Techniques for the Conversion of Loss Development Factors

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Abstract

It sometimes happens that accident year development factors are available and policy year factors are not and vice versa. The purpose of this paper is to formulate a mathematical technique for converting from one form into another under various assump-tions concerning the time during the calendar year that policies are written. The connection between the policy year factor and the influence of changing exposures on accident year development is then explored.

TECHNIQUES FOR THE CONVERSION OF LOSS DEVELOPMENT FACTORS

1. Overview

This paper begins by deriving a general formula to convert accident year factors into policy year age-to-age loss development factors. To help understanding, a first simplifying assumption that the policies are written uniformly over the policy year is made and then further generalized to situations where only the average written date is known. The inverse of this formula then gives the means of converting accident year factors back to policy year factors. An analogy to the effect on accident year factors from changes in exposure leads to a reformulation of the problem. A practical example taken from real data illustrates the techniques.

2. Notation and Analysis

It will be necessary to make a few definitions first. Let:

a_k = the (incremental) dollar amount expected to be reported for an individual risk at development period *k*.

$a_k^{(p)}$ = the corresponding dollar amount for a policy period.
g_k = the accident year factor that develops incurred losses from age *k* to age *k+1*.
f_k = the policy year factor that develops incurred losses from age *k* to age *k+1*.
n = the number of policies written in a policy year.

$$g_k = 1 + \frac{a_{k+1}}{\sum_{j=1}^{k} a_j}; \quad f_k = 1 + \frac{a_{k+1}^{(p)}}{\sum_{j=1}^{k} a_j^{(p)}} \qquad (1)$$

Which implies that

$$\frac{a_{k+1}}{a_k} = \begin{cases} \dfrac{(g_k - 1)g_{k-1}}{(g_{k-1} - 1)} & \text{when } k \ge 2 \\ g_1 - 1 & \text{when } k = 1 \end{cases} \qquad (2)$$

A similar relationship holds for $a_k^{(p)}$. The importance of this ratio will become evident after examining the process of policy creation and the future claims associated with them. The proof is in Appendix A.

Scenario 1 - *Policies are written uniformly over the calendar year.*

Assume that each policy has a development pattern that corresponds to an accident year and that, for *n* policies written during the year, the first policy is written at time *1/n*, the second policy at time *2/n*, the third at *3/n* etc. Then the last policy will be written on December 31 of the calendar period and will not contribute any losses to it. To avoid the use of multiples of 12, we shall let the integer 1 stand for the first 12 months, 2 for 24 months etc. Hence g_1 will stand for the 12 to 24 month accident year age-to-age factor. Since each policy has the development pattern of an accident year, the first policy will contribute $[(n-1)/n]a_1$ of losses to the first 12 months of the policy year. The second policy written will contribute $[(n-2)/n]a_1$ to the first 12 months of the policy year. By extension of this reasoning, the first 12 months of the policy year will experience losses reported of $(1/n)[1+2+...+(n-1)]a_1=(n-1)a_1/2$. The second year of the policy period will have losses reported equal to the first 12 months of an accident year for the last policy written to 1/n times the first 12 months of an accident year for the first policy written, in addition to the beginnings of the 24-month accident year development on policies as they begin to expire in the second year. The 12-month accident year con-tribution to the second year will be $(1/n)(n+...+1)a_1$, and the 24-month contribution will be $(1/n)(n-1+n-2+...+1)a_2$. We can now use the principle of induction to derive the following relation-ships:

$$a_1^{(p)} = (n-1)a_1/2$$
$$a_2^{(p)} = (n+1)a_1/2 + (n-1)a_2/2$$
$$\vdots$$
$$a_k^{(p)} = (n-1)a_k/2 + (n+1)a_{k-1}/2$$

Which together with equation(1) → that

$$f_k = 1 + \frac{[(n-1)a_{k+1} + (n+1)a_k]}{[(n-1)\sum_{i=1}^{k} a_i + (n+1)\sum_{i=1}^{k} a_{i-1}]}$$

$$= 1 + \frac{[(n-1)a_{k+1} + (n+1)a_k]}{\left[2n\sum_{i=1}^{k-1} a_i + (n-1)a_k\right]} \qquad (3)$$

.

