A SIMPLE TOOL FOR PRICING LOSS SENSITIVE FEATURES OF REINSURANCE TREATIES

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Abstract

Proportional reinsurance treaties may contain a number of loss sensitive features which require pricing by the reinsurance actuary. This paper extends the ideas presented in Bear and Nemlick's paper Pricing the Impact of Adjustable Features and Loss Sharing Provisions of Reinsurance Treaties (PCAS 1990, Volume LXXVII) to address this problem. An aggregate distribution is introduced which can be written in closed form for quick calculations. The model could be adapted for use in aggregate applications outside of reinsurance.

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Introduction

Proportional reinsurance treaties may contain a number of loss sensitive features which require pricing by the reinsurance actuary. This paper extends the ideas presented in Bear and Nemlick's paper Pricing the Impact of Adjustable Features and Loss Sharing Provisions of Reinsurance Treaties (PCAS 1990, Volume LXXVII) to address this problem. An aggregate model is introduced which can be written in closed form for quick calculations.

Loss Sensitive Features

The most common loss sensitive features used with proportional treaties are sliding scale commissions, profit sharing provisions and loss corridors.

Sliding scale commissions adjust the amount of commission paid based on actual loss experience. For example, there may be a provisional amount paid of 25%. For every point of loss ratio below 60%, a point of commission is added up to a maximum of 35%. That is, the commission "slides" 1:1 with the loss ratio. On the other hand, for every point that the loss ratio is above 60%, half a point of commission is subtracted, down to a minimum commission of 20%.

Profit sharing provisions are designed to return to the reinsured a certain portion of the reinsurer's profit. A typical plan might return 40% of profit where "profit" is defined as premium minus losses minus a margin (e.g. 15% of premium).

Loss Corridors are used to give additional protection to reinsurers. This provision may say that the first 70% of the reinsurer's loss ratio is retained by the reinsurer, the 10% from 70% to 80% is covered by the reinsured and the amount above 80% is again retained by the reinsurer.

The Reinsurance Actuary's Task

The reinsurance actuary may be asked to quantify the impact of one of the features, or to provide guidance when several options are being considered.

Given an expected loss ratio E[lr], and the loss sensitive feature as a function of the actual loss ratio, s(lr), the simple calculation of s(E[lr]) will not provide the correct answer. Instead of estimating the function at the expected loss ratio, the expected value E[s(lr)] over all possible loss ratios must be estimated. When the function s(lr) is linear then s(E[lr]) equals E[s(lr)], but this is generally not true for non-linear functions.

Fortunately, however, the above features are piece-wise linear and so can be estimated with a finite number of calculations. If the function is linear from 0 to L1, L1 to L2, and L2 to infinity then E[s(lr)] can be written as follows:

$$
\begin{aligned}
E[s(lr)] = \; & s(E[lr \mid lr<L1]) \times Prob(lr<L1) \\
& + s(E[lr \mid L1<lr<L2]) \times Prob(L1<lr<L2) \\
& + s(E[lr \mid L2<lr]) \times Prob(L2<lr)
\end{aligned}
$$

The Aggregate Distribution Model

To evaluate the impact of the loss sensitive feature, the distribution of possible loss ratios is needed. Bear and Nemlick have proposed using the lognormal distribution to model the aggregate distribution when there is insufficient data to use more sophisticated models, such as collective risk models. The lognormal is unimodal and the expected value and variance can be easily set. The major drawback of the lognormal is that the cumulative distribution and excess loss function cannot be written in closed form. This requires the actuary to perform numerical approximations or use table look-ups with interpolation routines.

The model used in this paper is a lognormal distribution mixed with a gamma distribution. This shares the qualities of the lognormal, with a somewhat thicker tail, but can be written in a closed form. The great advantage here is that a simple spreadsheet calculation can be performed very quickly.

