Concepts of the Financial Actuary
by Stephen T. Morgan, ACAS

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I used to think that I began learning the concepts during 1993 when I began to calculate durations for my employer. A leveraged buyout had just been completed and the new owners wanted us to monitor cash flows more closely. From then till now, much of my time has been spent learning subjects and doing work related to becoming, what I term, a *financial actuary*. While I still consider myself, in many respects, a neophyte, these concepts are starting to come together in terms of their interrelations. It once appeared the various concepts were somewhat disparate. As the realizations occur, there is a better understanding of the concepts.

This is a work in progress. The purpose of this paper is to discuss these concepts and their interrelations. The paper is intentionally not technical for two reasons. One, I do not consider myself to be technical. Second, in my opinion, concepts are best communicated with relatively simple examples. There was a paper written by Charles Hewitt that used to be on Part 9 that had many pages of formulae accompanied by an ongoing example using a die and spinner. I spent a lot of time with the example and a little with the formulae. I passed the exam.

It's Only Cash You Know But I Like It-apologies to the Rolling Stones

I now know I really began learning the concepts in late 1971. When I was a baby actuary, one of my first assignments was analyzing occurrence-based medical malpractice insurance in California. Abandonment. Patients falling off the operating table. Patients not liking the way the doctor treated them. Patients died. How is it best analyzed: policy year, accident year, report year, created year or the ever famous calendar year? At first, these concepts were mind boggling. However, the concepts started to make some sense after I saw a quotation by L. H. Longley-Cook that referred to cutting a cake so many times that you were left with a pile of crumbs. Those concepts were just ways to slice the cake. **Concept 1: Whatever the data (whether it is coupon income on bonds or paid losses or loss reserves, for example), get it organized into groups that make sense for analytical purposes.**

So much work was necessary in analyzing the reserves and paid losses (yes, occurrence malpractice losses eventually were paid) that rarely, if ever, was there a thought about where the cash came from to pay those losses. There was enough to do in estimating what the losses would ultimately cost without having to worry about how they would get paid. Pouring over paid and incurred triangles, I learned how to detect patterns and I learned the Law of Small Numbers: "it is harder to detect patterns in small numbers." I learned how to look at many averages for the factors. It seemed I was tracking the average number of averages I was using to project ultimates. Talk about a nebulous concept!

Even better, I never got a chance to look at the piles of cash being held in reserves. Talk about another nebulous concept! Where were those reserves kept, in the president's office, in the basement, in a bank or in bonds, whatever they are? **Concept 2: Investment income or paid losses are just cash; it is your job to predict how the payments will be made.**

Ultimates. This is the total amount estimated to be paid for all losses that belong to a defined group (accident year, for example). Paids, unlike reserves, are facts, which always appeared to me to act like squeezing a flaccid balloon. Reserves are a continually moving target. Why is it that when reserves are added to the paids (incurreds), a better estimate of ultimates generally results for casualty lines than an estimate by using paids? **Concept 3: Reserves are just estimates of future cash streams.**

IBNR (Incurred But Not Reported), (Incurred But Not Reserved) and my favorite (Interesting But Not Relevant). In making ultimate projections, this is what is left after paid and case reserves, at a particular point in time, are subtracted. IBNR gets paid too, just later. IBNR is for those losses that have already happened but the client was not considerate enough to let you know about. So, they are losses that have happened, but you do not know about them. By the way, IBNR does include amounts for losses you do know about but do not know what they ultimately will cost. Yet another nebulous concept.

Paids. These will be referred to numerous times in this paper. It is not possible to overestimate their importance in learning the concepts of the financial actuary. As indicated above, almost everything in an insurance company (reinsurance company too, but much later) will become cash. You just have to be patient. The first ever Workers' Compensation claim in New York was still open as of a few years ago. The first report of a hurricane loss to a reinsurer may come up to two years after the storm. I have seen losses from a hurricane still open after ten years, okay. **Concept 4: Almost everything on a balance sheet will eventually become cash.**

