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CASUALTY ACTUARIAL SOCIETY

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To: CAS Members

From: Charles A. Bryan

Re: **Fifth Issue of CAS Forum**

Dear CAS Members:

You have in your hands the fifth issue of the CAS Forum. For the second issue in a row, we have a Forum that is almost entirely devoted to new papers. We have ten new articles that have been generously provided by CAS members and others for the information of our membership.

This is my last issue as editor of the Forum. Thanks to the authors, the Editorial Committee, and everyone who has helped to launch and maintain this new service to our members. As this responsibility passes, I am gratified that I have been able to play a part in the beginning of this fine publication.

The next issue of the Forum is scheduled for May of 1990. Authors who would like to submit articles for the next issue of the Forum should submit these articles to the new Vice President - Continuing Education by March 1.

Your truly,

Charles A. Bryan

CAB:njg

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Don Cunningham

Toward a Certain Equivalent Discounted Cash Flow Model of Insurance Ratemaking in a Regulatory Environment

Donald F. Cunningham

A basic tenet of financial theory is the competitive market's ability to establish efficient and equitable prices for financial securities. The competitive price is a fair price because it generates a rate of return to stockholders that is adequate but not excessive. To understand how competitive markets establish prices, financial theorists have developed models that mimic competitive prices. Such models are relevant to insurance ratemaking because insurance is also a financial security. Although insurance contracts are not traded on organized exchanges, they are priced, in some states, under competitive conditions. In states where prices are regulated, a competitive pricing model would help regulators because of their responsibility to maintain adequate but not excessive prices. Rate models based on sound financial theory would also promote the public understanding of ratemaking procedures and establish a higher common ground for debate. Rather than debating the propriety of the ratemaking procedure, participants could focus on the best measures of model variables.

This paper presents a special application of the most basic of financial pricing models--the discounted cash flow model (DCFM)--to insurance ratemaking.[1] The issue of appropriate interest credits to policyholder supplied funds is addressed in the context of

[1] Recent examples of financial models applied to the ratemaking process include Doherty and Garven (1986), Garven (1988), Hill and Modigliani (1987), and Myers and Cohn (1987).

financial theory's "Theory of the Firm." In the end, we argue for a discount rate that approaches the risk-free rate, the special case of DCFMs known as the certainty equivalent approach.

A cornerstone of financial theory is the rule of present value. Every introductory finance textbook presents an example similar to the following: $105 accumulated in a savings account over the course of one year has the same value as $100 today if the interest rate is 5%. The difference between the $100 today and the $105 in one year is the so-called time value of money. The $100 is the present value of the expected cash flow of the savings account. It is also referred to as the discounted price or the capitalized price. The $105 is the future value of the expected cash flow.

The most important point, which is often lost in the mechanics of present value calculations, is that the discounted price is the market price today. The future value is "tomorrow's" market price. In a market where competition has established 5% to be a fair and equitable return for the suppliers of capital at this risk level, the only appropriate price for $105 to be received in one year is $100 today.

A DCFM of setting insurance rates is simply an application of this very basic financial principle. The insurance premium should be the present value (i.e. price) of expected future losses and expenses. Such a model is fundamentally sound and well known. The following issues, which habitually arise in regulatory debate, are addressed below in the context of the DCFM: 1) recognizing the appropriate credit claims of policyholders commensurate with their incurred risk, 2) determining the appropriate interest rate for discounting expected future losses and expenses, and 3) developing a measure of profit comparability between insurance companies and other service related firms.

<u>Credit Claims of Policyholders and Their Right to Fair Returns</u>:

In order to understand the claims of policyholders, it is helpful to consider the financial structure of all publicly traded firms, regardless of their line of business, and then draw an analogy between these firms and insurance firms specifically. Using this Theory of the Firm approach we want to draw the analogy between bondholders of all firms and policyholders of insurance firms.

The cash flow generated by the assets of publicly traded firms are paid to two classes of security holders--bondholders and stockholders. Stockholders own the residual claim to asset earnings, and bondholders own the first claim. Both claims are financial securities that are priced in competitive markets by discounting their expected cash flows with an appropriate risk-adjusted rate.

The risk of the cash flows to the stockholders, however, is fundamentally different from the risk of the cash flows to the bondholders. Bondholders worry about, and are therefore compensated for, default risk. If a company manages its assets with superior skill, a higher return is not paid to bondholders. They simply earn the rate of interest that was contractually established in the covenants of the bond. Likewise, if the assets are managed with inferior skills, a lower return is not paid to bondholders. With either superior or inferior management, the return to bondholders is the same. They simply earn the contract rate of interest. Only if inferior management leads to a long-term deterioration in the productive quality of the assets of the firm will bondholders incur losses--namely default.

Stockholders, in contrast, are worried about net income volatility. Unlike Bondholders, their return is not fixed. It is determined entirely by the skill with which the assets of the firm are managed. Stockholders have contracted with the bondholders to pay a specific rate of interest regardless of the skill with which the assets are managed. The contract does not allow the stockholders to renegotiate a lower rate of interest for the bondholders if management should generate a poor return on assets (ROA). If the bondholders are promised, for example, a return of 10%, but the assets generate an ROA of only 6%, the bondholders still receive their 10%. The stockholders, however, suffer from a return lower than even the ROA. The flip side, of course, is that stockholders will earn a return greater than ROA when ROA exceeds the bond rate of interest. This allocation of risk and return through financial leverage drives the pricing of bonds and stocks in the competitive capital markets.

Understanding the unique difference in the risks incurred by bondholders and stockholders is critical to understanding how their securities are priced (i.e. discounted). Where insolvency probabilities are low, the expected cash flow to bondholders is discounted at a rate approaching the risk-free rate of interest. However, under the same conditions, the expected cash flow to stockholders is discounted with a rate substantially higher than the risk-free rate of interest. Even when default risk is zero, a high probability of low returns to stockholders still exits because ROA may fall below the rate of return promised to bondholders.

The financial structure of an insurance company is similar to that of other publicly traded firms discussed above; it maintains assets (predominantly in its investment portfolio) which are financed with both debt and equity capital. However, the source of the debt financing is quite unique for an insurance firm. Rather than issuing bonds, insurance companies raise debt primarily by issuing insurance policies. Their customers, the policyholders, are also their bondholders. Policyholders advance the company prepaid insurance premiums which the company uses to fund the investment portfolio from which losses and expenses are paid.

When policyholders advance the premiums they are actually buying two securities: one is similar to an annuity and the other is pure insurance. Annuities are just a class of bonds that yield a constant payment which includes interest and principal. Policyholders could buy just insurance coverage, but this would require purchasing a new policy each day. Instead, they buy the first day's coverage and simultaneously buy the annuity-type obligation which, over the subsequent period, will pay the daily insurance premiums for the policyholders. Like bondholders, policyholders expect their annuity-type (bond-type) obligation to be fairly priced (i.e. earn a fair return).

The sum of the price of the annuity portion and the pure insurance coverage is the appropriate value of the insurance premium. The price of each is simply the present value of their respective expected cash flows. The critical issue is finding the appropriate interest rate to use for discounting.

<u>An Appropriate Interest Rate for Discounting Policyholder Claims</u>:

To determine the appropriate discount rate for pricing the annuity-type (bond-type) portion and the pure insurance portion of the insurance premium, we must ascertain the riskiness of their respective cash flows. Based on the Theory of the Firm analogy of policyholders to bondholders developed above, the relevant question for pricing the annuity-type portion of the insurance is whether the insurance company might become insolvent. Moreover, even if the insurance company does become insolvent, would policyholders expect unpaid claims or unearned premiums to be lost? Would not an industry backed or government backed organization fulfill the outstanding obligations of the firm?

For the "typical" insurance company, on which the ratemaking process is centered, it seems appropriate for policyholders to expect zero insolvency losses. This expectation is based on the regulatory signaling and numerous protective devices that are created for the benefit of policyholders. In most states, policyholders are backed by the industry's Guaranty Fund which comes to the policyholders' rescue in the event of insolvency. State insurance codes prescribe the riskclass of investments acceptable for insurance company acquisition. Adequacy of capital tests trigger audit reviews by state departments of insurance should leverage appear excessive. Indeed the ratemaking process itself sends a message of the state's concern for the financial security of policyholders. Therefore, to the "typical" policyholder of a rate regulated company the probability of firm insolvency appears extremely low. For all practical purposes then, the appropriate discount rate to price (discount) the annuity-type portion of the insurance premium is the risk-free rate of interest.

By instituting a credit equal to the risk-free rate of interest to policyholder accounts, regulatory agencies would affirm the fiduciary responsibility of insurance companies to safeguard the prepaid premiums and unpaid claims of policyholders. Moreover, the independence of investment returns (decisions) from funding sources would be implicitly acknowledged. The risk of unfavorable returns on the investment portfolio should be incurred by the stockholders, not the policyholders. Likewise, the chance of favorable results from the investment portfolio should accrue to the stockholders. The volatility with which the investment portfolio performance vacillates between favorable returns and unfavorable returns creates the risk incurred by stockholders. That risk is irrelevant to policyholders because their return is implicitly guaranteed in a regulatory environment.

Stockholders may magnify the volatility of investment portfolio returns further by operating the firm at a higher premium-to-surplus ratio. Those who prefer risk will invest in firms with high premium-to-surplus ratios. Risk-averse stockholders will invest in firms with low-premium-to-surplus ratios. The differences in ROEs of these firms will vary because the levered risk to stockholders varies. However, as long as policyholders expect zero insolvency losses, their rate of return (credits) should not vary among insurance firms. They should all receive the risk-free rate. Neither the risk or returns of the investment portfolio or the risk created by the firm's equity level has any effect on the policyholders' returns as long as insolvency is precluded.[2]

Without the supervision of the states' Departments of Insurance and the industry's Guaranty Funds, policyholders would be at risk to lose their unexpired coverage or their unpaid claims in the event of insolvency. Policyholders would understand this prospect and demand a higher credit than the risk-free rate on the bond portion of their insurance premium. To grant a higher rate than the risk-free rate of return to policyholders insinuates the possibility that insolvency losses may be inflicted on policyholders. Only if the regulatory body wishes to proclaim that possibility should policyholders be credited with a rate greater than the risk-free rate. Even then, the rate of interest should vary according to premium-to-surplus ratios, as well as business lines.

Expenses and losses, unlike interest credits to policyholders, are not guaranteed. They may be more or less than anticipated. This possibility of variance from their anticipated level makes expenses and losses a risky cash flow. Risky cash flows should be discounted with risky rates of return. Unfortunately, estimating the appropriate risky discount rate is a difficult task. This is

[2] The irrelevancy of leverage can be further demonstrated with the Modigliani and Miller (1958) self-made leverage proposition.

especially true when the cash flows are not directly generated by publicly traded securities. The capital asset pricing model (CAPM) and the option pricing model (OPM) were developed by financial theorists to discount the cash flows of publicly traded securities. Applying these models to the expenses and losses of insurance operations requires considerable indirect inference and is subject to dispute.

An alternative procedure for discounting risky cash flows is to discount their "certain equivalent" with the risk-free rate. The following is an example. A $100 risky annuity might be the equivalent of an $80 riskless annuity. Discounting the risky annuity at a risky rate of 10% results in a value of $1000 ($100/.10). Discounting the riskless annuity at the risk-free rate of 8% also results in a value of $1000 ($80/.08). The $80 is the "certain equivalent" of the $100 risky cash flow. One is indifferent between owning a risky $100 cash flow or an $80 riskless cash flow. Discounting a certain equivalent cash flow with the risk-free rate generates the same results as discounting the risky cash flow with a risky rate of interest.

For insurance companies, the "certain equivalent" of losses and expenses is "premiums minus profits". The insurance company is willing to underwrite the risk of uncertain losses and expenses because of the possibility for profits. Profits buffer the shareholders against net income variability caused by risky cash flows of expenses and losses. Profits are in essence the price of risk for being in the insurance business. Subtracting profits from premiums leaves the certain equivalent of expenses and losses. By extracting out the risk, only the certain equivalent remains which can be discounted by the risk-free rate. The certain equivalent method is appealing for ratemaking purposes because profits of service related companies can be observed and compared with the profits of insurance companies.

A Comparable Profit Measure:

Corporations arrive at a price for their products through price competition. The price is, in essence, the starting point for operations. From there, corporations can choose to "lever-up" with either a capital intensive asset base, or "lever-up" with a debt intensive capital structure. The decision reflects the objectives of the company and expresses management's preference for high risk or low risk.

Companies within the same industry may have entirely different management styles. Those with risky management styles "lever-up" with the expectation of achieving a higher return on equity. Risk-averse managers do the opposite. The only common thread that runs between these companies is the product price. Consumers force the product price to be the same because they will not pay more for an identical product offered elsewhere. In the insurance industry, an identical product within a business line is guaranteed coverage. The policyholders expect to be protected from lost coverage or lost claims in the event of firm insolvency because of the protective devices discussed above.

If we compare profit margins, return on assets, and return on equity by industry groups, the least variance should occur in the comparison of profit margin on sales. The greatest variance should occur in the return on equity comparisons because some industry groups choose to lever-up, while others choose the comfort of equity financing. If the purpose of ratemaking is to establish fair prices, then its focus should be on a measure that reflects the stability of prices unencumbered by leverage effects. Ratemaking procedures that focus on return on equity distort the issue of fair prices with issues of leverage. While leverage certainly alters expected ROE, it does not impact product price. If policyholders don't have to worry about insolvency losses, then they will pay insurance premiums that are independent of the issuing company's leverage.

Using either ROA or an adjusted profit margin on sales would eliminate distortions from financial leverage. However, profit margin has the advantage of being expressed in a form equivalent to the insurance industry's underwriting profit provisions ratio. Several ratios are generically referred to as profit margins. These include the gross profit margin, the operating profit margin, and the net profit margin. For comparative purposes, the best performance measure is an adjusted net profit margin that excludes financial leverage effects net of tax:

$$\text{net profit margin} + \frac{\text{interest expense (1-tax rate)}}{\text{Sales}}$$

This is the "unlevered" net profit margin of a levered firm. With adjusted net profit margins, the operating performance of firms can be compared without the distortions created by their respective managements' leverage choices.

The adjusted net profit margins of thirteen well-known service-firm groups appear in Table 1. Service firms are presented because of the service nature of insurance. The data was taken from the publications of Dun & Bradstreet (D&B), Roger Morris and Associates (RMA) and Statistics on Banking. D&B covers a greater number of firms and includes publicly traded companies. RMA compiles its data from commercial banks on the basis of companies to which they lend. Statistics on Banking is compiled by the FDIC and covers only financial data on commercial banks. The companies reported by RMA are typically smaller and often privately held. Although the breadth of D&B data is preferable, its presentation does not include interest expense information. Therefore, interest expense was taken from RMA and added (after taxes) to the net profit margins reported by D&B. A cursory comparison of these ratios suggests a profit margin in the 6% range appears fair for setting insurance premiums.

The purpose of this paper, however, is not to establish an acceptable profit margin, but to present the case for a DCFM of ratemaking that utilizes profit margins to estimate expense and loss ratios. Richard Woll (1987) offers such a procedure. Ratemaking debate can focus on the appropriate profit margin.

Table 1
PROFIT MARGINS

	1987	1986	1985	1984	1983	Avg.
Commercial Banks						
Net Profit Margin	1.3	6.3	6.4	5.6	6.2	5.2
Interest Expense	0	0	0	0	0	0
Adj. Profit Margin	1.3	6.3	6.4	5.6	6.2	5.2
5399/Genl Mdse Stores						
Net Profit Margin	4.7	4.6	4.9	5.0	4.8	
Interest Expense*	0.4	0.6	0.6	0.4	0.6	
Adj Profit Margin	5.0	4.9	5.2	5.2	5.1	5.1
5651/Family Clothing						
Net Profit Margin	5.9	5.1	5.5	5.8	6.6	
Interest Expense	0.4	0.6	0.6	0.4	0.6	
Adj Profit Margin	6.2	5.4	5.8	6.0	6.9	6.1
5712/Furniture						
Net Profit Margin	4.8	5.0	5.0	4.7	4.6	
Interest Expense	0.3	0.4	0.4	0.4	0.3	
Adj Profit Margin	5.0	5.2	5.2	4.9	4.8	5.0
5722/Appliances						
Net Profit Margin	4.5	5.3	4.5	5.2	5.9	
Interest Expense	-0.1	0.4	0.4	0.4	-0.2	
Adj Profit Margin	4.4	5.5	4.7	5.4	4.8	5.0
5812/Restaurants						
Net Profit Margin	4.2	4.0	3.9	4.2	4.4	
Interest Expense	1.9	1.7	1.9	2.1	1.6	
Adj Profit Margin	5.5	4.9	4.9	5.3	5.2	5.1
6311/Life Insurance						
Net Profit Margin	6.4	6.7	8.0	5.9	4.8	
Interest Expense*	0.0	0.0	0.0	0.0	0.0	
Adj Profit Margin	6.4	6.7	8.0	5.9	4.8	6.4
6321 Health Insurance						
Net Profit Margin	4.8	6.2	5.2	3.7	3.9	
Interest Expense*	0.0	0.0	0.0	0.0	0.0	
Adj Profit Margin	4.8	6.2	5.2	3.7	3.9	4.8

	1987	1986	1985	1984	1983	Avg.
8062/Hospitals						
Net Profit Margin	3.6	4.5	4.9	3.5	3.6	
Interest Expense *	1.9	1.7	1.9	2.1	1.6	
Adj Profit Margin	4.9	5.4	5.9	4.6	4.4	5.0
8111/Legal **						
Before Tax Margin	16.2	13.9	13.5	14.6	14.4	
Net Profit Margin	10.7	7.0	6.8	7.3	7.2	
Interest Expense	1.4	1.1	1.0	2.1	2.6	
Adj Profit Margin	11.6	7.7	7.4	8.7	8.9	8.9
8611 Bus. Associations						
Net Profit Margin	5.7	4.9	5.2	4.5	3.7	
Interest Expense *	0.5	0.7	0.7	0.5	0.7	
Adj Profit Margin	6.0	5.2	5.5	4.7	4.0	5.1
8911/Eng/Architectural						
Net Profit Margin	6.5	7.1	7.3	6.8	6.4	
Interest Expense	0.9	1.5	1.3	1.1	1.3	
Adj Profit Margin	7.1	7.9	8.0	7.4	7.1	7.5
8931/Accounting **						
Before Tax Margin	14.4	15.6	15.9	14.6	16.4	
Net Profit Margin	9.5	7.8	8.0	7.3	8.2	
Interest Expense*	0.4	0.5	0.5	0.4	0.5	
Adj Profit Margin	9.8	8.1	8.3	7.6	8.5	8.5

Net Profit Margin - Source: Industry Norms and Key Business Ratios published by Dun & Bradstreet.

Interest Expense - Source: Annual Statement Studies published by Robert Morris and Associates (RMA).

For Commercial Banks

Net Profit Margin - Source: Statistics on Banking

Adj Profit Margin - Equals: Net Profit + $\frac{\text{Interest Expense}}{\text{(1-Corp. Tax Rate)}}$

* Data not available in RMA. Interest Expense based on industry with similar Debt/Equity ratio.

**Denotes partnerships. Profit margins reported are before tax and must be adjusted for taxes.

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EASIER ALGORITHMS FOR AGGREGATE EXCESS

Gary Venter

EASIER ALGORITHMS FOR AGGREGATE EXCESS

BY GARY VENTER

Probabilities for aggregate claims can be calculated from frequency and severity probabilities using the characteristic function algorithm of Heckman-Meyers [1983], but the formulas are somewhat difficult to follow and program. Two easier algorithms are presented below. These are particularly efficient when there are only a small number of claims expected and when the severity distribution is fairly flat, as in many reinsurance situations.

The first algorithm is the recursion method introduced into actuarial literature in Panjer [1981] and into the PCAS in Venter [1983]. The severity density r(x) is approximated by a discrete function. That is, some interval size h is established, and claims come in lumps of h, 2h, ... At ih this discrete severity function will be denoted by g_i. The claim frequency distribution is assumed to have the following recursive property: there are constants a and b such that $p_{i+1}=[a+\frac{b}{i+1}]p_i$. For the Poisson, a=0 and $b=\lambda$. For the negative binomial with mean $\alpha(1-\beta)/\beta$ and variance $\alpha(1-\beta)/\beta^2$, the constants are $a=1-\beta$ and $b=(1-\beta)(\alpha-1)$. The aggregate density f can be calculated recursively:

$$f_0 = p_0, \quad f_k = \sum_{i=1}^{k}(a+b\tfrac{i}{k})g_i f_{k-i}, \quad k>0.$$

To implement this for a general severity, a rule for doing the discrete approximation is needed. A fairly arbitrary rule which appears reasonable is to match interval probabilities and means for each consecutive pair of points $\langle g_1,g_2\rangle$, $\langle g_3,g_4\rangle$, etc. Then the cumulative distribution and limited severity will match the original at every second point. For instance, the probability assigned to the points 3h and 4h would be determined by solving the two equations:

$$g_3+g_4=\int_{2.5h}^{4.5h} r(z)dz \qquad\qquad 3hg_3+4hg_4=\int_{2.5h}^{4.5h} zr(z)dz$$

The solution can be expressed with the aid of two auxiliary functions d and m:

$$d_i=\int_{0}^{ih} r(z)dz \qquad\qquad m_i=\frac{1}{h}\int_{0}^{ih} zr(z)dz$$

Note that d_i is just the distribution function. It can be verified that:

$$g_3=m_{2.5}-m_{4.5} + 4(d_{4.5}-d_{2.5})$$
$$g_4=m_{4.5}-m_{2.5} - 3(d_{4.5}-d_{2.5})$$

Also, the impact of the extra interval from 0 to h/2 should be incorporated into g_1 and g_2.

As an example, for the Pareto severity in b,c (mean $\frac{b}{c-1}$), the functions are $d_i=1-(1+\frac{ih}{b})^{-c}$ and $m_i=\frac{b}{h(c-1)}[1-(1+\frac{ih}{b})^{-c}(1+\frac{ich}{b})]$. Note that the h can be eliminated by expressing b in units of h. For c=1, $m_i=(b/h)\ln(1+ih/b)-i/(1+ih/b)$.

The second algorithm uses the same type of frequency distribution but assumes a continuous severity distribution. Panjer also derived a continuous recursion:

$$f(x)=p_1 r(x)+\int_0^x (a+b\tfrac{y}{x})r(y)f(x-y)dy, \quad x>0$$

This on the surface appears impractical, but it is in fact a Volterra integral equation of the second type, and can be solved numerically. Methods are in Baker [1977]. The following solution is based on the trapezoid rule. Solutions based on Simpson's rule and quadratic quadrature may be found in Ströter [1985], who also incorporates numerical integration of the severity functions.

A third auxiliary severity function is introduced:

$$v_i=\frac{1}{h^2}\int_0^{ih} z^2 r(z)dz$$

For instance, for the Pareto, $v_i=\frac{2(b/h)^2}{(c-1)(c-2)}\left[1-(1+\frac{ih}{b})^{-c}(1+\frac{ich}{b}+\frac{c(c-1)i^2(h/b)^2}{2})\right]$, except for $c=1$, $v_i=2(b/h)^2\left[\frac{i(.5i+b/h)}{(b/h)(i+b/h)}-\ln(1+ih/b)\right]$ and $c=2$, $v_i=2(b/h)^2\left[\ln(1+ih/b)-\frac{i(2i+b/h)}{(i+b/h)^2}\right]$.

The numerical approximation of f will involve first selecting an interval h, and then recursively approximating f at 0, h, 2h, ... First, set $f(0)=p_1 r(0)$. This is not $f_0=p_0$, the discrete probability lump at zero, but is rather the approximation of the aggregate density f at just above 0. Then f(kh) is approximated by:

$$f(kh)=\frac{p_1 r(kh)+\sum_{i=1}^{k} f((k-i)h)w_{i,k}}{1-w_{0,k}},$$

where the weights $w_{i,k}$ are defined as follows:

Let $s_i=2m_i-m_{i+1}-m_{i-1}+i[d_{i+1}+d_{i-1}-2d_i]+d_{i+1}-d_{i-1}$, and

let $t_i=2v_i-v_{i+1}-v_{i-1}+i[m_{i+1}+m_{i-1}-2m_i]+m_{i+1}-m_{i-1}$.

Then for $0<i<k$, define $w_{i,k}=bt_i/k + as_i$. Let $w_{0,k}=b(m_1-v_1)/k + a(d_1-m_1)$ and let $w_{k,k}=(b/k)[v_k-v_{k-1}-(k-1)(m_k-m_{k-1})]+a[m_k-m_{k-1}-(k-1)(d_k-d_{k-1})]$.

The error in this approximation is proportional to h^2, so it reduces quadratically as h gets smaller. This knowledge of the error structure is an advantage over other approximations. The fact that the s's and t's are not functions of k simplifies the calculation of the weights, so that a spreadsheet calculation is possible. If h is small enough, $w_{0,k}$ will be between 0 and 1, which is necessary to get reasonable results. This sometimes forces a smaller h than would otherwise be needed.

In application of these approximations, two functions are often calculated: the cumulative distribution F and the excess ratio (portion of loss dollars excess of x) R. With μ the aggregate mean, this is given by $R(x)=\frac{1}{\mu}\int_x^{\infty}(z-x)f(z)dz$, and so $\mu R(x)=\mu-\int_0^x zf(z)dz-x[1-F(x)]$. For the continuous case, care must be taken to include the point mass at zero, $f_0=p_0$, in the integral. Thus, using trapezium, $F(kh)=f_0+.5h(f(0)+f(kh))+h\sum_{i=1}^{k-1}f(ih)$, and $\mu R(kh)=\mu-.5hkhf(kh)-h\sum_{i=1}^{k-1}ihf(ih)-kh(1-F(kh))$. In the discrete approximation formula, $\mu R_k=\mu-\sum_{i=1}^{k}if_i-k(1-F_k)$, where F_k is just $\sum_{i=0}^{k}f_i$.

As an example, a Pareto severity with b=1,000,000 and c=3 was used, which has a mean of 500,000, along with a Poisson frequency. The Pareto was chosen with reinsurance in mind, as excess losses from a Pareto are themselves Pareto distributed. The exhibits compare the discrete and continuous approximations for intervals h of 100,000, $33{,}333\frac{1}{3}$, 10,000, and $3333\frac{1}{3}$ and λ's of .2, 1, and 5. For the continuous approximation, the errors in the distribution function seem generally smaller than those in the excess ratio. In both cases, the numerical integration is contributing to the error, and a better integration method may help. For the smallest h, the distribution function appears to be accurate to five places, and the errors appear to be increasing by a factor of 10 for each larger h, which agrees with the order of h^2 theory. The errors in the excess ratio also appear to increase proportionally to h^2, but they are considerably larger, especially for the higher probabilities, than those for the distribution function.

The discrete approximation was actually better in some ways. Because it is an exact calculation given the discrete severity, the F and R functions are reasonable for h up to 100,000. For even higher h's, the severity approximation can be negative for g_2, however. The main disadvantage of the discrete method is that the discretizing process used seems to make the estimated F_i a closer approximation to the true $F_{i+.5}$ than it is to the true F_i. To illustrate this for $\lambda=5$, an exhibit is included which compares the discrete approximation with h=100,000 at 500,000, 1,000,000, etc. to the continuous approximation with $h=3333\frac{1}{3}$ at 550,000, 1,050,000, etc. The coarser interval with the discrete approximation seems close enough for most purposes, if the half-shift in the interval is acceptable. For some reason, the half shift problem does not seem to arise with the excess ratios.

A significant advantage of the discrete form is that it works easily with limited severity, i.e., a point mass at the top of the severity distribution. Since a continuous severity density is required for positive values for the continuous approximation, it is not clear how it could be adapted for limited severity.

In order to see if an improved numerical integration method would improve the accuracy of the continuous approximation, an adjustment to the trapezoid rule was developed. Rather than approximating the function by line segments, as in the trapezoid, quadratic polynomials were fit through each combination of three consecutive points, and the area under these polynomials used to approximate the integral. The result is just a slight adjustment to the trapezoid rule: in calculating $\int_0^{ih} u(z)dz$ in steps of width h, add $(h/24)[u(2h)-u(0)+u(ih-h)-u(ih+h)]$ to the trapezoid approximation. The continuous approximation exhibits were redone using this quadratic integration rule. The errors at each h level were reduced substantially by this adjustment.

The solution of the integral equation is not necessarily optimal, and the continuous approximation may perform better with other approximations.

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Illana Leiser and Jack Barnett provided invaluable assistance in solving the integral equation and programming the exhibits, with the usual rejoinder.

COMPARATIVE APPROXIMATIONS
Distribution Function

	Continuous Severity Model				Discrete Severity Model			
Intrvl→ (000)	100	33 1/3	10	3 1/3	100	33 1/3	10	3 1/3
0	0.81873	0.81873	0.81873	0.81873	0.81873	0.81873	0.81873	0.81873
500	0.94157	0.94036	0.94022	0.94021	0.94585	0.94206	0.94076	0.94039
1000	0.97502	0.97362	0.97346	0.97345	0.97529	0.97407	0.97363	0.97351
1500	0.98774	0.98628	0.98612	0.9861	0.98693	0.98637	0.98618	0.98613
2000	0.99357	0.99208	0.99191	0.9919	0.99229	0.99203	0.99194	0.99191
2500	0.99658	0.99508	0.99491	0.9949	0.99512	0.99497	0.99492	0.9949
3000	0.99828	0.99678	0.99661	0.99659	0.99672	0.99663	0.9966	0.9966
3500	0.9993	0.99781	0.99763	0.99762	0.9977	0.99764	0.99763	0.99762
4000	0.99996	0.99846	0.99829	0.99827	0.99832	0.99829	0.99828	0.99827
4500	1.0004	0.9989	0.99873	0.99871	0.99874	0.99872	0.99871	0.99871
5000	1.0007	0.9992	0.99903	0.99901	0.99903	0.99902	0.99901	0.99901

Poisson: Lamda = 0.2
Pareto : b = 1,000,000 & c = 3

COMPARATIVE APPROXIMATIONS
Distribution Function

Intrvl→ (000)	Continuous Severity Model 100	33 1/3	10	3 1/3	Discrete Severity Model 100	33 1/3	10	3 1/3
0	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788
500	0.70744	0.70544	0.70521	0.70519	0.72602	0.71195	0.7072	0.70586
1000	0.84686	0.84411	0.84379	0.84376	0.85315	0.84683	0.84468	0.84407
1500	0.91393	0.91081	0.91045	0.91042	0.91523	0.91199	0.91089	0.91058
2000	0.94914	0.94583	0.94545	0.94542	0.94802	0.94627	0.94567	0.9455
2500	0.96886	0.96545	0.96506	0.96502	0.96653	0.96551	0.96517	0.96507
3000	0.98051	0.97704	0.97664	0.97661	0.97751	0.9769	0.97669	0.97663
3500	0.9877	0.98421	0.9838	0.98377	0.98434	0.98395	0.98382	0.98378
4000	0.99233	0.98881	0.98841	0.98837	0.98874	0.98849	0.9884	0.98838
4500	0.99541	0.99188	0.99147	0.99143	0.99168	0.99151	0.99145	0.99144
5000	0.99752	0.99398	0.99357	0.99354	0.99371	0.99359	0.99355	0.99354
5500	0.99901	0.99546	0.99505	0.99502	0.99514	0.99505	0.99502	0.99502
6000	1.00009	0.99653	0.99612	0.99609	0.99617	0.99611	0.99609	0.99608
6500	1.00088	0.99732	0.99691	0.99687	0.99694	0.99689	0.99688	0.99687
7000	1.00148	0.99791	0.9975	0.99747	0.99752	0.99748	0.99747	0.99747
7500	1.00193	0.99837	0.99796	0.99792	0.99796	0.99793	0.99792	0.99792
8000	1.00229	0.99872	0.99831	0.99827	0.9983	0.99828	0.99827	0.99827
8500	1.00257	0.999	0.99859	0.99855	0.99857	0.99856	0.99855	0.99855
9000	1.00279	0.99922	0.99881	0.99877	0.99879	0.99878	0.99877	0.99877
9500	1.00297	0.9994	0.99899	0.99895	0.99896	0.99895	0.99895	0.99895
10000	1.00312	0.99955	0.99914	0.9991	0.99911	0.9991	0.9991	0.9991

Poisson: Lamda = 1
Pareto : b = 1,000,000 & c = 3

COMPARATIVE APPROXIMATIONS
Distribution Function

Intrvl→ (000)	Continuous Severity Model 100	33 1/3	10	3 1/3	Discrete Severity Model 100	33 1/3	10	3 1/3
0	0.00674	0.00674	0.00674	0.00674	0.00674	0.00674	0.00674	0.00674
500	0.09445	0.09478	0.09481	0.09482	0.10537	0.09869	0.09599	0.09521
1000	0.22777	0.22873	0.22884	0.22885	0.2428	0.2336	0.23028	0.22932
1500	0.37055	0.3722	0.37239	0.3724	0.38654	0.37707	0.37381	0.37287
2000	0.50177	0.50406	0.50432	0.50434	0.51697	0.50844	0.50557	0.50475
2500	0.61275	0.61556	0.61588	0.61591	0.62643	0.61929	0.61692	0.61625
3000	0.70204	0.70528	0.70564	0.70567	0.71407	0.70835	0.70647	0.70594
3500	0.77173	0.77528	0.77568	0.77571	0.78223	0.77778	0.77633	0.77592
4000	0.82507	0.82886	0.82929	0.82933	0.8343	0.8309	0.8298	0.82949
4500	0.86546	0.86943	0.86987	0.86991	0.87368	0.8711	0.87027	0.87004
5000	0.89586	0.89996	0.90042	0.90046	0.90329	0.90135	0.90073	0.90055
5500	0.91869	0.92289	0.92336	0.9234	0.92553	0.92407	0.9236	0.92347
6000	0.93586	0.94012	0.9406	0.94065	0.94225	0.94116	0.9408	0.9407
6500	0.9488	0.95312	0.95361	0.95365	0.95487	0.95404	0.95377	0.95369
7000	0.9586	0.96297	0.96346	0.9635	0.96442	0.9638	0.96359	0.96354
7500	0.96607	0.97047	0.97096	0.97101	0.97171	0.97123	0.97108	0.97103
8000	0.9718	0.97622	0.97672	0.97676	0.9773	0.97694	0.97682	0.97678
8500	0.97623	0.98066	0.98116	0.9812	0.98163	0.98134	0.98125	0.98122
9000	0.97967	0.98412	0.98462	0.98466	0.98499	0.98477	0.9847	0.98468
9500	0.98237	0.98683	0.98733	0.98738	0.98764	0.98746	0.98741	0.98739
10000	0.98451	0.98898	0.98948	0.98952	0.98973	0.98959	0.98955	0.98953
10500	0.98621	0.99069	0.99119	0.99123	0.9914	0.99129	0.99125	0.99124
11000	0.98758	0.99206	0.99256	0.99261	0.99274	0.99265	0.99262	0.99262
11500	0.98868	0.99317	0.99367	0.99372	0.99383	0.99376	0.99373	0.99373
12000	0.98959	0.99408	0.99458	0.99463	0.99472	0.99466	0.99464	0.99463
12500	0.99033	0.99482	0.99533	0.99537	0.99545	0.9954	0.99538	0.99538
13000	0.99094	0.99544	0.99594	0.99599	0.99605	0.99601	0.996	0.996
13500	0.99145	0.99595	0.99646	0.9965	0.99656	0.99653	0.99651	0.99651
14000	0.99188	0.99638	0.99689	0.99694	0.99698	0.99695	0.99694	0.99694
14500	0.99225	0.99675	0.99726	0.9973	0.99734	0.99732	0.99731	0.99731
15000	0.99255	0.99706	0.99756	0.99761	0.99764	0.99762	0.99762	0.99762
15500	0.99282	0.99732	0.99783	0.99787	0.9979	0.99789	0.99788	0.99788
16000	0.99304	0.99755	0.99806	0.9981	0.99813	0.99811	0.99811	0.99811
16500	0.99324	0.99774	0.99825	0.9983	0.99832	0.99831	0.9983	0.9983
17000	0.99341	0.99791	0.99842	0.99847	0.99849	0.99848	0.99847	0.99847
17500	0.99355	0.99806	0.99857	0.99861	0.99863	0.99862	0.99862	0.99862
18000	0.99368	0.99819	0.9987	0.99874	0.99876	0.99875	0.99875	0.99875
18500	0.9938	0.99831	0.99881	0.99886	0.99887	0.99887	0.99886	0.99886
19000	0.99389	0.99841	0.99891	0.99896	0.99897	0.99897	0.99896	0.99896
19500	0.99398	0.99849	0.999	0.99905	0.99906	0.99905	0.99905	0.99905
20000	0.99406	0.99857	0.99908	0.99912	0.99914	0.99913	0.99913	0.99913

Poisson: Lamda = 5
Pareto : b = 1,000,000 & c = 3

COMPARATIVE APPROXIMATIONS
Excess Ratio

Intrvl→ (000)	Continuous Severity Model 100	33 1/3	10	3 1/3	Discrete Severity Model 100	33 1/3	10	3 1/3
0	1	1	1	1	1	1	1	1
500	0.47807	0.47169	0.47112	0.47108	0.46927	0.4709	0.47106	0.47107
1000	0.27994	0.27061	0.26987	0.26982	0.26924	0.26975	0.26981	0.26981
1500	0.18639	0.1741	0.17317	0.17311	0.17284	0.17308	0.1731	0.17311
2000	0.13637	0.12099	0.11988	0.11981	0.11968	0.11979	0.1198	0.1198
2500	0.10747	0.08894	0.08763	0.08755	0.08747	0.08753	0.08754	0.08754
3000	0.08996	0.06825	0.06674	0.06664	0.06659	0.06663	0.06663	0.06663
3500	0.07911	0.0542	0.05247	0.05237	0.05233	0.05235	0.05236	0.05236
4000	0.07239	0.04425	0.04232	0.04221	0.04218	0.04219	0.0422	0.0422
4500	0.06835	0.03699	0.03486	0.03473	0.0347	0.03471	0.03472	0.03472
5000	0.06614	0.03155	0.02921	0.02907	0.02905	0.02905	0.02905	0.02905