The cumulative distribution function is given as follows.

$$F(x) = \begin{cases} \dfrac{\lambda^2(\frac{1}{4}+2\lambda)^{-1}x^{-\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda}}}{(-\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda})}\left((\tfrac{1}{4}+2\lambda)^{-\frac{1}{2}}-\ln(x)+(-\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+2\lambda})^{-1}\right) & x\le 1 \\ 1-\dfrac{\lambda^2(\frac{1}{4}+2\lambda)^{-1}x^{-\frac{1}{2}-\sqrt{\frac{1}{4}+2\lambda}}}{(\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda})}\left((\tfrac{1}{4}+2\lambda)^{-\frac{1}{2}}+\ln(x)+(\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+2\lambda})^{-1}\right) & x>1 \end{cases}$$

The insurance charge function, which tells the portion of the total loss dollars above an aggregate amount, is given as:

$$\phi(x) = \begin{cases} (1-x)+\dfrac{\lambda}{2}(\tfrac{1}{4}+2\lambda)^{-1}x^{\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda}}\left((\tfrac{1}{4}+2\lambda)^{-\frac{1}{2}}-\ln(x)+\tfrac{1}{\lambda}\sqrt{\tfrac{1}{4}+2\lambda}\right) & x\le 1 \\ \dfrac{\lambda}{2}(\tfrac{1}{4}+2\lambda)^{-1}x^{\frac{1}{2}-\sqrt{\frac{1}{4}+2\lambda}}\left((\tfrac{1}{4}+2\lambda)^{-\frac{1}{2}}+\ln(x)+\tfrac{1}{\lambda}\sqrt{\tfrac{1}{4}+2\lambda}\right) & x>1 \end{cases}$$

$$\lambda = \frac{\sqrt{1+CV^2}}{-1+\sqrt{1+CV^2}}$$

where CV is the coefficient of variation

The functions shown are normalized to an expected value of 1. The parameter lambda can be set based on the coefficient of variation (standard deviation divided by the mean). The full details of the distribution are given in the appendix.

Exhibit 1 shows excess insurance charge values, generated by this distribution, compared with those in Table M. Table M lists aggregate excess charges to be used with workers compensation retrospectively rated programs. While this is not always applicable to reinsurance treaties, the fact that the entries are so close should indicate that this reasonably approximates an aggregate distribution.

Given the cumulative distribution function and the insurance charge function, the expected losses in any layer are calculated as follows:

$$E[x \mid X1<x<X2] = \frac{\phi(X1)-\phi(X2)+X1(1-F(X1))-X2(1-F(X2))}{F(X2)-F(X1)}$$

Using the Model

Exhibit 2 shows an example of this calculation applied to the sliding scale commission described above. The expected loss ratio and coefficient of variation are required from the user.

Sometimes loss sensitive features apply to multi-year blocks. For these cases, the annual coefficient of variable should be divided by the square root of the number of periods in the block. For treaties with unlimited carryforward provisions, the actuary will need to use his or her judgement as to how far to reduce the coefficient of variation.

The table shows loss ratios and the corresponding sliding scale commissions. Note that the commission for any intermediate value is just a linear interpolation between the loss ratios. The Entry Ratio is the loss ratio relative to the expected loss ratio. The CDF and Excess Charge are calculated from the lognormal/gamma model as described above.

The last three columns show the results of the model. The "Weight" is the difference between successive CDF values; the .211 indicates that there is a 21.1% chance that the loss ratio will be between 0.00% and 50.00%. The expected L/R of 43.27% says that, given the loss ratio is between 0.00% and 50.00%, it is expected to be 43.27%. The weighted average of these loss ratios will, of course, reproduce the expected loss ratio. The final column shows the commission amount corresponding to each loss ratio. The expected sliding scale commission is 27.14%.

The spreadsheet is self-contained; there are no side calculations needed beyond what is displayed. This allows us to perform a variety of sensitivity tests including deciding between different sliding scale options.

One other observation on this model is that as the coefficient of variation goes to zero, the expected sliding scale commission approaches the commission at the expected loss ratio.

Other Applications

Similar spreadsheets can be set up to show profit sharing provisions and loss corridors (see Exhibits 3 and 4). While this discussion has focused on proportional treaties, this same method could be used for excess of loss treaty features such as annual aggregate deductibles and swing plans.

