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An Extension to the Cape Cod Method with Credibility Weighted Smoothing

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Abstract

The Cape Cod method is a commonly used technique where the a priori loss ratio is calculated as the weighted average of the chain ladder ultimate loss ratios across all years with the "used" premium as the weights. It applies the same a priori loss ratio estimate (on a trended, current rates level) across all years, without consideration for any possible changes that may have occurred. A difficulty arises when the loss ratios show improvement or deterioration, which is a fairly common scenario. When this occurs, the amount of credibility that should be given to the shift is mostly left to guesswork.

This paper uses the Kalman Filter to automatically smooth the loss ratios based on the amount of credibility inherent in the data in a manner that is robust and that is consistent with the Cape Cod method. It is shown how this method can be thought of as a credibility weighting between the Cape Cod and Chain Ladder techniques, each of which are possible at the two extremes. It is then shown how external predictive information, such as the state of the economy or the insurance cycle, can be incorporated to help produce more accurate results. Simulation results are presented that illustrate the error reduction this method can provide to both historical years and to the latest year.

Keywords. Loss Reserving, Credibility, Smoothing, Kalman Filter, Trend

1. INTRODUCTION

The Cape Cod or Stanard-Buhlmann (Stanard 1985) method is a commonly used technique where the a priori loss ratio is calculated as the weighted average of the chain ladder ultimate loss ratios across all years with the "used" premium as the weights. It applies the same a priori loss ratio estimate (on a trended, current rates level) across all years, without consideration for any possible changes that may have occurred. A difficulty arises when the loss ratios show improvement or deterioration, which is a fairly common scenario. This can happen as a result of using rate changes or trends that are not completely accurate, changes in policy wording, or temporary shifts in the exposure to loss caused by economic or other factors. When this occurs, the amount of credibility that should be given to the shift is mostly left to guesswork. Being too slow to give credit to improving experience can cause the company to miss out on profitable opportunities and also frustrate underwriting management, and detecting deterioration too late can cause declines in profitability and capital that could have been avoided. Being too slow to detect any type of change can also contribute to diminished confidence in the entire reserving process. On the other hand, reacting to noise too quickly will cause faulty decisions to be made with negative results as well.

This paper presents a method that automatically smooths loss ratios based on the amount of credibility inherent in the data and that is consistent with the Cape Cod approach. If no credibility is given to any changes, a single loss ratio will be indicated for all years, and the results will match the Cape Cod method. On the other extreme, if full credibility is given to the chain ladder indications from each year, the final results will match the Chain Ladder method. Anywhere in between can be thought of as a credibility weighting between these two methods. It is then shown how external predictive information, such as the state of the economy or the insurance cycle, can be incorporated to help produce more accurate results.

1.1 Research Context

Gluck (1997) improved upon the original Cape Cod technique by adding a decay factor which gives increased weight to the more local experience, effectively smoothing the data. But there is still little guidance as to how much credibility or smoothness should be used. (There are formulas in the appendix for approximating this, but they are difficult to follow and implement, and the iterative approach suggested to solve the equations is not guaranteed to converge to the optimal solution, and likely will not.)

The Kalman Filter is a very popular smoothing algorithm used in many econometric applications. De Jong and Zehnwirth (1983) were the first to introduce its use into reserving and used it to help smooth development patterns. Both Zehnwirth (1996) and Wuthrich and Merz (2008) use the Kalman Filter to smooth the actual reserving estimates, but their formulations are much more complicated than a simple Cape Cod approach and will not be discussed here. Evans and Schmid (2007) use the Kalman Filter to derive smooth trend estimates but their approach is not suitable, nor intended, to apply directly to loss ratio estimates. None of these approaches demonstrate a simple, easy to understand framework that is in line with traditional actuarial practice, as the Cape Cod method does. The Kalman Filter formulas can also seem non-intuitive and hard to understand, making implementation of such an algorithm in the reserving context challenging. Finally, and also very critical, the indicated smoothness derived from the Kalman Filter or similar methods can be very volatile and inaccurate, essentially precluding its use in practice. As mentioned in Schmid et al. (2013), even the time series used for NCCI ratemaking is too short to reliably estimate the variance of the year-to-year changes, which is essential to determining the credibility. Having a smaller amount of data than NCCI would compound this problem. If this issue is not properly handled, such as by using the strategies that will be discussed in this paper, the Kalman Filter results cannot be relied upon.

1.2 Objective

The goal of this paper is to present a simple, easy to understand, and yet powerful and robust framework of applying the Kalman Filter to smooth loss ratio estimates that is consistent with the

Cape Cod method. This smoothing algorithm is applied to the on-level, trended ultimate Chain Ladder loss ratios with weights equal to the premiums divided by the LDFs, or the "used premiums". The results of this algorithm are the a priori loss ratios to apply to each year via a Bornhuetter-Ferguson method. If the algorithm determines that no credibility or smoothness should be given, the result for each year will be the weighted average across all years, and the method will be equivalent to the Cape Cod. On the other extreme, if full credibility or maximum smoothness is indicated, the a priori loss ratios for each year will match those of the Chain Ladder method, and so the final results will be identical to the Chain Ladder as well. Anywhere in the middle, the method can be thought of as a credibility weighting between these two methods.

This paper will also discuss the intuition behind the Kalman Filter formulas relating them to basic credibility theory. Many of the approaches mentioned apply the Kalman Filter on the logarithm of loss ratios, making it inconsistent with the Cape Cod approach and hard to determine the relative weights by year and requiring a messy bias correction if not using Bayesian software for calculation. Taylor and McGuire (2003) show a solution to this problem via what they call an EDF Filter, but the math required to implement it is complex. This paper applies the Kalman Filter on the loss ratios themselves but modifies the algorithm in a similar but simpler fashion to be able to handle multiplicative innovations, that is, the changes from year to year, and non-normally distributed errors. Strategies are also shown to make it robust so that it can be used in practice even with sparse, volatile data, and this is illustrated via simulation testing.

1.3 Outline

Section 2 discusses the intuition behind the Kalman Filter and shows how to apply it to model loss ratios, and section 3 shows how to make the algorithm more robust. Incorporating external predictive information is discussed in section 4, and examining multiple lines simultaneously is discussed in section 5. Finally, section 6 shows the results of running simulations using the methods discussed.

2. THE KALMAN FILTER

The method presented in this paper uses the Kalman Filter to determine the amount of smoothness or credibility that should be given to each year. The Kalman Filter was originally developed in 1960 for use in signal processing (Kalman 1960) but has become very common for solving time series econometric models. It is able to handle more complex types of models than are illustrated in this paper. For ease of understanding and implementation, a simplified version that contains only the needed components is discussed instead.

2.1 Intuition Behind the Kalman Filter

To understand how this algorithm works, assume that rate and trend are both flat and that we are attempting to predict the expected loss ratio for year 2 where we know (for certain) that the loss ratio for year 1 was 70%. Before observing any experience from the second year, our prediction would be 70%, the same as year 1. Assume that now we observe a (projected) loss ratio of 80% in the second year, which is still incomplete, and we want to estimate the expected loss ratio to be used in a Bornhuetter-Ferguson method to estimate the IBNR for the remainder of the year. If there was no loss volatility, we would assume that the 80% loss ratio will continue for the remainder of the year and this would be our estimate. On the other hand, if the loss volatility was extremely high such that the 80% prediction for this year had a large degree of uncertainty, we would give it almost no credibility, and our estimate would be the year 1 estimate, which is 70%. More practically, our estimate should fall somewhere in between these two extremes and take into account both the volatility of the losses and the volatility of the year-to-year changes. If these two variances were equal, we would select the midpoint, 75%. More generally, the optimal credibility to give to the second year's experience equals the variance of the year-to-year changes divided by the sum of the two variances, since this would produce the result with the lowest variance. Venter (2003) derives this result and shows that it is the basis for Buhlmann credibility. The variance of this estimate cannot be greater than each of the individual predictors; otherwise, we would just select one of them instead. The inverse of the variance equals the sum of the inverses of each of the variances. (Bolstad 2007)

If we now want to estimate the expected loss ratio for year 3, similar logic would apply, except that now the variance of the year 2 estimate needs to be taken into account as well. The total variance of using the year 2 estimate for year 3 would equal the variance of this estimate plus the variance of the year-to-year changes. This variance would then be compared to the loss volatility to calculate the optimal credibility to give to the third year's experience in the year 3 estimate. Once we have observed and predicted the loss ratio for the third year, this estimate can now be used to improve the prediction for the second year. To determine the amount of credibility to give to the year 3 estimate for the year 2 result, a similar formula is used except that the variance of this predictor is compared against the variance of the year-to-year changes instead of the variance of the losses.

This is essentially what the Kalman Filter does (the part that we are using, at least); the actual formulas are shown in the next section.

2.2 Kalman Filter Formulas

Similar logic is used to run the Kalman Filter. A first iteration is performed looking at the years (or quarters, etc.) going forwards. Then, once an initial estimate has been determined for each year, another iteration is performed, this time, starting at the end and traveling backwards by year. This is done to back-smooth the results and modify the earlier estimates taking into account what is known about the later years, since the first iteration only considers the reverse.

As alluded to in the previous section, two values are used in the first iteration for each prediction estimate and variance. The first represents the prediction for a particular year before observing the experience of that year, and the second represents a revised prediction that also takes into account the experience of that year.

There are three unknown parameters that are needed to run this algorithm: the starting loss ratio for the first year, the volatility of the experience, and the volatility of the year-to-year changes. Maximum likelihood is used to determine these values. Note that the likelihood is calculated using the initial loss ratio estimates, that is, the estimates before considering the experience for each year. This is done because otherwise, if the estimates after considering each year's experience were used, the algorithm would seek to minimize the differences between these actual and fitted loss ratios, which would result in indications that were completely smoothed to all of the noise in the experience. Then, after this forward iteration has been performed and after the values of all of the unknown parameters have been determined, another back-smoothing iteration is performed to calculate the final results.

The amount of credibility each new year is given in the rolling forward predictor is known as the Kalman gain and is equivalent to the credibility discussed in the previous section. This is shown as K in the formulas below. The formulas below show the predictor of year t before considering that year's experience as $X_{t|t-1}$, the predictor after considering the year's experience as $X_{t|t}$, and the final back-smoothed predictor as $X_{t|T}$. Similar notation is used for the variance. Note that these are <u>not</u> the final formulas, as some changes are needed to make the algorithm more suitable for loss ratios, which are shown later in section 2.3. Explanations are given by the formulas to relate it to the concepts discussed in the previous section. For the notation, Y are the observed loss ratios, X are the predicted loss ratios, P are the variances of the predictors, n is the forecast error, R is the loss volatility, Q is the volatility of the year-to-year changes, K is the Kalman gain, f is the total variance of the predictor including the volatility of the losses, and *loglik* is the log-likelihood. *Norm(a, b, c)* is used here to represent the log-likelihood of the normal distribution at *a,* with mean of *b,* and variance of *c.* (Kim and Nelson 1999)

The best estimate for the next year before observing the experience is the previous year's prediction. The variance of this prediction is the same as the previous year's variance plus the volatility of the year-to-year changes.

$$X_{t|t-1} = X_{t-1|t-1} \tag{2.1}$$

$$P_{t|t-1} = P_{t-1|t-1} + Q \tag{2.2}$$

The total error for the amount a prediction can differ from actual equals the prediction error (from the "true" value) plus the loss volatility.

$$f_t = P_{t|t-1} + R \tag{2.3}$$

To determine the amount of credibility to give to a year's experience, the variance of the rolling forward prediction is compared against the loss volatility. This is shown as *K,* and is called the Kalman gain.

$$K_t = P_{t|t-1} \,/\, f_{t,} = P_{t|t-1} \,/\, (\, P_{t|t-1} + R\,) \tag{2.4}$$

$$n_t = Y_t - X_{t|t-1} \tag{2.5}$$

$$X_{t|t} = X_{t|t-1} + K_t n_t = (1 - K_{t)})\, X_{t|t-1} + K_t\, Y_t \tag{2.6}$$

The variance of this rolling forward predictor decreases after observing and incorporating the experience, based on the formula mentioned that the inverse of the variance equals the sum of the inverses of the two variances. Simple algebra can show that this is equivalent to the below.

$$P_{t|t} = P_{t|t-1}\,(1 - K_t) = P_{t|t-1}\,R \,/\, (\, P_{t|t-1} + R\,) \tag{2.7}$$

$$1 \,/\, P_{t|t} = 1 \,/\, R + 1 \,/\, P_{t|t-1} = (\, P_{t|t-1} + R\,) \,/\, P_{t|t-1}\,R\,;\; P_{t|t} = P_{t|t-1}\,R \,/\, (P_{t|t-1} + R\,)$$

The likelihood is calculated on the prediction error before observing that year's experience using the variance calculated for the rolling forward predictor.

$$loglik_t = Norm(\, n_t\,,\, 0,\, f_t) \tag{2.8}$$

After the initial prediction of the last year has been calculated, the results are back-smoothed. This matches the result mentioned in the previous section.

$$X_{t|T} = X_{t|t} + (\, P_{t|t} \,/\, P_{t+1|t}\,)(\, X_{t+1|T} - X_{t|t}\,) = Z\, X_{t+1|T} + (1 - Z)X_{t|t}\,,\; where\; Z = P_{t|t} \,/ (P_{t|t} + R\,) \tag{2.9}$$

Even though a few modifications will be made to these formulas to apply more to loss ratios, an illustration is shown below using the numbers from the previous section. The *R* (loss volatility) and *Q* (volatility of year-to-year changes) parameters, which are determined via maximum likelihood, are assumed to be 1 and 0.5, respectively.

$X_{1|0} = 70\%$

$X_{2|1} = X_{1|0} = 70\%$

$P_{1|0} = 0$

$P_{2|1} = P_{1|0} + Q = 0 + 0.5 = 0.5$

$f_2 = P_{2|1} + R = 0.5 + 1 = 1.5$

$K_2 = P_{2|1} / f_2 = 0.5 / 1.5 = 0.333,$

$n_2 = Y_2 - X_{2|1} = 80\% - 70\% = 10\%$

$X_{2|2} = X_{2|1} + K_2 n_2 = 70\% + 0.333 \times 10\% = 73.33\%$

$P_{2|2} = P_{2|1} (1 - K_2) = 0.5 \times (1 - 0.333) = 0.333$

2.3 Modifications for Loss Ratios

As mentioned, this smoothing algorithm will be applied to determine the a priori loss ratios for use in a Bornhuetter-Ferguson method. The inputs are the chain ladder loss ratios, since these are the loss ratios that have been observed for incomplete years at the current point in time. The "used premiums" are used as the weights, since this represents the volume for the losses observed thus far. If no smoothness is indicated, the a priori loss ratios will match that of the Cape Cod technique. If, on the other hand, maximum smoothness is given, they will match the chain ladder estimates, and using these in a Bornhuetter-Ferguson method will yield identical results as this method. Anywhere in between can be thought of as a credibility weighting between these two methods as the IBNR predicted for the remainder of each year will only consider each year's experience to the extent that it is credible.

To apply this algorithm on loss ratio data, a couple of modifications are necessary. The first is to deal with years that have different premium volumes, and thus different expected loss volatility, since the original formulas assume that this is constant per year. To allow for different variances, a variance factor can be used as one of the parameters instead of the actual variance. Assuming that the variance of each year is inversely proportional to the premium volume, which is a good assumption if all policies are homogenous in terms of severity, the variance for each year is equal to this variance factor divided by the premium. For incomplete years, the "used" premium is used instead, as discussed.

Ideally, the factor applied to the premiums of incomplete years should reflect the additional variance of these years, which includes both the decreased volume as well as any uncertainty in the loss development patterns. Performing some algebra, it can be seen that the factor relating to the decreased volume is actually the claim count development factor and not the loss development factor, as used in the Cape Cod method. (The derivation is shown in Appendix A.) However, using the claim count development factor would be ignoring any uncertainty in the loss development pattern, so using the loss development factor, which is usually slightly higher than the claim count development factor, is recommended to account for this additional variance as an approximation. This will also make it consistent with the Cape Cod method, which is a desirable property. Alternatively, it is also possible to use the claim count development factors and estimate the uncertainty in the development patterns more exactly if desired.

Another modification is needed to handle non-normally distributed errors. Instead of calculating the likelihood using a normal distribution as the original algorithm does, a gamma or negative binomial distribution can be used instead. (A gamma distribution is appropriate for modeling on severity data and a negative binomial for modeling on frequency data.) The mean and variance resulting from the Kalman Filter algorithm can be used to solve for the two parameters of the appropriate distribution. If using a gamma distribution, for example, the variances calculated in the Kalman Filter algorithm will really be the variances divided by the means squared, and so it is assumed that the variance is proportional to the square of the mean. A negative binomial is not appropriate for modeling loss ratios, since this data often has a variance-to-mean ratio less than one, which this distribution does not allow. A Poisson distribution cannot be used since it does not have an additional parameter for the variance. An overdispersed Poisson has another parameter for the variance but is more difficult to implement. Similarly, implementing a Tweedie distribution, which is often used to model on loss ratios, is difficult as well.

But both a Poisson and Tweedie can be approximated fairly well. Calculating the log-likelihood as the average of the log-likelihoods of the normal and gamma distributions produces results that are very close to using a Generalized Linear Model with a Poisson distribution. Taking a weighted average between these two log-likelihoods with the weight to the gamma distribution equal to half the desired power of a Tweedie distribution also comes very close to using a Generalized Linear Model with a Tweedie distribution. So, for example, applying a weight of 1.67 / 2 = 0.835 to the gamma log-likelihood and a weight of 0.165 to the normal log-likelihood comes very close to using a Tweedie with a power of 1.67. When this is done, another parameter is needed as the constant factor to convert the variance to the coefficient of variation, which is needed to solve for the gamma parameters. (If only a gamma is used, this parameter is not needed, since the variance variable in the Kalman Filter formulas will already represent the variance divided by the mean squared.) Conducting a simulation[1] and comparing the results to a similar GLM when no smoothness resulted (which was about half the time) produced results that were very close. The gamma results matched the GLM results almost exactly. The Poisson and Tweedie results were within 0.05 percentage points of the GLM indications 89% and 98% of the time, respectively, and were within 0.1 percentage points 100% of the time. The results show that this method produces a fairly decent approximation.

If a gamma distribution is used, the yearly innovations are assumed to be multiplicative since it assumes that the variance is proportional to the square of the response, which works well with handling multiplicative relationships, similar to its use in Generalized Linear Models. If a normal distribution is used, the yearly innovations are assumed to be additive. If the approximation of the Poisson distribution is used, as described, the yearly innovations are assumed to be in between additive and multiplicative. It is difficult to say what the appropriate form these innovations should take[2], but if it is desired to have multiplicative innovations, the formulas can be modified to use the product of Q and the loss ratio for a Poisson distribution. For a Tweedie distribution with power p, the product of Q and the loss ratio to the power of two minus p is used instead. This change will cause the variance of the innovations to be related to the square of the mean.

The final formulas that take these modifications into account are shown below. *epow* is the exponential power used (0 for normal, 1 for Poisson, between 1 and 2 for Tweedie, and 2 for gamma), *EP* is the used premium, *Gamma(x, alpha, beta)* is the gamma log-likelihood at *x* with parameters *alpha* and *beta,* and *NB(x, n, p)* is the negative binomial log-likelihood at *x* with parameters *n* and *p*. These formulas assume that the year-to-year changes are multiplicative, although this may or may not be the case.

$$X_{1|0} = <\textit{Set from a parameter}> \tag{2.10}$$

$$P_{1|0} = 0 \tag{2.11}$$

$$X_{t|t-1} = X_{t-1|t-1} \tag{2.12}$$

$$P_{t|t-1} = P_{t-1|t-1} + Q\, X_{t|t-1}^{2\,-\,epow} \tag{2.13}$$

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1 Frequency was simulated using a negative binomial with a mean of 50 and a variance-to-mean ratio of 2.5. Severity was simulated from a lognormal distribution with mu and sigma parameters of 10 and 2, respectively, a retention of $100 thousand and limit of two million. Trend per year was 5%, autocorrelation was 10%, and variance of the year-to-year changes was 0.0001. Premium was set so that the expected loss ratio for the first year would be 70%. 500 simulations were run.

2 Looking at industry data using a Box-Cox test (which is out of scope of this paper produced conflicting results with a very large confidence interval depending on the line and the time period looked at.

$$f_t = P_{t|t-1} + R \,/\, EP_t \tag{2.14}$$

$$K_t = P_{t|t-1} \,/\, f_t, \tag{2.15}$$

$$n_t = Y_t - X_{t|t-1} \tag{2.16}$$

$$X_{t|t} = X_{t|t-1} + K_t n_t \tag{2.17}$$

$$P_{t|t} = P_{t|t-1}\,(1 - K_t) \tag{2.18}$$

$$loglik\text{-}norm_t = Norm(\, n_t\, ,\, 0, f_t\,) \tag{2.19}$$

$$alpha = X^2_{t|t-1} \,/\, (f_t \times <Parameter>) \tag{2.20}$$

$$beta = X_{t|t-1} \,/\, (f_t \times <Parameter>) \tag{2.21}$$

$$loglik\text{-}gamma_t = Gamma(\, Y_t\, ,\, alpha,\, beta\,) \tag{2.22}$$

$$loglik_t = (epow/2)\; loglik\text{-}gamma_t + (1 - epow/2)\; loglik\text{-}norm_t \tag{2.23}$$

Back-Smoothing:

$$X_{t|T} = X_{t|t} + (\, P_{t|t} \,/\, P_{t+1|t}\,)\,(\, X_{t+1|T} - X_{t|t}\,) \tag{2.24}$$

For a negative binomial:

$$n = X_{t|t-1} \,/\, (\, f_t \times <Parameter> - 1\,) \tag{2.25}$$

$$p = 1 \,/\, (\, f_t \times <Parameter>\,) \tag{2.26}$$

$$loglik_t = NB(\, Y_t\, ,\, n,\, p\,) \tag{2.27}$$

Since a gamma distribution is used which does not have any likelihood at zero, any zero loss ratios should be set to a very small number slightly above zero.

As general advice, when solving for the two variance parameters, it is recommended to use one parameter for the total variance and another parameter for the percentage of the total variance that is attributable to the year-to-year changes (a logit function can be used to ensure that this value is between zero and one). The noise variance parameter can then be set to the total variance parameter multiplied by one minus this percentage, and then multiplied by the average premium volume, or something similar, to make this parameter relative to the premium volume. If this strategy is not used, care should be taken as solving for these variance parameters directly can sometimes cause difficulty with optimization routines.

With volatile data, it is often helpful to cap losses at an appropriate point to make the data more stable. If there have been changes in retentions or policy limits, the premium should be adjusted appropriately as well. It is also possible to use this same algorithm on claim frequency and/or severity separately. For frequency, the premium should be adjusted if there have been changes in the retentions or policy limits by dividing out the average expected (conditional) severity. When looking at frequency, it is possible to include all claims, or to only include significant claims greater than a certain threshold.

3. ROBUSTIFYING THE METHOD

As mentioned, the indicated smoothness of the Kalman Filter can be unreliable with relatively few data points. It also struggles with data as volatile as loss ratios. Without addressing these issues, the algorithm cannot be used in practice.

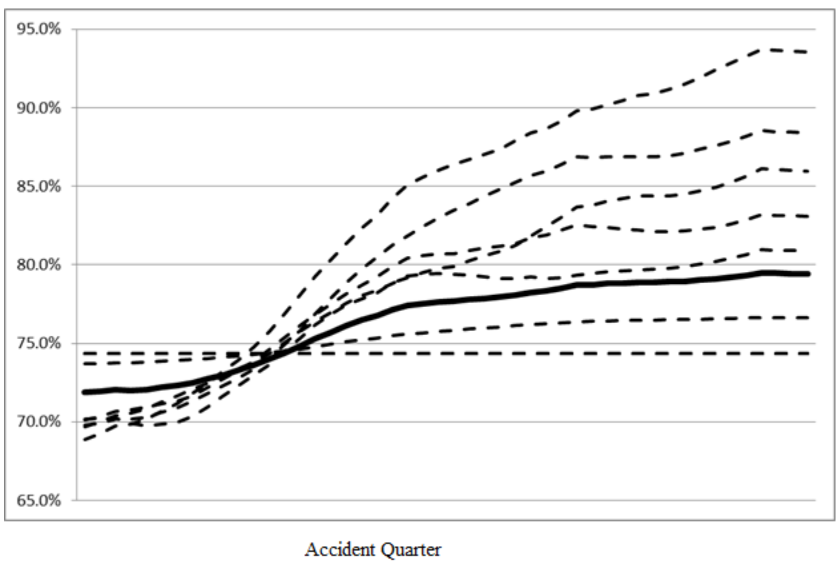
The number of available data points depends on how long the company's history is with the segment being analyzed. It also depends on how consistent processes and practices have been since this determines the relevant data that can be used. Even though the purpose of this algorithm is to address gradual shifts, it may still be beneficial to discard older information that is deemed less relevant and that does not add any value for prediction of the more recent data. If less than twenty years or so of data are available for analysis, it is strongly recommended to use quarterly data instead, which will increase the number of data points four-fold. Even with twenty years of data or more, using quarterly data can still greatly increase the accuracy of the method since it enables better estimation of the variance. If different loss ratios are expected in each quarter due to the effects of seasonality, this can be addressed similarly to the incorporation of external data, as described in section 4.1. (Credibility can be incorporated as well, as described in section 4.2.)

Another technique to make the algorithm more robust is to use bootstrapped aggregation, or "bagging", where multiple iterations of the algorithm are performed, each time on only a fraction of the years or quarters. The final indicated a priori loss ratios are then calculated as the average across all iterations. Each iteration will receive a varying amount of smoothness based upon which years/quarters are included, and averaging across all of these produces a much more stable and reliable result. (Just to be clear, the average of each indicated loss ratio should be used, and not the average of the smoothness parameters, since the former produces much more reliable results than the latter.) Using fifty iterations with selecting two thirds of the data each time seems to perform quite well both in simulation tests and on actual data. (When implementing, it is important to either explicitly set the random number generator seed or to ensure that the same bootstrapped simulations are used each time to avoid having the indications change slightly when rerun.)

To implement, if a data point is skipped, the Kalman gain should be set to zero to give it no credibility. This will result in the predictor variance being increased by the year-to-year variance reflecting the fact that the prediction interval is being extended by skipping this point. So even though the Kalman gain is artificially decreased at one point, this will cause it to be increased for the following point. The likelihood of this point should still be included in the overall likelihood so that it affects the average, however, since the bootstrapping is only needed for the amount of smoothness, and bootstrapping on this will only decrease stability slightly.

An example where the Kalman Filter was run fifty times from simulated data is shown in Figure 1. The first ten individual runs are shown as well as the run that resulted in the most smoothness (dotted lines). The average is shown as the thick solid line. Note how volatile the amount of smoothness can be from single runs, ranging from far too much credibility given to none at all, which occurred in 17 out of the 50 runs. The average incorporates all of these indications and results in a much more stable and reasonable result.

Figure 1



4. ADDING PREDICTIVE VARIABLES

4.1 Formulas

Predictive variables, such as the state of the economy or of the market cycle, can be incorporated to improve the accuracy of the predictions. The following formulas can be used, where *V* is the total impact of the predictive variables at each period, *v* are the predictive variables, and *coef* are fitted coefficients for each of these variables[3]:

$$b_{1|0} = <\textit{Set From a Parameter}> \tag{4.1}$$

$$P_{1|0} = 0 \tag{4.2}$$

$$V_t = exp(\textstyle\sum_i \; coef_i \times v_i) \tag{4.3}$$

$$b_{t|t-1} = b_{t-1|t-1} \tag{4.4}$$

$$X_{t|t-1} = b_{t|t-1} \; V_t \tag{4.5}$$

$$P_{t|t-1} = P_{t-1|t-1} + Q \; X_{t|t-1}{}^{2 \text{ - } epow} \tag{4.6}$$

$$f_t = P_{t|t-1} \; V_t^2 + R \; / \; EP_t \tag{4.7}$$

$$K_t = P_{t|t-1} \; V_t \; / \; f_t, \tag{4.8}$$

$$n_t = Y_t \text{ - } X_{t|t-1} \tag{4.9}$$

$$b_{t|t} = b_{t|t-1} + K_t n_t \tag{4.10}$$

$$P_{t|t} = P_{t|t-1} \; (1 \text{ - } K_t \, V_t) \tag{4.11}$$

$$loglik\text{-}norm_t = Norm(n_t \, , \, 0, f_t) \tag{4.12}$$

$$alpha = X^2{}_{t|t-1} \; / \; (f_t \times <Parameter>) \tag{4.13}$$

$$beta = X_{t|t-1} \; / \; (f_t \times <Parameter>) \tag{4.14}$$

$$loglik\text{-}gamma_t = Gamma(Y_t \, , \, alpha, \, beta) \tag{4.15}$$

$$loglik_t = (epow/2) \; loglik\text{-}gamma_t + (1 \text{ - } epow/2) \; loglik\text{-}norm_t \tag{4.16}$$

Back-Smoothing:

$$b_{t|T} = b_{t|t} + (P_{t|t} \; / \; P_{t+1|t}) \; (b_{t+1|T} \text{ - } b_{t|t}) \tag{4.17}$$

$$X_{t|T} = b_{t|T} \; V_t \tag{4.18}$$

An exponential function was used to calculate the impact of the predictive variables, similar to a log-link GLM, but other alternatives are possible as well. *b* is an intermediate variable similar to an intercept. Using this method is similar to using a GLM where the intercept can vary over time.

The predictive variables here function similarly to a GLM, in that their effect is calculated cumulatively, as opposed to being incremented by an additional amount for each year. This means

[3] These formulas are obtained by replacing the *H* matrix from the original formulas with the result of the predictive variables.

that if, for example, the change in GDP is judged to affect loss ratios, then the actual GDP should be used as a variable, and not the change in the GDP. This way, the incremental effect to each year will be the change in this variable. Similarly, if the change in the GDP growth rate is desired instead, then the GDP change should be used as a variable.

Using this method, it also is possible to fit a constant trend to the data by including the year as a predictive variable. This example is used to help illustrate this method. Loss ratios with a constant frequency trend per year were simulated[4]. Three methods were compared: a (Tweedie) GLM, the Kalman Filter model with no trend and the Kalman Filter model with the year as a predictive variable to represent the trend (both using the approximation for the Tweedie distribution that was discussed).

It is interesting to see the results of the Kalman Filter without trend model. Sometimes this model can do a fairly decent job of following the trend in the data, although it often needs to adapt too much to the data in order to do so, and as a result, produces some overfitting as in Figure 2. In this example, the Kalman Filter with trend model indicated no smoothness and so the result is very close to the Tweedie GLM. The dotted, "actual" line here is the "true" value for each year before volatility is added in the simulation, and the solid, "observed" line is the result with added volatility.

[4] Frequency was simulated using a negative binomial with a mean of 25 and a variance-to-mean ratio of 3. Severity was simulated from a lognormal distribution with mu and sigma parameters of 10 and 2, respectively, a retention of $100 thousand and limit of two million. Trend per year was 3%, autocorrelation was 40%, and variance of the year-to-year changes was 0.0025. Premium was set so that the expected loss ratio for the first year would be 70%. For the bagging, 25 iterations were used using ⅔ of the data on each iteration. 200 simulations were run. The models were fit using the approximation for the Tweedie family mentioned earlier.

Figure 2

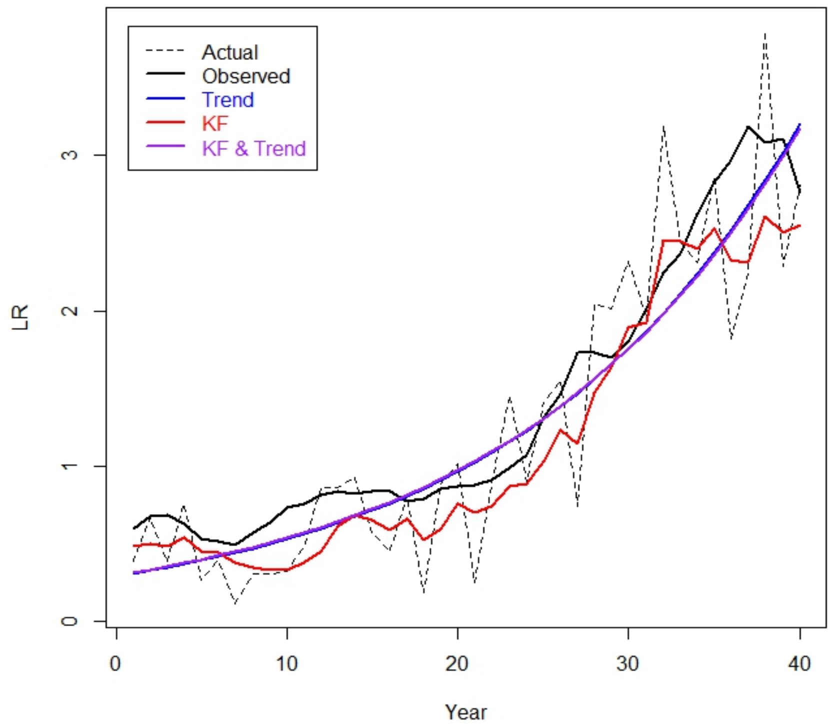
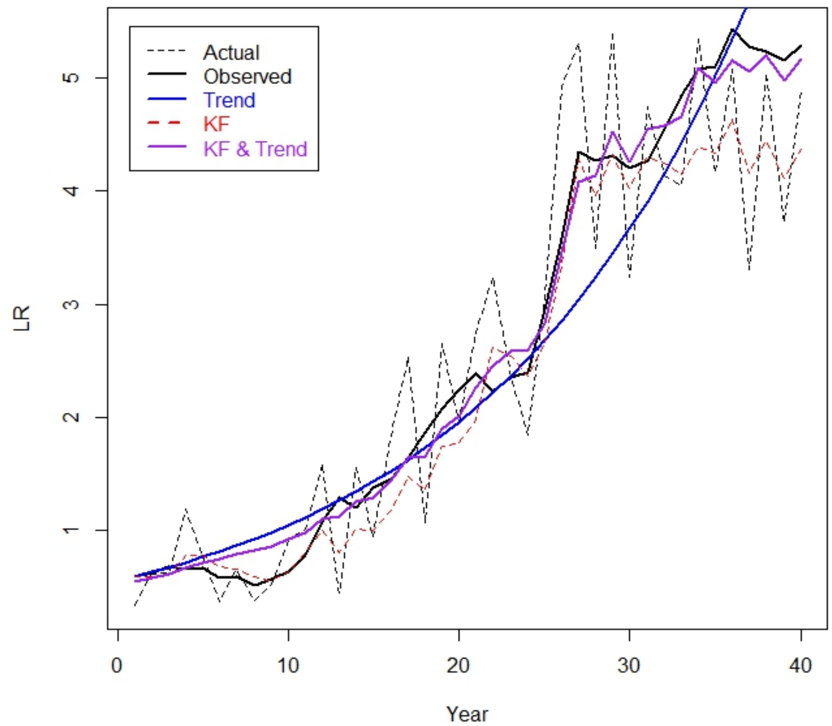


Figure 3 is another example where the Kalman Filter with trend model differs from the GLM and also smooths to the data. In this example, this model does a very good job of adapting to the changing loss ratios per year as well as to the trend in the data, much better than both the simpler trend model and Kalman Filter model (although, of course, this will not always be the case). (The Kalman Filter line is shown with a thinner line in the below graph, as it is not relevant in this example.)

Figure 3



The results of running many simulations are shown in Figure 4. As expected, the Kalman Filter with trend model outperforms both the Kalman Filter without trend and the GLM models.

Figure 4

Method	RMSE[5]
Kalman Filter Without Trend	0.201
GLM	0.167
Kalman Filter With Trend	0.157

[5] RMSE stands for Root mean squared error. It is the square root of the average error squared.

4.2 Further Robustifying This Method

Accidentally including a variable that has no true predictive value can degrade performance. Significance tests can also be unreliable. One way to address this issue and also to increase the accuracy even for truly predictive variables is to use penalized regression. This applies a penalty to large coefficient values, which helps to stabilize the model. With categorical/dummy variables, the effect is similar to credibility weighting, but this method can be used for all types of variables. Ridge regression, a type of penalized regression, will be illustrated. To implement, the logarithm of a normal probability density function with a mean of 0 is evaluated at each coefficient value (just for the predictive variables, that is), and this sum is added to the total log-likelihood. The variance of this normal distribution can be estimated using cross validation.

One simple way to perform cross validation is to test various candidate variance values and fit the model on only a fraction of the data. The remaining data is then used to calculate the mean squared error divided by the mean to the appropriate power (one for Poisson, two for gamma, etc.), multiplied by the used premium. This process should be repeated several times to gather a more reliable estimate. It also helps reduce the number of iterations needed if the same samplings are used for each value being testing, although this is not required. A graph of the average mean square errors can show whether enough iterations have been performed.

The same variance is usually used for all coefficients. Non-dummy variables should be standardized to all be on the same scale so that their variances are comparable; this can be done by subtracting out the mean and dividing by the standard deviation, or if dummy variables are being used as well, by dividing by two times the standard deviation (Gelman 2008). Using this method lessens the negative effect of noise variables and also improves the performance of predictive variables. There are other methods of performing cross validation that will not be discussed here.

5. MULTIPLE LINES

Multiple lines can be evaluated together using the same variance parameters, R and Q, but allowing different initial loss ratio parameters for each line. This will leverage the volatility estimation across all of the lines together.

Going one step further, it is possible to do the same, but have the initial loss ratios related to each other via credibility weighing. This can be done using Bayesian credibility, and this method can be implemented simply, without the use of specialized Bayesian software, as will be explained. If a normal distribution is used as the prior distribution for the initial loss ratios, this is a conjugate prior since a normal distribution is also being used for the loss ratios, and so, the posterior distribution will be normally distributed. This means that maximum likelihood estimation, which returns the mode, can be used to estimate the mean, since the mean is identical to the mode for a normal distribution. Performing credibility in this fashion will also match the Buhlmann-Straub credibility results (Herzog 1989). To implement, another parameter should be added for the complement of credibility. Then, the log-likelihood of a normal probability density function evaluated at each initial loss ratio with a mean of the credibility complement should be calculated for each line. Adding the sum of these log-likelihoods to the total log-likelihood will cause the loss ratios to shift towards the overall mean and credibility weighting will be performed. The variance of this normal prior distribution is equivalent to the between variance used in the Buhlmann-Straub method. One way to estimate it is to use the Buhlmann-Straub formulas (as described in Korn 2015 to apply to loss ratios, for example).

Using this approach to calculating the between variances, however, does not consider the loss ratio changes by year as calculated by the Kalman Filter and so is slightly inconsistent. As an alternative, cross validation can be used instead, similar to ridge regression, which was described earlier. Different between variances can be tested where the loss ratios are fit using only a fraction of the data and the remainder of the data is used to calculate the mean square error divided by the mean to the appropriate power, multiplied by the used premium. Using this will be consistent with the loss ratio changes by year.

There is still an issue, however, since credibility weighting the initial loss ratios towards the mean but then allowing the remaining ones to vary freely sometimes produces results that deviate away from the mean with time, even if this is not the case, especially if the between variance chosen is relatively small. Bayesian credibility was used to credibility weight the initial loss ratios, which has the formula:

$$f(Posterior \mid Data, Parameters) = f(Likelihood \mid Data, Parameters) \times f(Prior \mid Parameters).$$

Credibility weighting is performed since the prior component, *f(Prior | Parameters)*, applies a penalty to the parameters as they deviate away from the mean. This prior needs to be a function of the model parameters.

However, it is also possible to reparameterize the model so that instead of using the initial loss ratios as the parameters, the ending loss ratios are used instead. Note that it is possible to solve for the ending loss ratios given all of the Kalman Filter parameters including the initial loss ratios. Because of this, it is also possible to invert the equations and to solve for the initial loss ratios given the ending loss ratios. So, the ending loss ratios can be used as the parameters of the model, the initial loss ratios can be solved for, and then the Kalman Filter can be run as normal. To make the process simpler, instead of actually performing all of these calculations, we can run the Kalman Filter as normal using the initial loss ratios as the parameters, but still "pretend" that the ending loss ratios are the parameters and calculate the prior distribution credibility penalty using the ending loss ratios, since the result would be exactly the same. So, in summary, nothing needs to be changed, and the ending loss ratios can be used for credibility weighting.

This method produces better behaving models that do not artificially deviate either towards or away from the mean. Because the Kalman Filter iterates forwards through all of the loss ratios, and then conducts another iteration backwards to smooth the results, the ending loss ratio can almost be thought of as the midpoint of the iteration. Therefore, it is recommended to use the ending loss ratios for calculating the log-likelihoods of the normal prior distribution.

6. SIMULATION RESULTS

A simulation was run[6] to help illustrate the benefits this method can provide, although, of course, the exact benefit will vary from case to case. In this scenario, two random variables were combined to simulate the frequency per year, and it was assumed that one of these was known. This was done to simulate a scenario where a predictive variable is known that affects the frequency per year, such as the state of the economy, but that not everything about how the frequency changes is known.

The summary of the results are shown in Figure 5.

[6] Frequency was simulated using a negative binomial with a mean of 50 for complete years and a variance-to-mean ratio of 2.5. Severity was simulated from a lognormal distribution with mu and sigma parameters of 10 and 2, respectively, a retention of $100 thousand and limit of two million. Autocorrelation was 30% for each of the frequency variables and for the severity variable, variance of the year-to-year changes was 0.005 for each of the frequency variables and 0.00025 for the severity variable. Development factors were used that affected the frequency that decreased by 0.05 starting at the 22nd period. Premium was set so that the expected loss ratio for the first year would be 70%. For the methods that used bagging, 25 iterations were used using ⅔ of the data on each iteration. 500 simulations were run. The models were fit using the approximation for the Tweedie family mentioned earlier.

Figure 5

Method	RMSE All Years	RMSE Latest Year	RMSE All Years - Compared to Cape Cod	RMSE Latest Year - Compared to Cape Cod
Cape Cod	1.211	0.312	0%	0%
Kalman Filter	0.538	0.132	-55.5%	-57.6%
Kalman Filter with Bagging	0.508	0.126	-58.1%	-59.5%
Kalman Filter with Predictive Variable	0.485	0.120	-59.9%	-61.4%
Kalman Filter with Predictive Variable and Bagging	0.462	0.114	-61.9%	-63.3%
Kalman Filter with Predictive Variable, Penalized Regression and Bagging[7]	0.453	0.105	-62.6%	-66.3%
Tweedie GLM with Predictive Variable (Weighted by Used Premium per Year)	0.704	0.165	-41.8%	-47.2%

The main conclusion is the amount of benefit this method is capable of providing over the Cape Cod, which does not adapt to changing conditions and cannot include predictive variables. Each of these individually is also able to provide significant benefit.

7. CONCLUSIONS

The goal of this paper was to present a relatively simple method that can be implemented in spreadsheets to extend the Cape Cod and is capable of accounting for changes indicated in the data and from external predictive variables. Estimating expected loss ratios per year with volatile data can often be a confusing and difficult task, subject to a large degree of judgement. It is our hope to improve this process by adding some guidance from modern statistical techniques without losing the simple and intuitive nature of the Cape Cod method.

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[7] Only 100 iterations were performed for this method because of its longer running time. Also, only 10 iterations of bootstrapping were performed. Using a higher number is expected to further improve the performance of this method.

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APPENDIX A

The appropriate development factor to relate to the increased volatility of incomplete experience can be derived using the formula for the variance of aggregate losses, where A are the ultimate aggregate losses, F is the ultimate frequency, S is the ultimate severity, VTM_F is the variance-to-mean ratio for the frequency, CV_S is the coefficient of variation for the severity, RPT are the reported losses, and ULT are the ultimate losses:

$$V(A) = V(F)\, E(S^2) + E(F)\, V(S) = VTM_F\, F\, S^2 + F\, CV_s\, S^2 = F\, S^2\, (\, VTM_F + CV_S^2\,)$$

The variance of the reported losses is equal to the below, since the observed frequency is F / $CCDF$ (where $CCDF$ is the claim count development factor), and the observed severity is S / SDF (where SDF is the severity development factor, which is equal to the LDF divided by the $CCDF$):

$$V(RPT) = \frac{F}{CCDF} \times \frac{S^2}{SDF^2} \times (\, VTM_F \,+\, CV_S{}^2\,)$$

The variance of ultimate losses is then equal to:

$$\begin{aligned} V(ULT) &= V(RPT) \times\, LDF^2 \,=\, V(RPT) \times CCDF^2 \times SDF^2 \\ &= \frac{F}{CCDF} \times \frac{S^2}{SDF^2} \times (\, VTM_F \,+\, CV_S{}^2\,) \times CCDF^2 \times SDF^2 \\ &= F \times S^2 \times (VTM_F \,+\, CV_S{}^2\,) \,\times CCDF \\ &= ULT \,\times\, S \,\times\, (VTM_F \,+\, CV_S{}^2\,) \,\times CCDF \end{aligned}$$

Note how all SDF terms cancel out and the only development term remaining is the claim count development factor.