Sympathy for the Regulator - again apologies

I learned about the annual statement at an early age. I am dating myself but I can remember Schedule P on a policy-year basis. It is so much better now on an accident-year basis (chortle, chortle). The first page I remember seeing is Page 4, Underwriting and Investment Exhibit; Statement of Income; and Gains and (Losses) in Surplus. Earn the premiums, incur the losses and expenses, add in investment income and other income, mix with taxes and net income is the result. It is not quite cash and it is not quite reserves. The net income is added in with other items and you have the change in surplus for the year. Again, we have a mix of paids and reserves. **Concept 5: When performing a financial analysis, segregate the cash items from the non-cash items. Then convert the non-cash items and perform the analysis.**

On page 3, Liabilities, Surplus and Other Funds are shown. Here we see another mix of paids and reserves. Very confusing. Now on to Page 2, Assets, and the equation to end

all equations: Assets = Liabilities + Surplus. Very, very confusing. Assets and surplus are "helps" and liabilities are "hurts"; yet they equal each other. Since this equation is apparently true, part of the assets must equal liabilities and the remaining portion must equal surplus. Carried to its logical conclusion, this means the portion of assets that supports surplus can be held in assets that will not necessarily be needed soon. This means they can be held in longer term assets (like bonds). These assets will only need to be used if the assets allocated to the liabilities are not sufficient. That is the textbook definition of surplus. **Concept 6: Align a portion of assets to support the ability to pay liabilities and align the rest to surplus. When properly aligned, better financial management may take place.**

The rest of the assets support the liabilities. That is where the cash is to pay the losses. When liabilities come due, just liquidate the assets to pay them. It is just that simple. I think not. Statutory accounting is really run-off accounting. This is fine if your company is going out of business fairly soon. The way accounting is done, you will sell assets to pay losses. How much sense does it make to sell off 30 year bonds to pay losses when, as an ongoing company, you have a continual new supply of cash coming in the door. Why not use the cash from written premiums instead? It is not earning investment income. Only consider selling the assets if the rate being earned on them is less than the rate being paid on new investments or if cash inflows are insufficient to pay losses (a catastrophe, for example). Consequently, while **the** equation holds, its application in the real world is not as simple as it first appears. **Concept 7: Run-off accounting is generally not a fair**

representation of how a company operates. Realize that cash is continually coming in while it is continually going out.

You may have heard about asset liability management (ALM) or asset liability matching or cash flow matching (discussed below). The purpose is to track how assets and liabilities work together. Remember, the interrelations are not as straightforward as they may first appear. Statutory accounting rarely reflects how an ongoing company conducts its business. Remember, when you match assets with liabilities, different assets than you are matching will probably be used to actually pay the liabilities.

Duration (discussed below) is a run-off concept. It infers that the assets currently held are being related to the liabilities on the books. When the liabilities are finally paid, it is a different ball game. Other assets will most likely be used for cash.

Actuaries' Big Helpers [1]

Now, when looking back on those days when I was an Actuarial Trainee (oxymoronic, maybe), it is fairly easy to say what is important. You know what is said about hindsight, 20-20% of the time. This section's purpose is to talk about the papers and experiences that have contributed to developing the concepts of the financial actuary. In no particular order, they are:

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[1] **Author's Note**: Absolutely no slight is intended to any author of any paper I have read that is not included here. There have been too many papers read to list each one. Each had its own particular impact. Necessarily, the list here is somewhat brief.

- **Individual Risk Rating** The Hartford put me in a department doing this work until I passed the first two exams. At the time, I was upset because I was eager to do actuarial work and they were sticking me in a rating department. Little did I realize that experience rating and retrospective rating are the basis for all actuarial work. During this time, experience modifications, insurance charges (Table M) and Plan D basics were calculated by hand. I wish I had a nickel for every one I calculated. The knowledge learned here gave me an appreciation for credibility, patterns and the ability to pass Part 9.

- **Theory of Interest** This was on Part 3 when I took it. Never **discount** the importance of this area for the Concepts. I refined my knowledge of compound interest. I learned about bonds and how they are priced.

  As I look back on it, I do not recall any emphasis about bonds being a stream of cash flows. Also, I did not make a connection that **(Concept 8) the stream of coupons and the return of principal is equivalent to a payout pattern on losses**. Finally, **(Concept 9) a reason that values are discounted is to put them on a comparable economic basis with other values**. Both are important concepts.

