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Including the DFA Call Papers

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NOTICE

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The Summer 1998 Edition of the CAS *Forum* is a cooperative effort of the CAS Continuing Education Committee on the CAS *Forum* and the Dynamic Financial Analysis Task Force on Variables.

The Dynamic Financial Analysis Task Force on Variables is pleased to present for discussion nine papers prepared in response to its 1998 DFA Call Paper Program. These papers include papers that will be discussed by the authors at the 1998 CAS Seminar on Dynamic Financial Analysis, July 13-14, in Boston, Massachusetts.

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The Casualty Actuarial Society *Forum*
Summer 1998 Edition
Including the DFA Call Papers

To CAS Members:

This is the Summer 1998 Edition of the Casualty Actuarial Society *Forum*. It contains nine Dynamic Financial Analysis Call Papers and five new papers.

The Casualty Actuarial Society *Forum* is a non-refereed journal printed by the Casualty Actuarial Society. The viewpoints published herein do not necessarily reflect those of the Casualty Actuarial Society.

The CAS *Forum* is edited by the CAS Committee for the Casualty Actuarial Society *Forum*. Members of the committee invite all interested persons to submit papers on topics of interest to the actuarial community. Articles need not be written by a member of the CAS, but the paper's content must be relevant to the interests of the CAS membership. Members of the Committee for the Casualty Actuarial Society *Forum* request that the following procedures be followed when submitting an article for publication in the *Forum*:

1. Authors should submit a camera-ready original paper, and two copies.
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3. All exhibits, tables, charts, and graphs should be in original format and camera ready.
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5. Authors should submit an electronic file of their paper using a popular word processing software (eg., Microsoft Word and WordPerfect) for inclusion on the CAS Web Site.

The CAS *Forum* is printed periodically based on the number of call paper programs and articles submitted. The committee publishes two to four editions during each calendar year.

All comments or questions may be directed to the Committee for the Casualty Actuarial Society *Forum*.

Sincerely,

Robert G. Blanco, CAS *Forum* Chairperson

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A Portfolio Management System for Catastrophe Property Liabilities

by Adam J. Berger, Ph.D., John M. Mulvey, Kevin Nish, and Robert Rush

Abstract

As catastrophe modeling systems become more sophisticated, the property insurance portfolio manager can receive better account loss information than ever before. We describe a software system called SmartWriter which effectively processes this information for the portfolio manager. Specifically, the system determines:

- Appropriate pricing for an account
- Which accounts to remove from a portfolio to maximize risk-adjusted return
- How to merge two books of business
- Where to grow or shrink business geographically to achieve maximum diversification benefits

We utilize a number of optimization techniques to address these issues. We formulate the problem as a large mathematical program with numerous loss scenarios (10,000 or more). We then describe an algorithm to solve the resulting stochastic optimization problem in order to maximize risk-adjusted return, expected utility, or other user-defined performance measures.

The SmartWriter system is a PC-based Windows application. USF&G, a large property and casualty insurance company, currently employs SmartWriter as an integral part of its decision making process.

1. Introduction

The insurance portfolio manager and underwriter require sophisticated analytical tools to assist decision making. Just as an asset portfolio manager, such as a mutual fund director, can immediately see the effects of adding a security or option to his portfolio's risk and return profile, the insurance portfolio manager needs to understand the effects of adding an additional account to the business line. In addition, there are many other issues the manager must address, such as: (1) Should an existing account be renewed and, if so, at what price? (2) Where are the best areas to expand the current portfolio? (3) How can two books of business be merged profitably?

We have developed a decision support system, called SmartWriter, which answers these questions for one application area, the catastrophe property business. SmartWriter employs data from earthquake and hurricane modeling systems to show the effects of adding a new account or subtracting an existing account from the current portfolio. In addition, SmartWriter optimizes the portfolio composition to produce a portfolio meeting user-specified characteristics. Although we are describing SmartWriter in the context of catastrophe property, the methodology applies to Directors & Officers, Errors & Omissions, Workers Compensation, and other insurance lines.

The paper proceeds as follows. Section 2 describes the method for evaluating an incremental account and the return on capital methodology. Section 3 lays out the optimization model to address the questions raised above. The algorithm for solving the problem is described in Section 4, and results are presented in section 5. We conclude with some next steps in Section 6.

2. Modeling an incremental account

Suppose we have a portfolio of insurance liabilities. As an example, we look at a portfolio of commercial businesses insured against earthquakes in California by USF&G, a large property and casualty insurance company. A potential new piece of business is presented to the portfolio manager, who must decide whether to write the account or reject it. Of course, some negotiating with the insurance broker who presents the account is possible, so the portfolio manager would also like to know the required premium to meet a profitability hurdle. Before analyzing the incremental business, we need to define a profitability measure for the existing portfolio. Two measures are return on allocated capital and expected utility.

2.1 Return on Capital

In this method, capital is assigned to a portfolio (or business unit) based on the risk of the portfolio. Risk is calculated based on characteristics of the cumulative loss distribution and portfolio profitability. For the catastrophe property business, capital is often a function of points in the tail of the distribution, similar to Value at Risk (VAR). For simplicity, we focus on a single point of the loss distribution, the 99th percentile, and calculate the capital requirements as the funds needed to pay the loss incurred at the 99th percentile. This is referred to as the "1-in-100 year loss", since one would expect the loss to be as bad or worse than this level once every hundred years. More complex formulas based on multiple points of interest on the loss distribution are possible (see Mulvey et al, 1997). Equation 1 shows the allocated capital calculation:

$$\text{capital} = \rho\, F^{-1}(0.99) - (p - e) \qquad (1)$$

where ρ is a discount factor, F is the cumulative distribution function for the loss, p is the premium received and e is the non-catastrophe expenses. The discount factor ρ is necessary since we receive premiums and pay out expenses (e.g. commissions) at the beginning of the year, and losses are incurred during the year. Thus, we calculate the capital required at the beginning of the year, and discount losses, so that all terms are on the same basis.

To calculate return, we first define *expected catastrophe loss* as the expected value of the loss distribution. Expected margin is simply premium less expense less expected cat loss. Expected *return on capital* (ROC) is calculated by dividing expected margin by the allocated capital:

$$\text{ROC} = (p - e - Ef(x)) / \text{capital} \qquad (2)$$

where f(x) is the loss distribution and Ef(x) is the expected value of the distribution.

We define the *marginal capital* for an account as the difference in capital required for the portfolio with the account and the portfolio without the account. Return on marginally allocated capital (ROMAC) has the same expected margin for the numerator and marginal capital in the denominator. Define c_p as the capital required for the portfolio with the account and $\underline{c}_p$ as the capital for the portfolio excluding the account. Then the marginal capital m_o and the return on marginal capital (ROMAC) is defined as:

$$m_o = c_p - \underline{c}_p \qquad (3)$$

$$\text{ROMAC} = (p - e - Ef(x)) / m_o \qquad (4)$$

ROMAC captures the diversification benefit of the account with respect to the portfolio. An account with a high ROMAC doesn't require much additional capital allocation for the portfolio as a whole, and thus is a good diversifier. Conversely, an account may have a high return on capital on a stand-alone basis, but a low ROMAC, and thus is most likely located in an area of high concentration.

To facilitate combining loss distributions, we discretize the sample space and create numerous scenarios. Each scenario represents a year's worth of catastrophes. We can then determine losses for the account in each scenario and combine accounts scenario by scenario to determine portfolio losses. Although it is not necessary to have scenarios for the above calculations (since capital with and without an account can be calculated separately with no need for combining accounts), it will be important in performing the optimization described in Section 3.

2.2 Expected Utility

An alternative approach to allocated capital is expected utility. Given an asset position for a business line (or company) at the start of a year, define a von Neumann-Moregnstern utility function over the range of possible asset positions at the end of the year (see Bell [1995] for an example). Each portfolio will then have an expected utility value calculated from its loss distribution. Portfolios can be compared simply on expected utility, with higher expected utility being more desirable. To see whether to add an account to a portfolio, compare the expected utility before and after the addition.

There are advantages and disadvantages to the return on capital and expected utility approaches. Return on capital is a familiar financial ratio and is easily explained. Allocating capital based on points on the loss distribution is straightforward and captures, to some extent, the risk inherent in the business. Unlike expected utility, however, the return on capital does not provide a definitive answer on whether to add a new piece of business (e.g., if a new account has below average return on capital and above average ROMAC). The expected utility framework takes into account all points of the loss distribution whereas return on capital methods generally incorporate only a few. Expected utility provides a definitive recommendation, but does not have an immediately intuitive interpretation. For example, a portfolio manager can appreciate that adding a new account will boost return on capital from 12.0% to 12.5%, but may not as readily interpret increasing expected utility from 3.45 to 3.47. Depending on the model, expected utility can be easier to solve (see Berger [1995]) since it fits more easily into a mathematical programming framework than return on capital, which requires sorting a discrete distribution; Sections 3 and 4 discuss this issue further. This comparison is summarized in Table 1.

	Advantages	Disadvantages
Allocated Capital	- Easy to explain - Returns have intuitive meaning	- Extra work to sort discrete distributions - Limited points on loss distribution
Expected Utility	- Handle entire loss distn. at once - Convex math program	- Hard to determine utility function - Results not intuitive

Table 1: Comparison of allocated capital and expected utility objective functions

2.3 *Sample Decision*

We present SmartWriter analysis (Table 2) of an account recently offered to USF&G's commercial property business. Although we have altered the raw output to protect confidentiality, the returns are consistent with the actual analysis.

	New Account	Current Portfolio	Combined
Premium	$980	$3,800	$4,780
Expenses	$294	$1,140	$1,434
Expected Catastrophe Loss	$71	$615	$686
Expected Profit	**$615**	**$2,045**	**$2,660**
Loss at $99^{th}\% = F^{-1}(0.99)$	$5,200	$14,300	$18,100
Capital Required	$4,200	$11,600	$14,700
Return on Capital: ROC	**14.6%**	17.6%	18.1%
Ret. on Marginal: ROMAC	**19.8%**		

Table 2: New account analysis. All numbers in ($000), except where indicated.

The SmartWriter output is divided into three columns. The first column is the new account as a stand alone business. The expected income for the account, after taking expenses and expected catastrophe losses from the premium, is $615,000. The new account requires $4,200,000 in capital based on the 1-in-100 year loss of $5,200,000. This yields a return of 14.6%, which is below our hurdle rate of 15%.

The second column contains data on the portfolio as it stands today, and the final column is the portfolio performance if the new account were added. Note that the capital requirement for the combined portfolio is less than the sum of the new account and current portfolio capital: This indicates that the new account will diversify the business to some extent. Two additional items help quantify this diversification. The ROMAC for the new account is 19.8%, which means that the marginal return for adding the account divided by the marginal capital is significantly over the hurdle rate. The second item to note is the increase in the ROC for the portfolio from 17.6% to 18.1% if the account is added. For these reasons, the account was considered a good prospect, even though on a stand alone basis it was slightly below the hurdle rate.

3. Optimization Model

Optimization is the process of finding good solutions to a mathematical model. In the context of insurance underwriting, several problems are amenable to optimization. For a portfolio of large commercial accounts, the optimizer could locate the five accounts most in need of repricing, or the subset of the current portfolio which maximizes return. For a homeowners portfolio, the book of business is managed less on a home-by-home basis and more on a zip code, county, or state level; the optimizer can focus on which counties to expand market penetration and which zip codes to reduce premium volume. The next section describes SmartWriter optimization for commercial portfolios, and the following section for homeowner books.

3.1 Commercial Portfolios

Section 2 defined a method for comparing portfolios of accounts, either with return on capital or expected utility. We can now formulate an optimization model for choosing an optimal subset of accounts for the given portfolio. As mentioned above, we will define a discrete set of scenarios, where each scenario represents a number of catastrophes for a year. This facilitates the problem of combining loss distributions. For general continuous loss distributions, there is no simple method that can be used.

3.1.1 Variables and Objective

Define the following sets:

$\{1, 2, \ldots N\}$ – set of accounts in the portfolio
$\{1, 2, \ldots S\}$ – set of loss scenarios

Define the following input parameters:

p_i = premium for account i
e_i = non-catstrophe expense for account i
l_{is} = loss (in dollars) for account i in scenario s
π_s = probability of scenario s
ρ = discount factor

Define the following decision variables:

x_i, $i=1,\ldots,N$ – amount of account i in the portfolio

Our objective is to maximize return on capital:

$$\text{Max } \Sigma_{s=1,S} \Sigma_{i=1,N} \; \pi_s (x_i (p_i - e_i - l_{is})) / [\rho \, F^{-1}(0.99) - \text{sum } x_i (p_i - e_i)] \qquad (5)$$

where $F^{-1}(0.99)$ is calculated from the revised loss distribution $x_i{*}l_{is}$.

Note that correlations are implicitly captured in the analysis. Since the entire loss distribution is calculated for the objective function, the correlation among accounts will affect the return on capital.

3.1.2 Constraints

The following are constraints that can be added to the model.
An account can either be in the portfolio or out of the portfolio so we add a binary constraint

$$x_i \in \{0,1\}$$

If one or more properties must be retained, we add:

$$x_i = 1$$

The total premium for the portfolio can not be reduced past a specified level, MinPrem:

$$\Sigma_{i=1,N}(x_i * p_i) >= \text{MinPrem}$$

The expected income on the portfolio can not be reduced past a specified level, MinInc:

$$\Sigma_{i=1,N} (x_i *(p_i - e_i - l_{is}) > \text{MinInc}$$

3.2 Expansion problem

Another problem facing insurers is where to grow a portfolio of a large number of small accounts, for example the homeowners market in California. These portfolios can not be analyzed account by account, since underwriters do not have the flexibility of choosing to write one home and not another. Accounts must be aggregated to a meaningful level: not too large so that accounts within a group possess similar characteristics, but not too small so that they can be managed effectively, such as with target marketing. The following model chose the zip code level as a reasonable trade-off between these competing demands. The objective function remains the same, but we change a few variable and constraint definitions. Our emphasis now is determining how much premium to retrieve from each zip code. We assume that the loss characteristics within a zip code are constant. Zip codes where this is not the case can be broken down into smaller units.

Define the following sets:

{1, 2, ... Z} – set of zip codes in the region
{1, 2, ... S} – set of loss scenarios

Define the following input parameters:

e = non-catstrophe expense ratio
l_{zs} = loss per dollar of premium in zip code z in scenario s
π_s = probability of scenario s
ρ = discount factor

Define the following decision variables:

x_z, z=1,...,Z – amount of premium from zip code z in the portfolio

Our objective is to maximize return on capital:

$$\text{Max } \Sigma_{s=1,S} \Sigma_{z=1,Z} \ \pi_s (x_z - e^* x_z - l_{zs}{}^* x_z)) / [\rho F^{-1}(0.99) - \Sigma_{z=1,Z} (x_z - e^* x_z)]$$

where $F^{-1}(0.99)$ is calculated from the revised loss distribution $l_{zs}{}^* x_z$.

Constraints similar to the ones in the pruning example above can be added; we give a few examples here. The premium level across zip codes can be bounded between two values, MinPrem and MaxPrem:

$$\text{MinPrem} < x_z < \text{MaxPrem}$$

Alternatively, the total expansion of the portfolio can be limited to a dollar value, MaxPort:

$$\Sigma_{z=1,Z} x_z <= \text{MaxPort}$$

4. Solution Procedure

The models described in the previous section are not easily solved with traditional mathematical programming procedures, due to the necessity of the sorting during the capital allocation calculation. We employ a number of metaheuristic search procedures to find the global optimum value for the problem. For all of these, it is important to find good starting points, which we describe first, followed by the search algorithm.

4.1 Elite Solutions

Elite solutions are points in the decision space which are believed to be good locations for a local search (also called intensification, since the local area is being explored thoroughly). One method for generating elite solutions for this example depends on the profitability of the portfolio as a whole and on the individual accounts. If the portfolio is profitable, then a candidate elite solution would be the entire portfolio, or the portfolio with a small subset of poor performing accounts removed. Alternatively, for a poorly performing portfolio, a candidate elite solution could be a small subset of profitable accounts, or no accounts at all. Another approach ranks accounts by profitability and correlation with the portfolio as a whole; an account with high profitability and low correlation would be included in an elite portfolio.

A more profitable approach relies on problem-specific information. Suppose the optimization procedure is run monthly or quarterly. Optimal solutions from previous runs can be stored and will provide good elite solutions, even if the portfolio has changed measurably since the last run.

Of course, accounts no longer in the portfolio but in the previous optimal solution must be removed.

After a number of elite solutions have been generated using some or all of the methods above, the solutions are ranked in terms of attractiveness. This ranking will then determine the order for the local searches (see next section). Ranking can be based on objective function value alone, but to fully explore promising areas of the decision space we can use a weighted average of the objective function and the distance from higher ranked elite solutions. As more solutions are ranked, the benefit for diversification increases.

4.2 Tabu Search

Tabu search was originally developed by Glover and has proven highly effective for solving combinatorial optimization problems. (See Glover [1989] for an introduction). The procedure searches a feasible region by monitoring key attributes of the points that comprise the search history. Potential search iterates possessing attributes that are undesirable with respect to those already visited become tabu; appropriate penalties discourage the search from visiting them. We provide details below.

Consider a general non-convex optimization problem of the form:

$$\underset{x}{\text{minimize}}\ f(x),\ x \in X$$

where the function f(x) corresponds to the return on capital objective in Equation 5.

Our adaptation of tabu search has three basic elements:

- a function $g(x) = f(x) + d(x) + t(x)$. The function d(x) penalizes x for infeasibility. The function t(x) penalizes x for being labeled tabu.
- the current iterate x_c,
- a neighborhood of the current point N_c.

The procedure generates a new iterate x_{new} by selecting the element of N_c for which g(x) is smallest. The tabu restrictions represented in t(x), can address short-, intermediate-, and long-term components of the search history. Short-term monitoring is designed to prevent the search from returning to recently visited points, allowing the procedure to "climb out of valleys" associated with local minima. Short-term monitoring can also serve as a rudimentary diversification vehicle. Intermediate- and long-term monitoring techniques provide for a much more effective diversification of search over the feasible region. In addition, the elite solutions described previously also provide diversification. See Glover [1990] for additional details.

Details of four processes are required to define our adaptation of tabu search: formation of the neighborhood of the current point, assignment of tabu penalties, termination of search procedure, and *greedy selection* of the new iterate from the neighborhood of potential moves.

Neighborhood formation proceeds as follows. Let $x_c = (x_{1c}, \ldots, x_{nc})$ be the current point; the decision vector thus has n components. For the example in Section 3.1, this would be a vector of zeros and ones, where a "one" indicates the account is in the portfolio. Each member of the neighborhood of x_c, N_c, is formed by modifying one of its components either up or down by an amount equal to some value *step_size*. Note that this operation implicitly defines a discretization of the continuous feasible region. There are thus 2n members of N_c. We call each of these members a potential move; one of these will become the new iterate, i.e. - the actual move. Each potential move is characterized by two move attributes: *index changed* and *new value*. Attribute *index changed* is equal to j, where x_{jc} is the component of x_c whose value is changed by the potential move; *new value* is the value that the component being changed by the potential move assumes (formally: *new value* = x_{jc}, such that j = *index changed*).

The manner in which we assign tabu penalties -- and thus define the function t(x) -- to each potential move relies on exploitation of short-term search history; the methodology is based on the technique developed in Glover, Mulvey, and Hoyland [1996]. The assignment is based on a comparison of the move attributes of each potential move and those of the iterates comprising the recent search history. The maintenance of two data structures is necessary: 1) the *tabu list*, and 2) *time of last change list*. The *tabu list* is composed of the attributes of the T most recent search iterates; *tabu list* is thus a T x 2 array where T = TABU LIST SIZE. The *time of last change list* is an n x 1 array, where *time of last change list*$_j$ = the last iteration during which the actual move's *index changed* attribute equaled j. We also define f_{BEST} as the best objective value (in terms of minimization) found by the procedure at any point in the search process.

Three criteria govern our assessment of the tabu status of each potential move (x_p):

> Condition 1: do the move attributes of x_p match any of those in the *tabu list*?
> Condition 2: is length of stay < REQUIRED STAY, where length of stay
> = current iteration – *time of last change*$_j$,where j = *index changed*?
> Condition 3: is $f(x_p) < f_{BEST}$ <u>and</u> $x_p \in X$?

If either of the first two Conditions are true, we assign an appropriate tabu penalty to the potential move, discouraging the search from moving to x_p. Condition 1 prevents the search from revisiting a point whose move attributes match those of points recently visited. It is this operation that allows the search process to move away from local minima, as we described earlier. Condition 2 insures that a variable is not changed too soon after it becomes the basis for an actual move; it thus is a vehicle for short-term search intensification If the final condition is satisfied, we eliminate the tabu penalty for x_p; this allows the search to move to a tabu point if the objective value associated with this point is better than that of the best point found thus far. (This is our implementation of the concept of *aspiration criteria*; we refer the reader to Glover [1990] for details.)

We present three termination criteria:

1) Total time exceeds a preset maximum
2) Total iterations exceed a preset maximum
3) The amount of time spent without any improvement in the solution exceeds a preset maximum

Finally we address the greedy criterion for selecting from the set of potential moves the actual move, and thus the new iterate. The standard approach for selecting the new iterate is to find the point in the neighborhood of the current iterate for which g(x) = f(x) + d(x) + t(x) is smallest, a process that by definition requires evaluation of every member of the neighborhood. This strategy can degrade the effectiveness of the search when the computational effort required to evaluate f(x) is prohibitive. The greedy search strategy addresses this difficulty. It calls for the evaluation of the set of potential moves to cease when a neighbor fax is found which $f(x) < f(x_c)$ and $d(x) = t(x) = 0$, i.e. – x is feasible, not tabu, and shows improvement.

5. Results

Below is the SmartWriter output for a California earthquake portfolio with 173 accounts. The results are from real company data, but the numbers have been disguised to protect client confidentiality. We ran the analysis on a Windows 95-based PC with 64MB of memory, with run time between 5 and 10 minutes, depending on parameter settings.

The optimizer recommended the removal of 16 accounts from the portfolio. Table 3 shows summary information before and after the optimization for the portfolio as a whole.

On the whole, this was a profitable book of business, but there were a small number of poorly performing accounts. Not only did these accounts have a poor expected return, but they had a severe effect in the tail of the distribution. Expected income only decreased by $100,000 (3%), but the loss at the 99th percentile decreased by over $15MM. Return on capital jumped from 14.7% to 37.5%. We have seen this with other books of business as well: a small percentage of accounts represent a large portion of the tail of the loss distribution.

	Portfolio Today	Optimized Portfolio
Number of accounts	173	157
Premium	$5,600	$5,200
Expenses	$1,700	$1,600
Expected Cat Loss	$500	$300
Expected Income	$3,400	$3,300
Loss at 99th% = $F^{-1}(0.99)$	$28,600	$12,900
Capital Required	$23,200	$8,800
Return on Capital: ROC	14.7%	37.5%

Table 3: Portfolio before and after optimization. Unless otherwise noted, numbers are in (000).

Ideally, the portfolio manager should reprice these accounts upon renewal instead of terminating them. Although market conditions will determine the extent to which this is feasible, SmartWriter provides output on all the accounts targeted by the optimizer. Table 4 contains information for one of these accounts.

	Account A
Premium	$20
Expenses	$6
Expected Cat Loss	$12
Expected Profit	**$2**
Loss at $99^{th}\%$ = $F^{-1}(0.99)$	$780
Capital Required	$740
Return on Capital: ROC	**0.3%**
Ret. on Marginal: ROMAC	**0.4%**
Premium needed to meet 15% ROC hurdle	**$150**
Premium needed to meet 15% ROMAC hurdle	**$145**

Table 4: Account targeted for removal or repricing by optimizer.

For this example, the premium needed to meet the stand alone return on capital hurdle of 15% is $150,000, much greater than the current premium of $20,000. Repricing is most likely not an option for this account, but for examples where the current ROC is closer to the hurdle rate, repricing can be viable.

5.1 Portfolio Expansion

As with the pruning portfolio example above, portfolio financials are available before and after optimization. Rather than repeat the above tables, we display the graphical output available from SmartWriter. Since the analysis was conducted at a zip code level, financials can be displayed in map form for quick understanding. We show an example below.

Figure 1 shows profitability by zip code, if each zip code is evaluated on a stand alone basis, for the San Francisco Bay area. Dark green indicates zip codes with a high expected ROC per home, light green less profitable, and red low profitability. These maps can be generated for expected income, marginal capital, and for the results of the optimization: optimal concentration by zip code. For confidentiality reasons, we do not give the recommended map for concentration, but it overlaps the map below to a large extent. Most zips that are low profitability the optimizer recommends moving away from, and for zips with high profitability, the optimizer recommends a greater penetration. The optimizer takes into account, however, the problems with overproducing in a number of closely located zip codes which all may be affected by the same earthquake.

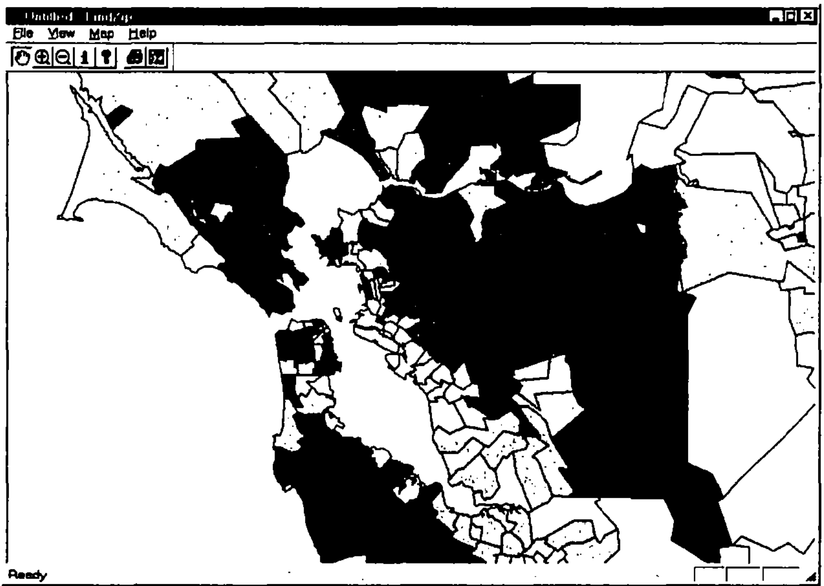

Figure 1: Expected return on capital by zip code for the San Francisco bay area. Dark green indicates most profitable zip codes, red indicates poor performing zip codes.

6. Next Steps

The portfolio management system can be readily extended to account for overlapping risks across business lines and asset investment categories. The concept is to develop a price of risk for each product-location under each scenario at each time period. These prices are available directly from the optimal decision variables for the strategic planning system. See Mulvey et al. (1998).

Ideally, one would like to link the liability decision with the asset investment strategy. In this paper we focused on the day-to-day underwriting decisions and take the asset return as a fixed input. In the future, one could tailor the asset portfolio in conjunction with the liability portfolio, such as purchasing catastrophe options or catastrophe-linked bonds for the property business line.

Another extension is the addition of multi-year contracts. As the catastrophe market continues to soften, these contracts may become more desirable for insurer and insured: They provide price protection for both parties. These can be linked with capital market projections which produce a range of possibilities (scenarios) a number of years ahead, such as the Towers Perrin CAP:Link system.

Finally, reinsurance decisions can be directly integrated into the optimization model. A desired profit distribution could be entered along with the current portfolio and a range of reinsurance options and treaties, and the optimizer would choose the best reinsurance options to match the desired profit distribution as closely as possible.

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Applying A DFA Model To Improve Strategic Business Decisions

by Salvatore Correnti, CFA,
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and Daniel B. Isaac, FCAS, MAAA

Published in response to the CAS Call for Papers

Applying a DFA Model to Improve Strategic Business Decisions

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ABSTRACT

Until recently, insurance companies were forced to evaluate business decisions at the functional level. With the advancement in computing power and understanding of advanced financial mathematics, company's are now able to integrate all of the various operational functions into a total company model, and evaluate the impact of various business decisions on the total company's risk/reward profile. This paper describes an approach for using "decomposition of risk" as part of a comprehensive ALM analysis for an insurance company. The objective is to identify and quantify the major factors that contribute to a company's total risk. Isolating each component of risk allows a company to better understand its total risk and thus develop strategies to improve its risk/reward profile. As a result, management can assimilate the relative and combined risk of assets, liabilities, and capital markets into a set of stochastic financial statements, thereby providing the information necessary to improve strategic investment, operating and capital allocation decisions.

Applying a DFA Model to Improve Strategic Business Decisions

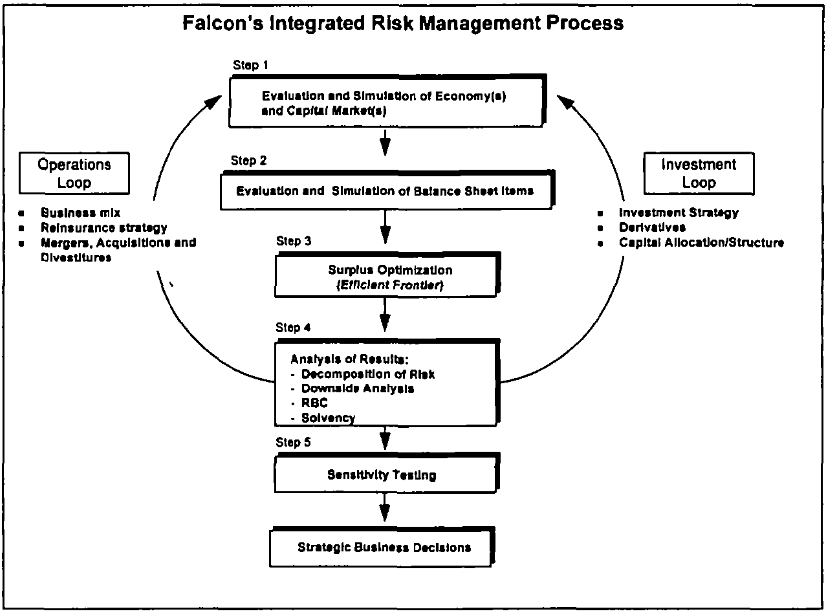
Until recently, insurance companies were forced to evaluate business decisions at the functional level. For example, Actuarial and Underwriting departments focused on the liability side of the operations, Investment departments concentrated on the risk and rewards of alternative asset strategies and asset classes, Treasury evaluated capital allocation decisions, and the Reinsurance unit explored the impact of various reinsurance treaties. With the advancement in computing power and understanding of advanced financial mathematics, company's are now able to integrate all of the various operational functions into a total company model, and evaluate the impact of various business decisions on the total company's risk/reward profile.

The risk management process developed at Falcon Asset Management, called Falcon Integrated Risk Management (FIRM™), is an example of a total company model that uses sophisticated techniques and gives management the ability to analyze problems at the total company level in a completely integrated framework (i.e., combining liabilities, assets and economic factors). As a result, management can analyze their key profit/cost centers, such as investment management, corporate finance/capital management, underwriting and reinsurance functions, on a consistent basis. An integrated risk management model uses simulation analysis of the aforementioned business functions and their key drivers to develop a comprehensive risk/reward profile for the company.

Many articles and papers have been written showing the benefits of including an insurance company's liabilities into its asset allocation decisions, including Sweeney and Correnti [1994] and Carino, et al. [1994]. Figure 1 expands on these concepts and gives a schematic view of an integrated risk management process. Total integrated risk management builds on traditional asset/liability analysis in that it explicitly considers strategic decisions impacting both operations and investment activities within a holistic framework . Once the key factors contributing to the overall risk of the company are identified and quantified, management has the ability to "loop" through the process by selecting either the investment loop (e.g., asset allocation, derivatives and capital allocation) or through the operations loop (e.g., business mix, reinsurance strategy and merger & acquisition analysis).

Traditional asset/liability analysis has been used to explore asset issues relating to asset allocation and derivative strategies only. An integrated risk management approach combines a more complete set of asset, liability, economic and capital market factors at the total company level giving management the ability to investigate the risk/reward tradeoffs of a wide range of alternative strategic business decisions. In addition the company is able to evaluate the joint impact of multiple strategic decisions through their interrelationships on the total company risk/reward profile.

Figure 1



For example, management can now evaluate various reinsurance strategies and quantify their impact on the company's financial objectives. The cost for the reinsurance protection can be compared to the reduction in risk provided by the reinsurance program and decisions concerning the appropriate level of reinsurance can be made. In addition, the integrated risk management approach provides management with a consistent framework to access the myriad of problems that they face. Whether deciding on an appropriate asset allocation strategy, reinsurance programs or corporate finance issues, management can use the integrated risk management process to perform the necessary analysis under a consistent risk/reward framework.

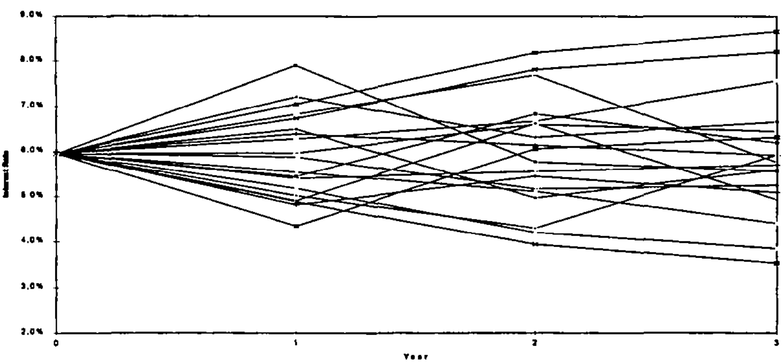
This paper will focus on the decomposition of risk step and how this information to assist a company with their strategic business decisions.

Economic and Capital Market Modeling

The first step in evaluating the asset allocation strategy for an insurance company is to evaluate the economy and the capital markets. This is Step 1 in the integrated risk management framework presented in Figure 1. For asset-only analysis over a single time period mean/variance models can be used effectively (see Markowitz [1987]). These models require inputs concerning the mean, standard deviation and correlations related to a particular set of asset categories being considered in the analysis. While effective for single period, asset-only analysis, these models are not adequate for more advanced asset/liability analysis or for use within a total integrated risk management framework. This is due to the fact that there is no explicit modeling of the underlying economic environment such as interest rates and inflation. The implicit economic environment that underlies a mean variance model can lead to interest rates that both explode to unreasonably high levels and even more undesirable, become negative.

Asset/liability management relies on the consistent relationship of both asset and liability movements to the underlying economic environment. Thus it is critical to model the economic variables explicitly to ensure reasonable future economic projections. The best models available for this purpose are models that utilize stochastic differential equations to describe the dynamics of the interest rate and inflation rate movements. For a more complete discussion of stochastic diffusion models see Mulvey and Thorlacius [1997]. Figure 2 shows twenty simulations corresponding to a three year projection of short-term interest rates that were generated from a stochastic diffusion model. This picture shows the year to year movements of the short-term interest rates together with the range of potential interest rate levels.

Figure 2



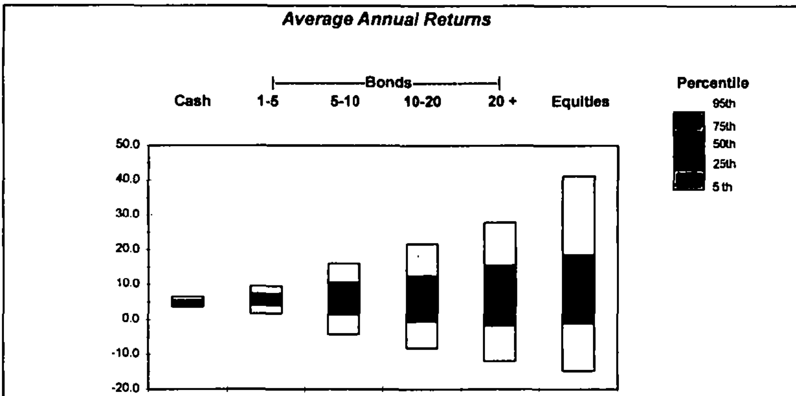
The economic and capital market diffusion model used employs a cascade, or top-down structure as described in Wilkie [1987]. The top of the cascade model involves generating price inflation rates. Future interest rates are modeled consistent with the previously generated inflation rates using a variant of the Heath-Jarrow-Morton interest rate model (see Heath, Jarrow and Morton [1988]). Once the future yield curves are determined, the cascade structure of the model produces asset class returns (both total returns and income returns) that behave consistently with the underlying economic scenario.

Asset classes are defined as homogeneous groups of individual investments such as fixed income of various maturities, equity, and cash. Fixed income categories are defined as a function of their anticipated yield, duration, convexity, and default or volatility risk. Equity returns are modeled as a function of their earnings yield and earnings growth. Asset classes, such as mortgage-backed securities, high yield bonds and property returns can be added to the analysis through the use of return generation tools available in the model. The modeled classes serve as a proxy for the assets currently held and/or expected to be held by the company.

The resulting returns can be summarized using the same mean, standard deviation and correlation statistics that are typically used as inputs to a mean/variance model. In addition, the same economic variables that are used to generate the capital market returns can be used to project the premium, loss and expense cash flows that will be required for the asset/liability analysis. This is the type of asset modeling system that we use in the integrated risk management system presented in Figure 1.

Figure 3 shows the 5th through the 95th percentile results corresponding to the average annual returns for each of six asset categories. As expected, over an annual holding period, cash returns show the smallest annual average return range while equities show the largest return range.

Figure 3



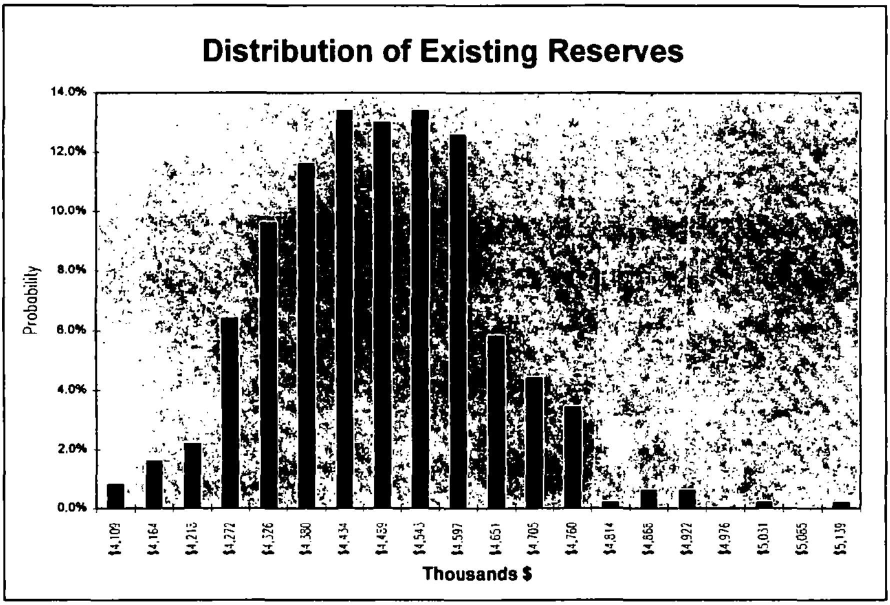
	Cash	1-5	5-10	10-20	20 +	Equities
Mean	4.86	5.65	5.88	6.29	7.10	9.87
Standard Dev	0.94	2.40	6.35	9.41	12.40	17.18
95th percentile	6.46	9.45	16.10	21.68	27.70	41.14
75th percentile	5.60	7.38	10.45	12.59	15.50	18.56
50th percentile	4.76	5.71	5.84	6.19	6.59	8.13
25th percentile	4.09	3.97	1.43	-0.51	-1.76	-1.15
5th percentile	3.56	1.64	-4.20	-8.24	-11.96	-14.81

The use of a stochastic economic and capital market simulation model of the type discussed above ensures that the asset class returns are consistent with the economic conditions that are being simulated. This is of critical importance to any application that is attempting to model assets and liabilities simultaneously.

Evaluation of Financial Statements

Since an integrated risk management process is dependent on an insurance company's liabilities, modeling the liability cash flows is critical for obtaining meaningful results. Liability simulation should consider both the existing reserves, and the company's business plan. Like asset categories, existing reserves and new business liabilities can be broken down into homogeneous lines of business to ensure that the unique characteristics of each line are captured. Historical experience and expected future trends need to be reflected in the assumptions to capture how the insurance company's liability structure will develop in the future.

Figure 4



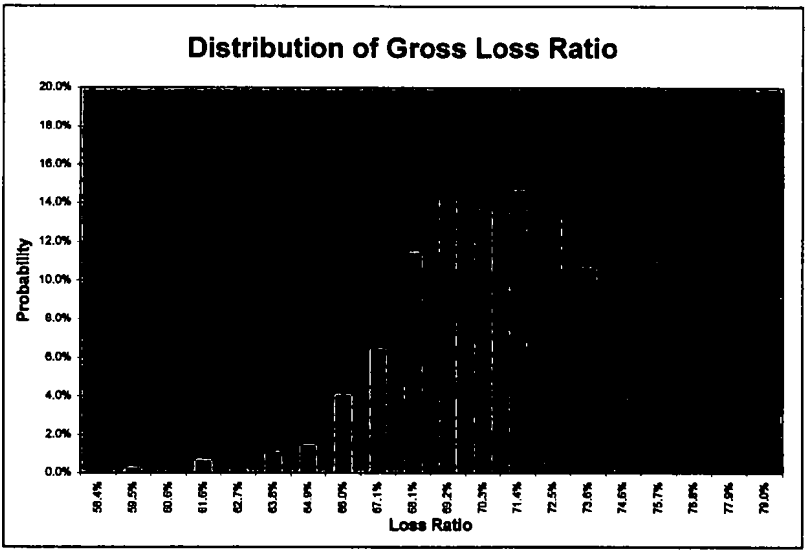
Projections of the existing loss reserves are generated stochastically by assuming an underlying distribution for the loss reserves and inputting an expected reserve runoff pattern. The loss reserve simulations should recognize that the magnitude of adverse loss development is potentially greater than the magnitude of beneficial loss development. Figure 4 illustrates the simulated distribution of the company's existing reserves.

Modeling the existing liabilities alone would imply that the company is in a liquidation, or runoff mode. Since most companies consider themselves a going concern, it is imperative to model the company's new business plan in order to accurately reflect the company's complete liability structure in the future. Typically companies budget three to five years of new business which can be layered on top of the existing reserve cash flows.

In order to project the new business liability cash flows, assumptions regarding written and earned premium, loss ratios, expected accident year payout patterns, IBNR factors and expenses are needed. Loss ratios should be modeled so as to reflect relationships with the underlying economic environment and should be general enough to allow the user to incorporate cycles and reversions.

The low frequency/high severity nature of catastrophes requires more precise modeling techniques to simulate catastrophic events and the resulting cash flows. There are several cat models available in the marketplace today (e.g. AIR, EQE, RMS, etc.). Loss ratios and cash flows attributed to catastrophes can be generated using one of these simulation models and merged with the non-cat losses described above to produce the company's overall loss ratio distribution. Figure 5 shows the distribution of simulated year 1 loss ratios for a hypothetical property/casualty company.

Figure 5



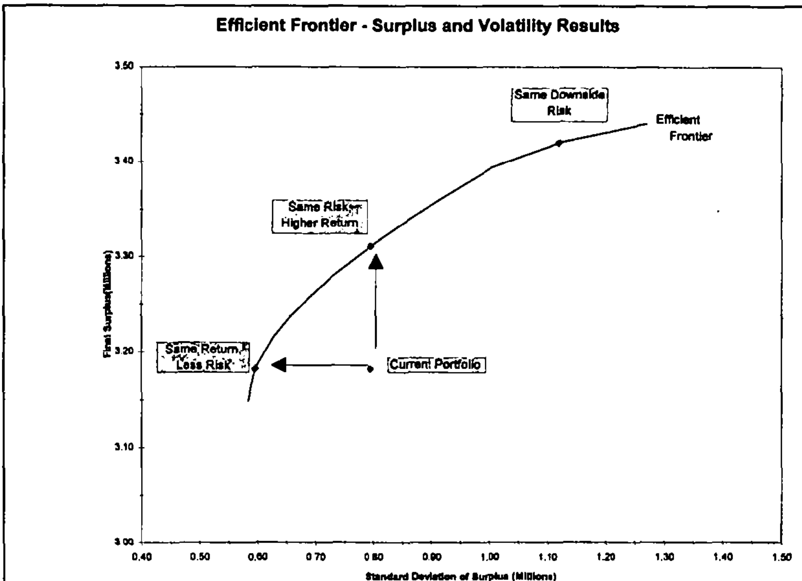
Once the projected loss ratios are determined, the total liability cash flows are calculated by multiplying the generated loss ratio by the forecasted earned premium and accident year payout pattern. The carried reserves can then be calculated as a function of the ultimate loss reserve, the expected loss reserve and the appropriate IBNR factor. It is important to recognize that since each line of business has its own characteristics, all of the above projections need to be performed on a line-by-line basis before being aggregated to a total company level.

To reconcile the model results to forecasted profit and loss statements, assumptions regarding taxes, premium collection patterns, and various other liability items (including non-cash flow items) are required. With this information, stochastic income statements and balance sheets can be produced on a statutory, GAAP and economic basis. Further information concerning asset and liability model requirements for property/casualty insurance companies can be found in Almagro and Sonlin [1996].

Consolidation and Analysis

In Step 3, from the integrated risk management flowchart, the liability and asset simulations are fed into an insurance optimization model to solve for an efficient frontier (a set of portfolios that provide the highest reward for a given level of risk). There are an unlimited number of objective functions that can be used for optimization. Some simple objective functions can be defined as mean ending surplus (statutory surplus, shareholders' equity, or economic value) for the reward measure, and the standard deviation of ending surplus for the measure of risk. Alternatively, we can look at various downside risk measures or company specific risk/reward functions.

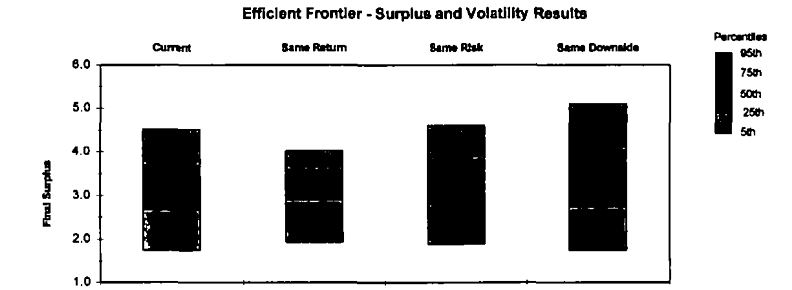
Figure 6



	Current	Same Return	Same Risk	Same Downside
Reward:	3.2	3.2	3.3	3.4
Risk:	0.8	0.6	0.8	1.0

Figure 6 shows an example of an efficient frontier using ending economic surplus as the reward measure, and the standard deviation of ending economic surplus as the risk measure. It is important to note that the efficient frontier plots expected results only. One must analyze the entire distribution of results to determine the optimal choice based on the company's risk tolerance. Figure 7 shows the distribution of results for three selected portfolios from the efficient frontier.

Figure 7



Distribution of Results:				
95th percentile	4.5	4.0	4.6	5.1
75th percentile	3.7	3.6	3.8	4.0
50th percentile	3.2	3.3	3.3	3.3
25th percentile	2.7	2.9	2.8	2.7
5th percentile	1.7	1.9	1.9	1.7
Mean:	3.2	3.2	3.3	3.4
Std Deviation:	0.8	0.6	0.8	1.0

Once efficient portfolios are identified, the "analysis of results" phase of the integrated risk management process (Step 4) can commence. Two of the more common types of analyses performed are decomposition of risk and downside risk analysis. These types of analyses identify the factors that have the greatest impact on the company's overall risk, and, as a result, require additional sensitivity testing (Step 5) or the identification of appropriate risk mitigating strategies. See Correnti and Sweeney [1994/1995], and Correnti, Nealon and Sonlin [1996/1997] for additional details on the process.

The end results of an integrated risk management process goes far beyond the objectives and goals of traditional ALM. Like traditional ALM, a primary use of integrated risk management is to determine an appropriate investment strategy. However, by being able to analyze a company in the aggregate and in a fully integrated framework (integrating liabilities, assets, and capital markets), the company has an invaluable tool which can help evaluate a wide range of business decisions and quantify various risk management strategies. For example, an integrated risk management process can be used to analyze a company's business mix and determine the optimal mix of premium to allocate to each of its lines of business. It could be used to evaluate possible acquisitions and divestitures in light of the impact these decisions would have on the total economic risk profile of the company. Alternatively, such a model could assist in determining the appropriate level of reinsurance from a total company viewpoint, and to determine the value/cost tradeoffs of various reinsurance strategies.

Decomposition of Risk

Variance analysis techniques are used to investigate the effects of two or more factors that influence an outcome. The method described below allows us to decompose the total risk facing an insurance company into its key components. In this framework, the total variance represents the volatility of ending surplus resulting from a particular asset portfolio chosen from the efficient frontier. To analyze this volatility further, one can break down the total risk into key drivers such as asset risk and liability risk. Identifying and comprehending the factors that contribute to the total risk for the company allows management to develop strategies to mitigate its risk exposure or to exploit market conditions. In either case, the company will have a better understanding of its risk profile and will be able to take proactive steps to improve that position in the future.

In general, recall that:

$$VAR(x+y) = VAR(x) + VAR(y) + 2COV(x,y) \quad (1)$$

$$= VAR(x) + VAR(y) + 2CORREL(x,y) \times$$

$$STDDEV(x) \times STDDEV(y)$$

where

$$VAR(x) = E\left[(x-\mu_x)^2\right] = \Sigma(x-\mu_x)^2 Pr(x),$$

$$COV(x,y) = E\left[(x-\mu_x)(y-\mu_y)\right] = \Sigma\Sigma(x-\mu_x)(y-\mu_y)Pr(x,y),$$

$$STDDEV(x) = \sqrt{VAR(x)}\,;\ STDDEV(y) = \sqrt{VAR(y)}\,;$$

$$STDDEV(x+y) = \sqrt{VAR(x+y)} \quad (2)$$

and

$$CORREL(x,y) = COV(x,y) \div \{STDDEV(x) \times STDDEV(y)\}$$

It is important to observe that if two variables are perfectly correlated (i.e., $CORREL(x,y) = 1$), then equation (2) reduces to:

$$STDDEV(x+y) = STDDEV(x) + STDDEV(y).$$

For correlations less than 1, the standard deviation of the sum of two variables will be less than the sum of the two standard deviations. In other words, if $CORREL(x,y) < 1$, then

$$STDDEV(x+y) < STDDEV(x) + STDDEV(y). \quad (3)$$

The covariance (or correlation) component of the total variance will reduce the overall standard deviation of a distribution unless the underlying variables are perfectly correlated. This fact is crucial to our risk management process. Additional factors (such as new asset classes or new lines of business) that in isolation appear to be risky, may improve the overall company risk profile when viewed in aggregate provided that the new factor is not perfectly correlated with all of the existing factors. This observation will be explored in further detail in the case study below.

For three variables, the formula for variance expands to:

$$VAR(x+y+z) = VAR(x) + VAR(y) + VAR(z) + 2COV(x,y) + 2COV(x,z) + 2COV(y,z) \quad (4)$$

and,

$$STDDEV(x+y+z) = \sqrt{VAR(x+y+z)} \quad (5)$$

As above, unless the factors are perfectly correlated, the resulting standard deviation of the sum of the variables will be less than the sum of the standard deviations, i.e.,

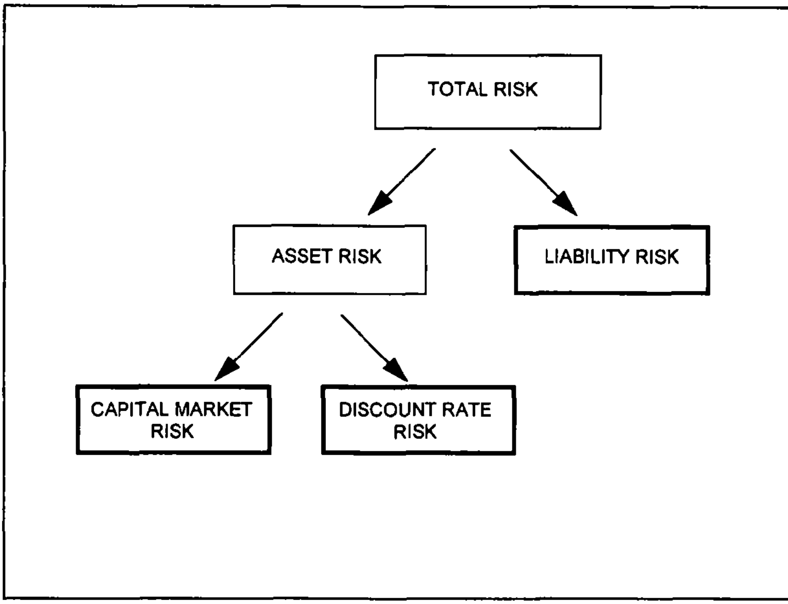
$$STDDEV(x + y + z) < STDDEV(x) + STDDEV(y) + STDDEV(z).$$

We are now ready to discuss the actual methodology of isolating individual risk factors.

Methodology

There are two main components that contribute to the total risk of an insurance company. They are the risk arising from the uncertainty in the economy and capital markets (asset risk) and the risk arising from the uncertainty in the ultimate loss payouts (liability risk). Further, the asset risk can be separated into the uncertainty surrounding the appropriate economic discount rate (discount rate risk) and the uncertainty in the asset class total returns (capital market risk). These risk breakdown components are outlined in Exhibit II.

Figure 8 **DECOMPOSITION OF RISK COMPONENTS**



This process can be used to isolate each of these risk components by holding two of the factors deterministic (constant), while allowing the third factor to be stochastic (variable). For example, to isolate the contribution to total risk from liability uncertainty, the model is run holding asset returns and interest rates constant while allowing liability cash flows to be stochastic. By running the model with deterministic liability cash flows and interest rates and stochastic asset returns, the capital market risk component can be identified. Finally, by making the liabilities and asset returns deterministic while allowing interest rates to be stochastic the model will identify the discount rate component of total risk. Table 1 outlines the eight runs necessary to complete a decomposition of risk analysis (S = Stochastic, D = Deterministic).

Table 1

Decomposition of Risk Runs

Run	Liabilities	Capital Market	Discount Rates
A	S	S	S
B	S	S	D
C	S	D	S
D	S	D	D
E	D	S	S
F	D	S	D
G	D	D	S
H	D	D	D

Run A, which assumes liabilities, asset returns and interest rates are all stochastic, represents the total risk to the company. By "turning off" discount rate and capital market volatility, we can determine the contribution to total risk arising from the liabilities (Run D). Similarly, making the liabilities deterministic allows us to quantify the impact of volatile capital market returns and discount rates (Run E). The other runs are necessary in order to calculate the covariance components of risk. Note that Run H, which assumes that all factors are deterministic, will have zero volatility and will represent the company's forecast as described earlier in this paper. The results of these runs will allow for the identification of each of the variance and covariance terms identified in equation 4.

The following case study illustrates the steps involved in decomposing the volatility of a property/casualty insurance company into its key risk components, namely liability risk, discount rate risk, and capital market risk and how this information can be used to make more informed decisions.

Case Study

As described above, decomposition of risk is an effective means for isolating and quantifying the key components of a company's total risk exposure. By identifying the major contributors of risk, management is better positioned to evaluate the consequences of strategic decisions that involve these components. Further, by identifying the covariance components between these risk factors, the company will be better able to evaluate the potential benefits of diversification and/or hedging activities.

The following case study shows how decomposition of risk can be used to help a property/casualty insurance company more effectively make business decisions. Property/Casualty Insurance Company (PCIC) is a hypothetical insurance company with rapid growth plans. PCIC writes primarily short-tailed property lines. As a result, PCIC has amassed a substantial amount of CAT exposure. In response to the large potential variability of their liabilities, PCIC has traditionally invested its assets very conservatively: their current investment strategy is 20% cash and 80% bonds. Even with their conservative investment strategy, PCIC's senior management team was concerned that a large CAT might force them to seek a capital infusion in order to avoid regulatory action. This analysis focuses on two basic questions. First, what is the probability that

PCIC will need a capital infusion during the next three years given its current business plan. Second, if necessary, what is the best way to combine reinsurance and/or a revised asset allocation to reduce this capital risk while minimizing the reduction in economic value at the end of the three-year time horizon.

PCIC's liabilities were modeled based on a thorough analysis of industry and PCIC historical loss ratio data and payout patterns. The historical information was combined with PCIC management's business plan and results from a commercially available CAT model to generate 500 simulations of future premiums, loss payments and expenses using the process described above. PCIC's investment options were broken down into the following five asset categories:

- Cash Equivalents
- Short Term Bonds - 1 to 5 Years
- Medium Term Bonds - 5 to 10 Years
- Long Term Bonds - 10 to 30 Years
- Large Capitalization Stocks

Five hundred simulations of income and total returns for each of these five asset classes were generated and merged with the previously generated liability scenarios. PCIC's current asset allocation is 20% to cash, 25% to short term bonds, 50% to medium term bonds and 5% to long term bonds.

In order to set the baseline values for the analysis, PCIC's three-year business plan and current asset allocation strategy were run through the system. The system calculated the economic value and the progression of statutory surplus for each of the 500 scenarios modeled. The major differences between PCIC's economic value, as defined in the system, and its projected statutory surplus are: 1. economic value reflects the market (not book) value of all assets, 2. economic value discounts the future liability cash flows at the projected market rates of interest and 3. economic value includes a component related to future business, even business renewed beyond the end of the time horizon. Based on these 500 simulations, PCIC's average economic value at the end of the three-year horizon was $919.9 thousand with a standard deviation of $186.8 thousand. Based on the assumption that PCIC would need a capital infusion in any simulation in which the premium to surplus ratio exceeded 3.0 at any time during the three-year time horizon, these same simulations indicated that there was roughly a 5% chance that PCIC would need to raise capital during that time frame.

PCIC's management was comfortable with both the average economic value and economic risk associated with their current asset allocation. What concerned them was having such high a probably of needing to raise capital, especially given the large uncertainties associated with the CAT model's loss predictions. In order to better understand the drivers of this risk, both the economic value and statutory surplus risk were decomposed into an underwriting and an asset component. Specifically, by holding the loss, expense and premium cash flows constant and letting the capital market returns and economic discount rates be stochastic, PCIC was able to identify the component of total risk that was the result of its current asset strategy. Further, by holding the capital market returns and economic discount rates constant while using stochastic liability cash flows, PCIC was able to identify the component of total economic risk attributable to their underwriting operations.

Tables 2 and 3, below, show the asset and liability components of risk, as well as the corresponding covariance between the assets and the liabilities.

Table 2

Decomposition of Total Economic Value Risk - Current Portfolio

Run	Liabilities	Capital Market	Discount Rates	Std Dev (in $000s)	Variance (in $000s)
A	S	S	S	186.6	34,814.3
D	S	D	D	183.5	33,674.8
E	D	S	S	44.6	1,992.3

COV (Liab, Cap Mkt + Disc Rates)			
A	VAR (Liab+Cap Mkt+Disc Rate)	34,814.3	
D	VAR (Liab)	33,674.8	
E	VAR (Cap Mkt+Disc Rate)	1,992.3	
	COV (Liab,Cap Mkt+Disc Rate) = (A - D - E) * .5	(426.4)	
	CORREL (Liab,Cap Mkt+Disc Rate)	(0.052)	
VAR (Liab + Cap Mkt + Disc Rates)			% Total
D	VAR (Liab)	33,674.8	96.7%
E	VAR (Cap Mkt+Disc Rate)	1,992.3	5.7%
	COV (Liab,Cap Mkt+Disc Rate)* 2	(852.8)	
	VAR (Liab+Cap Mkt+Disc Rate)	34,814.3	
	STDDEV (Liab+Cap Mkt+Disc Rate)	186.6	

Table 3

Decomposition of Total Statutory Surplus Risk - Current Portfolio

Run	Liabilities	Capital Market	Discount Rates	Std Dev (in $000s)	Variance (in $000s)
A	S	S	S	179.0	32,028.8
D	S	D	D	178.9	32,004.1
E	D	S	S	22.8	520.2

COV (Liab, Cap Mkt + Disc Rates)			
A	VAR (Liab+Cap Mkt+Disc Rate)	32,028.8	
D	VAR (Liab)	32,004.1	
E	VAR (Cap Mkt+Disc Rate)	520.2	
	COV (Liab,Cap Mkt+Disc Rate) = (A - D - E) * .5	(247.8)	
	CORREL (Liab,Cap Mkt+Disc Rate)	(0.061)	
VAR (Liab + Cap Mkt + Disc Rates)			% Total
D	VAR (Liab)	32,004.1	99.9%
E	VAR (Cap Mkt+Disc Rate)	520.2	1.6%
	COV (Liab,Cap Mkt+Disc Rate)* 2	(495.5)	
	VAR (Liab+Cap Mkt+Disc Rate)	32,028.8	
	STDDEV (Liab+Cap Mkt+Disc Rate)	179.0	

By decomposing risk into its asset and liability component parts, it could be seen that over 95% of PCIC's total economic and statutory risk, as measured by variance, was due solely to the uncertainty surrounding the liability loss cash flows. Both PCIC's asset strategy and the covariance component of risk were negligible. As a result, the next step was for PCIC to develop an alternative reinsurance plan. After this plan, which included a substantial quota share treaty on one of the more CAT-prone lines, had been developed, the liability and financial runs were updated with the revised information.

As expected, the probability of needing to raise capital was reduced to a more acceptable level (i.e., less than 1% over the three-year time horizon) as a result of the revised reinsurance. In addition, the overall economic risk was reduced from \$186.8 thousand to \$111.6 thousand. Unfortunately, the overall economic value was also reduced from \$919.9 thousand to \$823.0 thousand.

PCIC's management was uncomfortable giving away nearly 10% of their company's economic value even given the dramatic reduction in risk. Given the small amount of risk generated by the asset portfolio, which was confirmed by decomposing the risk of the revised reinsurance position in Tables 4 and 5, we were confident that PCIC's asset allocation strategy could be changed to improve the economic value without sacrificing the risk reduction achieved. In order to identify such a strategy, our proprietary insurance optimizer was employed. Figure 9 shows PCIC's asset allocation efficient frontier along with the risk/reward point corresponding to PCIC's current portfolio with and without the reinsurance.

Table 4

Decomposition of Total Economic Value Risk - Revised Reinsurance

Run	Liabilities	Capital Market	Discount Rates	Std Dev (in $000s)	Variance (in $000s)
A	S	S	S	111.5	12,429.6
D	S	D	D	104.5	10,924.0
E	D	S	S	39.9	1,594.4

COV (Liab, Cap Mkt + Disc Rates)			
A	VAR (Liab+Cap Mkt+Disc Rate)	12,429.6	
D	VAR (Liab)	10,924.0	
E	VAR (Cap Mkt+Disc Rate)	1,594.4	
	COV (Liab,Cap Mkt+Disc Rate) = (A - D - E) * .5	(44.4)	
	CORREL (Liab,Cap Mkt+Disc Rate)	(0.011)	
VAR (Liab + Cap Mkt + Disc Rates)			% Total
D	VAR (Liab)	10,924.0	87.9%
E	VAR (Cap Mkt+Disc Rate)	1,594.4	12.8%
	COV (Liab,Cap Mkt+Disc Rate)* 2	(88.8)	
	VAR (Liab+Cap Mkt+Disc Rate)	12,429.6	
	STDDEV (Liab+Cap Mkt+Disc Rate)	111.5	

Table 5

Decomposition of Total Statutory Surplus Risk - Revised Reinsurance

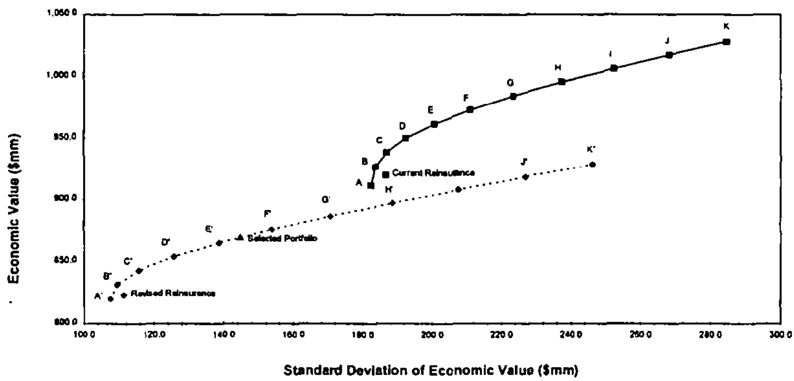
Run	Liabilities	Capital Market	Discount Rates	Std Dev (in $000s)	Variance (in $000s)
A	S	S	S	98.5	9,698.7
D	S	D	D	98.2	9,645.2
E	D	S	S	15.8	250.4

COV (Liab, Cap Mkt + Disc Rates)			
A	VAR (Liab+Cap Mkt+Disc Rate)	9,698.7	
D	VAR (Liab)	9,645.2	
E	VAR (Cap Mkt+Disc Rate)	250.4	
	COV (Liab,Cap Mkt+Disc Rate) = (A - D - E) * .5	(98.4)	
	CORREL (Liab,Cap Mkt+Disc Rate)	(0.063)	
VAR (Liab + Cap Mkt + Disc Rates)			% Total
D	VAR (Liab)	9,645.2	99.4%
E	VAR (Cap Mkt+Disc Rate)	250.4	2.6%
	COV (Liab,Cap Mkt+Disc Rate)* 2	(196.9)	
	VAR (Liab+Cap Mkt+Disc Rate)	9,698.7	
	STDDEV (Liab+Cap Mkt+Disc Rate)	98.5	

Figure 9

Economic Efficient Frontier

3-Year Time Horizon



Asset Allocation (%):	Current	Current Reinsurance A	C	E	G	K	Revised Reinsurance A'	C'	E'	G'	K'
Cash-U.S.A	20.0	79.2	0.0	0.0	0.0	0.0	30.6	0.0	0.0	0.0	0.0
Stck-U.S.A	0.0	8.5	22.7	44.8	64.7	100.0	5 4	24.5	45.9	65.3	100 0
Bonds 1-5	25.0	0.0	77.3	55 2	35.3	0.0	62.7	75 5	54.1	34.7	0.0
Bonds 5-10	50.0	0.0	0.0	0.0	0.0	0.0	1.3	0.0	0.0	0 0	0.0
Bond 10-30	5.0	12.2	0 0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Based on these results, PCIC was convinced that they could minimize the economic value reduction by taking on substantial additional risk on the asset side. Specifically, they were interested in a 50% stock, 50% short term bond allocation. This mix seemed to offer a reasonable trade-off between additional economic value (i.e., an increase from $823.0 to $869.5 thousand) and additional economic risk (i.e., an increase from $111.6 to $144.8 thousand) over just implementing the revised reinsurance. In addition, when we ran this strategy through the model, we discovered that the probability of needing a capital infusion was still roughly 1%. Finally, the decomposition of risk results for this asset allocation indicated a much better balance between liability and asset risks (see Tables 6 and 7).

Table 6

Decomposition of Total Economic Value Risk - Revised Asset Allocation

Run	Liabilities	Capital Market	Discount Rates	Std Dev (in $000s)	Variance (in $000s)
A	S	S	S	144.6	20,916.7
D	S	D	D	104.4	10,903.1
E	D	S	S	107.6	11,571.5

COV (Liab, Cap Mkt + Disc Rates)

A	VAR (Liab+Cap Mkt+Disc Rate)	20,916.7	
D	VAR (Liab)	10,903.1	
E	VAR (Cap Mkt+Disc Rate)	11,571.5	
	COV (Liab,Cap Mkt+Disc Rate) = (A - D - E) * .5	(779.0)	
	CORREL (Liab,Cap Mkt+Disc Rate)	(0.069)	

VAR (Liab + Cap Mkt + Disc Rates)

			% Total
D	VAR (Liab)	10,903.1	52.1%
E	VAR (Cap Mkt+Disc Rate)	11,571.5	55.3%
	COV (Liab,Cap Mkt+Disc Rate)* 2	(1,558.0)	
	VAR (Liab+Cap Mkt+Disc Rate)	20,916.7	
	STDDEV (Liab+Cap Mkt+Disc Rate)	144.6	

Table 7

Decomposition of Total Statutory Surplus Risk - Revised Asset Allocation

Run	Liabilities	Capital Market	Discount Rates	Std Dev (in $000s)	Variance (in $000s)
A	S	S	S	154.5	23,856.0
D	S	D	D	115.9	13,431.7
E	D	S	S	111.7	12,469.5

COV (Liab, Cap Mkt + Disc Rates)			
A	VAR (Liab+Cap Mkt+Disc Rate)	23,856.0	
D	VAR (Liab)	13,431.7	
E	VAR (Cap Mkt+Disc Rate)	12,469.5	
	COV (Liab,Cap Mkt+Disc Rate) = (A - D - E) * .5	(1,022.6)	
	CORREL (Liab,Cap Mkt+Disc Rate)	(0.079)	
VAR (Liab + Cap Mkt + Disc Rates)			% Total
D	VAR (Liab)	13,431.7	56.3%
E	VAR (Cap Mkt+Disc Rate)	12,469.5	52.3%
	COV (Liab,Cap Mkt+Disc Rate)* 2	(2,045.1)	
	VAR (Liab+Cap Mkt+Disc Rate)	23,856.0	
	STDDEV (Liab+Cap Mkt+Disc Rate)	154.5	

This outcome shows the importance of being able to analyze several different decisions (e.g., asset allocation and reinsurance) in a single, consolidated analysis. Specifically, PCIC would not have been able to assess this outcome using the traditional approach of evaluating these types of decisions independently. On a stand alone basis, PCIC's senior management would probably have rejected just the revised reinsurance structure since it gave up too much economic value. In addition, they would have never considered increasing PCIC's asset risk given their concern over requiring additional capital. As Figures 10 and 11 show, by combining the decisions, we have developed an economically viable alternative with substantially less downside exposure.

Figure 10

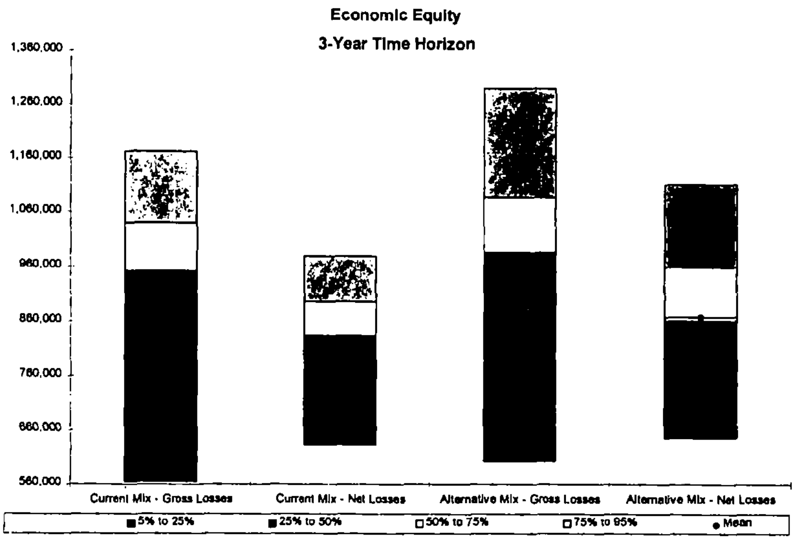
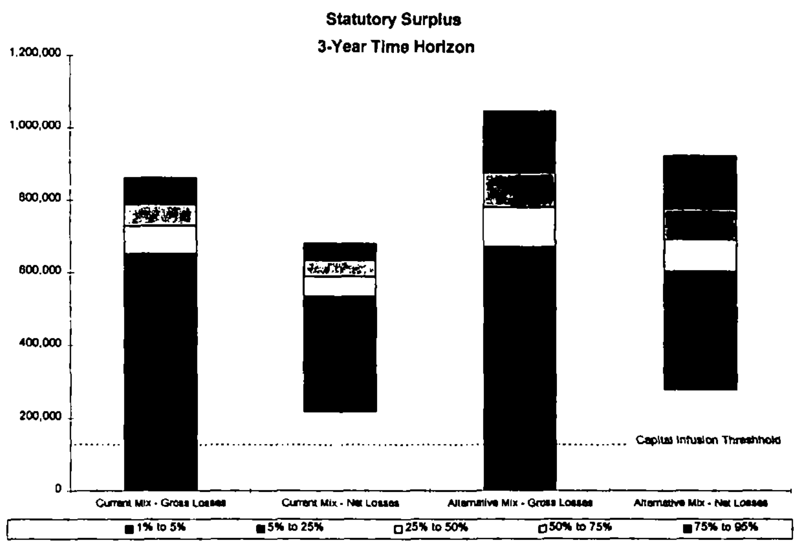


Figure 11



However, this combination is not without its own problems. One of its largest drawbacks is the large decrease in GAAP Net Operating Income and the resulting reduction in ROE. Specifically, when the business is profitable, the reinsurance cedes off a substantial amount of Operating Income. This is compounded by the fact that realized gains and losses, which comprise most of the total return for equities, are not included in Operating Income. One way to offset this impact would be for PCIC to swap its 150 million of debt from fixed to floating. While the model can be used to perform this type of analysis, the details of this strategy will be left for a subsequent paper.

Another issue is the impact this asset/reinsurance strategy would have on rating agency, regulatory and analysts' perceptions and views towards PCIC. Obviously, the strategies illustrated in this case study were extreme to demonstrate our point. Substantial work needs to be done to educate constituents on the benefits of a DFA type approach compared to the current piecemeal analysis which can be detrimental to the long term well being of the industry.

Conclusion

By undertaking this analysis, PCIC not only identified their asset and liability risk exposures, but, more importantly, their combined exposure. Armed with this information, they are able to revise both their reinsurance and asset allocation strategies to reduce their solvency concerns while minimizing the amount of decrease in expected economic value.

It must be made clear, however, that this analysis was based on a property/casualty insurance company with a large CAT exposure. Because the process is dependent on a company's general ALM characteristics (i.e., liability structure, surplus level) different companies will likely experience different results.

This paper presents only one possible application of decomposition of risk analysis within a total integrated risk management framework. PCIC could have performed a similar analysis on its business mix strategy to determine the optimal mix of premium to allocate to its different lines of business. It could have also evaluated possible acquisitions and divestitures in light of the impact these decisions would have on the total economic risk profile of the company. Finally, decomposition of risk could help PCIC better control volatility of shareholder's equity or statutory surplus over shorter time horizons.

The diverse characteristics of numerous risk elements at play within a large insurance company compound the difficulties of making appropriate decisions based on the overall benefit, or value, to the corporation. Management is often forced to make strategic and business decisions within the confines of each individual business or risk component. Moreover, even when individual decisions are correct, companies can still experience suboptimal financial results with respect to managing the overall risk/reward value of the total company. By using total integrated risk management and decomposition of risk to evaluate decisions within each subcomponent, management will be better positioned to make decisions that will benefit the company within a holistic decision making framework.

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Using the Public Access DFA Model:
A Case Study

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Using the Public Access DFA Model: A Case Study

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Using the Public Access DFA Model: A Case Study

Abstract

This paper describes the application of a publicly available property-liability insurance DFA model to an actual insurance company. The structure and key parameters of the model, as well as how to run the model, are explained in detail. A copy of the report to management of the company is included. The initial company reaction to this model was favorable. Management intends to use the model for such purposes as long term planning, capital allocation, reinsurance negotiations, competitor analysis and external communications with the regulatory and investment communities.

This paper describes the application of a DFA model to an actual insurance company. One goal of this work is to help actuaries learn about DFA by observing the use of a working model in a realistic setting. The model described in this paper is publicly available and accessible over the Internet. The company that generously allowed its data to be used in this exercise has asked to remain anonymous. Thus, minor modifications have been made to the data to help preserve the anonymity of this insurer. These changes do not affect the operation of the DFA model or obscure the data gathering process involved in running a DFA model.

Introduction

The DFA model used in this paper, termed Dynamo2, was developed by the actuarial consulting firm of Miller, Rapp, Herbers, & Terry, Inc. The model is accessible via their website (www.mrht.com) and requires only Microsoft Excel and @Risk in order to run. For those without access to @Risk, a limited version of the model can also be run solely in Excel. The Excel version is also useful for running a small number of iterations quickly to check the reasonableness of input values.

The general purpose of this model is to simulate a large number of possible outcomes from specific input data. By viewing the expected values and distributions of key variables, such as statutory surplus, premium-to-surplus ratios, and net income, the user can determine if these results are acceptable. If they are, then they validate the operating strategy of the company, subject to the general caveats of using DFA models. If not, then management can vary the input values to learn which changes would be effective in improving results to an acceptable level.

The model, when run using @Risk, allows the user to examine any of the stochastic parameters of interest determined as an @Risk function. Thus, users can view the randomly generated values for all of the unacceptable outcomes to see if any factor tended to be responsible for a significant number of these cases. For example, if a large percentage of the cases in which surplus falls below a minimum standard involved a high level of catastrophe losses, then the company may be able to reduce catastrophe exposure by revising its reinsurance arrangements or shifting its geographic distribution. Management could use the DFA model to test the effects these changes would have on the results by re-running the model with the revised input before deciding whether these approaches should be adopted.

The basic operation of the model is to generate insurance company cash flows and then evaluate the effect of these cash flows. The model integrates the cash flows from investments and underwriting, including catastrophes and taxes. The model consists of six different inter-related modules: underwriting, investments, catastrophes, taxation, an interest rate generator, and a payment pattern generator. Values generated in one module are shared with the other modules in subsequent calculations.

This paper focuses on an application of DFA. In order to obtain a fuller understanding of DFA modeling, including the limitations of this approach, readers should refer to additional sources. Some useful sources are: D'Arcy, Gorvett, et. al. (1997), D'Arcy, Gorvett, Herbers and Hettinger (1997), CAS Committee on Valuation and Financial Analysis (1995), CAS DFA Handbook (1996) and the multi-part Actuarial Review series "How DFA Can Help the Property-Casualty Industry" (1996-1998).

The Test Company

The company used to test this model is a mid-sized property-liability insurer that operates nationwide. The major lines are private passenger and commercial automobile, commercial multi-peril, workers compensation and homeowners. The company has standard reinsurance contracts: excess of loss, quota share and catastrophe coverage. Since the company has been in operation for more than twenty years, enough historical information is available to generate loss payout triangles, frequency and severity trends, loss ratios by age of business, and the other input required for the DFA model.[1]

Once the company's data were received, they were input into Dynamo2. Results from the model were generated, and incorporated in a report which was transmitted to the company. That report is included as an Appendix to this paper -- in order to follow the progression of this project, the reader is advised to read the Appendix at this point. This initial report served as the basis for discussions on DFA at a meeting between the authors of this paper and representatives from the company; company personnel involved in these discussions included actuaries, investment personnel, and business planning staff. This report provides both an introduction to DFA and a starting point for a detailed dynamic financial analysis of the firm. The questions raised at that meeting will be covered later in this paper, after a detailed explanation of this DFA model.

The Model

The DFA model used in this paper starts with detailed underwriting and financial data showing the historical and current positions of the company, randomly selects values for 4,387 (!) stochastic variables, calculates the effect on the company of each of these selected values, and then produces summary

[1] Generating and gathering the data needed to run this model required the efforts of many people at the company, including the Chief Financial Officer, the Chief Investment Officer and the Chief Actuary, as well as members of their staff. We are very grateful for their cooperation and willingness to supply us with their data; without their help, this paper could not have been written.

financial statements of the company for the next five years based on the combined effect of the random variables and other deterministic factors. All this represents a single iteration of the model. The model is set up to run multiple iterations of the model and analyze the distribution of the various outcomes.

Interest Rate Generator

The primary driver of this DFA model is the interest rate generator. Extensive work has been done in finance to develop sophisticated interest rate models. The interested reader is referred to Chan, Karolyi, Longstaff and Sanders (1992) and Hull (1997) for detailed descriptions of some of these models. In this DFA model, a relatively simple (in comparison with other interest rate models) single factor interest rate model is used, one derived by Cox, Ingersoll, and Ross (1985) (hereafter referred to as CIR). This simpler interest rate model was selected for two primary reasons. First, property-liability insurers are generally less exposed to interest rate risk than life insurers and banks, two industries for which much of the complex interest rate modeling has been performed. Thus, it is not quite as critical for property-liability insurers that interest rates be modeled as precisely. Second, and more importantly, it is vital that the users of the model fully understand the various components of the model. Actuaries are generally not very familiar with the terminology and approaches of interest rate modeling. Thus, beginning with a relatively straightforward interest rate model should allow the users to become more comfortable with the DFA model relatively quickly. Later, more sophisticated interest rate models can be incorporated and evaluated.

The CIR model describes the short term interest rate as a mean-reverting stochastic process. The CIR interest rate model was originally developed in a continuous-time framework; in that environment, the process *dr* for the instantaneous change in the level of the short-term risk-free interest rate is characterized by the equation

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz$$

where θ = the long-run mean to which the interest rate reverts,
κ = the speed of reversion of the interest rate to its long-run mean,
r = the current (instantaneous) short-term interest rate,
σ = the volatility of the interest rate process (as expressed by the standard deviation), and
dz = a standard Wiener process (essentially, a random walk).

For purposes of this DFA model, a discrete-time version of this model is required. According to Cox, Ingersoll, and Ross (1985), the short-term interest rate, in discrete-time, follows a (non-central) chi-squared distribution with degrees of freedom and non-centrality parameters being a function of the κ, θ, and σ parameters above. However, in this DFA model, we approximate the discrete-time form of the CIR model using the following formula:

$$\Delta r = a(b-r)\Delta t + s\sqrt{r}\,\epsilon$$

where Δr = the discrete-time (annual) change in the short-term interest rate,
Δt = the discrete time interval (one year), and
ϵ = a random sampling from a standard normal distribution.

The CIR model separates interest rate changes into two components, one deterministic component, ***a(b-r)***, and one stochastic component, $sr^{0.5}\epsilon$. The deterministic component moves the current interest rate part way (represented by ***a***) back toward the long term mean ***b***. The further the current interest rate is from this long term mean, the greater the deterministic component of the interest rate movement. The stochastic component causes the interest rate to jump around this otherwise level trend back toward the mean. Since the stochastic component is multiplied by the square root of the current interest rate, when interest rates are low, the stochastic component is small. This reduces the likelihood that interest rates will become negative. (In the continuous time application of this model, interest rates cannot become negative because if the interest rate were ever to become zero, which a continuous line must cross before becoming negative, then the interest rate will have no stochastic component and will simply be pulled back toward the long term mean (it will actually become ***a(b-r)***). However, in the discrete approximation of this model, negative interest rates can occasionally occur.)

In this interest rate model, the current interest rate is the actual short-term interest rate in the economy at the time the model is run. As of mid-March, 1998, 3 month Treasury bills, a common proxy for short term rates, were yielding 4.985%. Thus, in this model, ***r(0)*** is set to 5%. The long-run mean, ***b***, is also set at 5%. (Empirical tests of the CIR model on historical data indicate a value for the long-run mean of approximately 8%. These tests are based largely on data from the 1980s. When ***b*** is set at 8% in this model, any investment strategy based on long-term bonds tends to under-perform a shorter-term portfolio, since interest rates would tend to move upward, depressing bond prices. To avoid introducing this bias, the long term mean was selected to be the same as the initial value of the short term interest rate. However, this is a variable that can, and should, be altered by the user to reflect individual views of interest rate movements, and to test the sensitivity of results to this variable.)

Since, under the above parameter value selections, the value of ***b-r(0)*** is zero, the deterministic component of the interest rate change is zero in the first year. The stochastic component, then, determines the entire interest rate change.

In one run of the model, the value of ϵ in the first year was randomly selected by the model to be -1.00945. Thus, the calculation for the change in interest rates in that model run was:

$$\Delta r = s\sqrt{r}\,\epsilon = (0.0854)(\sqrt{0.05})(-1.00945) = -0.0193$$

Since the interest rate started at 0.05, the change of -0.0193 led to a new short-term interest rate of 0.0307, or 3.07%.

Once selected, the short term interest rate is used to generate the term structure of interest rates. Based on the interest rate model parameters selected, and upon the simulated short-term interest rate, rates on zero-coupon Treasury bonds are generated for each annual duration up to thirty years. This Treasury term structure is used to determine the market value of the company's bond holdings. The specific equations used to generate the term structure are taken from Cox, Ingersoll, and Ross (1985):

$$R(r,t,T) = \frac{rB(t,T) - lnA(t,T)}{T - t}$$

where R is the yield-to-maturity at time t on a discount bond that matures at time T, and

$$A(t,T) = \left[\frac{2\gamma e^{[(\kappa+\lambda+\gamma)(T-t)]/2}}{(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2}$$

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}$$

$$\gamma \equiv ((\kappa+\lambda)^2+2\sigma^2)^{1/2}$$

The short-term interest rate is also used to determine the general inflation rate, based on the following formula:

$$I_{CPI} = a+br+s\epsilon$$

where I_{CPI} is the general inflation rate,
a is a constant (set equal to 0),
b is a constant (set equal to .725),
r is the short term interest rate,
s is the standard deviation of the residuals (here 0.025), and
ϵ is a random sampling from the standard normal distribution.

The parameter values specified above were derived from regressions on the historical relationships between short-term interest rates and the consumer price index. Continuing the sample case illustrated above for the interest rate (3.07%), the value for $s\epsilon$ in one model run was randomly selected as -0.00459. Thus, the general inflation rate for this year was calculated as

$$I_{CPI} = 0.725(0.0307)-0.00459 = 0.0177$$

The inflation rate for each line of business is then calculated based on the simulated general inflation rate, according to the following formula:

$$I_{LOB} = a + b\, I_{CPI} + s\, \epsilon$$

where I_{LOB} is the line of business specific inflation rate,
a is a constant that varies by line,
b is a constant that varies by line,
I_{CPI} is the general inflation rate,
s is the standard deviation of the residuals, and
ϵ is a random sampling from the standard normal distribution.

The parameter values used to determine the line of business inflation rates in the DFA model are shown in the following table, along with a continuation of the sample model run described above, in which the short-term interest rate was 3.07% and the general inflation rate 1.77%. The parameter values were derived from regressions on the historical relationships between the consumer price index and line of business claims inflation rates.

Line of Business	Assumed Inflation in Payment Pattern	a	b	s	Sample Line of Business Inflation
Homeowners	0.052	0.032	.54	.0173	.037
PP Auto - Liability	0.067	0.047	.55	.0194	.060
PP Auto - Phys Dam	0.043	0.011	.88	.0307	.016
Comm Auto - Phys Dam	0.043	0.011	.88	.0307	.053
Comm Auto - Liab	0.067	0.047	.55	.0194	.074
CMP - Liab.	0.045	0.025	.55	.0147	.049
CMP - Prop.	0.045	0.025	.55	.0147	.028
Other Liab.	0.073	0.058	.40	.0206	.061
Other Liab. - Umbrella	0.073	0.058	.40	.0206	.101
WC	0.068	0.047	.58	.0250	.075

The line of business inflation rates are used for two purposes. First, they affect loss development. The initial loss reserves presume a specific inflation rate; the values selected for this run are listed on the above table. To the extent that the calculated line of business inflation rate differs from this value, loss payments will diverge from the initial loss reserves.

The second effect of the line of business inflation rates is on loss severity, which drives the need for future rate increases. In the present application of this model for this specific company, frequency was assumed to be stable, so the only factor that affects the projected pure premium is the severity trend. Thus, the line of business inflation rate determines the indicated rate level change.

Jurisdictional Risk

Each state poses unique advantages and disadvantages to the operation of an insurance company. Those advantages and disadvantages may take the form of judicial, legislative, or regulatory risk. For example, the likelihood of retroactive workers compensation benefit increases, mandated premium rebates, generous (for the policyholder) interpretations of contract provisions, and the ability to obtain rate increases all vary by state.

In this model, jurisdiction risk is reflected in two ways. First, each state has a range of "acceptable" rate changes -- that is, there is associated with each state a range of rate changes that can be implemented without extraordinary company cost (in terms of time or money) and/or additional insurance department scrutiny. Generally, these ranges limit rate increases more than they do rate decreases, and the ranges are smaller in states with more restrictive regulation. The obvious effect of strict rate regulation is to prevent insurers from increasing rates to the degree they feel is necessary. However, a side effect of capping rate increases is to make companies more reluctant to lower rates as much as would be otherwise indicated if pure premiums are improving.

The other effect of jurisdictional risk is to introduce a lag in implementing indicated rate changes. This lag, shown in the model in terms of years, is longer in states with restrictive rate regulation. The lags indicated on the jurisdictional risk exhibit included in the Appendix are estimated averages for rate increases and decreases; the average lags in the model are multiplied by 1.50 for rate increases and by 0.50 for rate decreases.