Poisson: Lamda = 0.2
Pareto : b = 1,000,000 & c = 3

COMPARATIVE APPROXIMATIONS
Excess Ratio

Intrvl-(000)	Continuous Severity Model 100	33 1/3	10	3 1/3	Discrete Severity Model 100	33 1/3	10	3 1/3
0	1	1	1	1	1	1	1	1
500	0.56701	0.56467	0.56444	0.56442	0.56232	0.56425	0.5644	0.56442
1000	0.35211	0.34824	0.34788	0.34785	0.34686	0.34776	0.34784	0.34784
1500	0.23478	0.22924	0.22873	0.22868	0.22817	0.22864	0.22868	0.22868
2000	0.16647	0.1591	0.15842	0.15836	0.15807	0.15833	0.15835	0.15836
2500	0.12471	0.11541	0.11454	0.11448	0.1143	0.11445	0.11447	0.11447
3000	0.09821	0.08691	0.08586	0.08577	0.08566	0.08575	0.08576	0.08576
3500	0.08093	0.06758	0.06633	0.06623	0.06615	0.06621	0.06622	0.06622
4000	0.06945	0.05403	0.05257	0.05246	0.0524	0.05244	0.05244	0.05244
4500	0.06176	0.04424	0.04259	0.04246	0.04241	0.04244	0.04244	0.04244
5000	0.05663	0.03701	0.03515	0.035	0.03496	0.03498	0.03498	0.03498
5500	0.05328	0.03154	0.02948	0.02932	0.02928	0.0293	0.0293	0.0293
6000	0.0512	0.02734	0.02507	0.02489	0.02486	0.02487	0.02487	0.02487
6500	0.05005	0.02406	0.02159	0.02139	0.02136	0.02137	0.02137	0.02137
7000	0.04958	0.02146	0.01879	0.01857	0.01854	0.01855	0.01855	0.01855
7500	0.04964	0.01938	0.0165	0.01628	0.01624	0.01625	0.01625	0.01625
8000	0.0501	0.01771	0.01462	0.01438	0.01435	0.01435	0.01435	0.01435
8500	0.05087	0.01635	0.01306	0.01279	0.01276	0.01276	0.01276	0.01276
9000	0.0519	0.01523	0.01174	0.01146	0.01142	0.01143	0.01143	0.01143
9500	0.05312	0.01432	0.01062	0.01032	0.01029	0.01029	0.01029	0.01029
10000	0.05451	0.01357	0.00966	0.00935	0.00931	0.00931	0.00931	0.00931

Poisson: Lamda = 1
Pareto : b = 1,000,000 & c = 3

COMPARATIVE APPROXIMATIONS
Excess Ratio

Intrvl-(000)	Continuous Severity Model 100	33 1/3	10	3 1/3	Discrete Severity Model 100	33 1/3	10	3 1/3
0	1	1	1	1	1	1	1	1
500	0.80893	0.80901	0.80902	0.80902	0.80845	0.80899	0.80902	0.80902
1000	0.64078	0.64095	0.64097	0.64097	0.64019	0.64092	0.64097	0.64097
1500	0.50077	0.50112	0.50116	0.50116	0.50034	0.50111	0.50116	0.50116
2000	0.3884	0.38905	0.38913	0.38913	0.38839	0.38908	0.38913	0.38913
2500	0.30032	0.30139	0.30151	0.30152	0.3009	0.30148	0.30152	0.30152
3000	0.23223	0.23384	0.23402	0.23403	0.23353	0.234	0.23403	0.23404
3500	0.17999	0.18221	0.18245	0.18248	0.18208	0.18245	0.18248	0.18248
4000	0.13998	0.14287	0.1432	0.14322	0.14292	0.14321	0.14323	0.14323
4500	0.10928	0.1129	0.1133	0.11334	0.11311	0.11333	0.11334	0.11334
5000	0.08561	0.08999	0.09048	0.09052	0.09035	0.09051	0.09053	0.09053
5500	0.06723	0.07239	0.07297	0.07302	0.07289	0.07301	0.07302	0.07302
6000	0.05282	0.05878	0.05944	0.0595	0.05941	0.0595	0.05951	0.05951
6500	0.04139	0.04817	0.04893	0.04899	0.04892	0.049	0.049	0.049
7000	0.03223	0.03983	0.04068	0.04075	0.0407	0.04076	0.04076	0.04076
7500	0.02478	0.03322	0.03416	0.03424	0.0342	0.03424	0.03425	0.03425
8000	0.01865	0.02792	0.02895	0.02904	0.02902	0.02905	0.02905	0.02905
8500	0.01352	0.02363	0.02476	0.02486	0.02484	0.02487	0.02487	0.02487
9000	0.00918	0.02013	0.02135	0.02146	0.02145	0.02147	0.02147	0.02147
9500	0.00545	0.01724	0.01856	0.01867	0.01867	0.01868	0.01869	0.01869
10000	0.0022	0.01483	0.01625	0.01637	0.01637	0.01638	0.01638	0.01639
10500	¯0.00067	0.01281	0.01432	0.01445	0.01446	0.01447	0.01447	0.01447
11000	¯0.00324	0.0111	0.0127	0.01284	0.01285	0.01286	0.01286	0.01286
11500	¯0.00555	0.00963	0.01133	0.01148	0.01149	0.0115	0.0115	0.0115
12000	¯0.00767	0.00837	0.01016	0.01031	0.01033	0.01033	0.01033	0.01033
12500	¯0.00962	0.00727	0.00915	0.00932	0.00933	0.00934	0.00934	0.00934
13000	¯0.01144	0.0063	0.00828	0.00846	0.00847	0.00848	0.00848	0.00848
13500	¯0.01315	0.00545	0.00752	0.00771	0.00773	0.00773	0.00773	0.00773
14000	¯0.01476	0.00469	0.00686	0.00705	0.00707	0.00707	0.00707	0.00707
14500	¯0.01629	0.00401	0.00628	0.00648	0.0065	0.0065	0.0065	0.0065
15000	¯0.01776	0.0034	0.00576	0.00597	0.00599	0.00599	0.00599	0.00599
15500	¯0.01917	0.00284	0.0053	0.00552	0.00554	0.00554	0.00554	0.00554
16000	¯0.02053	0.00234	0.00489	0.00511	0.00514	0.00514	0.00514	0.00514
16500	¯0.02184	0.00187	0.00452	0.00475	0.00478	0.00478	0.00478	0.00478
17000	¯0.02313	0.00144	0.00419	0.00443	0.00446	0.00446	0.00446	0.00446
17500	¯0.02438	0.00105	0.00389	0.00414	0.00417	0.00417	0.00417	0.00417
18000	¯0.0256	0.00068	0.00362	0.00388	0.00391	0.00391	0.00391	0.00391
18500	¯0.0268	0.00034	0.00337	0.00364	0.00367	0.00367	0.00367	0.00367
19000	¯0.02797	0.00001	0.00314	0.00342	0.00345	0.00345	0.00345	0.00345
19500	¯0.02913	¯0.00029	0.00294	0.00322	0.00325	0.00325	0.00325	0.00325
20000	¯0.03027	¯0.00058	0.00274	0.00304	0.00307	0.00307	0.00307	0.00307

Poisson: Lamda = 5
Pareto : b = 1,000,000 & c = 3

COMPARATIVE APPROXIMATIONS

Distribution Function

(000)	Continuous Model h=3,333 1/3		Discrete Model h=100,000		Continuous Model h=3,333 1/3
0	0.00674	0	0.00674	0	0.00674
550	0.10686	500	0.10537	500	0.09482
1,050	0.24324	1,000	0.24280	1,000	0.22885
1,550	0.38634	1,500	0.38654	1,500	0.37240
2,050	0.51648	2,000	0.51697	2,000	0.50434
2,550	0.62585	2,500	0.62643	2,500	0.61591
3,050	0.71352	3,000	0.71407	3,000	0.70567
3,550	0.78176	3,500	0.78223	3,500	0.77571
4,050	0.83392	4,000	0.83430	4,000	0.82933
4,550	0.87337	4,500	0.87368	4,500	0.86991
5,050	0.90306	5,000	0.90329	5,000	0.90046
5,550	0.92535	5,500	0.92553	5,500	0.92340
6,050	0.94212	6,000	0.94225	6,000	0.94065
6,550	0.95476	6,500	0.95487	6,500	0.95365
7,050	0.96435	7,000	0.96442	7,000	0.96350
7,550	0.97165	7,500	0.97171	7,500	0.97101
8,050	0.97726	8,000	0.97730	8,000	0.97676
8,550	0.98159	8,500	0.98163	8,500	0.98120
9,050	0.98497	9,000	0.98499	9,000	0.98466
9,550	0.98762	9,500	0.98764	9,500	0.98738
10,050	0.98971	10,000	0.98973	10,000	0.98952
10,550	0.99138	10,500	0.99140	10,500	0.99123
11,050	0.99273	11,000	0.99274	11,000	0.99261
11,550	0.99382	11,500	0.99383	11,500	0.99372
12,050	0.99471	12,000	0.99472	12,000	0.99463
12,550	0.99544	12,500	0.99545	12,500	0.99537
13,050	0.99604	13,000	0.99605	13,000	0.99599
13,550	0.99655	13,500	0.99656	13,500	0.99650
14,050	0.99697	14,000	0.99698	14,000	0.99694
14,550	0.99733	14,500	0.99734	14,500	0.99730
15,050	0.99764	15,000	0.99764	15,000	0.99761
15,550	0.99790	15,500	0.99790	15,500	0.99787
16,050	0.99812	16,000	0.99813	16,000	0.99810
16,550	0.99831	16,500	0.99832	16,500	0.99830
17,050	0.99848	17,000	0.99849	17,000	0.99847
17,550	0.99863	17,500	0.99863	17,500	0.99861
18,050	0.99876	18,000	0.99876	18,000	0.99874
18,550	0.99887	18,500	0.99887	18,500	0.99886
19,050	0.99897	19,000	0.99897	19,000	0.99896
19,550	0.99905	19,500	0.99906	19,500	0.99905
20,050		20,000	0.99914	20,000	0.99912

Lamda=5

COMPARATIVE APPROXIMATIONS
Distribution Function

Intrvl-(000)	Continuous Severity Model 100	33 1/3	10	3 1/3	Discrete Severity Model 100	33 1/3	10	3 1/3
0	0.81873	0.81873	0.81873	0.81873	0.81873	0.81873	0.81873	0.81873
500	0.9408	0.94024	0.94021	0.94021	0.94585	0.94206	0.94076	0.94039
1000	0.97408	0.97348	0.97345	0.97344	0.97529	0.97407	0.97363	0.97351
1500	0.98677	0.98614	0.9861	0.9861	0.98693	0.98637	0.98618	0.98613
2000	0.99257	0.99194	0.9919	0.9919	0.99229	0.99203	0.99194	0.99191
2500	0.99558	0.99494	0.9949	0.9949	0.99512	0.99497	0.99492	0.9949
3000	0.99728	0.99664	0.99659	0.99659	0.99672	0.99663	0.9966	0.9966
3500	0.99831	0.99766	0.99762	0.99762	0.9977	0.99764	0.99763	0.99762
4000	0.99896	0.99832	0.99827	0.99827	0.99832	0.99829	0.99828	0.99827
4500	0.9994	0.99875	0.99871	0.99871	0.99874	0.99872	0.99871	0.99871
5000	0.9997	0.99905	0.99901	0.99901	0.99903	0.99902	0.99901	0.99901

Poisson: Lamda = 0.2
Pareto : b = 1,000,000 & c = 3

Quadratic Integration

COMPARATIVE APPROXIMATIONS
Distribution Function

Intrvl→ (000)	Continuous Severity Model 100	33 1/3	10	3 1/3	Discrete Severity Model 100	33 1/3	10	3 1/3
0	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788
500	0.70636	0.70527	0.70519	0.70518	0.72602	0.71195	0.7072	0.70586
1000	0.84543	0.84391	0.84377	0.84376	0.85315	0.84683	0.84468	0.84407
1500	0.91237	0.9106	0.91043	0.91042	0.91523	0.91199	0.91089	0.91058
2000	0.94753	0.94561	0.94543	0.94542	0.94802	0.94627	0.94567	0.9455
2500	0.96722	0.96523	0.96504	0.96502	0.96653	0.96551	0.96517	0.96507
3000	0.97886	0.97682	0.97662	0.9766	0.97751	0.9769	0.97669	0.97663
3500	0.98606	0.98398	0.98378	0.98376	0.98434	0.98395	0.98382	0.98378
4000	0.99068	0.98858	0.98838	0.98837	0.98874	0.98849	0.9884	0.98838
4500	0.99376	0.99165	0.99145	0.99143	0.99168	0.99151	0.99145	0.99144
5000	0.99587	0.99375	0.99355	0.99353	0.99371	0.99359	0.99355	0.99354
5500	0.99736	0.99523	0.99503	0.99501	0.99514	0.99505	0.99502	0.99502
6000	0.99843	0.9963	0.9961	0.99608	0.99617	0.99611	0.99609	0.99608
6500	0.99922	0.99709	0.99689	0.99687	0.99694	0.99689	0.99688	0.99687
7000	0.99982	0.99769	0.99748	0.99747	0.99752	0.99748	0.99747	0.99747
7500	1.00028	0.99814	0.99794	0.99792	0.99796	0.99793	0.99792	0.99792
8000	1.00063	0.99849	0.99829	0.99827	0.9983	0.99828	0.99827	0.99827
8500	1.00091	0.99877	0.99857	0.99855	0.99857	0.99856	0.99855	0.99855
9000	1.00113	0.99899	0.99879	0.99877	0.99879	0.99878	0.99877	0.99877
9500	1.00131	0.99917	0.99897	0.99895	0.99896	0.99895	0.99895	0.99895
10000	1.00146	0.99932	0.99911	0.9991	0.99911	0.9991	0.9991	0.9991

Poisson: Lamda = 1
Pareto : b = 1,000,000 & c = 3

Quadratic Integration

COMPARATIVE APPROXIMATIONS
Distribution Function

	Continuous Severity Model				Discrete Severity Model			
Intrvl-(000)	100	33 1/3	10	3 1/3	100	33 1/3	10	3 1/3
0	0.00674	0.00674	0.00674	0.00674	0.00674	0.00674	0.00674	0.00674
500	0.09457	0.09479	0.09481	0.09482	0.10537	0.09869	0.09599	0.09521
1000	0.22802	0.22876	0.22884	0.22885	0.2428	0.2336	0.23028	0.22932
1500	0.37086	0.37224	0.37239	0.3724	0.38654	0.37707	0.37381	0.37287
2000	0.50211	0.5041	0.50432	0.50434	0.51697	0.50844	0.50557	0.50475
2500	0.61309	0.6156	0.61588	0.61591	0.62643	0.61929	0.61692	0.61625
3000	0.70238	0.70532	0.70565	0.70568	0.71407	0.70835	0.70647	0.70594
3500	0.77205	0.77532	0.77568	0.77571	0.78223	0.77778	0.77633	0.77592
4000	0.82539	0.8289	0.82929	0.82933	0.8343	0.8309	0.8298	0.82949
4500	0.86576	0.86946	0.86988	0.86991	0.87368	0.8711	0.87027	0.87004
5000	0.89615	0.89999	0.90042	0.90046	0.90329	0.90135	0.90073	0.90055
5500	0.91898	0.92292	0.92336	0.9234	0.92553	0.92407	0.9236	0.92347
6000	0.93614	0.94016	0.94061	0.94065	0.94225	0.94116	0.9408	0.9407
6500	0.94908	0.95316	0.95361	0.95365	0.95487	0.95404	0.95377	0.95369
7000	0.95888	0.963	0.96346	0.9635	0.96442	0.9638	0.96359	0.96354
7500	0.96635	0.9705	0.97097	0.97101	0.97171	0.97123	0.97108	0.97103
8000	0.97208	0.97625	0.97672	0.97676	0.9773	0.97694	0.97682	0.97678
8500	0.9765	0.98069	0.98116	0.98121	0.98163	0.98134	0.98125	0.98122
9000	0.97994	0.98415	0.98462	0.98466	0.98499	0.98477	0.9847	0.98468
9500	0.98264	0.98686	0.98734	0.98738	0.98764	0.98746	0.98741	0.98739
10000	0.98478	0.98901	0.98948	0.98952	0.98973	0.98959	0.98955	0.98953
10500	0.98648	0.99072	0.99119	0.99123	0.9914	0.99129	0.99125	0.99124
11000	0.98784	0.99209	0.99257	0.99261	0.99274	0.99265	0.99262	0.99262
11500	0.98895	0.9932	0.99368	0.99372	0.99383	0.99376	0.99373	0.99373
12000	0.98985	0.99411	0.99458	0.99463	0.99472	0.99466	0.99464	0.99463
12500	0.9906	0.99485	0.99533	0.99537	0.99545	0.9954	0.99538	0.99538
13000	0.99121	0.99547	0.99595	0.99599	0.99605	0.99601	0.996	0.996
13500	0.99172	0.99599	0.99646	0.9965	0.99656	0.99653	0.99651	0.99651
14000	0.99215	0.99642	0.99689	0.99694	0.99698	0.99695	0.99694	0.99694
14500	0.99251	0.99678	0.99726	0.9973	0.99734	0.99732	0.99731	0.99731
15000	0.99282	0.99709	0.99757	0.99761	0.99764	0.99762	0.99762	0.99762
15500	0.99308	0.99735	0.99783	0.99787	0.9979	0.99789	0.99788	0.99788
16000	0.99331	0.99758	0.99806	0.9981	0.99813	0.99811	0.99811	0.99811
16500	0.99351	0.99778	0.99825	0.9983	0.99832	0.99831	0.9983	0.9983
17000	0.99367	0.99795	0.99842	0.99847	0.99849	0.99848	0.99847	0.99847
17500	0.99382	0.99809	0.99857	0.99861	0.99863	0.99862	0.99862	0.99862
18000	0.99395	0.99822	0.9987	0.99874	0.99876	0.99875	0.99875	0.99875
18500	0.99406	0.99834	0.99882	0.99886	0.99887	0.99887	0.99886	0.99886
19000	0.99416	0.99844	0.99892	0.99896	0.99897	0.99897	0.99896	0.99896
19500	0.99425	0.99853	0.999	0.99905	0.99906	0.99905	0.99905	0.99905
20000	0.99433	0.99861	0.99908	0.99913	0.99914	0.99913	0.99913	0.99913

Poisson: Lamda = 5
Pareto : b = 1,000,000 & c = 3

Quadratic Integration

COMPARATIVE APPROXIMATIONS
Excess Ratio

	Continuous Severity Model				Discrete Severity Model			
	100	33 1/3	10	3 1/3	100	33 1/3	10	3 1/3
0	1	1	1	1	1	1	1	1
500	0.48036	0.47207	0.47116	0.47108	0.46927	0.4709	0.47106	0.47107
1000	0.28231	0.271	0.26991	0.26982	0.26924	0.26975	0.26981	0.26981
1500	0.18866	0.17448	0.17321	0.17312	0.17284	0.17308	0.1731	0.17311
2000	0.13857	0.12136	0.11992	0.11981	0.11968	0.11979	0.1198	0.1198
2500	0.10963	0.08931	0.08767	0.08755	0.08747	0.08753	0.08754	0.08754
3000	0.0921	0.06862	0.06677	0.06665	0.06659	0.06663	0.06663	0.06663
3500	0.08123	0.05456	0.05251	0.05237	0.05233	0.05235	0.05236	0.05236
4000	0.0745	0.04461	0.04236	0.04221	0.04218	0.04219	0.0422	0.0422
4500	0.07046	0.03735	0.03489	0.03473	0.0347	0.03471	0.03472	0.03472
5000	0.06825	0.03191	0.02925	0.02907	0.02905	0.02905	0.02905	0.02905

Poisson: Lamda = 0.2
Pareto : b = 1,000,000 & c = 3

Quadratic Integration

COMPARATIVE APPROXIMATIONS
Excess Ratio

	Continuous Severity Model				Discrete Severity Model			
	100	33 1/3	10	3 1/3	100	33 1/3	10	3 1/3
0	1	1	1	1	1	1	1	1
500	0.56809	0.56483	0.56445	0.56442	0.56232	0.56425	0.5644	0.56442
1000	0.35343	0.34843	0.3479	0.34785	0.34686	0.34776	0.34784	0.34784
1500	0.2361	0.22943	0.22874	0.22869	0.22817	0.22864	0.22868	0.22868
2000	0.16775	0.15929	0.15844	0.15837	0.15807	0.15833	0.15835	0.15836
2500	0.12596	0.1156	0.11456	0.11448	0.1143	0.11445	0.11447	0.11447
3000	0.09944	0.08709	0.08587	0.08577	0.08566	0.08575	0.08576	0.08576
3500	0.08214	0.06776	0.06635	0.06623	0.06615	0.06621	0.06622	0.06622
4000	0.07065	0.0542	0.05259	0.05246	0.0524	0.05244	0.05244	0.05244
4500	0.06296	0.04442	0.04261	0.04246	0.04241	0.04244	0.04244	0.04244
5000	0.05782	0.03718	0.03517	0.035	0.03496	0.03498	0.03498	0.03498
5500	0.05447	0.03171	0.0295	0.02932	0.02928	0.0293	0.0293	0.0293
6000	0.05239	0.02751	0.02509	0.0249	0.02486	0.02487	0.02487	0.02487
6500	0.05123	0.02423	0.0216	0.02139	0.02136	0.02137	0.02137	0.02137
7000	0.05076	0.02163	0.0188	0.01858	0.01854	0.01855	0.01855	0.01855
7500	0.05082	0.01956	0.01652	0.01628	0.01624	0.01625	0.01625	0.01625
8000	0.05128	0.01788	0.01464	0.01438	0.01435	0.01435	0.01435	0.01435
8500	0.05205	0.01652	0.01307	0.0128	0.01276	0.01276	0.01276	0.01276
9000	0.05308	0.01541	0.01176	0.01146	0.01142	0.01143	0.01143	0.01143
9500	0.0543	0.01449	0.01064	0.01033	0.01029	0.01029	0.01029	0.01029
10000	0.05569	0.01374	0.00968	0.00935	0.00931	0.00931	0.00931	0.00931

Poisson: Lamda = 1
Pareto : b = 1,000,000 & c = 3

Quadratic Integration

COMPARATIVE APPROXIMATIONS
Excess Ratio

	Continuous Severity Model				Discrete Severity Model			
	100	33 1/3	10	3 1/3	100	33 1/3	10	3 1/3
0	1	1	1	1	1	1	1	1
500	0.80888	0.80901	0.80902	0.80902	0.80845	0.80899	0.80902	0.80902
1000	0.64072	0.64094	0.64097	0.64097	0.64019	0.64092	0.64097	0.64097
1500	0.50075	0.50112	0.50116	0.50116	0.50034	0.50111	0.50116	0.50116
2000	0.38843	0.38906	0.38913	0.38913	0.38839	0.38908	0.38913	0.38913
2500	0.30038	0.3014	0.30151	0.30152	0.3009	0.30148	0.30152	0.30152
3000	0.23231	0.23385	0.23402	0.23403	0.23353	0.234	0.23403	0.23404
3500	0.18008	0.18222	0.18245	0.18248	0.18208	0.18245	0.18248	0.18248
4000	0.14007	0.14288	0.1432	0.14322	0.14292	0.14321	0.14323	0.14323
4500	0.10937	0.11291	0.1133	0.11334	0.11311	0.11333	0.11334	0.11334
5000	0.0857	0.09	0.09048	0.09052	0.09035	0.09051	0.09053	0.09053
5500	0.06731	0.0724	0.07297	0.07302	0.07289	0.07301	0.07302	0.07302
6000	0.05289	0.05879	0.05945	0.0595	0.05941	0.0595	0.05951	0.05951
6500	0.04147	0.04818	0.04893	0.04899	0.04892	0.049	0.049	0.049
7000	0.0323	0.03984	0.04068	0.04075	0.0407	0.04076	0.04076	0.04076
7500	0.02485	0.03322	0.03416	0.03424	0.0342	0.03424	0.03425	0.03425
8000	0.01872	0.02792	0.02895	0.02904	0.02902	0.02905	0.02905	0.02905
8500	0.01359	0.02363	0.02476	0.02486	0.02484	0.02487	0.02487	0.02487
9000	0.00924	0.02013	0.02135	0.02146	0.02145	0.02147	0.02147	0.02147
9500	0.00551	0.01724	0.01856	0.01867	0.01867	0.01868	0.01869	0.01869
10000	0.00226	0.01484	0.01625	0.01637	0.01637	0.01638	0.01638	0.01639
10500	¯0.00061	0.01282	0.01432	0.01445	0.01446	0.01447	0.01447	0.01447
11000	¯0.00318	0.0111	0.0127	0.01284	0.01285	0.01286	0.01286	0.01286
11500	¯0.0055	0.00964	0.01133	0.01148	0.01149	0.0115	0.0115	0.0115
12000	¯0.00761	0.00837	0.01016	0.01031	0.01033	0.01033	0.01033	0.01033
12500	¯0.00957	0.00727	0.00915	0.00932	0.00933	0.00934	0.00934	0.00934
13000	¯0.01138	0.0063	0.00828	0.00846	0.00847	0.00848	0.00848	0.00848
13500	¯0.01309	0.00545	0.00752	0.00771	0.00773	0.00773	0.00773	0.00773
14000	¯0.0147	0.00469	0.00686	0.00705	0.00707	0.00707	0.00707	0.00707
14500	¯0.01624	0.00401	0.00628	0.00648	0.0065	0.0065	0.0065	0.0065
15000	¯0.0177	0.0034	0.00576	0.00597	0.00599	0.00599	0.00599	0.00599
15500	¯0.01911	0.00285	0.0053	0.00552	0.00554	0.00554	0.00554	0.00554
16000	¯0.02047	0.00234	0.00489	0.00511	0.00514	0.00514	0.00514	0.00514
16500	¯0.02179	0.00188	0.00452	0.00475	0.00478	0.00478	0.00478	0.00478
17000	¯0.02307	0.00145	0.00419	0.00443	0.00446	0.00446	0.00446	0.00446
17500	¯0.02432	0.00105	0.00389	0.00414	0.00417	0.00417	0.00417	0.00417
18000	¯0.02554	0.00069	0.00362	0.00388	0.00391	0.00391	0.00391	0.00391
18500	¯0.02674	0.00034	0.00337	0.00364	0.00367	0.00367	0.00367	0.00367
19000	¯0.02792	0.00002	0.00314	0.00342	0.00345	0.00345	0.00345	0.00345
19500	¯0.02908	¯0.00028	0.00294	0.00322	0.00325	0.00325	0.00325	0.00325
20000	¯0.03022	¯0.00057	0.00274	0.00304	0.00307	0.00307	0.00307	0.00307

Poisson: Lamda = 5
Pareto : b = 1,000,000 & c = 3

Quadratic Integration

CALCULATING IBNR BASED ON CASE RESERVES

Rick Atkinson

CALCULATING IBNR BASED ON CASE RESERVES

by

Rick Atkinson

I. The Problem

Faced with the task of producing an estimate of incurred by not reported (IBNR) loss reserves, as of a particular evaluation date, given only

1. Case reserves as of the evaluation date;
2. Industrywide reported and paid loss development factors (LDFs) to ultimate; and
3. Sufficient evidence to believe that the industrywide LDFs are applicable

how should one proceed?

II. General Approach

Noting that

IBNR = ultimate loss - paid loss - case reserves

and that case reserves are known, an estimate of IBNR can be made if a reasonable estimate of ultimate loss and paid loss is available.

III. Estimating Ultimate Loss

An estimate of ultimate loss can be made using the known case reserves and the applicable industrywide LDFs. Noting that

case reserves = reported loss - paid loss

we have

$$\frac{\text{case reserves}}{\text{ultimate loss}} = \frac{\text{reported loss}}{\text{ultimate loss}} - \frac{\text{paid loss}}{\text{ultimate loss}}$$

$$= \frac{1}{\text{reported-to-ult LDF}} - \frac{1}{\text{paid-to-ult LDF}}$$

which implies

$$\text{ultimate loss} = \frac{\text{case reserves}}{\dfrac{1}{\text{reported-to-ult LDF}} - \dfrac{1}{\text{paid-to-ult LDF}}}$$

IV. Estimating Paid Loss

An estimate of paid loss in now readily obtainable.

$$\textbf{paid loss} = \frac{\textbf{ultimate loss}}{\textbf{paid-to-ult LDF}}$$

Finally, IBNR can be estimated using the formula

IBNR = ultimate loss - paid loss - case reserves

V. Conclusion

Exhibit 1 displays sample calculations of IBNR using this methodology.

In addition to being used to produce an estimate of IBNR, this method may also be used as a reasonableness check of case reserves or IBNR estimates developed using different methods.

Sample Calculation of IBNR Based on Case Reserves

Exhibit 1

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Age of Accident Year	Reported to Ultimate LDF	Paid to Ultimate LDF	Reported Completion Ratio	Paid Completion Ratio	Case Reserves	Case Reserves as a Ratio of Ultimate Loss	Estimate of Ultimate Loss	Estimate of Paid Loss	Estimate of IBNR
12	1.700	3.900	0.588	0.256	100,000	0.332	301,364	77,273	124,091
24	1.350	1.950	0.741	0.513	85,000	0.228	372,938	191,250	96,688
36	1.250	1.650	0.800	0.606	60,000	0.194	309,375	187,500	61,875
48	1.200	1.500	0.833	0.667	45,000	0.167	270,000	180,000	45,000

(2) Based on industry data.
(3) Based on industry data.
(4) 1 / (2)
(5) 1 / (3)
(6) Available from company.
(7) (4) - (5)
(8) (6) / (7)
(9) (8) / (3)
(10) (8) - (9) - (6)

Sample Calculation of IBNR Based on Case Reserves

(1) Age of Accident Year	(2) Reported to Ultimate LDF	(3) Paid to Ultimate LDF	(4) Reported Completion Ratio	(5) Paid Completion Ratio	(6) Case Reserves	(7) Case Reserves as a Ratio of Ultimate Loss	(8) Estimate of Ultimate Loss	(9) Estimate of Paid Loss	(10) Estimate of IBNR
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THE ACTUARIAL PARADIGM: A NONTECHNICAL EXPOSITION

Lee Smith

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THE ACTUARIAL PARADIGM: A NONTECHNICAL EXPOSITION

OVERVIEW

In his insightful and ground-breaking essay entitled The Structure of Scientific Revolutions, Thomas S. Kuhn discussed the notion of scientific advancement. In particular, he introduces the idea of a paradigm, a body of knowledge and beliefs which provides direction for research. The development and acceptance of a paradigm is a key factor in transforming a group of practitioners into a profession.

The beginning of a scientific revolution is tne discovery of an anomaly relative to expectations which the existing paradigm creates. A "crisis" occurs when standard analysis is not producing desired results. A search for an alternative paradigm commences, and its acceptance completes the revolution.

Developments in the liability insurance area in recent years have caused casualty actuaries to reevaluate the tools they use in solving problems in their domain. Simultaneously, the range of issues determined to be within the purview of the actuarial profession has expanded rapidly. As a result, the generally agreed upon body of knowledge of casualty actuaries have been undergoing something of a revolution.

The intent of this paper is to survey the elements which are impacting on the issues casualty actuaries face today and to discuss the implications of that on our ability to develop and maintain a paradigm which can respond to the challenges. Such a paradigm must help define the profession as well as provide a basis for problem solving. To the extent that the scope of actuarial activity is such that some aspects of that activity are not amenable to the notion of a paradigm, those aspects will be considered as separate from the primary activity of professional actuaries.

As actuaries approach the 100th anniversary of their profession in North America, they will pause to reflect on the status of their profession. The challenges faced today provide perspective on the larger issues to be faced in coming years. Socioeconomic changes of the type that have caused the liability insurance area to be revolutionized will continue to provide actuaries with opportunities to creatively resolve some of society's most perplexing problems.

ELEMENTS OF THE CURRENT ACTUARIAL PARADIGM

To evaluate the current core of actuarial science, it may be helpful to develop a notion of what an actuary is. Conventional wisdom uses phrases like "insurance mathematician," and "ratemaking and reserving specialist." A more objective way of defining an actuary's role might be to consider the body of knowledge encompassed by the professional examinations which an actuary must take. The following brief listing of key topics covered in the 1989 Syllabus of Examinations of the Casualty Actuarial Society provides a valuable framework:

- Calculus and Linear Algebra
- Probability and Statistics
- Operations Research
- Numerical Methods
- Interest and Life Contingencies
- Credibility Theory and Loss Distributions
- Economics
- Theory of Risk and Insurance
- Ratemaking
- Reserves

- Accounting
- Law
- Financial Operations
- Forecasting

A review of some of the topics covered under these broad headings provides additional perspective:

- Simulation
- Bayesian Credibility
- Loss Distributions
- Risk Management
- Exposure Bases
- Statistical Plans
- Litigation
- Risk Classification
- Increased Limits Pricing
- Experience Rating

From these elements an actuarial paradigm could arise.

Plan of Paper

This paper is structured so as to introduce some key actuarial concepts, survey elements from related subjects which are considered a part of actuarial science, and to discuss a possible paradigmatic structure for actuarial science. Related subjects discussed here include law, mathematics, probability, statistics, economics and accounting.

The first area of knowledge reviewed is that of tort law. In that review strands from other areas of law are reviewed to the extent they have impact on actuarial science. Whether law is a component of an actuarial paradigm is doubtful, but the law greatly impacts the values and phenomena being analyzed by actuaries.

The next area considered is that of accounting. Like law, accounting probably contributes little to the actuarial core of knowledge, but yet it must be understood for actuarial science to realize its full potential. Ultimately the results of actuarial calculations are filtered through accounting rules and conventions.

Economics is the third area surveyed. While, like law and accounting, economics is somewhat peripheral to actuarial science, it is proposed herein that economics does contribute directly to the actuarial paradigm. This is best seen in the context of pricing which requires economic elements, but is also apparent in terms of generally overlapping methodologies.

Physics is given attention because it provides the best example of the application of mathematical principles to the analysis of complex phenomena. Like actuaries, physicists are attempting to uncover patterns and rules which will help them better explain and predict real world phenomena. The models physicists have developed and the mathematics they have employed will provide direction to actuaries in search of their core models.

Philosophy is briefly covered to provide a sense of general importance to the work of the actuary. As a professional in search of procedures by which to analyze and quantify socioeconomic phenomena, actuaries are involved in activity with epistemological, ethical, and even metaphysical implications. While those may not be apparent in the day to day routine of actuarial work, an awareness of their existence provides useful perspective for the actuary in his/her long-term planning.

Mathematics is the subject which is identified as most likely to provide the key to development of a paradigm to cover key aspects of actuarial science. A brief survey of some of the broad areas of mathematics reveals the richness of that subject and the power it can bring to actuarial sciences. The subject matter of probability and statistics is covered as a separate area.

Utility theory and interest theory are briefly reviewed to provide perspective on how these specialized topics impact on actuarial science. Selected general actuarial topics are then discussed to illustrate some of the specific areas of current interest to casualty actuaries. Appendices are included to provide additional perspective on the elements to consider in developing an actuarial paradigm.

Risk Theory

Risk theory is the study of the random variation of insurance claims. The probability distribution of the amount of claims expected to arise from an insurance portfolio over a period of time is to be estimated. This distribution may be derived from distributions of claim counts and average claim sizes expected to arise from portfolio components in the time frame.

Estimation of losses arising from insurance portfolios is a key aspect of actuarial work. Risk theory provides a theoretical basis for loss estimation. As such, it must be considered to be a key aspect of the actuarial paradigm.

Complicating Elements

Analysis of losses arising from liability insurance coverages involves the consideration of a variety of factors. As opposed to the individual risk-theoretic approach which disaggregates the portfolio and the collective approach which assumes losses follow a particular distribution, the loss analysis performed by most actuaries in the course of their standard activity is based on aggregated data from an unknown mathematical structure. Historical losses for a group of occurrences or policies are evaluated and relationships derived from that data are used to estimate losses for the category under review.

For a typical group of policies covering liability insurance exposure, losses can be influenced by such things as policyholder characteristics, jurisdictional attributes, contract language, social and economic conditions, weather, and chance. Both the occurrence of a claim and its amount will be subject to wide variability depending upon the mix of factors involved.

The actuarial paradigm of loss estimation, then, will be grounded in the theory of mathematics and statistics. Certain aspects of the problems facing actuaries, on the other hand, will involve the evaluation of a number of factors for which the theoretical core of actuarial science has no component. It is in balancing this need for a theoretical core with the reality of the scope of the actuary's work that the challenge of providing a framework will arise.

Other Issues

Actuaries respond to a number of issues requiring tools not directly related to those used in the estimation of losses from a portfolio of insurance policies. Included among these issues are expense levels, profit margins, interest income, exposure bases, rate level indices, and credibility. In addition, many of the factors impacting on losses are such that incorporation into a standardized loss estimation model is prohibitively complex.

The implication is that the body of knowledge deemed to encompass the subject of actuarial science will consist of a number of paradigms as well as some elements that fall outside the paradigm structure. Any discussion of the paradigmatic structure of actuarial science will need to define the scope of issues and elements covered by the paradigms.

LAW

One of the factors which influences the aspects of liability insurance within the actuary's domain is the legal environment. The general area of law which has had the greatest impact in the liability insurance area is tort law. The evolution and interpretation of cases and statutes related to torts has been a significant factor in the insurance crises which have occurred in recent years.

Tort law, which provides the underlying structure from which liability losses emanate, is concerned with allocating losses which result from human activity. Unlike contract theory, which attempts to assess damages based on a very narrow range of criteria, tort theory attempts to assess the consequences of the tort actions. A balance is sought between the utility of a certain kind of conduct and the harm which may arise from that conduct.

Originally, liability under tort was grounded in actual intent or in a cause and effect relationship between the conduct and the damage. Today, the concept of social fault has arisen whereby consequences alone may determine liability. Tort liability generally falls into one of three classes--based on intent; based on negligence; or strict.

Negligence is roughly described as conduct which falls below the standard established by law. The standard of care requirement is that which a prudent person would follow under the circumstances. A professional is expected to have the same skill and learning as a typical member of his/her profession and to apply that skill and learning with the same care as is generally exercised.

In recent years a number of creative bases for allocating damages has arisen. This creativity creates problems for actuaries attempting to quantify potential losses. In a 1980 case, for example, in which the plaintiff could not identify the manufacturer of a drug which had injured his mother, the concept of "market share liability" arose whereby several companies could be held responsible.

In a 1975 case, a landlord was held liable for a criminal act by an unrelated party in his building. A 1969 case removed immunity from tort liability which a charitable institution had enjoyed, and a 1973 case removed the defense of governmental immunity from a local Board of Education.

In an article in the December 12, 1988 issue of the Wall Street Journal a brand new legal damage theory was described. Named "hedonistic damages," this approach advocates urges the award of large sums to the estates of victims to provide for the intrinsic value of life.

A range of the implied value of human life is calculated from statistics on amounts people will pay to reduce the risk of death. The range is from $66,000 which is the value of prompt coronary care to $12 million which is the value of safer airline travel.

Product liability has become a significant component of tort law. The notion of recovering from manufacturers for negligence was established in a 1916 case involving Buick Motor Co. A 1960 case involving Chrysler established that even a signed disclaimer by the plaintiff did not eliminate the liability. A 1979 Act of Congress established a wide range of remedies for consumers of products with implied warranties. All of this activity impacts on the problem solving activity of actuaries.

In addition to tort law, there are several other areas of law that impact on the work actuaries perform. Property law is concerned with the notion of ownership. Property law attempts to establish principles by which to evaluate situations involving property rights. Many colorful and historically revealing notions underlie the theoretical structure of property law.

A system of estates exists in property law which distinguishes types of ownership. The estate types include fee simple, fee tail, life and leasehold. A fee simple is the most commonly known type of ownership representing an estate that has no end point and has no limitations on inheritability. Defeasible fees are estates that may end if certain conditions arise.

A future interest is an interest in property which can inure in the future. The Statute of Uses created a type of future interest in a transferee called an executory interest. These interests may spring out of the transferor or shift to a transferee. The Rule against Perpetuities arose to curb indestructible executory interests.

Estates may be held concurrently. Commonly known forms of concurrent ownership include Tenancy in Common, joint tenancy and tenancy by the entirety. Co-tenants have various rights and obligations relative to the co-tenancy.