- **"Trend and Loss Development Factors"** This is a paper written by Charles F. Cook (*PCAS* LVII, 1970). I read it at a time when I was first leaning about loss

development factors. The "overlap" concept has never been forgotten. Knowledge of patterns is an important concept and this paper contributed greatly.

- **Insurance Statutory and GAAP Accounting** For years, I only saw the statutory side of accounting. I saw inconsistencies, as noted above. Why account for almost everything on a run-off basis if you are staying in business? Are assets really liquidated to pay losses? GAAP (Generally Accepted Accounting Principles) endeavors to correct some of these problems. GAAP is comparable to a function of discounting in that, under GAAP, values in insurance are put on an equal economic basis with other industries. **Concept 10: Relating assets to liabilities on an economic basis (GAAP) is an important step towards being able to match them for cash flow purposes.**

- **"Duration"** This is a paper written by Ronald E. Ferguson (*PCAS* LXX, 1984). When I started calculating durations, this was the first paper I read. His explanations of calculating duration, the time value of money and investment related risks (timing, credit and reinvestment) are excellent.

- **"The Measurement and Management of Interest Rate Risk"** This paper was presented at a special interest seminar on *Valuation Issues* in 1989 and was written by Linda A. Dembiec, James D. Pogorzelski and Vincent T. Rowland, Jr. This was the second paper I read. Like "Duration", I refer to it often. Their explanations of ALM, matching and applying ALM to property & casualty insurance are excellent.

- **"Asset/Liability Management: Beyond Interest Rate Risk"** This paper was written by William H. Panning and was published in the *Financial Analysis of Insurance Companies* 1987 *Discussion Paper Program.* His extended discussion of three posed questions related to ALM is excellent.

- **"An Investigation of Methods, Assumptions, and Risk Modeling for the Valuation of Property-Casualty Insurance Companies"** This paper was written by Robert S. Miccolis and was from the same program as Mr. Panning's. I read this when it was first released and it helped make all other related readings since that time easier to understand.

- **"Chapter 8: Investment Issues in Property-Liability Insurance" and "Chapter 9: Special Issues"** Both chapters were written by Stephen P. D'Arcy and are in the *Foundations of Casualty Actuarial Science* book. He is not only an excellent author, but I have enjoyed numerous presentations on related issues by Mr. D'Arcy.

- **"Dynamic Financial Analysis Handbook"** This was written by the DFA Handbook Committee as a subcommittee of the Valuation and Financial Analysis Committee of the CAS. It is a new publication and its purpose is to give guidance to actuaries in the financial analysis of an insurance company. It is intended to evolve over time as actuaries gain experience in DFA modeling.

- **Data Quality** I have often joked, "Nobody knows the data I've seen." I have worked for small companies and large companies. The lack of data quality affects them all to a certain extent. Most actuarial papers start with the tacit assumption you have perfect data with which to work. Often, however, this is not the case. It is not just a concept of the financial actuary, but rather any actuary: **Concept 11: Make sure you are familiar with the weaknesses as well as the strengths of your data.**

19th Nervous Breakdown - no apology needed.

There is a concept, which is extremely important, that affects almost all actuarial work, but is most likely not covered to any great extent in the Syllabus of Examinations. This is the oral and written communication of the results of your work The financial actuary will most likely come in contact with both insurance and non-insurance (board members or investment professionals, for example) professionals. Consequently, communication skills are extremely important. Everyone has heard the question: "If a tree falls in the woods and no one hears it, does it make a sound?" **Concept 12: If an actuary cannot relate the results of his/her work, was anything accomplished?**

I, like most actuaries, have dealt with the stereotype that actuaries are introverted, geeky and dull. Often, perception is reality. If enough people perceive you that way, it does not matter how you really are. We deal with a lot of complicated issues. Just think back to the first time you heard some of the things mentioned in this paper or the papers

referenced. You were probably confused too, at least for awhile. Translation and communication are the keys to success. A part of being successful includes the following:

- Put your audience at ease. Sometimes the communication deals with topics that may be perceived as "bad news", or even worse, actuarial. Always take the time to have the audience see you as a person, not an ACTUARY. Well-placed humor, especially of the self-deprecating kind, will go a long way towards putting your audience at ease. Also, spend a few moments talking about something completely unrelated to the topic at hand. Once the initial jitters are gone, yours and theirs, start the presentation.

