.

Starting with "basics," Line A has a mean surplus of 10,000,000 and a standard deviation of 1,000,000 and Line B has a mean surplus of 8,000,000 and a standard deviation of 2,000,000. There is a correlation of about 25% between the lines (if the functions were normal rather than lognormal, it would be exactly 25%). Each line is written with a premium equal to the mean loss plus 5%. We interpret this calculation as our estimate at time zero of the value at time 1 of the underwriting cash flows, including all investment returns on reserves and premiums.

The investment income on the surplus is taken directly. The investment is at a mean rate of 4% with a standard deviation of 10%. The total of the results, on which we will define our leverage functions, is then added to the beginning surplus of 9,000,000 to get the ending surplus. As a consequence of the input values, the mean return on surplus is 14%. We would all be happy to have such a company, provided it is not too risky.

The simulation ("Sim basics") shows the actual correlation of the lines and the coefficient of variation on the return, as well as the distribution of total ending surplus and return. From the "Sim basics" sheet we can see that the probability of ruin is less than one in a thousand, and the coefficient of variation on the return is better than on the investment, which is good. We can also see from comparing the simulated means and standard deviations of the income variables to their known underlying values that the simulation is running correctly.

Management has decided that it wants to consider not just ruin, but on-going risk measures. In particular, it wants to get the TVAR values at various percentiles. It wants to formulate its risk appetite as *"For the x percent of possibilities of net income that are less than \$(income corresponding to x%), we want the surplus to be a prudent multiple of the average value so that we can go on in business."* What we do not know yet is what is *x%*, and what is the "prudent multiple." Gary Venter has suggested that the prudent multiple could be such that the renewal book can still be serviced after an average bad hit.

The sheet "TVARs" has the calculations needed for TVAR simulation in cells G36:N42. Column G contains the percentage values from 10% to 0.1%, and Column H the values of the total net income corresponding to those percentages. These values come from the sheet "Sim basics." Column I answers the question if whether the income is less than the value in Column H. Columns J through M are either "FALSE" if Column I is FALSE, or contain respectively the total income, the Line A income, the Line B income, and the investment income. Column N is a variable that is 1 if Column I is TRUE, and zero if it is FALSE. Upon selecting these cells and simulating, the mean value of Column N (for each row) will be the percentage of the time that the condition was satisfied. This should be close to the percentage in Column G. During simulation, non-numeric values in the selected cells are ignored. The mean values of cells in Columns J through M are the conditional means of the income variables for different threshold values, as desired.

The result of simulation is:

	Income	Mean Value of TVAR and Allocation Percentages			
%	is Below	Total	Line A	Line B	Investment
0.1	(8,892,260)	(10,197,682)	12.30%	85.99%	1.71%
0.2	(7,967,851)	(9,326,936)	12.49%	85.73%	1.78%
0.4	(7,024,056)	(8,380,265)	12.89%	85.09%	2.02%
1	(5,749,362)	(7,129,796)	13.38%	84.67%	1.95%
2	*(4,732,795)*	*(6,159,564)*	*13.60%*	*84.30%*	*2.10%*
5	(3,309,641)	(4,811,947)	13.60%	84.20%	2.20%
10	(2,143,327)	(3,734,177)	13.26%	84.94%	1.80%

The allocation percentages are just the ratios of the means for the pieces to the mean for the total; they automatically will add to 100%. What is noticeable here is that the allocation percentages change very little with the TVAR level, and that Line B needs some six times the surplus of Line A. That it needs more is not surprising; that it needs so much more perhaps is. What these allocations say is that when the total result is in the worst 10% of cases, about 5/6 of it is from Line B.

Management decides to adopt the rule *"We want our surplus to be $1\frac{1}{2}$ times the average negative income in the cases where it is below the 2% level."* That row is in *italic*, and this rule means that the 9,000,000 surplus is sufficient.