In addition to providing perspective to the actuaries with regard to principles underlying insurance contract language, the study of property law provides insights into the structure of society. It also provides a fascinating analytical framework in the analysis of estate interests which has many features of a deterministic mathematical model. Along with contract law, property law provides the systematic underpinnings of much of the social structure which exists in modern democracies.

Contract law may be looked at as an abstraction. All particularities of person and subject matter are removed from pure contract theory. It is what is left after accounting for all the specialized areas of law.

For a contract to exist there must be mutual assent and consideration. Mutual assent often manifests itself in terms of offer and acceptance. Contract law attempts to uphold the reasonable expectations of parties relying on promises.

Determining whether a contract arises involves evaluating the consideration involved and deciding whether there was a meeting of minds. The two essential features of consideration are the item bargained for and its legal value. The requirements of a valid offer are intent, definitiveness and communication. Modes of acceptance are different for unilateral and bilateral contracts.

Once it is determined that a contact has been formed, there are various remedies if a party breaches. Among these are damages, specific performance, recission and restitution. Damages may be compensatory, punitive, or nominal.

Damages in contract are to be distinguished from those in tort. The contract breaker is not held liable for all consequences of his/her actions. As articulated in the 1854 Hadley v. Baxendale case, damages for breach of contract are "such as may fairly and reasonably be considered either arising naturally...or such as may reasonably be supposed to have been in the contemplation of both parties..." Doctrines such as promissory estoppel are causing changes in this scheme, however, and contract damages may ultimately be subsumed into the mainstream of tort law.

Constitutional law is another area of law which provides actuaries with perspective on the legal system which impacts on the phenomena they are modeling. Equal protection is a limitation on government regulation to guard against arbitrary discrimination. Discrimination by race and sex has been greatly scrutinized by recent court decisions. The notion of establishing reasonable classifications to be permitted has analogies in actuarial science.

Another area of constitutional law of interest to actuaries is that of state regulation. Under the tenth amendment the states have all power not specifically given to the Federal Government. Where there is a conflict, the Federal Government controls. Federal power with respect to the regulation of commerce is found in the Commerce Clause.

Actuarial science is faced with the problem of estimating losses which will arise from evolving legal theories. To the extent these legal theories are dynamic in the sense that future damages will not be similar to past damages, predictability is affected. The actuarial paradigm will be effective only to the extent it can quantify the effect of the dynamism.

In a 1972 Harvard Law Review article, George Fletcher set forth two paradigms for tort liability which would replace the fault and nonfault categories. The first paradigm, reciprocity, provides all individuals the same degree of security from risk and grants compensation whenever a disproportionate distribution of risk injures someone more than his/her fair share. The paradigm of reasonableness established the notion of a "reasonable man" as the basis of determining what ought to be done in a given set of circumstances.

Lawyers, like mathematicians, deduce conclusions from axioms about undefined terms. Chains of deductive legal reasoning give meaning to these terms only at the point where conclusions are drawn. These meanings are not necessarily consistent from one application to the next.

Should tort and other law become paradigmatic and should the paradigms lend themselves to a degree of predictability, the contingencies to be estimated by actuaries in the liability insurance area may become more quantifiable. Until then, the actuary must continue to use models which are constructed to estimate losses arising from a very subjective legal system. Appendix A provides a glossary of legal terms which are of relevance to the subject of actuarial science.

ACCOUNTING

Actuaries produce results which become elements of income statements and balance sheets of entities with insurance types of exposure. Understanding the basic accounting principles which affect the way actuarial results are interpreted is an essential aspect of the actuary's work. To provide perspective on the accounting implications of the actuarial paradigm, a review of key accounting principles will be useful.

Presenting a fair picture of the financial status of a business organization is the goal to which the various accounting rules relate. Some accounting principles which relate to actuarial calculations are:

- Recording the value of assets at historical cost.
- Recognizing revenue at the completion of a transaction.
- Matching costs to related revenues by period.
- Basing financial statement elements on objective evidence.
- Being conservative where evidence is unclear or conflicting.
- Determining materiality subjectively.
- Disclosing anything which could affect the financial position of the entity.
- Documenting changes in procedures.

The basic accounting equation which provides a framework for all financial accounting is Assets = Liabilities + Equity. The balance sheet shows the components of the equation at a point in time. Another key statement is the income statement which displays revenues and expenses. A statement which has become more prominent in recent years is the Statement of Changes in Financial Position which provides useful information on liquidity changes.

Valuation of assets is a topic of increasing interest in recent years. Traditionally assets have been valued based on historical or acquisition costs. Inflation levels of recent years, however, have caused concern regarding the relevance of such an approach. Most alternatives, however, require that subjective judgments be made. Different methods may provide a better picture in different situations.

Valuation of liabilities for many entities is not a significant problem. The notion is that a liability is a known amount expected to be paid in the future. Liabilities of financial institutions like insurance companies, however, are not so easy to estimate.

A firm's leverage position measures the degree to which it is obligated for fixed costs. As regards financial leverage, the interest rate on borrowed money should be less than the expected rate of return on the venture supported by the borrowing. Break-even analysis involves dividing fixed costs by the margin per unit to determine the volume of sales needed to cover fixed costs. Decisions as to desired degree of leverage must consider stability of earnings (risk).

Amortization involves spreading costs over time. Depreciation is a form of amortization which involves changing the value of an asset over time as its value changes.

Various methods have arisen to provide a basis for depreciation. The Tax Acts of 1981 and 1982 established the idea of accelerated cost recovery which allowed depreciation costs to be accelerated.

An effect of using accelerated depreciation for tax purposes relative to financial reporting services is to produce deferred taxes. The difference between tax expense reported and tax expense paid is shown as a liability. This deferred tax amount may never be paid unless the firm declines in size and is profitable as it declines.

The basic inventory equation is: Ending Inventory = Beginning Inventory plus Purchases less Sales. A question arises as to how to value cost of goods sold. Two well known methods are first-in, first-out (FIFO) and last-in, first-out (LIFO).

The analysis of long-term projects involves capital budgeting. Approaches to capital budgeting include:

1. Payback Method

2. Net Present Value Method

3. Internal Rate of Return Method

The idea is to evaluate alternative investment opportunities so as to choose the mix that will provide the best results.

The words "cost" and "expense" have specific meaning in accounting usage. The cost refers to the amount paid for an item at acquisition. Expenses are costs which are matched to revenues which have resulted from the expenditures. Before a cost becomes an expense it is an asset.

Financial analysis is often enhanced by the calculation of certain analytical measures. Certain ratios have provided helpful insights into various financial aspects of a company. Among these are:

1. Liquidity ratios: Ratios of current assets to liabilities.

2. Competition ratios: Relationships of key values for the entity with the same values for similar entities.

3. Efficiency ratios: Measures which show speed of turnover of key items.

4. Solvency ratios: Measures which show how operations and financial status support ongoing operations.

5. Profitability ratios: Measures of profit relative to various benchmarks.

Since actuaries are involved in the calculation of a number of balance sheet and income statement components, they must be aware of the accounting principles which must be met in recording actuarial items. Improper interpretation of the end product of an actuarial calculation will offset the value created by the mathematical ingenuity.

ECONOMICS

As was the case with the legal environment, the economic environment which impacts on the values the actuary is attempting to estimate is very dynamic. The impact of national economic policies and microeconomic forces on the liability losses the actuary is attempting to estimate can be dramatic. The actuary must understand these economic forces to be in a position to estimate the losses which will be dependent on those forces.

Because the economic paradigm which would help evaluate economic variables is in a state of revolution, the actuary gets little guidance from economic science. Macroeconomic variables like employment, inflation, and interest rates which can greatly impact insurance costs are becoming less and less predictable. As a result, loss estimation models must provide for the assimilation of a wide variety of economic scenarios.

The microeconomics of the insurance industry also adds to the complexity of actuarial modeling. Evolving marketing and underwriting strategies including changing of policy coverages, revised reinsurance programs, and changing target markets make historical data inadequate to the task of evaluating future loss patterns. Consumer demand and production theories provide an analytical framework for the evaluation of these topics.

The work of actuaries is inextricably bound up with that of economists. The development of rate levels and allocation of overall rate level to groupings of customers is an economic function. Elements of economic theory influence actuarial models. A full discussion of the underpinnings of actuarial science must account for the various economic elements.

Economics is concerned with the allocation of scarce resources. It evaluates the problems of what to produce, how to produce it, and how to distribute the results. The starting point in a market economy is generally an analysis of supply of and demand for products.

Microeconomics

A simple demand schedule relates units of a good which will be purchased with prices at which the good would be offered. For a normal good, the higher the price the lower the quantity demanded. Among the reasons for this pattern is the fact that consumers will shift spending to the point at which the total amount available to spend is allocated so as to produce the most utility. An increase in the price of a particular good makes other goods relatively more attractive.

A simple supply schedule relates units of a good which will be offered for sale with possible price levels. Because the presumption is that profit is a primary reason for production, and that costs of production are given, higher prices lead to greater profits and more output. Assumptions must be made as to the time frames in which output adjustments can be made.

The equilibrium price and quantity for a good occurs at the point where the supply and demand curves intersect. An equilibrium position is considered stable if a deviation from equilibrium brings into operation market forces which move the curves back to the same equilibrium position. Ceteris paribus, a shift in either the supply or demand curve, will cause the equilibrium point to move.

A concept of importance to actuaries in their pricing work is that of elasticity. Elasticity is a measure of the change in demand which results from a change in price level. When an actuary is attempting to estimate the policyholder mix to be written under a set of proposed rates, he/she must estimate both the volume and mix changes which will occur. A form of elasticity measure is needed.

Utility theory comes in many forms, but in basic demand theory it relates to the degree of satisfaction the consumer receives from consuming a good. To maximize the total satisfaction which can be derived from expenditure of a given amount of income, the utility of the last dollar spent on the commodities purchased must be the same. In other words, the quotient of the marginal utility and price of every good would be equal.

Graphically, indifference curves can be used to illustrate the notion of maximizing utility when two goods are involved. Indifference curves show combinations of two goods which yield equal utility. The marginal rate of substitution between the goods is the slope of the curve and represents the amount of one good which will be given up to gain a unit of the other.

The budget constraint line shows the various combinations of the goods which can be purchased given available expenditure by the consumer and the prices of the good. The point at which the budget line is tangent to the highest indifference curve is the point of maximum utility or equilibrium. To increase utility, the consumer must enter into exchange with another who has a different marginal rate of substitution.

These notions are important to the actuary in pricing work because they provide guidance as to reasons for market reactions to alternative price levels. Also important is the supply side of the market. As demand theory is the primary economic paradigm for demand, production theory provides the supply paradigm.

A production function shows levels of production that are possible in a given time frame given the available input factors. The law of diminishing returns is reflected in a marginal product curve which eventually declines. Each input factor provides for various production stages which reflect the relative average product level.

A graphical device in production theory somewhat analogous to the indifference curve in demand theory is the isoquant. An isoquant shows combinations of two input factors which produce a particular output level. The marginal rate of technical substitution of one factor for another is the amount of one a firm can release by increasing the other by a unit without changing the level of production. It is equal to the ratio of the marginal products of the factors.

An isocost shows combinations of input factors which a firm can purchase given its budget for such factors and factor prices. The equilibrium for a producer occurs where the ratio of the marginal products of the factors equals the ratio of their prices. At that point, the marginal product resulting from the last dollar spent on each input is the same.

Factor inputs in the insurance industry do not follow the patterns established in basic economic textbooks. Similarly, costs of production do not arise as a conventional textbook illustrates. There are enough analogies, however, that economic theory provides a sound context for a discussion of insurance pricing.

Cost curves show the minimum cost associated with production of various output levels. Total costs are composed of fixed and variable elements. In a given time frame, the shapes of the various cost curves follow patterns as follows relative to quantity produced:

1. Average fixed costs decline as quantity rises.

2. Average variable costs decline at first and then rise.

3. Marginal costs decline at first and then rise..

4. Total average costs decline at first and then rise.

Pricing an insurance product involves determining levels of fixed and variable costs in a given time frame. The primary cost associated with a casualty insurance product is loss potential. Loss potential is variable in the sense that increasing quantities sold at some point are correlated with relaxed underwriting standards.

Many expense elements of the cost associated with insurance are fixed in the time frame involved. As a result, insurance cost curves for a given product could follow patterns not unlike those in economic theory, ceteris paribus.

An issue of much interest in the insurance industry is that of the level of competition. Many aspects of the market model which theoretically results in optimal societal welfare depend on the assumption that pure competition prevails in the market of interest. Price and output in noncompetitive markets are not considered optimal.

Other market types often considered are monopoly, monopolistic competition, and oligopoly. Within each of these are significant variations. The germane question is not whether markets are perfectly competitive, but whether they are workably competitive.

For the market model to be fully operative, perfect competition must exist in the factor as well as the product markets. To maximize profit, a company will produce the optimal output level given the optimal combination of factor inputs. This level occurs where the ratio of marginal product to the price of each input equals the reciprocal of the marginal cost of a product and its price. Where perfect competition does not exist, marginal revenue differs from price, and equilibrium occurs at a nonoptimal point.

This fairly abstract theory of market equilibrium actually leads to a notion of general equilibrium for a society and a measure of welfare therein. A condition known as Pareto optimality can arise at which the marginal rates of transformation and the marginal rates of substitution associated with the goods in a society are equal. The fact that this can occur only where perfect competition exists is the basic argument in favor of pure competition. A complication not handled in this notion is that of externality which belies the optimality of results from a market economy.

Macroeconomics

To this point the focus of the discussion of economic theory has been primarily in the area generally thought of as microeconomics. The subject area of macroeconomics also has much to offer the actuary in terms of perspective on underpinnings of actuarial science. In addition to macroeconomic variables such as unemployment and interest rates which may enter directly into actuarial algorithms, basic macroeconomic theory has elements of importance to actuarial science.

It is generally becoming recognized that for macroeconomics to have a needed degree of scientific rigor it must develop linkages to microeconomic foundations. While this will severely strain existing macroeconomic modeling patterns, it will lead to models which are more demonstrably related to real world phenomena. The elements of microeconomic theory lend themselves to modeling techniques on aggregates which can have similar degrees of success.

The problem of general equilibrium in a market economy was tackled by Wald in 1936. An exchange economy consisting of individuals, commodities, and stocks of goods could be "solved" given a set of exchange equations. The question arises, however, as to whether such an abstract system has any real economic content.

Keynes theory of macroeconomics was not reconcilable with general equilibrium theory. Samuelson introduced a dynamic adjustment mechanism. Von Neumann introduced mathematical techniques including mini-max and fixed point notions which advanced the mathematical aspects of the analysis.

A model associated with Arrow, Debreu, and McKenzie has provided a new foundation for work on the microeconomic foundations of macroeconomic theory. They introduced production sets, preference structures, and a more vigorous treatment of the elements of a market economy.

Hicks introduced the notion of temporary equilibrium. Lange introduced some of the uncertainty, probability, and risk concerns which arose in the Keynesian system. Patinkin dealt with the idea that various financial assets have the function of providing a store of purchasing power.

The transition from the micro to the macro level involves determination of aggregation rules which allow choice-theoretic household demands to aggregate to a consumption function. In Microfoundations, Weintraub illustrates that general equilibrium theory is merely an example of general systems theory the latter of which applies to systems in general.

General systems theory (GST) finds structural similarities among fields of science. If economics is a science, GST is applicable. Making micro and macroeconomics compatible means defining them so a systematic structure can emerge. Rather than a single model providing for the economic paradigm, economics has become a body of developing knowledge responding to particular problems.

In this context, the rather abstract ideas of macroeconomics take on more meaning. Macroeconomics is generally thought of as dealing with behavior of aggregated variables like income, price, and growth levels. The simple two-sector model considers relationships among income, savings, investment, and consumption. Government and international sectors may be added.

The simple model provides a basis for determining the effects on national income of changes in levels of basic variables. This basis is the multiplier effect, the idea being that a change in the level of a variable, say investment, has a multiplicative effect on national income because of dependence effects.

Equilibrium in commodity markets is often analyzed using a graphical device known as our IS curve. In a market, equilibrium occurs where the volume of output equals planned spending. The IS schedule shows equilibrium income levels at various interest rates.

Equilibrium in money markets is often analyzed in terms of an LM curve. Money demand is a function of general types of demand. Equilibrium occurs where supply and demand of money are equal, and the LM curve gives combinations of interest and income consistent with equilibrium in the money markets.

When equilibrium occurs in the money and product markets at the same income and interest rate level, a form of general equilibrium arises. Monetary and fiscal policy can be judged according to their effect on this equilibrium point. The effectiveness of the various policies in this model are a function of the relative slopes of the curves.

Economic theory is obviously rich with tools which can be of value to the actuary. Like the other underpinnings of actuarial science, however, economics makes no claims of uncovering truth for the objects of its study. In an essay entitled <u>An Essay on the Nature and Significance of Economic Science</u>, Lord Robbins discusses the elements of economic reasoning.

The first step involves defining economics. "Economics is the science which studies human behavior as a relationship between ends and scarce means which have alternative uses." The nature of economic generalizations is discussed so as to identify the postulates underlying economic theory and challenge their realism.

Generalizations of economics are not only based upon assumptions regarding relative valuations but also on assumptions regarding human psychology. Unlike the natural sciences, transition from qualitative to quantitative aspects of social sciences is tenuous. Ultimately, economics deals with ethical and moral questions regarding optimal forms of society.

Actuarial Science and Economics

To the extent the actuarial paradigm does not reflect the dynamic economic environment which is impacting the problem which the paradigm is attempting to resolve, it may eventually face a crisis of the type discussed in Kuhn's book. Our concepts, models, and shared beliefs will ultimately be evaluated from the perspective of their success in solving problems which will have economic roots. Our ability to integrate the useful parts of economic theory into our body of knowledge will impact on our success in providing useful results.

A specific issue facing actuaries in the area of ratemaking which has a strong economic element is that of appropriate profit load. While economists have not resolved the issue of equilibrium or optimal profit levels in a dynamic industry in a market economy, there are some generally agreed upon notions. The appropriate profit level is that which attracts the amount of capital needed to produce the optimal level of output.

Profitability in the insurance industry has been a topic of significant interest for some time. The National Association of Insurance Commissioners (NAIC) conducted studies in the late 1960s, the results of which were published in their 1970 Proceedings. Six alternative measures of return were suggested for monitoring, but no concensus was reached on a particular approach.

In utility regulation where much work has been done in determining an appropriate profit standard, cost of capital has evolved as a key measure. Capital is separated into long-term debt, preferred stock and common stock, and the appropriate return on each component is determined. The degree of risk is a factor in making these determinations.

In the area of insurance, risk loads are becoming more of interest. Given the large uncertainties in the reserve setting process, for example, the issue of providing a margin beyond expected value to cover potential adverse developments is arising. The sense that standard reserve setting practices may have a downward bias in the type of dynamic environment insurers have recently faced gives additional impetus to this idea. Appendix B provides a glossary of economic terms which are of importance to actuarial science.

PHYSICS

Physics, like economics, provides examples of mathematical modeling which are of interest to the actuarial profession. Two key developments which have reshaped our world view in the twentieth century were Planck's Quantum and Einstein's Relativity theories. Quantum mechanics deals with probabilities for phenomena for which no conceptual framework can be designed. By

Heisenberg's uncertainty principle, we cannot know both the position and momentum of a particle. We cannot observe reality without changing it. Mathematics applied to real world phenomena has produced some startling results.

Subatomic particles can only be dealt with statistically. Activity of individual units cannot be predicted, but broad patterns can be. Just as Einstein uncovered a contant quantity at the macrolevel, Planck discovered a constant at the microlevel. Planck's constant relates the frequency of a light wave with its energy. Mathematical calculations have produced results which human reason could not have imagined.

Light was shown to be both wave-like and particle-like. This duality dealt a fatal blow to standard ways of considering causalty. Experiments showed that the paths of individual photons could not be determined. The notion of a probability wave arose which provided a bridge between the ancient metaphysical duality notions of the ideal and the real. Mathematics has provided the tools by which physicists have been able to unlock seemingly unknowable mysteries.

The physical world by this view is an interconnecting group of relationships rather than a structure made up of components. The notion of a wave function as developed by Schrodinger allows for the development of a probability function which allows for application of deterministic rules. Quantum theory as thus evolved provides a bridge to the real world of three spatial dimensions and time from a potential world of infinite dimension.

Concrete reality is sacrificed by this approach. Metaphysical issues related to uncovering ultimate reality remain. The basic duality of mind and matter becomes as much an epistemological as a metaphysical issue. Proof in the mathematical sense that a particular world view holds up becomes impossible. Mathematics has paradoxically led to a finding that mathematics is limited in power.

Heisenberg determined that since activity in the subatomic realm can never be known, observation should take precedence over mathematical modeling. He devised a means of organizing data which provided an alternative to wave equations in calculating transition probabilities for the effect of an experiment. Heisenberg's uncertainty principle arose from his discovery that nature presents itself in such a way that the accuracy of developments will always be limited. "What we observe is not nature itself, but nature exposed to our method of questioning."

Einstein's special theory of relativity merges space with time and energy with mass. Lorentz transformations were a key mathematical tool by allowing for relating observations from different frames of reference. Combined with the discovery of the constancy of the speed of light, this tool led Einstein to the development of the notion of a space-time continuum. Abstract mathematics led to a new world view.

Einstein's general theory of relativity attempted to generalize his prior results to apply to all frames of reference. The mathematical equations of the general theory describe the changing structure of gravitational fields. All reality could be described by space-time and motion.

Quantum Field Theory is an ad hoc theory which merges aspects of quantum mechanics and relativity. The complementarity of key aspects of the physical universe is reflected in quantum field theory which lacks mathematical and conceptual consistency. With fields hypothesized as providing the substance of the universe, matter becomes a transitory phenomenon.

Bells' theorem illustrated the dilemma posed by quantum theory. It showed that if the statistical predictions of quantum theory are correct, our normal way of thinking about the world is not. The principle of local causes is inconsistent with the statistical predictions of quantum theory.

Because actuaries, like physicists, attempt to build mathematical models to better understand and predict physical phenomena, they too must attempt to determine the extent to which their models fit reality. Actuaries study phenomena with significant subjective components for which mathematical modeling is difficult to apply. Because relativity is such a widely recognized and useful paradigm of physical science, a review of some of the mathematical elements of it illustrates how mathematics can support an area of applied mathematics like actuarial science.

The important point of this discussion as regards actuarial science relates to the complexity of the mathematics involved in analyzing real world phenomena. Whether the scope of the search for the core of our paradigm would include lead to such lengths is not known. Before we can determine the proper level of abstraction, however, it is important to consider the alternatives.

To conclude this digression on the use of mathematics in one of the more interesting areas of physical science, a review of the general theory of relativity and its mathematics is warranted. The special theory had a mathematical requirement that algebraic statements of fundamental laws be invariant under Lorentz transformations of coordinates. Einstein used tensor calculus to provide for invariance under general transformations.

General relativity is analytically formulated in terms of a G-tensor. Formulating the general theory involved finding field equations specifying the G-tensor in terms of the distributions of mass and energy in space-time. Every feature of the universe may eventually be reduced to a geometrical property of the space-time manifold.

A fundamental metric tensor is determined by the gravitational field the effect of which is expressed by taking the components of the Ricci tensor equal to zero. Given this mathematical structure, many physical phenomena can be modeled with great precision. The subject matters of vector and tensor analysis provide a rich structure of mathematics for this purpose. Their use in the area of modern physics provides an important illustration of the use of mathematics in an applied area.

PHILOSOPHY

Philosophy concerns itself with the ultimate questions of existence. While this may seem somewhat removed from the concerns of actuarial science, it must be noted that the subject matter underlying actuarial science and the impact of actuarial activity have implications which are ultimately philosophical in nature. Consider the welfare effects of classification of risks, for example, or the epistemological implications of the determinations as to how actuarial models are to be developed.

Metaphysics, the branch of philosophy which deals with reality can no longer be separated from physical science. As we saw in the survey of physics, reality is no longer considered a deterministic notion. Complementarity is the rule in physical as well as social science.

Epistemology is the branch of philosophy which deals with what we can know and how we can gain knowledge. As we have already seen, physical science has determined that the mere act of gathering knowledge impacts the content of the knowledge. As was the case with metaphysics, recent developments in physics have totally changed the way epistemological issues are posed.

Other branches of philosophy deal with large questions related to particular concerns. Ethics concerns itself with questions of morality and obligation. Government concerns itself with the ordering of human affairs so as to realize some agreed upon principles. Logic is concerned with principles of valid reasoning. The concern of aesthetics is beauty.

The work of actuaries has governmental, ethical, metaphysical, epistemological and even aesthetic elements. The nature of government impacts the nature and use of actuarial calculations. The economic implications of actuarial calculations have ethical aspects. The relation of actuarial models to reality is a metaphysical issue. Epistemological questions arise as actuaries determine what they can know and how they can know it. The aesthetic aspect of actuarial science has been commented upon by many nonactuaries.

The Socratic method of finding truth uses the technique of counter-example to develop questions which help clarify definitions. This method is widely used in law school to allow law students to refine their ability to argue a position according to a particular kind of logic and set of rules. Recent developments in physics would suggest that this method of pursuing truth continues to have great credibility.

The notion of a dualistic metaphysics was formed by Plato. He identified physical objects which are in a constant state of flux and ideas or essences which are permanent and unchangeable. His famous allegory of the cave provides a metaphor for the distinction between true and superficial knowledge.

Plato's theory of knowledge involves the construction of a hierarchy of knowledge. A basic distinction is made between knowledge and opinion. The four levels of knowledge are imagination, belief, understanding and reason. Understanding of the type reached by the mathematician is inferior to reason which uncovers true knowledge through the use of dialectic.

Aristotle's metaphysics was not dualistic, but considered matter and form inseparable. The universe has its own form. His ethics involved seeking for the natural role. The Aristotelian synthesis ruled philosophical thinking for over a thousand years.

Descartes provided a philosophical basis for modern science. He developed the modern duality between mind and matter. Newton followed with a physics which explained the universe.

New schools of thought arose to challenge the mechanistic view of the universe. Empiricists like John Locke felt knowledge arose from sense perception, and that metaphysical systems based on reason were of questionable value. David Hume carried this thought to its logical conclusion that only sense perceptions and feelings exist.

Kant developed a theory of knowledge partly in response to the radical conclusions of Hume. For Kant, reality conformed to the concepts of the human mind. The laws of nature became dependent upon categories in the mind.

Hegel provided a synthesizing philosophy, bringing in elements of romanticism, rationalism and Kantianism. Hegelian dialectic reaches higher levels of truth by synthesizing opposite notions. Ultimate reality becomes manifest in the nation state the values of which all its components would accept.

Existentialism is a modern philosophical school. Existentialism stresses the subjectivity of the individual and the need for the individual to accept responsibility for his/her own actions. As opposed to a systematic structure, existentialism tends to present its themes most effectively in literature of authors like Camus and Sartre.

Actuaries bring a world view to their work and that world view affects their approaches and expectations. Currently, standards of practice are being discussed which will provide some additional structure to the approach actuaries will take toward their work. The search for a structure from within with which to evaluate the work of an actuary relative to a paradigmatic structure will be an ongoing one.

MATHEMATICS

While actuaries must be knowledgeable about such diverse areas as economics, law, accounting, government, business, and finance, their unique identity is found within the context of the subject of mathematics. It is through mathematical tools that actuaries filter the various factors influencing the variables they are attempting to estimate, and it is these tools that provide the solutions the actuary presents. Our body of knowledge and the center of gravity of our discipline is ultimately an aspect of mathematical science.

As an applied field impacted by a wide variety of subjective factors, actuarial science, like physics, must ultimately define its scope. Physical science abstracts away many complicating elements of the real world in order to apply structured models to its problems. Actuarial science must do likewise, at least for aspects of its work which are determined to be covered by the paradigm.

To identify the mathematical core of the actuarial paradigm, then, we must evaluate the subject matter of mathematics and extract the elements most relevant to the actuarial model. From these elements, we will be in a position to construct a theoretical foundation for actuarial science. That theoretical foundation may or may not have application in a large percentage of the practical situations faced by an actuary in the course of normal business.

Foundations

In his book, The Philosophy of Mathematics, Korner discusses three modern schools of the philosophy of mathematics. They are the logicist school, the formalist school, and the intuitionist school.

The logicist school, associated with thinkers like Frege and Russell, attempts to confine pure mathematics to a manageable number of concepts and principles. Cantor's logic of classes, however, led to contradictions requiring that any logicist system have one postulate which is pragmatically grounded. Mathematical theories are existential in a sense in which logic is not.

A leading proponent of the formalist school is Hilbert. Hilbert attempts to reconcile finite mathematics with transfinite. The ideal is a consistent, complete, mechanical formalism of all mathematics.

The intuitionist school rejects the notion that mathematics requires the support of an extended logic or rigorous formalization. While the subject matter of the metamathematics of the formalists was perceptual objects, intuitionists deal with nonperceptual conceptions. Brouwer formulated the fundamental theses of the intuitionist philosophy.

Intuitionists consider mathematics to be independent of language and logic. A sequence of abstract entities, the natural numbers, is the starting point. Godel's proof of the inadequacy of the original metamathematics does not affect the intuitionist approach.

Mathematics, then, which provides a foundation for actuarial science, does not itself have a universal set of accepted results. When Godel proved that the consistency of mathematics could not be proven from existing principles, the hope of finding infallible objective laws and standards evaporated. This has not rendered mathematics impotent, however.

The work of Copernicus and Kepler, for example, while no longer the last word on astronomy, still influences our thinking about mathematical laws which underlie physical phenomena. Newton's laws of motion still explain mathematically most laws of nature not withstanding the fact that relativity and quantum theory have replaced the implied mechanistic determination of Newtonian physics. Actuarial science, then, while not responding adequately to all questions for which answers are desired, does provide a basis for structuring a response.

Mathematical reality cannot be unambiguously incorporated into axiomatic systems. Nonetheless, developments in such diverse areas of pure mathematics as non-Euclidean geometry, topology, group theory, tensor analysis, and matrix theory continue to yield rich rewards in many practical applications. Actuarial science can draw from the subject matter of mathematics in developing its paradigmatic structure.

Mathematics provides the basic tools of actuarial science. Because actuarial science covers a broad range of problems, it uses tools from a broad range of mathematical subjects. This section will survey broad areas of mathematical science which include elements of importance to actuaries in developing a core body of knowledge.

Calculus--Calculus deals with the real number system and the theory of limits. The limiting process is explored using differentiation and integration. Its study is preparatory to a more rigorous study of the real number system and the concept of a limit.

The derivative measures the instantaneous rate of change of a function. Marginal concepts which arise regularly in the social sciences can be expressed as the derivative of a respective total function. Marginal costs and revenues, for example, which are important concepts to actuaries, may be found by taking the derivative of associated total cost and revenue functions.

Integration involves finding the function for which the rate of change is known. An actuary could be faced with the need to determine the capital base into which a known investment is flowing. Capital level would be the integral with respect to time of net investment. Integration may be used to derive a total function from a marginal one, as well as to find probabilities from a given density function.

Special functions studied within the calculus include power, exponential, and logarithmic functions. These functions are widely used by actuaries. Exponential functions provide a basic tool for interest theory. In economics, production functions which measure output relative to input combinations often assume a power curve.

Multivariable functions create another layer of problems for applied mathematicians. The profit of an insurance company, for example, might be considered a function of the costs and revenues associated with several products. Partial derivatives of the proper degree would be used to determine optimum output levels of each product. Maximization or minimization under constraint can often be accomplished by the use of a Lagrangian function.

Differential equations may be used to find functions where the underlying growth rate is known. A formula for the total value of a sum of money compounding continuously for a defined period at a defined interest rate can be derived using differential equations. Differential equations are utilized in a large number of applications in the physical sciences.

Linear Algebra--Linear algebra involves the solution of simultaneous linear equations. This may be done in a variety of ways, often associated with the notions of elimination or determinant. In a more abstract sense, the idea is to find the kernel of a linear mapping and to characterize the subspace spanned by a set of vectors.

Matrices have arisen as a tool for expressing a complicated system of equations succinctly. A matrix is a rectangular array of numbers, and matrix algebra provides a shorthand method of solving equation systems. Determinants are values derived from a matrix using prescribed rules.

A matrix which provides perspective on the power of the tools of matrix algebra is the Leontief matrix which is used in economic input-output analysis. The level of total output for an economy, including intermediate as well as final output, can be solved by matrix algebra. A vector representing final demand for each product and another representing input values in the production of products provide the basis for the solution of the level of output of each product.

More pertinent to the immediate work of many actuaries is the value of linear algebra in evaluating variables more microeconomic in nature. Linear algebra provides a basis for the solution of a system of equations involving several variables. Linear statistical models are widely used for regression, analysis of variance, and design of experiments.

Certain determinants and matrices have special value to applied mathematicians. A Jacobian determinant allows testing for linear and nonlinear dependence of a system of equations. A Hessian is a determinant composed of all second order partial derivatives of a multivariable system and provides a second order test for optimization. Bordered Hessian determinants are valuable in solving problems involving constrained optimization.

Linear programming has as its objective the determination of an optimal allocation subject to constraints. One method often used to solve such problems is the simplex algorithm. The algorithm moves from one basic feasible solution to another until an optimal solution is found.

Profit maximization is a category of problem to which this approach may be applied. Profit is a function of output levels of products, the production of which is subject to a number of constraints. Constraint equations are established in matrix form, and a simplex tableau is set up by which to determine optimal allocations.

As was the case with calculus, we see that linear algebra has many tools which would be valuable to people solving problems of the type actuaries face. Because calculus and linear algebra are prominent mathematical topics in the actuarial syllabus, they have been given significant attention. Because actuarial science draws on many other areas of mathematics, however, those must also be reviewed to uncover any threads which may be important in the determination of an actuarial paradigm.

Vector and Tensor Analysis--A vector is a quantity having both magnitude and direction. Vector algebra includes the operations of addition, subtraction, and multiplication. The laws of association, distribution, and commutation apply. Vector functions are differentiable, integrable, and have many physical applications.

Vector analysis provides a basis for tensor analysis. Tensor analysis, which is of great consequence in such areas as general relativity theory and differential geometry, is a result of the study of consequences of the fact that physical laws are independent of any particular coordinate system.

A tensor is a system of numbers or functions whose components obey a certain law of transformation when the variables undergo a linear transformation. Tensors may be of any order. An aggregate of tensors at every point of a region of space is a tensor field.

The covariant derivative of a vector field is a mixed tensor. The divergence, curl, gradient, and LaPlacian notions all involve the covariant derivative. Intrinsic and tensor derivatives of a tensor reduce at a given point to ordinarily derivatives of the tensor.

Tensor calculus is a key aspect of the analysis of particle dynamics. It is also applicable to the theory of electricity and magnetism. Green's Theorem and Stoke's Theorem use tensor calculus to provide for transformations of volume and surface integrals to line integrals. This leads to special relativity.

The representation of the motion of a particle under the action of a force system in space-time is a curve. The equations of motion of the particle may be investigated using tensor calculus. The relativistic form of the equations of motion of a continuous medium may be expressed in terms of world tensors. Electrodynamical equations are also expressed in terms of world tensors.

The metric for special relativity affects the formulas for the LaPlacian and divergence. Maxwell's equations in vacuum may be extended to space-time via dual tensors under Lorentz Transformations. A tensor field may be defined on a manifold.

Algebraic Systems--Algebraic systems are sets of objects together with relationships and operations defined for the sets. Two systems are considered isomorphic if there exists one to one correspondence between the sets and if relations and operations defined on the sets are preserved in the correspondence. Well-known systems include the real numbers, natural numbers, integers, complex numbers, and rational numbers.

More abstract systems include groups, rings, integral domains, and fields, each of which have distinguishing characteristics. Notions introduced previously, including vector spaces, matrices, linear algebra, and polynomials are all algebraic systems. Boolean algebras involve binary operations and are widely used in computer science.

Topology--As was the case with tensor analysis, topology covers the analysis of concepts which are of importance to applied mathematicians like actuaries. The notions of countability, equivalence, boundaries, and completeness all arise within the area of topology. Technically, a topology is a class of subsets satisfying certain axioms.

Topology involves the investigation of properties like compactness and connectedness which have fairly concrete application. Metric spaces are topological spaces and include Hilbert spaces, which exhibit phenomena not occurring in Euclidean space. Threads from the subject matter of topology are found in many of the theoretical structures which have arisen in the area of actuarial science.

<u>Numerical Analysis</u>--Numerical analysis is another subject covered by the actuarial syllabus and intimately related to the work of actuaries. Numerical analysis involves developing and evaluating methods for arriving at numerical results. As such, it cuts across many other areas of mathematics.

Use of a polynominal to approximate a function is a common approach used in numerical analysis. A particular polynominal, called a collocation polynominal, can be found which coincides with the function being estimated at various points. As is the case with all approximations, minimization of error is a key criteria for the selection.

Other polynominals are also of value in solving numerical problems. Osculating polynominals match derivatives, as well as primary values with a given function at specified points. Least-squares polynominals minimize the squares of errors utilizing the idea of orthogonal projection in a vector space. Mini-max approximation polynominals minimize the maximum error (equal-error property).

Polynominal approximation is also a basis for a variety of integration solutions. Series approximation is another aspect of numerical analysis, as is the solution of differential equations. Finding roots of systems of equations utilizes numerical analysis techniques.

Solving systems of linear equations, as discussed previously, is a key area of applied mathematics. Reducing such a system using matrices and vectors with an algorithm from numerical analysis is a common activity of applied mathematicians. Gaussian elimination is a key algorithm used for this purpose.

Special problems often arise in applying mathematical principles to physical problems and numerical analysis techniques tend to provide useful approaches. A system of linear equations may be overdetermined in the sense that the coefficient matrix may have more rows than columns. Boundary conditions may make solutions of differential equations problemmatical. Numerical analysis techniques provide a way of handling such problems.

Monte Carlo methods solve certain types of problems by the use of random numbers. They may be used to provide early approximations for more cumbersome algorithms. Simulation and sampling are two such methods which provide for approximations to the description real phenomena.

Problems involving approximation often involve the selection of an element from a set which is close to an element not in the set. A decision as to how the distance between the elements is to be measured is needed. A metric space provides an abstract concept for such a problem.

Concepts from topology may be defined in metric space. Compactedness, closedness, denseness and boundedness are examples. Normed linear spaces are particular kinds of metric spaces, and Banach and Hilbert spaces are particular normed linear spaces.

The inner product of two vectors in linear space obeys established postulates. Vectors in an inner product space may be orthogonal, and if an orthogonal set of vectors is of unit norm, the set is orthonormal. Orthonormal sets of vectors allow for explicit solutions to problems involving the location of a point on a subspace of minimum distance from an external point, a problem faced by actuaries in regression analysis.

In the Cartesian Plane, a convex set is one where any of its point pairs can be joined by a line which is contained in the set. Convexity is a property which, when it exists, provides for better modeling. Convex programming has arisen as a way of locating the minimum points of a convex function defined on a convex set.

The Weierstrass theorem provides that there exists a sequence of polynomials which will converge to a continuous function uniformly on a closed bounded interval. A set of vectors in a space satisfies the Haar condition if every set of them is linearly independent. A Markoff system is a system of continuous functions on an interval which satisfies the Haar condition in its initial segments.

Least squares approximation involves quadratic norms, and uses orthogonal polynomials, Fourier series, and harmonic analysis. Orthogonal polynomials are utilized in the process of numerical integration. A transformation of the Fourier Series provides for uniform approximation of continuous functions.