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Summer 2016 e-Forum

Supplement for An Extension to the Cape-Cod Method with Credibility Weighted Smoothing

http://www.casact.org/pubs/forum/16sforum/Korn.xlsm

Hierarchical Compartmental Models for Loss Reserving

Jake Morris, FIA

Abstract

Motivation. This paper proposes a triangle-based stochastic reserving framework for parsimoniously describing insurance claims generation, reporting and settlement processes with intuitive parameters.

Method. Deterministic compartmental models are explored as extensible tools to describe and project the insurance claims process using a small number of parameters, including a measure of case reserve robustness. A Schedule-P reserving case study illustrates the application of a nonlinear hierarchical ("mixed-effects") framework to fit compartmental models to outstanding and cumulative paid claims development triangles, simultaneously. This allows one or more of the claims process parameters to vary by claims cohort in accordance with a statistical distribution. An optional Bayesian implementation facilitates the robust incorporation of external information and judgment into the projection of reserve uncertainty.

Results. A flexible stochastic reserving framework is established, with benefits including the ability to explicitly account for reporting and/or settlement rate changes, make inferences about components of the claims process and scenario test future process changes using information gathered across the business.

Conclusions. Hierarchical compartmental models can describe and project the insurance claims process in an optional level of detail for the purpose of setting reserves.

Availability. Frequentist model R code is contained in Appendix E, Bayesian model OpenBUGS code is contained in Appendix F and an illustration spreadsheet is available at: http://www.casact.org/pubs/forum/16sforum/.

Keywords. Stochastic loss reserving, compartmental reserving models, claims process modeling, hierarchical models, nonlinear mixed-effects, Bayesian modeling, MCMC.

1. INTRODUCTION

A variety of triangle-based stochastic reserving techniques have been proposed for estimating future general insurance claims payments, ranging from generalized linear models (England and Verrall, 2002) to nonlinear hierarchical models (Guszcza, 2008). Methods incorporating both paid and incurred information have been explored (Martínez-Miranda, Nielsen and Verrall, 2012; Quarg and Mack, 2004), which provide richer inference and improved interpretability. Furthermore, Bayesian methods (Zhang, Dukic and Guszcza, 2012; Meyers, 2007; England and Verrall, 2005; Verrall, 2004) have become increasingly ubiquitous; providing flexibility and the ability to robustly incorporate judgment into uncertainty projections.

This paper explores a new triangle-based (and optionally-Bayesian) stochastic reserving *framework* which considers the relationship between exposure, case reserves and paid claims. By doing so, it enables practitioners to build communicable models that are consistent with their understanding of the insurance claims process. Furthermore, it supports the identification and quantification of claims process characteristics to provide tangible business insights.

1.1 Research Context

Compartment(al) models (Sheppard, 1948) are extensible tools for describing the transfer of material between components of a system over time. For a sufficient volume of claims, the insurance claims process can be represented by a small number of compartments and intuitive parameters. The parameters describe aggregate claims movements between compartments and the ultimate loss ratio (ULR), decomposed into a reported loss ratio and a measure of case reserve robustness.

Motivated by Guszcza (2008), a nonlinear hierarchical modeling framework is proposed for fitting compartmental loss reserving models to claims triangles, allowing one or more of the model parameters (and hence development patterns) to vary by claims cohort in accordance with a statistical distribution. This enables flexible and parsimonious compartmental models to be fitted to reported outstanding claims and cumulative paid claims development triangles, simultaneously.

An optional Bayesian implementation (akin to Zhang, Dukic and Guszcza, 2012) allows external information and judgment to be incorporated into reserve uncertainty projections. Additionally, Markov chain Monte Carlo (MCMC) techniques facilitate model flexibility, and consequently, specific features such as the correlation between successive observations and calendar shocks can be accounted for.

1.2 Objective

Hierarchical compartmental reserving models have parallels with the hierarchical growth curves put forward by Guszcza (2008). In contrast to monotonic growth curves however, compartmental models can be fitted to cumulative paid claims *and* outstanding claims reserves, simultaneously. Since outstanding claims typically rise and fall over time, negative incurred claims development is supported. Furthermore, explicit modeling of outstanding claims may reduce the subjectivity inherent in the selection of a growth curve for tail extrapolation. Finally, relating compartmental model parameters back to the claims process provides intuitive control over the level of model complexity.

In contrast to Zhang, Dukic and Guszcza (2012), the corresponding Bayesian implementation enables prior beliefs to be more readily incorporated into process-based model parameters. This allows drivers of uncertainty to be isolated. Additionally, Bayesian hierarchical compartmental models have the flexibility to handle negative development for reserve uncertainty projections contrary to many existing GLM-type methods (England and Verrall, 2002).

Furthermore, compared to existing methods that utilize both paid and incurred data (e.g. Martínez-Miranda, Nielsen and Verrall, 2012; Quarg and Mack, 2004), a compartmental approach ensures consistency between estimated paid and incurred claims.

Although this paper proposes a triangle-based approach, methods incorporating individual claims data (e.g. Antonio and Plat, 2014; Parodi, 2013) exhibit a number of desirable properties, including the ability to reflect underlying claims processes. Such methods typically require a combination of models to be parameterized however, whereas a compartmental framework allows claims process characteristics to be quantified using a single structural model. Additionally, hierarchical model diagnostic tests can help to mitigate the risk of overfitting the data and reducing extrapolation validity.

1.3 Outline

The remainder of the paper proceeds as follows:

- Section 2 will introduce compartmental modeling theory, hierarchical compartmental models and Bayesian hierarchical compartmental models.
- Section 3 will define a compartmental model for the claims process. Parameter interpretations will be discussed and a number of practical extensions will be explored.
- Section 4 will contain a triangle reserving case study detailing the application of frequentist and Bayesian hierarchical compartmental models to a Schedule-P dataset.
- Section 5 will present a brief overview of future development areas.
- Section 6 will summarize the paper's findings.

Appendices will contain various supplementary materials including the case study data, frequentist modeling R code, and Bayesian modeling OpenBUGS code.

2. COMPARTMENTAL MODELS

A system is said to be a compartment(al) system when its entities can be grouped into a finite number of connected homogeneous components, known as compartments (Sheppard, 1948). They are often used to describe how entities/materials change location or state over time. The set of all possible compartments in a system is called the *state-space*, and the phenomena under study in each compartment are described by *state-variables* (Blomhøj, Kjeldsen and Ottesen, 2014).

Compartmental models can be deterministic or stochastic, containing discrete or continuous state-variables in discrete or continuous time. In deterministic models, the behavior of the quantities within the system is dictated solely by their past behavior and the rules that govern the model. In contrast, stochastic models imply a distribution of possible behaviors (Brauer, 2008). A useful feature of compartmental models is that complexity can be controlled by adjusting the number of compartments and/or their corresponding inflows and outflows.

The focus of this paper will be a practical claims reserving application of **deterministic, continuous state-variable and continuous-time compartmental models**. The rationale is as follows:

- Compartmental models describing exposure, reported outstanding claims and cumulative paid claims (where the latter two are simultaneously fitted) have not yet been introduced into the loss reserving literature.
- Deterministic models are practical to implement, and their simplicity results in clear and communicable claims process parameters.
- The hierarchical framework proposed in Section 4 increases mathematical complexity to the extent that at present, appropriate hierarchical stochastic compartmental reserving models are not easily implementable in conventional software.

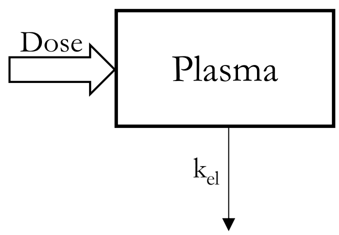
Sections 2.1 and 2.2 will contain overviews of deterministic and stochastic compartmental models, and Section 2.3 will introduce hierarchical compartmental models.

2.1 Deterministic compartmental models

Deterministic compartmental models have many possible applications. One of which is to describe the transport of material through biological systems, where compartments have physiological interpretations. For example, "compartmental pharmacokinetic models" are commonly used to describe the continuous transfer of an administered drug into, within and out of a patient. State-spaces typically comprise blood plasma and body tissues/organs, with state-variables denoting their drug concentration-time (or amount-time) profiles.

Deterministic, continuous-time models can be expressed as linear systems of ordinary differential equations (ODEs), with state-variables expressed as differentials of time. Analytical state-variable solutions are linear combinations of exponential terms describing the estimated amounts of material in each compartment at each time.

A one compartment pharmacokinetic model with state-space $\{Plasma(t)\}$ for a direct intravenous drug dose can be written schematically as follows:



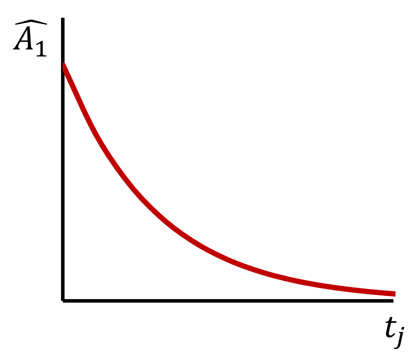
Alternatively, the model can be written as a single ODE, where the state-variable $\{A_1(t)\}$ denotes the amount of drug in the blood plasma at time t, and the positive "rate elimination constant" $\{k_{el}\}$ describes how quickly the drug is eliminated from the body. It is assumed that elimination of the drug is constant and directly proportional to its amount (first-order kinetics):

$$dA_1/dt = -k_{el}A_1$$
$$A_1(0) = Dose \tag{2.1}$$

A patient's blood plasma amount-time profile $A_1(t)$ can be measured by repeatedly sampling their blood over the time following a drug dose. The rate parameter k_{el} can then be estimated by solving the ODE and fitting the model to the patient's amount-time observations. Denoting y_j as the jth drug amount measurement for a patient, we can specify a nonlinear regression (Seber and Wild, 1989) as

$$A_1(t_j) = y_j = Dose \cdot e^{-k_{el}t_j} + \varepsilon_j$$
$$\varepsilon_j \sim N(0, \sigma^2) \tag{2.2}$$

where σ^2 is the variance of the discrepancy between the model fit and the drug amount measurements. For illustrative purposes, an estimated blood plasma amount-time profile $\widehat{A_1}(t_j)$ for a given dose and rate of elimination is as follows:



2.2 Stochastic compartmental models

In contrast to deterministic compartmental models, stochastic compartmental models introduce uncertainty external to the history of the modeled process by assuming that one or more of the states are random variables. This may be achieved, for example, by adding probabilistic state transfer mechanisms to an existing deterministic structure (Rescigno and Segre, 1966).

Three example forms of stochastic compartmental models and their corresponding properties are:

1) *Discrete-time Markov chain models:* discrete state-variables, discrete time steps
2) *Continuous-time Markov chain models:* discrete state-variables, continuous time scale
3) *Stochastic differential equation (SDE) models:* continuous state-variables and time scale

Hachemeister (1980) provides a loss reserving application of discrete-time Markov chain models. Analogously, Orr (2007), Hasselager (1994) and Norberg (1993) provide loss reserving applications of continuous-time Markov chain models.

2.3 Hierarchical compartmental models

Section 2.1 describes how a deterministic compartmental model can be used to estimate a drug amount-time profile for a single patient. However, in practice drug developers wish to make inferences about a population of individuals that may eventually take a particular drug. Assuming a drug has been administered to a group of individuals and expressing y_{ij} as the jth drug amount measurement ($j = 1\ to\ n_i$) for the ith individual ($i = 1\ to\ M$), we could use nonlinear regression to **fit a separate compartmental model to each individual:**

$$y_{ij} = Dose_i \cdot e^{-k_{el_i} t} + \varepsilon_{ij} \tag{2.3}$$

However, this modeling approach may result in many parameters relative to the number of data points available for modeling, reducing the credibility of each estimated parameter.

An alternative approach is to pool all individuals' concentration measurements and **fit one compartmental model with a single parameter to all individuals combined:**

$$y_{ij} = Dose_i \cdot e^{-k_{el} t} + \varepsilon_{ij} \tag{2.4}$$

Although k_{el} is likely to be estimated with greater precision than each k_{el_i} in Eq. (2.3), it is unlikely to result in an accurate fit to each individual due to between-patient variability e.g. differing metabolisms.

The approach commonly used in pharmacokinetic modeling in addition to other life and social sciences is **nonlinear hierarchical modeling**, which has previously been advocated for loss reserving by Guszcza (2008). **Hierarchical** (or *mixed-effects*) **models allow some model parameters to be fixed across individuals and some to vary by individual**. More generally, they allow parameters to vary by any natural data grouping. For example, for estimating insurance claims reserves Guszcza (2008) proposes claims cohorts (individual accident years) as a grouping for cumulative paid claims.

A hierarchical framework allows model parameters to vary by the assumed data grouping in accordance with a statistical distribution defined by a mean and variance only. This reduces the number of estimable parameters compared to the first modeling approach outlined above. Conversely, because the modeler can select which parameters vary by individual, each individual can be described in greater detail compared to the second modeling approach outlined above.

Hierarchical/mixed-effects models are said to allow data-sparse individuals to "borrow strength" from data-rich individuals. For parameters that vary by individual, an individual's parameter estimate is a weighted average of:

1) The estimated average parameter value across all individuals; and
2) The estimated individual parameter value for the individual.

The weight given to the individual parameter value is proportional to the individual's data volume. To illustrate how nonlinear hierarchical models are structured, Eq. (2.3) can be rewritten as

$$y_{ij} = Dose_i \cdot e^{-(\overline{k_{el}}+(k_{el_i}-\overline{k_{el}}))\cdot t} + \varepsilon_{ij} \tag{2.5}$$

where $\overline{k_{el}}$ represents the average rate of elimination across all individuals. Denoting $\overline{k_{el}}$ as β, and $k_{el_i} - \overline{k_{el}}$ as b_i (Pinheiro and Bates, 2000), this becomes

$$\begin{gathered} y_{ij} = Dose_i \cdot e^{-(\beta+b_i)\cdot t} + \varepsilon_{ij} \\ \varepsilon_{ij} \sim N(0, \sigma^2), \qquad b_i \sim N(0, \psi^2) \end{gathered} \tag{2.6}$$

where β is referred to as a *fixed-effect* and b_i as a *random-effect*, which has its own probability sub-model. A shared distribution for the random-effects induces a correlation between individuals, which may be an appropriate assumption if they are assumed to come from a wider population. σ^2 represents the within-subject variability, whereas ψ^2 represents the between-subject variability. For any number of individuals being modeled, only three parameters (β, σ and ψ) need to be estimated.

Two key reasons for using a hierarchical framework are **parsimony** and **flexibility**. These features may be useful for loss reserving where data are incomplete and sometimes limited for modeling purposes, requiring descriptive models that do not overfit.

Antonio and Zhang (2014) provide a detailed exploration of nonlinear hierarchical models for insurance data.

2.3.1 Bayesian hierarchical compartmental models

A modeler may want to incorporate external information and/or judgment into a compartmental model to account for information not contained within the modeled dataset. For example, in drug development there may be other data-rich drug administration studies from which to base parameter prior distributions. For the hierarchical model outlined in Eq. (2.6) it could be assumed that

$$\log(\beta) \sim N(\overline{\beta}, \gamma^2) \tag{2.7}$$

where β denotes the fixed-effect for the rate of drug elimination, and $\overline{\beta}, \gamma^2$ denote the prior mean and variance of $\log(\beta)$ respectively, which are specified by the modeler rather than estimated. Bayes' rule can then be used to estimate the posterior distribution of the fixed-effect as

$$\begin{aligned} p(\beta|y_{ij}) &\propto p(\beta)\, p(y_{ij}|\beta) \\ \equiv p(\beta|y_{ij}) &\propto p(\beta)\, \mathcal{L}(\beta; y_{ij}) \\ posterior &\propto prior \times likelihood \end{aligned} \tag{2.8}$$

where $p(\cdot)$ is a probability density function, β is the "random" parameter for which we wish to make inferences, y_{ij} is the "fixed" jth observation for individual i and $\mathcal{L}(\cdot)$ is the likelihood function. The posterior distribution is a *credibility weighting* of the prior distribution and likelihood function, where the weight placed on prior beliefs is inversely proportional to the volume of modeled data.

As highlighted by Zhang, Dukic and Guszcza (2012), this approach can be useful for loss reserving where it is often essential for a practitioner to **incorporate judgment** into reserve projections to allow for information not contained within the modeled data. Additionally, Bayesian methods allow us to **quantify reserve uncertainty** consistently with the definition stated by the 2005 Casualty Actuarial Society Working Party on Quantifying Variability in Reserve Estimates:

'Given . . . our current state of knowledge, what is the probability that [the entity's] final payments will be no larger than the given value'.

This can be framed mathematically using Bayesian statistics. Denoting ULR_i as the ultimate loss ratio (and parameter of interest) for the ith claims cohort and $Incurred_{ij}$ as the jth cumulative incurred claims observation for the ith claims cohort, the posterior density of ULR_i *given* $Incurred_{ij}$ is

$$p(ULR_i|Incurred_{ij}) \propto p(ULR_i)\, \mathcal{L}(ULR_i; Incurred_{ij}) \tag{2.9}$$

which provides an estimate of ULR parameter uncertainty. It is straightforward to incorporate process uncertainty into this posterior, from which a distribution for final payments can be derived consistently with the above definition. Finally, Bayesian models **increase flexibility** because they require only that model parameters and the relationships between them are specified.

A detailed exposition of Bayesian methods and their applications is given by Gelman *et al.* (2013).

3. COMPARTMENTAL MODELS FOR LOSS RESERVING

To specify a deterministic, continuous state-variable and continuous-time compartmental model for the insurance claims process, a state-space must be defined. The selection of a possible state-space is illustrated by considering the insurance claims process over development time for a *cohort* of claims e.g. an accident year:

1) Once a group of insurance policies have been written and incepted, they are exposed to the risk of making claims. Therefore an initial **"Exposed to Risk"** state is defined.
2) For some proportion of exposed policies, claim events will occur and be reported to the insurer. Claims are typically case reserved and classed as being outstanding until settled, defining a second state: **"Claims Outstanding"**.
3) A proportion of all reported outstanding claims will be settled with a payment amount, which defines a **"Claims Paid"** state.

The state-space is therefore $\{\text{Exposed to Risk}(t), \text{Claims Outstanding}(t), \text{Claims Paid}(t)\}$. The states-variables in turn denote the amount of remaining exposure, the monetary amount of claims outstanding, and the cumulative monetary amount of claims paid at development time t.

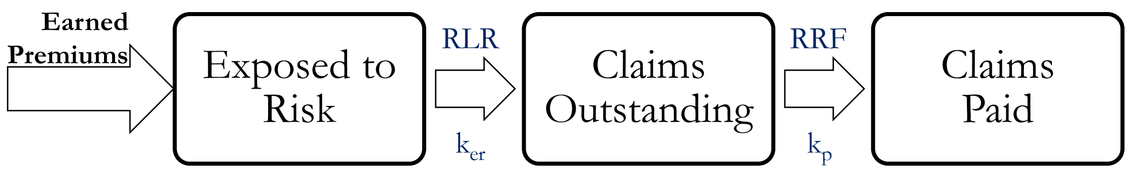
Assuming that the above process is a forward process only, i.e. no material re-opening of paid claims, a model schematic can be written as follows:

Exposure reduces over time as groups of claims are reported and become paid at some proportion of their outstanding amounts. This reduces the claim amounts outstanding (eventually to 0 as t becomes large) and ensures consistency between paid and incurred claims estimates. Although this model is for claim amounts, an adapted version could be defined for claim numbers (not shown).

To initialize the process, a suitable measure of exposure must be chosen as an input variable. For an accident year/quarter cohort of claims, earned premiums could be used (Guszcza, 2008; Clark, 2003). Alternatively, a pure exposure measure could be chosen in line with the original pricing basis (see Section 3.2).

Independently of the chosen exposure measure, the timing of policies coming on-risk during the claims cohort should be considered. If policies coming on-risk during an accident year/quarter are largely replaced by similar policies coming off-risk, i.e. steady-state conditions, a practitioner could input all exposures into to the system at development time 0. This is the approach taken for the case study presented in Section 4. Similarly, it would be acceptable to input all exposures at time 0 if all (or a large proportion of) policies that could give rise to a claim in the cohort are on-risk at the start of the cohort (e.g. accident quarter). However, if exposure materially fluctuates during the lifetime of the cohort, a more sophisticated approach is required to match the input exposures with the cohort's development times at which the policies come on-risk (see Section 3.2).

For an accident year/quarter cohort of claims, the use of (ultimate) earned premiums as an exposure measure provides an appealing parameter set. A schematic for what will be termed the "**baseline model**" is shown below, followed by its corresponding parameter definitions:



- **Reported Loss Ratio** ("RLR"): the *proportion* of premiums that become reported claims.
- **Rate of earning and reporting** ("k_{er}"): the *rate* at which claim events occur and are subsequently reported to the insurer.
- **Reserve Robustness Factor** ("RRF"): the *proportion* of outstanding claims that eventually become paid by the insurer.
- **Rate of payment** ("k_p"): the *rate* at which outstanding claims are paid by the insurer.

Therefore this model is defined in terms of proportions $\{RLR, RRF\}$ and rates $\{k_{er}, k_p\}$. For the continuous-time assumption to hold, a sufficient number of policies must be written to give rise to a steady "flow" of reported and paid claims over development time.

Denoting the state-space {Exposed to Risk(t), Claims Outstanding(t), Claims Paid(t)} as $\{EX(t), OS(t), PD(t)\}$, the above model can be written as follows:

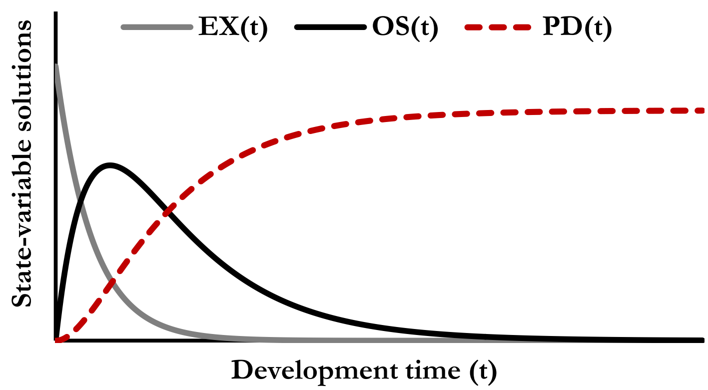
$$
\begin{aligned}
dEX/dt &= -k_{er} \cdot EX \\
dOS/dt &= k_{er} \cdot RLR \cdot EX - k_p \cdot OS \\
dPD/dt &= k_p \cdot RRF \cdot OS
\end{aligned}
\quad \textbf{(3.1)}
$$

Compartment initial conditions are $EX(0) = earned\ premiums = P$, $OS(0) = 0$ and $PD(0) = 0$, assuming steady-state exposure. Each parameter is assumed to be constant over development time t; however, this assumption is relaxed in Section 3.2.

Analytical state-variable solutions can be obtained using Laplace transforms (Gustav, 1974):

$$EX(t) = Pe^{-k_{er}t}$$
$$OS(t) = \frac{P \cdot RLR \cdot k_{er}}{k_{er} - k_p} \cdot \left(e^{-k_p t} - e^{-k_{er}t}\right) \qquad \textbf{(3.2)}$$
$$PD(t) = \frac{P \cdot RLR \cdot RRF}{k_{er} - k_p} \cdot \left(k_{er} \cdot \left(1 - e^{-k_p t}\right) - k_p \cdot \left(1 - e^{-k_{er}t}\right)\right)$$

The paid claims solution is analogous to a "growth curve", as put forward for loss reserving by Clark (2003). For a given set of parameters, the state-variables in the above system can be visualized over development time t as follows:



Although exposure may be impractical to track over time, outstanding and cumulative paid claims are typically observable at specific development time points, albeit incomplete for reserving purposes. Nonlinear regression techniques can be used to fit Eq. (3.2) to outstanding and cumulative paid claims simultaneously to derive parameter estimates and project the claims process to ultimate.

3.1 Parameter interpretations

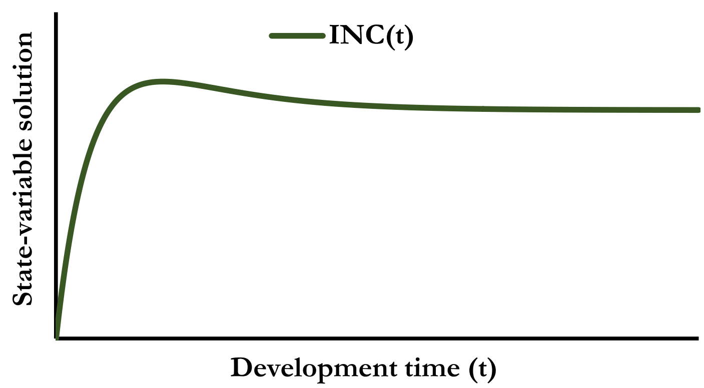
The two rate parameters k_{er} ($k_{er} > 0$) and k_p ($k_p > 0$) determine the monetary value of remaining exposures reported and outstanding claims paid respectively, per infinitesimal unit of time (with units t^{-1}). The term "k_{er}" is used to reflect that a policy exposed to risk must have a claim **e**vent occur before it is **r**eported, and may also be termed a rate of reporting (from exposure). It follows that higher magnitude rate parameters imply faster claims reporting/payment. However, if the model were to contain these rate parameters alone then all exposure would eventually convert to paid claims, resulting in a ULR equal to 100% (for a premium-based exposure measure).

To allow a range of possible ultimate loss ratios it is necessary to specify at least one proportion parameter, similar to Clark (2003). The two proportion parameters RLR ($RLR > 0$) and RRF ($RRF > 0$) determine the percentage of exposures that become reported claims and the percentage of outstanding claims that become paid claims respectively. The RRF parameter therefore indicates the average level of case over- or under-reserving for a cohort of claims. If case handlers persistently under-reserve, this would imply an RRF of over 100% and vice versa. An RRF of 100% indicates perfect case reserving on average amongst a cohort of claims. However, this may be the result of some claims being heavily over-reserved and some claims being heavily under-reserved, cancelling each other out in aggregate.

Persistent over-reserving is often associated with an incurred development pattern that rises and falls. Claims incurred at development time t $\{INC(t)\}$ can be derived under the model as

$$INC(t) = OS(t) + PD(t) \quad \textbf{(3.3)}$$

and visualized (for an RRF less than 100%):



Thus the model is able to capture negative incurred increments. Under the model, estimated cumulative incurred and paid claims tail development is defined by the extrapolation of estimated outstanding claims to zero (driven by k_p), and the estimated RRF. A convenient result is that the estimated ultimate loss ratio (ULR) can be directly obtained as

$$ULR = RLR \cdot RRF \quad \textbf{(3.4)}$$

The reason for this can be observed by equating the paid loss ratio (PLR) at development time t to the product of the RLR and RRF parameter definitions for a homogeneous group of claims, which are reported and subsequently paid together, i.e.

$$\frac{PD(t)}{P} = \frac{OS(t-v)}{P} \cdot \frac{PD(t)}{OS(t-v)} \quad \textbf{(3.5)}$$

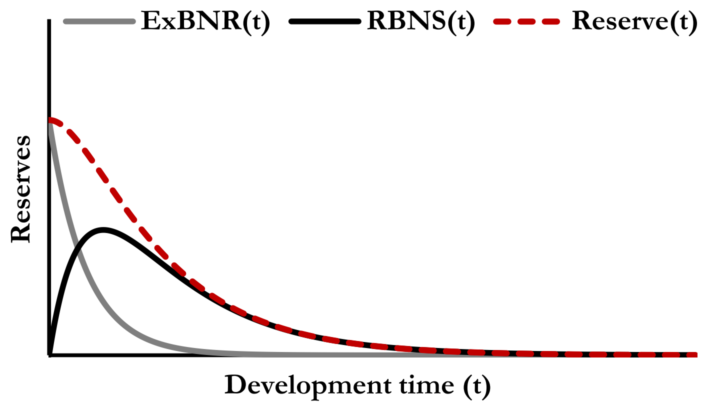
where t denotes development time within the claims cohort and v represents the elapsed time between the homogeneous group of claims being reported outstanding and paid. It is assumed that the premiums (P) for their underlying policies are written before time $t - v$. It follows that the RLR numerator and RRF denominator of the right hand side cancel out, and the PLR converges to the ULR for sufficiently large t.

3.1.1 ExBNR vs. RBNS

Using the compartmental model above it is possible to derive an exposed but not reported ("ExBNR") reserve and reported but not settled ("RBNS") reserve at development time t. The term "ExBNR" reflects the loss of claim event timing information once claims are grouped into outstanding and paid claims cohorts, and contains incurred but not reported ("IBNR") *plus* unearned claims:

$$\begin{aligned} ExBNR(t) &= EX(t) \cdot RLR \cdot RRF \\ RBNS(t) &= OS(t) \cdot RRF \\ Reserve(t) &= ExBNR(t) + RBNS(t) \end{aligned} \tag{3.6}$$

The reserves contain "IBNER" (incurred but not enough reported), indicated by the appearance of the reserve robustness factor (RRF). They can be visualized over development time for a given set of parameters as follows:



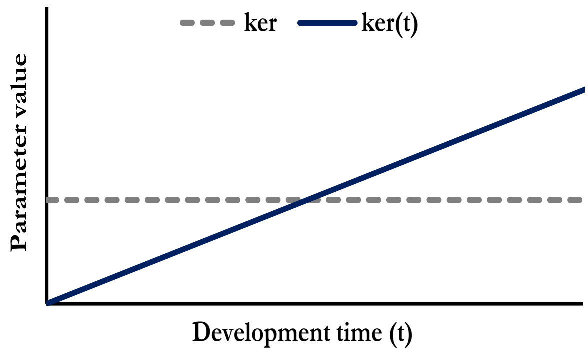
Taking the definition of IBNR to be ultimate losses less incurred losses to date, Eq. (3.4) can be used to define ultimate losses as $P \cdot RLR \cdot RRF$, and hence $IBNR(t) = P \cdot RLR \cdot RRF - INC(t)$. When $EX(0) = P$, Eq. (3.6) provides an alternative derivation: $IBNR(t) = Reserve(t) - OS(t)$.

3.2 Model extensions

Compartmental models are extensible, allowing practitioners to adjust them in line with their understanding of the claims process for the class of business being modeled. Matching model extensions to underlying processes may also enable models to be more easily communicated to stakeholders.

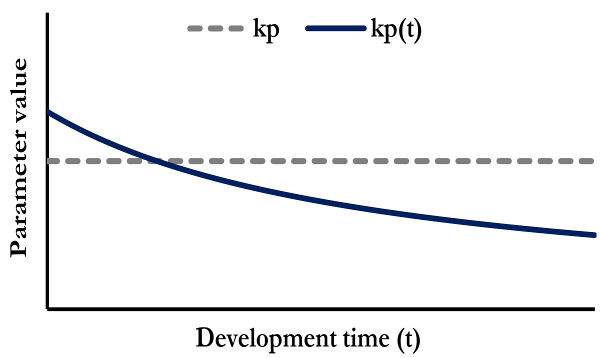
Parameters within the model have thus far been assumed to be constant and independent of development time. However, it may be desirable for one or more of the model parameters to depend on development time. For example, allowing the rate of reporting k_{er} to increase with development time t could capture delays between claim events and reports:

$$k_{er}(t) = \beta_{er} \cdot t \tag{3.7}$$



Alternatively, liability claims outstanding in later development periods may be those in/awaiting litigation. To reflect a potentially slower rate of settlement and subsequent payment for such claims, a nonlinear rate of payment $k_p(t)$ could be specified as follows:

$$k_p(t) = \frac{\beta_{p,1}}{\beta_{p,2} + t} \tag{3.8}$$



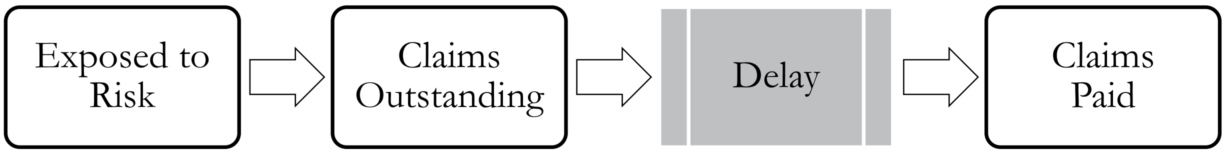
This function assumes that rate of payment reductions *decrease* over development time. Substituting the above rate functions into Eq. (3.1) gives

$$\begin{aligned} dEX/dt &= -\beta_{er} \cdot t \cdot EX \\ dOS/dt &= \beta_{er} \cdot t \cdot RLR \cdot EX - \{\beta_{p,1}/(\beta_{p,2} + t)\} \cdot OS \\ dPD/dt &= \{\beta_{p,1}/(\beta_{p,2} + t)\} \cdot RRF \cdot OS \end{aligned} \tag{3.9}$$

Implied development patterns for the baseline model and extended model incorporating the above functions are outlined in Appendix A.

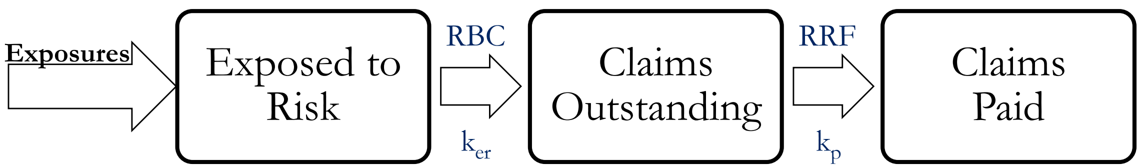
The proportion parameters could also be expressed as functions. For example, case reserves may be less robust at later development times for claims facing uncertain litigation. In practice, there are numerous plausible functions for describing how the claims process parameters are observed or perceived to behave; however, these will not be explored further in this paper.

It is also possible to increase the number of compartments to reflect claims sub-processes. For example, a bodily injury claims sub-class may exhibit a marked delay between claims being reported and subsequently being settled while damages are being quantified. This could be modeled using a "delay" state as follows:



Other potentially modelable sub-processes include calendar shocks (Section 4.2.2), reopened claims, third party claims payment recoveries, reinsurance recoveries, latent claims etc. However, available data may limit the degree to which complexity can be increased.

As noted in Section 3, it may be appropriate to initialize a compartmental reserving model with a non-premium measure of exposure. In which case the baseline schematic can be rewritten as follows:



The parameter interpretations for this model are largely unchanged; however, the reported loss ratio is replaced by a reported burning cost:

- **Reported Burning Cost** ("RBC"): the *proportion* of exposures that become reported claims.

The ultimate burning cost (UBC) can be obtained from the RBC and RRF parameters (analogously to the ULR in Eq. (3.4)) as

$$UBC = RBC \cdot RRF \tag{3.10}$$

This parameterization could be useful for pricing. The anticipated exposure for a cohort of new business could be multiplied by a selected UBC (allowing for changes in underwriting, claims environment, reserve robustness etc.) to derive an estimated loss cost. This may form the risk premium or be a precursor to a full frequency-severity analysis, for example.

Finally, compartmental reserving models can be generalized to describe exposure accumulation for cases where steady-state conditions do not hold (see Section 3). This can be achieved by continuously inputting portions of premium/exposure into the system over a period of time.

For example, if claims are grouped into an underwriting cohort then ultimate premiums can be projected to derive a discrete incremental writing pattern. For a cohort's premium and claims data observed at *discrete* development times $r\Delta, r \in \{0,1,2, \ldots\}$ after the commencement of the underwriting cohort, $PPN[r]$ can be defined as the proportion of ultimate premiums written uniformly over the period $r\Delta \rightarrow r\Delta + \Delta$. It follows that $\sum_0^\infty PPN[r] = 1$. The input to the exposure compartment (denoted by $\overrightarrow{EX}$) over each *continuous* time increment $t \rightarrow t + \delta t$ (where δ is infinitesimally small) can then be set to

$$\overrightarrow{EX}(t \rightarrow t + \delta t) = ultimate\ premiums \cdot \delta t \cdot PPN\left(\left\lfloor \frac{t}{\Delta} \right\rfloor\right) \quad \textbf{(3.11)}$$

where $\lfloor \cdot \rfloor$ denotes the floor or "next smallest integer value". If there is substantial time between policies being written and subsequently incepting (i.e. bound but not incepted "BBNI" policies), then the aforementioned writing pattern could be replaced by an inception pattern.

3.3 Limitations

As discussed in Section 3, all exposures can be input to the compartmental reserving system at time 0 under steady-state conditions. However, if steady-state conditions do not hold and material exposure fluctuations are not taken into account (e.g. using the approach outlined above), these will be absorbed into the reporting rate parameter k_{er}. This could lead to misleading k_{er} comparisons if the model is fitted to multiple claims cohorts, and additionally, may result in poor model fits.

Equation (3.5) illustrates a key assumption of deterministic compartmental reserving models: at a given time, claims within each compartment are assumed to be well-mixed and homogeneous i.e. they are assumed to behave uniformly and in accordance with a single set of parameters. In reality, each *individual* claim is likely to have a distinct RLR, k_{er}, RRF and k_p from every other claim. However, for an aggregated cohort of claims values it is only necessary for the *average* behavior of the cohort to be in line with the model parameters at each time, which may be a reasonable assumption for a high volume of claims within a particular claim size range.

A limitation of this approach is that a cohort with many heterogeneous individual claims (e.g. low-frequency high-value claims) or erratic case reserve fluctuations may not be well reflected by a deterministic compartmental model. To model a heterogeneous cohort, one could cap claims values within the cohort at a specified threshold and apply a frequency-severity or alternative approach for losses above the threshold. Other data segmentation techniques may be appropriate or, alternatively, the differing behavior of individual claims may be more accurately reflected by a stochastic or semi-stochastic compartmental model, as outlined in Appendix B.

A practical limitation is that some claims cohorts will have limited development histories, preventing a credible deterministic compartmental reserving model from being fitted due to a high ratio of parameters relative to data points. This limitation is addressed in Section 4.

3.4 Illustration

A spreadsheet containing a parameter-adjustable discretized compartmental reserving model is available at: http://www.casact.org/pubs/forum/16sforum/. This illustrates the dynamics of how the amounts in each compartment are determined over time for both constant and non-constant rate parameters. Additionally, it allows both steady-state and accumulating exposure.

4. HIERARCHICAL COMPARTMENTAL RESERVING CASE STUDY

The preceding Sections explore a deterministic compartmental reserving model for a single cohort of claims (e.g. an accident year). However, reserves are typically set for several cohorts of claims, often grouped into triangles. Cohorts are likely to have some shared characteristics; for example, due to the same underwriters and claims handling philosophy. However, they are also likely to exhibit differences; for example, due to changes in underlying risk profiles and differing claims environments.

The nonlinear hierarchical approach outlined in Section 2.3 allows for individual claims cohort characteristics when the data are credible, while allowing less mature cohorts to borrow strength from more mature cohorts. This can help to achieve parsimony. Following Guszcza (2008), triangles are viewed as "longitudinal" datasets, where claims cohorts are individuals and cumulative losses at various development times are a series of observations for each individual.

Frequentist and Bayesian hierarchical compartmental models will be fitted to a sample loss reserving dataset obtained at: http://www.casact.org/research/index.cfm?fa=loss_reserves_data. The selected workers' compensation dataset comprises both outstanding and cumulative paid claims development data grouped by accident years 1988-1997 and development years 1-10, together with earned premiums by accident year. The dataset contains both upper triangles (calendar years 1988-1997) and lower triangles of data (calendar years 1998-2006). The upper triangles and earned premiums as at 12/31/1997 are as follows:

Outstanding Claims ($'000s)

AY	Prem	1	2	3	4	5	6	7	8	9	10
1988	104	53	41	32	25	17	13	10	7	2	1
1989	89	54	37	27	20	14	10	7	3	2	
1990	86	55	37	28	18	11	7	4	3		
1991	99	61	42	26	15	9	6	4			
1992	105	66	46	31	22	12	8				
1993	119	68	51	40	22	17					
1994	111	62	47	32	24						
1995	78	57	49	35							
1996	64	57	42								
1997	48	41									

Cumulative Paid Claims ($'000s)

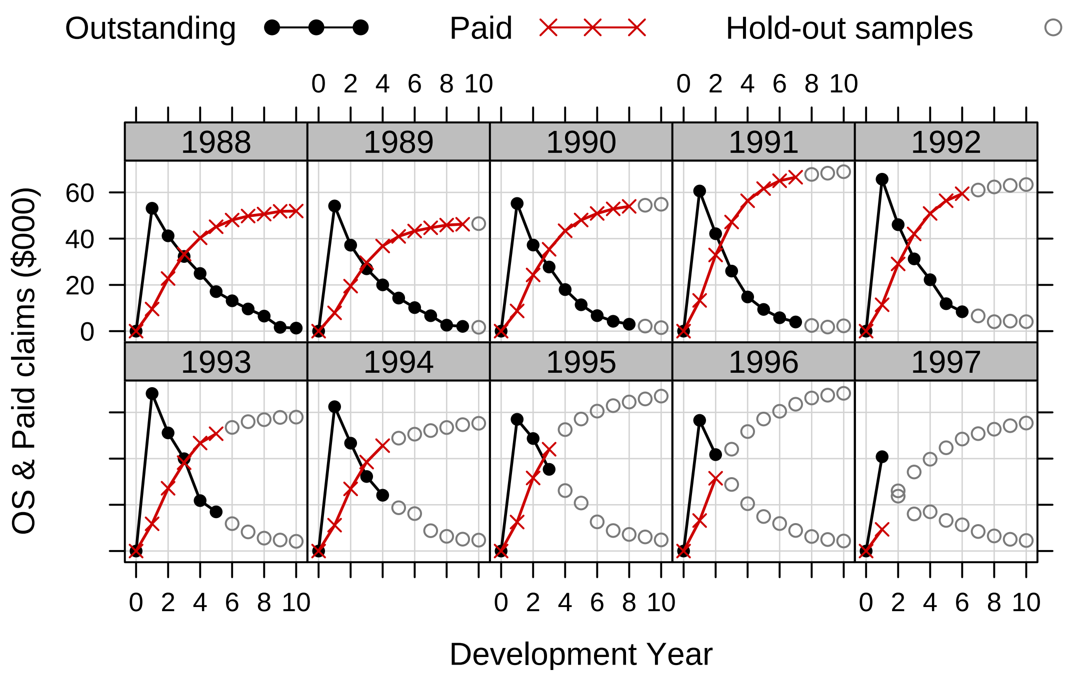
AY	Prem	1	2	3	4	5	6	7	8	9	10
1988	104	10	23	33	40	45	48	50	51	52	52
1989	89	8	19	30	37	41	43	45	46	46	
1990	86	9	24	35	43	48	51	53	54		
1991	99	13	33	47	56	62	65	67			
1992	105	11	29	42	51	56	59				
1993	119	12	27	38	47	51					
1994	111	11	27	38	46						
1995	78	13	32	44							
1996	64	13	31								
1997	48	9									

Claims development data should initially be visualized by accident year to establish whether:

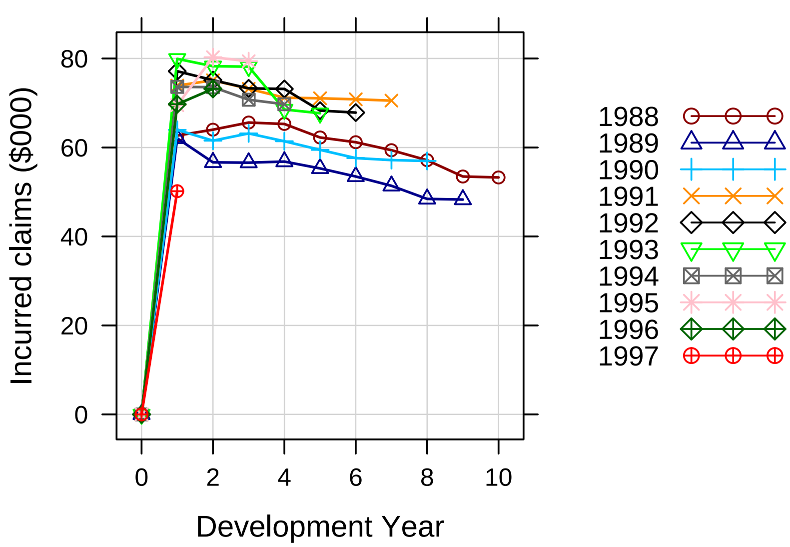
1) A compartmental model is appropriate i.e. whether there is a detectable process; and

2) There are any claims process characteristics that can be identified from the outset.