Lognormal/Gamma Model Compared with Table M
Insurance Charges by Entry Ratio

Exhibit 1

	Expected Loss Group 30		Expected Loss Group 25		Expected Loss Group 20		Expected Loss Group 15		Expected Loss Group 10	
WC Expected Loss		500,000		1,000,000		2,500,000		7,500,000		35,000,000
Lambda :	2.915		4.3		6.8		12.3		28	
C.V. :	1.148		0.835		0.612		0.430		0.275	
Entry Ratio	LN/Gamma Model	Table M	LN/Gamma Model	Table M	LN/Gamma Model	Table M	LN/Gamma Model	Table M	LN/Gamma Model	Table M
0.25	0.760	0.759	0.755	0.752	0.752	0.750	0.750	0.750	0.750	0.750
0.50	0.560	0.556	0.538	0.533	0.520	0.510	0.507	0.502	0.501	0.501
0.75	0.407	0.405	0.368	0.365	0.330	0.322	0.296	0.289	0.267	0.276
1.00	**0.300**	**0.300**	**0.250**	**0.250**	**0.200**	**0.200**	**0.150**	**0.150**	**0.100**	**0.100**
1.25	0.228	0.228	0.175	0.172	0.124	0.127	0.075	0.078	0.033	0.052
1.50	0.179	0.179	0.128	0.121	0.081	0.085	0.041	0.043	0.012	0.029
1.75	0.145	0.146	0.097	0.086	0.056	0.059	0.023	0.026	0.005	0.017
2.00	0.119	0.122	0.075	0.063	0.040	0.042	0.014	0.017	0.002	0.011
2.25	0.100	0.104	0.060	0.046	0.029	0.031	0.009	0.010	0.001	0.007
2.50	0.086	0.089	0.049	0.034	0.022	0.023	0.006	0.007	0.001	0.005
2.75	0.074	0.076	0.041	0.025	0.017	0.019	0.004	0.005	0.000	0.003
3.00	0.065	0.065	0.034	0.019	0.014	0.015	0.003	0.004	0.000	0.003

Model is a lognormal with a gamma (alpha=2) prior
Model parameter set to match insurance charge at an entry ratio of 1.00

Exhibit 2

Aggregate Model for Sensitivity Analysis
Lognormal with Gamma Prior
Sliding Scale Commission

Expected	60.00%
Annual Coef of Var	0.4
# Years in block	3
Block Coef of Var	0.231
Lambda	38.99351

L/R	Sliding Scale Comm.	Entry Ratio	CDF	Excess Charge	Weight	Expected L/R	Expected SS Comm
0.00%	35.00%	0.0000	0.0000	1.0000			
					0.211	43.27%	35.00%
50.00%	35.00%	0.8333	0.2112	0.1903			
					0.331	55.32%	29.68%
60.00%	25.00%	1.0000	0.5424	0.0847			
					0.273	64.38%	22.81%
70.00%	20.00%	1.1667	0.8157	0.0340			
					0.184	81.07%	20.00%
Infinity	20.00%	Infinity	1.0000	0.0000			
				Total :	1.000	60.00%	27.14%

Exhibit 3

Aggregate Model for Sensitivity Analysis
Lognormal with Gamma Prior
Profit Sharing Provision

Expected	80.00%
Annual Coef of Var	0.4
# Years in block	3
Block Coef of Var	0.231
Lambda	38.99351
Reinsurer's Margin	15.0%
% of Returned	40.0%

L/R	Entry Ratio	CDF	Excess Charge	Weight	Expected L/R	Profit Share
0.00%	0.0000	0.0000	1.0000			
				0.669	70.32%	5.87%
85.00%	1.0625	0.6687	0.0602			
				0.331	99.53%	0.00%
Infinity	Infinity	1.0000	0.0000			
			Total :	1.000	80.00%	3.93%

Exhibit 4

Aggregate Model for Sensitivity Analysis
Lognormal with Gamma Prior
Impact of Loss Corridor

Expected	70.00%
Annual Coef of Var	0.5
# Years in block	1
Block Coef of Var	0.500
Lambda	9.472136
% of L.C. Returned	100.0%

L/R	Entry Ratio	CDF	Excess Charge	Weight	Expected L/R	L/R Net of Corridor
0.00%	0.0000	0.0000	1.0000			
				0.585	49.61%	49.61%
70.00%	1.0000	0.5852	0.1704			
				0.132	74.73%	70.00%
80.00%	1.1429	0.7172	0.1211			
				0.283	109.99%	99.99%
Infinity	Infinity	1.0000	0.0000			
			Total :	1.000	70.00%	66.55%