Tchebycheff approximation may be appropriate for approximation problems where parameters to be determined have not occurred linearly. Existence, unicity, characterization and computation of the approximations are issues of primary concern. The minimax algorithm may apply in rational approximation.

Integral Theory--Because the functions actuaries work with are not always well behaved, standard integrals do not always produce optimal results. In addition to techniques from numerical analysis, the actuary may utilize fairly recent developments in integration theory. These developments have proven especially valuable in the continuous advancement of measure theory.

Classical integration theory has proven inadequate to many problems which have arisen in physics and other applied sciences. The standard integral does not provide for integration over sets with a large number of parameters, applies only to a fairly small number of well defined functions, and requires that the domain of integration be homogeneous. The modern theory of integration, which traces back to Lebesgue, has provided for a much wider range of application.

Measure Theory

Because actuaries use probability theory (to be discussed later) in solving problems in their domain, and because probability theory is based on a rich and extensive mathematical structure, that structure must be an aspect of the actuarial paradigm. Probability has the formal properties characterizing a class of functions of sets known as measures. Measure theory which has a rich mathematical foundation, then, is a component of actuarial science.

Miscellaneous Topics

LaPlace transforms are useful in the solution of linear differential equations. Situations arise where a function can be expressed as the product of two functions each of which is a transform of a known function. One of the properties of LaPlace transforms is that the product is an integral called the convolution of the functions.

Orthogonality is a notion that often arises in actuarial science. Two functions are said to be orthogonal over a region if the integral of their product vanishes over the region. Sets of characteristic functions corresponding to boundary value problems are orthogonal with respect to a weighting function. A set of normalized orthogonal functions is said to be orthonormal.

In addition to ordinary integrals defined along a line segment, there is a corresponding integral defined along a curve which is given the name line integral. Similarly, corresponding to an ordinary double integral defined on a region in a plane, there are surface integrals defined on a surface. Line and surface integrals have found many applications in theoretical work done in the physical and social sciences.

In Riemann integration, the functions and regions of integration are assumed to be bounded. Stieltjes integrals involve two functions defined on a closed interval. Stieltjes integrals play an important part in the formulation of such key notions in probability and statistics as first moment, variance, and mathematical expectation.

Point-set theory provides additional grounding for the study of continuous functions, integration, and series. The Bolzano-Weierstross Theorem states that there is at least one point of accumulation in a bounded infinite set. Cauchy's Convergence Theorem provides conditions for convergence of a sequence. The Heine-Borel Theorem states that a finite number of open sets may be chosen from a collection which covers a bounded and closed point set in such a way that the set is covered by the new collections.

The gamma function is an improper integral which has some useful properties. It provides a convenient way of interpolating between factorial values. There is a connection between the gamma and beta functions.

Fourier Series play a key role in the solution of many problems in the physical sciences. Lebesgue integrals are often used in dealing with Fourier Series where discontinuous and unbounded functions are involved. Bessel's inequality arises as a result of the orthogonality relations among the terms in a Fourier Series.

Because functions in mathematics are exact, but those in the physical and social sciences are not, the determination of a function by applied scientists will be different from that for a mathematician. Methods known as relaxation methods have arisen in the applied sciences which allow for computational facility. These methods allow for evaluation of approximate values of the required function for a finite set of values.

Modern developments in quantum physics indicate that statistical laws are the fundamental laws of nature. Probability calculus of the type utilized by actuaries is based on empirical assumptions about the stability of relative frequencies of events. The pure and applied aspects of mathematics are inextricably bound, and actuaries are in a position to utilize developments in mathematical science to push the frontiers of their profession.

Mathematics builds on itself. Algebra builds upon arithmetic. Geometry builds on both. Calculus builds on all three. Topology builds on geometry, set theory, and algebra. Noncantorian set theory takes the axioms of restricted set theory and adds a form of the negation of the axiom of choice such as the negation of the continuum hypothesis. Nonstandard analysis takes the standard universe of mathematics and adds nonstandard models.

To introduce the notion of complex numbers, it is instructive to consider vectors. Rotations in space lead to a new algebra of quaternions. Quantities which require more than three numbers for their identification may be handled by tensor algebra.

Rates of change of derived magnitudes with respect to length are called gradients. Because there is a tendency in nature toward uniformity, gradients enter into may physical laws. The divergence of a vector also has many physical applications and is the limit of a surface integral. The curl is a vector often used in physical science and, like the divergence, is usually expressed in terms of derivatives of functions.

Many of the fundamental equations of physics are formulated in terms of partial differential equations. These include the field equations which include a group of terms called the LaPlacian of the function. The LaPlacian measures the difference between local and average values of the function in an infintessimal neighborhood.

<u>Summary</u>--This section provides a sketch of certain areas of the subject matter of mathematics which may be useful to the actuary in the course of practical or theoretical work. The actuary must borrow extensively from the subject matter of mathematics and, in moving toward the definition of paradigms, must identify elements which may be part of the paradigm structure.

While there are many mathematical tools which cannot be covered in a concise but broad survey of the type which is attempted here, this survey may at least provide some perspective on the breadth of mathematical tools available to the actuary. Appendix C provides a glossary of mathematical terms which are related to actuarial considerations.

PROBABILITY AND STATISTICS

Actuarial science is best described as an area of applied mathematics. Like physics, which attempts to analyze and explain physical phenomena using mathematical models, actuarial science attempts to solve problems arising in the real world using mathematical techniques. Because the areas of inquiry of actuarial science (particularly in the liability insurance area) involve phenomena for which the modeling involves a wide range of possible variables, probabilistic as opposed to deterministic models tend to be favored.

Probability theory originally developed in response to some very practical problems involving gambling and commerce. The basic idea was that if there are 'n' possible ways in which an event 'E' can happen, and 'm' of those ways are favorable, the probability of a favorable outcome is 'm/n,' if all the ways are equally probable of happening. In a more general sense, probability is the limit of the frequency ratio as the number of trials of a repetitive event increases without limit.

As in the case with mathematics, probability theory has not led to truth or certainty for the class of problems in its area. No system of axioms adequately reflects behavior of real world phenomena. Whatever randomness exists in the phenomena being analyzed by applied mathematicians using available tools is too varied to be handled axiomatically.

Overview

A lack of rigorous foundation for pure mathematics and probability theory provides the backdrop for a discussion of another key tool of actuaries, statistical science. Statistics involves collecting, organizing, summarizing and analyzing data and making decisions based on that analysis.

Information about a population may be inferred by analyzing samples drawn therefrom. Statistical inference involves the estimation of population parameters from sample statistics. R. A. Fisher suggested three criteria for such estimation as consistent, efficient, and sufficient.

When a relationship is thought to exist between two or more variables, a mathematical form describing that relationship can often be found. When a data set is analyzed by finding equations or curves which fit the data, the term curve fitting is used. The method of least squares involves finding the curve which minimizes the sum of the squares of the distances of the data points from the curve.

Curve fitting is a technique used extensively by actuaries. To model the phenomena they analyze, it is often useful and necessary to define an expected relationship between components of the phenomena. Once that relationship is defined, various assertions can be made about the expected behavior of the phenomena and the implications of that to the problem being addressed by the actuary.

The Bayesian approach to inference regards probability as a degree of reasonable belief rather than a frequency. Bayes' Theorem provides for posteriori probabilities based on prior probabilities and the impact of new data on the likelihood function. The need for the use of unknown prior probability creates an additional problem for this branch of probability theory.

One attempt to deal with this problem was provided by Wald. The notion was to modify the problem as Lebesgue had modified the ordinary integral to solve a class of problems which had previously been open. Wald came up with the Theory of Statistical Decision Functions, which unified a number of prior theories.

Classical probability theory, then, which provides a key foundation for actuarial science, has some fairly serious limitations. A finite set of possible outcomes is required. The outcomes must be equally likely.

A more general model which incorporates features of real world problems of the type faced by actuaries which is abstract at the core but which provides for adaption to concrete problems will be even less rigorous.

In addition to uncertainty related to the mathematical models themselves, actuaries face uncertainties related to the data to which the models apply. Oskar Morgenstern provided a study of the properties of economic data which discussed problems related to the "accuracy of economic observations." A key point made was that the acceptability of a measurement is inextricably tied to the use to which it is put.

Among the sources of inaccuracies involved in the use of economic statistics, Morgenstern listed the lack of designed experiments, observation errors, deliberate falsification, lack of definition, and interdependence. He suggests, at a minimum, that economic statistics be published only in conjunction with an estimate of their error.

The Subject Matter

The subject matter of statistics involves the collection, analysis, presentation, and utilization of numerical information. The purpose is to make inferences and reach decisions in the face of uncertainty.

A common way to organize data in performing statistical analysis is to establish a frequency distribution. Such a distribution breaks data into groups and totals the number of observations in each group. A cumulative frequency distribution shows for each class the total number of observations in all classes up to and including the class itself.

The mean or average value of the elements in the population is the most common measure of central tendency. The standard deviation is the most widely known measure of dispersion.

A random variable is one whose values are associated with some probability of being observed. The set of all possible values of a random variable is called a probability distribution. A number of probability distributions figure prominently in actuarial science.

The normal distribution is a probability distribution commonly used in statistical science. Given the mean and standard deviation, the normal distribution is completely determined. The normal distribution is a continuous probability distribution.

The Poisson distribution is a discrete probability distribution. It is useful in determining the probability of a certain number of events in a given time frame. The assumption is that the events are independent and that the mean per unit of time is constant. Actuaries have found these assumptions tolerable.

Probability Theory

Modern Probability Theory, which builds upon a mathematical foundation, considers outcomes as points in a sample space. An event is a set of points in the space. A random variable is a function defined at each point of the sample space.

Because the models used by actuaries to approximate the phenomena they are evaluating have characteristics which are mathematically complex, an understanding of them requires solid grounding in the core elements of probability theory. One such element is the Moment Generating Function (MGF) of the random variable in question. Since the MGF is a complete characteristic, conclusions reached in terms of MGFs can be translated into corresponding properties of the probability distribution.

Since many of the distributions utilized by actuaries are related, the shorthand provided by the use of MGFs is enhanced. The geometric distribution is a special case of the negative binomial. The exponential distribution is a special case of the gamma. The gamma distribution is the continuous analogue of the negative binomial.

By developing a theoretical distribution of losses expected to occur for a model portfolio of policies, actuaries have a starting point for the various types of loss analysis they perform, Even if they are working with an aggregated database and attempting to estimate losses from statistics not conformable to the assumptions in the theoretical model, the results of the analysis can still be judged in the context of how well they fit what the corresponding idealized model would have produced.

Two models used in developing a loss distribution for a group of policies relate to the probability of the occurrence of the insured event and the amount of loss which arises once a loss occurs. Models also exist by which to determine the aggregate loss distribution directly. Like all models, those which attempt to reflect insurance contingencies of the type faced by actuaries must find a balance between computational facility and accuracy of representation of the phenomena being modeled.

One probability distribution which is often selected to model the occurrence of insured events is the Poisson. The Poisson Model can be used to calculate the probability of the occurrence of specified number of losses from groups of exposure units in a given time frame. When the probability of occurrence is not constant over time, or not uniform among members of the group, a compound Poisson or negative binomial distribution is often used.

Given the distribution representing loss occurrence, a distribution of amount of loss is needed to derive an aggregate loss distribution for a group of exposures. Size of loss distributions are obviously very sensitive to the nature of underlying insurance coverage. Among the distributions often discussed as providing reasonable approximations to size of loss distributions for common insurance coverages are the Lognormal, Pareto, and gamma.

Once distributions of loss occurrence and amount of loss are determined, aggregate claim distributions can be developed. The notion of convolution sometimes arises in this regard. Convolutions allow for the development of cumulative distribution functions given occurrence and amount distributions.

The notion of convolution arises in the mathematical theory of generalized functions as well. This subject is based on the theory of topological linear spaces. Convolution is the operation that maps an ordered pair of elements into the product element. Recent research in the area of integral transformations has involved investigation of the conventional convolution transformation. Extension to the N-dimensional case has been explored.

Because compound distributions adequate to the task of estimating aggregate claims in the collective model become so complex as to make calculation impractical for many applications, approximation methods have been developed. Among these are the normal approximation graduation by orthogonal polynomials, the Esscher approximation, the Cornish-Fisher Expansion, and the Edgeworth Series.

In addition to direct determination of distribution functions, indirect methods are available. Simulation is an approach widely used when the problem to be solved has an analytical structure which is too complex to be modeled effectively. The simulation model describes the process in terms of individual events.

The first step in performing a simulation is to develop a model which adequately represents the phenomena being analyzed. Elements of the model which are unknown may be simulated using random numbers to obtain random observations from a selected probability distribution. Conventional methods are available by which random numbers may be generated by computers.

Monte Carlo techniques are often utilized to increase the efficiency of the simulation and increasing the precision of sample estimators. Stratified sampling is a particularly effective technique for a number of actuarial applications. The statistical theory underlying simulations is the same as that which applies to physical experimentation.

The essence of a Monte Carlo study is the specification of sets of parameter values for postulated distributions. Drawings from the distributions provide samples to which estimation techniques may be applied. The sampling distributions of the estimates are then analyzed in relation to expectations.

The key actuarial tools for developing loss estimates and variability estimates for exposure related to a group of insurance policies are fairly well established. The problems faced by actuaries in their everyday work, however, do not always fit the model specifications very well. As a result, techniques used in practice often have at best a tenuous relationship to the mathematical underpinnings.

Consider a typical actuarial problem, the need to establish reserves for a line of insurance for a company which has written the line in question for a period of 10 years. The standard actuarial approach is to consider payment and reserve patterns over the relevant history to determine the ultimate value of losses expected to arise for each segment. Building a model which considers development and trend factors as well as the basic frequency and severity components at each point in time would be a significant mathematical challenge.

Once a loss distribution for an insurance portfolio is established, the theory of risk and ruin theory come into play. An insurance company is "ruined" if the aggregate amount of claims in a period equals or exceeds the funds available to provide for such claims. Current models are limited in the sense that not all the risk elements facing an insurer are incorporated, but they provide perspective on the very important solvency issue.

In addition to the complexity associated with the establishment of mathematical models to complex physical phenomena, actuaries have developed the notion of credibility to apply to the particular situations they encounter. Because information available for a category of risk being evaluated by an actuary may be sparse, appeal to the law of large numbers is not always sufficient. Credibility procedures allow an actuary to incorporate consideration of the impact of the relative volume of data available on the results of the calculations performed.

Credibility standards can be established based on an assumed distribution of claim counts. Adding the variability in the size of claim distribution allows for the calculation of more refined standards. Given the credibility standard, Bayesian or parameter-free methods of utilizing partial credibilities may be considered.

Statistics

Given the uncertainty involved in the mathematical modeling of physical phenomena, various statistical methods have arisen by which to make inferences based upon available information. Sampling theory studies relationships between populations and samples drawn from the population. Given the known characteristics of the population, information about random samples drawn from the population is determined. Statistical inference uses principles of sampling theory to infer information about populations from samples drawn from the population.

One of the more common uses of statistical estimation theory is the development of confidence intervals. By making assumptions about the underlying distribution, confidence levels for population parameters can be estimated based on samples drawn therefrom. Hypothesis testing involves evaluation of a particular hypothesis in the light of sample information and statistical decision rules.

Where samples are small and it cannot be assumed sampling distributions are approximately normal, distributions often utilized are the "students" and chi-square distributions. To complete statistics for such distributions, sample observations and population parameters must be obtained. The number of degrees of freedom of such a statistic is the number of independent observations in the sample less the number of population parameters which must be estimated.

Tests have been developed based on the chi-square distribution which have much significance for actuaries. Chi-square tests for hypotheses regarding goodness of fit and independence of variables have been widely utilized. Another technique widely used by actuaries involves analysis of variance.

Analysis of variance is used to test mean differences. The one way analysis of variance model may be represented by a linear equation, as may the two-way model. Testing hypotheses regarding populations from which sample means emerge is a common aspect of many actuarial problems.

Regression analysis is another area of statistics widely used by actuaries. Regression analysis attempts to predict the value of one variable from the values of associated variables. Among the assumptions of the linear regression model are the randomness of the dependent variable, the linear relationship of the dependent and independent variable, and homoscedasticity.

The least squares regression line is that for which the sum of the squared deviations between the estimated and actual values of the dependent variable is minimized. The means of the probability distributions of the dependent variable at each value of the independent variable have a systematic (linear) relationship to each other value. Inferences can be made regarding the parameters of the regression line.

Correlation analysis measures the degree of relationship between the variables. The correlation coefficient is a common value used in evaluating the relationship because it is included in a test statistic which approximates the distribution and is therefore amenable to statistical testing. In particular, the significance of the coefficient can be tested.

The notion of a probability weighted average is a critical one in probability theory. Integrals which are generalized sums provide a basis for expression of these sums. The abstract integral is a key tool in the evaluation of mathematical expectation.

The mean value is the coordinate of the center of mass of the probability distribution. The standard deviation is the radius of gyration of the mass distribution, and the variance is the second moment or moment of inertia about the center. Moment generating and characteristic functions involve LaPlace and Fourier transforms from Classical Integration Theory.

Random processes are infinite families of random variables. The covariance and convolution functions which play roles in such areas of importance to actuarial science as extrapolation and smoothing of time series are important to the analytical treatment of many random processes. Random (stochastic) processes generalize the notion of random variables allowing for the extension of probability models to dynamic systems.

Empirical work with random processes involves finding a function which is typical over an interval of sufficient length. The mathematical problem is to express conditions such that one can use physical information to determine whether a useful probability model can be developed.

In addition to standard regression analysis techniques, a number of variations have arisen to deal with specific problems. Qualitative explanatory variables can be introduced by the use of dummy variables. Where the dependent variable is influenced by past as well as current values of the independent variable, the distributed lag model is available.

Problems which often arise when economic data is being analyzed include multicollinearity, heteroscedasticity, auto correlation, and measurement errors. Also, the dependent variable in one system may be an explanatory variable in another in which case the system is a simultaneous equation system.

The notion of Bayesian analysis plays a role with regard to much actuarial work. Decision making based on probabilities, economic consequences, and expected utility involves the use of Bayesian techniques. Criteria for the use of and value of sample information also fall in this area. Bayesian procedures permit decision makers to enter opinions in a formal way.

By pointing out the problem in arguing from the particular to the general in statistical inference, Bayes created a challenge for future statisticians. Fisher developed a postulate requiring the selection from among all possible values of a population parameter that value which maximizes the likelihood that the sample obtained would arise. Wald, concerned about continued assumptions about prior probabilities in Fisher's approach, developed an alternative standard involving minimizing the maximum risk of loss.

Given the complexity and uncertainty involved in the mathematical theory of probability and statistics, actuaries could find it difficult to pick models applicable to problems at hand. This has not generally been the case for a variety of reasons. One is that actuaries are not always aware of or concerned about the mathematical rigor of the model being used. Another is that many of the problems faced by actuaries are best handled by approaches which are fairly simple in mathematical structure.

Operations Research and Forecasting

Because actuaries are often involved in business decisions, in addition to being concerned with reducing numerical chaos, they are concerned about reducing the chaotic aspects of management decision making. An area of applied mathematics which has grown dramatically in recent years to deal with such problems is known as operations research. The various techniques falling in this area are involved in a quest for organizing activities so as to provide for desired results.

Like the other areas of applied mathematics surveyed in this paper, aspects of operations research arise in the contexts of other areas of mathematics and have been commented upon elsewhere herein. Linear programming has been introduced as a tool by which resource allocation may be judged. It provides an algorithm by which to optimize a function subject to constraints.

Network analysis allows for a graphic representation of the activities involved in completing a project. Game theory provides a basis for decision making under uncertainty. Queueing theory allows for systematic analysis of service time. Markov chains can be developed with discount factors allowing for optimization of a functional equation with unbounded horizon.

Forecasting is a particular area of statistical science which is a significant aspect of the actuary's work. A stable mathematical or statistical data structure allows for specification of a model from which forecasts may be made. Models used in forecasting have analogues in other areas of statistics, and a brief review will provide valuable perspective.

Perhaps the simplest model is the constant mean model which assumes values of the dependent variable vary from a mean value only by a random error element. The linear trend model assumes the dependent variable is a function of time. Regression models consider variables in addition to time.

In stochastic models the random element plays the dominant role in determining the model structure. Seasonal models incorporate the idea of periodic variation effects. Probabilistic models estimate future probabilities of events.

Game Theory is an area in which significant theoretical progress is currently being made. A game is characterized by a set of rules having a certain formal structure governing the behavior of certain components. Game Theory allows the reduction of any game to a simple or normal form. The aim of each player in a game is to maximize expected utility.

Perfect-information games are such that the players know the choices and outcomes of all prior moves and they have a pure value. The matrix of a finite game of perfect information has a saddle point. Each player has a good pure strategy. Every finite game has a value and each player has at least one good mixed strategy.

Any finite game can be solved by solving a finite number of linear equation systems. Each statistical game postulates a sample space which describes all outcomes of an experiment. A decision function provides a rule which associates with each outcome a point in the strategy space.

Statistical games utilize the notions of loss and risk functions. A priori and a posteriori probability distributions are also involved in the development of game strategies. Preference patterns are described by utility functions. Choice principles include Minimax, Bayes, and Maximin.

Game Theory provides an example of a subject designed to produce very pragmatic results using abstract mathematical concepts. The basic theorem of Game Theory is an assertion about convex sets in Cartesian n-Space. Convex sets and functions involve topological notions of boundedness, closedness, denseness, continuity, and uniformity.

Statistical science is clearly rich with tools for use in handling actuarial problems. As is the case with more abstract mathematical subjects, the tools afforded by statistical science are subject to challenge both as to internal consistency and external application. The continuing challenge of actuarial science is to explore ways to optimize the use of these tools in areas where they can be profitably applied.

MATHEMATICS AND ACTUARIAL SCIENCE

Given this uncertainty at the abstract core of actuarial science, what can we say about the results of actuarial calculations which blend the uncertainty of mathematical models with the subjectivity of the social and economic forces impacting the topic of evaluation? The discipline of statistics, as discussed above, helps bridge the gap from the mathematical models to the real world.

The statistical view of nature is that while deterministic laws may be developed which summarize phenomena which occur in the real world, these laws merely provide probabilistic, as opposed to deterministic, information. Real world phenomena do not consistently follow deterministic patterns and therefore these phenomena can only be evaluated statistically.

In actuarial science, this view manifests itself in a variety of ways. To make calculations manageable, actuaries often assume that the phenomena they are evaluating follow specified patterns. As an example, it is often assumed that the claims arising in a particular time interval for a particular insurance product will follow a Poisson distribution.

Computer models are now being developed, the purpose of which is to gain insight into the complexities of the real world by simulating natural phenomena. Because the mathematical equations used by scientists to describe simple phenomena become overwhelmed when attempting to describe the countless interactions involved in real world phenomena, the new simulations attempt to minimize the mathematical structure imposed initially so as to allow the phenomena themselves to determine the evolution of the structure. In this way, a better fit is expected to arise.

The core of the actuarial paradigm, then, must be found in the subject matter of mathematics. Given the core, various aspects of the related field of statistics would be brought in to expand the range of problems which can be solved. Finally, adaptation to the various subjective areas such as law, economics, business, operations, and the physical world itself, is necessary. Appendix D provides a glossary of econometric terms which would be of interest to actuaries.

UTILITY THEORY

Utility theory provides insights into problems involving decision making under uncertainty. Actuaries are often involved in calculating expected values related to various types of transactions. Decisions are not always based on the expected outcome of an event. The value associated with various outcomes may be related to, but not equal to, the expected value.

A utility function relates level of wealth to utility given defined levels of uncertainty in the decision problem. For a utility function to exist, certain conditions must be met. Different forms of utility functions reflect different attitudes as to risk aversion. The greater the risk aversion, the more the decision maker will pay to eliminate the risk.

Utility theory requires that decision makers can express preferences for outcomes which affect wealth levels. Given a choice between alternative outcomes, the decision makers' choices determine the utility function. The simplest function, a linear one, assumes risk neutrality, i.e., that the level of risk does not affect the value placed on an outcome by the decision maker. Risk neutral decision makers make decisions based on expected values.

A form of utility function which is felt to provide a basis for analyzing the behavior of many decision makers under risk conditions is the exponential one. It assumes a constant aversion to risk in the sense that changes in wealth do not result in changes in behavior. Exponential utility functions provide many computational advantages. Other types of utility functions reflect other risk attitude structures.

Another way of viewing a utility function is as an indifference curve showing combinations of risk and return to which an investor is indifferent. The higher the risk involved the higher return needed to provided the same utility level.

In 1979 the Casualty Actuarial Society had a Call Paper Program on the subject Total Return Due A Property-Casualty Insurance Company. Several papers from that program discussed the notion of a Capital Asset Pricing Model (CAPM). CAPM provides a framework for determining the relationship between risk, return, and reward for taking risk.

Modern Portfolio Theory which incorporates the ideas of risk and utility into investment strategy develops the notion of an efficient portfolio. Efficient portfolios maximize return for a given amount of risk or minimize risk for a given level of return. Identification of efficient portfolios requires information regarding each investment's expected return, variance of return, and covariance of return with that of other investments under consideration.

CAPM assumes all investors are risk averse and that for efficient portfolios the standard deviation of return is an appropriate measure of risk. A linear relationship between risk and return is hypothesized. For individual investments, risk is measured by covariance between the investment's return and that of the market as well as the relative volatility of the return on the investment. Return on an investment becomes a function of the riskless rate of return, the market rate of return, and the sensitivity of the investment's return to that of the market.

The measure of systematic risk (beta) can enter into valuation theory by impacting the discount rate used to value the income stream anticipated from the asset being valued. The value of the various utility-based models to casualty actuaries will be related to success in measuring the needed parameters.

THE THEORY OF INTEREST

In addition to estimating the loss and expense elements arising from the provision of insurance coverage, the actuary is involved in the estimation of cash flows which such coverage produces. Such estimation includes a number of elements. Key among those are the timing of the various cash payments made and the interest rate which properly reflects the valuation of cash deposits held by one or the other party.

In liability insurance, payments made under an insurance contract and their timing are influenced by the same kind of factors as influence the total amount of losses arising for a portfolio of policies. The data available to an actuary might include payments which have arisen historically under portfolios considered to be similar. From that data, the actuary must attempt to estimate future cash flows and assess a relative value on those given an interest rate which allows for appropriate evaluation of the cash flows.

As we have seen in the discussion regarding the syllabus of material to be studied by actuaries, the theory of interest is a key area of study. The theory of interest uses some of the tools of growth and decay analysis which apply to the physical sciences to analyze the impact of timing differences in cash flows given a discount or interest rate. Given the period of investment, the interest rate, and an amount at a point in time, the actuary can use existing tools to resolve any problem involving the time value of money.

AN ACTUARIAL PARADIGM

Actuarial science is ripe for, and probably in the midst of, a scientific revolution. This is to some degree a result of the fact that it is attempting to deal with a wide range of issues which are impacted by a wide variety of factors.

Standard approaches have often been unsuccessful in solving problems they were expected to address, and adjustments to these approaches have been evolving rapidly. This paper will suggest that the actuarial profession is at a crossroad which can lead to a new paradigm structure of significantly greater breadth than one we generally recognize today.

As is the case with economics, any description of a paradigm of actuarial science must recognize that a number of components exist. Some of the better known components of the economic paradigm are covered by the categories microeconomics, macroeconomics and welfare economics. In actuarial science the categories are not so well defined.

While actuarial science deals with areas of law, accounting, economics, and management, it is fundamentally a mathematical discipline. While a broad number of issues are dealt with by actuaries, the most fundamental is the estimation of liabilities arising from contingencies covered by insurance contracts. Once these liabilities are estimated, the values can be put into various actuarial models for the development of rate level, rate structure, reserves, and other "actuarial" items.

Life and casualty actuaries evaluate different types of contingencies. Losses associated with many life insurance products are felt to be predictable enough to fit some fairly well defined patterns which are quantified in various types of mortality tables. Losses associated with property/casualty insurance products, particularly those covering some of the more significant exposure to liability losses, are not felt to be likely to fall into any easily predetermined patterns.

Casualty losses for a particular exposure for a particular entity must be estimated by considering the unique characteristics of the exposure. Variables affecting casualty losses for a category at a point in time are numerous and include the nature of the coverage itself, the operating and socioeconomic environment, chance, and the insured population.

The actuarial paradigm may be defined as the methodology or methodologies generally agreed upon by members of the actuarial community to solve the problems being addressed by that community. In the evolving area of actuarial methodology relating to estimation of losses arising from liability insurance, there is often a dichotomy between methodology used in practice and methodology developed in theory.

The model often used in practice when adequate historical data is available utilizes data organized in the form of development tables. These tables organize claim count and claim amount information for a grouping of policies or incidents in such a way that changes in these values over time can be analyzed.

A theoretical approach to the problem of loss estimation has been discussed under the general title of risk theory. The idea is that the occurrence of an insured event is a random variable, and given that occurrence, the amount of loss which will be generated will fit a probability curve.

A BRIEF SURVEY OF CURRENT ACTUARIAL TOPICS

We have now reviewed some of the factors impacting on the problems actuaries are asked to resolve. We have also identified mathematics as the primary tool which the actuary applies to resolution of those problems. Of interest now is a discussion of some topics often associated with actuarial science.

Ratemaking--An area generally associated with actuarial science is that of development of rates or premiums to be charged for defined insurance coverages. In addition to estimating losses expected to arise for the portfolio of risks to be written at the rates being calculated, the actuary is faced with the issues of determining a price which covers other costs associated with the insurance product. Those costs include the cost of capital required to provide the product and the appropriate profit level associated therewith.

The ratemaking activity of the actuary, therefore, includes many elements of modification to whatever general paradigm is established for actuarial science. Those include estimating future social and economic factors which will influence needed rate level, determining a profit load which provides adequate return to capital, estimating how elasticity of demand and other factors will affect the mix of policyholders expected to purchase the coverage being priced, and estimating nonloss costs associated with provision of the product.

Reserving--Estimating loss reserves for defined categories of business may be the area of actuarial activity most directly related to an underlying paradigm. Unlike ratemaking, the estimation of loss reserves can be done from a particular data framework using a defined mathematical model. Peripheral elements like expense components and profit loads need not be considered in estimating loss reserves for a defined category.

Where deviation from a standard model may arise in the reserve estimation process is in the specification of the category for which reserves are being estimated and in the nature of the data which is available. The type of exposure involved, including policy limits, retention, policyholder mix, coverage mix, and volume of claim potential may result in modifications to the core model.

Other--In addition to standard pricing and reserving work, actuaries are called upon to provide analysis related to a wide variety of other issues. For purposes of paradigm evaluation, these other issues are considered peripheral or subsumed. A review of some of them, however, provides perspective on the texture and richness of actuarial activity which emanates from an abstract core.

One area receiving a good deal of actuarial attention currently is that of valuation. Actuaries are expanding the role they play in the determination of value of an insurer's assets and liabilities. This led to much more attention to cash flows associated with these assets and liabilities, as well as more direct recognition of possible effects of economic and operating conditions.

Additional knowledge regarding accounting rules also becomes important, as does evaluation of the uncertainties attached to the various components of the analysis. An understanding of the components of asset categories and their characteristics also proves to be important.

A second "secondary" area of actuarial science falls under the heading risk classification. While losses for a portfolio may be estimable given certain assumptions, that portfolio may consist of heterogeneous components. Risk classification attempts to evaluate differences in average loss costs associated with different groupings of risks.

Methodologies for identifying and evaluating risk differences are varied. Each individual exposure unit has unique characteristics, so any classification scheme must abstract away certain elements of difference. Once the scheme is determined, the problem of developing statistically sound differentials given the general credibility and homogeneity characteristics of the database remains.

Actuaries thus deal with a wide range of issues which may not fit neatly into a core model. Topics such as reinsurance, excess limits, individual risk rating, utility theory, interest theory, and graduation which the actuary faces all may be dealt with in the context of a core model or a component thereof, but the realism of the assumptions which must be made to achieve results is always a significant challenge.

THE NATURE OF ACTUARIAL WORK

While the mathematical models which apply to actuarial science are rich and robust, the work done by casualty actuaries is often difficult to transform into problems to which the models are well adapted. The data available by which to perform a typical ratemaking calculation, for example, would consist of a multitude of elements for the category under consideration. These would include:

- Paid and incurred losses for the coverage for a period of years.
- Various expense elements for the coverage for a period of years.
- Exposure measures for the coverage for a period of years.
- Premium measures for the coverage for a period of years.

The loss data can be structured in a variety of ways depending upon desired accounting period and time lag structure. Expense elements can be broken in a number of ways. Effects of such items as residual markets, excess limits, coverage variations, individual risk rating, and elasticity of demand can be incorporated in various ways.

Determination of target profit level and the way of reflecting it in the price level is another complicating element. Determination of trend factor and weights given to various historical periods add to the complexity, as does reflection of credibility.

Once overall rate level is determined, allocation to various risk classifications is normally done. The classifications themselves must be determined first. Once a satisfactory classification scheme has been developed, actuarial methodologies for spreading loss costs must be determined.

The introduction of claims made coverages in recent years has added to the challenges faced by actuaries in rate and reserve analysis as well as in the other areas in which they work. By breaking the standard occurrence coverage into seemingly endless potential components, the implementation of claim-made policies has provided additional challenges in the areas of credibility, discounting, data structure, policy limits, risk classification, expense and profit loads, trend analysis, payment and reporting patterns, and reinsurance analysis.

Actuarial work, then, like the work of other applied scientists, encompasses a myriad of processes, elements, and interrelationships to which mathematical modeling can provide only rough approximations. The challenge is to improve on the mathematical foundations of actuarial science while simultaneously developing optimal approaches to real world problems. Appendix E provides a glossary of insurance terms which relate to aspects of the actuary's role in the business world.

CONSIDERATIONS IN DEVELOPING AN ACTUARIAL PARADIGM

The first step in determining whether actuarial science is a body of knowledge for which there is or can be developed a core model or paradigm from which research can emanate is to define the group of problems this model will attempt to address. In the area of risk theory actuaries have developed a model which can guide research related to loss estimation in a way consistent with Kuhn's notion of the concept of a paradigm. It is in adapting that model to the practical situations which actuaries face that the question of the nature of the body of knowledge known as actuarial science arises.

In his book, Kuhn indicated that "by focusing attention upon a small range of relatively esoteric problems, the paradigm forces scientists to investigate some part of nature in a detail and depth that would otherwise be unimaginable." If actuaries could function as pure scientists, they could take well defined models from probability theory, interest theory, utility theory, and number theory, and develop core models for the resolution of well defined problems. Actuarial science as currently defined, however, is an applied discipline, and results therefrom must resolve real, as opposed to idealized, problems.

The real problems to be solved fit the theoretical models available to actuaries only loosely. Once the cumulative loss distribution for a portfolio of exposures has been determined from risk theory, for example, fitting that portfolio to an actual portfolio which exists in the dynamic insurance world becomes difficult. Even if the portfolio presented a fairly good fit at a point in time, adjusting it to conditions expected to occur at a future point in time presents serious problems.

Even if actuarial paradigms can be developed for several of the abstract problems facing actuaries, the degree of certainty produced from those paradigms should not be overestimated. We have seen that even in that most abstract of disciplines (mathematics) for which truths were at one time felt secure no certainty exists. As we move away from mathematics toward the body of knowledge underlying actuarial science, the uncertainty levels increase.

Godel proved that the mathematics which had developed up to his time was fatally flawed. The concept of a universally accepted, infallible body of reason was found to be an illusion. Mathematicians have retreated from the search for the solution to large problems to narrow topics where proofs are felt to be safe.

Physics and the other sciences have undergone similar revolutions. Quantum mechanics has replaced classical physics in explaining the phenomenon of the world in which we live. The mechanistic determination of Newtonian physics has given way to the probabilistic perspective of quantum mechanics. Heisenberg's uncertainty principle shows that we cannot know both the position and the momentum of a particle. The metaphysical implication is that the world philosophers have been trying to understand and explain is not fully comprehensible.

The epistemological quandary faced by physicists is magnified for persons dealing with less abstract problems. The complementarity which haunts physicists is overwhelming for social scientists. Shakespeare's plays often poignantly illustrate the complementarity of values faced by ordinary and extraordinary people in their everyday existence.

Actuarial science ultimately must fit into some larger scheme of things. As an applied social science, actuarial science is involved with a very diverse group of phenomena. To adequately address the problems it faces, the profession must have a perspective which encompasses those phenomena. Appendix F provides some statistical data which illustrates the range of topics actuaries handle.

CONCLUSION

This paper was conceived from the notion that, as it approaches its 100th anniversary, the actuarial profession in North America is faced with profound choices. This is particularly the case in the area of liability insurance. The tools available to the actuary and the results expected from the actuary may not be ideally matched at any point in time, but the challenges provided by the continuing need to model complex phenomena produce great opportunities for the profession.

This paper has surveyed a number of subjects which are felt to provide perspective on the body of knowledge from which actuaries draw. Mathematics has been emphasized as the area from which the core of the actuarial paradigm is expected to be found. Other subjects, such as economics and law, are surveyed because they impact on the structure of the modeling which must be done and because they provide perspective on the use to which actuarial analysis is put.

The nature of a synopsis presented in a paper such as this is such that none of the bodies of knowledge which are surveyed are rigorously treated. The need addressed in this paper is not the need for rigorous treatment of a narrow area of actuarial science, but the need for a consolidation of existing knowledge.

As quantifiers of significant social phenomena, actuaries face many of the issues facing any applied mathematician. Because physicists have developed very complex mathematical models by which to evaluate the phenomena in their domain, some attention was paid to some of the more significant models developed in the physical sciences. Those models may provide actuaries with insights into more refined theoretical approaches which can be developed with respect to the type of problems they face.

It is suggested that for actuarial science to be a body of knowledge as described by Kuhn, the scope of problems to be resolved by its paradigm must be defined. Loss estimation, a key aspect of the actuary's work, is best viewed as mathematically based. Similarly, interest theory and utility theory, which underlie certain actuarial problems, have a mathematical foundation. To the extent legal, regulatory, operational, accounting, economic, or other phenomena interact with the subject matter of actuarial science, they can be integrated without changing the basic paradigms.

No claim to originality is made. The intent is to provide a glimpse of the universe of information which an actuary must sift through to determine the nature of his/her subject. Hopefully, it provides many of the elements which will be needed in developing a paradigm for our profession.

APPENDIX A: SELECTED GLOSSARY OF LEGAL TERMS OF INTEREST TO ACTUARIES

a fortiori	All the more; for a stronger reason.
Abatement	A making less; a suspension of action.
Accretion	Addition to property by natural causes.
Additur	A remedy by which a new trial is denied if the defendant agrees to an increased judgment.
Adhesion Contract	A standard form prepared by one party.
Adverse Possession	The holding of land under a claim of right inconsistent with that of the true owner.
Aleatory Contract	An agreement in which performance of a party is dependent upon the occurrence of an uncertain event.
Assumpsit	An action for damage for breach of contract.
Bailment	Agreement created by delivery of property by the owner.
Casualty	An unforeseen circumstance occasioning loss.
Certiorari	An action whereby a cause is moved to a superior court.
Collateral	Property which is subject to a security interest.
Color of Title	Apparent ownership of land based upon a written instrument.