The below plots suggest a clear process between claims being reported and subsequently paid, therefore a compartmental model may be appropriate. The data are also relatively stable, suggesting that the baseline compartmental model outlined in Section 3 is an appropriate starting point.



Incurred development has a clear downwards trend typical of over-stated case reserves at some point during the development history, i.e. $RRF < 1$:



In Section 4.1 a frequentist hierarchical compartmental model will be fitted, assessed for goodness of fit and improved as necessary. A Bayesian implementation will be explored in Section 4.2. For both exercises, model predictability will be tested against the lower triangle hold-out samples.

4.1 Frequentist modeling

The motivations for exploring frequentist hierarchical compartmental models (and point estimates) before their Bayesian counterparts are as follows:

- Best estimate reserves are of principle stakeholder interest, followed by reserve uncertainty;
- Fewer modeling assumptions are required and thus model building is less time consuming; and
- Model run times are relatively quick, allowing models to be tested, interpreted and improved upon relatively quickly.

The baseline compartmental model ODEs (Section 3) are

$$\begin{aligned} dEX/dt &= -k_{er} \cdot EX \\ dOS/dt &= k_{er} \cdot RLR \cdot EX - k_p \cdot OS \\ dPD/dt &= k_p \cdot RRF \cdot OS \end{aligned} \tag{4.1}$$

with initial conditions $EX(0) = earned\ premiums = P$, $OS(0) = 0$ and $PD(0) = 0$ (assuming steady-state exposure – see Section 3). To ensure that compartmental model parameter estimates are positive, we reparameterize using the logarithm of the parameters $\{lk_{er} = \log(k_{er}), lRLR = \log(RLR), lk_p = \log(k_p), lRRF = \log(RRF)\}$ to give an initial "structural" model:

$$\begin{aligned} dEX/dt &= -\exp(lk_{er}) \cdot EX \\ dOS/dt &= \exp(lk_{er}) \cdot \exp(lRLR) \cdot EX - \exp(lk_p) \cdot OS \\ dPD/dt &= \exp(lk_p) \cdot \exp(lRRF) \cdot OS \end{aligned} \tag{4.2}$$

This model can be specified in a format compatible with the R software (R Core Team, 2016) package "nlmeODE" (Tornøe *et al.*, 2004a) and combined with a grouped data object (see Appendices D and E). The data comprise upper triangles of outstanding and cumulative paid claims together with compartment initial conditions (earned premiums) by accident year, as outlined above.

To fit a hierarchical compartmental model based on the above ODEs, it must be decided which of the model parameters should have random-effects and therefore vary by accident year. For this case study, the components of the ultimate loss ratio (the reported loss ratio and reserve robustness factor) will be assumed to vary by accident year to define a baseline hierarchical model.

The Eq. (4.2) outstanding and cumulative paid claims state-variable solutions for accident year $i = 1 \text{ to } 10$ and development year $j = 1 \text{ to } 11 - i$ can be denoted $f_{OS}(P_i, \boldsymbol{\phi}_i, t_j)$ and $f_{PD}(P_i, \boldsymbol{\phi}_i, t_j)$ respectively, where P_i is the earned premium for accident year i. Stacking response variables for outstanding claims OS_{ij} and cumulative paid claims PD_{ij} into a single response variable $\boldsymbol{y}_{ij} = (OS_{ij}, PD_{ij})^T$ enables a nonlinear hierarchical "statistical" model to be specified (Model 1):

$$\boldsymbol{y}_{ij} = \boldsymbol{f}(P_i, \boldsymbol{\phi}_i, t_j) + \boldsymbol{\varepsilon}_{ij}$$

$$\boldsymbol{y}_{ij} = \begin{bmatrix} OS_{ij} \\ PD_{ij} \end{bmatrix}, \quad \boldsymbol{f}(P_i, \boldsymbol{\phi}_i, t_j) = \begin{bmatrix} f_{OS}(P_i, \boldsymbol{\phi}_i, t_j) \\ f_{PD}(P_i, \boldsymbol{\phi}_i, t_j) \end{bmatrix}, \quad \boldsymbol{\varepsilon}_{ij} = \begin{bmatrix} \varepsilon_{ij}{}^{OS} \\ \varepsilon_{ij}{}^{PD} \end{bmatrix}$$

$$\boldsymbol{\phi}_i = \begin{bmatrix} \phi_{1i} \\ \phi_{2i} \\ \phi_{3i} \\ \phi_{4i} \end{bmatrix} = \begin{bmatrix} lk_{er} \\ lRLR_i \\ lk_p \\ lRRF_i \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} + \begin{bmatrix} 0 \\ b_{1i} \\ 0 \\ b_{2i} \end{bmatrix} = \boldsymbol{\beta} + \boldsymbol{b}_i \tag{4.3}$$

$$\boldsymbol{b}_i \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \psi_1^2 & 0 \\ 0 & \psi_2^2 \end{bmatrix}\right), \qquad \boldsymbol{\varepsilon}_{ij} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & \lambda^2 \end{bmatrix}\right)$$

The fixed-effects $\boldsymbol{\beta}$ represent the mean values of the *logarithm* of the claims process parameters across a theoretical "population" of accident years, and the random-effects $\boldsymbol{b}_i$ represent the deviations of the individual accident year parameters $\boldsymbol{\phi}_i$ from their mean values. The random-effects are assumed to be independent for different accident years and the within-accident-year errors $\boldsymbol{\varepsilon}_{ij}$ are assumed to be independent for different (i, j), and independent of the random-effects (Pinheiro and Bates, 2000). The variance of random-effect $b_{qi} \in \boldsymbol{b}_i$ is denoted $\psi_q{}^2$. The within-accident-year variances are denoted σ^2 and $\lambda^2\sigma^2$ for outstanding and cumulative paid claims respectively.

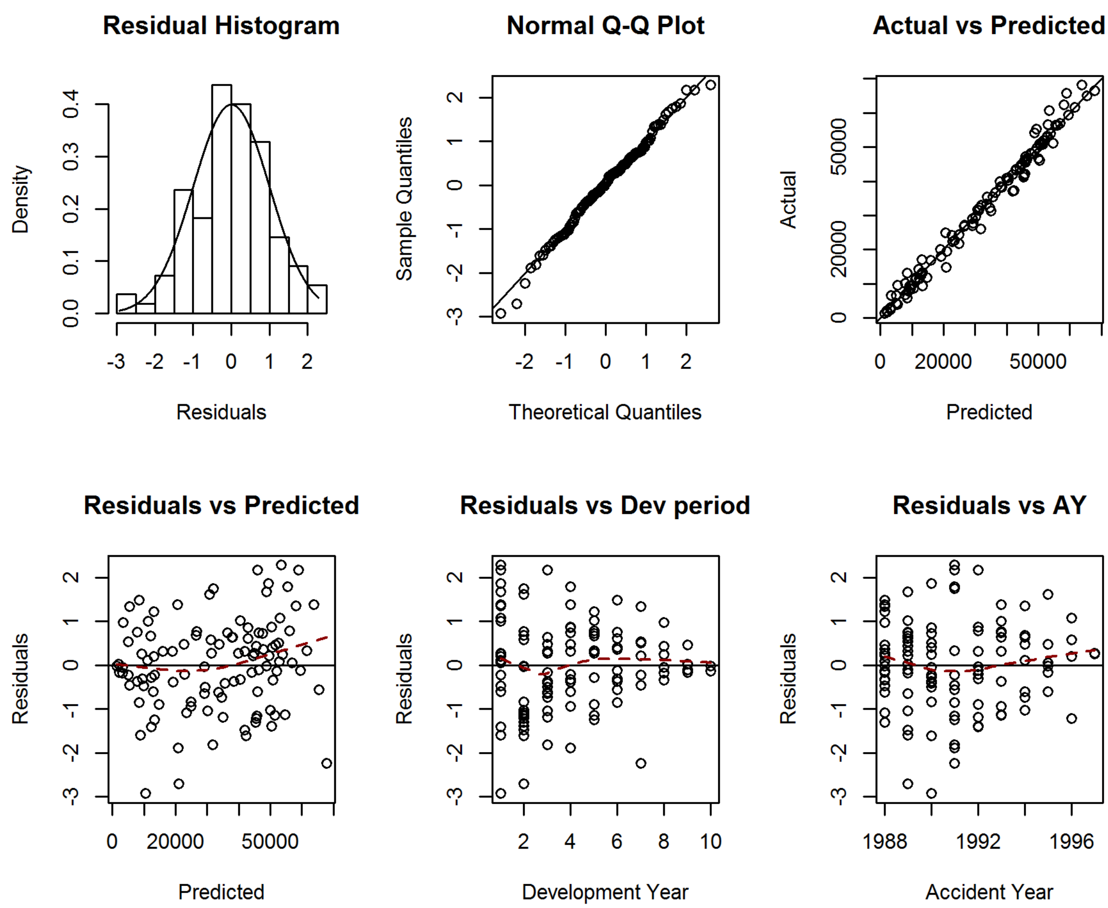
Initial fixed-effect parameter estimates are required to begin optimization, which could be obtained using a self-starting algorithm (Appendix C) or selected judgmentally as follows:

- Development year 1 outstanding claims are observed to be a high proportion of earned premiums. Therefore the reported loss ratio initial value has been selected as 100%, i.e. all premiums are assumed to convert to reported claims.
- The early outstanding loss peaks indicate a fast rate of reporting, so an initial value of 1.5 has been selected. This results in a value of claims reported in the first development year equal to $(1 - e^{-1.5}) \cdot P \cdot RLR = 78\% \cdot P \cdot RLR$.
- The downwards incurred development trend indicates large case reserve redundancies ($RRF < 1$), therefore a value of 0.75 has been selected.
- The rate of payment is observed to be slower than the rate of reporting, justifying a selected initial value equal to half the rate of reporting (0.75).

The above model can be combined with the previously outlined ODE system and fitted to the outstanding and cumulative paid triangles concurrently using the R package "nlme" (Pinherio *et al.*, 2016). Convergence is achieved in seconds. Appendix E contains the R model code and output.

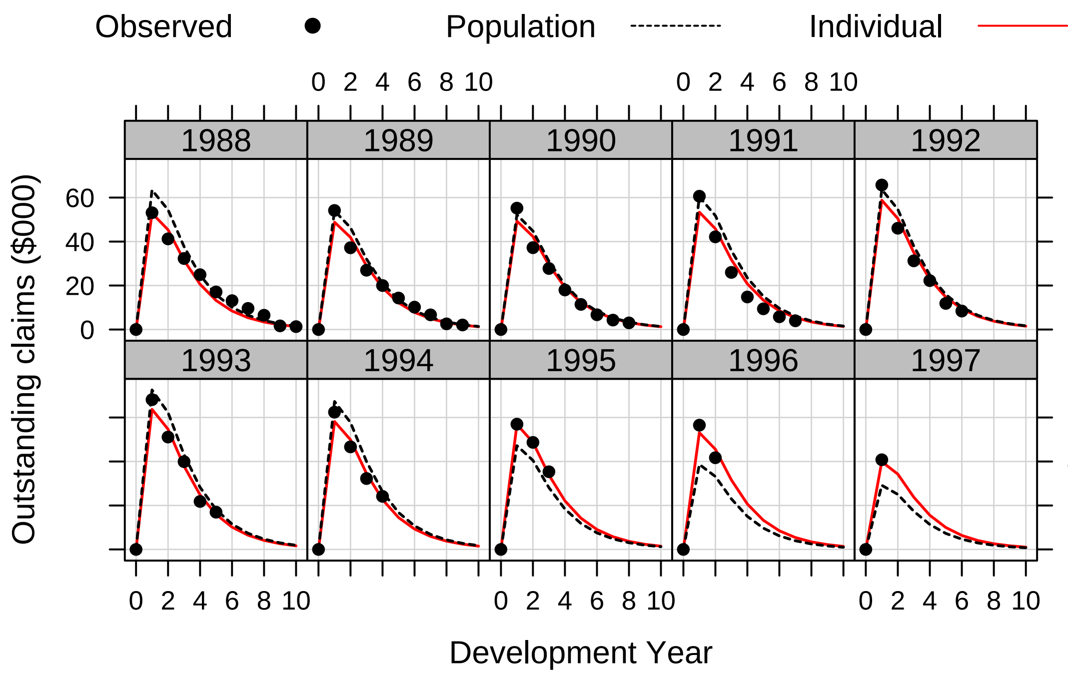
The estimated random-effect standard deviations ($\hat{\psi}_q$) relative to the fixed-effects ($\hat{\beta}_p \in \widehat{\boldsymbol{\beta}}$) for $lRLR_i$ ($\hat{\psi}_1 = 0.19$; $\hat{\beta}_2 = 0.03$) and $lRRF_i$ ($\hat{\psi}_2 = 0.13$; $\hat{\beta}_4 = -0.41$) indicate significant variation by accident year, justifying the inclusion of the random-effects. The within-accident-year error standard deviation for paid claims fits is estimated to be $\hat{\lambda} = 18\%$ of the within-accident-year error standard deviation for outstanding claims fits, which seems reasonable since paid claims development is comparatively stable.

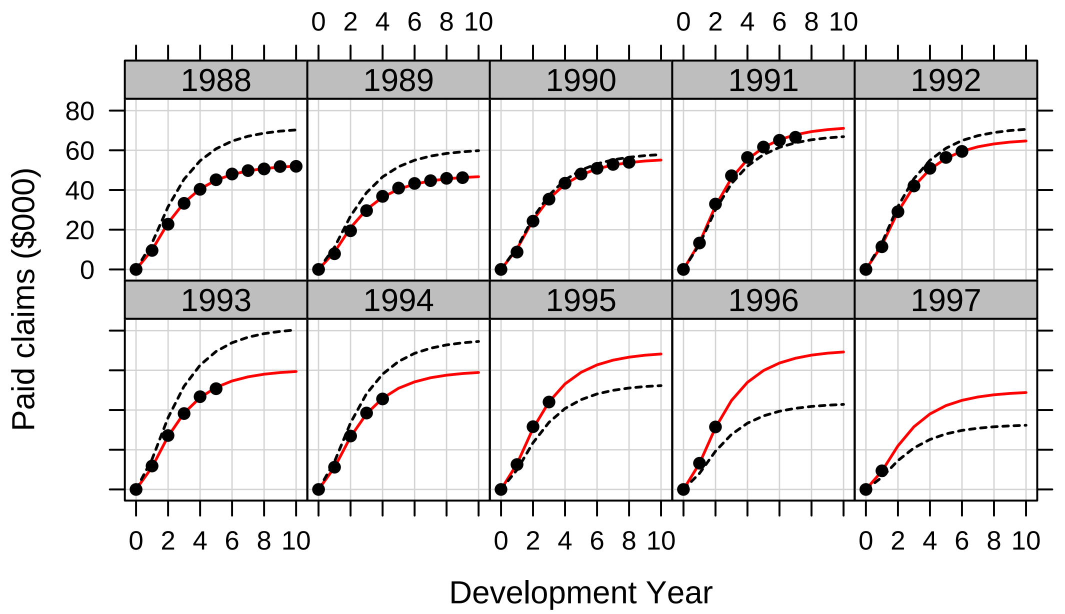
A set of diagnostic plots can be inspected to verify modeling assumptions and assess model fit:



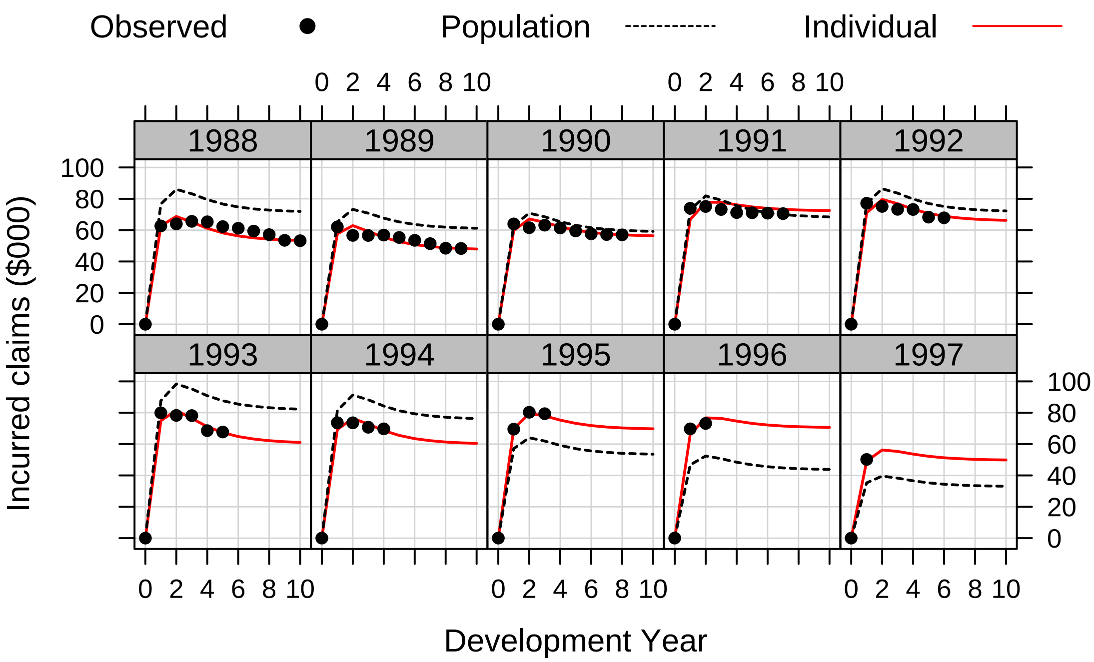
The upper left two plots indicate that the standardized residuals are approximately normal for this model, and the "Actual vs. Predicted" plot shows that the model fits the data reasonably well for most of the data range. However, some higher valued observations are under-predicted by the model, and the "Residuals vs. Predicted" plot highlights this. The remaining residual plots mostly lie between [-2, 2] and overlaid LOESS smoothers (Cleveland, 1979) suggest they are absent of trends.

To assess how well this model describes each accident year, we can plot the observed development data by accident year (circles) and superimpose the individual model fits (solid lines). To highlight the between-accident-year variability, the population-level fits (based on the fixed-effects and replicating a pooled model fit – see Section 2.3) are also shown (dashed lines):





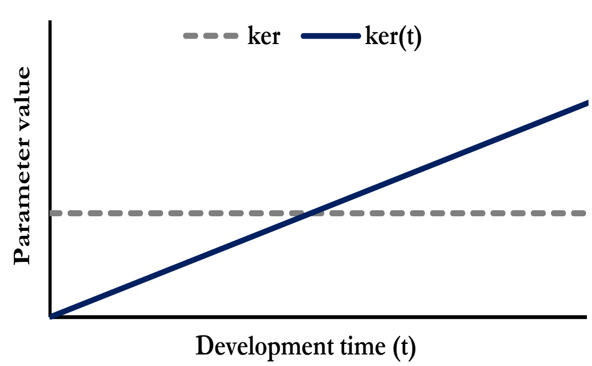
The population fits demonstrate that the model would not accurately describe claims development if parameters were fixed across accident years. The individual cumulative paid claims fits are reasonable, but the outstanding claims fits systematically under-predict the peak observations. It appears that claims are modeled to be reported over a longer time period than the data suggests.



4.1.1 Development time-dependent reporting rate

To attempt to improve the fits, we can adjust the structural model. Selecting a rate of reporting that speeds up over time may reduce the overall modeled reported time and reflect any reporting delays (see Section 3.2):

$$k_{er}(t) = \beta_{er} \cdot t$$



To incorporate this rate of reporting into the model, we can define $l\beta_{er} = \log(\beta_{er})$ and re-specify the compartmental model's ODE system as follows:

$$\begin{aligned} dEX/dt &= -\exp(l\beta_{er}) \cdot t \cdot EX \\ dOS/dt &= \exp(l\beta_{er}) \cdot t \cdot \exp(lRLR) \cdot EX - \exp(lk_p) \cdot OS \\ dPD/dt &= \exp(lk_p) \cdot \exp(lRRF) \cdot OS \end{aligned} \tag{4.4}$$

This structural model can be specified in R using the code in Appendix E.

Revising the definition of $\boldsymbol{f}(P_i, \boldsymbol{\phi}_i, t_j)$ to reflect the state-variable solutions of Eq. (4.4), we can write down a second hierarchical model (Model 2):

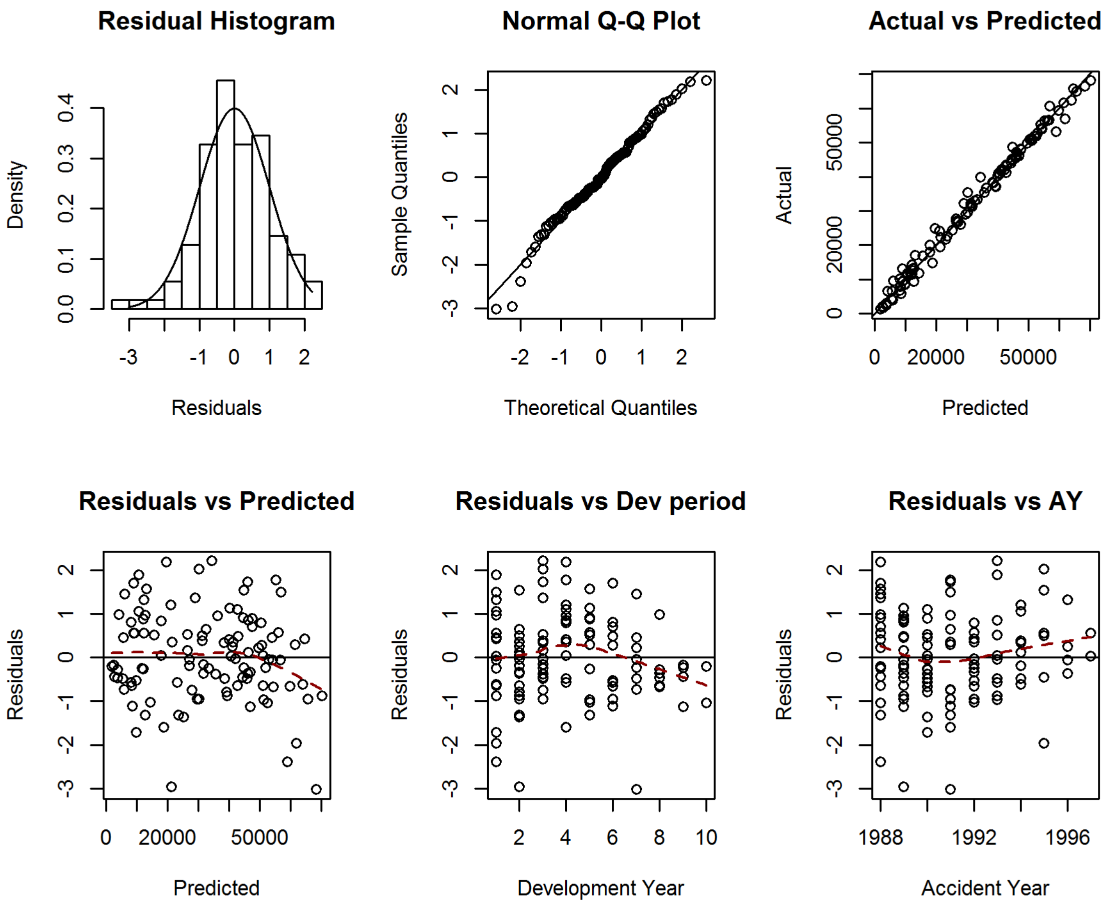
$$\begin{gathered} \boldsymbol{y}_{ij} = \boldsymbol{f}(P_i, \boldsymbol{\phi}_i, t_j) + \boldsymbol{\varepsilon}_{ij} \\ \boldsymbol{y}_{ij} = \begin{bmatrix} OS_{ij} \\ PD_{ij} \end{bmatrix}, \quad \boldsymbol{f}(P_i, \boldsymbol{\phi}_i, t_j) = \begin{bmatrix} f_{OS}(P_i, \boldsymbol{\phi}_i, t_j) \\ f_{PD}(P_i, \boldsymbol{\phi}_i, t_j) \end{bmatrix}, \quad \boldsymbol{\varepsilon}_{ij} = \begin{bmatrix} \varepsilon_{ij}{}^{OS} \\ \varepsilon_{ij}{}^{PD} \end{bmatrix} \\ \boldsymbol{\phi}_i = \begin{bmatrix} \phi_{1i} \\ \phi_{2i} \\ \phi_{3i} \\ \phi_{4i} \end{bmatrix} = \begin{bmatrix} l\beta_{er} \\ lRLR_i \\ lk_p \\ lRRF_i \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} + \begin{bmatrix} 0 \\ b_{1i} \\ 0 \\ b_{2i} \end{bmatrix} = \boldsymbol{\beta} + \boldsymbol{b}_i \\ \boldsymbol{b}_i \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \psi_1^2 & 0 \\ 0 & \psi_2^2 \end{bmatrix}\right), \qquad \boldsymbol{\varepsilon}_{ij} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & \lambda^2 \end{bmatrix}\right) \end{gathered} \tag{4.5}$$

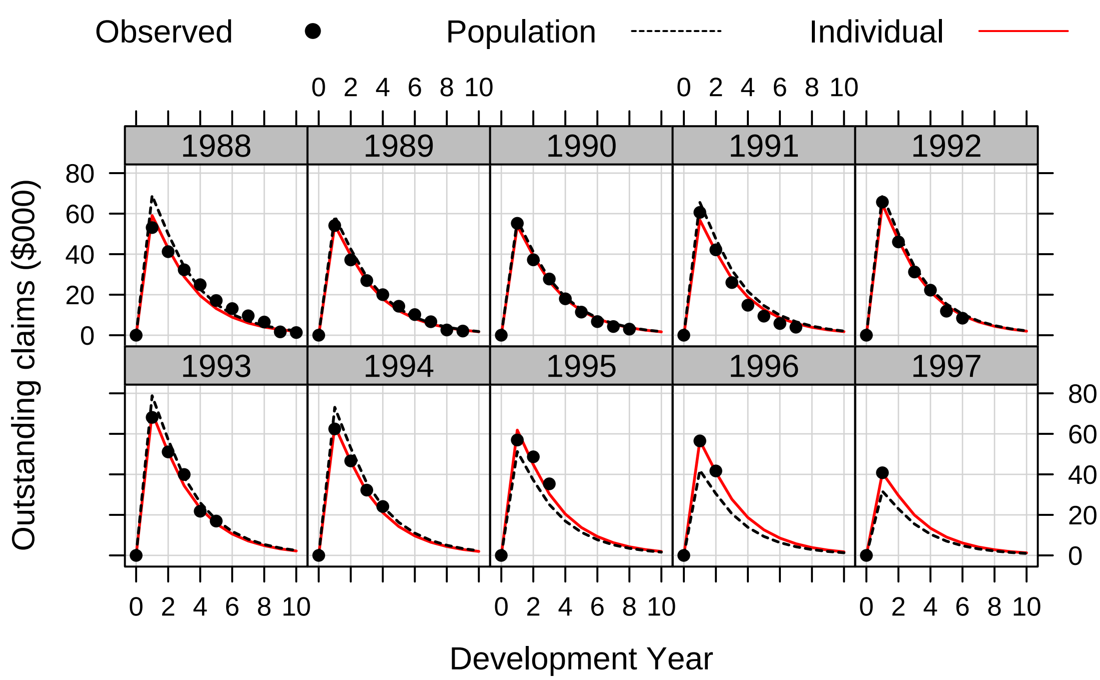
The model form, number of estimable parameters and statistical assumptions are unchanged from the previous model. However, rather than estimating the logarithm of the rate of reporting, we are estimating the logarithm of the linear coefficient for how the rate of reporting increases over development time, i.e. $l\beta_{er}$ replaces lk_{er}.

To ensure that outstanding claims are modeled to be reported over a shorter time-frame than previously, the starting value for $l\beta_{er}$ has been set to 5. This implies a reporting rate that is approximately 1.5 times faster than the Model 1 estimated rate at development year 0.5. The remaining initial parameter values have been set to the estimated fixed-effects in the previous model (to 2 decimal places). The model code and numerical output is contained in Appendix E.

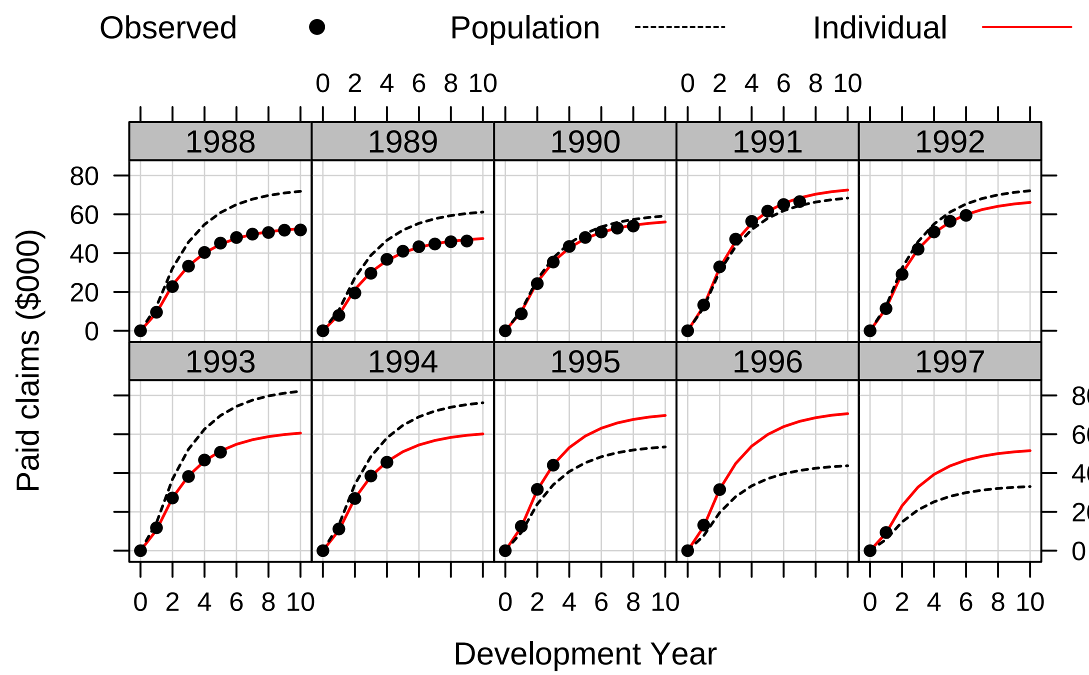
Under Model 2, the within-accident-year error standard deviation for cumulative paid claims fits is estimated to be $\hat{\lambda} = 25\%$ of the within-accident-year error standard deviation for the outstanding claims fits (up from $\hat{\lambda} = 18\%$), which may be due to an improvement in outstanding claims model fits.

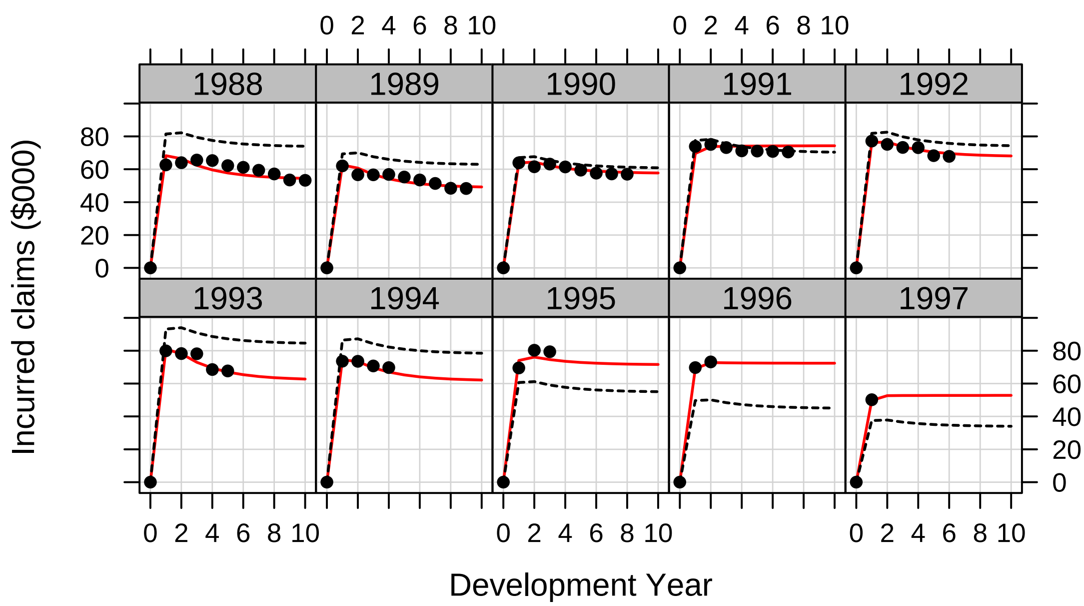
The "Actual vs. Predicted" plot below suggests that this model fits the data more closely than the last; however, the residuals exhibit a minor violation of normality. Furthermore, the "Residuals vs. Development Year" plot has a downwards trend across later development periods, indicating a small degree of over-prediction. Few data points drive this trend however, and therefore it may not be significant.


Residual Histogram
Density
Residuals
Normal Q-Q Plot
Sample Quantiles
Theoretical Quantiles
Actual vs Predicted
Actual
Predicted
Residuals vs Predicted
Residuals
Predicted
Residuals vs Dev period
Residuals
Development Year
Residuals vs AY
Residuals
Accident Year


Observed
Population
Individual
1988
1989
1990
1991
1992
1993
1994
1995
1996
1997
Outstanding claims ($000)
Development Year

The inclusion of a time-dependent rate of reporting has resulted in a more accurate description of the outstanding claims peaks. However, for the 1991 accident year there is evidence of continued over prediction, perhaps due to a differing rate of payment for this year.





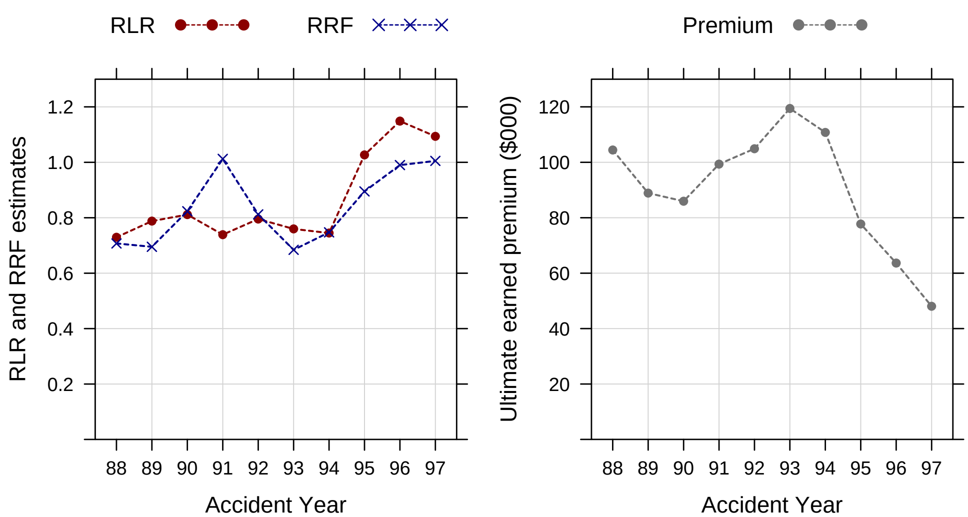
Paid claims are slightly over-predicted for later development periods, consistent with the residual plots. However, the incurred fits are improved due to the more accurate description of outstanding claims. A statistical comparison of this model against the last shows that the information criterion statistics (AIC and BIC) have both reduced. Therefore we may deduce that Model 2 is preferred to Model 1 and inspect it in greater detail (see Appendix E)

Approximate 95% confidence intervals show that the fixed-effects $\boldsymbol{\beta}$ (the mean-level logarithm of the compartmental reserving model parameters) are statistically significantly different from zero at the 1% level. Furthermore, the estimated fixed-effects correlation matrix contains a strong negative correlation between the rate of reporting and rate of payment parameters (-0.72). This seems intuitive: if few claims are reported over a given time period, a case handling team is likely to be better equipped to handle each payment more quickly than if many claims are reported over an equivalent time period.

At this stage the structural model could justifiably be selected as final. However, for other datasets further modifications may be required, such as those outlined in Section 3.2.

4.1.2 Random-effects correlation

In a hierarchical framework there are various possible statistical model modifications. For example, correlations between random-effects can be explored. The graphs below show the Model 2 estimated RLR_i and RRF_i parameters for each accident year alongside earned premiums for illustration:



The first plot suggests a positive correlation between the reported loss ratio and reserve robustness factor parameters by accident year, indicative of a case reserving cycle effect, i.e. more conservative case reserves (low RRF_i) in a hard market (low RLR_i) to create cushions for the future (Line *et al.*, 2003). The model estimates market softening between 1994 and 1997 (increasing RLR_i); a conclusion supported by reducing premium volumes across these years.

Additionally, case reserves are estimated to be increasingly robust between 1993 and 1997, which corroborates the reducing downward trend for incurred model fits across these years. There is a discrepancy for the 1991 accident year where the data appear to be exhibiting over-reserving, yet the model does not recognize this.

To estimate the correlation between the random-effects for $lRLR_i$ and $lRRF_i$, we can update the random-effects variance-covariance matrix to define a third model (Model 3 in Appendix E):

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_1^2 & \psi_{12} \\ \psi_{21} & \psi_2^2 \end{bmatrix}$$

The covariance between random-effect q and random-effect $r \neq q$ is denoted ψ_{qr}. The updated model estimates a strong and statistically significant positive correlation between the estimated reported loss ratio and reserve robustness factor random-effects (0.78).

To assess whether this model is significantly improved from the last, a likelihood ratio test can be carried out. The resultant p-value of 0.013 indicates that the hypothesis that the correlation between the random-effects is zero can be rejected at the 5% level (but not at the 1% level). We may therefore marginally prefer Model 3 to Model 2, particularly if we wish to make inferences about the correlation between $lRLR_i$ and $lRRF_i$ to assess case reserve cycle strength.

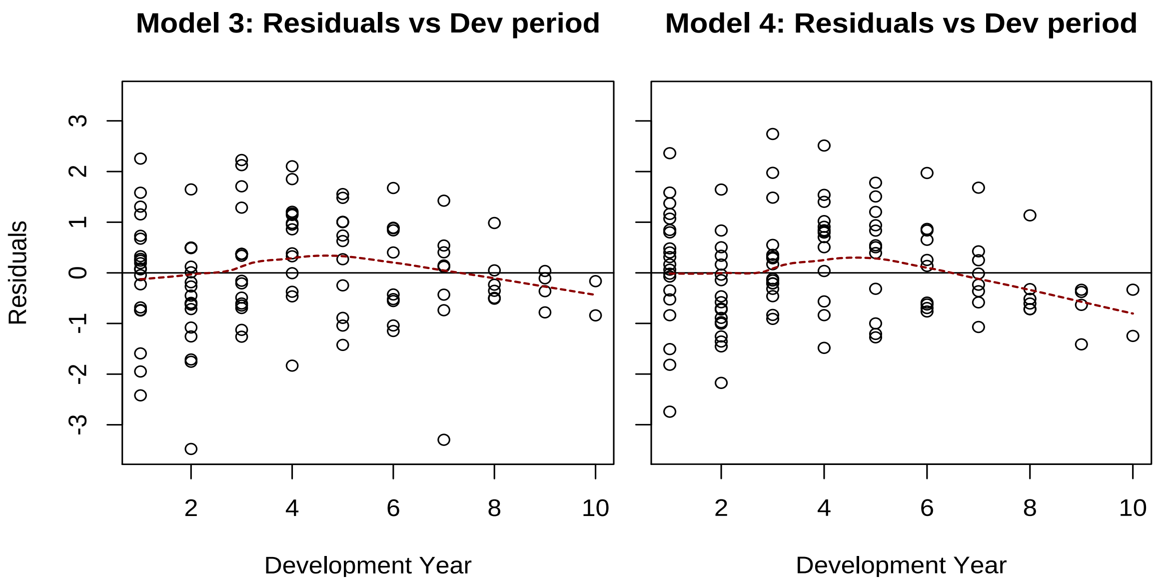
We could add random-effects for the remaining compartmental model parameters to define a fourth model. For example, a "block-diagonal" random-effects variance-covariance structure (Pinheiro and Bates, 2000) allows the rate of payment to vary by accident year independently of $lRLR_i$ and $lRRF_i$, resulting in differing payment patterns by accident year:

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_1^2 & \psi_{12} & 0 \\ \psi_{21} & \psi_2^2 & 0 \\ 0 & 0 & \psi_3^2 \end{bmatrix}$$

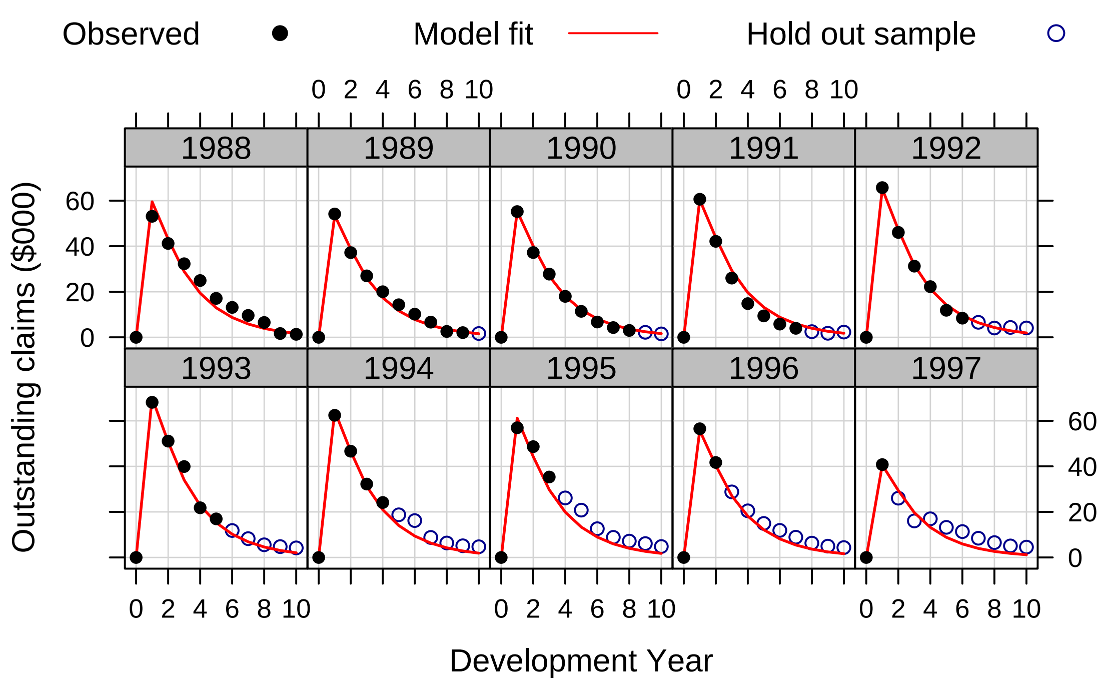
The statistical comparisons for Model 4 against the previous models (Appendix E) show a reduced BIC and significant likelihood ratio test for the new random-effect, suggesting that Model 4 should be preferred. However, the "Residuals vs. Development Year" diagnostic comparisons below tell a different story. Although Models 3 and 4 both produce a downwards residual trend which indicates a degree of over-prediction, Model 4's trend is stronger.