Appendix :
Technical Description of Aggregate Distribution
Lognormal with Gamma Prior

We begin with the lognormal distribution. It is a unimodal curve which is often considered a useful model for aggregate distributions. The probability density function (PDF) is given as:

$$f(x) = \frac{\exp\left(-\frac{1}{2}(\ln(x)-\mu)^2/\sigma^2\right)}{x\,\sigma\,\sqrt{2\pi}}$$

$$\textit{Where}\quad E[x^n] = \exp\left(n\mu+\tfrac{1}{2}n^2\sigma^2\right)$$

For simplicity, the parameters are changed such that the mean of the distribution is always unity.

$$\textit{Let}\quad c = -2\mu = \sigma^2$$

$$f(x;c) = \frac{x^{-\frac{3}{2}}\,c^{-\frac{1}{2}}}{\sqrt{2\pi}}\exp\left(-\frac{\ln(x)^2}{2c}-\frac{c}{8}\right)$$

$$\textit{Then}\quad E[x^n] = \exp\left(\frac{n(n-1)\,c}{2}\right)$$

The major problem with this distribution is that the cumulative distribution function (CDF) and the limited expected value function are intractable. As a compromise to this problem, a prior gamma distribution for the single parameter "c" is introduced.

$$g(c;\alpha,\lambda) = \frac{\lambda^\alpha c^{\alpha-1}e^{-\lambda c}}{\Gamma(\alpha)}$$

The resulting distribution, in its general form is given by:

$$f(x;\alpha,\lambda) = \int_0^\infty f(x;c)\;g(c;\alpha,\lambda)\;dc$$

For the lognormal with gamma prior this is rewritten as:

$$f(x;\alpha,\lambda) = \int_0^\infty \frac{x^{-\frac{3}{2}}c^{-\frac{1}{2}}}{\sqrt{2\pi}} \exp\left(-\frac{\ln(x)^2}{2c}-\frac{c}{8}\right) \frac{\lambda^\alpha c^{\alpha-1}}{\Gamma(\alpha)} \exp(-\lambda c)\, dc$$

$$= \left(\frac{x^{-\frac{3}{2}}\lambda^\alpha}{\Gamma(\alpha)}\right)\int_0^\infty \frac{c^{\alpha-\frac{3}{2}}}{\sqrt{2\pi}} \exp\left(-\frac{\ln(x)^2}{2c}-\left(\frac{1}{8}+\lambda\right)c\right) dc$$

This expression can be simplified with reference to the reciprocal inverse guassian distribution (see Insurance Risk Models by Panjer and Willmot, Society of Actuaries 1992).

$$PDF: \quad g_{rig}(c) = (2\pi\beta c)^{-\frac{1}{2}}\exp\left(-\frac{(c-\mu)^2}{2\beta c}\right)$$

$$E[c] = \mu+\beta \qquad \mu,\beta > 0$$

$$E[c^n] = 2\beta\left(n-\frac{1}{2}\right)E[c^{n-1}]+\mu^2E[c^{n-2}]$$

This may be rearranged into the following form:

$$g_{rig}(c) = \left(\frac{e^{\frac{\mu}{\beta}}}{\sqrt{\beta}}\right)\frac{c^{-\frac{1}{2}}}{\sqrt{2\pi}}\exp\left(-\frac{\mu^2}{2\beta c}-\frac{c}{2\beta}\right)$$

With the following substitution, the integral portion of the lognormal with gamma prior is shown to be proportional to the (alpha-1)th moment of the reciprocal inverse guassian distribution.

$$\beta = \left(\frac{1}{4}+2\lambda\right)^{-1} \qquad \mu = |\ln(x)|\left(\frac{1}{4}+2\lambda\right)^{-\frac{1}{2}}$$

This substitution allows the PDF for an integer value of alpha to be written in closed form. For the case of alpha=2, as is used in this paper, the final distribution after some algebra can be written as below.