This holds for k=1 by letting the summation term be zero for this case. Dividing top and bottom by na_k, letting n approach infinity and substituting our expression for a_{k+1}/a_k we get the following transformation:

$$f_k = \frac{g_{k-1}(1+g_k)}{(1+g_{k-1})} = \frac{(1+g_k)}{(1+g_{k-1}^{-1})} \qquad (4)$$

By allowing g_0 to be infinity this formula will be true for all integer values of $k \geq 1$. The algebraic details are again left to Appendix B.

Scenario 2 - *The policies aren't written evenly over the calendar year but the average written date is known.*

Let T be the average written date as a percentage of the year. Let t_k be the time the k-th policy is written as a percentage of the whole year. Generalizing the argument above, we get that:

$$a_1^{(p)} = [\sum_{i=1}^{n} (1 - t_i)]\, a_1$$

$$a_2^{(p)} = [\sum_{i=1}^{n} t_i]\, a_1 + [\sum_{i=1}^{n} (1 - t_i)]\, a_2$$

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$$a_k^{(p)} = [\sum_{i=1}^{n} t_i]\, a_{k-1} + [\sum_{i=1}^{n} (1 - t_i)]\, a_k$$

$$= nTa_{k-1} + n(1 - T)\, a_k \qquad (5)$$

Repeating the previous analysis gives us the following modification:

$$f_k = g_{k-1} \frac{\left[g_k(1 - T) + T\right]}{\left[g_{k-1}(1 - T) + T\right]}$$

$$= \frac{\left[g_k(1 - T) + T\right]}{\left[1 - T + Tg_{k-1}^{-1}\right]} \qquad (6)$$

Notice that T=1/2 is the same as assuming uniform writings over the whole year. (This also follows by letting $t_k = k/n$ and finding T as n approaches infinity). Also, by using T instead of an assumption about when the policies are written, the n term will cancel from the ratio, making the limiting value the same as the finite value for the same T.

The inverse problem of finding the accident year factors from the policy year factors follows immediately from (6) and induction: where, for the sake of convenience, $\alpha = T/(1-T)$ and $f_0 = 1$.

$$g_1 = f_1 - \alpha$$

$$g_2 = \frac{f_1 f_2}{f_1 - \alpha} - \alpha$$

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$$g_k = \frac{\prod_{i=1}^{k} f_i}{\sum_{j=0}^{k-1} (-\alpha)^{(k-j-1)} \prod_{i=0}^{j} f_i} - \alpha \qquad (7)$$

The assumption in this approach is that the losses reported in successive years are proportional to the time the policy has been in force. This, in turn, depends on the written date. If T=1, then all of the policies are written at the end of the calendar year and $f_k = g_{k-1}$. This means that the policy year is exactly the same as an accident evaluation at one period earlier. If T=0, all policies are written at the beginning of the period, then $f_k = g_k$ and the policy year and accident year are identical. The fundamental assumption necessary to this approach is that there be a policy year of exactly one year and that the average date of the policies written during that year is known. Also the accident year factors should begin at the 12 to 24 month development age and increase in 12 month increments. If the accident year factors are known at other development ages, a simple approach would be to fit a curve to the known factors and then use the curve to get the year end factors. Equation (6) would give the corresponding year-end factors for the policy year. A new curve fit to these factors would then give the policy year factors at the desired development ages. Table 1 illustrates these concepts and the effect that the average written date has on the derived policy year factors.