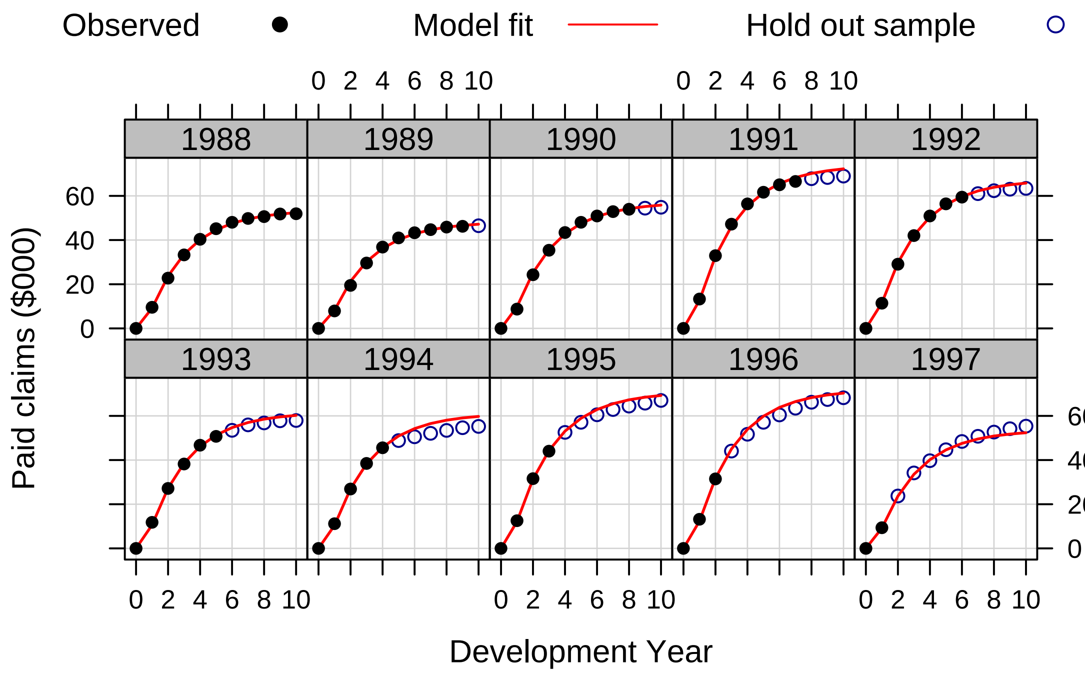
A double-log transformation $\log\{\boldsymbol{y}_{ij}\} = \log\{\boldsymbol{f}(P_i, \boldsymbol{\phi}_i, t_j)\} + \boldsymbol{\varepsilon}_{ij}$ reduces the downwards trend for both models, but residual normality is consequently violated (not shown).

Although the residual plot for Model 3 has two outliers, we may judge this model more suitable for best estimate reserving purposes if it is considered less likely to over-project ultimate losses. On this basis Model 4 will be rejected in favor of Model 3 (noting that either model could be justifiably selected).

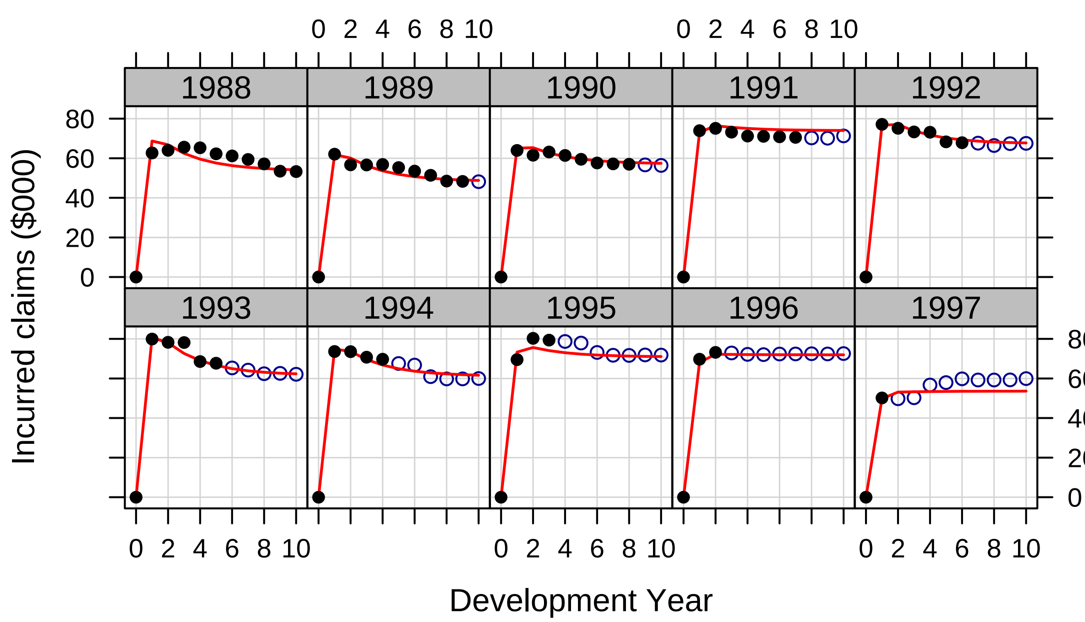


While it may be possible to further improve model fits by experimenting with alternative initial parameter values, Model 3 appears adequate based on the residual plot above and individual accident year fits (see below). We can therefore select Model 3 as final and compare its projections against the lower triangle hold-out samples (open circles) as follows:





Outstanding claims extrapolations have generally under-estimated actual outstanding development, while the cumulative paid claims extrapolations have generally over-estimated actual cumulative paid development.

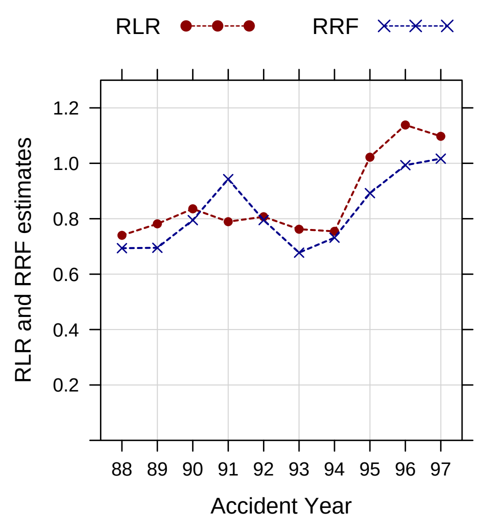


The under- and over-projections largely offset each other for the incurred extrapolations, although the aforementioned 1991 accident year development fit issue has propagated into the extrapolation.

In contrast to the historical upper triangle development, hold-out sample outstanding claims have taken longer to converge to zero and hold-out sample cumulative paid claims increases have tapered relative to their initial "growth". These characteristics suggest a **slow-down in the rate of payment in the hold-out sample**, perhaps consistent with the nonlinear rate of payment function defined in Section 3.2. It could be that the estimated softening market (see graph below) led to tighter cashflow and slower payments (Line *et al.*, 2003). There may also be reserve robustness improvements in later hold-out development years. Additionally, the residual plots at the fitting stage displayed some evidence of over-prediction, which could partially account for the paid claims over-projections.

Although the modeled dataset showed insufficient evidence of a payment rate reduction over time, had cashflow tightening been anticipated as a result of the estimated softening market, a practitioner could have scenario tested slowdowns in the rate of payment for the purpose of setting reserves.

In addition to payment rate reductions, case reserve robustness appears to have increased between the 1993 and 1996 accident years, shown by negative incurred development flattening across these years. Furthermore, the 1997 accident year appears to have exhibited under-reserving (or late reporting/claim re-openings) in contrast to the over-reserving trend seen in previous years. The compartmental model estimated increasing reserve robustness between 1993 and 1996, and a small amount of under-reserving for 1997. This is despite there being only two observations available for modeling 1997, resulting in a fit principally reliant on data-rich years which exhibited over-reserving:



The hierarchical compartmental reserving (CR) modeled development time 10 and ultimate incurred claims (time ∞, given by $P_i \times \widehat{RLR}_i \times \widehat{RRF}_i$) are shown below, alongside the Munich chain ladder (MCL; Quarg and Mack, 2004) and basic chain ladder (CL) incurred method results (without tail factors) by accident year.

To compare the predictability of each method, the percentage differences from the actual time 10 incurred claims are shown with the closest estimate(s) highlighted:

AY	Time 10 Incurred	CR Incurred t=10	CR Incurred t=∞	MCL Incurred	CL Incurred	var(CR)	var(MCL)	var(CL)
1988	53,261	54,149	*53,611*	53,261	53,261	2%	0%	0%
1989	48,162	48,769	*48,288*	47,640	48,109	1%	-1%	0%
1990	56,368	57,447	*57,112*	57,132	54,697	2%	1%	-3%
1991	71,274	74,028	*73,926*	72,016	65,550	4%	1%	-8%
1992	67,515	67,718	*67,323*	66,276	61,847	0%	-2%	-8%
1993	62,122	62,331	*61,664*	60,035	60,658	0%	-3%	-2%
1994	59,974	61,670	*61,160*	59,663	60,521	3%	-1%	1%
1995	71,829	71,073	*70,878*	69,426	66,815	-1%	-3%	-7%
1996	72,573	71,970	*71,959*	69,680	61,118	-1%	-4%	-16%
1997	59,939	53,597	*53,617*	49,977	42,242	-11%	-17%	-30%
Total	**623,017**	**622,751**	***619,537***	**605,106**	**574,819**	**0%**	**-3%**	**-8%**

The following conclusions can be drawn:

- The compartmental model produces the closest time 10 incurred loss estimates in total;
- The superior estimation accuracy of the compartmental approach for less mature accident years can be accredited to the model estimating increasingly robust case reserve setting (driven by a softening market – see above); and
- The Munich chain ladder method recognizes a shift in case reserve robustness by utilizing paid claims development. However, the basic chain ladder method does not, resulting in heavily under-estimated time 10 incurred losses.

In practice the Bornhuetter-Ferguson method (1972) may be used for the less mature years, possibly closing the estimation accuracy gap. Although not shown, the compartmental modeled ultimate paid and incurred estimates are equal whereas the Munich chain ladder estimates differ.

Thus far we have only considered point estimates. However, a compartmental framework enables scenario testing of one or more of the claims process parameters to generate a range of possible ultimate claims. For example, case reserving philosophy or settlement approaches could be discussed with the relevant case handlers/claims teams to establish a range of plausible RRF and/or k_p parameters.

Prediction errors could be assessed analytically or using bootstrapping techniques (England and Verrall, 1999). Additionally or alternatively, a hierarchical compartmental reserving model could be specified in a fully Bayesian framework, which will be explored in the following Sections.

4.2 Bayesian modeling

We may wish to implement the selected frequentist model within a Bayesian framework for the reasons outlined in Section 2.3.1. In particular:

- Judgment and information external to the claims triangle data can be robustly incorporated;
- Reserve uncertainty can be quantified as part of the fitting process; and
- Flexibility enables additional model features to be incorporated with relative ease.

We can specify a Bayesian hierarchical model by rewriting the paid and outstanding compartmental model differential equation solutions in Eq. (4.5) to explicitly state parameters with random-effects ($\boldsymbol{\phi}_i$) and without random-effects ($\boldsymbol{\eta}$). The Bayesian implementation will incorporate autoregressive sub-models for outstanding and cumulative paid claims residuals to reduce recurrent under/over prediction (Zhang, Dukic and Guszcza, 2012):

$$OS_{ij} = f_{OS}\left(P_i, \boldsymbol{\phi}_i, \boldsymbol{\eta}, t_j\right) + \varepsilon_{ij}^{OS}$$

$$PD_{ij} = f_{PD}\left(P_i, \boldsymbol{\phi}_i, \boldsymbol{\eta}, t_j\right) + \varepsilon_{ij}^{PD}$$

$$\boldsymbol{\phi}_i = \begin{bmatrix}\phi_{1i}\\ \phi_{2i}\end{bmatrix} = \begin{bmatrix}lRLR_i\\ lRRF_i\end{bmatrix} \qquad \boldsymbol{\eta} = \begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} = \begin{bmatrix}lk_{er}\\ lk_p\end{bmatrix}$$

$$\begin{bmatrix}\phi_{1i}\\ \phi_{2i}\end{bmatrix} \sim N_2\left(\boldsymbol{\theta} = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}, \boldsymbol{\Psi} = \begin{bmatrix}\psi_1^2 & \psi_{12}\\ \psi_{21} & \psi_2^2\end{bmatrix}\right) \tag{4.6}$$

$$\varepsilon_{ij}^{OS} = \rho_{OS}\varepsilon_{ij-1}^{OS} + \delta_{ij}^{OS} \qquad \varepsilon_{ij}^{PD} = \rho_{PD}\varepsilon_{ij-1}^{PD} + \delta_{ij}^{PD}$$

$$\delta_{ij}^{OS} \sim N\{0, \sigma_{OS}^2(1-\rho_{OS}^2)\} \qquad \delta_{ij}^{PD} \sim N\{0, \sigma_{PD}^2(1-\rho_{PD}^2)\}$$

$$\varepsilon_{i1}^{OS} \sim N(0, \sigma_{OS}^2) \qquad \varepsilon_{i1}^{PD} \sim N(0, \sigma_{PD}^2)$$

The statistical assumptions are analogous to the selected frequentist model and similarly, $lRLR_i$ and $lRRF_i$ are assumed to vary by accident year with co-dependency. Residual autocorrelation terms are denoted ρ_{OS} and ρ_{PD}, and model process error is captured by the residual error terms ε_{ij}^{OS} and ε_{ij}^{PD}.

Normal prior distributions have been assigned to the implied *fixed-effects*. Similarly to the frequentist model these are the means of $lRLR_i$ and $lRRF_i$ (denoted $\boldsymbol{\theta}$), together with $l\beta_{er}$ and lk_p (denoted $\boldsymbol{\eta}$):

$$\boldsymbol{\theta} \sim N_2(\overline{\boldsymbol{\theta}}, \boldsymbol{\Omega})$$
$$\boldsymbol{\eta} \sim N_2(\overline{\boldsymbol{\eta}}, \boldsymbol{\Pi}) \tag{4.7}$$

In Eq. (4.7), $\overline{\boldsymbol{\theta}}$ and $\boldsymbol{\Omega}$ denote the prior mean and variance-covariance matrix of $\boldsymbol{\theta}$, whereas $\overline{\boldsymbol{\eta}}$ and $\boldsymbol{\Pi}$ denote the prior mean and variance-covariance matrix of $\boldsymbol{\eta}$.

The prior means for the fixed-effects have been set to the estimated fixed-effects in the selected frequentist model, and the prior variance-covariance matrices describing uncertainty in the fixed-effects have been set to replicate the frequentist estimated fixed-effects confidence intervals:

$$\bar{\boldsymbol{\theta}} = \{-0.15, -0.21\}^T \qquad \boldsymbol{\Omega} = \begin{bmatrix} 0.0513^2 & 0 \\ 0 & 0.0506^2 \end{bmatrix}$$
$$\bar{\boldsymbol{\eta}} = \{1.7, -0.9\}^T \qquad \boldsymbol{\Pi} = \begin{bmatrix} 0.0392^2 & 0 \\ 0 & 0.0124^2 \end{bmatrix} \tag{4.8}$$

These priors imply fixed-effect independence; however, their posterior distributions can demonstrate dependence. A prior distribution has also been assigned to the variance-covariance matrix of $\boldsymbol{\phi}_i$ ($\boldsymbol{\Psi}$) i.e. the variance of the implied *random-effects*, describing the magnitude of variability for the accident year varying (log) proportion parameters $lRRF_i$ and $lRLR_i$:

$$\boldsymbol{\Psi} \sim W_2^{-1}(\boldsymbol{\Sigma}, \nu) \tag{4.9}$$

W_2^{-1} is an inverse-Wishart distribution (and conjugate prior) with 2×2 scale matrix $\boldsymbol{\Sigma}$ and ν degrees of freedom (Gelman *et al.*, 2013). The frequentist analysis results have not been used to inform this prior. Instead, a vague prior has been set to allow the variance-covariance matrix to be principally estimated from the data. The prior *inverse* scale matrix and degrees of freedom have been set as

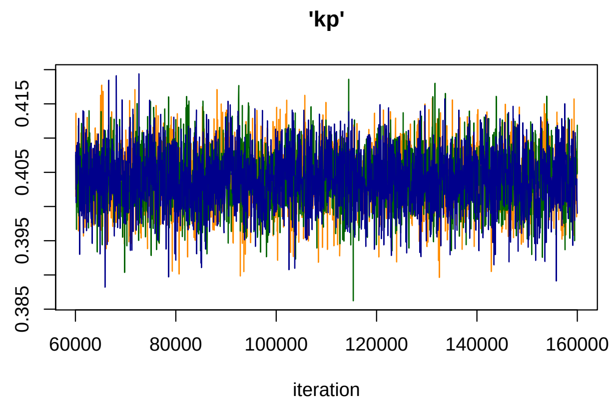
$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}, \nu = 2 \tag{4.10}$$

where the degrees of freedom are as low as possible while still maintaining a proper distribution (Johnson and Kotz, 1972). Although this prior is vague in its description of accident year variability magnitude, the off-diagonal elements have been set to give a 0.80 positive correlation between $lRLR_i$ and $lRRF_i$ (recall that the estimated correlation in the selected frequentist model was 0.78).

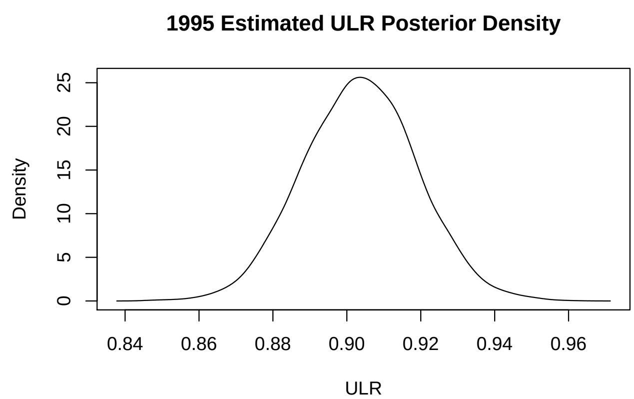
Vague priors have been assigned to the remaining model parameters. Priors for the standard deviations of the within-accident-year errors have been selected to comfortably cover the standard deviations estimated in the selected frequentist model. Finally, priors for the correlation terms of the autoregressive processes have been set to cover the minimum and maximum correlation values:

$$\sigma_{OS} \sim U(0,10000) \qquad \rho_{OS} \sim U(-1,1)$$
$$\sigma_{PD} \sim U(0,5000) \qquad \rho_{PD} \sim U(-1,1) \tag{4.11}$$

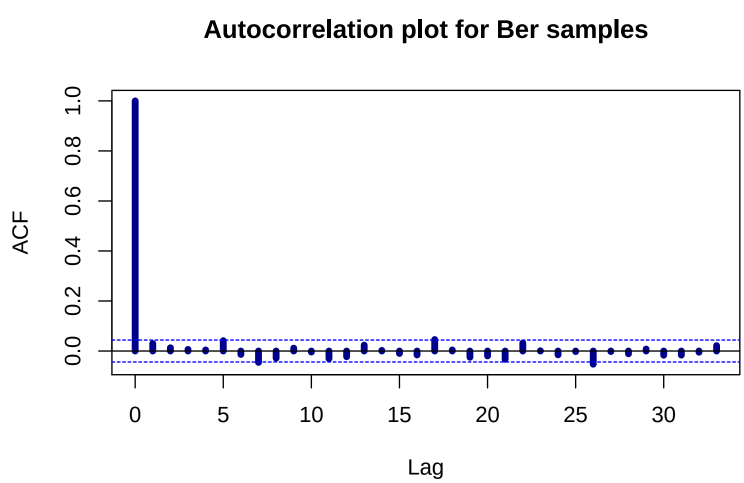
Using OpenBUGS (**B**ayesian inference **U**sing **G**ibbs **S**ampling; Lunn *et al.*, 2000), three parallel Markov chains were run with 60,000 burn-in iterations per chain, followed by 100,000 iterations per chain. To reduce sample autocorrelation, every 50th iteration of each chain was used, resulting in 2,000 simulated draws per chain and 6,000 samples in total. Various diagnostics checks were carried out to ensure that the simulation had converged to its approximate stationary distribution. Individual parameter estimation convergence was initially assessed and, as an example below, the values of k_p have been plotted over MCMC iterations by Markov chain.



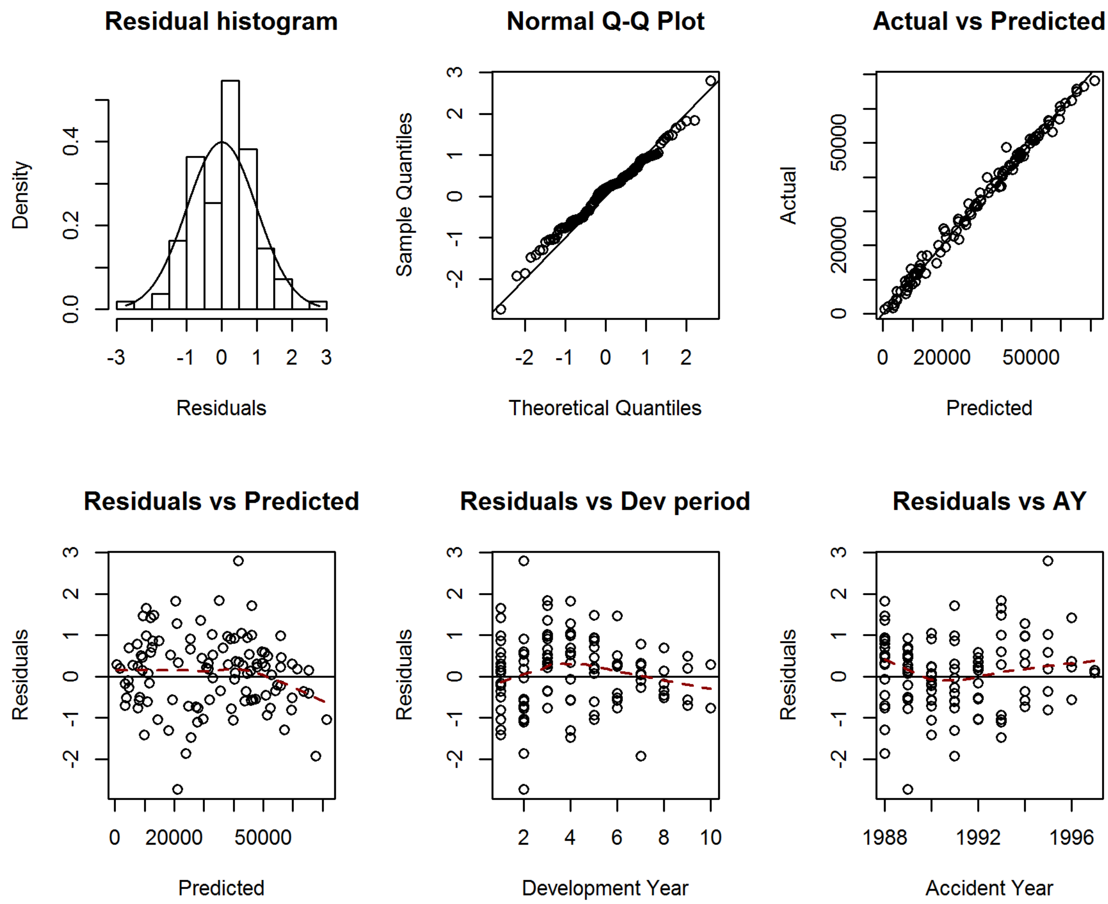
The similar and stable chains indicate that the posterior distribution of k_p has approximately converged to its stationary distribution. Densities for model parameters were also inspected. The estimated ultimate loss ratio posterior density for the 1995 accident year is as follows:



The density is smooth and bell-shaped, suggesting that convergence has been achieved. Finally, checks were carried out to assess sample autocorrelation. The plot below shows that the autocorrelation of the second chain β_{er} samples is not statistically different from zero:

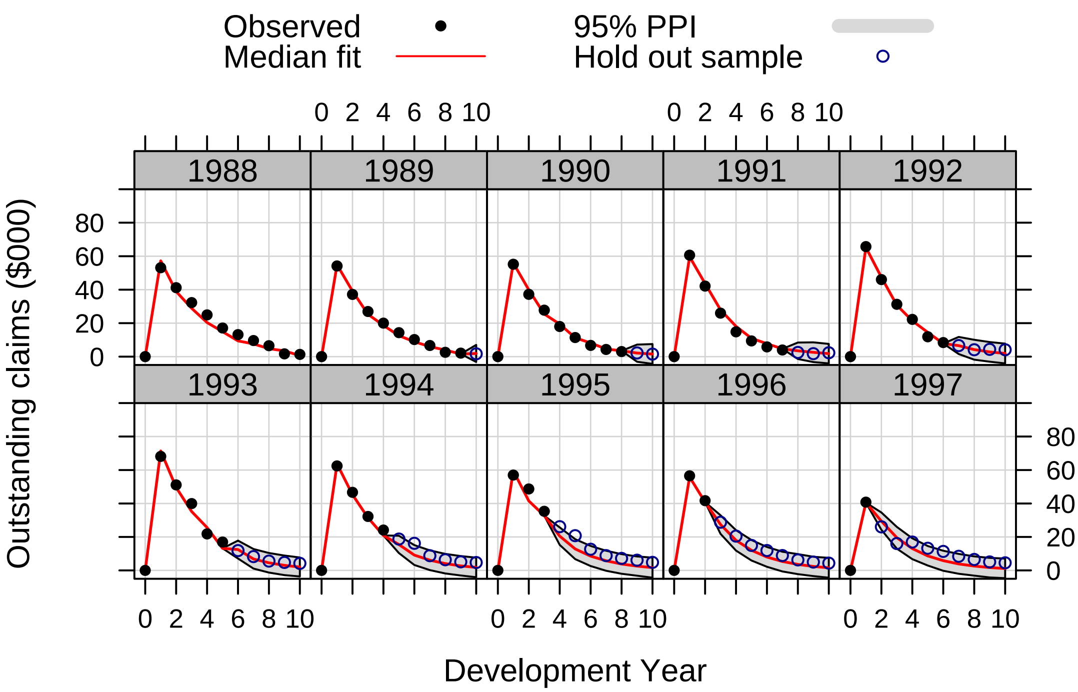


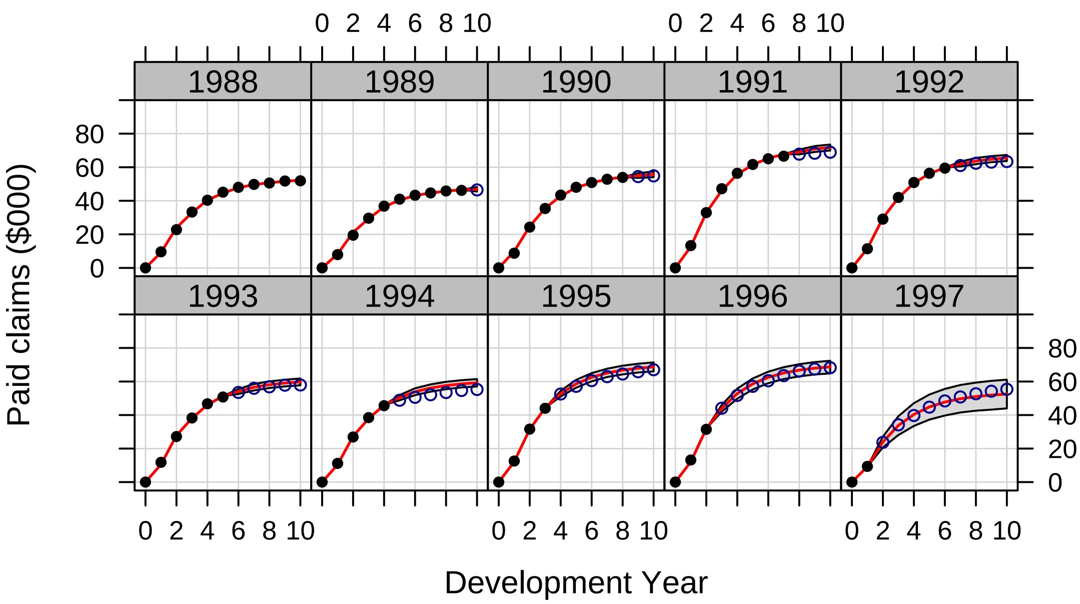
Given the diagnostics above (and various others not shown) the simulation appears to have reached approximate convergence and we can proceed to inspect the model diagnostic plots. Posterior densities are estimated for all parameters of interest, and therefore the diagnostic plots are based on estimated posterior density *medians*:



Residual normality approximately holds and the model fits are close to the observations. However, similarly to the frequentist model there is a downward trend in the "Residuals vs. Development Year" plot across later development years.

Similarly to Section 4.1, the individual accident year fits can be inspected. In the Bayesian setting however, for unobserved development years ($t_j \in i + j \geq 12$) 95% posterior predictive intervals ("PPIs") can be plotted (Gelman *et al.*, 2013). These show a range of prediction uncertainty due to both parameter and process uncertainty. Since this model is a Bayesian implementation of the selected frequentist model, we will compare the median fits, extrapolations and PPIs to the observed and hold-out sample development together:





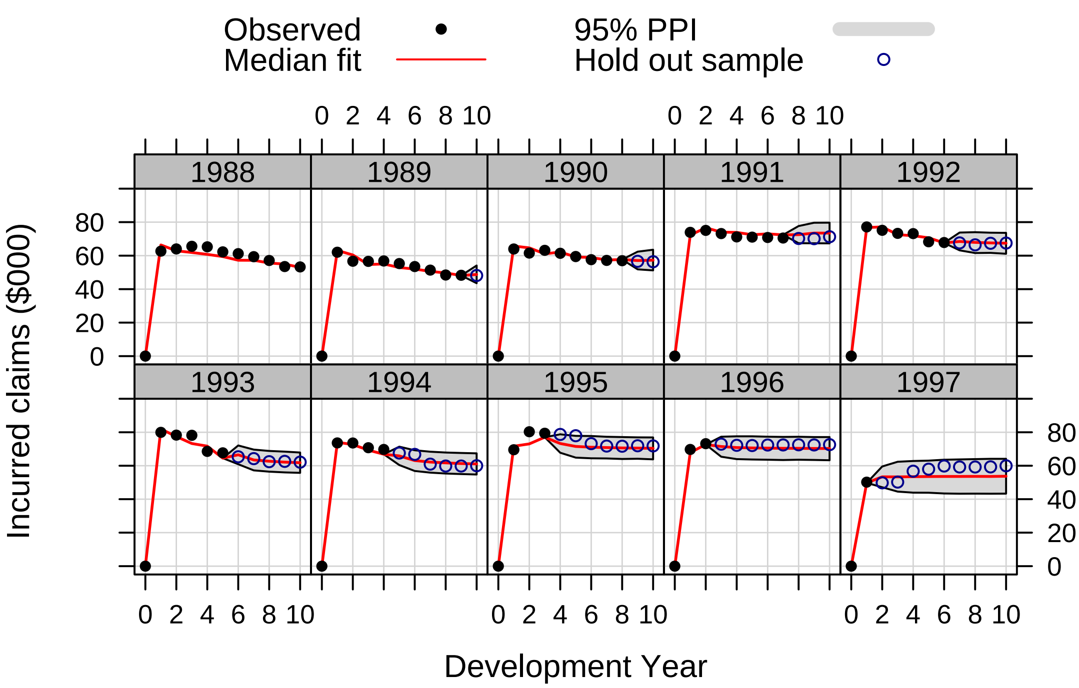
The median fits are similar to the selected frequentist model with some minor improvements. The PPIs are slightly wider for less mature accident years and contain the possibility of both under- and over- reserving. However, the outstanding claims PPIs do not converge to zero and even fall below zero in later development periods because of the residual normality assumption in Eq. (4.6).

To address this shortfall, a double-log transformed model form was tested:

$$\log(OS_{ij}) = \log\{f_{OS}(P_i, \boldsymbol{\phi}_i, \boldsymbol{\eta}, t_j)\} + \varepsilon_{ij}^{OS}$$
$$\log(PD_{ij}) = \log\{f_{PD}(P_i, \boldsymbol{\phi}_i, \boldsymbol{\eta}, t_j)\} + \varepsilon_{ij}^{PD} \quad \textbf{(4.12)}$$

Similarly to the equivalent frequentist model, this transformation resulted in a violation of residual normality. In particular, there were too many small residuals relative to larger residuals, which is characteristic of an overfitted model. Therefore the model was rejected.

The paid claims PPIs are generally narrower than the outstanding claims PPIs due to closer model fits: median $\hat{\sigma}_{PD} = 760$ and median $\hat{\sigma}_{OS} = 3151$. However, paid claims are over-projected similarly to the frequentist model, suggesting that a smaller residual error variance could instil false extrapolation confidence if possible future development period claims process shifts are not considered.



To assess posterior parameter uncertainty, we can review median parameter estimates and their 95% central posterior intervals {median [2.5%ile, 97.5%ile]} (Gelman *et al.*, 2013). For the 1997 accident year $\widehat{RLR}_{10} = 1.10\ [0.95, 1.25]$ and $\widehat{RRF}_{10} = 1.02\ [0.83, 1.23]$, suggesting that case reserve robustness is the main driver of ULR uncertainty ($\widehat{ULR}_{10} = 1.12\ [0.93, 1.30]$).

The estimated residual autocorrelations are $\hat{\rho}_{OS} = 0.58\ [0.30, 0.83]$ and $\hat{\rho}_{PD} = 0.55\ [0.27, 0.75]$, indicating moderate to strong serial correlation. The estimated accident year correlation between $lRLR_i$ and $lRRF_i$ is $\hat{\rho}_{lRLR_i lRRF_i} = \hat{\psi}_{12}/\{\hat{\psi}_1\hat{\psi}_2\} = 0.77\ [0.38, 0.93]$, indicating a strong case reserving cycle effect. However, the 95% posterior interval is quite wide and the extent of the estimated effect is significantly influenced by the variance-covariance matrix prior in Eq. (4.10).

Model predictive power can be evaluated by inspecting the 95% PPI hold-out sample coverages as follows:

95% PPI Coverage	1-year ahead	10-years ahead	Total
Outstanding	89%	100%	93%
Paid	100%	67%	82%
Incurred	100%	100%	98%

The outstanding and incurred claims PPI coverages are close to the nominal 95% rate across all time horizons. The poor coverage for the 10-year ahead and total paid claims hold-out samples can be attributed to over-projection, particularly for the 1994 accident year. Removing this year from the coverage calculations gives a 10-year coverage of 78% and total coverage of 93%.

The over-projections may be the result of hold-out sample rate of payment reductions (see Section 4.1.2) and/or differing rates of payment by accident year not reflected by the structural model and PPIs. Similarly to the frequentist setting, we could have scenario tested slow-downs in the rate of payment over development time. PPI coverage could possibly have been improved in practice by using informative priors for the random-effects or, alternatively, by increasing the number of random effects. The latter option will be explored in the following scenario.

4.2.1 Scenario 1: Fully random structure

We may be able to achieve a more accurate description of historical claims development by allowing *all* claims process parameters to vary by accident year:

$$\boldsymbol{\phi}_i = \begin{bmatrix} \phi_{1i} \\ \phi_{2i} \\ \phi_{3i} \\ \phi_{4i} \end{bmatrix} = \begin{bmatrix} lk_{er,i} \\ lRLR_i \\ lk_{p,i} \\ lRRF_i \end{bmatrix}$$

$$\begin{bmatrix} \phi_{1i} \\ \phi_{2i} \\ \phi_{3i} \\ \phi_{4i} \end{bmatrix} \sim N_4 \left(\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}, \boldsymbol{\Psi} = \begin{bmatrix} \psi_1^2 & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_2^2 & \psi_{23} & \psi_{24} \\ \psi_{31} & \psi_{32} & \psi_3^2 & \psi_{34} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_4^2 \end{bmatrix} \right) \quad \textbf{(4.13)}$$

$\boldsymbol{\Psi}$ contains ten estimable parameters, which may not be supported by this dataset. However, negligible posterior covariance terms can be enforced by setting a prior assumption that $lk_{er,i}$ and $lk_{p,i}$ vary independently of all other parameters (see below). The assigned prior distributions and assumed parameter values are unchanged from the previous model, yet fewer priors are required because we do not need to distinguish between those parameters that do and do not vary by accident year.

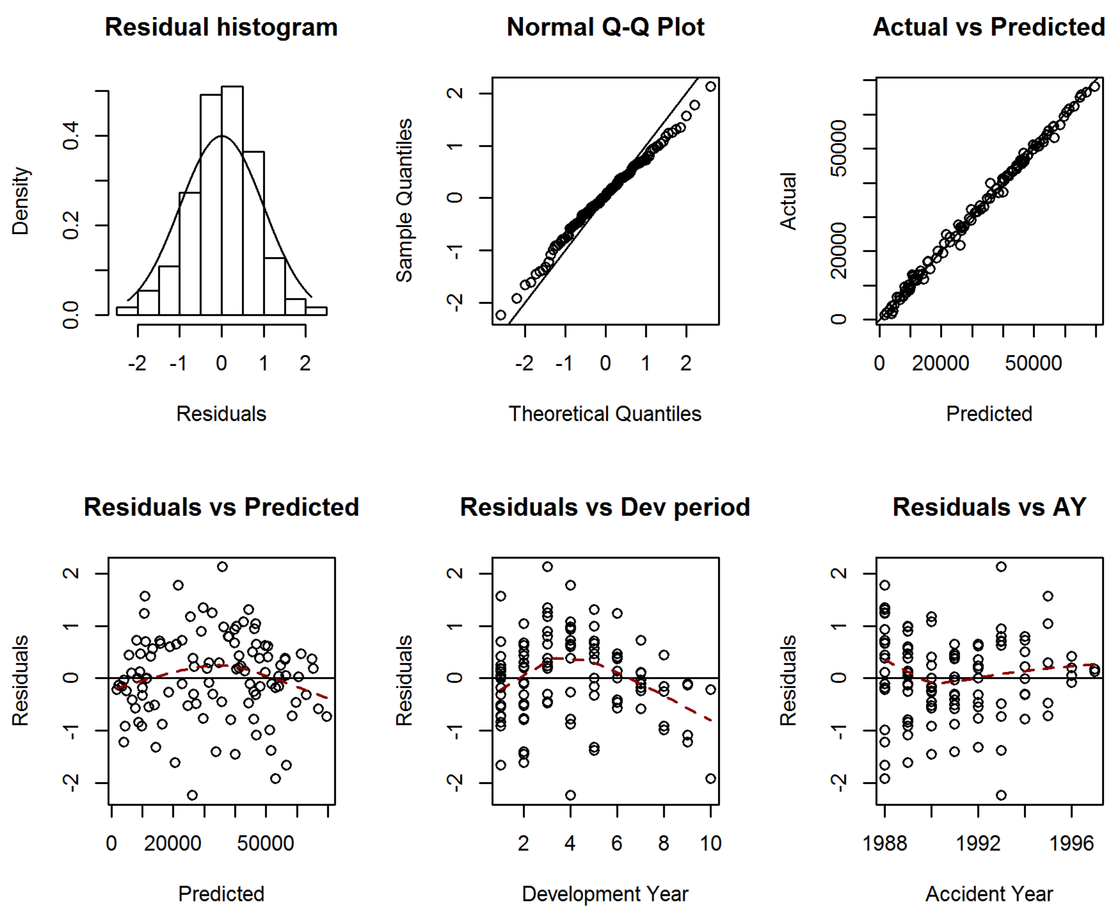
The fixed-effects prior assumptions are as follows:

$$\boldsymbol{\theta} \sim N_4(\overline{\boldsymbol{\theta}}, \boldsymbol{\Omega})$$
$$\overline{\boldsymbol{\theta}} = \{1.7, -0.15, -0.9, -0.21\}^T$$
$$\boldsymbol{\Omega} = \begin{bmatrix} 0.0392^2 & 0 & 0 & 0 \\ 0 & 0.0513^2 & 0 & 0 \\ 0 & 0 & 0.0124^2 & 0 \\ 0 & 0 & 0 & 0.0506^2 \end{bmatrix} \tag{4.14}$$

The random-effects variance-covariance matrix prior assumptions are as follows:

$$\boldsymbol{\Psi} \sim W_4^{-1}(\boldsymbol{\Sigma}, \nu)$$
$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.8 \\ 0 & 0 & 1 & 0 \\ 0 & 0.8 & 0 & 1 \end{bmatrix}, \nu = 4 \tag{4.15}$$

This assumes independence of the random-effects for $lk_{er,i}$ and $lk_{p,i}$. The remaining parameter priors, statistical assumptions and convergence arguments are unchanged from the previous model.

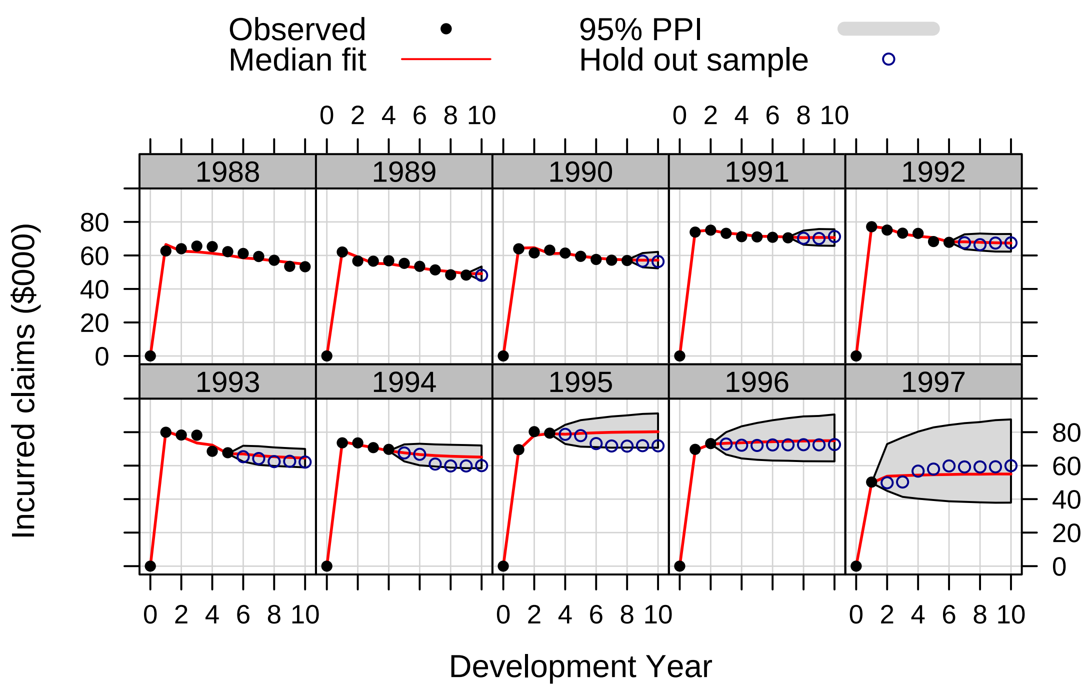


The diagnostics reveal that the fits are closer to the observations than the previous model. However, the residuals appear to be in violation of normality, indicating a degree of overfitting. Additionally, the downwards residual vs. development year trend is worsened (analogously to when complexity was increased in the frequentist modeling).

To assess this model against the last, we can compare each model's deviance information criterion (DIC) as follows:

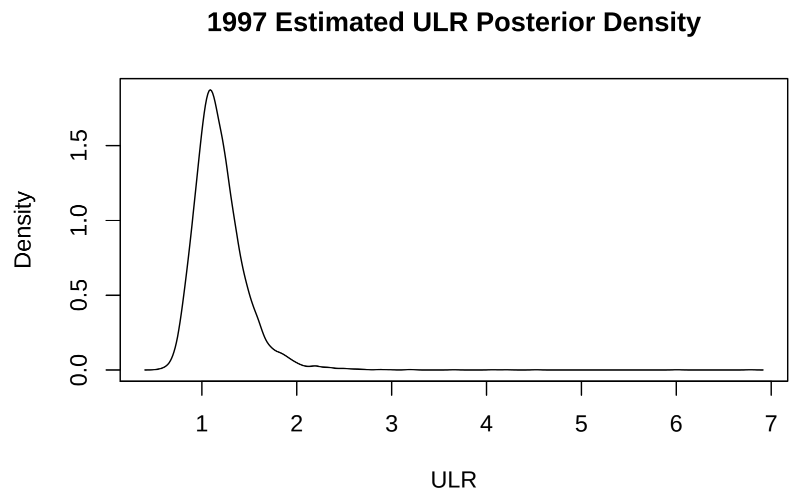
DIC	Outstanding	Paid
Bayesian Model 1	1031.0	879.6
Bayesian Model 2	1003.0	890.1

The DIC has decreased for the outstanding fits, indicating an improvement. However, it has increased for the paid fits which suggests that the model could be over-parameterized. There is an overall DIC reduction, and given the diagnostic plots a practitioner may select this model. In which case we will compare this model's incurred extrapolations against the hold-out samples as follows:



The fits more closely describe each individual year's incurred development relative to the previous model. Additionally, despite a number of years exhibiting over-reserving, the 1995 accident year fit assumes under-reserving on average (median $\widehat{RRF}_8 = 1.08$). This difference could be a feature of allowing all of the compartmental model parameters to vary by accident year according to a vague prior for $\boldsymbol{\Psi}$, enabling the model to place weight on the sharp incurred increase between development years 1 and 2. We could reduce the degree to which parameters vary across years (particularly less mature years where priors carry greater weight) by setting an informative variance-covariance prior.