Consideration	The matter of inducement of a contract.
Contingency	An unforeseeable, but possible event.
Curtesy	The estate of the husband in wife's fee simple or fee tail estates after her death.
Defeasance	A condition for the determination of an estate or interest based upon a performance or occurrence.
Detinue	A personal action for recovery of goods or value.
Discovery	Method by which parties to a lawsuit gather facts.
Easement	A right which the owner of real property has with regard to another property.
Equality	The condition of persons when none has an unfair advantage.
Estate	The condition and circumstance in which a person stands with regard to those around him and her property.
Estate in Fee Simple	An estate free of restrictions, conditions, or limitations.
Estate in Reversion	An estate remaining in the grantor.
Estoppel	A declaration by which a person is precluded from bringing controverting evidence.
Executory	Not completed.

Fair Cash Value	The instrinsic value of shares of stock in a corporation.
Fair Market Value	The amount an article would bring if sold in the market under normal conditions.
Feoffment	The transfer of possession of a freehold estate.
Fiduciary	One who has a duty to act in another's best interest.
Gift Intervivos	An irrevocable gift from a live donor.
Hornbook	A one-volume work containing elementary principles of an area of law.
Hypothecation	The deposit of securities to secure repayment of a loan.
Implead	To bring a new party into a lawsuit.
Insurable Interest	A concern in the subject of insurance entitling the possessor to obtain insurance.
Insurance	The act of providing against a possible loss by entering into a contract with a party willing to make good the loss should it occur.
Interlocutory	Incident to a suit still pending.
Interpleader	A procedure whereby persons with claims against another may be joined as parties to a lawsuit.
Intestacy	The condition of a person who dies without a valid will.

Joint and Several Liability	A situation where one or more liable parties, or all of them, may be sued by the plaintiff.
Jurisdiction	The authority of a court to hear and decide an action or lawsuit.
Leading Question	An inquiry which suggests an answer to a witness.
Letter of Credit	An irrevocable credit issued by a financial institution.
Liability	A present or potential duty.
Livery of Seisen	An act by which real property is transferred.
Malpractice	Improper performance by a professional of duties incumbent on account of professional status.
Mortality Tables	Statistical charts showing life expectancy.
Negligence	Failure to use ordinary care or to follow the standard established by law under the circumstances.
Nuisance	Activities which harm others under the facts of the suit.
Pleadings	Opposing written statements of parties to a lawsuit.
Premium	The consideration paid for issuance of an insurance policy.

Product Liability	Law dealing with responsibility of manufacturers to buyers concerning quality of merchandise and consequences resulting from substandard quality.
Remainder	Residue of interest in real property.
Remittitur	Disposal of a lawsuit by ordering a new trial unless a party agrees to damages different from that from trial court.
Replevin	Lawsuit to recover possession of specific chattels.
Reversion	Residue of estate left in grantor.
Stare Decisis	To follow precedent.
Subrogation	The substitution of a person to the rights of another concerning a claim the former has paid.
Surety	A person who makes himself responsible for the obligations of another.
Title	A valid claim of right.
Tort	A legally recognized private wrong other than a breach of contract.
Trover	A special form of trespass.
Usury	Interest in excess of the legal maximum.
Voir Dire	A preliminary examination of a witness to determine competence.
Writ	A written court order or judicial process.

APPENDIX B: SELECTED GLOSSARY OF ECONOMIC TERMS

Adjustment Lag	The time taken for a variable to adjust to charges in its determinants.
Agglomeration Economics	Cost savings resulting from enterprises locating near one another.
Aggregation Problem	The problem of deriving predictable macroeconomic behavior from that of underlying units.
Automatic Stabilizers	Relationships which reduce the volatility of cyclical fluctuations in the economy.
Average Cost Pricing	A pricing rule whereby the firm adds a mark-up onto average variable costs to cover average total costs including a fair net profit margin.
Average Revenue Product	Average product of an input factor multiplied by average revenue (price).
Averch-Johnson Effect	Profit maximizing response of a regulated firm faced with a determined rate of return.
Axioms of Preference	Axioms individuals are assumed to obey in consumer demand theory, including transitivity, completeness, selection, dominance, continuity, and convexity.

Balance Budget Multiplier	The ratio of the change in real income to a change in government expenditure which is matched by a change in tax revenue.
Balanced Growth	A dynamic condition where all real variables are growing at the same rate.
Cobb Douglas Production Function	A homogeneous production function.
Collective Good	A commodity with the characteristic of nonexcludability.
Comparative Advantage	The idea behind the notion that specialization and free trade lead to a higher level of general welfare.
Comparative Statics	The comparison of equilibrium positions.
Coefficient of Concentration	A statistical measure of the degree to which an economic activity is concentrated.
CES Production Function	A linearly homogeneous production function.
Consumer Equilibrium	The point at which the consumer maximizes utility.
Consumer's Surplus	A measure of consumer benefit resulting from the ability to buy a good at the market price.
Consumption Function	The relationship between consumption and income.

Contract Curve	The locus of points where the marginal rate of substitution between goods is the same for the individuals.
Cross Elasticity of Demand	The responsiveness of the quantity demanded of one good to a change in the price of another.
Cyclical Unemployment	Short run unemployment caused by deficient demand.
Devaluation	A fall in the exchange rate of a currency.
Distributive Judgment	A value judgment as to appropriate distributional effects of economic policies.
Economics of Scale	Reduction in long-run average cost resulting from expanded output.
Elasticity	A measure of the percentage change in the value of one variable with respect to a change in another.
Externalities	Side effects of production or consumption for which the market is ineffective in providing for.
Fair Rate of Return	That which allows for continued capital attraction.
Fiscal Policy	The use of taxation and government expenditure to affect the level of economic activity.
General Equilibrium	A condition whereby all markets in an economy are simultaneously in equilibrium.

Gini Coefficient	A measure of distribution equality.
Harrod-Domar Growth Model	A one sector growth model.
Herfindahl Index	A measure of market concentration.
Indifference Curve	The locus of combinations of goods to which the individual is indifferent.
Input-Output	A method of analysis in which the economy is represented by a set of linear production functions.
IS-LM Diagram	The diagram detailing simultaneous equilibrium in money and product markets.
Isoquant	Combinations of inputs which produce a particular output.
Iso Profit Curves	The focus of combinations of independent variables of the profit function which yield equal profit.
Laffer Curve	An illustration of the thesis that there exists an optimal tax rate for maximizing government tax revenue.
Laspeyres Price Index	A base year weighted price index.
Lerner Index	An indicator of monopoly power.

Leverage	The relationship between long-term debt and capital.
Loss Function	A disutility function to be minimized.
Marginal Efficiency of Capital	The rate of discount at which the present value of net returns from an asset equal its cost.
Marginal Rate of Substitution	The amount of one good required to compensate a consumer for the loss of another.
Opportunity Cost	The value of foregone alternatives.
Paasche Price Index	A current year weighted price index.
Pareto Optimum	A condition in the economy where to make someone better off someone else must be made worse off.
Permanent Income Hypothesis	The idea that consumption depends on lifetime income including asset depletion.
Wealth Effect	Increase in aggregate expenditure due to fall in prices and interest rates.
Welfare Economics	The normative area of economics.

APPENDIX C: SELECT GLOSSARY OF MATHEMATICAL TERMS

Abelian Group	A group in which the binary operation obeys the commutative law.
Accumulation Point	The limit of a sequence of points of a set.
Algorithm	A standardized procedure for the solution of a particular type of problem.
Applied Mathematics	The area of mathematics concerned with solution of physical problems by supporting empiricism with logically deduced solutions.
Boolean Algebra	A ring with an identity element which reproduces each element.
Calculus of Variations	The study involving the finding of a function for which a given expression attains an extreme value.
Cauchy's Test	Ratio test for convergence or divergence of an infinite series.
Center of Mass	The point at which the entire mass may be concentrated in theory to produce the same effect as when originally distributed.
Cramer's Rule	A rule using determinant notation for the solution of simultaneous equations.

DioPhantine Analysis	The determination of integral solutions of certain algebraic equations.
Equilibrium	The state of a body when the resultant of all forces acting on it is zero.
Group	A mathematical system consisting of a set of elements subject to a binary operation satisfying certain axioms.
Hyperbolic Space	A non-Euclidean space based on the postulate that through a point external to a given line several lines parallel to the given line exist.
Integral Domain	A set of elements subject to two binary operations satisfying certain axioms.
Invariant Property	That property of a function which remains unaltered under a transformation.
Isomorphism	Two mathematical systems in one-to-one correspondence.
Klein Bottle	A bottle with a single surface and properties of a three-dimensional strip.
Linear Algebra	The study of the algebraic properties of vector spaces.
Linear Transformation	A transformation produced by the use of algebraic linear equations.

Manifold	A class with subclasses.
Mathematical Programming	The process of finding an optimum value of a function where the variables are subject to constraints which often take the form of equations.
Mathematics	A system of organized thinking of an analytic and synthetic nature.
Measure	The size of something expressed in standard units.
Neighborhood	Part of a line which contains a given point.
Number Field	A set of numbers which is arithmetically closed.
Number Theory	The study of integers and the relationships between them.
Order of Group	The number of elements in a finite group.
Orthogonal	Right-angled.
Paradox	An apparent contradiction between two reasonable conclusions.
Parametric Equations	Equations in which the parameters are expressed as functions.
Postulate	As assumption upon which a logical argument is based.

Probability	A numerical measure of the likeliness of an event occurring.
Quadrature	The process by which a square equal in area to that of a given surface is found.
Range	The set of all possible values.
Relativity	The theory that all natural processes take place in a four dimensional space-time continuum.
Resolution	The process of determining two vectors equivalent to a given one.
Resultant	The vector quantity equivalent to two or more vector qualities.
Sampling	The process of obtaining a sample of population for the purpose of inference.
Set Theory	The Boolean algebra of sets or classes.
Simplex	A topological terms for the simplest geometric figure in a space.
Space	A set of elements which satisfy a set of postulates.
Standard Form	An accepted way of writing an expression.
Statistics	The study of the methodology of collecting and analyzing data.

Surd	A numerical expression containing an irrational number.
Surface	A set of points forming a two-dimensional space.
Symmetric Relation	A relation which is identical to its own inverse.
System	A set of elements which have a common property.
Theory	The principles involved in the development of a central concept.
Topological Property	A property of a geometric figure which remains invariant under a topological transformation.
Transcendental	Nonalgebraic.
Transfinite Number	An infinite cardinal or ordinal number.
Vector Quantity	A quantity with magnitude and direction.
Wave Length	The distance between successive points on a wave which represent the same phase of disturbance.

APPENDIX D: SELECTED GLOSSARY OF ECONOMETRIC TERMS

Analysis of Variance	The breakdown of total variation in a dependent variable into the proportions accounted for by variation in explanatory variables.
Arima Forecasts	Forecasts generated from models which capture time series characteristics of a variable by relating current to lagged values.
Asymptotic Distribution	The probability distribution to which a statistic tends as the sample size increases indefinitely.
Autoregression	The regression of a variable on its own lagged values.
Bayesian Techniques	Methods of statistical analysis in which prior information is formally combined with sample data to produce estimates or test hypotheses.
BLUE	The best linear unbiased estimator has the smallest variance of all unbiased linear estimators.
Binary Variable	A variable which can take two values and is normally used to provide for qualitative factors.
Blus Residuals	Best linear unbiased estimators with a scalar covariance.
Central Limit Theorem	A theorem which states that the sum and mean of a set of random variables will follow a normal distribution if the sample is large enough.

Coefficient Determination	A statistic which summarizes the explanatory power of an equation.
Coefficient of Variation	A measure of the degree to which a variable is distributed around its mean value.
Contingency Table	A device for measuring the degree of association between variables.
Correlation	The degree to which two variables are linearly related.
Correlogram	A plot of the correlation coefficient between the current value of a variable and lagged values agains the length of the lag.
Covariance	A measure of the degree to which two variables are linearly related.
Critical Value	The value in a probability distribution above or below which a specified percentage of probability lies.
Cross Sectional Analysis	Analysis of a set of observations taken at a point in time.
Degrees of Freedom	The number of pieces of information which can vary independently of each other.

Deseasonalization	The process of removing seasonal influences from data.
Detrending	The process by which a time trend is removed from data.
Distributed Lags	The result of formulating mathematical relationships for coefficients of lagged values.
Disturbance Term	The error term in a regression equation.
Dummy Variable	A binary variable which accounts for shifts in econometric relationships.
Durbin-Watson Statistic	A statistic which diagnoses serial correlation in error terms in a regression.
Estimation	The quantitative determination of the parameters of economic models through statistical manipulation of data.
Fourier Analysis	A method by which time series data can be transformed to the frequency domain.
Full Information Maximum Likelihood	A technique for estimating systems of simultaneous equations.
General Linear Model	A model which specifies the dependent variable as a linear function of independent variables.

Heteroscedasticity	A situation where the variance of the error term in a regression equation is not constant between observations.
Identification Problem	A condition arising in the estimation of parameters of simultaneous equations where the equation being estimated cannot be uniquely identified.
Indirect Least Squares	A procedure for estimating parameters of simultaneous equations which avoids bias.
Joint Probability Distributions	Probability distributions which give the probability with which two or more variables take certain values.
Kalman Filtering	An optimal method of predicting endogenous variables and updating parameter estimates.
Klein-Goldberger Model	An econometric model of the economy.
Lagrangian Technique	A method for solving constrained optimization problems.
Linear Programming	A technique for formulation and analysis of constrained optimization of problems.
Logit Analysis	A linear probability model where values of the dependent variable are constrained within probability limits.

Markov Process	A process which relates the current value of a variable to previous values and an error term.
Maximum Likelihood	An estimation technique involving maximization of the likelihood function of the observations with respect to the parameters being estimated.
Moments	Summary statistics which characterize a distribution.
Multicollinearity	A condition in which two or more explanatory variables in a regression are highly correlated.
Orthogonal	A condition arising from uncorrelated variables.
Parameter	A quantity which remains constant in a given context.
Random Walk	A time series model in which the current value of a variable is equal to the most recent value plus a random element.
Rank Correlation	A measure of the rank relationship between variables.
Recursive Model	Current and lagged values of one variable determine the value of another values of which determines the value of the former.
Residual	The difference between an actual data point and its estimated value.

Simulation	The generation of a range of estimates based on alternative assumptions.
Simultaneous Equation Bias	A bias resulting from feedback effects between equations in the model.
Singular Matrix	A matrix whose determinant is zero because of linear dependence.
Spectral Analysis	A technique by which cyclical properties of a variable can be established from time series data.
Stochastic Process	A time related series subject to random variation.
Systems Estimator	An estimator used to gain estimates of all the parameters in a simultaneous equation system.
Two Stage Least Squares	A procedure for estimation of structural form parameters of simultaneous equation systems.
Von Neumann Ratio	A test statistic for detecting serial correlation of residuals in regression analysis.

APPENDIX E: SELECTED GLOSSARY OF ACTUARIALLY ORIENTED INSURANCE TERMS

Actuary	A person concerned with the application of mathematical theories to the practical problems of insurance related fields.
Classification	The underwriting or rating group into which a particular risk is placed.
Combined Ratio	The sum of the loss and expense ratios.
Deductible	The amount of loss paid by the insured.
Deficiency Reserve	Supplemental reserve required when the gross premium for a class of life insurance policies is less than the net premium.
Defined Contribution Plan	A pension plan where an individual account is maintained for each participant.
Deviation Rate	A premium change that differs from the scheduled rate.
Discrimination	Treating a given class of risks differently from other similar risks.
Earned Premium	That part of a policy premium for protection already provided.

Excess Limit	A coverage limit above the basic one established.
Experience Rating	Ratemaking based on specific experience of the risk.
Facultative Reinsurance	A type of reinsurance available on a per risk basis.
Governing Classification	The main operation of any employer for workers compensation insurance.
Graduated Life Table	An actuarial table derived from a mortality experience curve.
Gross Net Premiums	Gross premiums less return premiums.
Insurance	A contractual relationship whereby one party for consideration agrees to reimburse another for loss from designated contingencies.
Lapse Ratio	The ratio of policies surrendered to those in force.
Law of Large Numbers	Theory of probability that is the basis of insurance.
Liability Insurance	Coverage whereby the insured is protected against claims from other parties.
Loss Conversion Factor	A factor applied to losses to provide for expenses.
Loss Reserve	The portion of assets available to pay probable claims.

Manual Rate	Cost of a unit of insurance.
Maximum Probable Loss	The longest loss expected for a given risk.
Merit Rating	A system for measuring differences of a specific risk from a standard risk.
Mortality Table	A statistical table showing the death rate at each age.
Net Retention	The amount of insurance a ceding company keeps.
Portfolio Reinsurance	A type of reinsurance whereby the reinsurer assumes all obligations of a certain type.
Prospective Reserve	A reserve base on the present value of future claims less the present value of future net premiums.
Pure Loss Cost	The ratio of reinsurance losses incurred to the ceding company's subject premium.
Pure Premium	The amount of money needed to pay the loss portion of the insurance coverage.
Quota Share Treaty	A reinsurance arrangement whereby each company accepts a stated proportion of premium and losses for covered risks.
Reinsurance	An agreement between insurance companies to spread the risk of loss.

Retrocession	A cession of reinsurance by a reinsurer.
Retrospective Rating	A rating method by which the final premium is based on experience of the risk for the covered period.
Self-Insurance	The systematic provision of a fund to cover losses the entity may suffer.
Standard Premium	Premium before adjustment for discounts or retrospective adjustments.
Tontine	A reverse form of life insurance.
Tort	A wrongful act not involving a breach of contract.

APPENDIX F: SELECTED STATISTICS OF INTEREST TO ACTUARIES

I. Federal Government Receipts and Expenditures

Year	Receipts	Expenditures	Deficit
1977	$357 Billion	$ 402 Billion	$ 45 Billion
1987	854 Billion	1,002 Billion	148 Billion

II. Federal Reserve Board Discount Rates; Annual CPI Change

Date	Discount Rate	Year	CPI Change
May 5, 1981	14	1980	13.5%
September 1, 1988	6.5	1987	3.6

III. Unemployment Insurance Data for 1987

Claims	Benefits	Average Weekly Benefit
$17 million	$14 billion	$140

IV. Productivity and Unit Labor Costs

Year	Output Per Hour	Unit Labor Costs
1975	92.9	91.7
1987	132.4	139.7

V. Foreign Trade

Year	Exports	Imports	Deficit
1980	$220 Billion	$241 Billion	$ 21 Billion
1987	253 Billion	424 Billion	171 Billion

VI. Pollution Abatement and Control Expenditures

Year	Amount
1975	$30 Billion
1985	$74 Billion

VII. Probabilities

Full House (Poker)	12 (Dice)	13-0-0-0 (Bridge)	Heads (Fair Coin)
1/694	1/36	1/159 Billion	1/2

VIII. World Population; U.S. Population

Year	World Population	U.S. Population
1960	3 Billion	179 Million
1987	5 Billion	243 Million

IX. Social Security Statistics

	OASI Trust Fund		DI Trust Fund		HI Trust Fund	
Year	Receipts	Expenditures	Receipts	Expenditures	Receipts	Expenditures
1970	$ 32 Billion	$ 30 Billion	$ 5 Billion	$ 3 Billion	$ 6 Billion	$ 5 Billion
1987	211 Billion	188 Billion	20 Billion	21 Billion	63 Billion	51 Billion

X. Births and Deaths in the U.S.

Year	Births	Total Deaths	Motor Vehicle Deaths	Home Accident Deaths
1970	3.7 Million	1.9 Million	54,633	27,000
1987	3.8 Million	2.1 Million	48,700	20,500

XI. Life Insurance Statistics

Year	Life Insurance Purchases	Insurance In Force	Life Expectancy
1970	$ 193 Billion	$1,402 Billion	70.8
1986	1,309 Billion	6,720 Billion	74.9

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EXCESS LOSS DISTRIBUTIONS OVER AN UNDERLYING ANNUAL AGGREGATE

Joseph Schumi

Excess Loss Distributions over an Underlying Annual Aggregate

by Joseph R. Schumi

Abstract

The purpose of the paper is to develop a method of calculating the aggregate loss distribution for a policy covering excess claims over occurrence limit plus claims arising from the primary losses over an underlying annual aggregate.

Usually, when working with losses from more than one source you would determine the aggregate distributions of each component and convolute the result to get the overall distribution. The problem is that the two distributions - the excess over occurrence limits and the excess over the retained annual aggregate are not independent.

Using results developed in my earlier note, I develop the conditional probability distribution of the number of non-excess claims based on the number of excess claims. It is argued that, in the probability subspace defined by a particular number of excess claims, the random variables describing the distributions of excess and retained losses are independent and thus so are the distributions of the excess losses and the excess of the retained losses over an annual aggregate. Thus the distribution of their sum can be determined by convoluting the respective distributions.

The conditional results for zero, one, two etc, excess claims are then summed using the probabilities of that number of excess claims.

Finally I outline a computer implementation of the process. I have created a simple demonstration version in Turbo Pascal for the Macintosh. It is limited in that I used a simple loss distribution to limit the number of points required for the calculations.

While not developed explicitly in this paper, this approach could also be used to determine increased limits factors as a function of the expected number of claims when an underlying aggregate is involved.

Introduction

In many situations we are asked to analyze the loss distribution of an excess policy which includes coverage for retained losses that exceed some aggregate accumulation. While it is possible to determine the distributions of both the retained losses and the excess losses, it is not readily apparent how to combine the distributions because the distributions are not independent, since a claim could contribute to each distribution.

The procedure described in this note decomposes the problem in such a way as to make the two distributions independent.

The key to this decomposition is to determine the excess and retained loss distributions for a given and fixed number of excess claims. The first step is to determine the claim frequency distribution of non-excess claims conditioned on the number of excess claims. This problem is solved under the assumption of a negative binomial claim distribution using Bayes Theorem.

The next step is to determine the aggregate loss distributions for the excess and retained losses. For a given number of excess claims, say N_x, the distribution of excess losses is given by the N_x-fold convolution of the excess loss distribution, which will be denoted $f_{xL}(X)^{N_x}$. The distribution of losses below the underlying per occurrence loss limit, L, is given by the Compound Distribution of the non-excess claims plus N_xL, the amount contributed by the excess claims. From this, the distribution of underlying losses over an annual aggregate, Agg, can be determined. I will call this $f_{Agg}(X|N_x)$.

The key observation is that in the subspace defined by the number of excess claims, the aggregate distribution of the excess claims and the aggregate distribution of the retained losses are independent.

The argument is as follows: The excess distribution is determined by the N_x-fold convolution of the excess claim distribution. The distribution of the aggregate retained losses is made up of two components. The first is the aggregate distribution of the non-excess claims. Under the usual assumptions the individual claims are independent, thus the size of the non-excess claims is not influenced by the size of the excess claims and vise versa. The other component is the retained portion of the excess claims. In the entire probability space, this isn't independent of the number of excess claims, but for a given number of excess claims it is a fixed amount. Thus, in each subspace defined by the number of excess claims, the distribution of the retained amounts is independent of the distribution of amount of the excess losses.

Finally, since the excess losses and the retained losses are independent, so are the excess losses and the excess of the retained losses over the annual aggregate.

This is discussed in more detail in Appendix A.

Thus for any number of excess claims, say N_x, the total loss distribution is given by the convolution of $f_{xL}(X)^{N_x}$ and $f_{Agg}(X|N_x)$.

Finally the total distribution, $f_{xL,Agg}(X)$ is obtained by summing the conditional distributions $f_{xL}(X)^{N_x}*f_{Agg}(X|N_x)$ weighted by the probability distribution of N_x, that is

$$f_{xL,Agg}(X) = \sum f_{xL}(X)^{k*}f_{Agg}(X|k)\ P(N_x = k).$$

The Conditional Distribution of the Number of Claims

Let π stand for the probability that a claim is an excess claim. Recall that $({}_nC_k)$ stands for the binomial coefficient $(n)!/((n-k)!(k)!)$

If the basic claim process is Negative Binomial with parameters h and n, then for $k \geq 0$ we have that the probability of k claims is given by

$$P_k = ({}_{h+k-1}C_k)\ [h/(n+h)]^h\ [n/(n+h)]^k .$$

In this case, the excess claims are also a Negative Binomial with parameters h and πn and again for $k \geq 0$, the probability of k claims is given by

$$P_k = ({}_{h+k-1}C_k)\ [h/(\pi n+h)]^h\ [\pi n/(\pi n+h)]^k.$$

Consider the probability that the total number of claims is N and that the number of excess claims is N_x, call this $P(N_x \cap N)$. On the one hand,

$P(N_x \cap N) =$ P(nr of excess claims $= N_x|$ nr of claims $= N$)P(nr of claims $= N$)

or in algebraic terms

$$P(N_x \cap N) = {}_NC_{N_x}\ \pi^{N_x}\ (1-\pi)^{(N-N_x)}({}_{h+N-1}C_N)\ [h/(n+h)]^h\ [n/(n+h)]^N.$$

On the other hand,

$P(N_x \cap N) =$ P(nr of claims $= N|$nr of excess claims $= N_x$)P(nr of excess claims$= N_x$)

$$P(N_x \cap N) = P(N|N_x)\ ({}_{h+N_x-1}C_{N_x})\ [h/(\pi n+h)]^h\ [\pi n/(\pi n+h)]^{N_x}.$$

Combining the equations we have

$$P(N|N_x) = \frac{{}_NC_{N_x}\ \pi^{N_x}\ (1-\pi)^{(N-N_x)}\ ({}_{h+N-1}C_N)\ [h/(n+h)]^h\ [n/(n+h)]^N}{({}_{h+N_x-1}C_{N_x})\ [h/(\pi n+h)]^h\ [\pi n/(\pi n+h)]^{N_x}}.$$

Let $\eta = N-N_x$, the number of non-excess claims.

Considering the combinatorial terms, we have

$$\frac{{}_{N}C_{N_x}\,({}_{h+N-1}C_{N})}{({}_{h+N_x-1}C_{N_x})}$$

which equals

$$\frac{\dfrac{(N!)}{(N-N_x)!(N_x)!}\cdot\dfrac{(h+N-1)!}{(h-1)!(N)!}}{\dfrac{(h+N_x-1)!}{(h-1)!(N_x)!}}$$

and by multiplying through we have

$$\frac{(N!)(h+N-1)!\;\;(h-1)!(N_x)!}{(N-N_x)!(N_x)!\;\;(h-1)!(N)!(h+N_x-1)!}$$

which reduces to

$$\frac{(h+N-1)!}{(N-N_x)!(h+N_x-1)!}.$$

Which can be rewritten as

$$\frac{(h+N_x+\eta-1)!}{(h+N_x-1)!(\eta)!}.$$

and which equals $({}_{h+N_x+\eta-1}C_{\eta})$.

The rest of the equation is

$$\frac{\pi^{N_x}\,(1-\pi)^{(N-N_x)}\,[h/(n+h)]^{h}\,[n/(n+h)]^{N}}{[h/(\pi n+h)]^{h}\,[\pi n/(\pi n+h)]^{N_x}}$$

Rearranging terms

$$\frac{\pi^{N_x}\;\;[n/(n+h)]^{N_x}\;\;[h/(n+h)]^{h}\;\;[n/(n+h)]^{(N-N_x)}(1-\pi)^{(N-N_x)}}{[\pi n/(\pi n+h)]^{N_x}\;\;[h/(\pi n+h)]^{h}}$$

Summarizing

$$[(\pi n+h)/(n+h)]^{N_x} \quad [(\pi n+h)/(n+h)]^{h} \quad [n(1-\pi)/(n+h)]^{(N-N_x)}$$

And finally

$$[(\pi n+h)/(n+h)]^{(h+N_x)} [n(1-\pi)/(n+h)]^{(N-N_x)}$$

or

$$[(\pi n+h)/(n+h)]^{(h+N_x)} [n(1-\pi)/(n+h)]^{\eta}$$

Now $(\pi n+h)/(n+h)$ can be expressed as $(h+N_x)/(h+N_x+X)$ where $X = (h+N_x)\, n(1-\pi)/(h+\pi n)$. Thus $n(1-\pi)/(n+h) = X/(h+N_x+X)$

and the full expression becomes

$$({}_{h+N_x+\eta-1}C_{\eta})[(h+N_x)/(h+N_x+X)]^{(h+N_x)}[X/(h+N_x+X)]^{\eta}.$$

This is the conditional probability that the total number of claims is N and also the conditional probability that the number of non-excess claims is $\eta = N-N_x$.

Thus the distribution of non-excess claims is a Negative Binomial with $h' = h+N_x$ and $n' = (h+N_x)\, n(1-\pi)/(h+\pi n)$.

Recall that this is a distribution with mean n', i.e. $(h+N_x)\, n(1-\pi)/(h+\pi n)$. In particular, if $N_x = \pi n$, the expected number of excess claims, then the expected number of non-excess claims is $n(1-\pi)$ and the total expected number of claims is n.

The Conditional Loss Distributions

Assuming that we know the loss severity distribution we can now determine the conditional excess and primary aggregate loss distributions.

For a given number of excess claims N_x, the aggregate distribution of the excess claims is the N_x-fold convolution of the loss severity distribution truncated below at the loss retention L. While this does not generally have a closed-form solution, it can be easily determined numerically for for any given N_x.

For the primary layer, the loss distribution of the non-excess claims is a compound distribution with claim frequency distribution as determined above and with a conditional loss severity distribution derived from the original severity distribution restricted to losses up to the occurrence limit. This distribution does not have a point mass at its upper limit for the losses over the occurrence retention. Finally, this distribution is shifted to the right by $N_x L$ to account for the retained portion of the excess losses.

There are several numerical tools available to determine the aggregate distribution, the choice of which depends on the parameters of the frequency and severity.

Now, since under the conditional assumption of a known fixed number of excess claims the distribution of the excess losses and the retained losses are independent, the distribution of the sum of the retained and excess losses is given by the convolution of their respective distributions.

The Total Loss Distribution

Finally having determined $f_{xL,Agg}(X|N_x)$ [$= f_{xL}(X)^{N_x} * f_{Agg}(X|N_x)$] for each N_x, we can determine $f_{xL,Agg}(X)$ by multiplying each $f_{xL,Agg}(X|N_x)$ by $P(N_x)$, where $P(N_x)$ is the probability that the number of excess claims is N_x. The number of terms to be calculated is determined by specifying a stopping probability parameter. The stopping probability is compared to

$F_{xL,Agg}(X) = \sum_{\text{all } X} f_{xL,Agg}(X)$ after each step. If $F_{xL,Agg}(X)$ is less than the stopping probability, the process is repeated for N_x+1.

Implementation

I created a simple program to carry out these calculations. In the calculations in this program, I based the severity distribution on a simple Pareto distribution of the form $(B^q/(q-1)/(X+B)^{q+1}$, for $X \geq 0$ and computed the aggregate distributions using the method I described in my previous note.

In general, the various distributions could be determined using any of tools at one's disposal; all that is needed is a device to obtain the aggregate distributions of the primary and excess losses, translate it and convolute it. In particular, if the expected number of primary claims is large, the primary aggregate distribution would probably best be obtained using one of the Fourier transform methods.

In the example, the starting assumptions were that we expected 10 claims with a variance of 11, that the primary claims represented the first five points of the severity distribution and that the primary aggregate retention was ten loss units. This was set up in the parameter file read by the program. The parameters of the frequency distributions for the excess claims and non-excess claims for a given number of excess claims were derived as described above. In particular, h = 100 and n = 10 for the primary distribution and h = 100 and n = 0.688 for the excess distribution.

In the main part of the program I loop on the number of excess claims.

I use the method of calculating aggregate distributions discussed in my previous note to determine the aggregate non-excess losses using the normalized probabilities for first five points from the loss severity distribution as the non-excess severity distribution. The non-excess distribution is translated by $500N_x$ to include the retained part of the excess claims in the aggregate primary loss distribution.

Finally, the excess of the annual aggregate retention distribution is determined by adding the probabilities of all of the points below the aggregate retention and assigning this probability to zero and assigning the probabilities of the points above the aggregate retention to the corresponding point shifted downward by the aggregate retention.

The excess convolutions are performed using the normalized probabilities of the upper 15 points of the original severity distribution.

The two distributions are then convoluted to yield the total distribution for the current number of excess claims.

This distribution is then added to the previous distributions by weighting the current distribution by the probability of the current number of excess claims.

$$f_{XL,Agg}(X) \text{ [this step]} = f_{XL,Agg}(X) \text{ [last step]} + f_{XL,Agg}(X|k)\ P(N_X = k).$$

The cumulative distribution obtained from this distribution is evaluated at the maximum aggregate loss value, X_{max}. This value is compared to the stopping value. That is, if

$$F(X_{max}) \text{ [this step]} < \text{aggregate stopping parameter}$$

then increase the value of the excess claim count by one and repeat the calculations. In some situations the memory constraints of a PC may be such that it is not possible to make X_{max} large enough for $F(X_{max})$ to exceed the stopping value.

In each step, the aggregate distribution calculations have been stopped when the probability first exceeds a given stopping probability parameter. The overall calculated aggregate probability cannot exceed this value by more than a slight amount. A slight amount because each step will exceed it by some amount. Thus, in general, the overall stopping value must be less than or equal to the value used for the individual steps.

Another possible problem is that as the number of excess claims increases, the convoluted excess distribution will require additional points to satisfy its stopping parameter. It is possible that these subcalculations might be truncated because of array size constraints which would cause the probability to be understated and making the aggregate stopping value unattainable. While this is easy to deal with given sufficient computing resources, the outcome of the individual steps should be monitored.

Closing Comments

As a check of the calculations, I performed these calculations using an annual aggregate of zero, that is, the insurance company assumes all of the losses. The resulting distribution, within the precision controlled by the stopping probabilities, turned out be the same as the aggregate distribution calculated directly from the frequency parameters and the total severity distribution. I consider this to be a check both of the program and the algorithm. Selected data from these runs is included as Appendix B

Also attached as Appendix C is a copy of part of the output file of the program. In each step I included a number of statistics that allowed me to determine if the calculations were correct. For example, the mean and variance of the convoluted distribution should be the sum of those statistics for the input distributions, the mean and variance of the aggregate distribution should relate to the mean and variance of the frequency and severity distributions by the well known formulas, etc.

Appendix A

Following is a sketch of the proof of the independence of the the primary retained losses over the annual aggregate and the excess losses in the probability subspace defined by the number of excess claims.

Recall the definition of and some facts about independent random variables.

Definition. A set of random variables is independent if every finite subset is independent.

Theorem A. Random variables are independent if and only if their joint distribution function factors into a product of their individual distributions.

Theorem B. Any Borel measurable functions of independent random variables are again independent random variables.

Theorem C. Any Borel measurable functions of disjoint sets of independent random variables are independent random variables.

The total probability space can be thought of as a set of Cartesian products of the interval $I = (0, \infty)$, where the number of terms in the product correspond to the number of claims. This is the total probability space can be expressed as $\Omega = \bigcup I^N$ where I^0 is a single point. Let Ω_{N_x} stand for the subspace of N_x excess claims. In any I^N, under the usual assumptions, the N claims are independent random variables. Thus the joint distribution $g(x_1, \ldots, x_N) = f(x_1) \ldots f(x_N)$.

Now define two new random variables. $X_{ex} = L$ if $X \le L$ and $X_{ex} = X$ if $X > L$ and $X_{pri} = X$ if $X \le L$ and $X_{pri} = L$ if $X > L$, where L is the occurrence limit. These are Borel measurable functions with respect to the sigma-algebra generated by the original random variables. Thus, if these random variables are substituted for any of the original random variables, the resulting set of random variables is still independent.

Now restrict attention to the subspace I^N and assume we are in the subspace of I^N where the first N_x claims are excess claims and the remaining N-N_x are non-excess claims. In this subspace, the probability density of these claims is identical with the joint probability distribution given by substituting X_{ex} for the first N_x claims and X_{pri} for the remaining N-N_x claims, since it is zero outside of this space. Since this distribution factors on the entire space, it factors on the subspace. Obviously any other configuration of excess claims would yield the same result.

Next consider, the two random variables defined, respectively, as the sums of the excess and primary claims. By Theorem C, these are independent random variables. Finally, in the intersection of subspace I^N and Ω_{N_x}, the random variable defined as the sum of the excess claims less N_xL and the random variable defined to be the maximum of zero and the sum of the non-excess claims plus N_xL less the annual aggregate are again Borel measurable and independent.

Appendix B

Following are two copies of the output data sets. The first is from a run that calculates the total aggregate distribution corresponding to the underlying claim frequency and severity distribution in the normal fashion.

The second performs the calculation as described in the paper by separately calculating the primary and excess components, performing the convolution of these terms for each excess loss and finally calculating the weighted sum with the annual aggregate retention set to zero.

Note that the means and standard deviations are equal and the probabilities at the loss amounts shown are very close.

First Method

The mean and std dev of the aggregate distribution is:

2082 1270

loss amount	probability	Cumm Prob	Pure Prem Ratio
0	0.000073	0.000073	1.000000
100	0.000520	0.000593	0.951974
200	0.001932	0.002524	0.903974
300	0.004962	0.007486	0.856066
400	0.009942	0.017429	0.808396
500	0.016618	0.034046	0.761204
600	0.024190	0.058236	0.714811
700	0.031618	0.089854	0.669579
800	0.037971	0.127825	0.625865
900	0.042648	0.170473	0.583975
1000	0.045443	0.215916	0.544134
2000	0.025905	0.590415	0.258203
3000	0.018752	0.793156	0.106955
4000	0.007832	0.919984	0.039573

Second Method

The mean and std dev of the aggregate distribution is:

2082 1270

loss amount	probability	Cumm Prob	Pure Prem Ratio
0	0.000073	0.000073	1.000000
100	0.000520	0.000593	0.951973
200	0.001932	0.002524	0.903970
300	0.004962	0.007486	0.856060
400	0.009942	0.017428	0.808388
500	0.016617	0.034046	0.761195
600	0.024189	0.058235	0.714799
700	0.031618	0.089853	0.669565
800	0.037970	0.127823	0.625850
900	0.042647	0.170471	0.583958
1000	0.045443	0.215913	0.544116
2000	0.025905	0.590410	0.258186
3000	0.018752	0.793154	0.106957
4000	0.007830	0.919947	0.039601

Appendix C

The following is a sample of the output data set from the computer program. The first section echoes the input parameters - a label for the run (Test Data),the mean (10.0) and variance (11.0) of the first dollar claims process, the unit loss value (100), the number of points in the primary distribution (5), the number of points in the aggregated retention (10) and the total number of points in the severity distribution (20). The last line shows the probabilities controlling the aggregate loss calculation, the main loop and the printing of the excess pure premium table and the maximum number of points to be used in the loss arrays.