Using those allocation percentages, the mean returns on allocated surplus are Total: 14%; Line A: 40.9%; Line B: 5.3%; Investment: 190.6%. The total is a weighted average of the pieces. One needs to be careful in interpreting these return numbers, because they are dependent on both the relative volume of the lines and on the allocation method. But in any case, because Line B needs so much of the surplus, its return is depressed and the other returns are enhanced.

The next sheet, "change volume," looks at the case where we can change the underwriting volumes of Lines A and B. Clearly we want to reduce Line B and increase Column A, so the example has Column A increased by 60% and Column B decreased by 75%. This keeps the same mean net income. The standard deviations have been taken as proportional to volume, thinking of each line as a sum of independent policies.

Running the simulations, the allocations for Line A, Line B, and Investments now are respectively 32.8%, 60.9%, and 6.4%. Their implied returns change to 27.1%, 1.8%, and 62.8%. Line B is still bad, but because there is less of it, there is not such a large contribution at the 2% level. The 2% level, which was (4,732,795), is now better at about (3,250,000).

We also see that according to the management rule, we can release surplus of about 2,500,000. Alternatively, we can keep the same surplus and have a more conservative rule, with the prudent ratio being 2 instead of $1\frac{1}{2}$.

However, it may not be possible to change line volume, for various reasons. For example, these may be two parts of an indivisible policy, like property and liability from homeowners. Regulatory requirements may make it difficult to exit Line B. In addition, it takes time to switch the portfolio and requires a major underwriting effort. Management may decide to look at the possibility of buying reinsurance to improve the picture, since that is a decision that can be implemented quickly and easily changed next year.

The sheet "reinsurance" has an excess reinsurance contract on Line B, with a limit of 5,000,000 and an attachment of 10,000,000. It is priced with a load of 25% of its standard deviation. Once again, note that in the spreadsheet the results are calculated because we used easy forms, but that we could have complex forms and just simulate. The reinsurance results flow into the total net income.

Running the simulations, the allocations for Lines A and B, Investments, and now Reinsurance are respectively 36.3%, 73.9%, 14.2%, and −24.4%. The negative value for the reinsurance allocation reflects that the hedge is working, effectively supplying capital in these events. However, because of the positive net average cost of reinsurance, the return on the total is reduced to 12.1%. The implied returns on the pieces are 15.3%, 6.0%, 28.3%, and 7.9%. Line B is still bad, but because of the reinsurance there is not such a large contribution at the 2% level. Again, the 2% level has gone from (4,732,795) to (3,300,000). If we were to combine the reinsurance into Line B the combined allocation would be 49.5% and the return would be 5.1%.

There is also some 3,000,000 in surplus that the management rule would allow to be released. In the sheet "reinsurance (2)" the starting surplus has been reduced to 7,250,000 in order to bring the mean return on the total back up to 14%. Running the simulations, the 2% level on income is actually (3,237,000) but we ran the TVAR at (3,300,000). The essential point is that the results look reasonable, and the rule would allow release of still more surplus.

What is omitted in the calculation is the value of the 1,750,000 already released from the original 9,000,000 surplus. What this is worth depends on how the released surplus is going to be used. At the very least, this should be worth the risk-free income from it. Classical financial theory would suggest that it should be evaluated at the firm's cost of borrowing.

Measures other than TVAR were also run on the same basic situation, but are not shown in the spreadsheet. They were of two types. One was VAR measures, using a 1% interval around the VAR values. This measure says, given that the total loss is *at* a particular level, how much of it is from the different contributions. The other class of measures is the power measures, as in Equation (3.10). Each measure is a power of $(\mu - x)$ for $\mu > x$, and zero otherwise. In other words, these are downside measures.[16] The powers 0 and 1 are respectively the mean downside deviation and the semivariance. The others could be called "semiskewness," "semikurtosis," and so on—but why bother?

The results for VAR are quite similar to TVAR, except at the 10% level. This is because of the particular conditions we have for variability and correlation, and will not be true in general.

[16]Note that in contrast to the earlier discussion on losses where the downside is outcomes greater than the mean, here on return to surplus the downside is outcomes less than the mean.