The jurisdictional risk parameters are based on a Conning & Company study that ranks all states with respect to regulatory restrictiveness. States ranked as most restrictive were assigned the lowest acceptable rate ranges and the longest lags. The actual values were selected primarily based on the judgement of individuals with experience with rate filings in those states.

As an example of jurisdictional risk in this DFA model, the range of Homeowners rate changes in Massachusetts is from .85 to 1.06 (rates could be lowered by 15% or increased by 6% without significant additional company cost or regulatory scrutiny). Since the average lag is estimated to be ½ year, it would take 3 months to implement a decrease and 9 months to implement an increase. The company's distribution of writings countrywide is used to determine the overall impact of jurisdictional risk.

Aging Phenomenon

The model reflects the aging phenomenon by separating writings for each line of business into new business, first renewals, and then second and subsequent renewals. Under the aging phenomenon, loss ratios gradually decline with the length of time the policies have been in force with the company. For more details on this experience, see Woll (1987), D'Arcy and Doherty (1989), D'Arcy and Doherty (1990) and Feldblum (1996). One requirement that this approach introduces is the need for the company to supply exposures and losses broken down by age of the business. Although this allocation is not needed for any statutory or accounting reports, many firms maintain this information for internal reports, although not necessarily in the detail required for the DFA model. In this case, estimates of the loss frequency and severity by age of business can be tried and the resulting loss ratio indications checked for reasonableness, before finalizing these values. The overall result is that new business should have the highest loss ratio, first renewal business should have a slightly lower loss ratio, and the remainder (second and subsequent renewals) should have the lowest loss ratio. Based on data published in D'Arcy and Doherty (1990), the loss ratio on new business ranged from 8 to 42 percentage points above the loss ratio on second and subsequent renewals.

In the model, the distribution of exposures by renewal category is determined as follows. For each line of business, renewal ratios are input that show what percentage of new, first renewal, and second and subsequent renewal business is renewed in the following year. Each renewal rate is applied to the appropriate business from the prior year to determine how many exposures are renewed. For example, for Homeowners, the new business renewal ratio is 60 percent, the first renewal business renewal ratio is 90 percent, and the second and subsequent renewal business is 95 percent. Thus, 60 percent of the exposures that were new business in 1997 become first renewal business in 1998 and 90 percent of first renewal exposures become second and subsequent renewal business in 1999. Thus, policy renewals are deterministic in this model. Since the company has a target growth rate, the number of new policies written in a given year is simply the number needed to achieve the growth target.

Underwriting Cycles

The premium level at which policies are written depends on the targeted growth rate and the position in the underwriting cycle. The property-liability insurance industry underwriting cycle has been the subject of extensive study and is recognized as being quite complex. In line with the goal of keeping this model as straightforward as possible, especially for this early version, the underwritng cycle is simpifled. However, it still reflects the different relationships of growth rates and price levels depending on the position of the cycle.

In this model, the underwriting cycle, which can vary /by line, is characterized as being in one of four conditions: mature hard, mature soft, immature hard and immature soft. In a hard market, rates can generally be increased somewhat and growth may still be obtainable. In a soft market, rates generally have to be reduced in order to grow. For each of the four cycle conditions, the probability of moving to another condition in the cycle (e.g., from mature soft to immature hard) is specified as an input. Thus, over the course of the simulation, the company moves through different phases in the underwriting cycle.

In the simulation described in the Appendix, Homeowners is initially in a soft market. Based on the parameters selected, there is a 70 percent chance of remaining in a soft market and a 30 percent chance of moving to an immature hard market in the next year. If the soft market continued and the company wanted to achieve a high growth rate, then the company would have to lower rates, or at least not fully implement any indicated rate increases, in the next year.

Catastrophes

A catastrophe is defined as any natural disaster causing in excess of $25 million in insured losses. The total number of catastrophes countrywide is simulated based on a Poisson distribution, and then assigned to a "focal point" state based on historical catastrophe experience. The size of each catastrophe is then simulated based on a lognornal distribution, the parameters of which vary according to the identity of the focal point state. For each simulated catastrophe, the contagion effect of the catastrophic losses from the focal point to other states, and by property line of business, is determined based on historical relationships. Finally, the effect of these catastrophes on the company is determined by the market share of the company in each state, by line of business.

For example, in Florida the probability of any number of catastrophes occurring is determined based on a Poisson distribution with a mean of 0.6667. This value, relative to the parameters for all other states, determines the likelihood of a catastrophe being assigned to Florida. For each simulated catastrophe, the size is then determined based on the lognormal distribution with a mean parameter of 2.7697 (in millions) and a variance parameter of 1.1563. For each catastrophe in which Florida is the focal point, 86 percent of the loss is assumed to be incurred in Florida, with the remaining 14 percent distributed to nearby states. All of these parameters were calculated based on data from Property Claim Services over the period 1949-1995. As an example, in one iteration of the model, no catastrophes occurred in Florida in 4 of the 5 years simulated; in the fifth year (2001), two catastrophes occurred, one causing $143 million in insured losses and the other $269 million in losses.

It should be noted that the catastrophe module in this DFA model is meant to produce reasonable estimates, and is not intended to replace the more rigorous catastrophe models that are available. In fact, it is possible that the results from other commercially available catastrophe packages could be used in this DFA model.

Investment Results

Investment results for both fixed income securities and equities are determined in the investment module. For bonds, both the statutory value and the market values are calculated for each category of bond (Government, corporate, municipal) and for each maturity segment indicated in the Annual Statement (e.g., one year or less, one to five years, etc.). The market value is determined based on the term structure of interest rates obtained in the interest rate generator module. The cash flows on bonds consider interest rates, coupon rates and default rates, generated stochastically based on historical patterns.

The market value of equities is determined from a simulation based on the Capital Asset Pricing Model. The rate of return on equities is determined in a two step approach. The initial expected market return is the risk free rate, as obtained in the interest rate generator, plus a market risk premium of 8.5% (historical average for 1926-1996). The adjusted market return is the initial expected return minus 4 times the simulated change in the short term interest rate. A random component based on a normal distribution with a mean of 0 and a standard deviation of 15 percent is generated and added to the adjusted market return to determine the overall market return for each year. The return for the company is then determined by applying the equity beta, which is an input value.

Collecting Data

One decision that needs to be made is how to deal with multiple companies operating under the same management. Many insurers have subsidiaries, but operations are coordinated within the group. In this case, the model should be run on the group as a whole, rather than for each individual company. However, if more detail is needed, then each company can be modeled separately.

The primary source of input data for the model is the Annual Statement. However, additional information is also necessary, which requires the company to provide, or generate, some internal management reports. In addition, the company needs to provide information about exposure growth anticipated, by line for the next five years, and any shift in investment allocations that are contemplated.

Examples of the specific data requirements are illustrated on the exhibits included in the Appendix. In a typical application of this model, some of the more problematic data areas might potentially include exposures and rates by renewal category, historic loss ratios by renewal category, and various aggregation issues (the trade-off between data volume and its homogeneity when examining lines and types of business). Also, in order to generate more credible cash flows, or to deal with homogeneous data, Annual Statement lines of business can be aggregated or split into separate components, as needed.

Running the Model

The first step in running the model (after the company-specific data has been input) is to determine where the industry stands in the underwriting cycle for each line of business. It is presumed that the insurance industry follows a time dependent cycle of competitiveness. In a soft market, premium increases tend to significantly reduce market share. Conversely in a hard market, policyholders find it difficult to obtain insurance, so it is easier for an insurer to increase market share.

The next step is to determine the number of iterations to be run. The higher the number of iterations, the more stable the distribution of outcomes is likely to be, but the program will take correspondingly longer to run. As a word of advice, when beginning to learn the program, this number should be kept small (5-10) to minimize the time needed to complete the run. Frequently, it will be apparent from even that limited output that something is amiss. After adjusting the input data and the parameters until the user feels confident that they are reasonable, a larger number of iterations (e.g., 1,000 or more) should be run to obtain the full benefit of the DFA model.

At this point, reasonability checks should be performed to make sure the input values are realistic. One check is to multiply frequency by severity and divide the product by the average premium, for each age of business, to see if the implied loss ratios had the appropriate relationship (new business highest, second and subsequent renewal the lowest). Another check is that the average catastrophe losses are within expected bounds.

The next step is to determine exactly what output is desired. Any value that appears in the sections of the model where calculations are performed, or any parameter generated by the model, is a potential output value. Premiums, surplus, loss and operating ratios, investment returns, catastrophe losses, interest rates, inflation rates, and regulatory ratios are all potentially useful output values. In some cases additional detail might be desired. For example, the loss ratio by line, by year and by age of business, direct, ceded, or net, could all be listed as output variables. To determine the cause of a potentially high loss ratio, the frequencies, severities, number of exposures and average premiums could also be listed. However, at some point the magnitude of the output data could become unmanageable. Since the model provides for ten lines of business forecasted for the next five years, and exposures are maintained for new business, first renewals, and second and subsequent renewals, if each value were shown for direct, ceded and net values, there would be 450 loss ratios (plus frequencies, severities, and exposures) for each iteration. Finding the cause of any adverse indications would be a major chore. Thus, care needs to be exercised to keep the output manageable, especially when the model is being fine-tuned. The exhibits included in the Appendix are indicative of the types of output that can be helpful.

Changing the Model's Parameters

Since the DFA model is built in a spreadsheet environment, changing the model's parameters is straightforward. The user merely needs to know which input screen contains the key variables. The following table lists some of these key variables, and their locations in the spreadsheet model.

Variable	Description	Sheet Location	Cell Reference
U/W Cycle Position	Users viewpoint on current market conditions.	General Input	C6 to C15
Growth Rates	Expected growth rates in exposures	Premium Input	Row 22
Renewal Ratio		Premium Input	Rows 30-32
Expense Provisions	Commissions, General, Other Acq., Taxes, Dividends, and Nonrecurring Expenses	Premium Input	Rows 42, 46, 50, 54, 57, 59
Q/S Ceding Commission		Premium Input	Row 62
Exposure Changes	Use to Change Exposures and Market Shares by State	Exposure Input	
Selected 1997 Severities		Loss Input	Rows 167 to 169
Selected 1997 Frequencies		Loss Input	Rows 196 to 198
Selected ULAE Provisions		Loss Input	Rows 227 to 233
Q/S Arrangements		Loss Input	Rows 255-259
XOL Arrangements	Includes Attachment Points and Cost of Reinsurance	Loss Input	Rows 268 to 297
Stop Loss Arrangements	Includes Attachment Points and Cost of Reinsurance	Loss Input	Rows 349 to 353
Cat. Re Arrangements	Includes Attachment Points and Cost of Reinsurance	Loss Input	Rows 359 to 363
Stock Betas		Investment Input	Rows 95 to 98
Capital Infusions		Investment Input	Rows 86 to 91
Reinvestment Allocations	How Investment Income is Reinvested	Investment Input	Rows 109 to 125
Long-Run Interest Rate		Interest Generator	C27
Current Interest Rate		Interest Generator	C29
General Inflation Parameters		Interest Generator	C35 to C37
LOB Inflation Parameters		Interest Generator	Rows 54 to 56

U/W Cycle Parameters	Includes Probability of Changing Market Condition and Supply/Demand Curves	U/W Cycle Generator	C7 to H34

Initial Reaction of the Company to the DFA Report

First Impressions

The company's first direct exposure to the DFA model occurred at a meeting between the authors and representatives of the company's actuarial, investment, and business planning departments. At this time the report included in the Appendix was delivered and a detailed explanation of the DFA model was presented. Many questions were raised at that point, a majority of which related to asking for an explanation of how the model worked. However, there were also a number of questions that will lead to model improvements and enhancements. Overall, company personnel were enthusiastic about the model and have hopes of using it in the future for strategic planning purposes. They also saw it as a tool to help the different divisions of the company -- actuarial, financial, investment, and planning -- work together. Finally, the company liked the software platform on which the DFA model is based. The Excel spreadsheet format makes the model user-friendly and simple to change and enhance, and allows the user to examine the inner workings of the model in a non-black box environment.

Concerns

The company expressed certain concerns regarding the model and the results that were initially supplied to them. It was evident that the **Base Case** indications were unacceptable (primarily due to the high growth goals of the company); however, the managers felt that constraining growth was not a viable alternative. Other options were explored, including increasing the new business renewal rate. For Homeowners this value was 60 percent. Raising it to 80-90 percent caused some improvement, but not enough to turn results around completely. Another change was to modify the maximum ceded under the aggregate reinsurance contract. This also had a favorable effect on forecasted results.

In order to gain a better understanding of what was causing the results, two additional values, the short term interest rate and catastrophe losses, were added to the output page and the simulation re-run during the meeting. The ability to modify the model and quickly see the impact of the changes was viewed very favorably.

Some of the questions raised indicated the need for enhancements in future versions of the model. One question related to prepayments on bonds and CMOs as a function of interest rate changes. Another wanted to examine the effect of changing growth patterns by state, to examine the effect on the company of growing in a particular area, in this case a high catastrophic-risk state.

The company would like to use a DFA model for capital allocation. The current model examines the riskiness of the company as a whole. It was suggested that separate runs could be performed for separate business segments (commercial/personal lines or by regions) in order to determine capital needs.

Another question related to the ability to plug in output from sophisticated fixed income security and catastrophe software into the DFA model. When Dynamo2 was originally designed, it was anticipated that many users would have access to different catastrophe models and might want to use those instead of the catastrophe module built in to this model. It is apparent from this question that similar issues relate to the investment modules.

Several questions related to the investment allocation. Currently the investment allocation applies to new money. If the cash flow requires assets to be sold, this is done proportionally. The investment managers would like to be able to reallocate the entire investment portfolio and indicate which assets should be sold, if necessary.

Another issue raised was the ability to focus on the difference between the expected values indicated by running the model and actual results. Managers wanted to be able to see why results differed from what was projected, so that they could better understand what they did right if a year was better than projected, or what went wrong if actual results were worse than expected. This DFA model allows this to occur, but requires the user to retain detailed output from the projections.

In examining the DFA runs, many questions were raised about what might have been causing adverse experience. It was suggested that the program be revised to capture detailed financial data on any simulation where surplus fell below a certain level. Thus, the managers could look at what caused the problems in order to better avoid them.

Applications

In addition to expressing the desire to use the DFA model for capital allocation purposes, the company also discussed the possibility of using the model to look at other companies. This might allow them to gain insights into their competitive position in the industry. The company also sees the model as a significant strategic planning tool -- for example, in evaluating how growth in one particular state affects the overall company. Another use was in reinsurance contract negotiations, where the expected effect of different limits or other contract terms could be evaluated. Finally, the CFO of the company expressed an interest in using the model, not only internally, but also in external communications. The investment community was specifically mentioned in this regard, but other possibilities also include regulators, rating agencies, and reinsurers.

Variable Adjustments

During the presentation, several different computers loaded with the DFA model were available, allowing the managers to break into groups and test different DFA scenarios. For example, one group of managers adjusted the interest rate parameters. Specifically, they raised the long-run mean interest rate level to 10 percent and reduced the volatility parameter to 0, to observe the effect of increasing interest rates for a small sample of runs. Other groups ran the model after adjusting one or more of exposures, losses, the reinsurance program, catastrophe parameters, exposure growth assumptions, and investment variables. In still other cases, certain stochastic variables were "shut off" -- e.g., by setting the volatility parameter of the variable equal to zero. This allowed the user the opportunity to see the impact of certain stochastic variables without introducing additional "noise" from those variables that were turned off.

In general, this exercise was seen as beneficial by all the groups, not just the actuaries. Having a viable DFA model will serve to help the different areas of the company work more closely together, and facilitate coordinating the efforts of the various areas.

Presentation to Upper Management

Members of the group raised several questions about how this model should be presented to the upper management of the company. In addition to needing to get comfortable with the model, they also wanted to be able to focus on how actual results differed from the projections. To do this, it was suggested that they might use the model to project results for last year (run the model without including data for the latest year and then compare the actual results with the output from the model). In addition, they wanted to print out key financial exhibits for the situations that were unacceptable, so that they could focus on what went wrong in those cases. This feature is available in the @Risk version of the model, but currently not in the Excel version.

Examining the effect of a company's use of a DFA model is a long term prospect. Modifications and enhancements to the model would be expected, as the company asks new questions after seeing initial indications. While it is too early to provide any information about the final effect of this process, the initial meeting and response suggest that the DFA model will provide a very useful management tool.

Future Enhancements

Enhancement of the public-access DFA model is an on-going process. Input and suggestions from users and other interested parties are welcomed and encouraged. The following items represent some of the enhancements to the model which are currently being considered.

- Determine the impact of callability provisions and other options embedded in insurer bond holdings. This will require identification of those bonds in the insurer's portfolio that have such options, information regarding when during the life of the bond the option is exercisable, and the call premium or other parameters associated with the embedded option. The valuation framework already incorporated within the DFA model -- i.e., market valuation of fixed-income securities based on the simulated term structure of interest rates -- will form the basis for the endogenous decision whether or not to exercise the option.
- Explicitly value mortgage-backed securities. These securities are comprising ever-larger proportions of insurer portfolios. In particular, for example, the prepayment risk associated with collateralized mortgage obligations will be simulated using the Public Securities Association (PSA) model of monthly prepayments on residential mortgages, with the parameters of the PSA model being impacted by simulated general economic conditions.
- Add state and/or regional detail in the underwriting module to facilitate measuring the effect of, for example, a change in the growth rate for a particular state.
- Continue to develop the underwriting cycle module and the associated demand curves, including their impact on business retention rates and jurisdictional risk.
- Implement correlations for the frequency and severity figures for business of different ages within a given line and between lines of business.
- Add tax-loss carry-forwards and carry-backs to the tax module.
- Add a module which produces risk-based capital results.

Conclusion

DFA is becoming an important concept for property-liability insurers, and it is likely that actuaries will be called upon to participate in, if not lead, this endeavor. This paper describes one DFA model. This model is publicly available and its use is encouraged, and comments on its effectiveness, limitations and potential improvements are actively solicited. While DFA for property-liability insurers is in a nascent stage, the intial reaction of company management to the application of this model to their operations was very favorable and provided evidence that DFA will prove valuable to the industry.

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Application of a Dynamic Financial Analysis Model to the Test Company: Report to Management

Introduction

The purpose of this report is to describe and explain a Dynamic Financial Analysis (DFA) model that represents a new management tool for insurance companies. The attached exhibits should be viewed as illustrative examples of output from running this model. These results are not a full blown dynamic financial analysis of the company, but represent a starting point for performing an analysis.

DFA, in essence, represents an enhanced approach to the traditional planning function undertaken by insurance companies. It provides a far more effective tool for forecasting future financial and operating conditions of an insurance company than prior methods for two primary reasons. First, the interactions between the underwriting and investment sides of the insurance business are formally integrated. Second, this approach utilizes advances in computer technology and modeling techniques to provide almost instantaneous feedback to decision makers, allowing for the evaluation of numerous operating alternatives.

The specific innovations to the planning process that are incorporated in DFA modeling are:

1) DFA provides a probability distribution of likely outcomes, rather than a single expected value forecast

2) DFA incorporates the correlations among lines of business, between loss reserve adequacy and rate adequacy, and between the investment and underwriting sides of insurance operations

3) by utilizing the technology of personal computers and common software, DFA models can be run by the users many times with different assumptions and different parameters, in order to see the effect that changes in the model or in operations can have on the results

Caveats

Although the output generated by a DFA model can look impressive, with detailed exhibits indicating the expected results for years into the future, and other exhibits indicating the probabilities of financial distress, the user must keep in mind that the output is only as good as the model and the underlying assumptions. DFA modeling has several specific limitations. First, models are simplified representations of reality. Models must be simplified in order to be useful; if all the factors that could possibly affect an insurer were included in a model, then it would just be too complex to be a useful model. When developing a model, the most relevant factors at that time are included. However, if conditions were to change markedly, which is entirely possible, then other factors that were omitted from the model could become important, affecting the accuracy of the results of the model. For example, during the 1920s, insurance profit margins were established that effectively ignored investment income. At this time interest rates were low (1-2%) and most business was in the short-tailed property lines. However, by the 1960s, interest rates were much higher and long-tailed lines accounted for almost 2/3rds of written premiums. Thus, it was no longer feasible to ignore the effect of interest rates on underwriting profit margins.

Second, some factors are important, but because they are beyond the scope of an actuarial analysis, they are omitted from the model. For example, fraud by managers is a leading cause of insurance insolvency. However, all insurers are not equally exposed to fraudulent behavior. Whether fraud is likely to occur (or is currently occurring) at a particular insurer, is not something an actuary is qualified to ascertain. Thus, any financial effects from fraudulent behavior are simply omitted from the model. Other examples of omitted factors that definitely could have a significant effect on insurance operations include a change in the tax code, repeal of the McCarran-Ferguson Act, a major shift in the application of a legal doctrine or the risk of a line of business being socialized by a state, province or federal government. Thus, the range of possible outcomes from operating an insurance company is actually greater than a DFA model would indicate; the model is designed to account only for risks that can be realistically quantified.

Finally, the values used as input in the model are derived from past experience and current operational plans. To the extent that something happens in the future that is completely out of line with past events, the model will be inaccurate. For example, the size of a specific catastrophe is based on a lognormal distribution with the parameter values based on experience over the period 1949-1995 (adjusted for inflation). However, if this process had been used just prior to 1992, the chance of two events occurring within the next 2 ½ years, both of which exceeded the largest previous loss by a factor of more than 2, would have been extremely small. However, Hurricane Andrew caused $15.5 billion in losses in August 1992 and the Northridge earthquake caused $12.5 billion in insured losses in January 1994. The largest insured loss prior to that was Hurricane Hugo, which had caused $4.2 billion in losses in 1989. Also, if changes in any operations occur, then the results would not be valid. Thus, the proper use of a DFA model is to continue to update the model as conditions or operations change.

With these caveats in mind, let's proceed to a description of the DFA model.

Dynamo2

The specific DFA model that is applied to the company's financial data is termed **Dynamo2**, which is a public access DFA model developed by the actuarial consulting firm Miller, Rapp, Herbers & Terry, Inc. This model is designed to be run on personal computers with Microsoft Excel and @Risk, two widely available software programs. The model operates by running a large number of iterations, with each iteration representing a single possible outcome. Each iteration, in turn, reflects the results of hundreds of different, but sometimes correlated, random factors that affect different parts of the insurer's operations. Selected values from each simulation are stored and used to calculate the mean and the distribution of the indicated results.

The model consists of several different modules, each of which calculates a component of the model indications. Separate modules are included for investments, catastrophes, underwriting, taxation, the interest rate generator and loss reserve development. The model allows for ten different lines of business:

- Homeowners
- Private Passenger Auto Liability
- Private Passenger Auto Physical Damages
- Commercial Auto Liability
- Commercial Auto Physical Damage
- Commercial Multi-Peril - Liability (which includes Professional Liability)
- Commercial Multi-Peril - Property (including Special Property)
- Other Liability
- Other Liability - Umbrella
- Workers Compensation

For each line of business, the underwriting gain or loss is calculated separately for: 1) new business, 2) 1st renewal business and 3) 2nd and subsequent renewals. This division is provided to reflect the aging phenomenon, in which loss experience improves with the length of time a policyholder has been with a company. These three categories are then added to calculate underwriting results on a direct, ceded and net basis.

The values for each simulation are shared among the different modules. Thus, if the random number generator produces a high value for the short term interest rate, this high interest rate is used in the investment module as well as the underwriting module. Similarly, a high value for catastrophes in the catastrophe module carries through to the reinsurance and underwriting modules.

The primary risks that are reflected in the model are:

1) Pricing risk
2) Loss reserve development risk
3) Catastrophe risk
4) Investment risk

Pricing risk is composed of a number of interrelated components. First, loss frequency and severity are both subject to random variation. Second, inflation affects loss severity. This effect is correlated with the short term interest rate, and is line of business specific. The indicated rate level change depends on the relationship between the current premiums and the premium indicated by inflationary impact on loss severity by line. However, jurisdictional risk (which is state specific) affects the ability of an insurer to make rate changes. Jurisdictional risk is reflected in both a range of allowable rate changes (lower increases would be allowed in jurisdictions with stringent regulation) and the time lag for incorporating new rates (it would take longer to raise rates in a state with restrictive regulation).

Finally, pricing risk is subject to the underwriting cycle. The underwriting cycle is simplified to be represented by four distinct phases: mature hard market (price increases can be taken with a minimal effect on market share), mature soft market (price increases significantly reduce market share), immature hard market (the market is starting to harden) and immature soft market (the market is beginning to soften). For each phase, the supply/demand function for insurance is different. Also, for each phase, there is a different probability distribution that represents the chance of remaining in that stage or of moving to another stage for the next year.

The loss reserves input into the model should be the reserves indicated based on an actuarial analysis of loss development, not necessarily the carried reserves. For this project, we relied on the reserve analysis performed by the company without independent audit, review or verification. Assuming the reserve levels are accurate, the expected reserve development would be zero. However, reserve development is still subject to random variation and to inflation. The indicated loss reserves contain an implied inflation factor. To the extent that inflation differs from this level, there will be a systematic effect on reserve development. Even if inflation were to occur at the expected level, then remaining random errors will affect the development.

Catastrophe risk is included in the model by the use of a two step approach. A poisson distribution is used to generate the number of catastrophes (of all types) that occur in a given year. Then, each catastrophe that occurs is assigned, based on historical patterns, to a specific geographical area (one state that is the primary focus of the loss). Next, the size of each catastrophe is determined based on a lognormal distribution, with the parameters determined based on the primary state in which the loss occurs. Finally, the contagion effect of the loss on other states, again based on historical patterns, is determined so that the total catastrophe loss for the year in each state can be determined. The amount of each loss that is ceded is determined based on the company's catastrophe insurance program, which allows calculation of the direct, ceded and net experience.

The investment risk reflects the combined effect of bonds and stocks. Statutory bond values are determined based on the interest rates in effect when the bond was purchased and the amortization schedule, plus defaults that occur randomly based on historical patterns. Market values of bonds are a function of the current interest rates as simulated. Stock market values are based on the starting values and the randomly generated rates of return. Equity returns are based on simulated changes in interest rates, and include significant random variation, with the parameters determined based on historical rates of return.

Model Input

The model requires extensive financial data as input. Some of the historical data required for input can be obtained from the Annual Statement, but in other cases direct, rather than net, data are preferable, which must be drawn from additional reports. In this case, the input was provided by the company, including reports on direct and net premiums, exposures by line and by age of business, and premium level, loss frequency, loss severity, market share and renewal rates by line. In addition, planned growth by line of business and the user's perception of the phase of the underwriting cycle by line is input. From the Annual Statement the input values include the statutory value of assets and liabilities and the current investment allocations. The expense provisions were taken from the Insurance Expense Exhibit. Loss development was developed based on direct triangles provided by the company. The company also provided a detailed listing of reinsurance contracts and the beta for equities.

Attached are copies of the data input for this program for the company as a whole and for the Homeowners line of business. This line of business data illustrates the by line information required to run this model. These exhibits include:

- **General Input** - selections for the current market conditions by line
- **Loss Triangle Input** - historical direct paid loss development by line
- **Underwriting Module Input** - new and renewal exposures written and premium levels for the last two years, projected growth rates for the next five years, renewal ratios by age of business and expense factors, all by line of business

- **Exposure Distribution** - current number of exposures written by state, by line and historic exposures written by line

- **Market Share** - market share estimates for property coverage (for catastrophe losses)

- **Loss Development Factor Selection** - the selected paid loss development factors based on the historic loss development patterns (used to generate cash flows)

- **Loss Information Input** - selected ultimate losses and allocated loss adjustment expenses and claim counts, direct and net paid losses and earned premium, loss frequencies and severities (in total and by age of business), unallocated loss adjustment expense factors, and reinsurance treaties, all by line of business

- **Investment Input** - statutory and market values of assets by annual statement category, coupon and dividend rates and equity betas

Model Output

The ability to generate an almost infinite number of reports from a DFA model is both a strength and a weakness of this approach. Care has to be taken to assure that the user is not overwhelmed with information and, therefore, unable to utilize the results of the model in any reasonable manner. Thus, the initial report focuses on a limited number of key variables for an insurer, and indicates the expected values as well as the distribution of outcomes from the model. Also, examples of more detailed reports for a few selected outcomes are shown to illustrate the potential of a DFA model to troubleshoot particular problems that contributed to adverse financial results.

The true benefit of a DFA model is the ability it gives to the decision makers in an insurance company to test out various financial and operating strategies and see what the indicated effect is on both expected returns and the distribution of results. Unlike the planning process that has previously been used by many insurers, which tended to be done annually or on some other regular schedule, a DFA model can be a regular management tool that can be rerun whenever a major decision needs to be made. Thus, the goal of our first meeting will be to demonstrate the use of this DFA model so that management can decide what values to change.

The output from the DFA model based on the initial input values (as shown on the input exhibits) for a run with 50 iterations using the Excel option are shown in the exhibits marketed **Base Case**. The results for each simulation, and the average values, are shown for statutory surplus, the premium to surplus ratio, the operating ratio and the net loss ratio for all lines combined for each year 1998-2002. In this run, the average value of the surplus over all 50 iterations was $177 million for 1998, $173 million for 1999, $167 million for 2000, $150 million for 2001 and $133 million for 2002. Since the simulation included 50 iterations, it is difficult to draw conclusions from the individual results. The distribution of these results for surplus, premium to surplus ratio, operating ratio and loss ratio for the year 2002 are shown in the graphs. These illustrate the distribution of outcomes to allow the user to determine the likelihood of specific outcomes, either bad (surplus below a minimum level, premium to surplus ratio over an acceptable target, etc.) or favorable (operating ratio below a target level).

In addition, detailed data can be analyzed for selected outcomes. For example, the statutory balance sheet, the IRIS test results and the loss ratios on a direct, ceded and net basis by age of business are shown for an example of a single iteration. If desired, even more detailed data (frequency and severity, interest rate level, number, size and distribution of catastrophes, etc.) can be examined. This allows the user to troubleshoot the unfavorable outcomes to determine what strategies would work best to reduce the likelihood of their occurrence.

It is obvious from looking at the average values and the distributions from this initial run that the results are very unfavorable. The statutory surplus declines, on average, and the premium to surplus ratios increase to unacceptable levels. Loss ratios, especially in the latter years of the forecast period, increase to over 75 percent. These indications, while causing concern, are actually exactly what is needed to illustrate the potential benefits of a DFA model. Since the forecasted values are unacceptable, then changes should be made to generate more favorable indications. What changes should be made are up to management, and DFA is the tool to help management access the effect of particular changes.

For example, one cause of the increase in loss ratios is the amount of new business that is written to meet the growth rates initially input into the model. This growth, coupled with relatively low retention rates, requires the company to write a large amount of new business each year, with its corresponding high loss ratios. The **Base Case** model projects exposure growth of 5-10% for all lines of business for the years 2000-2002. This compares with a negative growth forecast for 1998 and low growth, 1-3.5%, for 1999. In this example, detailed loss and exposure results are shown for new Homeowners business so that the effect of rapid growth in exposures can be examined. In an effort to grow at a 10% rate, the number of new Homeowners exposures in 2002 is 16,119. (See the exhibit on New Business for Homeowners) Since the loss ratio on this new business is expected to be 26 percentage points higher than long term business (see last line on this sheet), this high growth imposes a significant penalty on the company.

The effect of reducing these growth rates can be seen in the exhibits marked **Constrained Growth**. The only difference between the initial run and this run is that the growth rates were held to a maximum of 2 percent per year. The indications are much more favorable in this situation. In this case the average values of surplus are $176 million, $177 million, $183 million, $192 million and $203 million, for 1998-2002 respectively. Although the distributions illustrated on the graphs for 2002 still show unacceptable results in some situations, the average values are much more feasible than in the **Base Case**. The effect of constraining the growth can be seen on the New Business for Homeowners exhibit. In this case, the number of new exposures is only 7,177, compared to 16,119 at the 10 percent growth rate.

The output illustrated in the two cases discussed above was based on runs of 50 iterations each using the Excel option. The model also can be run using @Risk, which provides significant additional capabilities. The **Base Case** model was also run using @Risk with 1000 iterations. The numerical values of statutory surplus, displayed both in percentiles and graphically for 1998-2002, are shown as additional exhibits.

What other changes could or should be made? Such items as policy renewal rates, expense provisions, the rate at which premium is earned (which reflects policy term), exposure distribution by state, projected average frequencies and severities by age of business, reinsurance provisions (including attachment points, costs and ceding commissions) and investment provisions (including allocation of new investments, stock betas and surplus additions) can all be easily manipulated and evaluated by the use of this DFA model.

The primary point of this report is that DFA is a management tool. The decision makers in the company should take the initiative in proposing changes and analyzing the effects. The goal of the meeting with the company is to explain and demonstrate the DFA model so that managers can effectively use this tool. Much of the meeting will be devoted to hands-on work with the model so you can evaluate its effectiveness and we can see what works for you and in what ways the model needs to be improved to facilitate its use as a management planning tool.

Index of Exhibits

Exhibit A-1

Company Name: ABC Insurance Company

First Year to be Modeled: 1998

Current Market Conditions:

HMP	Mature Soft
PPAL	Mature Soft
APD-P	Mature Soft
APD-C	Mature Soft
CAL	Mature Soft
CMP-L	Mature Soft
CMP-P	Mature Soft
OL	Mature Soft
OL-U	Mature Soft
WC	Mature Hard

Simulation Technique

○ @Risk ◉ Excel

☐ Cat Module De-activated

General Input

Loss Triangle Input
Paid Losses & ALAE Direct & Assumed

Line of Business: HMP

Accident Year	Evaluations in Months 12	24	36	48	60	72	84	96	108	120	132
1986	-	7,390,982	7,667,373	7,831,090	7,834,571	7,840,897	7,841,882	7,841,882	7,843,008	7,843,296	7,843,296
1987	4,782,601	5,948,892	6,074,429	6,200,184	6,503,498	6,210,370	6,210,489	6,211,047	6,212,269	6,212,269	6,212,269
1988	3,429,881	4,540,502	4,682,931	4,776,067	4,775,599	4,777,092	4,776,204	4,775,904	4,775,654	4,775,304	-
1989	4,428,674	6,216,163	6,302,820	6,338,508	6,320,451	6,319,874	6,320,461	6,278,231	6,278,447	-	-
1990	4,905,508	6,491,617	6,672,882	7,304,431	7,341,614	7,371,753	7,401,759	7,433,900	-	-	-
1991	6,136,783	8,546,891	8,735,593	8,828,725	8,868,053	8,875,065	8,875,733	-	-	-	-
1992	6,623,741	9,339,087	9,578,819	9,803,573	9,825,756	9,821,798	-	-	-	-	-
1993	9,318,694	12,752,572	13,100,827	13,345,650	13,355,820	-	-	-	-	-	-
1994	9,675,280	12,400,427	12,631,087	12,720,083	-	-	-	-	-	-	-
1995	10,819,650	15,166,286	15,813,794	-	-	-	-	-	-	-	-
1996	14,372,636	17,806,453	-	-	-	-	-	-	-	-	-
1997	19,593,642	-	-	-	-	-	-	-	-	-	-

Underwriting Module Input Page

Exhibit A-3

Homeowners Multiple Peril

	2nd Prior Year 1996	1st Prior Year 1997	1st Year 1998	2nd Year 1999	3rd Year 2000	4th Year 2001	5th Year 2002
Premiums Input							
1. Written Exposure Input							
a. New Business	10,740	9,569					
b. 1st Renewal	9,095	9,591					
c. 2nd & Subsequent Renewal	37,541	42,166					
d. Total	57,376	61,326					
2. Average Annual Rate Input							
a. New Business	388	377					
b. 1st Renewal	432	421					
c. 2nd & Subsequent Renewal	432	421					
3. Exposure Growth Rate							
a. Enter Growth Objectives			-1.0%	2.0%	7.5%	10.0%	10 0%
4. % of Premiums Earned in Year Written							
a. New Business	50%	50%	50%	50%	50%	50%	50%
b. 1st Renewal	50%	50%	50%	50%	50%	50%	50%
c. 2nd & Subsequent Renewal	50%	50%	50%	50%	50%	50%	50%
5. Renewal Ratio							
a. New Business	60%	60%	60%	60%	60%	60%	60%
b. 1st Renewal	90%	90%	90%	90%	90%	90%	90%
c. 2nd & Subsequent Renewal	95%	95%	95%	95%	95%	95%	95%
6. % of Written Premiums Held By Agents	13%	13%	13%	13%	13%	13%	13%
Expense Input							
1. Commissions (a. ● % of Written Premium; b. ○ % of Earned Premium) 1	14.1%	13.5%	14.0%	14.0%	14.0%	14.0%	14.0%
2. General Expense (a. ○ % of Written Premium; b. ● % of Earned Premium) 2	6.5%	6.3%	6.5%	6.5%	6.5%	6.5%	6.5%
3. Other Acquisition (a. ○ % of Written Premium; b. ● % of Earned Premium) 2	12.6%	11.8%	11.8%	11.8%	11.8%	11.8%	11.8%
4. Premium Taxes							
a. % of Written Premium	3.2%	3.3%	3.4%	3.4%	3.4%	3.4%	3.4%
5. Policyholder Dividends							
a. % of Earned Premium	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
6. Other Nonrecurring Expenses	-	-	931,848	-	-	-	-
7. Ceding Commission							
a. % of Earned Premium	0%	0%	0%	0%	0%	0%	0%

Premium Input

Exhibit A-4

Exposure Input

1. Enter Your Distribution By State by Line:

State	HMP
AK	-
AL	-
AR	128
AZ	67
CA	529
CO	802
CT	2,492
DC	-
DE	-
FL	60
GA	-
HI	-
IA	511
ID	15
IL	1,886
IN	1,625
KS	390
KY	1,100
LA	2,436
MA	-
MD	-
ME	-
MI	-
MN	409
MO	1,866
MS	1,059
MT	-
NC	4
ND	-
NE	68
NH	-
NJ	-
NM	-
NV	-
NY	1,536
OH	-
OK	2,868
OR	251
PA	38
RI	-
SC	8
SD	-
TN	531
TX	3,980
UT	-
VA	-
VT	-
WA	416
WI	204
WV	-
WY	-
CW	25,279

2. Enter Historic Written Exposures By Line

Year	HMP
1985	-
1986	-
1987	-
1988	-
1989	-
1990	30,915
1991	34,730
1992	39,599
1993	44,257
1994	49,513
1995	53,563
1996	57,376
1997	61,326

3. Enter Your Market Share By State by Line:

State	HMP
AK	0.000%
AL	0.000%
AR	0.054%
AZ	0.016%
CA	0.017%
CO	0.165%
CT	0.521%
DC	0.000%
DE	0.000%
FL	0.003%
GA	0.000%
HI	0.000%
IA	0.240%
ID	0.016%
IL	0.172%
IN	0.294%
KS	0.122%
KY	0.375%
LA	0.341%
MA	0.000%
MD	0.000%
ME	0.000%
MI	0.000%
MN	0.089%
MO	0.354%
MS	0.406%
MT	0.000%
NC	0.001%
ND	0.000%
NE	0.044%
NH	0.000%
NJ	0.000%
NM	0.000%
NV	0.000%
NY	0.170%
OH	0.000%
OK	0.721%
OR	0.094%
PA	0.003%
RI	0.000%
SC	0.002%
SD	0.000%
TN	0.107%
TX	0.180%
UT	0.000%
VA	0.000%
VT	0.000%
WA	0.077%
WI	0.055%
WV	0.000%
WY	0.000%

Exposure Input

Exposure Input

Jurisdictional Risk Worksheet

Exhibit A-5

State	HMP Low	HMP Hi	HMP Lag
AK	0.85	1.10	0.25
AL	0.85	1.10	0.25
AR	0.85	1.10	0.25
AZ	0.85	1.10	0.25
CA	0.85	1.06	0.50
CO	0.85	1.10	0.25
CT	0.85	1.10	0.25
DC	0.85	1.10	0.50
DE	0.85	1.10	0.25
FL	0.85	1.05	0.50
GA	0.85	1.099	0.50
HI	0.85	1.10	0.25
IA	0.75	1.20	-
ID	0.75	1.20	-
IL	0.75	1.20	-
IN	0.75	1.20	-
KS	0.85	1.10	0.50
KY	0.85	1.10	0.25
LA	0.85	1.06	0.50
MA	0.85	1.06	0.50
MD	0.85	1.10	0.25
ME	0.85	1.10	0.25
MI	0.85	1.06	0.50
MN	0.85	1.10	0.25
MO	0.85	1.10	0.25
MS	0.85	1.10	0.25
MT	0.75	1.20	-
NC	0.85	1.10	0.50
ND	0.75	1.20	-
NE	0.85	1.10	0.25
NH	0.85	1.10	0.25
NJ	0.85	1.06	0.50
NM	0.85	1.10	0.25
NV	0.85	1.10	0.25
NY	0.85	1.06	0.50
OH	0.85	1.08	0.25
OK	0.85	1.10	0.25
OR	0.85	1.10	0.25
PA	0.85	1.08	0.50
RI	0.85	1.10	0.50
SC	0.85	1.06	0.50
SD	0.85	1.10	0.25
TN	0.85	1.10	0.25
TX	0.75	1.20	0.50
UT	0.75	1.20	-
VA	0.85	1.10	0.25
VT	0.85	1.10	0.25
WA	0.85	1.10	0.50
WI	0.75	1.20	-
WV	0.85	1.10	0.25
WY	0.75	1.20	-
CW	0.82	1.13	0.30

Exposure Input

Loss Development Factor Selection
Homeowners

Exhibit A-6

Paid Losses and ALAE

Accident Year	Evaluations in Months 12	24	36	48	60	72	84	96	108	120	132
1986		7,390,982	7,667,373	7,831,090	7,834,571	7,840,897	7,841,882	7,841,882	7,843,008	7,843,296	7,843,296
1987	4,782,601	5,948,892	6,074,429	6,200,184	6,503,498	6,210,370	6,210,489	6,211,047	6,212,269	6,212,269	6,212,269
1988	3,429,881	4,540,502	4,682,931	4,776,067	4,775,599	4,777,092	4,776,204	4,775,904	4,775,654	4,775,304	
1989	4,428,674	6,216,163	6,302,820	6,338,508	6,320,451	6,319,874	6,320,461	6,278,231	6,278,447		
1990	4,905,508	6,491,617	6,672,882	7,304,431	7,341,614	7,371,753	7,401,759	7,433,900			
1991	6,136,783	8,546,891	8,735,593	8,828,725	8,868,053	8,875,065	8,875,733				
1992	6,623,741	9,339,087	9,578,819	9,803,573	9,825,756	9,821,798					
1993	9,318,694	12,752,572	13,100,827	13,345,650	13,355,820						
1994	9,675,280	12,400,427	12,631,087	12,720,083							
1995	10,819,650	15,166,286	15,813,794								
1996	14,372,636	17,806,453									
1997	19,593,642										

Report to Report Factors

Accident Year	12 - 24	24 - 36	36 - 48	48 - 60	60 - 72	72 - 84	84 - 96	96 - 108	108 - 120	120 - 132	132 - Ult.
1986		1.037	1.021	1.000	1.001	1.000	1.000	1.000	1.000	1.000	
1987	1.244	1.021	1.021	1.049	0.955	1.000	1.000	1.000	1.000	1.000	
1988	1.324	1.031	1.020	1.000	1.000	1.000	1.000	1.000	1.000		
1989	1.404	1.014	1.006	0.997	1.000	1.000	0.993	1.000			
1990	1.323	1.028	1.095	1.005	1.004	1.004	1.004				
1991	1.393	1.022	1.011	1.004	1.001	1.000					
1992	1.410	1.026	1.023	1.002	1.000						
1993	1.368	1.027	1.019	1.001							
1994	1.282	1.019	1.007								
1995	1.402	1.043									
1996	1.239										
1997											
3 Yr Simple Average	1.307	1.030	1.016	1.002	1.001	1.001	0.999	1.000	1.000	0.667	
5 Yr Simple Average	1.340	1.027	1.031	1.002	1.001	1.001	1.000	1.000	1.000	0.667	
Selected LDF's:	1.350	1.030	1.016	1.002	1.001	1.000	1.000	1.000	1.000	1.000	1.000
Cumulative LDF's:	1.417	1.050	1.019	1.003	1.001	1.000	1.000	1.000	1.000	1.000	1.000
Expected % Paid:	70.6%	95.3%	98.1%	99.7%	99.9%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Incremental % Paid:	70.6%	24.7%	2.9%	1.6%	0.2%	0.1%	0.0%	0.0%	0.0%	0.0%	0.0%

Loss Information Input

Exhibit A-7-a

1. Selected Ultimate Losses & ALAE For Prior Years

Year	HMP
1985	7,489,604
1986	7,843,366
1987	6,212,235
1988	4,775,322
1989	6,279,003
1990	7,434,451
1991	8,880,880
1992	9,824,747
1993	13,356,276
1994	12,830,617
1995	16,136,340
1996	18,397,611
1997	24,422,398

2. Selected Ultimate Counts For Prior Years (Direct &

Year	HMP
1985	6,637
1986	6,331
1987	4,741
1988	3,353
1989	3,526
1990	4,427
1991	4,707
1992	5,281
1993	5,860
1994	6,336
1995	6,642
1996	8,956
1997	10,097

3. Direct Paid Loss and ALAE

Year	HMP
1985	
1986	7,843,296
1987	6,212,269
1988	4,775,304
1989	6,278,447
1990	7,433,900
1991	8,875,733
1992	9,821,798
1993	13,355,820
1994	12,720,083
1995	15,813,794
1996	17,806,453
1997	19,593,642

4. Net Ultimate Losses & ALAE

Year	HMP
1985	
1986	
1987	
1988	5,547,149
1989	7,171,961
1990	8,031,336
1991	10,616,599
1992	10,849,921
1993	12,916,739
1994	13,946,706
1995	17,177,712
1996	17,442,599
1997	23,968,647

5. Net Paid Losses & ALAE

Year	HMP
1985	
1986	
1987	
1988	5,547,212
1989	7,171,995
1990	8,030,936
1991	10,613,127
1992	10,849,326
1993	12,916,608
1994	13,832,267
1995	16,857,224
1996	16,785,003
1997	19,179,809

6. Earned Premiums (Direct & Assumed)

Year	HMP
1985	9,800,926
1986	926,774
1987	9,471,742
1988	8,326,726
1989	8,342,498
1990	9,345,453
1991	10,922,208
1992	13,022,473
1993	15,227,428
1994	17,168,201
1995	19,117,232
1996	22,694,060
1997	24,754,064

7. Net Earned Premium

Year	HMP
1985	
1986	
1987	
1988	8,950,000
1989	8,664,000
1990	10,428,000
1991	12,229,000
1992	14,007,000
1993	15,609,000
1994	17,027,350
1995	17,858,925
1996	19,982,000
1997	21,976,961

8. Selected Severities (Direct & Assumed)

Year	HMP
1985	1,870
1986	1,969
1987	1,996
1988	2,080
1989	2,494
1990	2,255
1991	2,429
1992	2,296
1993	2,697
1994	2,298
1995	2,643
1996	2,143
1997	2,419

Selected Severities

a. New Business	2,000
b. 1st Renewal	2,000
c. 2nd & Subsequent Renewal	2,000
d. Weighted Average Severity	2,000
e. Standard Deviation of Historic Severities	192
	2,416
	2,339

9. Selected Frequency

Year	HMP
1985	
1986	
1987	
1988	
1989	
1990	0.143
1991	0.136
1992	0.133
1993	0.132
1994	0.128
1995	0.124
1996	0.156
1997	0.165

Selected Frequencies

a. New Business	0.157
b. 1st Renewal	0.143
c. 2nd & Subsequent Renewal	0.130
d. Weighted Average Severity	0.136
e. Standard Deviation of Historic Frequencies	0.01
	0.140

Reasonability Check

a. Loss Cost = Severity x Frequency	
New Business	315
1st Renewal	286
2nd & Subsequent Renewal	260
Total	273
b. Average Premium (From Premium Input Sheet)	
New Business	377
1st Renewal	421
2nd & Subsequent Renewal	421
Total	414
c. Implied Loss Ratios = Loss Cost / Average Pre	
New Business	0.834
1st Renewal	0.679
2nd & Subsequent Renewal	0.618
Total	0.658
	0.704

10. Paid ULAE as a % of Paid Losses & ALAE

Calendar Year	HMP
1997	7.0%
1998	7.0%
1999	7.0%
2000	7.0%
2001	7.0%
2002	7.0%
2002 and Subsequent	7.0%

11. Quota Share Reinsurance Treaties

Year	HMP
1985	
1986	
1987	
1988	
1989	
1990	
1991	
1992	
1993	
1994	
1995	
1996	
1997	
1998	
1999	
2000	
2001	
2002	

Loss Input

Exhibit A-7-b

12. Excess of Loss Treaties

Year	HMP
Estimated Cost of XOL as a % of Earned Premium	0.041
1st Retention per Occurrence	
1996	500,000
1997	500,000
1998	500,000
1999	500,000
2000	500,000
2001	500,000
2002	500,000
Max Coverage From Reinsurers	
1996	26,000,000
1997	26,000,000
1998	26,000,000
1999	26,000,000
2000	26,000,000
2001	26,000,000
2002	26,000,000

Coefficient of Variation	3

Mean	HMP
1996	2,563
1997	3,125
1998	2,098
1999	2,357
2000	2,575
2001	2,567
2002	2,706

Standard Deviation	HMP
1996	7,688
1997	9,375
1998	6,294
1999	7,072
2000	7,726
2001	7,700
2002	8,119

Mu	HMP
1996	6.70
1997	6.90
1998	6.50
1999	6.61
2000	6.70
2001	6.70
2002	6.75

Sigma	HMP
1996	1.52
1997	1.52
1998	1.52
1999	1.52
2000	1.52
2001	1.52
2002	1.52

Excess Percentage	HMP
1996	0.0011%
1997	0.0020%
1998	0.0006%
1999	0.0009%
2000	0.0012%
2001	0.0012%
2002	0.0013%

13. Stop Loss Reinsurance	Cost as a % of Premium	Max. Loss & ALAE Ratio	Max. Amount Ceded
1998	1.00%	77.50%	10,000,000
1999	1.00%	77.50%	10,000,000
2000	1.00%	77.50%	10,000,000
2001	1.00%	77.50%	10,000,000
2002	1.00%	77.50%	10,000,000

14. Catastrophic Reinsurance	Cost as a % of Premium	1st Retention Per Occ.	Max. Amount Per Occ.
1st Retention Per Occurrence			
1998	5.00%	5,000,000	125,000,000
1999	5.00%	5,000,000	125,000,000
2000	5.00%	5,000,000	125,000,000
2001	5.00%	5,000,000	125,000,000
2002	5.00%	5,000,000	125,000,000

Loss Input

Investments Input

Exhibit A-8-a

1. Statutory Values as of 12/31/1997:

	Total	Bond Maturity: 1 Year or Less	1 - 5 Years	6 - 10 Years	10 - 20 Years	20 + Years
a. U.S. Government Bonds	91,134,188	5,530,471	35,797,026	37,150,075	6,026,081	6,630,535
b. Bonds Exempt From U.S. Tax	97,647,732	1,119,454	5,731,002	16,196,148	72,240,409	2,360,719
c. Other Bonds (Unaffiliated)	184,436,496	13,168,046	45,292,782	88,371,852	23,668,997	13,934,819
d. Bonds (Affiliated)	-	-	-	-	-	-
e. Preferred Stocks (Unaffiliated)	12,222,841					
f. Preferred Stocks (Affiliated)						
g. Common Stocks (Unaffiliated)	19,967,926					
h. Common Stocks (Affiliated)	72,455,000					
i. Mortgage Loans	196,144					
j. Real Estate	16,880,795					
k. Collateral Loans						
l. Cash on hand and on Deposit	30,951,773					
m. Short Term Investments						
n. Other Invested Assets	446,683					
o. Derivative Instruments						
p. Aggregate Write-Ins						
q. Subtotal	526,339,678					

2. Market Values as of 12/31/1997:
Please Enter Par Values for Bonds

	Total	Bond Maturity: 1 Year or Less	1 - 5 Years	6 - 10 Years	10 - 20 Years	20 + Years
a. U.S. Government Bonds	91,455,926	5,536,473	35,942,332	37,264,001	6,025,474	6,687,646
b. Bonds Exempt From U.S. Tax	98,063,451	1,122,383	5,747,795	16,218,170	72,602,239	2,372,864
c. Other Bonds (Unaffiliated)	184,723,128	13,125,017	45,372,699	88,641,989	23,701,093	13,882,330
d. Bonds (Affiliated)	-	-	-	-	-	-
e. Preferred Stocks (Unaffiliated)	12,325,625					
f. Preferred Stocks (Affiliated)						
g. Common Stocks (Unaffiliated)	19,967,926					
h. Common Stocks (Affiliated)	60,732,786					
i. Mortgage Loans	196,144					
j. Real Estate	16,880,795					
k. Collateral Loans						
l. Cash on hand and on Deposit	30,951,773					
m. Short Term Investments						
n. Other Invested Assets	446,683					
o. Derivative Instruments						
p. Aggregate Write-Ins						
q. Subtotal	515,744,247					

2. Number of Units as of 12/31/1997

	Total	Bond Maturity: 1 Year or Less	1 - 5 Years	6 - 10 Years	10 - 20 Years	20 + Years
a. U.S. Government Bonds	214	14	60	100	20	20
b. Bonds Exempt From U.S. Tax	132	10	20	24	69	9
c. Other Bonds (Unaffiliated)	171	7	47	66	27	24
d. Bonds (Affiliated)	-	-	-	-	-	-
e. Preferred Stocks (Unaffiliated)	440,000					
f. Preferred Stocks (Affiliated)	-					
g. Common Stocks (Unaffiliated)	920,987					
h. Common Stocks (Affiliated)	832,000					
i. Mortgage Loans	1					
j. Real Estate	7					
k. Collateral Loans	-					
l. Cash on hand and on Deposit	1					
m. Short Term Investments	-					
n. Other Invested Assets	2					
o. Derivative Instruments	-					
p. Aggregate Write-Ins	-					
q. Subtotal	2,193,515					

3. Bond Coupon Rates:

	Total	Bond Maturity: 1 Year or Less	1 - 5 Years	6 - 10 Years	10 - 20 Years	20 + Years
a. U.S. Government Bonds	7.495%	6.913%	7.160%	7.315%	9.000%	9.435%
b. Bonds Exempt From U.S. Tax	6.735%	5.750%	6.831%	6.497%	6.773%	7.425%
c. Other Bonds (Unaffiliated)	7.742%	7.735%	7.878%	7.317%	8.652%	8.452%
d. Bonds (Affiliated)	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
e. Subtotal	7.418%	7.394%	7.513%	7.223%	7.341%	8.631%

Investment Input

Exhibit A-8-b

4. Capital & Surplus

	As of Year End					
	1997	1998	1999	2000	2001	2002
a. Surplus as Regards to Policyholders		[redacted]				
b. Contributed Surplus						
c. Unassigned Surplus						
d. Special Surplus Funds						
e. Additions to Capital						
f. Contributions to Surplus						

5. Stock Betas

	1997	1998	1999	2000	2001	2002
a. Preferred Stocks (Unaffiliated)	-	-	-	-	-	-
b. Preferred Stocks (Affiliated)	-	-	-	-	-	-
c. Common Stocks (Unaffiliated)	0.70	0.70	0.70	0.70	0.70	0.70
d. Common Stocks (Affiliated)	-	-	-	-	-	-

6. Dividends as a % of Market Value

	1997	1998	1999	2000	2001	2002
a. Preferred Stocks (Unaffiliated)	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%
b. Preferred Stocks (Affiliated)	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
c. Common Stocks (Unaffiliated)	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%
d. Common Stocks (Affiliated)	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

7. Reinvestment Allocations

	1998	1999	2000	2001	2002
a. U.S. Government Bonds	20.9%	20.9%	20.9%	20.9%	20.9%
b. Bonds Exempt From U.S. Tax	22.4%	22.4%	22.4%	22.4%	22.4%
c. Other Bonds (Unaffiliated)	42.3%	42.3%	42.3%	42.3%	42.3%
d. Bonds (Affiliated)	0.0%	0.0%	0.0%	0.0%	0.0%
e. Preferred Stocks (Unaffiliated)	2.8%	2.8%	2.8%	2.8%	2.8%
f. Preferred Stocks (Affiliated)	0.0%	0.0%	0.0%	0.0%	0.0%
g. Common Stocks (Unaffiliated)	4.6%	4.6%	4.6%	4.6%	4.6%
h. Common Stocks (Affiliated)	0.0%	0.0%	0.0%	0.0%	0.0%
i. Mortgage Loans	0.0%	0.0%	0.0%	0.0%	0.0%
j. Real Estate	0.0%	0.0%	0.0%	0.0%	0.0%
k. Collateral Loans	0.0%	0.0%	0.0%	0.0%	0.0%
l. Cash on hand and on Deposit	7.1%	7.1%	7.1%	7.1%	7.1%
m. Short Term Investments	0.0%	0.0%	0.0%	0.0%	0.0%
n. Other Invested Assets	0.0%	0.0%	0.0%	0.0%	0.0%
o. Derivative Instruments	0.0%	0.0%	0.0%	0.0%	0.0%
p. Aggregate Write-Ins	0.0%	0.0%	0.0%	0.0%	0.0%
q. Total	100.0%	100.0%	100.0%	100.0%	100.0%

Base Case
50 Iterations Using Excel

Exhibit B-1-a

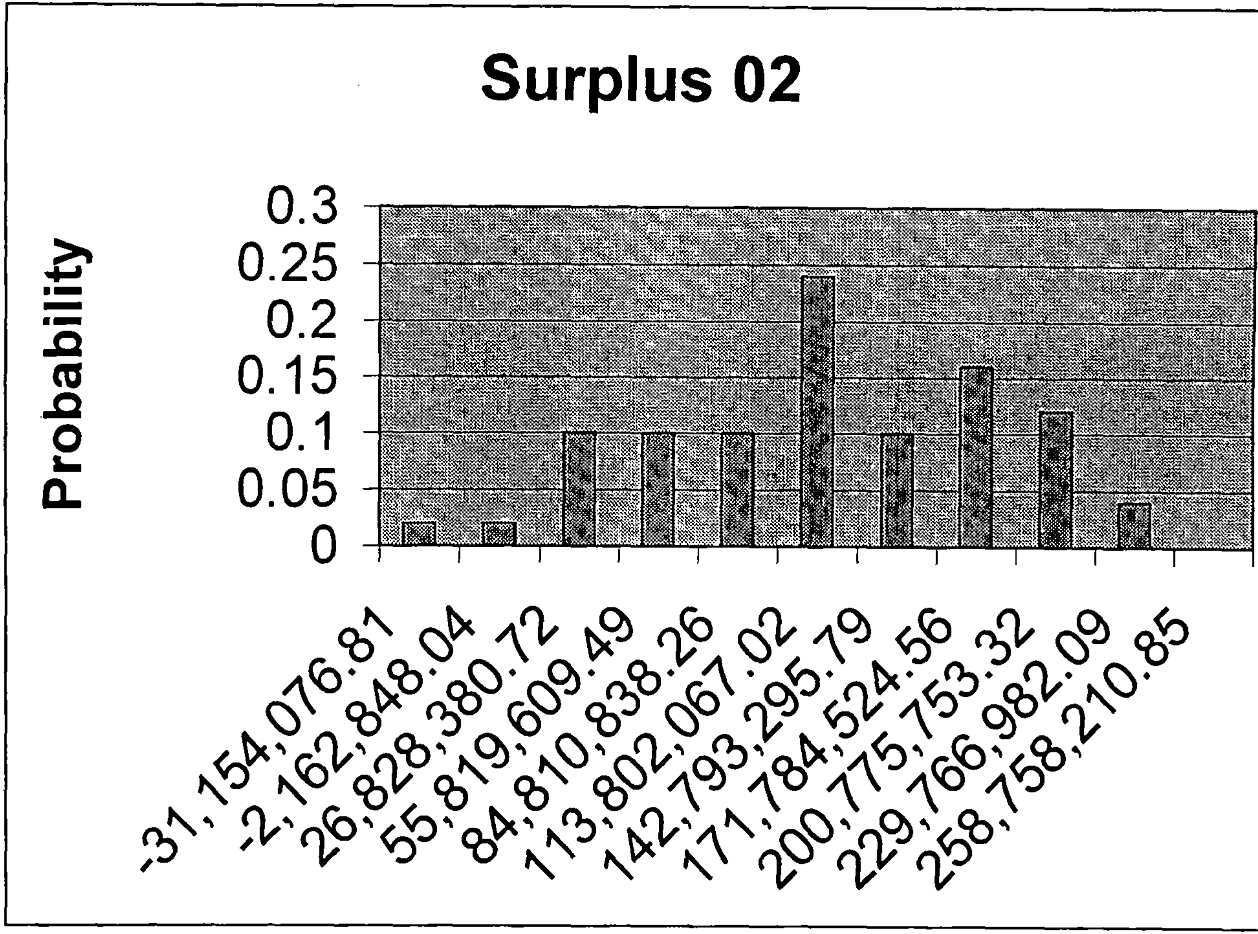
	Output	Output	Output	Output	Output	Output	Output	Output	Output	Output
	N19	o19	p19	q19	r19	n34	o34	p34	q34	r34
	No	No	No	No	Yes	No	No	No	No	Yes
	176,885,913	172,931,113	166,756,714	149,824,294	132,577,715	2.119	2.381	2.856	2.931	5.405
rial #	Surplus 98	Surplus 99	Surplus 00	Surplus 01	Surplus 02	P/S 98	P/S 99	P/S 00	P/S 01	P/S 02
1	155,056,836	136,285,644	125,884,389	104,845,572	98,080,612	2.440	3.080	3.882	5.383	6.504
2	163,754,761	164,156,925	144,163,294	132,106,526	114,403,000	2.307	2.527	3.275	4.008	5.245
3	151,004,758	143,401,131	123,512,074	82,674,485	43,919,839	2.506	2.900	3.894	6.710	14.021
4	158,066,626	150,059,705	130,380,788	87,954,798	31,342,920	2.387	2.706	3.375	5.500	17.071
5	180,817,284	160,571,798	116,639,827	127,270,793	106,137,941	2.049	2.541	4.047	4.279	5.962
6	176,983,630	182,229,470	190,321,674	198,105,573	226,929,915	2.074	2.159	2.342	2.596	2.549
7	185,484,837	195,865,894	196,972,524	170,517,586	137,203,363	2.019	2.080	2.247	2.852	4.001
8	187,493,323	187,965,994	210,107,085	227,230,199	226,201,991	1.990	2.168	2.190	2.268	2.584
9	190,786,885	194,824,479	209,767,197	196,742,871	174,560,266	1.964	2.097	2.201	2.647	3.322
10	181,063,632	179,046,371	188,441,033	191,629,135	163,677,384	2.085	2.292	2.432	2.732	3.634
11	189,282,966	187,621,214	185,556,317	192,830,503	144,997,562	1.940	2.106	2.416	2.658	4.010
12	179,097,951	162,374,018	155,255,449	140,497,037	107,179,405	2.083	2.508	2.949	3.672	5.599
13	206,334,170	194,066,840	197,795,911	185,398,822	198,001,975	1.809	2.102	2.341	2.838	3.023
14	192,256,361	206,374,790	225,257,082	244,173,354	258,758,211	1.914	1.891	1.935	2.055	2.198
15	158,996,375	109,413,710	76,951,202	83,456,665	30,107,692	2.323	3.612	5.742	5.792	17.571
16	192,084,556	182,903,839	156,457,909	113,565,527	42,527,958	1.951	2.237	2.905	4.450	13.188
17	171,950,680	175,119,712	149,272,187	139,904,590	153,390,005	2.171	2.311	2.990	3.548	3.570
18	207,597,698	198,800,845	190,262,222	185,336,538	190,838,068	1.828	2.104	2.483	2.881	3.181
19	176,493,821	180,685,552	174,268,247	176,909,483	132,801,741	2.136	2.321	2.823	3.233	4.770
20	182,658,307	184,898,173	193,440,919	207,590,082	229,871,495	1.999	2.110	2.273	2.315	2.307
21	187,487,132	214,477,613	206,325,925	190,953,858	175,161,950	1.984	1.871	2.240	2.794	3.462
22	163,680,557	149,439,027	144,444,143	141,325,295	138,566,626	2.223	2.566	2.954	3.364	3.790
23	193,520,995	184,894,747	182,415,965	150,931,468	130,138,577	1.926	2.174	2.473	3.357	4.238
24	153,782,040	127,467,851	115,007,869	97,667,136	70,325,247	2.443	3.224	4.044	5.382	8.355
25	183,487,333	198,015,640	192,622,658	176,697,964	221,398,154	2.018	2.021	2.382	3.035	2.798
26	169,338,793	161,756,742	168,158,619	144,618,216	115,463,692	2.222	2.581	2.844	3.723	5.140
27	185,015,192	209,155,500	195,231,303	(14,024,799)	(31,154,077)	1.986	1.860	2.193	(33.486)	(16.814)
28	181,845,495	174,372,734	176,337,321	163,885,783	133,105,197	2.045	2.325	2.583	3.169	4.435
29	168,615,142	169,250,925	173,305,072	178,631,918	227,036,866	2.179	2.313	2.514	2.775	2.472
30	198,122,884	211,810,902	217,971,663	216,242,550	197,739,136	1.829	1.801	1.942	2.120	2.555
31	155,731,666	170,520,332	157,007,686	124,886,640	73,787,220	2.414	2.385	2.852	3.994	7.672
32	183,408,593	175,962,461	214,637,325	225,412,309	229,507,272	2.033	2.317	2.160	2.302	2.508
33	182,025,276	200,334,032	186,474,096	210,463,473	227,212,542	2.059	2.024	2.468	2.513	2.623
34	161,045,484	164,901,955	148,997,362	122,288,617	120,832,684	2.302	2.419	3.031	4.253	4.928
35	167,276,583	166,237,313	144,692,366	155,948,891	129,097,889	2.232	2.426	3.203	3.379	4.464
36	168,512,868	149,471,240	136,427,445	70,512,535	26,147,276	2.235	2.761	3.381	7.309	21.718
37	179,375,319	188,935,446	189,557,632	172,942,609	145,737,563	2.062	2.118	2.321	2.805	3.711
38	190,009,685	194,852,965	185,751,255	144,019,398	133,396,459	1.975	2.087	2.504	3.729	4.517
39	166,521,814	138,654,639	143,176,677	112,504,436	94,448,703	2.290	3.026	3.260	4.603	6.131
40	163,610,201	153,976,150	130,036,669	97,867,105	98,148,871	2.269	2.578	3.410	5.127	5.863
41	183,157,866	174,488,792	158,635,585	151,195,898	131,481,945	2.032	2.330	2.892	3.423	4.464
42	173,326,400	173,494,789	178,721,577	123,152,381	64,039,502	2.149	2.330	2.513	4.094	8.943
43	168,080,564	138,058,395	141,283,166	130,695,817	120,033,406	2.219	2.917	3.186	3.845	4.705
44	174,260,205	157,411,765	131,399,467	130,187,575	81,084,619	2.110	2.496	3.397	3.910	7.141
45	193,984,194	188,461,064	190,868,617	182,792,446	174,750,214	1.916	2.114	2.355	2.822	3.398
46	177,859,480	154,758,503	118,209,155	97,034,841	66,015,851	2.074	2.561	3.733	5.114	8.292
47	175,667,670	175,577,383	138,132,421	88,572,711	48,797,283	2.109	2.269	3.316	5.930	11.959
48	171,160,931	161,253,876	172,409,914	155,134,017	154,034,124	2.169	2.457	2.547	3.188	3.623
49	160,233,927	168,965,639	192,132,697	210,109,166	189,041,905	2.341	2.441	2.487	2.644	3.426
50	150,113,410	143,524,775	144,600,010	163,104,227	195,157,803	2.501	2.849	3.235	3.402	3.414

Base Case
50 Iterations Using Excel

Exhibit B-1-b

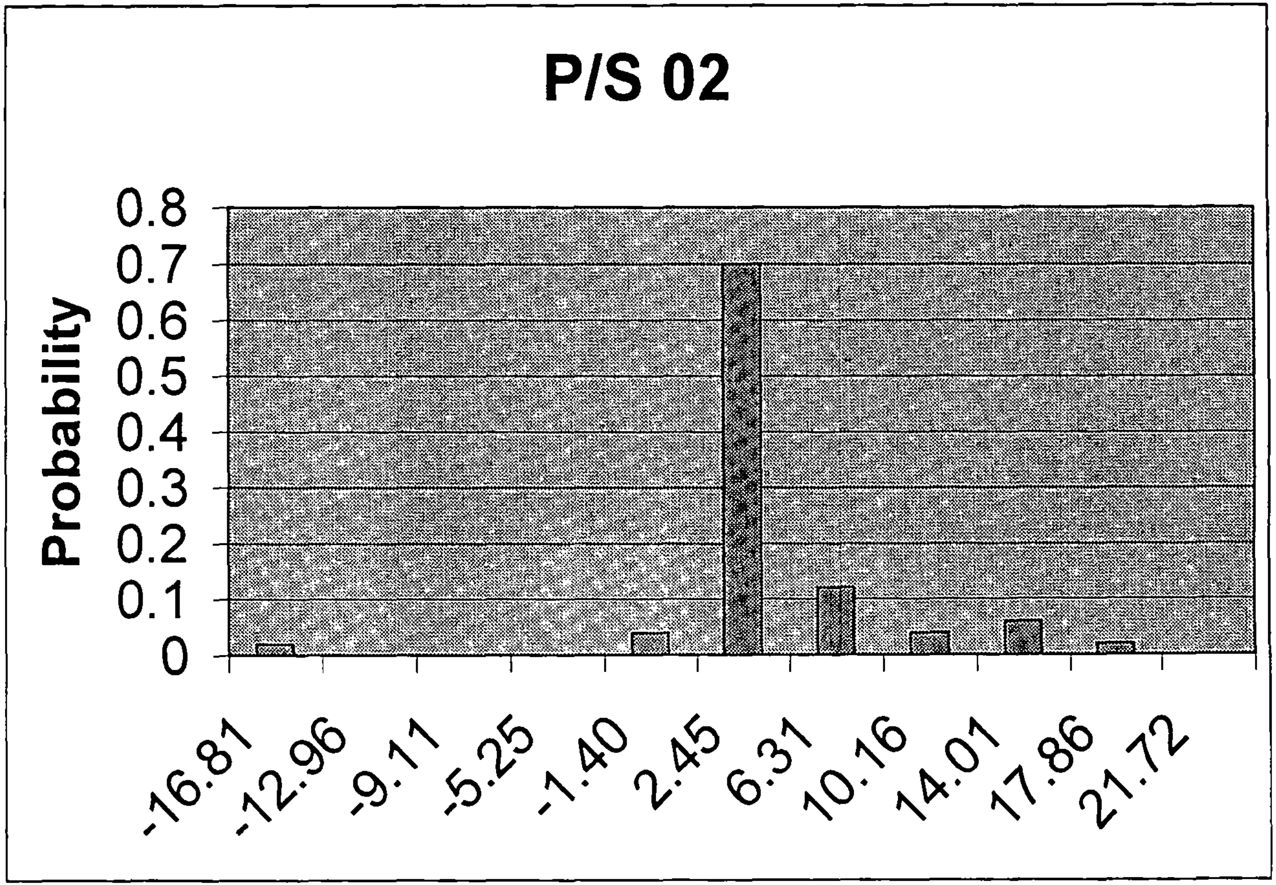
	Output	Output	Output	Output	Output	Output	Output	Output	Output	Output
	N27	o27	p27	q27	r27	W8	x8	y8	z8	aa8
	No	No	No	No	Yes	No	No	No	No	Yes
	1.035	1.004	1.005	1.030	1.027	0.726	0.745	0.747	0.771	0.766
Trial #	OR 98	OR 99	OR 00	OR 01	OR 02	Net LR 98	Net LR 99	Net LR 00	Net LR 01	Net LR 02
1	1.110	1.065	1.055	1.076	1.033	0.774	0.800	0.771	0.819	0.753
2	1.074	0.997	1.061	1.004	1.001	0.745	0.749	0.775	0.744	0.724
3	1.115	1.021	1.037	1.084	1.081	0.769	0.753	0.758	0.807	0.803
4	1.082	1.010	1.022	1.047	1.083	0.768	0.749	0.758	0.777	0.807
5	1.024	1.084	1.134	1.009	1.036	0.696	0.773	0.876	0.747	0.774
6	1.030	0.972	0.967	0.969	0.921	0.748	0.732	0.729	0.716	0.691
7	0.993	0.958	1.002	1.058	1.072	0.688	0.715	0.783	0.799	0.812
8	0.993	0.981	0.950	0.975	1.041	0.684	0.701	0.693	0.738	0.790
9	0.982	0.953	0.927	1.034	1.038	0.670	0.690	0.667	0.773	0.773
10	1.020	1.004	0.963	1.029	1.107	0.696	0.759	0.714	0.761	0.837
11	0.985	1.005	0.998	0.960	1.071	0.707	0.747	0.771	0.704	0.794
12	1.022	1.041	1.020	1.006	1.050	0.703	0.774	0.756	0.749	0.762
13	0.926	1.035	0.952	0.987	0.907	0.623	0.776	0.700	0.728	0.655
14	0.970	0.939	0.900	0.895	0.914	0.660	0.713	0.649	0.643	0.682
15	1.092	1.128	1.082	0.963	1.121	0.788	0.844	0.810	0.709	0.828
16	0.993	1.019	1.035	1.065	1.118	0.682	0.750	0.770	0.783	0.825
17	1.045	0.984	1.064	1.016	0.948	0.740	0.723	0.818	0.774	0.713
18	0.914	1.012	1.024	1.012	0.997	0.606	0.756	0.754	0.754	0.745
19	1.046	0.985	1.053	1.011	1.106	0.722	0.706	0.781	0.730	0.844
20	1.040	0.978	0.974	0.997	0.971	0.752	0.737	0.736	0.759	0.731
21	0.992	0.915	1.011	1.022	1.013	0.693	0.676	0.746	0.754	0.750
22	1.071	1.013	1.001	1.000	1.034	0.772	0.746	0.758	0.751	0.791
23	0.972	1.047	1.020	1.077	1.019	0.689	0.780	0.767	0.819	0.772
24	1.104	1.084	1.024	1.015	1.025	0.774	0.802	0.747	0.757	0.759
25	1.018	0.965	1.053	1.096	0.999	0.721	0.709	0.768	0.833	0.762
26	1.065	1.011	0.948	1.040	1.052	0.768	0.739	0.688	0.784	0.790
27	0.998	0.876	1.003	1.448	1.014	0.700	0.649	0.735	1.212	0.741
28	1.015	1.015	1.020	1.072	1.067	0.708	0.760	0.757	0.801	0.803
29	1.071	0.993	0.993	0.998	0.907	0.773	0.755	0.744	0.750	0.673
30	0.956	0.966	1.005	0.983	1.024	0.686	0.728	0.766	0.761	0.771
31	1.096	0.915	0.995	1.026	1.038	0.751	0.650	0.737	0.744	0.764
32	1.015	1.041	0.931	1.016	1.015	0.713	0.766	0.705	0.779	0.771
33	1.015	0.906	1.032	0.929	0.976	0.697	0.661	0.778	0.677	0.743
34	1.090	0.964	1.022	1.058	0.973	0.767	0.718	0.754	0.794	0.729
35	1.071	0.990	1.052	0.970	1.082	0.758	0.724	0.779	0.726	0.822
36	1.074	1.059	1.038	1.133	1.075	0.750	0.783	0.778	0.852	0.802
37	1.030	0.967	0.962	1.022	1.039	0.746	0.723	0.722	0.769	0.788
38	0.983	1.003	1.022	1.090	0.977	0.677	0.752	0.756	0.816	0.726
39	1.078	1.069	0.922	1.039	0.997	0.733	0.803	0.658	0.770	0.709
40	1.075	1.024	1.043	1.080	1.000	0.747	0.769	0.772	0.810	0.732
41	1.022	1.045	1.026	1.004	1.034	0.735	0.750	0.767	0.735	0.752
42	1.054	0.988	0.947	1.116	1.127	0.744	0.727	0.720	0.845	0.859
43	1.065	1.096	0.995	1.023	1.036	0.753	0.809	0.747	0.766	0.775
44	1.056	1.016	0.975	0.918	1.037	0.762	0.754	0.715	0.665	0.766
45	0.976	1.013	0.966	0.994	1.020	0.666	0.762	0.715	0.731	0.766
46	1.047	1.045	1.055	1.008	1.026	0.747	0.777	0.791	0.742	0.779
47	1.046	0.999	1.094	1.102	1.044	0.767	0.756	0.798	0.843	0.779
48	1.057	1.013	0.927	1.032	0.973	0.763	0.779	0.671	0.774	0.720
49	1.084	0.966	0.914	0.936	1.026	0.755	0.706	0.666	0.687	0.747
50	1.108	1.035	1.037	1.033	1.073	0.778	0.769	0.780	0.764	0.816

Exhibit B-2



Surplus 02
Probability
0.3
0.25
0.2
0.15
0.1
0.05
0
-31,154,076.81
-2,162,848.04
26,828,380.72
55,819,609.49
84,810,838.26
113,802,067.02
142,793,295.79
171,784,524.56
200,775,753.32
229,766,982.09
258,758,210.85

Exhibit B-3


P/S 02
Probability
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-16.81
-12.96
-9.11
-5.25
-1.40
2.45
6.31
10.16
14.01
17.86
21.72

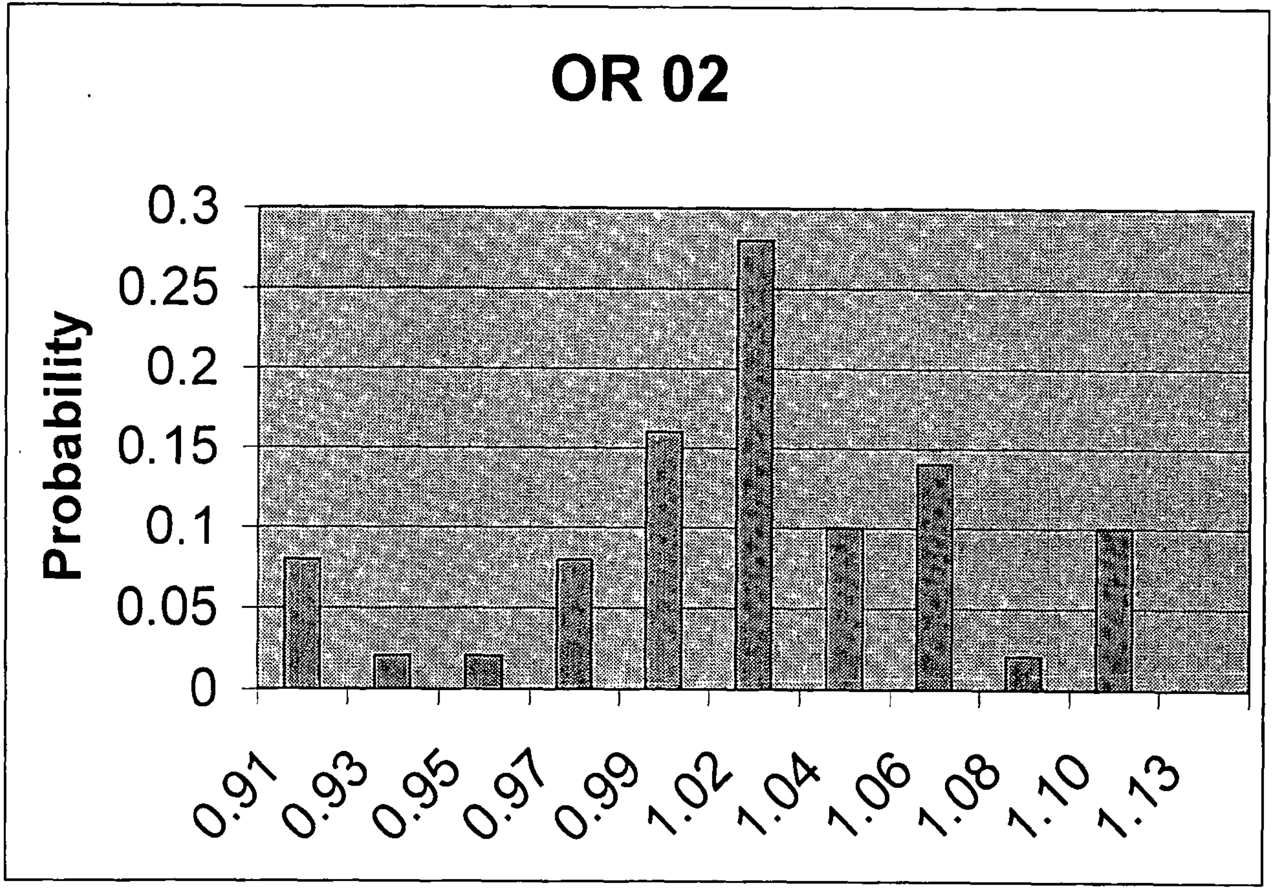
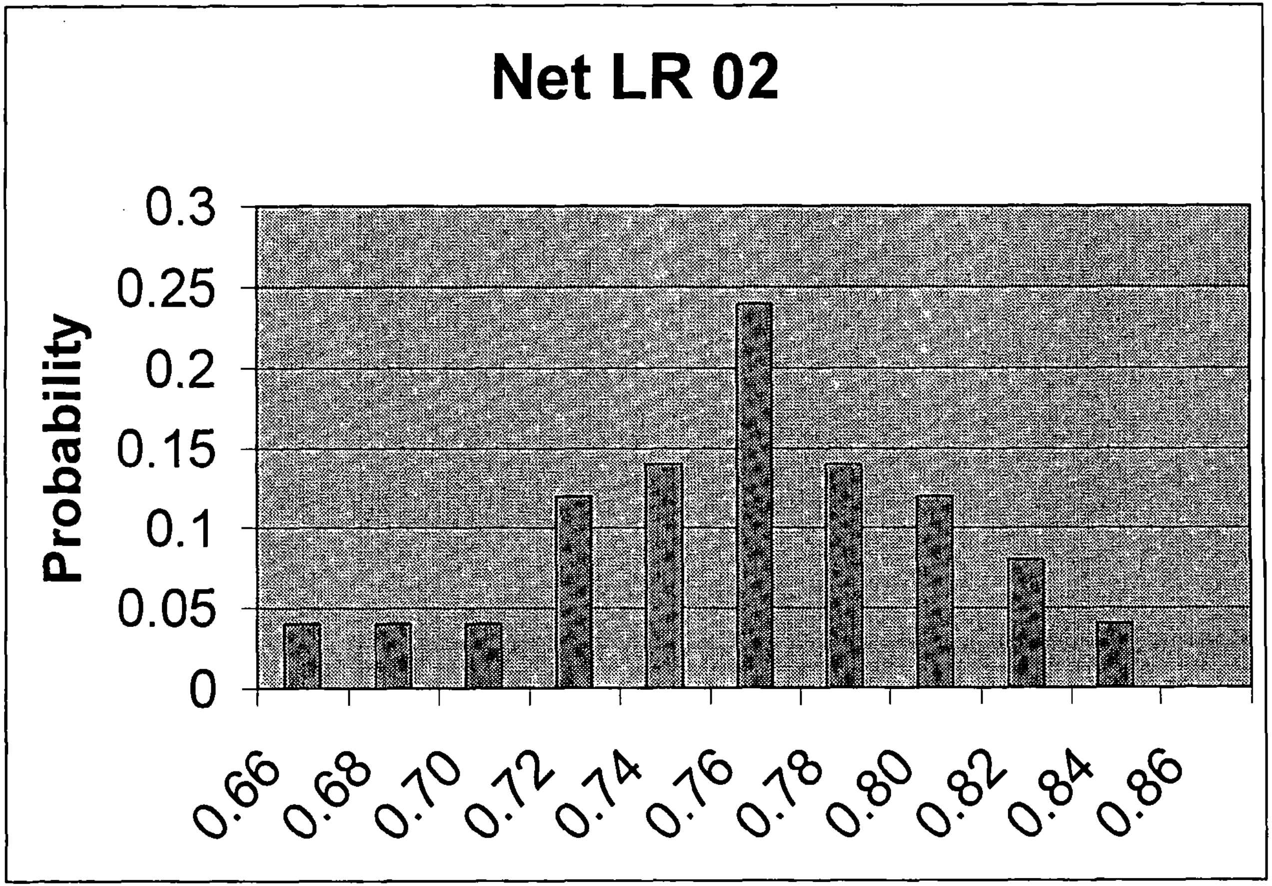

OR 02
Probability
0.3
0.25
0.2
0.15
0.1
0.05
0
0.91
0.93
0.95
0.97
0.99
1.02
1.04
1.06
1.08
1.10
1.13

Exhibit B-5



Net LR 02
Probability
0.3
0.25
0.2
0.15
0.1
0.05
0
0.66
0.68
0.70
0.72
0.74
0.76
0.78
0.80
0.82
0.84
0.86

Base Case
50 Iterations Using Excel

Exhibit B-6-a

ABC Insurance Company
Statutory Balance Sheet

		1998	1999	2000	2001	2002
ASSETS						
1.	Bonds	397,289,391	417,079,942	415,876,272	441,770,059	477,281,188
2.	Stocks:					
	2.1 Preferred stocks	14,414,504	16,591,024	18,437,526	21,489,014	24,487,902
	2.2 Common stocks	99,977,356	110,700,421	120,104,398	130,486,269	139,994,606
3.	Mortgage loans on real estate	196,144	196,144	196,144	196,144	196,144
4.	Real estate	16,880,795	16,880,795	16,880,795	16,880,795	16,880,795
5.	Collateral loans	-	-	-	-	-
6.	Cash	34,578,453	38,340,296	40,785,861	45,596,660	51,935,531
7.	Other Invested assets	446,683	446,683	446,683	446,683	446,683
8.	Aggregate write-ins					
9.	Subtotals, cash & invested assets	563,783,325	600,235,305	612,727,680	656,865,624	711,222,850
10.	Agents' balances or uncollected pr	48,846,694	53,406,225	59,581,118	68,346,149	78,074,692
11.	Funds held by reinsurer	210	210	210	210	210
12.	Bills receivable	-	-	-	-	-
13.	Reinsurance recoverables	5,818,016	6,999,378	6,873,290	7,867,660	9,239,345
14.	Federal income tax collectable		-	-	-	-
15.	Electronic data processing	2,992,030	2,992,030	2,992,030	2,992,030	2,992,030
16.	Interest, dividends & real estate	6,344,827	6,344,827	6,344,827	6,344,827	6,344,827
17.	Receivable from parent	1,107,674	1,107,674	1,107,674	1,107,674	1,107,674
18.	Equities and deposits in pools	-	-	-	-	-
19.	Amounts receivable relating to A&	-	-	-	-	-
20.	Other assets nonadmitted	-	-	-	-	-
21.	Aggregate write-ins	4,956,493	4,956,493	4,956,493	4,956,493	4,956,493
22.	Total assets	633,849,268	676,042,142	694,583,322	748,480,667	813,938,121

Base Case

Exhibit B-6-b

50 Iterations Using Excel

			1998	1999	2000	2001	2002
LIABILITIES							
1.	Losses & LAE:	###	290,900,796	349,968,894	343,664,523	393,382,978	461,967,235
2.	Unearned premiums:		152,532,149	166,061,916	183,427,198	209,716,288	239,005,263
3.	Other expenses		6,041,971	6,451,916	7,163,253	8,111,683	9,268,363
4.	Taxes, licenses and fees		6,264,517	6,854,283	7,640,894	8,751,580	9,974,421
5.	Federal income taxes		149,581	-	1,215,947	128,632	386,520
6.	Other liabilities						
7.	Total liabilities		455,889,014	529,337,009	543,111,815	620,091,161	720,601,801
SURPLUS							
8.	Additions to surplus		-	-	-	-	-
9.	Surplus as regards to policyholders		177,960,255	146,705,134	151,471,507	128,389,506	93,336,320
	Net Income (Before taxes)		6,697,898	(15,003,510)	40,535,320	1,738,279	4,700,000
	Underwriting Gain/(Loss)		(42,530,250)	(70,175,534)	(20,340,994)	(62,290,287)	(60,828,051)
	Combined Ratio		1.111	1.167	1.035	1.110	1.093
	Operating Ratios		1.032	1.091	0.964	1.046	1.038
	Investment Income / Surplus		0.165	0.206	0.208	0.249	0.334
	Investment Income / Earned Premium		0.079	0.076	0.071	0.064	0.055