So in this example, the primary layer includes losses up to 500 with an annual aggregate of 1000. The excess layer is losses from 500 to 2000.

```
   Test Data
  10.00      11.00
   100.0         5         10         20
0.99990   0.99990   0.99990       200

The Mean of the Primary Severity Distribution is          128
The Stdev of the Primary Severity Distribution is          77
The Mean of the Excess Severity  Distribution is         794
The Std Dev of the Excess Severity Distribution is        560

The number of excess claims is   0 with Prob 0.50383
the expected number of non-excess claims is  9.249
the variance number of non-excess claims is 10.104
the nr of points in the primary agg dist is 36
the mean of the primary aggregate is        1184
the stdev of the primary aggregate is        469

the nr of points in the primary excess agg dist is 26
the mean of the primary excess aggregate is        312
the stdev of the primary excess aggregate is        369
the nr of excess claims is 0 and the nr of xsPts is 1
the xs number of points is 1 with total prob 1.0000000
the mean of the excess distr is          0
the stdev  of the excess distr is          0

the total number of points in the int dist is 27
the mean of the intermediate distr is        312
the stdev of the intermediate distr is        369

writing out the Total Probability
0.1858302   0.0480659   0.0457648   0.0418939   0.0370323
0.0317229   0.0264131   0.0214297   0.0169782   0.0131599
  ...

finished with case Nx equals 0
total probability equals 0.5038258
the mean of the cumm distr is        157
```

Since the number of points in the primary aggregate distribution did not exceed 200, another input parameter, the step was completed because the probability stopping parameter was exceeded. This information is reported for one, two etc. excess claims.

The next section is from the step where N_X equals five.

The number of excess claims is 5 with Prob 0.00069
the expected number of non-excess claims is 9.711
the variance number of non-excess claims is 10.609
the nr of points in the primary agg dist is 37
the mean of the primary aggregate is 1243
the stdev of the primary aggregate is 480

the nr of points in the primary excess agg dist is 52
the mean of the primary excess aggregate is 2743
the stdev of the primary excess aggregate is 480

the nr of excess claims is 5 and the nr of xsPts is 76
the xs number of points is 76 with total prob 1.0000002
the mean of the excess distr is 3972
the stdev of the excess distr is 1251

the total number of points in the int dist is 128
the mean of the intermediate distr is 6716
the stdev of the intermediate distr is 1340

(Note: the mean of the intermediate distribution is the sum of the means of the primary excess aggregate and the excess distribution, and you can verify by squaring the standard deviations to see that the variances add.)

writing out the Total Probability

0.1858302	0.0497242	0.0493546	0.0477042	0.0452040
0.0422148	0.0390128	0.0357931	0.0326833	0.0297591

...

finished with case Nx equals 5
total probability equals 0.9999080
the mean of the cumm distr is 1156

Since the loop has reached the stopping probability of 0.9999, the program exits the main loop, calculates the excess pure premium ratios for the aggregate distribution, writes it to the output file and stops.

ANALYSIS OF SURETY RESERVES

Al Weller

ANALYSIS OF SURETY RESERVES

Alfred O. Weller

Abstract *This paper is an introduction to the actuarial analysis of surety reserves. Its main goal is to help students and those new to surety business appreciate fundamental considerations and properly recognize these considerations in the determination and evaluation of reserves.*

I. Introduction

As for any actuarial project, the first step in the analysis of surety reserves is to identify and define the purpose and scope of the actuarial engagement. Within this framework, possible objectives include:

1. Evaluation of statutory (SAP) reserves;

2. Evaluation of reserves determined in accordance with generally accepted accounting principles (GAAP reserves);

3. Evaluation of reserves for income tax purposes (tax basis reserves);

4. Evaluation of reserves for purposes of merger, acquisition or divestiture; for financial planning; or for other financial purposes.

Evaluation is used here in a general sense to communicate both the determination of reserves and the assessment of the propriety of reserves set by others. The paper emphasizes general guidelines and particular cases may require modification of these guidelines to accommodate unusual circumstances or data limitations.

II. General Description of Surety Business

The term "surety" refers to both fidelity bonds and pure surety bonds. Fidelity bonds guarantee a collateral contract between two parties. For example, an employer may purchase a fidelity bond covering fraudulent or dishonest acts of employees. Such bonds typically cover direct loss attributable to covered acts of employees and entitle the surety to recover its losses from the employee causing the loss. Because recovery might not be possible, such bonds are more akin to insurance, than, say, contract bonds.

Surety bonds, as opposed to fidelity, also apply to a broad spectrum of situations - contracts, judicial proceedings, and licenses to name a few. Compared to fidelity bonds, surety bonds come closer to pure suretyship, i.e., the issuer of the bond lends its name and credit to guarantee the obligation, but has minimal risk of ultimate net loss.

The word "surety" is also used to refer to a company that underwrites these bonds. However, the word "surety" need not appear in the name of a company for the company to function as a surety.

Surety companies are regulated by the insurance laws of the various states, as well as laws applicable to companies in general. Although bonds do protect against loss, the general purpose of surety bonds is best described as the guarantee of performance by one party of an obligation to another. The applicable terminology is:

Principal	=	One who undertakes to perform, to fulfill a contract, or meet an obligation, e.g., a contractor for a construction bond.
Obligee	=	The second party to the agreement with the principal, the individual whose interests will be protected by the surety bond.
Surety	=	The company (or individual) who guarantees the performance of the obligation of the principal to the obligee, e.g., a client whose reserves are being reviewed.

Within this context, the principal causes the loss to the obligee and the surety settles the loss by fulfilling the obligation. Usually, the surety may settle the claim by payment of an amount determined in accordance with the bond or by completing performance of the obligation. The surety is then entitled to recover its loss from the principal.

Because recovery of loss is possible and the probability of recovery is a key consideration in underwriting surety bonds, surety companies are generally deemed to assume minimal insurance risk.

Depending on the type of surety bond, the assumed risks are largely moral hazard and credit risks. In this context, surety premiums are interpreted to be primarily a fee for financial service.

III. What do Surety Reserves Represent?

We can now relate the scope of actuarial services in reviewing surety reserves to the above objectives.

The nature of recoveries and collateral supporting potential recoveries vary and are limited only by the combined imagination of the surety and principal. Possible sources of recovery include cash deposits, contract monies, letters of credit, mortgages on property, shares in projects, and more.

The nature of possible recoveries is a key consideration in evaluating reserves. Allowable credits against direct loss reserves vary with type of reserve as follows:

SAP reserves	-	Cash and cash equivalent collateral usually construed to include contract monies, cash deposits, letters of credit, and the like.
GAAP reserves	-	Recoveries for which a legal right to collection exists as of the date reserves are evaluated and there is a reasonable probability of collection.
Tax reserves	-	All probable sources of recovery recognized on a discounted basis.

The interpretation of these guidelines may vary by jurisdiction.

In general, SAP reserves will be greater than or equal to GAAP reserves, which in turn will be greater than or equal to tax reserves.

It should also be noted that the actuarial review normally encompasses both loss and loss adjustment reserves. And, in cases, there are statutory requirements for additional reserves (e.g., 5% of earned premium).

IV. Collateral

There are both general considerations and considerations specific to particular types of collateral. Also, the evaluation of collateral for open known claims will generally be different than the evaluation of collateral for incurred but not reported claims.

The first general issue is whether the collateral is dedicated to specific surety bonds, applies to several surety bonds (e.g., a line of bonds automatically issued to a single principal without reunderwriting), or applies to multiple obligations of the principal to several potential creditors. A second general issue is the relation of the amount of collateral to the size of the obligation - is there sufficient leeway to allow for fluctuation in the value of the collateral as the obligation is performed. And, the third general issue is whether the collateral can indeed by called upon and collected.

With regard to specific types of collateral:

A. <u>Contract Money</u> - The obligee and principal may have agreed to payment on an installment basis with holdback of a portion (the "retained percentage") of payments due the principal until the obligations are satisfied. The value of this contract money will vary with the extent of completion of the project and prior payments.

   Also, at the time that the principal fails to perform, there will be future payments due on uncompleted work at the time of default. These contracted amounts are payable by the Obligee in exchange for performance.

   The phrase contract monies includes both these future payments and the retained percentage described in the first paragraph. The contract monies are in the possession of the obligee at the time the principal defaults and may be netted against the amount of loss covered by a bond.

   For open claims, the amount of available contract money can be computed on a claim by claim basis. For IBNR claims, the value of contract money must be estimated.

It is possible for the surety to hire contractors for less than the contract monies available and achieve a profit in some cases.

B. Letters of Credit - Letters of credit commit a financial institution to afford funds on behalf of the principal to designated parties under specified conditions. For example, a letter of credit may become payable upon payment of direct losses by a surety. Letters of credit should remain in force until discovery periods or other bond provisions determining the ability to file claims have expired.

C. Project Shares - A surety bond might be secured by a contingent share in construction projects available for sale. Because such collateral cannot be collected until the project is actually sold and sale cannot be guaranteed, it should not be considered cash equivalent in determining SAP reserves. Depending on collectibility, it may or may not be appropriate for GAAP reserves.

D. Mortgages - A surety bond might be secured by a contingent mortgage obligation. Until cash for this obligation is received, it should not be considered cash equivalent. However, the contractual right exists throughout the course of the project or obligation. Such mortgages typically would qualify as uncollected recoveries for GAAP, but not SAP, reserves.

These examples do not cover all possible forms of collateral, but they are sufficient to illustrate the applicable principles.

V. Coverage Considerations and Company Operations

The potential for loss (in particular, IBNR loss) varies with coverage. For example, fidelity bonds may incorporate a discovery period after expiration of a bond for discovery and reporting of claims incurred during the bond period. Contract bonds may have terms that coincide with projects and have wording minimizing the potential for IBNR claims after the obligee accepts the project and releases contract monies.

Review of coverage, growth, and mix of business is therefore necessary input to the evaluation of reserves. Actuarial analysis should reflect the groupings in which the client monitors its business and their respective rates of growth. For example, the mix of contract and bid bonds, the size of bonded projects and contractors, and the underwriting standards employed by the client have implications regarding carried reserves.

Of special importance are claims operations and the flow of claims information. For example, a surety can advance funds to a contractor to avoid a default or await a report of claim before acting to satisfy the obligation to the obligee. In the first case, a claim has not technically occurred, but in the second case one has. Claim frequency can vary significantly for sureties with similar books of business.

Sureties will also approach recoveries differently - for example, a large surety may forego a recovery that is important to a small surety. The company placing less emphasis on recoveries will, other things being equal, have a larger percentage of closed claims. Surety claims in a sense have two closing dates - the first is when direct payments to the obligee are complete, and the second is when the surety has completely recovered its funds from the principal. The actual closing date is, of course, when the company's claim department decides that a claim is no longer active.

Lastly, it should be noted that allocated and unallocated claim expense will vary with the election to perform or indemnify on defaulted obligations, emphasis on recoveries, and use of inside or outside attorneys.

VI. Reinsurance

Reinsurance agreements must be reflected in reserve analyses. Depending on the situation, this can require review of contracts or cover notes, investigation of whether reinsurers are authorized, modeling of fronting agreements, review of side agreements, etc.

In reviewing SAP reserves it should be noted that reinsurance of surety business is often uncollectible, not because the reinsurer is financially weak, but because collateral and other recoveries reduce the overall size of losses covered by the reinsurance. Thus, if gross claims less cash and cash equivalent collateral generate indicated reinsurance recoveries, but gross claims less GAAP recoveries do not, the statutory reserve will reflect uncollectible reinsurance. Such reinsurance can be valuable to a surety because it increases statutory surplus.

VII. Other Accounting Conventions

In applying loss reserving methods that rely on exposures, actuaries must understand how the exposures are earned. For example, if a loss ratio approach is employed for an audit client, actuaries must verify that premiums are properly interpreted or adjusted, as appropriate, before using them in reserve reviews.

Unearned premium reserves and earned premiums need not be tied to surety bond terms on a contract by contract basis. For example, premiums for six month bonds may be earned over twelve months periods. Such conventions can distort loss reserve estimates derived using loss ratios or other ratio methods.

Sureties have few "loss driven" reserves. For example, retrospectively rated business is uncommon. However, reinstatement premiums do apply to some types of bonds. A reinstatement premium is paid to reinstate coverage after a loss has been incurred and reduced the limits of coverage. Thus, on a low frequency line, frequency for reinstated bonds should be higher than frequency for bonds in general.

Bond counts and bond penalty amounts merit similar considerations.

VIII. Example of Typical Transactions

This example illustrates the general accounts to be modelled and estimated in the actuarial review of reserves for a surety.

The principal "P" contracts with the obligee "O" to perform an obligation in exchange for consideration. The surety "S" guarantees "O" that the obligation will be satisfied for the agreed consideration and charges "P" a premium for this service.

The penalty of the surety bond is the value of the obligation identified in the surety bond. Typically, the obligee will retain a percentage of this value. For example, if payments are to be made in installments of 20% subject to 15% retention, then

A. Until the obligation is 15% complete, the obligee will retain funds equal to the completed value.

B. After the obligation is 15% complete, the obligee will retain between 15% and 35% of the value, depending on the extent to which the project is complete and whether installments have been paid.

If the principal defaults, the contract monies at the time of default will correspond to the uncompleted portion of the project plus the amount retained by the obligee on completed work. This money is available to offset the costs to the surety of completing the work or indemnifying the obligee. Depending on who retains a contractor to complete the work, the funds may or may not pass through the surety and be classified as gross loss payments.

The loss to the surety is the difference between the cost of fulfilling the obligation and the available contract monies[1]. This amount is collateralized by a letter of credit, deposit, or other form of security to the extent that the surety expects contract monies might prove insufficient to cover losses. Insofar as this collateral is liquid (aka cash equivalent) the surety may offset its loss on SAP, GAAP and tax accounting bases by the amount of contract monies and collateral.

If collateral is insufficient to fully indemnify the surety, the surety has a right of action against the principal and can recover its losses in court assuming that the principal can be served and has sufficient assets.

Reinsurance will also affect the net loss amounts to be recorded. Typically, the reinsurance will only apply after other sources of recovery have been exhausted. Therefore, the recoveries allowed by the applicable accounting conventions will also determine the estimated amount of reinsurance recoveries.

Loss adjustment expense is often not recoverable. On the other hand, foregone interest earnings may be.

IX. Actuarial Objectives

The amount of carried reserve is:

A. The Gross Reserve - the difference between gross incurred losses and gross payments to

1 Contract monies do not apply to all bonds.

date (thus, possibly including or excluding contract monies),

less

B. <u>Allowable Uncollected Recoveries</u> - collateral, reinsurance, contract monies, etc. This amount varies with the applicable accounting conventions.

Basic decisions in determining actuarial approaches are whether to estimate this reserve amount net or gross, and how to identify corresponding payments. In general, it will be expedient to evaluate each component separately because that will facilitate putting pieces together in consistent fashion in order to agree with SAP, GAAP and tax accounting. Thus, the actuarial review will often include the following estimates:

A. Estimate gross incurred loss

B. Estimate gross reserves by subtracting paid amounts.

C. Estimate recoveries (both incurred and uncollected) by type of recovery.

D. Results of combining steps A, B, and C to generate "best" reserve estimates.

E. Analysis of the uncertainty in estimates and determination of a range of reasonable reserves.

F. Comparison to benchmarks (e.g., industry experience, underwriting targets, consistency with changes in operations).

X. Selection of Actuarial Methods

Various actuarially sound loss reserving methods may be used. Selection will vary with size of company, available data, nature of surety business, etc. Considerations that influence selection should include:

A. <u>Analysis of Controls and Reporting</u> - In order to understand the recording of losses (including loss adjustment expenses, and recoveries), the actuary must understand:

1. How losses are recorded as to account, amount and period.

2. How available data are recorded and maintained?

3. How sources of recovery (reinsurance, salvage, subrogation, contract monies) are identified and matched to loss records? In particular what is the relative timing of these transactions?

B. CAS Loss Reserving Principles[2] - In particular,

1. Homogeneity - Is data in appropriate groupings to render reasonable estimates in accordance with the purposes of the actuarial evaluation?

2. Credibility - Is the best use of available data made? Could better, more certain estimates be derived by incorporating additional information or selecting alternative data for analyses?

3. Data Availability - What data is available? What are alternatives? Can available data be reconciled to carried reserves and audited data sources?

   Is the data understood? How does the client define a claim? Do recorded amounts for gross loss include or exclude contract monies? What about contract monies and recoveries in excess of costs?

4. Patterns - Emergence, Settlement, Reopenings, and Development in General - Are operations and books of business sufficiently stable to warrant analysis of development patterns? Are net or gross patterns better suited to actuarial objectives? What changes in operations have occurred and how do they affect development patterns? Are actuarial measurements and assumptions regarding development consistent with company operations and coverages?

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2 Casualty Actuarial Society Statement of Principles Regarding Property and Casualty Loss and Loss Adjustment Expense Reserves 5/24/88

5. Frequency and Severity - Is analysis of claim frequency and severity appropriate? Does severity correlate with bond penalty so that bond penalty can be used to estimate severity? Do particular large claims merit special attention? Is a model of distribution by size of loss appropriate?

6. Coverage and Limits (Specific and Aggregate) - Are the nature of coverage and applicable limits appropriately reflected in the actuarial analyses?

7. Reinsurance & Portfolio Transfers - Is reinsurance properly reflected? Should ceded and assumed business be separately analyzed? What about individual contracts?

8. Accounting Principles - What accounting principles apply to the estimates? Are estimates consistent with applicable accounting principles?

9. Salvage, Subrogation, and other Forms of Recovery - Recoveries are especially important in the accurate determination of surety reserves. Recognition of recoveries varies with applicable accounting principles. Is the actuary able to distinguish and estimate the different types of recoveries? Is reporting and recording clear? Are gross and net estimates of loss logically consistent?

10. Discounting - Should reserves reflect the time value of money? Is appropriate discounting employed?

11. External Influences - For example, if there is a downturn in the construction industry, an increase in surety claims might be expected? Is the analysis of reserves consistent with external indications or are special adjustments appropriate?

12. Loss-Related Balance Sheet Items - Are there unearned premiums, collateral amounts, deposits or other accounts that are functions of the amount of loss or otherwise directly related to loss

estimates? Are they within the scope of the actuarial review? Are they consistent with actuarial estimates?

13. <u>Reasonableness and Uncertainty</u> - What benchmarks have been established for testing the reasonableness of estimates? What about uncertainty and the reasonable range for estimates?

XII. Conclusion

The actuary's professional obligation lies in the integrity of the overall evaluation and not the algorithmic execution of a particular calculation. As described above, methods should be selected in light of the purposes of the actuarial analysis, client and industry characteristics, and approaches used for comparable clients. Professional attention to pertinent considerations identified in this paper will help ensure both quality and responsiveness of actuarial analyses of surety reserves.

But, a help is not a guarantee. Each individual practicing actuary is responsible for the value of professional services and the accomplishment of the goals of an actuarial engagement. It is the individual commitments of professional actuaries to this responsibility that guarantees quality and responsiveness.

CREDIBILITY: PRACTICAL APPLICATIONS

Howard Mahler

There are many fine papers on the theory behind credibility. However, today we are concentrating on the uses of credibility theory, but only if they are important for credibility practice. My talk will be from the point of view of a Bureau actuary or an actuary with a primary insurer.

Let X be the quantity we wish to estimate. For example, X might be the expected losses for a Workers' Compensation class relative to the statewide, i.e., X is the class relativity.

Let Y_1, Y_2, Y_3, etc. be various estimates for X. For example, Y_1 might be the current relativity, Y_2 might be the observed relativity for the latest data, Y_3 might be the relativity based on national data suitably adjusted, then we might estimate X by taking a weighted average of the different estimates Y_i.

$$X = \sum_{i=1}^{n} Z_i Y_i$$

where X = quantity to be estimated
Y_i = ith estimate of X
Z_i = weight assigned to ith estimate of X

Usually the weights Z_i are restricted to the closed interval between 0 and 1. In the most common situation we have two estimates, i = 2. In that case we usually write:

$$X = Z\, Y_1 + (1-Z)\, Y_2$$

where Z is called the credibility and 1-Z is called the complement of credibility. However, it is important to note that the usual terminology tempts us into making the mistake of thinking of the two weights and two estimates differently. The actual mathematical situation is symmetric.

We can now discuss those rules I think will aid you in using credibility for practical applications.

Rule 1A: Spend a lot of time and effort deciding on or choosing the Y_i. Each Y_i should be a reasonable estimate of X.

So for example, if trying to estimate a medical claim cost trend it may not make much sense to assign the complement of credibility to an estimate based on the general rate of inflation.

Rule 1B: Spend a lot of time and effort computing, collecting data on, or estimating each Y_i.

If you are going to include a value in your weighted average, it makes sense to try to carefully quantify that value.

Rule 2A: The procedure is generally forgiving of small "errors" in the weights. Therefore, do not worry overly much about getting the weights exactly right.

This is discussed in my paper "An Actuarial Note on Credibility Parameters" in PCAS 1986.

Rule 2B: The concept of credibility is a relative concept. A relative weight can only be assigned to any single estimator, if you know what all the other estimators are.

For example, assume you have two estimators each of which has been assigned "only" 50% credibility. This merely indicates that the two estimators are equally good or equally bad, not whether they are good or bad in some absolute sense.

Rule 2C: The less random variation in an estimate the more weight it should be given. In other words, the more useful information and the less noise, the more the weight.

Rule 2D: The more closely related to the desired quantity, the more weight an estimator should receive.

For example, observations more distant in time usually deserve less weight.

Rule 3: Cap the changes in relativities that result from the use of credibility.

A properly chosen cap may not only add stability, but may even make the methodology more accurate by eliminating extremes.

Let's go back to the question of choosing the estimators to use for a given task. The estimators Y_i can have many sources. For example:

1. The recent observation(s) of X.

2. The recent observation(s) of the same quantity as X, but for a superset.

3. The recent observation(s) of a similar quantity to X; there may be an adjustment necessary.

4. Past estimates(s) of X. There may be an adjustment for the intervening period of time.

5. The result of a model.

6. The result of judgement.

In the following real world example of the use of credibility to determine Workers' Compensation classification relativities, I used two estimators: the relativities in the current rates and the relativities indicated by the recent experience of each class. How much weight should be given to each estimator for each class?

In the actual methodology used, three pieces of the pure premium were each estimated separately and then added together. The three pieces were: serious, non-serious, and medical as per Roy Kallop's paper "A Current Look at Workers' Compensation Ratemaking" in PCAS 1975.

For each class, the relativity indicated by the most recent experience was given credibility $0 \leq Z \leq 1$

$$Z = \left(\frac{E}{F}\right)^{2/3} \qquad F > 0$$

where E was the expected losses for that class and F is the so-called standard for full credibility. (F would vary by serious, non-serious, and medical.) This is formula 1 on Exhibit 4.

The problem was to estimate an appropriate value of F to use for this purpose. For various values of F, the resulting estimates of the class relativities were compared to the observed future relativities.

In order to do this, I needed an extra year of data not used to make any of the estimates. To quantify how close a match was obtained between the predicted relativities and the relativities observed in this additional year, I calculated the mean squared error (using payrolls to weight the squared errors by class.)

The result for serious losses appears in a graph in Exhibit 2. F is in thousands of dollars of expected losses, and appears on a logarithmic scale. The minimum MSE occurs when the standard for full credibility is about $15 million.

There are a few points worth making:

1. Values of F between $10 million and $20 million all perform quite well.

2. For values of F within a factor of 10 of optimal, the graph of the mean square error is approximately symmetric when F is put on a log scale. For values of F differing from $15 million by the same factor less than or equal to 10, the MSE is roughly the same.

3. The reduction in mean squared error in this case due to the use of credibility is not that large, but neither is it atypical. (Not only is difficult to estimate the expected relative losses, but the actual observation varies randomly around the expected result.)

In this case, using the current relativities ($F=\infty$) gives a MSE of 2.385. Using the observed relativities ($F=0$) gives a MSE of 2.503. Thus in this case by this relatively simple use of credibility, we have reduced the mean square error by about 8% from .2385 to 2.200.

The basic reason the credibility procedure provides some improvement is that the "observed" relativities generally worked well for larger classes, while the current relativities were generally a better estimate for the smaller classes. The credibility method puts more weight on the generally better estimate.

The results of this study were used to modify the standards for full credibility.

	Standards for Full Credibility	
	Previous	New
Serious	25 average serious losses	175 average serious losses
Non-Serious	300 average non-serious losses	120 average non-serious losses
Medical	240 average non-serious losses	190 average non-serious losses

The credibilities using the previous and new full credibility standards are shown in Exhibit 3. As we expect, the change in full credibility standards substantially lowered the credibilities assigned to serious losses, raised the credibilities assigned to non-serious losses, and raised slightly the credibilities assigned to medical losses.

The MSE using the two sets of standards for full credibility are as follows:

	Mean Squared Error	
	Previous	New
Serious	2.267	2.200
Non-Serious	.337	.334
Medical	.454	.453

Thus we see that even for serious losses, where there was the most substantial revision to the standard for full credibility, the improvement in mean squared error is not overwhelming. While it is worthwhile to review the manner in which the credibilities are determined, provided the method automatically adjusts for inflation, such reviews need only be made infrequently, ex., every decade or two. Usually such reviews will show that the current method works reasonably well, but can be improved somewhat. It is generally more useful to spend your efforts on improving the estimates that are to be weighted together.

In this study, besides examining the parameter used in the formula for credibility, I also examined the use of other formulas. Some of these formulas shown in Exhibit 4 should be very familiar, while some may be new to some of you.

For practical uses of credibility, it's sufficient to know of the existence of these different formulas for the dependence of credibility on the size of the class or the risk. If one of them works significantly better, use it, without agonizing over the theoretical mumbo jumbo that lies behind the formula.

In the particular example here, using another formula produces no significant improvement. However in other situations, such as Workers' Compensation Experience Rating, it turns out that there is a significant improvement obtained by using formula 6.

Exhibit 5 shows an example of the different behavior you get with size of risk for formulas 3 through 6. (These four formulas are all based on the ideas of "Bayesian" as opposed to classical credibility.) Formula 3 goes to zero at zero and 1 at infinity. Formula 4 has some minimum credibility greater than zero for small risks. Formula 5 has some maximum credibility less than one for large risks. Formula 6 combines the behavior of Formulas 4 and 5.

A practical actuary might just select a curve and set of parameters that produce credibilities that seem reasonable.

The next example of a practical use of credibility involves revising the definitions of automobile insurance territories in Massachusetts. Each town's relative loss potential is determined based on 4 years of data and a relatively complicated credibility methodology. Then towns with similar loss potential are grouped together. Here we will ignore the details of the procedure which are explained in Robert Conger's paper in PCAS 1987, and focus on the results of the latest review conducted for 1989 rates.

The predictions of the methodology as used in the review of 1986 rates were compared with the subsequent observations. Using the methodology reduced the mean squared error to .0091 from .0117 if the observations had been relied on solely. Thus the credibility methodology performed its task of reducing the mean squared error, in this case by 22%.

However, such summary statistics do not tell the whole story. Credibility is a linear process, and thus the extreme cases may not be dealt with as well as they might.

For example, let's look at the results of applying the same methodology consistently over time to two small towns.

	Estimated Loss Potential Relative to Statewide Average			
	1984 Review	1986 Review	1988 Review	1989 Review
Acushnet	.84	.87	.88	.87
Brewster	.74	.84	.70	.61

	Indicated Territory (Prior to Capping)			
	1984 Review	1986 Review	1988 Review	1989 Review
Acushnet	5	6	6	6
Brewster	3	6	2	1

The results for the first town Acushnet, with 6000 exposures per year are typical. The relative loss potential varies somewhat from review to review, with a change in indicated territory of plus or minus one from time to time.

The results for the second town Brewster, with 5000 exposures per year, are not typical. In fact, Brewster was chosen as the most extreme case of fluctuating experience over this period of time. As you can see the estimated relative loss potential swung up and then down. This in turn resulted in large changes in the indicated territories. This occurred in spite of relying on four years of data, so that the data periods used in the reviews overlap. This occurred in spite of the use of credibility, which ameliorated the effect of the large fluctuations in the experience of this town.

Such large swings are unlikely. However, when dealing with 350 towns, something that only has only a .3% chance of happening per town, on average occurs for one town.

This problem is dealt with by capping movements. The actual cap chosen was to restrict movements to at most one territory either up or down. This is an example of the third rule I discussed earlier.

I have tried to describe a number of practical applications of credibility. I've given a number of general rules which you should find useful in your own work with credibility.

The theory behind the use of credibility can be complex. However, the use of credibility itself is set up precisely so that it can be understood by a layman. While ratemakers may differ in their knowledge of credibility theory, all ratemakers should be completely familiar with credibility practice.

Exhibit 1

RULE 1A: SPEND A LOT OF TIME AND EFFORT DECIDING ON OR CHOOSING THE Y_i. EACH Y_i SHOULD BE A REASONABLE ESTIMATE OF X.

RULE 1B: SPEND A LOT OF TIME AND EFFORT COMPUTING, COLLECTING DATA ON, OR ESTIMATING EACH Y_i.

RULE 2A: THE PROCEDURE IS GENERALLY FORGIVING OF SMALL "ERRORS" IN THE WEIGHTS. THEREFORE, DO NOT WORRY OVERLY MUCH ABOUT GETTING THE WEIGHTS EXACTLY RIGHT.

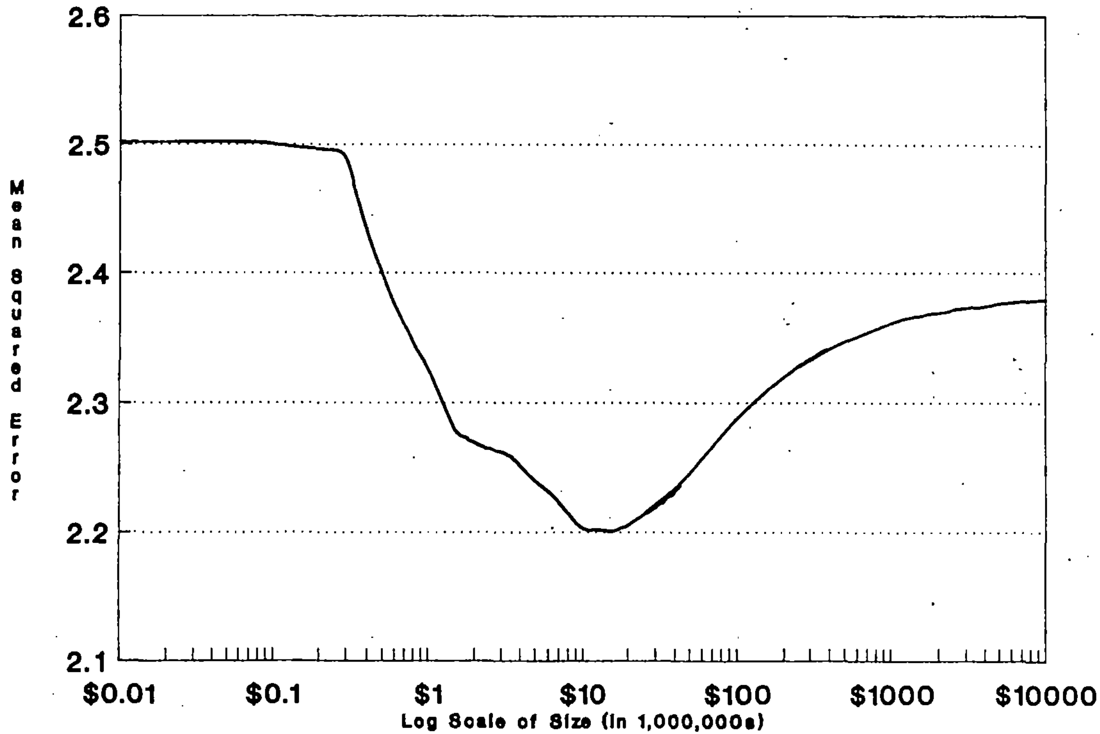
RULE 2B: THE CONCEPT OF CREDIBILITY IS A RELATIVE CONCEPT. A RELATIVE WEIGHT CAN ONLY BE ASSIGNED TO ANY SINGLE ESTIMATOR, IF YOU KNOW WHAT ALL THE OTHER ESTIMATORS ARE.

RULE 2C: THE LESS RANDOM VARIATION IN AN ESTIMATE THE MORE WEIGHT IT SHOULD BE GIVEN. IN OTHER WORDS, THE MORE USEFUL INFORMATION AND THE LESS NOISE, THE MORE THE WEIGHT.

RULE 2D: THE MORE CLOSELY RELATED TO THE DESIRED QUANTITY, THE MORE WEIGHT AN ESTIMATOR SHOULD RECEIVE.

RULE 3: CAP THE CHANGES IN RELATIVITIES THAT RESULT FROM THE USE OF CREDIBILITY.

Workers' Compensation Serious Pure Premiums



Standard for Full Credibility (F)

Exhibit 3

Credibility for Various Expected Losses

SERIOUS

Expected Losses (000's)	Current F 2,175,000	Indicated F 15,200,000
0	0%	0%
40	7	2
80	11	3
160	18	5
320	28	8
640	44	12
1,280	70	19
2,560	100	30
5,120	100	48
10,240	100	77
20,480	100	100

NON-SERIOUS

Expected Losses (000's)	Current F 1,260,000	Indicated F 496,000
0	0%	0%
10	4	7
20	6	12
40	10	19
80	16	30
160	25	47
320	40	75
640	64	100
1,280	100	100

MEDICAL

Expected Losses (000's)	Current F 1,008,000	Indicated F 800,000
0	0%	0%
15	6	7
30	10	11
60	15	18
120	24	28
240	38	45
480	61	71
960	97	100
1,920	100	100

$$Z = (E/F)^{2/3}$$

Exhibit 4

Various Credibility Formulas

(1)*	$Z = (\frac{E}{F})^{2/3}$	$F > 0$	Traditional Workers' Compensation
(2)**	$Z = (\frac{E}{F})^{\frac{1}{2}}$	$F > 0$	Classical/Limited Fluctuation
(3)***	$Z = \frac{E}{E+K}$	$K \geq 0$	Bayesian/Buhlmann
(4)****	$Z = \frac{E+I}{E+K+I}$	$I \geq 0$ $K \geq 0$	Risk Inhomogeneity
(5)****	$Z = \frac{E}{EJ+K}$	$K \geq 0$ $J \geq 1$	Parameter Uncertainty
(6)****	$Z = \frac{E+I}{EJ+K+I}$	$I \geq 0$ $J \geq 1$ $K \geq 0$	Risk Inhomogeneity and Parameter Uncertainty

* This formula is used for example in R. Kallop, "A Current Look at Workers' Compensation Ratemaking," PCAS LXII, 1975, p. 62.

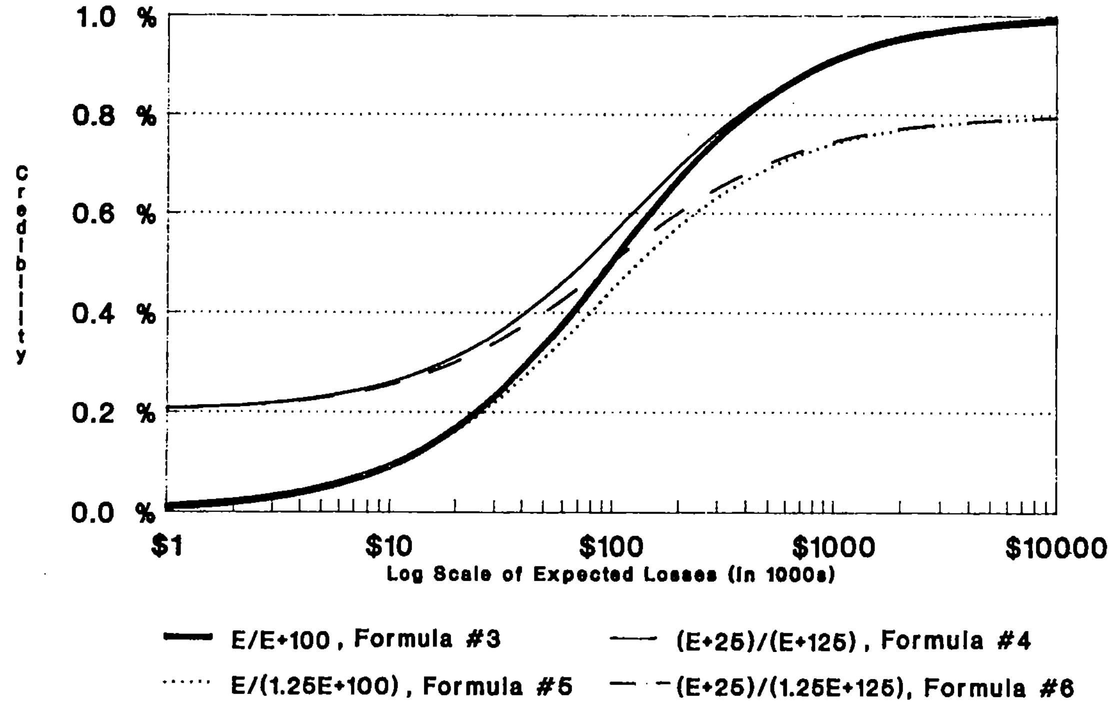
** This formula is explained for example in L.H. Longley-Cook, "An Introduction to Credibility Theory," PCAS XLIX, 1962, p. 194.

*** This formula is explained for example in A.L. Mayerson, "A Bayesian View of Credibility," PCAS LI, 1964, p. 85.

**** These formulas are explained for example in H.C. Mahler, Discussion of G.G. Meyers, "An Analysis of Experience Rating," PCAS LXXIV 1987, p. 119.

Exhibit 5

Comparison of Credibility Formulas



GENERALIZED EARNED PREMIUM RATE ADJUSTMENT FACTORS

Richard Bill

ABSTRACT

The loss ratio method of ratemaking requires the Actuary to adjust earned premiums to reflect all rate changes that have been implemented. Many Actuaries use the parallelogram method of finding the portion of the exposures earned from a given rate change. The assumption is made that exposures are being written at a constant level.

Computers are becoming an indispensable tool for the Actuary and the parallelogram method is cumbersome for computer applications. The purpose of this paper is to derive a simple general formula for finding the earned portion of a rate increase for any given policy term, rate effective date, and evaluation period (period during which the premiums are earned). In the latter portion of the paper a formula is given based on the assumption that exposures are increasing at a given growth rate. Finally, a comparison is made of results produced by the constant exposure model versus the constant growth rate model.

GENERALIZED EARNED PREMIUM RATE ADJUSTMENT FACTORS

The loss ratio method of ratemaking requires the actuary to adjust earned premiums to reflect all rate changes that have been implemented. Several papers in the Proceedings have addressed the calculation of earned premium at current rates. If there has been just one rate change, the formula for this adjustment is:

Equation (1.1) $$F = \frac{1+r}{1+Pr}$$

where F is the rate adjustment factor, r is the rate change and P is the portion of the earned exposure that was subject to the new rate. The purpose of this paper is to derive a simple general formula for calculating P for any given policy term, rate effective date, and evaluation period. The evaluation period is defined as the period stated in years during which the premium was earned. For example, if one wanted to convert the first quarter of 1988 earned premiums to current rates, the evaluation period would be .25 regardless of whether the policy term was semiannual, annual, quarterly, etc.

TRADITIONAL METHODS OF FINDING P

Roy Kallop's[1] paper on Workers' Compensation Ratemaking gives a description of the parallelogram method of finding P. This method assumes that exposures are being written at a constant rate. This assumption will be made throughout the remainder of the paper unless otherwise noted.

Calculations are relatively easy, but the Parallelogram method does not lend itself to computer applications. Another method of finding P is to develop precalculated tables based on effective date, policy term, and evaluation period. These tables can become rather large and cumbersome for computer applications if several policy terms and/or evaluation periods are involved. Also, they are usually calibrated to rate changes effective at the beginning of the month and those effective later in the month are not calculated correctly.