- Always use examples when communicating. Even actuaries enjoy examples. It gives the audience something they can relate to and follow.

- Explain why the topic is important and useful to the audience, even if not at present.

- Relate the subject of the presentation to concepts that the audience is familiar with.

- To be sure the audience is still with you, look for continual feedback during the presentation. Pose unassuming questions to see if they are.
- Take turns addressing each person, by name, if possible. If the audience is large, be sure to scan all portions of the audience.

- Take several mental breaks during the presentation. It is easier to tackle a tough subject if one gets away, if only briefly, from it. Your audience will appreciate it and be more likely to stay with you.

- Try to use words the audience knows the meaning of. If you do not, a vituperative and obstreperous reaction may result. After that, even obsequious and propitiatory behavior will not help.

- Be aware of time constraints, especially around noon and the end of the day.

- It is human nature to enjoy being appreciated or thanked. Do so for your audience.

Now, explanations of duration, matching, and stress testing.

Jumpin' Jack Cash

Insurance companies are in the business of paying losses that insureds, or reinsureds, have incurred. Setting the value of these incurreds in advance is difficult, and determining when they will be paid is not any easier. There are numerous variables that enter the process that add to the uncertainty.

As discussed earlier, assets back up the ability to pay losses. They need to be there at the appropriate time and in the appropriate amount in order to meet this responsibility. One of the main uncertainties on the asset side, even with a large fixed income portfolio, is the

movement of interest rates or, maybe better, interest rate risk. In certain circumstances, the ability to pay losses could be affected.

Insolvency is the ultimate risk that you will not be able to pay all of your liabilities. Therefore, protection of solvency is paramount. Answering to your shareholders and understanding the amount of risk they are willing to take is a key ingredient.

Concept 13: The management of interest rate risk through the management of assets and liabilities will help maintain solvency. Strictly matching the assets and liabilities is not the key. It is the prudent management of the mismatch that is important. Duration and convexity are some of the tools that help in this work. More on mismatch later.

When performing a duration analysis for the first time, plan to spend an extensive amount of time with the accountants as well as reserving and pricing actuaries. Attach payout patterns to nearly every asset and liability on the balance sheet. If a holding company is involved, be sure to model any assets or liabilities (debt, for example) at that level too. Anything that is or will become cash should get a duration. Rely on investment professionals to help model investable assets. Other assets, like Agents' Balances, can be modeled by sampling.

The implied duration for economic surplus falls out of these calculations. This calculation makes a statement as to how susceptible surplus is to interest rate changes. In a perfect

world, it would be a small number. I would like to go over some examples and definitions to help explain duration and convexity. Then, I will briefly relate some of the key issues. Look at Chart 1.

Column 3 displays a hypothetical overall calendar year future payout pattern. Remember similar calculations are done for bonds too. The stream of coupon income is similar to the loss payout pattern. It is derived from accident-year patterns by line of business. The calendar-year pattern is an accumulation of the projected contributions to each calendar year of the various accident-year payouts. This pattern is then applied to the reserves as of December 31, 1994 ($1,000). The payments are assumed to be made in the middle of each year. That is why the first payment is as of time .5 (six months after December 31, 1994). The payments shown in column 4 are discounted to the present using a selected discount factor (6% in this example). The discount factors are in column 5 and the present values are in column 6. You can think of the present value of the liabilities as their market value or price. Conceptually, that makes the liabilities like the market value of the bonds or their price.