A word needs to be said about the assumption that the development of an individual risk resembles the development of an accident year. To see that this is so, it is only necessary to develop the accident year expected losses in terms of the expected losses for each risk. If A_i represents the reported incurred (incremental) losses at development period *i*, a little thought will demonstrate that $A_i = \sum(1-t_j)a_i$. Briefly, the reason is that the development of losses that occurred in the calendar year in which the policies were written depends only on the length of time that the policies were in force. A policy written on December 31 would have no impact on accident year development , although it will have an effect on the policy year development.

Thus $g_k = 1 + A_{k+1}/\sum a_i = 1 + a_{k+1}/\sum a_j$.

Another assumption, that the expected losses for each risk is the same, is necessary to make the formulation of the problem more tractable. To know the actual risk parameters at the time the risk is written would require information virtually impossible to obtain. Each risk can be regarded as having the same distribution as the aggregate distribution. Since the number of risks drops out of the ratios for the factors, this assumption does no harm.

It is often assumed that because the average accident date of the policy year is December 31 and that the average accident date of the accident year is July 1, the 12-month policy year development factor is the same as the 6- month accident year factor. Underlying this is actually two assumptions: (1) that the date of loss is exactly 1/2 of the policy period and (2) that the average written date is July 1. The approach taken above accepts the average date of loss implied by the accident year factors and permits a more flexible assumption about the average written date.

Table 1

T = 0.25

Age	AY Factors	Age	Fitted AY Factors	PY Factors as per (6)	Age	Fitted PY Factors
9	1.889	12	1.500	1.833	9	2.612
21	1.163	24	1.125	1.193	21	1.243
33	1.066	36	1.056	1.071	33	1.089
45	1.036	48	1.031	1.037	45	1.044
57	1.022	60	1.020	1.023	57	1.026
69	1.015	72	1.014	1.015	69	1.017
81	1.011	84	1.010	1.011	81	1.012
93	1.008	96	1.008	1.008	93	1.009

T = 0.75

Age	AY Factors	Age	Fitted AY Factors	PY Factors as per (6)	Age	Fitted PY Factors
9	1.889	12	1.500	4.500	9	5.679
21	1.163	24	1.125	1.193	21	1.443
33	1.066	36	1.056	1.071	33	1.126
45	1.036	48	1.031	1.037	45	1.053
57	1.022	60	1.020	1.023	57	1.028
69	1.015	72	1.014	1.015	69	1.016
81	1.011	84	1.010	1.011	81	1.010
93	1.008	96	1.008	1.008	93	1.007

3. An Alternate Interpretation

The policy year is similar to the situation in which the exposure for each accident year is increasing. This is because each policy written is an increase in exposure for the calendar accident year. If we can succeed in translating the concept of policies written into exposures assumed we could use (7) to adjust the accident year factors for an increase in exposure.

To do this, let E_{i-1} represent the exposure at the beginning of accident year i where E_0 is the exposure at the beginning of the first year. This situation is different from the beginning of a policy year in that, for a policy year, the exposure always begins at zero. The average "written" date for accident year i now includes a mass of "policies" at T = 0 equal to E_{i-1}. We now rewrite (5) for accident year 1 as follows:

$$a_1' = E_0 a_1 + (E_1 - E_0)(1 - T') a_1$$
$$a_2' = (E_1 - E_0)\left[T' a_1 + (1 - T') a_2\right] + E_0 a_2$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$a_k' = (E_1 - E_0)\left[T' a_{k-1} + (1 - T') a_k\right] + E_0 a_k \qquad (8)$$

As before, T' is the average exposure date for the increase, but now the a's stand for the reported cost per unit of exposure. To use (6) and (7) without modification, we find a T that is equivalent to the expressions in (8) by setting $E_0 = 0$ in the last line of (8), replacing T' by T and letting it equal the original expression. *Equating the coefficients of* a_{k-1} *or* a_k *gives* $T = (1-E_0/E_1)T'$. What it means to have an average date for the new exposures needs some clarification. If the exposures are new stores or new employees, the average opening date or average hire date is the correct interpretation. However, if the exposure is payroll or sales, a natural assumption of uniform increase over the year means that T' = 1/2. So if $E_1 = 2E_0$ then T = 1/4.