IRIS Ratios

	1998	1999	2000	2001	2002
1. **Premium to Surplus**	2.11	2.80	3.03	4.09	6.43
2. Change in Writings	1.3%	9.3%	11.6%	14.7%	14.2%
3. Surplus Aid to Surplus	3.6%	5.2%	5.6%	7.7%	12.3%
4. Two Year Overall Operating Ratio		108%	105%	104%	108%
5. Investment Yield	5.4%	5.0%	5.1%	4.9%	4.4%
6. Change in Surplus	11.1%	-15.0%	2.7%	-12.3%	-20.7%
7. Liabilities to Liquid Assets	66%	72%	72%	76%	82%
8. Agents Balances to Surplus	27%	36%	39%	53%	84%
9. One Year Development	5.5%	1.3%	-2.1%	13.8%	-5.8%
10. Two Year Development		7.1%	1.5%	2.4%	11.3%
11. Estimated Current Reserve Deficiency to Surplus		#N/A	15.0%	27.4%	-8.1%

Base Case
50 Iterations Using Excel

Exhibit B-7

		Apriori Loss & ALAE Ratios				
		Accident Years				
Coverage	Subdivision	1998	1999	2000	2001	2002
All	Direct	0.65	0.78	0.72	0.68	0.72
	Ceded	0.13	0.39	0.76	0.11	0.26
	Net	0.72	0.83	0.72	0.75	0.78
HMP	New	0.70	0.72	0.83	1.09	1.11
	Renewal	0.42	0.48	0.77	0.91	1.08
	Renewal (2)	0.55	0.56	0.49	0.63	0.72
	Direct	0.59	0.70	0.62	0.75	0.84
	Ceded	0.00	0.16	0.47	0.00	0.00
	Net	0.65	0.75	0.63	0.83	0.92
PPAL	New	0.84	0.84	0.93	0.85	0.95
	Renewal	0.87	0.73	0.84	0.81	0.73
	Renewal (2)	0.95	0.89	0.71	0.67	0.68
	Direct	0.93	0.87	0.76	0.73	0.75
	Ceded	0.00	0.00	0.00	0.00	0.00
	Net	0.97	0.91	0.80	0.76	0.79
APD-P	New	0.71	0.84	0.75	0.81	0.74
	Renewal	0.57	0.61	0.61	0.84	0.83
	Renewal (2)	0.61	0.70	0.54	0.59	0.69
	Direct	0.65	0.84	0.63	0.69	0.73
	Ceded	0.00	0.23	0.36	0.00	0.00
	Net	0.69	0.88	0.65	0.73	0.78
APD-C	New	0.62	1.35	0.78	0.97	0.52
	Renewal	0.42	0.37	0.50	0.51	0.52
	Renewal (2)	0.59	0.37	0.45	0.56	0.44
	Direct	0.59	0.52	0.59	0.63	0.47
	Ceded	0.00	0.14	0.87	0.00	0.00
	Net	0.63	0.55	0.57	0.68	0.50
CAL	New	0.96	2.01	1.44	1.17	0.77
	Renewal	0.65	0.55	0.55	1.22	0.92
	Renewal (2)	0.50	0.99	0.69	0.38	0.39
	Direct	0.55	1.04	0.75	0.57	0.51
	Ceded	0.01	0.02	0.02	0.01	0.01
	Net	0.58	1.08	0.79	0.60	0.54
CMP-L	New	0.61	0.93	0.67	1.01	0.84
	Renewal	0.42	0.66	0.79	0.66	0.63
	Renewal (2)	0.61	0.52	0.45	0.63	0.67
	Direct	0.59	0.58	0.51	0.70	0.69
	Ceded	0.00	0.00	0.00	0.00	0.00
	Net	0.62	0.60	0.54	0.73	0.73
CMP-P	New	0.52	1.05	0.65	0.75	1.54
	Renewal	0.54	0.25	1.24	0.75	0.68
	Renewal (2)	0.49	0.74	0.70	0.52	0.74
	Direct	0.55	0.99	1.07	0.68	0.89
	Ceded	0.15	0.48	1.57	0.19	0.25
	Net	0.61	1.08	0.98	0.76	1.00
OL	New	0.56	0.39	0.49	0.54	0.31
	Renewal	0.38	0.20	0.34	0.29	0.39
	Renewal (2)	0.42	0.11	0.03	0.09	0.26
	Direct	0.43	0.14	0.13	0.20	0.29
	Ceded	0.00	0.00	0.00	0.00	0.00
	Net	0.45	0.15	0.14	0.21	0.30
OL-U	New	0.24	0.12	0.10	0.02	0.12
	Renewal	0.10	0.01	0.09	0.23	0.05
	Renewal (2)	0.35	0.06	0.17	0.24	0.24
	Direct	0.32	0.06	0.15	0.19	0.19
	Ceded	0.32	0.06	0.15	0.19	0.19
	Net	0.33	0.06	0.16	0.20	0.19
WC	New	0.63	1.02	0.77	0.77	0.61
	Renewal	0.58	0.50	0.81	0.99	0.71
	Renewal (2)	0.60	0.44	0.43	0.64	0.57
	Direct	0.60	0.49	0.50	0.70	0.60
	Ceded	0.00	0.00	0.00	0.01	0.00
	Net	0.62	0.50	0.51	0.72	0.61

Output

New Business
Homeowners Multiple Peril
Direct Underwriting Module

Exhibit B-8

Description	2nd Prior Year 1996	1st Prior Year 1997	1st Year 1998	Accident Years 2nd Year 1999	3rd Year 2000	4th Year 2001	5th Year 2002
1. Premiums:							
a. Exposure Growth Rate			-1%	2%	8%	10%	10%
b. Number of Exposures	10,740	9,569	6,282	6,736	10,287	13,788	16,119
c. Average Rate Growth Rate			5%	11%	3%	2%	8%
d. Average Rate per Exposure	387.61	377.37	397.38	440.83	455.88	463.75	499.31
e. Written Premiums	4,162,984	3,610,877	2,496,361	2,969,449	4,689,613	6,394,149	8,048,344
f. Earning Ratio	0.50	0.50	0.50	0.50	0.50	0.50	0.50
g. Earned Premiums	4,162,984	3,886,930	3,053,619	2,732,905	3,829,531	5,541,881	7,221,247
h. Unearned Premium Reserves	2,081,492	1,805,438	1,248,180	1,484,724	2,344,807	3,197,075	4,024,172
i. Renewal Ratio	60%	60%	60%	60%	60%	60%	60%
2. Expenses:							
a. Commissions	585,760	486,894	349,491	415,723	656,546	895,181	1,126,768
b. General Expense	272,033	243,112	198,485	177,639	248,920	360,222	469,381
c. Other Acquisition	523,786	458,017	360,327	322,483	451,885	653,942	852,107
d. Premium Taxes	133,330	117,821	84,876	100,961	159,447	217,401	273,644
e. Policyholder Dividends	-	-	-	-	-	-	-
f. Other Nonrecurring Expenses	-	-	931,848	-	-	-	-
g. Subtotal (Expenses)	1,514,908	1,305,843	1,925,027	1,016,806	1,516,797	2,126,746	2,721,900
3. Losses:							
a. Initial Severity Mean	2,000	2,000	2,000	2,000	2,000	2,000	2,000
b. Initial Severity Std.	192	192	192	192	192	192	192
c. Severity Trend	0.959	1.000	1.043	1.115	1.105	1.171	1.248
d. U/W & Rate Adjustments							
e. Modeled Severity	1,719	1,781	1,846	2,633	1,894	2,219	3,002
f. Initial Frequency Mean	0.157	0.157	0.157	0.157	0.157	0.157	0.157
g. Initial Frequency Std.	0.014	0.014	0.014	0.014	0.014	0.014	0.014
h. Frequency Trend	1.000	1.000	1.000	1.000	1.000	1.000	1.000
i. U/W & Rate Adjustments	1.000	1.000	1.000	1.000	1.000	1.000	1.000
j. Modeled Frequency	0.15	0.15	0.17	0.13	0.17	0.13	0.16
k. a Priori Ultimate Losses & ALAE	2,795,926	2,561,872	1,914,826	2,390,620	3,275,774	4,068,293	7,605,820
l. a Priori Loss & ALAE Ratio	0.67	0.66	0.63	0.87	0.86	0.73	1.05
m. New Business Penalty	(0.14)	(0.33)	(0.03)	0.30	0.28	0.11	0.26

Constained Growth Case

50 Iterations Using Excel

Exhibit C-1-a

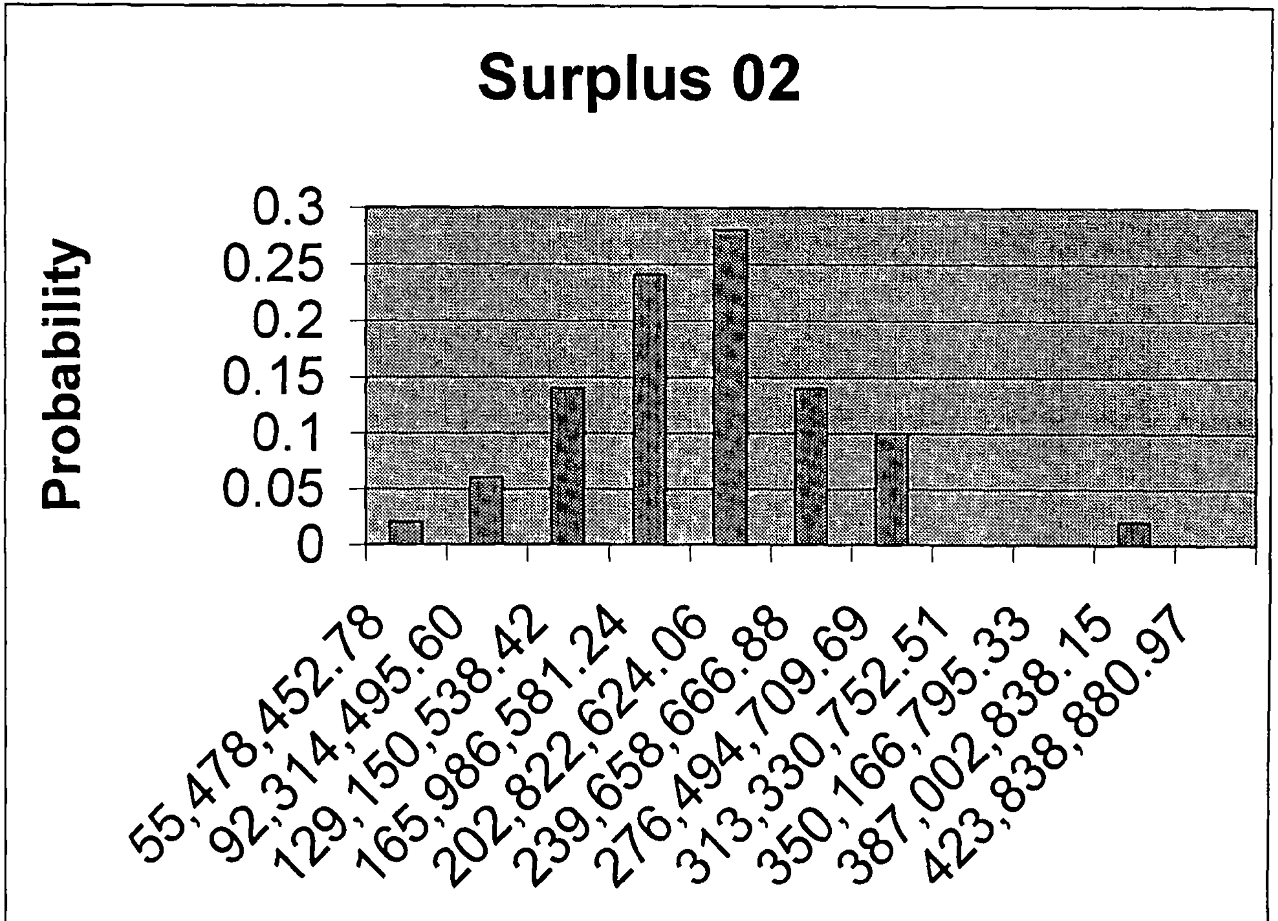
	Output N19 No 175,804,947	Output o19 No 177,290,291	Output p19 No 182,504,379	Output q19 No 192,153,293	Output r19 Yes 203,398,666	Output n34 No 2.128	Output o34 No 2.302	Output p34 No 2.458	Output q34 No 2.572	Output r34 Yes 2.737
Trial #	Surplus 98	Surplus 99	Surplus 00	Surplus 01	Surplus 02	P/S 98	P/S 99	P/S 00	P/S 01	P/S 02
1	177,841,779	167,912,331	168,844,782	179,295,863	212,021,245	2.069	2.323	2.462	2.466	2.178
2	181,059,869	187,173,731	185,287,185	200,046,953	221,523,361	2.032	2.096	2.300	2.272	2.218
3	166,966,791	149,474,454	140,715,199	147,634,907	121,635,939	2.270	2.780	3.164	3.218	4.308
4	192,635,204	209,507,227	209,709,103	220,156,166	264,713,480	1.884	1.820	1.924	1.959	1.712
5	171,116,002	154,671,307	167,894,973	151,567,761	140,269,784	2.158	2.560	2.576	3.040	3.553
6	165,479,513	184,319,389	173,127,374	168,013,654	203,180,933	2.264	2.185	2.528	2.848	2.523
7	183,332,162	183,675,934	189,911,787	233,546,717	277,009,685	2.013	2.176	2.333	2.127	1.989
8	170,990,389	165,399,172	156,883,024	157,326,334	163,498,520	2.175	2.438	2.825	3.092	3.128
9	166,219,582	176,014,335	194,302,088	187,946,530	195,213,865	2.247	2.311	2.286	2.613	2.740
10	167,549,849	170,272,178	163,200,039	157,833,112	133,012,541	2.213	2.316	2.536	2.804	3.582
11	181,989,043	175,721,582	181,291,022	193,981,472	219,868,778	2.044	2.234	2.323	2.403	2.277
12	171,173,909	171,489,352	165,795,558	188,518,302	204,182,589	2.181	2.338	2.640	2.518	2.529
13	187,218,190	202,499,309	215,812,659	227,802,009	233,368,315	1.966	1.908	1.893	1.931	2.087
14	174,554,551	178,557,958	183,769,330	186,220,807	234,376,888	2.119	2.234	2.412	2.660	2.315
15	163,605,439	157,051,627	161,536,361	176,364,972	191,007,728	2.268	2.525	2.663	2.658	2.673
16	174,687,514	169,203,255	191,150,328	250,344,258	288,986,500	2.136	2.424	2.387	2.007	1.889
17	186,363,932	198,662,672	212,179,904	189,277,418	182,227,624	1.978	1.983	2.011	2.493	2.818
18	180,357,986	173,817,775	189,970,611	206,837,028	239,280,898	2.066	2.337	2.299	2.287	2.181
19	199,678,781	213,265,100	236,976,237	259,981,632	247,071,935	1.842	1.817	1.758	1.720	1.937
20	154,142,665	162,509,036	185,054,667	205,015,116	231,133,529	2.423	2.500	2.347	2.279	2.249
21	155,221,003	118,718,197	92,886,218	78,669,964	55,478,453	2.394	3.421	4.875	6.342	9.896
22	175,247,353	184,870,219	190,839,173	192,707,442	196,359,498	2.120	2.164	2.243	2.354	2.462
23	165,738,288	129,422,352	146,643,228	165,293,737	172,175,738	2.257	3.155	3.017	2.824	2.853
24	188,367,238	212,822,053	227,044,964	265,479,686	296,200,982	1.983	1.893	1.938	1.820	1.796
25	179,013,070	184,596,877	195,657,567	184,165,528	181,694,094	2.045	2.134	2.208	2.476	2.604
26	162,119,150	162,399,400	156,024,363	118,100,922	116,516,157	2.304	2.438	2.688	3.804	4.070
27	197,293,489	214,106,548	224,820,661	232,681,325	233,141,728	1.910	1.931	1.996	2.098	2.237
28	166,821,963	161,029,300	154,493,557	156,015,553	145,959,338	2.236	2.501	2.865	3.083	3.595
29	192,919,927	194,954,825	201,447,824	232,288,838	285,751,895	1.922	2.045	2.148	2.028	1.770
30	167,274,874	153,646,410	152,881,964	177,887,239	206,060,179	2.237	2.634	2.794	2.513	2.338
31	161,172,142	171,102,588	191,640,480	216,988,209	263,704,092	2.317	2.390	2.385	2.307	2.057
32	158,229,833	147,982,570	158,106,954	156,149,224	152,057,763	2.396	2.774	2.753	2.939	3.149
33	190,861,707	221,047,612	254,435,227	261,244,340	260,722,050	1.947	1.829	1.673	1.678	1.757
34	176,386,326	186,603,421	220,228,631	228,437,423	254,584,475	2.105	2.135	1.935	1.980	1.913
35	188,698,228	196,972,298	201,538,134	195,616,576	157,858,918	1.981	2.071	2.233	2.432	3.200
36	152,363,410	140,578,639	157,474,700	157,070,679	156,579,559	2.450	2.913	2.867	3.106	3.311
37	190,039,899	207,857,645	232,091,596	248,277,009	250,383,091	1.968	1.971	1.941	1.968	2.169
38	184,363,321	186,687,410	186,932,473	205,737,487	203,362,673	1.976	2.030	2.180	2.172	2.398
39	178,958,213	208,166,934	199,078,688	217,182,852	214,074,617	2.072	1.918	2.204	2.212	2.421
40	188,911,722	175,829,627	171,477,115	184,375,749	209,054,244	1.974	2.259	2.539	2.637	2.543
41	163,221,155	175,395,082	180,854,923	157,710,730	169,908,087	2.295	2.324	2.489	3.175	3.235
42	180,852,903	175,439,975	192,045,808	175,663,109	187,755,230	2.045	2.257	2.213	2.564	2.532
43	182,714,124	180,445,572	180,889,033	199,618,725	200,390,796	2.066	2.281	2.481	2.472	2.708
44	166,435,215	172,961,456	186,831,678	249,789,424	311,488,334	2.245	2.344	2.356	1.904	1.635
45	198,691,266	194,024,781	183,725,795	183,243,534	167,210,103	1.901	2.108	2.407	2.552	2.888
46	156,146,480	149,628,366	139,393,175	128,571,645	115,688,005	2.427	2.763	3.218	3.662	4.214
47	181,822,715	176,811,631	199,487,704	224,282,332	243,874,670	2.036	2.223	2.094	1.992	1.977
48	191,453,368	200,425,847	167,487,942	177,224,928	183,034,215	1.911	1.883	2.392	2.403	2.468
49	156,140,880	151,498,912	122,842,747	157,326,157	171,881,494	2.424	2.718	3.654	3.122	3.045
50	189,913,004	225,044,188	245,884,995	302,084,201	423,838,881	1.970	1.796	1.802	1.634	1.295

Constained Growth Case
50 Iterations Using Excel

Exhibit C-1-b

	Output p27 No	Output q27 No	Output r27 Yes	Output W8 No	Output x8 No	Output y8 No	Output z8 No	Output aa8 Yes
	0.977	0.968	0.970	0.727	0.728	0.727	0.720	0.723
Trial #	OR 00	OR 01	OR 02	Net LR 98	Net LR 99	Net LR 00	Net LR 01	Net LR 02
1	0.993	0.988	0.933	0.724	0.769	0.747	0.757	0.705
2	1.013	0.974	1.030	0.720	0.726	0.770	0.736	0.768
3	1.006	0.969	1.075	0.720	0.774	0.755	0.705	0.786
4	0.965	0.959	0.897	0.701	0.668	0.732	0.714	0.690
5	0.956	1.025	0.994	0.741	0.792	0.707	0.781	0.715
6	1.028	1.002	0.868	0.758	0.687	0.766	0.753	0.635
7	1.017	0.945	0.959	0.730	0.756	0.767	0.688	0.725
8	1.006	0.986	0.982	0.735	0.729	0.745	0.721	0.743
9	0.908	0.964	0.956	0.758	0.652	0.672	0.699	0.703
10	0.975	0.981	1.024	0.757	0.714	0.722	0.716	0.750
11	0.951	0.934	0.909	0.704	0.772	0.714	0.685	0.684
12	1.005	0.927	0.963	0.737	0.737	0.731	0.693	0.711
13	0.962	0.983	1.040	0.711	0.691	0.725	0.747	0.787
14	0.995	1.011	0.936	0.735	0.714	0.736	0.742	0.704
15	1.009	0.977	1.029	0.772	0.759	0.738	0.738	0.769
16	0.983	0.927	1.037	0.750	0.774	0.734	0.691	0.796
17	0.958	1.016	0.969	0.698	0.698	0.706	0.748	0.714
18	0.981	0.973	0.959	0.716	0.763	0.729	0.727	0.717
19	0.903	0.901	1.041	0.683	0.671	0.669	0.681	0.792
20	0.952	0.983	1.006	0.767	0.697	0.721	0.736	0.758
21	1.070	1.038	1.063	0.771	0.829	0.800	0.775	0.773
22	0.951	0.980	0.954	0.716	0.687	0.711	0.724	0.706
23	0.954	0.924	0.927	0.759	0.832	0.707	0.679	0.685
24	1.003	0.964	0.954	0.717	0.683	0.761	0.724	0.715
25	0.932	1.022	0.965	0.757	0.738	0.679	0.777	0.712
26	0.955	1.045	0.941	0.772	0.736	0.695	0.760	0.702
27	0.964	0.994	0.950	0.640	0.678	0.723	0.736	0.692
28	0.992	0.957	0.984	0.741	0.737	0.731	0.711	0.698
29	0.952	0.874	0.799	0.689	0.704	0.711	0.632	0.580
30	1.004	0.938	0.961	0.719	0.769	0.746	0.696	0.721
31	0.969	0.969	0.922	0.780	0.706	0.716	0.731	0.699
32	0.954	0.997	0.975	0.776	0.757	0.696	0.755	0.715
33	0.861	0.965	1.013	0.671	0.627	0.663	0.735	0.783
34	0.895	0.980	0.951	0.727	0.719	0.665	0.742	0.709
35	1.011	1.023	1.076	0.693	0.734	0.756	0.794	0.810
36	0.974	1.019	0.956	0.780	0.783	0.711	0.747	0.707
37	0.936	0.922	0.951	0.675	0.665	0.702	0.690	0.691
38	0.991	0.955	1.006	0.734	0.726	0.737	0.699	0.770
39	0.993	0.876	0.963	0.721	0.625	0.742	0.626	0.705
40	1.006	0.988	0.983	0.663	0.798	0.747	0.731	0.754
41	0.979	1.041	0.937	0.775	0.707	0.712	0.780	0.676
42	0.895	1.014	0.902	0.713	0.765	0.662	0.756	0.661
43	0.980	0.904	0.952	0.714	0.742	0.726	0.653	0.702
44	0.975	0.872	0.886	0.752	0.709	0.737	0.638	0.669
45	1.012	0.971	1.041	0.655	0.788	0.746	0.720	0.790
46	1.033	1.033	1.038	0.765	0.770	0.776	0.777	0.772
47	0.943	0.948	0.996	0.732	0.758	0.699	0.715	0.755
48	1.096	0.966	1.009	0.696	0.729	0.832	0.735	0.770
49	1.066	0.892	0.982	0.774	0.742	0.768	0.656	0.742
50	0.947	0.896	0.852	0.669	0.617	0.711	0.676	0.655

Exhibit C-2



Surplus 02
Probability
0.3
0.25
0.2
0.15
0.1
0.05
0
55,478,452.78
92,314,495.60
129,150,538.42
165,986,581.24
202,822,624.06
239,658,666.88
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313,330,752.51
350,166,795.33
387,002,838.15
423,838,880.97

Exhibit C-3

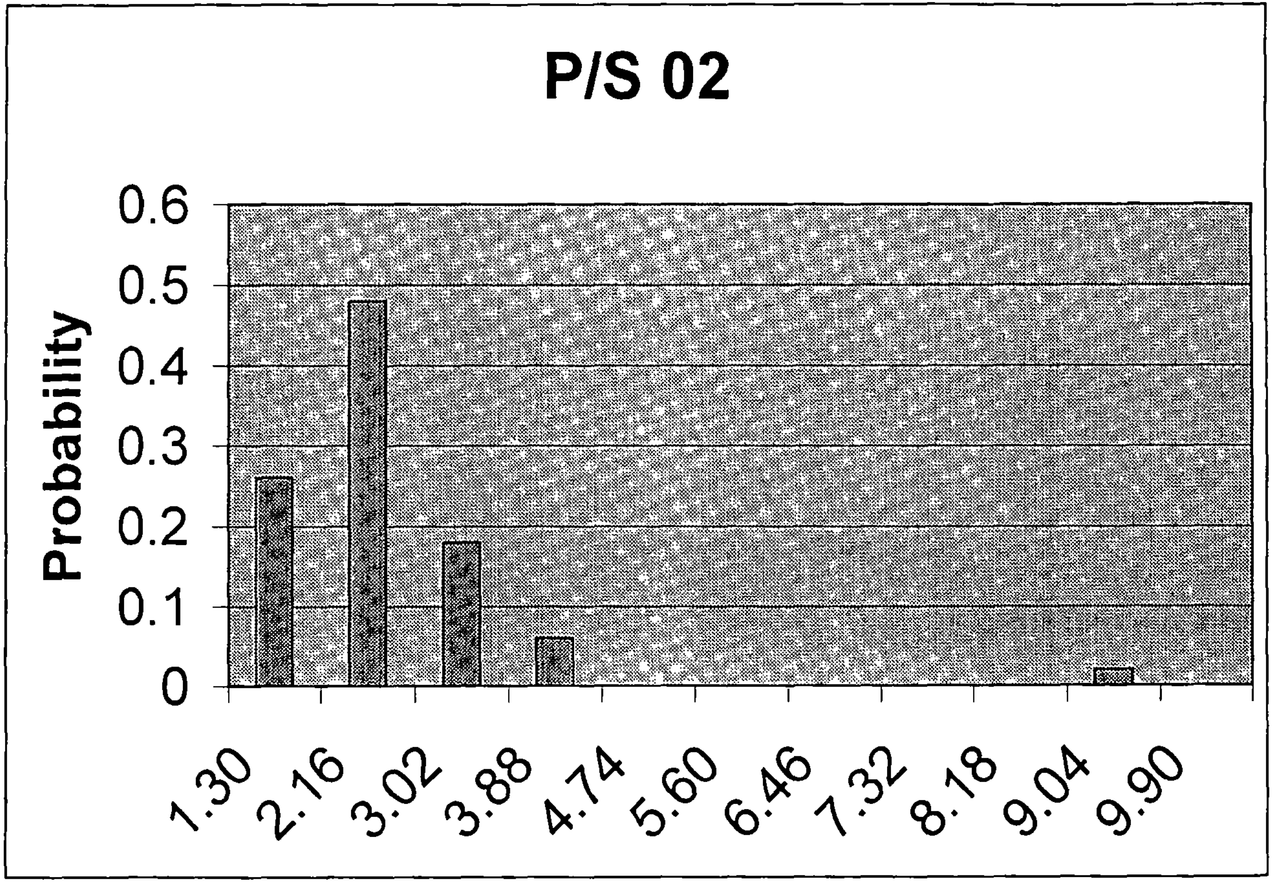
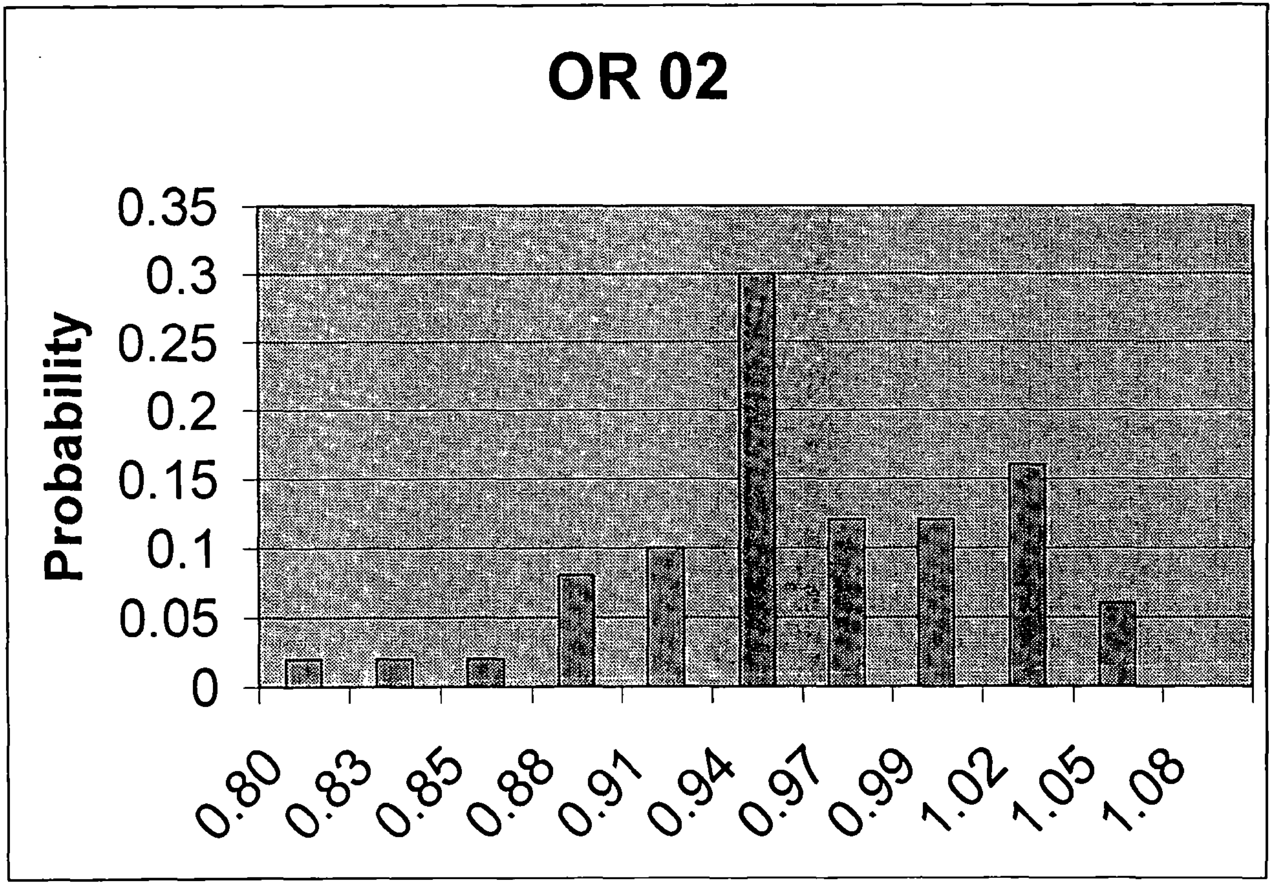
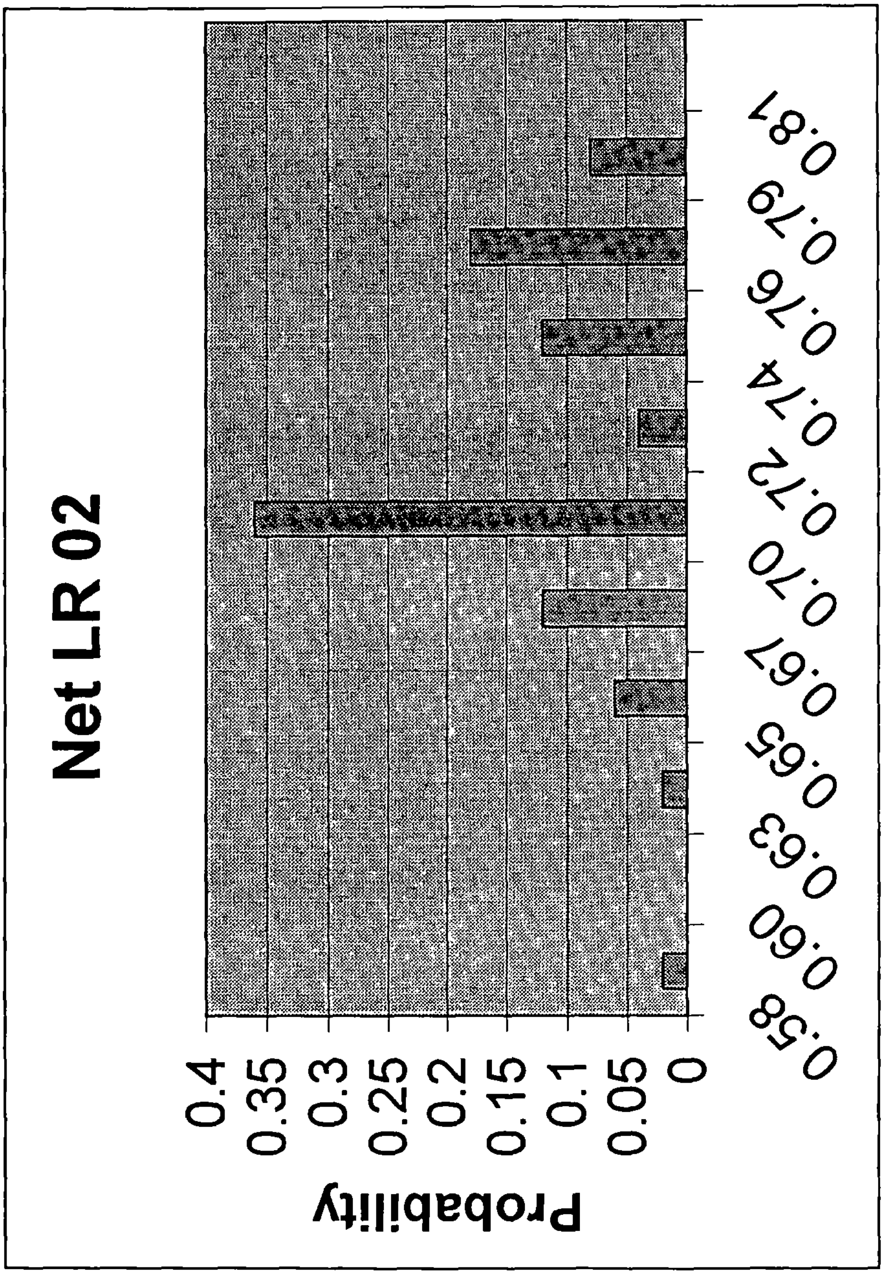

P/S 02
Probability
0.6
0.5
0.4
0.3
0.2
0.1
0
1.30
2.16
3.02
3.88
4.74
5.60
6.46
7.32
8.18
9.04
9.90

Exhibit C-4



OR 02
Probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.80
0.83
0.85
0.88
0.91
0.94
0.97
0.99
1.02
1.05
1.08

Exhibit C-5


Net LR 02
Probability
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.58
0.60
0.63
0.65
0.67
0.70
0.72
0.74
0.76
0.79
0.81

Constrained Growth Case
50 Iterations Using Excel

Exhibit C-6-a

ABC Insurance Company
Statutory Balance Sheet

		1998	1999	2000	2001	2002
ASSETS						
1.	Bonds	396,499,803	446,898,745	497,829,153	589,700,288	718,720,887
2.	Stocks:					
	2.1 Preferred stocks	14,023,733	17,061,531	20,785,554	26,423,310	34,323,460
	2.2 Common stocks	97,994,321	105,550,614	112,460,582	126,104,943	147,248,793
3.	Mortgage loans on real estate	196,144	196,144	196,144	196,144	196,144
4.	Real estate	16,880,795	16,880,795	16,880,795	16,880,795	16,880,795
5.	Collateral loans	-	-	-	-	-
6.	Cash	34,512,971	40,507,351	46,442,206	55,121,920	65,046,227
7.	Other Invested assets	446,683	446,683	446,683	446,683	446,683
8.	Aggregate write-ins					
9.	Subtotals, cash & invested assets	560,554,450	627,541,863	695,041,118	814,874,083	982,862,990
10.	Agents' balances or uncollected pr	48,628,153	52,552,186	57,615,575	64,149,153	71,373,787
11.	Funds held by reinsurer	210	210	210	210	210
12.	Bills receivable	-	-	-	-	-
13.	Reinsurance recoverables	5,497,330	5,921,323	6,645,300	7,603,519	8,196,091
14.	Federal income tax collectable		-	-	-	-
15.	Electronic data processing	2,992,030	2,992,030	2,992,030	2,992,030	2,992,030
16.	Interest, dividends & real estate	6,344,827	6,344,827	6,344,827	6,344,827	6,344,827
17.	Receivable from parent	1,107,674	1,107,674	1,107,674	1,107,674	1,107,674
18.	Equities and deposits in pools	-	-	-	-	-
19.	Amounts receivable relating to A&	-	-	-	-	-
20.	Other assets nonadmitted	-	-	-	-	-
21.	Aggregate write-ins	4,956,493	4,956,493	4,956,493	4,956,493	4,956,493
22.	Total assets	630,081,167	701,416,606	774,703,227	902,027,989	1,077,834,103

Output

Exhibit C-6-b

Constrained Growth Case

50 Iterations Using Excel

		1998	1999	2000	2001	2002
LIABILITIES						
1.	Losses & LAE: ###	274,866,487	296,066,154	332,264,999	380,175,936	409,804,567
2.	Unearned premiums:	152,246,049	164,553,572	180,545,169	200,946,307	222,552,191
3.	Other expenses	6,017,656	6,359,261	6,925,775	7,660,788	8,533,674
4.	Taxes, licenses and fees	6,233,649	6,738,378	7,387,003	8,224,718	9,145,463
5.	Federal income taxes	804,323	2,655,053	1,695,285	2,936,039	3,959,327
6.	Other liabilities					
7.	Total liabilities	440,168,163	476,372,419	528,818,231	599,943,787	653,995,222
SURPLUS						
8.	Additions to surplus	-	-	-	-	-
9.	Surplus as regards to policyholders	189,913,004	225,044,188	245,884,995	302,084,201	423,838,881
	Net Income (Before taxes)	27,769,271	78,821,663	51,800,669	87,013,290	131,537,893
	Underwriting Gain/(Loss)	(21,458,877)	20,982,474	(21,902,348)	(7,733,303)	(2,955,578)
	Combined Ratio	1.055	0.938	1.041	1.004	0.994
	Operating Ratios	0.976	0.855	0.947	0.896	0.852
	Investment Income / Surplus	0.155	0.143	0.163	0.170	0.177
	Investment Income / Earned Premium	0.079	0.082	0.094	0.108	0.142
IRIS Ratios						
1.	**Premium to Surplus**	**1.97**	**1.80**	**1.80**	**1.63**	**1.30**
2.	Change in Writings	0.9%	8.1%	9.6%	11.3%	11.3%
3.	Surplus Aid to Surplus	3.6%	3.4%	3.4%	3.0%	2.3%
4.	Two Year Overall Operating Ratio		93%	93%	95%	90%
5.	Investment Yield	5.4%	5.1%	5.8%	6.3%	7.6%
6.	Change in Surplus	17.5%	16.0%	8.1%	20.0%	35.8%
7.	Liabilities to Liquid Assets	64%	63%	63%	62%	56%
8.	Agents Balances to Surplus	26%	23%	23%	21%	17%
9.	One Year Development	5.1%	0.8%	1.3%	1.2%	2.9%
10.	Two Year Development		6.6%	1.9%	2.0%	3.1%
11.	Estimated Current Reserve Deficiency to Surplus		#N/A	-3.7%	-4.4%	4.0%

Output

Constrained Growth Case
50 Iterations Using Excel

Exhibit C-7

		Apriori Loss & ALAE Ratios				
		Accident Years				
Coverage	Subdivision	1998	1999	2000	2001	2002
All	Direct	0.61	0.55	0.64	0.61	0.59
	Ceded	0.16	0.05	0.11	0.06	0.06
	Net	0.67	0.62	0.71	0.68	0.66
HMP	New	0.67	0.83	1.03	0.68	0.91
	Renewal	0.49	0.58	0.82	0.72	0.67
	Renewal (2)	0.81	0.49	0.71	0.67	0.64
	Direct	0.79	0.57	0.78	0.68	0.68
	Ceded	0.10	0.02	0.04	0.00	0.00
	Net	0.86	0.63	0.86	0.75	0.75
PPAL	New	0.99	1.07	1.07	0.92	1.00
	Renewal	0.68	0.90	0.94	0.79	0.84
	Renewal (2)	0.73	0.70	0.88	0.80	0.85
	Direct	0.76	0.77	0.92	0.82	0.88
	Ceded	0.00	0.00	0.00	0.00	0.00
	Net	0.79	0.81	0.96	0.86	0.92
APD-P	New	0.75	0.69	0.74	0.66	0.57
	Renewal	0.59	0.60	0.74	0.63	0.62
	Renewal (2)	0.57	0.51	0.60	0.62	0.49
	Direct	0.62	0.57	0.69	0.63	0.52
	Ceded	0.07	0.02	0.10	0.00	0.00
	Net	0.65	0.61	0.73	0.68	0.56
APD-C	New	0.76	0.82	0.82	0.76	0.78
	Renewal	0.34	0.39	0.63	0.70	0.45
	Renewal (2)	0.49	0.38	0.62	0.56	0.42
	Direct	0.52	0.44	0.68	0.60	0.46
	Ceded	0.10	0.02	0.08	0.00	0.00
	Net	0.55	0.47	0.72	0.64	0.49
CAL	New	1.08	1.63	0.85	1.56	1.25
	Renewal	0.67	0.79	0.87	0.67	0.88
	Renewal (2)	0.86	0.93	0.36	0.47	0.72
	Direct	0.86	0.97	0.45	0.60	0.79
	Ceded	0.02	0.02	0.01	0.02	0.03
	Net	0.89	1.02	0.48	0.63	0.83
CMP-L	New	0.84	0.87	0.76	0.64	0.65
	Renewal	0.55	0.59	0.67	0.57	0.48
	Renewal (2)	0.48	0.53	0.46	0.51	0.40
	Direct	0.52	0.57	0.52	0.53	0.44
	Ceded	0.00	0.00	0.00	0.00	0.00
	Net	0.54	0.59	0.54	0.56	0.46
CMP-P	New	0.83	0.94	0.79	0.66	0.37
	Renewal	0.55	0.51	0.69	0.65	0.58
	Renewal (2)	0.39	0.16	0.36	0.45	0.51
	Direct	0.54	0.35	0.58	0.52	0.52
	Ceded	0.33	0.13	0.26	0.14	0.14
	Net	0.57	0.39	0.63	0.58	0.59
OL	New	0.66	0.03	0.48	0.25	0.19
	Renewal	0.27	0.62	0.45	0.24	0.12
	Renewal (2)	0.47	0.24	0.12	0.69	0.08
	Direct	0.47	0.25	0.20	0.59	0.10
	Ceded	0.00	0.00	0.00	0.02	0.00
	Net	0.49	0.26	0.21	0.61	0.10
OL-U	New	0.20	0.02	0.37	0.07	0.18
	Renewal	0.18	0.03	0.20	0.04	0.12
	Renewal (2)	0.14	0.02	0.01	0.03	0.03
	Direct	0.15	0.02	0.08	0.04	0.06
	Ceded	0.15	0.02	0.07	0.04	0.06
	Net	0.16	0.02	0.08	0.04	0.06
WC	New	0.52	0.81	0.71	0.98	0.65
	Renewal	0.45	0.51	1.04	0.77	0.63
	Renewal (2)	0.55	0.47	0.51	0.49	0.37
	Direct	0.53	0.50	0.57	0.58	0.43
	Ceded	0.00	0.00	0.00	0.00	0.00
	Net	0.55	0.51	0.58	0.60	0.45

Output

New Business
Homeowners Multiple Peril
Direct Underwriting Module

Exhibit C-8

Description	2nd Prior Year 1996	1st Prior Year 1997	1st Year 1998	2nd Year 1999	3rd Year 2000	4th Year 2001	5th Year 2002
			Accident Years				
1. Premiums:							
a. Exposure Growth Rate			-1%	2%	2%	2%	2%
b. Number of Exposures	10,740	9,569	6,282	6,736	6,881	7,031	7,177
c. Average Rate Growth Rate			3%	4%	4%	10%	13%
d. Average Rate per Exposure	387.61	377.37	389.81	403.87	421.91	463.85	521.94
e. Written Premiums	4,162,984	3,610,877	2,448,776	2,720,447	2,903,131	3,261,302	3,745,942
f. Earning Ratio	0.50	0.50	0.50	0.50	0.50	0.50	0.50
g. Earned Premiums	4,162,984	3,886,930	3,029,826	2,584,612	2,811,789	3,082,216	3,503,622
h. Unearned Premium Reserves	2,081,492	1,805,438	1,224,388	1,360,224	1,451,565	1,630,651	1,872,971
i. Renewal Ratio	60%	60%	60%	60%	60%	60%	60%
2. Expenses:							
a. Commissions	585,760	486,894	342,829	380,863	406,438	456,582	524,432
b. General Expense	272,033	243,112	196,939	168,000	182,766	200,344	227,735
c. Other Acquisition	523,786	458,017	357,519	304,984	331,791	363,702	413,427
d. Premium Taxes	133,330	117,821	83,258	92,495	98,706	110,884	127,362
e. Policyholder Dividends	-	-	-	-	-	-	-
f. Other Nonrecurring Expenses	-	-	931,848	-	-	-	-
g. Subtotal (Expenses)	1,514,908	1,305,843	1,912,393	946,342	1,019,702	1,131,512	1,292,957
3. Losses:							
a. Initial Severity Mean	2,000	2,000	2,000	2,000	2,000	2,000	2,000
b. Initial Severity Std.	192	192	192	192	192	192	192
c. Severity Trend	0.959	1.000	1.056	1.082	1.165	1.278	1.403
d. U/W & Rate Adjustments							
e. Modeled Severity	2,228	2,021	2,098	2,120	2,227	1,946	3,050
f. Initial Frequency Mean	0.157	0.157	0.157	0.157	0.157	0.157	0.157
g. Initial Frequency Std.	0.014	0.014	0.014	0.014	0.014	0.014	0.014
h. Frequency Trend	1.000	1.000	1.000	1.000	1.000	1.000	1.000
i. U/W & Rate Adjustments	1.000	1.000	1.000	1.000	1.000	1.012	1.022
j. Modeled Frequency	0.16	0.15	0.15	0.15	0.19	0.15	0.15
k. a Priori Ultimate Losses & ALAE	3,744,409	2,934,173	2,038,890	2,136,907	2,882,135	2,081,766	3,195,043
l. a Priori Loss & ALAE Ratio	0.90	0.75	0.67	0.83	1.03	0.68	0.91
m. New Business Penalty	0.09	(0.23)	(0.12)	0.25	0.24	(0.01)	0.23

Exhibit D-1

@RISK Simulation of DYNAMO2E.XLS

Run on 3/19/98
Simulations = 1
Iterations = 1,000

	1998 Surplus	1999 Surplus	2000 Surplus	2001 Surplus	2002 Surplus
Minimum =	(461,984,300)	(464,044,400)	(658,655,200)	(3,981,046,000)	(4,109,432,000)
Maximum =	219,620,400	232,325,600	247,397,700	279,958,500	349,451,900
Mean =	175,183,300	172,729,100	162,437,000	140,325,500	119,960,000
Std Deviation =	25,367,610	35,172,020	58,886,520	158,989,100	170,083,700
Skewness =	(16)	(8)	(8)	(19)	(17)
Kurtosis =	399	135	96	464	393
Errors Calculated =	0	0	0	0	0
Mode =	183,323,400	189,186,500	157,942,800	146,236,600	144,654,100
5% Perc =	150,111,300	133,530,300	110,952,500	69,588,900	17,046,880
10% Perc =	156,466,800	143,562,100	126,444,200	93,083,060	47,894,900
15% Perc =	160,428,200	150,650,900	134,350,400	109,960,300	69,858,660
20% Perc =	163,369,400	156,111,900	141,608,300	118,977,200	83,737,990
25% Perc =	166,067,800	160,462,300	147,622,400	125,549,900	92,905,990
30% Perc =	168,222,300	164,026,900	153,066,700	133,642,000	102,295,100
35% Perc =	170,358,300	167,002,500	157,766,700	140,360,400	110,937,800
40% Perc =	172,559,400	170,754,200	161,036,400	144,735,900	120,214,500
45% Perc =	174,956,500	172,904,100	165,105,400	148,773,700	126,096,300
50% Perc =	177,070,800	175,912,900	168,157,400	154,535,400	132,922,300
55% Perc =	178,402,200	178,036,100	171,592,500	160,116,100	141,605,100
60% Perc =	180,724,700	180,663,500	175,279,600	164,656,600	147,768,700
65% Perc =	182,738,400	183,921,300	179,008,700	168,188,100	155,598,400
70% Perc =	184,302,400	187,031,800	183,095,600	172,875,500	162,953,700
75% Perc =	186,266,200	189,848,800	186,883,400	179,016,000	171,141,000
80% Perc =	188,621,500	193,273,200	190,873,800	185,600,500	181,757,900
85% Perc =	190,977,800	197,860,700	197,096,300	193,082,800	192,261,800
90% Perc =	194,243,200	202,111,600	204,055,200	205,672,300	209,546,100
95% Perc =	199,679,300	208,505,600	213,341,300	219,849,100	233,545,300

Exhibit D-2

@RISK Simulation of DYNAMO2E.XLS

Run on 3/19/98
Simulations = 1
Iterations = 1,000

	1998 NWP/Surplus Ratio	1999 NWP/Surplus Ratio	2000 NWP/Surplus Ratio	2001 NWP/Surplus Ratio	2002 NWP/Surplus Ratio
Minimum =	(0.808)	(5.412)	(94.557)	(189.958)	(634.106)
Maximum =	3.736	104.599	3808.938	78.941	346.398
Mean =	2.132	2.458	6.631	3.471	4.851
Std Deviation =	0.231	3.285	120.382	7.519	30.270
Skewness =	(0.918)	30.076	31.510	(15.803)	(10.783)
Kurtosis =	32.101	935.062	995.272	455.744	255.953
Errors Calculated =	0.000	0 000	0.000	0.000	0.000
Mode =	2.091	2.379	2.803	2.950	3.645
5% Perc =	1.852	1.881	2.067	2.261	2.212
10% Perc =	1.901	1 966	2.187	2.445	2.609
15% Perc =	1.937	2.011	2.272	2.604	2.895
20% Perc =	1.967	2.062	2.351	2.708	3.085
25% Perc =	1.990	2.108	2.415	2.816	3.288
30% Perc =	2.011	2.149	2.461	2.929	3.442
35% Perc =	2.034	2.187	2.518	3.008	3.629
40% Perc =	2.061	2.220	2 573	3.105	3.786
45% Perc =	2.082	2.260	2.647	3.192	3.988
50% Perc =	2.103	2.282	2.705	3.286	4.180
55% Perc =	2.131	2.320	2.766	3.386	4.396
60% Perc =	2.155	2.354	2.813	3.503	4.666
65% Perc =	2.183	2.402	2.875	3.608	4.996
70% Perc =	2.214	2.464	2.952	3.787	5.359
75% Perc =	2.244	2.529	3.069	3.996	5.793
80% Perc =	2.281	2.606	3.207	4.269	6.435
85% Perc =	2.327	2.689	3.366	4.552	7.259
90% Perc =	2.390	2.817	3.588	5.183	9.074
95% Perc =	2.488	3.021	3.989	6.361	14.287

Exhibit D-3

@RISK Simulation of DYNAMO2E.XLS

Run on 3/19/98
Simulations = 1
Iterations = 1,000

	1998 Net Loss Ratio	1999 Net Loss Ratio	2000 Net Loss Ratio	2001 Net Loss Ratio	2002 Net Loss Ratio
Minimum =	0.587	0.569	0.601	0.629	0.624
Maximum =	2.567	1.904	2.612	9.949	1.368
Mean =	0.730	0.742	0.759	0.781	0.772
Std Deviation =	0.072	0.064	0.107	0.318	0.056
Skewness =	16.268	7.583	12.696	24.984	2.537
Kurtosis =	413.071	122.695	199.693	696.739	27.796
Errors Calculated =	0.000	0.000	0.000	0.000	0.000
Mode =	0.750	0.773	0.777	0.766	0.773
5% Perc =	0.654	0.666	0.683	0.690	0.693
10% Perc =	0.670	0.684	0.699	0.706	0.708
15% Perc =	0.682	0.695	0.712	0.718	0.720
20% Perc =	0.692	0.703	0.719	0.726	0.733
25% Perc =	0.700	0.712	0.725	0.736	0.743
30% Perc =	0.706	0.719	0.733	0.742	0.750
35% Perc =	0.712	0.725	0.739	0.750	0.756
40% Perc =	0.719	0.731	0.745	0.757	0.762
45% Perc =	0.724	0.736	0.752	0.762	0.766
50% Perc =	0.729	0.741	0.756	0.766	0.771
55% Perc =	0.736	0.746	0.761	0.769	0.774
60% Perc =	0.741	0.751	0.765	0.772	0.777
65% Perc =	0.748	0.757	0.769	0.775	0.783
70% Perc =	0.754	0.762	0.773	0.779	0.791
75% Perc =	0.759	0.766	0.776	0.786	0.799
80% Perc =	0.765	0.771	0.779	0.798	0.809
85% Perc =	0.771	0.775	0.790	0.808	0.822
90% Perc =	0.776	0.782	0.801	0.821	0.833
95% Perc =	0.790	0.807	0.825	0.841	0.853

Exhibit D-4

@RISK Simulation of DYNAMO2E.XLS

Run on 3/19/98
Simulations = 1
Iterations = 1,000

	1998 Combined Ratio	1999 Combined Ratio	2000 Combined Ratio	2001 Combined Ratio	2002 Combined Ratio
Minimum =	0.954	0.889	0.905	0.928	0.942
Maximum =	2.978	2.260	2.958	10.278	1.705
Mean =	1.118	1.080	1.092	1.112	1.102
Std Deviation =	0.077	0.070	0.111	0.320	0.063
Skewness =	13.829	6.105	11.660	24.671	1.882
Kurtosis =	333.794	92.078	178.080	684.653	18.775
Errors Calculated =	0.000	0.000	0.000	0.000	0.000
Mode =	1.143	1.100	1.123	1.091	1.082
5% Perc =	1.030	0.994	1.007	1.009	1.011
10% Perc =	1.049	1.016	1.023	1.029	1.029
15% Perc =	1.062	1.025	1.035	1.041	1.043
20% Perc =	1.075	1.033	1.047	1.052	1.055
25% Perc =	1.083	1.044	1.055	1.062	1.064
30% Perc =	1.092	1.053	1.061	1.070	1.075
35% Perc =	1.098	1.060	1.068	1.077	1.082
40% Perc =	1.106	1.067	1.075	1.084	1.089
45% Perc =	1.111	1.072	1.081	1.089	1.094
50% Perc =	1.118	1.078	1.086	1.093	1.099
55% Perc =	1.124	1.083	1.091	1.099	1.105
60% Perc =	1.132	1.090	1.098	1.105	1.111
65% Perc =	1.139	1.096	1.103	1.111	1.117
70% Perc =	1.145	1.102	1.109	1.117	1.125
75% Perc =	1.150	1.108	1.116	1.124	1.134
80% Perc =	1.158	1.116	1.122	1.133	1.145
85% Perc =	1.166	1.123	1.133	1.146	1.158
90% Perc =	1.176	1.133	1.147	1.161	1.175
95% Perc =	1.191	1.165	1.171	1.189	1.197

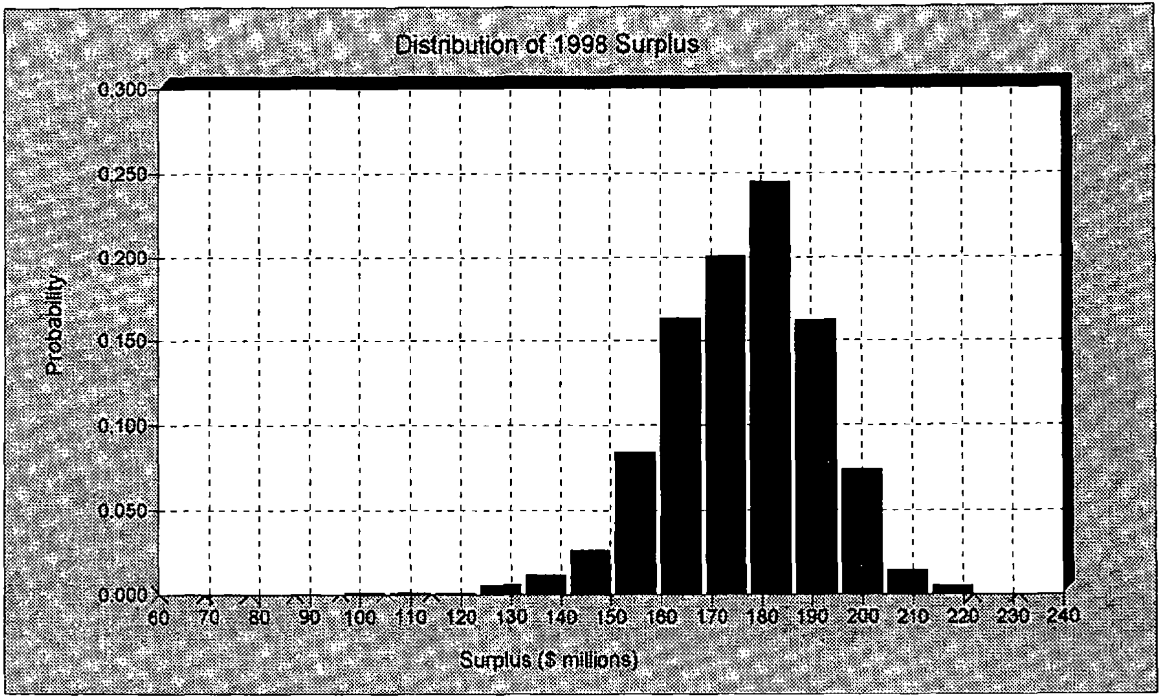
Exhibit D-5

@RISK Simulation of DYNAMO2E.XLS

Run on 3/19/98
Simulations = 1
Iterations = 1,000

	1998 Operating Ratio	1999 Operating Ratio	2000 Operating Ratio	2001 Operating Ratio	2002 Operating Ratio
Minimum =	0.874	0.806	0.838	0.871	0.874
Maximum =	2.899	2.177	2.879	10.206	1.657
Mean =	1.039	1.000	1.016	1.040	1.034
Std Deviation =	0.078	0.070	0.111	0.320	0.063
Skewness =	13.774	6.108	11.774	24.704	1.909
Kurtosis =	332.041	91.546	180.029	685.955	19.120
Errors Calculated =	0.000	0.000	0.000	0.000	0.000
Mode =	1.066	1.025	1.029	1.039	1.040
5% Perc =	0.950	0.916	0.931	0.936	0.942
10% Perc =	0.969	0.936	0.947	0.956	0.961
15% Perc =	0.983	0.946	0.959	0.972	0.973
20% Perc =	0.995	0.954	0.972	0.981	0.987
25% Perc =	1.003	0.965	0.979	0.990	0.995
30% Perc =	1.012	0.975	0.985	0.999	1.004
35% Perc =	1.018	0.981	0.992	1.005	1.011
40% Perc =	1.026	0.987	0.999	1.012	1.020
45% Perc =	1.032	0.993	1.005	1.019	1.026
50% Perc =	1.038	0.999	1.010	1.024	1.031
55% Perc =	1.044	1.004	1.016	1.029	1.038
60% Perc =	1.053	1.010	1.022	1.033	1.044
65% Perc =	1.059	1.016	1.027	1.039	1.051
70% Perc =	1.065	1.023	1.033	1.044	1.059
75% Perc =	1.071	1.028	1.040	1.051	1.068
80% Perc =	1.078	1.036	1.047	1.061	1.077
85% Perc =	1.087	1.044	1.055	1.073	1.089
90% Perc =	1.097	1.056	1.069	1.088	1.106
95% Perc =	1.112	1.084	1.093	1.116	1.128

Exhibit D-6


Distribution of 1998 Surplus
Probability
0.300
0.250
0.200
0.150
0.100
0.050
0.000
60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240
Surplus ($ millions)

On the Cost of Financing Catastrophe Insurance

by Glenn G. Meyers, FCAS, MAAA, and
John J. Kollar, FCAS, MAAA

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Insurance Services Office, Inc.

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Dynamic Financial Analysis - Applications And Uses

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Abstract

After surveying various instruments used to finance catastrophe insurance, this paper demonstrates a method for analyzing the cost of financing catastrophe insurance with the following instruments: (1) insurer capital; (2) reinsurance; and (3) catastrophe options. The procedure first quantifies the cost of financing in terms of the cost of those instruments. The method then permits searching for a mix of instruments that minimizes the cost.

Using a catastrophe model, we create a distribution of simulated losses for each of fifty insurers that report their exposure to ISO. We then create an illustrative catastrophe index based on the combined simulated losses of the fifty insurers. We perform a sample analyses for three insurers.

The analyses show that the best mix of capital, reinsurance, and catastrophe options depends on how well an insurer's losses correlate with the index – that is, on the basis risk. Some insurers can significantly reduce their cost of financing catastrophe insurance by using catastrophe options. To illustrate the effect on premiums of the cost of financing catastrophe insurance, we convert those costs into risk loads.

1. Introduction

Hurricane Andrew caused $15.5 billion of insured property losses in 1992. And it missed Miami, otherwise losses could have been in the $50 billion range. The Northridge Earthquake resulted in $12.5 billion of losses in 1994. And it was only of magnitude 6.7.

In a recent study[1], ISO used the Risk Management Solutions, Inc. (RMS) catastrophe model to simulate possible catastrophic events for the insurers who report their exposure to ISO. The study concluded that losses from a severe hurricane along the east cost could exceed $150 billion. Similarly a severe earthquake in California could generate losses of $50 billion or more.

Losses from such a megacatastrophe could have severe adverse effects on property/casualty insurers and their policyholders. Many insurers could become insolvent or seriously impaired and, therefore, unable to continue insuring the same volume of business. The recognition of this risk has stimulated industry efforts to address the problem of megacatastrophes. Insurance regulators, legislators, government agencies, investment bankers, and others have also contributed to the public policy debate on this critical issue.

Catastrophe Management

A property/casualty insurer can measure the extent of its catastrophe risk by conducting a portfolio analysis to determine the expected distribution of losses from possible events such as hurricanes or earthquakes. This distribution of losses is created by analyzing the company's catastrophe exposure with a computer simulation model, which provides an estimate of losses that would result from a representative set of catastrophic events. Where potential catastrophe losses are too high, the insurer might take steps to reduce its concentration of exposures. Some insurers have given up some business in overly exposed areas to reduce their catastrophe risk to a more manageable level. An insurer could also diversify its catastrophe risk by writing more exposures in areas where it has a lower concentration of exposures or in areas not subject to catastrophes. A concern about that strategy is that the insurer could be taking on a different risk by writing new business in areas where it lacks expertise and an effective distribution network.

Many insurers have opted for loss-reduction measures such as increasing deductible sizes, imposing special wind/earthquake deductibles and offering discounts for loss mitigation activities by policyholders (such as the addition of storm shutters).

Property/casualty insurers have pursued many loss mitigation efforts, such as the ISO Building Code Effectiveness Grading Schedule (BCEGS). The BCEGS program evaluates a community's building code and its enforcement. Insurers can offer discounts for structures built in municipalities with good enforcement of an effective loss mitigating building code.

Financing Catastrophe Risk

Insurers have also been looking at ways of financing their catastrophe risk. One approach is adding capital to the balance sheet. Many insurers have benefited from recent stock market gains as a source of additional capital. Because of their improved capital positions, some insurers have elected to retain more catastrophe risk.

The surge in catastrophes that began in 1989 with Hurricane Hugo, resulted in an increased demand for reinsurance., The rising demand, in turn, produced substantial price increases which led to the formation of new catastrophe reinsurers. That increase in reinsurer capital coupled with improved catastrophe experience has led to more plentiful and less expensive catastrophe coverage.

Traditional reinsurance is not the only approach to financing catastrophes. Those active in capital markets activities, reinsurers, reinsurance intermediaries and property/casualty

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[1]Insurance Services Office, Inc., *Managing Catastrophe Risk*, May 1996.

insurers themselves have come to recognize the possibility of securitizing risk – that is, using other financial instruments to transfer catastrophe risks to the broader capital markets.

All of the instruments for financing catastrophe risk have a cost, but they also have benefits. It takes sophisticated analysis to find an efficient mix of risk financing instruments that provides the greatest benefit for the least cost. Providing an example of such an analysis is the goal of this paper.

This analysis is part of what casualty actuaries call dynamic financial analysis, or DFA. It is similar to other aspects of DFA because it views the various risk financing instruments as assets, with the returns on these assets being positively correlated to insurer losses.

A key factor for delivering an efficient mix of risk financing instruments is the cost of the individual instruments. This cost ultimately becomes part of the price of insurance. This price will be sensitive to the variation in results – many years with small catastrophe losses and occasional years with very large catastrophe losses. Actuaries have traditionally called this part of the price the risk load. We must expand the definition of traditional risk load to include the various instruments available to finance catastrophe insurance.

The intense competitive forces in the marketplace may cause insurers to focus on short-term operating results at the expense of long-term solidity. This amounts to insurers ignoring the possibility of rare catastrophes in their decision making. Insurers may not adequately reflect risk load in pricing, nor make sufficient provision for catastrophe risk financing.

The capital markets can bring an immense amount of financing into the insurance industry, and perhaps significantly lower the cost of financing for the long term. Our challenge is to figure out how to efficiently bring these resources into the insurance industry.

2. A Survey of the Instruments Used in Financing Insurance

Raising Insurer Capital

An insurer always has the option of raising sufficient capital to cover its potential losses, but to raise capital, the insurer must increase its net income to justify this capital. There is also the lost opportunity since the capital committed to an insurer is not available for another venture.

Compared with other industries, property/casualty insurance has not generally achieved high historic returns. Competition from the large number of suppliers has been a major contributing factor. Furthermore, regulation has in some cases also acted to keep insurance rates below actuarially indicated levels.[2]

If an insurer has a heavy concentration of exposures in catastrophe-prone areas, the amount of capital needed can be relatively large compared with the insurer's existing surplus. Furthermore, the additional capital may only be needed occasionally when catastrophe losses are unusually large – perhaps every 100 years. Committing a large amount of additional capital to cover infrequent losses is extremely inefficient and virtually impossible to sustain in a highly competitive marketplace.

Those considerations drive an insurer to seek alternatives to raising capital.

Reinsurance

The capital of US reinsurers was $13.2 billion in 1992. It grew to $26.2 billion by the end of 1997. With the increased demand for reinsurance following the catastrophes in the early 1990s, new offshore reinsurers provided additional capacity. But that capacity is also relatively small compared with the size of potential catastrophe losses.

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[2] Insurance Services Office, *Risk and Returns: Property/Casualty Insurance Compared with Other Industries*. December 1995.

Reinsurers provide modest layers of coverage which are usually sufficient to protect small insurers but not larger insurers.

The availability of reinsurance varies considerably over the life of an insurance cycle. The price may also vary substantially depending on supply and demand as well as recent experience.

Reinsurance pays for the primary insurer's losses that exceed certain amounts, or on a quota share basis. The reinsurance coverage follows the fortunes of the primary insurer. On the other hand, reinsurance can also have high and variable transaction costs for the customized coverage provided.

It is important to remember that a reinsurer may not be able to meet its obligation if a large catastrophe occurs.

One possible solution to the problem of large catastrophes is proposed legislation under which the federal government would provide excess reinsurance. The trade-off for providing this coverage may be increased regulation.

Securitization

The property/casualty insurance industry does not have enough capital to handle a very large catastrophe. By contrast, the broader capital markets have trillions of dollars to invest. The returns on many of these investments are correlated – that is their value is influenced by the same economic conditions. To diversify their portfolios, investors are always looking for investment opportunities not correlated with the economy. Catastrophe risk is independent of the economic conditions that affect other financial instruments.

Many types of financial instruments to transfer catastrophe risk have emerged in recent years. They treat catastrophe risk in various fashions, but all offer the investor a way to profit in exchange for accepting some risk.

Catastrophe bonds have already gained a level of acceptance with several successful deals. A catastrophe, or contingency, bond represents a loan (principal) over a specified term in exchange for fixed interest payments. The occurrence of a qualified catastrophic event during the term of the bond may result in the reduction or elimination of interest payments and for some bonds the loss of some or all of the principal that the investor has loaned to the insurer. If no qualifying catastrophe occurs, the investor receives his principal plus interest. The interest rate usually reflects a premium to reward the additional risk.

Catastrophe bonds generally reflect the catastrophe experience of the insurer selling the bond, although covered losses can be based on an index of industry catastrophe losses. If an industry index is used, then the bond may not mirror the catastrophe experience of the selling insurer.

Securitization of risk has also involved contingent equities. In an agreement developed by Aon Corporation, called a CatEPut℠, an insurer purchases the option of selling a prearranged amount of its stock if a qualifying catastrophe occurs.

This arrangement provides the insurer with immediate access to equity in the event that a loss impairs its surplus. The additional equity increases the likelihood that the insurer will maintain its ratings and will be able to continue its business operations virtually uninterrupted in the wake of such a loss. The seller of the CatEPut℠ has the option to eventually convert the preferred shares to common stock. The insurer can refinance and redeem the shares at any time[3]. Also, there is a provision that the investor does not have to purchase the stock if the catastrophe results in a serious impairment of the insurer, in other words, if the investor's capital infusion would not be sufficient to continue the financial viability of the insurer.

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[3] Reported by William Jewett "Converging Roles Within the Insurance and Finance Marketplace" at the web site: http://www.centrere.com/insights/converge.htm on April 3, 1998.

A third kind of securitization deal involves trading options on a catastrophe index. The index is based on the catastrophe experience of (at least a sample of) the property/casualty industry. An insurer or reinsurer can purchase catastrophe call options that are exercisable if the catastrophe index exceeds a specified strike price. When the index value exceeds the strike price, the contract pays either a specified flat amount, or the amount by which the index exceed the strike price.

These options are traded on an exchange. For example, the Property Claims Service (PCS) index is traded on the Chicago Board of Trade (CBOT). The Guy Carpenter Catastrophe Index (GCCI) is traded on the Bermuda Commodities Exchange (BCE). In addition to public trading, these indices may also be used in private placements. The Risk Management Solutions (RMS) catastrophe index, which is based on the RMS catastrophe model, is used for this specific purpose.

From an individual insurer's perspective, a critical element when considering the use of a catastrophe index is basis risk – that is, how well the index correlates with the insurer's experience. For example, an insurer with exposure concentrated in a small geographic area may suffer high losses if a catastrophe occurs in that area. But that catastrophe may not trigger options based on a national index. An insurer can improve the potential correlation by purchasing options based on smaller geographic areas, such as regions, states or even ZIP-codes, that match the insurer's own portfolio.

Many investors favor the use of an industry index because the losses are not a function of an individual insurer's underwriting and claim settlement practices. Furthermore, the provisions of an option contract are standardized. This increases liquidity, as standardized contracts are easier to trade than customized contracts. Because of standardization, options can have smaller transaction costs than reinsurance or catastrophe bonds which require individual analysis and negotiation.

Catastrophe options provide certain challenges that insurers must recognize. As noted earlier, basis risk provides a measure of how well catastrophe options will meet an insurer's need to hedge risk. An insurer may collect substantial funds on catastrophe options when its actual catastrophe losses are small. More importantly, an insurer may collect little or no funds on catastrophe options but still suffer a substantial catastrophe loss. An insurer must carefully analyze basis risk before deciding if catastrophe options are a good way of hedging catastrophe risk.

Another critical element in the success of securitization is the regulatory acceptance of catastrophe options and other securitization instruments as reinsurance – an offset to an insurer's direct losses. Some insurers have established offshore companies to reinsure their catastrophe risk. The insurers then sell catastrophe bonds or use other financial instruments to finance the offshore reinsurers.

Rating agencies' evaluation of an insurer's financial strength is a critical element in attracting and retaining business. If rating agencies do not view an insurer's securitization measures as financially sound, the insurer may receive a poor rating – and therefore suffer a loss of business. Consequently, rating agencies' acceptance of a catastrophe securitization approach may be important to its success.

3. The Cost of the Instruments Used in Financing Insurance

So far, this paper has surveyed the various instruments available to finance catastrophe risk. The remainder of the paper will focus on one promising form of securitization – options on a catastrophe index – and see how insurers can combine them with capital and reinsurance to finance catastrophe risk.

We classify the various instruments for financing catastrophe insurance into the following elements:

1. Insurer Capital – This is money put up by investors in the insurance company. The company can use its capital to pay losses if current income is insufficient.
2. Reinsurance – This is money provided by outside entities that agree to pay losses in accordance with a predetermined function of the insurer's loss. Some securitization deals fall into this category.
3. Catastrophe Options – This is money provided by outside entities that agree to pay money contingent on the occurrence of a catastrophic event recorded on an index. That payment may or may not correspond with the insurer's loss. That is, catastrophe options do present basis risk.

Each instrument has a cost and a benefit. The insurer's problem is to find the combination of instruments that provides adequate financing for the least cost.

We define:
The cost of financing insurance =

the expected loss (net of reinsurance recoveries and recoveries from catastrophe options)
+ the cost of capital
+ the cost of reinsurance
+ the cost of catastrophe options

Our purpose in using reinsurance and catastrophe options is to reduce the expected loss and the cost of capital – and ultimately the cost of financing insurance.

Although this definition covers the insurer's entire operation, we will focus on catastrophes. Thus, our discussion of the cost of financing insurance will reflect *only the catastrophe losses*, with one exception – the cost of capital. The insurer's other assets and liabilities affect that cost. This discussion will ignore the remaining elements of the insurer's operation.

Quantifying the Cost of Financing Insurance

To perform this analysis, we will need to quantify the cost of financing insurance in terms of the probability of a catastrophic loss. We give some sample costing formulas below. The formulas have the advantage of being simple, but they are by no means unique or necessary to the examples given below.

For any random variable, Z, we define:

μ_Z = the expected value of Z

σ_Z = the standard deviation of Z.

See the appendix for the formulas for the various means and standard deviations used below.

Quantifying the Cost of Capital

We employ a probabilistic capital requirements formula as the starting point for this methodology. In the United States, insurers are not subject to an official probabilistic capital requirements formula. However, most actuaries believe that capital requirements should have probabilistic input. Actuaries generally accept the idea of a formula, but any particular formula will spark a debate. While we use one such formula here, an insurer can use another formula that suits the needs and perceptions of its management.

Let X be a random variable representing the insurer's total loss, net of recoveries from reinsurance and catastrophe options. Our formula for the cost of capital is:

$$\text{Cost of Capital} = K \times T \times \sigma_X$$

where:

T is a factor reflecting the insurer's risk aversion; and

K is the required return needed to attract sufficient capital.

We can link T to the insurer's probability of insolvency. For example, if we assume the insurer's losses follow a normal distribution, a choice of T = 2.32 corresponds to a one-in-one-hundred chance of insolvency. If the insurer is more risk averse, or if it feels that the distribution of insurer results is unusually skewed, the insurer can select a higher value of T.

The insurer will select K so that its rate of return is close to that obtained by other investments with similar risk. K will vary with market conditions.

In the examples below, we will let

$$X = X_O + X_C$$

where:

X_C = All catastrophe losses net of recoveries from reinsurance and index contracts; and

X_O = All other net losses.

When we partition X is this manner, the formula for the cost of capital becomes

$$\text{Cost of Capital} = K \times T \times \sqrt{\sigma^2_{X_O} + \sigma^2_{X_C}}$$

under the assumption that X_O and X_C are independent.

Quantifying the Cost of Reinsurance

The cost of catastrophe reinsurance depends upon market conditions. After a large catastrophe, the demand for reinsurance usually rises and reinsurer capital falls. Therefore, catastrophe insurance is in short supply and the reinsurance available fetches a high price. High prices attract new capital to reinsurers, and prices generally fall until the next catastrophe occurs.

The benefit of the reinsurance treaty is to reduce the insurer's cost of capital by reducing its expected loss, μ_{X_C}, and its standard deviation of loss, σ_{X_C}.

To develop a strategy for using reinsurance, an insurer needs to know its reinsurance costs. Those costs depend upon the retention and the limit of the reinsurance treaty, and each reinsurer has its own prices.

Let X_R be a random variable representing the reinsurance recovery. We will use the following formula for the cost of reinsurance in the examples below:

$$\text{Reinsurance Cost} = (\mu_{X_R} + \lambda \cdot \sigma^2_{X_R}) \times (1 + e)$$

where λ is a risk load multiplier, and e is an acquisition expense factor.

Quantifying the Cost of Catastrophe Options

In this paper, we will work with binary options on a catastrophe index. The holders of those options exercise them for a fixed amount, such as $1,000, when the index exceeds a predetermined strike price. Otherwise the options expire worthless.

To the seller of such options, the expected return should be competitive with other available investments of comparable risk. One way of gauging comparable risk is the analysis of bond defaults. For example, Moody's Investors Service has a web site that publishes bond default rates and interest rate spreads. In browsing Moody's web pages one finds the following statements about default rates:

- "Moody's trailing 12-month default rate for speculative-grade issuers ended 1997 at 1.82% -- up from last year's 1.64%, but well below its average since 1970 of 3.38%."
- "Moody's expects its speculative-grade 12-month default rate to rise toward the 2.5% level in 1998."[4]

With respect to interest rate spreads, Moody's states the following:

- "The spread of the median yield-to-maturity of intermediate-term speculative-grade bonds over seven-year US Treasuries climbed just 3 basis points to 267 basis points -- 92 basis points below its January 1993 to January 1997 average of 359 basis points."[5]

When comparing speculative-grade bonds to catastrophe options, the investor might consider the following:

- The projected 12-month default rate of speculative-grade bonds is 2.5%.
- We can estimate the probability of exercising the catastrophe options (as we will show below). We can compare that probability with estimated default rates for bonds.
- Catastrophe options can require posting a 100% margin at the time of sale. The money in the margin account earns a risk-free rate of return. Thus, the price of the option should be comparable to the interest rate spread for a bond of comparable risk over risk-free investments.
- The average spread of speculative-grade bonds over intermediate-term risk-free investments is about 3.5%. The spread could be lower over a 12-month term, but it should not be lower than the projected default rate.

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[4] The web site URL is http://www.moodys.com/defaultstudy/index.html. We obtained this quote on April 3, 1998.

[5] The web site URL is http://www.moodys.com/economic/1QDFLT97.htm. We obtained this quote on April 3, 1998.

- The exercise of a catastrophe option is not correlated with the other economic risks. That fact makes the catastrophe options more attractive to investors and should lower their price.

With all this information, one can compare the posted price of catastrophe options with bonds of equivalent risk. Investors will have varying interpretations of the information, but our point is that information relevant to the pricing of catastrophe options is publicly available.

4. An Illustrative Example

As an illustration of the kind of analysis investors and insurers can do, we used a catastrophe model to quantify the cost of financing insurance in terms of the costs of attracting capital, buying reinsurance, and buying catastrophe options. We compared the insurer's losses – generated by the catastrophe model – to the benefits provided by the various instruments.

To do the analysis, we took a sample of fifty insurers that report their personal lines exposure to ISO. We then analyzed the personal lines exposure for each of the fifty insurers using a hurricane model provided by Risk Management Solutions, Inc.[6] The analysis provided loss estimates and annual rates of occurrence for about 9,000 events for the insurers in the sample. We created "index" events by summing the losses for each event over all the insurers. We then multiplied the loss for each event by a factor that set the largest event equal to 100.

We then produced Table 4.1 below. The table contains the illustrative index values and the model-generated losses for one of the fifty insurers from the sample. We produced a similar exhibit for each of the fifty insurers.

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[6] All hurricane loss estimates incorporated in this paper were developed by ISO's use of Risk Management Solutions' (RMS) proprietary IRAS hurricane technology. However, development of the individual company exposure data and the analyses were performed by ISO. Therefore the loss projections and conclusions presented in this paper are the responsibility of ISO.

With information like that provided in the exhibit, we can adjust insurer losses for any recoveries from a reinsurance contract or from catastrophe options. Since the model gives us the probability[7] of any loss and/or recovery, we can calculate any summary statistics needed to determine the cost and benefits of the various instruments used in financing insurance.

Table 4.1
Illustrative Index and Insurer Information

Event	Event Probability	Illustrative Index Value	Direct Insurer Loss
1	0.000001210	100.000	1,212,550,269
2	0.000001210	89.041	1,509,161,589
3	0.000001810	87.558	1,303,694,653
4	0.000007020	83.480	761,956,629
5	0.000007020	83.197	734,137,782
6	0.000004660	82.153	735,660,852
7	0.000007910	80.948	1,004,861,128
8	0.000050600	80.548	1,071,076,934
9	0.000007020	79.187	688,269,904
10	0.000001810	77.481	1,652,933,116
11	0.000002590	76.217	741,327,246
12	0.000005760	75.547	654,930,780
13	0.000009060	75.175	1,450,085,508
14	0.000022900	75.108	1,148,344,417
15	0.000001210	75.046	1,003,713,967
16	0.000007020	74.142	718,320,849
17	0.000000460	73.670	612,322,934
18	0.000002590	72.964	607,625,092
19	0.000000767	72.303	1,035,338,915
20	0.000000460	72.180	564,886,456
21	0.000001810	72.050	1,269,991,504
22	0.000021000	71.547	921,203,300
23	0.000000738	71.478	582,199,078
24	0.000018700	71.246	757,962,586
25	0.000000202	70.661	1,078,827,927
26	0.000001210	70.567	1,017,469,903
27	0.000001210	70.289	1,162,380,661
28	0.000001810	68.992	1,273,618,722
29	0.000007250	68.731	966,395,280
30	0.000007020	68.640	598,955,192
⇓	⇓	⇓	⇓

[7] Event probabilities can be calculated from the RMS model output. The RMS model provides annual rates of occurrence for individual events.

Illustrative Catastrophe Options

Using the illustrative catastrophe index, we set up illustrative catastrophe options that pay $1,000 if the largest single event loss in the year exceeds a specified strike price. If no single event exceeds the strike price, the option is not exercised and the buyer receives $0. In the examples that follow, we consider trades on options with strike prices of 5, 10, 15, . . . , 95, 100. The following table gives the probabilities that each option will be exercised. See the appendix for the formula for calculating those probabilities.

Table 4.2

Strike Price	Exercise Probability
0	1.00000000
5	0.16313724
10	0.07855957
15	0.04006306
20	0.02321354
25	0.01387626
30	0.00816229
35	0.00440132
40	0.00296168
45	0.00187601
50	0.00100615
55	0.00070126
60	0.00040197
65	0.00028771
70	0.00018975
75	0.00013880
80	0.00008846
85	0.00001125
90	0.00000121
95	0.00000121
100	0.00000121

The catastrophe options used in this example have a structure similar to those traded on the Guy Carpenter Catastrophe Index (GCCI),[8] with four important differences:

1. The scale of the indices is different. The illustrative index has 100 as its highest value whereas the GCCI has 700 as its highest value.
2. The sets of insurers that make up the indices are different.
3. The illustrative index simply sums the losses for each insurer, whereas the GCCI uses a complex set of rules designed to keep a single insurer from having too much influence at the ZIP-code level.
4. The illustrative index is an annual index, whereas the GCCI is semiannual and overlaps with the normal hurricane season in either one or five months.

The following table gives the costs used in the examples below. To calculate the price of the option, we added 0.035% of the variance of the contract payoff to the expected payoff. We arrived at the 0.035% figure by comparing the exercise probability of an option with a strike price of 20, against the price of a speculative-grade bond, as discussed above.

[8] For information about the options traded on the Guy Carpenter Catastrophe Index, visit the Bermuda Commodities Exchange web site at http://www.bcoe.bm

Table 4.3

Strike Price	Expected Payout	Contract Price
0	1000.000	1000.000
5	163.137	210.920
10	78.560	103.895
15	40.063	53.523
20	23.214	31.150
25	13.876	18.666
30	8.162	10.996
35	4.401	5.935
40	2.962	3.995
45	1.876	2.531
50	1.006	1.358
55	0.701	0.947
60	0.402	0.543
65	0.288	0.388
70	0.190	0.256
75	0.139	0.187
80	0.088	0.119
85	0.011	0.015
90	0.001	0.002
95	0.001	0.002
100	0.001	0.002

Insurer Examples

The following analysis of three insurers shows how those insurers can reduce the cost of financing insurance through the proper use of reinsurance and catastrophe options. The insurers are three members of the sample of fifty insurers that we selected above. We randomly adjusted the losses of each insurer to protect their anonymity.

- Insurer #1 is a medium sized national insurer with exposure that tracks relatively well with the exposure underlying the illustrative index.
- Insurer #2 is a large national insurer with exposure that tracks less well with the exposure underlying the index than Insurer #1.
- Insurer #3 is a regional insurer with exposure that does not track well with that of the index.

We provide summary statistics for the insurers' catastrophe losses.

Table 4.4

	Insurer #1	**Insurer #2**	**Insurer #3**
Expected Catastrophe Loss	34,839,348	95,417,229	2,385,629
Std. Dev. Of Catastrophe Loss	81,044,318	196,767,192	18,098,024
Coef. of Correlation with Index	0.93	0.75	0.35

We now provide the economic assumptions underlying our estimate of the cost of financing insurance. The assumptions made here are not specific to the particular insurer, but we could modify the assumptions and/or make them specific after a discussion with an insurer's management.

The Cost of Financing Insurance

As discussed above, we use the following formula for the cost of insurer capital:

$$\text{Cost of Capital} = K \times T \times \sqrt{\sigma_{X_0}^2 + \sigma_{X_C}^2}$$

with K = 20%; T = 3.00 and σ_{X_0} = the insurer's initial σ_{X_C}. In a real case, we would estimate σ_{X_0} by analyzing the insurer's other assets and liabilities.

In the examples that follow, we use the following formula for the cost of reinsurance:

$$\text{Reinsurance Cost } = (\mu_{X_R} + \lambda \cdot \sigma_{X_R}^2) \times (1 + e)$$

with $\lambda = 1.5 \times 10^{-7}$ and e = 10%. The selected value of λ is close to what ISO uses in its risk load formula for increased limits ratemaking.

If the insurer buys N_S contracts for strike price S at cost C_S, the total cost of the index contracts is:

$$\sum_S N_S \cdot C_S$$

Table 4.3 gives the values of C_S for each strike price, S.

The insurer's management has to make three key decisions to minimize the cost of financing insurance:

1. How much capital should the insurer retain?
2. What layer of reinsurance does the insurer buy?
3. How many index contracts, N_S, does the insurer buy at a given strike price, S?

Now, for a given reinsurance layer and a given set of index contracts, we can calculate the quantities μ_{X_R}, $\sigma^2_{X_R}$, μ_{X_C}, and $\sigma^2_{X_C}$ using formulas given in the appendix.

Thus our expression for the cost of financing insurance becomes

$$\mu_{X_C} + K \times T \times \sqrt{\sigma^2_{X_O} + \sigma^2_{X_C}} + (\mu_{X_R} + \lambda \cdot \sigma^2_{X_R}) \times (1+e) + \sum_S N_S \cdot C_S$$

We seek to minimize this expression by choosing the right layer of reinsurance and the right numbers, N_S, of catastrophe options.

We do not now have an analytic solution to this minimizing problem. That is because of the effort involved in deriving one and because we do not feel that the assumptions we made in calculating the cost of financing insurance are final.[9] Instead, we used a numerical search algorithm, Excel Solver™. As it is difficult to ascertain that the numerical search solution is indeed the optimum, we should characterize the results as "the best solution we could find."

In order to reduce the computing time, we restricted the reinsurance retention and limit to multiples of $1,000,000 and the number of catastrophe options to multiples of 100. In addition we forced the number of catastrophe options to be the same for each of the

[9] For an analytic solution to a simpler problem, see "A Buyer's Guide to Options on a Catastrophe Index" by Glenn Meyers. The paper has been accepted for publication in the *Proceedings of the Casualty Actuarial Society*.

following groups of strike prices: 5, 10, 15, and 20; 25, 30, 35 and 40; 45,50, and 55; 60, 65, and 70; 75, 80, and 85; and 90, 95, and 100.