Take a quick look at Chart 2, Page 1, then we will return to Chart 1. It shows the formula for calculating the price of a bond. Remember, the price of a bond is the sum of the present values of the coupon payments and the present value of the redemption of principal. The individual present values are multiplied by the appropriate time period (.5,1.0,1.5) and then summed. The resulting product sum ($3,420.19) is divided by the present value of the payments ($744.51). This value is called the Macaulay duration (4.59). In this context, it can be thought of as the weighted average time to payment. The

weights are the individual present values. This is DM on Chart 2, Page 1. Then the convexity (36.12) is the weighted average time squared to payment using the same weights.

The "TOTAL" line in column 6 displays the present value of the payments ($744.51) at 6%. I have also displayed the present value when the interest rate rises 1% (in investment parlance, 100 basis points) to 7%. The present value drops to $713.53, or a decrease of 4.16%.

Instead of rerunning the present values, we can use duration and convexity to give a fairly good approximation of that decrease in present values. First, calculate the modified duration. This is the Macaulay duration of 4.59 divided by 1 plus the interest rate of 6%. The relationship is obtained by taking the first derivative of the present value function with respect to the yield divided by the present value function. Thus, it is the percentage change in present values when the interest rate changes 100 basis points. This change is equal to the modified duration of 4.33 multiplied by -.01 (the change in the interest rate). -.0433 is fairly close to the change in present values of -.0416. Again, the value of -.0433 is the estimate of the change in present value when interest rates change one percentage point. Remember that an increase in rates decreases present values. Modified duration is the first term in a Taylor series expansion.

You may not feel this is close enough. This is where convexity increases the accuracy. If I take one-half times .01 squared times the convexity, I get a value of .0018. If this is

added to the estimation of the value change from looking only at duration, -.0433, we get -.0415 which is very close to the real change of -.0416. The convexity expression is derived from the second term of the Taylor series for the present value function. It is the rate of change of duration as interest rates change. It is also termed "duration drift". For background on convexity as well as duration, I refer you to "The Measurement of Interest Rate Risk" by Dembiec, Pogorzelski and Rowland, mentioned earlier.

For those of you that remember calculus and love formulas, I will give you a slightly more rigorous explanation of modified duration. Please look at Chart 2, Page 2.

It is somewhat confusing, but people often use Macaulay and modified duration interchangeably. For example, a duration of 4.5 would mean both a 4.5 year duration and present values will decrease 4.5% if interest rates rise 100 basis points (1%). Since the two values are usually close, it does not present much of a problem.

Satisfaction

You would probably say "great" if I told you the assets and liabilities were matched or that the durations are equal. What does this really mean? Look at Chart 3.

Assume we are going to pay a total of $1,000 in losses over the next 5 years and we have reasonable estimates of the individual payments. This is line one. We can buy 5 bonds that will fund each of the 5 payments. We design the first bond so that the last coupon payment ($8.04) and the return of principal ($66.96) equal the last loss payment ($75.00). The other four coupon payments are used to fund part of the other four loss payments. A second bond with a 4 year term is selected so its last coupon and principal equal $91.96 and so forth. The combined duration of the bonds equals the liability duration. Durations are combined using their respective prices as weights. This calculation makes intuitive sense. Since the individual payments or coupons are the weights and the resulting durations are equal, this implies the weights or payments are equal for each asset and each liability payment.

Why do we care about interest rate changes? I will recap what I said earlier. On the asset side, the market value of bonds decreases as interest rates increase and vice versa. Since the purpose of assets is to back up the ability to pay our liabilities, it is important to keep these values in step with each other or to intelligently know the risk of a mismatched position. The purpose of surplus is to protect the company from extreme changes in the assets or liabilities. These extreme changes might come from investments, for example. If surplus was allowed to move at some multiple of interest rate changes, it would not be in a

very good position to protect assets or liabilities. It is being affected by the thing it is supposed to protect against.

As an example, look at Chart 4, Page 1. We all know that assets (A) equal liabilities (L) plus surplus (S). Then the same formula works for their respective market values or present values. Suppose that total liabilities have a duration of 3.5 and are 90% of the market value assets. Assume that the assets have a duration of 5.15. Solving for the surplus duration, we see it is 20.0. That means if interest rates rise 1% then the market value surplus falls 20%. It can also work the other way. Page 2 of this chart shows that asset matching with the liabilities is not the complete answer. The assets backing the liabilities are what is needed to be matched.