The plots also show that the 1997 incurred density's mean is greater than its median. This is because the ULRs are assumed to be log-normally distributed (recall from Eq. (4.13) that $lRLR_i$ and $lRRF_i$ are assumed to be normally distributed).



The 95% PPI hold-out sample coverages (*with the previous model's stated in brackets*) are as follows:

95% PPI Coverage	**1-year ahead**	**10-years ahead**	**Total**
Outstanding	**100% (*89%*)**	100% (*100%*)	**100% (*93%*)**
Paid	**89% (*100%*)**	67% (*67%*)	**69% (*82%*)**
Incurred	100% (*100%*)	100% (*100%*)	**100% (*98%*)**

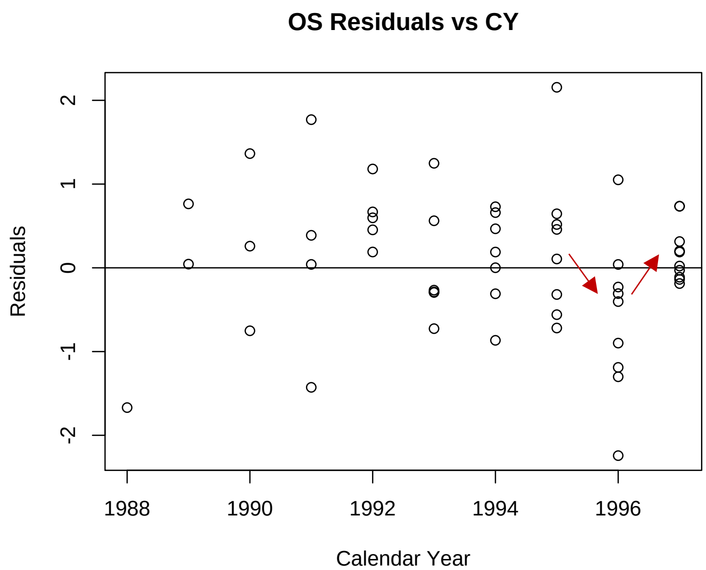
PPI 1-year ahead coverage has marginally improved for outstanding claims but worsened for paid claims. Outstanding and incurred claims coverages have improved to 100% across all time horizons. However, paid claims coverage has deteriorated owing to the model estimated (average) under-reserving for the 1995 accident year not materializing. As with the previous model's extrapolations, rate of payment reductions are not projected.

The OpenBUGS code for this model is contained in Appendix F.

The final scenario and area of model improvement that will be considered concerns the model fits by calendar year.

4.2.2 Scenario 2: Calendar shock sub-model

The outstanding claims residuals vs. calendar year plot for the previous model (shown below) displays a moderate downwards positional shift of residuals between the 1995 and 1996 calendar years, and upwards positional shift between 1996 and 1997.



This appears to be the result of outstanding claims exhibiting a step change at the same point in calendar time, perhaps due to a case reserve review during 1996. To capture the 1996 calendar shock within the model, we can define an indicator variable C_{ij} (for accident year $i = 1$ to 10 and development year $j = 1$ to $11 - i$) to mark the time before and after the calendar shock for each accident year:

$$C_{ij} = \begin{cases} 1, & i + j < 10 \\ 0, & i + j \geq 10 \end{cases} \tag{4.16}$$

To quantify the impact of the apparent case reserve review, we can then define an estimable proportional calendar shock impact variable, a_i, and restate OS_{ij} in Eq. (4.6):

$$OS_{ij} = f_{OS}(P_i, \boldsymbol{\phi}_i, t_j) \cdot (1 - C_{ij} \cdot a_i) + \varepsilon_{ij}^{OS} \tag{4.17}$$

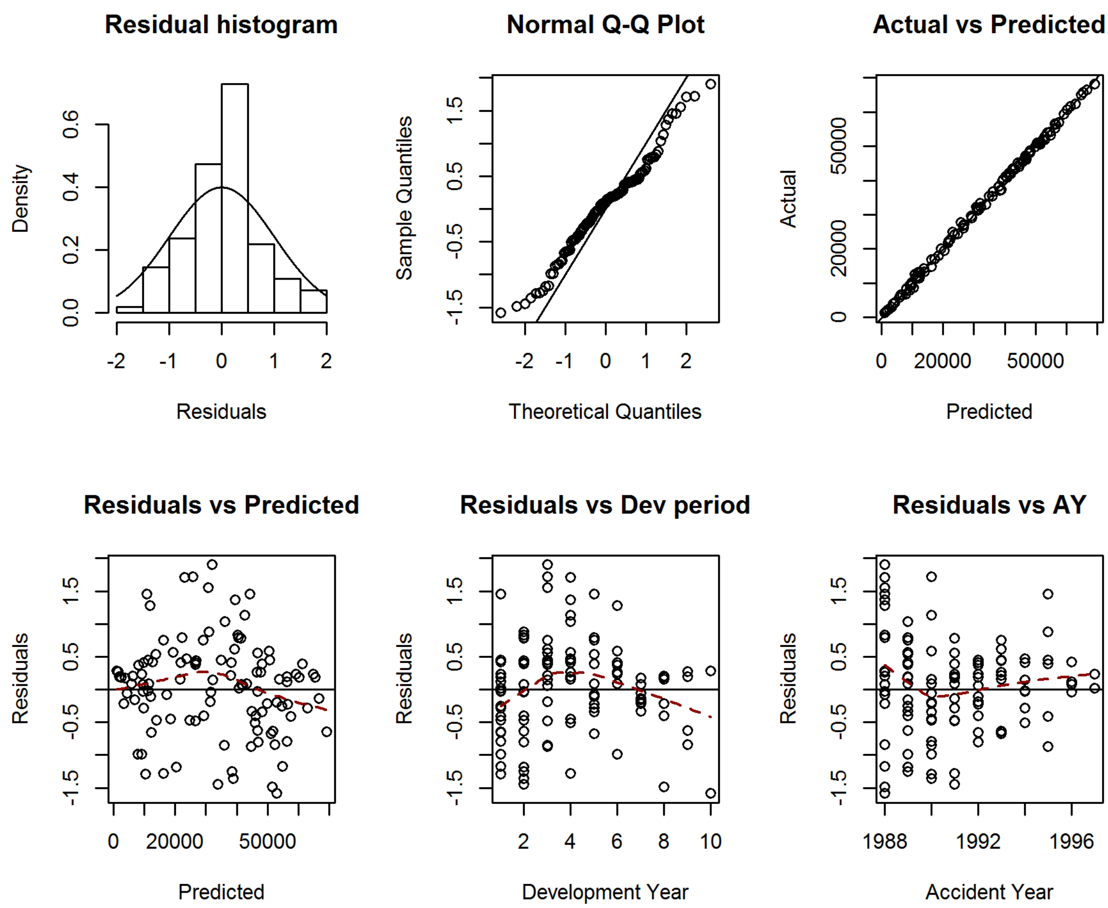
Therefore up until the end of the 1995 calendar year, the expected outstanding claims for accident year i and development year j are equal to $f_{OS}(P_i, \boldsymbol{\phi}_i, t_j)$ as before. However, once the review has taken place, outstanding claims are estimated to be $(1 - a_i)$% of their pre-shock values. Subsequent claims payments can be modeled to account for the shock as follows:

$$\begin{aligned} dEX/dt &= -\exp(l\beta_{er}) \cdot t \cdot EX \\ dOS/dt &= \exp(l\beta_{er}) \cdot t \cdot \exp(lRLR) \cdot EX - \exp(lk_p) \cdot OS \\ dPD/dt &= \exp(lk_p) \cdot \exp(lRRF) \cdot (1 - C_{ij} \cdot a_i) \cdot OS \end{aligned} \tag{4.18}$$

Upper and lower bounds for the estimated change in outstanding claims following the calendar shock have been set at 1% and 199% of the outstanding claims prior to the shock and assumed to be accident year independent. Candidate a_is are selected from a uniform distribution in the optimization:

$$a_i \sim U(-0.99, 0.99) \quad \textbf{(4.19)}$$

The remaining parameter priors, statistical assumptions and convergence arguments are unchanged from the previous model.



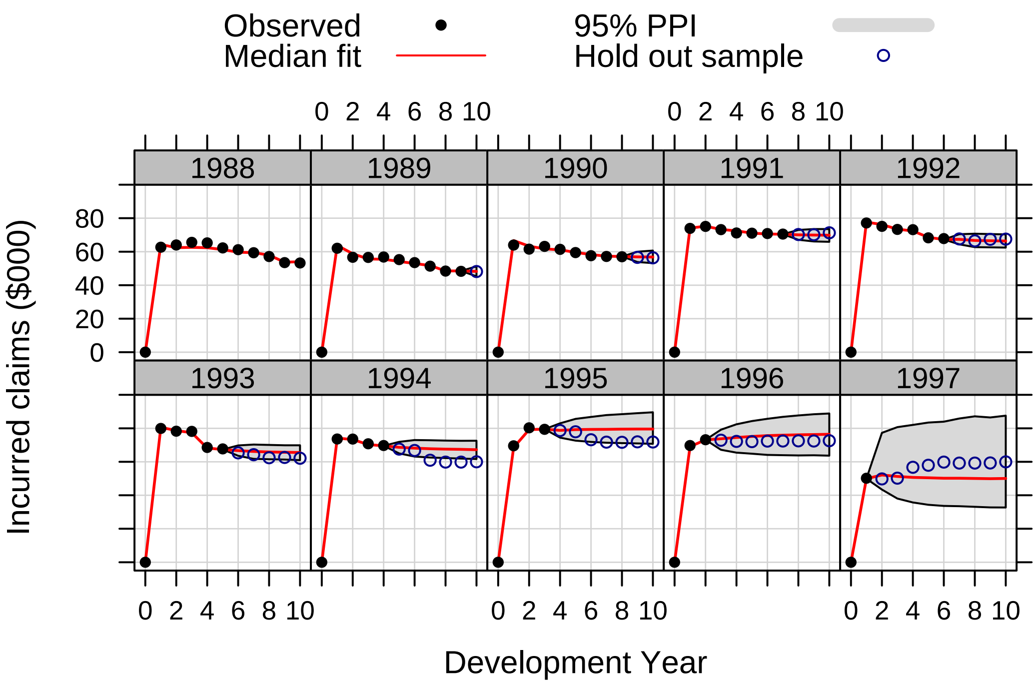
While the model fits are very close to the observations, there appears to be a serious violation of residual normality. The residual histogram shows that this model has far too many small magnitude residuals relative to mid-size residuals than expected under a standard normal distribution; an indication that claims are being *overfitted* by the model (similarly to the double-log model in Eq. (4.12)).

DIC	Outstanding	Paid
Bayesian Model 1	1031.0	879.6
Bayesian Model 2	1003.0	890.1
Bayesian Model 3	930.3	899.4

The DIC has substantially reduced for outstanding claims but has increased again for paid claims.

Combined with the above diagnostics, this suggests that the model is over-parameterized.

Although this model is not advisable for reserving purposes, its incurred extrapolations have been compared to the hold-out samples for illustrative purposes as follows:



The 95% PPI hold-out sample coverages (*with the previous model's stated in brackets*) are as follows:

95% PPI Coverage	**1-year ahead**	**10-years ahead**	**Total**
Outstanding	100% (*100%*)	100% (*100%*)	98% (*100%*)
Paid	89% (*89%*)	67% (*67%*)	**71% (*69%*)**
Incurred	100% (*100%*)	**89% (*100%*)**	**91% (*100%*)**

This model describes historical claims development more accurately than all previous models, yet incurred claims PPI coverage has reduced to its lowest level. It appears that explicitly modeling the outstanding claims calendar shock removes it from the modeled process error. Consequently, potential future calendar shocks are less likely to be adequately covered by the PPIs (such as the apparent 1999 shock affecting the 1994, 1995 and 1997 accident years).

The OpenBUGS code for this model is contained in Appendix F.

Although a more complex model could be built for future calendar shocks, this would not resolve the existing overfitting issue.

5. DISCUSSION

A hierarchical framework draws statistical strength across individuals, which can facilitate parsimony. However, as the case study demonstrates, this does not imply that the resultant model *will* be parsimonious. Diagnostic scrutiny is essential when selecting a hierarchical model for estimating reserves and their uncertainty.

Clark and Rangelova (2015) illustrate the importance of capturing accident year/development year interactions, and recommend that statistical methods allow intervention points for adjustment of intermediate results. In a hierarchical compartmental framework, an optional number of random-effects describe accident year development pattern differences based on intuitive parameters. The parameters themselves can be modeled to vary over development time. This flexibility allows the description of accident year/development year interactions such as changes in reporting/settlement rates and case reserve robustness, in addition to calendar shocks.

Although not demonstrated in the case study, continuous calendar trends such as inflation can be modeled within a compartmental framework. If a continuous "force of inflation" δ is assumed then expected claims payments $f_{PD}(P_i, \boldsymbol{\phi}_i, t_j)$ can be revised to include the inflation factor:

$$f'_{PD}(P_i, \boldsymbol{\phi}_i, t_j) = f_{PD}(P_i, \boldsymbol{\phi}_i, t_j) \cdot e^{(i+j-2)\delta} \quad \textbf{(5.1)}$$

As detailed by Zhang, Dukic and Guszcza (2012), the first calendar year $i + j = 2$ is treated as a "base" and subsequent expected calendar year payments are inflated by a factor $e^{(i+j-2)\delta}$, where δ is estimated or pre-specified. A similar approach could be taken to inflate outstanding claims or, alternatively, the differential equation system itself could be adjusted.

The deterministic compartmental model assumption of a smooth and detectable claims process relies upon claims cohort homogeneity for a volume of claims. This may not always be the case, and therefore further research is required to establish the validity and value of hierarchical semi-stochastic compartmental reserving models (Appendix B).

Other possible areas for future research include:

- The use of compartmental models to capture specific sub-processes such as legal shocks, catastrophes, latent claims, reopened claims, reinsurance recoveries and salvage/subrogation, to name but a few.
- Exploring the value of covariate models based on separate data sources. For example, if a claims handling team increased in size then one might expect the rate of payment to increase also.

- Establishment of a library of reporting/payment rate vs. development time functions along with their corresponding development profile properties.
- Simultaneous compartmental reserving for multiple insurance companies, e.g. by adding an extra level of hierarchy to describe company variation (Zhang, Dukic and Guszcza, 2012).

Many of the aforementioned extensions could be naturally incorporated within a Bayesian framework. Additionally, the Bayesian implementation itself could be further refined by considering alternative prior distributions. For example, prior dependence of random-effect variance and correlation terms could be controlled by using the separation strategy proposed by Barnard, McCulloch and Meng (2000).

Further work is required to evaluate the benefits of a compartmental approach compared to established methods, particularly for the estimation of reserve uncertainty.

6. CONCLUSIONS

This paper introduces a practical compartmental modeling framework for describing cumulative claims development. In particular, by considering the claims process over time as **Exposed to Risk → Claims Outstanding → Claims Paid**, an intuitive set of parameters have been defined which include a measure of case reserve robustness.

Cumulative paid claims model solutions are analogous to Clark's growth curve approach to loss reserving (2003). In contrast to growth curves which contain implicit tail factors, compartmental reserving model tail factors and hence ultimate projections are dictated by the extrapolation of outstanding losses to zero and estimated case reserve robustness. A number of possible model extensions have been explored to describe the nuances of the class of business being modeled, including changing reporting and/or settlement rates.

Following Guszcza (2008), a flexible nonlinear hierarchical framework is proposed to describe claims triangle data. Claims cohorts are viewed as individuals and cumulative losses are viewed as a series of observations for each individual. In contrast to Guszcza, cumulative paid triangles *and* outstanding claims triangles are fitted to, which enhances inference and interpretability. A probability sub-model allows a selection of the compartmental model parameters to vary by cohort and describe claims cohort pattern heterogeneity. Claims process trends can be identified and scenario tested, and parameter interpretability facilitates model discussion across the wider business.

A Bayesian implementation (similar to Zhang, Dukic and Guszcza, 2012) enables the robust incorporation of judgment and/or external information into claims projections. In addition to quantifying reserve uncertainty consistently with its definition, it offers additional model flexibility so that features such as residual autocorrelation and calendar effects can be explicitly accounted for.

Acknowledgments

I would like to thank Matt Locke for enduring numerous extremely useful conversations regarding the method and paper, Robert Ruiz for his advocacy, musings and mathematical rigor and Markus Gesmann for his valuable feedback and continual enthusiasm. I also thank Karl Goring and Matthew Killough for their insightful reviews. Additionally, I would like to thank Rob Murray for his encouragement and Lane Clark & Peacock for sponsoring the intermediate development of this research. Finally, I thank Jeff Courchene for approaching me to write this paper, and Nicki Power for graciously supporting several months' preoccupation. Any remaining errors are entirely my own.

Supplementary Material

A single-cohort compartmental reserving model illustration spreadsheet is available at http://www.casact.org/pubs/forum/16sforum/. The frequentist models in this paper were fitted using statistical software 'R', freely available at http://www.r-project.org. The R packages "nlmeODE" and "nlme" can be installed from the base R program. The Bayesian models in this paper were fitted using Bayesian Gibbs Sampling software 'OpenBUGS', freely available at http://www.openbugs.net. The case study dataset is freely available at http://www.casact.org/research/reserve_data/wkcomp_pos.csv (NAIC company code 337).

Appendix A: Implied development patterns

Implied continuous-time patterns of development are obtainable from the baseline compartmental reserving model solutions. Recall Eq. (3.2), which describes the claims process assuming that all exposure is input at time 0 and all model parameters are constant over development time t:

$$OS(t) = \frac{P \cdot RLR \cdot k_{er}}{k_{er} - k_p} \cdot (e^{-k_p t} - e^{-k_{er} t})$$

$$PD(t) = \frac{P \cdot RLR \cdot RRF}{k_{er} - k_p} \cdot \left(k_{er} \cdot (1 - e^{-k_p t}) - k_p \cdot (1 - e^{-k_{er} t})\right)$$

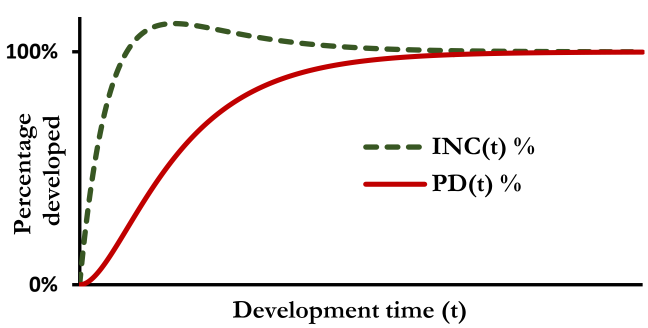
Given that $ULR = RLR \cdot RRF$, it follows that $P \cdot RLR \cdot RRF$ from the third ODE equates to the estimated ultimate losses. To derive the implied pattern of paid development at time t, we can therefore divide $PD(t)$ by $P \cdot RLR \cdot RRF$ to give

$$PD(t)\ \% = \frac{1}{k_{er} - k_p} \cdot \left(k_{er} \cdot (1 - e^{-k_p t}) - k_p \cdot (1 - e^{-k_{er} t})\right) \qquad \textbf{(A.1)}$$

Similarly, by summing $OS(t)$ and $PD(t)$, dividing by $P \cdot RLR \cdot RRF$ and simplifying, the implied incurred pattern of development can be derived as

$$INC(t)\ \% = \frac{k_{er} \cdot (e^{-k_p t} - e^{-k_{er} t}) + RRF \cdot \left(k_{er} \cdot (1 - e^{-k_p t}) - k_p \cdot (1 - e^{-k_{er} t})\right)}{RRF \cdot (k_{er} - k_p)} \qquad \textbf{(A.2)}$$

For a given set of parameters (with an $RRF < 1$), Eq. (A.1) and (A.2) can be visualized over development time as follows:



For perfect case reserving on average across a cohort of claims i.e. $RRF = 1$ (resulting in all claim amounts outstanding becoming paid claims), the incurred pattern in Eq. (A.2) simplifies and can be interpreted as an Exposed to Risk ("EtR") to reporting pattern:

$$INC(t)\ \% = EtR\ to\ Report(t)\ \% = 1 - e^{-k_{er} t} \qquad \textbf{(A.3)}$$

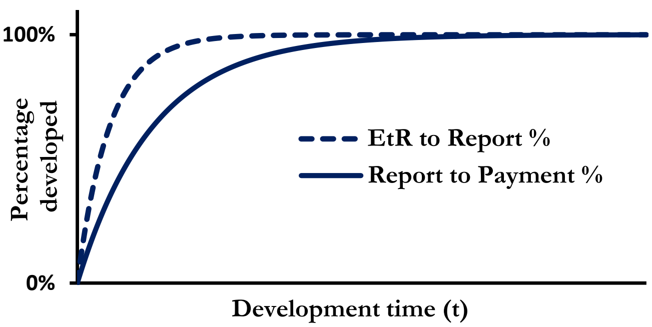
This result can also be obtained by letting $k_p \to 0$ in Eq. (3.2), and dividing $OS(t)$ by $P \cdot RLR$. To derive a report to payment pattern, it could be assumed that all exposures are initialized into the outstanding compartment at time 0. This results in a model that is defined in terms of two parameters only: a rate of payment and a reserve robustness factor. We can write the state-variable solutions as

$$OS(t) = Pe^{-k_p t}$$
$$PD(t) = P \cdot RRF \cdot \left(1 - e^{-k_p t}\right) \quad \textbf{(A.4)}$$

Similarly to above, the payment pattern $PD(t)\,\%$ can be derived by dividing $PD(t)$ by ultimate claims, which in this instance is $P \cdot RRF$. Given that we are only considering the claims process from reporting onwards, the resulting pattern can be interpreted as a report to payment pattern:

$$PD(t)\ \% = Report\ to\ Payment(t)\ \% = \ 1 - e^{-k_p t} \quad \textbf{(A.5)}$$

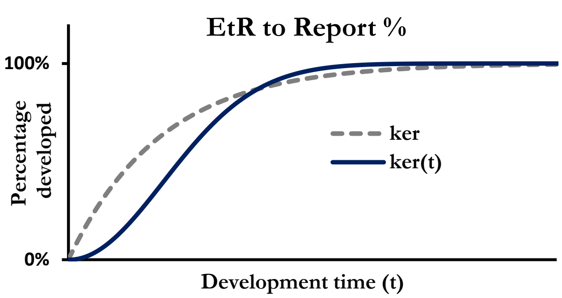
This result can also be obtained by letting $k_{er} \to \infty$ in Eq. (3.2), and dividing $PD(t)$ by $P \cdot RRF \cdot RLR$. For a given set of parameters, the EtR to report and report to payment development patterns can be visualized over development time as follows:

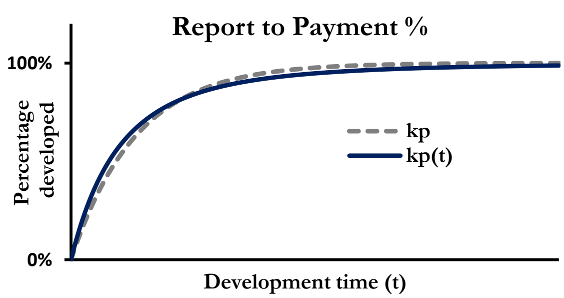


The development patterns are based on rate parameters which are constant over development time. If the rate parameters varied over time however, development patterns would also be expected to vary. Equations (A.3) and (A.5) can be generalized to allow for variable rates by writing

$$EtR\ to\ Report(t)\ \% = 1 - e^{-\int_0^t k_{er}(t)dt}$$
$$Report\ to\ Payment(t)\ \% = 1 - e^{-\int_0^t k_p(t)dt} \quad \textbf{(A.6)}$$

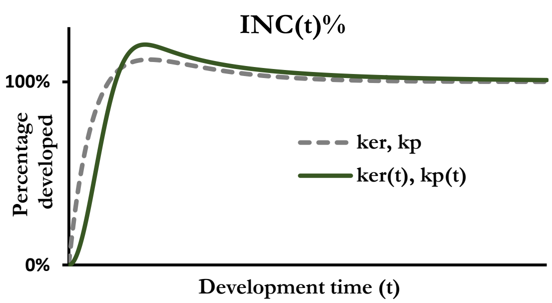
The graphs below show implied EtR to report and report to payment development patterns both for constant rate parameters (dashed lines), and parameters that vary over development time in accordance with the functions outlined in Section 3.2 (solid lines):

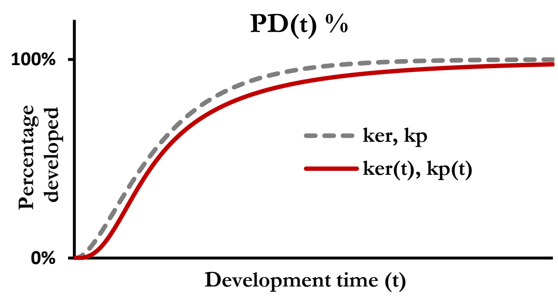




In contrast to a constant rate of reporting, a reporting rate that linearly increases over development time results in a slower pattern of reported claims development initially, which speeds up over time. This could be used to reflect a delay between claim events and claim reports for an accident cohort of claims. Allowing the rate of payment to decrease over development time results in a faster pattern of payment initially, which slows down over time. This is reflective of a slower settlement rate for claims outstanding in later development periods, perhaps due to litigation.

Corresponding incurred and payment patterns for both constant and non-constant rate parameters (obtained using numerical methods) can also be compared as follows:





The impact of altering parameters values/functions on development patterns can be seen in the illustration spreadsheet available at: http://www.casact.org/pubs/forum/16sforum/.

Appendix B: Semi-stochastic compartmental reserving models

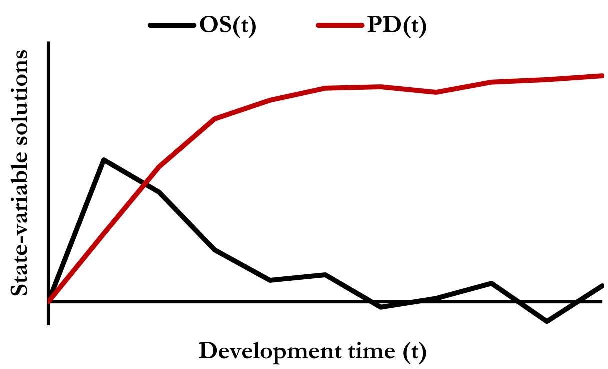
The deterministic compartmental model outlined in Section 3 assumes the same average claims behavior throughout the lifetime of a cohort. However, there are many reasons why there may be additional variability in the process, e.g. erratic case reserve fluctuations, claims payment backlogs etc. It may therefore be appropriate to re-specify the baseline model as a semi-stochastic (or "grey box"; Tornøe *et al.*, 2004b) model by introducing a Wiener process (or multiple processes) into the model's structural form. To do this we must first re-write Eq. (3.1) by moving the time increment (dt) terms to the right hand side of the ODEs, giving

$$\begin{aligned} dEX &= (-k_{er} \cdot EX)dt \\ dOS &= (k_{er} \cdot RLR \cdot EX - k_p \cdot OS)dt \\ dPD &= (k_p \cdot RRF \cdot OS)dt \end{aligned} \quad \textbf{(B.1)}$$

To incorporate a Wiener process for outstanding claims we can write

$$\begin{aligned} dEX &= (-k_{er} \cdot EX)dt \\ dOS &= \left(k_{er} \cdot RLR \cdot EX - k_p \cdot OS\right)dt + \sigma_{OS} dW \\ dPD &= (k_p \cdot RRF \cdot OS)dt \end{aligned} \quad \textbf{(B.2)}$$

where W is a standard (and additive) Wiener process such that $W(t_2) - W(t_1) \sim N(0, |t_2 - t_1|)$, and σ_{OS} is the estimable element of the Wiener process standard deviation (the diffusion coefficient), representing volatility in outstanding claims not captured by the deterministic ODEs. For illustration, this allows model solutions (plotted at yearly time steps) to look as follows:



An issue with the model outlined above is that the volatility in outstanding claims is assumed to be constant, and therefore the Wiener process can cause outstanding claims to fall below zero. Although this is plausible for classes of business where salvage/subrogation is material, Eq. (B.2) assumes that large outstanding claims fluctuations can persist at later development times where they would typically be expected to be zero. This can lead to negative paid increments, as shown above. To address this, the above Wiener process can be assumed to be a multiple of the amount in the outstanding claims compartment (i.e. state-dependent), giving

$$
\begin{aligned}
dEX &= (-k_{er} \cdot EX)dt \\
dOS &= \left(k_{er} \cdot RLR \cdot EX - k_p \cdot OS\right)dt + \sigma_{OS} OS dW \\
dPD &= (k_p \cdot RRF \cdot OS)dt
\end{aligned} \tag{B.3}
$$

The volatility introduced to the claims process is therefore proportional to amounts outstanding at each development time, which may be a more realistic assumption. Although this model can be fitted to a single cohort, for the multiple cohort case using hierarchical models (Section 4) it is not straightforward to implement Eq. (B.3) in conventional software (at the time of writing). However, Eq. (B.2) can be implemented in a hierarchical framework using the R package "PSM" (Klim *et al.*, 2009).

A key benefit of using SDEs is that they can account for residual autocorrelation (see Section 4.2) in a flexible manner. Furthermore, SDEs can describe claims process mechanisms that are too complex to include in the structural model (Overgaard *et al.*, 2005). A similar approach could be used to model low-frequency high-severity losses. As an alternative to the semi-stochastic model above, probability transfer mechanisms between compartments could be incorporated (Rescigno and Segre, 1966).

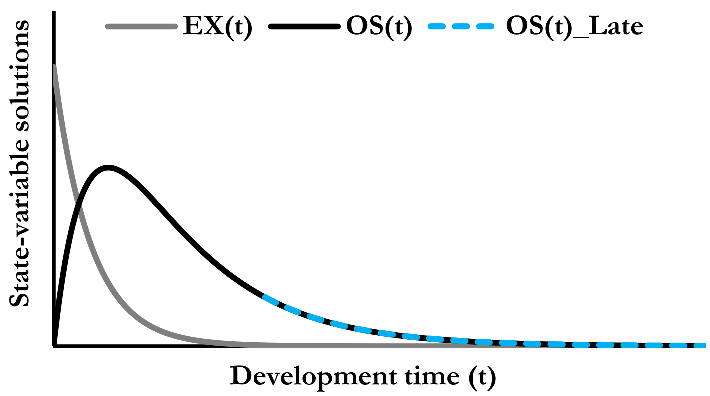
Appendix C: Nonlinear regression self-starting algorithm

Nonlinear regression models require parameter starting values for optimization to take place. Although visual inspection and judgment can be used to select reasonable starting values (see Section 4.1), inappropriate estimates can result in the model converging to a local rather than global likelihood maximum. A starting value algorithm is therefore outlined below for a single-cohort baseline compartmental reserving model, based on the "method of residuals" (Macheras, 1987).

We reexamine the baseline compartmental model defined by Eq. (3.1) and (3.2) and note that by some development time point, most claims will have been reported i.e. $EX(t) \to 0$. From this point onwards, only the claims payment phase of the process will remain. Provided that k_{er} is sufficiently larger than k_p, we can ignore the reporting term $e^{-k_{er}t}$ and obtain the following expression for later development time outstanding claims, $OS(t)^{LATE}$:

$$OS(t)^{LATE} = \frac{P \cdot RLR \cdot k_{er}}{k_{er} - k_p} \cdot e^{-k_p t} \quad \textbf{(C.1)}$$

This can be viewed graphically as follows:



Denoting $\boldsymbol{\beta} = \{\beta_1, \beta_2, \beta_3, \beta_4\}^T = \{k_{er}, RLR, k_p, RRF\}^T$ and OS_j as the jth outstanding claims observation, we can write down the following regression model:

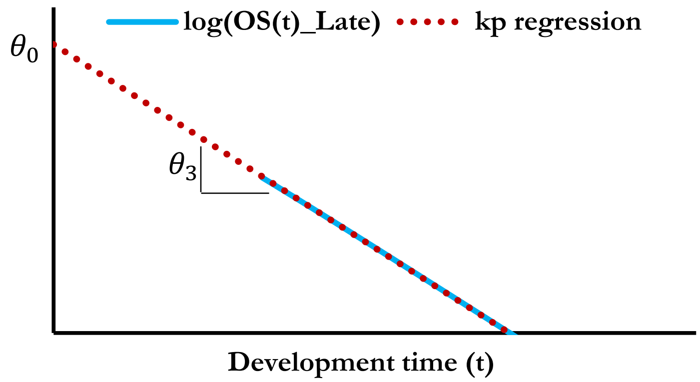
$$OS_j^{LATE} = \frac{P \cdot \beta_2 \cdot \beta_1}{\beta_1 - \beta_3} \cdot e^{-\beta_3 t_j} + \varepsilon_j^{OS} \quad \textbf{(C.2)}$$

This phase of the solution has only one exponential term, enabling us to take logarithms of both sides to linearize the model:

$$\log(OS_j^{LATE}) = \log\left(\frac{P \cdot \beta_2 \cdot \beta_1}{\beta_1 - \beta_3}\right) - \beta_3 t_j + \epsilon_j^{OS} \quad \textbf{(C.3)}$$

Denoting $\theta_0 = \log\left(\frac{P \cdot \beta_2 \cdot \beta_1}{\beta_1 - \beta_3}\right)$, $\theta_3 = -\beta_3$, a linear regression can be specified and carried out:

$$\log(OS_j^{LATE}) = \theta_0 + \theta_3 t_j + \epsilon_j^{OS} \quad \textbf{(C.4)}$$



This regression should be carried out for the logarithm of outstanding claims development values from the point at which the exposure is assumed to be negligible. However, this time point is not likely to be known. Even if it was, there may be practical restrictions to carrying out regression C.4 and subsequent regressions from this time point onwards (discussed at the end of this Appendix).

Once estimates $\hat{\theta}_0$ and $\hat{\theta}_3$ have been found, we establish that $\frac{\widehat{P \cdot \beta_2 \cdot \beta_1}}{\beta_1 - \beta_3} = e^{\hat{\theta}_0}$ and $\hat{\beta}_3 = -\hat{\theta}_3$.

This gives an estimate of the rate of payment, k_p. The next step is to identify that

$$OS_j = OS_j^{LATE} - \frac{P \cdot \beta_2 \cdot \beta_1}{\beta_1 - \beta_3} \cdot \left(e^{-\beta_1 t_j}\right) \quad \textbf{(C.5)}$$

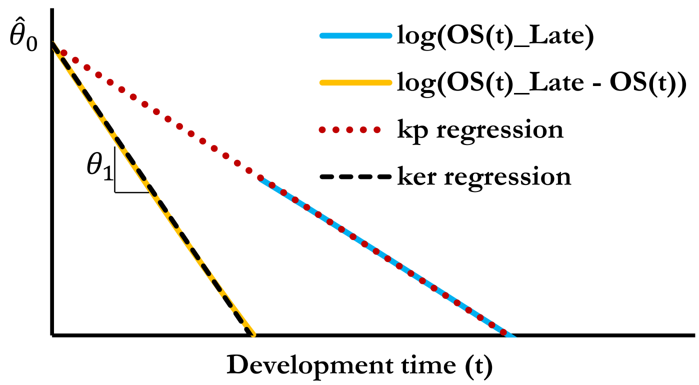
This can be rearranged and linearized as follows for $OS_j^{LATE} - OS_j > 0$:

$$OS_j - OS_j^{LATE} = -\frac{P \cdot \beta_2 \cdot \beta_1}{\beta_1 - \beta_3} \cdot \left(e^{-\beta_1 t_j}\right)$$
$$\log(OS_j^{LATE} - OS_j)\Big|_{OS_j^{LATE} - OS_j > 0} = \log\left(\frac{P \cdot \beta_2 \cdot \beta_1}{\beta_1 - \beta_3}\right) - \beta_1 t_j \quad \textbf{(C.6)}$$

OS_j^{LATE} can be taken as its estimated value in the previous regression, $\widehat{OS}_j^{LATE}$, and the intercept $\log\left(\frac{P \cdot \beta_2 \cdot \beta_1}{\beta_1 - \beta_3}\right)$ can be fixed to the previously estimated intercept, $\hat{\theta}_0$.

Denoting $\theta_1 = -\beta_1$ and rearranging, a second linear regression can be specified through the origin (Turner, 1960):

$$\log(\widehat{OS}_j^{LATE} - OS_j)\Big|_{\widehat{OS}_j^{LATE} - OS_j > 0} = \hat{\theta}_0 + \theta_1 t_j + \xi_j$$
$$\log(\widehat{OS}_j^{LATE} - OS_j)\Big|_{\widehat{OS}_j^{LATE} - OS_j > 0} - \hat{\theta}_0 = \theta_1 t_j + \xi_j \quad \textbf{(C.7)}$$



Once an estimate of $\hat{\theta}_1$ of θ_1 has been found, we establish that $\hat{\beta}_1 = -\hat{\theta}_1$, thus providing an estimate of the rate of reporting, k_{er}. Given our estimates of k_{er} and k_p, we can infer an estimate of the RLR. To see how, we recall the definition of θ_0 in Eq. (C.4) and rearrange as follows:

$$\log\left(\frac{P\cdot\beta_2\cdot\beta_1}{\beta_1-\beta_3}\right)=\theta_0$$

$$\beta_2=\frac{e^{\theta_0}\cdot(\beta_1-\beta_3)}{P\cdot\beta_1} \quad \textbf{(C.8)}$$

$$\beta_2=\frac{e^{\theta_0}\cdot(-\theta_1+\theta_3)}{P\cdot-\theta_1}$$

We can therefore substitute in the previously estimated parameters to get an estimate of β_2:

$$\hat{\beta}_2=\frac{e^{\hat{\theta}_0}\cdot(-\hat{\theta}_1+\hat{\theta}_3)}{P\cdot-\hat{\theta}_1} \quad \textbf{(C.9)}$$

This is an estimate of the RLR. Finally, to estimate the RRF we note that the above procedure generates parameter estimates for all elements of the paid claims solution in Eq. (3.2) except the RRF:

$$PD(t)=\frac{P\cdot RLR\cdot RRF}{k_{er}-k_p}\cdot\left(k_{er}\cdot\left(1-e^{-k_pt}\right)-k_p\cdot\left(1-e^{-k_{er}t}\right)\right)$$

$$PD(t)=\frac{P\cdot RLR}{k_{er}-k_p}\cdot\left(k_{er}\cdot\left(1-e^{-k_pt}\right)-k_p\cdot\left(1-e^{-k_{er}t}\right)\right)\cdot RRF \quad \textbf{(C.10)}$$

Rewriting as a regression as per above gives:

$$PD_j=\frac{P\cdot\beta_2}{\beta_1-\beta_3}\cdot\left(\beta_1\cdot\left(1-e^{-\beta_3t_j}\right)-\beta_3\cdot\left(1-e^{-\beta_1t_j}\right)\right)\cdot\beta_4+\omega_j \quad \textbf{(C.11)}$$

Substituting in the estimates of each parameter apart from β_4, we can denote $\theta_4=\beta_4$ and rewrite Eq. (C.11) as follows:

$$PD_j=f\left(P,\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3,t_j\right)\cdot\theta_4+\omega_j \quad \textbf{(C.12)}$$

This regression is linear in the parameter that we are interested in estimating, θ_4 (RRF), and therefore a linear regression through the origin can be carried out to derive an estimate of θ_4: $\hat{\theta}_4$.

The vector of parameter starting values, $\boldsymbol{\beta}^0$, can then be set to be:

$$\boldsymbol{\beta}^0 = \{\beta_1^0, \beta_2^0, \beta_3^0, \beta_4^0\}^T = \left\{k_{er}^0, RLR^0, k_p^0, RRF^0\right\}^T = \left\{-\hat{\theta}_1, \hat{\theta}_2, -\hat{\theta}_3, \hat{\theta}_4\right\}^T \qquad \textbf{(C.13)}$$

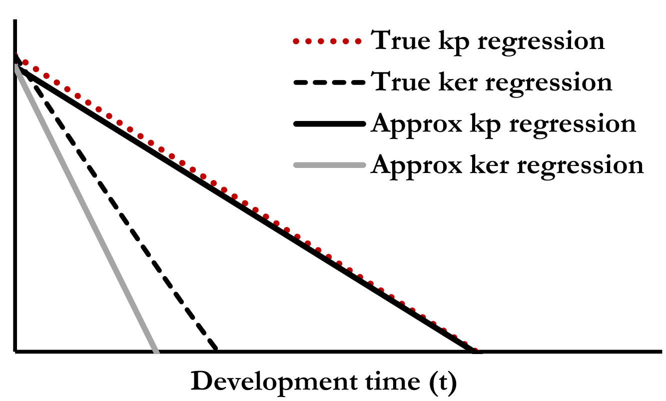
Although this algorithm is based on a single cohort of claims, for the multiple cohort case (e.g. using hierarchical models as outlined in Section 4), the algorithm could be used to derive parameter estimates for each individual cohort. To derive fixed-effect starting values, one could then calculate weighted average parameters based on the number of data points within each cohort, for example.

Selecting t_j^{LATE}

As stated above, a practitioner is unlikely to be able to identify when exposure has fallen close to zero for a particular claims cohort. Furthermore, a claims cohort might not have a long enough development history to be able to fit a regression from t_j^{LATE} onwards. This issue is more prevalent if the rate of reporting is slow because by definition, exposures will convert to reported claims and tend to zero at a slower rate.

Being that the goal is to specify starting values for the parameters being estimated and not to derive final estimates, it may be acceptable to compromise on the point at which t_j^{LATE} is defined at the cost of reducing the accuracy of the initial parameter estimates. One possibility is to calculate peak outstanding claims from the data, MAX_OS, and define the corresponding development time point as $t_j^{MAX\_OS}$. For the regressions outlined above, t_j^{LATE} could then be defined as $t_j \geq t_j^{MAX\_OS}$.

In some instances (e.g. when k_{er} much faster than k_p) this will be a close approximation to when exposure is close to zero. In others however, it is likely to be a less accurate approximation due to a high probability of new non-negligible value claims being subsequently reported. The graph below illustrates the discrepancy in regression slopes, i.e. initial parameter estimates, for $k_{er} = 2.33k_p$:



This approach to defining t_j^{LATE} will result in a degree of starting value parameter estimation error (more predominantly for k_{er}), with a magnitude inversely proportional to the underlying rate of reporting. On the other hand, if k_{er} is too fast, there won't be early phase data to derive its estimate in the first place (in which case a large estimate can be selected arbitrarily). Additionally, the less mature the cohort, the less reliable the parameter estimates will be. However, this approach should initialize the nonlinear regression optimization process at a sensible point in the parameter space.

It's worth noting that $t_j^{MAX\_OS}$ may appear long before the true payment phase if outstanding claims development is volatile. Therefore in practice, judgment will be necessary to decide from which development time point the observed logarithm of the outstanding claims can be considered linear. The degree of linearity must be balanced with the number of development data points available to carry out the regression for k_p. In cases where there are no observations subsequent to the maximum outstanding claims value, this algorithm cannot be used.

In the case of development time-dependent parameters (Section 3.2), the parameter starting value algorithm could be used to find approximate parameter starting values by setting nonlinear rate *functions* equal to the parameter estimates above. However, identifiability will be an issue for rate functions with more than one parameter (unless at least one of the parameters is arbitrarily fixed).