The search for the minimum cost of financing insurance produced the following results:

Table 4.5

Contract	Number of Index Contracts		
Range	**Insurer #1**	**Insurer #2**	**Insurer #3**
5-20	47,400	93,100	0
25-40	74,400	118,100	6,300
45-55	59,500	67,900	0
60-70	47,600	28,600	0
75-85	81,400	545,100	0
90-100	37,200	634,800	0
	Reinsurance		
Retention	73,000,000	457,000,000	54,000,000
Limit	13,000,000	36,000,000	105,000,000

The elements of the cost of financing insurance are as follows:

Table 4.6

Best Solution Obtained for the Cost of Financing Insurance

	Insurer #1	**Insurer #2**	**Insurer #3**
Expected Net Loss	16,315,629	62,086,995	1,464,410
Cost of Capital	47,905,407	143,662,761	12,914,922
Cost of Reinsurance	2,132,070	1,848,530	1,726,342
Cost of Index Contracts	22,252,015	42,409,101	249,427
Cost of Financing Insurance	88,605,121	250,007,387	16,355,100

We compared the "best solution" with two alternative solutions:

Table 4.7

Cost of Financing Insurance without Reinsurance or Index Contracts

	Insurer #1	**Insurer #2**	**Insurer #3**
Expected Net Loss	34,839,348	95,417,229	2,385,629
Cost of Capital	62,095,747	166,962,499	15,356,683
Cost of Reinsurance	0	0	0
Cost of Index Contracts	0	0	0
Cost of Financing Insurance	96,935,095	262,379,728	17,742,312

Table 4.8

Cost of Financing Insurance after Dropping the Smallest Element from the Best Solution

	Insurer #1	Insurer #2	Insurer #3
Expected Net Loss	17,945,994	63,198,145	1,648,555
Cost of Capital	48,508,962	145,045,517	13,023,441
Cost of Reinsurance	0	0	1,726,342
Cost of Index Contracts	22,252,015	42,409,101	0
Cost of Financing Insurance	88,706,971	250,652,763	16,398,337

We can make two observations:

- The introduction of catastrophe options and reinsurance can significantly reduce the cost of financing insurance. In the examples the cost was reduced by 8.6 % for Insurer #1, 4.7% for Insurer #2, and 7.8% for Insurer #3.
- The role of catastrophe options was more significant for the insurers whose catastrophe losses were better correlated with the index. Conversely the role of reinsurance was more significant for the insurer whose catastrophe losses were poorly correlated with the index.

The Marginal Cost of Financing Catastrophe Insurance

The examples illustrate that reinsurance and catastrophe options can significantly reduce the cost of financing insurance. However the analysis does not address the question of how much the insurer needs to build the cost of financing into its premiums. Actuaries usually refer to that cost as the risk load.[10]

To answer the question, we calculate the cost of financing insurance, with and without the catastrophe lines. We call the difference between those costs the marginal cost of

[10] See "The Competitive Market Equilibrium Risk Load Formula for Catastrophe Ratemaking" by Glenn Meyers, *Proceedings of the Casualty Actuarial Society LXXXIII*, 1997, for background on risk loads for catastrophe ratemaking. That paper goes beyond the current paper by allocating the risk load to individual insureds. However it accounts only for the cost of capital, and does not account for reinsurance and catastrophe options.

financing catastrophe insurance. If the insurer can recover that cost in the premiums it charges, it should write the insurance.

Continuing our example, the cost of financing insurance without catastrophe insurance[11] is: $K \times T \times \sigma_{X_0}$. Thus the marginal cost of financing catastrophe insurance becomes

$$\mu_{X_C} + K \times T \times \left(\sqrt{\sigma^2_{X_O} + \sigma^2_{X_C}} - \sigma_{X_O}\right) + (\mu_{X_R} + \lambda \cdot \sigma^2_{X_R}) \times (1+e) + \sum_S N_S \cdot C_S$$

We summarize the results for the three insurers in our illustrative example:

Table 4.9

The Marginal Cost of Financing Catastrophe Insurance Using the Best Solution

	Insurer #1	Insurer #2	Insurer #3
Cost of Financing without Cats	43,908,324	103,258,865	10,764,807
Cost of Financing with Cats	88,605,121	250,007,387	16,355,100
Marginal Cost of Cats	44,696,797	146,748,522	5,590,293
Marginal Cost/Expected Loss	1.283	1.538	2.343

We do a similar calculation without considering reinsurance or contracts on a catastrophe index.

Table 4.10

The Marginal Cost of Financing Catastrophe Insurance Without Reinsurance or Index Contracts

	Insurer #1	Insurer #2	Insurer #3
Cost of Financing without Cats	43,908,324	103,258,865	10,764,807
Cost of Financing with Cats	96,935,095	262,379,728	17,742,312
Marginal Cost of Cats	53,026,771	159,120,863	6,977,505
Marginal Cost/Expected Loss	1.522	1.668	2.925

Here we see that the proper use of reinsurance and catastrophe options can have a significant effect on premiums, as the marginal cost of financing catastrophe insurance is substantially lower for each insurer using a mix of reinsurance and catastrophe options.

[11] Technically, we should include the expected value of the losses without the catastrophe insurance. But the focus of this paper is on catastrophes, and the expected loss for the noncatastrophe exposure will cancel out when we compute the marginal cost of financing catastrophe insurance.

5. The Next Steps

This paper has taken a first step beyond the insurer capital and reinsurance paradigm, by showing how to incorporate instruments with basis risk to reduce the cost of financing catastrophe insurance. Having taken this first step, there are a number of directions that can be taken. We list a few.

- The insurer could consider buying catastrophe options on a regional or state index, as well as a national index. The additional flexibility could decrease the cost of providing insurance for some insurers – such as Insurer #3 above.
- Returns from catastrophe options could be imbedded within the reinsurance. That is, the reinsurance would cover the difference between the insurer's actual loss and the index recovery.
- We could create a customized index to form the basis of settlement between the insurer and a reinsurer. Such an index would be based on the industry data, but with a customized set of ZIP-codes. With such an arrangement, adverse selection by the primary insurer would no longer be an issue.
- A reinsurer could use the catastrophe options as a hedge for its combined exposure. To do this, the reinsurer would have to combine the exposure of all its treaties and do an analysis similar to that done above. The options could give the reinsurer increased capacity to write more catastrophe coverage.

Appendix

The Calculation of the Statistics for a Maximum Event Index Contract

This appendix gives the formulas for the statistics used in calculating the cost of financing insurance. The calculations are complicated by the fact that the catastrophe index recovery for an event depends upon whether or not the event was the largest event. We solve this by calculating conditional statistics based on the event being the largest – and then calculate global statistics by summing over the conditional probabilities.

We are given n (about 9000) events from the catastrophe model and the index values associated with each event. We assume that the events are independent and that they can only happen once in a year[12]. The events are sorted in decreasing order of the index value. Table A.3 gives the first 30 rows of the of the calculation. The following table gives the formulas used in this exhibit.

Table A.1
Formulas for Table A.3

ith Row of Column	Description and Formula
Event	The ith event specified by the catastrophe model
Index Value	The value of the index if the ith event is the largest
Event Probability, p_i	The probability of the ith event as specified by the catastrophe model
Max Event Probability, ${}_M p_i$	The probability that the ith event happens and all larger events do not happen $${}_M p_i = p_i \cdot \prod_{j=1}^{i-1} (1 - p_j)$$
Contract Value, v_i	The amount paid by the insurer's portfolio of catastrophe options *given that the ith event is the maximum event*
Direct Insurer Loss, x_i	The loss generated by catastrophe model for the ith event on the insurer's exposure
Reinsurance Recovery, r_i	The amount recovered from the reinsurance contract for the ith event
Event Loss Given Max, e_i	$e_i = x_i - v_i - r_i$

[12] The RMS model provides annual rates of occurrence for events. Because rates are so small, making the assumption that events can only happen once per year is not unreasonable.

Table A.1 – Continued

ith Row of Column	Description and Formula
E[Loss | Event is the Max], E_i	$E_i = e_i + \sum_{j=i+1}^{n} E[(x_j - r_j)]$ $= e_i + \sum_{j=i+1}^{n} (x_j - r_j) \cdot p_j$ $= e_i + E_{i+1} - e_{i+1} + (x_{i+1} - r_{i+1}) \cdot p_{i+1}$
E[Loss^2 | Event is the Max], ${}_2E_i$	${}_2E_i = E_i^2 + \sum_{j=i+1}^{n} Var[(x_j - r_j)]$ $= E_i^2 + \sum_{j=i+1}^{n} (x_j - r_j)^2 \cdot p_j \cdot (1 - p_j)$ $= E_i^2 + {}_2E_{i+1} - E_{i+1}^2 + (x_{i+1} - r_{i+1})^2 \cdot p_{i+1} \cdot (1 - p_{i+1})$

Table A.2

Cost of Financing Insurance Statistics

Overall Statistic	Formula
E[Reinsurance Recovery], μ_{X_R}	$\mu_{X_R} = \sum_{i=1}^{n} p_i \cdot r_i$
Var[Reinsurance Recovery], $\sigma_{X_R}^2$	$\sigma_{X_R}^2 = \sum_{i=1}^{n} r_i^2 \cdot p_i \cdot (1 - p_i)$
E[Net Catastrophe Loss], μ_{X_C}	$\mu_{X_C} = \sum_{i=1}^{n} {}_M p_i \cdot E_i$
Var[Net Catastrophe Loss], $\sigma_{X_C}^2$	$\sigma_{X_C}^2 = \sum_{i=1}^{n} {}_M p_i \cdot {}_2E_i - \mu_{X_C}^2$

Exercise Probabilities

Let PE_i denote the probability that maximum event catastrophe option at the level of event i will be exercised. The option will be exercised if either the ith or a lower numbered (higher loss) event happens. That is:

$$PE_1 = p_1, \ PE_i = p_i + PE_{i-1} \cdot (1 - p_i)$$

Table A.3 Preliminary Calculations for the Cost of Financing Insurance Statistics

Event	Index Value	Event Probability	Max Event Probability	Contract Value	Direct Insurer Loss	Reinsurance Recovery	Event Loss Given Max	E|Loss|Max|	E|Loss^2|Max|
1	100.00	0.000001210	0.000001210	1,125,200,000	1,212,550,269	16,000,000	71,350,269	105,039,888	1.06712E+16
2	89.04	0.000001210	0.000001210	1,021,700,000	1,509,161,589	16,000,000	471,461,589	505,149,400	2.33194E+17
3	87.56	0.000001810	0.000001810	1,021,700,000	1,303,694,653	16,000,000	265,994,653	299,680,134	7.95274E+16
4	83.48	0.000007020	0.000007020	939,300,000	761,956,629	16,000,000	(193,343,371)	(159,663,127)	4.17510E+16
5	83.20	0.000007020	0.000007020	939,300,000	734,137,782	16,000,000	(221,162,218)	(187,487,015)	5.30470E+16
6	82.15	0.000004660	0.000004660	939,300,000	735,660,852	16,000,000	(219,639,148)	(185,967,298)	5.23874E+16
7	80.95	0.000007910	0.000007910	939,300,000	1,004,861,128	16,000,000	49,561,128	83,225,155	8.84949E+15
8	80.55	0.000050600	0.000050598	939,300,000	1,071,076,934	16,000,000	115,776,934	149,387,575	2.02818E+16
9	79.19	0.000007020	0.000007019	856,900,000	688,269,904	16,000,000	(184,630,096)	(151,024,174)	3.88460E+16
10	77.48	0.000001810	0.000001810	856,900,000	1,652,933,116	16,000,000	780,033,116	813,636,074	6.19226E+17
11	76.22	0.000002590	0.000002590	856,900,000	741,327,246	16,000,000	(131,572,754)	(97,971,674)	2.23955E+16
12	75.55	0.000005760	0.000005759	856,900,000	654,930,780	16,000,000	(217,969,220)	(184,371,820)	5.20551E+16
13	75.18	0.000009060	0.000009059	856,900,000	1,450,085,508	16,000,000	577,185,508	610,769,915	3.42608E+17
14	75.11	0.000022900	0.000022898	856,900,000	1,148,344,417	16,000,000	275,444,417	309,002,893	8.34181E+16
15	75.05	0.000001210	0.000001210	856,900,000	1,003,713,967	16,000,000	130,813,967	164,371,248	2.37695E+16
16	74.14	0.000007020	0.000007019	774,500,000	718,320,849	16,000,000	(72,179,151)	(38,626,801)	1.07551E+16
17	73.67	0.000000460	0.000000460	774,500,000	612,322,934	16,000,000	(178,177,066)	(144,624,990)	3.68535E+16
18	72.96	0.000002590	0.000002590	774,500,000	607,625,092	16,000,000	(182,874,908)	(149,324,364)	3.85299E+16
19	72.30	0.000000767	0.000000767	774,500,000	1,035,338,915	16,000,000	244,838,915	278,388,677	6.68006E+16
20	72.18	0.000000460	0.000000460	774,500,000	564,886,456	16,000,000	(225,613,544)	(192,064,034)	5.58109E+16
21	72.05	0.000001810	0.000001810	774,500,000	1,269,991,504	16,000,000	479,491,504	513,038,744	2.37731E+17
22	71.55	0.000021000	0.000020997	774,500,000	921,203,300	16,000,000	130,703,300	164,231,531	2.34399E+16
23	71.48	0.000000738	0.000000738	774,500,000	582,199,078	16,000,000	(208,300,922)	(174,773,109)	4.83588E+16
24	71.25	0.000018700	0.000018697	774,500,000	757,962,586	16,000,000	(32,537,414)	976,524	6.73762E+15
25	70.66	0.000000202	0.000000202	774,500,000	1,078,827,927	16,000,000	288,327,927	321,841,651	9.01151E+16
26	70.57	0.000001210	0.000001210	774,500,000	1,017,469,903	16,000,000	226,969,903	260,482,415	5.82464E+16
27	70.29	0.000001210	0.000001210	774,500,000	1,162,380,661	16,000,000	371,880,661	405,391,786	1.45612E+17
28	68.99	0.000001810	0.000001810	726,900,000	1,273,618,722	16,000,000	530,718,722	564,227,570	2.89618E+17
29	68.73	0.000007250	0.000007249	726,900,000	966,395,280	16,000,000	223,495,280	256,997,239	5.66513E+16
30	68.64	0.000007020	0.000007019	726,900,000	598,955,192	16,000,000	(143,944,808)	(110,446,942)	2.59361E+16

Linking Strategic and Tactical Planning Systems for Dynamic Financial Analysis

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Linking Strategic and Tactical Planning Systems

For Dynamic Financial Analysis

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Abstract

Total enterprise risk management involves a systematic approach for evaluating/controlling risks within a large firm such as a property-casualty insurance company. The basic idea is to coordinate planning throughout the organization, from traders and underwriters to the CFO, in order to maximize the company's economic surplus at the desired level of enterprise risk. At present, it is difficult to link strategic systems, such as asset allocation, to tactical systems for pricing securities and selecting new products. We propose two solutions. First, we develop a "price of risk" for significant decisions possessing correlated factors. Second, we create a set of dynamic investment categories, called hybrid assets, for use in an asset and liability management framework. We illustrate the concepts via an insurance planning problem, whereby the goal is to optimize the company's surplus.

1. Introduction to Dynamic Financial Planning

Dynamic financial analysis (DFA) assumes that a large financial company can benefit by coordinating its operations across diverse business lines, such as insurance, banking, and investment management. The goal of DFA is to maximize the firm's surplus wealth, while keeping within desired risk tolerances. Several barriers exist to achieving this goal. First, the deregulation of financial markets has not kept pace with the explosion of new products and the merging of businesses. Second, organizational constraints limit the ability of firms to improve profitability. The firm may have the best information regarding risk-adjusted profit, but it may not act fast enough to grow the profitable activities (and shrink unprofitable activities).

A third barrier involves the linkage of information within the firm. In this paper, we describe a systematic approach for linking tactical and strategic planning systems for large financial organizations. The goal is to establish a total integrated risk management system (TIRM). Prominent applications include insurance companies, banks, mutual funds, and pension plans. We propose an approach for transmitting signals from the optimal solution of the strategic system to the individual decision-makers who must carry out the optimal strategy. A key concept is the price of risk, as defined within the context of a dynamic investment strategy. In addition, we develop the concept of a hybrid asset security. These securities involve considerable dynamic intervention, and they serve as benchmarks for the tactical components of the risk management system.

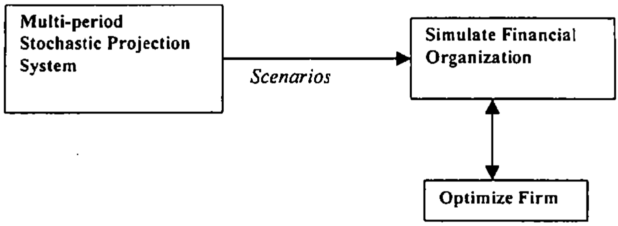
At present, there are a number of successful asset and liability management systems. There has been considerable work on the strategic aspects of asset allocation, for example, in the area of pension planning. See the recent book "World Wide Asset and Liability Modeling," by Ziemba and Mulvey (1998) and the references therein.

Rung 5:	Total integrated risk management
Rung 4:	Dynamic asset and liability management
Rung 3:	Dynamic asset-only (multi-period)
Rung 2:	Static asset-only portfolios
Rung 1:	Pricing single securities

The Risk Ladder
Figure 1

Economic theory assumes that firms maximize their shareholder value. An enterprise risk management system helps the company achieve this objective in a systematic fashion. We employ strategic planning systems to address critical questions for an institution's long term survival. Some prominent issues include the company's leverage structure, investment for research, the amount of assets in riskier categories, such as growth equity. In addition, transaction and market impact costs may be high when trying to pull out of an activity. Last, there are often autocorrelations in markets, and these intertemporal dependencies should be addressed.

The fundamental approach for analyzing long-term issues is asset allocation (and its extension to asset and liability management – see book by Ziemba and Mulvey). A dynamic financial analysis requires three primary elements (Figure 2). First, we must be able to generate scenarios for the future across a multi-period horizon.



Primary Elements of a Strategic Financial Planning System
Figure 2

1.1. Stochastic Projection System

The purpose of the stochastic model is to estimate the uncertain parameters in the firm-wide simulation. A critical issue is to link the uncertain parameters to a small set of essential economic factors – the driving factors. Figure 3 illustrates the idea. We first estimate factors such as interest rates and inflation over the T-time periods by means of a stochastic difference equation, approximating a diffusion equation. For example, we might use the Ornstein Ulenbeck process for the short interest rates:

$$dr_t = a\,(r_0 - r_t) + s\, r_t\, dZ.$$

This series displays mean reversion to the parameter r_0, has volatility s, and drift a. These three parameters must be determined by calibration tools (see Campbell et al. 1997, and Mulvey et al. 1996). The White noise term, dZ, represents the standard Normal (0,1) distribution function. Discrete samples are taken from this stochastic equation in order to derive representative set of scenarios. Each scenario depicts a single plausible path for all of the uncertain parameters over the planning period. Employing variance reduction methods, in concert with the stochastic optimization model can reduce the number of scenarios (see Campbell et al. 1997, and Mulvey and Rush 1997).

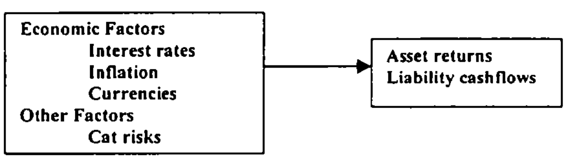


Figure 3
Simulation of Driving Economic Factors

A number of scenario generators exist for projecting economic variables and asset returns. Some prominent examples include Towers Perrin's CAP:Link/OPT:Link (Mulvey 1996), Wilkie's investment system in the UK (Wilkie 1987, 1995), Frank Russell's VAR (Carino 1994, 1998), and ORTEC in the Netherlands (Boender 1995).

There are several scenario generators for projecting liabilities. For example, catastrophic (CAT) modeling firms (e.g. AIR, Dames and Moore, EQE, RMS, and Tillinghast) estimates catastrophic risks for losses under earthquake and hurricane events. Monte Carlo simulation techniques derive these estimates, whereby the number of scenarios must be a large due to the rarity of the worst CAT events. Over 10,000 scenarios are required in most studies.

Loss ratios for non-CAT lines of business are also modeled in the scenario generators. In many cases, there is adequate historical data on the losses so that estimates can be calculated in a reliable fashion.

1.2. Simulate the Enterprise or Activity

Given the stochastic scenarios, we can simulate the financial organization over the planning period, up to the horizon at period T. For this simulation, we must identify the dynamic decision rules and the market forces that will drive the firm. It is critical to focus on the company's or the investor's surplus. We define surplus wealth as:

Market value (assets minus liabilities) – Present value (goals)

The simulation of the core economic factors over time provides a linkage across business activities. For example, asset returns and liability cashflows are dependent on changes in interest rates and inflation. The degree of overlapping risks depends upon a combination of the decision strategies and the uncertainties. It is often under control of the firm.

1.3. *Control and Optimize*

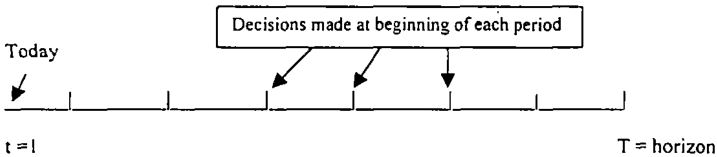
Once a simulation is conducted, we can improve the company's performance by employing stochastic optimization techniques. For example, we can maximize the growth of the company's economic surplus by maximizing the expected utility of wealth, wherein utility equals log(wealth).

We stress the concept that stochastic optimization algorithms are now feasible and available. We can solve a stochastic program with a large number of decision nodes (tens of thousands), or by means of a set of decision rules (and the resulting solution to the non- convex program.) See Mulvey and Ruszczynski 1995. Next, we define the primary equations for a strategic financial planning system.

1.4 Model Structure

The investment process consists of T time stages. The first stage represents the current date. The end of the planning period, T is called the planning horizon. Typically, it depicts a point in which the investor has a critical planning purpose, such as the repayment date of a substantial liability, or a natural juncture as the annual board of director's meeting. Strategic systems look out over several years or even decades – for insurance companies and pension plans. Tactical systems have much shorter time horizons – weeks, days, or even minutes.

At the beginning of each period, the investor renders decisions regarding the asset mix, the liabilities, and the financial goals. Between time steps, uncertainties take over. For example, the stock market and bond returns occur. As mentioned, we employ a system of stochastic differential equations for modeling the stochastic parameters over time. These relate a set of key economic factors to remaining components, such as asset and liability returns. For an example, see the CAP:Link system (Mulvey 1996, and Mulvey and Thorlacius 1998). The alternative modeling approaches address the integration of the stochastic and the optimization models in a different manner.



The Planning Period (t = 1,2, ..., T)
Figure 4

The primary decision variables designate asset proportions, liability-related decisions, and goal payments:

$x_{j,t}^s$	investment in asset j, time t, scenario s
$y_{k,t}^s$	liability or product k, time t, scenario s
$u_{l,t}^s$	goal payment l, time t, scenario s.

At each time period, t, the model maximizes its objective function, f(x), by moving money between asset categories, adjusting liabilities, and paying off goals. There are numerous candidates for the objective function; see the next section. In addition, we impose constraints on the process such as limiting borrowing to certain ratios, addressing transactions costs whenever assets are bought or sold, or taking advantage of investment opportunities. There are several modeling approaches for including constraints. Our goal is to find a feasible point, which maximizes a temporal objective function. Since we are dealing with uncertainty in a temporal setting, the optimal solution, like all points, will encompass a set of paths -- trajectories -- for the investor's wealth (or other measures such as surplus wealth). Ranking these paths is the subject of the next subsection.

There are two basic equations for the flow of funds at each time-period, and scenario:

Equation [1] for j^{th} asset category:

$$x^s_{j,t} = (x^s_{j,t-1} + r^s_{j,t-1}) - p^s_{j,t} + q^s_{j,t}(1-t_j) \qquad \text{for asset j, time t, scenario s.}$$

where $r_{j,t}^s =$ return for asset j, time t, scenario s,
$p_{j,t}^s =$ sales of asset j, time t, scenario s,
$q_{j,t}^s =$ purchase of asset j, time t, scenario s,
$t_j =$ transaction costs for asset j.

Equation [2] for the cash flows:

$$x^{s}_{1,t} = (x_{1,t-1}^{s} + r_{1,t-1}^{s}) - \sum_{j} q_{j,t}^{s} + \sum_{j} p_{j,t}^{s}(1 - t_j) + w_t^{s} - \sum_{k} y_{k,t}^{s} - \sum_{l} u_{l,t}^{s}$$

where w_t^s = cash inflows at time t, scenario s, cash is asset category 1.

The multi-stage investment model avoids looking into the future in an inappropriate fashion. The model cannot optimize over scenarios that do not represent a range of plausible outcome for the future. To prevent this occurrence, we add constraints to the model, called non-anticipatory conditions. The general form of the constraints is:

$$x_{j,t}^{s1} = x_{j,t}^{s2}$$

for all scenarios s1 and s2 which inherit a common past up to time t.

The financial planning system addresses non-anticipatory conditions, either explicitly or implicitly, and special purpose algorithms are available for solving the resulting stochastic optimization model.

In addition to the economic surplus, market value of assets and liabilities, we must address the regulatory environment. The simulation model should set constraints on the regulatory measures, such as STAT and GAAP, while maximizing the economic surplus. This effort requires a complex set of issues when the model cuts across a multi-national company with many tax and cultural concerns.

1.5 Financial Objectives

A major element of enterprise risk management involves trading off risks and rewards. It is natural to expect that investments possessing more volatility will often generate greater expected returns than assets with lower levels of volatility. The temporal issue complicates the decision since longer term horizons dictate a longer time span to recoup losses, thus the more volatile assets may be, in fact, safer in terms of contextual risks. An example is the stock/cash comparison: stocks provide higher *expected* returns but are more volatile than cash. We must consider the time horizon in measuring contextual risks.

There are numerous ways to evaluate financial risks, just as there are alternative measures of profitability. We might consider the *chance* of a loss over the next year, such as 15% -- value at risk. Or, we might set a profitability target and evaluate the probability of missing the target. In both cases, risk increases as a function of probability. An improved alternative for evaluating risks is to estimate the full probability distribution of shareholders equity, along with other measures of financial well being for the company. The scenario generators in conjunction with the firm simulation system provide this information.

Calculating these curves requires a comprehensive approach for linking all major activities and uncertainties in a financial organization. Given a distribution, we can evaluate not only risks but also compare it against reward potential. Typically, we equate reward with expected value. We might be interested in profit or loss over the next year per dollar of allocated capital:

$$\textit{Expected profit} = \sum_{s \in S} p_s * z^s \; / \; (\textit{Allocated capital})$$

where p_s is the probability of scenario s,
z^s is the profit or loss under scenario s,
S is a set of representative scenarios,
Allocated capital depends upon the loss distribution (VAR).

Comparing alternative distributions on a direct basis can be difficult for most decision-makers. To aid in the process, we can employ the concepts of stochastic dominance. For example, if two cumulative distributions cross only once and the decision-maker is risk averse, she will take the curve with the highest expected value if its variance is less than the alternative. Other dominance tests are possible, but these tests are unlikely to apply in a wide set of circumstances.

There are two primary theories for setting up an objective function under uncertainty. First, we can transform random variables to deterministic values, such as the value at risk or a certainty equivalent. Alternatively, we can fit a classical utility function to the characteristics of the output of the model. An example is to define risk as the volatility of the return of a por tfolio. There are numerous variants of each theory.

After 50 years, the von Neumann Morgenstern [VM] theory remains the pre-eminent approach for making decisions in the face of uncertainty. The resulting optimization model can be stated simply as follows:

[VM] Max $E(v(z_T))$

where $E(v(z_T)) = \sum_{s} p_s * v(z^s)$
where $v(z^s)$ is the VM preference function
z^s_T = investors wealth under scenario s, time T
p_s = probability of scenario s.

Once the solution of model VM is found, z^*, we determine its certainty equivalent (CE) by computing the inverse function at the recommended solution $CE = v^{-1}(z^*)$. This value represents the exact amount that we would take in order to sell (or buy) the random variable z. While the VM theory is generally accepted as a theoretical measure, there are several difficulties. First, most executives are unable to come up with an acceptable level of risk aversion. Second, the temporal aspects of decision making are ignored in the VM theory. Thus, we are generally unable to decide upon a high-risk asset that will pay off in several years versus a lower returning but safer asset. Generally, we focus on the expected utility at the end of the planning horizon, period T. The intermediate points are constrained to achieving acceptable results.

There are several heuristic approaches to decision making under uncertainty. Two of the most popular are value at risk (VAR), and the risk adjusted return on allocated capital. In both cases, we set a level of confidence in the return distribution as a reference point. Profits and risks are measured with respect to this assumed point. For instance, we might decide that the 1/100 loss point is the reference. Capital allocation rules are then generated by the amount of losses at this point. The concepts are easy to understand. But they can lead to errors since they are not considering the entire distribution of gains and losses. In addition, these methods do not easily address the issue of overlapping risks.

1.6 Limitations of Strategic Systems for Large Organizations

There are several limitations to the use of a strategic financial planning system within a large financial institution. The first issue involves the lack of detailed information regarding the risks and most importantly overlapping and correlated risks. If the organization could separate activities that are independent of each other, they could allocate capital on a straightforward risk adjusted basis, such as some function of value at risk (VAR). However, during the 1990s, financial organizations are merging diverse activities – traditional banking, insurance, mutual funds, and trust and wealth management. It is difficult to design these operations so that the risks are independent of each other. In addition, we discount projected future cashflows by Treasury interest rates when computing the market value of assets and liabilities. Therefore, even seemingly independent activities are linked by their dependence on interest rate movements.

Second, the asset allocation approach runs into difficulties when portfolio managers do not possess well-defined investment benchmarks, or when the managers stray from the benchmarks. The risks for the individual tactical investors will certainly increase when correlated elements exist in their portfolios. Yet many financial companies decompose their activities into loosely managed divisions; they pay scant attention to overlapping risks. The problem is especially difficult when the issues involve the rare events – tails of the loss distribution. For example, several investors may decide to move into a single asset at the same time, and the asset drops dramatically. In other cases, there is a more subtle relationship between the degree of overlapping risks. The scenario generators should be equipped to handle this factor.

Another challenge occurs when the strategic plan needs modifying. A tactical system can assist in the change of course decisions. Yet, there needs to be close coordination of the affected systems. The tactical system by necessity works at a more detailed level of information, such as individual stocks, as compared with generic asset categories. This offers great opportunities. The prices of risks and target benchmarking can play a pivotal role as we show in the next section.

2. Linking Strategic and Tactical Planning Systems

This section discusses the linkage of the strategic planning system with one or more tactical investment systems. As before, the critical issue entails overlapping risks, across product lines, and investments. We suggest three possible approaches. The first involves the creation of target benchmarks based on hybrid securities. Dynamic asset and liability processes form the hybrids. In the second approach, we generate prices of risks for each product-location, or asset category. We add these prices to the profit calculations for the business units. In the third approach, we track the degree of overlapping risks and allocate capital based on risks. This approach requires a relatively conservative allocation rule or a closely monitored organization.

Figure 5 illustrates the flow of information between the strategic system and the tactical systems. Herein, the target benchmarks and/or prices of risks are sent to the tactical investors – traders, underwriters and asset managers -- along with their capital allocation. A straightforward benchmark might be the Morgan Stanley Capital International Index; the goal is to exceed the benchmark return, while investing under the same risk profile as the MSCI index.

Hybrid securities can play a distinguishing role in the construction of benchmarks for tactical asset managers. A prototypical example is the principal-protected equity bond discussed in the next section. For this example, the asset manager must beat this index over an assigned time-period. The manager has several options. First, he could attempt to replicate the security by following a delta or gamma neutral strategy (Hull, 1997). Alternatively, he could increase the equity proportions in order to gain additional returns, at the costs of additional risks. However, the investor must be careful when taking on increased risks. Here is where the price of risk comes in. The tactical system should evaluate the marginal costs of adding risks by modify the excess profit computations (over and above the target benchmark). The prices of risks should be included in the calculations. In some cases, there is adequate independence of the activities so that overlapping risks can be ignored. Whenever possible, the organizational design should attempt to reduce overlapping risks by setting up units that are independent on a risk basis, such as giving a manager a separate asset category. Alternatively, the tactical manager can simply replicate the target benchmark at a minimum cost, thus eliminating the price of risks requirements.

In a similar fashion, a product manager or insurance underwriter can be assigned a benchmark. An example is the amount of allocated capital for the manager's businesses along with the risk adjusted profit values. As on the asset side, risk profiles should depend on the projected movements of the core economic factors. Moreover, as before, we can compute the price of risk for the activities by referring to the dual variables from the optimal solution to the strategic ALM system. Any decision (investment/product/line) possessing a positive margin profit will benefit the company and is worthy of further analysis. The formula for adjusting profit is:

$$profit = net \quad revenue - \sum_{s \in S} l_{s,t} * \pi_{s,t}$$

where $\pi_{s,t}$ = optimal dual solution from strategic system,

$l_{s,t}$ = loss under scenario s, time t.

Another approach is to approximate the prices of risks by computing the historical correlations and assume that the profits and losses are derived from a multi-normal or other suitable distribution. Herein, the time horizon is relatively short, one day to several weeks, and the model is generally single period.

We illustrate the prices of risks via a generic tactical tool for insurance underwriters. This system takes in loss estimates for catastrophic events, such as hurricanes and earthquakes, and generates risk adjusted profitability values for the properties in the underwriter's book of business. It also can optimize on the parameters of the book, such as deductibles, identify properties to eliminate, etc., and find the best set of properties when two books are combined. A sample output of the system might show the expected profits displayed per zip-code, adjusted by the prices of risks. The underwriter can quickly gain insights into the relative areas of profitability on a geographic basis.

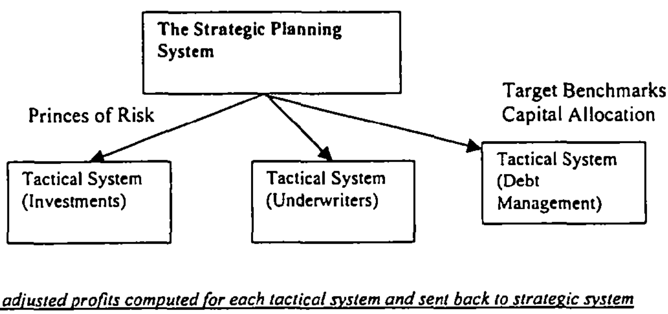


Figure 5
Coordinating the Strategic and Tactical Planning Systems

3. Empirical Results

In this section, we illustrate the advantages of hybrid securities for a real world strategic planning model involving a large insurance company[1]. The goal is to maximize the company's surplus over a five-year horizon. Re-balancing decisions occur annually. We employ the CAP:Link scenario generator for constructing 500 scenarios for the economic factors and the asset returns. Tillinghast-Towers Perrin actuaries performed liability projections under these same 500 scenarios. At each period, the model revises the asset mix, according to the target mix values, pays out the necessary liabilities and taxes, and distributes dividends and interest as appropriate.

Thirteen asset categories were selected by the insurance client. These asset categories form the basis for many asset allocation studies. We include two categories of Treasury inflation protected bonds (TIPs), mid-term and long-term, in addition to the standard assets. These assets protect the insurance company's liabilities against unexpected inflation.

A strategic planning model was developed for the insurance company, in which the company paid out required obligations each year as dictated by the actuarial estimates, under each of the 500 scenarios. The goal was to maximize the company's surplus at the end of the 5-year horizon.

To solve the model, we employed a nonlinear optimization system, called OPT:Link, to generate the surplus efficient frontier at the end of the 5-year planning period. Figure 6 and Table 1 show the company's surplus expected values and standard deviations for the resulting mixes. Eleven points on the efficient frontier are displayed, from the low risk portfolio consisting of cash and bonds, to the high-risk portfolio consisting of smaller capitalized US stock.

Asset Mix %:	1	2	3	4	5	6	7	8	9	10	11
Cash-U.S.A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Eqty-U.S.A	5.5	7.4	9.0	10.3	11.0	5.6	5.2	5.1	0.2	0.0	0.0
US Real Es	9.0	10.9	11.9	12.4	13.2	16.4	18.1	17.8	12.5	1.0	0.0
High Yld B	10.2	13.9	18.4	22.7	27.4	39.3	42.7	48.2	51.5	54.6	22.5
LT TIPS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
MT TIPS	45.8	47.1	41.7	36.0	30.9	17.6	9.8	3.0	0.0	0.0	0.0
Sht G/C	9.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Mid G/C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Long G/C	15.7	15.3	11.0	8.2	4.6	0.0	0.0	0.0	0.0	0.0	0.0
US20YrZero	0.0	0.0	0.0	0.0	0.2	2.4	1.8	0.8	0.0	0.0	0.0
US SmCap	0.0	0.1	1.7	4.1	7.1	13.8	16.0	18.2	27.9	37.7	77.5
EAFE	4.4	5.3	6.2	6.3	5.6	5.0	6.6	7.0	7.9	6.7	0.0
WrldBndXUS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Reward	7.3	7.6	7.9	8.1	8.4	9.0	9.2	9.5	9.7	9.9	10.2
Risk	0.3	0.3	0.3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	1.0

Table 1
Asset Mixes for 11 Selected Points on Surplus Efficient Frontier

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[1] The details of the insurance company example are disguised.

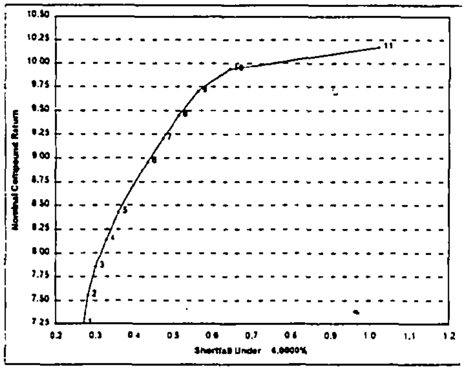


Figure 6
Surplus Efficient Frontier for Sample Insurance Company

Next, we construct two hybrid securities. The first is a dynamic combination of equity and cash, similar to constant proportional portfolio insurance (CPPI). See Perold and Sharpe (1988). The basic idea is to set a minimum level for the asset wealth, which we call the floor. Based on this constant, we compute the difference between current wealth and the floor -- called the cushion. The hybrid security sets the stock/cash proportions equal to a linear function of the cushion value at the beginning of each period. We update the proportions each month (rather than the annual re-balancing carried out in the strategic model). The resulting hybrid stock/cash security is called dynamic-equity-protection (DEP). For the purpose of this study, we established the floor = 100 and the multiplier parameter = 1.1. Figure 7 depicts the compound returns of the DEPs over the five year planning period, as generated by the CAP:Link investment system.

Instead of following a dynamic replication strategy, we can purchase securities with the desired properties. Several mutual fund companies market stock/cash hybrid securities, including Salomon/Smith/Barney, and Merrill Lynch's Mitts. These securities trade on the New York and other stock exchanges. The term of the security is typically five years; they trade as non-dividend paying stocks.

We construct a second hybrid security geared towards the fixed income marketplace. Again, we combine two traditional asset categories. Instead of stock/cash, however, we dynamically allocate between mortgage backs and cash in a proprietary fashion. The mix shifts towards cash when interest rates are dropping, whereas the mix shifts towards bonds when interest rates are increasing. We label this hybrid category MBS, to indicate the association of this strategy with mortgage backed securities. Figure 7 lists the nominal returns for the MBS hybrid over the 5-year horizon.

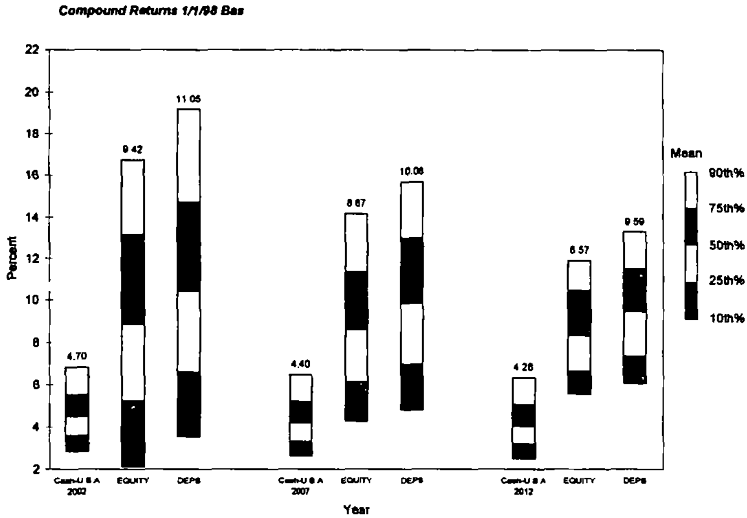
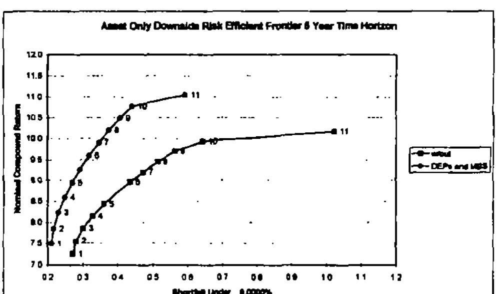


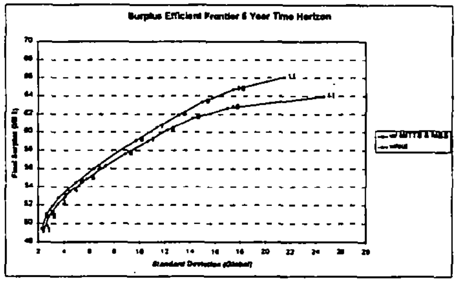
Figure 7
Distribution of Returns for Hybrid Security versus Equities and Cash

We combine the two hybrid securities with the other 13 asset categories and solve the resulting surplus optimization problem. The advantages of the DEP hybrid for risk averse investors can be readily seen in Figure 8. Here, we plot an asset-only efficient frontier with downside risks at the 6% target level. The efficient frontier solutions are considerably improved by adding the DEPs. They give upside gains, but limit the downside losses during downswings in the equity markets.

The surplus efficient frontiers are displayed in Figure 9 and Table 2, with and without the two hybrids. By adding these securities, we improve the surplus returns and reduce the surplus risks. The stock/cash hybrid (DEPs) occurs at the higher risk levels, whereas the mortgage back/cash hybrid (MBS) occurs at the lower risk levels. One of these two assets is present in all of the efficient points. The advantages of the dynamic financial strategy are clear-cut in this real-world case.

Figure 8
Asset Only Efficient Frontiers with and without Hybrid Securities





Surplus Efficient Frontiers with & without Hybrid Securities
Figure 9

Numerous variations on the hybrid security apply to insurance companies and pension plans. The floor and multiplier parameters are available for modifications. Alternatively, we could implement other decision rules, replacing the CPPI strategy with combination strategies. Due to computational bounds for the nonlinear stochastic program, there is a limit on the number of asset categories that can be included in a strategic planning study. Still, we can readily solve models possessing several hundred hybrid securities with high performance PCs in mid-1998. The optimization model can readily accommodate linear constraints on the optimal asset mix, such as lower and upper bounds.

Overall, the hybrid securities give the strategic planning system greater realism. They also can serve as target benchmarks for the tactical systems, in a more innovative manner than the traditional fixed asset mix or weighted indices. The target benchmarks can link to the prices of risks, so that the tactical manager can move away from the benchmark in a manner that continues to optimize the company's surplus wealth. The amount of allocated capital determines the amount of movement that is possible for each tactical manager.

Asset Mix %:	1	2	3	4	5	6	7	8	9	10	11
Cash-U.S.A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Eqty-U.S.A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
US Real Es	6.3	5.6	7.2	4.4	3.6	4.5	5.1	4.6	0.0	0.0	0.0
High Yld B	6.7	8.8	12.0	13.1	15.6	21.6	23.1	26.1	26.7	28.1	0.0
LT TIPS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
MT TIPS	22.4	21.4	19.2	13.3	10.9	5.7	0.0	0.0	0.0	0.0	0.0
Sht G/C	8.5	5.7	0.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Mid G/C	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Long G/C	7.4	6.2	3.7	0.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
US20Yr Zero	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
MBS	33.2	31.3	32.2	35.6	31.2	20.1	18.3	11.2	7.1	0.0	0.0
US SmCap	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
EAFE	1.5	1.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
WrldBnd XUS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
DEPS	14.0	19.7	25.2	33.1	38.6	48.0	53.4	58.2	66.2	71.9	100.0
Reward	7.5	7.8	8.2	8.6	8.9	9.6	9.9	10.2	10.5	10.8	11.1
Risk	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.4	0.4	0.4	0.6

Table 2
Selected Points on Surplus Efficient Frontier with Hybrid Securities

4. Conclusions

Enterprise risk management requires a coordinated program of financial planning throughout the institution. Traders and arbitrageurs search out mispriced securities by option analysis and other tools. Portfolio managers attempt to beat popular financial benchmarks via mean/variance optimization. Insurance underwriters aim to exceed risk adjusted profit targets. Pension planners carry out asset allocation strategies to insure the soundness of their assets with respect to the pension liabilities. CFO's identify the optimal leverage factors to maximize shareholder value. In each of these cases, there must be well-defined target benchmarks for the decision-makers. A strategic financial planning system generates these targets.

We have described a systematic technique for combining strategic and tactical financial planning systems. First, we define a price of risk for overlapping risks. These prices depend upon the optimal shadow prices of the strategic system. In the second step, we develop hybrid asset categories, such as the stock/cash example shown in Section 3. We extend traditional asset categories to encompass many forms of embedded options and dynamic investment strategies.

The benchmark targets and possibly the prices of risk are transmitted to the tactical systems. If the tactical manager stays relatively close to the target risk profile, he can ignore the prices of risk and maximize the excess returns. Otherwise, the prices of risks must be considered when the investor decides to take on increased risks. Considering historical correlations can approximate the price of risk, but there is no guarantee that backward looking data will be appropriate for the future.

An example of strategic planning is the capital management strategy for an insurance company presented in the previous section. We showed that the hybrid assets improve the company's risk adjusted returns. The solution to the strategic problem serves as targets for the tactical planning systems.

Applying these techniques will enhance a financial institution's ability to maximize its shareholder value. In addition, enterprise risk management applies to institutions with diverse operations, such as a combined bank, insurance company, and mutual fund. Correlated risks are present in these organizations. Identifying and pricing these correlated risks will be allow the institution to grow its surplus in an optimal fashion, while maintaining the desired level of risks at the enterprise level.

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Applications of Resampling Methods in Dynamic Financial Analysis

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Applications of Resampling Methods in Dynamic Financial Analysis

May 13, 1998

Abstract

Dynamic Financial Analysis can be viewed as the process of studying profitability and solvency of an insurance firm under a realistic and integrated model of key input random variables such as loss frequency and severity, expenses, reinsurance, interest and inflation rates, and asset defaults. Traditional models of input variables have generally fitted parameters for a predetermined family of probability distributions. In this paper we discuss applications of some modern methods of non-parametric statistics to modeling loss distributions, and possibilities of using them for modeling other input variables for the purpose of arriving at an integrated company model. Several examples of inference about the severity of loss, loss distributions percentiles and other related quantities based on data smoothing, bootstrap estimates of standard error and bootstrap confidence intervals are presented. The examples are based on real-life auto injury claim data and the accuracy of our methods is compared with that of standard techniques. Model adjustment for inflation and bootstrap techniques based on the Kaplan-Meier estimator, useful in the presence of policies limits (censored losses), are also considered.

1 Introduction

D'Arcy, Gorvett, Herbers and Hettinger (1997) discuss Dynamic Financial Analysis (DFA) for insurance firms and point out the following two sets of key variables involved in the process.

Financial Variables:

- Short-term interest rates;
- Term premiums;
- Default premiums;
- Default risk;
- Equity premiums;
- Inflation.

Underwriting variables:

- Rate level;
- Exposures;
- Loss frequency;
- Loss severity;
- Expenses;
- Catastrophes;
- Jurisdiction;
- Payment patterns;
- Reinsurance.

In that classification, the financial variables generally refer to asset-side generated cash flows of the business, and the underwriting variables relate to the cash flows of the liabilities side. The process of developing a DFA model begins with the creation of a model of probability distributions of the input variables, including the establishment of the proper range of values of input parameters. The use of parameters is generally determined by the use of parametric families of distributions.

Fitting of those parameters is generally followed by either Monte Carlo simulation and integration of all inputs for profit testing and optimization, or by the study of the effect of varying the parameters on output variables in sensitivity analysis and basic cash flow testing. Thus traditional actuarial methodologies are rooted in parametric approaches which fit prescribed distributions of losses and other random phenomena studied (e.g., interest rate or other asset return variables) to the data. The experience of the last two decades has shown greater interdependence of basic loss variables (severity, frequency, exposures) with asset variables (interest rates, asset defaults, etc.), and sensitivity of the firm to all input variables listed above. Increased complexity has been accompanied by increased competitive pressures, and more frequent insolvencies. This situation is precisely the reason why DFA has come to the forefront of new actuarial methodologies. In our opinion, in order to properly address the DFA issues one must carefully address the weaknesses of traditional methodologies. These weaknesses can be summarized as originating either from ignoring the uncertainties of inputs, or mismanaging those uncertainties. While early problems of DFA could be attributed mostly to ignoring uncertainty, we believe at this point the uncertain nature of model inputs is generally acknowledged. Derrig and Ostaszewski (1997) used fuzzy set techniques to handle the mixture of probabilistic and non-probabilistic uncertainties in asset/liability considerations for property-casualty claims. In our opinion it is now time to proceed to deeper issues concerning the actual forms of uncertainty. The Central Limit Theorem and its stochastic process counterpart provide clear guidance for practical uses of the normal distribution and all distributions derived from it. But one cannot justify similarly fitting convenient distributions to, for instance, loss data and expect to easily survive the next significant change in the marketplace. What does work in practice, but not in theory, may be merely an illusion of applicability provided by powerful tools of modern technology. If one cannot provide a justification for the use of a parametric distribution, then a nonparametric alternative should be studied, at least for the purpose of understanding firm's exposures. In this work, we will show such a study of nonparametric methodologies as applied to loss data, and will advocate the development of an integrated company model with the use of nonparametric approaches.

1.1 Loss Distributions for DFA

We begin by addressing the most basic questions concerning loss distributions. The first two parameters generally fitted to the data are claims average size (*claims average severity*), and the number of claim occurrences per unit of exposure (*claims frequency*). Can we improve on these estimates by using nonparametric methods?

Consider the problem of estimating the severity of a claim, which is, in its most general setting, equivalent to modeling the probability distribution of a single claim size. Traditionally, this has been done by means of fitting some parametric models from a particular continuous family of distributions (cf. e.g., Daykin, Pentikainen, and Pesonen 1994, chapter 3). While this standard approach has several obvious advantages, we should also realize that occasionally it may suffer some serious drawbacks.

- Some loss data has a tendency to cluster about round numbers like \$1,000, \$10,000, etc., due to rounding off the claim amount and thus in practice follows a mixture of continuous and discrete distributions. Usually, parametric models simply ignore the discrete component in such cases.
- The data is often truncated from below or censored from above due to deductibles and/or limits on different policies. Especially, the presence of censoring, if not accounted for, may seriously compromise the goodness-of-fit of a fitted parametric distribution. On the other hand, trying to incorporate the censoring mechanism (which is often random in its nature, especially when we consider losses falling under several insurance policies with different limits) leads to a creation of a very complex model, one often difficult to work with.
- The loss data may come from a mixture of distributions depending upon some known or unknown classification of claim types.
- Finally, it may happen that the data simply does not fit any of the available distributions in a satisfactory way.

It seems, therefore, that there are many situations of practical importance where the traditional approach cannot be utilized, and one must look beyond parametric models. In this work we point out an alternative, nonparametric approach to modeling losses and other random parameters of financial analysis originating from the modern methodology of nonparametric statistics. Especially, we analyze possible inroads by the fairly recent statistical methodology known as *bootstrap* into dynamic financial analysis. To keep things in focus we will be concerned here only with applications to modeling the severity of loss, but the methods discussed may be easily applied to other problems like loss frequencies, asset returns, asset defaults, and combining those into models of Risk Based Capital, Value at Risk, and general DFA, including Cash Flow Testing and Asset Adequacy Analysis.

1.2 The Concept of Bootstrap

The concept of bootstrap was first introduced in the seminal piece of Efron (1979) and relies on the consideration of the discrete empirical distribution generated by a random sample of size n from an unknown distribution F. This empirical distribution assigns equal probability to each sample item. In the sequel we will write $\widehat{F}_n$ for that distribution. By generating an independent, identically distributed (iid) random sequence (resample) from the distribution $\widehat{F}_n$ or its appropriately smoothed version, we can arrive at new estimates of various parameters and nonparametric characteristics of the original distribution F. This idea is at the very root of the bootstrap methodology. In particular, Efron (1979) points out that the bootstrap gives a reasonable estimate of standard error for any estimator, and it can be extended to statistical error assessments and to inferences beyond biases and standard errors.

1.3 Overview of the Article

In this paper, we apply the bootstrap methods to two data sets as illustrations of the advantages of resampling techniques, especially when dealing with empirical loss data. The basics of bootstrap are covered in Section 2 where we show its applications in estimating standard errors and calculating confidence intervals. In Section 3, we compare bootstrap and traditional estimators for quantiles and excess losses using some truncated wind loss data. The important concept of smoothing the bootstrap estimator is also covered. Applications of bootstrap to auto bodily injury liability claims in Section 4 show loss elimination ratio estimates together with their standard errors in a case of lumpy and clustered data (the data set is enclosed in Appendix B). More complicated designs that incorporate data censoring and adjustment for inflation appear in Section 5. Sections 6 and 7 provide some final remarks and conclusions. The Mathematica 3.0 programs used to perform bootstrap calculations are provided in Appendix A.

2 Bootstrap Standard Errors and Confidence Intervals

As we have already mentioned in the Introduction, the idea of bootstrap is in sampling the empirical cumulative distribution function (cdf) $\widehat{F}_n$. This idea is closely related to the following, well known statistical principle, henceforth referred to as the "plug-in" principle. Given a parameter of interest $\theta(F)$ depending upon an unknown population cdf F, we estimate this parameter by $\hat{\theta} = \theta(\widehat{F}_n)$. That is, we simply replace F in the formula for θ by its empirical counterpart $\widehat{F}_n$ obtained from the observed data. The plug-in principle will not provide good results if $\widehat{F}_n$ poorly approximates F or if there is information about F other than that provided by the sample. For instance, in some cases we might know (or be willing to assume) that F belongs to some parametric family of distributions. However, the plug-in principle and the bootstrap may be adapted to this latter situation as well. To illustrate the idea, let us consider a parametric family of cdf's $\{F_\mu\}$ indexed by a parameter μ (possibly a vector) and for some given μ_0 let $\widehat{\mu}_0$ denote its estimate calculated from the sample. The plug-in principle in this case states that we should estimate $\theta(F_{\mu_0})$ by $\theta(F_{\widehat{\mu}_0})$. In this case, bootstrap is often called parametric, since a resample is now collected from F_{μ_0}. Here and elsewhere in this work we refer to any replica of $\hat{\theta}$ calculated from a resample as "a bootstrap estimate of $\theta(F)$" and denote it by $\hat{\theta}^*$.

2.1 The Bootstrap Methodology

Bickel and Freedman (1981) formulated conditions for consistency of bootstrap, which resulted in further extensions of the Efron's (1979) methodology to a broad range of standard applications, including quantile processes, multiple regression and stratified sampling. They also argued that the use of bootstrap did not require theoretical derivations such as function derivatives, influence functions, asymptotic variances, the Edgeworth expansion, etc.

Singh (1981) made a further point that the bootstrap estimator of the sampling distribution of a given statistic may be more accurate than the traditional normal approximation. In fact, it turns out that for many commonly used statistics the bootstrap is asymptotically equivalent to the one-term Edgeworth expansion estimator, usually having the same convergence rate, which is faster then normal approximation. In many more recent statistical texts the bootstrap is recommended for estimating sampling distributions and finding standard errors, and confidence sets. The bootstrap methods can be applied to both parametric and non-parametric models, although most of the published research in the area is concerned with the non-parametric case since that is where the most immediate practical gains might be expected. Let us note though that often a simple, non-parametric bootstrap may be improved by other bootstrap methods taking into account the special nature of the model. In the iid non-parametric models for instance, the smoothed bootstrap (bootstrap based on some smoothed version of $\widehat{F}_n$) often improves the simple bootstrap (bootstrap based solely on $\widehat{F}_n$). Since in recent years several excellent books on the subject of resampling and related techniques have become available, we will not be particularly concerned here with providing all the details of the presented techniques, contenting ourselves with making appropriate references to more technically detailed works. Readers interested in gaining some basic background in resampling are referred to Efron and Tibisharani (1993), henceforth referred to as ET. For a more mathematically advanced treatment of the subject, we recommend Shao and Tu (1995).

2.2 Bootstrap Standard Error Estimate

Arguably, one of the most important applications of bootstrap is providing an estimate of standard error of $\hat{\theta}$ ($se_F(\hat{\theta})$). It is rarely practical to calculate it exactly. Instead, one usually approximates $se_F(\hat{\theta})$ with the help of multiple resamples. The approximation to the bootstrap estimate of standard error of $\hat{\theta}$ (or $BESE$) suggested by Efron (1979) is given by

$$\widehat{se}_B = \left\{ \sum_{b=1}^{B} [\hat{\theta}^*(b) - \hat{\theta}^*(\cdot)/(B-1)]^2 \right\}^{1/2} \tag{2.1}$$

where $\theta^*(\cdot) = \sum_{b=1}^{B} \hat{\theta}^*(b)/B$, B is the total number of resamples (each of size n) collected with replacement from the plug-in estimate of F (in parametric or non-parametric setting), and $\hat{\theta}^*(b)$ is the original statistic $\hat{\theta}$ calculated from the b-th resample ($b = 1, \ldots, B$). By the law of large numbers

$$\lim_{B \to \infty} \widehat{se}_B = BESE(\hat{\theta}),$$

and for sufficiently large n we expect

$$BESE(\hat{\theta}) \approx se_F(\hat{\theta}).$$

Let us note that B, total number of resamples, may be taken as large as we wish, since we are in complete control of the resampling process. It has been shown that for estimating the standard error, one should take B to be about 250, whereas for different resampled statistics this number may have to be significantly increased in order to reach the desired accuracy (see ET).

2.3 The Method of Percentiles

Let us now turn to the problem of using the bootstrap methodology to construct confidence intervals. This area has been a major focus of theoretical work on the bootstrap and several different methods of approaching the problem have been suggested. The "naive" procedure described below is by far the most efficient one and can be significantly improved in both rate of convergence and accuracy. It is, however, intuitively obvious and easy to justify and seems to be working well enough for the cases considered here. For a complete review of available approaches to bootstrap confidence intervals, see ET.

Let us consider $\hat{\theta}^*$, a bootstrap estimate of θ based on a resample of size n from the original sample $X_1, \ldots, X_n$, and let G_* be its distribution function given the observed sample values

$$G_*(x) = P\{\hat{\theta}^* \leq x | X_1 = x_1, \ldots, X_n = x_n\}.$$

Recall that for any distribution function F and $p \in (0,1)$ we define the p-th quantile of F (sometimes also called p-th percentile) as $F^{-1}(p) = \inf\{x : F(x) \geq p\}$. The *bootstrap percentiles method* gives $G_*^{-1}(\alpha)$ and $G_*^{-1}(1-\alpha)$ as, respectively, lower and upper bounds for the $1-2\alpha$ confidence interval for $\hat{\theta}$. Let us note that for most statistics $\hat{\theta}$ the distribution function of the bootstrap estimator $\hat{\theta}^*$ is not available. In practice, $G_*^{-1}(\alpha)$ and $G_*^{-1}(1-\alpha)$ are approximated by taking multiple resamples and then calculating the empirical percentiles. In this case the number of resamples B is usually much larger than for estimating $BESE$; in most cases it is recommended that $B \geq 1000$.

3 Bootstrap and Smoothed Bootstrap Estimators vs Traditional Methods

In making the case for the usefulness of bootstrap in modeling loss distributions we would first like to compare its performance with that of the standard methods of inference as presented in actuarial textbooks.

3.1 Application to Wind Losses: Quantiles

Let us consider the following set of 40 losses due to wind-related catastrophes that occurred in 1977. These data are taken from Hogg and Klugman (1984) (henceforth referred to as HK) where they are discussed in detail in Chapter 3. The losses were recorded only to the nearest \$1,000,000 and data included only those losses of \$2,000,000 or more. For convenience they have been ordered and recorded in millions.

2,	2,	2,	2,	2,	2,	2,	2,	2,	2
2,	2,	3,	3,	3,	3,	4,	4,	4,	5
5,	5,	5,	6,	6,	6,	6,	8,	8,	9
15,	17,	22,	23,	24,	24,	25,	27,	32,	43

Using this data set we shall give two examples illustrating the advantages of applying bootstrap approach to modeling losses. The problem at hand is a typical one: assuming that all the losses recorded above have come from a single unknown distribution F we would like to use the data to obtain some good approximation for F and its various parameters.

First, let us look at an important problem of finding the approximate confidence intervals for the quantiles of F. The standard approach to this problem relies on the normal approximation to the sample quantiles (order statistics). Applying this method, Hogg and Klugman have found the *approximate 95% confidence interval for the .85-th quantile of F to be between* X_{30} and X_{39} which for the wind data translates into the observed interval

$$(9, 32).$$

They *also have noted that "..This is a wide interval but without additional assumptions this is the best we can do. "* Is that really true ? To answer this question let us first note that in this particular case the highly skewed binomial distribution of the .85-th sample quantile is approximated by a symmetric normal curve. Thus, it seems reasonable to expect that normal approximation could be improved here upon introducing some form of correction for skewness. In the standard normal approximation theory this is usually accomplished by considering, in addition to the normal term, the first non-normal term in the asymptotic Edgeworth expansion of the binomial distribution. The resulting formula is messy and requires the calculation of a sample skewness coefficient as well as some refined form of the continuity correction (cf. e.g., Singh 1981). On the other hand, the bootstrap has been known to make such a correction automatically (Singh 1981) and hence we could expect that a bootstrap approximation would perform better here[1]. Indeed, in this case (in

[1]This turns out to be true only for a moderate sample size (here: 40); for binomial distribution with large n (i.e., large sample size) the effect of the bootstrap correction is negligible. In general, the bootstrap approximation

the notation of Section 2) we have $\theta(F) = F^{-1}(.85)$ and $\hat{\theta} = \widehat{F}_n^{-1}(.85) \approx X_{(34)}$ –the 34-th order statistic which for the wind data equals 24. For sample quantiles the bootstrap distribution G_* can be calculated exactly (Shao and Tu 1995, p.10) or approximated by an empirical distribution obtained from B resamples as described in Section 2. Using either method, the $1 - 2\alpha$ confidence interval calculated using the percentile method is found to be between $X_{(28)}$ and $X_{(38)}$ (which is also in this case the exact confidence interval obtained by using binomial tables). For the wind data this translates into the interval

$$(8, 27)$$

which is considerably shorter then the one obtained by Hogg and Klugman.

3.2 Smoothed Bootstrap. Application to Wind Losses: Excess Losses

As our second example, let us consider the estimation of the probability that a wind loss will exceed a $29,500,000 threshold. In our notation that means that we wish to estimate the unknown parameter $1-F(29.5)$. A direct application of the plug-in principle gives immediately the value 0.05, the nonparametric estimate based on relative frequencies. However, note that the same number is also an estimate for $1 - F(29)$ and $1 - F(31.5)$, since the relative frequency stays the same for all the threshold values not present in reported data. In particular, since the wind data were rounded off to the nearest unit, the nonparametric method does not give a good estimate for any non-integer threshold. This problem with the same threshold value of $29,000,000 was also considered in HK (Ex.4 p. 94 and Ex.1 p. 116). As indicated therein, one reasonable way to deal with the non-integer threshold difficulty is first to fit some continuous curve to the data. The idea seems justified since the clustering effect in the wind data has most likely occurred due to rounding off the records. In their book Hogg and Klugman have used standard techniques based on method of moments and maximum likelihood estimation to fit two different parametric models to the wind data: the truncated exponential with cdf

$$F_\mu(x) = 1 - e^{-(x-1.5)/\mu} \qquad 1.5 < x < \infty \tag{3.1}$$

performs better than normal one for large sample sizes only for continuous distributions.

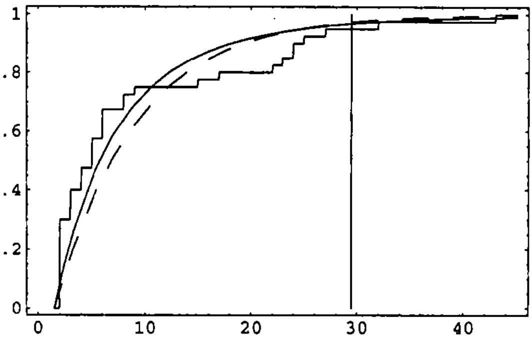

Figure 1: Empirical cdf for the wind data and two parametric approximations fitted by the maximum likelihood method. The solid smooth line represents the curve fitted from the exponential family (3.1); the dashed line represents the curve fitted from the Pareto family (3.2). The vertical line is drawn for reference at x=29.5.

for $\mu > 0$, and the truncated Pareto with cdf

$$F_{\alpha,\lambda}(x) = 1 - \left(\frac{\lambda}{\lambda + x - 1.5}\right)^{\alpha} \qquad 1.5 < x < \infty \tag{3.2}$$

for $\alpha > 0, \quad \lambda > 0$.

For the exponential distribution the method of moments as well as maximum likelihood estimator of μ was found to be $\hat{\mu} = 7.725$. The MLE 's for the Pareto distribution parameters were $\hat{\lambda} = 28.998$ and $\hat{\alpha} = 5.084$. Similar values were obtained using the method of moments. The empirical distribution function for the wind data along with two fitted maximum likelihood models are presented in Figure 1. It is clear that the fit is not good at all, especially around the interval $(16, 24)$. The reason for the bad fit is the fact that both fitted curves are consistently concave down for all the x's and F seems to be concave up in this area. The fit in the tails seems to be a little better.

Once we determined the values of the unknown model parameters, MLE estimators for $1 - F(29.5)$ may be obtained from (3.1) and (3.2). The numerical values of these estimates, their

Fitted Model	Estimate of $1-F(29.5)$	Approx. s.e.	Approx. 95% c.i. (two sided)
Non-parametric (Plug-in)	0.05	0.034	(-0.019, 0.119)
Exponential	0.027	0.015	(-0.003, 0.057)
Pareto	0.036	0.024	(-0.012, 0.084)
3-Step Moving Average Smoother	0.045	0.016	(0.013, 0.079)

Table 1: Comparison of the performance of estimators for $1 - F(29.5)$ for the wind data. All the confidence intervals and variances for the first three estimates are calculated using the normal theory approximation. The variance and confidence intervals for the estimate based on the moving-average smoother are calculated by means of the approximate BESE and bootstrap percentile methods described in Section 2.

respective variances and 95% confidence intervals are summarized in the second and third row of Table 1. In the first row the same characteristics are calculated for the standard non-parametric estimate based on relative frequencies. As we may well see, the respective values of the point estimators differ considerably from model to model and, in particular, both MLE's are quite far away from the relative frequency estimator. Another thing worth noticing is that the confidence intervals for all three models have negative lower bounds - they are obviously too long, at least on one side. This also indicates that their true coverage probability may be in fact greater than 95%.

In order to provide a better estimate of $1 - F(29.5)$ for the wind data we will first need to construct a smoothed version of the empirical cdf. In order to do so we employ the following data transformation widely used in image and signal processing theory where a series of raw data $\{x_1, x_2, \ldots, x_n\}$ is often transformed to a new series of data before it is analyzed. The purpose of this transformation is to smooth out local fluctuations in the raw data, so the transformation is called *data smoothing* or a *smoother*. One common type of smoother employs a linear transformation and is called a linear filter. A linear filter with weights $\{c_0, c_1, \ldots, c_{r-1}\}$ transforms the given data to weighted averages $\sum_{j=0}^{r-1} c_j x_{t-j}$ for $t = r, r+1, \ldots, n$. Notice that the new data set has length $n - r - 1$. If all the weights c_k are equal and they sum to unity, the linear filter is called a r-term moving average. For an overview of this interesting technique and its various applications see e.g., Simonoff (1997). To create a smoothed version of the empirical cdf for the wind data we have first used a 3-term moving average smoother and then linearized in-between any two consecutive data

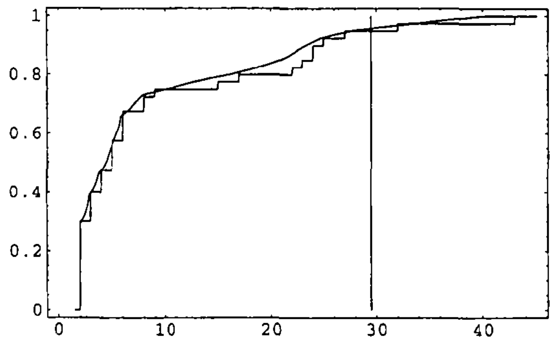

Figure 2: Empirical cdf for the wind data and its smoothed version obtained using the 3-term moving average smoother. The vertical line is drawn for reference at x=29.5.

points. The plot of this linearized smoother along with the original empirical cdf is presented in Figure 2. Let us note that the smoother follows the "concave-up-down-up" pattern of the data, which was not the case with the parametric distributions fitted from the families (3.1) and (3.2).

Once we have constructed the smoothed empirical cdf for the wind data we may simply read the approximate value of $1 - F(29.5)$ off the graph (or better yet, ask the computer to do it for us). The resulting numerical value is 0.045. What is the s.e. for that estimate? We again may use the bootstrap to answer that question without messy calculations. An approximate value for $BESE$ (with B=1000, but the result is virtually the same for B=100) is found to be 0.016, which is only slightly worse then that of exponential model MLE and much better then the s.e. for the Pareto and empirical models. Equivalently, the same result may be obtained by numerical integration. Finally, the 95% confidence interval for $1 - F(29.5)$ is found by means of the bootstrap percentile method with the number of replications, B=1000. Here the superiority of bootstrap is obvious, as it gives an interval which is the second shortest (again exponential MLE model gives a shorter interval) but, most importantly, is bounded away from 0. The results are summarized in Table 1. Let us note that the result based on a smoothed empirical cdf and bootstrap dramatically improves that based on the relative frequency (plug-in) estimator and standard normal theory. It is perhaps of interest to note also that the MLE estimator of $1 - F(29.5)$ in the exponential model is nothing else but a parametric bootstrap estimator. For more details on the connection between MLE estimators and bootstrap, see ET.

4 Clustered Data

In the previous section we have assumed that the wind data were distributed according to some continuous cdf F. Clearly this is not always the case with loss data and in general we may expect our theoretical loss distribution to follow some mixture of discrete and continuous cdf's.

4.1 Massachusetts Auto Bodily Injury Liability Data

In the Appendix B we present the set of 432 closed losses due to bodily injuries in car accidents under bodily injury liability (BI) policies reported in the Boston Territory (19) for the calendar year of 1995, as of mid-1997. The losses are recorded in thousands and are subject to various policy limits but have no deductible. Policy limits capped 16 out of 432 losses which are therefore considered right-censored. The problem of bootstrapping censored data will be discussed in the next section; here we would like to concentrate on another interesting feature of the data. Massachusetts BI claim data are of interest because the underlying behavioral processes have been analyzed extensively. Weisberg and Derrig (1992) and Derrig, Weisberg and Chen (1994) describe the Massachusetts claiming environment after a tort reform as a "lottery" with general damages for non-economic loss (pain and suffering) as the prize. Cummins and Tennyson (1992) showed signs of similar patterns countrywide while RAND (1995) and the Insurance Research Council (1996) documented the pervasiveness of the lottery claims in both tort and no-fault state injury claim payment systems. The overwhelming presence of suspected fraud and buildup claims[2] allow for distorted relationships between the underlying economic loss and the liability settlement. Claim negotiators can greatly reduce the usual non-economic damages when exaggerated injury and/or excessive treatment are claimed as legitimate losses. Claim payments in such a negotiated process with discretionary injuries

[2] In auto, fraudulent claims are those in which there was no injury or the injury was unrelated to the accident whereas buildup claims are those in which the injury is exaggerated and/or the treatment is excessive.

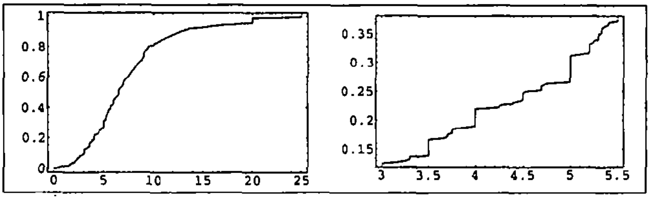

Figure 3: Approximation to the empirical cdf for the BI data adjusted for the clustering effect. Left panel shown the graph plotted for the entire range of observed loss values (0,25). Right panel zooms in on the values from 3.5 to 5. Discontinuities can be seen here as the graph's "jumps" at the observed loss values of high frequency: 3.5, 4, 4.5, 5.

tend to be clustered at some usual mutually acceptable amounts, especially for the run-of-the-mill strain and sprain claims. Conners and Feldblum (1997) suggest that the claim environment, rather than the usual rating variables, are the key elements needed to understand and estimate relationships in injury claim data. All the data characteristics above tend to favor empirical methods over analytic ones.

Looking at the frequencies of occurrences of the particular values of losses in Massachusetts BI claim data we may see that several numerical values have especially high frequency. The loss of $5,000 was reported 21 times (nearly 5% of all the occurrences), the loss of $20,000 was reported 15 times, $6,500 and $4,000 losses were reported 14 times, a $3,500 loss was only slightly less common (13 times), and the losses of size $6,000 and $9,000 occurred 10 times each. There were also several other numerical values that have occurred at least 5 times. The clustering effect is obvious here and it seems that we should incorporate it into our model. This may be accomplished for instance by constructing an approximation to the empirical cdf which is linearized in between the observed data values except for the ones with high frequency where it behaves like the original, discrete cdf. In Figure 3 we present such an approximate cdf for the BI data. We have allowed our adjusted cdf to have discontinuities at the observed values which occurred with frequencies of 5 or greater.

4.2 Bootstrap Estimates for Loss Elimination Ratios

To give an example of statistical inference under this model, let us consider a problem of eliminating part of the BI losses by purchasing a re-insurance policy that would cap the losses at some level d. Since the BI data is censored at \$20,000 we would consider here only values of d not exceeding \$20,000. One of the most important problems for the insurance company considering purchasing re-insurance is an accurate prediction of whether such a purchase would indeed reduce the experienced severity of loss and if so, by what amount. Typically this type of analysis is done by considering the loss elimination ratio (LER) defined as

$$LER(d) = \frac{E_F(X,d)}{E_F X}$$

where $E_F X$ and $E_F(X,d)$ are, respectively, expected value and limited expected value functions for a random variable X following a true distribution of loss F. Since LER is only a theoretical quantity unobservable in practice, its estimate calculated from the data is needed. Usually, one considers empirical loss elimination ratio ($ELER$) given by the obvious plug-in estimate

$$ELER(d) = \frac{E_{F_n}(X,d)}{E_{F_n} X} = \frac{\sum_{i=1}^{n} \min(X_i, d)}{\sum_{i=1}^{n} X_i} \tag{4.1}$$

where $X_1, \ldots, X_n$ is a sample.

The drawback of $ELER$ is in the fact that (unlike LER) it changes only at the values of d being equal to one of the observed values of $X_1, \ldots, X_n$. It seems, therefore, that in order to calculate approximate LER at different values of d some smoothed version of $ELER$ ($SELER$) should be considered. $SELER$ may be obtained from (4.1) by replacing the empirical cdf $\widehat{F}_n$ by its smoothed version obtained for instance by applying a linear smoother (as for the wind data considered in Section 3) or a cluster-adjusted linearization. Obviously, the $SELER$ formula may become quite complicated and its explicit derivation may be tedious (and so would be the derivation of its standard error). Again, the bootstrap methodology can be applied here to facilitate the computation of an approximate value of $SELER(d)$, its standard error and confidence interval for any given value of d.

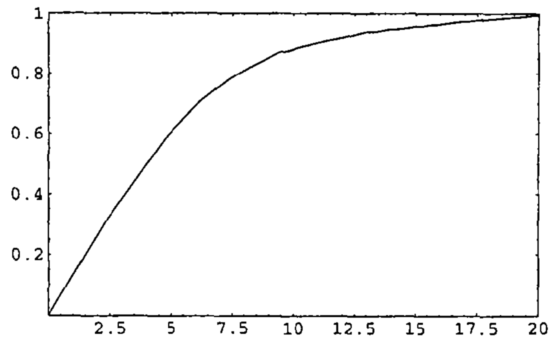

Figure 4: Approximate graph of $SELER(d)$ plotted for the values of d between 0 and the first censoring point (20) for the BI data.

In Figure 4 we present the graph of the $SELER$ estimate for the BI data calculated for the values of d ranging from 0 to 20 (lowest censoring point) by means of a bootstrap approximation. This approximation was obtained by resampling the cluster-adjusted, linearized version of the empirical cdf (presented in the left panel of Figure 3) a large number of times ($B = 300$) and replicating $\hat{\theta} = SELER$ each time. The resulting sequence of bootstrap estimates $\hat{\theta}^*(b)$ ($b = 1, \ldots, B$) was then averaged to give the desired approximation of $SELER$. The calculation of standard errors and confidence intervals for $SELER$ was done by means of $BESE$ and the method of percentiles, as described in Section 2. The variances and 95% confidence intervals of $SELER$ for several different values of d are presented in Table 2.

5 Extensions to More Complicated Designs

So far in our account we have not considered any problems related to the fact that often in practice we may have to deal with truncated (e.g., due to deductible) or censored (e.g., due to policy limit) data. Another frequently encountered difficulty is the need for inflation adjustment, especially with data observed over a long period of time. We will address these important issues now.

d	$SELER(d)$	s.e.	95% c.i. (two sided)
4	0.505	0.0185	(0.488,0.544)
5	0.607	0.0210	(0.597,0.626)
10.5	0.892	0.0188	(0.888,0.911)
11.5	0.913	0.0173	(0.912,0.917)
14	0.947	0.0127	(0.933,0.953)
18.5	0.985	0.00556	(0.98,0.988)

Table 2: Numerical values of $SELER(d)$ for the BI data tabulated for several different d along with the standard errors and 95% confidence intervals calculated by means of the approximate BESE and bootstrap percentile methods described in Section 2.

5.1 Policy Limits and Deductibles. Bootstrapping Censored Data

Let us consider again the BI data presented in Section 4. There were 432 losses reported out of which 16 were at the policy limits[3]. These 16 losses may therefore be considered censored from above (or right-censored) and the appropriate adjustment for this fact should be made in our approach to estimating the loss distribution F. Whereas 16 is less then 4% of the total number of observed losses for the BI data, these censored observations are crucial in order to obtain a good estimate of F for the large loss values.

Since the problem of censored data arises naturally in many medical, engineering, and other settings, it has received considerable attention in statistical literature. For the sake of brevity we will limit ourselves to the discussion of only one of the several commonly used techniques, the so-called Kaplan-Meier (or product-limit) estimator.

The typical statistical model for right-censored observations replaces the usual observed sample $X_1 \dots, X_n$ with the set of ordered pairs $(X_1, \delta_1), \dots, (X_n, \delta_n)$ where

$$\delta_i = \begin{cases} 0 & \text{if } X_i \text{ is censored,} \\ 1 & \text{if } X_i \text{ is not censored} \end{cases}$$

and the recorded losses are ordered $X_1 = x_1 \leq X_2 = x_2 \leq \dots \leq X_n = x_n$ with the usual convention that in the case of ties the uncensored values x_i ($\delta_i = 1$) precede the censored ones ($\delta_i = 0$). The

[3]Fifteen losses were truncated at $ 20,000 and one loss was truncated at $25,000.

Kaplan-Meier estimator of $1 - F(x)$ is given by

$$\widehat{S}(x) = \prod_{i:x_i \leq x} \left(\frac{n-i}{n-i+1}\right)^{\delta_i} \tag{5.1}$$

The product in the above formula is that of i terms where i is the smallest positive integer less or equal n (the number of reported losses) and such that $x_i \leq x$. The Kaplan-Meier estimator, like the empirical cdf, is a step function with jumps at those values x_i that are uncensored. In fact, if $\delta_i = 1$ for all i, $i = 1, \ldots, n$ (i.e., no censoring occurs) it is easy to see that (5.1) reduces to the usual empirical cdf. If the highest observed loss x_n is censored, the formula (5.1) is not defined for the values of x greater then x_n. The usual practice is then to add one uncensored data point (loss value) x_{n+1} such that $x_n < x_{n+1}$ and to define $\widehat{S}(x) = 0$ for $x \geq x_{n+1}$. For instance, for the BI data the largest reported loss was censored at 25 and we had to add one artificial "loss" at 26 to define the Kaplan-Meier curve for the losses exceeding 25. The number 26 was picked quite arbitrarily, in actuarial practice more precise guess of the maximal possible value of loss (e.g. based on past experience) should be easily available. The Kaplan-Meier estimator enjoys several optimal statistical properties and can be viewed as a generalization of the usual empirical cdf adjusted for the fact of censoring losses. Moreover, truncated losses or truncated and censored losses may be easily handled by some simple modifications of (5.1). For more details and some examples see for instance Klugman, Panjer and Willmot (1998 chap.2).

In the case of loss data coming from a mixture of some discrete and continuous cdf's, like, for instance, the BI data, the linearization of Kaplan-Meier estimator with adjustment for clustering seems to be appropriate. In Figure 5 we present the plots of a linearized Kaplan-Meier estimator for the BI data and the approximate empirical cdf function, which was discussed in Section 4, not corrected for the censoring effect. It is interesting to note that the two curves agree very well up to the first censoring point (20), where Kaplan-Meier estimator starts to correct for the effect of censoring. It is thus reasonable to believe that for instance the values of $SELER$ calculated in Table 2 should be close to the values obtained by bootstrapping the Kaplan-Meier estimator. This, however, does not have to be the case in general. The agreement between the Kaplan-

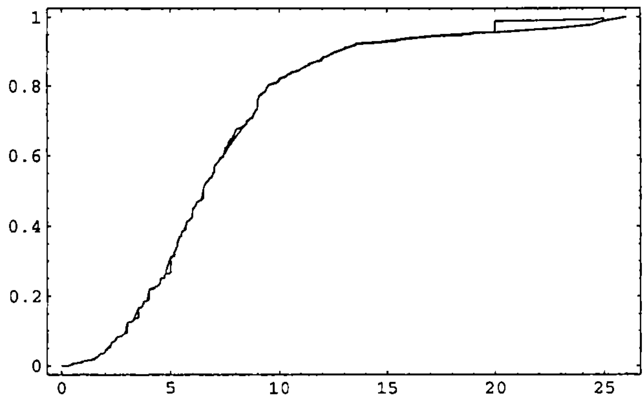

Figure 5: Linearized and adjusted for clustering Kaplan-Meier estimator of the true loss distribution F for the BI data plotted along with the empirical cdf described in Section 4 which was adjusted for the clustering effect but disregarded censoring. The two curves agree very well up to the first censoring point (20), where Kaplan-Meier estimator (lower curve) starts to correct for the effect of censoring.

Meier curve and the smoothed cdf of the BI data is mostly due to the relatively small number of censored values. The estimation of other parameters of interest under the Kaplan-Meier model (e.g. quantiles, probability of exceedance, etc) as well as their standard errors may be performed using the bootstrap methodology outlined in the previous sections. For more details on the problem of bootstrapping censored data, see for instance Akritas (1986).

5.2 Inflation Adjustment

The adjustment for the effect of inflation can be handled quite easily in our setting. If X is our random variable modeling the loss which follows cdf F, when adjusting for inflation we are interested in obtaining an estimate of the distribution of $Z = (1 + r)X$, where r is the uniform inflation rate over the period of concern. If Z follows a cdf G then obviously,

$$G(z) = F\left(\frac{z}{1+r}\right)$$

and the same relation holds when we replace G and F with the usual empirical cdf's or their smoothed versions.[4] In this setting bootstrap techniques described earlier should be applied to the empirical approximation of G.

6 Some Final Remarks

Although we have limited the discussion of resampling methods in DFA to modeling losses, even with this narrowed scope we have presented only some examples of modern statistical methods relevant to the topic. Other important areas of applications which has been purposely left out here include kernel estimation and the use of resampling in non-parametric regression and auto-regression models. The latter includes for instance such important problems as bootstrapping time series data, modeling time correlated losses and other time-dependent variables. Over the past several years some of these techniques, like non-parametric density estimation, have already found their way into actuarial practice (cf. e.g., Klugman at al. 1998). Others, like bootstrap, are still waiting. The purpose of this article was not to give a complete account of the most recent developments in non-parametric statistical methods but rather to show by example how easily they may be adapted to the real-life situations and how often they may, in fact, outperform the traditional approach.

7 Conclusions

Several examples of the practical advantages of the bootstrap methodology were presented. We have shown by example that in many cases bootstrap provides a better approximation to the true parameters of the underlying distribution of interest then the traditional, textbook approach relying on the MLE and normal approximation theory. It seems that bootstrap may be especially

[4]Subclasses of losses may inflate at different rates, soft tissue vs hard injuries for the BI data as an example. The theoretical cdf G may be then derived using multiple inflation rates as well.

useful in the statistical analysis of data which do not follow any obvious continuous parametric model (or mixture of models) or/and contain a discrete component (like the BI data presented in Section 4). The presence of censoring and truncation in the data does not present a problem for the bootstrap which, as seen in Section 5, may be easily incorporated into a standard non-parametric analysis of censored or truncated data. Of course, most of the bootstrap analysis is typically done approximately using a Monte Carlo simulation (generating resamples), which makes the computer an indispensable tool in the bootstrap world. Even more, according to some leading bootstrap theorists, automation is the goal: "One can describe the ideal computer-based statistical inference machine of the future. The statistician enters the data ... the machine answers the questions in a way that is optimal according to statistical theory. For standard errors and confidence intervals, the ideal is in sight if not in hand" (quoted from page 393 of ET).

The resampling methods described in this paper can be used (possibly after correcting for time-dependence) to handle the empirical data concerning all DFA model input variables, including interest rates and capital market returns. The methodologies also apply to any financial intermediary, such as a bank or a life insurance company. It would be interesting, indeed it is imperative, to make bootstrap-based inferences in such settings and compare their effectiveness and applicability with classical parametric, trend-based, Bayesian, and other methods of analysis. The bootstrap computer program (using Mathematica 3.0 programming language, see Appendix A) that we have developed here to provide smooth estimates of an empirical cdf, BESE, and bootstrap confidence intervals could be easily adapted to produce appropriate estimates in Dynamic Financial Analysis, including regulatory calculations for Value at Risk and Asset Adequacy Analysis. It would also be interesting to investigate further all areas of financial management where our methodologies may hold a promise of future applications. For instance, by modeling both the asset side (interest rates and capital market returns) and the liabilities side (losses, mortality, etc.), as well as their interactions (crediting strategies, investment strategies of the firm) one might create nonparametric models of the firm, and use such a whole-company model to analyze value optimization and solvency protection in an integrated framework. Such whole company models are more and more commonly used by financial intermediaries, but we propose an additional level of complexity by adding the bootstrap estimation of their underlying random structures. This methodology is immensely computationally intensive, but it holds great promise not just for internal company models, but also for regulatory supervision, hopefully allowing for better oversight avoiding problems such as insolvencies of savings and loans institutions in the late 1980s, life insurance firms such as Executive Life and Mutual Benefit, or catastrophe-related problems of property-casualty insurers.

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Appendices

Appendix A

The computer program written in *Mathematica* 3.0 programming language used to calculate bootstrap replications, bootstrap standard errors estimates (BESE) and bootstrap 95% confidence intervals using the method of percentiles.