$$f(x;2,\lambda) = \begin{cases} \lambda^2 (\frac{1}{4}+2\lambda)^{-1} x^{-\frac{3}{2}+\sqrt{\frac{1}{4}+2\lambda}} ((\frac{1}{4}+2\lambda)^{-\frac{1}{2}} - \ln(x)) & when\ x \le 1 \\ \lambda^2 (\frac{1}{4}+2\lambda)^{-1} x^{-\frac{3}{2}-\sqrt{\frac{1}{4}+2\lambda}} ((\frac{1}{4}+2\lambda)^{-\frac{1}{2}} + \ln(x)) & when\ x > 1 \end{cases}$$

While this may look complex, it is not too difficult to work with and the CDF and insurance charge function can be written in closed form.

$$F(x;2,\lambda) = \begin{cases} \dfrac{\lambda^2 (\frac{1}{4}+2\lambda)^{-1} x^{-\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda}}}{(-\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda})} ((\frac{1}{4}+2\lambda)^{-\frac{1}{2}} - \ln(x) + (-\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda})^{-1}) & x \le 1 \\ 1 - \dfrac{\lambda^2 (\frac{1}{4}+2\lambda)^{-1} x^{-\frac{1}{2}-\sqrt{\frac{1}{4}+2\lambda}}}{(\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda})} ((\frac{1}{4}+2\lambda)^{-\frac{1}{2}} + \ln(x) + (\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda})^{-1}) & x > 1 \end{cases}$$

$$\phi(x;2,\lambda) = \begin{cases} (1-x) + \dfrac{\lambda}{2} (\frac{1}{4}+2\lambda)^{-1} x^{\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda}} ((\frac{1}{4}+2\lambda)^{-\frac{1}{2}} - \ln(x) + \frac{1}{\lambda}\sqrt{\frac{1}{4}+2\lambda}) & x \le 1 \\ \dfrac{\lambda}{2} (\frac{1}{4}+2\lambda)^{-1} x^{\frac{1}{2}-\sqrt{\frac{1}{4}+2\lambda}} ((\frac{1}{4}+2\lambda)^{-\frac{1}{2}} + \ln(x) + \frac{1}{\lambda}\sqrt{\frac{1}{4}+2\lambda}) & x > 1 \end{cases}$$

The general form of the insurance charge function is

$$\phi(x) = \int_x^\infty (t-x) f(t)\, dt$$

The moments and coefficient of variation can be written simply

$$E[X^n] = \left(\frac{\lambda}{\lambda - \frac{n(n-1)}{2}}\right)^2 \qquad for\ \alpha = 2$$

$$CV = \frac{\sqrt{E[X^2] - E[X]^2}}{E[x]} = \sqrt{\left(\frac{\lambda}{\lambda-1}\right)^2 - 1} \qquad \lambda = \frac{\sqrt{1+CV^2}}{-1+\sqrt{1+CV^2}}$$

The resulting distribution is unimodal with a mean of one (see Figure 1).

The distribution resulting from a prior gamma with an alpha of 1 is given below.

$$f(x;1,\lambda) = \begin{cases} \lambda\left(\frac{1}{4}+2\lambda\right)^{-\frac{1}{2}} x^{-\frac{3}{2}+\sqrt{\frac{1}{4}+2\lambda}} & x \le 1 \\ \lambda\left(\frac{1}{4}+2\lambda\right)^{-\frac{1}{2}} x^{-\frac{3}{2}-\sqrt{\frac{1}{4}+2\lambda}} & x > 1 \end{cases}$$

$$F(x;1,\lambda) = \begin{cases} \dfrac{\lambda\left(\frac{1}{4}+2\lambda\right)^{-\frac{1}{2}} x^{-\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda}}}{-\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda}} & x \le 1 \\ 1-\dfrac{\lambda\left(\frac{1}{4}+2\lambda\right)^{-\frac{1}{2}} x^{-\frac{1}{2}-\sqrt{\frac{1}{4}+2\lambda}}}{\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda}} & x > 1 \end{cases}$$

$$\phi(x;1,\lambda) = \begin{cases} (1-x)+\frac{1}{2}\left(\frac{1}{4}+2\lambda\right)^{-\frac{1}{2}} x^{\frac{1}{2}+\sqrt{\frac{1}{4}+2\lambda}} & x \le 1 \\ \frac{1}{2}\left(\frac{1}{4}+2\lambda\right)^{-\frac{1}{2}} x^{\frac{1}{2}-\sqrt{\frac{1}{4}+2\lambda}} & x > 1 \end{cases}$$

An interesting side result is the relationship between the alpha=1 case and the single parameter pareto distribution.

$$\frac{F(x;1,\lambda) - F(1;1,\lambda)}{1 - F(1;1,\lambda)} = 1 - x^{-\frac{1}{2}-\sqrt{\frac{1}{4}+2\lambda}} \qquad \text{for } x \ge 1$$

The only disadvantage of the alpha=1 case is that the PDF has a sharp corner at x=1 (see Figure 2). The alpha=2 case produces a much smoother PDF.