Appendix D: Frequentist case study data

<u>Data key</u>

“Cohort” = accident year
“t” = development year
“Claims” = outstanding claims for “Type”=1 and cumulative paid claims for “Type”=2
“Dose” = exposure/earned premium
“Cmt” = exposure compartment number

```
> Data <- groupedData(Claims ~ t | Cohort/Type, data = Data)

Grouped Data: Claims ~ t | Cohort/Type
    Cohort  t Claims Type   Dose Cmt
1     1988  0      0    1 104437   1
2     1988  0      0    2      0   1
3     1988  1  53121    1      0   1
4     1988  1   9558    2      0   1
5     1988  2  41222    1      0   1
6     1988  2  22778    2      0   1
7     1988  3  32309    1      0   1
8     1988  3  33298    2      0   1
9     1988  4  24944    1      0   1
10    1988  4  40348    2      0   1
11    1988  5  17104    1      0   1
12    1988  5  45146    2      0   1
13    1988  6  13137    1      0   1
14    1988  6  48048    2      0   1
15    1988  7   9605    1      0   1
16    1988  7  49782    2      0   1
17    1988  8   6515    1      0   1
18    1988  8  50623    2      0   1
19    1988  9   1661    1      0   1
20    1988  9  51812    2      0   1
21    1988 10   1322    1      0   1
22    1988 10  51939    2      0   1
23    1989  0      0    1  88883   1
24    1989  0      0    2      0   1
25    1989  1  54145    1      0   1
26    1989  1   7913    2      0   1
27    1989  2  37188    1      0   1
28    1989  2  19472    2      0   1
29    1989  3  26976    1      0   1
30    1989  3  29622    2      0   1
31    1989  4  20015    1      0   1
32    1989  4  36816    2      0   1
33    1989  5  14319    1      0   1
34    1989  5  40975    2      0   1
35    1989  6  10179    1      0   1
36    1989  6  43302    2      0   1
37    1989  7   6672    1      0   1
38    1989  7  44707    2      0   1
39    1989  8   2575    1      0   1
40    1989  8  45871    2      0   1
41    1989  9   2071    1      0   1
```

42	1989	9	46229	2	0	1
43	1990	0	0	1	85956	1
44	1990	0	0	2	0	1
45	1990	1	55211	1	0	1
46	1990	1	8744	2	0	1
47	1990	2	37221	1	0	1
48	1990	2	24302	2	0	1
49	1990	3	27760	1	0	1
50	1990	3	35406	2	0	1
51	1990	4	17990	1	0	1
52	1990	4	43412	2	0	1
53	1990	5	11417	1	0	1
54	1990	5	48057	2	0	1
55	1990	6	6716	1	0	1
56	1990	6	50897	2	0	1
57	1990	7	4282	1	0	1
58	1990	7	52879	2	0	1
59	1990	8	3015	1	0	1
60	1990	8	53956	2	0	1
61	1991	0	0	1	99339	1
62	1991	0	0	2	0	1
63	1991	1	60617	1	0	1
64	1991	1	13301	2	0	1
65	1991	2	42144	1	0	1
66	1991	2	32950	2	0	1
67	1991	3	25987	1	0	1
68	1991	3	47201	2	0	1
69	1991	4	14805	1	0	1
70	1991	4	56394	2	0	1
71	1991	5	9406	1	0	1
72	1991	5	61650	2	0	1
73	1991	6	5792	1	0	1
74	1991	6	65039	2	0	1
75	1991	7	3966	1	0	1
76	1991	7	66566	2	0	1
77	1992	0	0	1	104897	1
78	1992	0	0	2	0	1
79	1992	1	65719	1	0	1
80	1992	1	11424	2	0	1
81	1992	2	46047	1	0	1
82	1992	2	29086	2	0	1
83	1992	3	31250	1	0	1
84	1992	3	42034	2	0	1
85	1992	4	22245	1	0	1
86	1992	4	50910	2	0	1
87	1992	5	11878	1	0	1
88	1992	5	56406	2	0	1
89	1992	6	8408	1	0	1
90	1992	6	59437	2	0	1
91	1993	0	0	1	119427	1
92	1993	0	0	2	0	1
93	1993	1	68133	1	0	1
94	1993	1	11792	2	0	1
95	1993	2	51102	1	0	1
96	1993	2	27161	2	0	1
97	1993	3	39934	1	0	1

98	1993	3	38229	2	0	1
99	1993	4	21824	1	0	1
100	1993	4	46722	2	0	1
101	1993	5	16955	1	0	1
102	1993	5	50742	2	0	1
103	1994	0	0	1	110784	1
104	1994	0	0	2	0	1
105	1994	1	62434	1	0	1
106	1994	1	11194	2	0	1
107	1994	2	46661	1	0	1
108	1994	2	26893	2	0	1
109	1994	3	32248	1	0	1
110	1994	3	38488	2	0	1
111	1994	4	24140	1	0	1
112	1994	4	45580	2	0	1
113	1995	0	0	1	77731	1
114	1995	0	0	2	0	1
115	1995	1	56971	1	0	1
116	1995	1	12550	2	0	1
117	1995	2	48677	1	0	1
118	1995	2	31604	2	0	1
119	1995	3	35336	1	0	1
120	1995	3	44045	2	0	1
121	1996	0	0	1	63646	1
122	1996	0	0	2	0	1
123	1996	1	56526	1	0	1
124	1996	1	13194	2	0	1
125	1996	2	41707	1	0	1
126	1996	2	31474	2	0	1
127	1997	0	0	1	48052	1
128	1997	0	0	2	0	1
129	1997	1	40799	1	0	1
130	1997	1	9372	2	0	1

Appendix E: Frequentist modeling R code

Baseline structural model (Section 4.1)

```
> DEmodel <- list(
+                 DiffEq=list(
+                     dy1dt = ~ -lker*y1,
+                     dy2dt = ~ lker*lRLR*y1 - lkp*y2,
+                     dy3dt = ~ lkp*lRRF*y2),
+                 ObsEq=list(
+                     EX   = ~ 0,
+                     OS   = ~ y2,
+                     PA   = ~ y3),
+ States=c("y1","y2","y3"),
+ Parms=c("lker","lRLR","lkp","lRRF"),
+ Init=list(0,0,0))
```

Model 1 (Section 4.1)

```
> ReservingModel <- nlmeODE(DEmodel,Data)   ### "Data" = data in Appendix D

> nlmeModel <- nlme(Claims ~
ReservingModel(lker,lRLR,lkp,lRRF,t,Cohort,Type),

+ data = Data,
+ fixed = lker+lRLR+lkp+lRRF ~ 1,           ### fixed-effect parameters
+ random = pdDiag(lRLR + lRRF ~ 1),         ### parameters with random-effects
+ groups = ~Cohort,                         ### data grouping (accident years)
+ weights = varIdent(form = ~1 | Type),     ### residual error functions: OS&PD
+ start = c(lker = log(1.5), lRLR = log(1),
     lkp = log(0.75), lRRF = log(0.75)),    ### parameter starting values
+ control=list(returnObject=TRUE,msVerbose=TRUE,
+ msMaxIter=10000,pnlsMaxIter=10000,
+ pnlsTol=0.4),                             ### tolerance for PNLS convergence
+ verbose=TRUE)

> nlmeModel
Nonlinear mixed-effects model fit by maximum likelihood
 Model: Claims ~ ReservingModel(lker, lRLR, lkp, lRRF, t, Cohort, Type)
 Data: Data
 Log-likelihood: -1164.386
 Fixed: lker + lRLR + lkp + lRRF ~ 1
       lker        lRLR         lkp        lRRF
 0.40824328  0.02575157 -0.79246675 -0.40644353
                                            ### estimated fixed-effects: β̂
Random effects:
 Formula: list(lRLR ~ 1, lRRF ~ 1)
 Level: Cohort
 Structure: Diagonal
            lRLR      lRRF Residual
StdDev: 0.1870103 0.1318661 3171.213
                                            ### estimated random-effect &
Variance function:                          ### residual std dev terms: {ψ̂_ik, σ̂}
 Structure: Different standard deviations per stratum
 Formula: ~1 | Type
 Parameter estimates:
       1         2
1.0000000 0.1790677                         ### OS&PD residual std deviation
Number of Observations: 130                 ### multipliers: {1, λ̂}
Number of Groups: 10
```

Extended structural model – development time-dependent reporting rate (Section 4.1.1)

```
> DEmodel2 <- list(
+               DiffEq=list(
+                   dy1dt = ~ -lBer*t*y1,
+                   dy2dt = ~ lBer*t*lRLR*y1 - lkp*y2,
+                   dy3dt = ~ lkp*lRRF*y2),
+               ObsEq=list(
+                   EX   = ~ 0,
+                   OS   = ~ y2,
+                   PA   = ~ y3),
+ States=c("y1","y2","y3"),
+ Parms=c("lBer","lRLR","lkp","lRRF"),
+ Init=list(0,0,0))
```

Model 2 (Section 4.1.1)

```
> ReservingModel2 <- nlmeODE(DEmodel2,Data)

> nlmeModel2 <- nlme(Claims ~
ReservingModel2(lBer,lRLR,lkp,lRRF,t,Cohort,Type),
+ data = Data,
+ fixed = lBer+lRLR+lkp+lRRF ~ 1,
+ random = pdDiag(lRLR + lRRF ~ 1),
+ groups = ~Cohort,
+ weights = varIdent(form = ~1 | Type),
+ start=c(lBer = log(5), lRLR = log(1.03),
     lkp = log(0.45), lRRF = log(0.67)),
+ control=list(returnObject=TRUE,msVerbose=TRUE,
+ msMaxIter=10000,pnlsMaxIter=10000,
+ pnlsTol=0.4),
+ verbose=TRUE)

> nlmeModel2
Nonlinear mixed-effects model fit by maximum likelihood
 Model: Claims ~ ReservingModel2(lBer, lRLR, lkp, lRRF, t, Cohort, Type)
 Data: Data
 Log-likelihood: -1156.344
 Fixed: lBer + lRLR + lkp + lRRF ~ 1
      lBer       lRLR        lkp       lRRF
 1.7637739 -0.1608870 -0.9339032 -0.1886841

Random effects:
 Formula: list(lRLR ~ 1, lRRF ~ 1)
 Level: Cohort
 Structure: Diagonal
             lRLR      lRRF Residual
StdDev: 0.1684008 0.1469151 2491.433

Variance function:
 Structure: Different standard deviations per stratum
 Formula: ~1 | Type
 Parameter estimates:
        1         2
1.0000000 0.2509692
Number of Observations: 130
Number of Groups: 10
```

```
> anova(nlmeModel,nlmeModel2)
           Model df      AIC      BIC    logLik
nlmeModel      1  8 2344.771 2367.711 -1164.386
nlmeModel2     2  8 2328.688 2351.628 -1156.344

> intervals(nlmeModel2)
Approximate 95% confidence intervals

 Fixed effects:
          lower       est.       upper
lBer  1.6599011  1.7637739  1.86764663
lRLR -0.2696058 -0.1608870 -0.05216819
lkp  -0.9671036 -0.9339032 -0.90070283
lRRF -0.2873897 -0.1886841 -0.08997844

> summary(nlmeModel2)

Correlation:
     lBer   lRLR   lkp
lRLR -0.110
lkp  -0.723  0.143
lRRF  0.142 -0.077 -0.253
```

Model 3 – random-effects correlation (Section 4.1.2)

```
> nlmeModel3 <- update(nlmeModel2,random=list(lRLR+lRRF~1))

> intervals(nlmeModel3)
Approximate 95% confidence intervals

Random Effects:
 Level: Cohort
                    lower      est.     upper
sd(lRLR)       0.09864002 0.1571791 0.2504587
sd(lRRF)       0.09475350 0.1517442 0.2430128
cor(lRLR,lRRF) 0.34584978 0.7795638 0.9387946

> anova(nlmeModel,nlmeModel2,nlmeModel3)
           Model df      AIC      BIC    logLik   Test  L.Ratio p-value
nlmeModel      1  8 2344.771 2367.711 -1164.386
nlmeModel2     2  8 2328.688 2351.628 -1156.344
nlmeModel3     3  9 2324.543 2350.351 -1153.272 2 vs 3 6.144368  0.0132
```

Model 4 – block-diagonal random-effects structure (Section 4.1.2)

```
> nlmeModel4 <- update(nlmeModel3,random=pdBlocked(list(lRLR + lRRF~1, lkp
~ 1)))

> anova(nlmeModel,nlmeModel2,nlmeModel3,nlmeModel4)
           Model df      AIC      BIC    logLik   Test   L.Ratio p-value
nlmeModel      1  8 2344.771 2367.711 -1164.386
nlmeModel2     2  8 2328.688 2351.628 -1156.344
nlmeModel3     3  9 2324.543 2350.351 -1153.272 2 vs 3  6.144368  0.0132
nlmeModel4     4 10 2305.500 2334.175 -1142.750 3 vs 4 21.043472  <.0001
```

Supplementary code – for structural model ***x***, hierarchical model ***y***

```
> residuals(nlmeModely, type="normalized")  ### standardized model residuals

> fitted(nlmeModely)                        ### model predictions

> IndCoef <- coef(nlmeModely)               ### individual accident year (log)
                                            compartmental parameter estimates

> ReservingModelx(                          ### model projections to time 10
+ rep(IndCoef[,1],each=2*11),
+ rep(IndCoef[,2],each=2*11),
+ rep(IndCoef[,3],each=2*11),
+ rep(IndCoef[,4],each=2*11),
+ Data_Full$t,Data_Full$Cohort,Data_Full$Type)
```

Appendix F: Bayesian modeling OpenBUGS code

Scenario 1: Fully random structure model (Section 4.2.1)
Replace red code with *blue code* to switch to Scenario 2: Calendar shock sub-model (Section 4.2.2).

```
model {
        for (i in 1:n.ind) {
                for (j in 1:1) {
                        data_O[i, j] ~ dnorm(mean_O[i, j] , tau_O)
                        data_P[i, j] ~ dnorm(mean_P[i, j] , tau_P)
                        data_I[i, j] <- data_O[i, j] + data_P[i, j]

                        mean_O[i, j] <- solution[i,j,2]
                        mean_P[i, j] <- solution[i, j, 3]
                        mean_I[i, j] <- mean_O[i, j] + mean_P[i, j]
                                                                }
                for (j in 2:n.grid) {
                        data_O[i, j] ~ dnorm(mean_O[i, j] , tau_O2)
                        data_P[i, j] ~ dnorm(mean_P[i, j] , tau_P2)
                        data_I[i, j] <- data_O[i, j] + data_P[i, j]

                        mean_O[i, j] <- solution[i, j, 2] + rho2 * (data_O[i, j-1] - mean_O[i, j-1])

                        #Calendar shock substitution
                        #mean_O[i, j] <- solution[i,j,2] * (1 - C[i,j] * a[i]) + rho2 *(data_O[i, j-1] -
                        #mean_O[i, j-1])

                        mean_P[i, j] <- solution[i, j, 3] + rho3 * (data_P[i, j-1] - mean_P[i, j-1])
                        mean_I[i, j] <- mean_O[i, j] + mean_P[i, j]
                                                                }
                theta[i, 1:p] ~ dmnorm(mu[1:p], omega.inv[1:p, 1:p])

                param[i, 1] <- theta[i, 1]
                param[i, 2] <- theta[i, 2]
                param[i, 3] <- theta[i, 3]
                param[i, 4] <- theta[i, 4]
                param[i, p+1] <- prem[i]

                Ber[i] <- exp(theta[i, 1])
                RLR[i] <- exp(theta[i, 2])
                kp[i] <- exp(theta[i, 3])
                RRF[i] <- exp(theta[i, 4])

                ULR[i] <- RLR[i] * RRF[i]
                ILR10[i] <- data_I[i, 10] / prem[i]

                solution[i, 1:n.grid, 1:dim] <- ode(inits[i, 1:dim],
                                        grid[1:n.grid], D(A[i, 1:dim], t[i]), origin, tol)

                D(A[i, 1], t[i]) <- -Ber[i] * t[i] * A[i, 1]
                D(A[i, 2], t[i]) <- Ber[i] * t[i] * RLR[i] * A[i, 1] - kp[i] * A[i, 2]
```

```
        D(A[i, 3], t[i]) <- kp[i] * RRF[i] * A[i, 2]

        #Calendar shock substitution
        #D(A[i, 3], t[i]) <- kp[i] * RRF[i] * (1 - V[i]*a[i]) * A[i, 2]
        #V[i] <- step((i + t[i]) - 10)
        #a[i] ~ dunif(-0.99,0.99)

        inits[i, 1] <- prem[i]
        inits[i, 2] <- 0
        inits[i, 3] <- 0
                                            }

mu[1:p] ~ dmnorm(mu.prior.mean[1:p], mu.prior.prec[1:p, 1:p])
omega.inv[1:p, 1:p] ~ dwish(omega.inv.matrix[1:p, 1:p], omega.inv.dof)

omega[1:p, 1:p] <- inverse(omega.inv[1:p, 1:p])
ResC <- omega[2, 4] / (sqrt(omega[2, 2]) * sqrt(omega[4, 4]))

sigma_O ~ dunif(0, 10000)
tau_O <- pow(sigma_O, -2)

sigma_O2 <- sigma_O * sqrt(1 - pow(rho2, 2))
tau_O2 <- pow(sigma_O2, -2)

sigma_P ~ dunif(0, 5000)
tau_P <- pow(sigma_P, -2)

sigma_P2 <- sigma_P * sqrt(1 - pow(rho3, 2))
tau_P2 <- pow(sigma_P2, -2)

rho2 ~ dunif(-1,1)
rho3 ~ dunif(-1,1)


#Standardized residuals
for (i in 1:n.ind) {
        for (j in 1:1) {
                r_O[i,j] <- (data_O[i, j] - mean_O[i, j] ) * sqrt(tau_O)
                r_P[i,j] <- (data_P[i, j] - mean_P[i, j] ) * sqrt(tau_P)

                                                    }
        for (j in 2:n.grid) {

                r_O[i,j] <- (data_O[i, j] - mean_O[i, j] ) * sqrt(tau_O2)
                r_P[i,j] <- (data_P[i, j] - mean_P[i, j] ) * sqrt(tau_P2)

                                                    }
                                            }
                                }
```

Data and prior parameters

```
list(
p = 4, dim = 3,
origin = 0.0,
tol = 1.0E-6,
n.ind = 10, n.grid = 10,
grid = c(1,2,3,4,5,6,7,8,9,10),
prem = c(104437, 88883, 85956, 99339, 104897, 119427, 110784, 77731, 63646, 48052),
mu.prior.mean = c(1.7, -0.15, -0.9, -0.21),
mu.prior.prec = structure(
.Data = c(
650, 0, 0, 0,
0, 380, 0, 0,
0, 0, 5400, 0,
0, 0, 0, 390),
.Dim = c(4, 4)),
omega.inv.matrix = structure(
.Data = c(
1, 0, 0, 0,
0, 1, 0, 0.8,
0, 0, 1, 0,
0, 0.8, 0, 1),
.Dim = c(4, 4)),
omega.inv.dof = 4,

data_O = structure(.Data = c(
53121, 41222, 32309, 24944, 17104, 13137, 9605, 6515, 1661, 1322,
54145, 37188, 26976, 20015, 14319, 10179, 6672, 2575, 2071, NA,
55211, 37221, 27760, 17990, 11417, 6716, 4282, 3015, NA, NA,
60617, 42144, 25987, 14805, 9406, 5792, 3966, NA, NA, NA,
65719, 46047, 31250, 22245, 11878, 8408, NA, NA, NA, NA,
68133, 51102, 39934, 21824, 16955, NA, NA, NA, NA, NA,
62434, 46661, 32248, 24140, NA, NA, NA, NA, NA, NA,
56971, 48677, 35336, NA, NA, NA, NA, NA, NA, NA,
56526, 41707, NA, NA, NA, NA, NA, NA, NA, NA,
40799, NA, NA, NA, NA, NA, NA, NA, NA, NA),
.Dim = c(10,10)),
data_P = structure(.Data = c(
9558, 22778, 33298, 40348, 45146, 48048, 49782, 50623, 51812, 51939,
7913, 19472, 29622, 36816, 40975, 43302, 44707, 45871, 46229, NA,
8744, 24302, 35406, 43412, 48057, 50897, 52879, 53956, NA, NA,
13301, 32950, 47201, 56394, 61650, 65039, 66566, NA, NA, NA,
11424, 29086, 42034, 50910, 56406, 59437, NA, NA, NA, NA,
11792, 27161, 38229, 46722, 50742, NA, NA, NA, NA, NA,
11194, 26893, 38488, 45580, NA, NA, NA, NA, NA, NA,
12550, 31604, 44045, NA, NA, NA, NA, NA, NA, NA,
13194, 31474, NA, NA, NA, NA, NA, NA, NA, NA,
9372, NA, NA, NA, NA, NA, NA, NA, NA, NA),
.Dim = c(10,10))
)
```

```
#Calendar shock substitution
#,C = structure(
#.Data = c(
#0, 0, 0, 0, 0, 0, 0, 0, 1, 1,
#0, 0, 0, 0, 0, 0, 0, 1, 1, 1,
#0, 0, 0, 0, 0, 0, 1, 1, 1, 1,
#0, 0, 0, 0, 0, 1, 1, 1, 1, 1,
#0, 0, 0, 0, 1, 1, 1, 1, 1, 1,
#0, 0, 0, 1, 1, 1, 1, 1, 1, 1,
#0, 0, 1, 1, 1, 1, 1, 1, 1, 1,
#0, 1, 1, 1, 1, 1, 1, 1, 1, 1,
#1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
#1, 1, 1, 1, 1, 1, 1, 1, 1, 1),
#.Dim = c(10,10))
#)
```

Initial values (1)

```
list(
rho2 = 0.5,
rho3= 0.5,
sigma_O = 5000,
sigma_P = 500,
mu = c(1.7, -0.15, -0.9, -0.21),
omega.inv = structure(
.Data = c(
10, 0, 0, 0,
0, 10, 0, 0.8,
0, 0, 10, 0,
0, 0.8, 0, 10),
.Dim = c(4, 4)),
theta = structure(
.Data = c(
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21,
1.7, -0.15, -0.9, -0.21),
.Dim = c(10, 4))
)
```

Initial values (2)

```
list(
rho2 = 0.6,
rho3= 0.2,
sigma_O = 3000,
sigma_P = 700,
mu = c(1.4, -0.07, -0.2, -0.51),
omega.inv = structure(
.Data = c(
15, 0, 0, 0,
0, 15, 0, 0.5,
0, 0, 15, 0,
0, 0.5, 0, 15),
.Dim = c(4, 4)),
theta = structure(
.Data = c(
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51,
1.4, -0.07, -0.2, -0.51),
.Dim = c(10, 4))
)
```

Initial values (3)

```
list(
rho2 = 0.2,
rho3= 0.6,
sigma_O = 1500,
sigma_P = 1000,
mu = c(1.1, 0, 0, -0.29),
omega.inv = structure(
.Data = c(
5, 0, 0, 0,
0, 5, 0, 0.3,
0, 0, 5, 0,
0, 0.3, 0, 5),
.Dim = c(4, 4)),
theta = structure(
.Data = c(
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29,
1.1, 0, 0, -0.29),
.Dim = c(10, 4))
)
```

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Abbreviations and notations

AY, accident year (row dimension of triangle)
AIC, Akaike information criterion
BBNI, bound but not incepted
BIC, Bayesian information criterion
CY, calendar year
DIC, deviance information criterion
EtR, Exposed to Risk
EX, exposure
ExBNR, exposed but not reported
GLM, generalized linear model
IBNR, incurred but not reported
OS, outstanding claims
MCMC, Markov chain Monte Carlo
NLME, nonlinear mixed-effects
ODE, ordinary differential equation
PD, paid claims (cumulative)
PLR, paid loss ratio
PPI, posterior predictive interval
RBNS, reported but not settled
RBC, reported burning cost
RLR, reported loss ratio
RRF, reserve robustness factor
SDE, stochastic differential equation
UBC, ultimate burning cost
ULR, ultimate loss ratio

Biography of the Author

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Supplement for Hierarchical Compartmental Models for Loss Reserving

http://www.casact.org/pubs/forum/16sforum/Compartmental_reserving.xlsx

The Actuary & Enterprise Risk Management: Integrating Reserve Variability

Mark R. Shapland, FCAS, FSA, MAAA
Jeffrey A. Courchene, FCAS, MAAA

Abstract

Motivation. The development of a wide variety of reserve variability models has been primarily driven by the need to quantify reserve uncertainty. This quantification can serve as the basis for satisfying a number of Solvency II requirements in Europe, can be used to enhance Own Risk Solvency Assessment (ORSA) reports, and is often used as an input to DFA or Dynamic Risk Models, to name but a few. Moving beyond quantification, the purpose of this paper is to explore other aspects of reserve variability which allow for a more complete integration of these key risk metrics into the larger Enterprise Risk Management framework.

Method. This paper will primarily use a case study to discuss and illustrate the process of integrating the output from periodic reserve and reserve variability analysis into the enterprise risk management process. Consequences of this approach include the production of valuable performance indicators and an increase in the lines of communication between the actuarial function and other insurance functional departments, both of which are valuable to management.

Results. By expanding the regular reserving process to include regular variability analysis and expanding the dialogue with management, the actuary can greatly contribute to the understanding of risks related to claim management within an enterprise.

Conclusions. The value of this process is not limited to reserving as it can logically and directly be extended into pricing, reinsurance optimization, etc.

Availability. In lieu of technical appendices, companion Excel workbooks are included that illustrate the calculations described in this paper. The companion materials are summarized in the Supplementary Materials section and are available at [CAS to fill in location].

Keywords. Reserve variability, enterprise risk management, actual versus expected, back-testing, deviations from expectation, one-year time horizon, validation, reserve distribution testing, assumption consistency, run-off analysis, key performance indicator.

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1. Introduction

Never has it been more important for actuaries to improve their understanding of reserve variability. Updated International Financial Reporting Standards (IFRS Phase II) will likely require all insurance companies to record an independently measured and updated risk margin. In Europe, Solvency II directives already require the recognition of a risk margin and validation standards require the Actuarial Function to comment on material deviations from prior expectations.

A range of reasonable estimates can be selected based on the results of deterministic methods, some scenario testing, and a few basic rules of thumb. Such a range, together with some heroic assumptions, can provide an unsophisticated aid to management in selecting a risk margin. More commonly, however, the calibration of risk margins makes use of modern stochastic modelling techniques, resulting in a distribution of possible outcomes,[1] with the outcomes providing the ability to measure statistical properties such as the mean, mode, percentiles, etc. There are a number of uses for the results of stochastic modelling techniques beyond the calibration of a risk margin, many of which can be incorporated for use within the Enterprise Risk Management ("ERM") process such that "new" information can be quickly used to assess performance. For example, key performance indicators ("KPIs") can be developed based on a range of percentiles around the expected outcomes.

Back-testing is a validation technique that enables the reserving actuary to assess the "new" information inherent in the loss triangles, relative to "known" information and future expectations inherent in the prior analysis. However, without an analysis of reserve variability, an assessment of the ***significance of deviations*** from expectations on both a granular level (individual accident periods) and an aggregate level (by reserving segment, by line of business, or by Company) is not quantifiable. Even with an analysis of reserve variability, bifurcating significant deviations as being the result of mean estimation error, variance estimation error, and/or random error is difficult.

[1] A distribution of possible outcomes is an expression of the "full breadth" of the possibilities of the future payouts. Note that the estimation of unpaid claims involves significant uncertainties that cannot be completely estimated, so "full breadth" should be thought of as a reasonable estimate of the distribution to the extent that it can be estimated using historical data (for independent risk) and a subjective adjustment to account for variability attributable to systemic risk. Further, the available historical data may be limited such that an adjustment to account for events not in the data ("ENID") may also be necessary. For this reason, a distribution of possible outcomes may not be possible using the most sophisticated actuarial techniques available.

A systematic back-testing process as part of a comprehensive ERM system, which uses the output of prior reserve variability analyses, significantly increases the ability of the actuary to assess deviations from expectations and provides management with an early indication of the current period's performance relative to the actuary's expectations. Further, a systematic back-testing process allows for the evaluation of the universe of deviations, relative to the distributional expectations for the current period.

Within the comprehensive ERM solution, assumption consistency becomes an important consideration. When selecting a central estimate[2] for an unpaid claim estimate, the practicing actuary commonly weights the results from multiple methods. By assigning weights to multiple methods, the actuary is partially accepting or rejecting the assumptions inherent in each method that contributes to the selection of their central estimate.[3]

Therefore the future expectation for each data element (e.g., incremental paid losses) is a weighted average of the respective expected data element from each of the methods which received weight. Likewise, the inherent uncertainty in the selected estimate is more appropriately modeled as a weighted average of the expected uncertainty in the methodology which underlies each model used to estimate uncertainty as this also helps to address model risk.[4] In contrast, an approach which uses a single model (e.g., Mack or an ODP bootstrap of the paid chain ladder method alone) to estimate the uncertainty around a point estimate based on multiple methods, uses an assumption set for the variance which is at best partially rejected during the selection of the point estimate and at worst involves assumptions which are completely different from those used for the point estimate.

This paper will develop and examine a framework for reserve distribution testing and validation and demonstrate its use with real datasets within an Enterprise Risk Management framework. It will also illustrate how stochastic results based on a one-year time horizon (as specified in Solvency II) can be used in the subsequent year's process of estimating reserves

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2 This paper uses the term "central estimate," consistent with Actuarial Standard of Practice No. 43, "Property/Casualty Unpaid Claim Estimates," promulgated by the Actuarial Standards Board [1]. With respect to Solvency II and IFRS Phase II, regulations and guidance use the term "best estimate" to mean the same thing.

3 Accepting or rejecting assumptions is a simplification of the entire process and all considerations. For example, not giving weight to a method for a specific year is not rejection of the method or any specific assumption within the method as the method may be given some weight for another year. Thus, this description of the process of weighing methods to arrive at a central estimate should be interpreted as including all considerations an actuary uses.

4 Weighting deterministic methods is also a way to address model risk. The entire process of weighting multiple models is outside the scope of this paper, but common issues (like consistency of variances between models) are assumed to have been resolved when selecting weights.

to get an early indication of the expected reserve changes due to the emergence of new information.

1.1 Research Context

The importance of assumption consistency should not be underestimated. Paragraph 3.6.2 of ASOP 43 [1] states that an actuary "should use assumptions that, in the actuary's professional judgment… are not internally inconsistent." Also note that Article 122.2 of the Solvency II Framework Directive [10] ("FD") states that models "used to calculate the probability distribution forecast shall… be consistent with the methods used to calculate technical provisions." Finally, section C from Technical Actuarial Standards: Modelling ("TAS-M") [11] states that assumptions should be consistent in "a model or in a suite of models." TAS-M further suggests that different assumptions (i.e., use of multiple methods that use different assumptions) are "not always inconsistent. For example, if several independent models are used in conjunction to provide better estimates than any one model could provide on its own, different assumptions might be chosen deliberately." If however, inconsistent assumptions are used, TAS-M requires a disclosure statement.

Actuarial literature includes a number of approaches to quantifying the uncertainty of reserve estimates based on the variability observed in the actual historical development of the claims under consideration. In practice, the most frequently used approaches are statistical approximations to relatively simple regression models. Such approaches have the advantages of being (relatively) straightforward to implement, interpret, and explain. They can be applied equally well to accident or underwriting period data to generate results on the same basis. Two regression models in particular tend to dominate: the Mack [18] linear regression model and the ODP bootstrap model originally developed by England & Verrall [7, 8].

In both cases, the expected values of the reserve estimate are equal to the results of the deterministic paid chain ladder method using the all-year volume-weighted average development factors, which is rarely the sole basis for the central estimate, especially for immature accident periods. Some practitioners of such models get around this limitation by "shifting" the modelled distribution such that the mean of the distribution is equal to the central estimate and the standard deviation from the model is maintained. The "shift" is usually implemented in an additive fashion by adding to each iteration the difference between the central estimate and the result of the paid chain ladder method (using the all-year volume-weighted average link ratios) by accident period. In order to get to the expected payments by development period, the "shift" will also need to be allocated to the incremental payments, which is often done in proportion to the overall expected average incremental payments before the shift.

As originally framed, the Mack [18] model (and by extension, the Merz-Wüthrich [19] model) provides a method for estimating a coefficient of variation ("CoV") for the reserve estimate. In order to convert the CoV into an estimate at a specific confidence level, however, it is necessary to select a particular parametric probability distribution whose parameters can be determined by the CoV together with the central estimate.

The ODP Bootstrap model originally developed by England & Verrall [7, 8] is often used in a similar manner to Mack [18] in the sense that the distributional output for the basic "chain ladder" model with paid data is "shifted" so the mean matches the central estimate. However, the ODP bootstrap approach can be extended to simulate any number of methods without requiring the selection of a particular parametric probability distribution as described in Shapland [27]. It is this approach which enables the actuary to maximize the assumption consistency between the central estimate of loss reserves and the calibration of reserve variability.

1.2 Objective

The objective of integrating loss reserve variability into the ERM process is to improve the estimation and management of loss reserves and reserving risk.

In order to manage reserve risk, one needs to measure it first. Integrating reserve risk into a continuously monitored ERM process ensures that assumptions are tracked and validated over time and that changes in assumptions are justified relative to the performance of prior assumptions.

Back-testing is a validation technique which can provide insight which improves a reserving process in that inevitable deviations from expectations are forced to be understood and future decision points (i.e., assumptions and expert judgement) can be based on the performance of past decision points.

2. Notation

The notation in this paper is from the CAS Working Party on Quantifying Variability in Reserve Estimates Summary Report [5] since it is intended to serve as a basis for further research.

Many models visualize loss data as a two-dimensional array, (w,d) with accident period or policy period w, and development age d (think w = "when" and d = "delay"). For this discussion, it is assumed that the loss information available is an "upper triangular" subset for rows $w = 1, 2, \ldots, n$ and for development ages $d = 1, 2, \ldots, n-w+1$. The "diagonal" for which $w+d$ equals the constant, k, represents the loss information for each accident period w as of accounting period k.[5]

For purposes of including tail factors, the development beyond the observed data for periods $d = n+1, n+2, \ldots, u$, where u is the ultimate time period for which any claim activity occurs – i.e., u is the period in which all claims are final and paid in full – must also be considered.

The paper uses the following notation for certain important loss statistics:

$c(w,d)$: cumulative loss from accident year w as of age d.[6]

$q(w,d)$: incremental loss for accident year w from d - 1 to d.

$c(w,n) = U(w)$: total loss from accident year w when claims are at ultimate values at time n, or with tail factors[7]

$c(w,u) = U(w)$: total loss from accident year w when claims are at ultimate values at time u.

$R(w)$: future development after age d for accident year w, i.e., = $U(w) - c(w,d)$.

$f(d)$: factor applied to $c(w,d)$ to estimate $q(w,d+1)$ or can be used more generally to indicate any factor relating to age d.

$F(d)$: factor applied to $c(w,d)$ to estimate $c(w,d+1)$ or $c(w,n)$ or can be

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5 For a more complete explanation of this two-dimensional view of the loss information, see the *Foundations of Casualty Actuarial Science* [12], Chapter 5, particularly pages 210-226.

6 The use of accident year is for ease of discussion. All of the discussion and formulas that follow could also apply to underwriting year, policy year, report year, etc. Similarly, year could also be half-year, quarter or month.

7 This would imply that claims reach their ultimate value without any tail factor. This is generalized by changing n to $n+t=u$, where t is the number of periods in the tail.

used more generally to indicate any cumulative factor relating to age d.

$G(w)$: factor relating to accident year w – capitalized to designate ultimate loss level.

$h(k)$: factor relating to the diagonal k along which $w+d$ is constant.[8]

$e(w,d)$: a random fluctuation, or error, which occurs at the w,d cell.

$E(x)$: the expectation of the random variable x.

$Var(x)$: the variance of the random variable x.

$Dist(x)$: the distribution of the random variable x.

$P_y(x)$: the y percentile of the distribution of the random variable x.

$\hat{x}$: an estimate of the parameter x.

What are called factors here could also be summands, but if factors and summands are both used, some other notation for the additive terms would be needed. The notation does not distinguish paid vs. incurred, but if this is necessary, capitalized subscripts P and I could be used.

3. Back-Testing

Back-testing is a process of comparing actual results with the expected results in order to answer the question "are the actual results better or worse than expected?" This simple question has many important nuances and ramifications, including psychological implications in the sense that people naturally tend to assume or hope for more "better than expected" back-tests than "worse than expected". While people also intuitively understand that a "worse than expected" back-test is "normal" the tendency to want more "better than expected" back-tests can creep into the initial expected results in the form of a bias for setting expectations higher than they may have otherwise been set. On the other hand, pressure to publish better financial results can push initial expectations lower.

In its simplest form a back-test can be formulated as in (3.1) for a particular incremental

[8] Some authors define $d=0,1,\ldots,n-1$ which intuitively allows $k=w$ along the diagonals, but in this case the triangle size is $n \times n-1$ which is not intuitive. With $d=1,2,\ldots,n$ defined as in this paper, the triangle size $n \times n$ is intuitive, but then $k=w+1$ along the diagonals is not as intuitive. A way to think about this which helps tie everything together is to assume the w variables are the beginning of the accident periods and the d variables are at the end of the development periods. Thus, if years are used then cell $c(n,1)$ represents accident year n evaluated at 12/31/*n*, or essentially 1/1/*n+1*.

value.

$$q(w,d) - E[\hat{q}(w,d)] \tag{3.1}$$

By subtracting the expected result from the actual result a "better than expected" result means that the actual result was less than the expected result. Somewhat counterintuitively, however, this "better than expected" result is actually a negative number.

The term "run-off" or a run-off analysis is often used interchangeably with "back-test" as the goal is to watch how actual results compare to the initial expectations. However, the run-off outcome is generally formulated as in (3.2) for a particular incremental value.

$$E[\hat{q}(w,d)] - q(w,d) \tag{3.2}$$

For the run-off test a "better than expected" result also means that the actual result was less than the expected result, but in this case the value is positive and perhaps more intuitive. Even as "back-test" and "run-off" can be used interchangeably, formulas (3.1) and (3.2) could also be interchanged between terms. For simplicity, from this point forward the paper will only refer to "back-testing" and will assume the reader can transition between terms and formulas (3.1) and (3.2) as preferred.

A back-test can be performed at either a granular or at a higher level. At a granular level, this would involve testing a single method or even a specific assumption within a method, with the goal of understanding the efficacy of that method or assumption. At a higher level the back-test will provide insight into the sum total of all methods and assumptions used to produce a final estimate. Granular level back-testing tends to be more of an academic or technical review whereas the higher level back-testing tends to focus at a management level, which is where the remainder of this paper will focus.

Within the ERM vernacular, the output of back-testing can be considered a KPI. As with other KPIs within an ERM system, information about deviations from expected outcomes provides valuable information for management.

3.1 Deterministic Back-Testing

For deterministic methods, the resulting point estimate is the sole source of the "expectation" from which to test deviations.[9] Consider, for example, the back-test results in Table 3.1. While a final back-test of the ultimate projection will be useful when all the claims

[9] For a deterministic analysis the point estimate does not contain any specific statistical meaning such as a mean, mode or median, so the term "expectation" likewise does not have any statistical connotation other than being a convenient reference to the central estimate.

are completely settled, the value of the back-test is typically drawn from the interim evaluations in order to check whether the incremental amounts are consistent with the development to date with respect to the ultimate projection.

In Table 3.1, actual accruals for accident year ("AY") 2015 are shown but expected accruals for AY 2015, and therefore differences, are not shown. This is because the 2015 calendar year ("CY") experience includes payments and case reserve changes attributable to AY 2015 and prior. The expectations, on the other hand, are based on the reserve analysis as of the prior year-end, in this case for AY 2014 and prior (i.e., as of 31 December 2014). In this paper the term "AY < CY" is used to denote the subtotal of all accident years not including the current accident year and "AY = CY" is used to denote the experience for the most recent AY which does not have a comparable expectation based on the prior reserve analysis alone.

Table 3.1 Back-Testing Example: Deterministic Actual vs. Expected

Sample Insurance Company Consolidation of All Segments Deterministic Actual vs. Expected as of December 31, 2015							
AY	Age	Actual Paid	Expected Paid	Difference	Actual Incurred	Expected Incurred	Difference
2006	120	3,069	3,701	(632)	1,863	2,158	(295)
2007	108	5,905	7,405	(1,500)	3,145	2,794	351
2008	96	8,986	10,073	(1,087)	3,553	6,142	(2,589)
2009	84	18,992	19,027	(35)	9,872	11,285	(1,413)
2010	72	51,003	47,151	3,852	25,942	26,873	(931)
2011	60	105,067	103,127	1,940	52,012	54,534	(2,522)
2012	48	202,932	194,479	8,453	106,624	106,020	604
2013	36	334,434	325,644	8,790	189,908	192,143	(2,235)
2014	24	841,484	833,793	7,691	454,217	479,073	(24,856)
2015	12	1,798,138			2,528,235		
Totals		3,370,010			3,375,371		
AY<CY		1,571,872	1,544,400	27,471	847,136	881,022	(33,886)

The "Difference" columns in Table 3.1 are calculated using formula (3.1), but like all deterministic back-tests the amounts reveal more about the direction of the outcome than the significance. Similar comparisons of actual and "expected" values are not difficult to compile for a number of other data elements (e.g., closed claims, reported claims, etc.), but while the total numbers of positive and negative deviations may be instructive it does not overcome the lack of a measure of significance. The only area where care needs to be exercised is in the calculation of the expected incremental amounts. For this, each method used should be converted into the incremental value being tested (e.g., paid claims) and then weighted together to arrive at an expectation which is consistent with the overall assumptions used to determine the selected estimate by accident period.[10] A typical short cut of multiplying the selected estimate by a selected development pattern will create a disconnection between assumptions at the macro and micro levels and should therefore be avoided.

A logical extension of this back-test is to check if the actual outcome falls within the reasonable range that was used to develop and select the central estimate. With a range, the formulation of the back-test can take the form of a percent, with a result between 0 and 100% indicating the outcome was within the range, a result greater than 100% indicating the outcome was above the range, and a result less than zero indicating the outcome was below the range.

$$\frac{q(w,d) - Min[\hat{q}(w,d)]}{Max[\hat{q}(w,d)] - Min[\hat{q}(w,d)]} \tag{3.3}$$

Continuing the example above, the back-test using a range is illustrated in Table 3.2, with the "Range Percent" columns calculated using formula (3.3).

Table 3.2 Back-Testing Example: Actual to Deterministic Range of Estimates

Sample Insurance Company Consolidation of All Segments Deterministic Actual vs. Method Range as of December 31, 2015									
AY	Age	Actual Paid	Paid Minimum	Paid Maximum	Range Percent	Actual Incurred	Incurred Minimum	Incurred Maximum	Difference
2006	120	3,069	3,701	3,704	-21075%	1,863	2,158	2,162	-6790%
2007	108	5,905	5,827	8,983	2%	3,145	1,210	4,380	61%
2008	96	8,986	9,887	10,277	-231%	3,553	5,955	6,356	-599%
2009	84	18,992	17,726	20,381	48%	9,872	9,981	12,657	-4%
2010	72	51,003	44,889	49,487	133%	25,942	24,600	29,236	29%
2011	60	105,067	100,495	106,278	79%	52,012	51,856	57,857	3%
2012	48	202,932	191,183	198,745	155%	106,624	102,222	110,845	51%
2013	36	334,434	310,031	338,355	86%	189,908	174,120	205,898	50%
2014	24	841,484	794,706	853,821	79%	454,217	436,298	503,306	27%
2015	12	1,798,138				2,528,235			
Totals		3,370,010				3,375,371			
AY<CY		1,571,872	1,481,602	1,586,896	86%	847,136	811,568	929,564	30%

The range used for this test can vary based on preferences or testing criteria. For example, the range could include only methods given some weight by accident year (the "weighted range"), the range could include all methods given weight for any accident year (the "method range"), or the range could be expanded to include methods not given any weight or scenario testing (the "possible range").

[10] The "Results – Deterministic" sheet in the "LOB Backtest.xlsm" file illustrates the process of combining weighted estimates of the incremental values consistently with the overall unpaid estimates by accident year.

While the relationship between the actual outcome and the range is a bit more instructive than the back-test of actual to "expected", unfortunately it is still more about direction than significance.

3.2 Stochastic Back-Testing

The only way to test the significance of the deviations from expected is to start with a reserve variability analysis to estimate the distribution of possible outcomes – i.e., instead of simply reviewing whether the outcome is better or worse than expected, the question becomes "is the outcome significantly different than expected?" As with a deterministic back-test, the calculation of expected values will reflect the models employed during the analysis and requires assumption consistency with the methods contributing to the selected unpaid claim estimate. More importantly, in order to dissect the efficacy of the models and assumptions used in a stochastic analysis of unpaid claims, consistency of assumptions for both mean and variance is important. As noted in Section 1.1, using multiple methods to select a point estimate and then using a single "shifted" model approach is quite inconsistent in the sense that the assumptions for the mean and variance are completely different.