	Mean Value and Allocation Percentages			
%	Total	Line A	Line B	Investment
0.1	(8,892,557)	13.51%	84.01%	2.48%
0.2	(7,969,738)	13.41%	84.74%	1.85%
0.4	(7,021,936)	15.32%	83.22%	1.46%
1	(5,746,279)	13.94%	84.18%	1.88%
2	(4,731,425)	14.20%	83.43%	2.38%
5	(3,308,824)	13.38%	83.64%	2.98%
10	(2,143,340)	11.16%	88.07%	0.76%

The downside power measure simulation results are:

	Mean Values $\hat{}(1/(N+1))$ and Allocations from Simulation			
Power	Total	Line A	Line B	Investment
0	2,183,834	22.44%	65.52%	12.04%
1	2,839,130	20.63%	69.79%	9.58%
2	3,424,465	19.42%	72.30%	8.28%
3	3,985,058	18.35%	74.30%	7.35%
4	4,510,337	17.43%	75.97%	6.60%
5	5,018,663	16.55%	77.45%	6.00%
6	5,514,616	15.69%	78.79%	5.51%

As the power increases and the measure is increasingly sensitive to the extreme values, the allocations move toward the TVAR allocations. This is probably not surprising.

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APPENDIX A

SOME MATHEMATICAL ASIDES

$\theta(x)$ is the step function: zero for negative argument and 1 for positive. It is also referred to as the index function.

$\delta(x)$ is the Dirac delta function. It can be heuristically thought of as the density function of a normal distribution with mean zero and standard deviation arbitrarily small compared to anything else in the problem. This makes it essentially zero everywhere except at zero but it still integrates to 1.

The index function can also be thought of as the cumulative distribution function of the same normal distribution, and it is in this sense that the delta function can be thought of as the derivative of the index function. All the usual calculus rules about derivatives apply without modification.

Always, we are implicitly taking the limit as the standard deviation of this distribution goes to zero. This whole usage can be justified in the theory of linear functionals, but the author has no idea where.

These notions lead to some fundamental properties of the delta function. For any continuous function $f(x)$

$$f(a) = \int f(x)\delta(x-a)dx, \tag{A.1}$$

and for $c > b$

$$\int_b^c f(x)\delta(x-a)dx = \theta(c-a)\theta(a-b)f(a). \tag{A.2}$$

If $h(a) = 0$ then

$$\int f(x)\delta(h(x))dx = \frac{f(a)}{|h'(a)|}. \tag{A.3}$$

The density function $f(x)$ for the total sum of variables can most easily be written as

$$f(x) = \int \overline{dF}\,\delta\left(x - \sum_{k=1}^{n} x_k\right)$$
$$\equiv \int dx_1 \ldots dx_n f(x_1,\ldots,x_n)\delta\left(x - \sum_{k=1}^{n} x_k\right). \qquad \text{(A.4)}$$

For calculation this is often a convenient form, as in the derivation of Equation (2.7):

$$\int \overline{dF}\left(\sum_{k=1}^{n} x_i - \mu\right) g\left(\sum_{k=1}^{n} x_k\right)$$
$$= \int dx \int \overline{dF}\,\delta\left(x - \sum_{k=1}^{n} x_k\right)(x-\mu)g(x)$$
$$= \int f(x)(x-\mu)g(x)dx. \qquad \text{(A.5)}$$

Similarly, the marginal density for any variable can be written

$$f_k(y) = \int \overline{dF}\,\delta(y - x_k). \qquad \text{(A.6)}$$

The cumulative distribution function for the total is

$$F(x) = \int \overline{dF}\,\theta\left(x - \sum_{k=1}^{n} x_k\right)$$
$$\equiv \int dx_1 \ldots dx_n f(x_1,\ldots,x_n)\theta\left(x - \sum_{k=1}^{n} x_k\right), \qquad \text{(A.7)}$$

and

$$f(x) = \frac{d}{dx}F(x) \qquad \text{(A.8)}$$

emerges from simple differentiation rules.