(* Here we include the standard statistical libraries to be used in our bootstraping program *)

```
<< Statistics`DataManipulation`
<< Statistics`ContinuousDistributions`
```

(* Here we define resampling procedure "boot[]" as well as empirical cdf functions: usual empirical cdf "empcdf[]" and its smoothed version "cntcdf[]" . Procedure "inv[]" is used by "boot[]" *)

(* Arguments for the procedures are as follows:
"boot[]" has two arguments: "lst" (any data list of numerical values) and ,"nosam" (number of resamples, usually nosum=Length[lst]
"empcdf[]" and "cntcdf[]" both have two arguments "lst" (any data list of numerical values) and "x" -the numerical argument of function *)

```
inv[x_, lstx_] :=
Module[{nlx = Length[lstx]},
 If [x == 0 , lstx[[1]],
  If[x == 1, lstx[[nlx]] ,  k = Floor[(nlx - 1) x];
   ((nlx - 1) x - k ) (lstx[[k + 2]] - lstx[[k + 1]]) + lstx[[k + 1]]
  ]
 ]
];

boot[lx_, nosam_] := Module[{tt, i, a, n, lstx}, lstx = Sort[lx]; n = Length[lx];
lstx = Flatten[{{2 lstx[[1]] - lstx[[2]]}, lstx, {2 lstx[[n]] - lstx[[n - 1]]}}];
 tt = RandomArray[UniformDistribution[0, 1], nosam] ;
  For[i = 1, i <= nosam , i++, a[i] = inv[tt[[i]], lstx]];
   Table[a[i], {i, 1, nosam}]
   ];

cntcdf[lst_, x_] := Module[{ll = Sort[lst], n = Length[lst], i = 1},
  ll = Flatten[{{2 ll[[1]] - ll[[2]]}, ll, {2 ll[[n]] - ll[[n - 1]]}}];
While[i <= n + 2 && x > ll[[i]], i++];
 If[i == 1, 0, If[i == n + 3, 1, ((x - ll[[i - 1]]) / (ll[[i]] - ll[[i - 1]]) + (i - 2)) / (n + 1)]]
 ];


empcdf[lst_, x_] := Module[{ll = Sort[lst], n = Length[lst], i = 1},
While[i <= n && x > ll[[i]], i++];
 If[i == 1, 0, (i - 1) / n]
 ];
```

(* Here we define the bootstrap replications of statistic theta[]
Procedure "theta[]" calculates a statistic from the list of data "lst".
Procedure "replicate[]" replicates the statistic "theta[]" "norep" number of times using procedure "boot []" with parameters "lst" and "nosam". As a result of this procedure we obtain a list of replicated values of "theta[]" *)

```
theta[lst_] := 1; (* define your Theta statistic here*)

replicate[lst_, norep_, nosam_] := Module[{i, ll = {}}, For [i = 1, i <= norep, i++,
  ll = Flatten[{ll, theta[boot[lst, nosam]]}]
  ]; ll
 ];
```

(*Here we calculate BESE and 95% confidence interval based on the method of percentiles for 1000 replications *)

```
(* run "replicate[]" procedure, store the results in variable "listofrep" *)

listofrep = replicate[lst, norep, nosam];

(* BESE*)
Variance[listofrep]

(* 95 % confidence interval for number of replications (norep)=1000 *)

95 ci = {listofrep[[25]], listofrep[[975]]}
```

No	Injury Type	Total Amt Paid	Policy Limit
1	05	$393	$20,000
2	01	$500	$20,000
3	06	$500	$20,000
4	08	$900	$20,000
5	06	$1,000	$20,000
6	05	$1,000	$20,000
7	05	$1,250	$20,000
8	05	$1,500	$20,000
9	05	$1,500	$20,000
10	05	$1,525	$20,000
11	05	$1,631	$100,000
12	04	$1,650	$20,000
13	05	$1,700	$20,000
14	05	$1,700	$20,000
15	05	$1,800	$20,000
16	05	$1,950	$20,000
17	05	$2,000	$20,000
18	05	$2,000	$25,000
19	05	$2,007	$20,000
20	05	$2,100	$20,000
21	05	$2,100	$20,000
22	05	$2,100	$20,000
23	05	$2,250	$20,000
24	05	$2,250	$20,000
25	05	$2,250	$20,000
26	05	$2,250	$20,000
27	05	$2,270	$20,000
28	05	$2,300	$20,000
29	06	$2,300	$20,000
30	05	$2,375	$20,000
31	05	$2,450	$20,000
32	05	$2,500	$20,000
33	05	$2,500	$100,000
34	05	$2,500	$20,000
35	06	$2,500	$20,000
36	01	$2,600	$20,000
37	05	$2,750	$20,000
38	05	$2,800	$20,000
39	05	$2,813	$20,000
40	05	$2,900	$20,000
41	05	$3,000	$20,000
42	05	$3,000	$20,000
43	05	$3,000	$20,000
44	05	$3,000	$20,000
45	05	$3,000	$20,000
46	05	$3,000	$20,000
47	05	$3,000	$20,000
48	06	$3,000	$20,000
49	06	$3,000	$50,000
50	99	$3,000	$20,000
51	06	$3,000	$20,000
52	05	$3,000	$20,000
53	05	$3,000	$20,000
54	04	$3,000	$20,000
55	05	$3,150	$20,000
56	05	$3,250	$20,000
57	05	$3,300	$20,000
58	05	$3,300	$20,000
59	05	$3,300	$20,000
60	04	$3,500	$20,000
61	04	$3,500	$1,000,000
62	05	$3,500	$20,000
63	01	$3,500	$20,000
64	05	$3,500	$20,000

No	Injury Type	Total Amt Paid	Policy Limit
65	05	$3,500	$20,000
66	05	$3,500	$20,000
67	05	$3,500	$20,000
68	05	$3,500	$20,000
69	04	$3,500	$20,000
70	05	$3,500	$20,000
71	05	$3,500	$50,000
72	99	$3,500	$20,000
73	05	$3,650	$20,000
74	05	$3,700	$20,000
75	05	$3,700	$20,000
76	05	$3,700	$20,000
77	05	$3,750	$20,000
78	05	$3,750	$20,000
79	05	$3,750	$20,000
80	05	$3,750	$20,000
81	06	$3,900	$20,000
82	05	$4,000	$20,000
83	05	$4,000	$1,000,000
84	05	$4,000	$20,000
85	05	$4,000	$20,000
86	05	$4,000	$20,000
87	04	$4,000	$20,000
88	06	$4,000	$20,000
89	05	$4,000	$20,000
90	05	$4,000	$20,000
91	05	$4,000	$20,000
92	05	$4,000	$20,000
93	05	$4,000	$20,000
94	01	$4,000	$20,000
95	05	$4,000	$25,000
96	05	$4,250	$20,000
97	06	$4,250	$20,000
98	06	$4,278	$50,000
99	05	$4,396	$25,000
100	05	$4,400	$20,000
101	05	$4,476	$20,000
102	05	$4,500	$20,000
103	05	$4,500	$20,000
104	05	$4,500	$25,000
105	05	$4,500	$20,000
106	10	$4,500	$20,000
107	05	$4,500	$20,000
108	05	$4,521	$20,000
109	05	$4,697	$20,000
110	05	$4,700	$20,000
111	05	$4,700	$20,000
112	05	$4,700	$20,000
113	04	$4,725	$20,000
114	05	$4,750	$20,000
115	05	$5,000	$20,000
116	05	$5,000	$100,000
117	05	$5,000	$20,000
118	05	$5,000	$20,000
119	05	$5,000	$20,000
120	05	$5,000	$20,000
121	05	$5,000	$20,000
122	04	$5,000	$20,000
123	05	$5,000	$20,000
124	05	$5,000	$20,000
125	05	$5,000	$20,000
126	05	$5,000	$20,000
127	05	$5,000	$20,000
128	06	$5,000	$20,000
129	04	$5,000	$20,000

No	Injury Type	Total Amt Paid	Policy Limit
130	01	$5,000	$20,000
131	05	$5,000	$20,000
132	05	$5,000	$20,000
133	05	$5,000	$20,000
134	05	$5,000	$100,000
135	05	$5,000	$20,000
136	06	$5,100	$20,000
137	05	$5,200	$20,000
138	05	$5,200	$20,000
139	05	$5,200	$20,000
140	05	$5,200	$20,000
141	05	$5,200	$20,000
142	05	$5,200	$20,000
143	05	$5,200	$20,000
144	05	$5,225	$20,000
145	05	$5,250	$20,000
146	05	$5,250	$20,000
147	05	$5,292	$20,000
148	05	$5,296	$20,000
149	05	$5,300	$20,000
150	05	$5,300	$20,000
151	04	$5,300	$20,000
152	05	$5,333	$20,000
153	05	$5,333	$20,000
154	05	$5,333	$20,000
155	05	$5,333	$20,000
156	04	$5,344	$20,000
157	05	$5,366	$20,000
158	04	$5,400	$30,000
159	05	$5,400	$20,000
160	05	$5,415	$20,000
161	05	$5,497	$100,000
162	04	$5,500	$20,000
163	05	$5,500	$20,000
164	05	$5,500	$20,000
165	05	$5,500	$20,000
166	06	$5,500	$20,000
167	05	$5,566	$20,000
168	05	$5,600	$25,000
169	05	$5,714	$20,000
170	05	$5,714	$20,000
171	05	$5,714	$20,000
172	05	$5,714	$20,000
173	05	$5,714	$20,000
174	05	$5,714	$20,000
175	05	$5,714	$20,000
176	05	$5,725	$20,000
177	06	$5,750	$20,000
178	05	$5,750	$100,000
179	05	$5,750	$20,000
180	05	$5,852	$20,000
181	06	$5,898	$20,000
182	05	$5,900	$20,000
183	05	$5,964	$20,000
184	06	$5,990	$20,000
185	05	$6,000	$25,000
186	05	$6,000	$20,000
187	05	$6,000	$20,000
188	05	$6,000	$20,000
189	01	$6,000	$20,000
190	05	$6,000	$20,000
191	05	$6,000	$20,000
192	05	$6,000	$20,000
193	05	$6,000	$20,000
194	05	$6,000	$20,000

No	Injury Type	Total Amt Paid	Policy Limit
195	04	$6,077	$20,000
196	05	$6,078	$20,000
197	05	$6,131	$20,000
198	05	$6,166	$20,000
199	05	$6,166	$20,000
200	05	$6,169	$20,000
201	05	$6,171	$20,000
202	05	$6,208	$20,000
203	05	$6,243	$20,000
204	05	$6,318	$20,000
205	05	$6,399	$20,000
206	05	$6,413	$20,000
207	05	$6,500	$20,000
208	05	$6,500	$20,000
209	05	$6,500	$20,000
210	05	$6,500	$20,000
211	05	$6,500	$20,000
212	05	$6,500	$20,000
213	05	$6,500	$20,000
214	05	$6,500	$20,000
215	99	$6,500	$20,000
216	05	$6,500	$20,000
217	05	$6,500	$50,000
218	05	$6,500	$25,000
219	05	$6,500	$20,000
220		$6,500	$50,000
221	05	$6,519	$20,000
222	04	$6,536	$20,000
223	05	$6,549	$20,000
224	01	$6,558	$25,000
225	06	$6,600	$20,000
226	05	$6,600	$20,000
227	06	$6,620	$20,000
228	05	$6,700	$20,000
229	06	$6,703	$20,000
230	01	$6,743	$25,000
231	05	$6,750	$20,000
232	05	$6,800	$20,000
233	04	$6,870	$20,000
234	05	$6,893	$50,000
235	05	$6,898	$50,000
236	05	$6,907	$20,000
237	05	$6,933	$20,000
238	05	$6,935	$100,000
239	05	$6,977	$100,000
240	05	$7,000	$100,000
241	05	$7,000	$20,000
242	05	$7,000	$20,000
243	05	$7,000	$20,000
244	05	$7,000	$20,000
245	05	$7,000	$20,000
246	05	$7,000	$20,000
247	05	$7,014	$20,000
248	04	$7,043	$20,000
249	05	$7,079	$20,000
250	05	$7,118	$20,000
251	05	$7,163	$20,000
252	05	$7,191	$20,000
253	05	$7,200	$20,000
254	05	$7,200	$20,000
255	05	$7,250	$20,000
256	04	$7,252	$20,000
257	05	$7,304	$20,000
258	01	$7,412	$25,000
259	01	$7,425	$100,000

No	Injury Type	Total Amt Paid	Policy Limit
260	05	$7,432	$20,000
261	05	$7,444	$50,000
262	05	$7,447	$20,000
263	05	$7,500	$20,000
264	05	$7,500	$20,000
265	05	$7,500	$25,000
266	05	$7,500	$20,000
267	05	$7,500	$20,000
268	05	$7,500	$20,000
269	99	$7,500	$20,000
270	01	$7,564	$20,000
271	05	$7,620	$20,000
272	18	$7,629	$20,000
273	05	$7,657	$20,000
274	01	$7,670	$20,000
275	05	$7,671	$20,000
276	04	$7,696	$100,000
277	04	$7,700	$100,000
278	05	$7,750	$20,000
279	05	$7,754	$20,000
280	05	$7,820	$20,000
281	04	$7,859	$20,000
282	05	$7,868	$20,000
283	01	$7,873	$25,000
284	05	$7,920	$100,000
285	05	$7,922	$20,000
286	05	$7,945	$20,000
287	05	$7,954	$20,000
288	05	$7,961	$20,000
289	05	$8,000	$100,000
290	05	$8,000	$100,000
291		$8,000	$20,000
292	10	$8,013	$50,000
293	05	$8,073	$20,000
294	05	$8,200	$20,000
295	01	$8,298	$25,000
296	06	$8,300	$20,000
297	01	$8,420	$20,000
298	05	$8,485	$20,000
299	05	$8,500	$50,000
300	05	$8,500	$20,000
301	99	$8,500	$20,000
302	05	$8,500	$20,000
303	05	$8,515	$20,000
304	05	$8,612	$20,000
305	05	$8,634	$100,000
306	05	$8,686	$20,000
307	05	$8,785	$20,000
308	05	$8,786	$20,000
309	05	$8,794	$20,000
310	05	$8,805	$20,000
311	05	$8,815	$20,000
312	05	$8,856	$20,000
313	05	$8,861	$20,000
314	06	$8,882	$20,000
315	05	$8,911	$20,000
316	05	$8,914	$20,000
317	05	$8,988	$20,000
318	05	$9,000	$100,000
319	05	$9,000	$20,000
320	05	$9,000	$20,000
321	05	$9,000	$20,000
322	05	$9,000	$20,000
323	05	$9,000	$0
324	05	$9,000	$20,000

No	Injury Type	Total Amt Paid	Policy Limit
325	05	$9,000	$20,000
326	05	$9,000	$20,000
327	05	$9,000	$20,000
328	05	$9,009	$20,000
329	05	$9,020	$20,000
330	05	$9,030	$25,000
331	05	$9,051	$20,000
332	05	$9,053	$20,000
333	05	$9,073	$100,000
334	05	$9,100	$20,000
335	01	$9,129	$20,000
336	05	$9,200	$20,000
337	05	$9,208	$20,000
338	05	$9,300	$20,000
339	05	$9,355	$20,000
340	05	$9,356	$20,000
341	05	$9,392	$20,000
342	05	$9,395	$100,000
343	05	$9,423	$20,000
344	05	$9,428	$20,000
345	05	$9,451	$100,000
346	05	$9,500	$20,000
347	05	$9,500	$20,000
348	05	$9,602	$20,000
349	05	$9,710	$20,000
350	04	$9,881	$25,000
351	05	$9,909	$20,000
352	08	$10,000	$20,000
353	06	$10,000	$20,000
354	05	$10,000	$100,000
355	06	$10,000	$20,000
356	04	$10,106	$20,000
357	05	$10,229	$20,000
358	05	$10,330	$20,000
359	05	$10,331	$20,000
360	05	$10,400	$20,000
361	05	$10,505	$100,000
362	04	$10,555	$20,000
363	01	$10,645	$20,000
364	08	$10,861	$20,000
365	05	$10,968	$20,000
366	05	$11,000	$50,000
367	04	$11,000	$100,000
368	05	$11,032	$20,000
369	05	$11,144	$20,000
370	05	$11,166	$20,000
371	01	$11,262	$25,000
372	05	$11,344	$50,000
373	99	$11,353	$20,000
374	05	$11,385	$20,000
375	01	$11,500	$20,000
376	05	$11,626	$20,000
377	05	$11,835	$20,000
378	99	$11,986	$20,000
379	05	$11,991	$20,000
380	04	$12,000	$20,000
381	05	$12,000	$20,000
382	05	$12,000	$20,000
383	05	$12,214	$100,000
384	05	$12,274	$20,000
385	05	$12,374	$20,000
386	99	$12,380	$20,000
387	03	$12,500	$20,000
388	05	$12,509	$20,000
389	05	$12,621	$100,000

No	Injury Type	Total Amt Paid	Policy Limit
390	05	$12,756	$20,000
391	05	$12,859	$20,000
392	05	$12,988	$20,000
393	07	$13,000	$20,000
394	05	$13,009	$20,000
395	05	$13,299	$50,000
396	04	$13,347	$20,000
397	05	$13,500	$20,000
398	05	$13,570	$20,000
399	99	$13,572	$100,000
400	04	$14,181	$20,000
401	05	$14,700	$20,000
402	05	$14,953	$20,000
403	05	$15,500	$20,000
404	05	$15,500	$100,000
405	05	$15,765	$20,000
406	18	$16,000	$20,000
407	05	$16,668	$20,000
408	05	$16,794	$20,000
409	04	$17,267	$100,000
410	99	$18,500	$20,000
411	99	$18,500	$20,000
412	18	$19,000	$20,000
413	05	$19,012	$20,000
414	99	$20,000	$20,000
415	05	$20,000	$20,000
416	07	$20,000	$20,000
417	08	$20,000	$20,000
418	08	$20,000	$20,000
419	07	$20,000	$20,000
420	07	$20,000	$20,000
421	03	$20,000	$20,000
422	06	$20,000	$20,000
423	16	$20,000	$20,000
424	05	$20,000	$20,000
425	06	$20,000	$20,000
426	05	$20,000	$20,000
427	09	$20,000	$20,000
428	05	$20,000	$20,000
429	01	$22,692	$100,000
430	05	$24,500	$50,000
431	99	$25,000	$25,000
432	02	$25,000	$100,000

Appendix B

Massachusetts BI Data

Injury Type	Description
01	MINOR LACERATIONS/CONTUSIONS
02	SERIOUS LACERATION
03	SCARRING OR PERMANENT DISFIGUREMENT
04	NECK ONLY SPRAIN/STRAIN
05	BACK OR NECK & BACK SPRAIN/STRAIN
06	OTHER SPRAIN/STRAIN
07	FRACTURE OR WEIGHT BEARING BONE
08	OTHER FRACTURE
09	INTERNAL ORGAN INJURY
10	CONCUSSION
11	PERMANENT BRAIN INJURY
12	LOSS OF BODY PART
13	PARALYSIS/PARESIS
14	JAW JOINT DYSFUNCTION
15	LOSS OF A SENSE
16	FATALITY
17	DENTAL
18	CARTILAGE/MUSCLE/TENDON/LIGAMENT INJURY
19	DISC HERNIATION
20	PREGNANCY RELATED
21	PRE-EXISTING CONDITION
22	PSYCHOLOGICAL CONDITION
30	NO VISIBLE INJURY
99	OTHER

Stochastic Modeling and Error Correlation in Dynamic Financial Analysis

by Son T. Tu, Ph.D., ACAS, MAAA

STOCHASTIC MODELING AND ERROR CORRELATION IN DYNAMIC FINANCIAL ANALYSIS

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ABSTRACT

New treatments of stochastic modeling and error correlation in dynamic financial analysis are introduced. The former refers to the methods for modeling individual insurance operations. The latter refers to the technique for considering the interactions and correlations among those operations. The stochastic chain ladder model, a new technique for loss development, is also introduced and is shown to be an integral part of DFA.

1. INTRODUCTION

Dynamic financial analysis is now segregated into two different philosophies: that of stochastic simulation, and that of scenario testing. Feldblum[1] discussed the strengths and weaknesses of the two approaches. We believe that the two need not be separate and competing, but indeed need to be complementary. With the model proposed in this paper, we hope to narrow the gap between these two approaches.

The construction of this model is motivated principally by four factors. The first two, qualitative in nature, are understandability and flexibility. The other two, quantitative in nature, are the stochastic modeling of individual insurance components and the error correlation among those components.

Many users and interested parties of a dynamic financial analysis model are not actuaries and technical analysts. Therefore, it is important that these users can gain, relatively easily, a good understanding and confidence in the model.

Secondly, the usage to which a model will be applied varies widely. In some studies, the analyst may only be interested in the overall picture; and relatively few insurance operations need be modeled. In another study, a relatively large number of operations needed to be included, because more detailed quantitative analyses are required. A model should have sufficient flexibility to suit both extremes.

To satisfy the above two factors, our model is controlled by a set of governing equations. One or more of these equations describes each operation. The model has understandability, because each equation is usually a readily accepted insurance formula. The model has flexibility, because the number of equations can be expanded or contracted, depending on the needs and objectives of the analyst.

There can be several stochastic models for generating observations of an insurance variable, such as the loss ratio or the investment return. By far, the most popular among actuaries is the averaging technique, where the observations are assumed to be random about some average. In a subsequent section, we present two other alternatives, which we name the current-value and current-change models. We show that they fit the historical data used in this study better than the averaging technique. These two models have analogies in time-series analysis.

An important consideration in any DFA model is the correlation among the variables considered in the analysis. Depending upon the sign of the correlation between two variables, the correlation can be either stabilizing or destabilizing, a concept that we will elucidate in section 4. The correlation coefficients among the variables will be measured. As a natural and necessary by-product, we present a technique for the generation of correlated random numbers.

In this paper, we aim only to demonstrate the concept and potential of the model. We have simplicity as one of the objectives of the paper; therefore, the number of operations has been kept to a relative few. We will study a hypothetical insurer, which is assumed to have written only Workers Compensation for the last ten years. Our study projects five years into the future. At the end of that time frame, among other quantities, we want to examine the probability of ruin. To work with realistic data, all of the relevant data has been taken from the 1997 Best's Aggregates and Averages publication.

In section 2, we present the governing system of equations used in this study. In section 3, we present the hypothetical initial state of the company

In section 4, we present the stochastic modeling of the insurance random variables. In section 5, we model paid losses. For this purpose, we will introduce our research on the stochastic chain ladder and Bornhuetter-Ferguson loss reserving models.

In section 6, a technique for the generation of correlated random numbers will be introduced. In section 7, we pull together the materials in all the preceding sections to generate simulated solutions for the next five years.

In section 8, we show that the simulated results can be assumed normally distributed. In section 9, we outline the many potential extensions to the model.

In the concluding section, we summarize and discuss the criteria by which a user of dynamic financial analysis would evaluate one strategy or decision as being superior to another.

[1] Sholom Feldblum, "Forecasting the Future: Stochastic Simulation and Scenario Testing." *Incorporating Risk Factors in Dynamical Financial Analysis*, 1995 CAS Discussion Paper Program.

2. Governing System of Equations

Following is the list of random variables that we consider for this hypothetical study:

AS = Assets,
LI = Liabilities,
SU = Surplus,
PL = Paid Losses,
IL = Incurred Losses,
II = Investment Income,
EP = Earned Premiums,
PD = Paid Policyholders' Dividends,
PE = Paid Underwriting Expenses,
WP = Written Premiums,
IIR = Investment Income Ratio,
UER = Calendar-Year Underwriting Expense Ratio,
PDR = CY Policyholders' Dividend Ratio,
EPR = CY Earned Premium Ratio,
LR = Accident-Year Loss and LAE Ratio,
F = value of the loss cumulative distribution function, and
ε = process error of the paid loss.

Each of the variables takes the argument of time. It is an understood that a variable refers to the value during that year (such as written premiums and paid losses) or at year-end (such as assets and liabilities). For simplicity, we will consider only yearly intervals.

Consider the following system of nine equations:

(2.1) $$AS(t) = LI(t) + SU(t)$$

(2.2) $$LI(t) = LI(t-1) - PL(t) + IL(t)$$

(2.3) $$AS(t) = AS(t-1) + WP(t) + II(t) - PL(t) - PE(t) - PD(t)$$

(2.4) $$EP(t) = EPR(t) * [WP(t-1) + WP(t)]$$

(2.5) $$II(t) = IIR(t) * AS(t)$$

(2.6) $$PE(t) = UER(t) * WP(t)$$

(2.7) $$PD(t) = PDR(t) * EP(t)$$

(2.8) $$IL(t) = LR(t) * EP(t)$$

(2.9) $$PL(t) = \sum_{all\ AY} LR(i) * EP(i) * [F(k+1) - F(k)] * (1+\varepsilon)$$

Even though there are many variables, many of them are inter-related. In fact, only five of them are independent. They are the investment income ratio (IIR), the U/W expense ratio (UER), the dividend ratio (PDR), the earned premium ratio (EPR), and the loss ratio (LR). In section 4, we will model the stochastic behaviors of these ratios from historical data and calculate their correlations. In section 6, we will simulate correlated random numbers for the ratios.

The most complex equation in the above set is (2.9), which is the sum of the paid losses for all accident years up to the evaluation date. In section 5, we will explain our stochastic loss reserving models and the workings of (2.9).

This is only an example of a set of governing equations. The analyst designs the exact set to meet his own needs. This offers great generality and flexibility.

3. THE INITIAL STATE OF THE COMPANY

Our insurer has been in existence for the last ten years, and writes only Workers Compensation. In the following table, loss ratios, earned premiums, paid losses, and liabilities for the past ten years are listed.

Table 1: The Initial State of the Company					
AY	LR	Ear. Prm	Inc. Loss	Paid Loss	Liability
1987	91.19%	5,002	4,562	4,209	353
1988	92.91%	5,403	5,020	4,558	462
1989	93.76%	5,835	5,471	4,865	606
1990	91.72%	6,302	5,780	4,997	783
1991	85.15%	6,806	5,795	4,819	976
1992	74.40%	7,350	5,469	4,299	1,170
1993	72.50%	7,938	5,755	4,150	1,605
1994	72.14%	8,573	6,185	3,859	2,326
1995	74.21%	9,259	6,871	3,232	3,639
1996	75.77%	10,000	7,577	1,593	5,984

The loss ratios were obtained from Schedule P – Part 1D of the Best's Aggregates & Averages publication. The incurred loss is the product of the loss ratio and the earned premium. The paid loss is a function of the incurred loss and the cumulative distribution function, which will be explained in section 5. The liability is the difference between the incurred and paid losses.

We assume the following initial liabilities, surplus and assets, with the initial year being 1996:

$$(3.1) \qquad LI(0) = 17{,}904; \quad SU(0) = 6{,}667; \quad AS(0) = 24{,}570.$$

The total liability is the sum of the last column in Table 1, and (3.1) satisfies (2.1). We assume that the insurer has the following target written premiums for the next five years:

Table 2: Target WP	
Year	WP
1997	10,800
1998	11,664
1999	12,597
2000	13,605
2001	14,693

We could as easily assume that the written premium is a product of the premium-to-surplus ratio and the surplus:

$$(3.2) \qquad WP(t) = PSR(t) * SU(t-1).$$

In such case, we would add (3.2) to the set of governing equations in the previous section. We elect not to follow this route, primarily because of a lack of historical data for the Workers' Compensation premium-to-surplus ratios. The analytical treatment of the two cases is similar.

4. STOCHASTIC MODELING OF THE INSURANCE RATIOS

In this section, we present the modeling of the loss and LAE ratio, the U/W expense ratio, the paid dividend ratio, the investment income ratio, and the earned premium ratio. The first is on an accident-year basis; the others on a calendar-year basis. All are taken from Best's Aggregates & Averages – Cumulative by Line Underwriting Experience.

Table 3: The Historical Insurance Ratios					
YEAR	LR	UER	PDR	IIR	EPR
1987	91.19%	15.00%	7.20%	10.20%	
1988	92.91%	13.40%	9.40%	10.90%	51.20%
1989	93.76%	13.00%	7.10%	11.40%	51.30%
1990	91.72%	13.40%	5.60%	10.80%	51.40%
1991	85.15%	14.60%	6.00%	11.70%	52.80%
1992	74.40%	16.50%	6.50%	16.60%	50.60%
1993	72.50%	17.20%	6.60%	14.60%	48.10%
1994	72.14%	18.60%	9.20%	13.90%	46.40%
1995	74.21%	20.30%	9.50%	16.70%	46.30%
1996	75.77%	23.30%	9.00%	16.90%	47.60%

The corresponding ratios for future years are, of course, random. The simulation of the random numbers is determined by the historical patterns. There are two things to consider in these patterns: the pattern within each set of ratios, and the correlation between any two sets of ratios.

To determine the pattern within each set of ratios, we consider three models: the average-value model, the current-value model, and the current-change model. For a given set of data, we pick the model that gives the least error deviation.

The average-value, or the averaging, model states that a random number is normally distributed about some average:

$$(4.1) \qquad x_i = \bar{x} + \varepsilon_i .$$

The first term on the right-hand side (RHS) of (4.1) is the average; the second is the uncorrelated errors of mean zero and some standard deviation. If we apply (4.1) to the loss ratios of Table 3, we have:

$$(4.2) \qquad \bar{x} = 82.4\%, \quad \sigma(\varepsilon_i) = \left\{ \frac{1}{n-1} \sum_i \varepsilon_i^2 \right\}^{1/2} = 9.37\% .$$

Therefore, the loss ratios have a mean of 82.4%, and the standard deviation of the errors is 9.37%. Note that n is the number of observations, and the degree of freedom is one less than that value since an average has to be estimated.

There are two sources of error in the average-value model. There is the parameter error, associated with the uncertainty in the estimation of the average. Also, there is the process error, which is associated with the random errors.

If we take a closer look at the loss ratios in Table 3, the average-value model does not seem to be appropriate. In the earlier years, 1987-91, all the ratios are greater than the average. In the later years, 1992-96, they are all smaller. Therefore, we next propose the current-value model:

$$(4.3) \qquad x_{i+1} = x_i + \varepsilon_{i+1} .$$

This model says that a random number tends to stay about its current value. The errors are assumed to be uncorrelated and of mean zero. If we apply (4.3) to the loss ratios, we get:

$$(4.4) \qquad \sigma(\varepsilon_i) = \left\{ \frac{1}{n} \sum_i \varepsilon_i^2 \right\}^{1/2} = 4.44\% .$$

The deviation of the average-value model is much greater than that of the current-value model, indicating that the latter is a much better fit for the observed loss ratios.

Contrary to the average-value model, there is only one source of error in (4.3), the process error, since no parameter has to be estimated in that equation.

If we look even more closely at the loss ratios in Table 3, we notice that an increase in the ratio tends to be followed by another increase, a decrease tends to be followed by another decrease. Therefore, we propose a third model, the current-change model:

$$(4.5) \qquad z_{i+1} = x_{i+1} - x_i, \quad z_{i+1} = z_i + \varepsilon_{i+1}.$$

This model says that the next change tends to be equal to the current change. And, like the current-value model, it only has process error. If we apply (4.5) to the loss ratios, we get:

$$(4.6) \qquad \sigma(\varepsilon_i) = \left\{\frac{1}{n}\sum_i \varepsilon_i^2\right\}^{1/2} = 4.09\%.$$

Since the current-change deviation is smallest, it represents the best fit, and we choose it to model the loss ratios in our analysis.

It makes a great deal of difference which model is chosen to represent a set of random variables. For instance, if we choose the average-value model for the loss ratios, then the simulated 1997 loss ratios have a mean of 82.4% and deviation of 9.37%, as shown in (4.2). If we choose the current-value model, they have a mean of 75.8% and deviation of 4.44%. If we choose the current-change model, they have a mean of 77.4% and deviation of 4.09%.

For the other four ratios, we will use the current-change model. The error terms have the following deviations:

Table 4: The Standard Deviations of the Ratios				
LR	UER	PDR	IIR	EPR
4.09%	1.57%	1.49%	2.06%	1.49%

We now turn to the calculation of the correlation between any two sets of errors. Let's consider the loss and the dividend ratios. They have the following errors:

Table 5: Values of two Error Sets		
YEAR	LR	PDR
1988		.022
1989	-.009	-.023
1990	-.029	-.015
1991	-.045	.004
1992	-.042	.005
1993	.089	.001
1994	.016	.026
1995	.024	.003
1996	-.005	-.005

The correlation coefficient of the two sets in Table 5 equals .185. The correlation coefficient is defined as:

$$(4.7) \qquad \rho(A,B) = \frac{Cov(A,B)}{\sigma(A)\sigma(B)}.$$

The correlation coefficients among the five ratios are found to be:

Table 6: Correlation Coefficients of the Five Insurance Ratios					
	LR	UER	PDR	IIR	EPR
LR	1.000	0.000	0.185	-0.528	-0.486
UER		1.000	0.000	0.000	0.132
PDR			1.000	0.000	-0.429
IIR				1.000	0.000
EPR					1.000

For any coefficient with an absolute value smaller than .1, we assume it to be statistically insignificant and set it equal to zero.

For this study, we use the empirical coefficients. For a genuine study, the analyst should decide whether the observed correlations seem reasonable. He may decide to override them if they do not.

The significance of Table 6 is this: not only should the simulated ratios have the deviations in Table 4, but they should have the correlations shown there. These have real consequences regarding the stability of the insurance process. For instance, that the loss and investment ratios have a negative correlation is destabilizing. The negative correlation means that a higher-than-average loss ratio tends to be coupled with a lower-than-average investment ratio, and vice versa. Taking the former case, the higher-than-average loss ratio means more loss payments, and the lower-than-average investment ratio means less investment income. If two quantities in conjunction tend to have the same effects on the balance sheet, then the correlation is destabilizing. Conversely, if they tend to impart opposite effects, then the correlation is stabilizing.

Every correlation in Table 6 destabilizes, except for the positive correlation between the expense and the earned premium ratios. In this case, if the insurer experiences higher-than-average expenses, then it also experiences higher-than-average earned premiums. The two have opposite impacts on the balance sheet, because the higher outgo (expenses) counteracts the higher income (earned premiums).

We emphasize that there are other reasonable stochastic models for the variables. This aptly demonstrates the tremendous flexibility and variety available to the analyst. The bottom line is that he should have confidence that the underlying model is representative of the future.

We are grateful to a review who pointed out that the current-value and current-change models have analogies in time-series forecasting.

5. Stochastic Modeling of Paid Losses

We have developed two stochastic loss reserving models: one based on the traditional chain ladder method, and the other on the Bornhuetter-Ferguson method. We have written a paper on each of these models.[2] [3]. The interested reader should contact the author for copies of the papers.

Basically, we model the stream of paid losses for an accident year as a function of a cumulative distribution function. The function that we use for Workers Compensation loss payment is the transformed lognormal:

$$(5.1) \qquad F(t;\mu,\sigma,\tau) = \Phi\left\{sign(\ln t)|\ln t|^{\tau};\mu,\sigma\right\}.$$

In (5.1), Φ is the normal distribution of mean μ and deviation σ. The argument t is measured in years.

Let an accident year have earned premium EP and loss ratio LR. Let Y_t be the incremental loss payment for that accident year between the report years t and $t+1$. Then the stochastic Bornhuetter-Ferguson model gives the following relationship:

[2] Son T. Tu, "The Application of Cumulative Distribution Functions in the Stochastic Chain Ladder Model," Scruggs Consulting Research Paper. This paper is in the process of publication in the *Casualty Actuarial Society Forum*.

[3] Son T. Tu, "The Stochastic Bornhuetter-Ferguson Model," Scruggs Consulting Research Paper.

(5.2) $$Y_{t+1} = EP * LR * [F(t+1) - F(t)] * (1 + \varepsilon_{t+1}),$$

where the process error ε_t is a normal distributed random variable of zero mean and some deviation.

In our paper on the stochastic Bornhuetter-Ferguson model, we demonstrate how to fit (5.2) to a triangle of incremental payments and come up with estimates of the function parameters. From an actual Best's Aggregates and Averages paid loss triangle, we derive the following estimates:

Table 7: The Estimates for the Parameters

	Estimate	Deviation
μ	.7840	.0591
σ	.9733	.0360
τ	.9286	.0352

If we use the estimates in Table 7 in (5.1), then we can obtain the following values for the distribution function:

Table 8: The Values of the Cumulative Distribution Function

time	1	2	3	4	5	6	7	8	9	
F	.2103	.4703	.6239	.7211	.7861	.8316	.8646	.8892	.9080	

Table 8 says that, after one year, 21.03% of payments for any accident year has been paid. After ten years, 92.27% has been paid, and therefore 7.73% has yet to be paid.

The function parameters also have the following matrix of correlation coefficients:

Table 9: The Matrix of Parameter Correlation Coefficients

	μ	σ	τ
μ	1.000	.9815	-.7633
σ		1.000	-.8180
τ			1.000

The way that we use (5.2) in the DFA model is as follows. For any calendar year, the loss payments are the sum of the paid losses for all accident years. The paid loss for each accident year is modeled by (5.2).

For the ten accident years in the past, we assume that the earned premiums and loss ratios are fixed, given by the values in Table 1. For the five accident years in the future, the earned premiums and loss ratios are stochastic quantities, given by numerical simulation. For this exercise, the process error in (5.2) has a standard deviation of 10.36%.

6. Generation of Correlated Random Numbers

Section 4 shows the necessity to generate five correlated insurance ratios. Section 5 shows the necessity to generate three correlated function parameters. In this section, we present a general technique to generate correlated random numbers. For instance, from Table 4, the errors of the loss ratios and the investment income ratios have expected deviations of 4.09% and 2.06%, respectively. But additionally, from Table 6, those errors have an expected correlation coefficient of -.528. In this section, we will introduce a technique to generate errors with the desired correlation characteristics. We will present some very technical work, which is needed for the sake of stochastic realism. But the reader may decide to skip this section without fear of losing the continuity among the other sections.

We will work with three variables. The technique can be easily generalized to any number of variables. Let's suppose that we need to generate three normally distributed random numbers X, Y, Z, and together they have the following variance matrix:

$$(6.1) \qquad Var(\mathrm{X}) = \begin{bmatrix} Var(X) & Cov(X,Y) & Cov(X,Z) \\ Cov(X,Y) & Var(Y) & Cov(Y,Z) \\ Cov(X,Z) & Cov(Y,Z) & Var(Z) \end{bmatrix}.$$

Instead of this problem, we are going to generate three uncorrelated normally distributed numbers A, B, C such that:

$$(6.2) \qquad Var(\mathrm{A}) = \begin{bmatrix} Var(A) & 0 & 0 \\ 0 & Var(B) & 0 \\ 0 & 0 & Var(C) \end{bmatrix}$$

Note that the second problem is much simpler than the original one.

We express the two sets of numbers as:

$$(6.3) \qquad \begin{aligned} A &= X, \\ B &= Y + b_1 X, \\ C &= Z + c_1 X + c_2 Y, \end{aligned}$$

where b_1, c_1, c_2 are unknown variables to be found. We will use the condition of no correlation among A, B, C to determine these unknowns. The condition that

$$(6.4) \qquad Cov(A,B) = 0.$$

We apply (6.4) to the first two equations of (6.3) to derive:

$$(6.5) \qquad b_1 Var(X) = -Cov(X,Y).$$

The conditions that

$$(6.6) \qquad Cov(A,C) = Cov(B,C) = 0,$$

yield

$$(6.7) \qquad \begin{aligned} & c_1 Var(X) + c_2 Cov(X,Y) = -Cov(X,Z), \\ & c_2 \left[Var(Y) + b_1 Cov(X,Y) \right] = -Cov(Y,Z) - b_1 Cov(X,Z). \end{aligned}$$

The first equation of (6.6) and the first and third equations of (6.3) give the first of (6.7). The second equation of (6.6) and the last two equations of (6.3) yield the second of (6.7). Equations (6.5) and (6.7) give the values of the unknown b_1, c_1, c_2. Taking the variance of (6.3), we have:

(6.8) $Var(A) = Var(X),$

$Var(B) = Var(Y) + b_1^2 Var(X) + 2b_1 Cov(X,Y),$

$Var(C) = Var(Z) + c_1^2 Var(X) + c_2^2 Var(Y) + 2[c_1 Cov(X,Z) + c_2 Cov(Y,Z) + c_1 c_2 Cov(X,$

Now we generate three uncorrelated random numbers with the variances in (6.8). Then we can invert (6.3) to obtain:

(6.9) $X = A,$

$Y = B - b_1 X,$

$Z = C - c_1 X - c_2 Y.$

In summary, we generate three uncorrelated random numbers A, B, C. Their variances are given by (6.8). Then we derive X, Y, Z from (6.9). The latter set of random numbers has an expected variance matrix of (6.1).

This technique can be used for any number of correlated variables. The equations for the unknown coefficients, corresponding to (6.5) and (6.7), become quite long and involved, but they fit a very regular and predictable pattern.

7. Numerical Simulation

In this section, we outline the numerical simulation scheme to obtain quantitative results of the modeled insurance process. In this scheme, we conduct 200 trials. For a genuine analysis, at least 1000 should be done.

We want to project the study five years into the future. For each year and for each trial, we generate five random numbers for the five insurance ratios discussed in section 4. In the generation of the random ratios, we take into account the correlation coefficients in Table 6. For instance, the 200 loss ratios have an expected deviation of 4.09%, and the 200 dividend ratios have an expected deviation of 1.49%. Moreover, the 200 pairs of loss and dividend ratios have an expected correlation coefficient of .185. As we mentioned earlier, the loss ratios and earned premiums for the past ten accident years are considered non-stochastic, and shown in Table 1.

For each trial, we generate a set of three function parameters, for use in the lognormal cumulative distribution function, having the variances and covariances shown in section 5. This accounts for the parameter errors in the paid losses. For each incremental payment, we also generate the process error in (5.2).

For each trial, we substitute the simulated numbers into equations (2.1)-(2.9). Therefore, for each random variable at each time t, we have a series of 200 realized values. Then we can simply take the mean and deviation of these values, which represent the mean and deviation of the random variable.

8. Normal Distribution of Numerical Results

From the numerical simulation, we can obtain the estimate and deviation of any random variable. Ideally, we would want to approximate every random variable as being normally distributed, because then the percentiles for the variable can be readily estimated. In this section, we will use the chi square goodness-of-fit test to show that the variables are approximately normally distributed.

Among the numerical details, in this section we look only at the surplus. The following table gives the means and deviations of the surplus for the next five years. It also includes the probability of ruin, (defined as the insurer having negative surplus), the number of expected ruins, and the number of observed ruins, among the 200 trials.

Table 10: Numerical Results of the Surplus					
Year	Mean	Deviation	% ruin	Expected	Observed
0	6,667	0	-	-	-
1	10,455	830	0.0	0	0
2	15,071	1,871	0.0	0	0
3	20,199	4,372	0.0	0	0
4	26,356	8,595	0.1	.2	0
5	33,770	14,699	1.1	2.2	2

To establish the ruin probabilities in Table 10, we assume the distribution of the surplus to be normal with the given means and deviations. We then compute the probability that the surplus reach zero in any given year. The expected number of ruins is then the product of that probability and 200. The fact that the expected and observed values are very comparable indirectly validates the merit of our approach.

To establish percentiles, we can of course take the distribution found among 200 trials. But a more desirable and convenient way would be to establish that the simulated results are approximately normally distributed. We note that the assumption of normal distribution cannot be taken for granted, since, even though the simulated random numbers are assumed normally distributed, equations (2.1)-(2.9) contain products of normally distributed simulated numbers, which generally do not follow that distribution.

For a numerical example, we take the surplus of the fifth year, and see if the simulated results could be reasonably approximated as being normally distributed. We use the chi-square goodness-of-fit test to either validate or reject this assumption. For the fifth-year surplus of mean 33,770 and deviation 14,699, we divide the whole spectrum of $(-\infty, \infty)$ into ten intervals of equal probability. For instance, the second interval runs from 14,933 to 21,399, representing the 10th and 20th percentiles, respectively. If the distribution is normally distributed, 20, or 10%, of the outcomes would be expected to fall into this interval. The following table presents the observed and expected frequencies for our simulated set:

Table 11: The Chi Square Test for the Fifth-Year Surplus										
	1	2	3	4	5	6	7	8	9	10
Obs.	17	21	25	19	16	28	18	17	19	20
Exp.	20	20	20	20	20	20	20	20	20	20

The chi-square value is:

$$(8.1) \qquad \chi^2 = \sum_{i=1}^{10} \frac{(O_i - E_i)^2}{E_i} = 6.50.$$

χ^2 should follow the chi-square distribution with nine degrees of freedom, giving a probability of 69%. In other words, if the 200 simulated fifth-year surplus values are normally distributed, there is a 69% probability that their chi-square value would be greater than 6.50. Therefore, the normal-distribution assumption is accepted.

We use the chi-square test on many of the random variables, and, by and large, the assumption of normal distribution is reasonably satisfied.

9. Extensions of the Model

In conducting the study, we use historical data. In other words, we assume that our insurer would continue on the same trends as found in the past. But we can also use scenario or assumed data in the model. For instance, after looking at the probabilities of ruin in Table 11, management finds them too great, and decides on two simultaneous changes in operations. First, written premiums could be curtailed. Secondly, underwriting standards could be strengthened, so as to decrease the level and variability of the loss ratios. If the analyst can quantify these changes, they can be built into the model. The model can in turn quantify the degrees of the necessary changes, in order to decrease the probabilities of ruin to acceptable levels.

The model can be used as a tool of scenario testing. For instance, the analyst may discard the historical loss ratios, and decides a future loss ratio of 75% with a deviation of .05 is reasonable. He can then carry out the simulation and analysis with these scenario values.

For the study, we chose a situation as simple as possible. But the model offers great flexibility. As more and more operations are added to the analysis, the number of governing equations would increase. Below we list some of the many other operations that the model can readily accommodate.

Multi-lines insurer: We expect that as more lines are added, the financial results would stabilize. This is especially true if the loss ratios of the various lines have no or negative correlation.

Differing investment strategies: We can allocate the available investment assets into different segments, such as bonds, stocks, and real estate. We can also consider sub-segments within each major category: such as, taxable versus tax-exempt bonds. The model can give us an idea of the optimal investment strategy, given a corporate objective, such as growth versus stability.

Tax liabilities: This item can be readily built into the model.

Interest rates and inflation: These two affect the investment income and the loss payments. There are many theories concerning how inflation affects the stock and bond markets. Once the analyst decides to use a particular theory or model, it can be readily integrated into the framework of our DFA model. Regarding the loss payments, things are not so apparent. There are many ad-hoc techniques to account for inflation. But to our knowledge, there is no mathematically rigorous model that can explain how inflation affects insurance loss payments.

Reinsurance: Two aspects of this item may be considered. One is the default rate of the reinsurers. A default occurrence can be modeled as a Poisson process. Secondly, we can consider different reinsurance strategies, such as excess versus quota-share, and their effects on the balance sheet.

Catastrophes: If the insurer has much property exposures, we have to consider this aspect. An existing software package can be incorporated into this model.

Varying payment patterns: For the same line of business, the payment patterns for the different accident years may vary. We analyze this situation in our loss-reserving papers. For this study, we simplify, and elect not to account for the varying patterns.

SAP/GAAP bases: The model can be used in either basis. In the latter, unrealized capital gains, deferred acquisition costs, etc. have to be considered.

10. CONCLUSION

We have presented a dynamic financial analysis with two key ingredients: stochastic modeling of the individual operations and the error correlation of the operations in concert. One of its strengths lies in its use of the set of governing equations. This set can be contracted or expanded, depending on whether the actuary wants a simpler or more extensive analysis.

A user of dynamic financial analysis can evaluate the desirability of a strategy over another on several criteria: stability, profitability, and growth, among others. For stability, he should determine that the variability of the results and the probability of ruin are kept to acceptably low levels. For profitability, he should look at the overall income, which in our simplified example is:

$$(10.1) \qquad EP(t) + II(t) - IL(t) - PE(t) - PD(t).$$

In our example, we assume built-in growth. But we can certainly model it as a function of other variables, such as equation (3.2).

We note that the three aforementioned criteria are in many ways conflicting. But with dynamic financial analysis, the user has a better idea of where the best compromise lies, given the objectives and constraints of the company.

Implications of Reinsurance and Reserves on Risk of Investment Asset Allocation

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IMPLICATIONS OF REINSURANCE AND RESERVES ON RISK OF INVESTMENT ASSET ALLOCATION

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ABSTRACT

DFA makes possible a greater integration of asset management with underwriting management. This paper looks at how investment risk and reinsurance can affect the overall risk to the company, and how the two can be managed simultaneously. A significant underwriting variable is the risk of loss development, and models of the development risk are presented, with some methodology for determining which one is most appropriate given the data at hand. Term-structure models are key to asset risk modeling, and a test of these models is proposed.

Implications of Reinsurance and Reserves on Risk of Investment Asset Allocation

Asset-Liability Management

Property- liability insurers have traditionally managed investment and underwriting functions separately, except for some forays into duration matching and perhaps to set goals for their investment risk that recognize that they do have some underwriting exposure. Dynamic Financial Analysis (DFA), by jointly modeling asset and liability risks, provides a means to more closely integrate the management of investment and underwriting risk, and thereby directly manage the total risk of bottom line results. This paper will focus on modeling GAAP pre-tax surplus change, which includes the effect of unrealized gains and losses, but any income statement or balance sheet item could be modeled similarly.

The principal risk elements to pre-tax surplus change are asset risk, reserve development, and current year underwriting results. These each have separate modules in the model described below, but some common economic elements, such as inflation and interest rates, feed all the modules.

Looking at assets alone, higher yielding assets generally bear more risk of adverse deviation, with short-term treasury securities usually regarded as having the least risk and least expected return. However adding liabilities - even fixed liabilities - to this picture changes the risk profile. If liabilities are of medium term, then holding short-term assets could be of higher risk, as interest rates may decrease and generate less than enough investment income to cover the liabilities. Long-term investment also increases in risk in this case, as interest rates could go up, requiring liquidation of depressed assets to meet the liabilities. Long-term investments may still have higher expected returns than medium term, but the insurer with medium-term liabilities will be exposed to more risk than the asset-only investor for using those instruments. On the other hand, medium-term assets could be carried at a greater reduction in risk than for the usual investor in this case. This is the rationale for duration matching. Uncertain liabilities and payout timing complicates the matching process, and can render perfect matching impossible. Simulation of loss payment requirements against asset fluctuations can be used to estimate the risk of different investment strategies in this case.

But the real world keeps intruding: if a company with medium or long-term liabilities grows just with inflation, it tends to have positive cash flow. If positive cash flow were a certainty, assets would never have to be liquidated to pay liabilities; the risk-return situation reverts back to the asset-only situation. Add to this accounting for bonds at amortized values and long-term investments suddenly become low-risk high-return opportunities. In this paper bonds will be evaluated at market, which records more risk for long-term bonds, but the same approach could work with amortized costs - with different results expected.

It is when cash flow is also risky that the DFA approach to asset/liability management really shows its merits. Without the shield of reliable positive cash flows, the uncertainty about interest rates and loss payout requirements are back, complicated by the fact that cash flows will often but not always be positive. All of these elements can be simulated simultaneously to quantify their interactions. This would allow the measurement of the effect of different reinsurance strategies, through their impacts on cash flow, on the combined asset/liability risk. Inflation can affect both asset values, through the interest rates, as well as premium volume and loss payments, and so its impact is complex. Reserves may be inflation sensitive as well, which would add yet another impact on the surplus change. All of these effects can be captured using a DFA approach to asset-liability management.

MODELING ISSUES

SCENARIOS AND PROBABILITY

Prior to DFA modeling, risk was measured by scenario testing. A few scenarios were selected and the financial outcomes under those scenarios were computed. This enabled management to have some confidence that their strategies would bear up under various sorts of adverse developments. It did not, however, allow for an assessment of the probability of achieving various earnings targets. Without knowing the probabilities of the various scenarios arising, management could have been sacrificing overall profitability to guard against some exceedingly rare eventualities.

DFA can do more than merely increase the number of scenarios tested. With good models of the underlying processes it can generate a set of scenarios that in some sense reflects the probability of occurrence of the various outcomes. There are of course issues of how well the model represents the processes being modeled - there is both art and science to modeling. The criterion to which a model should be judged is not its ability to generate a wide variety of scenarios, but rather its ability to generate scenarios according to their likelihood of occurrence.

ASSET MODELS

The asset modeling approach adopted here is to first generate a series of treasury yield curves using diffusion models. This is detailed in Appendix 1. Many other economic variables, such as the inflation rate and security prices, have historically correlated to the current and past yield curves, so these variables can be modeled by regression and simulated from the regression models and the simulated yield curves. This builds in the correlations among these variables with appropriate levels of random fluctuation.

A portfolio of assets and liabilities is subject to risk from complex changes to the shape of the yield curve – not just simple upward and downward movements. Thus a yield curve model has to be able to generate curves of different shapes, and in accordance with the probability that they might arise. In Appendix 1 we introduce measures of yield curve shapes and compare some yield curve models and historical data as to the distribution of the shape of the yield curve conditional on the short term rate. It is shown there that some yield curve models, although they can generate yield curves of different shapes, tend to generate only very restricted shapes of yield curves for any given short-term rate. This is not consistent with historical data, and so those models could not be expected to produce scenarios in accord with occurrence probabilities.

Reserve Development Models

Many different assumptions can be made about the processes that generate loss development. In Appendix 2 a classification scheme is outlined that groups reserve development processes into 64 different classes. This is based on answering 6 yes-no questions about the development process. Empirical methods of answering these questions based on triangulated data are also discussed. Once a process is identified that plausibly could have generated the loss triangle in question, this process can be used to simulate scenarios of future development. Doing this study has implications for loss reserving as well, as each process of generating loss emergence implies a reserving methodology. The implied methodology is essentially the one that provides the best estimates of the parameters of the process that is generating the development, and is explored in Appendix 2.

In the examples below it is assumed that this study has been completed, and only two of these classes of processes are illustrated. The first starts by generating ultimate losses, and then uses random draws around expected percentages of payment to generate the paid losses at each age. This is essentially the process used by Stanard (1985) to testing development methods through loss simulation. It turns out that a parameterized form of the Bornheutter-Ferguson method is optimal for this process.

The second process is similar, but the paid losses at each age are then adjusted up or down by the difference between actual and expected reserve inflation. In this case the paid losses in each year will depend on the inflation for the year, and the final ultimate losses will end up different from the initial ultimate originally drawn. That sounds like a more realistic process for the generation of actual loss histories, but empirical tests of loss development do not always identify an effect of post-event inflation. If losses are sensitive to inflation after the loss date, the risks to holding a given set of assets will be different from what they would be otherwise. The optimal reserving method in this case involves estimating the impact of calendar-year inflation (i.e., diagonal trend) on the loss triangle.

Mack (1994) showed that the chain ladder is optimal for the process that generates each age's emerged loss as a factor times the cumulative emerged-to-date for the accident year, plus a random element. This process could be used to generate losses in a DFA model, but it is not illustrated here.

Underwriting Risk Models

Models of current year underwriting risk can be intricate, but are usually straightforward. The approach here is to simulate large individual losses from models of frequency, severity, and parameter uncertainty and smaller losses in the aggregate from a single aggregate distribution for each line. Then the difference between simulated and expected inflation is applied, followed by application of the reinsurance program.

SIMULATION ASSUMPTIONS

COMMON ASSUMPTIONS

A few simplifying assumptions will be made in all the simulations in order to highlight the essential elements being tested. These are not intrinsic to the model, however. First, it will be assumed that all cash flows take place at year-end or an infinitesimal time later at the beginning of the next year. Thus premiums are all written, expenses are paid out, and the remaining unearned premiums are invested at this instant. A year later the payments to be made for losses for that year and all previous accident years are paid out, any bonds mature, coupon payments are made, etc. All losses are assumed to pay out over a 10-year period with an average payout lag of three years after policy issuance, but the actual payout pattern may be stochastic. The following investment strategies will be tested: short term - everything is in one-year treasuries; medium term - all in three-year treasuries; long term - all in ten-year treasuries; and stocks plus - 50% in stocks and 50% in ten-year treasuries. Surplus is assumed to be one-fourth of assets.

COMPANY RISK FACTORS

Several different hypothetical companies will be simulated to test how various underwriting risks interact with the investment scenarios above. The first will be a what-if test of surplus only - the reserves and other assets are ignored. The second will assume the company has a fixed known payout pattern - i.e., no reserve risk. The third will be a company with stochastic reserves - there is a distribution around each payout - but with no inflation risk - the payouts have a random element but not correlated with inflation. Fourthly, the payouts will be assumed correlated with inflation. In this case the reserves will be adjusted at year-end by the ratio of the actual to expected inflation factor. All these tests will be based on a reinsurance program with a moderately high retention. The final test will repeat the fourth with a more conservative approach to reinsurance.

A Few Details

For each set of company risk assumptions and each investment strategy, the distribution of year-end pre-tax GAAP surplus is simulated. Comparisons are made of the mean, standard deviation, and 99th, 90th, 10th and 1st percentile of each distribution. These percentiles correspond to the upper and lower 1-in-10 and 1-in-100 probability of exceeding levels.

The strategies and risk profiles tested below are not completely realistic. They are intended to illustrate the capabilities of DFA modeling in the asset-liability management arena, and the interaction of that with reserving and reinsurance. Because of this and for the sake of simplicity, the CIR (Cox, Ingersoll and Ross) model from Appendix 1 is used for the examples, but with different parameters. The initial short-term interest rate r is assumed to be 0.05, and its change is generated by the following process:

$$dr = 0.2(0.06 - r)dt + 0.075r^{1/2}dz.$$

The CPI and Wilshire 5000 stock index are simulated as measures of inflation and stock market performance. These are generated by regression on the yield curve and lags of the yield curve. The regressions were done on quarterly data, so for notational purposes the time periods will be expressed as quarters. Notation such as 3L40:12 will denote the third lag of the difference between the 40 quarter and 12 quarter interest rates, i.e., the 10 year rate less the 3 year rate seen 9 months ago. Without the colon 0L40 is just the 10 year rate for the current quarter.

The inflation variable estimated here, denoted qccpi, is the ratio of the CPI for a quarter to that for the previous quarter. The variables used in the fit along with indications of their significance are shown in the table below. The data used is from the fourth quarter of 1959 to first quarter 1997, as this was available from pointers within the CAS website.

Change in CPI

Variable	Estimate	T-statistic	Significance Level
1:4Lqccpi	0.9994	1649.4	<.01%
0L40:4	-0.2668	-5.3349	<.01%
2L40:20	0.8486	4.6411	<.01%
3L2:1	0.7182	3.4663	.07%

The most important indicator of inflation is recent inflation. The variable used to represent this, denoted 1:4Lqccpi, is the average of qccpi for the past four quarters. The coincident variable, 0L40:4 has a negative coefficient. This may be due to inflation influencing current interest rates, but with a greater impact on short term than long term rates, thus flattening the yield curve. At lag 2 quarters, the coefficient for 2L40:20 is positive and at lag 3 quarters that for 3L2:1 is positive. These indicate a general tendency for a steeper yield curve to anticipate future inflation. The r-squared, adjusted for degrees of freedom, is 65%. The standard error of the estimate is 0.0051. Thus the typical predicted quarterly change is accurate to about half a percentage point. The standard error is the standard deviation of a residual normally distribution around the predicted point, which can be used to draw the scenario actually simulated. The actual vs. fit is graphed in Appendix 3. The series can be seen to be fairly noisy, but the model does pick up the general movements over time. The residuals are graphed on a normal scale. Normality looks to be reasonably consistent with the observed residuals.

The stock market variable modeled, qcw5, is the ratio of the Wilshire 5000 index W5 at the end of a quarter to that at the previous quarter end. In this case the CPI percentage change variable qccpi was included in the regression as an explanatory variable. This allows creation of scenarios that have simulated values of W5 that are probabilistically consistent with the CPI value for the scenario.

The fitted equation for quarter ending data 1971 through first quarter 1997 is shown in the table below. In this regression only two variables were used, but they are composite series. The first, denoted 0-4Lqccpi, is the increase in qccpi over the last year, i.e., the current rate less the rate a year earlier. This variable has a negative coefficient, indicating that an increase in inflation is bad for equity returns. The other variable is denoted qcrelsprd. It represents the previous quarter's increase in the long-term spread less this quarter's increase in the short-term spread. Here the long-term spread is the difference between 10-year and 5-year rates, and the short-term spread is the difference between 6-month and 3-month rates. The increases noted are the quarter-to-quarter arithmetic increases in these spreads.

The coefficient on qcrelsprd is positive. This variable is positive if the increase in the short-term spread is less than the previous increase in the long-term spread, or if its decrease is greater. Either could suggest moderating inflation and interest rates, and thus be positive for equity returns.

Quarterly Change in Wilshire 5000

Variable	Estimate	T-statistic	Significance Level
0-4Lqccpi	-2.7113	-3.1936	0.2%
qcrelsprd	11.869	4.5273	<.01%
constant	1.02316	145.311	<.01%

The adjusted-r-squared is only 24% for this regression, indicating that the fit is not particularly good. The residual standard deviation is .0721, which allows a fairly wide deviation from the model. The fit is graphed in Appendix 3.

RESULTS

The table below shows the mean surplus, the ratio of mean to standard deviation, and several percentiles of the surplus for the case in which there are no losses, just investment of surplus.

Surplus Only

	Mean	Mean/SD	1%	10%	90%	99%
Short	3048	-	3048	3048	3048	3049
Medium	3053	45.3	2867	2967	3125	3227
Long	3071	19.5	2706	2861	3284	3407
Stocks+	3136	13.1	2577	2829	3422	3760

The ratio of mean to standard deviation is chosen as a risk measure for which higher is better, as is the case with all the other figures in the table. This table is consistent with the idea that riskier investments have higher expected return, but could have more adverse developments as well. The one-year bonds have no risk in this case, as they are held a year and then mature.

The next table shows the results of adding fixed liabilities to the mix.

Fixed Liabilities

	Mean	Mean/SD	1%	10%	90%	99%
Short	3419	-	3419	3419	3419	3419
Medium	3434	14.8	2798	3104	3705	3953
Long	3492	7.3	2031	2848	4094	4409
Stocks+	3581	4.6	1951	2656	4630	5282

Here the mean surplus is higher, due to the expected profits from the insurance business. However, the risk is considerably greater, due to the larger investment portfolio compared to the same surplus. This works at both the low and high end of the probability distribution.

Adding variability to the liabilities further increases the risk, as shown below. Here the change in the extreme percentiles is greater for the short-term investments, showing that the increase in risk over fixed liabilities is greater when investing short.

Variable Liabilities -No Inflation on Reserves

	Mean	Mean/SD	1%	10%	90%	99%
Short	3422	21.0	3085	3220	3626	3821
Medium	3443	11.4	2600	3024	3786	4096
Long	3470	6.8	2182	2801	4117	4784
Stocks+	3540	4.1	1762	2287	4661	6120

If reserves are subject to post-event inflation, risk increases more:

With Post-Event Inflation

	Mean	Mean/SD	1%	10%	90%	99%
Short	3429	20.2	3021	3205	3635	3859
Medium	3438	10.6	2589	2972	3816	4289
Long	3538	6.3	1899	2848	4242	4879
Stocks+	3569	3.9	1358	2294	4613	6197

Stocks may pose too much of a risk at the 1% level in this case, where they may have been an acceptable risk without post-event inflation. This illustrates the value of understanding the reserve-generating process when setting investment strategy.

Finally, buying more reinsurance reduces the expected surplus but also the variability of surplus.

No Post-Event Inflation with More Reinsurance

	Mean	Mean/SD	1%	10%	90%	99%
Short	3227	55.3	3271	3351	3500	3618
Medium	3255	14.9	2884	3156	3749	3951
Long	3345	6.9	2197	2865	4202	4630
Stocks+	3473	4.4	1773	2564	4909	5642

For this company, buying more reinsurance with long-term investments has lower expected return and more downside risk than buying less reinsurance with medium term investments. This strategy would give up considerable upside potential, however.

CONCLUSION

The risks to the various investment strategies that an insurer may follow will change depending on underwriting risk and reserve development risk. To quantify this risk the process generating reserve development needs to be identified. Once that is done, the trade-offs between different investment strategies and different underwriting strategies - including alternative reinsurance programs - can be quantified by dynamic financial analysis.

Appendix 1 — Simulating Asset Performance

Most asset classes and many economic series have been found to correlate to the treasury yield curve. Realistic simulation of the yield curve is an involved undertaking, and a subject of ongoing research among academics and all sorts of financial practitioners. If any researchers have gotten this absolutely right, they're keeping it a secret, and probably getting wealthy. Some of the progress in this area is discussed below, along with some proposed tests of yield curve simulation methods for DFA modeling.

Once the yield curves have been generated, the other assets and economic values can be simulated by regressions against the yield curve and lags of the yield curve (and perhaps against the other economic variables already simulated). In each case, a random draw from the error term of the distribution should be added to the regression estimate in order to keep the correlations from being perfect (unless they happen to be, which is rare).

A good deal of the work in yield-curve simulation is done for the purpose of pricing or evaluating the pricing of interest-rate options. For this purpose it is important that the model captures the current yield curve and its short-term dynamics as precisely as possible. This would be important to insurers who are actively trading bond options. However, the usual emphasis in DFA modeling is a little different. The risks inherent in different investment strategies over a longer time frame are more of a concern. A wide variety of yield curves should be produced to test this, but the model producing the widest variety is not necessarily the most useful - the different yield curves should be produced in relative proportion to their probability of occurring. It would be nice if the short-term forecasts were very close to the current curve, but this is less important for DFA than it is for option trading.

Historical data on the distribution of yield curves can be used to test the reasonability of the distribution of curves being produced by any given model. However, it is not reasonable to expect that the probability of yield curves in a small given range showing up in the next two or three years is the same as their historical appearance. Some recognition needs to be given to the current situation and the speed at which changes in the curve tend to occur. Care also needs to be exercised in the selection of the historical period to which comparisons are to be made. The years 1979-81 exhibited dramatic changes in the yield curve, and the analyst needs to consider how prominent these years will be in the history selected. It seems reasonable that using a period beginning in the 1950's will give this unusual phase due recognition without over-emphasizing it.

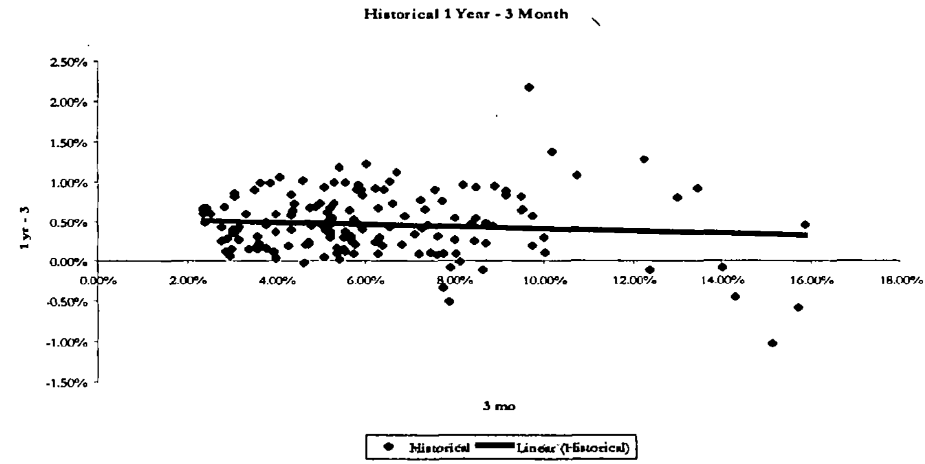
The following are proposed as general criteria that a model of the yield curve should meet:

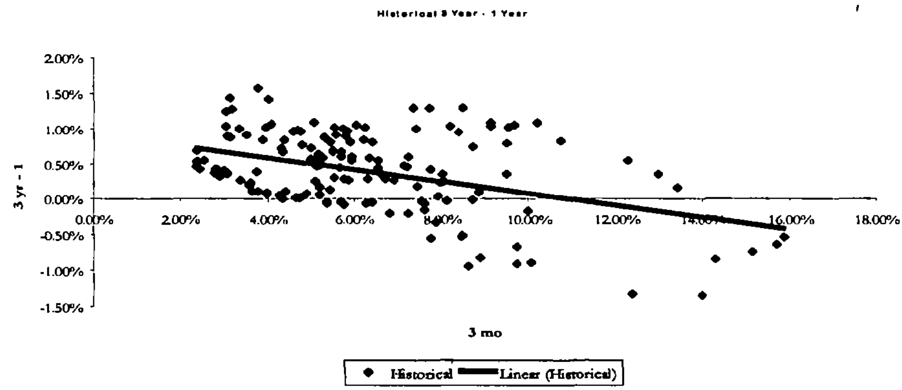
- It should closely approximate the current yield curve.
- It should produce patterns of changes in the short-term rate that match those produced historically.
- Over longer simulations, the ultimate distributions of yield curve shapes it produces, given any short-term rate, should match historical results.

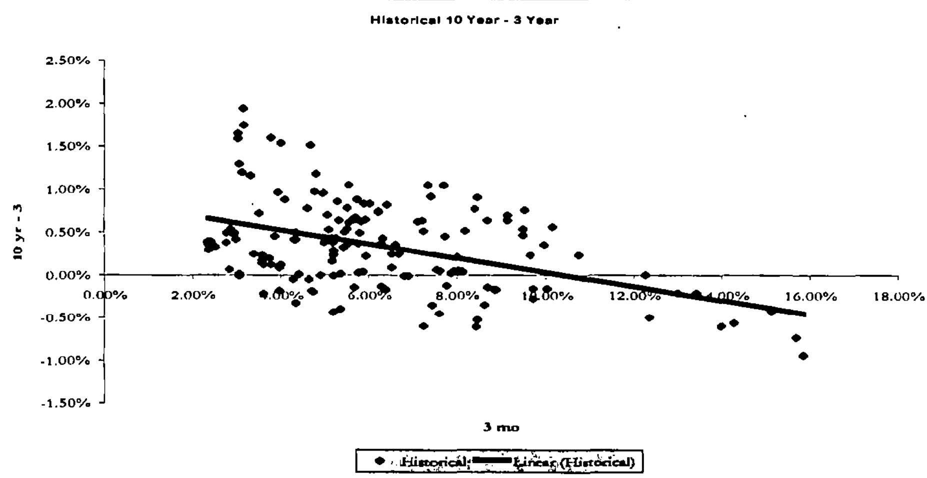
This last criterion looks at the contingent distributions of yield curve shapes given the short-term rate. Thus it allows for the possibility that the distribution of short-term rates simulated even after several years will not match the diversity of historical rates. But it does require that for any given short-term rate the distribution of yield curves should be as varied as seen historically for that short-term rate. It could be argued that somewhat less variability would be appropriate, and this may be so. How much less would be a matter of judgment, but too little variation in this conditional distribution would seem ill-advised when generating scenarios to test investment strategies against.

To measure the distribution of yield curve shapes, some shape descriptors are needed. The ones used here are based on differences of interest rates of different maturities. The first measures are just the successive differences in yield rates for 3-month, 1-year, 3-year, and 10-year instruments. Then the differences in these differences are taken, and finally the differences of those second differences. The first differences quantify the steepness of different parts of the yield curve. These would be zero for a flat curve. The second differences quantify the rate of change in the steepness as you move up the curve. These would be zero for a linearly rising curve. The third difference would be zero for a quadratic curve, and so quantifies the degree to which the curve is not quadratic.

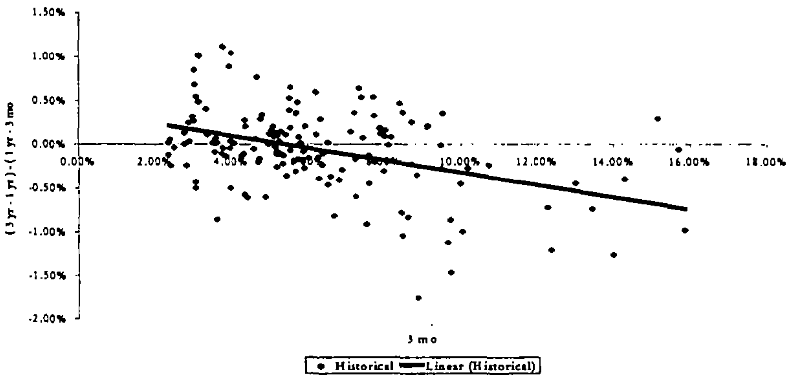
These shape measures will be reviewed historically as a function of the 3-month rate. The patterns for these six measures are graphed below along with the regression lines against the 3-month rate. It is interesting to note that the 1-year / 3-month yield spread appears to be independent of the 3-month rate, but the longer-term spreads appear to decline slightly with higher 3-month rates. At least in the US economy, when the short-term rates are high, the long-term rates tend to show less response, perhaps because investors expect the short-term rates to come down, and so the yield curve flattens out or even shows reversals (i.e., short-term rates higher than long-term). It might be argued that the slopes of the regression lines are small enough compared to the noise that they should not be considered significant. It turns out, however, that in testing models against this data the non-significance of the slope is a most significant issue - most models tend to produce more steeply falling slopes than the data shows.


Historical 1 Year - 3 Month
1 yr - 3
3 mo
Historical
Linear (Historical)

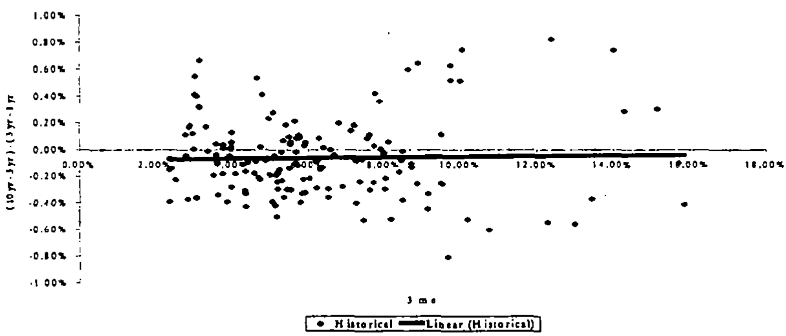

3 yr - 1
3 mo
Historical
Linear (Historical)


Historical 10 Year - 3 Year
10 yr - 3
3 mo
Historical
Linear (Historical)

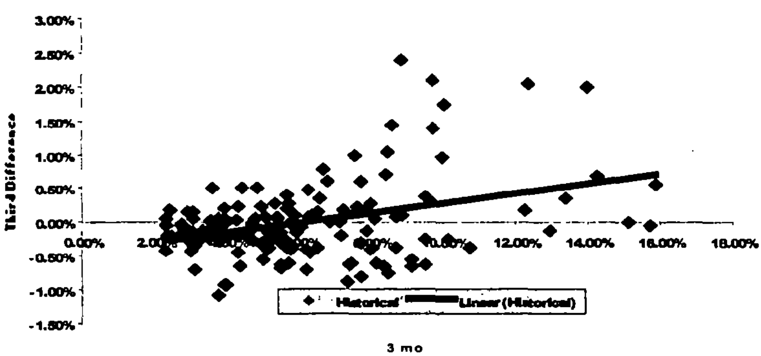
Historical Second Difference - Short



Historical Second Difference - Long



Historical Third Difference



Typically the short-term interest rate, denoted as r, is modeled directly, and longer-term rates are inferred from the implied behavior of r, along with market considerations. The modeling of r is usually done as a continuously fluctuating diffusion process. This is based on Brownian motion. A continuously moving process is hard to track, and processes with random elements do not follow a simple formula. These processes are usually described by the probability distribution for their outcomes at any point in time. A Brownian motion has a simple definition for the probabilities of outcomes: the change from the current position between time zero and time t is normally distributed with mean zero and variance $\sigma^2 t$ for some σ. If r is the short-term interest rate and it follows such a Brownian motion, it is customary to express the instantaneous change in r by $dr = \sigma dz$. Here z represents a Brownian motion with $\sigma=1$. If r also has a trend of bt during time t, this could be expressed as $dr = bdt + \sigma dz$.

Cox, Ingersoll and Ross (1985) provided a model of the motion of the short-term rate that has become widely studied. In the CIR model r follows the following process:

$$dr = a(b - r)dt + sr^{1/2}dz.$$

Here b is the level of mean reversion. If r is above b, then the trend component is negative, and if r is below b it is positive. Thus the trend is always towards b. The speed of mean reversion is expressed by a, which is sometimes called the half-life of the reversion. Note that the volatility depends on r itself, so higher short-term rates would be associated with higher volatility. The period 1979-81 had high rates and high volatility, and studies that emphasize this period have found that the power of ½ on r is too low. It appears to be about right in longer studies however.

Nonetheless, the CIR model fails to capture other elements of the movement of short-term rates. There have been periods of high volatility with low interest rates, and the rates sometimes seem to gravitate towards a temporary mean for a while, then shift and go towards some other. One way to account for these features is to let the volatility parameter s and the reversion mean b both be stochastic themselves.

Andersen and Lund (Working Paper No. 214, Northwestern University Department of Finance) give one such model:

$$dr = a(b - r)dt + sr^k dz_1 \qquad k>0$$

$$d\ln s^2 = c(p - \ln s^2)dt + v dz_2$$

$$db = j(q - b)dt + wb^{1/2} dz_3$$

Here there are three standard Brownian motion processes, z_1, z_2, and z_3. The rate r moves subject to different processes at different times. It always follows a mean-reverting process, with mean b. But that mean itself changes over time, following a mean-reverting process defined by k, q, and w. The volatility parameter s^2 also varies over time via a mean reverting geometric Brownian motion process (i.e., Brownian motion on the log). In total there are eight parameters: a, c, j, k, p, q, v, and w and three varying factors r, b, and s.

Models of the short-term rate can lead to models of the whole yield curve. This is done by modeling the prices of zero-coupon bonds with different maturities all paying $1. If P(T) is the current price of such a bond for maturity T, the implied continuously compounding interest rate can be shown to be $-\ln[P(T)]/T$. P(T) itself is calculated as the risk adjusted discounted expected value of $1. Here "discounted" means continuously discounted by the evolving interest rate r, and "expected value" means that the mean discount is calculated over all possible paths for r. This can be expressed as:

$$P(T) = E^*[\exp(-\int r_t dt)]$$

Where r_t is the interest rate at time t, the integral is over the time period 0 to T, and E^* is the risk-adjusted expected value of the results of all such discounting processes.

If E were not risk adjusted, P(T) could be estimated by many instances of simulating the r process to time T over small increments and then discounting back over each increment. The risk-adjusted expected value is obtained by using a risk-adjusted process to simulate the r's. This process is like the original process except that it tends to produce higher r's over time. These higher rates provide a reward for bearing the longer-term interest rate risk. Increasing the trend portion of the diffusion process produces the adjusted process. In the CIR model it is increased by λr, where λ is called "the market price of risk." Andersen and Lund add λrs, and also add a similar risk element to the b diffusion.

However, in the case of the CIR model a closed form solution exists which simplifies the calculation. The yield rate for a zero coupon bond of maturity T is given by $Y(T) = A(T) + rB(T)$ where:

$A(T) = -2(ab/s^2T)\ln C(T) - 2aby/s^2$

$B(T) = [1 - C(T)]/yT$

$C(T) = (1 + xye^{T/x} - xy)^{-1}$

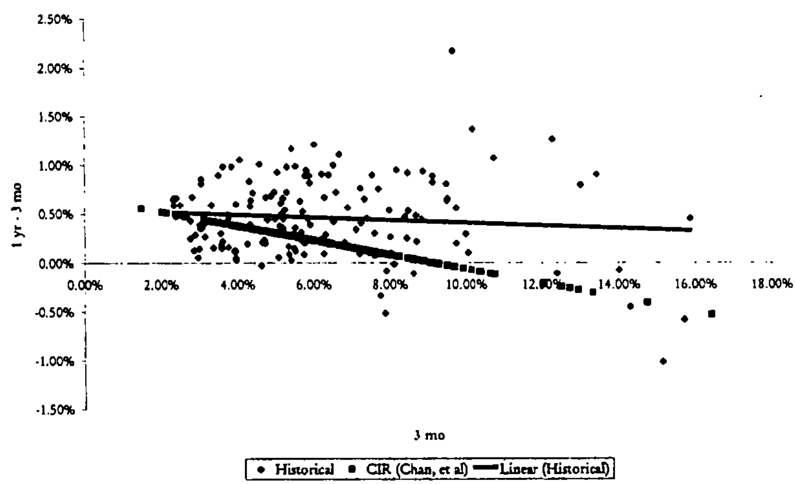
$x = [(a - \lambda)^2 + 2s^2]^{-1/2}$

$y = (a - \lambda + 1/x)/2.$

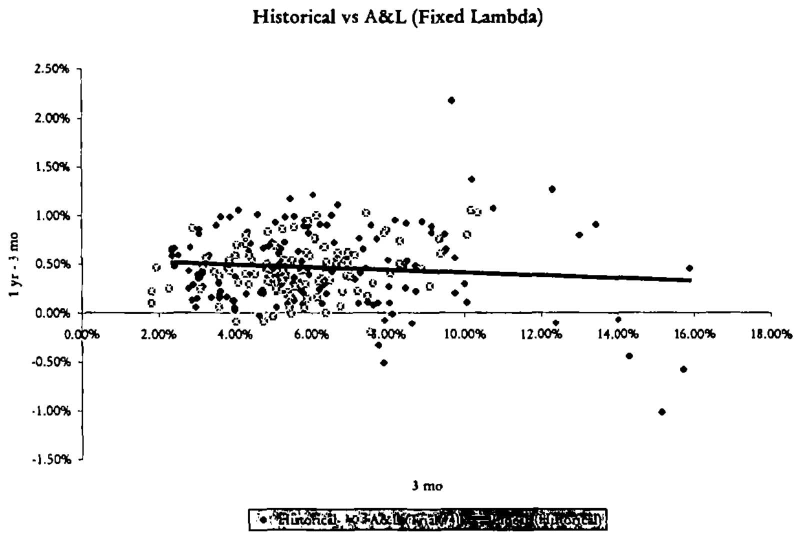
Note that neither A nor B is a function of r, so Y is a linear function of r (but not of T of course). Thus for the CIR model, all the yield curve shape measures defined above are linear functions of r, and as the three-month rate is as well, the shape measures are strictly linear in the three-month rate. This is in contrast to the historical data, which shows a dispersion of the shape measures around a perhaps linear relationship. The graph below as an example shows the historical and CIR implied 1 year less 3 month spread as a function of the 3-month rate, along with the historical trend line.

The parameters used here for the CIR model, from Chan etal. (1992) are: a=.2339; b=.0808; s=.0854, with λ set to .03. Different parameter values could possibly get the slope closer to that of the historical data, but the dispersion around the line cannot be achieved with this model. Experimentation with different parameter values suggests that even getting the slopes to match historical for all three of the first-difference measures may be difficult as well.

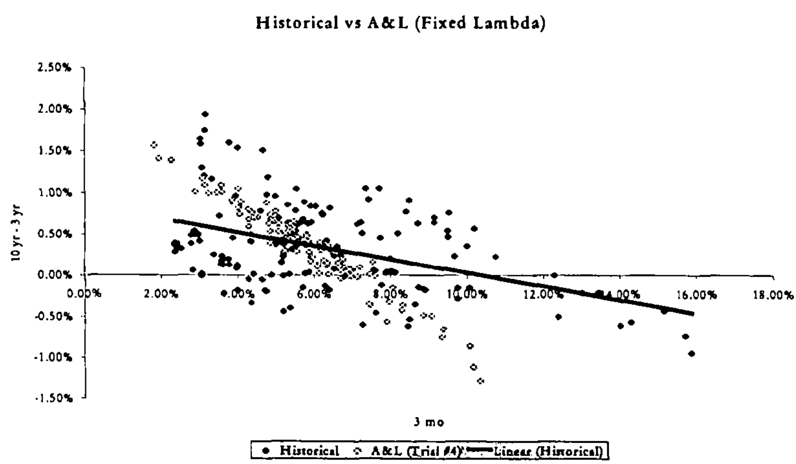
Historical vs CIR (Fixed Lambda)



Another potential problem with the CIR model is that the very long-term rates do not vary with r at all, but it's not clear how long the rates have to be for this.

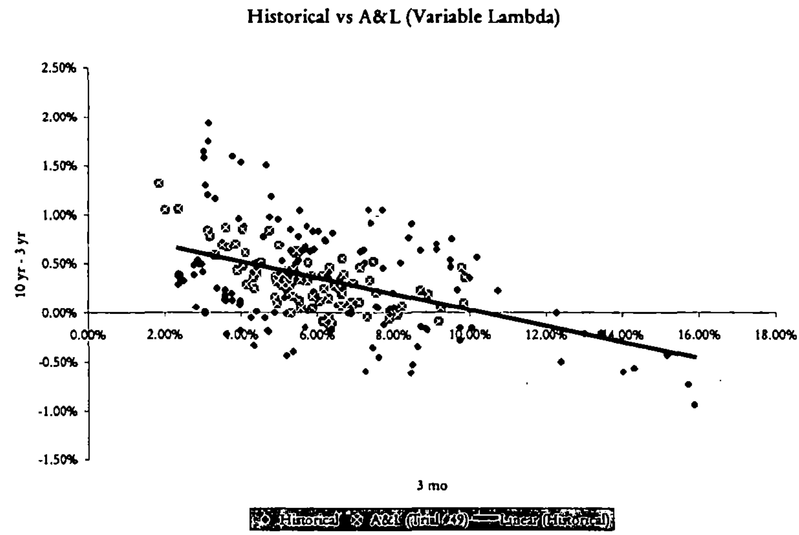


The Andersen-Lund model does provide more dispersion around the trend line, and also has about the right slope for the 3-month to 1-year spread, as the graph above shows. It does not do as well with the 3-year to 10-year spread in either



slope or dispersion, as shown here.

One approach that seems to give a degree of improvement is to let the market price of risk vary as well, through its own stochastic process. This would allow the same short-rate process to generate different yield curves at different times due to different market situations. This approach is capable of fixing the slope and dispersion problem for the long spread, as shown below.



Allowing stochastic market price of risk may improve the CIR model's performance on these tests as well, but it's not clear how to do this and still maintain a closed-form yield curve, which is the main advantage of CIR.

The table below summarizes some of the comparisons of model and historical results discussed above. For each of the models and each of the yield spreads, the linear relationship between the yield spread and the three-month rate is summarized by three statistics: the slope of the regression line of the spread on the three-month rate, the value on that line for r = .06, and the standard deviation of the points around the line. The value at r = .06 was compared instead of the intercept of the line to show how the model matched historical values for a typical interest rate.

The values were based on simulations of rates about three years beyond the initial values. Thus perhaps less variability of the residuals might be justifiable than in the historical data, which were quarterly values from 1959 through 1997. The whole variety of yield curve shapes from this nearly forty-year period may not be likely in just three years. A longer simulation period would thus give a better test of these models, and a somewhat lower residual standard deviation than historical may be acceptable for the test actually performed.

1 yr - 3 mo	**Historical**	**CIR**	**AL Fixed**	**AL Variable**
Slope	1.17%	-7.31%	2.49%	1.66%
Predicted @ 6%	0.48%	0.23%	0.44%	0.43%
Std Dev of Residuals	0.35%	0.00%	0.25%	0.43%
3 yr - 1 yr				
Slope	-8.41%	-16.58%	-5.68%	-2.57%
Predicted @ 6%	0.42%	0.49%	0.39%	0.40%
Std Dev of Residuals	0.52%	0.00%	0.12%	0.36%
10 yr - 3 yr				
Slope	-8.17%	-34.23%	-29.22%	-9.86%
Predicted @ 6%	0.35%	0.89%	0.29%	0.32%
Std Dev of Residuals	0.48%	0.00%	0.12%	0.50%

All the models tested had a lower residual standard deviation for the 3-year to 1-year spread than seen historically, but not unreasonably so for the variable price of risk model. The slopes of the 10-year to 3-year spread were all steeper than historical, but again the variable model was best.

This methodology gives an indication of a method of testing interest rate generators. There are quite a few of these in the finance literature, so none of the generators tested above can be considered optimal. In addition some refinement of the testing methodology may be able to tighten the conclusions discussed above.

APPENDIX 2 - SIMULATING LOSS DEVELOPMENT

The principal task in simulating a company's loss development is identifying the stochastic process that generates that development. Testing different processes against the historical development data is a way to approach this task. The second task is to model how the company's carried reserves respond to the loss emergence scenarios generated. One assumption for this may be that the company knows the process that produces its development, and uses a reserving methodology appropriate for that process. The simulation would proceed by generating loss emergence scenarios stochastically and then applying the selected reserving method to produce the carried reserves for each scenario. On the other hand, if the company has a fixed reserve methodology that it is going to use no matter what, then that methodology can be used to produce the carried reserves from the simulated emergence.

For this discussion, "emergence" could either mean case emergence or paid emergence, or both. The main concern here is simulating the emerging losses by period. This may or may not involve simulating the ultimate losses. For instance, one way to generate the losses to emerge in a period is to multiply simulated ultimate losses times a factor drawn from a percentage emerged distribution. This is appropriate when the process producing the losses for each period works by taking a randomized percent of ultimate losses. This method might involve some quite complicated methods of simulating ultimates, but all those that take period emergence as a percentage of ultimate will be considered to be using the same type of emergence pattern. Several other emergence patterns will be considered below, and the reserving methods appropriate for each will be discussed. Then methods for identifying the emergence patterns from the data triangles will be explored.