The interpretation so far has only been for an increase in exposure. However, (8) would hold without modification under conditions of declining exposure. The expression for T would be negative since it was derived under the assumption of a beginning exposure of zero. Under this condition, no decline in exposure is possible. However, the algebra is equivalent even though allowing T to be negative makes no conceptual sense.

How do we use this information? We want to use equation (7) to factor out the increase in the development factors due to the increase in exposure. First note that the relevant term in (7) is $\alpha = T/(1-T)$. Since we know what happens when T is zero or 1 we restrict our discussion to the case where $0<T<1$. If it is true that $E_1/E_0 = E_2/E_1 = \ldots = E_i/E_{i-1}$ (i.e., the increase in exposure is a constant percentage of the previous exposure), then it is easy to check that $T_1 = T_2 = \ldots = T_i$ if T', the average date of the exposure increase, is the same for all accident years where T_i is the adjusted date for accident year *i*. Thus, as long as there is an increase at the same *rate*[1], there should be no overall change in the factors after the *first* increase. However we might want to adjust the new factors to develop a new year where the changes have stopped. Also, the most common situation, where exposure is changing but at different rates introduces the problem of how to adjust the factors to be appropriate for the exposure level of each accident year. To begin, we will keep the same notation but let the *f*'s stand for the growth accident year factors (the "policy year")and the *g*'s will be the accident year factors with growth removed (the "accident year"). As an additional refinement, we will add a superscript to distinguish the accident year being adjusted. Thus, $f_j^{(i)}$will be the unadjusted factor for the *i-th* year and the *j-th* development period. The flattened factor will be $g_j^{(i)}$. Finally, we will add primes to represent the factors adjusted to the growth level of a different year. If we desire to adjust the *i-th* year to the level of the *n-th* year, we first flatten year *i* and then re-inflate to the level of year *n*. If *n* is the latest year having only the 12-24 month development factor, each year *i* will have its 12-24 month factor adjusted to year *n*. First deflating, we get:$g_1^{(i)} = f_1^{(i)} - \alpha_i$ and then inflating: $f_1^{(i)'} = g_1^{(i)} + \alpha_n = f_1^{(i)} - \alpha_i + \alpha_n$. Similarly, the 24-36 month factor will be adjusted for all accident years as illustrated in the derivation of equation (9) below. We proceed in this fashion for each year *i*. The alphas are calculated from the adjusted average

[1]*The same rate* in both percentage dollar increases and at the same time as measured by the average date.

exposure date from the beginning to the end of the year corresponding to the subscript:
$\alpha_i = T_i/(1-T_i)$ where $T_i = (1-E_{i-1}/E_i)T'$ and $f_0^{(1)}=1$.

An example of the method is in Appendix C. Sheet 1 shows a real Workers Compensation incurred loss development triangle. The

$$g_2^{(i)} = \frac{f_1^{(i)} f_2^{(i)}}{f_2^{(i)} - \alpha_i} - \alpha_i$$

$$f_2^{(i)'} = \frac{(g_2^{(i)} + \alpha_n)}{(g_1^{(i)} + \alpha_n)} g_1^{(i)}$$

$$= \frac{\left[\frac{f_1^{(i)} f_2^{(i)}}{f_1^{(i)} - \alpha_i} + \alpha_n - \alpha_i\right]}{(f_1^{(i)} - \alpha_i + \alpha_n)} (f_1^{(i)} - \alpha_i)$$

$$= \frac{f_1^{(i)} f_2^{(i)} + (\alpha_n - \alpha_i)(f_1^{(i)} - \alpha_i)}{(f_1^{(i)} - \alpha_i + \alpha_n)} \quad \text{or}$$