For an ongoing concern, I do not believe perfect matching is desirable. As long as the slope of the yield curve is positive, meaning longer term bonds have higher yields, it is possible to pick up yield on assets by purchasing bonds with longer durations. The emphasis is then on the prudent management of mismatch. It does not matter whether your duration is 1.0 or 5.0 or 10.0. It is the management of mismatch that is important. The Dembiec etal paper discusses mismatch and surplus duration.

Now that we have seen the theory and the mechanics, here are some of the issues. When you construct the accident-year payouts, be sure you know what is included. It is probably not necessary to do each and every accident year. When the reserves are small enough, you can add them together to make one combined year. For example, assume

you have some losses open for accident years 1974 and prior. They could be paid out with accident year 1975 losses without materially affecting the overall accuracy. As another example, assume that some of your automobile accident years are substantially affected by PIP losses, and a separate payout pattern for these losses is not available. I would suggest using a workers' compensation payout rather than an automobile payout. If you have a stop loss cover, truncate the payout to the extent you expect to utilize the cover. If any reserves are discounted, like workers' compensation, the reserves should be grossed up before applying payout percentages. While this is certainly not an exhaustive list, it is hoped that it gives you a flavor for what to consider.

Doing a duration on only the carried loss reserves is not complete. The losses contemplated in the unearned premium reserve (UPR) need to be considered too. The duration of the UPR is not its earning period. As the UPR is earned, losses will be incurred. I recommend calculating ultimate losses in the UPR by using ultimate loss and loss expense ratios, by line for the most recent accident year, applied to the UPR. These ultimates can then be paid out with accident-year patterns. The accident-year payments can be accumulated into one UPR calendar year payout. The UPR payout is blended in with the carried reserve payout and an overall duration is calculated.

For the carried reserves, it is fairly safe to assume the average payment is in the middle of the year. Assuming quarterly payments would be closer to reality, but I have found that estimating quarterly payments for a reinsurer is subject to considerable variation. This leads to an issue on the UPR payouts: when is the first payment made? We have

discussed it internally and agreed to assume that the average UPR earns off in six months and the first payments will average coming six months later. Therefore, the first payments are shown twelve months after the date of the UPR, which, while not perfect, is what we are assuming today while we reconsider the issues.

You Can't Always Get What You Want

Sometimes, you find, you get what you need. ***Concept 14: In order to totally protect the company, cash flow or scenario testing is needed.*** (I prefer to call it "stress testing".) **Cash flow testing involves developing several scenarios, from best case to a catastrophic case, on timing and amounts of losses.** Please note: comparable comments also apply to the stressing of assets.

- Lengthen and shorten the payouts. It is best to apply the stress on payouts to the next couple of calendar years rather than at some distant point. In this manner, multiple stresses can be applied for maximum impact to roughly the same time frame. The simultaneous application of multiple stresses is in itself, a stress.

- Assume the reserves are deficient or redundant by various percentages. This can be considered as a type of catastrophe. Note, this will have the effect of changing the overall accident-year payout pattern.
- Assume the deficiencies or redundancies emerge at different times. This should still be done in the relative near term.

- If there is a possibility of natural catastrophes occurring in the upcoming time period, multiple occurrences should be included as a part of the stressing.

- For discounting, test different interest rates over the time period of the payout pattern. In real life, interest rates do change. Therefore, in modeling, why not do the same?

- Also, benefit can be derived from testing different interest rates where each is applied over the entire period.

- Consider doing the payouts separately between gross and ceded. Then you could test the effect if some of your reinsurance is not collectible.

- Test annual versus quarterly, or maybe even monthly, payments. This is particularly important for short-tailed lines.

- Prior financials should be stressed for comparison purposes.

- Run scenarios on future premiums and accompanying losses as well as current and future investments. This has particular application for valuations for a possible acquisition.

Several possibilities for stresses have been listed. Obviously, each is not applicable in a given situation. Concentrate on those stresses that will have an impact on the overall projected results. A future large increase in other underwriting expenses may be feasible; but it will likely not have a noticeable impact on future financials. Reasonableness of the occurrence of the stress is integral. Applying the equivalent effect of two Andrew like storms may be interesting, but not applicable in a given situation.