Given the PDF for alpha=1 and alpha=2 cases, the PDF for other integer values of alpha can be generated using an iterative formula.

$$f(x;\alpha,\lambda) = \left(\frac{2\alpha-3}{\alpha-1}\right)\frac{\lambda}{\left(\frac{1}{4}+2\lambda\right)} f(x;\alpha-1,\lambda) + \frac{\lambda^2 \ln(x)^2}{(\alpha-1)(\alpha-2)(\frac{1}{4}+2\lambda)} f(x;\alpha-2,\lambda)$$

The general form of the moments can be written as follows:

$$E[x^n] = \left(\frac{\lambda}{\lambda-\frac{n(n-1)}{2}}\right)^\alpha$$

$$Let \quad \frac{\alpha}{\lambda} = c$$

$$Then \quad \lim_{\alpha\to\infty}\left(\frac{\alpha/c}{\alpha/c-\frac{n(n-1)}{2}}\right)^\alpha = \exp\left(\frac{n(n-1)c}{2}\right)$$

As alpha becomes larger, the resulting PDF approaches the lognormal. The functions for alpha>2 become more cumbersome to work with, leaving the alpha=2 case as the best alternative.

Lognormal with Gamma Prior

Alpha=2 Case

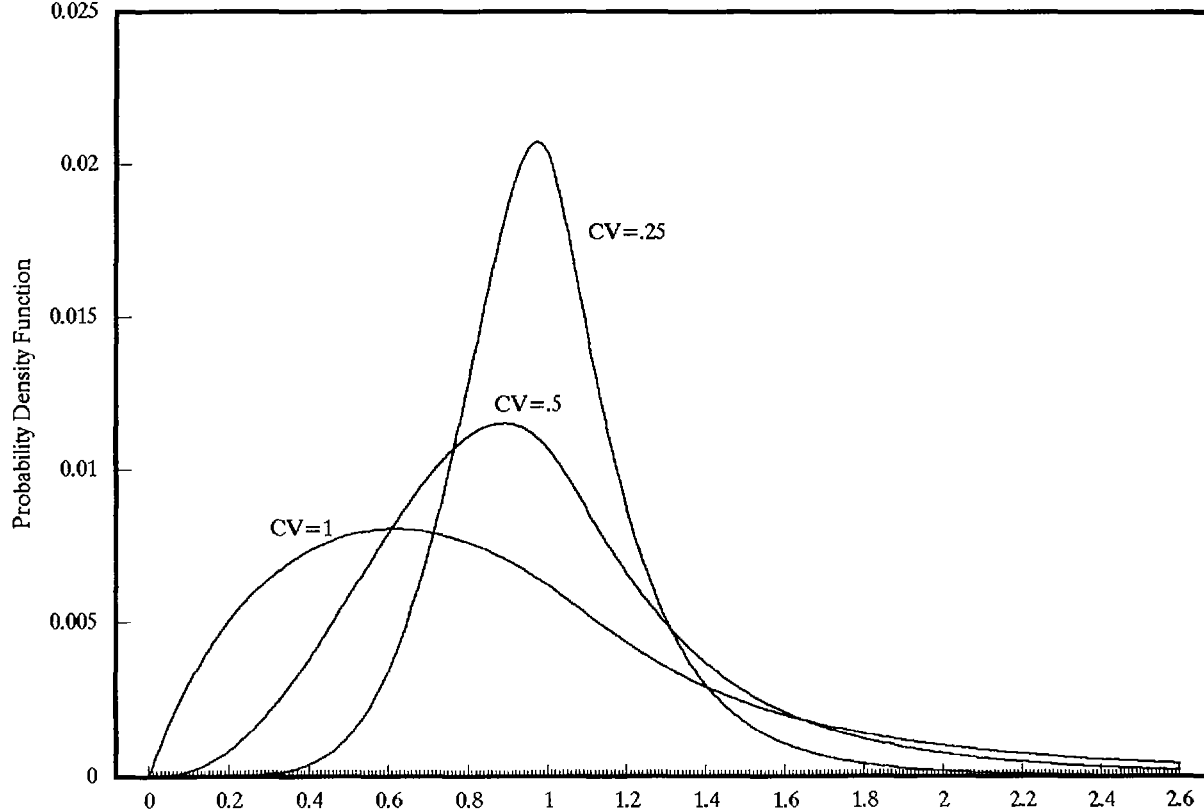

Figure 1

Lognormal with Gamma Prior

Coefficient of Variation = .500

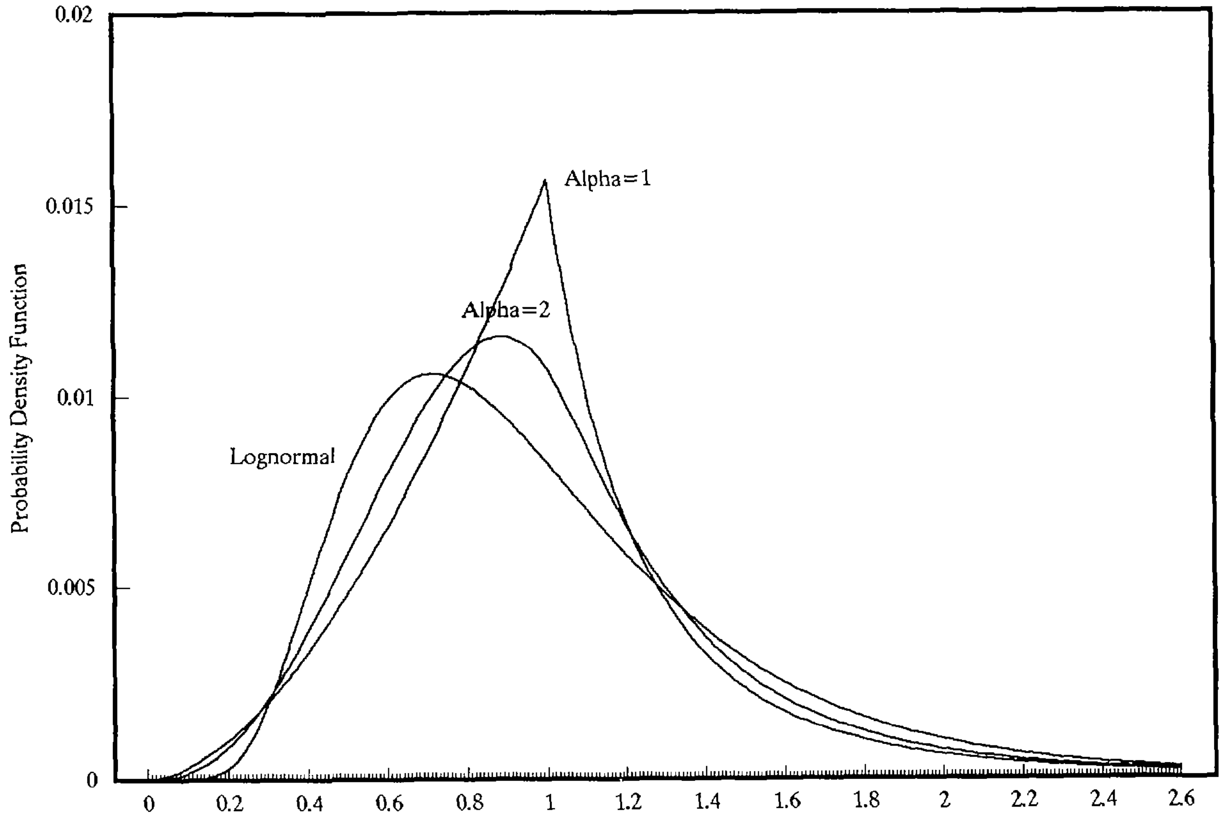

Figure 2