Assuming that model and assumption consistency is maintained within a reserve variability analysis, the assessment of the significance or materiality of the resulting differences is a straightforward process using a percentile function. Formula (3.4) uses the Excel PERCENTRANK.INC function, but percentile functions for other software would be similar.[11]

$$P_x[q(w,d)] = \text{PERCENTRANK.INC}\{Dist[\hat{q}(w,d)], q(w,d)\} \tag{3.4}$$

Like for the deterministic back-test, the only area where care needs to be exercised is in the development of the distributions for each incremental value. The output of stochastic models may only include the simulations for the totals by year, but most software will include the simulations of incremental amounts as an output option. Assuming the incremental simulations are available, then the only issue remaining is to insure that the incremental output has been weighted and shifted consistently with the overall model

[11] In Excel, the =PERCENTRANK.INC(*Array,X*) function has two required parameters, *Array*, which is the range of values which can be used to determine relative standing within the range and, *X*, which is the value for which you want to determine the rank. The function returns the rank of *X* within the *Array* as a percentage (0, 1, inclusive) of the range of values.

assumptions.[12]

For the examples used in this paper a reserve variability analysis was completed using four variations of the ODP bootstrap model (i.e., Paid Chain Ladder, Incurred Chain Ladder, Paid Bornhuetter-Ferguson, Incurred Bornhuetter-Ferguson), including weighting and shifting to match the assumptions and unpaid claim estimates for a deterministic analysis using the same methods in order to estimate the expected distribution of possible outcomes. The approach was used for three sample reserving segments and correlated to derive an aggregate distribution in order to illustrate the process for a whole company.[13]

Table 3.3 Back-Test Example: Stochastic Actual vs. Expected

Sample Insurance Company Aggregation of All Segments Stochastic Actual vs. Expected as of December 31, 2015							
AY	Age	Actual Paid	Expected Paid	Percentile	Actual Incurred	Expected Incurred	Percentile
2006	120	3,069	4,077	31.8%	1,863	2,115	49.8%
2007	108	5,905	6,163	47.9%	3,145	1,819	80.6%
2008	96	8,986	10,176	33.6%	3,553	6,026	20.9%
2009	84	18,992	20,033	39.0%	9,872	10,399	46.3%
2010	72	51,003	48,298	71.6%	25,942	25,562	55.3%
2011	60	105,067	104,415	54.3%	52,012	53,101	44.8%
2012	48	202,932	196,083	74.2%	106,624	104,075	61.7%
2013	36	334,434	331,701	57.1%	189,908	185,173	64.0%
2014	24	841,484	839,689	52.8%	454,217	469,822	29.3%
2015	12	1,798,138			2,528,235		
Totals		3,370,010			3,375,371		
AY<CY		1,571,872	1,560,637	61.2%	847,136	858,093	37.6%

Large (small) deviations between actual and expected values are expected when a reserve variability analysis concludes that uncertainty is high (low). The use of an expected distribution of possible outcomes for each accident period and in total (i.e. AY < CY) implies that the use of percentiles automatically adjusts for differences in uncertainty by year or segment as illustrated in Table 3.3.

Note that for simplicity the examples and case study do not include an expected distribution of possible outcomes for most recent accident period (i.e., AY = CY), as this would require modeling that is generally not included in the reserving analysis for the prior period. However, if the reserving analysis is extended to include a distribution of the next

[12] For a useful reference see Shapland [27]. The "RawSimResults" sheets in the "LOB Backtest.xlsm" file assume that the incremental output by year and by iteration has been weighted and shifted as described in Shapland [27].

[13] While the terms can be used interchangeably, in this paper "consolidation" is used to mean a deterministic sum of the parts or segments whereas "aggregation" is used to mean the stochastic correlation of the parts or segments.

accident year (perhaps in a "pricing risk" calibration) then this could be included with the back-test. The only caveat to the inclusion of pricing risk is that it will be based on expectations of future exposures, so any back-test should first adjust the distribution for the actual exposures prior to calculation of percentiles in order to more properly compare these once future exposures to all the prior years which were based on actual exposures.

Deviations expressed as a percentile provide an indication as to the materiality. Note that deviations expressed as extreme percentiles do not necessarily indicate a problem with the methodology employed during the prior analysis, as observations at the extreme levels of a distribution of possible outcomes should occur.

3.3 Stochastic Key Performance Indicators

Reviewing a single percentile is instructive, but hardly useful. In the greater scheme of determining materiality, the single observation is more about random noise than materiality. Only with a large number of observations can the analyst start to detect material issues by observing patterns or biases in the percentiles. It is in the detection of patterns that the key performance indicators add value to the stochastic analysis. Consider for example Figure 3.1 which graphically displays pre-defined thresholds which are used to define stochastic KPI thresholds.

Figure 3.1 Pre-defined KPI thresholds

0%	5%	25%	75%	95%	100%

As illustrated in Figure 3.1, the case study in this paper uses thresholds at the 25th and 75th percentile, the 5th and 95th percentile, as well as the simulated minimum and maximum of the distribution of possible outcomes to denote material deviations from expected. Such deviations can be communicated visually using a table of numbers (see Tables 3.3 and 5.10), a chart of individual accident periods (see Figures 3.2a and 3.2b), or a chart of the total calendar year – i.e., all accident years combined (see Figures 3.3a and 3.3b).

Figure 3.2a Paid KPI Thresholds by Accident Year

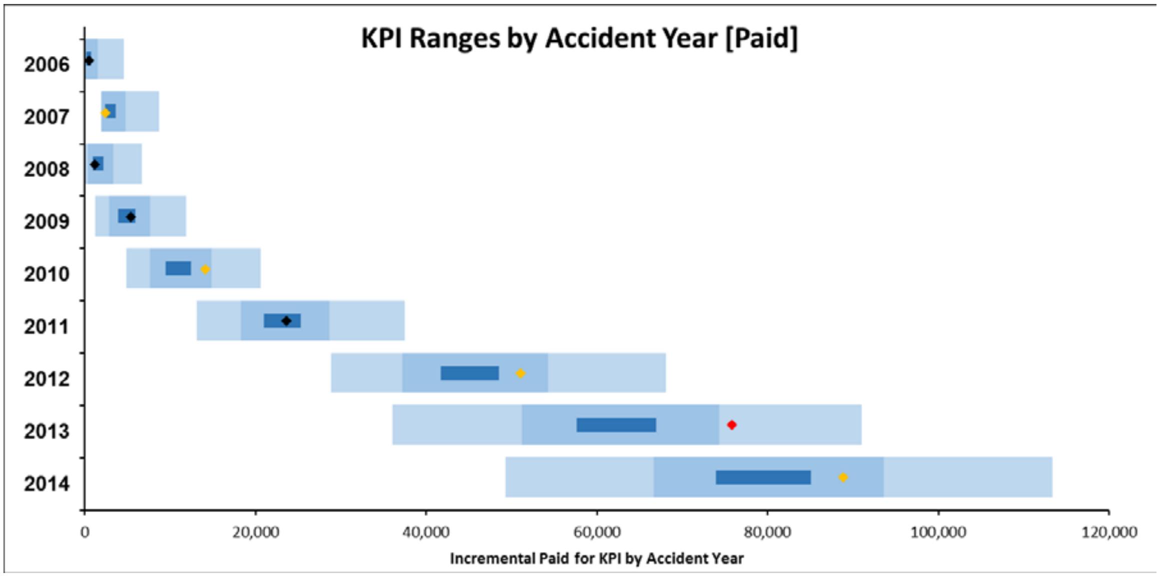
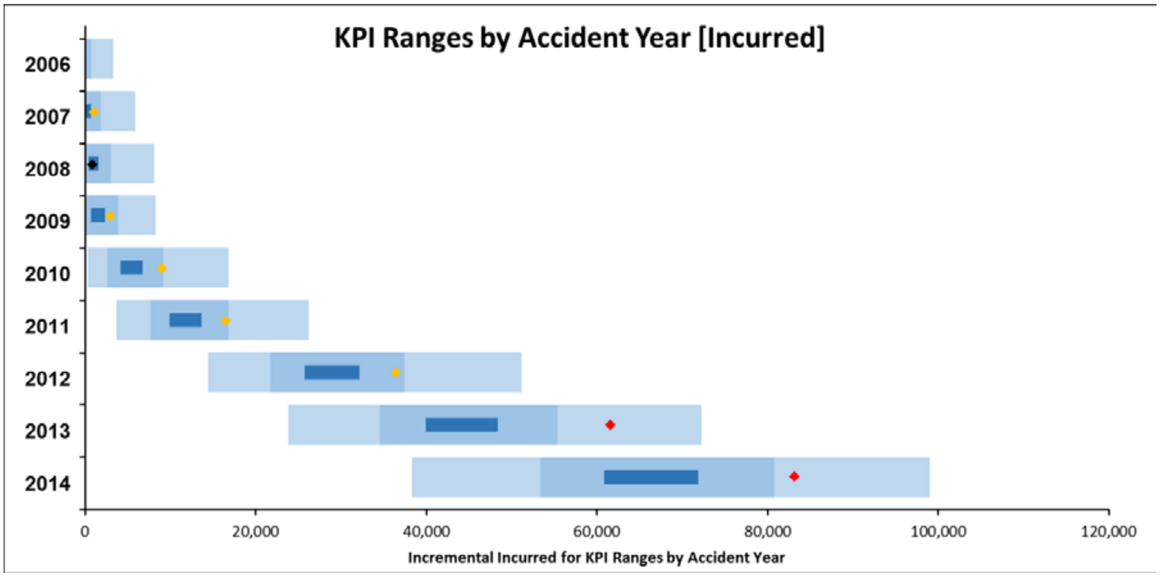


Figure 3.2b Incurred KPI Thresholds by Accident Year



Figures 3.2a and 3.2b show where the actual incremental paid and actual incremental incurred by accident year for a single reserving segment; the black, orange, and red points, fall within the thresholds of the expected distribution of possible outcomes. Note that the blue color coded areas represent the areas defined by the pre-defined thresholds as defined in Figure 3.1.

Figures 3.3a and 3.3b show where the actual incremental paid and actual incremental incurred for the calendar year (i.e., all accident years AY < CY) for a Segment; the orange and red points, fall within the expected distribution of possible outcomes. Again, the blue color coded areas represent the areas defined by the pre-defined thresholds.

Figure 3.3a Calendar Year Paid KPI

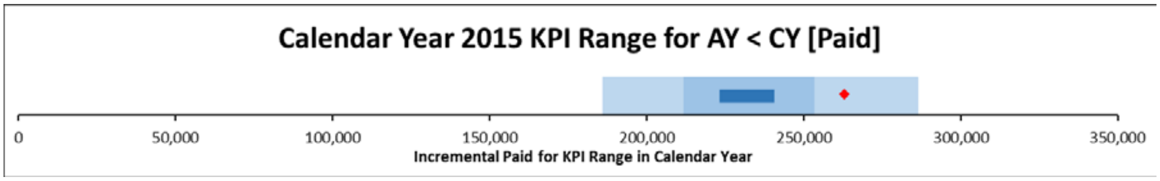
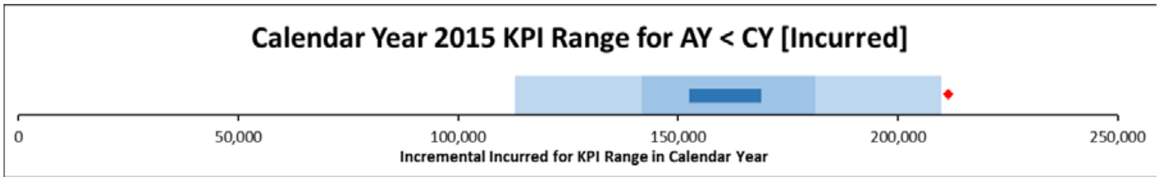


Figure 3.3b Calendar Year Incurred KPI



When using tables or charts, the materiality of the deviation can be better understood by using color coded fonts (see Tables 3.3 and 5.10) or color coded areas representing breaches of pre-defined thresholds (see Figure 3.1) within the distribution of possible outcomes.

There are caveats to this approach such as:

1. Various assumptions (each requiring validation) need to be made in order to produce a distribution of possible outcomes (distributional predictions);
2. The approach tends to work well for high frequency segments on a gross of reinsurance basis but not necessarily for low frequency segments or on a net or ceded basis; and
3. Analysis of industry performance over the past few decades show that some ODP bootstrap model variations, absent adjustment for model weaknesses, may underestimate reserve risk (i.e. the distribution of possible outcomes could be wider).

4. Reserving Within an ERM Framework

There are numerous definitions of ERM. The common themes and principles that emerge from the various definitions, as summarized by the 2016 International Actuarial Association paper [16] "Actuarial Aspects of ERM for Insurance Companies," are:

1. ERM is a continuous process;
2. ERM adopts a holistic view to risk and assesses risk from the perspective of the company's aggregate position as well as from a standalone perspective;

3. ERM is concerned with all risks, including those that are unquantifiable or difficult to quantify;
4. ERM considers uncertainty from both a positive and negative viewpoint;
5. ERM aims to achieve greater value for all stakeholders by assisting in achieving an appropriate risk-reward balance; and
6. ERM considers both the short term and the long term aspects of risk.

Key components of a company's ERM system include risk governance, risk strategy, and the steps that make up the core risk management process consisting of risk identification, risk assessment, risk measurement, risk response, risk monitoring and risk reporting.

Risk governance generally includes the assignment of roles and responsibilities, the establishment of risk policies and procedures, robust internal control systems, and risk culture. For the assignment of roles and responsibilities, many companies adopt a "three lines of defense" model. The first line is responsible for the regular operations of the business. The second line is responsible for overseeing of the operations of the first line. Finally, the third line is responsible for independent review (i.e., audit) and assurance of the operations of the first and second lines.

Once risk has been identified, analyzed and measured then management is faced with responding to the risks. Responses are often characterized as avoiding, accepting, mitigating, or sharing.

The ERM process does not change the way that an actuarial function manages loss reserves and the corresponding reserving risk. Rather, the ERM process formalizes the governance around the process and ensures a consistent and continuous approach. In the case study below, one such approach is described. With or without an ERM process, the actuarial function within an insurance entity is responsible for the reliability and adequacy of the calculation of loss reserves, including:

- Promptly reporting major deviations from expectations such that management has the relevant information necessary for the decision-making process; and
- Investigating the causes of deviations such that changes to the assumptions and methodologies can be suggested in order to improve the central estimate of loss reserves.

The ERM process adds a change control process such that unauthorized changes to the model are restricted and changes are documented.

Risk monitoring is linked to risk measurement and reporting in that the quality of measurement and reporting often determines the extent of monitoring possible. In the case study below, a high quality measurement process which increases the scope of typical monitoring of loss reserves is described, including:

- Clear assignment of risk ownership and establishment of timely automatic reporting mechanisms;
- Consistent, accurate, and auditable controlling of both the deterministic method(s) and methodology supporting the selected central estimate, and the stochastic model(s) supporting the corresponding reserve uncertainty conclusion in the form of an expected distribution of possible outcomes;
- Producing metrics than an actuarial function can use to identify deviations from prior expectations and efficiently allocate analysis resources, prior to commencing with the current analysis;
- Allowing for analysis resources to hypothesize and monitor whether deviations from expectations are the result of mean estimation error, variance estimation error, or random error;
- Producing performance indicators that management can use to anticipate the conclusions of the actuarial analyses, based on how the prior assumptions have held up; and
- Expanding the discussion to interested parties outside of the actuarial function, regarding major deviations from expectations.

Monitoring would normally be done with a frequency that is appropriate to the risk in question. Monitoring should be sufficiently frequent to allow decisions to be made and for action to be taken on an informed basis. In the case study below, a process that uses annual analyses is described, which is typical, but a more frequent basis can be similarly achieved as long as the data and processes are established accordingly.

5. Enterprise Risk Management in Action – A Case Study

With the foundation established, the rest of the paper will illustrate the advantages of integrating reserve variability into the Enterprise Risk Management system by using a case study. Summary tables and graphs for each LOB and the aggregate results are shown in Appendices C, D, E, and F, respectively.

5.1 Introduction

The case study presents the work cycle for an actuarial function within a sophisticated ERM system, including a more robust estimation process for the unpaid claim estimates (i.e., loss reserves) as of 31 December 2015. To set the stage, a general timeline of activity is established before presenting the details.

- Prior to year-end 2015: Levels of back-testing granularity are defined[14] to be Entity Total, Segment Total (where Entity Total = ΣSegment), and AY for each Segment (where Segment Total = Σ AY for each Segment).[15]
- Prior to year-end 2015: Two levels of thresholds are defined,[16] such that observations in the 5% tail areas (i.e., less than the 5th percentile and greater than the 95th percentile) and 25% tail areas initiate action.[17]
- Prior to year-end 2015: Elements included in the automatic back-testing system are defined to include paid loss and incurred loss. ***Other elements, such as reported and closed claim counts, could be included in a live system but they are excluded here for simplicity.***
- Prior to year-end 2015: Enhanced documentation standards[18] of assumptions and expert judgement are established for the analysis and validation of each reserving segment.[19]

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[14] Note that changes in the segmentation, and the ramifications to the ERM system, need to be thoroughly addressed prior to the year-end.

[15] Note that it is often more practical to exclude special Segments and very mature AYs, such that "Entity Total = ΣSegment + excluded segments" and "Segment Total = Σ AY for each Segment + prior AYs".

[16] Note that thresholds could be nominal (e.g., differences larger than $1 million), relative (e.g., differences 150% larger than the mean expected), or distributional (e.g., observations above the 95th percentile of possible future outcomes).

[17] Note that the identification of a threshold breach does not imply that an error in the prior calculation has been identified. Rather, a breach brings attention to large deviations such that the assumptions and methodology underlying the expectation can be reviewed.

[18] Note that enhanced documentation includes a list of relevant and material assumptions for each segment, the results of sensitivity testing material assumptions, segment specific diagnostics with qualitative descriptions supporting the conclusions, and justification (if available) for material expert judgement exercised.

[19] Note that enhanced documentation together with the automated back-testing ensures that a change in employee personnel does not unnecessarily render the historical assumption set and rationale less transparent or understandable (i.e., the institutional memory stays intact.)

- 4 January 2016: The accounting function closes the books such that all data elements as of the 31 December 2015 valuation date are available on an AY and CY basis.
- 5 January 2016: Granular results of automated back-testing of the current CY (i.e., CY 2015) and deviations[20] from the predictions for CY 2015 (based on the loss reserve analysis as of 31 December 2014) are available.
  - Previously identified segments (or previously identified data elements from a segment) are included in the automated back-testing procedure where a robust validation of the CY 2015 accruals can be achieved.
  - AY 2014 and prior incremental accruals (i.e., AY < CY) are compared to the expectations as of 31 December 2014, based on the final distribution of possible outcomes estimated by the actuarial function in the prior reserving analysis. ***The process can be expanded to include specific models, but that is not done here only for simplicity***.
  - AY 2015 incremental accruals (i.e., AY = CY) can be compared to the expectations for losses related to the unearned premium as of 31 December 2014, with adjustment for actual new business written during 2015. ***For simplicity, these amounts are not included in the details of the case study shown below, although it should be noted that deviations from expectations can be described as a mixture of reserve risk and premium risk.***
- 5 January 2016: The actuarial function determines an efficient allocation of analysis resources so that segments and/or AYs which exhibit a large number of significant deviations receive additional attention.
- 5 January 2016: Breaches in the 25% tail areas initiate additional hindsight analysis including hypothesis testing as to whether the breach could have been caused by an assumption error in either the deterministic or stochastic analysis, a systematic effect (e.g., an explainable change in the internal or external environment), or random variation.
- 5 January 2016: Breaches in the 5% tail areas initiate an alert system intended to collect relevant information from other departments (e.g., data quality, underwriting, claims, and reinsurance).

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[20] The automated back-test identifies areas where the deviations from predictions breach a pre-defined threshold (for multiple levels of granularity and for multiple data elements.)

- 5 January 2016: Conditional reserve estimates using the 1-year time horizon analysis as of 31 December 2014 are available to management as an early indication of the reserve changes that will occur for the 31 December 2015 evaluation. (See Appendix A for an overview of the one-year time horizon.)
- 5 January 2016: Armed with a view of how each segment performed during CY 2015, relative to the expectations inherent in the actuarial methodology as of 31 December 2014, the actuarial function can commence with its valuation analysis as of 31 December 2015.
- 5-26 January 2016: During the analysis, diagnostics and statistical tools are used to review assumptions and calibrate the parameters of each of the methods and models which comprise the segment's methodology. Such diagnostics and tests are retained in a log so that they can be referenced in the actuarial report. Also interaction with interested parties outside of the actuarial function provide a critical sounding board for expert judgement exercised.
- 27 January 2016: At the conclusion of the analysis a recommendation for the loss reserve is sent to management, taking the form of an actuarial function report.
- 10 February 2016: After the dust settles, the expectations for CY 2016 are compiled by the actuarial function, based on the expectations inherent in the analysis as of 31 December 2015. Further analyses of change are completed and documented. Suggestions for the enhancement of the robust estimation process for CY 2016 (levels of granularity, thresholds, data elements, diagnostic retention and other enhanced documentation) are considered, based on the performance and the collective findings of the analysis.

5.2 Basis of Underlying Data

In producing this case study real industry data was used.[21] To ensure confidentiality, triangular data for 10 accident years was aggregated from a small number of insurance entities writing Commercial Auto ("CA"), Private Passenger Auto ("PPA"), and Homeowners ("HO"), as of consecutive year-ends. This produced a data set for a fictitious entity.

By performing a deterministic and stochastic analysis on the annual data for this fictitious

[21] The data comes from historical Schedule P triangles, as compiled by SNL Financial.

entity, an exercise which is often undertaken by actuarial departments every year-end, the case study attempts to highlight the wealth of information that is ripe for integration within an ERM framework to enhance the understanding of the underlying dynamics, including the production of KPIs for reserving risk.

The deterministic analysis was limited to four methods, namely: the paid and incurred chain ladder ("Pd CL" and "Inc CL") methods and the paid and incurred Bornhuetter-Ferguson ("Pd BF" and "Inc BF") methods. The selected ultimate loss estimates for each accident year are a weighted average of the four methods. To maximize assumption consistency, four ODP bootstrap models consistent with each of the deterministic methods were used. The selected distribution of possible outcomes for each accident year are a weighted average of the four ODP bootstrap models (using the same weights as for the deterministic methods),[22] shifted such that the mean of the distribution for each accident year is equal to the selected unpaid loss.

It is reasonable to expect that the underlying data within the fictitious entity would be available by the first Monday of the year (4 January 2016) and that the generous management of the fictitious entity allows the actuarial department to spend three weeks in completing its work. Within such tight schedules, the importance of activity before the year-end is emphasized, which calibrates the framework such that diagnostics and KPIs are produced as soon as the underlying data is available.

In the case study, the diagnostics and KPIs focus on the performance of the most recent period (i.e., the past CY). The framework and approach can just as easily focus on multiple periods, which for some reserving segments would be appropriate. The multiple period approach provides insight that could be used to reduce unnecessary adjustments in the underlying actuarial assumptions (i.e., additional volatility caused by overreaction to single period observations).

5.3 Validation of the Prior Analysis

As noted above, enhanced documentation standards of assumptions and expert judgement are established for the analysis and validation of each reserving segment. A non-

[22] Note that weighting distributions together requires that possible outcomes mean the same thing in each model. For example, the unadjusted output for an ODP bootstrap model applied to a paid (an incurred) loss triangle would result in a distribution of possible unpaid loss (IBNR) outcomes. Prior to weighting, the incurred ODP bootstrap models implemented were adjusted such that the outputs were distributions of possible unpaid loss outcomes as described in Shapland [27].

exhaustive list of assumptions that require validation and examples of enhanced documentation could include the following:

5.3.1 Selected Loss Development Factors ("LDFs")

The Mack [18] paper introduced three assumptions which underlie the chain ladder method, the first two of which are validated as part of the enhanced documentation for the fictitious entity.

$$E[c(w,d+1) \mid c(w,1),...,c(w,d)] = c(w,d) \times F(d) \quad (5.1)$$

$$\{c(i,1),...,c(i,n)\} \;\&\; \{c(j,1),...,c(j,n)\} \text{ are independent for } i \neq j \quad (5.2)$$

$$Var[c(w,d+1) \mid c(w,1),...,c(w,d)] = c(w,d) \times \sigma_d^2 \quad (5.3)$$

Assumption (5.1) says that the all year loss weighted average ("AYLWA") multiplied by the value in the last diagonal is equivalent to the expected value of the next diagonal given the observations to date. The validation test for this assumption (shown in Figures 5.1 and 5.2) compares the LDF which is a regression through the origin (red line) relative to an alternative approach that uses an intercept term (green line).[23] If the regression with an intercept is not significantly different than the regression through the origin, then the LDF is validated.

[23] A more complete exposition of tests which can be used to validate the three Mack assumptions are provided in Venter [29]. The graphs in Figures 5.1, 5.2, 5.3 and 5.4 were created using the "Bootstrap Models.xlsm" companion Excel file for Shapland [27].

Table 5.1 Commercial Auto: Chain Ladder Methods

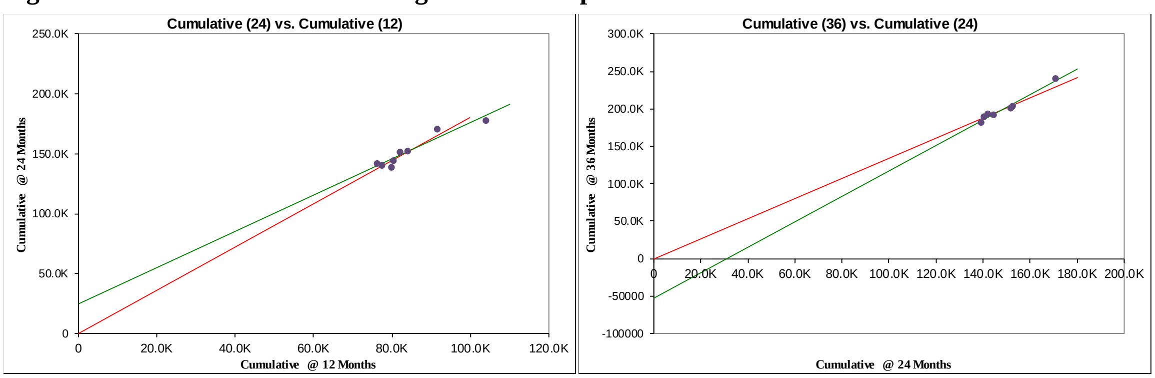
Sample Insurance Company
Commercial Auto -- Paid Data
Chain Ladder Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120
2006	77,401	140,425	189,316	223,326	243,182	250,182	254,305	256,672	257,689	
2007	76,085	142,122	193,196	224,406	246,220	257,226	263,698	264,871		
2008	79,850	139,041	181,905	209,366	228,012	237,792	240,300			
2009	80,323	144,482	192,134	227,723	249,165	259,339				
2010	83,919	152,487	203,761	245,150	270,525					
2011	82,001	151,768	201,189	245,541						
2012	91,514	170,696	240,652							
2013	103,957	177,709								
2014	105,547									
	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-132
ATA	1.805	1.347	1.184	1.095	1.039	1.018	1.007	1.004	1.002	1.002
CDF	3.385	1.875	1.392	1.176	1.074	1.033	1.015	1.008	1.004	1.002
Unpaid	0.705	0.467	0.282	0.149	0.069	0.032	0.015	0.008	0.004	0.002

Sample Insurance Company
Commercial Auto -- Incurred Data
Chain Ladder Development as of December 31, 2014

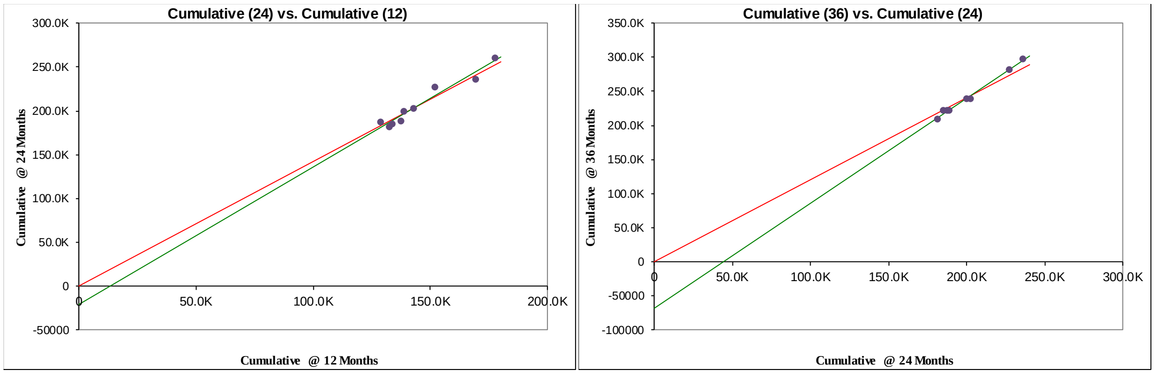
AY	12	24	36	48	60	72	84	96	108	120
2006	133,521	185,161	221,635	241,420	251,646	255,508	256,596	258,041	258,524	
2007	128,727	187,403	222,093	247,345	258,712	265,636	269,558	270,758		
2008	132,567	181,263	209,262	226,237	236,863	241,107	242,171			
2009	137,295	188,962	222,624	247,335	258,856	265,496				
2010	142,862	202,363	239,239	269,940	281,376					
2011	138,650	199,791	239,719	266,101						
2012	151,778	227,353	282,394							
2013	169,171	235,983								
2014	177,611									
	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-132
ATA	1.418	1.193	1.106	1.045	1.022	1.008	1.005	1.002	1.001	1.001
CDF	2.029	1.431	1.200	1.085	1.038	1.016	1.008	1.003	1.001	1.001
Unrptd	0.507	0.301	0.166	0.078	0.037	0.016	0.008	0.003	0.001	0.001

Figure 5.1 Commercial Auto: Testing the first two paid LDFs



Corr. = 0.952 P-Value = 0.000 Int. P-Value = 0.045 Corr. = 0.985 P-Value = 0.000 Int. P-Value = 0.004

Figure 5.2 Commercial Auto: Testing the first two incurred LDFs



Corr. = 0.974 P-Value = 0.000 Int. P-Value = 0.005 Corr. = 0.996 P-Value = 0.000 Int. P-Value = 0.000

For the fictitious entity, the LDFs were validated, so the CL methods using the AYLWA are reasonable. Note that each ODP bootstrap model is 100% consistent with using the AYLWA for the deterministic method, so none of the residuals were removed (i.e., no outliers were selected in the calibration of the ODP bootstrap models). The a priori loss ratios and tail factors used in the ODP bootstrap models were also consistent, except that variance assumptions were also added.

Note that the implementation of a "picker approach" (to reflect observable trends) in selecting LDFs would necessitate additional validation of each "pick" and consideration of consistent treatment of the residuals in the calibration of the ODP bootstrap model, but that was not done in the case study in keeping with the theme of simplicity.

5.3.2 Accident Year Independence

Regarding assumption (5.2), the independence of the accident years can be validated using a table of the individual LDFs and color coding the LDFs which are smaller (green shading) or larger (red shading) than the median LDF for each development period, as illustrated in Figure 5.3. This color coding aids in searching for patterns in the LDFs which could indicate that they are not independent. For example, the independence assumption could be violated if there were a strong diagonal trend, or clustering, of one of the colors.

Figure 5.3 Commercial Auto: Testing independence of accident years

Test of the Independence Between Accident Years (Paid)

AY	12	24	36	48	60	72	84	96	CY Small	CY Large
2006	1.81	1.35	1.18	1.09	1.03	1.02	1.01	1.00	1	0
2007	1.87	1.36	1.16	1.10	1.04	1.03	1.00		0	2
2008	1.74	1.31	1.15	1.09	1.04	1.01			2	1
2009	1.80	1.33	1.19	1.09	1.04				4	0
2010	1.82	1.34	1.20	1.10					3	2
2011	1.85	1.33	1.22						1	3
2012	1.87	1.41							1	5
2013	1.71								4	3
Median	1.82	1.34	1.18	1.09	1.04	1.02	1.01	1.00		

Test of the Independence Between Accident Years (Incurred)

AY	12	24	36	48	60	72	84	96	CY Small	CY Large
2006	1.39	1.20	1.09	1.04	1.02	1.00	1.01	1.00	1	0
2007	1.46	1.19	1.11	1.05	1.03	1.01	1.00		0	2
2008	1.37	1.15	1.08	1.05	1.02	1.00			2	0
2009	1.38	1.18	1.11	1.05	1.03				3	1
2010	1.42	1.18	1.13	1.04					3	1
2011	1.44	1.20	1.11						2	4
2012	1.50	1.24							1	6
2013	1.39								4	2
Median	1.41	1.19	1.11	1.05	1.02	1.00	1.01	1.00		

In practice, the independence of the accident years can be distorted by certain calendar year effects like major changes in the claims handling process or in case reserve strengthening.

5.3.3 A Priori BF Loss Ratios ("IELR")

In the case study, the a priori or initial expected loss ratios ("IELR") used in the BF methods were based on published figures (i.e., selected ultimate loss amounts from Schedule P), expressed as a percentage of premium. IELRs are an important assumption and an example of expert judgement which requires additional validation.

Table 5.2 Commercial Auto: IELRs

Sample Insurance Company Commercial Auto				
AY	Paid CL ULR	Inc CL ULR	Management IELR	Selected ULR
2006	73.2%	73.2%	73.3%	73.2%
2007	76.0%	77.3%	77.4%	76.7%
2008	64.5%	64.5%	64.6%	64.5%
2009	62.8%	63.2%	63.2%	63.0%
2010	60.4%	60.7%	60.8%	60.6%
2011	53.2%	53.2%	53.4%	53.2%
2012	57.9%	58.5%	58.5%	58.2%
2013	54.5%	55.3%	54.7%	54.9%
2014	57.3%	57.7%	52.9%	54.7%

Validation, in this case, would likely take the form of sensitivity testing the important assumptions underlying the IELR. The common sources of expert judgement in this case would be renewal studies performed by the underwriting department and actuarial analyses summarizing average premium levels achieved relative to the expected premium level.

5.3.4 Weighting Scheme

No single method is perfect. For this reason, it has become best practice for actuaries estimating an insurer's unpaid claim estimate to review and assess the merits of multiple methods for each reserving segment in the actuarial analysis.

Traditional unpaid claim projection methods are generally based on averages that produce an indication of the unpaid claims reserves or a "reasonable estimate" for each accident period and in total. The results of these methods, being based on different data and assumptions, give different answers. For example, chain ladder approaches applied to aggregate paid losses and aggregate incurred losses will produce different estimates of ultimate losses for each accident period and in total.

Expert judgement supported by tangential information (e.g., expected loss ratios, severities, and frequencies from underwriting and claims experts) can be helpful in the reconciliation of the results from various methods. The reconciliation of the method results is a process where an actuary investigates and rationalizes large differences at a granular level (i.e., by reserving segment and accident period) in the results from multiple methods.

Although the reconciliation process is generally a source of significant insight, a common outcome is that a subset of implemented methods each produce different but reasonable outcomes for a given accident period. In this case, the actuary often chooses to credibility weight the results of the methods which have produced reasonable results, rather than selecting a single method for that accident period.

Estimates for immature accident periods benefit from expert judgement supported by tangential information. For these accident periods, payments are few and case reserves are based on incomplete information, which means that chain ladder methods can be easily distorted by the behavior of a few claims. As accident periods mature, the actuary tends to rely more on period-specific information as found in chain ladder methods. This is because settlement amounts are known for closed claims and future payments for open claims become more predictable as more claim specific information is collected (e.g., loss survey, repair estimates, details of injury).

Table 5.3 Commercial Auto: Weighting scheme

Sample Insurance Company
Commercial Auto
Calculation of Weighted Ultimate as of December 31, 2014

		Ultimate Values by Method				Weights by Method				Weighted
AY	Age	Paid CL	Inc CL	Paid BF	Inc BF	Paid CL	Inc CL	Paid BF	Inc BF	Ultimate
2006	108	258,835	258,835	258,837	258,836	50.0%	50.0%	0.0%	0.0%	258,835
2007	96	267,103	271,591	267,143	271,592	50.0%	50.0%	0.0%	0.0%	269,347
2008	84	243,981	244,137	243,991	244,141	50.0%	50.0%	0.0%	0.0%	244,059
2009	72	267,942	269,784	267,999	269,783	50.0%	50.0%	0.0%	0.0%	268,863
2010	60	290,475	292,079	290,608	292,092	50.0%	50.0%	0.0%	0.0%	291,277
2011	48	288,645	288,592	288,785	288,669	50.0%	50.0%	0.0%	0.0%	288,618
2012	36	335,023	338,775	335,956	338,702	25.0%	25.0%	25.0%	25.0%	337,114
2013	24	333,220	337,698	333,662	336,635	0.0%	0.0%	50.0%	50.0%	335,149
2014	12	357,305	360,286	338,097	344,953	0.0%	0.0%	50.0%	50.0%	341,525
Totals		2,642,529	2,661,779	2,625,078	2,645,402					2,634,788

As illustrated in Table 5.3, the selection of a weighting scheme is an example of exercising expert judgement, which should be adequately documented, including: the inputs on which the judgement is based; the objectives and decision criteria; the materiality of the expert judgement made; any material limitations and the steps taken to mitigate the effect of these limitations; and the validation carried out for the expert judgement. Other selections based on expert judgment should also be adequately documented.

Article 77 of the Solvency II FD states that the "value of technical provisions shall be equal to the sum of a best estimate and a risk margin." Ignoring discounting and the risk margin for the purposes of this case study, the best estimate is further defined to correspond to the "probability weighted average of future cash flows."[24] Note that Article 122.2 of the

[24] A strong interpretation of the required correspondence to a probability weighted average of future cash flows is that a "distribution of possible outcomes" needs to be modelled. Note that deriving such a distribution of possible outcomes may not be possible using even the most sophisticated actuarial techniques available. The best attempt at such, however, would require the consideration of multiple (deterministic) methods and multiple (stochastic) models in order to calibrate a distribution of possible outcomes. In addition, such a distribution would require consideration of systemic risks that may not have been adequately modelled otherwise. A weaker interpretation of the required correspondence to a probability weighted average of future cash flows is that each actuarial method produces future cash

FD ensures that models "used to calculate the probability distribution forecast shall… be consistent with the methods used to calculate technical provisions." Consistency would include elements of expert judgement exercised by the actuary during the calculation of technical provisions, including the use of shorter term average development factors, adjustment for trends, etc.

5.3.4 Other Manual Adjustments

It can happen that adjustments to the ultimate loss estimate are implemented based on (i.e., after) the weighting of multiple methods or models. In the case study, the weighting of paid and incurred chain ladder methods for accident year 2007 results in an IBNR value less than 0 for Commercial Auto. Such a scenario implies that the case reserve may be redundant. The suggested course of action is to interact directly with the claims team, if possible, to determine the likelihood of this conclusion. For purposes of the case study, a small IBNR has been added and the consequences of this decision is included in the expected values of the subsequent year's back-test as illustrated in Table 5.4. Throughout the tables in the "LOB Backtest.xlsm" file, deviations from the weighted results are highlighted in green.

Table 5.4 Commercial Auto: Manual Adjustment of Accident Year 2007

Sample Insurance Company Commercial Auto Total Unpaid Reconciliation as of December 31, 2014										
AY	Age	Paid to Date	Incurred to Date	Weighted Ultimate	Case Reserve	IBNR	Total Unpaid	Selected Ultimate	Selected IBNR	Total Unpaid
2006	108	257,689	258,524	258,835	835	311	1,146	258,835	311	1,146
2007	96	264,871	270,758	269,347	5,887	(1,411)	4,476	**271,500**	**742**	**6,629**
2008	84	240,300	242,171	244,059	1,871	1,888	3,759	244,059	1,888	3,759
2009	72	259,339	265,496	268,863	6,157	3,367	9,524	268,863	3,367	9,524
2010	60	270,525	281,376	291,277	10,851	9,901	20,752	291,277	9,901	20,752
2011	48	245,541	266,101	288,618	20,560	22,517	43,077	288,618	22,517	43,077
2012	36	240,652	282,394	337,114	41,742	54,720	96,462	337,114	54,720	96,462
2013	24	177,709	235,983	335,149	58,274	99,166	157,440	335,149	99,166	157,440
2014	12	105,547	177,611	341,525	72,064	163,914	235,978	341,525	163,914	235,978
Totals		2,062,173	2,280,414	2,634,788	218,241	354,374	572,615	2,636,941	356,527	574,768

5.3.5 Coefficient of Variation of the IELR

In the case study, the uncertainty in the IELR is required as an input to the ODP bootstrap for the BF models and was calibrated to follow a lognormal distribution with a Coefficient of Variation ("CoV") of 8%. The purpose of this assumption is to include uncertainty in the IELR by simulating from a lognormal distribution a different IELR for each iteration.

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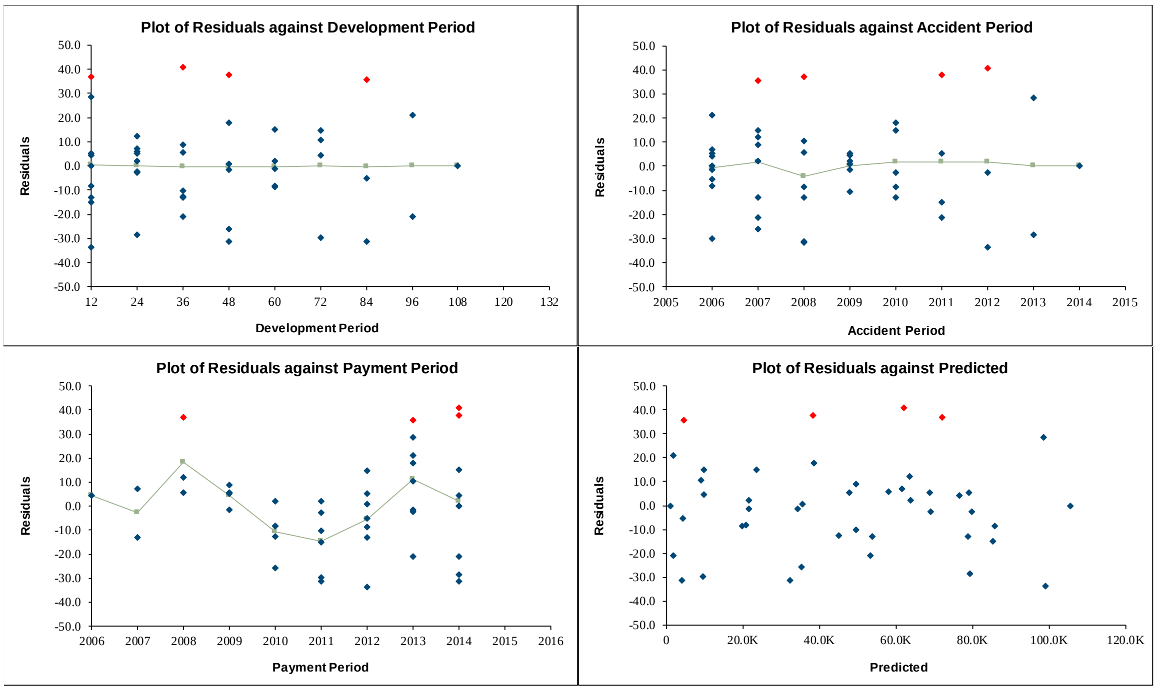
flows unique to the assumptions underlying the respective method as applied to an accident period and reserving segment. These competing cash flow projections are weighted together based on the subjective credibility assigned to each accident period of each method.

5.3.6 Heteroscedasticity

An analysis of residuals by itself is an example of a validation technique. For the case study, the residuals are analyzed to identify trends or other features in the data that may not be completely modeled by the chain ladder approach.

Figure 5.4 Commercial Auto: Plots of Residuals (Paid)



Particularly important are the identification of heteroscedasticity and outliers. In the ODP bootstrap model,[25] residuals are resampled with replacement – that is, they are taken from any location in the residual triangle, and placed in another random location to form the sample triangle. Therefore, the residuals should all be independent, identically distributed random numbers (i.e., homoscedastic). Heteroscedasticity occurs when the residuals are not identically distributed. By looking at the variability of the residuals by period (e.g., by accident year) you can visually inspect them to make sure the variability is consistent between periods. If they are not consistent, this is an indication that heteroscedasticity is present in the residuals and additional parameters may be needed to adjust for the different variances by period.[26]

The adjustment for heteroscedasticity is typically made by focusing on the Plot of

[25] The typical ODP bootstrap model is semi-parametric, but conditions could exist for the implementation of a fully parametric ODP bootstrap, which allows for the sampling of residuals from a distribution (a more robust solution).

[26] For a more complete discussion, see Shapland [27] section 4.6 and section 5.

Residuals against Development Period (see Figure 5.4) and identifying columns with similar dispersion of residuals. While it is tempting to add hetero groupings to force additional consistency of the residuals (e.g., at 60 months where the dispersion appears low), this will be done at the expense of adding more parameters to an already highly parameterized model. This is not to say that trying other hetero groups is never justified, just that the ODP bootstrap already has one parameter for every development period and one parameter for every accident period (minus one), so adding parameters for heteroscedasticity must be decided carefully.

5.3.7 Process Variance adjustment to the ODP Bootstrap

One of the last steps in the ODP bootstrap is the use of a distributional assumption in order to add process variance to the simulated future incremental values. Without this step the projected incremental values would be point estimates rather than possible outcomes. In the case study, the Gamma distribution was used as this is the most common choice. The Normal or Lognormal distributions are possible alternative distributions which could be tested to see if they produce material differences in results, but that is outside the scope of the case study.