The stresses listed here are being applied in a deterministic manner. By this I mean that the stresses are applied discretely. Stresses applied in this manner are easier to explain than if stochastic methods are used. Stochastic methods involve using probability distributions in which the random variables are the stresses listed above. Stochastic modeling has technical appeal because of the virtually unlimited values that can be given to the variables. Clustering of outcomes can be noted and credence can thus be given to the particular values generating the clustered outcomes.

Deterministic modeling is a subset of stochastic modeling. Its advantages are ease of explanation and the controlling of the values given to the variables. Stochastic modeling has the advantages of giving practically innumerable values to variables and the sense of impartiality when applying stresses.

As I indicated, this is very much a work in progress. The entire area is evolving quickly and very little of what is done has yet to be cast in granite. That is what makes it so fascinating.

RESERVES AT 12/31/94 1,000.00 CHART 1
DISCOUNT RATE 6.0%

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
						(2)*(6)	(6)*(2)^2
		CAL. YR.	CAL. YR.	DISCOUNT	DISC	DP*	DP*
YEAR	TIME(T)	PAYOUT	PYMTS.	FACTOR	PYMTS(DP)	TIME	TIME^2
1995	0.5	0.100	100.00	0.9713	97.13	48.57	24.28
1996	1.5	0.170	170.00	0.9163	155.77	233.66	350.48
1997	2.5	0.120	120.00	0.8644	103.73	259.33	648.31
1998	3.5	0.090	90.00	0.8155	73.40	256.90	899.15
1999	4.5	0.080	80.00	0.7693	61.55	276.98	1,246.39
2000	5.5	0.050	50.00	0.7258	36.29	199.60	1,097.77
2001	6.5	0.049	49.00	0.6847	33.55	218.08	1,417.49
2002	7.5	0.047	47.00	0.6460	30.36	227.70	1,707.75
2003	8.5	0.046	46.00	0.6094	28.03	238.26	2,025.17
2004	9.5	0.045	45.00	0.5749	25.87	245.77	2,334.77
2005	10.5	0.044	44.00	0.5424	23.86	250.53	2,630.57
2006	11.5	0.042	42.00	0.5117	21.49	247.14	2,842.05
2007	12.5	0.041	41.00	0.4827	19.79	247.38	3,092.19
2008	13.5	0.040	40.00	0.4554	18.22	245.97	3,320.60
2009	14.5	0.036	36.00	0.4296	15.47	224.32	3,252.57
TOTAL		1.000	1,000.00		744.51	3,420.19	26,889.54

DURATION AT 6% =(7)/(6)=3420.19/744.51	4.593880539
DURATION AT 7% =(7)/(6)=3179.08/713.53	4.455425841
CONVEXITY AT 6% =(8)/(6)=26889.54/744.51	36.11709715
CONVEXITY AT 7% =(8)/(6)=24488.87/713.53	34.32072933
PRESENT VALUE OF PYMTS AT 6%	744.51
PRESENT VALUE OF PYMTS AT 7%	713.53
% CHANGE IN VALUE FOR 100BP CHANGE IN INTEREST RATE =713.53/744.51-1	-4.16%
MODIFIED DURATION -4.59/1.06	-4.33%
CHANGE IN VALUE MD+.5*CON*.01^2	-4.15%

BOND PRICE AND MACAULAY DURATION

P = price of a t-year bond

R = redemption value

t = time in years

c = coupon

i = yield

D_M = Macaulay duration

$$v = \frac{1}{(1+i)}$$

$$P = c \bullet v^1 + c \bullet v^2 + \cdots cv^t + R \bullet v^t$$

$$D_M = \frac{1 \bullet c \bullet v^1 + 2 \bullet c \bullet v^2 + \cdots + t \bullet c \bullet v^t + t \bullet R \bullet v^t}{P}$$

DERIVATION OF MODIFIED DURATION

A. Take the first derivative of the price function (P) with respect to i

$$\frac{dP}{di} = -1 \bullet c \bullet v^{2} - 2 \bullet c \bullet v^{3} - \cdots - t \bullet c \bullet v^{t+1} - t \bullet R \bullet v^{t+1}$$