TYPES OF EMERGENCE PATTERNS

Six characteristics of emergence patterns will be considered here. Each will be treated as a binary choice, thus producing 64 types of emergence patterns. However there will be sub-categories within the 64, as not all of the choices are actually binary. The six basic choices for defining loss emergence processes are:

Do the losses that emerge in a period depend on the losses already emerged? Mack has shown that the chain ladder method assumes an emergence pattern in which the emerged loss for a period is a constant factor times the previous emerged, plus a random disturbance. Other methods, however, might apply factors only to ultimate losses, and then add a random disturbance. The latter is the emergence pattern assumed by the Bornheutter-Ferguson (BF) method, for example.

Is all loss emergence proportional? Both the chain ladder and BF methods use factors to predict emergence, and so are based on processes where emergence is proportional to something - either ultimate losses in the BF case or previously emerged in the chain ladder. However, the expected loss emergence for a period could be constant - not proportional to anything. Or it could be a factor times something plus a constant. If this is the emergence pattern used, then the reserving methodology should also incorporate additive elements.

Is emergence independent of calendar year events? Losses to emerge in a period may depend on the inflation rate for the period. This is an example of a calendar year or diagonal effect. Another example is strong or weak development due to a change in claim handling methods. Thus this is not a purely binary question - if there are diagonal effects there will be sub-choices relating to what type of effect is included. The Taylor separation method is an example of a development method that recognizes calendar year inflation. In many cases of diagonal effects, the ultimate losses will not be determined until all the development periods have been simulated.

Are the parameters stable? For instance a parameter might be a loss development factor. A stable factor could lead to variable losses due to randomness of the development pattern, but the factor itself would remain constant. The alternative is that the factor changes over time. There are sub-cases of this, depending on how they change.

Are the disturbance terms generated from a normal distribution? The typical alternative is lognormal, but the possibilities are endless. Clearly the loss development method will need to respond to this choice.

Are the disturbance terms homoskedastic? Some regression methods of development assume that the random disturbances all have the same variance, at least by development age. Link ratios are often calculated as the ratio of losses at age j+1 divided by losses at age j, which assumes that the variance of the disturbance term is proportional to the mean loss emerged. Another alternative is for the standard deviation to be proportional to the mean. The variance assumption used to generate the emerging losses can be employed in the loss reserving process as well.

Notation

Losses for accident year w evaluated at the end of that year will be denoted as being as of age 0, and the first accident year in the triangle is year 0. The notation below will be used to specify the models.

$c_{w,d}$: cumulative loss from accident year w as of age d

$c_{w,\infty}$: ultimate loss from accident year w

$q_{w,d}$: incremental loss for accident year w to emerge in period d

f_d: factor used in emergence for age d

h_w : factor (dollar amount) used in emergence for year w

g_{w+d}: factor used in emergence for calendar year w+d

a_d: additive term used in emergence for age d

QUESTION 1

The stochastic processes specified by answering the six questions above can be numbered in binary by considering yes=1 and no=0. Then process 111111 (all answers yes) can be specified as follows:

$$q_{w,d} = c_{w,d-1} f_d + e_{w,d} \qquad (1)$$

where $e_{w,d}$ is normally distributed with mean zero. Here f_d is a development factor applied to the cumulative losses simulated at age d-1. A starting value for the accident year is needed which could be called $c_{w,-1}$. For each d it might be reasonable to assume that $e_{w,d}$ has a different variance. Note that for this process, ultimate losses are generated only as the sum of the separately generated emerged losses for each age.

Mack has shown that for process 111111 the chain ladder is the optimal reserve estimation method. The factors f_d would be estimated by a no-constant linear regression. In process 111110 (heteroskedastic) the chain ladder would also be optimal, but the method of estimating the factors would be different. Essentially these would use weighted least squares for the estimation, where the weights are inversely proportional to the variance of $e_{w,d}$. If the variances are proportional to $c_{w,d-1}$, the resulting factor is the ratio of the sum of losses from the two relevant columns of the development triangle.

In all the processes 1111xx Mack showed that some form of the chain ladder is the best linear estimate, but when the disturbance term is not normal, linear estimation is not necessarily optimal.

Processes of type 0111xx do not generate emerged losses from those previously emerged. A simple example of this type of process is:

$$q_{w,d} = h_w f_d + e_{w,d} \qquad (2)$$

Here h_w can be interpreted as the ultimate losses for year w, with the factors f_d summing to unity. For this process, reserving would require estimation of the f's and h's. I call this method of reserving the parameterized BF, as Bornheutter and Ferguson estimated emergence as a percentage of expected ultimate. The method of estimating the parameters would depend on the distribution of the disturbance term $e_{w,d}$. If it is normal and homoskedastic, a regression method can be used iteratively by fixing the f's and regressing for the h's, then taking those h's to find the best f's, etc. until both f's and h's converge. If heteroskedastic, weighted regressions would be needed. If a lognormal disturbance is indicated, the parameters could be estimated in logs, which is a linear model in the logs.

QUESTION 2

Additive terms can be added to either of the above processes. Thus an example of a 0011xx process would be:

$$q_{w,d} = a_d + h_w f_d + e_{w,d} \qquad (3)$$

If the f's are zero, this would be a purely additive model. A test for additive effects can be made by adding them to the estimation and seeing if significantly better fits result.

QUESTION 3

Diagonal effects can be added similarly. A 0001xx model might be:

$$q_{w,d} = a_d + h_w f_d g_{w+d} + e_{w,d} \qquad (4)$$

Again this can be tested by goodness of fit. There may be too many parameters here. It will usually be possible to reasonably simulate losses without using so many distinct parameters. Specifying relationships among the parameters can lead to reduced parameter versions of these processes. For instance, some of the parameters might be set equal, such as $h_w=h$ for all w. Note that the 0111xx process $q_{w,d}= hf_d + e_{w,d}$ is the same as the 0011xx process $q_{w,d}= a_d + e_{w,d}$, as a_d can be set to hf_d. The resulting reserve estimation method is an additive version of the chain ladder, and is sometimes called the Cape Cod method.

Another way to reduce the number of parameters is to set up trend relationships. For example, constant calendar year inflation can be specified by setting $g_{w+d}=(1+j)^{w+d}$. Similar trend relationships can be specified among the h's and f's. If that is too much parameter reduction to adequately model a given data triangle, a trend can be established for a few periods and then some other trend can be used in other periods.

QUESTION 4

Rather than trending, the parameters in the loss emergence models could evolve according to some more general stochastic process. This could be a smooth process or one with jumps. The state-space model is often used to describe parameter variability. This model assumes that observations fluctuate around an expected value that itself changes over time as its parameters evolve. The degree of random fluctuation is measured by the variance of the observations around the mean, and the movement of the parameters is quantified by their variances over time. The interplay of these two variances determines the weights to apply, as in credibility theory.

To be more concrete, a formal definition of the model follows where the parameter is the 2nd to 3rd development factor. Let:

β_i=2nd to 3rd factor for ith accident year

y_i=3rd report losses for ith accident year

x_i=2nd report losses for ith accident year

The model is then:

$y_i=x_i\beta_i+\varepsilon_i$. (5)

The error term ε_i is assumed to have mean 0 and variance σ_i^2.

$\beta_i=\beta_{i-1}+\delta_i$. (6)

The fluctuation δ_i is assumed to have mean 0 and variance ν_i^2, and to be independent of the ε's.

In this general case the variances could change with each period i. Usually some simplification is applied, such as constant variances over time, or constant with occasional jumps in the parameter - i.e., occasional large ν_i's.

If this model is adopted for simulating loss emergence, the estimation of the factors from the data can be done using the Kalman filter.

Questions 5 and 6

The error structure can be studied and usually reasonably understood from the data triangles. The loss estimation method associated with a given error structure will be assumed to be maximum likelihood estimation from that structure. Thus for normal distributions this is weighted least squares, where the weights are the inverses of the variances. For lognormal this is the same, but in logs.

Identifying Emergence Patterns

Given a data triangle, what is the process that is generating it? This is useful to know for loss reserving purposes, as then reserve estimation is reduced to estimation of the parameters of the generating process. It is even more critical for simulation of company results, as the whole process is needed for simulation purposes.

Identifying emergence patterns can be approached by fitting different ones to the data and then testing the significance of the parameters and the goodness of fit. As more parameters often appear to give a better fit, but reduce predictive value, a method of penalizing over-parameterization is needed when comparing competing models. The method proposed here is to compare models based on sum of squared residuals divided by the square of the degrees of freedom, i.e., divided by the square of observations less parameters.

This measure gives impetus to trying to reduce the number of parameters in a given model, e.g., by setting some parameters the same or by identifying a trend in the parameters. This seems to be a legitimate exercise in the effort of identifying emergence patterns, as there are likely to be some regularities in the pattern, and simplifying the model is a way to uncover them.

Fitting the above models is a straightforward exercise, but reducing the number of parameters may be more of an art than a science. Two approaches may make sense: top down, where the full model is fit and then regularities among the parameters sought; and bottom up, where the most simplified version is estimated, and then parameters added to compensate for areas of poor fit.

To illustrate this approach, the data triangle of reinsurance loss data first introduced by Thomas Mack will be the basis of model estimation.

Table 1 shows incremental incurred losses by age for some excess casualty reinsurance. As an initial step, the statistical significance of link ratios and additive constants was tested by regressing incremental losses against the previous cumulative losses. In the regression the constant is denoted by a and the factor by b. This provides a test of question 1 - dependence of emergence on previous emerged, and also one of question 2 – proportional emergence. Here they are being tested by looking at whether or not the factors and the constants are significantly different from zero, rather than by any goodness-of-fit measure.

Table 1 - Incremental Incurred Losses

0	1	2	3	4	5	6	7	8	9
5012	3257	2638	898	1734	2642	1828	599	54	172
106	4179	1111	5270	3116	1817	-103	673	535	
3410	5582	4881	2268	2594	3479	649	603		
5655	5900	4211	5500	2159	2658	984			
1092	8473	6271	6333	3786	225				
1513	4932	5257	1233	2917					
557	3463	6926	1368						
1351	5596	6165							
3133	2262								
2063									

Table 2 - Statistical Significance of Link Ratios and Constants

	0 to 1	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8
'a'	5113	4311	1687	2061	4064	620	777	3724
Std a	1066	2440	3543	1165	2242	2301	145	0
'b'	-0.109	0.049	0.131	0.041	-0.100	0.011	-0.008	-0.197
std b	0.349	0.309	0.283	0.071	0.114	0.112	0.008	0

Table 2 shows the estimated parameters and their standard deviations. As can be seen, the constants are usually statistically significant (parameter nearly double its standard deviation, or more), but the factors never are. The lack of significance of the factors shows that the losses to emerge at any age d+1 are not proportional to the cumulative losses through age d. The assumptions underlying the chain ladder model are thus not met by this data. A constant amount emerging for each age usually appears to be a reasonable estimator, however.

Figure 1 illustrates this. A factor by itself would be a straight line through the origin with slope equal to the development factor, whereas a constant would give a horizontal line at the height of the constant.

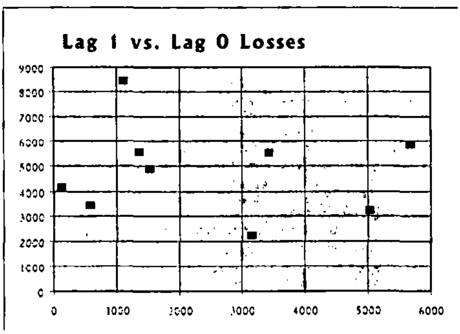


Figure 1

Although emerged losses are not proportional to previous emerged, they could be proportional to ultimate incurred. To test this, the parameterized BF model (2) was fit to the triangle. As this is a non-linear model, fitting is a little more involved. A method of fitting the parameters will be discussed, followed by an analysis of the resulting fit.

To do the fitting, an iterative method can be used to minimize the sum of the squared residuals, where the w,d residual is $[q_{w,d}-f_d h_w]$. Weighted least squares could also be used if the variances of the residuals are not constant over the triangle. For instance, the variances could be proportional to $f_d{}^p h_w{}^q$, in which case the regression weights would be $1/f_d{}^p h_w{}^q$.

A starting point for the f's or the h's is needed to begin the iteration. While almost any reasonable values could be used, such as all f's equal to 1/n, convergence will be faster with values likely to be in the ballpark of the final factors. A natural starting point thus might be the implied f_d's from the chain ladder method. For ages greater than 0, these are the incremental age-to-age factors divided by the cumulative-to-ultimate factors. To get a starting value for age 0, subtract the sum of the other factors from unity. Starting with these values for f_d, regressions were performed to find the h_w's that minimize the sum of squared residuals (one regression for each w). These give the best h's for that initial set of f's. The standard linear regression formula for these h's simplifies to:

$$h_w = \sum_d f_d q_{w,d} \,/\, \sum_d f_d^2 \qquad (7)$$

Even though that gives the best h's for those f's, another regression is needed to find the best f's for those h's. For this step the usual regression formula gives:

$$f_d = \sum_w h_w q_{w,d} \,/\, \sum_w h_w^2 \qquad (8)$$

Now the h regression can be repeated with the new f's, etc. This process continues until convergence occurs, i.e., until the f's and h's no longer change with subsequent iterations. Ten iterations were used in this case, but substantial onvergence occurred earlier. The first round of f's and h's and those at convergence are in Table 3. Note that the h's are not the final estimates of the ultimate losses, but are used with the estimated factors to estimate future emergence. In this case, in fact, h(0) is less than the emerged to date. A statistical package that includes nonlinear regression could ease the estimation.

Standard regression assumes each observation q has the same variance, which is to say the variance is proportional to $f_d^p h_w^q$, with p=q=0. If p=q=1 the weighted regression formulas become:

$$h_w^2 = \sum_d [q_{w,d}^2 / f_d] \,/\, \sum_d f_d \text{ and}$$

$$f_d^2 = \sum_w [q_{w,d}^2 / h_w] \,/\, \sum_w h_w$$

Table 3 - BF Parameters

Age d	0	1	2	3	4	5	6	7	8	9
f_d 1st	0.106	0.231	0.209	0.155	0.117	0.083	0.038	0.032	0.018	0.011
f_d ult	0.162	0.197	0.204	0.147	0.115	0.082	0.037	0.030	0.015	0.009
Year w	0	1	2	3	4	5	6	7	8	9
h_w 1st	17401	15729	23942	26365	30390	19813	18592	24154	14639	12733
h_w ult	15982	16501	23562	27269	31587	20081	19032	25155	13219	19413

For comparison, the development factors from the chain ladder are shown in Table 4. The incremental factors are the ratios of incremental to previous cumulative. The ultimate ratios are cumulative to ultimate. Below them are the ratios of these ratios, which represent the portion of ultimate losses to emerge in each period. The zeroth period shown is unity less the sum of the other ratios. These factors were the initial iteration for the f_d's shown above.

Table 4 - Development Factors

	0 to 1	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9
Incremental	1.22	0.57	0.26	0.16	0.10	0.04	0.03	0.02	0.01
	0 to 9	1 to 9	2 to 9	3 to 9	4 to 9	5 to 9	6 to 9	7 to 9	8 to 9
Ultimate	6.17	2.78	1.77	1.41	1.21	1.10	1.06	1.03	1.01
0.162	0.197	0.204	0.147	0.115	0.082	0.037	0.030	0.015	0.009

Having now estimated the BF parameters, how can they be used to test what the emergence pattern of the losses is?

A comparison of this fit to that from the chain ladder can be made by looking at how well each method predicts the incremental losses for each age after the initial one. The sum of squared errors adjusted for number of parameters is the comparison measure, where the parameter adjustment is made by dividing the sum of squared errors by the square of [the number of observations less the number of parameters], as discussed earlier. Here there are 45 observations, as only the predicted points count as observations. The adjusted sum of squared residuals is 81,169 for the BF, and 157,902 for the chain ladder. This shows that the emergence pattern for the BF (emergence proportional to ultimate) is much more consistent with this data than is the chain ladder emergence pattern (emergence proportional to previous emerged).

The Cape Cod (CC) method was also tried for this data. The iteration proceeded similarly to that for the BF, but only a single h parameter was fit for all accident years. Now:

$$h = \sum_{w,d} f_d q_{w,d} \, / \, \sum_{w,d} f_d^2 \qquad (9)$$

The estimated h is 22,001, and the final factors f are shown in Table 5. The adjusted sum of squared errors for this fit is 75,409. Since the CC is a special case of the BF, the unadjusted fit is of course worse than that of the BF method, but with fewer parameters in the CC, the adjustment makes them similar. This formula for h is the same as the formula for h_w except the sum is taken over all w.

Intermediate special cases could be fit similarly. If, for instance, a single factor were sought to apply to just two accident years, the sum would be taken over those years to estimate that factor, etc.

Table 5 - Factors in CC Method

0	1	2	3	4	5	6	7	8	9
0.109	0.220	0.213	0.148	0.124	0.098	0.038	0.028	0.013	0.008

This is a case where the BF has too many parameters for prediction purposes. More parameters fit the data better, but use up information. The penalization in the fit measure adjusts for this problem, and shows the CC to be a somewhat better model. Thus the data is consistent with random emergence around an expected value that is constant over the accident years.

The CC method would probably work even better for loss ratio triangles than for loss triangles, as then a single target ultimate value makes more sense. Adjusting loss ratios for trend and rate level could increase this homogeneity.

In addition, a purely additive development was tried, as suggested by the fact that the constant terms were significant in the original chain ladder, even though the factors were not. The development terms are shown in Table 6. These are just the average loss emerged at each age. The adjusted sum of squared residuals is 75,409. This is much better than the chain ladder, which might be expected, as the constant terms were significant in the original significance-test regressions while the factors were not. The additive factors in Table 6 differ from those in Table 2 because there is no multiplicative factor in Table 6.

Table 6 - Terms in Additive Chain Ladder

1	2	3	4	5	6	7	8	9
4849.3	4682.5	3267.1	2717.7	2164.2	839.5	625	294.5	172

As discussed above, the additive chain ladder is the same as the Cape Cod method, although it is parameterized differently. The exact same goodness of fit is thus not surprising.

Finally, an intermediate BF-CC pattern was fit as an example of reduced parameter BF's. In this case ages 1 and 2 are assumed to have the same factor, as are ages 6 and 7 and ages 8 and 9. This reduces the number of f parameters from 9 to 6. The number of accident year parameters was also reduced: years 0 and 1 have a single parameter, as do years 5 through 9. Year 2 has its own parameter, as does year 4, but year 3 is the average of those two. Thus there are 4 accident year parameters, and so 10 parameters in total. Any one of these can be set arbitrarily, with the remainder adjusted by a factor, so there are really just 9. The selections were based on consideration of which parameters were likely not to be significantly different from each other.

The estimated factors are shown in Table 7. The accident year factor for the last 5 years was set to 20,000. The other factors were estimated by the same iterative regression procedure as for the BF, but the factor constraints change the simplified regression formula. The adjusted sum of squared residuals is 52,360, which makes it the best approach tried. This further supports the idea that claims emerge as a percent of ultimate for this data. It also indicates that the various accident years and ages are not all at different levels, but that the CC is too much of a simplification. The actual and fitted values from this, the chain ladder, and CC are in Exhibit 1. The fitted values in Exhibit 1 were calculated as follows. For the chain ladder, the factors from Table 4 were applied to the cumulative losses implied from Table 1. For the CC the fitted values are just the terms in Table 6. For the BF-CC they are the products of the appropriate f and h factors from Table 7.

Table 7 - BF-CC Parameters

Age d	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
f_d	*	**0.230**	**0.230**	**0.160**	**0.123**	**0.086**	**0.040**	**0.040**	**0.017**	**0.017**
Year w	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
h_w	**14829**	**14829**	**20962**	**25895**	**30828**	**20000**	**20000**	**20000**	**20000**	**20000**

Calendar Year Impacts - Testing Question 3

One type of calendar year impact is high or low diagonals in the loss triangle. Mack suggested a high-low diagonal test which counts the number of high and low factors on each diagonal, and tests whether or not that is likely to be due to chance. Here another high-low test is proposed: use regression to see if any diagonal dummy variables are significant. An actuary will often have information about changes in company operations that may have created a diagonal effect. If so, this information could lead to choices of modeling methods - e.g., whether to assume the effect is permanent or temporary. The diagonal dummies can be used to measure the effect in any case, but knowledge of company operations will help determine how to use this effect. This is particularly so if the effect occurs in the last few diagonals.

A diagonal in the loss development triangle is defined by w+d = constant. Suppose for some given data triangle, the diagonal w+d=7 is found to be 10% higher than normal. Then an adjusted BF estimate of a cell might be:

$$q_{w,d}=1.1f_d h_w \text{ if } w+d=7, \text{ and } q_{w,d}=f_d h_w \text{ otherwise} \quad (10)$$

1	2	5	4
3	8	9	
7	10		
7			

The small sample triangle of incremental losses here will be used as an example of how to set up diagonal dummies in a chain ladder model. The goal is to get a matrix of data in the form needed to do a multiple regression. First the triangle (except the first column) is strung out into a column vector. This is the dependent variable. Then columns for the independent variables are added. The second column is the cumulative losses at age 0 for the loss entries that are at age 1, and zero for the other loss entries. The regression coefficient for this column would be the 0 to 1 cumulative-to-incremental factor. The next two columns are the same for the 1 to 2 and 2 to 3 factors. The last two columns are the diagonal dummies. They pick out the elements of the last two diagonals. The coefficients for these columns would be additive adjustments for those diagonals, if significant.

2	1	0	0	0	0
8	3	0	0	1	0
10	7	0	0	0	1
5	0	3	0	1	0
9	0	11	0	0	1
4	0	0	8	0	1

This method of testing for diagonal effects is applicable to many of the emergence models. In fact, if diagonal effects are found significant in chain ladder models, they probably are needed in the BF models of the same data, so goodness-of-fit tests should be done with those diagonal elements included. Some examples are given in Appendix 2.

Another popular modeling approach is to consider diagonal effects to be a measure of inflation (e.g., see Taylor 1977). In a payment triangle this would be a natural interpretation, but a similar phenomenon could occur in an incurred triangle. In this case the latest diagonal effects might be projected ahead as estimates of future inflation. An understanding of what in company operations is driving the diagonal effects would help address these issues.

As with the BF model, the parameters of the model with inflation effects, $q_{w,d} = h_w f_d g_{w+d} + e_{w,d}$, can be estimated iteratively. With reasonable starting values, fix two of the three sets of parameters, fit the third by least squares, and rotate until convergence is reached. Alternatively, a non-linear search procedure could be utilized. As an example of the simplest of these models, modeling $q_{w,d}$ as just $6756(0.7785)^d$ gives an adjusted sum of squares of 57,527 for the reinsurance triangle above. This is not the best fitting model, but is better than some, and has only two parameters. Adding more parameters to this would be an example of the bottom up fitting approach.

Testing Question 4 - Stability of parameters

If a pattern of sequences of high and low residuals is found when plotted against time, instability of the parameters may be indicated. This can be studied and a randomness in the parameters incorporated into the simulation process, e.g., through the state-space model.

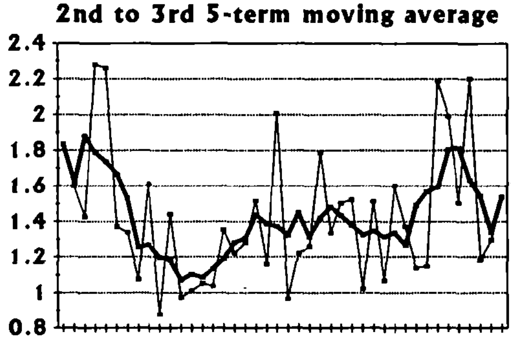


Figure 2

Figure 2 shows the 2nd to 3rd factor by accident year from a large development triangle (data in Exhibit 2) along with its five-term moving average. The moving average is the more stable of the two lines, and is sometimes in practice called "the average of the last five diagonals." There is apparent movement of the mean factor over time as well as a good deal of random fluctuation around it. There is a period of time in which the moving average is as low as 1.1 and other times it is as high as 1.8.

The state-space model assumes that observations fluctuate around a mean that itself changes over time. The degree of random fluctuation is measured by variance around the mean, and the movement of the mean by its variance over time. The interplay of these two variances determines the weights to apply, as in credibility theory.

The state-space model thus provides underlying assumptions about the process by which development changes over time. With such a model, estimation techniques that minimize prediction errors can be developed for the changing development case. This can result in estimators that are better than either using all data, or taking the average of the last few diagonals. For more details on the state space models see the Verrall and Zehnwirth references.

QUESTIONS 5 & 6: VARIANCE ASSUMPTIONS

Parameter estimation changes depending on the form of the variance. Usually in the chain ladder model the variance will plausibly be either a constant or proportional to the previous cumulative or its square. Plotting or fitting the squared residuals as a function of the previous cumulative will usually help decide which of these three alternatives fits better. If the squared residuals tend to be larger when the explanatory variable is larger, this is evidence that the variance is larger as well.

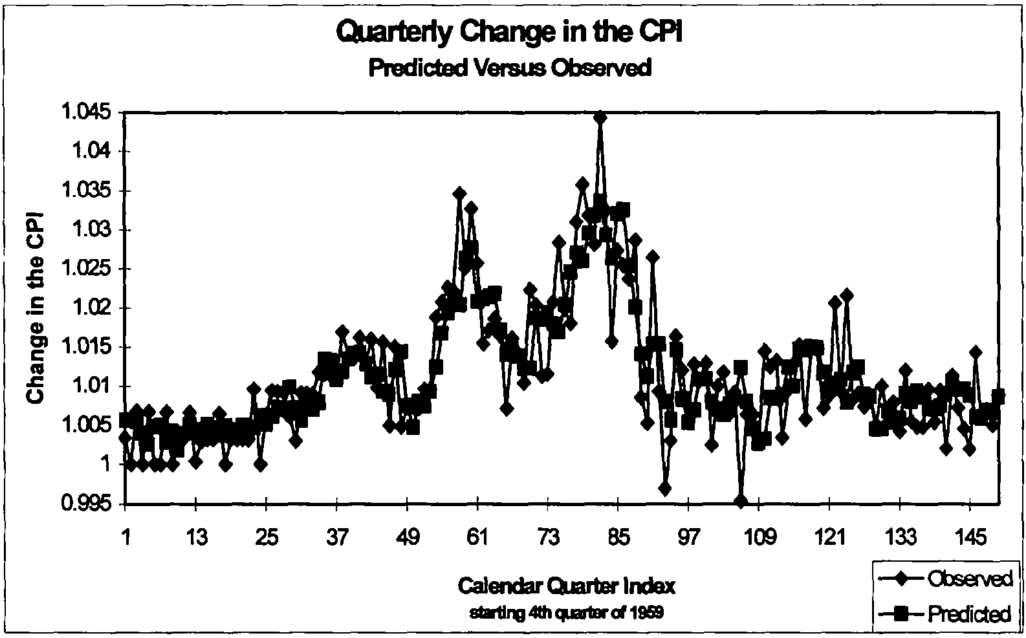
Another variance test would be for normality of the residuals. Normality is often tested by plotting the residuals on a normal scale, and looking for linearity. This is not a formal test, but it is often considered a useful procedure. If the residuals are somewhat positively skewed, a lognormal distribution may be reasonable. The non-linear models discussed are all linear in logs, and so could be much easier to estimate in that form. However, if some increments are negative, a lognormal model becomes awkward. The right distribution for the residuals of loss reserving models seems an area in which further research would be helpful.

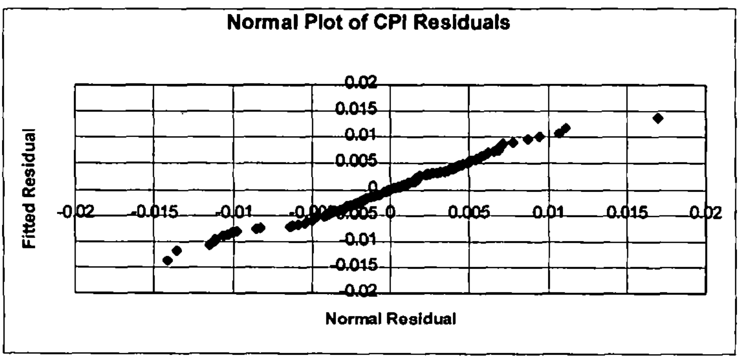
CONCLUSION

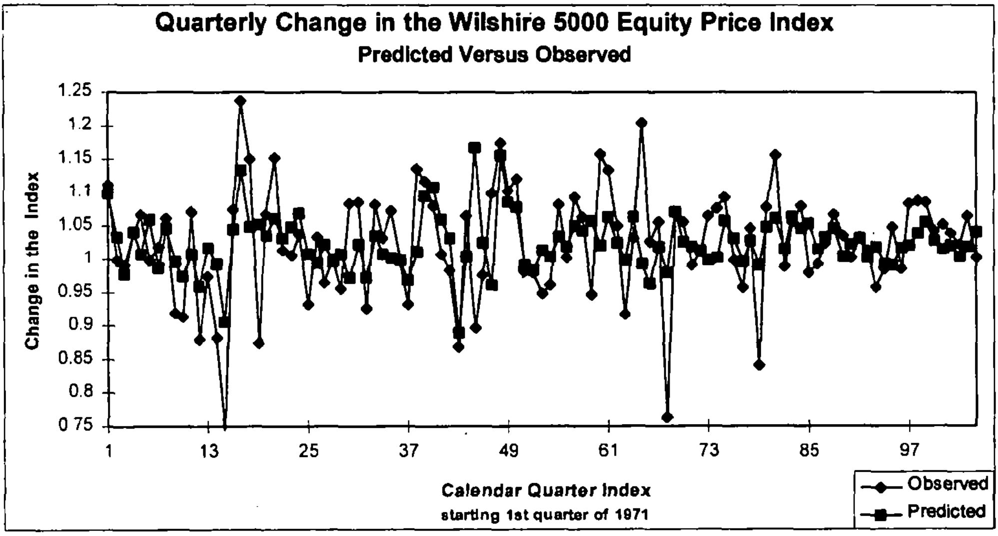
The first test that will quickly indicate the general type of emergence pattern faced is the test of significance of the cumulative-to-incremental factors at each age. This is equivalent to testing if the cumulative-to-cumulative factors are significantly different from unity. When this test fails, the future emergence is not proportional to past emergence. It may be a constant amount, it may be proportional to ultimate losses, as in the BF pattern, or it may depend on future inflation.

The addition of an additive component may give an even better fit. Reduced parameter models could also give better performance, as they will be less responsive to random variation. If an additive component is significant, converting the triangle to on-level loss ratios may improve the model. Tests of stability and for calendar-year effects may lead to further improvements.

APPENDIX 3 – REGRESSION GRAPHS






Quarterly Change in the Wilshire 5000 Equity Price Index
Predicted Versus Observed
Change in the Index
1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
0.75
1
13
25
37
49
61
73
85
97
Calendar Quarter Index
starting 1st quarter of 1971
Observed
Predicted

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Profitability Targets: DFA Provides Probability Estimates

by Susan E. Witcraft, FCAS, MAAA

Profitability Targets: DFA Provides Probability Estimates

Abstract

This paper will discuss the analysis we undertook to address the questions described below:

Background

During each of the past several years, an insurance company's actual experience has been much worse than the plan provided to its Board. A dynamic financial analysis was performed to address the following questions:

Questions

1. What is the probability that the insurance company will meet or exceed the earnings estimates for the following year provided to its Board?

2. Are the assumptions underlying the earnings estimates overly optimistic, or has the company had a run of bad luck?

3. What elements of the company's business are its source of greatest risk?

This paper will discuss the type of model we developed to address these questions, which risk variables (e.g., catastrophe losses, investment yield, expense ratios, etc.) were addressed in the model, the type of information that we collected from the company and from external sources for the model, and how the model results were interpreted to develop answers to the questions.

Results

The paper concludes with a presentation of the results of the analyses and a summary of management's actions. Briefly, these actions were:

1. Changed underwriting guidelines and pricing for general liability business.

2. Revised plan to be closer to findings of our analysis.

3. Developed monthly monitoring statistics reflecting key drivers identified in analysis.

Profitability Targets: DFA Provides Probability Estimates

Dynamic financial analysis (DFA) is currently used in many applications and will probably be used to address an even wider range of issues in the coming years. One application for which we[1] have used DFA is the evaluation of the likelihood that an insurer will achieve the profit levels projected in its financial plan. In this paper, we will describe the model and types of data used in the analysis, identify the risks that were specifically addressed by the model and those that were specifically considered outside of the scope of the project, and present illustrative model results. Finally, we will provide a discussion of how management used the findings of the analysis in its decision making process.[2]

Background Regarding the Company

The company for whom this engagement was performed is a medium-sized insurer that writes nationally, but has a regional focus. Its business is approximately 65% personal lines and 35% commercial lines. The company maintains excess of loss and catastrophe reinsurance to protect itself against large claims and property catastrophes. In addition, for one line of business (general liability for this discussion), it maintains an underlying quota share with a significant sliding scale commission.[3]

In recent years, the company has experienced a number of unexpected events, primarily affecting the general liability book of business, that have caused it to be unprofitable. The company maintains a net-written-premium-to-surplus ratio of about 1.5, so capitalization and solvency are not of serious

[1] The author would like to thank David Appel for his contributions to this paper and his review of the draft.

[2] We note that, throughout this paper, the data, insurer characteristics, amounts and findings have been disguised to protect the confidentiality of the company for whom the actual project was performed.

[3] The ceding commission can range from 18% to 40% depending on the ceded loss ratio.

concern. The consistent lack of profitability, however, has led to a loss of credibility with the Board and with rating agencies.

The Questions

Company management was interested in increasing the credibility of its financial plan and the presentation thereof. We therefore performed a dynamic financial analysis to evaluate the probability that the net income and statutory surplus projections would come to fruition. If our findings were that it was unlikely that plan results would be achieved, management was interested in (1) the differences between our best estimate of the future results and its plan and (2) factors that are projected to lead to the most significant variation from our expected results.

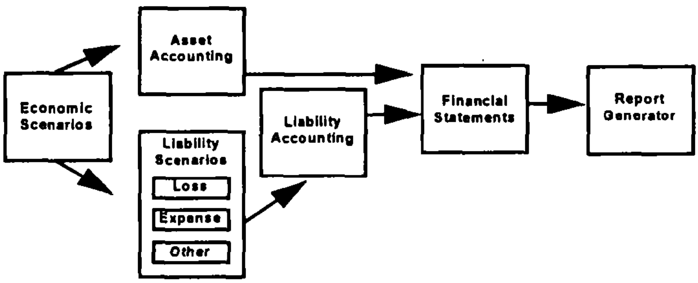
As will be discussed later, there were significant differences between the initial plan and our best estimates. Reconciliation of those differences (including additional information being provided, changes in strategy and changes in projected results) was a significant portion of the engagement. Identification of the factors that are projected to lead to the most significant variation from expected results served two purposes: (1) identification of possible strategies to reduce the variability and (2) selection of statistics for monitoring interim results to determine whether actual experience was as expected or whether the adverse experience was continuing.

The Model Used

Overview

The model used to perform this analysis was a customized, early version of Milliman & Robertson, Inc.'s dynamic financial model software, FINANS©. The foundation of that model is a spreadsheet that maintains the computations for the liability projections and the financial statements. This spreadsheet is similar to the financial projection models that are typically used by many property-casualty insurance companies for financial planning and/or valuation. It includes projections of statutory, GAAP, cash and tax financial statements and estimates risk-based capital and the IRIS tests.

In addition to the spreadsheet portion of the model, FINANS has a macroeconomic scenario generator, an asset accounting model and a report generator. The schematic below illustrates the major modules of the model:



The *macroeconomic scenario generator* is a multi-equation econometric model which develops quarterly projections of six economic and financial variables, namely, gross domestic product growth, inflation, long and short term interest rates, and stock returns and dividends. These projections are then used to drive both the asset and liability sides of the balance sheet.

The econometric model begins with a two stage autoregressive model of gross domestic product growth, where gross domestic product growth is a function of two lagged values of itself and a random error term. The remainder of the model is recursive, in that each subsequent variable is estimated as a function of a previously derived variable (and generally lagged values of itself). Thus, inflation is estimated as a function of gross domestic product growth (and lagged inflation), short term interest rates are a function of inflation (and lagged interest rates), long term rates are a function of short term rates, and so on.

The asset accounting model combines the output of the macroeconomic scenario generator with information regarding (1) the assets owned by the company on the valuation date, (2) the cash flows from underwriting derived from the spreadsheet, and (3) the company's investment strategy to project market, book and par valuations of assets by class at each year end, as well as interest, dividends, capital gains (realized and unrealized), amortization, maturities and other income and cash transactions occurring during each year. The output of the asset accounting model is fed to the spreadsheet portion of the model and is integrated into the financial statements.

At the time this project was performed, the report generator module simply collected information regarding each of the dynamic inputs and selected financial statement values and placed them in a data base. Analysis of results was accomplished using an Excel spreadsheet.

Inputs

The key inputs to the model can be separated into those related to the invested asset portfolio, those related to underwriting and other balances specific to the company as a whole.

With respect to the invested asset portfolio, the model requires information regarding:

(1) The book, acquisition and par values of each of government, municipal, corporate and high yield bonds by maturity and coupon.

(2) The book, acquisition and market values of other investment classes (stocks, real estate, mortgages, cash and short-term investments).

(3) The investment strategy - either the desired distribution of cash generated during the year among classes or the desired mix of assets at the end of the year. If the former approach is taken, the user must specify the manner in which assets are to be disposed in situations in which cash flows are negative.

For each modeled line of business, the user inputs information regarding premiums, losses, expenses and reinsurance. The company's business was divided into the following lines for modeling:

- Property. (Commercial and personal property exposures were combined due to limitations on catastrophe modeling in this version of the software.)

- General liability, including other liability, products liability and special liability.

- Workers' compensation.

- Commercial automobile, including liability and physical damage.

- Personal automobile, including liability and physical damage.

For premiums, information regarding direct written premium, earning patterns and collection lags are provided. For losses, information regarding loss, loss adjustment expense (LAE) and salvage/subrogation ratios, reserve strengthening (calendar year by accident year), and payment patterns are required. Expenses can be broken down into commissions, premium taxes, other variable expenses, fixed expenses and policyholder dividends. Information regarding each of quota share, excess of loss, catastrophe and annual aggregate reinsurance is provided to the model. Other information regarding the company as a whole, such as other income, stockholder dividends and capital infusions, can also be entered into the model.

Risks Modeled

The number of risks that can be made dynamic for any given company is endless. One of the important roles of the DFA actuary, in conjunction with company management, is to identify those risks that warrant inclusion in the model. For this application, many risks were identified, several of which were modeled dynamically as discussed below.

Investment Yields and Returns: Investment returns were derived from the macroeconomic scenario generator. That is, interest and dividends from investments held at the valuation date and through the projection period were calculated based on the characteristics of the assets. Market values of high yields bonds and stocks were calculated in the asset accounting model using standard valuation formulas. Bond defaults were derived based on the economic conditions as described by the output of the macroeconomic scenario generator.

Premium: Uncertainty regarding the growth of premium (combined exposure growth and rate changes) was introduced.

Losses: For each line, losses were modeled in three categories: catastrophes (only for property lines), large claims and the loss ratio resulting from small (all other) claims. For catastrophes, the number of catastrophes in excess of a certain size was modeled using a Poisson distribution. The sizes were drawn from a distribution derived from catastrophe modeling software. For large claims, the number of claims in excess of a selected threshold was modeled using a Poisson distribution with the average sizes (ground up) being selected from Pareto distributions.

Fixed expenses: The ratio of fixed expenses to direct earned premium was assumed to vary using a Normal error term. This error term was assumed to be constant across all lines of business (i.e., there was 100% correlation among lines) because the parameters of the error term distribution were derived from companywide historical expense data.

Statutory Assessments: With the relatively recent payment by some companies of Proposition 103 rollbacks, the risks emanating from statutory assessments were considered important by the company.

Risks Not Modeled

There were a large number of risks that were not modeled, as described below.

Mass torts: The company has not written any exposures that have generated claims from mass torts in the past. Having reviewed its current book of business, it does not believe that it has material exposure to any mass torts. This risk was therefore not explicitly modeled.

Loss payment patterns: Loss payment patterns were assumed to vary by line, but not accident year. As such, the model did not reflect the volatility in payment patterns from changes in inflation, mix of claims or other factors affecting payment patterns.

Reserve strengthening: The company has historically experienced favorable development of ultimate losses and ALAE between their initial report and the final estimates. For conservatism, the model assumed that the booked reserves as of December 31, 1996 did not contain any such margin. Because of the consistency of the reserve estimates, the risk related to changes in estimates was considered relatively small and was not modeled.

LAE ratios: Ratios of ALAE to loss and ULAE to loss, by line, were held constant across accident years and scenarios.

Reinsurance pricing: Reinsurance premium rates and contingent premium terms were held constant across the three-year projection period for all scenarios. With the relatively short time period covered by the analysis, it was believed that changes in reinsurance rates and terms would not be a significant factor relative to many of the other risks that were modeled.

Illiquid assets: The company has a number of illiquid invested assets, though they comprise only a small proportion of invested assets. The expected value of the interest income from these assets was used in all scenarios and the book value of these assets was held constant.

Reduction in Best's rating: A serious concern of the company is that it's Best's rating might be reduced in light of the recent unprofitability. A reduction in Best's rating could have a significant impact on the company's ability to maintain is current premium volumes and its ability to select risks in the marketplace. The company chose not to model the impact of this risk, so all results are conditional on the assumption that the company maintains its current Best's rating.

Data Used in Analysis

The data provided for our analysis included:

(1) Management's three-year financial plan.

(2) Five years of statutory annual statements.

(3) The company's analysis of direct ultimate losses and LAE by accident year and subline, along with corresponding payment triangles and earned premium. These estimates were accepted as best estimates. An independent evaluation of reserves was outside of the scope of the engagement.

(4) Development triangles of individual paid and incurred losses in excess of $500,000.

(5) Probability distributions of catastrophe losses for all property exposures in the aggregate.

(6) Policy limits profiles.

(7) A list of catastrophe losses exceeding $2 million for the past 10 years.

These data were used to develop the expected value assumptions for all inputs to the model and to derive the parameters of the distributions for each of the modeled risks.

Premium

For the expected value case, we accepted management's premium growth assumptions which anticipated approximately 5% per annum growth for personal lines and 0% per annum for commercial lines. A common premium growth rate was used for all commercial lines and a separate growth rate was used for personal lines. The premium growth rates were assumed to be Normally distributed with a standard deviation of 2.5%, a minimum of 0% for personal lines and -5% for commercial lines and a maximum of 10% for personal lines and 5% for commercial lines. The base case assumptions regarding direct written premium by line for each of the three projections years are shown in Exhibit 1. Also shown in that exhibit are the projected percentages of premium earned and collected in the year written.

Losses

As discussed previously, the model separates losses into the following categories: (1) catastrophes, (2) ground up losses on claims exceeding a selected size ($500,000 per claim for this analysis) and (3) small losses.

The historical loss experience by line and accident year was first decomposed into the three components. As indicated previously, data were available to remove the impact of catastrophe losses. The development of individual claims in excess of $500,000 per claim was used to derive projections of the ultimate cost of large claims. These projections and the catastrophe losses were subtracted from direct ultimate losses to estimate small losses. Exhibit 2 shows the decomposition of property and general liability losses into the three components. Similar analyses were performed for the other lines.

The expected number of catastrophe losses in excess of $5 million per event per year (0.25) was derived from the catastrophe model output. The distribution of these events was also derived from the catastrophe model output, as shown on Exhibit 3. Because there are only relatively small variability in premium volume projected, no adjustments were made to the catastrophe loss parameters across iterations. These assumptions correspond to a ratio of catastrophe losses to property premium of approximately 9%.

The historical frequency and size of large claims was reviewed to derive assumptions for use in the projection period. Exhibit 4 shows the number, projected frequency and projected average cost of large general liability claims for Accident Years 1987 through 1996. Initially, we selected a frequency of large general liability claims of 0.30 claims per $1 million of general liability premium and an average cost per large claim of $1.2 million. These assumptions were much higher than those implicit in management's assumptions (which anticipated that the recent large claim experience reflected a run of bad luck, not a precursor of future losses) and much higher than would have been expected based on the excess of loss reinsurance pricing. In light of the relatively small number of claims, the lack of available industry information regarding large claims from the particular niches written by the company and the reinsurer's evaluation of the company's large loss exposure, we introduced uncertainty with respect to the expected frequency of large general liability claims. That is, the model assumed a 20% chance that the expected frequency of large general liability claims is 0.225, a 50% chance that it is 0.30 and a 30% chance that it is 0.35.

For all other lines, the frequency of large claims was much more stable, so a single expected frequency of claims was selected. A Poisson distribution was used to model the actual number of large claims for each line in each scenario using a mean equal to the expected number of large claims (frequency times direct earned premium). The expected frequencies of large claims for lines other than general liability are shown in Table 1.

Table 1: Large Claim Assumptions

Line	Expected Frequency	Expected Severity
Property	0.15	$1 million
Workers compensation	0.05	1.5 million
Commercial auto	0.25	700 thousand
Personal auto	0.01	600 thousand

Pareto size of loss distributions were used to model the cost of individual claims. For each line, the parameters of the Pareto distribution were selected after reviewing:

(1) The historical experience regarding large claims by size.

(2) The average claim cost implicit in reinsurance pricing (after consideration of the historical distribution of policy limits).

(3) Changes in the distribution of policy limits.

(4) The average claim costs implicit in insurance industry increased limits factors (assuming the company's large claim frequency is appropriate).

To simplify modeling, the Pareto parameters were selected so that the claim size distribution implicitly incorporated the policy limit distribution. That is, the claim sizes selected from the Pareto distribution are assumed to have already been capped by any applicable policy limits. The occurrence and size of large losses was assumed to be independent across lines and time.

Using cascading regression and applying judgment, models of the small loss ratios were derived. The formulas for the small loss ratios are as follows:

$$l/r_{j,k} = a + b(l/r_{j-1,k}) + \sum_{x \neq k} c_x(l/r_{j,x}) + \sum_{x \neq k} d_x(l/r_{j-1,x}) + f(i_j) + e_j$$

where l/r is loss ratio
j is the year,
x is line of business,
k is the specific line of business being modeled,
i is the interest rate,
a, b, c, d and f are constants and
e is a Normal random variable.

The resulting loss ratios (small, large, catastrophe and total) are summarized on Exhibit 5.

Expenses

We reviewed historical ratios of ALAE and ULAE to loss by accident year and line to select these ratios for use in the model. The selected ratios are shown on Exhibit 6.

For the base case, we accepted the company's assumptions regarding commissions and premium taxes. The base case assumptions are presented in Table 2.

Table 2: Base Case Assumptions

Type of Expense	Ratio to Written Premium
Commissions	17.3%
Premium Taxes	2.7%

Fixed expenses were projected from 1996 levels assuming that fixed expenses increased (1) with CPI inflation and (2) with 50% of any increase in direct earned premium. In addition, the ratio of fixed expenses to direct earned premium was assumed to have a random component. To incorporate this random component, we added a percentage drawn from a Normal distribution with a mean of 0 and a standard deviation of 1% of direct earned premium to the expenses otherwise derived for each line of business. (The same percentage was added for each line.) The standard deviation of the error term was derived after reviewing ten years of expense ratios (excluding premium taxes and agents' commissions) after adjustment for a change in accounting and a significant one-time expenditure.

Statutory Assessments

A discrete distribution of statutory assessments (including assigned risk and guaranty fund assessments, rollbacks, excess profits refunds and the like) was derived after considering the distribution of premium by state and a probability distribution of assessments as a percentage of direct premium in a state. The resulting probability distribution of statutory assessments as a percentage of countrywide direct written premium is shown on Exhibit 7.

Reinsurance

The company purchases primarily excess of loss reinsurance. The attachment point is $1 million per claim for all lines, except general liability for which it is $5 million per claim. There is no ceding commission in any of the excess of loss contracts. It is assumed for modeling purposes that premiums are ceded and losses are recovered quarterly in arrears. The 1997 ceded premium for the excess of loss coverage is shown in Table 3.

Table 3: 1997 Ceded Excess of Loss Premium

Line	1997 Ceded Premium (000s)
Property	$ 360
General liability	1,440
Workers' compensation	600
Commercial auto	360
Personal auto	2

For general liability, the company also entered into a quota share agreement under which 75% of losses and premium are ceded. This contract has a significant slide on the ceding commission. The provisional commission is 25%. For each point increase in the pure ceded loss ratio above 55%, the commission is decreased by 0.8 percentage points, subject to a minimum of 18% and a maximum of 40%. The commission provision applies to each accident year individually.

For property, catastrophe reinsurance is also purchased in the layer $50 million excess of $10 million. The cost of the catastrophe reinsurance is $4.5 million. There are two reinstatements available at a rate on line[4] of 5%.

All reinsurance is assumed to be collectible; that is, credit risk from reinsurers is not modeled.

Illustrative Results

As was discussed earlier in this paper, the scope of the engagement entailed:

(1) Evaluation of the likelihood that actual results would equal or exceed those in the company's plan.

(2) Identification of differences in assumptions between us and the company.

(3) Identification of key drivers of results.

The dynamic financial model was used to derive 2,000 possible results based on the assumptions presented previously. The results of these iterations were used to address the company's questions.

Probability of Attaining Plan Results

Exhibit 8 shows the probability distribution of net income by year and 1999 projected surplus. Figure 1 shows the distribution of 1997 net income graphically. Also shown on Exhibit 8 are the income and surplus amounts in the company's three-year financial plan and our estimates of the probability of attaining those results. As can be seen, the analysis indicated that there is a relatively low probability that the company's targets will be attained.

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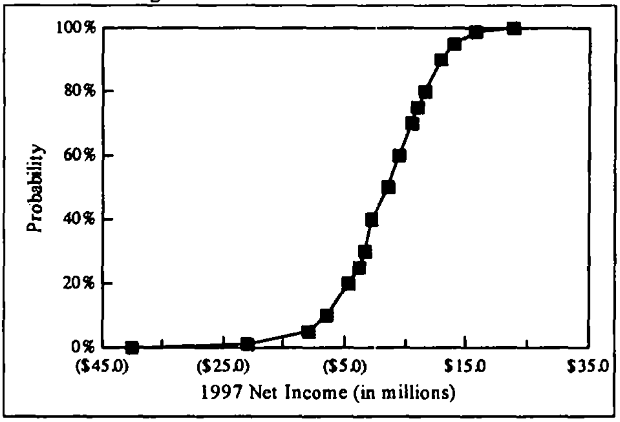
[4] For every dollar recovered from the catastrophe reinsurer for the first two catastrophes in excess of the attachment point, 5¢ is paid as reinstatement premium.

Figure 1: Distribution of Net Income



Differences in Assumptions

As was indicated in the discussion of assumptions, one significant difference in assumptions was the frequency of large general liability claims. We pointed out that the company had entered a new type of business in the early 1990s and that the earlier favorable experience with large general liability claims was not indicative of the future. We therefore calculated the probability that the actual number of large claims for 1994 through 1996 would have been observed using expected values of management's assumptions of 3 large claims per year and our three assumptions regarding the number of large claims of 7.2, 9.2 and 10.7. These probabilities are shown in Table 4.

Table 4: Probabilities

Expected Number of Large Claims	Probability of Last Three Year's Results
6.0	0.1%
7.2	1.3%
9.2	17.4%
10.7	46.0%

The other significant difference in assumptions relates to fixed expenses. The company had projected that fixed expenses related to commercial lines would remain constant, but planned to keep the same level of personnel. That is, the company did not reflect the impact of wage inflation on salaries and related expenses. After reviewing our model and seeing the impact of inflation, the company revised its expense projections.

Key Drivers

The process used to identify key drivers was:

(1) Identify all of the independent variables monitored in the analysis, as shown in Exhibit 9.

(2) Use a t-test to determine whether there was a statistically significant correlation between each variable and calendar year net income. (Several approaches, including stepwise regression, were used to ensure that correlation among independent variables did not distort the findings.) Those variables whose correlation with net income were not statistically significant were dropped from this list.

(3) Calculate the impact on net income if each of the statistically significant independent variables were at its 90th and 99th percentile. Those variables that were found to have statistically significant correlation with net income, but had much less than a $1 million impact on net income at the 90th percentile were excluded.

The remaining variables and several measures of their impact on net income are shown in Table 5.

Table 5: Impact on Net Income

Variable	Average 1997 Value	Net Income Impact if 10% Worse than Expected (thousands)	Probability of 10% Worse than Expected	Net Income Impact of 90th Percentile Adverse Deviation (millions)
Small Loss Ratio - General Liability	25.0%	$ 775	16%	$-1.0
Small Loss Ratio - Commercial Auto	45.0%	1,739	19%	-2.6
Small Loss Ratio - Personal Auto	68.0%	3,877	3%	-2.8
Small Loss Ratio - Workers' Compensation	67.5%	1,457	22%	-2.6
Small Loss Ratio - Property	43.0%	2,790	15%	-4.0
Number of Large Property Claims	9.7	970	36%	-3.3
Number of Large General Liability Claims	9.3	1,116	34%	-4.4
Number of Large Commercial Auto Claims	9.7	679	31%	-2.3
Number of Large Workers' Compensation Claims	1.1	165	30%	-2.9
Number of Catastrophes	0.25	141	25%	-2.5
Underwriting Expenses (Deviation from Expected)	0%	N/A	N/A	-2.8

Management Use of Results

Company management made a number of changes to its plan, its underwriting and its monitoring tools in response to our findings. The company first reviewed our report to identify those assumptions for which our best estimate significantly differed from the assumptions underlying its plan. Three or four such assumptions were found, most of which related to the general liability book of business. The company therefore carefully reviewed its current book of business and made numerous changes to its underwriting guidelines. It also made several changes to the manner in which individual accounts are rated and will make increased use of facultative reinsurance to limit its exposure to large claims. The company presented these changes to us and its analyses supporting its estimates of the impact of these changes on the key assumptions underlying our model.

In addition to making these changes to operations, the company revised its plan to make it somewhat less favorable. We then evaluated the analyses and revised the assumptions underlying our model. Although we still project that there is less than a 50% change of attaining the plan results, our projections are much closer to the plan than was displayed on Exhibit 9.

Finally, management is using the information regarding key drivers to monitor results on a monthly basis. With the importance of attaining the results in the financial plan, the company wants to identify possible sources of adverse deviation as quickly as possible.

Exhibit 1

Sample Insurance Company

SUMMARY OF PREMIUM DATA

Line	Direct Written Premium 1997	1998	1999	Collection Lag	Percent Earned in Year
Property	$64,889	$65,668	$68,951	2.4	46.4%
General Liability	31,000	31,000	31,000	2.1	53.6%
Workers' Compensation	21,586	21,586	21,586	1.8	60.1%
Commercial Auto	38,638	38,638	38,638	2.2	51.0%
Personal Auto	57,018	60,636	64, 435	2.1	53.4%

Notes: 1. Dollar amounts are in thousands.
2. Premium collection lag is stated in months.

Sample Insurance Company

SUMMARY OF HISTORICAL LOSS DATA

General Liability and Property

Accident Year	(1) Ultimate Direct Losses	(2) Losses on Large Claims	(3) Catastrophe Losses	(4) Direct Earned Premium	(5) Small Loss Ratio [(1)-(2)-(3)]/(4)
			General Liability		
1987	$ 7,316	$ 0	$ 0	$28,640	25.5%
1988	9,668	0	0	32,736	29.5%
1989	10,752	2,800	0	36,340	21.9%
1990	14,000	4,000	0	41,396	24.2%
1991	11,368	0	0	42,244	26.9%
1992	15,240	4,000	0	38,992	28.8%
1993	13,860	3,200	0	36,240	29.4%
1994	19,788	12,000	0	36,636	21.2%
1995	16,276	7,200	0	35,124	25.8%
1996	21,012	13,200	0	32,336	24.2%
			Property		
1987	$13,172	$ 0	$ 0	$31,893	41.3%
1988	13,654	0	0	37,408	36.5%
1989	18,904	1,929	0	38,580	44.0%
1990	23,952	3,870	0	43,002	46.7%
1991	29,352	6,174	2,460	47,038	46.7%
1992	24,484	4,356	0	46,459	43.3%
1993	27,086	5,561	0	49,427	43.6%
1994	41,806	12,059	9,750	53,597	46.4%
1995	33,618	6,401	0	60,247	45.2%
1996	35,466	7,012	0	62,330	45.7%

Note: Dollar amounts are in thousands.

Exhibit 3

Sample Insurance Company

DISTRIBUTION OF CATASTROPHE LOSSES

Probability	Amount
0.5%	$200
3.0%	130
1.5%	110
1.5%	90
1.5%	70
2.5%	60
2.5%	50
2.5%	44
2.5%	38
2.5%	32
2.5%	26
2.5%	20
5.0%	18
5.0%	16
5.0%	14
5.0%	12
5.0%	10
9.5%	9
10.0%	8
10.0%	7
10.0%	6
10.0%	5

Note: Dollar amounts are in millions.

Exhibit 4

Sample Insurance Company

SUMMARY OF GENERAL LIABILITY LARGE CLAIMS

Accident Year	(1) Number of Large Claims	(2) Projected Frequency	(3) Projected Average Cost	(4) Losses on Large Claims (1)x(3)
1987	0	0.00	--	--
1988	0	0.00	--	--
1989	4	0.11	$ 700	$2,800
1990	4	0.10	1,000	4,000
1991	0	0.00	--	--
1992	4	0.10	1,000	4,000
1993	4	0.11	800	3,200
1994	12	0.33	1,000	12,000
1995	8	0.23	900	7,200
1996	12	0.37	1,100	13,200

Notes: 1. Large claims are those that exceed $500,000.
2. Frequency is per $1 million premium.

Sample Insurance Company

SUMMARY OF LOSS RATIO ASSUMPTIONS

Line	(1) Small Loss Ratio	(2) Large Loss Ratio	(3) Catastrophe Loss Ratio	(4) Direct Loss Ratio (1)+(2)+(3)
Property	43.0%	15.0%	8.7%	66.7%
General Liability	25.0%	36.0%	0%	61.0%
Workers' Compensation	67.5%	7.5%	0%	75.0%
Commercial Auto	45.0%	17.5%	0%	62.5%
Personal Auto	68.0%	0.6%	0%	68.6%

Exhibit 6

Sample Insurance Company

SUMMARY OF LOSS ADJUSTMENT EXPENSE RATIO ASSUMPTIONS

Line	ALAE/Loss Ratio	ULAE/Loss Ratio
Property	10.5%	6.0%
General Liability	15.0%	5.0%
Workers' Compensation	8.0%	4.5%
Commercial Auto	8.5%	7.0%
Personal Auto	8.0%	7.0%

Exhibit 7

Sample Insurance Company

STATUTORY ASSESSMENTS

Probability	Statutory Assessments/ Direct Written Premium
95%	0.5%
3%	1.0%
1%	2.0%
1%	5.0%

Sample Insurance Company

STATUTORY RESULTS

	Net After-Tax Income			1999
	1997	1998	1999	Surplus
Mean	$2,020	$1,740	$ 855	$120,852
Probability				
(Min) 0%	$-40,231	$-40,456	$-41,342	$ 64,729
1%	-21,026	21,320	22,116	86,912
5%	-10,998	-11,201	-12,089	101,731
10%	-8,020	-8,213	-9,118	106,444
20%	-4,305	-4,558	-5,508	112,337
25%	-2,754	-3,012	-3,887	114,765
30%	-1,647	-1,892	-2,808	116,562
40%	-432	-667	-1,589	119,668
50%	2,213	2,070	1,137	122,115
60%	3,874	3,609	2,707	125,816
70%	5,879	5,616	4,696	127,994
75%	6,992	6,612	5,698	128,275
80%	7,963	7,716	6,833	134,001
90%	10,720	10,529	9,628	136,349
95%	12,952	12,689	11,754	136,981
99%	16,341	16,028	15,117	142,560
(Max) 100%	22,616	22,327	21,472	147,783
Plan	4,000	4,500	5,000	131,500
P{x>Plan}	38%	35%	28%	15%

Note: Dollar amounts are in thousands.

Exhibit 9

Sample Insurance Company

LIST OF VARIABLES TESTED

Gross Written Premium
- Commercial Lines
- Personal Lines

Underwriting Expense Deviation

Statutory Assessments

Number of Catastrophes

Size of Each Catastrophe

Small Loss Ratio
- Property
- Commercial Auto
- General Liability
- Workers' Compensation
- Personal Auto

Number of Large Claims
- Property
- Commercial Auto
- General Liability
- Workers' Compensation
- Personal Auto

Average Cost of Large Claims
- Property
- Commercial Auto
- General Liability
- Workers' Compensation
- Personal Auto

Inflation

Short and Long Term Rates

Pricing Catastrophe Reinsurance with Reinstatement Provisions Using a Catastrophe Model

by Richard R. Anderson, FCAS, MAAA, and
Wemin Dong, Ph.D.

Pricing Catastrophe Reinsurance With Reinstatement Provisions Using a Catastrophe Model

Richard R. Anderson, FCAS, MAAA
Weimin Dong, Ph.D.

Abstract

In recent years catastrophe reinsurers' use of catastrophe models has been increasing until currently virtually all of the catastrophe reinsurers in the world use a catastrophe model to aid them in their pricing and portfolio management decisions.

This paper explicitly models various types of reinstatement provisions, including reinstatements that are limited by the number of occurrences and by the aggregate losses; and reinstatement premiums based on the size of loss and by the time elapsed to the first occurrence. The paper also investigates the effects on the fair premium of a catastrophe treaty when various reinstatement provisions are considered.

This is an expansion of the methods developed in papers by Leroy J. Simon and Bjorn Sundt, which were written before the widespread use of catastrophe models.

The catastrophe model used for this paper is the Insurance / Investment Risk Assessment System (IRAS) produced by Risk Management Solutions, Inc.

Pricing Catastrophe Reinsurance With Reinstatement Provisions Using a Catastrophe Model

Introduction

In recent years catastrophe reinsurers' use of catastrophe models has been increasing until currently virtually all of the catastrophe reinsurers in the world use a catastrophe model to aid them in their pricing and portfolio management decisions.

Leroy Simon's 1972 paper [1] on catastrophe reinsurance investigated the relationships between various provisions of catastrophe reinsurance treaties to ensure consistency in pricing between contracts. In his paper he assumes that each loss causes a total loss to the layer of reinsurance. Bjorn Sundt expanded on this theme in his paper in 1991 [2], focusing on reinstatements based on aggregate losses. This paper applies the methods outlined in these previous works to the output of a catastrophe model to calculate a fair premium for a catastrophe treaty when reinstatement premium is considered.

The paper develops the fair premium for catastrophe reinsurance with various types of reinstatement provisions, including reinstatements that are limited by the number of occurrences and by the aggregate losses; and reinstatement premiums based on the size of loss and by the time elapsed to the first occurrence. The paper also investigates the effects on the fair premium of a catastrophe treaty when various reinstatement provisions are considered.

The catastrophe model used for this paper is the Insurance / Investment Risk Assessment System (IRAS) produced by Risk Management Solutions, Inc.

As background, we start with some descriptions of reinstatement provisions and how they are applied. We then describe an event loss table, the output of the catastrophe model that gives us all of the information that we need to perform the calculations. Next we turn our attention to the calculation of the fair premium of catastrophe treaties with various types of reinstatement provisions. First we discuss reinstatement provisions that limit the number of occurrences, then reinstatement provisions that limit the aggregate losses. Finally we investigate reinstatement premiums that are pro rata as to time.

Reinstatement Provisions

A common feature of many catastrophe reinsurance contracts is a reinstatement provision. A reinstatement provision puts a limit on either the number of occurrences or the aggregate losses that will be paid under the contract. For example, if a contract has a provision for one reinstatement based on the number of occurrences, then the reinsurer will be responsible for at most two occurrences (original occurrence plus one reinstatement). If the contract has a provision for one reinstatement based on aggregate losses, and the limit is $1 million, then the reinsurer will be responsible for at most $2 million in aggregate, regardless of the number of occurrences.

The reinstatements may be free or paid. If the reinstatements are free, then all of the premium is paid up front. For paid reinstatements, a portion of the premium is paid following the occurrence of an event. For example, if a contract has a provision for one paid reinstatement, then after the first event the cedant will pay some premium to the reinsurer to reinstate the coverage for a second occurrence. This additional premium is called the reinstatement premium. The reinstatement premium may vary based on the amount of reinstatement (pro rata to full limit) or the time remaining in the contract (pro rata to full time). In this paper we will limit the discussion to reinstatement premium that is pro rata to full limit, and is either 100% to time or pro rata to full time.

Event Loss Table

An "event" as we use it in this paper is a scenario taken from the set of all possible outcomes. For example, event *e* might be an earthquake of magnitude 7.3 on the San Andreas fault centered two miles off the coast of San Francisco; and event *h* might be a category 3 hurricane making landfall in Dade county Florida with a specific track, central pressure, etc. The final product from an IRAS analysis is a table of events with their expected losses and annual occurrence rates. The set of events in the Event Loss Table (ELT) represents the full range of possible outcomes that can occur to a portfolio.

Suppose that we have a catastrophe treaty of *LMT* excess *ATT*, where *LMT* is the limit of the treaty, and *ATT* is the attachment point of the treaty. Denote the gross loss for the j^{th} event as $GLOSSj$ and the expected loss to the catastrophe treaty as Lj. We have

$$L_j = \int_{ATT}^{ATT+LMT} (GLOSS_j - ATT) f_j(GLOSS_j) dGLOSS_j + LMT[1 - F_j(ATT + LMT)] \quad (1)$$

where

$f_j(GLOSSj)$ = probability density of the gross loss given that event j has occurred
$F_j(ATT + LMT)$ = cumulative probability that the gross loss ≤ ATT + LMT, given that event j has occurred

In the ELT shown below in Table 1, λ_j is the annual rate of occurrence for event j, and L_j is the expected loss to the catastrophe treaty for event j, calculated from equation (1).

Table 1 Event-Loss Table (ELT)

Event	Rate	Expected Loss
1	λ_1	L_1
2	λ_2	L_2
:	:	:
j	λ_j	L_j
:	:	:

J	λ_J	L_J

We assume here that each event is an independent random variable, each with a Poisson frequency distribution[1]. We assume that the occurrence of one event will have no effect on the rate or the expected loss of any other event. We look at these multi-events (the occurrence of one or more events) as a compound Poisson process[2] with a total rate equal to:

$$\lambda = \sum_j \lambda_j \tag{2}$$

Hence, the probability of exactly n occurrences in a year for this process is given by

$$p(n) = \frac{\lambda^n e^{-\lambda}}{n!} \tag{3}$$

The average annual loss (AAL) for event j is given by the expected frequency times the expected severity, which, given our Poisson frequency assumption, is $\lambda_j L_j$. Because we assume that each event is an independent random variable, the total AAL is the sum of the AAL's for all events:

$$AAL = \sum_j \lambda_j L_j \tag{4}$$

This represents the pure premium of a treaty with unlimited free reinstatements.

With all of this as background, we now turn our attention to the calculation of the fair premium of catastrophe treaties with various types of reinstatement provisions. First we discuss reinstatement provisions that limit the number of occurrences, then reinstatement provisions that limit the aggregate losses. For each of these cases we assume that the reinstatement premiums are pro rata as to limit and 100% as to time. Finally we investigate reinstatement premiums that are pro rata as to time, where we calculate the expected arriving time for the occurrence of an event.

Reinstatements Limited by the Number of Occurrences

The reinstatement premium will be paid whenever an event occurs with losses to the catastrophe treaty and the reinstatements are not already used up. The amount of reinstatement premium (P_{reinst}) is

[1] Other frequency distributions, such as Negative Binomial, may be appropriate for some perils or regions. The use of these distributions is beyond the scope of this paper.

[2] For more information on Poisson processes, see references [4], [5], and [6]

$$P_{reinst} = R \cdot c \cdot L \tag{5}$$

where

R = Premium rate paid up front (rate on line)
c = fraction of reinstatement premium rate versus up-front premium rate
L = loss to the catastrophe treaty, which is a random variable.

It can be seen that the reinstatement premium formula (5) is pro rata as to limit by noting that R, the rate on line, equals the premium (P) divided by the limit (LMT):

$$P_{reinst} = \frac{P}{LMT} \cdot c \cdot L = P \cdot c \cdot \frac{L}{LMT} \tag{6}$$

We calculate the fair up-front premium rate (ignoring expense and risk load charges) by setting the expected premium collections equal to the expected loss payments.

First we calculate the expected loss to the catastrophe treaty as the expected severity times the expected frequency.

The expected loss given an event has occurred (expected severity) is given by

$$S(L) = \frac{\sum_j \lambda_j L_j}{\lambda} \tag{7}$$

To calculate the expected frequency, we make use of the limited expected value function[3]. The expected number of occurrences limited to k occurrences is given by

$$\begin{aligned} E(n;k) &= \sum_{n=1}^{\infty} \min(n,k) \cdot p(n) \\ &= \sum_{n=1}^{k-1} n \cdot p(n) + k \cdot (1 - F(k-1)) \end{aligned} \tag{8}$$

where

$p(n)$ = the probability that exactly n events will occur, as calculated by equation (3)
$F(k\text{-}1)$ = the cumulative probability that k-1 or fewer events will occur.

Let *nor* be the number of occurrence reinstatements allowed. The total number of occurrences covered by the contract is *nor*+1 (one original occurrence + *nor* additional occurrences). We define $E_O(L;nor + 1)$ to be the expected loss limited to *nor*+1 occurrences, which is the expected severity times the expected frequency:

$$E_O(L;nor+1) = S(L) \cdot E(n;nor+1) \tag{9}$$

[3] For more information on the limited expected value function, see Hogg and Klugman[3]

The subscript O in $E_O(L;nor + 1)$ stands for occurrence, to differentiate this from the case where the reinstatements are limited by the aggregate losses, which we will discuss later.

The expected premium collected is equal to the up front premium plus any reinstatement premiums collected.

$$E(P) = R \cdot LMT + R \cdot c \cdot E_O(L;nor) \tag{10}$$

where

R = rate on line for the contract

$E_O(L;nor)$ = expected loss limited to *nor* occurrences (no reinstatement premium is collected following the $nor+1^{th}$ occurrence).

Setting the expected premium equal to the expected losses, we get

$$R \cdot LMT + R \cdot c \cdot E_O(L;nor) = E_O(L;nor+1) \tag{11}$$

Solving for R, we get the fair up front rate on line:

$$R = \frac{E_O(L;nor+1)}{\left(LMT + c \cdot E_O(L;nor)\right)} \tag{12}$$

For example, assume that we have a simple event loss table (ELT) with ATT = \$2 million and LMT = \$2 million as shown in Table 2:

Table 2 Sample ELT

Event	Annual Rate	Gross Loss	Cat. Loss[4]
1	0.1	5 million	2 million
2	0.2	3 million	1 million

For this case, the expected severity for the catastrophe treaty is

$$S(L) = \frac{0.1 \cdot 2 + 0.2 \cdot 1}{0.1 + 0.2} = 1.333 \text{ million}$$

The expected losses and premium rates with various numbers of reinstatements are given in Table 3 for c = 1.0:

Table 3 Expected Losses and Fair Premium Rates when c = 1

Number of Reinst.	Expected Loss (in \$million)	Rate on Line

[4] For simplicity, the losses to the catastrophe treaty in this table are calculated assuming that the gross losses are constant. The actual output from the computer model calculates the catastrophe losses using equation (1).

0	0.34558	0.17279
1	0.39482	0.16833
2	0.39962	0.16687
3	0.39998	0.16668
∞	0.40000	0.16667

In this example, as the number of reinstatements increases, the up-front premium decreases, because the expected additional reinstatement premiums outweigh the higher expected losses. There can be situations where this is not the case, and the up-front premium increases as the number of reinstatements increases. This can happen, for example, when the expected severity is very low relative to the limit.

If $c = 0$ (free reinstatements), then equation (12) reduces to

$$R = \frac{E_O(L;nor+1)}{LMT} \tag{13}$$

and the fair up-front premium rates are shown in Table 4:

Table 4 Expected Losses and Fair Premium Rates when $c = 0$

Number of Reinst.	Expected Loss (in $million)	Rate on Line
0	0.34558	0.17279
1	0.39482	0.19741
2	0.39962	0.19981
3	0.39998	0.19999
∞	0.40000	0.20000

In this case, as the number of reinstatements increases, the up-front premium also increases, since the losses would be higher (because losses for more occurrences are paid), but there are no additional reinstatement premiums (because $c = 0$).

It is not uncommon to set the pure premium to the average annual loss from equation (4), which is $0.4 million in this example. If the rate on line is based on this pure premium, then it is equivalent to collecting up-front premium with unlimited free reinstatements, as shown in the last row of Table 4.

<u>**Reinstatements Limited by Aggregate Losses**</u>

When the reinstatements are limited by the number of occurrences, there can be some situations in which the buyer of the reinsurance will have a difficult decision to make. For example, suppose that one event has occurred with a very small loss to the catastrophe treaty. If the insurer makes a claim, it will use up one reinstatement for a small recovery. If it doesn't make a claim, then perhaps no other events follow, and it loses a chance of recovery. To avoid this dilemma, it is common practice to limit the reinstatement by aggregate losses rather than by number of occurrences.

Here the number of reinstatements refers not to the number of occurrences, but to the number of limits. Thus, a contract with *nlr* reinstatements will pay at most *nlr*+1 times *LMT*, regardless of the number of occurrences.

We calculate the fair up-front premium rate by again setting the expected premium collections equal to the expected loss payments.

To calculate the expected losses, we must first calculate the aggregate loss distribution. In this compound Poisson process, the probability of exactly n occurrences is given in equation (3). Given n events have occurred, the aggregate loss is calculated in equation (14):

$$A = L_1 + L_2 + \cdots + L_n \tag{14}$$

The distribution of A can be obtained by Panjer's recursive approach [4] and [5], by the use of Fourier transforms as described by Heckman & Meyers [6], or by a simulation approach. Let $f(A)$ and $F(A)$ be the probability density function and cumulative probability distribution of the aggregate losses obtained by one of these approaches. Note that this distribution is for the aggregate losses, not separated into the frequency and severity pieces as we did for the reinstatements based on the number of occurrences.

For a continuous aggregate loss distribution, the limited expected value of A limited to A_m is:

$$\begin{aligned} E(A;A_m) &= \int_0^\infty \min(A, A_m) f(A) dA \\ &= \int_0^{A_m} A f(A) dA + A_m \left(1 - F(A_m)\right) \end{aligned} \tag{15-C}$$

For a discrete aggregate loss distribution, the limited expected value of A limited to A_m is:

$$\begin{aligned} E(A;A_m) &= \sum_{i=1}^\infty \min(A_i, A_m) \cdot f(A_i) \\ &= \sum_{i=1}^{m-1} A_i \cdot f(A_i) + A_m \cdot \left(1 - F(A_{m-1})\right) \end{aligned} \tag{15-D}$$

where the A_i's are sorted in ascending order.

Because a contract with *nlr* reinstatements will pay at most *nlr*+1 times *LMT*, the expected loss for a treaty with *nlr* reinstatements is then the limited expected value of the aggregate losses limited to $(nlr+1)\cdot LMT$. We define the expected loss for the treaty as $E_A(L;nlr+1)$:

$$E_A(L;nlr+1) = E(A;(nlr+1)\cdot LMT) \tag{16}$$

The expected reinstatement premium is proportional to the aggregate losses capped at the treaty limit. If *nlr* reinstatements are allowed, then the expected reinstatement premium is proportional to the aggregate loss capped at *nlr* limits. Adding the up-front premium, we get the total expected premium:

$$E(P) = R\cdot LMT + R\cdot c\cdot E_A(L;nlr) \tag{17}$$

Setting the expected premium equal to the expected loss, we get:

$$R\cdot LMT + R\cdot c\cdot E_A(L;nlr) = E_A(L;nlr+1) \tag{18}$$

Solving for *R*, we get the fair up-front premium rate with *nlr* reinstatements:

$$R = \frac{E_A(L;nlr+1)}{\left(LMT + c\cdot E_A(L;nlr)\right)} \tag{19}$$

For an example, we used the same event loss table as for the occurrence-limited example (Table 2), and calculated the aggregate loss distribution using Panjer's approach[5] (see Appendix A for the calculations). The probability distribution is shown in Table 5:

Table 5 Aggregate Loss Distribution

Aggregate loss *A* (in $million)	Probability *f*(*A*)	Cumulative *F*(*A*)
0	0.7408182	0.7408182
1	0.1481636	0.8889818
2	0.0888982	0.9778800
3	0.0158041	0.9936841
4	0.0052351	0.9989192
5	0.0008416	0.9997608
6	0.0002026	0.9999634
7	0.0000298	0.9999932
8	0.0000058	0.9999990

[5] Here we make a simplifying assumption that the losses to the catastrophe treaty, given that an event has occurred, are constant. The actual output from the computer model shows not only the expected loss, but the coefficient of variation of the losses, from which a distribution can be assumed.

9	0.0000008	0.9999998
10	0.0000001	0.9999999

For $c = 1.0$, we have results as shown in Table 6:

Table 6 Expected Losses and Fair Premium Rates when $c = 1$

Number of Reinst.	Expected Loss (in $million)	Rate on Line
0	0.37020	0.18510
1	0.39864	0.16819
2	0.39996	0.16674
3	0.40000	0.16667
∞	0.40000	0.16667

Comparing Table 6 with Table 3, notice that the expected loss for occurrence-based with no reinstatement is lower than the expected loss for aggregate-based with no reinstatement. This is because for aggregate-based, more than one occurrence will be paid if the aggregate loss of the first occurrence is less than the limit. For example, if a contract has a provision for nr reinstatements, then the occurrence-based reinstatements provide $nr+1$ occurrences which have loss values less than or equal to the limit; the aggregate-based reinstatements provide $nr+1$ limits of coverage for as many occurrences as needed (at least $nr+1$) to reach the aggregate limit. Also note that for one or more reinstatements, the aggregate-based rate on line is less than the occurrence-based rate on line. This again is because the expected additional reinstatement premiums outweigh the higher additional losses.

And for $c = 0$, we have results as shown in Table 7:

Table 7 Expected Losses and Fair Premium Rates when $c = 0$

Number of Reinst.	Expected Loss (in $million)	Rate on Line
0	0.37020	0.18510
1	0.39864	0.19932
2	0.39996	0.19998
3	0.40000	0.20000
∞	0.40000	0.20000

There is a significant difference between the fair premium rate for no reinstatement and the fair premium rate based on AAL, which is equivalent to unlimited free reinstatements. In the above examples, the up-front premium rates are 0.1729 and 0.1851 for occurrence-based and aggregate-based, respectively, versus 0.2 based on the AAL. The difference increases with the increase of the total occurrence rate λ, particularly for occurrence-based contracts. Table 8 shows the impact of the total occurrence rate on the premium rates, keeping the severity distribution unchanged.

Table 8 Impact of Total Occurrence Rate on Premium

λ	Occurrence-Based*	Aggregate-Based*	Unlimited Free Reinstatements
0.03	0.0197	0.0199	0.02
0.3	0.1729	0.1851	0.20
3.0	0.6335	0.9004	2.00
3000	0.6667	1.0000	2000

* No reinstatements

The limiting case of the premium rate for an occurrence-based contract with no reinstatements as $\lambda \to \infty$ is the expected severity divided by the limit, since it is a certainty that an event will occur, and when it does occur the expected loss is equal to the expected severity. The limiting case of the premium rate for an aggregate-based contract with no reinstatements as $\lambda \to \infty$ is unity, since it is a certainty that the full aggregate limit will be paid.

Reinstatement Premiums Pro Rata for Time

Often, the reinstatement premium is proportional to the remaining time in the reinsurance contract after an occurrence. Given a loss, the reinstatement premium would be

$$P_{reinst} = R \cdot c \cdot L \cdot (1 - t) \tag{20}$$

And the total collected reinstatement premium for a contract limited by *nor* number of occurrences[6] is

$$TotP_{reinst} = \sum_{i=1}^{\min(n,nor)} R \cdot c \cdot L_i \cdot (1 - t_i) \tag{21}$$

where t is the time of the loss in years (assuming a one-year contract) and n is the number of occurrences in the year. The time remaining in the contract is $1 - t$. For example, if a loss occurs on October 1st of an annual contract with an effective date of January 1st, then $t = 0.75$, and the time remaining is 0.25.

The expected value of the total collected reinstatement premium is

$$E[TotP_{reinst}] = E\left[\sum_{i=1}^{\min(n,nor)} R \cdot c \cdot L_i \cdot (1 - t_i)\right] \tag{22}$$

$$= R \cdot c \cdot E\left[\sum_{i=1}^{\min(n,nor)} L_i \cdot (1 - t_i)\right] \tag{23}$$

[6] Reinstatements limited by the aggregate losses are left for further study.

Since the L_i's are independent of the t_i's,

$$= R \cdot c \cdot \sum_{i=1}^{\min(n,nor)} E(L_i) \cdot E(1-t_i) \tag{24}$$

Since the L_i's are independent of each other, $E(L_i)$ equals the expected severity:

$$= R \cdot c \cdot \sum_{i=1}^{\min(n,nor)} S(L) \cdot E(1-t_i) \tag{25}$$

$$= R \cdot c \cdot S(L) \cdot \sum_{i=1}^{\min(n,nor)} E(1-t_i) \tag{26}$$

Since $E_O(L;nor) = S(L) \cdot E(n;nor)$,

$$= R \cdot c \cdot E_o(L;nor) \cdot \frac{\sum_{i=1}^{nor} RT_i}{E(n;nor)} \tag{27}$$

where RT_i is the expected time remaining after the i^{th} occurrence.

Adding the up-front premium, we get the total expected premium collections:

$$E(P) = R \cdot c \cdot E_o(L;nor) \cdot \frac{\sum_{i=1}^{nor} RT_i}{E(n;nor)} \tag{28}$$

To calculate the fair premium amount, we set the expected premium collections from equation (28) equal to the expected losses from equation (9). Letting $\theta_k = \frac{\sum_{i=1}^{k} RT_i}{E(n;k)}$, we get

$$R \cdot LMT + R \cdot c \cdot E_o(L;nor) \cdot \theta_{nor} = E_o(L;nor+1) \tag{29}$$

Solving for R, we get the fair up front rate on line:

$$R = \frac{E_O(L;nor+1)}{\left(LMT + c \cdot E_O(L;nor) \cdot \theta_{nor}\right)} \tag{30}$$

We calculate the expected remaining time RT_k by integrating the distribution of the arriving time. Given the assumption of a Poisson process, the distribution of the arriving time for the k^{th} occurrence is given by a Gamma distribution, as shown in equation (31):

$$f_k(t) = \frac{\lambda(\lambda t)^{(k-1)} e^{-\lambda t}}{(k-1)!} \tag{31}$$

The expected time remaining after the k^{th} occurrence is

$$RT_k = \int_0^1 (1-t) f_k(t) dt \tag{32}$$

For k = 1, this reduces to equation (33). See Appendix B for the derivation.

$$RT_1 = \frac{\lambda + e^{-\lambda} - 1}{\lambda} \tag{33}$$

Table 9 shows the expected remaining time after the first occurrence for various λ values.

Table 9 Expected Remaining Times

λ	RT_1	θ_1
0.003	0.0015	0.5002
0.03	0.0149	0.5025
0.3	0.1361	0.5250
3.0	0.6833	0.7191
30	0.9667	0.9667
3000	0.9997	0.9997

The limiting case of θ_1 as $\lambda \to 0$ is 0.5, and the limiting case of θ_1 as $\lambda \to \infty$ is unity.

The expected losses and premium rates with various numbers of reinstatements are given in Table 10 for $c = 1.0$, using the event loss table from Table 2:

Table 10 Expected Losses and Fair Premium Rates When $c = 1$

Number of Reinst.	Expected Loss (in $million)	Rate on Line
0	0.34558	0.17279
1	0.39482	0.18090
2	0.39962	0.18176
3	0.39998	0.18180
∞	0.40000	0.18182

It is interesting to observe that the summation of the remaining time for a one year period, $\sum_{k=1}^{nr} RT_k$, converges to $\lambda / 2$ when nr approaches infinity (see Appendix C for the proof). Since $E(n;\infty) = \lambda$, θ_k converges to 0.5. Hence, the fair premium converges to

$$R = \frac{E_O(L;\infty)}{(LMT + c \cdot E_O(L;\infty) \cdot \theta_k)} = \frac{0.4}{(2.0 + 1 \cdot 0.4 \cdot 0.5)} = 0.18182$$

Comparing Table 10 with Table 3, the up-front premium when considering the remaining time is higher because the cost of a reinstatement after an occurrence is lower.

It should be noted that although earthquakes occur uniformly throughout the year, hurricanes and tornadoes are seasonal. Particularly, along the Atlantic coast, most hurricane landfalls are in September or October. Thus, the above derivation would need to be modified to account for this seasonality. The consideration of seasonality is beyond the scope of this paper.

Summary

This paper has shown how to use the output from a catastrophe model to calculate the fair premium of catastrophe treaties with reinstatement provisions. The basis for the analysis is the catastrophe model's event loss table, which contains all of the information needed to make the calculations.

The paper also investigated the effects on the fair premium of a catastrophe treaty when various reinstatement provisions are considered. Some of the findings:

- Basing the up-front premium on the average annual loss to a treaty, disregarding reinstatements, is equivalent to assuming that there are unlimited free reinstatements. If, on the other hand, reinstatements are limited and paid, then the up-front premium will be lower because fewer losses will be covered (because the reinstatements are limited) and some of the premium will be paid after an event has occurred (because the reinstatements are paid).

- Unless the expected severity is very small relative to the limit, the more paid reinstatements allowed the lower the up-front premium will be. This is because the additional reinstatement premiums expected to be collected will outweigh the additional expected losses.

- Reinstatement provisions based on aggregate losses will have higher expected losses than those based on the number of occurrences. In general, if the number of reinstatements is one or more, the up-front premiums will be less for aggregate-based reinstatements than for occurrence-based reinstatements. This again is because the additional expected reinstatement premiums will outweigh the higher expected losses.

- If the reinstatement premium is proportional to the remaining time in the reinsurance contract after an occurrence, then the up-front premium should be higher because less reinstatement premiums will be collected.

In this paper we did not consider expenses or risk loads, which are areas for further study. Other areas that deserve further study are reinstatement provisions that are limited by aggregate losses and have reinstatement premiums pro rata for time; and the effect of seasonality on the expected reinstatement premiums.

Acknowledgment

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Appendix A

Calculation of the aggregate loss distribution

We use the recursive method as described in "The Aggregate Claims Distribution and Stop Loss Reinsurance" by Harry H. Panjer. Mr. Panjer uses for his examples fixed benefit life insurance claims. Here we make the translation that an event that causes loss to the catastrophe treaty is one claim.

Using Mr. Panjer's notation, our event loss table (Table 2) is show below as Table A1:

Table A1 Event Loss Table

j	Loss Amount $= jU$	Rate $= \theta_j$	$j\theta_j = E_j$
1	$1,000,000	0.2	0.2
2	$2,000,000	0.1	0.2
Total		0.3	0.4

U is the greatest common divisor of the loss amounts for the claims, in this case $1,000,000. Then j is the loss amount divided by U.

Note that the sum of the E_j's is the average annual loss.

Let P_i represent the probability that the aggregate loss will be exactly iU, and n be the number of events in the event loss table. Mr. Panjer derives the recursive formula for P_i:

$$P_i = \frac{1}{i} \sum_{\substack{j=1 \\ E_j \neq 0}}^{\min(i,n)} E_j P_{i-j} \tag{A1}$$

where

$$P_0 = \exp\left(-\sum_{\substack{j=1 \\ E_j \neq 0}}^{n} \theta_j \right) \tag{A2}$$

Applying these formulas to the values in our event loss table, we get:

$P_0 = \exp(-0.3) = 0.7408$
$P_1 = 0.2 * 0.7408 = 0.1482$
$P_2 = (1/2) * (0.2 * 0.1482 + 0.2 * 0.7408) = 0.0889$
$P_3 = (1/3) * (0.2 * 0.0889 + 0.2 * 0.1482) = 0.0158$
$P_4 = (1/4) * (0.2 * 0.0158 + 0.2 * 0.0889) = 0.0052$
etc.

These are the probabilities $f(A)$ shown in Table 5.