FORMULA METHOD FOR FINDING P

Set the beginning of the evaluation period at 0. Let T equal the policy term, E equal the evaluation period, and D equal the effective date of a rate revision relative to the beginning of the evaluation period using years as the unit of measurement. P can be calculated by the following formula:

Equation (1.2) $$P = 1 - \frac{A^2-B^2-C^2}{2ET}$$

Where:

$A = D + T$

$B = \text{MAX }(A-E,\ 0)$

$C = \text{MAX }(D,\ 0)$

Although one can easily memorize this formula, the time saved versus the parallelogram method is probably minimal if there is only one rate change involved. The advantage of the formula method is that it can be easily programmed on a personal computer or a mainframe and multiple rate change factors for several years can be computed quickly and automatically.

If a complete computerized ratemaking system is being developed, the formula is much easier to program than alternative methods. Also, the formula is generalized so that it can be used for any policy term, rate change date, and evaluation period.

EXAMPLE

Assume that one wishes to convert 1988 year-to-date premiums as of August 31, 1988, to current rates. One rate revision was implemented on November 15, 1987, and all policies are on a quarterly policy term. Then:

$D = -1.5/12 = -.125 \qquad T = .25 \qquad E = 8/12 = .66667$

$A = D + T = -.125 + .25 = .125$

$B = \text{Max } (.125 - .66667, 0) = 0$

$C = \text{Max } (D, O) = 0$

$$P = 1 - \frac{(.125)^2 - (0)^2 - (0)^2}{(2)\ (.66667)\ (.25)}$$

$P = .953125$

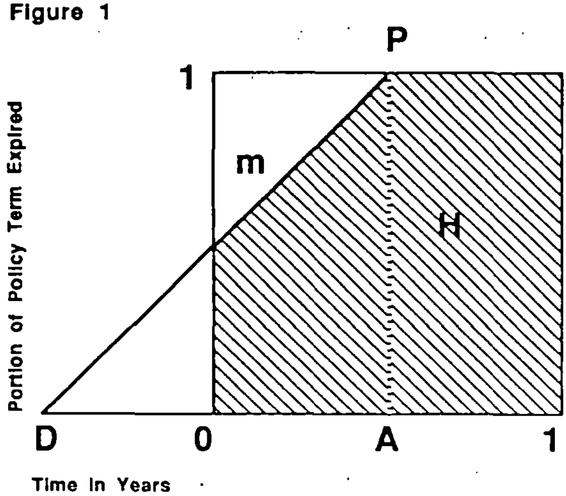
DERIVATION OF THE FORMULA

Jim Ross, in "Generalized Premium Formulas,"[2] and Miller and Davis, in "A Refined Model for Premium Adjustment,"[3] did an excellent job of developing formulas to find P. They showed several formulas, but did not derive a general formula to calculate P in a straightforward way. By expanding on their work, the formula described above was derived.

PARALLELOGRAM METHOD

Figure 1 gives an illustration of the traditional parallelogram approach assuming we are working with annual premium (E=1):

Figure 1



D = effective date of rate change

A = time when rate change is fully earned

As in the Miller and Davis paper, the x-axis is time in years and the y-axis is the portion of policy term expired. For the example shown in Figure (1), line m indicates the portion of the policy term expired for a policy effective at time D. The origin corresponds to the beginning of the evaluation period with the point (1,0) corresponding to the end of the year. D is the effective date of the rate revision and A is the point in time when the rate revision becomes fully earned for a policy effective at time D. Line m is drawn between D and P.

Figure (1) is an illustration of the case where D is less than 0 and 0<A<E. To find the portion of the rate increase that is earned, we must find the area

of the shaded portion of the rectangle in Figure 1 (area H) and compare it to the total area of the rectangle. Using traditional methods, the shaded area is subdivided into triangles and rectangles and the area is calculated manually.

GENERALIZED FORMULA FOR THE AREA

The shaded area can be calculated by integrating the line m from 0 to A and integrating 1 from A to 1. If the equation for line m is f(x), then the area is:

Equation (2.1) $H = \int_{0}^{A} f(x)\,dx + \int_{A}^{1} 1\,dx$

Let T be the policy term. Note that A = D + T since a policy effective at time D becomes fully earned at time D + T. The slope of line m would be the change in Y of 1 divided by the change in X of A - D, or D + T - D, which yields $\frac{1}{T}$.

Using the general equation of a line given the slope and one point, the equation for line m becomes:

Equation (2.2) $y = \frac{1}{T} \cdot x - \frac{D}{T} = \frac{x-D}{T} = f(x)$

and Equation (2.1) becomes:

Equation (2.3) $H = \int_{D}^{A} \frac{(x-D)}{T}\,dx + \int_{A}^{1} 1\,dx$

Where A = D+T

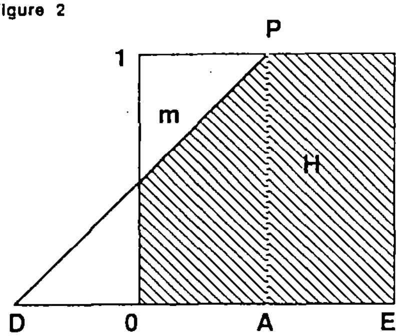
The total area of the rectangle is 1 and the percentage of the rate increase earned is equation (2.3) divided by 1.

To arrive at the general case, one must consider all possible values for D and

A. Also, one must allow evaluation periods other than one year.

To consider the latter, define E as the policy evaluation period, the length of time in years of the earned premiums to be converted to current rates. This could be one year of earned premiums (E=1) or one month of earned premiums (E=1/12). The only change to our diagram in Figure 1 is that instead of the rectangle ending at 1, it would end at E as shown below:

Figure 2



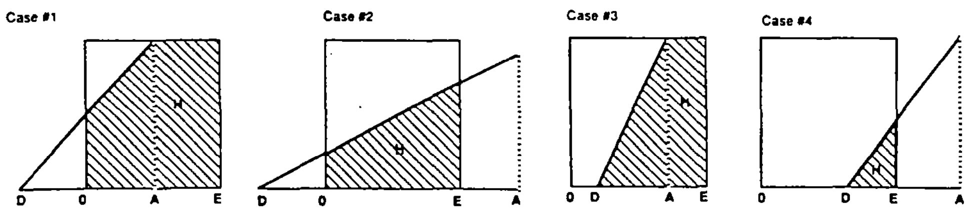
The total area of the rectangle is E which is calculated by multiplying the length E by the height 1. The percent of the rate increase that is earned is the area H divided by the area E.

With respect to points D and A, there are two trivial and four nontrivial possibilities not considering cases where D or A exactly equals zero or E. The two trivial cases are:

1. D and A are both less than zero and the rate increase is fully earned and;
2. D and A are both greater than E and none of the rate increase is earned.

Illustrated in Figure 3 are the four nontrivial cases.

Figure 3



Case 1 is the example we have been using and the area of the rectangle is:

Equation (2.4) $$H = \int_{0}^{A} \frac{(x-D)}{T}\,dx + \int_{A}^{E} 1\,dx$$

For case 2, A is greater than E. The formula would be the same except the second term in equation 2.4 drops out and the integration of the first term extends from 0 to E. To include both cases 1 and 2 equation 2.4 becomes:

Equation (2.5) $$H = \int_{0}^{Min(A,E)} \frac{(x-D)}{T}\,dx + \int_{Min(A,E)}^{E} 1\,dx$$

Note that for A > E the second term is the integral from E to E. This equals zero as expected.

For cases 3 and 4, D is greater than 0 and the integration begins at D rather than 0. Therefore, in the general case we begin our integration at 0 or D, whichever is greater and formula 1.5 becomes:

Equation (2.6) $$H = \int_{Max(D,0)}^{Min(A,E)} \frac{(x-D)}{T}\,dx + \int_{Min(A,E)}^{E} 1\,dx$$

See appendix 1 for the actual evaluation of this integral. The formula for P is found by evaluating this integral and dividing by the area of the total rectangle which is the length E multiplied by the height of one. Using appendix 1 and considering the trivial cases the formula for P is:

Equation (2.7)

$P = 1$ $\qquad\qquad\qquad\qquad A < 0$

$P = 0$ $\qquad\qquad\qquad\qquad D > E$

$P = 1 - \dfrac{A^2-B^2-C^2}{2ET}$ $\qquad A > 0$ and $D < E$

Where:

$A = D + T$

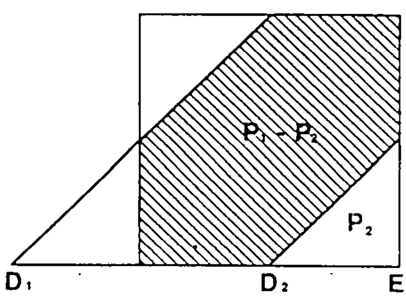
$B = \text{Max}\ (A-E,\ 0)$

$C = \text{Max}\ (D,0)$

MULTIPLE RATE CHANGES

Assume there are two rate changes during the period, effective at D_1 and D_2 as shown below.

Figure 4



Let P_1 be the portion of the total earned exposure from policies written between D_1 and E and let P_2 be the portion of the earned exposure from policies written between D_2 and E. It follows that the portion of the earned exposure from policies written between D_1 and D_2 is $P_1 - P_2$. The general equation for the average rate level during the evaluation period E is:

APPENDIX II
DERIVATION OF RATE ADJUSTMENT FACTORS

$$AR = (1 - P_1) + (P_1 - P_2)R_1 + (P_2 - P_3)R_2 + \cdots + (P_{n-1} - P_n)\, R_{n-1} + P_nR_n$$

Where AR equals the average rate level with rates R_1, R_2, $R_3, \cdots R_n$ rates indexed to the initial rate.

$$AR = 1 - P_1 + P_1R_1 - P_2R_1 + P_2R_2 - P_3R_2 + P_3R_3 - P_4R_3 + \cdots + P_{n-1}\, R_{n-1} - P_nR_{n-1} + P_nR_n$$

$$AR = 1 + P_1\, [R_1 - 1] + P_2\, [R_2 - R_1] + P_3\, [R_3 - R_2] + P_4\, [R_4 - R_3] + \cdots + P_{n-1}\, [R_{n-1} - R_{n-2}] + P_n\, [R_n - R_{n-1}]$$

$$R_i = R_{i-1}\,(1 + r_i)$$

Where r_i is the ith rate change

$$AR = 1 + P_1\, [1 + r_i - 1] + P_2\, [(1 + r_2)\, R_1 - R_1] + P_3\, [(1 + r_3)\, R_2 - R_2] + P_4\, [(1 + r_4)\, R_3 - R_3] + \cdots + P_{n-1}\, [(1 + r_{n-1})\, R_{n-2} - R_{n-2}] + P_n\, [(1 + r_n)\, R_{n-1} - R_{n-1}]$$

$$AR = 1 + P_1r_1 + P_2R_1r_2 + P_3R_2r_3 + P_4R_3r_4 + \cdots + P_{n-1}\, R_{n-2}\, r_{n-1} + P_nR_{n-1}r_n$$

Using Equation (3.2)

$$AR = 1 + P_1r_1 + P_2(1 + r_1)\, r_2 + P_3\, (1 + r_1)(1 + r_2)\, r_3 + \cdots + P_n(1 + r_1)(1 + r_2)\cdots(1 + r_{n-1})\, r_n$$

The rate level adjustment factor equals the current rate R_n divided by the average rate level or:

$$= \frac{Rn}{AR} = \frac{(1 + r_1)(1 + r_2)\cdots(1 + r_n)}{1 + P_1r_1 + P_2\, (1 + r_1)\, r_2 + \cdots + P_n\, (1 + r_1)(1 + r_2)\cdots(1 + r_{n-1})\, r_n}$$

Equation (3.1)

$$(1-P_1)\ R_0+(P_1-P_2)\ R_1+(P_2-P_3)\ R_2 + \dots + (P_{n-1}-P_n)\ R_{n-1}+P_nR_n$$

Where R_0 is the initial rate with R_1, R_2, R_3, ... R_n subsequent rates.

Please note that any P_i would be 1 if the rate change were fully earned before the evaluation period and would be 0 if the rate change were effective after the end of the evaluation period.

If one indexes all rates to the initial rate R_0, then $R_0 = 1$. By definition $R_i = (R_{i-1})\ (1+r_i)$ where r_i is the ith rate change implemented at time D_i. It follows that

Equation (3.2)

$$R_i = (1+r_1)(1+r_2).....(1+r_i)$$

Using equations 3.1 and 3.2, the following formula for the rate level adjustment factor can be derived (derivation shown in appendix 2):

Equation (3.3)

$$F = \frac{(1+r_1)(1+r_2)\dots(1+r_n)}{1+P_1r_1+P_2(1+r_1)r_2+\dots+P_n(1+r_1)(1+r_2)\dots(1+r_{n-1})r_n}$$

This formula is probably somewhat easier to program in a computer than equation 3.1. Note that if there is only one rate change during the period equation 3.3 yields:

$$\frac{1+r_1}{1+P_1r_1}$$

which is the familiar formula shown in equation (1.1).

EXPOSURES NOT WRITTEN AT A CONSTANT RATE

Up to this point we have assumed that exposures are being written at a constant rate throughout the period. In practice, the exposures are not written at a constant rate and the formulas above do not produce exactly the correct results. In most cases the error is negligible. Jim Ross in "Generalized Premium Formulas"[2] developed an integral for calculating P based on the assumption that exposures are growing at a constant compound growth rate. However, each individual rate change had to be integrated to calculate P for that rate change. By evaluating his integrals using generalized limits as we did in the constant exposure formula, a truly generalized formula can be developed for this model.

Assume that exposures are increasing at a constant rate and let V equal one plus the growth rate ($V \neq 1$). The formula for P, excluding the trivial cases, is:

Equation (4.1)

$$P = \frac{\ln V[T(1-V^{C})+BV^{B+E-T}-(A-C)V^{D-C}]+V^{D-C}+V^{E}(1-V^{B-T})-1}{(V^{E}-1)(1-V^{-T})}$$

Where:

$A = D + T$

$B = \text{Max}(A-E,0)$

$C = \text{Max}(D,0)$

EXAMPLE OF CONSTANT GROWTH RATE MODEL

Assume that the evaluation period is calendar year 1988 with annual policies and that a rate change was implemented on July 1, 1987. Also, assume that exposures are increasing at the rate of 60 percent per year.

$D = -.5$

$V = 1.6$

$E = 1$

$T = 1$

Therefore:

$A = .5$

$B = 0$

$C = 0$

Then P equals:

$$p = \frac{\ln(1.6)[-(.5)(1.6)^{-.5}]+(1.6)^{-.5}+(1.6)(1-(1.6)^{-1})-1}{(1.6-1)(1-(1.6)^{-1})}$$

$P = .91$

The following table illustrates the magnitude of the error when exposures are growing at a constant rate and the traditional parallelogram method is used (which assumes that exposures are being written at a constant rate). Assume we are converting calendar year 1988 earned premiums to current rates.

ACA12/053089

PORTION OF RATE INCREASE EARNED

Effective Date	Fixed Exposure	Growth Rate 20%	40%	60%
		Annual Policy Term		
4/01/87	.969	.973	.976	.979
7/01/87	.875	.890	.901	.910
10/01/87	.719	.744	.764	.781
1/01/88	.500	.530	.556	.578
4/01/88	.281	.307	.330	.351
7/01/88	.125	.141	.155	.168
10/01/88	.031	.036	.041	.045
		Semiannual Policy Term		
10/01/87	.938	.944	.949	.953
1/01/88	.750	.769	.784	.797
4/01/88	.500	.525	.545	.563
7/01/88	.250	.269	.286	.301
10/01/88	.063	.069	.076	.081

CONCLUSION

Computers are becoming an indespensible tool for the actuarial profession. The formulas derived in this paper can greatly simplify the task of programming the computer to convert earned premiums to current rates. These formulas are particularly helpful when one is dealing with multiple policy terms, and evaluation periods, or evaluation periods of different lengths.

FOOTNOTES

[1]Kallop, R., "A Current Look at Workers' Compensation Ratemaking," PCAS LX11 (1975)

[2]Ross, J. P., "Generalized Premium Formulas," PCAS LX11 (1975)

[3]Miller, D. L. and Davis, G. E., "A Refined Model for Premium Adjustment," PCAS LX111, (1976)

APPENDIX 1
INTEGRATION OF RATE ADJUSTMENT FORMULA

From Equation (2.6):

$$H = \int_{\mathrm{Max}(D,0)}^{\mathrm{Min}(A,E)} \frac{x-D}{T}\,dx + \int_{\mathrm{Min}(A,E)}^{E} 1\,dx$$

$$H = \left[\frac{x^2}{2T} - \frac{Dx}{T}\right]_{\mathrm{Max}(D,0)}^{\mathrm{Min}(A,E)} + \Big[\,X\,\Big]_{\mathrm{Min}(A,E)}^{E}$$

$$H = \frac{\mathrm{Min}(A,E)^2}{2T} - \frac{D\cdot \mathrm{Min}(A,E)}{T} - \frac{\mathrm{Max}(D,0)^2}{2T} + \frac{D\cdot \mathrm{Max}(D,0)}{T} + E - \mathrm{Min}(A,E)$$

Note that $D\cdot \mathrm{Max}(D,0) = \mathrm{Max}(D,0)^2$

$$H = E + \frac{\mathrm{Min}(A,E)^2 - 2D\cdot \mathrm{Min}(A,E) - \mathrm{Max}(D,0)^2 + 2\,\mathrm{Max}(D,0)^2}{2T} - \frac{2T\cdot \mathrm{Min}(A,E)}{2T}$$

$$H = E + \frac{\mathrm{Min}(A,E)^2 - 2D\cdot \mathrm{Min}(A,E) - 2T\cdot \mathrm{Min}(A,E) + \mathrm{Max}\;(D,0)^2}{2T}$$

Let $C = \mathrm{Max}(D,0)$

$$H = E + \frac{\mathrm{Min}(A,E)^2 - 2\,\mathrm{Min}(A,E)(D+T) + C^2}{2T}$$

Since $A = D+T$ and completing the square of the first two terms in the numerator we have:

$$H = E + \frac{(\mathrm{Min}(A,E) - A)^2 - A^2 + C^2}{2T}$$

Note that $\mathrm{Min}(A,E) - A = \mathrm{Min}(0,E-A)$ and for all E and A, $\mathrm{Min}(0,E-A) = (-1)(\mathrm{Max}(A-E,0)$

Therefore:

$$H = E + \frac{\mathrm{Max}(A-E,0)^2 - A^2 + C^2}{2T}$$

Let $B = \mathrm{Max}(A-E,0)$ *and factoring out a negative one we have:*

$$H = E - \frac{A^2 - B^2 - C^2}{2T}$$

The formula for P is found by dividing H by the area of the rectangle which is the length E multiplied by the height 1. Therefore, dividing the above equation by E yields

$$P = \frac{H}{E} = 1 - \frac{A^2 - B^2 - C^2}{2ET}$$

SEMINAR ON VALUATION ISSUES

Robert Miller
Robert Miccolis

WHAT IS VALUATION?

AN INTRODUCTION TO FUNDAMENTALS AND ISSUES

Miller: We've known for quite a while that the "title" of our portion of the program is "What is valuation? - An introduction to fundamentals and issues." Now we have to figure out what we're going to say. Do you have any ideas?

Miccolis: At least we're going to have to try to answer the question: "What is valuation?". And then we're going to have to identify some fundamentals and discuss them; and identify the important issues and discuss them.

Miller: One of those issues is "Why is valuation important to casualty actuaries?"

Miccolis: I agree. I think the answer to that question lies in the close relationship between valuation on the one hand and ratemaking and reserving on the other. I think we ought to discuss that point in some detail, but before we do that we ought to discuss the definition of valuation in depth and try to describe the fundamental concepts of the valuation process.

I. DEFINITION OF VALUATION

Miller: If we look at the dictionary definition of valuation, some of the phrases used are: "Act of valuing; appraisal; appraised price".

Miccolis: Notice the words "appraisal" and "appraised" in that definition.

Miller: I hear them. But I also notice the word "valuing". Let's see what the dictionary says about that.

Miccolis: "Value" can be used as a verb or as a noun. The definition of the verb is: "To estimate the value or worth of; to appraise."

Miller: That definition brings up a couple of points.

First, our whole discussion deals only with valuations where <u>property/casualty contingencies</u> are involved.

Second, the word "appraise" in the definition reminds me that for the most part, casualty actuaries think of an <u>appraisal</u> as being made in connection with a merger or acquisition. That's a narrow use of the word. In fact, used in that sense, an appraisal is just one form of what we mean by a valuation.

Miccolis: Your point about property/casualty contingencies should not limit our focus only to dealing with property/casualty losses. In an appraisal situation the valuation could encompass assets, liabilities other than losses, future profitability, and future growth in business.

Miller: That means that casualty actuaries need to learn more about analyzing the investment, operations, and production sides of the insurance business.

Miccolis: Good point. Another point to notice is that the definition of "value" as a verb says: "To <u>estimate</u> the value ... of". The use of the word "estimate" implies there is <u>uncertainty</u> in valuation. Also, the definition of "value" as a verb is based on the definition of "value" as a noun.

Miller: Let's go to the dictionary again. The definition of "value" as a noun includes: "Monetary worth of a thing; marketable price." So I'd like to say that for purposes of this discussion, "<u>value</u>" is something that is measurable in terms of money.

Miccolis: Let's see how all these ideas fit with the definition of valuation that our Committee on Valuation Principles has decided on after taking into account the comments and suggestions we have received about the exposure draft of the Statement of Principles.

The definition in the revised Statement of Principles reads as follows:

> Valuation is the process of <u>determining and comparing</u>, for the purpose of assessing a <u>risk bearer's</u> <u>financial condition</u> as of a given date, called the <u>valuation date</u>, the values of part or all of the risk bearer's <u>obligations</u> and the <u>assets</u> and <u>considerations</u> designated as <u>supporting</u> those obligations.
>
> A valuation is carried out in accordance with specified <u>rules and assumptions</u> <u>selected or prescribed</u> in accordance with the <u>purpose</u> of the valuation to produce an assessment of the risk bearer's financial condition.

Miller: That definition is a gold mine of subjects to be discussed. One of those subjects is that the definition doesn't even mention property/casualty contingencies.

Miccolis: That's because the definition is embedded in a Statement that begins by saying that its purpose is to "identify and describe principles applying to property and casualty valuations". So this point doesn't need to be repeated in the definition.

Otherwise the definition in the Statement seems to be generally consistent with the dictionary definitions. Of course, the Statement definition is considerably narrower than the very general definitions given in the dictionary.

Miller: As I said a little while ago, there are a lot of things in the definition that we need to discuss. But I think that before we discuss those points we ought to go over some of the fundamental concepts underlying the valuation process.

PURPOSE OF VALUATION

Miccolis: A little while ago you said that: "An appraisal is just one form of what we mean by a valuation." The purpose of an appraisal is to determine the net worth of the target of a merger or acquisition. Once determined the net worth can be used in agreeing on the financial arrangements for completing the deal.

Other forms of valuation can be made for the purpose of:

- analyzing an insurer's financial strength, i.e. determining solvency;
- planning for future growth and profitability;
- determining how well an insurer is performing; or
- managing an insurer's risks

Miller: The forms you mentioned all relate to the valuation of an insurance company. Couldn't a self-insurer use a form of valuation to determine the assets it would need to prudently support the retention of its own property/casualty claims.

Miccolis: You're absolutely right. A valuation for a self-insurer could also be used to assess how successful its risk management program had been in the past and to project how successful it might be in the future.

Miller: Notice that all of the "other forms" of valuation that you have listed relate to matters that are - or should be - of great interest to the management of an insurance company or a self-insurer. Relatively few actuaries are engaged in appraisals; but a great many actuaries are involved in work that is at least related to the other forms of valuation. These other forms of valuation are a large part of the reason why the subject of valuation is important to casualty actuaries.

Miccolis: A little later we can talk about how traditional actuarial work like ratemaking and reserving is related to valuation, but for now let's talk about several different forms of valuation and let's start with appraisals for mergers and acquisitions.

Miller: Presumably there should be at least two distinct appraisals in connection with each merger or acquisition - one for the buyer and one for the seller. That would imply that there could be at least two distinct results for valuations carried out as of the same date with respect to the target of the merger or acquisition.

Miccolis: I'm not sure that these are necessarily two distinct appraisals. Certainly the buyer and seller would use different assumptions. Of course, the buyer would be interested in paying the lowest possible price and the seller would be interested in getting the highest possible price. But even in the absence of these reasons, there could be real differences in aversion to risk, earnings objectives, synergy of operations, economies of scale, efficiencies of distribution systems, etc.

Miller: Wouldn't these differences produce different valuations by actuaries representing each party. In the end, if the merger or acquisition is to be consummated, both parties will have to agree on a single result which may be different from the results of the valuations.

Miccolis: On the other hand, every valuation involves uncertainty. This implies that actual results are quite likely to differ from those anticipated in determining the most likely value of the target by either the buyer or the seller.

Miller: An appraisal is one form of valuation in which an actuary must consider the views of the participants in the merger or acquisition as to the reasonableness of the assumptions used to estimate value.

Miccolis: Let's turn from appraisals to balance sheets and income statements produced internally by an insurer to assess its financial strength and earning performance. Here, the rules and assumptions for the valuation will be approved by management, hopefully after considering the recommendations of actuaries, investment managers, tax planners, marketing managers, legal counsel and other professionals.

Miller: Where the actuary will have to consider input form other types of professional, the actuary will still have the ultimate responsibility for the reasonableness of the assumptions and the results of valuation. He is usually going to have to evaluate the reasonableness of the assumptions developed by the other professionals in making recommendations on rules and assumptions.

Miccolis: Let's turn now to the usefulness of valuations in assessing earning performance. We ought to point out that there may be some difficulty in using the results of valuations for this purpose. The definition of valuation in the Statement of Principles focuses on an insurer's financial condition as of a given date. This is the same thing as focusing on the balance sheet. In general, valuation processes that focus on the balance sheet are not ideally suited to developing good information as to the insurer's earning performance. The statutory annual statement is a good illustration of this point.

Miller: But, if it's possible to produce a valuation that gives a reasonably good measure of economic net worth, it should be possible to develop the same kind of measure for the purpose of determining how well an insurer is performing. If a valuation as of a given date provides a good estimate of an insurer's economic net worth as of that date, then the economic earnings for a period would be equal to the difference between the amounts of net worth developed by valuations made at the beginning and end of the period. The rate of return on equity could be obtained by dividing the economic earnings for the period by the economic net worth as of the beginning of the period. An insurer might be considered to be performing well if it is achieving a satisfactory rate of return on economic net worth with a satisfactory degree of year-to-year stability.

Miccolis: Valuation can also be used as an aid to planning for future growth and profitability. However, to do this, more than one type of valuation must be used.

We've just talked about using valuations for assessing earning performance. Planning would involve projecting valuations of this type into the future. It would also be necessary to project statutory balance sheets to answer questions such as whether rapid growth may bring about some regulatory intervention. In addition, it would be necessary to project taxable earnings so as to take into account the effects of federal income taxes and GAAP earnings for stock companies so as to determine how the company's performance will look to securities analysts and investors.

Miller: The valuations used to assess earnings performance would have to be carried out using the same set of assumptions from period to period so as to assure that the projected results are consistent with each other in that respect. The need for consistency puts a substantial constraint on the assumptions to be used, but the constraint has to be accepted if truly useful results are to be achieved.

Miccolis: This need for consistency carries over into making appraisals because it's important to know such things as how much of a company's net worth will be available for supporting growth and how much of its earnings will be available for distribution to policyholders or shareholders.

Miller: One of the objectives of an insurer's management should be to manage the risks assumed by the insurer. The process of risk management involves identification, control, financing and transfer of risks. It would be an impossible task for management to identify and control every risk faced by an insurer. So it becomes necessary to simplify the task.

Miccolis: In the financial world the value or price of an asset is affected by the perceived uncertainty involved in the income produced by the asset and the future market value of the asset. The degree of that uncertainty is affected by the nature of the asset. For example, the value of a common stock is affected by greater uncertainty than the value of a Treasury bond. The degree of uncertainty is also affected by the environment; for example, the values of stocks are less stable in an unsettled economy.

The values of debt are affected by the same sources of uncertainty.

Miller: One way of simplifying the job of identifying and controlling an insurer's risks is to sort all assets and liabilities into broad classes, so that all the elements in each class have about the same degree of uncertainty. The next steps would be to determine the degree of uncertainty of each of the classes as a whole and then to determine the degree of uncertainty of the aggregate of all of the classes.

Miccolis: With this information in hand, the management would be able to determine whether the degree of uncertainty of the aggregate of all of the classes was acceptable in the light of its goals in the areas of stability and profitability. The information should also be helpful in identifying ways in which to reduce uncertainty so as to improve stability or to accept somewhat greater uncertainty in such a way as to improve earnings.

Miller: Maybe we could bring this part of the discussion to an end by explicitly pointing out that both statutory and GAAP balance sheets are forms of valuation. They allow some latitude for making assumptions but either of them can produce an assessment of financial condition that is a long way from a valuation of economic net worth. Furthermore, balance sheets and other financial statements do not give any indication of the degree of uncertainty in the result.

Miccolis: We have talked about the use of valuations in appraisals and in performance measurement. One of the other areas we mentioned was in the determination of solvency and the related issue of capital/surplus requirements.

Miller: This is an important point. In the life insurance field the term "valuation actuary" has become almost synonymous with an actuarial opinion of an insurer's solvency. Of course, the need for actuarial loss reserve opinions for property/casualty companies has been principally a result of solvency issues. For life companies the solvency concern has been particularly acute where a company can be significantly affected by interest sensitive products, guaranteed interest rate contracts, and policy loan provisions. By requiring a valuation actuary to render an opinion as to the solidity of the company, the regulatory authorities hope to decrease the frequency of insolvencies and to limit the extent of the insolvencies.

Miccolis: For Canadian property/casualty companies I believe that an opinion from an actuary is currently required that addresses the adequacy of both the liabilities and the assets. I do believe, however, that many casualty actuaries feel that an opinion for loss reserves is all that is necessary. This may be due to a perception that much of the industry's problems are the result of insufficient loss reserves and that interest rates do not have a material effect on loss payments or on the company's ability to pay the losses.

Miller: I believe that those property/casualty companies that discount their reserves are just now becoming aware of their need to more closely monitor their cashflow, investment income and asset values. In addition, because of the number of recent insolvencies, especially where there is a significant amount of assumed or ceded reinsurance, the interest in cashflow analyses has increased substantially.

II. FUNDAMENTALS

CASH FLOWS

Miccolis: Given the definition of valuation and the purposes of valuation that we have discussed, I think we should mention the basic, fundamental elements used in determining economic value.

Miller: It's possible to look at an insurer's business as being made up of a sequence of potential cash flows spread out over time after the valuation date. These cash flows are receipts or disbursements of cash. An example of receipts would be the interest and dividends received from investments; disbursements would be the payment of losses. There are many other potential cash flows associated with the operation of an insurance company.

Miccolis: We can group cash flows according to the activity that generated the cash flows. For example, municipal bonds are assets whose cash flows are: (1) the payment of cash to purchase the bonds, (2) the interest received in cash, and (3) the cash flow received from the sale or maturity of the bonds.

Miller: This means that every asset, obligation or consideration is associated with one or more items of cash flow. In fact, the definitions of assets, obligations and considerations in the Statement of Principles are stated in terms of receipts and disbursements.

Miccolis: Before we go any further perhaps we should review what we mean by assets, obligations and considerations. We could use loss reserves as an example of obligations, premiums as an example of considerations, and stocks and bonds as an example of assets.

Miller: That's fine but I think we should point out that those are only examples and that there are many other type of transactions that should be classified into one of these categories.

Miccolis: We should also note that there can be significant interdependencies among cash flows connected with the same asset, with assets belonging to the same class, or with assets that are affected by similar kinds of environmental change. Similar statements apply to cash flows connected with obligations or with considerations.

Miller: In fact, there are even interdependencies among assets, obligations and considerations.

The savings and loan industry provides some interesting examples of interdependency between assets and obligations. When market interest rates rise above the rates being credited by savings and loan associations there is a tendency for the public to reduce the rate of flow of deposits or increase the rate of withdrawals. At the same time there is a tendency for borrowers to cut as far back as possible on prepayments of loans, which are mostly long term mortgages. Together, these two tendencies shorten the life of S&L obligations and lengthen the life of assets. The S&Ls have the least cash when they need it most.

When market interest rates go down the flow of deposits tends to strengthen, withdrawals drop in volume and the flow of prepayments increases. This lengthens the life of obligations and shortens the life of assets. The S&Ls have the most cash when they need it least.

Miccolis: Property/casualty insurers are not nearly as much affected by this particular type of change in the environment. However, when interest rates drop there is a tendency for borrowers to exercise the call provisions in their bonds. This has the effect of giving insurers more cash to invest when the available rates of return are less attractive.

Miller: Property/casualty insurers are significantly affected by inflation, however, and the timing of changes in inflation versus changes in interest rates can have a material impact on the economic value of an insurer.

Miccolis: The impact of inflation would change the amount of the losses to be paid in the future. The changes in interest rates would affect the investment returns on the insurer's investments.

Miller: Asset/liability management is certainly closely involved in the valuation process. The same methods and techniques used to effectively manage the cash flow risks associated with cash inflow from investments and cash outflow from paid losses can also be used in a valuation. The management of interest rate risk and inflationary risk also focuses on the impact that these risks would have on the value or earnings of an insurer.

Miccolis: We should also remember that the timing of the payment of losses can vary significantly and thereby affect the asset/liability matching and the overall economic value of the company.

Miller: A valuation is intended to produce an assessment of an insurer's financial condition. Conceptually, an insurer's "true" net worth - if there is such a thing - is equal to the combined value of all of its potential cash flows. In determining the combined value it would be necessary to take into account the offsetting nature of receipts and disbursements. It would also be necessary to take into account the uncertainty inherent in each potential cash flow as well as the interdependencies among the potential cash flows. In other words, the combined value of the cash flows is not a simple algebraic sum.

Miccolis: Traditionally, actuaries have been concerned with reserving and ratemaking. Today actuaries have to be concerned with the <u>financing of risk</u> also. In other words we have to become familiar with the risks involved in investments. This part of the business has become vastly more complicated than it was a few years ago when interest rates were stable and many of the types of investments now readily available, such as "junk bonds", were either non-existent or hardly ever heard of. Asset defaults have been around for a long time, but these "junk bonds" may increase default rates.

Miller: One implication of what you have just said is that there will have to be more communication between the underwriting and investment sectors of the insurance business. This doesn't mean that actuaries will be deciding on investment strategy or even that they will have to become familiar with the process of assigning credit ratings to investments. However, they will have to become familiar with the risks involved in investment and how investment performance is related to credit rating. In addition, actuaries will need to tell investment managers about the loss payment and variation characteristics of the risks being underwritten by the insurer. This will enable the investment managers to develop suitable strategies for financing those risks.

RISK BEARER

Miccolis: We have been talking about applications of valuations to insurance companies and possibly to self-insurers that are subject to property/casualty contingencies.

Perhaps we should rely on a more general understanding of what type of entities are subject to property and casualty contingencies and we can refer to such an entity as a "risk bearer". Hence a self-insurance pool, risk retention group, or individual self-insurer could be considered a risk bearer.

Miller: That's sounds good. Let's define a "risk bearer" as a person or other entity that is exposed to the risk of financial losses that may arise out of the occurrence of specified contingent events during a specified period of exposure.

Miccolis: The words "specified period of exposure" imply that a given person or other entity can be a risk bearer at some times and not a risk bearer at other times.

Miller: The words "specified contingent events" mean any types of event you care to specify as being subject to property and casualty valuations.

Miccolis: The words "person or other entity" are very broad too. It would seem to include individuals who knowingly decide they'll take the risk of "going bare" for certain types of coverage, such as professional liability, for example.

Miller: Some persons will object that there is no practical way to perform a valuation for an individual but I feel that practical considerations would limit the application to an individual.

Miccolis: However, as a conceptual matter, a person who "goes bare" has undertaken an obligation to pay whatever losses may arise while he is "bare". The individual's assets supporting his self-insured's obligations are probably inadequate because of the degree of uncertainty involved in the losses that might occur.

SEGMENTATION

Miller: The definition of valuation refers to the determination and comparison of the values of part or all of a risk bearer's obligations and the assets and considerations designated as supporting those obligations.

This means that a valid valuation can be made for some segment of an insurer's obligations as long as the assets and considerations supporting those obligations are specifically identified.

Miccolis: The idea of obligations being "supported" by assets and considerations is related to the financing of risk. In order to be able to finance a risk, a risk bearer has to have funds available to pay a loss arising out of the risk at the time the loss occurs. We have said that assets and considerations are intended to provide funds for paying those obligations. In the valuation process, it is necessary to identify the assets and considerations that support the obligations being considered in the valuation.

Miller: The concepts of obligations and the supporting assets and considerations should include "commitments", where a commitment is a promise to do something.

For example, a reinsurer can promise to reinsure all of the policies written by a direct writer within a treaty year which may not end until after the valuation date. Policies as yet unwritten will create cash flows arising out of periods of exposure that have not begun as of the valuation date. This commitment, made before the valuation date, should be taken into account in a valuation.

Miccolis: Another example relates to the asset side of the balance sheet. An insurer may make a commitment before the valuation date to buy a certain security after the valuation date. The values of the cash flows arising out of this transaction should be taken into account in a valuation.

Miller: In appraisals, the value of future new business that may be developed after the valuation date is often taken into account. Future new business involves commitments that have not been made as of the valuation date but are projected to be made after the valuation date.

Miccolis: This is an interesting point because it helps to make clear what periods of time may be taken into account in valuation. Some valuations take into account only cash flows that are related to commitments made on or before the valuation date. On the other hand, an appraisal often takes into account not just the cash flows arising out of commitments made on or before the valuation date but also takes into account cash flows arising out of commitments projected to be made after the valuation date.

DETERMINING AND COMPARING

Miller: Let's be clear on what we mean by valuation as the process of determining and comparing values of obligations and supporting assets and considerations.

Miccolis: This makes it sound as though the determination and comparison are separate sub-processes with determination coming first. Conceptually, it would be possible to determine the value of the obligations without reference to the supporting assets and considerations and vice versa. Such determinations would not directly reflect the interdependencies among the cash flows associated with the obligations, assets and considerations. By comparing values we mean that the process will account for the interdependencies between obligations, assets and consideration.

Miller: This is the practical way to go. In fact, this is the way that the Society of Actuaries' Committee on Valuation and Related Areas attacked the problem. This mode of attack resulted in the description of the C-risks relative to assets, obligations and interest rates.

However, the problem of assessing the combined effect of these types of risk or uncertainty remains substantially unresolved despite the publication of an extensive, sound and interesting report by the Society of Actuaries' Combination of Risks Task Force.

Miccolis: The comparison step is the one that provides the basis for assessing the financial condition of the risk bearer. In effect, this step compares the positives with the negatives after taking interdependencies into account and comes up with a combined value. As we have said before this is not a simple algebraic difference.

Miller: The financial condition of the risk bearer depends on both net worth and the degree of uncertainty in the result of the valuation.

RULES AND ASSUMPTIONS

Miccolis: The application of valuation concepts requires the selection of rules and assumptions to be applied in carrying out the valuation. We should talk about what kinds of assumptions have to be made.

Miller: In theory, an assumption has to be made about each of the variables affecting every item of cash flow. This would be an impossible task. One solution is to make assumptions about the variables affecting classes of assets or classes of obligations or perhaps even all assets or all obligations. Of course, there are many other ways to select assumptions.