$$f_1^{(i)\prime} f_2^{(i)\prime} = f_1^{(i)} f_2^{(i)} + (\alpha_n - \alpha_i)(f_1^{(i)} - \alpha_i)$$

$$\text{or} \prod_{j=1}^{k} f_j^{(i)\prime} = \prod_{j=1}^{k} f_j^{(i)} + (\alpha_n - \alpha_i) \sum_{j=0}^{k-1} (-\alpha_i)^{k-j} \prod_{m=0}^{j} f_m^{(i)} \qquad (9)$$

selected factors are the average of the overall average and the average after removing the largest and smallest values for the years for the years for which the latter exists and the overall average for the remaining years. Sheet 2 shows the exposure which is number of employees hired. The assumed average hire date is in the middle of the fiscal policy year (T'=1/2). The adjustment for each accident year is for the change in the number of employees from the beginning to the end of the year. Thus the adjustment is from year *i* to year *n=i+1* in the following derivation.

Although not true in this example, an examination of the variance in the factors by column sometimes reveals that the adjustment actually increases the variance for some ages while decreasing it for others. The obvious explanation would be that there is a lag in the influence of new exposures. New employees would not have the linear influence on the incidence of new claims as the derivation of the formulas would imply. It would generally be several years after employment before a claim would be filed. However if the exposure is increasing at the same rate every year the influence of the increase in the older years would nullify this argument. The increase at a faster rate would make the adjustment too large, but an increase at a slower rate would make it too small. Another explanation would be that a change in hiring practices or safety programs would make new employees have different loss potential from older ones. Also the average date of hire will vary from year to year, making the T'=1/2 assumption invalid. There are other complications such as the change in operations or reserving practices that would distort the results as well.

4. Summary

A formulaic approach to transform policy year age-to-age development factors into accident year age-to-age factors has been found that helps to clarify the relationship hidden in the definitions. The formulas derived from the investigation of that relationship led to a better understanding of the effect of the changes in exposure on the development of accident year factors.

APPENDIX A

Proof of equation (2) in the text:

$$g_k = 1 + \frac{a_{k+1}}{\sum_{i=1}^{k} a_i}$$

$$= 1 + \frac{a_{k+1}}{\sum_{i=1}^{k-1} a_i + a_k}$$

$$= 1 + \frac{a_{k+1}/a_k}{\sum_{i=1}^{k-1} a/a_k + 1}$$

$$= 1 + \frac{a_{k+1}/a_k}{\dfrac{1}{(g_{k-1} - 1)} + 1}$$ which implies

$$g_k - 1 = \frac{a_{k+1}}{a_k}\left(\frac{g_{k-1} - 1}{g_{k-1}}\right)$$ or:

$$\frac{a_{k+1}}{a_k} = g_{k-1}\left(\frac{g_k - 1}{g_{k-1} - 1}\right)$$

APPENDIX B
Sheet 1

$$f_k = 1 + \frac{(n-1)a_{k+1} + (n+1)a_k}{2n\sum_{i=1}^{k-1} a_i + (n-1)a_k}$$

$$= 1 + \frac{(1 - 1/n)a_{k+1}/a_k + (1 + 1/n)}{2\sum_{i=1}^{k-1} a_i/a_k + (1 - 1/n)} \quad \text{let} \quad n \to \infty \quad \text{we get}$$

$$= 1 + \frac{a_{k+1}/a_k + 1}{2\sum_{i=1}^{k-1} a_i/a_k + 1}$$

$$= 1 + \frac{\dfrac{(g_k - 1)g_{k-1}}{(g_{k-1} - 1)} + 1}{\dfrac{2}{(g_{k-1} - 1)} + 1} \quad \text{from (2) and (1) in the text} \qquad \textbf{(11)}$$

$$= 1 + \frac{g_k g_{k-1} - 1}{g_{k-1} + 1}$$

$$= \frac{g_{k-1}(1 + g_k)}{(1 + g_{k-1})}$$

Appendix B
Sheet 2

Proof of Equation (6) in the text:

$$a_{k+1}^{(p)} / \sum_{i=1}^{k} a_i^{(p)} = \frac{nTa_k + n(1-T)a_{k+1}}{nT\sum_{i=1}^{k} a_{i-1} + n(1-T)\sum_{i=1}^{k} a_i}$$

$$= \frac{T + (1-T)(a_{k+1}/a_k)}{T\left(\sum_{i=1}^{k-1} a_i/a_k\right) + (1-T)(a_{k+1}/a_k)\left(\sum_{i=1}^{k} a_i/a_{k+1}\right)} \quad \text{[Since } a_0 = 0\text{]}$$

$$= \frac{T + (1-T)g_{k-1}\left(\frac{g_k - 1}{g_{k-1} - 1}\right)}{\frac{T}{(g_{k-1}-1)} + \frac{(1-T)g_{k-1}(g_k - 1)}{(g_{k-1}-1)}\left(\frac{1}{g_k - 1}\right)}$$

$$= \frac{T(g_{k-1} - 1) + (1-T)g_{k-1}(g_k - 1)}{T + g_{k-1}(1-T)}$$

$$1 + \frac{a_{k+1}^{(p)}}{\sum_{i=1}^{k} a_i^{(p)}} = \frac{g_{k-1} + (1-T)g_{k-1}(g_k - 1)}{T + (1-T)g_{k-1}}$$

$$= g_{k-1}\frac{(1-T)g_k + T}{(1-T)g_{k-1} + T}$$

$$= f_k$$

APPENDIX C

WORKERS COMPENSATION

Incurred Loss Development

	Incurred as of____months						
Policy Period	12	24	36	48	60	72	84
86-87	127,543	237,609	255,255	261,471	293,234	286,390	292,540
87-88	238,622	336,699	447,647	474,771	657,817	673,145	
88-89	413,446	629,368	692,177	743,374	759,378		
89-90	483,344	755,863	815,980	717,682			
90-91	441,426	551,559	610,788				
91-92	592,559	832,558					
92-93	649,736						

Policy Period	12-24	24-36	36-48	48-60	60-72	72-84	84- Ult.
86-87	1.863	1.074	1.024	1.121	0.977	1.021	
87-88	1.411	1.330	1.061	1.386	1.023		
88-89	1.522	1.100	1.074	1.022			
89-90	1.564	1.080	0.880				
90-91	1.249	1.107					
91-92	1.405						
Average	1.502	1.138	1.010	1.176	1.000	1.021	
Wtd. Avg	1.456	1.124	0.994	1.156	1.009	1.021	
Avg-HI/LO	1.476	1.096	1.042	1.121			
Selected	1.476	1.124	1.010	1.156	1.009	1.021	

	12-ult.	24-ult.	36-ult.	48-ult.	60-ult.	72-ult.	84-ult.
Cumulative	1.994	1.352	1.203	1.191	1.031	1.021	1.000

Appendix C

Sheet 2

	Exposure	T	Alpha	T' = 1/2
85-86	193			
86-87	228	0.075	0.0814	
87-88	239	0.023	0.0236	
88-89	317	0.123	0.1405	
89-90	340	0.034	0.0354	
90-91	339	-0.001	-0.0011	
91-92	419	0.095	0.1050	
92-93	444	0.029	0.0298	

Adjusted to 93-94 accident year level

Year	12-24	24-36	36-48	48-60	60-72	72-84	84- Ult.
86-87	1.811	1.054	1.024	1.126	0.970	0.995	0.998
87-88	1.417	1.330	1.062	1.384	1.025		
88-89	1.412	1.078	1.076	1.015			
89-90	1.558	1.078	0.879				
90-91	1.280	1.111					
91-92	1.330						
Variance	0.1200	0.0898	0.0774	0.1315	0.0274	0.0000	
	1.468	1.130	1.010	1.175			
	1.502	1.138	1.010	1.176			