Appendix B

Expected time remaining after the first occurrence

$$RT_1 = \int_0^1 (1-t) f_1(t)dt \tag{B1}$$

$$= \int_0^1 f_1(t)dt - \int_0^1 t f_1(t)dt \tag{B2}$$

$$= \int_0^1 \lambda e^{-\lambda t}dt - \int_0^1 t\lambda e^{-\lambda t}dt \tag{B3}$$

$$= \lambda\left[\frac{-e^{-\lambda t}}{\lambda}\right]_0^1 - \lambda\left[\frac{e^{-\lambda t}}{\lambda^2}(-\lambda t - 1)\right]_0^1 \tag{B4}$$

$$= \lambda\left(\frac{-e^{-\lambda}}{\lambda} + \frac{1}{\lambda}\right) + \lambda\left(\frac{e^{-\lambda}}{\lambda^2}(\lambda + 1) - \frac{1}{\lambda^2}\right) \tag{B5}$$

$$= \frac{\lambda + e^{-\lambda} - 1}{\lambda} \tag{B6}$$

Appendix C

Proof of the convergence of the summation of remaining times

Assuming the contract period T is one year, we have

$$\sum_{k=1}^{nr} RT_k = \int_0^1 \left\{ \sum_{k=1}^{\infty} (1-t) f_{T_k}(t) \right\} dt \qquad \text{(C1)}$$

$$= \int_0^1 \left\{ \sum_{k=1}^{\infty} (1-t) \frac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!} \right\} dt \qquad \text{(C2)}$$

Set $k' = k-1$, and we have

$$\sum_{k=1}^{nr} RT_k = \int_0^1 \lambda(1-t) \left\{ \sum_{k'=0}^{\infty} \frac{(\lambda t)^{k'} e^{-\lambda t}}{k'!} \right\} dt \qquad \text{(C3)}$$

$$= \int_0^1 \lambda(1-t) dt = 0.5\lambda \qquad \text{(C4)}$$

Catastrophe Risk Mitigation: A Survey of Methods

by Lewis V. Augustine, ACAS, MAAA

INTRODUCTION

Until the late 1980's, insurers typically handled catastrophe risk through the purchase of a cat reinsurance treaty. Despite its low retention, cat losses were not expected to pierce this layer. In fact, from 1980 through 1988, aggregate industry cat losses averaged only $1.5 billion annually with a standard deviation of $0.7 billion. However, these statistics deteriorated immensely in the following years, due to Hurricanes Andrew, Hugo, and Iniki, the Loma Prieta and Northridge Earthquakes, and years of poor winter weather. Average annual cat losses in these years increased seven-fold to $9.8 billion. Even more shocking was the volatility around this average, with the standard deviation increasing to $7.4 billion[1].

Following Hurricane Andrew in 1992, the cat reinsurance market hardened, due to "payback" for the hurricane, insolvencies, and a general reluctance to write reinsurance at any price. Out of this capacity shortage emerged a host of products aimed at tapping new sources of capital to help insurers and reinsurers mitigate their cat risk. The capital markets with trillions of dollars invested in stocks, bonds, and real estate, seemed the likely candidates to lead this charge. In fact, the Chicago Board of Trade (CBOT) developed and began trading options and futures contracts based on ISO property losses in late 1992. Since that time, the following products have also emerged:

1. The Catastrophe Risk Exchange (CATEX)
2. PCS Cat Options
3. Contingent surplus notes / Act of God Bonds / Cat Equity Puts
4. Special purpose reinsurers

In this paper, I will analyze these "non-traditional" methods of reducing and/or transferring cat risk; "traditional" reinsurance mechanisms will also be examined. None of the reinsurance concepts are new. However, they may not have been viewed in light of cat mitigation in the past. With the property reinsurance market the softest in five years, it is essential to consider these traditional products whenever we evaluate any of the alternatives.

[1] These statistics are based on Property Claim Services (PCS) loss estimates. It should be recognized that I performed these calculations based on cats greater than $5 Million. Now, PCS only records cats greater than $25 Million.

TRADITIONAL

I. PER OCCURRENCE CAT EXCESS OF LOSS TREATY

Perhaps the most common form of reinsurance for handling cat risk is the per occurrence excess of loss cat treaty. These treaties typically apply after all other reinsurance, protecting the insurer's net line. They are usually split into five to seven layers, each with a retention, limit, and co-participation. Division into layers is done for several reasons. First, it allows reinsurers flexibility to participate on the layers of their choice. Some prefer the higher premiums associated with the lower layers. Other would rather sacrifice premium for the lower probability of loss in the upper layers.

Second, per program capacity is usually several times greater than per layer capacity.

Example
A reinsurer offers a maximum layer limit of $1 million and a maximum program limit of $5 million. An insurer looks to place a cat treaty of $100 million x $50 million, split into five equal layers of $20 million. Considering its maximum limits, the reinsurer can offer $1 million limits on each layer for a total of $5 million. If the program was not split into layers, the reinsurer could only offer $1 million in total limits (The program would be viewed as one layer).

Third, it allows the insurer more flexibility in establishing co-participation percentages by layer. This is similar to the first point above. An insurer may have different preferences for risk at various layers. Through the use of co-participation, this variability of risk appetite can be more easily satisfied.

In the years prior to Hurricane Andrew, cat treaty retentions were set at relatively low levels, such as $15 million - $25 million. When reinsurers realized the destruction that could be caused by cats, the markets tightened. Cat treaty retentions moved upwards toward $100M, rates increased, and cat capacity was difficult to obtain. Today, rates are softening, but not to the levels seen before Andrew.

From a reinsurer's standpoint, cat treaties are viewed as pure risk reinsurance. Neither the insurer nor the reinsurer expect to use the treaty, except possibly the first layer. Even then, only under remote circumstances. As such, the reinsurer should expect no payback for losses, if losses do occur.

A typical cat treaty covers one occurrence above the retention. If the contract contains an automatic reinstatement clause, the insurer must immediately pay a premium to reinstate the limit when the retention is breached. This provides coverage for a second occurrence in the reinsured layer. For this reason, they are usually viewed favorably by insurers. However, if the first cat occurs towards the end of the treaty period, reinstatement premium is a cost with little potential benefit. Reinstatement premiums can be proportional to the amount of limit used, the time remaining in the treaty period, or a combination of both.

II. QUOTA SHARE REINSURANCE

Quota share is one of the oldest forms of reinsurance and simplest to understand. Deals are transacted between the insurer and reinsurer directly or through a broker. In its purest form, the insurer agrees to cede X% of all premiums and losses to the reinsurer. The reinsurer will pay the cedant a ceding commission, which is loosely equal to the expense of writing and servicing the risk directly. The financial impacts of a simple quota share treaty can be seen in Appendix 1. If the direct expense ratio equals the ceding commission, the direct, ceded and net financial ratios will mirror each other.

Although it is possible to get an earnings enhancement with a quota share, it is an inefficient means to that end. However, it is an effective way to reduce the probable maximum loss [2] in a region, state, or country. A quota share treaty may be structured to function as a cat treaty. Suppose a company has the following underwriting expectations:

1997 direct accident year loss + ALAE ratio = 60%
1997 direct calendar year earned premium = $500 million
1997 direct expense ratio = 35%
1997 direct North Atlantic PML = $400 million

The 60% loss ratio only covers budgeted cat and non-cat losses.

Since the goal is to reduce the North Atlantic PML, a 25% quota share treaty for the North Atlantic only, having a 35% ceding commission and a 125% occurrence limit is purchased. In addition, there will be a loss corridor from 50% to 70% where the cedant is responsible for 100% of the losses. Since we expect to be within the corridor and, therefore, share underwriting results with the reinsurer below it, the treaty will mainly function as cat protection against a large event. To determine the amount of cat protection available, it is best to translate these treaty terms into those commonly found in a cat treaty.

We are expecting a 60% loss ratio for the accident year, which is in the middle of the corridor. The 10 points over this plan to the top of the corridor may be viewed as retention on the PML. For our plan this will be a $50 million retention on the $400 million PML. Above the $50 million, we can start ceding 25% of the PML. This is similar to co-participation, which is present in most cat treaties. In this case, we will have a 75% co-participation on the $350 million remaining loss. In cat treaty terminology, this is 25% part of 350 million x 50 million. The ceded portion of the PML would be 25% of $350 million or $87.5 million. As you can see, the net PML is reduced to $312.5 million.

Besides the PML protection, one other less obvious aspect of this treaty compared to a cat treaty is the relatively low price. In this example, we expect to pay a 15% margin or $18.75 million and receive an occurrence limit of $156.25 million. This is a 12% rate on line, which would be an attractive rate for a cat treaty with similar limits. In addition, there is usually room to cede at least part of a second occurrence with no associated reinstatement costs. On the other hand, there is usually an aggregate limit less than two times the occurrence limit.

A summary of some of the advantages of a quota share to an insurer is as follows:

[2] The probable maximum loss (hereafter referred to as "PML") is the maximum loss that will occur under normal circumstances. One example could be a large Homeowners fire loss, where the sprinkler system works to specifications. The home may be partially salvageable. This is in contrast to the maximum possible loss, which is the absolute worst loss that could occur.

1. PML reduction
2. Allows an insurer to grow in areas where cat risk is not fully known. Under this scenario, the insurer could purchase a quota share treaty the first year and reduce it in subsequent years as more is learned about the true risk in the area.
3. Immediate Statutory surplus relief equal to the amount of the ceding commission; minimal GAAP equity relief
4. Protection against non-cat losses

On the other hand, some of the disadvantages are as follows:

1. May be ceding a portion of our narrow direct Underwriting profit margin in a good year
2. If an insurer becomes too dependent on reinsurance, it will become costly when prices harden
3. Potentially, a false crutch for unmanaged, excessive growth
4. Should expect to pay back reinsurer in the long run
5. Credit risk of the reinsurer, especially during the most critical time -- following a large event
6. Giving away a small cash flow benefit

III. AGGREGATE EXCESS OF LOSS TREATY (XOL)

In the early 1990's, the NAIC and FASB began revising and implementing new regulations regarding reinsurance risk transfer. To qualify as reinsurance, a treaty must transfer underwriting and timing risk to the reinsurer; otherwise, no credit on losses can be taken and the transaction must be accounted for as a deposit. With these new regulations, finite risk reinsurance initially shrunk in popularity, but is growing again. It provides a good middle ground for insurers seeking a balance between reinsurance and straight financing

Aggregate excess of loss covers have been around for many years. One of the primary functions of a typical treaty is stabilizing current year earnings, while transferring a small amount of risk. If the company's goal is to achieve their accident year plan, it would purchase an aggregate XOL treaty that attached at the plan loss ratio or dipped down into the plan. Some insurers choose to accept a small amount of volatility in their plan and set the retention a few points above plan. In either case, the reinsurer provides a limit above the retention, which acts as a buffer against adverse results. Finite deals of this type are often characterized by one or more of the following features:

- Additional premiums based on a multiple of ceded losses
- Multi-year structure
- Sublimits
- Co-participation
- Funds Withheld accounting, limiting the actual cash flow to the margin paid

Essentially, the treaty provides acceleration of future investment income into the current period. In other words, we give up part of an uncertain future to lock in a benefit today. Although the reinsurer may incur losses soon after the treaty period begins, the reinsurer will not begin paying losses until direct paid losses exceed the insurer's retention. Therefore, the reinsurer often sacrifices current period accounting results for an economic gain.

Appendix 2 shows an example of the accounting and cash flow of an accident year aggregate excess of loss treaty attaching four points above plan. In this example, the incurred loss ratio ends up seven points above plan. All losses are incurred in the 1997 calendar year and there is no adverse or favorable development. Direct losses are paid equally over a ten year period. The investment income given up is roughly equal to the "Funds Withheld Investment Credit". These investment income amounts are cumulative. As you can see, the volatility of the accident year loss ratio is mitigated. The 85% direct loss ratio is reduced to 80.5% on a net basis. The reinsurer, on the other hand, suffers a 1997 loss ratio in excess of 200%.

One investment income benefit is not shown here. By ceding premium to the reinsurer, the insurer can release the surplus supporting this premium and use it for general business purposes. These opportunities may provide greater returns than the narrowly defined investments of surplus as stated by statutory guidelines.

Cat risk is one of the major threats to the reinsurer's economic gain under an aggregate XOL. Under expected circumstances the reinsurer will pay nothing for ceded losses since the plan will be achieved. If there is adverse development due to poor Worker's Compensation or other long-tailed lines, the reinsurer will book an incurred loss, but the payments to the insurer will not begin for several years. When a cat occurs, the reinsurer becomes more exposed to timing risk. Cats are usually substantially paid within a few months of occurrence. This can significantly shorten the duration of the liability stream, leading not only to an accounting gain for the insurer, but also possibly an economic gain.

From the insurer's perspective, an aggregate XOL treaty is a good way to accomplish the dual result of locking in current period profits while securitizing cat risk. Because of the timing risk cats present to the reinsurer, these treaties often have a sublimit capping the amount of cat losses subject to the treaty. However, for a large, diversified book the reinsurer would be more willing to set the sublimit fairly high.

The following shows some of the advantages and disadvantages of an aggregate excess of loss treaty:

<u>Advantages</u>

1. Current period income stability
2. Cat protection
3. Surplus protection
4. Favorable stock analyst response, possibly leading to "buy" recommendations
5. Structure passes reinsurance accounting guidelines on a conceptual basis
6. Should be favorable to rating agencies

<u>Disadvantages</u>

1. Giving up future investment income for present underwriting income
2. Cat losses are paid quickly on the direct side, but may not be reimbursed on a paid basis for many years (i. e. no cash flow benefit)
3. Specific features dictated by the market may cause failure of risk transfer tests
4. Credit risk, compounded by the long reserve tail
5. Could have large income and surplus hits if commuted early
6. Accident year help only; no coverage for prior years' reserve strengthening
7. Difficult to administer

IV. THE CATASTROPHE RISK EXCHANGE (CATEX)

CATEX became operational on October 1, 1996. It is a facility where insurers, reinsurers, and brokers can buy, sell, and trade insured risks. Subscribers to the exchange anonymously post potential deals on a highly secure CATEX E-mail system. Other parties do not have access to the system. CATEX is completely neutral to the deal. However, they help facilitate deals through providing standardized contracts and even arranging collateral if necessary.

Charter: CATEX is a for profit entity, licensed as a neutral reinsurance intermediary by the New York Insurance Department. The Department has the right to oversee and examine them in accordance with regulations.

Potential Members: Any insurer, reinsurer, or broker licensed or approved in New York. Unlicensed companies can also trade on the Exchange through a licensed intermediary.

Idea: CATEX was originally conceived as a facility for diversifying one's book of business. For example, a company heavily concentrated in Florida Hurricane could trade some of this exposure through CATEX to another insurer for Vermont Freeze. There are thousands of other possibilities. Recognizing that some of these exposures are not equivalent, the New York Insurance Department allows cash as part of the deal.

Interest in the original Exchange was not great, delaying the opening of it by over a year. In 1996, however, the New York Insurance Department approved cash only transactions (effectively reinsurance) on CATEX making the Exchange a lot more popular. Some well known companies are now part of the Exchange, including Travelers/Aetna, USF & G, Gerling Global, Employers Re, Everest Re, Lloyds of London. In addition, many of the major reinsurance brokers and all the Lloyds syndicates are members.

The Trade - An Example:

- Company A posts $10 million of insured values subject to Florida hurricane it wants to trade away
- Company A remains anonymous
- Company B has a large exposure to Kobi earthquake
- Florida Hurricane is 15 times riskier than Kobi earthquake
- Company B decides it is interested in beginning a negotiation at which point both parties mutually agree to reveal their identities
- After reviewing their book, B decides it can take on this Florida exposure, but insists on a co-participation and $1 million cash
- A will agree to a 25% co-part., but no cash; in addition, they insist on a riskiness relativity of 10
- B finds this acceptable and the deal is completed
- CATEX runs a computer program to randomly generate $10 million of insured values in Florida and $100 million of insured values in Kobi from the two books of business; this will minimize the risk of adverse selection

Accounting: transactions are recorded according to Statutory reinsurance accounting procedures. An imputed premium is agreed upon by the two parties, which will be the ceded and assumed premium for both parties. Losses are reduced for recoveries in the same manner as reinsurance. If hurricane losses are $40,000 and earthquake losses are $10,000, the accounting would look like the following for the two companies:

	Company A	Company B
Earned Premium - HO	($100,000)	$100,000
Earned Premium - EQ	100,000	(100,000)
Incurred Loss & LAE - HO	10,000	30,000
Incurred Loss & LAE - EQ	10,000	0
Commissions	16,500	16,500

Premium: An imputed premium of $100,000 was agreed upon by the two parties. Company A cedes $100,000 of hurricane premium and assumes an equal amount of earthquake premium.

Losses: Net losses are shown above. Company A incurs $40,000 of direct hurricane losses. They have a 25% co-participation, so they retain $10,000 and cede $30,000. Company B incurs $10,000 of direct earthquake losses. They have no co-participation, so they cede the entire $10,000 to A.

Commissions: In swap deals, CATEX charges $150 per $1 million in insured values traded, to each party. In the transaction above, there are $110 million of insured values, so the commission expense is $16,500 for each party. For cash deals, 75 basis points of the cash premium is charged. This is comparable to a reinsurer's brokerage fee.

Loss Occurrence: Following a cat occurrence, as defined by PCS or AM Best, CATEX will determine if the loss pierces the layer. If it does, both parties will be notified. Like reinsurance, the ceding company determines proximate cause, pays and settles all losses. The cat remains open for 18 months following occurrence. Once the cat is paid, proof of payment is presented to the assuming company, which will then reimburse the cedant.

Advantages

1. Geographical diversification of the portfolio
2. Diversification of perils
3. Greater diversification leads to greater spread of risk, creating capacity
4. Alternative sources of reinsurance
5. Benefits flow through underwriting income
6. State of the art P/C provided with internet capabilities
7. May be able to package trades into an asset-backed security to tap financial markets
8. PML reduction

Disadvantages

1. Under swaps, risk is not transferred; it is traded for an equal amount of risk
2. Need a dedicated phone line to realize full capability of internet
3. Although the ceding company determines the loss, disputes are bound to occur; how will they be settled?
4. Need more participation from major insurers to create liquidity

NON-TRADITIONAL

V. PROPERTY CLAIMS SERVICES (PCS) CATASTROPHE OPTIONS

PCS Cat Options grew out of ISO Cat Futures, which were first offered in December, 1992. PCS Cat Options were originally conceived as a way to tap into the trillions of dollars available in the financial markets. The standardized contracts are traded on the Chicago Board of Trade (CBOT), which guarantees their financial integrity. There has never been a default on the exchange. There are nine PCS industry loss indices tracked: National, Eastern, Northeastern, Southeastern, Midwestern, Western, Florida, Texas, and California.

On each index, two different sized contracts are traded. The small cap contract tracks industry cat losses between $0 and $20 billion. These are appropriate for hedging against high frequency cats, such as hail and tornadoes. The other contract is for high severity losses, those ranging from $20 billion to $50 billion. A company purchases PCS Cat Options as a hedge against direct cat losses.

Accounting period: the indices track cats occurring either in an accident quarter or accident year. These were developed to get at the seasonal nature of cats. Since hurricanes usually only occur in the third calendar quarter, a Florida accident quarter contract could be purchased. For California, on the other hand, only accident year contracts are offered, since earthquakes are not seasonal. In addition to length of contract, the parties to the contract must decide on a development period, which runs either six-months or twelve-months after the end of the coverage period.

Index valuation: the index value equals the industry cat losses during the loss period divided by $100 million. Quotes are in the following format: ###.# and each point is worth $200. Reported losses within the contract period and developed through the development period enter the index.

How can a company use options? One obvious function is for buying a layer of reinsurance. This is accomplished by buying an Option Call Spread. A Call is purchased because the buyer wants to lock in a price for losses in the event that the loss index increases. An Option Call Spread is done by buying a Call Option at the retention and simultaneously selling a Call Option at the (limit + retention). These points on the index are referred to as strike prices.

Example - Perfect Hedge
We want to hedge against California Earthquake. We have a 1% market share and would have an equivalent share of all losses. We have a cat treaty starting at $50 million, but would like to purchase protection below it, between $30 million and $50 million. We must answer the following questions:

- What are the industry strike prices?
- How many options should we purchase to be perfectly hedged?
- How much should we pay?

Let's answer each question. To determine the strike prices, we must calculate the industry limits corresponding to the layer we desire to purchase. Since we are 1% of the industry, these amounts are as follows:

Retention = \$30 million / 1% = \$3 billion; Strike price = \$3 billion / \$100 million = 30
Limit + Retention = \$50 million / 1% = \$5 billion; Strike price = \$5 billion / \$100 million = 50

This shows that we must purchase 30/50 call options, to provide coverage for industry losses between \$3 billion and \$5 billion. Each contract will provide a \$4,000 (20 points x \$200 / point) vertical strip of protection in this layer, but how many of these strips will we need? We want \$20 million in overall protection and each option provides \$4,000. Therefore we will need to purchase 5,000 30/50 call options to be perfectly protected in the layer.

What will this cost? The premium is a negotiated item. Since insurers and reinsurers are the primary participants, the pricing has thus far followed reinsurance rates. As supply of capital from financial markets increases, prices may decline from these levels.

For the Option Call Spread we just purchased there are three possible loss outcomes:

1. The index ends up < 30 - the spread expires worthless and the purchaser only loses the premium paid for it.
2. The index ends up > 50 - the purchaser realizes a gain of 20 points on each contract. The total gain will be 20 points x \$200 per point x 5,000 contracts = \$20 million, less the premium paid.
3. The index ends up between 30 and 50, say at 40. The total gain will be (40 - 30) points x \$200 per point x 5,000 contracts = \$10 million, less the premium paid.

Unlike typical option contracts, PCS cat options can only be exercised at expiration. Example one above expires worthless, while two and three are "in the money". This is one possible structure of a PCS Option. There are many others.

The greatest risk facing insurers buying Option Call Spreads is <u>basis risk</u>. An imperfect hedge can result if:

- The company experiences a large cat loss, but the industry does not
- The industry experiences a large cat loss, but the company does not

In these cases the recovery from the contracts will be less than and greater than the needed recovery, respectively.

Anyone opening an account is eligible to buy and sell options. To date, however, there has not been much trading activity in Cat Options. Most of the participants have been members of the insurance industry. One encouraging statistic is over 3,000 contracts were traded on September 5, 1996, providing \$6.6 million in limits. On the other hand, this amount equaled the prior quarter's <u>total</u> activity. Lack of appeal is due in part to the fact that results flow through investment income, not underwriting income, as is the case with nearly all of the capital markets solutions. From an economic perspective, Cat Options offer the same benefits as reinsurance. Rather than go through advantages and disadvantages of Options, it is instructive to compare and contrast them to reinsurance:

Cat Options	Reinsurance
Standardized contracts	Customized contracts
Reimbursed for incurred losses	Reimbursed for paid losses
6 to 12 month tail	Indefinite tail
No implied payback	May have implied payback
Basis risk	Perfect hedge
No credit risk	Credit risk
Limited Market	Large, international market
Flows through investment income	Flows through underwriting income
Large potential capital supply	Limited capital supply
No coverage disputes	Disputes/Arbitration part of the business
Anyone can become a "reinsurer"	May need approval for accounting advantages
Real risk transfer	Real risk transfer
Industry loss trigger	Company loss trigger

VI. CONTINGENT SURPLUS NOTES / "ACT OF GOD" BONDS / CAT EQUITY PUTS

Although these products come in many forms, they have one overriding purpose: to protect the company's surplus in the event of a catastrophe. Usually investment banks or brokers arrange their placement. Each of the products will be discussed followed by their common advantages and disadvantages.

A. Contingent Surplus Notes

The most well-known deal (and only one as of 8/96) was done by Nationwide. In early 1996, Nationwide determined that they needed a pool of funds to draw upon in case surplus was threatened. The product acquired the name "Contingent" because surplus notes were not issued immediately. There was the possibility of issuing them sometime in the future. Cat risk was the most important risk Nationwide was guarding their surplus against, but not the only one. There is no direct link between occurrence of a catastrophe and issuance of the Notes.

The deal works as follows:

- Nationwide Mutual establishes Nationwide Trust subsidiary
- The Trust sells corporate bonds to investors worth $400 million; coupons = Treasury + 240 basis points
- With the proceeds, the Trust purchases US Treasuries, that act as collateral for the bonds

At this juncture, Nationwide conducts business as usual. At some point in time, they could exercise their option to issue surplus notes. The transactions would be:

- Nationwide Mutual issues surplus notes to Nationwide Trust
- The Treasuries are sold to purchase the surplus notes
- The surplus notes replace the Treasuries as collateral on the corporate bonds
- Investors are still owed full principal; coupon rate remains unchanged

The costs to Nationwide are two-fold. First, they are paying a 240 basis point premium over the Treasuries they have purchased as collateral. Second, if they draw upon the capital by liquidating the Treasuries and interest rates have risen, they face a loss on the face value of the

Treasuries. Investors face the credit risk of replacing (risk-free) Treasuries with Nationwide surplus notes.

As a final point, all principal and interest payments to note holders require approval from the domiciliary commissioner before they are paid. This is a way that the commissioner will be sure that certain obligations are taken care of before the notes are paid. These obligations may include payments to policyholders after a large event.

B. "Act of God" Bonds

Unlike Contingent Surplus Notes, there is a direct relationship between occurrence of a cat and repayment of the bond. These deals are a little more common and typically work like this:

- Alpha Insurance Company issues five-year bonds to investors at a coupon rate above treasuries
- The coupons are guaranteed for a fixed amount of time, say three years
- If no cats occur, Alpha pays investors five annual coupons as well as the principal at the end of five years
- If a cat occurs and losses reach the coverage trigger a number of things could occur, depending on the wording of the deal:

    1. Reduced coupon payments following the guarantee period
    2. Reduced principal payments
    3. Risk of loss to principal and interest

As would be expected, the more the investor puts at risk, the greater the return over the Treasury rate. In deals where principal is guaranteed, a portion of the proceeds is invested in Treasuries that will mature to the face value of the bonds. In another actual deal where coupons and part of the principal were put at risk, the investor received 1,000 basis points over Treasuries.

C. Cat Equity Puts (CatEPuts)

A unique sort of cat financing product was developed by AON, a well-known insurance and reinsurance intermediary. The first deal involved Centre Re of New York and RLI Corporation of Illinois in the latter half of 1996. RLI had suffered major losses from the Northridge Earthquake in January of 1994 and sought traditional and non-traditional solutions in case a similar event happened in the future. They ended up with the following deal:

- Centre Re sells a Put option to RLI for three years
- The option allows RLI to put $50 million of non-voting RLI preferred stock to Centre Re in the event of a California earthquake
- RLI pays Centre Re $1 million per year for the Put option, for a total of $3 million

Note the specific coverage trigger, unlike Contingent Surplus Notes. This limit sits on top of all existing cat coverage. Relating the cost to reinsurance produces an annual rate on line of $1 million / $50 million = 2%. However, this is too simplistic a view. With reinsurance, the reinsurer provides capital in the event of a loss and the deal is done. This is an exchange of uncertainty for certainty. With CatEPuts, the "reinsurer" provides capital and could obtain an equity stake in the insurer in return. This is an exchange of uncertainty for equity. The equity is in the form of convertible preferred stock. Half of the stock is convertible to common stock three years after the event and the other half in four years. Unlike the preferred stock, common stock has voting rights.

Under GAAP accounting, CatEPuts are considered a part of surplus, not a liability like debt would be.

There are two contractual features worth noting. First, RLI has three to four years to buy back the shares at market rates and avoid giving up the equity stake in the company. It was acknowledged that Centre Re does not want to become a shareholder in RLI. Second, If the loss were so large as to cause surplus to fall below a threshold, the deal would be null and void.

AON is working on similar deals ranging in size from $100 million to $500 million.

The following lists show the advantages and disadvantages for the three products:

Advantages

1. Surplus protection
2. Lack of correlation with stock and bond markets
3. No basis risk - you get what you pay for
4. Possibly tap into alternate sources of capital within the insurance industry, namely life insurers and pension funds
5. Surplus notes are accounted for as equity, but are treated like debt for tax purpose, since their interest is tax deductible
6. Easier to construct multi-year deals than reinsurance
7. A. M. Best has promoted CatEPuts as "...an effective way to secure extra cat coverage"
8. Could be effective second event products
9. No reinstatement costs

Disadvantages

1. Liquidity risk, as evidenced by the failed USAA deal in the summer of 1996
2. Education - investors know about asset risks, but how many understand cat risk? Adverse selection may result
3. Cat risk may not be something an investor wants to have in his/her portfolio, especially with a limited upside in exchange for possible loss of principal and interest
4. These products are virtually junk bonds, subordinated to policyholder, stockholder, and debt-holder obligations
5. Credit risk
6. Results do not flow through underwriting income

VII. SPECIAL PURPOSE REINSURERS / SECURITIZATION

Special purpose reinsurers are established to provide reinsurance to one client. Often they are formed in places like Bermuda to take advantage of favorable regulation and to keep the transaction off the parent company's books. One deal completed towards the end of 1996 was done by Goldman Sachs for St. Paul Reinsurance. The deal works as follows:

- St. Paul Re establishes George Town Re
- George Town Re issues two types of securities to investors:

    1. Notes maturing in ten years - $44.5 million
    2. Preference shares maturing in three years - $24 million

- George Town Re becomes a quota share retrocessionaire for St. Paul Re under a ten-year reinsurance treaty
- George Town Re invests $23.2 million of the Notes in zero-coupon bonds to provide collateral for the Note principal maturing in ten years
- The rest of the proceeds ($45.3 million) will be used as collateral for reinsuring St. Paul Re

Please see Appendix 3 for a graphical portrayal of this transaction.

One of the unique features of this deal is the multiple tranche structure. The Notes are highly rated by S & P and Moody's, while the Preference Shares are unrated. The Notes provide a highly securitized principal because they are collateralized. However, interest payments are contingent on the reinsurance results. The Preference Shares, on the other hand, have no associated collateral. Therefore, not only is the interest at risk, so is the principal.

To mitigate the investment risk transferred to investors, the business reinsured is a diversified portfolio of low-frequency, high-severity reinsurance business. There are also sublimits on the different classes of business assumed by George Town Re, similar to finite risk reinsurance.

The initial transaction between the insurer and the special purpose reinsurer is considered reinsurance, assuming the risk transfer tests (FAS 113, Chapter 22) are passed. However, the deal between the reinsurer or trust fund and the bondholders shall not be construed as insurance or reinsurance. This portion is fully subject to investment laws.

Advantages

1. Keeps financing transactions off parent's books
2. Varying levels of risk offered by multiple tranches may attract a wider audience of investors
3. Company specific trigger, not industry
4. No basis risk
5. Less regulation with offshore reinsurer
6. Increased reinsurance capacity for St. Paul Re
7. Locks in pricing for a number of years
8. Benefits flow through underwriting income

Disadvantages

1. Both securities offer a large amount of risk; the reward is not specified
2. Liquidity risk
3. Credit risk to investors
4. Structure is untested thus far, since there have been no major catastrophes
5. Not much feedback from regulators

VIII. MISCELLANEOUS

In addition to the items listed above, a few other forms of securitization should be mentioned:

1. Bermuda Cat Reinsurers - these reinsurers arose in the wake of Hurricane Andrew as another source of cat reinsurance capacity. Many were formed through investment banks, such as J. P. Morgan and GE Capital. These markets offered no cat capacity as of 1989. However, they currently provide 36% of the total cat capacity in the reinsurance markets. Much of this was reallocated away from the London and domestic reinsurance markets. Some characteristics of these reinsurers are:

- Write property reinsurance only
- Use many of the industry cat models to evaluate risk
- Generally reinsure limits up to their capital and surplus level; this results in Premium: Surplus levels less than 50%

2. Lines of Credit - credit lines are one of the oldest capital sources. An insurer or reinsurer, based on its credit rating, pays a bank a percentage of the credit line to allow it to draw upon under a variety of circumstances.

Insurers could use lines of credit as a bridge loan following a catastrophe. Since cats present a tremendous timing risk to insurers, cash flow may not be available when a cat hits. However, it may be known that earnings throughout the year will be sufficient to pay for the cat. If the insurer had purchased a line of credit, it could draw down the funds necessary to pay for the cat. The cost to the insurer will be the initial fee and the interest accrued when paying back the line of credit. Since some of the companies will be able to pay this back in under a year, the latter cost should be minimal.

CONCLUSION

This paper has surveyed some of the core products on the market today geared to mitigate cat risk. There are numerous other products that retain some of the major features of one or more of the items listed above, but are tailored for individual customers. Reinsurance and alternative products share the characteristic that each contract is unique.

Reinsurance continues to be the primary means of handling cat risk. However, the new products are showing up more and more in the insurance periodicals as companies use them for deals. I believe education is the key to unlocking some of the capital routinely being invested in the financial markets. Not many people outside of the insurance industry truly understand insurance, let alone insurance contracts. This problem is exacerbated when we start talking about specifics, such as catastrophes, paid versus incurred losses, and reinsurance. Like anything new, there will be a learning curve. Once more people begin looking into these new forms of "reinsurance" and understanding them, I believe they will become more common, leading to greater liquidity and competitive pricing.

APPENDIX 1

Accounting for a Quota Share Treaty

Assumptions

Direct Premium = $1000	Direct PML = $200
Direct L/R = 60%	Quota Share = 25%
Direct E/R = 35%	Ceding Commission = 35%

	Direct	Ceded	Net
Premium	1000	250	750
Losses	600	150	450
PML	200	50	150
Expenses	350	87.5	262.5
U/W Margin	50	12.5	37.5
L/R	60%	60%	60%
Combined Ratio	95%	95%	95%

This transaction shows a year-end $12.5 decrease in Statutory and GAAP earnings, due to ceding profitable business. At intermediate points during the year, GAAP earnings will be better.

There is immediate Statutory surplus relief in a quota share transaction. This stems from the fact that we cede an unearned premium reserve (liability) and an equal amount of cash (asset). However, we also receive a ceding commission (cash), so Statutory surplus is increased by this amount. This benefit goes away under GAAP, since we are ceding DAE (asset) equal to the ceding commission.

APPENDIX 2

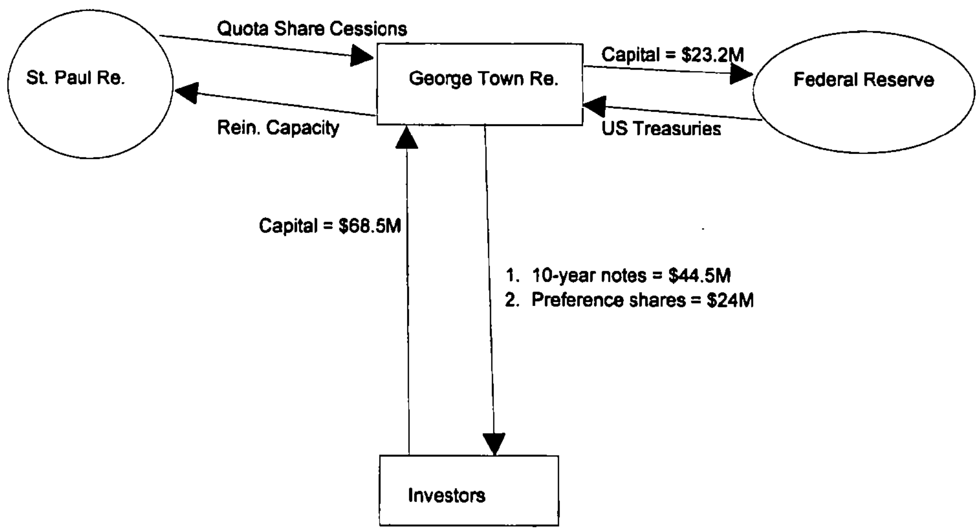
AGGREGATE EXCESS OF LOSS EXAMPLE - ACCOUNTING AND CASHFLOW

Subject Premium	5,600,000,000	
Plan Loss Ratio	74%	
Retention	78%	4,368,000,000
Aggregate Limit	500,000,000	
Leverage Factor	2.25	

Direct	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
Earned Premium	5,600,000,000									
Incurred Loss Ratio	**85%**									
Incurred Losses	4,760,000,000									
Paid Losses	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000
Cume Paid Losses	476,000,000	952,000,000	1,428,000,000	1,904,000,000	2,380,000,000	2,856,000,000	3,332,000,000	3,808,000,000	4,284,000,000	4,760,000,000
Cashflow	5,124,000,000	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)
Ceded										
Earned Premium	174,222,222									
Incurred Losses	392,000,000									
Paid Losses	-	-	-	-	-	-	-	-	-	392,000,000
Margin	10,000,000									
Funds Withheld Inv. Credit	0	12,316,667	25,557,083	39,790,531	55,091,488	71,540,016	89,222,184	108,230,514	128,664,470	150,630,971
Funds Withheld	164,222,222	176,538,889	189,779,306	204,012,753	219,313,710	235,762,238	253,444,406	272,452,737	292,886,692	(77,146,806)
Cashflow	10,000,000	-	-	-	-	-	-	-	-	(77,146,806)
Net										
Earned Premium	5,425,777,778									
Incurred Losses	4,368,000,000									
Incurred Loss Ratio	**80.5%**									
Paid Losses	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	476,000,000	84,000,000
Cume Paid Losses	476,000,000	952,000,000	1,428,000,000	1,904,000,000	2,380,000,000	2,856,000,000	3,332,000,000	3,808,000,000	4,284,000,000	4,368,000,000
Cashflow	5,114,000,000	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(476,000,000)	(398,853,194)

APPENDIX 3

SPECIAL PURPOSE REINSURER



Workers' Compensation D-Ratios, An Alternative Method of Estimation

by Howard C. Mahler, FCAS, MAAA

WORKERS' COMPENSATION D-RATIOS, AN ALTERNATIVE METHOD OF ESTIMATION

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Abstract

This paper presents a new method of estimating D-Ratios by class based on estimated average claim costs by class, that is being used in Massachusetts Workers' Compensation.

WORKERS' COMPENSATION D-RATIOS, AN ALTERNATIVE METHOD OF ESTIMATION

This paper will present a new method of estimating D-Ratios by class that is being used for Massachusetts Workers' Compensation.[1] This method based on average claim cost is compared to the prior method in Table 8.

Background

In Workers' Compensation Experience Rating claims are generally split into a Primary and Excess portion. In Massachusetts and most other states, the portion of each claim below $5,000 is Primary. The portion above $5,000 is Excess, but all the dollars above a certain limit (which is currently $175,000 in Massachusetts) are excluded from Experience Rating.

The D-Ratio (Discount Ratio) is defined as the ratio of the future Expected Primary Losses to the Expected Primary plus Excess Losses.[2] A separate D-Ratio for each classification in each state is needed. For Massachusetts the D-Ratios are generally between 10% and 30%.[3]

The effect on the Experience Modification of a difference in D-Ratios is discussed in the Appendix. All other things being equal the higher the D-Ratio the lower the Experience

[1] This method turns out to be similar to one presented by Arthur Bailey [1].

[2] In Workers' Compensation experience rating Expected Losses are obtained by multiplying payrolls by class times the corresponding Expected Loss Rates by class. Then for each class the Expected Primary Losses are the product of the Expected Losses times the D-Ratio for that class.

[3] In a state with lower average claim costs but using the same $5,000 dividing point, the percentage of primary losses would be higher and thus the D-Ratios would be higher.

Modification.[4] In order to get an accurate Experience Modification one desires the best estimate of D-Ratios.[5]

There are two basic problems in estimating D-Ratios. First, we are interested in the expected future value. Therefore, we need to adjust the past data to reflect future conditions. This is relatively routine and involves the usual severity trend and on-level factors for law amendments used elsewhere in ratemaking.[6] An example is shown in Table 1. Note that the factors in Table 1 adjust the data[7] available at the time of the rate indication to the expected level of the data that will be used to experience rate insureds during the policy effective period.[8]

Overview of Methodology

This paper will focus on the second and more difficult problem. The volume of data by class in a state is insufficient in most cases to allow a good estimate of the D-Ratio directly from the data for that class.

However, one can work with the larger groupings.[9] Currently, there are five Industry Groups generally used for Workers' Compensation for ratemaking: Manufacturing, Contracting,

[4] A .10 higher D-Ratio will result in a .04 to .07 lower Experience Modification, as discussed in the Appendix.

[5] As well as the best estimate of other inputs such as Expected Loss Rates, credibilities, etc.

[6] See for example Kallop [2] or Feldblum [3].

[7] Unit Statistical Plan data is usually compiled into a report called Schedule Z.

[8] Generally one would use three years of data to experience rate insureds. For example, during 1996 one would generally use 1994 at first report, 1993 at second report, and 1992 at third report. At the time one was estimating D-Ratios for 1996, one might have available 1992 at first report, 1991 at second report and 1990 at third report. In that case one would adjust the 1992 data at first report to level expected for the 1994 data at first report, etc.

[9] Hazard Groups were tried, but the use of Industry Groups did a better job of estimating D-Ratios. A major problem is that over 90% of the experience is concentrated in Hazard Groups 2 and 3.

Goods and Services, Office and Clerical, and Miscellaneous. In Massachusetts (and states with a similar or larger volume of data[10]) each Industry Group has a sufficient amount of data to estimate its D-Ratio directly from the data. (See Table 2.)

The Construction Industry Group stands out from the other four as having a very significantly lower D-Ratio.[11] Thus this breakdown splits out many of the classes with the lowest D-Ratios. Also, as will be seen, much of the remaining variation within Industry Group can be captured via relative average claim costs by class.

The methodology consists of estimating the D-Ratio of each class relative to the D-Ratio of its Industry Group. (These estimated relativities will be balanced to unity.) This relative D-Ratio for each class will in turn be estimated from the relative average claim cost for that class. Classes with higher than average severities will be estimated to have lower than average D-Ratios. In other words, if the average claim size is larger, more of the claim is excess and less is primary.

Estimated Relative Average Claim Costs by Class

The estimated Relative Average Claim Costs by class are calculated based on the most recent seven years of Unit Statistical Plan data at second report.[12] Average Claim Costs are calculated based on data excluding fatal, permanent total, and medical-only claims, as was used

[10] In states with very small amounts of data one could calculate a statewide D-Ratio and spread it to Industry Group based on the relativities over a longer period of time or in other states.

[11] More large claims apparently lead to a smaller percent of primary losses.

[12] Second report is the approximate average maturity of data used for experience rating. Unit Statistical Plan Data is submitted on every individual claim of size $2,000 or more. Evaluations are currently on a paid plus case reserve basis at the first five reports. First report is 18 months from policy inception. Subsequent reports are at 12 month intervals.

in the development of the model discussed subsequently. Table 3 is an example for the Office and Clerical Industry Group for Composite Policy Year 91/92.[13]

For each year, for each Industry Group, the Relative Average Claim Cost for a class is the ratio of the Class Average Claim Cost to the Industry Group Average Claim Cost. Figure 1 shows the results for two classes in the Office and Clerical Industry Group. For each class, the seven years of Observed Relative Average Claim Costs are combined by taking a weighted average using claim counts as weights. (See Table 4.)

However, there are only limited data for smaller classes. Therefore, Credibility has been used to combine the Observed Relative Average Claim Cost by class with unity. (Unity corresponds to the Industry Group average.) Credibility is taken equal to:

$$Z = \sqrt{\frac{\text{number of claims}}{2{,}500}}$$

A class with 2,500 or more claims is assigned a credibility of 1. The classical full credibility criterion of 2,500 claims for severity was selected based on adjusting a criterion for frequency of about 1,000 claims by multiplying by the square of the coefficient of variation of about 2.5.[14] The results herein are relatively insensitive to the precise choice of the full credibility criterion.[15] While a more "sophisticated" credibility method might have been employed, in the author's opinion classical credibility is more than adequate for this particular

[13] Composite Policy Year 91/92 includes all experience on policies with effective dates between 7/1/91 and 6/30/92.

[14] See Longley-Cook [4]. 1082 and 683 are common criterion for full credibility for frequency mentioned by Longley-Cook. The Appendix of Longley-Cook's paper recommends multiplying by the square of the coefficient of variation to get a criterion for average claim costs. The observed square of the coefficient of variation for the severity for permanent partial and temporary total claims is about 2.5. The square of the coefficient of variation = variance/mean^2.

[15] For a discussion of this subject see Mahler [5].

application.[16] The range of estimated D-Ratios is so small that minor changes in the estimated relative claim costs have relatively little final impact.[17]

For example, the relative average claim costs by year for class 4361 are shown in Table 4. There is sufficient fluctuation from year to year that any reasonable credibility method would assign significantly less than full weight to this data. For example, suppose instead of 35.9% credibility, 20% credibility were assigned. The relative average claim cost would be .957 rather than .923. The estimated relative D-Ratio would be 1.029 rather than 1.051. The resulting estimated D-Ratio would be .24 rather than .25 as shown in Table 7. This difference is well within the inherent error of the whole estimation procedure.

The relative average claim cost is estimated for each class as seen in Column 11 of Table 4:

Estimated Relative Average Claim Cost = 1 + Z (Observed Relative Average Claim Cost - 1)

These estimated Relative Average Claim Costs[18] are then used in the model, that will be described next, in order to derive estimated Relative D-Ratios.[19]

Model of Average Claim Cost vs. D-Ratio

As seen in Column 12 of Table 4, within industry groups, the overall average D-Ratio is spread to each classification using the following model:

---

[16] For a comparison of the practical impact of using classical credibility versus Bayesian/Bühlmann credibility see Mahler [5]. Mahler [6] discusses the use of different criteria to select optimal credibilities. Mahler [6] and Mahler [7] discuss the possible impact of shifting parameters over time. Taking into account the impact of shifting parameters over time here is a possible area of future research.

[17] See the Appendix.

[18] See Column (11) of Table 4.

[19] See Column (12) of Table 4.

$$(\text{Relative D-Ratio} - 1) = (-2/3)\ (\text{Relative Average Claim Cost} - 1)$$

The form of the model is based on the fact that larger claims contribute a smaller percentage to primary losses than do smaller claims. For example, a $3,000 claim has 100% of its losses as primary, while a $100,000 claim has 5/100 = 5% of its losses as primary. Thus classes with higher than average claim sizes will be expected to have a smaller percent of their losses as Primary, and therefore, have lower than average D-Ratios.

The particular coefficient used in the model was selected in Table 5, based on an examination of the historical relationship between average claim costs and D-Ratios.[20] Separately for each Industry Group weighted least squares regressions were performed on Relative Average Claim Costs and Relative D-Ratios by class. Table 6, Page 1 shows the Office and Clerical Industry Group.[21]

The most recent Unit Statistical Plan data (1st, 2nd, and 3rd report combined) by class is used (without adjustment for law amendment or trend). An Observed D-Ratio is calculated in Column 4 of Table 6 for each class as the ratio of Losses Limited to $5,000 to Losses Limited to $175,000. The Relative D-Ratio in Column 5 of Table 6 for each class is the class D-Ratio divided by the average for the Industry Group.

As was done previously, the Average Claim Cost by class is calculated for other than fatal, permanent total, and medical-only claims. The fatal and permanent total claims are rare

[20] In some sense the proportionality constant is a second use of credibility. The proportionality constant measures how much of a deviation from average one would expect in D-Ratio based on a certain deviation from the average severity.

[21] Table 6, Pages 2 and 3 shows the similar calculation for the Construction Industry Group.

and usually very large, and therefore would introduce undesirable random fluctuations.[22] The medical-only claims are very numerous but due to their very small size, account for a very small percent of total losses.[23] Based on the author's attempts to devise a method, apparently the medical only claims mask the important differences between classes which would be expected to lead to different D-Ratios.

Potentially valuable information has been "thrown away" in the calculation of the relative average claim costs by excluding fatal, permanent total and medical-only claims. However, the resulting relative average claim costs by class showed a strong correlation with the relative D-Ratios[24] by class. As in any actuarial computation, it would be possible to devise some way to incorporate this additional information in some manner to some extent. This is an area of potential future research, although given the small range of D-Ratios it is unlikely in the author's opinion to have much practical impact. There is some advantage to simple practical methods that work, without unnecessary technical refinements of no practical importance to the particular application.

The Relative Average Claim Cost by class in Column 9 in Table 6 is the class Average Claim Cost in Column 8 divided by the Industry Group Average Claim Cost. For purposes of the regression, the Relative Average Claim Cost by class is constrained to be between 0.5 and 2.0.

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[22] An alternative would have been to include fatal and permanent total claims, but to cap their size as is done for purposes of experience rating. In that case, the standard for full credibility of the observed relative average claim cost would be adjusted upwards.

[23] The medical onlys usually account for a significant proportion of primary losses.

[24] Which include the impact of claims of all injury kinds.

This prevents a small class with an extreme observed average claim cost over these three years, from unduly influencing the regression results.[25]

The weights used in the regression are the number of claims by class in Column 7. Then as stated previously, a weighted least squares regression between Relative Average Claim Costs and Relative D-Ratios by class is performed separately for each Industry Group. Figure 2 shows the regression for the Office and Clerical Industry Group.

These regressions yield five different estimates of an appropriate proportionality constant to be used in the model. As shown in Table 5, a single proportionality constant is selected within the indicated range.[26] The choice of a single proportionality constant is not a necessity for application of the method. That was the author's judgment given the ability to only examine data from one state over a limited period of time. Given data from more states or more years a different choice might have been made. In any case, each user of the method could select appropriate proportionality constants at this stage of the procedure based on the available information and his own judgment.

Then a Relative Average D-Ratio for each class in the Industry Group is calculated in Column 12 of Table 4, using the selected proportionality constant.

Table 7 shows the calculation of the D-Ratios for these classes. The relative D-Ratios are balanced to unity in Column 4 using the Expected Losses by class. In Column 5 the Indicated D-

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[25] As seen in Table 5, the results of capping were quite significant for the Miscellaneous Industry Group in this review.

[26] A similar range was indicted in a prior review. However, there is considerable fluctuation in the slopes of the regressions. Performing similar regressions in additional states and over more periods of time might allow one to select different proportionality constants by Industry Group. Again, given the small range of D-Ratios, it is unclear how much impact such a refinement could have on the estimated D-Ratios.

Ratios by class are the product of the balanced relativity D-Ratio for each class times the indicated D-Ratio for the Industry Group, in this case .2355 from Table 2 for the Office and Clerical Group.

For class 8742 (Salespersons) its estimated relative claim cost is 1.143, higher than average for the Office and Clerical Group. This yields an estimated relative D-Ratio of 1 - (2/3) (1.143 - 1) = .905, lower than average for the Office and Clerical Group. After balancing to unity the relative D-Ratio becomes .911. Then the estimated D-Ratio for class 8742 is (.911) (.2355) ≈ .21.

Similarly, for every class its observed relative average claim cost will be used to estimate its claim costs relative to its Industry Group. Then this in turn is used to estimate for each class its relative average D-Ratio. Then the estimated D-Ratio for each class is the product of its relative D-Ratio and the estimated D-Ratio for its Industry Group. Table 7 shows the final estimated D-Ratios for each class in the Office and Clerical Industry Group.[27]

Comparison to a Prior Method

The prior method used in Massachusetts was generally along the lines described in Gillam [8],[28] although some of the details differed. As shown in Table 8 in the prior method one calculated three "partial D-Ratios" as follows.

$$\text{D (Serious)} = \frac{\text{Primary Serious Losses (Indemnity \& Medical)}}{\text{Serious Indemnity Losses}}$$

$$\text{D (Non - Serious)} = \frac{\text{Primary Non - Serious Losses (Indemnity \& Medical)}}{\text{Non - Serious Indemnity Losses}}$$

[27] Similar exhibits would be produced for the other four Industry Groups.

[28] See Pages 238-239, 249-251 of PCAS 1992.

$$\text{D (Medical)} = \frac{\text{Medical Only Losses}}{\text{Total Medical Losses}}$$

The above statewide partial D-Ratios were used to calculate the D-Ratios by using the following formula:

$$\text{D-Ratio} = \frac{(P_s)(D_s) + (P_n)(D_n) + (P_m)(D_m)}{P_s + P_n + P_m} LEF,$$ where P_s, P_n, and P_m

are the adopted partial pure premiums underlying the rate for a class for the serious, non-serious, and medical losses, respectively; D_s, D_n, and D_m are the statewide partial D-Ratios; and LEF is the appropriate loss elimination factor.[29]

For example, in the filing for 1/1/95 Massachusetts Workers' Compensation rates, the partial D-Ratios were:

$$D_s = .089$$

$$D_n = .521$$

$$D_m = .110$$

For example, for Class 8810 (Clerical Risks) the partial pure premiums from the classification ratemaking process were

$$P_s = .10 \qquad P_n = .07 \qquad P_m = .08.$$

Thus, the estimated ratio of Primary Losses to Total Losses for this class was:

$$\frac{(.10)(.089) + (.07)(.521) + (.08)(.110)}{.10 + .07 + .08} = .217.$$

[29] Loss Elimination Factors (LEF's) varied by hazard group. Multiplication by the LEF was necessary since actual losses used in individual risk experience ratings are limited. The LEF removed that portion of the pure premium which is excluded in the individual risk experience rating.

The Loss Elimination Factor (LEF)[30] for Hazard Group 2[31] was 1.035. So for Class 8810 the estimated ratio of Primary Losses to limited losses entering experience rating was the product (.217) (1.035) = .22. Thus, the proposed D-Ratio for Class 8810 was .22. The D-Ratios for every other class were calculated similarly, with P_s, P_n and P_m differing by class and LEF varying by Hazard Group.

The concept of this prior method is that those classes with more serious losses and fewer non-serious losses would tend to have a corresponding higher proportion of large claims resulting in more excess and less primary losses. In practice, there are a number of potential difficulties.

First, the division between serious and non-serious losses is not always clear cut; it may depend on individual insurers statistical coding practices particularly at early reports.[32] Combined with the limited data available for smaller classes and/or smaller states, this can lead to uncertainty in the relative sizes of the partial pure premiums P_s, P_n and P_m.[33]

Second, the Medical Pure Premium P_m is being multiplied only by a ratio of medical only losses to total medical losses. Since this ratio is generally smaller than the average D-Ratio, the more medical losses a class has compared to similar classes the lower the estimated D-Ratio.

[30] This factor takes into account the limit on the dollars of claims that enter into experience rating. While the concept is used in the new alternative method, a separate such factor is not calculated.

[31] Class 8810 is in Hazard Group 2.

[32] A claim reported as Temporary Total is non-serious while one reported as Partial Disability (including the possibility of total benefits prior to partial benefits) is either serious or non-serious. At early reports prior to any partial disability payments, carrier judgments may determine whether a claim is reported as Temporary Total or Partial Disability.

[33] This can occur even if their sum: $P_s + P_n + P_m$ is fine for estimating class relativities.

Yet, a larger proportion of medical losses from both large and small accidents is not obviously a determinant of the proportion of primary dollars of loss.

Third, for the determination of primary and excess losses the medical and indemnity pieces of a claim are summed together rather than divided apart and treated separately.[34] Thus, the prior method employed a split not inherently present in the specific real world phenomena we are trying to measure and/or estimate.

In spite of all these potential problems, this prior method did a reasonable job. To some extent this is due to the relatively small range of D-Ratios compared to the large range of classification rates.[35] One step that could have been added to the prior method was to balance the final estimated D-Ratios by class back to those observed in the (adjusted) data either by Industry Group or overall. This would have removed any bias or off-balance introduced.

Conclusions

The method presented employs a series of relatively simple techniques to estimate D-Ratios by class from D-Ratios by Industry Group. This differs from the prior methodology which for each class weighted together "partial D-Ratios" using formula pure premiums broken down into serious, non-serious, and medical. These two methods are contrasted in Table 8. The method presented has the advantage of taking into account the actual severity data for each class (to the extent it is credible) in estimating the D-Ratio for each class.

[34] In addition, no specific distinction is made in most states based on injury kind for experience rating.

[35] While class rates could easily vary from 20 cents to 100 dollars, D-Ratios might range from about .10 to .30.

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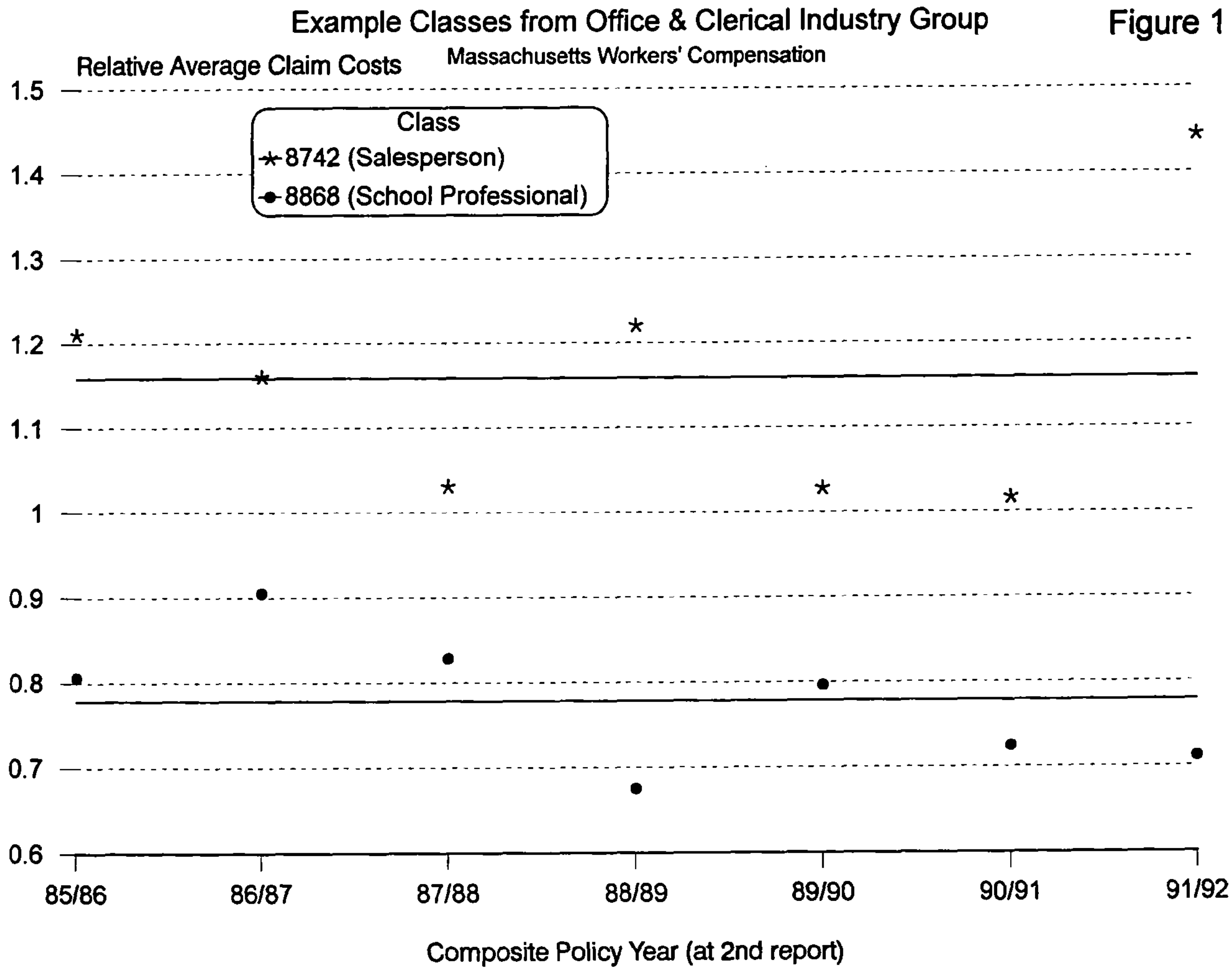
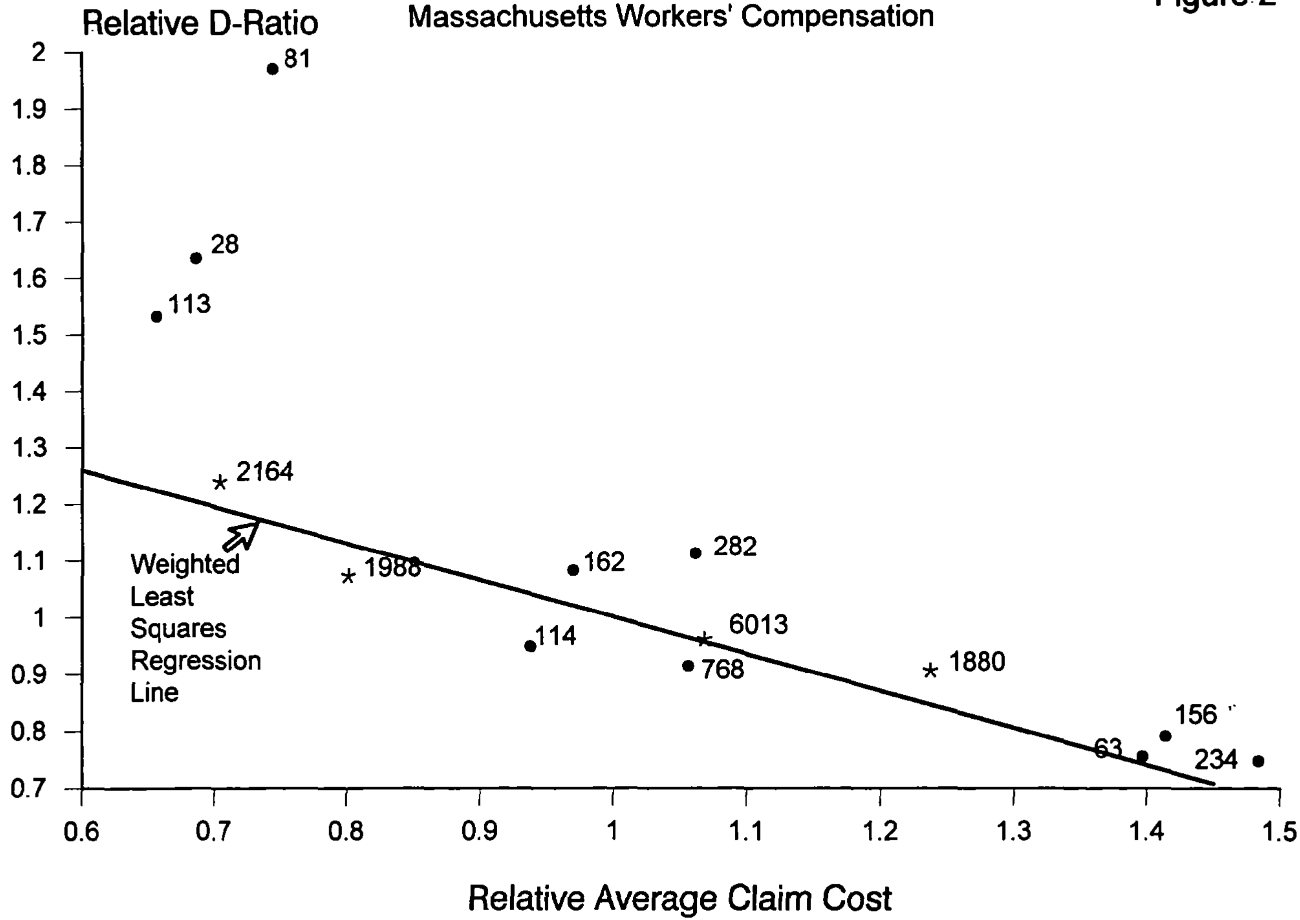

Example Classes from Office & Clerical Industry Group
Figure 1
Relative Average Claim Costs
Massachusetts Workers' Compensation
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
Class
8742 (Salesperson)
8868 (School Professional)
85/86
86/87
87/88
88/89
89/90
90/91
91/92
Composite Policy Year (at 2nd report)

Figure 2

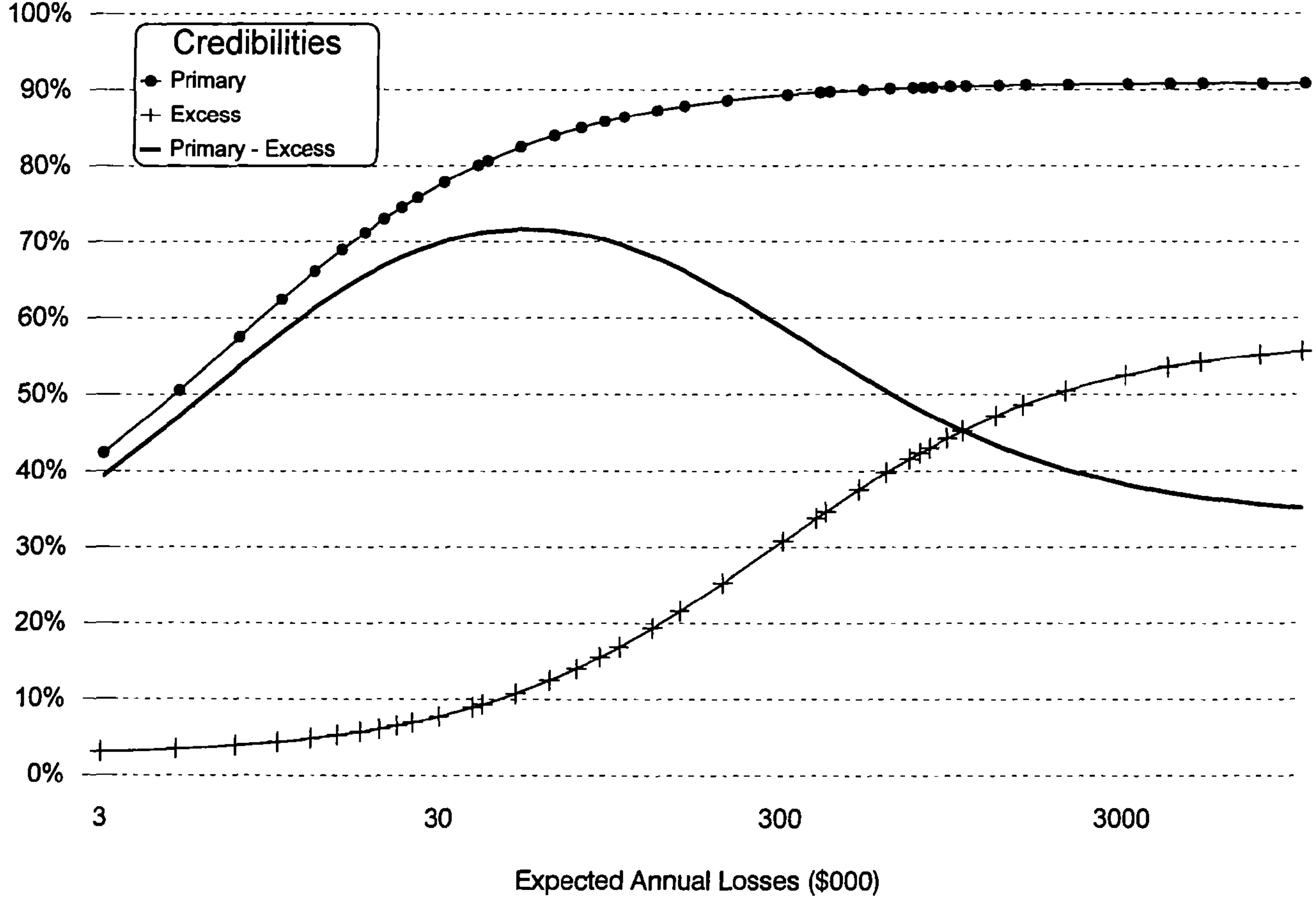
Office & Clerical Industry Group by Class
Massachusetts Workers' Compensation



Points for each class labeled by number of claims (3 years). CPY 92/93 @1st, CPY 91/92 @ 2nd, CPY 90/91 @ 3rd, combined. Permanent partial and temporary total claims.

Experience Rating Credibilities, Primary vs Excess
Massachusetts Workers' Compensation

Figure 3



Revised Experience Rating Plan, with g = 7

Table 1

Massachusetts Workers' Compensation

Combined Severity Trend and Law Amendment Factors

A. Severity Trends*

Composite Policy Year	Indemnity Injury Kind					Medical Injury Kind					
	1	2	3	4	5	1	2	3	4	5	6
90/91	1.0827	1.0827	1.0827	1.0827	1.0827	1.1746	1.1746	1.1746	1.1746	1.1746	1.1746
91/92	1.0699	1.0699	1.0699	1.0699	1.0699	1.1170	1.1170	1.1170	1.1170	1.1170	1.1170
92/93	1.0756	1.0756	1.0756	1.0756	1.0756	1.1077	1.1077	1.1077	1.1077	1.1077	1.1077

B. Law Amendment Factors

90/91	0.771	0.692	0.848	0.961	0.745	1.007	1.007	1.007	1.007	1.007	1.007
91/92	0.979	0.959	0.999	1.038	0.965	1.012	1.012	1.012	1.012	1.012	1.012
92/93	1.026	1.015	1.022	1.040	1.007	1.021	1.021	1.021	1.021	1.021	1.021

C. Combined Severity Trend and Law Amendment Factors (A x B)

90/91	0.8348	0.7492	0.9181	1.0405	0.8066	1.1828	1.1828	1.1828	1.1828	1.1828	1.1828
91/92	1.0474	1.0260	1.0688	1.1106	1.0325	1.1304	1.1304	1.1304	1.1304	1.1304	1.1304
92/93	1.1036	1.0917	1.0993	1.1186	1.0831	1.1310	1.1310	1.1310	1.1310	1.1310	1.1310

* The trend factors are adjusting for the effects of inflation expected during the two year period between the Schedule Z data used in the calculation of D-Ratios and the data that will be used to calculate Experience Modifications during the policy year effective period 7/1/96 to 6/30/97.
(This data corresponds to C.P.Y. 92/93, 93/94, and 94/95.)

Table 2

MASSACHUSETTS WORKERS' COMPENSATION

Observed D-Ratios by Industry Group

(1) Industry Group	(2) Adjusted Schedule Z Losses limited to $5,000	(3) Adjusted Schedule Z Losses limited to $175,000	(4) = (2) / (3) Observed D-Ratio
Manufacturing	107,469,897	431,334,977	0.2492
Construction	52,105,826	351,216,628	0.1484
Office & Clerical	55,821,603	237,007,928	0.2355
Goods & Services	160,437,682	629,524,720	0.2549
Miscellaneous	47,147,170	200,469,410	0.2352

(2), (3): Schedule Z losses (1st, 2nd, and 3rd report combined, includes all injury kind)
Losses are adjusted using the Law and Trend Factors shown in Table 1.

Table 3

MASSACHUSETTS WORKERS' COMPENSATION
Relative Average Claim Costs
Industry Group: Office & Clerical
Composite Policy Year 91/92 @2nd Report

(1)	(2)	(3)	(4) = (2) /(3)	(5) = (4)/TT(4)
Class	Losses (Indemnity+Med)	Number of Claims	Average Claim Cost	Relative Average Claim Cost
4361	512,291	33	15,524	1.002
7610	771,191	58	13,296	0.858
8601	1,290,543	91	14,182	0.915
8742	14,203,155	635	22,367	1.444
8748	847,220	45	18,827	1.215
8800	469,388	42	11,176	0.721
8803	597,359	17	35,139	2.268
8810	31,745,677	2,039	15,569	1.005
8820	2,075,642	87	23,858	1.540
8832	4,516,909	266	16,981	1.096
8833	8,752,453	730	11,990	0.774
8868	7,753,183	704	11,013	0.711
8901	47,799	8	5,975	0.386
9156	416,680	21	19,842	1.281
Total	73,999,490	4,776	15,494	

(2),(3): Losses and Number of Claims are as reported, but excluding any Fatal, Permanent Total, and Medical Only Claims. (Losses are neither limited nor adjusted.)

MASSACHUSETTS WORKERS' COMPENSATION
Estimated Relative D-Ratio
Industry Group: Office & Clerical

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11) = 1+(10)x[(9)-1]	(12) = 1-(2/3)x[(11)-1
Class Code	85/86 Relative ACC	86/87 Relative ACC	87/88 Relative ACC	88/89 Relative ACC	89/90 Relative ACC	90/91 Relative ACC	91/92 Relative ACC	Combined Relative ACC	Credibility	Estimated Relative ACC	Estimated Relative D-Ratio
4361	0.680	0.920	0.640	0.708	1.087	0.428	1.002	0.785	0.359	0.923	1.051
7610	1.625	1.351	0.839	0.934	1.127	0.969	0.858	1.059	0.382	1.023	0.985
8601	0.983	1.440	1.169	1.069	1.026	0.919	0.915	1.100	0.613	1.061	0.959
8742	1.211	1.161	1.031	1.221	1.028	1.017	1.444	1.143	1.000	1.143	0.905
8748	2.065	1.747	2.151	1.967	2.130	1.626	1.215	1.895	0.425	1.380	0.747
8800	0.626	0.725	1.025	0.830	0.883	1.365	0.721	0.889	0.361	0.960	1.027
8803	0.416	1.124	0.472	1.893	0.830	1.109	2.268	1.029	0.274	1.008	0.995
8810	0.982	1.021	1.044	1.040	1.068	1.113	1.005	1.040	1.000	1.040	0.973
8820	1.800	1.307	1.630	1.639	1.236	1.216	1.540	1.450	0.413	1.186	0.876
8832	1.031	1.233	1.536	1.176	1.051	1.037	1.096	1.150	0.769	1.115	0.923
8833	0.952	0.773	0.814	0.792	0.863	0.884	0.774	0.837	1.000	0.837	1.109
8868	0.806	0.905	0.828	0.675	0.796	0.724	0.711	0.774	1.000	0.774	1.151
8901	1.019	0.556	1.128	1.068	0.788	0.567	0.386	0.817	0.263	0.952	1.032
9156	0.490	0.668	1.005	1.066	0.701	0.604	1.281	0.803	0.261	0.949	1.034

(8): See Table 3.
(9): Seven Years of relative average claim costs are combined by taking a weighted average using claim counts as weights.
(10): Credibility = square root of (7-yrs-claim-count by class / 2,500) limited to unity.
(11): Relative Average Claim Costs are credibility weighted with unity.
(12): Relative D-Ratio = 1 - (2/3) (Relative ACC - 1), where the proportionality constant is selected based on separate regressions fit to data for each industry group. See Table 5.

Table 5

MASSACHUSETTS WORKERS' COMPENSATION

Determining the Proportionality Constant for Relative D-Ratio

(1) Industry Group	(2) Computed Proportionality Constant	(3) Computed Proportionality Constant (Capped)
Manufacturing	-0.568	-0.694
Construction	-0.719	-0.737
Office & Clerical	-0.650	-0.650
Goods & Services	-0.523	-0.540
Miscellaneous	-0.374	-0.898
Selected		- 2/3

(2) The proportionality constant is selected based on separate regressions fit to the relative average claim costs (for Permanent Partial and Temporary Total Claims) versus relative D-Ratios by class.
Data is from Schedule Z for first, second, and third report combined. It is not adjusted.

(3) The Relative Average Claim Cost by class is constrained to be between 0.5 and 2.0.

MASSACHUSETTS WORKERS' COMPENSATION
Determining the Proportionality Constant for Relative D-Ratio
Industry Group: Office & Clerical

(1) Class Code	(2) Losses Limited to $5,000	(3) Losses Limited to $175,000	(4) D-Ratio	(5) Relative D-Ratio*	(6) Total Losses (Ind.+Med.)	(7) Number of Claims	(8) ACC	(9) Relative ACC*
4361	417,428	1,169,121	0.357	1.532	1,198,587	113	10,607	0.656
7610	703,047	2,787,015	0.252	1.082	2,538,887	162	15,672	0.969
8601	1,233,601	4,762,304	0.259	1.112	4,836,242	282	17,150	1.061
8742	7,650,471	36,311,049	0.211	0.906	37,593,218	1,880	19,996	1.237
8748	659,224	3,592,017	0.184	0.790	3,566,061	156	22,859	1.414
8800	395,163	1,791,027	0.221	0.948	1,727,521	114	15,154	0.937
8803	258,653	1,468,291	0.176	0.755	1,423,213	63	22,591	1.397
8810	23,295,475	103,767,838	0.224	0.961	103,870,417	6,013	17,274	1.068
8820	1,007,946	5,777,652	0.174	0.747	5,615,590	234	23,998	1.484
8832	2,884,430	13,527,932	0.213	0.914	13,111,606	768	17,072	1.056
8833	6,682,957	26,771,313	0.250	1.073	25,744,078	1,988	12,950	0.801
8868	7,705,442	26,671,767	0.289	1.240	24,622,778	2,164	11,378	0.704
8901	130,497	342,235	0.381	1.635	310,662	28	11,095	0.686
9156	568,573	1,237,821	0.459	1.970	974,157	81	12,027	0.744
Total	53,592,907	229,977,382	0.233		227,133,017	14,046	16,171	

The weighted-least-squares solution for the straight-line regression (i.e., y= b+mx) is given by the formulas*:

$$m = \frac{\sum WXY - (\sum WX)(\sum WY)/\sum W}{\sum WX^2 - (\sum WX)^2/\sum W} \quad \text{and} \quad b = \frac{\sum WY}{\sum W} - m\left(\frac{\sum WX}{\sum W}\right)$$

where X = Relative ACC - 1, Y = Relative D-Ratio - 1, and W = Claim Count

Regression Result:

Uncapped Result	=> Y = 0.020 - 0.650 X	=> *Proportionality Constant*	=	-0.650
Capped Result	=> Y = 0.020 - 0.650 X	=> *Proportionality Constant*	=	-0.650

(2), (3): Latest schedule Z data, 1st Report (PY92/93), 2nd Report (PY91/92), and 3rd Report (PY90/91).
(6), (7): Latest Schedule Z data at 1st, 2nd, and 3rd report and injury kinds 3,4, and 5, (permanent partial and temporary total claims).
* The proportionality constants are calculated based on two separate weighted least squares regressions of Relative Average Claim Costs and Relative D-Ratios by class.
For the capped result, the Relative Average Claim Cost by class is constrained to be between 0.5 and 2.0.
The proportionality constants will be used to determine the slope of the line, Relative D-Ratio = 1 - m (Relative ACC -1).
Weights for the regression are the number of claims for the three years used to compute the relative ACC.

MASSACHUSETTS WORKERS' COMPENSATION
Determining the Proportionality Constant for Relative D-Ratio
Industry Group: Construction

(1) Class Code	(2) Losses Limited to $5,000	(3) Losses Limited to $175,000	(4) D-Ratio	(5) Relative D-Ratio*	(6) Total Losses (Ind.+Med.)	(7) Number of Claims	(8) ACC	(9) Relative ACC*
0050	0	0	0.000	0.000	0	0	0	0.000
3365	290,960	1,356,389	0.215	1.463	1,338,905	73	18,341	0.645
3724	2,021,485	10,603,943	0.191	1.299	9,381,940	519	18,077	0.636
3726	473,608	3,361,013	0.141	0.959	3,539,978	99	35,757	1.258
5020	184,446	710,580	0.260	1.769	688,540	44	15,649	0.550
5022	1,890,905	18,396,114	0.103	0.701	18,147,377	473	38,367	1.350
5037	59,546	878,054	0.068	0.463	872,591	12	72,716	2.558
5040	282,117	2,423,818	0.116	0.789	2,464,765	61	40,406	1.421
5057	179,242	1,832,777	0.098	0.667	2,065,949	45	45,910	1.615
5059	197,854	2,806,431	0.071	0.483	2,766,448	47	58,861	2.071
5069	0	0	0.000	0.000	0	0	0	0.000
5102	348,074	2,560,599	0.136	0.925	2,546,968	80	31,837	1.120
5146	418,689	2,866,815	0.146	0.993	4,275,866	100	42,759	1.504
5160	523,541	3,121,718	0.168	1.143	2,547,110	114	22,343	0.786
5183	5,468,919	30,309,688	0.180	1.224	30,120,877	1,364	22,083	0.777
5188	514,284	2,657,775	0.194	1.320	2,711,444	130	20,857	0.734
5190	4,846,811	26,036,511	0.186	1.265	28,324,142	1,231	23,009	0.809
5213	2,298,598	20,666,128	0.111	0.755	22,559,557	498	45,300	1.593
5215	540,583	2,818,636	0.192	1.306	3,650,903	151	24,178	0.850
5221	2,041,603	14,044,385	0.145	0.986	14,136,217	478	29,574	1.040
5222	323,049	2,458,801	0.131	0.891	2,445,396	72	33,964	1.195
5223	135,945	721,686	0.188	1.279	706,148	43	16,422	0.578
5348	313,358	3,019,976	0.104	0.707	3,302,631	78	42,341	1.489
5402	12,711	130,765	0.097	0.660	128,604	3	42,868	1.508
5403	1,007,087	6,336,159	0.159	1.082	6,510,484	250	26,042	0.916
5437	2,844,361	18,628,306	0.153	1.041	19,259,851	712	27,050	0.952
5443	4,503	4,503	1.000	6.803	4,503	1	4,503	0.158
5445	1,582,566	12,917,375	0.123	0.837	12,756,711	383	33,307	1.172
5462	428,823	2,839,970	0.151	1.027	2,872,793	102	28,165	0.991
5472	10,044	51,123	0.196	1.333	51,079	2	25,540	0.898
5473	56,427	243,086	0.232	1.578	237,221	13	18,248	0.642
5474	1,919,025	13,372,654	0.144	0.980	14,356,636	504	28,485	1.002
5479	1,404,906	8,703,239	0.161	1.095	8,862,747	363	24,415	0.859
5480	240,153	1,758,259	0.137	0.932	1,811,186	56	32,343	1.138
5491	0	0	0.000	0.000	0	0	0	0.000
5506	473,963	3,770,454	0.126	0.857	3,539,603	113	31,324	1.102
5507	606,373	5,260,851	0.115	0.782	5,017,185	131	38,299	1.347
5508	15,586	170,900	0.091	0.619	170,314	3	56,771	1.997
5509	399,010	1,498,814	0.266	1.810	1,527,072	129	11,838	0.416
5538	1,820,815	11,952,588	0.152	1.034	11,691,223	451	25,923	0.912
5545	92,886	931,065	0.100	0.680	933,751	24	38,906	1.369
5547	1,053,311	8,262,439	0.127	0.864	8,101,468	264	30,687	1.079
5606	1,675,797	12,448,637	0.135	0.918	13,197,169	384	34,368	1.209
5610	293,814	2,738,318	0.107	0.728	2,704,521	67	40,366	1.420
5645	4,247,167	25,132,765	0.169	1.150	27,353,628	1,125	24,314	0.855
5651	733,647	5,371,526	0.137	0.932	4,861,738	180	27,010	0.950
5701	0	0	0.000	0.000	0	0	0	0.000
5703	28,082	67,179	0.418	2.844	54,059	4	13,515	0.475
5705	7,784	74,872	0.104	0.707	74,872	2	37,436	1.317
6003	120,541	1,013,355	0.119	0.810	1,009,488	27	37,388	1.315

MASSACHUSETTS WORKERS' COMPENSATION
Determining the Proportionality Constant for Relative D-Ratio
Industry Group: Construction

(1) Class Code	(2) Losses Limited to $5,000	(3) Losses Limited to $175,000	(4) D-Ratio	(5) Relative D-Ratio*	(6) Total Losses (Ind.+Med.)	(7) Number of Claims	(8) ACC	(9) Relative ACC*
6005	0	0	0.000	0.000	0	0	0	0.000
6204	428,238	3,353,397	0.128	0.871	3,334,883	103	32,378	1.139
6217	2,835,335	22,451,780	0.126	0.857	23,588,377	628	37,561	1.321
6229	123,324	513,513	0.240	1.633	486,348	33	14,738	0.518
6233	147,313	1,319,626	0.112	0.762	1,394,680	29	48,092	1.692
6251	301,721	1,808,839	0.167	1.136	1,558,398	46	33,878	1.192
6252	46,410	178,983	0.259	1.762	162,315	7	23,188	0.816
6306	346,744	3,057,488	0.113	0.769	3,324,338	80	41,554	1.462
6319	459,265	3,943,188	0.116	0.789	4,298,717	108	39,803	1.400
6325	68,290	404,803	0.169	1.150	219,621	14	15,687	0.552
6400	190,150	1,031,940	0.184	1.252	993,606	53	18,747	0.659
7538	78,217	524,010	0.149	1.014	515,923	19	27,154	0.955
7601	133,924	1,114,583	0.120	0.816	1,100,339	37	29,739	1.046
7855	33,647	569,450	0.059	0.401	565,803	6	94,301	3.317
8227	720,884	4,705,343	0.153	1.041	4,511,813	193	23,377	0.822
9530	0	0	0.000	0.000	0	0	0	0.000
9534	55,093	611,613	0.090	0.612	610,404	10	61,040	2.147
9545	24,015	32,868	0.731	4.973	26,532	7	3,790	0.133
9549	41,916	100,213	0.418	2.844	97,394	11	8,854	0.311
9552	227,315	1,298,740	0.175	1.190	1,284,387	57	22,533	0.793
9553	15,104	145,338	0.104	0.707	145,234	3	48,411	1.703
Total	50,709,904	344,496,754	0.147		353,900,770	12,449	28,428	

The weighted-least-squares solution for the straight-line regression (i.e., y= b+mx) is given by the formulas*:

$$m = \frac{\sum WXY - (\sum WX)(\sum WY)/\sum W}{\sum WX^2 - (\sum WX)^2/\sum W} \quad \text{and} \quad b = \frac{\sum WY}{\sum W} - m\left(\frac{\sum WX}{\sum W}\right)$$

where X = Relative ACC - 1, Y = Relative D-Ratio - 1, and W = Claim Count

Regression Result:

Uncapped Result	=>	Y = 0.057 - 0.719 X	=> *Proportionality Constant*	=	-0.719
Capped Result	=>	Y = 0.057 - 0.737 X	=> *Proportionality Constant*	=	-0.737

(2), (3): Latest schedule Z data, 1st Report (PY92/93), 2nd Report (PY91/92), and 3rd Report (PY90/91).
(6), (7): Latest Schedule Z data at 1st, 2nd, and 3rd report and injury kinds 3,4, and 5, (permanent partial and temporary total claims).
* The proportionality constants are calculated based on two separate weighted least squares regressions of Relative Average Claim Costs and Relative D-Ratios by class.
For the capped result, the Relative Average Claim Cost by class is constrained to be between 0.5 and 2.0.
The proportionality constants will be used to determine the slope of the line, Relative D-Ratio = 1 - m (Relative ACC -1).
Weights for the regression are the number of claims for the three years used to compute the relative ACC.

MASSACHUSETTS WORKERS' COMPENSATION
D-Ratios, Adjusted for Trend and Law Factors
Industry Group: Office & Clerical
D-Ratios Balanced to: 0.2355

Phraseology	(1) Class Code	(2) Expected Losses ($ million)	(3) Estimated Relative D-Ratio	(4) Balanced Relative D-Ratio	(5) Indicated D-Ratio
Photographer-All Emp-Clerical,Sales-& Dr	4361	1.1	1.051	1.058	0.25
Radio or TV Broadcast-All Emp,Cler-& Dr	7610	2.0	0.985	0.992	0.23
Engineer or Architect-Consulting	8601	3.7	0.959	0.966	0.23
Salesperson,Collector,Messenger-Outside	8742	24.5	0.905	0.911	0.21
Auto Sales or Service Agcy-Salesperson	8748	2.8	0.747	0.752	0.18
Mailing or Addressing Co-& Clerical	8800	1.2	1.027	1.034	0.24
Auditor,Accountant,Etc-Traveling	8803	2.1	0.995	1.002	0.24
Clerical Office Employees NOC	8810	72.8	0.973	0.980	0.23
Attorney-All Emp-Clerical,Messenger & Dr	8820	3.9	0.876	0.882	0.21
Physician-& Clerical	8832	10.5	0.923	0.930	0.22
Hospital-Professional Employees	8833	19.3	1.109	1.117	0.26
School-Professional Emp & Clerical	8868	20.2	1.151	1.159	0.27
Telephone/Telegraph Co-Office Emp & Cl	8901	0.2	1.032	1.039	0.24
Theatre-Players,Entertainers,Musicians	9156	0.9	1.034	1.041	0.25
		Weighted Average =	0.993	1.000	0.23

(2): Expected Losses are the three years of payrolls times the Indicated Expected Loss Rates.

(3): From Table 4.

(4): Relative D-Ratios are balanced to unity using the expected losses as weights, where
Balanced Relative D-Ratio = (Estimated Relative D-Ratio) / (Estimated Relative D-Ratio Weighted Average)

(5): Proposed D-Ratio = (Balanced Relative D-Ratio) x (Industry Group Observed D-Ratio)
Industry Group Observed D-Ratio is from Table 2.