B. Factor out -v or $\frac{-1}{(1+i)}$ from right side of equation

$$\frac{dP}{di} = \frac{-1}{(1+i)} \bullet \left[1 \bullet c \bullet v + 2 \bullet c \bullet v^{2} + \cdots + t \bullet c \bullet v^{t} + t \bullet R \bullet v^{t}\right]$$

C. Divide both sides by P

$$\frac{\frac{(dp)}{(di)}}{P} = \frac{-1}{(1+i)} \bullet \left[\frac{(1 \bullet c \bullet v^{1} + 2 \bullet c \bullet v^{2} + \cdots + t\, c\, v^{t} + t \bullet R \bullet v^{t}}{P}\right]$$

$$\frac{\frac{(dp)}{(di)}}{P} = \frac{-1}{(1+i)} \bullet DM$$

DISCOUNT RATE 6.0%

CASH FLOW MATCHING

CHART 3

	COUPON	PRINC-IPAL	PRICE	TERM IN YRS.	DURATION	1	2	3	4	5	TOTALS
LIABILITY PAYOUT		$1,000.00	$880.71		2.136	$375.00	$275.00	$ 175.00	$100.00	$ 75.00	$1,000.00
BOND 1	12.0%	$ 66.96	$ 83.90	5	4.146	$ 8.04	$ 8.04	$ 8.04	$ 8.04	$ 75.00	$ 107.16
NET						$366.96	$266.96	$ 166.96	$ 91.96	$ 0.00	
BOND 2	11.0%	$ 82.85	$ 97.19	4	3.489	$ 9.11	$ 9.11	$ 9.11	$ 91.96		$ 119.29
NET						$357.85	$257.85	$ 157.85	$ 0.00	$ 0.00	
BOND 3	10.0%	$ 143.50	$158.84	3	2.749	$ 14.35	$ 14.35	$ 157.85			$ 186.55
NET						$343.50	$243.50	$ 0.00	$ 0.00	$ 0.00	
BOND 4	9.0%	$ 223.39	$235.69	2	1.920	$ 20.11	$243.50				$ 263.61
NET						$323.39	$ 0.00	$ 0.00	$ 0.00	$ 0.00	
BOND 5	0.0%	$ 323.39	$305.08	1	1.000	$323.39					$ 323.39
NET						$ 0.00	$ 0.00	$ 0.00	$ 0.00	$ 0.00	
TOTALS FOR BONDS		$ 840.09	$880.71		2.136						$1,000.00

DURATIONS MATCH

ASSUMPTIONS:

1. LIABILITY PAYOUT STREAM IS ASSUMED TO BE: 37.5%, 27.5%, 17.5%, 10.0% AND 7.5%.
2. ALL PAYMENTS AT END OF YEAR
3. BOTH LIABILITIES AND BONDS ARE DISCOUNTED AT 6%
4. SINCE DISCOUNT RATES ARE EQUAL, BOTH MACAULAY AND MODIFIED DURATIONS ARE EQUAL FOR LIABILITIES AND BONDS

Surplus Duration

Example Assets = Liability + Surplus
$PV_A * D_A = PV_L * D_L + PV_S * D_S$

Assume
$PV_A = \$100$
$PV_L = \$90$
$PV_S = \$10$
$D_A = 5.15$
$D_L = 3.50$

$100 * 5.15 = 90 * 3.5 + 10 * Ds$
$10Ds = 100 * 5.15 - 90 * 3.5$
$Ds = 20.0$ years

if interest rates rise 1%, surplus drops 20%:

Revised PV_A = \$100 * .9485	= \$94.85
Revised PV_L = \$90 * .9650	= \$86.85
Revised PV_S =	\$8.00

Surplus Duration

Use amounts from prior example but $D_A = D_L$

$$(100)(3.5) = (90)(3.5) + 10 * Ds$$
$$350 = 315 + 10Ds$$
$$1.0Ds = 35$$
$$Ds = 3.5$$

The assets that support liabilities should be matched