Miccolis: Environmental conditions also affect the degree of uncertainty in the valuation. Accordingly, assumptions have to be made as to whether and how to recognize this fact.

In short, assumptions have to be made about all the factors that:

- are used in the determination of value, or
- would produce uncertainty in a basic cash flows used for the valuation.

Assumptions that may restrict the recognition of uncertainty may simplify the process of carrying out a valuation. However, at the same time such assumptions may result in a valuation that is too removed from reality.

Miller: Valuations made for management purposes should reflect reality to the fullest extent practical. Valuations made for other purposes might be based on assumptions that deliberately move away from reality; for example this is true of the statutory statement which is supposed to produce a conservative assessment of an insurer's financial condition.

VALUATION VARIABLES

Miccolis: While we recognize that many assumptions are needed to carry out a valuation, we should identify the basic variables that are the building blocks of each assumption. The four valuation variables that we have discussed many times are:

- the occurrence of the cash flow,
- the amount of the cash flow,
- the interval of time between the valuation date and the date of occurrence of the cash flow, and
- a rate of interest related to the interval of time between the valuation date and the date of occurrence of the cash flow.

INTEREST RATE

Miller: For one of those variables, the interest rate, a question that comes up frequently is what interest rate should be used in a valuation - assuming that a rate different from zero is to be used. Over the years, a great many opinions have been expressed on this point. You have described "a rate of interest related to the interval of time between the valuation date and the date of occurrence of the cash flow". This doesn't say whether the rate should be a risk adjusted rate, a risk free rate, an after tax rate or any other type of rate.

Miccolis: You're right. The choice would be dependent upon the actuary's purpose in making the valuation. His choice may be affected by the opinions of others or by regulations.

Miller: In particular, it should be noted that because the rate is described as "related to (an) interval of time" the effects of the yield curve can be taken into account. The statement puts no limit on the actuary's ability to recognize differences in yield resulting from differences in asset quality or from differences in taxability. The description has been deliberately kept very general.

Miccolis: Deciding on the rate to be used in discounting cash flows related to obligations and considerations involves answering some other questions. Should an average rate be used? If so, what happens when an important shift in the mix of business occurs and the length of the payment tails is significantly lengthened or shortened? Does the company have a policy of matching the durations of assets and obligations? Does the insurer have a policy of trying to predict trends in interest rates? If so, how much risk is the insurer willing to take in this area? How may net investment income be affected by this risk? Does the insurer have a policy of maximizing net income by maximizing net after-tax investment income? How is the discount rate related to whatever rate may be recognized in pricing?

Miller: You bring up many important issues that actuaries will need to deal with in performing valuations. I don't think there will be any hard and fast rules that would apply in all circumstances.

DEGREE OF UNCERTAINTY

Miccolis: In evaluating a company's cash flow in terms of each valuation variable associated with a given asset, obligation or consideration it is important to reflect the degree of uncertainty involved. The uncertainty will depend upon:

- o the nature of the asset, obligation or consideration,
- o the various environments (e.g. regulatory, judicial, social, financial and economic environments) within which the valuation is being performed, and
- o the predictive reliability of the data on which assumptions relative to the variables are based.

Miller: In the light of the uncertainty and interdependencies affecting cash flows, assets, obligations and considerations and the further uncertainty arising out of the process of combining them, it is plain that the assessment of financial condition produced by a valuation involves both a measure of net worth and uncertainty. Some of methods for measuring and controlling the degree of uncertainty are addressed in the papers presented at this meeting.

RESULT OF A VALUATION

Miccolis: This would probably be a good time to switch to a discussion of what is the result of a valuation. First, we ought to discuss what we mean by "a valuation".

Making a valuation requires the development of a model. The model may be probabilistic or deterministic. If it is probabilisitic, for practical reasons such as those we have already discussed, it cannot take into account all the sources and degrees of variability in cash flows and combinations of cash flows. Thus, it will be necessary to make assumptions to make it feasible to use the model. Such a model would run several times to determine the average result and the degree of variability in the results. The actuary would need to test the effects of changes in assumptions on the average result and the degree of variability in the results.

Miller: If one approaches the valuation problem in this way, what is the result of a valuation?

Is each run of the model a valuation? It could be, but probably isn't. At the very least it suffers from the fact there is no experimental evidence as to what the degree of variability of the result might be.

Is the result of a valuation the average result of several runs of the model with a given set of assumptions? Once again, it could be but probably isn't. This result would suffer from a failure to have examined the effects of changes in the assumptions.

Is the result of a valuation a value and a degree of variability selected from the results of probabilistic runs of the model with different assumptions? This could be the result. But notice how much it depends upon judgment in selecting assumptions and in selecting a representative value and degree of variability on the basis of the outputs of the model under different sets of assumptions.

Miccolis: Deterministic models have been used to develop answers to practical valuation questions. Typically these models have made strongly constraining assumptions about the probability of occurrence of cash flows and the time of their occurrence. Interdependencies of cash flows have been built into them on a deterministic basis. Insurer reactions to cash shortfalls have been prescribed in advance. Several interest rate scenarios have been defined in advance. Examination of the results produced by the model using each set of assumptions in turn has been the basis of statements about the combined value of given sets of obligations and assets. Some idea of the effect of changing assumptions has been derived from arranging the values in order of size and noting the differences and making judgments as to the probability of occurrence of conditions roughly similar to those assumed to underlie each result.

Miller: Still, when all is said and done, we have to select a single value and a single measure of the degree of uncertainty in the value to describe the result of what we have done. But why shouldn't a valuation be any of the things we have described?

Miccolis: If we have done all that is needed to select a value and a measure of uncertainty, then the result has to be examined to determine whether risks embodied in the system have been satisfactorily controlled and financed or whether some increase or decrease in value or uncertainty would be desirable.

Miller: The association of a measure of uncertainty with the result of the valuation characterizes the principal difference between the result of a valuation and the surplus shown in a conventional balance sheet. Conventional surplus is simply the difference between assets on the one hand and liabilities on the other.

Miccolis: Of course, with suitable choices of assumptions the valuation process can produce a statutory balance sheet. The most important assumptions would be that there is no risk connected with admitted assets; that actual losses will not differ from those reflected in the reserves in the statement; and that the value of loss payments is not discounted for investment income. Notice that these assumptions effectively remove uncertainty from the picture.

VALUE

Miller: A valuation based entirely on cash flow analysis would give a good estimate of economic net worth. This makes the idea of using this type of valuation to assess an insurer's financial condition sound deceptively straightforward and attractive.

You and I both know that it's extremely difficult, if not impossible, to produce a valuation based entirely on cash flow analysis. This should be evident from our discussion of the complications involved in combining the values of cash flows and the assets, obligations and considerations with which they are connected. But, for now we can say that suitable choices of assumptions will enable the production of a valuation that comes close to estimating economic net worth. Even so, there is real difficulty in getting an adequate measure of the degree of uncertainty or risk involved in the valuation.

Miccolis: This is a critical point because to a great extent the assessment of an insurer's financial condition depends upon the stability and profitability of its operating results. In order to achieve stability and profitability, an insurer must be a competent manager of risk. The less the uncertainty, the greater the stability. But reductions in uncertainty tend to reduce profitability. As usual, the trick is to achieve a good balance between risk and profit.

Miller: A question similar to "What is valuation" is "What is the meaning of value". For any item of cash flow, the answer to this question is that the meaning depends upon the assumptions made as to the valuation variables related to that item of cash flow. In other words, value for the purposes of a valuation is a function of the assumptions made in carrying out the valuation.

Miccolis: It is conceivable that the only assumptions as to any item of cash flow could be descriptions of:

- a probability density function for each of the valuation variables associated with the item of cash flow;
- the interdependencies (correlations) of each of those variables with the other variables associated with the item of cash flow; and
- the interdependencies associated with each of the other items of cash flow involved in the valuation.

I can't think of another set of assumptions that would come closer to developing the "true" value of each item of cash flow. At the same time, the idea of trying to carry out a valuation on the basis of such assumptions boggles the mind with its complexity.

Miller: On the other hand, I'd say that the closer you can come to producing a valuation on this basis the closer you can come to developing a good estimate of the economic value of the insurer or self-insurance system being valued. This once again highlights the importance of selecting assumptions that will most closely imitate reality and at the same time be completed for reasonable amounts of time and effort.

RELATIONSHIP TO RESERVING AND RATEMAKING

Miccolis: The definition of valuation refers to determining and comparing the values of obligations and the corresponding assets and considerations. The comparison step draws an important distinction between the valuation process on the one hand and the reserving and ratemaking processes on the other.

Miller: There is a question as to how the valuation process is related to the reserving and ratemaking processes.

Let's discuss the relationship with the reserving process first. Loss payment triangles are regularly used as a basis from which to project future loss payments. Conceptually, each of the elements in the triangle is the product of a number of payments and the average amount of payment. The aggregate of the amounts of the payments in an element of the triangle is related to a particular interval of time which can be assumed to be concentrated in a point. This enables the determination of a time interval between that point and any other point in time such as a valuation date. Then, if the person developing the reserve wishes to do so he can associate an interest rate with interval of time from the valuation date. Finally, it can be assumed that the numbers, amount, time interval and interest rate are all "experimental" values resulting from some random processes which can be summarized in some formula that can be used to project future values for corresponding intervals of development. This formula might even provide a way to estimate the degree of uncertainty in expected values that could be calculated with the help of the formula.

Miccolis: Plainly, the information and assumptions used in the reserving process could be used in valuation. It's plain also that the same "valuation variables" are involved in both processes. It's common knowledge that the degree of uncertainty in reserves is influenced by the nature of the coverage to which the reserves are related, the various environments in which the coverage is provided and the predictive validity of the data used to make the projections of payments. Finally, there are interdependencies between the amounts of payments. For example, it is a common experience that the average amount of a loss payment that is long delayed after the date of occurrence tends to be larger than the amount of a loss that is paid quickly.

In other words there is a very strong relationship between valuation and reserving. The key difference is that "comparing" step in the valuation process.

Miller: Let's take a quick look at the ratemaking process.

Assume a block of policies providing a specified form of liability insurance is to be issued over a given interval of time, such as a policy year. The assumption about liability insurance is made simply to convey the idea that there will be available some information about the distribution over time of loss payments and possibly even numbers of claims. The payments can be assumed to be concentrated in a series of points in time one year apart. The issue of the policies can be assumed to be concentrated at the mid-point of the policy year. These assumptions help to define the intervals of time between the date of premium payment and the date of loss payment. Knowledge of current market interest rate conditions and the insurer's investment and tax strategies will help in the choice of an interest rate if interest is to be taken into account explicitly. Finally, just as in the case of the reserving process we described earlier, it can be assumed that the numbers, amount, time interval and interest rate are all subject to some random processes which can be summarized in some formula. This formula might even provide a way to estimate the degree of uncertainty in expected values of occurrence, amount, timing and interest rate.

Miccolis: Plainly, the information and assumptions used in ratemaking are closely related to the information and assumptions used in reserving. Accordingly these kinds of ratemaking information and assumptions could be used in valuation. The projections would be subject to the same kinds and sources of uncertainty as those encountered in the reserving and valuation processes.

In other words there is a very strong relationship between valuation and ratemaking. The key difference is that "comparing" step in the valuation process.

Miller: The importance of these relationships is that large numbers of the membership of the CAS are already actively involved in work that is closely related to valuation. That work is directed at the determination of the aggregate amounts of loss costs and expenses. Valuation is directed at the determination of the values of those amounts once the fundamental ratemaking and reserving work has been done. Valuation further involves the comparison of those values with the values of the supporting assets.

Miccolis: We should also say that the inclusion of interest as one of the valuation variables is not intended to mean that we are advocating changes in rating and reserving practices. What we are doing is recognizing what has been a fact of financial life for centuries. Money has time value. In practice, ratemakers and reservers have assumed, for reasons of conservatism, that rates and reserves should not explicitly take interest into account.

Miller: This is just one of the many kinds of important assumptions that can be made about how specified items of cash flow may be affected by any of the valuation variables. Other examples are the common assumptions that all policies are written in the middle of the year or that all cash flows occur either in the middle of the year or at the beginning or end of the year or at a fixed period of time after a policy is written. These assumptions are made to simplify analysis without significantly reducing the usefulness of projections based on them. Projections are most useful when they closely imitate reality without creating undue expense.

RESEARCH

Miccolis: Listening to these questions raises another question - "What valuation questions are most in need of research?"

Surely the question of what discount rates should be used is one of them. There is currently a considerable amount of work being done on this question.

Another is the development of a method for using cash flow analysis to assess risk in a portfolio of obligations, especially loss reserves. There is a method for determining the variance of a pure premium but it does not deal with the variability of the distribution of loss payments over time nor the change over time in the average amount of loss payments arising out of a given period of exposure. Likewise there is little, if any, published research on the question of how the random variability of timing of payments may affect present values of losses. The papers presented at this seminar, however, definitely will contribute to this research.

Miller: Until recently there has been surprisingly little research reported in the area of methods for using cash flow analysis to assess risk in a portfolio of assets. Financial analysts have done some work in this field and life actuaries have also published some papers. However, much remains to be done, especially in the area of assessing differences in risk by quality of assets. Another question is how to optimize the risk/reward balance in a portfolio of assets.

Miccolis: As we have already said, the Combination of Risks Task Force of the Society of Actuaries Committee on Valuation and Related Areas has done some analysis on the question of the interdependencies of the asset, obligation and interest risks. But a lot more could be done.

Miller: Interest rate risk has been the subject of a great deal of research and a large volume of literature has been produced in recent years.

So, to sum up, the greatest present need and opportunity for research seems to lie in the fields of the obligation risks and the asset risks. Perhaps after this work has matured some, it will be more useful to make further investigations into the problem of combination of risks.

Miccolis: We could go on to discuss the individual principles of valuation and the relationship between principles and standards of practice. However, we probably should take some questions at this point.

THE VALUE OF LIABILITIES IS EQUAL TO THE COST OF ASSETS NEEDED TO OFFSET THEM

John C. Burville

THE VALUE OF LIABILITIES IS EQUAL TO THE COST OF ASSETS NEEDED TO OFFSET THEM

By John C. Burville

BIOGRAPHY:

John Burville is a Principal and Consulting Actuary with Tillinghast, a Towers Perrin company, in their Bermuda office. Prior to joining Tillinghast in 1986, he was Vice President of AIRCO (Bermuda), a subsidiary of AIG. He has a Ph.D. and first class honors degree in mathematics from Leicester University in England. Mr. Burville became a Fellow of the Institute of Actuaries in 1975, until 1986 most of his actuarial career on life insurance and reinsurance, since that time has been involved with P&C Insurance consulting in Bermuda. He became a member of the American Academy of Actuaries in 1977.

ABSTRACT:

Current accounting techniques for P&C Insurance companies do not represent the real values for assets and liabilities. Discounting is now a major issue, which has been brought more to the fore with the Tax Reform Act of 1986. Apart from some special situations* unpaid liabilities are represented at their undiscounted value.

Cash flow techniques are becoming recognized as realistic methods of valuation. Methods which use a discount rate to determine the value of liabilities can be enhanced by establishing a model portfolio of assets which match the projected liability cash flow. The value of the unpaid liabilities can then be measured as the market value of the model assets.

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*Some medical malpractice reserves are discounted, as are workers' compensation pension cases.

THE VALUE OF LIABILITIES IS EQUAL TO THE COST OF ASSETS NEEDED TO OFFSET THEM

Accounting rules for insurance companies reflect hypothetical values for assets[1] and liabilities. There are many reasons for this posture. However, with the prevalence of computerized information, techniques which were considered impractical ten years ago are now very possible. For instance, a company can now determine the market value of held quoted assets monthly, if not daily. Consequently, the market value of assets can be easily estimated.

For the most part, liabilities have been accounted for on an undiscounted basis. No credit is given for future investment income. Discounting of the liabilities to recognize investment income before a loss is finally settled, has been a strongly debated issue for the industry. The most recent resolution to this debate was the requirement to discount for the calculation of taxable income (Tax Reform Act of 1986).

This paper presents the view that the real value of the insurance liabilities is equal to the market value, or current cost, of assets needed to offset those liabilities.

<u>Example</u>:

> In a simplified form, a liability for $5 million to be paid in exactly five years' time, has a value equal to the cost of a five year zero coupon bond (plus credit risk costs) with the same maturity date as the liability.

This example can be extended for each estimate of liability payments each year. This example may seem simple but this basic offset of values is often ignored. Indeed, some reinsurers of financial reinsurance products use this method to price a portfolio transfer.

This form of matching would be considered immunization. If the liability cash flow was as predicted, then the assets selected to match the cash flow would precisely offset the liabilities and therefore, the liabilities would be met regardless of any changes in the interest rates. Durational concepts are not relevant if cash flows are matched. However, they can be important for investment strategies which do not precisely match assets and liabilities. It will be demonstrated here that a fairly precise match of cash flows is possible.

Depending on the purpose of a valuation, (Statutory, GAAP, management, or acquisition) an alternative view of the value of the assets and liabilities will exist. This paper presents methods to enhance both the management perspective, as well as provide meaningful insight into the value of a company for acquisition purposes.

It is not the intent here to discuss the features of asset/liability management, but more that the creation of a model portfolio by comparing cash flows can provide insight into the real value of the predicted liabilities. Cash flow techniques are recognized as a realistic alternative view for valuation purposes.

Selection of a model portfolio is dependent on the available assets, as well as the predicted liability cash flow. The intent of the model portfolio is to match the cash flows of assets and liabilities.

The development of asset/liability management techniques are hampered by the effects of the variance in liabilities. In order to develop the discussion, it is necessary, initially, to assume that the actuarial estimates for unpaid losses and payment patterns are correct and will not change. The section on variance in the liability assumptions will discuss the impact of changing reserve and payment pattern estimates.

There will be five parts to this discussion:

- Valuation Treatment of Assets Versus Liabilities
- Cash Flow Comparison
- Selection of Model Portfolio
- Variance in the Liability Assumptions
- Market Value

This paper does not recommend a corporate investment policy of matching assets and liabilities. Such a strategy is not always appropriate. Alternative investment strategies are available which provide a greater benefit than a straightforward matched strategy. Nevertheless, a model asset portfolio which is matched to the liability cash flow can provide insight into the value of the liabilities, as well as the benefits of the investment strategy selected.

VALUATION TREATMENT OF ASSETS VERSUS LIABILITIES

Discounting is the technique which estimates the credit for accumulation of investment income during the period that unpaid liabilities are settled. Common techniques use the estimates of unpaid losses, a payout pattern and a discount rate. Many authors have discussed various methods of determining the discount rate based upon either a company's own assets, or possibly, yields on treasury bonds, municipal bonds, or some other alternative form of investment.

All these methods may be appropriate, depending on the circumstances of the guarantor of the liabilities.

The amounts of liabilities, and assets as well as the nature of the assets should be determined by the corporate philosophy of the insurer.

Casualty loss reserve estimates are not precise. A company may select a value from within a range (although the company should be consistent in this selection process from year to year). Therefore, some companies can take a conservative posture and others an optimistic posture on the amounts of liabilities.

Equally well, companies can have opposing postures on the assets and their use of the investable reserve funds. Corporate investment policy which requires only AAA rated bonds and a minimal amount of surplus in equities is far more conservative than a company which invests in "junk bonds". Of course, the latter company needs to establish a greater MSVR (Mandatory Security Valuation Reserve).

These opposing views on both the liabilities and the assets create differing values for the liabilities and therefore, the company. Discounting the liabilities at different rates of discount has the same effect and reflects two corporate views of the potential investment earnings from reserves.

To illustrate these opposing views liabilities will be valued using investments with credit ratings from treasuries to "junk bonds". Obviously, liabilities valued by comparison with a matched model portfolio using "junk bonds" has a lower value than using treasuries.

CASH FLOW COMPARISON

In their discussions to R. W. Sturgis[2], Rothman and Deutsch[3] proposed the use of cash flows. R. W. Sturgis[3] reconsidered the arguments presented in the discussions and presented a reconciliation of the comparison of present value of earnings versus present value of cash flows.

The technique proposed herein attempts to reduce the reliance on a discount rate assumption. This alternative approach for the value of liability cash flows, which relieves the use of a discount assumption, is to establish a model portfolio of assets which precisely matches the liability outflows (net of tax and investment costs). Assets should be nominally selected which produce a similar cash flow to the liabilities. If, however, settlements are skewed during the year, then the cash flows should be estimated on a more frequent basis than yearly.

It may not be possible to find assets which closely match the timing of the liability payments. Consequently, consideration must be given to an adjustment for mismatch. J. S. Bradley[4] presents a method for calculating this amount by offsetting cash flows.

Exhibit A contains five samples of matching the same liabilities with various alternative types of assets. In this instance, assets have been found which match the cash flow, except that in some years the assets mature earlier than mid-year, and in others later than mid-year. The mismatch adjustment included in these exhibits is in respect of this difference in timing of loss payments and asset payments during each year.

In this manner, an estimate of the value of the liabilities is determined as the cost (market value) of the model portfolio of assets which offsets the liability cash flow, plus a cost for any mismatch of cash flows. The assets valued, are those assets which provide a cash flow as close to the liabilities as is possible in practice. Obviously, no discount rate assumption is needed for determining the market value of the assets.

Although the mismatch portion of the reserve needs interest assumptions, the relative impact of these assumptions are much less than with a standard discounting technique. The use of interest assumptions is only needed for timing differences between the model portfolio and the liability cash flow. In the following example, the amount of mismatch reserve is less than 1% of the estimated value of the liabilities. Obviously, the extent of the mismatch reserve will be dependent on the availability of assets to match the projected liability cash flow. If liabilities extend out beyond 12 years, there will be a shortage of assets maturing in the later years. However, as asset/liability management techniques become increasingly used, a market should develop for suitable securities.

SELECTION OF MODEL PORTFOLIO

There are many alternative methods for selecting a model portfolio with which to value the liability cash flow. In Exhibit A (sheets 1 - 5) five types of selections are shown. The first four use fixed interest bearing bonds, the fifth uses zero coupon bonds (stripped treasuries, for example). The first four examples illustrate the costs using fixed-interest bonds of different quality (treasuries versus high grade corporate bonds versus other corporate bonds versus "junk bonds"). Sheets 1 and 5, however, illustrate the difference in cost between zero coupon bonds and fixed interest bonds (both treasuries).

It has been assumed that all loss payments occur at mid-year. If this is not the case, then maturity dates of the asset selections should reflect this. The maturity dates should be as close to the expected liability as possible.

With sufficiently large liability cash flow payments, several assets may be selected for a particular maturity year. In which case the timing of the liability cash flow for each year should be examined more closely.

To select a matched portfolio using fixed interest securities, the asset with the longest maturity date should be selected first to meet the cash flow at the latest duration. The reader will note in Exhibit A, sheet 1, allowance for one year's interest is made in the year 2000. In 1989, only six months' interest is included.

This selection method would not be appropriate if the liability cash flow increases dramatically in later years, or has years with no liability cash flows. In which case, zero coupon bonds could be used, or a mismatch reserve considered.

Leibowitz and Weinberger[5] refer to optimal cash flow matching. This is the selection of a least cost portfolio. Obviously, there are a variety of alternative portfolios which will be closely matched to the liability cash flow. The choice of the most appropriate portfolio, would depend on corporate philosophy, and investment policy. Within these guidelines there would always be a least cost portfolio.

In our example, it has been necessary to include a nominal reserve for mismatch. This is due to bonds maturing at times other than June 30. If the asset cash flow is in advance of the liability cash flow a reinvestment rate of 5% has been used. However, if the asset cash flow is after the liability cash flow, a borrowing rate of 10% has been used.

VARIANCE IN LIABILITY ESTIMATES

Property and casualty insurance liabilities can be considered as uncertain payments of cash in the future. The total amount of the payments is uncertain, as is the time when they will be made. The actuary's role is to use professional means and, where necessary, judgment in deriving estimates for the liabilities.

An insurance company maintains a surplus to ensure against adverse variation of losses. Consequently, in the valuation of a company, two views should be considered. The first is an expected scenario, and the second an adverse scenario.

When using cash flow techniques for valuation purposes, expected and adverse scenarios estimates should be made of both the unpaid liabilities, and the payment pattern. With these two assumptions, the methods presented in this paper can be applied, to derive a market value of liabilities for each of the two scenarios.

The method based on expected results is an estimation of the worth of the liabilities for a going concern. Whereas the results based upon adverse loss assumptions, is an indication of the ability of the company to settle its insurance liabilities, if it ceased writing business.

This paper will not present methods for estimating the adverse variation of losses. However, the reader is referred to the May 1988 Casualty Loss Reserve Seminar where several papers were written on this subject. In particular, the methods presented by A. Halpert and D. Oliver[6] provide a means of deriving a margin for adverse loss development as well as a methodology for estimating an adverse scenario payment pattern.

MARKET VALUE

The undiscounted liabilities used in the example in Exhibit A are $6,930, and indeed this value would be the balance sheet amount for statutory valuation purposes*. Nevertheless, it is clear from Exhibit A, that an undiscounted liability of $6,930 with the indicated cash flow stream, would have different market values depending on the types of investments selected. The appropriate selection of assets obviously is dependant on corporate philosophy and investment policy. Consequently, so is the value of the liabilities.

The table below summarizes the results of matching the sample liability cash flows with various types of assets.

Bond Types (all figures in $000's)	Market Cost	Equivalent Discount Rate	Mismatch Portion of Cost
U.S. Treasuries	$5,284	8.72%	($7)
Corporate (>=AA)	$5,295	8.65%	$44
Corporate (A)	$5,098	10.05%	($22)
"Junk Bonds" (<=BBB)	$4,710	13.16%	$15
Stripped Treasuries	$5,196	9.34%	($44)

*Except for reserves which are discounted on a statutory basis, such as some medical malpractice and workers' compensation pension cases.

Some of the mismatch portion of costs are positive and some negative. This represents the costs from borrowing or gains from reinvestment respectively. In view of the mismatch portion of the costs, the market value of the U.S. Treasuries is less than the market value of the high grade corporate bonds. The market value of treasuries versus high grade corporate bonds are extremely close.

The stripped treasuries produce a lower result than the corporate or the regular treasuries. This is because of the different yield curve between the two types of securities (interest bearing and zero coupon). Comparisons of yield curves between interest bearing and zero coupon are not straightforward, as interest bearing securities have cash payments during earlier years.

The market prices are as of September 30, 1988. However, for purposes of this exercise, the prices and equivalent discount rate have been assumed to be as at December 31, 1988.

The technique illustrated here provides a market value, or present value, without the variability resulting from selection of a discount rate. Furthermore, margins in the investment assumptions are easily reflected through investment selection (higher grade) or conservative assumptions in the liability estimates (ultimate losses and payment pattern).

This method of valuation has no implicit margins for adverse deviation of losses, early payment of losses, or investment income. Use of this method with a slower than actual payment pattern would create a value with an optimistic credit for future investment income.

If the purpose of the valuation is for acquisition, then margins are essential for both elements of adverse results. A risk margin for adverse deviation of losses is needed, as is a payment pattern assumption that is not slower than reality.

This paper has developed a method for generating a market value for the liabilities based on the market value of the model portfolio assets. The market value of the assets is a simple exercise. Readers will readily discern the comparison to P.D. Noris[7] where the term Market Value Surplus (MVS) is used. This represents the difference between the Market Value of Assets (MVA) and the Market Value of Liabilities (MVL). Algebraically this would be:

$$MVS = MVA - MVL$$

Quite simple!

By way of example, examine this simplified form of a balance sheet for an insurance entity:

Assets:

Fixed Interest Securities	8,000
Equities	800
Cash	300
Other Assets	1,500
Total Assets	$ 10,600
Liabilities:	
Loss and Loss Expense Reserves:	6,930
Other Liabilities	1,000
Total Liabilities	$ 7,930
Capital & Surplus	
Capital	1,000
Retained Earnings	1,670
Total Capital & Surplus	$ 2,670

The assets are presented at an amortized value, and equities are valued at cost.

The unpaid liabilities are the same as in Exhibit A, though the fixed interest bonds are of various types.

Assuming that the liabilities are as shown in the above model, and, assuming that the company's investment policy is to invest reserve funds in treasuries (or strips), then the model gives a market value of $5,196 (using strips) for the liabilities.

On the asset side, assuming that the assets were purchased when interest rates were lower, then the market value may be less than the amortized value. Suppose, therefore, that the market value of the total assets is $9,800; then the MVS is $3,604 ($9,800 less $6,196), versus the balance sheet value of $2,670.

For an ongoing entity a margin may be necessary if the actual assets held do not reasonably match the maximum assumption payment pattern. The reason for this is that with the passage of time, if the assets do not reasonably match the payout of the liabilities, then there is additional liability if the market value of assets (initially equal to the market value of liabilities) diverges from the market value of the liabilities.

In the case of an acquisition, the buyer has the option to adjust the asset portfolio to match the liability model portfolio and, therefore, would not be so concerned with this divergence, unless market conditions prevented such a shift in assets.

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Asset/Liability Matched Portfolio
Bond Allocation Method

Exhibit A
Sheet 1

Fixed Interest Treasuries
(assumed annual interest payments)

Bond	Maturity Date	Coupon	Par Value	Market Price	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
					[Cash Flow in Year											]
	Estimated Liability Cash Flow				1200	1300	1100	850	750	500	400	300	200	130	100	100
U.S. Treasuries:																
	15-Aug-2000	8.625%	92	88	4	8	8	8	8	8	8	8	8	8	8	100
	15-May-99	8.500%	85	83	4	7	7	7	7	7	7	7	7	7	92	
	15-May-98	9.000%	105	106	5	9	9	9	9	9	9	9	9	115		
	15-May-97	8.500%	162	158	7	14	14	14	14	14	14	14	175			
	15-May-96	7.375%	244	225	9	18	18	18	18	18	18	262				
	15-Jul-95	8.875%	316	318	14	28	28	28	28	28	344					
	15-Jul-94	8.000%	385	374	15	31	31	31	31	416						
	15-Jul-93	7.250%	592	562	21	43	43	43	635							
	15-Jun-92	8.250%	639	634	26	53	53	692								
	15-Jun-91	7.875%	824	814	32	65	889									
	15-Jun-90	8.000%	948	944	38	1024										
	15-Jun-89	7.375%	988	984	1024											
	Totals (Asset Cash Flow)			$5,291	1200	1300	1100	850	750	500	400	300	200	130	100	100
Offset (days) of asset payments (-ve indicates need to borrow)					14	13	13	13	-11	-10	-8	44	42	40	39	-46
Mismatch Adjustment (5% reinvestment, 10% borrowing)				-7	-2	-2	-2	-1	2	1	1	-2	-1	-1	-1	1
Total cost for assets and mismatch =				$5,284												

Note: (a) Par value of bonds and interest amounts have not been rounded.
(b) Mismatch adjustment is calculated assuming all interest payments take place annually.
(c) Offset days is the weighted difference between June 30, and the maturity date.

Asset/Liability Matched Portfolio
Bond Allocation Method

Exhibit A
Sheet 2

Fixed Interest Corporate Bonds Rated AA or better
(assumed annual interest payments)

Bond	Maturity Date	Coupon	Par Value	Market Price	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
					[Cash Flow in Year											]
Estimated Liability Cash Flow					1200	1300	1100	850	750	500	400	300	200	130	100	100
Allocation of asset cash flow:																
Xerox	15-Oct-2000	9.625%	91	92	4	9	9	9	9	9	9	9	9	9	9	100
Philip Morris	15-Nov-99	6.000%	86	66	3	5	5	5	5	5	5	5	5	5	91	
Belgium	10-Jul-98	9.625%	106	107	5	10	10	10	10	10	10	10	10	116		
Norsk Hydro	09-Apr-97	8.250%	162	149	7	13	13	13	13	13	13	13	176			
Liberty Mutual	08-Jul-96	8.500%	242	235	10	21	21	21	21	21	21	262				
Aetna Life	11-Oct-95	9.500%	312	318	15	30	30	30	30	30	342					
Prudential	15-Jul-94	8.750%	379	379	17	33	33	33	33	412						
Sarah Lee	25-Aug-93	8.375%	580	588	24	49	49	49	629							
IBM O/S	01-Jun-92	8.625%	626	625	27	54	54	680								
Prudential	22-Jul-91	7.750%	813	794	32	63	876									
GMAC	15-Jul-90	8.250%	936	927	39	1013										
GMAC	07-Jul-89	8.250%	978	971	1018											
Totals (Asset Cash Flow)				$5,251	1200	1300	1100	850	750	500	400	300	200	130	100	100
Offset (days) of asset payments (-ve indicates need to borrow)					-9	-17	-23	15	-53	-20	-90	-9	63	-22	-135	-107
Mismatch Adjustment (5% reinvestment, 10% borrowing)				44	3	6	7	-2	11	3	10	1	-2	1	4	3
Total cost for assets and mismatch =				$5,295												

Note: (a) Par value of bonds and interest amounts have not been rounded to the value shown.
(b) Mismatch adjustment is calculated assuming all interest payments take place annually.
(c) Offset days is the weighted difference between June 30, and the maturity date.

Asset/Liability Matched Portfolio
Bond Allocation Method

Exhibit A
Sheet 3

Fixed Interest Corporate Bonds Rated A
(assumed annual interest payments)

Bond	Maturity Date	Coupon	Par Value	Market Price	[---- Cash Flow in Year ---- 1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000 ----]
	Estimated Liability Cash Flow				1200	1300	1100	850	750	500	400	300	200	130	100	100
Allocation of asset cash flow:																
Carolina Power	01-Jun-2000	8.375%	92	82	4	8	8	8	8	8	8	8	8	8	8	100
Pacific Gas	01-Jun-99	6.625%	87	67	3	6	6	6	6	6	6	6	6	6	92	
Michigan Gas	15-Jul-98	5.500%	110	79	3	6	6	6	6	6	6	6	6	117		
Michigan Gas	01-Jul-97	8.125%	167	148	7	14	14	14	14	14	14	14	180			
Gulf Power	01-Jun-96	7.625%	248	218	9	19	19	19	19	19	19	267				
Alabama Power	01-Sep-95	6.000%	328	264	10	20	20	20	20	20	348					
Alabama Power	01-May-94	4.875%	408	325	10	20	20	20	20	428						
Alabama Power	01-May-93	4.625%	629	519	15	29	29	29	658							
Alabama Power	01-Jun-92	4.375%	699	592	15	31	31	729								
Hawiian Elec	01-Apr-91	4.650%	907	815	21	42	949									
Alabama Power	01-Aug-90	5.000%	1,054	982	26	1107										
Alabama Power	01-May-89	4.875%	1,051	1029	1077											
	Totals (Asset Cash Flow)			$5,120	1200	1300	1100	850	750	500	400	300	200	130	100	100
Offset (days) of asset payments (-ve indicates need to borrow)					56	-22	81	28	54	51	-53	27	1	-10	29	29
Mismatch Adjustment (5% reinvestment, 10% borrowing)				-22	-9	8	-12	-3	-6	-3	6	-1	-0	0	-0	-0
Total cost for assets and mismatch =				$5,098												

Note: (a) Par value of bonds and interest amounts have not been rounded to the value shown.
(b) Mismatch adjustment is calculated assuming all interest payments take place annually.
(c) Offset days is the weighted difference between June 30, and the maturity date.

Asset/Liability Matched Portfolio
Bond Allocation Method

Exhibit A
Sheet 4

Fixed Interest Corporate Bonds Rated BBB or worse
(assumed annual interest payments)

Bond	Maturity Date	Coupon	Par Value	Market Price	Cash Flow in Year 1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Estimated Liability Cash Flow					1200	1300	1100	850	750	500	400	300	200	130	100	100
Allocation of asset cash flow:																
Price Comm	01-Sep-2000	14.625%	87	82	6	13	13	13	13	13	13	13	13	13	13	100
Gillette	01-Aug-99	13.875%	77	70	5	11	11	11	11	11	11	11	11	11	87	
PCPI Fund	01-Apr-98	15.500%	92	86	7	14	14	14	14	14	14	14	14	107		
Allegheny	01-Jun-97	10.000%	148	113	7	15	15	15	15	15	15	15	162			
Zeus Components	01-Jul-96	12.500%	220	197	14	28	28	28	28	28	28	248				
Wickes Cos	01-May-95	15.000%	278	293	21	42	42	42	42	42	320					
Magnatrek	01-Jun-94	11.875%	338	320	20	40	40	40	40	378						
UDC-Universal	01-Aug-93	12.250%	524	502	32	64	64	64	588							
Zale Corp	01-Jun-92	11.500%	560	558	32	64	64	624								
Castle Cooke	01-Sep-91	12.000%	723	724	43	87	810									
Gulf State	01-Jul-90	4.875%	880	803	21	923										
Amer Medical	15-Jul-89	9.000%	947	947	990											
Totals (Asset Cash Flow)				$4,695	1200	1300	1100	850	750	500	400	300	200	130	100	100
Offset (days) of asset payments (-ve indicates need to borrow)					-13	-2	-43	24	-19	28	49	1	24	65	-36	-63
Mismatch Adjustment (5% reinvestment, 10% borrowing)				15	4	1	13	-3	4	-2	-3	-0	-1	-1	1	2
Total cost for assets and mismatch =				$4,710												

Note: (a) Par value of bonds and interest amounts have not been rounded to the value shown.
(b) Mismatch adjustment is calculated assuming all interest payments take place annually.
(c) Offset days is the weighted difference between June 30, and the maturity date.

Asset/Liability Matched Portfolio
Bond Allocation Method

Exhibit A
Sheet 5

Zero Coupon or Stripped Treasuries
(assumed annual interest payments)

Bond	Maturity Date	Coupon	Par Value	Market Price	Cash Flow in Year											
					1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
	Estimated Liability Cash Flow				1200	1300	1100	850	750	500	400	300	200	130	100	100
Stripped Treasuries:																
	15-May-2000	0%	100	36	0	0	0	0	0	0	0	0	0	0	0	100
	15-May-99	0%	100	37	0	0	0	0	0	0	0	0	0	0	100	
	15-May-98	0%	130	56	0	0	0	0	0	0	0	0	0	130		
	15-May-97	0%	200	95	0	0	0	0	0	0	0	0	200			
	15-May-96	0%	300	155	0	0	0	0	0	0	0	300				
	15-May-95	0%	400	228	0	0	0	0	0	0	400					
	15-May-94	0%	500	312	0	0	0	0	0	500						
	15-May-93	0%	750	511	0	0	0	0	750							
	15-May-92	0%	850	631	0	0	0	850								
	15-May-91	0%	1,100	889	0	0	1100									
	15-May-90	0%	1,300	1144	0	1300										
	15-May-89	0%	1,200	1147	1200											
	Totals (Asset Cash Flow)			$5,240	1200	1300	1100	850	750	500	400	300	200	130	100	100
Offset (days) of asset payments (-ve indicates need to borrow)					46	46	46	46	46	46	46	46	46	46	46	46
Mismatch Adjustment (5% reinvestment, 10% borrowing)				-44	-8	-8	-7	-5	-5	-3	-3	-2	-1	-1	-1	-1
Total cost for assets and mismatch =				$5,196												

Note: (a) Par value of bonds have not been rounded.
(b) Mismatch adjustment is calculated assuming all interest payments take place annually.
(c) Offset days is the weighted difference between June 30, and the maturity date.