TABLE 8

OVERVIEW OF TWO METHODS OF ESTIMATING D-RATIOS

Prior Massachusetts Method and/or Gillam [8], PCAS 1992[1]	Current Massachusetts Method (Alternative Method)
1. Adjust the reported data for changes expected between the data available now and that to be used for experience rating in the future.	1. Adjust the reported data for changes expected between the data available now and that to be used for experience rating in the future.
2. Calculate 3 Partial D-Ratios. $\text{Serious Partial D-Ratio} = \frac{\text{Serious Primary Losses}}{\text{Serious Indemnity Losses}}$ $\text{Non-Serious Partial D-Ratio} = \frac{\text{Non-Serious Primary Losses}}{\text{Non-Serious Indemnity Losses}}$ $\text{Medical Partial D-Ratio} = \frac{\text{Medical Only Losses}}{\text{Medical Losses}}$	2. Calculate D-Ratios by Industry Group.[2]
3. For each class take the estimated Serious, Non-Serious and Medical Partial Pure Premiums used to determine classification rate relativities.	3. Estimate Average Relative Claim Cost by Class within Industry Group.
4. Weight the Partial D-Ratios from Step 2 using the Partial Pure Premiums from Step 3.	4. Spread to each class the Average D-Ratio for each Industry Group from Step 2 using Relative Average Claim Costs in Step 3.
5. Adjust for the impact on the D-Ratio of those losses excluded from Experience Rating.[3]	

[1] The National Council on Compensation Insurance has been updating their methodologies every few years. Details have changed and continue to change, but the over-all approach has remained the same.

[2] The denominator of the D-Ratios is total losses minus those excluded from experience rating. The numerator is Primary Losses.

[3] The denominator of the D-Ratios should be expected total losses minus those expected to be excluded from experience rating.

APPENDIX

Let the experience modification be given by:

$$M = \frac{(1 - Z_p)\, E_p + Z_p\, A_p + (1 - Z_x)\, E_x + Z_x\, A_x}{E}$$

$$= (1 - Z_p)\, D + Z_p\, (A_p/E) + (1 - Z_x)\,(1 - D) + Z_x\,(A_x/E)$$

where:

A_p	=	Actual Primary Losses
A_x	=	Actual Excess Losses
E_p	=	Expected Primary Losses
E_x	=	Expected Excess Losses
E	=	$E_p + E_x$
Z_p	=	Primary Credibility
Z_x	=	Excess Credibility
D	=	$E_p/(E_p + E_x) = E_p/E$

then for all the other inputs fixed, for a change in the D-Ratio the change in the experience modification is

$$\frac{\partial M}{\partial D} = (1 - Z_p) - (1 - Z_x) = -(Z_p - Z_x)$$

Thus, the sensitivity of the modification to the D-Ratio depends on the difference between Z_p and Z_x. Since $Z_p > Z_x$, the larger D, the smaller the experience modification. Primary credibilities are usually 40% to 70% higher than excess credibilities with the result varying by size of risk. For example, for Massachusetts[1] the differences in credibility are shown in Figure 3.

Therefore, a .10 difference in D-Ratio (holding everything else equal) will produce between a .04 and .07 difference in the Experience Modification depending on the size of the insured. A very large difference in D-Ratios[2] produces only a relatively modest difference in the Experience Modification. This is why D-Ratios are rounded to two decimal places. This is also why detailed technical refinements to a methodology to estimate D-Ratios are unlikely to have much practical impact.

[1] Revised Experience Rating Plan with g = 7. See, for example, Mahler [9].

[2] For example, in Massachusetts the D-Ratios range from about .10 to about .30.

Techniques for the Conversion of Loss Development Factors

by Louis Spore, ACAS, MAAA

Abstract

It sometimes happens that accident year development factors are available and policy year factors are not and vice versa. The purpose of this paper is to formulate a mathematical technique for converting from one form into another under various assump-tions concerning the time during the calendar year that policies are written. The connection between the policy year factor and the influence of changing exposures on accident year development is then explored.

TECHNIQUES FOR THE CONVERSION OF LOSS DEVELOPMENT FACTORS

1. Overview

This paper begins by deriving a general formula to convert accident year factors into policy year age-to-age loss development factors. To help understanding, a first simplifying assumption that the policies are written uniformly over the policy year is made and then further generalized to situations where only the average written date is known. The inverse of this formula then gives the means of converting accident year factors back to policy year factors. An analogy to the effect on accident year factors from changes in exposure leads to a reformulation of the problem. A practical example taken from real data illustrates the techniques.

2. Notation and Analysis

It will be necessary to make a few definitions first. Let:

a_k = the (incremental) dollar amount expected to be reported for an individual risk at development period k.

$a_k^{(p)}$ = the corresponding dollar amount for a policy period.
g_k = the accident year factor that develops incurred losses from age k to age $k+1$.
f_k = the policy year factor that develops incurred losses from age k to age $k+1$.
n = the number of policies written in a policy year.

$$g_k = 1 + \frac{a_{k+1}}{\sum_{j=1}^{k} a_j}; \quad f_k = 1 + \frac{a_{k+1}^{(p)}}{\sum_{j=1}^{k} a_j^{(p)}} \qquad (1)$$

Which implies that

$$\frac{a_{k+1}}{a_k} = \begin{cases} \dfrac{(g_k - 1)g_{k-1}}{(g_{k-1} - 1)} & \text{when } k \ge 2 \\ g_1 - 1 & \text{when } k = 1 \end{cases} \qquad (2)$$

A similar relationship holds for $a_k^{(p)}$. The importance of this ratio will become evident after examining the process of policy creation and the future claims associated with them. The proof is in Appendix A.

Scenario 1 - *Policies are written uniformly over the calendar year.*

Assume that each policy has a development pattern that corresponds to an accident year and that, for *n* policies written during the year, the first policy is written at time *1/n*, the second policy at time *2/n*, the third at *3/n* etc. Then the last policy will be written on December 31 of the calendar period and will not contribute any losses to it. To avoid the use of multiples of 12, we shall let the integer 1 stand for the first 12 months, 2 for 24 months etc. Hence g_1 will stand for the 12 to 24 month accident year age-to-age factor. Since each policy has the development pattern of an accident year, the first policy will contribute $[(n-1)/n]a_1$ of losses to the first 12 months of the policy year. The second policy written will contribute $[(n-2)/n]a_1$ to the first 12 months of the policy year. By extension of this reasoning, the first 12 months of the policy year will experience losses reported of $(1/n)[1+2+...+(n-1)]a_1=(n-1)a_1/2$. The second year of the policy period will have losses reported equal to the first 12 months of an accident year for the last policy written to 1/n times the first 12 months of an accident year for the first policy written, in addition to the beginnings of the 24-month accident year development on policies as they begin to expire in the second year. The 12-month accident year con-tribution to the second year will be $(1/n)(n+...+1)a_1$, and the 24-month contribution will be $(1/n)(n-1+n-2+...+1)a_2$. We can now use the principle of induction to derive the following relation-ships:

$$a_1^{(p)} = (n-1)a_1/2$$
$$a_2^{(p)} = (n+1)a_1/2 + (n-1)a_2/2$$
$$\vdots$$
$$a_k^{(p)} = (n-1)a_k/2 + (n+1)a_{k-1}/2$$

Which together with equation(1) → that

$$f_k = 1 + \frac{[(n-1)a_{k+1} + (n+1)a_k]}{[(n-1)\sum_{i=1}^{k} a_i + (n+1)\sum_{i=1}^{k} a_{i-1}]}$$

$$= 1 + \frac{[(n-1)a_{k+1} + (n+1)a_k]}{\left[2n\sum_{i=1}^{k-1} a_i + (n-1)a_k\right]} \qquad (3)$$

.

This holds for k=1 by letting the summation term be zero for this case. Dividing top and bottom by na_k, letting n approach infinity and substituting our expression for a_{k+1}/a_k we get the following transformation:

$$f_k = \frac{g_{k-1}(1+g_k)}{(1+g_{k-1})} = \frac{(1+g_k)}{(1+g_{k-1}^{-1})} \qquad (4)$$

By allowing g_0 to be infinity this formula will be true for all integer values of $k \geq 1$. The algebraic details are again left to Appendix B.

Scenario 2 - *The policies aren't written evenly over the calendar year but the average written date is known.*

Let T be the average written date as a percentage of the year. Let t_k be the time the k-th policy is written as a percentage of the whole year. Generalizing the argument above, we get that:

$$a_1^{(p)} = [\sum_{i=1}^{n} (1 - t_i)]\, a_1$$

$$a_2^{(p)} = [\sum_{i=1}^{n} t_i]\, a_1 + [\sum_{i=1}^{n} (1 - t_i)]\, a_2$$

.

.

.

$$a_k^{(p)} = [\sum_{i=1}^{n} t_i]\, a_{k-1} + [\sum_{i=1}^{n} (1 - t_i)]\, a_k$$

$$= nTa_{k-1} + n(1 - T)\, a_k \qquad (5)$$

Repeating the previous analysis gives us the following modification:

$$f_k = g_{k-1}\frac{\left[g_k(1 - T) + T\right]}{\left[g_{k-1}(1 - T) + T\right]}$$

$$= \frac{\left[g_k(1 - T) + T\right]}{\left[1 - T + Tg_{k-1}^{-1}\right]} \qquad (6)$$

Notice that T=1/2 is the same as assuming uniform writings over the whole year. (This also follows by letting $t_k=k/n$ and finding T as n approaches infinity). Also, by using T instead of an assumption about when the policies are written, the n term will cancel from the ratio, making the limiting value the same as the finite value for the same T.

The inverse problem of finding the accident year factors from the policy year factors follows immediately from (6) and induction: where, for the sake of convenience, $\alpha = T/(1-T)$ and $f_0 = 1$.

$$g_1 = f_1 - \alpha$$

$$g_2 = \frac{f_1 f_2}{f_1 - \alpha} - \alpha$$

.

.

.

$$g_k = \frac{\prod_{i=1}^{k} f_i}{\sum_{j=0}^{k-1} (-\alpha)^{(k-j-1)} \prod_{i=0}^{j} f_i} - \alpha \qquad (7)$$

The assumption in this approach is that the losses reported in successive years are proportional to the time the policy has been in force. This, in turn, depends on the written date. If T=1, then all of the policies are written at the end of the calendar year and $f_k=g_{k-1}$. This means that the policy year is exactly the same as an accident evaluation at one period earlier. If T=0, all policies are written at the beginning of the period, then $f_k=g_k$ and the policy year and accident year are identical. The fundamental assumption necessary to this approach is that there be a policy year of exactly one year and that the average date of the policies written during that year is known. Also the accident year factors should begin at the 12 to 24 month development age and increase in 12 month increments. If the accident year factors are known at other development ages, a simple approach would be to fit a curve to the known factors and then use the curve to get the year end factors. Equation (6) would give the corresponding year-end factors for the policy year. A new curve fit to these factors would then give the policy year factors at the desired development ages. Table 1 illustrates these concepts and the effect that the average written date has on the derived policy year factors.

A word needs to be said about the assumption that the development of an individual risk resembles the development of an accident year. To see that this is so, it is only necessary to develop the accident year expected losses in terms of the expected losses for each risk. If A_i represents the reported incurred (incremental) losses at development period i, a little thought will demonstrate that $A_i = \sum(1-t_j)a_i$. Briefly, the reason is that the development of losses that occurred in the calendar year in which the policies were written depends only on the length of time that the policies were in force. A policy written on December 31 would have no impact on accident year development , although it will have an effect on the policy year development.

Thus $g_k = 1 + A_{k+1}/\sum a_i = 1 + a_{k+1}/\sum a_j$.

Another assumption, that the expected losses for each risk is the same, is necessary to make the formulation of the problem more tractable. To know the actual risk parameters at the time the risk is written would require information virtually impossible to obtain. Each risk can be regarded as having the same distribution as the aggregate distribution. Since the number of risks drops out of the ratios for the factors, this assumption does no harm.

It is often assumed that because the average accident date of the policy year is December 31 and that the average accident date of the accident year is July 1, the 12-month policy year development factor is the same as the 6- month accident year factor. Underlying this is actually two assumptions: (1) that the date of loss is exactly 1/2 of the policy period and (2) that the average written date is July 1. The approach taken above accepts the average date of loss implied by the accident year factors and permits a more flexible assumption about the average written date.

Table 1

T = 0.25

Age	AY Factors	Age	Fitted AY Factors	PY Factors as per (6)	Age	Fitted PY Factors
9	1.889	12	1.500	1.833	9	2.612
21	1.163	24	1.125	1.193	21	1.243
33	1.066	36	1.056	1.071	33	1.089
45	1.036	48	1.031	1.037	45	1.044
57	1.022	60	1.020	1.023	57	1.026
69	1.015	72	1.014	1.015	69	1.017
81	1.011	84	1.010	1.011	81	1.012
93	1.008	96	1.008	1.008	93	1.009

T = 0.75

Age	AY Factors	Age	Fitted AY Factors	PY Factors as per (6)	Age	Fitted PY Factors
9	1.889	12	1.500	4.500	9	5.679
21	1.163	24	1.125	1.193	21	1.443
33	1.066	36	1.056	1.071	33	1.126
45	1.036	48	1.031	1.037	45	1.053
57	1.022	60	1.020	1.023	57	1.028
69	1.015	72	1.014	1.015	69	1.016
81	1.011	84	1.010	1.011	81	1.010
93	1.008	96	1.008	1.008	93	1.007

3. An Alternate Interpretation

The policy year is similar to the situation in which the exposure for each accident year is increasing. This is because each policy written is an increase in exposure for the calendar accident year. If we can succeed in translating the concept of policies written into exposures assumed we could use (7) to adjust the accident year factors for an increase in exposure.

To do this, let E_{i-1} represent the exposure at the beginning of accident year *i* where E_0 is the exposure at the beginning of the first year. This situation is different from the beginning of a policy year in that, for a policy year, the exposure always begins at zero. The average "written" date for accident year *i* now includes a mass of "policies" at T = 0 equal to E_{i-1}. We now rewrite (5) for accident year 1 as follows:

$$a_1' = E_0 a_1 + (E_1 - E_0)(1 - T')a_1$$
$$a_2' = (E_1 - E_0)\left[T'a_1 + (1 - T')a_2\right] + E_0 a_2$$
$$\vdots$$
$$a_k' = (E_1 - E_0)\left[T'a_{k-1} + (1 - T')a_k\right] + E_0 a_k \qquad (8)$$

As before, T' is the average exposure date for the increase, but now the *a*'s stand for the reported cost per unit of exposure. To use (6) and (7) without modification, we find a T that is equivalent to the expressions in (8) by setting $E_0 = 0$ in the last line of (8), replacing T' by T and letting it equal the original expression. *Equating the coefficients of* a_{k-1} *or* a_k *gives* $T = (1-E_0/E_1)T'$. What it means to have an average date for the new exposures needs some clarification. If the exposures are new stores or new employees, the average opening date or average hire date is the correct interpretation. However, if the exposure is payroll or sales, a natural assumption of uniform increase over the year means that $T' = 1/2$. So if $E_1 = 2E_0$ then $T = 1/4$.

The interpretation so far has only been for an increase in exposure. However, (8) would hold without modification under conditions of declining exposure. The expression for T would be negative since it was derived under the assumption of a beginning exposure of zero. Under this condition, no decline in exposure is possible. However, the algebra is equivalent even though allowing T to be negative makes no conceptual sense.

How do we use this information? We want to use equation (7) to factor out the increase in the development factors due to the increase in exposure. First note that the relevant term in (7) is $\alpha = T/(1-T)$. Since we know what happens when T is zero or 1 we restrict our discussion to the case where $0<T<1$. If it is true that $E_1/E_0 = E_2/E_1 = \ldots = E_i/E_{i-1}$ (i.e., the increase in exposure is a constant percentage of the previous exposure), then it is easy to check that $T_1 = T_2 = \ldots = T_i$ if T', the average date of the exposure increase, is the same for all accident years where T_i is the adjusted date for accident year *i*. Thus, as long as there is an increase at the same *rate*[1], there should be no overall change in the factors after the *first* increase. However we might want to adjust the new factors to develop a new year where the changes have stopped. Also, the most common situation, where exposure is changing but at different rates introduces the problem of how to adjust the factors to be appropriate for the exposure level of each accident year. To begin, we will keep the same notation but let the *f*'s stand for the growth accident year factors (the "policy year")and the *g*'s will be the accident year factors with growth removed (the "accident year"). As an additional refinement, we will add a superscript to distinguish the accident year being adjusted. Thus, $f_j^{(i)}$will be the unadjusted factor for the *i-th* year and the *j-th* development period. The flattened factor will be $g_j^{(i)}$. Finally, we will add primes to represent the factors adjusted to the growth level of a different year. If we desire to adjust the *i-th* year to the level of the *n-th* year, we first flatten year *i* and then re-inflate to the level of year *n*. If *n* is the latest year having only the 12-24 month development factor, each year *i* will have its 12-24 month factor adjusted to year *n*. First deflating, we get:$g_1^{(i)} = f_1^{(i)} - \alpha_i$ and then inflating: $f_1^{(i)'} = g_1^{(i)} + \alpha_n = f_1^{(i)} - \alpha_i + \alpha_n$. Similarly, the 24-36 month factor will be adjusted for all accident years as illustrated in the derivation of equation (9) below. We proceed in this fashion for each year *i*. The alphas are calculated from the adjusted average

[1]*The same rate in both percentage dollar increases and at the same time as measured by the average date.*

exposure date from the beginning to the end of the year corresponding to the subscript:
$\alpha_i = T_i/(1-T_i)$ where $T_i = (1-E_{i-1}/E_i)T'$ and $f_0^{(1)}=1$.

An example of the method is in Appendix C. Sheet 1 shows a real Workers Compensation incurred loss development triangle. The

$$g_2^{(i)} = \frac{f_1^{(i)} f_2^{(i)}}{f_2^{(i)} - \alpha_i} - \alpha_i$$

$$f_2^{(i)'} = \frac{(g_2^{(i)} + \alpha_n)}{(g_1^{(i)} + \alpha_n)} g_1^{(i)}$$

$$= \frac{\left[\frac{f_1^{(i)} f_2^{(i)}}{f_1^{(i)} - \alpha_i} + \alpha_n - \alpha_i\right]}{(f_1^{(i)} - \alpha_i + \alpha_n)} (f_1^{(i)} - \alpha_i)$$

$$= \frac{f_1^{(i)} f_2^{(i)} + (\alpha_n - \alpha_i)(f_1^{(i)} - \alpha_i)}{(f_1^{(i)} - \alpha_i + \alpha_n)} \quad \text{or}$$

$$f_1^{(i)'} f_2^{(i)'} = f_1^{(i)} f_2^{(i)} + (\alpha_n - \alpha_i)(f_1^{(i)} - \alpha_i)$$

$$\text{or} \prod_{j=1}^{k} f_j^{(i)'} = \prod_{j=1}^{k} f_j^{(i)} + (\alpha_n - \alpha_i) \sum_{j=0}^{k-1} (-\alpha_i)^{k-j} \prod_{m=0}^{j} f_m^{(i)} \qquad (9)$$

selected factors are the average of the overall average and the average after removing the largest and smallest values for the years for the years for which the latter exists and the overall average for the remaining years. Sheet 2 shows the exposure which is number of employees hired. The assumed average hire date is in the middle of the fiscal policy year (T'=1/2). The adjustment for each accident year is for the change in the number of employees from the beginning to the end of the year. Thus the adjustment is from year *i* to year *n=i+1* in the following derivation.

Although not true in this example, an examination of the variance in the factors by column sometimes reveals that the adjustment actually increases the variance for some ages while decreasing it for others. The obvious explanation would be that there is a lag in the influence of new exposures. New employees would not have the linear influence on the incidence of new claims as the derivation of the formulas would imply. It would generally be several years after employment before a claim would be filed. However if the exposure is increasing at the same rate every year the influence of the increase in the older years would nullify this argument. The increase at a faster rate would make the adjustment too large, but an increase at a slower rate would make it too small. Another explanation would be that a change in hiring practices or safety programs would make new employees have different loss potential from older ones. Also the average date of hire will vary from year to year, making the $T'=1/2$ assumption invalid. There are other complications such as the change in operations or reserving practices that would distort the results as well.

4. Summary

A formulaic approach to transform policy year age-to-age development factors into accident year age-to-age factors has been found that helps to clarify the relationship hidden in the definitions. The formulas derived from the investigation of that relationship led to a better understanding of the effect of the changes in exposure on the development of accident year factors.

APPENDIX A

Proof of equation (2) in the text:

$$g_k = 1 + \frac{a_{k+1}}{\sum_{i=1}^{k} a_i}$$

$$= 1 + \frac{a_{k+1}}{\sum_{i=1}^{k-1} a_i + a_k}$$

$$= 1 + \frac{a_{k+1}/a_k}{\sum_{i=1}^{k-1} a_i/a_k + 1}$$

$$= 1 + \frac{a_{k+1}/a_k}{\frac{1}{(g_{k-1} - 1)} + 1} \quad \text{which implies}$$

$$g_k - 1 = \frac{a_{k+1}}{a_k}\left(\frac{g_{k-1} - 1}{g_{k-1}}\right) \quad \text{or:}$$

$$\frac{a_{k+1}}{a_k} = g_{k-1}\left(\frac{g_k - 1}{g_{k-1} - 1}\right)$$

APPENDIX B
Sheet 1

$$f_k = 1 + \frac{(n-1)a_{k+1} + (n+1)a_k}{2n\sum_{i=1}^{k-1} a_i + (n-1)a_k}$$

$$= 1 + \frac{(1-1/n)a_{k+1}/a_k + (1+1/n)}{2\sum_{i=1}^{k-1} a_i/a_k + (1-1/n)} \quad \text{let } n\to\infty \text{ we get}$$

$$= 1 + \frac{a_{k+1}/a_k + 1}{2\sum_{i=1}^{k-1} a/a_k + 1}$$

$$= 1 + \frac{\dfrac{(g_k-1)g_{k-1}}{(g_{k-1}-1)} + 1}{\dfrac{2}{(g_{k-1}-1)} + 1} \quad \text{from (2) and (1) in the text} \qquad \textbf{(11)}$$

$$= 1 + \frac{g_k g_{k-1} - 1}{g_{k-1} + 1}$$

$$= \frac{g_{k-1}(1+g_k)}{(1+g_{k-1})}$$

Appendix B
Sheet 2

Proof of Equation (6) in the text:

$$a_{k+1}^{(p)} / \sum_{i=1}^{k} a_i^{(p)} = \frac{nTa_k + n(1-T)a_{k+1}}{nT\sum_{i=1}^{k} a_{i-1} + n(1-T)\sum_{i=1}^{k} a_i}$$

$$= \frac{T + (1-T)(a_{k+1}/a_k)}{T\left(\sum_{i=1}^{k-1} a_i/a_k\right) + (1-T)(a_{k+1}/a_k)\left(\sum_{i=1}^{k} a_i/a_{k+1}\right)} \quad \text{[Since } a_0 = 0\text{]}$$

$$= \frac{T + (1-T)g_{k-1}\left(\frac{g_k - 1}{g_{k-1} - 1}\right)}{\frac{T}{(g_{k-1} - 1)} + \frac{(1-T)g_{k-1}(g_k - 1)}{(g_{k-1} - 1)}\left(\frac{1}{g_k - 1}\right)}$$

$$= \frac{T(g_{k-1} - 1) + (1-T)g_{k-1}(g_k - 1)}{T + g_{k-1}(1-T)} \Rightarrow$$

$$1 + \frac{a_{k+1}^{(p)}}{\sum_{i=1}^{k} a_i^{(p)}} = \frac{g_{k-1} + (1-T)g_{k-1}(g_k - 1)}{T + (1-T)g_{k-1}}$$

$$= g_{k-1}\frac{(1-T)g_k + T}{(1-T)g_{k-1} + T}$$

$$= f_k$$

APPENDIX C

WORKERS COMPENSATION

Incurred Loss Development

	Incurred as of____months						
Policy Period	12	24	36	48	60	72	84
86-87	127,543	237,609	255,255	261,471	293,234	286,390	292,540
87-88	238,622	336,699	447,647	474,771	657,817	673,145	
88-89	413,446	629,368	692,177	743,374	759,378		
89-90	483,344	755,863	815,980	717,682			
90-91	441,426	551,559	610,788				
91-92	592,559	832,558					
92-93	649,736						

Policy Period	12-24	24-36	36-48	48-60	60-72	72-84	84- Ult.
86-87	1.863	1.074	1.024	1.121	0.977	1.021	
87-88	1.411	1.330	1.061	1.386	1.023		
88-89	1.522	1.100	1.074	1.022			
89-90	1.564	1.080	0.880				
90-91	1.249	1.107					
91-92	1.405						
Average	1.502	1.138	1.010	1.176	1.000	1.021	
Wtd. Avg	1.456	1.124	0.994	1.156	1.009	1.021	
Avg-HI/LO	1.476	1.096	1.042	1.121			
Selected	1.476	1.124	1.010	1.156	1.009	1.021	

	12-ult.	24-ult.	36-ult.	48-ult.	60-ult.	72-ult.	84-ult.
Cumulative	1.994	1.352	1.203	1.191	1.031	1.021	1.000

Appendix C

Sheet 2

	Exposure	T	Alpha	T' = 1/2
85-86	193			
86-87	228	0.075	0.0814	
87-88	239	0.023	0.0236	
88-89	317	0.123	0.1405	
89-90	340	0.034	0.0354	
90-91	339	-0.001	-0.0011	
91-92	419	0.095	0.1050	
92-93	444	0.029	0.0298	

Adjusted to 93-94 accident year level

Year	12-24	24-36	36-48	48-60	60-72	72-84	84- Ult.
86-87	1.811	1.054	1.024	1.126	0.970	0.995	0.998
87-88	1.417	1.330	1.062	1.384	1.025		
88-89	1.412	1.078	1.076	1.015			
89-90	1.558	1.078	0.879				
90-91	1.280	1.111					
91-92	1.330						
Variance	0.1200	0.0898	0.0774	0.1315	0.0274	0.0000	
	1.468	1.130	1.010	1.175			
	1.502	1.138	1.010	1.176			

The Application of Cumulative Distribution Functions in the Stochastic Chain Ladder Model

by Son T. Tu, Ph.D., ACAS, MAAA

THE APPLICATION OF CUMULATIVE DISTRIBUTION FUNCTIONS IN THE STOCHASTIC CHAIN LADDER MODEL

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ABSTRACT

A new stochastic model based on the traditional chain ladder is introduced. It makes explicit use of cumulative distribution functions and payment patterns. It incorporates a mathematical rationale for non-stochastic variations in the age-to-age factors. Perturbation methods are used to obtain and justify the solution. Estimation of liabilities in the tail is a natural product of the model. All stochastic variables are assumed to be normally distributed, and the assumption is then confirmed with the chi square goodness-of-fit test. Extensive numerical solutions of an actual problem are given. Several new avenues of related research are suggested.

KEYWORDS

Chain ladder; Loss reserving; Cumulative distribution functions; Tail factor; Stochastic models; Perturbation theory.

Problem #2: This problem is one of tail development. One common solution is to consider the incurred and paid amounts for a number of accident years preceding the earliest year in the triangle and to *assume that the incurred-to-paid ratio for these years constitutes an accurate reflection of the tail* development. But development for some long-tailed lines such as Workers' Compensation can continue over thirty or more years. Data may not be available or credible over such a long period of time.

A second common technique to account for tail development is Bondy development. However, it is difficult to estimate the errors associated with this technique. Furthermore, it does not have a very firm mathematical or practical foundation.

Our model gives a mathematical rationale for non-stochastic variations in the ata factors. It estimates tail factors, and, whenever warranted by the data, allows these factors to vary by accident years. It also emphasizes the tendency of the traditional approach to rely on unnecessarily limited information for the estimation of critical quantities. As a case in point, take the 108:120 factor of 1.017 in Table 2. This is in fact only a realization of a random variable. But the traditional approach would carry it over to all accident years. In contrast, based on all the information available in the triangle, our model estimates the mean and standard deviation at each individual point.

In section 2, we relate how cumulative distribution functions with time-varying parameters can lead to non-stochastic variations. In section 3, we introduce the scaling and proportionality functions, and motivate and present the stochastic model. To obtain the solution, we use perturbation methods, in section 4, to develop an iterative regression procedure. The mathematical formulas for the estimates of the ultimate liability amounts and the standard deviations are derived in section 5; we also show that the losses are in theory normally distributed, to the leading order.

In section 6, numerical solutions to the Table 1 triangle are given. We use the chi square test, in section 7, to show that the numerical results are indeed normally distributed. In section 8, we discuss the different orders of error magnitudes apparent in any loss reserving exercise. In section 9, we compare the effects of using different scaling and proportionality functions. We define and discuss parametrization in section 10. We discuss some related topics in the concluding section.

2. *CUMULATIVE DISTRIBUTION FUNCTIONS*

Let the cumulative distribution function $F(t)$ *represent the payment pattern for an accident year,* with t in years. Figure 1 illustrates:

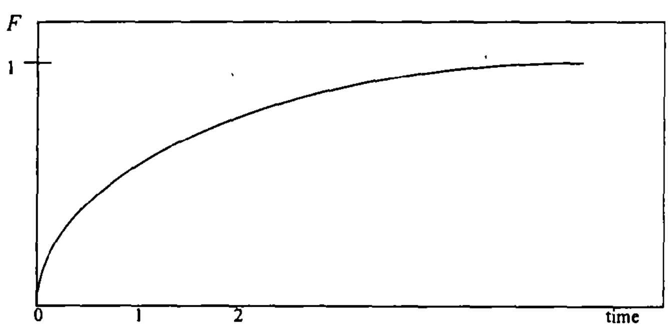

Figure 1: The cumulative distribution function $F(t)$.

At $t = 0$, $F = 0$ when no payments have been made. At $t = \infty$, $F = 1$ when all payments have been made. At any intervening time, the distribution gives the fraction that has been paid. Any ata factor

1. INTRODUCTION

In this paper, we introduce an innovation into the traditional chain ladder by making explicit use of cumulative distribution functions. We show how that can broaden the scope and usefulness of the chain ladder.

We begin with an example in Table 1, which contains the paid losses and allocated loss adjustment expenses (in millions of dollars) for Workers' Compensation from the 1992 Best's Aggregates and Averages. Best's data include most of the net insurance volume in the United States. The accident years are from 1982 to 1991.

AY	1	2	3	4	5	6	7	8	9	10
1	2409	4825	6173	7003	7578	7990	8295	8551	8747	8894
2	2602	5429	7010	8037	8738	9230	9589	9866	10060	
3	3105	6495	8617	9978	10886	11503	11959	12296		
4	3316	7241	9634	11202	12217	12915	13439			
5	3416	7701	10439	12216	13330	14105				
6	3831	8934	12063	14023	15266					
7	4527	10547	14309	16588						
8	4934	11881	16070							
9	5300	12901								
10	5488									

Table 1: Workers' Compensation paid loss triangle.

Table 2 contains the corresponding age-to-age (ata) factors, also called the linked ratios:

AY	1	2	3	4	5	6	7	8	9
1	2.003	1.279	1.134	1.082	1.054	1.038	1.031	1.023	1.017
2	2.086	1.291	1.147	1.087	1.056	1.039	1.029	1.020	
3	2.092	1.327	1.158	1.091	1.057	1.040	1.028		
4	2.184	1.330	1.163	1.091	1.057	1.041			
5	2.254	1.356	1.170	1.091	1.058				
6	2.332	1.350	1.162	1.087					
7	2.330	1.357	1.159						
8	2.408	1.353							
9	2.434								
slope	0.055	0.011	0.004	0.001	0.001	0.001	-0.001	-0.003	

Table 2: The actual age-to-age factors.

We discuss in general two problems evident in Table 2, and how the integration of cumulative distribution functions into the reserving scheme alleviates them.

Problem #1: The factors in the first few columns exhibit marked variations, as indicated by the regression slopes in the last row. The fact that most of the changes in the factors in the first few columns are increases raises the possibility that at least some of the variations are non-stochastic. We mention two common techniques to deal with this problem.

The first technique is to consider only the factors in the last few available years. For instance, the 12:24 estimate may be the average of the last four factors in the first column. The assumption is that the recent past is more reflective of the future than the distant past. While the assumption is generally sound, there are a few deficiencies with the approach. First, not all data, and information, are being used. Secondly, averaging precludes the continuation of a trend, and this preclusion may not be desirable.

The second common technique to resolve the trending problem is similar to the methods in BERQUIST and SHERMAN (1977). This approach is the regression fitting of the factors to a line or curve, from which the missing factors can be estimated. One difficulty of this approach is that the last several columns have few factors, so any single one, which after all is just a realization of a random variable, has an inordinately large amount of influence on the process. Secondly, fitting to a line compels the continuation of a trend. It cannot address a situation where the factors in a column attain either a minimum or maximum. As will be shown, that is a common occurrence for this set of data. Fitting to a curve on the other hand requires much information and therefore can result in high variability.

can be represented in terms of the distribution values. For example, the factor between the development periods j and $j+1$ is:

$$(2.1) \qquad r_j = F(j+1)/F(j).$$

We work with one function in particular, the transformed log-normal:

$$(2.2) \qquad F(t) = F(t;\mu,\sigma,\tau) = \Phi\left[\operatorname{sgn}(\ln t)\left|\ln t\right|^{\tau};\mu,\sigma\right]; \quad t>0,$$

where Φ is the normal distribution with mean μ and standard deviation σ. The parameter τ is the exponent of the function. Our selection of this function is dictated by the fact that, of all the functions that we tested, it best describes the WC payment patterns. We have more to say on this in section 6.

To simulate the non-stochastic variations in the ata factors, we allow the three function parameters to vary by accident year. Let k denote an accident year and (μ_k,σ_k,τ_k) denote the corresponding function parameters. We assume that these parameters can be expressed in the following polynomial forms:

$$(2.3) \qquad \begin{aligned} \mu_k &= \mu + (k-1)\alpha_1 + (k-1)^2\alpha_2 + (k-1)^3\alpha_3 + \ldots \\ \sigma_k &= \sigma + (k-1)\beta_1 + (k-1)^2\beta_2 + (k-1)^3\beta_3 + \ldots \\ \tau_k &= \tau + (k-1)\chi_1 + (k-1)^2\chi_2 + (k-1)^3\chi_3 + \ldots \end{aligned}$$

The right hand side (RHS) of (2.3) has the following interpretations: (μ,σ,τ) are the base parameters, $(\alpha_1,\beta_1,\chi_1)$ are the linear annual changes, $(\alpha_2,\beta_2,\chi_2)$ are the quadratic annual changes, and so on. We also refer to the first set as the base coefficients, the second the linear coefficients, the third the quadratic coefficients, and so forth. All coefficients are assumed independent. For any given problem, only some are statistically significant. A major part of the analysis is the determination of all those.

If all annual changes are statistically insignificant, the parameters and the ata factors do not vary by accident years. In that case, we retrieve somewhat the traditional chain ladder, but the methodology to estimate the factors differs from the traditional approach.

If any of the annual changes is statistically significant, the parameters and the ata factors vary by accident years, and thus exhibit non-stochastic variations. In such case, each accident year in effect has its own payment pattern, different from those of any other year.

There are many reasons for non-stochastic variations in the parameters. A major one is that the insurance operations are changing. Another is that the environmental climate in which the insurance operates is changing. Clearly, there can be many other factors. Equations (2.3) are simply our attempt to measure the extent to which all these factors affect the payment patterns. It is important to realize that (2.3) does not compel the accident years to have different payment patterns. It simply allows that possibility. If it turns out that the accident years have a statistically similar payment pattern, then the annual changes should be statistically insignificant.

We define the partial derivatives as follows:

$$(2.4) \qquad G = \frac{\partial F}{\partial \mu}, \quad H = \frac{\partial F}{\partial \sigma}, \quad K = \frac{\partial F}{\partial \tau}.$$

3. THE GOVERNING EQUATION

Let y_{kj} be the cumulative paid loss amount for accident year k at development year j. Then the *loss* factor is:

$$(3.1) \qquad q_{kj} = \frac{y_{k,j+1}}{y_{kj}},$$

and the *development* factor is:

$$(3.2) \qquad r_{kj} = \frac{F(j+1;\mu_k,\sigma_k,\tau_k)}{F(j;\mu_k,\sigma_k,\tau_k)}.$$

We define the *governing* equation as:

$$(3.3) \qquad q_{kj} = r_{kj} + \ln(r_{kj})\varepsilon_{kj}, \qquad \varepsilon_{kj} \sim N(0,s^2).$$

The ε_{kj}'s are assumed independent and normally distributed with mean zero and standard deviation s. We call ε_{kj} and $\ln(r_{kj})\varepsilon_{kj}$ the error and the error term, respectively. Equation (3.3) simply says that successive payments should obey the payment pattern, with some stochastic error. As its name implies, a governing equation contains the basic assumptions and governs the behavior of the model. Given it, everything else should logically follow.

We note that the development factors are in essence a proxy for the payment pattern. If all actual and estimated ata factors closely match, then we infer that the distribution accurately depicts the real payment pattern. If there is a consistent mismatch in some of the factors, then we cannot make that inference.

Undoubtedly, the most unusual feature of (3.3) is the scaling function $\ln(r_{kj})$. It is needed because the magnitudes of the error terms change drastically throughout the development. The magnitude is large at the beginning of development, it is small near full development, it is zero at full development, and it goes through the whole continuum in between. In section 8, we discuss this subject in detail with numerical examples.

We mention two mathematical anomalies which the scaling function prevents. First, suppose that we are very far in the tail where all development has definitively ceased. Therefore, $q_{kj} = 1$. If our model is any good, it would also predict $r_{kj} = 1$. Hence the error term must be zero, and the presence of the scaling function ensures that equality.

Secondly, suppose we want to compute the variance of the ultimate loss amount. As will be shown in section 5, that includes the sum of an infinite series, each term of which corresponds to the error term in (3.3). If the scaling function were absent, the infinite series and the variance would have no finite limits. But if it were present, the terms in the series would approach zero asymptotically, and the series would have a finite limit.

In section 7, we show that, without a scaling function, the model cannot satisfy the normality assumption. In section 8, we demonstrate that, in such case, the error terms are not properly scaled.

We will primarily work with (3.3), but the general form of the governing equation is:

$$(3.4) \qquad q_{kj} = r_{kj} + w(y_{kj})b(r_{kj})\varepsilon_{kj}.$$

ε_{kj} is the error, $b(r_{kj})$ is the scaling function, $w(y_{kj})$ is the proportionality function, and the product of all three, $w(y_{kj})b(r_{kj})\varepsilon_{kj}$, is the error term. The scaling function must satisfy the following conditions:

$$(3.5) \qquad b(r) > 0, \ \text{for} \ r > 1; \text{and } b(1) = 0.$$

We call $w(y_{kj})$ the proportionality function, because it dictates the loss amount proportionality of the error term. If we multiply (3.4) by y_{kj}, then:

$$(3.6) \qquad y_{k,j+1} = y_{kj}\left[r_{kj} + w(y_{kj})b(r_{kj})\varepsilon_{kj}\right].$$

For the particular form of (3.3), $w(y_{kj}) = 1$, and the error term is proportional to y_{kj}. We therefore call that the linear proportionality function. Similarly, when $w(y_{kj}) = 1/y_{kj}^{1/2}$ and $\ln(y_{kj})/y_{kj}$, the proportionality functions are square root and logarithmic, respectively.

A priori, we do not have any reference to prefer one set of scaling and proportionality functions over another. In section 9, we test a number of them and compare their numerical results, with the deviations and chi square values as the measuring sticks. Our conclusion is that the most appropriate model has the logarithmic scaling function and the linear proportionality function, as in (3.3).

MURPHY (1995) presented three models which can be written as follows:

(3.7) $y_{k,j+1} = y_{kj}r + \varepsilon_{kj}$, Least Squares Multiplicative (LSM);
$y_{k,j+1} = y_{kj}(r + \varepsilon_{kj})$, Simple Average Development (SAD); and
$y_{k,j+1} = y_{kj}r + y_{kj}^{1/2}\varepsilon_{kj}$, Weighted Average Development (WAD).

There are two major differences between (3.6) and (3.7). Murphy's models do not have a scaling function. *And they assume the development factor r to be constant in any given development period,* whereas we allow r_{kj} to vary within a development period. The forms of (3.7) have different proportionality functions. With our terminology, SAD takes the linear function, and WAD takes the square root function. For LSM, $w(y_{kj}) = 1/y_{kj}$.

For their chain-ladder stochastic models, many authors (VERRALL, 1990; ZEHNWIRTH, 1990) have assumed that the loss amounts are log-normally distributed. STANARD (1985) and HALLIWELL (1996) have shown that such models have inherent upward bias. MACK (1995) argued that they suffer higher variability. Our model bypasses these difficulties, because (3.3) implies that the loss quantities are normally distributed, as will be shown in section 5.

The normal distribution for the loss amounts has two additional advantages. First, if the liability for an accident year is normally distributed, the sum for all accident years is also normally distributed., and the variance of the sum can be calculated. Secondly, suppose we have another model which also gives normally distributed estimates, the combination of estimates from the two models is normally distributed.

The governing equation (3.3) is to be used in two ways, matching and estimation:

a) Matching. There are forty-five points (ata factors) in Table 2. We apply (3.3) to every point. From this matching, we obtain estimates for the n variables so as to minimize the sum of squares of errors.

b) Estimation. For the particular case of (3.3), (3.6) becomes:

$$(3.8) \qquad y_{k,j+1} = y_{kj}\left[r_{kj} + \ln(r_{kj})\varepsilon_{kj}\right].$$

Equation (3.8) gives the estimate at the next period based on the actual or estimated value at the previous period. If y_{kj} is an actual amount, we assume that there is no error associated with it; actually, this assumption is a direct consequence of the governing equation itself. And all the variance of the estimate $y_{k,j+1}$ comes from the parameter error in r_{kj} and the process error in ε_{kj}. If y_{kj} is an estimated amount, then its variance also contributes to that of the next estimate. Our convention is that the RHS of (3.6) and (3.8) should take the actual y_{kj} whenever available.

MACK (1995) made an important distinction. In many models (VERRALL, ZEHNWIRTH), y_{kj} in (3.8) is the expected value; whereas, in the traditional chain ladder, it is the actual value. In this paper, the latter is the case.

To recapitulate, our entire model consists of equations (2.2), (2.3), (3.1), (3.2) and (3.3). For a given set of data, we have to find all the statistically significant coefficients of (2.3) such that the sum of squares of errors in (3.3) are minimized, given that the payment patterns are specified by the cumulative distribution function in (2.2). The governing equation of (3.3) deserves its name because it has the central role of linking together all the different elements of the system.

4. THE ITERATIVE REGRESSION PROCESS

As described in the previous section, the system is a highly non-linear one; therefore it is impossible to obtain the solution in closed form or in one step. Instead, we apply the methods of perturbation theory to derive an iterative regression process, the application of which systematically leads to the solution.

To minimize the algebra, all derivations in this section are for the model in which only the base coefficients (μ, σ, τ) are variables. In the general model, we have to solve the regression system for n variables.

We begin by perturbing every variable:

$$(4.1) \qquad \mu \to \mu + \Delta\mu, \quad \sigma \to \sigma + \Delta\sigma, \quad \tau \to \tau + \Delta\tau .$$

We may think of a perturbation as the replacement of a value (μ, for instance) by the sum of that value and an infinitesimal increment ($\Delta\mu$). The value is a known quantity, and the increment is an unknown quantity to be found. The reason why a perturbation is helpful is that, since the increment is assumed infinitesimal, we may retain only the linear terms in the Taylor's series expansions. Instead of a non-linear system, we in effect solve a series of linear systems. The successive solutions of the linear systems lead us closer and closer to the solution of the non-linear system.

First, we supply a guess (μ, σ, τ). Based on that guess, the regression process gives us the incremental $(\Delta\mu, \Delta\sigma, \Delta\tau)$. The sum of the guess and the increment provides the next guess. We keep up the iteration process until it converges to the solution.

Using the definition of the derivatives in (2.4), the perturbation of F has the following form:

$$(4.2) \qquad F(j;\mu+\Delta\mu, \sigma+\Delta\sigma, \tau+\Delta\tau) = F(j;\mu,\sigma,\tau) + G(j;\mu,\sigma,\tau)\Delta\mu + H(j;\mu,\sigma,\tau)\Delta\sigma + K(j;\mu,\sigma,\tau)\Delta\tau .$$

In a more general case where, for instance, α_1 and α_2 are also variables, the RHS of (4.2) would include the terms $(k-1)G\Delta\alpha_1$ and $(k-1)^2 G\Delta\alpha_2$.

Using (4.2) in (3.2), we have the following for the perturbation of the development factor:

(4.3)
$$r_{kj}(\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau)=r_{kj}(\mu,\sigma,\tau)\left[1+\left(\frac{G_{k,j+1}}{F_{k,j+1}}-\frac{G_{kj}}{F_{kj}}\right)\Delta\mu+\left(\frac{H_{k,j+1}}{F_{k,j+1}}-\frac{H_{kj}}{F_{kj}}\right)\Delta\sigma+\left(\frac{K_{k,j+1}}{F_{k,j+1}}-\frac{K_{kj}}{F_{kj}}\right)\Delta\tau\right].$$

where $F_{kj}=F(j;\mu_k,\sigma_k,\tau_k)$, and so on. We give the full derivation of (4.3) in Appendix A. If we substitute (4.3) into (3.3), the result is:

(4.4)
$$r_{kj}\left(\frac{G_{k,j+1}}{F_{k,j+1}}-\frac{G_{kj}}{F_{kj}}\right)\Delta\mu+r_{kj}\left(\frac{H_{k,j+1}}{F_{k,j+1}}-\frac{H_{kj}}{F_{kj}}\right)\Delta\sigma+r_{kj}\left(\frac{K_{k,j+1}}{F_{k,j+1}}-\frac{K_{kj}}{F_{kj}}\right)\Delta\tau+\ln(r_{kj})\varepsilon_{kj}=q_{kj}-r_{kj}\ .$$

After division by the scaling function, (4.4) yields exactly a regression system for the increment variables $(\Delta\mu,\Delta\sigma,\Delta\tau)$. In matrix form, we have:

(4.5)
$$M\Delta+\varepsilon=b,$$
$$\Delta=(\Delta\mu,\Delta\sigma,\Delta\tau)^T\ ,\quad \varepsilon=\{\varepsilon_{kj}\}\ ,\quad b=\{(q_{kj}-r_{kj})/\ln(r_{kj})\}.$$

M is a $45\text{x}3$ coefficient matrix. This matrix changes after every iteration, since after each one, we have a new set of coefficients. For example, the row of M for accident year k and development period j has the following elements:

(4.6)
$$M^1_{kj}=\frac{r_{kj}}{\ln(r_{kj})}\left[\frac{G_{k,j+1}}{F_{k,j+1}}-\frac{G_{kj}}{F_{kj}}\right],$$
$$M^2_{kj}=\frac{r_{kj}}{\ln(r_{kj})}\left[\frac{H_{k,j+1}}{F_{k,j+1}}-\frac{H_{kj}}{F_{kj}}\right],$$
$$M^3_{kj}=\frac{r_{kj}}{\ln(r_{kj})}\left[\frac{K_{k,j+1}}{F_{k,j+1}}-\frac{K_{kj}}{F_{kj}}\right].$$

In the more general case of n variables, M would be a $45\text{x}n$ matrix, and Δ an n-dimensional vector.

The solution of (4.5) so as to minimize the sum of squares of errors is well known in multiple regression analysis. It is,

(4.7)
$$\Delta=(M^TM)^{-1}M^Tb\ .$$

The sum of (4.7) and the current guess constitutes the succeeding guess. When the process is stable and leads toward the solution, the sum of squares of errors of the succeeding guess is always smaller than that of the current guess. Therefore, if we continue the iteration until the guesses no longer vary, the resultant solution is guaranteed to have the smallest possible sum. We can see from (4.7) that the solution must satisfy:

(4.8) $\left(M^T M\right)^{-1} M^T b = 0.$

This actually may only be a local solution. Globally, the possibility of multiple solutions cannot be discounted. In practice, however, we have never encountered multiple solutions.

The vector in (4.7) indicates the general direction in which the solution lies. When the initial guess is very far from the solution, if we take full steps as indicated by (4.7), the guesses may quickly become unstable. In such cases, we should take steps that are fractions of the full steps. The steps should be sufficiently small until the iterative process enters some stable mode, then the step size may be increased. We have even encountered situations in which, by taking full steps, the guesses spiral stably toward the solution, but very slowly. In such cases, the full steps overstep the solution, and the convergence can be accelerated by taking smaller steps.

Finally, there exists the possibility of no convergence at all. This may be the result of either of two scenarios. First, the distribution being used may not be stable in the iterative regression scheme. For instance, we find any Pareto-type distribution to be highly unstable. Secondly, the distribution may not be the right one for the loss data being considered.

When the distribution is the transformed log-normal, the convergence is quite fast, and the initial guess need not be close to the solution.

The estimate for the variance of errors is:

$$(4.9) \qquad s^2 = \frac{1}{45-n}\sum_{k=1}^{9}\sum_{j=1}^{10-k}\varepsilon_{kj}^2, \quad \varepsilon_{kj} = \frac{q_{kj} - r_{kj}}{\ln(r_{kj})}.$$

The denominator in (4.9) is the number of degrees of freedom: 45 is the number of data points, and n is the number of variables. From (4.1), we have the following relationship for the variances of the coefficients:

$$(4.10) \qquad Var(\mu) = Var(\Delta\mu) \ ,$$

and so on. From standard regression analysis, we obtain the parameter variance matrix as:

$$(4.11) \qquad Var(\mathrm{P}) = \begin{bmatrix} Var(\mu) & Cov(\mu,\sigma) & Cov(\mu,\tau) \\ Cov(\mu,\sigma) & Var(\sigma) & Cov(\sigma,\tau) \\ Cov(\mu,\tau) & Cov(\sigma,\tau) & Var(\tau) \end{bmatrix} = s^2\left(M^T M\right)^{-1}.$$

5. THE ULTIMATE ESTIMATES

In this section, we assume that the iterative regression process has found all the coefficients and we have to obtain the estimates of the ultimate loss amounts and their variances. In particular, consider the k-th accident year, which has $y_{k,11-k}$ as the last actual cumulative paid amount. Using (3.1) and (3.3), the estimate for the loss amount at the next period is:

$$(5.1) \qquad y_{k,11-k+1} = y_{k,11-k}q_{k,11-k} = y_{k,11-k}\left[r_{k,11-k} + \ln(r_{k,11-k})\varepsilon_{k,11-k}\right].$$

After another iteration, the estimate for the succeeding period is:

(5.2) $$y_{k,11-k+2} = y_{k,11-k} q_{k,11-k} q_{k,11-k+1} = y_{k,11-k}\left[r_{k,11-k} + \ln(r_{k,11-k})\varepsilon_{k,11-k}\right]\left[r_{k,11-k+1} + \ln(r_{k,11-k+1})\varepsilon_{k,11-k+1}\right].$$

After repeated iterations, the estimate for the ultimate amount can be expressed as:

(5.3) $$y_k = y_{k,11-k}\prod_{j=0}^{\infty} q_{k,11-k+j} = y_{k,11-k}\prod_{j=0}^{\infty}\left[r_{k,11-k+j} + \ln(r_{k,11-k+j})\varepsilon_{k,11-k+j}\right].$$

In (5.3), we have an infinite variety of error terms. There are the linear error terms, containing ε_{kj}. There are the quadratic error terms, containing $\varepsilon_{kj}\varepsilon_{kl}$; and so on. A linear term is proportional to s, a quadratic term to s^2. Since s is generally small, the linear terms dominate in absolute value over the other error terms. We are thus justified in retaining only the linear terms, and (5.3) becomes:

(5.4) $$y_k = \frac{y_{k,11-k}}{F_{k,11-k}}\left[1 + \sum_{j=0}^{\infty}\frac{\ln(r_{k,11-k+j})}{r_{k,11-k+j}}\varepsilon_{k,11-k+j}\right].$$

Equation (5.4) is correct to the leading order. The ultimate loss is normally distributed, since it is the sum of normally distributed quantities. Taking the expected value of (5.4), we have:

(5.5) $$E\{y_k\} = y_{k,11-k} / F_{k,11-k}.$$

$F_{k,11-k} = F(11-k; \mu_k, \sigma_k, \tau_k)$ is the percent paid to date, and its reciprocal is the age-to-ultimate factor. Equation (5.5) says that the expected ultimate amount is the product of the paid-to-date amount and the age-to-ultimate factor, as we would expect. In the rest of this section, for the sake of brevity, we write y_k to denote the expected value of the same quantity.

To obtain the variance from (5.4), we use the following formula. Let $W = XY$ be the product of two independent stochastic quantities, then

(5.6) $$Var(W) = \bar{X}^2[Var(Y)] + \bar{Y}^2[Var(X)] + [Var(X)][Var(Y)],$$

where the bars denote expected values. If we apply (5.6) to (5.4), then we have:

(5.7) $$Var(y_k) = y_{k,11-k}^2\left[Var\left(\frac{1}{F_{k,11-k}}\right) + \frac{s^2}{F_{k,11-k}^2}\sum_{j=0}^{\infty}\left(\frac{\ln(r_{k,11-k+j})}{r_{k,11-k+j}}\right)^2 + s^2 Var\left(\frac{1}{F_{k,11-k}}\right)\sum_{j=0}^{\infty}\left(\frac{\ln(r_{k,11-k+j})}{r_{k,11-k+j}}\right)^2\right].$$

In the derivation of (5.7), we assume that there is no error associated with the actual $y_{k,11-k}$. The variance in (5.7) is the sum of three terms. The first is the parameter error, which is just the variance of the age-to-ultimate factor.

The second term is the process error. It is the sum of an infinite series, because, at each development period, an additional amount of error contributes to the total, and theoretically there are an infinite number of periods. In Appendix B, we prove that the series possesses a finite limit. We can also see the pivotal role of the scaling function: without it, the series would have no finite limit.

The third term is the product of the parameter and process errors. Every variance is proportional to s^2; every product of variances is proportional to s^4, and hence negligible. In the following derivations and calculations, we ignore those terms altogether. To the leading order, the variance in (5.7) is therefore the sum of the parameter and process errors.

To estimate the parameter error in (5.7), again we resort to perturbation:

$$\text{(5.8)} \qquad \frac{1}{F_{k,11-k}(\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau)} = \frac{1}{F_{k,11-k}(\mu,\sigma,\tau)}\left[1-\frac{G_{k,11-k}}{F_{k,11-k}}\Delta\mu - \frac{H_{k,11-k}}{F_{k,11-k}}\Delta\sigma - \frac{K_{k,11-k}}{F_{k,11-k}}\Delta\tau\right].$$

The derivation of (5.8) is in Appendix A.

Taking the variance of (5.8), we have:

$$\text{(5.9)} \qquad Var\left[\frac{1}{F_{k,11-k}}\right] = \frac{1}{F_{k,11-k}^2}\left\{\left[\frac{G_{k,11-k}}{F_{k,11-k}}\right]^2 Var(\mu) + 2\frac{G_{k,11-k}H_{k,11-k}}{F_{k,11-k}^2}Cov(\mu,\sigma) + \Lambda\right\},$$

where the dots represent the other variances and covariances. Finally, after collecting terms, (5.7) becomes:

$$\text{(5.10)} \quad Var(y_k') = y_k^2\left\{\left[\frac{G_{k,11-k}}{F_{k,11-k}}\right]^2 Var(\mu) + 2\frac{G_{k,11-k}H_{k,11-k}}{F_{k,11-k}^2}Cov(\mu,\sigma) + \Lambda + s^2\sum_{j=0}^{\infty}\left[\frac{\ln(r_{k,11-k+j})}{r_{k,11-k+j}}\right]^2\right\}.$$

We define the sum total of all the ultimate losses as:

$$\text{(5.11)} \qquad y_T = \sum_{k=1}^{10} y_k .$$

We note that y_T is normally distributed. It can be shown that the variance is:

$$\text{(5.12)} \qquad Var(y_T) = Var(\mu)\left[\sum_{k=1}^{10} y_k \frac{G_{k,11-k}}{F_{k,11-k}}\right]^2 + 2Cov(\mu,\sigma)\left[\sum_{k=1}^{10} y_k \frac{G_{k,11-k}}{F_{k,11-k}}\right]\left[\sum_{k=1}^{10} y_k \frac{H_{k,11-k}}{F_{k,11-k}}\right] + \Lambda + s^2\sum_{k=1}^{10} y_k^2 \sum_{j=0}^{\infty}\left[\frac{\ln(r_{k,11-k+j})}{r_{k,11-k+j}}\right]^2 .$$

Comparing (5.10) to (5.12), we observe that:

$$\text{(5.13)} \qquad \sum_{k=1}^{10}\left[y_k \frac{G_{k,11-k}}{F_{k,11-k}}\right]^2 \neq \left[\sum_{k=1}^{10} y_k \frac{G_{k,11-k}}{F_{k,11-k}}\right]^2,$$

and so on. Therefore, we conclude that:

$$(5.14) \qquad \sum_{k=1}^{10} Var(y_k) \neq Var(y_T).$$

In words, the sum of the variances does not equal the variance of the sum. If the ultimate losses have negative correlation, the former is greater. If they have positive correlation, the latter is greater. We expect the second scenario, because, whatever the realization of a random variable, it most likely affects the ultimate losses in the same direction. In the next section, we show with a numerical example that such is the case. To numerically evaluate each series, we sum up the first fifty terms.

6. NUMERICAL RESULTS

In the previous two sections, we present the mathematical formulas, for the most part assuming only the base coefficients are statistically significant. In this section, we present the numerical solution to the Table 1 triangle. For this problem, four of the annual changes are significant; therefore, the reader will have to modify the formulas in the previous sections to obtain the numerical solutions in this one. The statistically significant coefficients for the Table 1 triangle are:

	μ	σ	τ	β_1	χ_1	β_2	χ_2
estimate	.7582	1.0838	.8988	-.0459	.0450	.0028	-.0057
s.d.	.0051	.0085	.01114	.0047	.0079	.0005	.0011

Table 3: Estimates and standard deviations of the coefficients.

In the parlance of section 10, the solution has the correct parametrization. Our criterion for statistical significance is that an estimate must be at least twice as large in absolute value as its standard deviation. This criterion translates into: if the true value of a variable were indeed zero, we have a 4.6% probability of accepting it as a non-zero variable.

With the values in Table 3, the equations of (2.3) simplify to:

$$(6.1) \qquad \mu_k = \mu, \quad \sigma_k = \sigma + (k-1)\beta_1 + (k-1)^2\beta_2,$$
$$\tau_k = \tau + (k-1)\chi_1 + (k-1)^2\chi_2, \quad 1 \le k \le 10.$$

The process with which we obtain (6.1) is as follows. We begin with the model in which all coefficients up to and including the cubic ones are variables. In such model, we have twelve coefficients to estimate. We apply the iterative regression process to obtain the solution. If there are at least two statistically insignificant coefficients in the solution, we eliminate the most obviously insignificant one. We continue the process until all remaining coefficients are statistically significant.

In going from the estimation of twelve variables to that of seven variables, we have to examine six permutations of the model. Each permutation has a unique set of variables to be estimated. Given the assumptions that the base coefficients are always significant, which may not be true for the mean, and that all fourth- and higher-power coefficients are always insignificant, there are $2^9 = 512$ permutations, for a distribution of three parameters. In a format like EXCEL, which we use for this paper, we have to construct a separate spreadsheet for each permutation. We have fortunately systematized the process, so that a complete conversion from one permutation to another takes only a few minutes. We construct permutations as needed; we do not construct all at the same time. In some languages such as APL, which we have used in the past, one set of computer code suffices for all possible permutations, including different sizes of the data. Despite this obvious advantage of APL, we highly recommend EXCEL, given the choice between the two mediums. A programming error, especially a subtle one that does not result in an unreasonable solution, is much more likely to escape detection in APL. And it is much easier to build additional features into an EXCEL spreadsheet than into an APL code.

From (6.1), we estimate the mean to be a constant, and both the deviation and the exponent to be quadratic curves. Substituting the values in (6.1) into the formula for the transformed log-normal of (2.2), we obtain the values of the c.d.f.'s:

AY	1	2	3	4	5	6	7	8	9	10
1	.2421	.4857	.6196	.7047	.7629	.8048	.8352	.8604	.8795	.8949
2	.2331	.4812	.6259	.7180	.7802	.8243	.8567	.8812	.9001	.9151
3	.2249	.4776	.6315	.7293	.7947	.8402	.8730	.8972	.9157	.9299
4	.2475	.4751	.6363	.7384	.8058	.8521	.8848	.9087	.9265	.9401
5	.2111	.4739	.6402	.7450	.8135	.8599	.8925	.9159	.9332	.9462
6	.2059	.4740	.6429	.7487	.8174	.8637	.8960	.9190	.9360	.9487
7	.2019	.4756	.6445	.7495	.8174	.8633	.8953	.9182	.9350	.9477
8	.1994	.4787	.6447	.7472	.8135	.8586	.8902	.9131	.9301	.9429
9	.1983	.4834	.6437	.7418	.8057	.8494	.8805	.9032	.9207	.9340
10	.1986	.4896	.6412	.7332	.7935	.8354	.8658	.8885	.9060	.9198

Table 4: The values of the cumulative distribution functions.

Let's give an example of how one value in Table 4 is calculated. From (6.1), we obtain the parameters for the fifth accident year as (.7582,.9446,.9872). From (2.2), we get that $\bar{F}$(4;.7582,.9446,.9872)=.7450.

Table 4 says that, for the first accident year, 24.2% of the ultimate amount has been paid after one year, and 88.5% after 10 years. We then calculate the tail factor as the reciprocal of 88.5%. We note that the tail factors vary by accident years.

To obtain the estimated ata factors, we compute the quotients of successive values in Table 4.

AY	1	2	3	4	5	6	7	8	9
1	2.006	1.276	1.137	1.083	1.055	1.039	1.029	1.022	1.018
2	2.064	1.301	1.147	1.087	1.057	1.039	1.029	1.021	1.017
3	2.124	1.322	1.155	1.090	1.057	1.039	1.028	1.021	1.016
4	2.185	1.339	1.160	1.091	1.057	1.038	1.027	1.020	1.015
5	2.245	1.351	1.164	1.092	1.057	1.038	1.026	1.019	1.014
5	2.245	1.351	1.164	1.092	1.057	1.038	1.026	1.018	1.014
6	2.303	1.356	1.165	1.092	1.057	1.037	1.026	1.018	1.014
7	2.355	1.355	1.163	1.091	1.056	1.037	1.026	1.018	1.014
8	2.401	1.347	1.159	1.089	1.055	1.037	1.026	1.019	1.014
9	2.438	1.331	1.152	1.086	1.054	1.037	1.026	1.019	1.014
10	2.465	1.310	1.143	1.082	1.053	1.036	1.026	1.020	1.015
slope	0.055	0.011	0.004	0.002	0.001	0.000	-0.001	-0.001	

Table 5: The estimated age-to-age factors.

The match between Tables 2 and 5 is generally quite close. The slopes in the two tables match almost exactly. To make them directly comparable, those in the latter are calculated using only factors above the diagonal. The close match of the factors implies that the transformed log-normal adequately describes the payment patterns of the data.

We also consider using one of the following three distributions as the c.d.f.: the transformed normal, the transformed log-gamma, and the transformed gamma. For every of these functions, either a solution cannot be found, or there is a consistent mismatch of the ata factors somewhere in the triangle. We therefore believe that none of the three functions describes well the payment patterns of Workers' Compensation.

The table below presents the estimated cumulative paid amounts:

AY	1	2	3	4	5	6	7	8	9	10
1	2409	4833	6155	7021	7581	7994	8302	8535	8741	8900
2	2602	5370	7062	8042	8734	9232	9593	9863	10078	10227
3	3105	6595	8588	9952	10872	11509	11952	12292	12548	12744
4	3316	7245	9697	11180	12224	12918	13412	13802	14072	14278
5	3416	7669	10403	12148	13339	14091	14639	15023	15307	15520
6	3831	8821	12117	14048	15309	16131	16733	17164	17480	17717
7	4527	10662	14292	16641	18093	19107	19814	20321	20693	20973
8	4923	11846	16001	18624	20279	21401	22188	22759	23182	23503
9	5300	12921	17178	19796	21501	11668	23499	24110	24570	24925
10	5488	13528	17716	20257	21925	23082	23921	24549	25032	25412

Table 6: The estimated paid loss amounts.

Generally, we have quite good agreement between Tables 1 and 6. Let's give an example of how one value in the latter table is calculated. From Table 5, we have that $r_{54} = 1.092$. Therefore, the estimate for $y_{55} = 1.092 * 12216 = 13339$.

The estimates for the ultimate paid amounts and the corresponding standard deviations are:

AY	Ultimate	S.D.
1	9939	45
2	11176	40
3	13704	51
4	15188	62
5	16403	73
6	18676	91
7	22132	124
8	24925	166
9	26688	288
10	27629	583
total	186459	980

Table 7: Ultimate estimates.

Comparison of Tables 6 and 7 indicates that a considerable amount of liabilities lies in the tail. The standard deviation of the total is computed using (5.12). Under the assumption of mutual independence of ultimate estimates, the deviation of the total would only be 700. Since the variance of the sum is considerably greater than the sum of the variances, we infer that the ultimate estimates have a high degree of positive correlation.

7. THE CHI SQUARE TEST

In this section, we apply the chi square goodness-of-fit test to demonstrate the normality of the results. We also show that the model without the scaling function does not satisfy the normality assumption.

We define the *normalized error* as the quotient of the error and its standard deviation. From (3.3), we have:

$$(7.1) \qquad e_{kj} = \frac{\varepsilon_{kj}}{s} = \frac{q_{kj} - r_{kj}}{s \ln(r_{kj})}.$$

The normalized errors should follow the standard normal distribution. To test if that indeed is the case, we divide the real line into five intervals: $(-\infty, -.842)$, $(-.842, -.253)$, $(-.253, .253)$, $(.253, .842)$, and $(.842, \infty)$. We note that, if a random variable is normally distributed, each interval should contain 20% of the observations. We define the following two quantities: U_i as the number of observed e_{kj}'s in interval i, and V_i as the number of expected e_{kj}'s. Then the quantity χ^2 is defined as:

$$(7.2) \qquad \chi^2 = \sum_{i=1}^{5} \frac{\left(U_i - V_i\right)^2}{V_i}.$$

It is well known that χ^2 follow the chi square distribution with four degrees of freedom.

With $s = .0298$, we obtain the following normalized errors from Tables 2 and 5:

AY	1	2	3	4	5	6	7	8	9
1	-0.160	0.502	-0.740	-0.198	-0.366	-0.723	2.252	1.073	-1.349
2	1.044	-1.219	-0.158	0.199	-0.131	-0.336	0.388	-2.872	
3	-1.434	0.538	0.702	0.556	-0.345	0.547	0.431		
4	-0.056	-1.005	0.515	-0.253	-0.144	1.873			
5	0.383	0.525	1.446	-0.286	0.621				
6	1.183	-0.668	-0.451	-1.184					
7	-0.997	0.182	-0.817						
8	0.269	0.656							
9	-0.145								

Table 8: The normalized errors.

In Table 8, if there are either large positive or negative values grouped in at least one column, then the distribution does not fit well the payment patterns. We do not detect such a scenario in Table 8. The expected and observed frequencies are:

Interval	1	2	3	4	5
V_i	9	9	9	9	9
U_i	7	9	10	13	6

Table 9: The error frequencies.

The values in Table 9 give χ^2 =3.33. If the normalized errors come from the standard normal distribution, then there is a 50% probability that the chi square distribution with four degrees of freedom exceed 3.33. The normality assumption for the governing equation of (3.3) is therefore accepted.

If we use the model without a scaling function, many of the normalized errors, 26 to be exact, are bunched together in the middle interval, and χ^2 =43.56. The probability of the chi square distribution exceeding that value is nil. We therefore reject the normality assumption. In the next section, we discuss the reason why the model fails the test in that case.

8. ORDERS OF ERROR MAGNITUDES

In this section, we consider the differing orders of error magnitudes, and how the proper recognition of them is inextricably linked to the scaling function. We also indicate the reason that the model without the scaling function does not pass the normality test. We begin by considering the variances of the two points at opposite ends of the triangle, q_{19} and q_{91}.

From (3.3), the quantity q_{19}, which has the realized value of 1.0168, has the following formula:

$$(8.1) \qquad q_{19} = r_{19} + \ln(r_{19})\varepsilon_{19}.$$

Taking the expected value of (8.1), we get:

$$(8.2) \qquad E\{q_{19}\} = E\{r_{19}\} = F_{1,10}/F_{19} = 1.0175.$$

We obtain from (8.1) the variance as:

(8.3) $$Var(q_{19}) = Var(r_{19}) + s^2\left[\ln(r_{19})\right]^2.$$

The process error has the value:

(8.4) $$s^2\left[\ln(r_{19})\right]^2 = \left[.0298 * \ln(1.0175)\right]^2 = 2.7 * 10^{-7}.$$

To obtain the parameter error, we use the perturbation form for an ata factor of (4.3):

(8.5) $$r_{19}(\mu+\Delta\mu, \sigma+\Delta\sigma, \tau+\Delta\tau) = r_{19}\left[1+\left(\frac{G_{1,10}}{F_{1,10}}-\frac{G_{19}}{F_{19}}\right)\Delta\mu+\left(\frac{H_{1,10}}{F_{1,10}}-\frac{H_{19}}{F_{19}}\right)\Delta\sigma+\left(\frac{K_{1,10}}{F_{1,10}}-\frac{K_{19}}{F_{19}}\right)\Delta\tau\right].$$

Taking the variance of (8.5), we have:

(8.6) $$Var(r_{19}) = r_{19}^2\left[\left(\frac{G_{1,10}}{F_{1,10}}-\frac{G_{19}}{F_{19}}\right)^2 Var(\mu)+\Lambda\right] = 3.2 * 10^{-8}.$$

Combining terms, (8.3) gives a deviation of .0005. A normally distributed random variable with mean 1.0175 and deviation .0005 has a 10% probability of being less than 1.0168, the realized value.

The estimate for $y_{1,10}$ can be obtained from (3.1) as:

(8.7) $$y_{1,10} = y_{19}q_{19}.$$

The expected value of (8.7), given that $y_{19} = 8747$, is:

(8.8) $$E\left\{y_{1,10}\middle|y_{19} = 8747\right\} = y_{19}E\left\{q_{19}\right\} = 8747 * 1.0175 = 8900.$$

And the variance of $y_{1,10}$ is:

(8.9) $$Var\left(y_{1,10}\middle|y_{19} = 8747\right) = y_{19}^2 Var(q_{19}) = (8747 * .0005)^2 = 4.79^2.$$

A normally distributed random variable of mean 8900 and deviation 4.79 has a 10% probability of being less than 8894, the realized value.

The point q_{91}, which has the realized value of 2.434, has the following formula:

(8.10) $$q_{91} = r_{91} + \ln(r_{91})\varepsilon_{91}.$$

The mean and the variance are:

(8.11) $$E\{q_{91}\} = E\{r_{91}\} = F_{92}/F_{91} = 2.438,$$

(8.12) $$Var(q_{91}) = Var(r_{91}) + s^2\left[\ln(r_{91})\right]^2.$$

The process and parameter variances are:

(8.13) $$s^2\left[\ln(r_{91})\right]^2 = \left[.0298 * \ln(2.438)\right]^2 = 7.1 * 10^{-4},$$

(8.14) $$Var(r_{91}) = r_{91}^2\left[\left(\frac{G_{92}}{F_{92}} - \frac{G_{91}}{F_{91}}\right)^2 Var(\mu) + \Lambda\ + 8^4\left(\frac{K_{92}}{F_{92}} - \frac{K_{91}}{F_{91}}\right)^2 Var(\chi_2)\right] = 2.7 * 10^{-4}.$$

Combining terms in (8.12) gives a deviation of .031. A normally distributed random variable of mean 2.438 and deviation .031 has a 45% probability of being less than 2.434, the realized value.

The mean and variance of y_{92} are:

(8.15) $$E\{y_{92}|y_{91} = 5300\} = y_{91} * E\{r_{91}\} = 5300 * 2.438 = 12921.$$

(8.16) $$Var(y_{92}|y_{91} = 5300) = y_{91}^2 Var(q_{91}) = (5300 * .031)^2 = 165^2.$$

We note that the variances of q_{19} and q_{91} have very different orders of magnitudes. Their ratio is:

(8.17) $$\frac{Var(q_{91})}{Var(q_{19})} = \left(\frac{.028}{.0007}\right)^2 \approx 3200.$$

The variance of q_{91} is therefore several thousand times that of q_{19}. This is not surprising, because there is much more development, and variability, at the former than at the latter. The relative values of 2.438 and 1.018 attest to this. We can carry this further by saying that the error of a point at full development is zero. This conclusion is not only reasonable, but also inescapable, if we think with this perspective: given a realized value at full development, the estimate at the succeeding period is known with absolute certainty, namely that very same value. We note that, at full development, the parameter error is also zero, because an infinitesimal perturbation of the parameters cannot nudge the c.d.f. from unity.

The role of the scaling function is imperative in that it is the mechanism through which the different orders of errors are recognized. Without it, the process errors of (8.4) and (8.13) would be exactly equal, and the ratio in (8.17) would be very close to unity. In effect, the model would not be able to differentiate the widely divergent orders of errors. For points far from full development, the variances are understated; and for points close to full development, they are overstated. Because the normalized error is essentially the ratio of the actual error to the expected average error, the effect on it is just the opposite. For points far from full development, the normalized errors are overstated; and for points close to full development, they are understated. It is therefore not surprising that the model without the scaling function cannot pass the normality test.

9. COMPARISON OF SCALING AND PROPORTIONALITY FUNCTIONS

In this section, we want to compare the effects of different scaling functions. First, we set the proportionality function $w(y_{kj}) = 1$ in (3.4), then calculate the solutions to the Table 1 triangle using four different scaling functions, in addition to the logarithmic. In Table 10, we show the total liabilities, the deviations, the chi square values, and the implied percentages.

	Name	$b(r_{kj})$	y_T	deviation	χ^2	percent	comment
1	logarithmic	$\ln(r_{kj})$	186,459	980	3.33	50.4%	good
2	linear	$r_{kj}-1$	186,469	1095	5.11	27.6%	good
3	square root	$(r_{kj}-1)^{1/2}$	186,794	1331	11.11	2.5%	fair
4	square	$(r_{kj}-1)^2$	179,448	10146	10.00	4.0%	poor
5	no scaling	1	187,037	divergent	43.56	0.0%	invalid

Table 10: Comparison of scaling functions.

For the no scaling function, the infinite series for the process errors of (5.10) have no limits. The two best scaling functions are the logarithmic and the linear. Their estimates are identical, for all intents and purposes. The logarithmic gives the slightly lower deviation and chi square value.

Incidentally, as their argument approaches unity, these two functions have the same asymptotic behavior. Symbolically,

$$(9.1) \qquad \ln(r) \sim r-1, \ r \to 1.$$

No other function in Table 10 shares this property. We believe that the two functions do indeed have the correct error scaling.

We have analyzed quite a number of different loss triangles. The logarithmic and the linear invariably give nearly identical estimates and deviations, but the former consistently gives the lower chi square values. We therefore select the logarithmic as the most appropriate scaling function.

In Table 11, with the logarithmic scaling function, we compare three proportionality functions: the linear, logarithmic and square root.

	Name	$y_{kj}w(y_{kj})$	y_T	deviation	χ^2	percent	Comment
1	Linear	y_{kj}	186,459	980	3.33	50.4%	good
2	Logarithmic	$\ln(y_{kj})$	186,902	1182	9.78	4.4%	fair
3	Square root	$y_{kj}^{1/2}$	186,626	1041	5.56	23.5%	fair

Table 11: Comparison of proportionality functions.

Among the three proportionality functions, the linear gives the least deviation and chi square value.

From the results in Tables 10 and 11, we choose the logarithmic scaling function and linear proportionality function as the best combination.

10. PARAMETRIZATION

A solution is overparametrized if it quantifies at least one statistically insignificant coefficient. A solution is underparametrized if it omits at least one statistically significant coefficient. A solution has correct parametrization if it is neither overparametrized nor underparametrized.

In general, overparametrization leads to a smaller sum of squares of errors. But this does not lead to greater accuracy. This is manifested in two ways. First, the number of degrees of freedom, the denominator in (4.9), decreases, counteracting the smaller numerator. Secondly and more importantly, since more variables have to be estimated, the mutual interference among them increases and the elements of the inverse matrix in (4.11) generally increase in absolute value.

Underparametrization has the reverse effects: the sum of squares of errors increases, the number of degrees of freedom increases, and the elements of the inverse matrix generally decrease.

For the Table 1 triangle, Table 12 quantifies the results of parametrization. The overparametrization quantifies all function parameters up to and including the quadratic coefficients. The underparametrized solution quantifies only the base coefficients.

	Solution	y_T	deviation
1	Correct parametrization	186,459	980
2	Overparametrization	182,097	5490
3	Underparametrization	188,852	2535

Table 12: Parametrization.

The overparametrized solution has a much larger deviation, And, in this particular example, it is consistent. That is, if the solution of mean 182,097 and deviation 5490 indeed is correct and unbiased, there is considerable probability of attaining at least the correctly parametrized value of 186,459.

Our experience indicates that overparametrization invariably leads to higher deviations. The reason for this is simple: the more variables there are to be estimated, the less accuracy with which they can be estimated. The decreased accuracy translates into higher parameter errors. In our numerical tests, we usually find the overparametrized solutions to be consistent.

In this case, the underparametrized solution yields a consistent estimate and a higher deviation. But experience tells us that underparametrization can lead to inconsistent estimates and lower deviations. An underparametrized estimate is inconsistent when the difference between itself and the correctly parametrized estimate is well outside the range of the underparametrized deviation. The reason for lower deviations due to underparametrization is: the fewer variables there are to be estimated, the more accuracy with which they can be estimated. The reason for inconsistent results is: some statistically significant variables are being omitted.

The discussion in this section indicates that, if high deviations and misleading results are to be avoided, we must insure correct parametrization.

11. DISCUSSION

A) Cumulative distribution functions. We only use functions of three parameters, because we believe only they must have at least that number of parameters to have the flexibility to describe real payment patterns. We identify four such candidates: the transformed log-normal, transformed normal, transformed log-gamma, and transformed gamma. We find all Pareto-type functions to be unstable in our iterative regression scheme.

Of the four functions, only the log-normal works well for Workers Compensation, Products Liability and Medical Malpractice, the longest tailed liability lines. But none works well for Commercial Auto Liability, Personal Auto Liability and Commercial Multiple Peril, the shorter tailed liability lines. Fortunately, we have developed a class of functions for the latter lines. We will present it in another paper.

B) Type of data. The data on which we tested these models have always been paid loss. The question is whether the model could work as well on reported data. As formulated in this paper, the answer is negative. We give two reasons for this and suggest a possible remedy. The two reasons are related.

First, a c.d.f. is by construction monotonic from zero to unity. Often a reported pattern is not, surpassing unity at some intervals. This happens because of over-reserving: case reserves were set higher than actual payments. In an ideal world with perfect case reserving, this would not happen, because, when reported, case reserves would be set at exactly the future paid amounts. Therefore the ideal incurred pattern would also be monotonic. And the flip side, under-reserving, must also be prevalent. The inference is that actual incurred amounts have errors, because the case reserves cannot be set with perfect foresight.

And that brings us to the second reason. For this model, we assume that the actual paid amounts have no errors. While this certainly is not entirely true, it is much less true of reported data. Therefore, in working with reported data, it is imperative that we account for the errors associated with the actual data. Since the two reasons are more or less related, one remedy may rectify both.

For the y_{kj} in (3.6), we use the actual amount, and assume there is no associated error. Instead, we could use the *estimated* value, which has a quantifiable error. In effect, we are saying that since actual reported amounts have unknown errors, we should instead work with estimated amounts, for which the errors can be estimated. This also has an additional advantage that addresses the first problem: even if the actual reported pattern is not monotonic, the theoretical pattern could still be. This very distinction between using the actual and estimated values goes back to the point made by MACK (1995). The model based on estimated, as opposed to actual, estimates, is another interesting avenue of research.

With the above discussion in mind, an analysis of reported data, assuming that every theoretical obstacle can be overcome, may yield much higher parameter and process errors than that on the corresponding paid data. If such is indeed the case, there may not be much additional value in the consideration of reported data.

If paid data have considerable amounts of salvage and subrogation, they can also be non-monotonic. In such cases, it may be best to analyze the data gross of salvage and subrogation.

C) Advantages of the model. We generalize the difficulties of the traditional chain ladder fall into three categories: non-stochastic variations in the ata factors, limited information, and tail factor. We recapitulate how our model addresses each category.

i. The model simulates the non-stochastic variations in the ata factors. The statistical significance of all parameters is systematically determined. We have tested six liability lines, those mentioned in the first segment of this section. We have considered loss triangles for both individual companies and industry-wide data in the United States. And we have yet to encounter a single triangle in which only the base coefficients are statistically significant. In every case, at least some non-stochastic variations are evident.
ii. Limited information, as used in the traditional chain ladder, surfaces in a few instances. One is that averaging may only use the last few available years. Secondly, to estimate the ata factor in any development period, only information in that period is used. In contrast, to make the estimation at any single point, our regression scheme uses information available everywhere. This should decrease the parameter errors.
iii. Our model gives the tail factor for each accident year. In addition, it yields the variance of an ultimate loss, and it clearly divides that variance into parameter and process errors.

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Appendix A:
The Derivation of
a Perturbation Expression

In this appendix, we derive the perturbation expression in (4.3). All the other perturbation expressions can be obtained in a similar fashion.

In perturbation theory, our objective is to express any quantity, such as the LHS of (4.3):

$$(A.1) \qquad r_{kj}(\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau),$$

as a linear function of the increments $(\Delta\mu,\Delta\sigma,\Delta\tau)$.

We use (3.2) to write (A.1) as:

$$(A.2) \qquad r_{kj}(\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau)=\frac{F(j+1;\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau)}{F(j;\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau)}.$$

We need to express the denominator of (A.2) as a perturbation expression, which is just (4.2).

$$(A.3) \qquad \begin{aligned} F(j;\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau) &= F(j;\mu,\sigma,\tau)+ \\ &\quad G(j;\mu,\sigma,\tau)\Delta\mu+H(j;\mu,\sigma,\tau)\Delta\sigma+K(j;\mu,\sigma,\tau)\Delta\tau \\ &= F_{kj}+G_{kj}\Delta\mu+H_{kj}\Delta\sigma+K_{kj}\Delta\tau. \end{aligned}$$

In (A.3), we expressly recognize that the function parameters may vary by accident years. Similarly,

$$(A.4) \qquad F(j+1;\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau)=F_{k,j+1}+G_{k,j+1}\Delta\mu+H_{k,j+1}\Delta\sigma+K_{k,j+1}\Delta\tau.$$

If we now put (A.3) in the denominator, then we have:

$$(A.5) \qquad \begin{aligned} \frac{1}{F(j;\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau)} &= \frac{1}{F_{kj}+G_{kj}\Delta\mu+H_{kj}\Delta\sigma+K_{kj}\Delta\tau} = \\ &\frac{1}{F_{kj}}\frac{1}{1+\left(G_{kj}/F_{kj}\right)\Delta\mu+\left(H_{kj}/F_{kj}\right)\Delta\sigma+\left(K_{kj}/F_{kj}\right)\Delta\tau} = \\ &\frac{1}{F_{kj}}\left[1-\frac{G_{kj}}{F_{kj}}\Delta\mu-\frac{H_{kj}}{F_{kj}}\Delta\sigma-\frac{K_{kj}}{F_{kj}}\Delta\tau\right]. \end{aligned}$$

In (A.5), we retain only the linear term of the following Taylor's series expansion:

$$(A.6) \qquad \frac{1}{1+x}=1-x+x^2-\Lambda .$$

When we substitute (A.4) and (A.5) into (A.2), we get precisely (4.3):

$$(A.7)\qquad r_{kj}(\mu+\Delta\mu,\sigma+\Delta\sigma,\tau+\Delta\tau)=r_{kj}(\mu,\sigma,\tau)\left[1+\left(\frac{G_{k,j+1}}{F_{k,j+1}}-\frac{G_{kj}}{F_{kj}}\right)\Delta\mu+\left(\frac{H_{k,j+1}}{F_{k,j+1}}-\frac{H_{kj}}{F_{kj}}\right)\Delta\sigma+\left(\frac{K_{k,j+1}}{F_{k,j+1}}-\frac{K_{kj}}{F_{kj}}\right)\Delta\tau\right].$$

Every perturbation expression can be derived in a similar manner, and is simply the result of repeated and appropriate applications of the Taylor's series expansions.

Appendix B: The Finite Limit of the Process-Error Infinite Series

In this appendix, we prove that the infinite series in (5.10) has a finite limit. It is sufficient to show that:

$$(B.1)\qquad \sum_{t=A}^{\infty}\left[\frac{\ln(r_t)}{r_t}\right]^2<\infty,$$

where, without loss of generality, we suppress the parameter dependency on time. A is some positive integer, which can be as large as we wish.

We rewrite the equation for the transformed log-normal of (2.2) as:

$$(B.2)\qquad F(t)=\Phi\left[(\ln t)^{\tau};\mu,\sigma\right]=1-\int_t^{\infty}f(t)dt,$$

where we use the partial density function:

$$(B.3)\qquad f(t)=\frac{dF}{dt}=\frac{\tau}{\sqrt{2\pi}}\frac{(\ln t)^{\tau-1}}{t}\exp\left[-\tfrac{1}{2}\left(\frac{(\ln t)^{\tau}-\mu}{\sigma}\right)^2\right].$$

To obtain the asymptotic form of the development factor, we use the definition of (2.1), and perform successive approximations:

$$(B.4)\qquad r_t=\frac{F(t+1)}{F(t)}=\frac{1-\int_{t+1}^{\infty}f(t)dt}{1-\int_t^{\infty}f(t)dt}\approx\left[1-\int_{t+1}^{\infty}f(t)dt\right]\left[1+\int_t^{\infty}f(t)dt\right]$$

$$\approx 1+\int_t^{t+1}f(t)dt\le 1+f(t)$$

Taking the log of (B.4) and retaining only the first term of the resultant Taylor's series, we obtain:

$$(B.5) \qquad \ln(r_t) \le \ln\left[1 + f(t)\right] \approx f(t).$$

We simplify the elements of the series in (B.1) as:

$$(B.6) \qquad \left[\frac{\ln(r_t)}{r_t}\right]^2 \le \left[\ln(r_t)\right]^2 \le f^2(t) = \frac{\tau^2}{2\pi}\frac{\left(\ln t\right)^{2(\tau-1)}}{t^2}\exp\left[-\left(\frac{\left(\ln t\right)^\tau - \mu}{\sigma}\right)^2\right].$$

Therefore, (B.1) is satisfied if we have the following equality:

$$(B.7) \qquad \sum_{t=A}^{\infty}\frac{\left(\ln t\right)^{2(\tau-1)}}{t^2}\exp\left[-\left(\frac{\left(\ln t\right)^\tau - \mu}{\sigma}\right)^2\right] < \infty,$$

where we drop all multiple constants. Equation (B.7) is in turn satisfied if we have the following integral inequality:

$$(B.8) \qquad \int_A^{\infty}\frac{\left(\ln t\right)^{2(\tau-1)}}{t^2}\exp\left[-\left(\frac{\left(\ln t\right)^\tau - \mu}{\sigma}\right)^2\right]dt < \infty.$$

We can certainly pick an A such that:

$$(B.9) \qquad \frac{\left(\ln t\right)^{2(\tau-1)}}{t} < 1, \quad A < t.$$

Therefore, we have the following inequality:

$$(B.10) \qquad \int_A^{\infty}\frac{\left(\ln t\right)^{2(\tau-1)}}{t^2}\exp\left[-\left(\frac{\left(\ln t\right)^\tau - \mu}{\sigma}\right)^2\right]dt < \int_A^{\infty}\exp\left[-\left(\frac{\left(\ln t\right)^\tau - \mu}{\sigma}\right)^2\right]\frac{dt}{t}.$$

We make the following substitutions:

$$(B.11) \qquad x = \frac{\left(\ln t\right)^\tau - \mu}{\sigma}, \quad \frac{\sigma}{\tau}\frac{dx}{\left(\sigma x - \mu\right)^{1-1/\tau}} = \frac{dt}{t}, \quad p = 1 - \frac{1}{\tau}.$$

With (B.11), equation (B.10) can be written as:

$$(B.12) \qquad \int_B^{\infty}\frac{e^{-x^2}}{\left(\sigma x - \mu\right)^p}dx < \infty, \quad B = \frac{(\ln A)^\tau - \mu}{\sigma}.$$

The inequality in (B.12) holds, irrespective of the value of p, since the exponential decays much faster than any power of x.

We thus prove the inequality of (B.1). This line of argument is applicable to any distribution, the partial density function of which decays exponentially.