5.3.8 Correlation Between Segments

Thus far the list of assumptions which could be tested has been focused at the segment or model level. As the case study is intended to replicate a complete ERM system, correlation to derive an aggregate distribution is also included.

In general, the aggregate distribution of unpaid claims can be materially narrower than the sum of the individual distributions, after considering correlation between the segments. This difference between the correlated aggregate and the sum of the segments would not be as material in cases where the segments are all strongly positively correlated, where there is little variability in the individual distributions, or where one segment is far larger than the rest.

For the case study, correlation was measured using a pairwise approach.[27] A more robust solution, e.g., a maximum likelihood estimation ("MLE") copula, could be used to solve for all correlations at once since it is done analyzing all of the data at once. However, the MLE copula approach can be less than ideal when data is excluded or missing for one or more

[27] The pairwise approach is used in the "Aggregation.xlsm" companion file for the Shapland [27] paper, which was used to create Tables 5.5 and 5.6.

segments.[28,29] The measurement of correlation could be done using paid residuals and/or incurred residuals, both before and after heteroscedasticity adjustments. The resulting correlation matrices for paid loss residuals before heteroscedasticity are shown in Table 5.5.

Table 5.5 Pairwise Rank Correlation of Residuals and P-values– Paid Loss

Rank Correlation of Residuals prior to Hetero Adjustment - Paid

	PPA	CA	HO
PPA	1.000	0.276	-0.142
CA	0.276	1.000	0.027
HO	-0.142	0.027	1.000

P-Values of Rank Correlation of Residuals prior to Hetero Adjustment - Paid

	PPA	CA	HO
PPA	0.000	0.066	0.352
CA	0.066	0.000	0.860
HO	0.352	0.860	0.000

In order to aggregate distributions of possible outcomes for the entity, one needs to evaluate the inherent correlation by segment. For this, the p-values can be reviewed to assess the significance of the correlation between each pair of segments. In this test, the smaller the p-value the more significant the calculated correlation and a larger p-value (e.g., greater than 0.05 is a typical threshold) indicates that the correlation is not significantly different than zero. Therefore, the p-values of 0.352 (HO x PPA) and 0.860 (HO x CA) imply that the measured correlation is not significantly different from zero, while the p-value of 0.066 implies that the measured correlation is close to the true correlation. The selected correlation in Table 5.6 reflects the consideration of the p-values.

[28] For example, if you are only using two year average age-to-age ratios for one segment, then only the data for the last three diagonals can be used in the estimation process. The maximum likelihood copula only uses data points that are common for every segment, so it is possible to have a problematic situation where there are no common data points for all segments.

[29] It is important to note any adjustments to the ODP bootstrap model (i.e., anything less than the AYLWA for the link ratios or exclusion of outliers) will result in some of the residuals (that would otherwise be included) being excluded from the correlation matrix calculations.

Table 5.6 Selected Correlation Matrix

Assumed Correlation Matrix	PPA	CA	HO
PPA	1.000	0.276	0.000
CA	0.276	1.000	0.000
HO	0.000	0.000	1.000

The validation of correlation assumptions is a challenge. Monitoring both the measured rank correlation and corresponding p-values over time can provide some insight as to the stability of the correlation assumptions. Even so, the selected correlation assumption may also consider the impact of issues not in the measured coefficients, such as contagion or lack of prior catastrophe losses.

5.4 Implied Expected Values from Multiple Methods

Future expected incremental values (i.e., paid loss, reported claims, etc.) could be produced in a number of ways. For example, they could be independently calculated based on an independent analysis or they could be calculated based on consecutive differences of cumulative estimates which result from a curve fit. Although such practice is common, a continuous ERM process intends to improve the models and methods employed in the estimation process. Therefore, the approach used here is to estimate the future incremental values that arise from the methods (and models) which have received weight and any subsequent adjustments. The idea is that deviations can be traced back to the underlying deterministic calculations, for which validated assumptions with enhanced documentation is available and subsequent adjustments, for which documentation of decision points are available.

One challenge that immediately arises from this approach is that expected future incremental paid (and incurred) loss values must be gleaned from the expectations inherent in incurred (and paid) methods. In the extreme case where the incurred chain ladder method receives 100% of the weight for all accident years, expected incremental paid losses still need to be produced even though no paid method received weight. In order to address this challenge, the collection of methods as a whole is considered in order to rely on analogous paid methods. Continuing the example from the case study (see above for LDF validation and weighting scheme), the formulas (5.4) to (5.7) are used to derive expected cumulative amounts, for a particular method, from which incremental amounts follow.[30]

$$E[\hat{c}_P(w,d)]_{P-Method} = E[\hat{c}_P(w,d-1)]_{P-Method} \times F(d-1)_{P-Method} \quad (5.4)$$

$$E[\hat{c}_P(w,d)]_{I-Method} = E[\hat{c}_P(w,d)]_{P-Method} \times \frac{U(w)_{I-Method}}{U(w)_{P-Method}} \quad (5.5)$$

$$E[\hat{c}_I(w,d)]_{I-Method} = E[\hat{c}_I(w,d-1)]_{I-Method} \times F(d-1)_{I-Method} \quad (5.6)$$

$$E[\hat{c}_I(w,d)]_{P-Method} = E[\hat{c}_I(w,d)]_{I-Method} \times \frac{U(w)_{P-Method}}{U(w)_{I-Method}} \quad (5.7)$$

Note that a consequence of this approach is that any IBNR adjustment made subsequent to the weighting of methods will have an impact on both expected paid and incurred amounts. With cumulative paid and incurred amounts by development period so derived for each method, the weighting scheme can be applied to determine the weighted cumulative paid and incurred amounts, from which the incremental amounts can be derived. Examples of the next diagonal of incremental values (i.e., for Calendar Year 2015 during the year end 2014 analysis) are shown in Tables 5.7 and 5.8.

Table 5.7 Commercial Auto: Implied Expected Paid Losses

Sample Insurance Company Commercial Auto Expected Paid Losses during CY 2015						
AY	Paid CL	Inc CL	Paid BF	Inc BF	Weighted	Selected
2006	572	572	573	572	572	572
2007	1,049	5,518	1,068	5,497	3,284	4,863
2008	1,642	1,797	1,647	1,796	1,720	1,720
2009	4,560	6,375	4,590	6,348	5,468	5,468
2010	10,624	12,177	10,695	12,130	11,401	11,401
2011	23,280	23,230	23,355	23,247	23,255	23,255
2012	44,341	47,533	44,779	47,112	45,941	45,941
2013	61,648	64,865	61,823	63,957	62,890	62,890
2014	85,007	86,597	78,521	82,254	80,388	80,388
AY<CY	232,723	248,663	227,052	242,913	234,917	236,497

[30] Formulas (5.4) and (5.6) may seem redundant in the sense that the expected incremental development for the paid and incurred methods, respectively, are derived directly from the method itself. The formulas are included for completeness of exposition and as a link to the calculations in the "LOB Backtest.xlsm" file.

Table 5.8 Commercial Auto: Implied Expected Incurred Losses

Sample Insurance Company Commercial Auto Expected Incurred Losses during CY 2015						
AY	Paid CL	Inc CL	Paid BF	Inc BF	Weighted	Selected
2006	155	155	157	156	155	155
2007	(3,976)	507	(3,937)	507	(1,735)	**912**
2008	1,062	1,217	1,070	1,220	1,140	1,140
2009	288	2,116	345	2,115	1,202	1,202
2010	4,482	6,061	4,608	6,067	5,271	5,271
2011	11,967	11,915	12,068	11,956	11,941	11,941
2012	26,520	29,980	27,409	29,941	28,462	28,462
2013	41,780	45,513	42,556	45,037	43,797	43,797
2014	72,073	74,156	63,052	67,932	65,492	65,492
AY<CY	154,351	171,620	147,327	164,931	155,725	**158,372**

5.5 Advantages of Using the ODP Bootstrap

In the case study, the ODP bootstrap approach is relied on to model uncertainty. A main advantage of this approach is that the assumption set in the uncertainty calibration is largely consistent with the assumption set in the point estimate calibration, while areas of inconsistency (or adjustment) are identified, documented, and (to the extent possible) validated for reasonableness. Of course the uncertainty calibration required additional assumptions to be made, each of which required documentation and validation.[31]

Alternatively, the Mack [18] method could be used for the uncertainty calibration, but in doing so a number of additional challenges arise, only some of which can be overcome.

1. The variance assumptions in the Mack method would be largely inconsistent with the assumptions used to calibrate a point estimate. Recall that the selected weights imply a full rejection of the chain ladder methods for the most recent accident years.

2. The Mack method produces a variance estimate for each accident year and in total, but a distribution needs to be postulated in order to translate this variance estimate into a distribution of outcomes. The likelihood is low that such a distribution includes all possible outcomes and validation of such may not be possible.

3. The Mack formula and resulting variance estimate (on an ultimate basis) would need to be bifurcated such that variance estimates would be available for each development period between the valuation date and the date at which time the losses are fully

[31] This does not imply that the ODP bootstrap model is the only model suited for this process. In actual practice many other models can be considered with their assumptions validated, documented, etc.

developed (at ultimate).

4. The practicing actuary learns very little about the data and underlying uncertainty when using a closed form model such as Mack. This follows because such models require limited calibration to get a result and limited diagnostics regarding the underlying assumptions. Further, the uncertainty is highly dependent on the observable loss development factors, relative to the AYLWA, which in the tail area can be limited.
5. The practicing actuary has little ability to adjust the results of the Mack method in cases where the output from the closed form solution is inconsistent with expectations.

5.6 ERM Governance Elements and Automatic Alert System

The manipulation and validation of methods and models, while interesting and attractive to actuaries, is only a small part of the case study. The real benefit of a well-defined ERM process results from a governance structure that allows the actuary to actively manage resources and to escape the confines of their office to actively engage with professionals from other departments.

5.6.1 Governance

The ERM system used in the case study includes several KPIs that result from the reserving process. For each KPI, the risk owner and risk reviewer are defined. At the highest level, the KPIs for aggregate (i.e., entity-wide) paid loss and aggregate incurred loss could be defined such that the Chief Actuary is the Risk Owner and the Chief Executive Officer ("CEO") is the Risk Reviewer.

In discussing governance, KPIs, and thresholds, it is important to remember that 1 in 100 realizations is expected to fall above the 99th percentile. Stated differently, just because a deviation is large does not necessarily mean that the prior methods and models were calibrated incorrectly. On the contrary, there are three possible explanations which can be investigated:

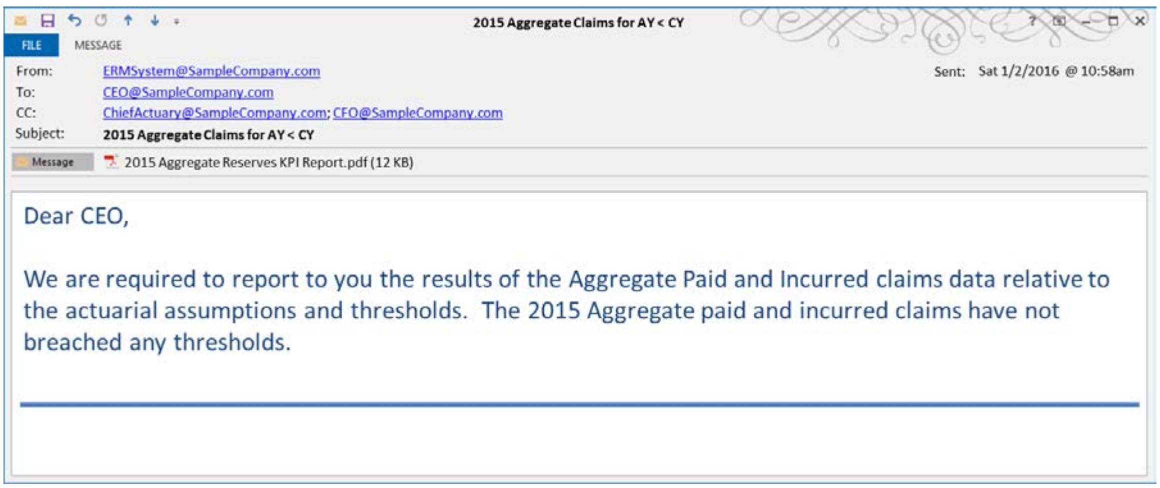
1. There could be a change in an internal process which was unknown at the time of the prior analysis contributing to the large deviation;
2. One or more of the prior modelling assumptions, with respect to the deterministic methods and stochastic models, may be causing the large deviation; or
3. A large deviation could simply be the result of a random occurrence.

5.6.2 Automatic Alert System

Further, the realized values are subject to thresholds, each with well-defined consequences in case of a breach. The case study uses thresholds at the 25th and 75th percentile, the 5th and 95th percentile, as well as the simulated minimum and simulated maximum of the distribution of possible outcomes to denote material deviations from expected, as illustrated in Figure 3.1.

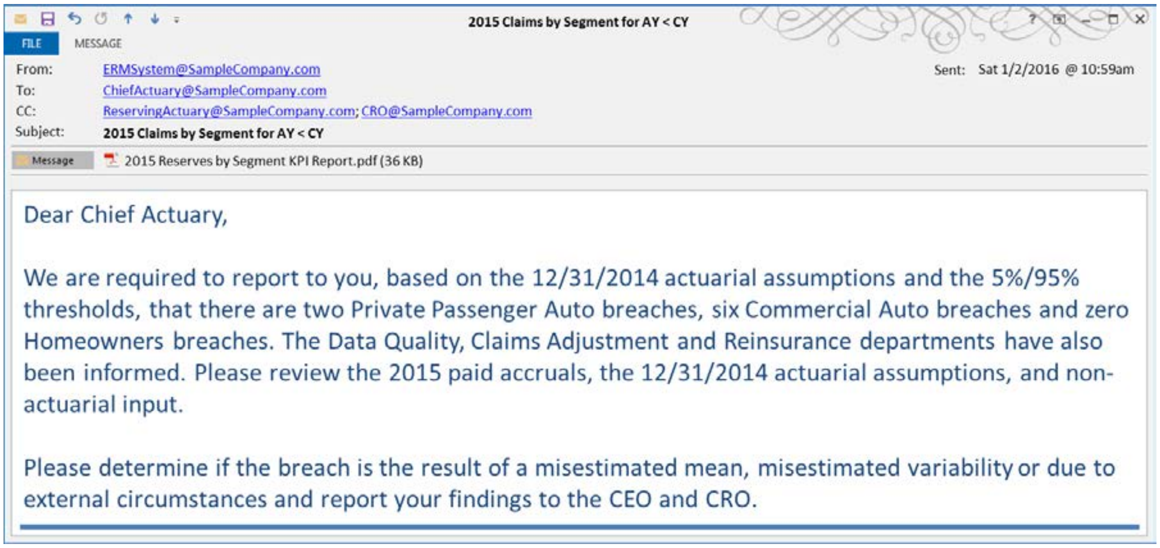
The CEO receives an immediate and automatic email from the ERM system on the first day of the analysis period confirming whether the 5% or 95% thresholds were breached by the aggregate paid loss or aggregate incurred loss.

Figure 5.5 Sample Automated E-Mail #1 to the CEO



The automatic alert system will send as many emails as needed based on the pre-defined thresholds to the appropriate Risk Owners and Risk Reviewers. For example, while the CEO is the risk reviewer and the Chief Actuary is the risk owner of the aggregate results, for the results by segment the Chief Actuary is the risk reviewer and the Reserving Actuary is the risk owner.

Figure 5.6 Sample Automated E-Mail #2 to the Chief Actuary



2015 Claims by Segment for AY < CY

FILE MESSAGE

From: ERMSystem@SampleCompany.com Sent: Sat 1/2/2016 @ 10:59am
To: ChiefActuary@SampleCompany.com
CC: ReservingActuary@SampleCompany.com; CRO@SampleCompany.com
Subject: **2015 Claims by Segment for AY < CY**

Message | 2015 Reserves by Segment KPI Report.pdf (36 KB)

Dear Chief Actuary,

We are required to report to you, based on the 12/31/2014 actuarial assumptions and the 5%/95% thresholds, that there are two Private Passenger Auto breaches, six Commercial Auto breaches and zero Homeowners breaches. The Data Quality, Claims Adjustment and Reinsurance departments have also been informed. Please review the 2015 paid accruals, the 12/31/2014 actuarial assumptions, and non-actuarial input.

Please determine if the breach is the result of a misestimated mean, misestimated variability or due to external circumstances and report your findings to the CEO and CRO.

For the emails illustrated in Figures 5.5 and 5.6 there is also a report attached which the recipients can open to review the specific results. The reports attached to the email, which also highlight any breached thresholds, are shown in Appendix B. For higher levels of management a more aggregate view will tend to be the first priority and at lower levels of management a more detailed view will be important as the automated system will reflect the responsibilities of the individuals.

5.6.3 One-Year Time Horizon as Preliminary Monitoring Tool

On the first day of the analysis, the Actuarial Function is capable of sharing even more information with the CEO & CFO, which is a valuable early warning system related to both the direction and potential magnitude of aggregate reserve changes on financial results. The value comes from estimating the one-year time horizon reserves which are conditional on the possible outcomes of the ultimate time horizon distribution. No matter whether the early warning is positive or negative, management as a whole can keep their eye on the risk management issues related to reserve changes from the beginning of the reserving exercise instead of reacting to surprises toward the end of the exercise, just prior to the publishing of financial results.

The one-year time horizon has been developed and promoted by entities subject to the Solvency II regime in Europe using both an ODP bootstrap approach and as a modification to the Mack model developed by Merz & Wüthrich [19]. Essentially, because entities are required to hold sufficient capital to be 99.5% certain of staying solvent over a one-year time horizon, actuaries have developed techniques which bifurcate measures of reserving risk into two pieces, the reserving risk over a single year and the reserving risk over all subsequent years.

The calibration of reserving risk over a one-year time horizon using the ODP bootstrap approach produces a conditional reserve at each probability level and involves a two-step process:[32]

1. Possible outcomes are simulated as usual but only the simulations of the first calendar year cash flows are retained (the one-year time horizon). These simulated diagonals are used to re-parameterize the ODP bootstrap model based on the original data plus the simulated diagonals;

2. Point estimates for the remainder of the unpaid claims subsequent to the one-year time horizon are created for each possible outcome of the original triangle plus the simulated one-year diagonal. Note that point estimates in this case have not been adjusted for process variance as they are intended to represent a reserve estimate which is conditional on the outcome of the one-year time horizon.

Table 5.9 Differences between Expected and Conditional Reserves

Sample Insurance Company Aggregation of All Segments Summary of Conditional Reserves as of December 31, 2015												
	Private Passenger Auto			Commercial Auto			Homeowners			Total (Sum)		
AY	Conditional Reserve	Expected Reserve	Change	Conditional Reserve	Expected Reserve	Change	Conditional Reserve	Expected Reserve	Change	Conditional Reserve	Expected Reserve	Change
2006	2,680	2,991	(311)	643	603	40	-	747	(747)	3,323	4,341	(1,018)
2007	7,248	5,498	1,750	3,257	4,242	(985)	164	721	(557)	10,669	10,461	208
2008	8,654	10,061	(1,406)	1,675	2,582	(907)	1,367	1,640	(272)	11,697	14,283	(2,586)
2009	15,635	19,472	(3,836)	5,593	4,121	1,472	(1,153)	1,793	(2,946)	20,075	25,386	(5,311)
2010	31,595	38,066	(6,470)	13,946	6,632	7,313	3,722	340	3,381	49,263	45,039	4,224
2011	73,359	71,302	2,057	20,073	19,441	632	3,979	6,894	(2,915)	97,412	97,638	(227)
2012	151,670	156,061	(4,390)	57,978	45,442	12,536	12,839	9,468	3,370	222,487	210,971	11,516
2013	292,882	322,812	(29,930)	110,701	81,627	29,075	21,590	26,615	(5,024)	425,174	431,054	(5,880)
2014	581,448	574,019	7,430	170,589	147,146	23,442	59,458	80,333	(20,875)	811,496	801,499	9,997
2015												
Totals	1,165,174	1,200,281	(35,107)	384,456	311,837	72,619	101,967	128,553	(26,586)	1,651,596	1,640,671	10,926
AY<CY	1,159,897	1,200,281	(40,385)	390,213	311,837	78,376	96,676	128,553	(31,876)	1,646,786	1,640,671	6,115

By calculating the percentile of the actual calendar year paid within the distribution of expected calendar year paid using (3.4), then the conditional reserve would be the same percentile of the distribution of point estimates subsequent to the one-year time horizon using formula (5.8). The expected reserve for the new analysis is equal to the expected reserve for the prior analysis less the actual amount paid during the year as shown in (5.9). In other words, the new expected reserve is equal to the prior expected reserve if the estimate

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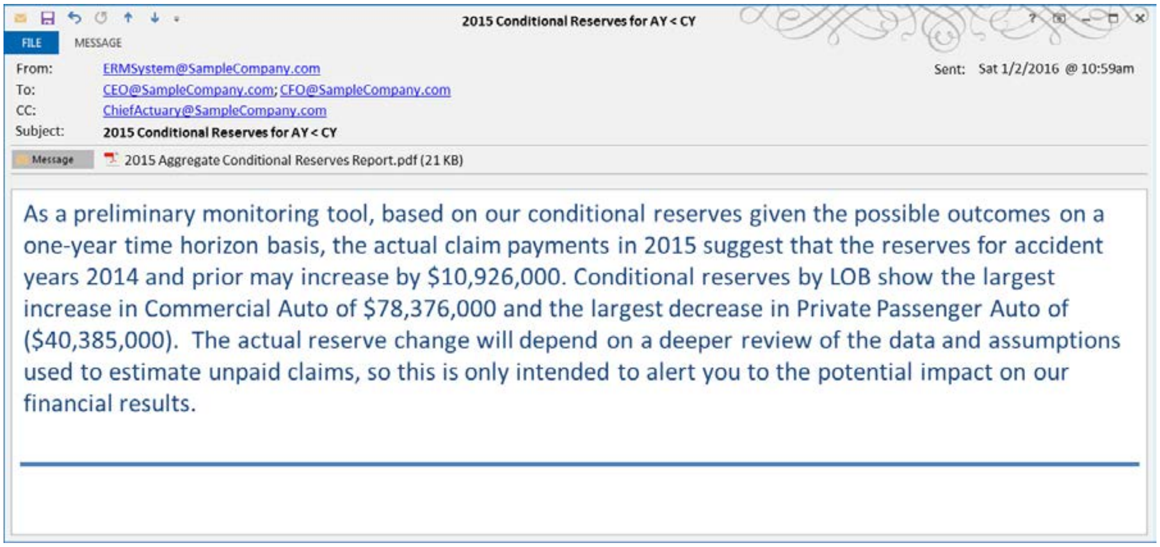
[32] See Appendix A for a graphical overview of the one-year time horizon calculations using the ODP bootstrap model.

of ultimate loss did not change at all. The estimated reserve change, therefore, is represented by the difference between conditional reserve and the expected reserve, i.e., (5.8) minus (5.9).

$$E[\hat{R}(w,d+1)\,|\,x] = PERCENTILE.INC\{Dist[\textstyle\sum_{d=t+1}^{u}\hat{q}(w,d)], P_x[q(w,d)]\} \qquad (5.8)$$

$$E[\hat{R}(w,d+1)] = E[\hat{R}(w,d)] - q(w,d) \qquad (5.9)$$

Figure 5.7 Automated E-Mail #3 to the CEO and CFO


2015 Conditional Reserves for AY < CY

FILE MESSAGE

From: ERMSystem@SampleCompany.com Sent: Sat 1/2/2016 @ 10:59am

To: CEO@SampleCompany.com; CFO@SampleCompany.com

CC: ChiefActuary@SampleCompany.com

Subject: 2015 Conditional Reserves for AY < CY

Message | 2015 Aggregate Conditional Reserves Report.pdf (21 KB)

As a preliminary monitoring tool, based on our conditional reserves given the possible outcomes on a one-year time horizon basis, the actual claim payments in 2015 suggest that the reserves for accident years 2014 and prior may increase by $10,926,000. Conditional reserves by LOB show the largest increase in Commercial Auto of $78,376,000 and the largest decrease in Private Passenger Auto of ($40,385,000). The actual reserve change will depend on a deeper review of the data and assumptions used to estimate unpaid claims, so this is only intended to alert you to the potential impact on our financial results.

The CEO and CFO receive an immediate and automatic email from the ERM system on the first day of the analysis period stating a preliminary estimate for the change in reserves, based on the conditional reserves given the possible outcomes under a one-year time horizon and the actual paid loss observed during the most recent calendar year. The report attached to the email is shown in Appendix B. Based on the conditional reserves, the aggregate increase of $10.9 million may not be of immediate concern, but the Commercial Auto increase of $78.4 million will certainly draw attention.

5.6.4 Allocating Resources

In addition to the conditional reserves by segment, it is possible to quantify and rank the deviation from expected for each of the outcomes. For the case study, 80 outcomes include 10 paid observations and 10 incurred observations, calculated as 9 AYs and Segment Total (i.e. AY < CY), for 3 Segments and the Aggregate (i.e., after correlation).

A ranked list of deviations allows for an alternative approach to managing actuarial resources. Actuarial managements often use an approach that assigns individuals to segments. An advantage of this approach is that an individual develops an area of expertise and relationships with corresponding claims and underwriting professionals. A disadvantage of this approach is that the methodology and corresponding documentation may receive less external challenge, increasing the risk that business will be disrupted in case the current expert needs to be replaced.

An alternative approach, using the ranked list of deviations, includes the allocation of resources based on the quantitative deviation from expected. This alternative approach envisions assigning resources based on need. If the methods and models are producing large deviations from expected, assignment of a resource with a proven ability to "put out fires" may be advantageous. This approach pre-supposes that the department manager has a strong sense of the strengths and weaknesses of their team.

5.6.5 Additional Indicators of Performance

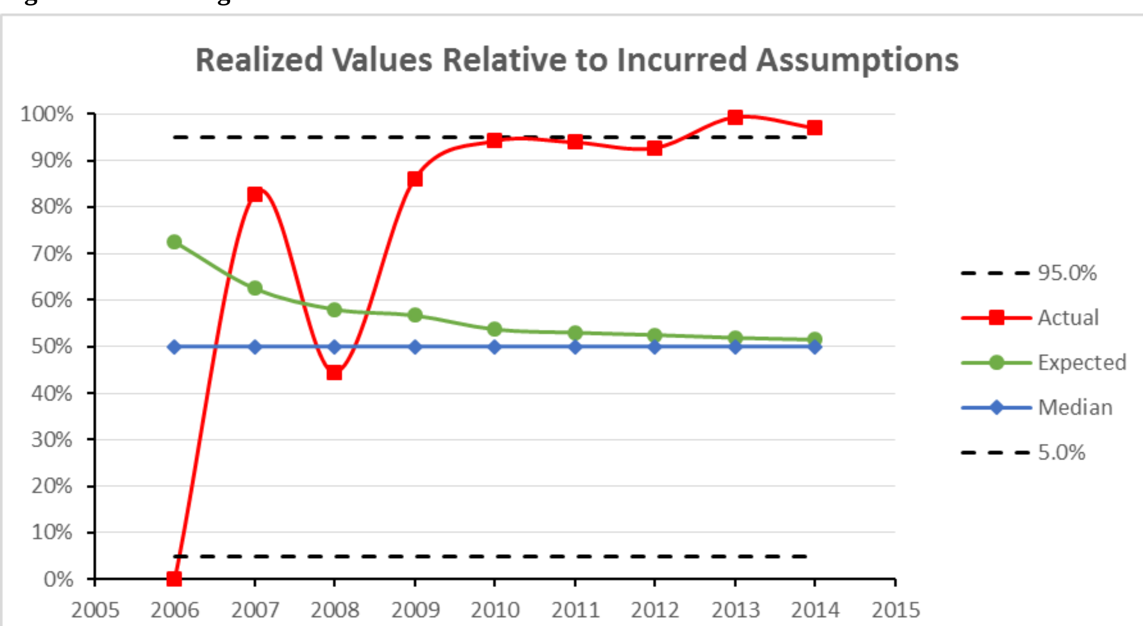
In the case of the Commercial Auto segment, the experience observed on day one of the analysis is quite poor so immediately digging into the drivers will be important. As shown in Table 5.10, two of the incurred observations (highlighted with grey shading) have breached the minimum and maximums defined by the prior models. A further two incurred and two paid observations have breached the 5%/95% threshold (highlighted with red font); and 5 incurred and 4 paid observations have breached the 25%/75% threshold (highlighted with orange font). Only 5 observations sit comfortably in the core 50%, from 25% to 75% of the distribution of possible outcomes. Absent changes in the methodology and modelling, the one-year time horizon exercise implies a deterioration of more than 13% (equal to 78,376 / [262,931 + 311,837], referring to values found in Tables 5.9 and 5.10).

Table 5.10 Assessing the 20 Observations for Commercial Auto

Sample Insurance Company Commercial Auto Stochastic Actual vs. Expected as of December 31, 2015							
AY	Age	Actual Paid	Expected Paid	Percentile	Actual Incurred	Expected Incurred	Percentile
2006	120	543	571	57.9%	(47)	154	0.0%
2007	108	2,387	3,131	21.8%	1,040	448	82.8%
2008	96	1,177	1,665	33.5%	851	1,167	44.5%
2009	84	5,403	5,044	63.1%	2,954	1,669	86.1%
2010	72	14,120	11,061	91.1%	9,035	5,606	94.2%
2011	60	23,636	23,276	56.1%	16,524	11,960	93.9%
2012	48	51,020	45,272	86.7%	36,454	29,103	92.7%
2013	36	75,813	62,481	96.5%	61,541	44,392	99.3%
2014	24	88,832	79,698	86.1%	83,154	66,555	97.0%
2015	12	99,123			178,539		
Totals		362,054			390,045		
AY<CY		262,931	232,199	98.9%	211,506	161,054	100.0%

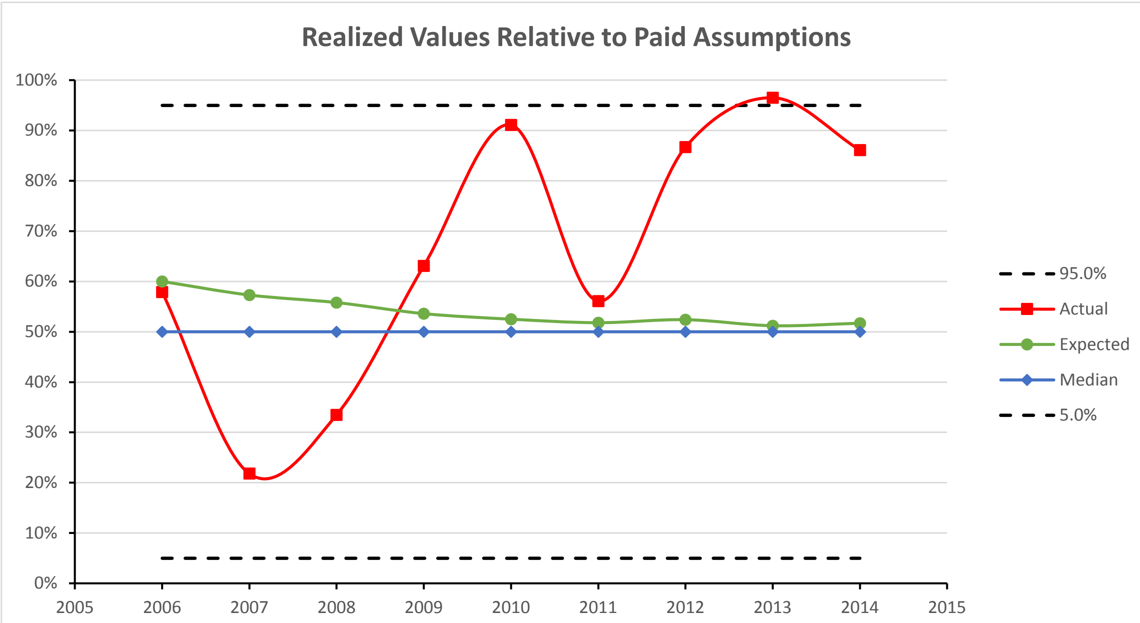
Looking closer at the incurred observations in Table 5.10 and Figure 5.8, notice that immature AYs appear to have been significantly underestimated. Though not conclusive, the realized values imply there may have been a problem with the deterministic methods underlying the prior analysis. Although the minimum and maximum have been breached, the prior uncertainty estimates may have been too narrow or the mean was too low or a combination of both, as 8 of the 10 realizations are above the 75th percentile of the distribution.

Figure 5.8 Assessing the Incurred AY Observations for Commercial Auto



Looking closer at the paid observations in Table 5.10 and Figure 5.9, notice that immature AYs appear to have again been significantly underestimated. Though not conclusive, the realized values imply again that there may have been a problem with the deterministic methods underlying the prior analysis. Again the prior uncertainty estimates may have been too narrow or the means too low or both (but to a lesser extent than observed in the incurred KPIs).

Figure 5.9 Assessing the Paid AY Observations for Commercial Auto



Note the skewness across AYs in the models underlying both the incurred and paid expectations by observing the differences between the expected values or means (the green line) and median values (the blue line) in the Figures 5.8 and 5.9.

An ERM system also has pre-defined actions, which are conditional on the breaching of the 95th percentile threshold. For Commercial Auto, these actions include immediate and automatic emails from the ERM system to the Data Quality Manager, Claims Manager, and Reinsurance Manager, among others; as illustrated in Figures 5.10 to 5.12. This presupposes some training of non-actuarial professionals so that they understand that 5 of the 100 observations should breach the 95th percentile and that a breach does not necessarily indicate that the methods and models were calibrated incorrectly. However, as part of the risk management collaboration that is being cultivated, these emails move all concerned to action.

Figure 5.10 Automated E-Mail #4 to the Data Quality Manager

2015 Claims by Segment for AY < CY

FILE MESSAGE

From: ERMSystem@SampleCompany.com
To: DataQualityMgr@SampleCompany.com
CC: ChiefActuary@SampleCompany.com
Subject: **2015 Claims by Segment for AY < CY**

Sent: Sat 1/2/2016 @ 10:59am

Message | 2015 Reserves by Segment KPI Report.pdf (36 KB)

Dear Data Quality Manager,

We are required to report to you, based on the 12/31/2014 actuarial assumptions and the 5%/95% thresholds, that there are two Private Passenger Auto breaches, six Commercial Auto breaches and zero Homeowners breaches. Please review the 2015 accruals and report to the Chief Actuary any changes in procedure, backlogs, anomalies or errors that might explain the breach.

Your qualitative feedback is expected by the Chief Actuary within 3 days.

Figure 5.11 Automated E-Mail #5 to the Claims Manager

2015 Claims by Segment for AY < CY

FILE MESSAGE

From: ERMSystem@SampleCompany.com
To: ClaimsMgr@SampleCompany.com
CC: ChiefActuary@SampleCompany.com
Subject: **2015 Claims by Segment for AY < CY**

Sent: Sat 1/2/2016 @ 10:59am

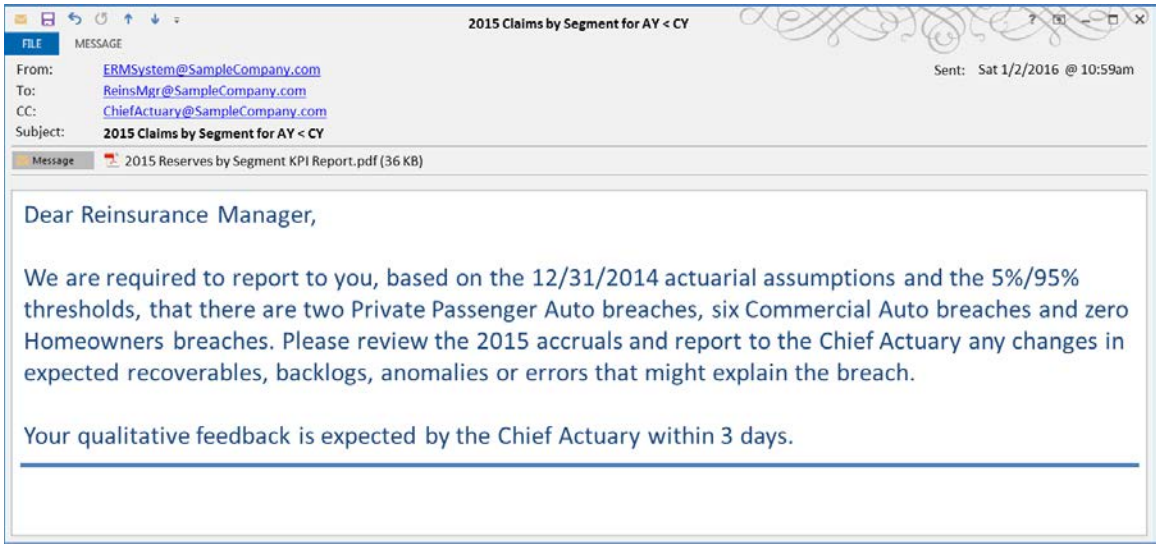
Message | 2015 Reserves by Segment KPI Report.pdf (36 KB)

Dear Claims Manager,

We are required to report to you, based on the 12/31/2014 actuarial assumptions and the 5%/95% thresholds, that there are two Private Passenger Auto breaches, six Commercial Auto breaches and zero Homeowners breaches. Please review the 2015 accruals and report to the Chief Actuary any changes in procedure, deterioration in specific accounts, anomalies or errors that might explain the breach.

Your qualitative feedback is expected by the Chief Actuary within 3 days.

Figure 5.12 Automated E-Mail #6 to the Reinsurance Manager



For the emails illustrated in Figures 5.10, 5.11, and 5.12 there is also a report attached which the recipients can open to review the specific results. The reports attached to the email, which also highlight any breached thresholds, are shown in Appendix B.

5.7 Using Back-testing Diagnostics to Assess Uncertainty

As noted above, a single observation has limited value related to assessing the overall quality of the variability estimates. However, it can be a value added exercise to review a large number of observed percentiles relative to the expectations. For the example in Table 5.11, 50% of the observations are expected to manifest within the 25th to 75th percentile. Likewise, 90% of the observations are expected to manifest within the 5th to 95th percentile and 10% of the observations are expected to manifest either below the 5th or above the 95th percentiles.

Table 5.11 Assessing Uncertainty in the 80 Observations

Sample Insurance Company
Summary of Theshold Activity by Segment as of December 31, 2015

	Number						Percentage					
	25% < X < 75%		5% < X < 95%		5% > X < 95%		25% < X < 75%		5% < X < 95%		5% > X < 95%	
	Expected	Actual	Expected	Actual	Expected	Actual	Expected	Actual	Expected	Actual	Expected	Actual
PPA	10	14	18	18	2	2	50.0%	70.0%	90.0%	90.0%	10.0%	10.0%
CA	10	**5**	18	**14**	2	**6**	50.0%	**25.0%**	90.0%	**70.0%**	10.0%	**30.0%**
HO	10	12	18	20	2	0	50.0%	60.0%	90.0%	100.0%	10.0%	0.0%
AGG	10	18	18	20	2	0	50.0%	90.0%	90.0%	100.0%	10.0%	0.0%
Total	40	49	72	72	8	8	50.0%	61.3%	90.0%	90.0%	10.0%	10.0%

Based solely on the 80 observations, the Commercial Auto line of business appears to need attention (which is consistent with the conditional reserves). Further, the Homeowners and Private Passenger Auto lines of business appear to be behaving with less uncertainty than expected. While not definitive, this process provides clues as to where the ODP bootstrap models may have been underestimating or overestimating the inherent uncertainty.

While it is tempting to draw conclusions, restraint is required as random noise can easily have a larger or smaller number of extreme observations than witnessed in Table 5.11. Nevertheless, evidence is mounting that Commercial Auto deserves the most attention.

5.8 The Feedback Loop

A critical and common part of reserving and ERM is the feedback loop. Reviewing and re-evaluating models and assumptions is a healthy part of any reserve analysis and an open discussion of risks within the ERM framework naturally leads back to the original assumptions. In the case study, all assumptions discussed in Section 5.3 were systematically reviewed and alternative assumptions tested to determine if there was a material difference in the back-test with the benefit of hindsight.

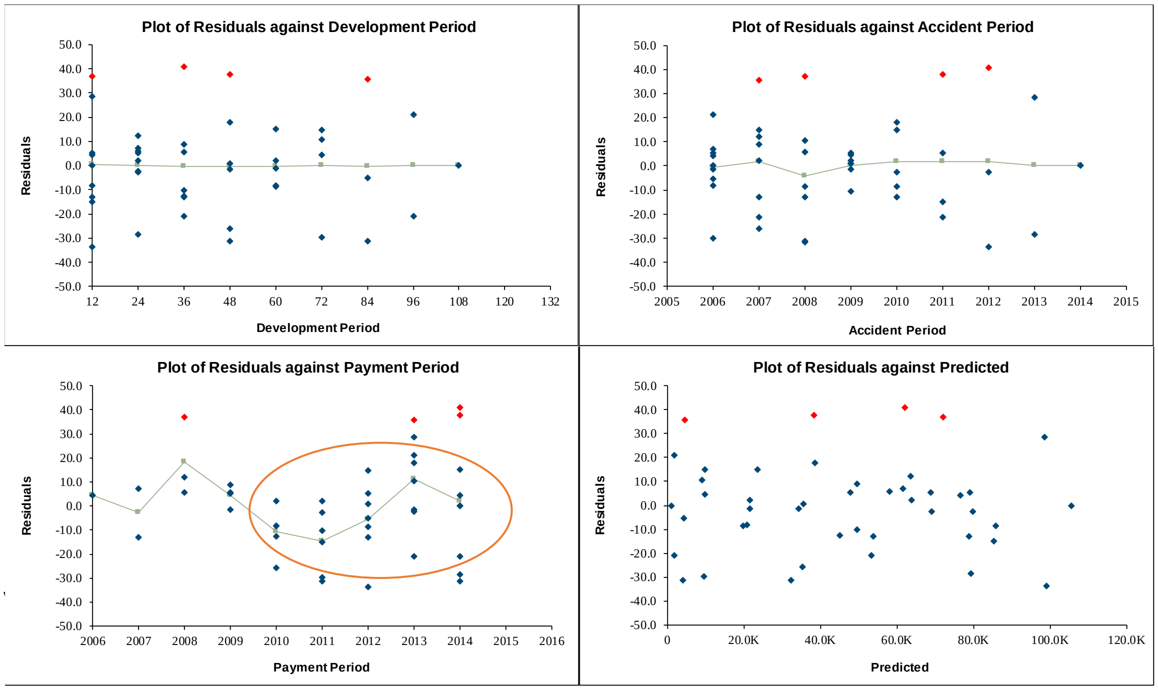
The only assumption that proved to have more than an insignificant impact on the back-test was the a priori loss ratio assumption for the Bornhuetter-Ferguson models. As shown in Table 5.2, the management IELR of 52.9% for 2014 is a bit low compared to the projected loss ratios from the Pd CL and Inc CL models, so for the back-test the 2014 IELR was changed to 57.5%. Comparing Table 5.12 with Table 5.10, the back-test of this assumption has a significant impact on the paid results for 2014, but the incurred results for 2014 are not as significant and the impact on the AY < CY results were insignificant.

Table 5.12 Revised Observations for Commercial Auto after A Priori Adjustment for 2014

Sample Insurance Company Commercial Auto Stochastic Actual vs. Expected as of December 31, 2015							
AY	Age	Actual Paid	Expected Paid	Percentile	Actual Incurred	Expected Incurred	Percentile
2006	120	543	571	57.9%	(47)	154	0.0%
2007	108	2,387	3,131	21.8%	1,040	448	82.8%
2008	96	1,177	1,665	33.5%	851	1,167	44.5%
2009	84	5,403	5,044	63.1%	2,954	1,669	86.1%
2010	72	14,120	11,061	91.1%	9,035	5,606	94.2%
2011	60	23,636	23,276	56.1%	16,524	11,960	93.9%
2012	48	51,020	45,272	86.7%	36,454	29,103	92.7%
2013	36	75,813	62,481	96.5%	61,541	44,392	99.3%
2014	24	88,832	85,603	65.4%	83,154	73,782	85.3%
2015	12	99,123			178,539		
Totals		362,054			390,045		
AY<CY		262,931	238,104	96.7%	211,506	168,281	99.9%

While the assumed loss ratios over the past few years have been decreasing, in the light of the back-testing it seems more likely that the loss ratios have remained constant at best or have been increasing.

Figure 5.13 Commercial Auto: Plots of Residuals (Paid)

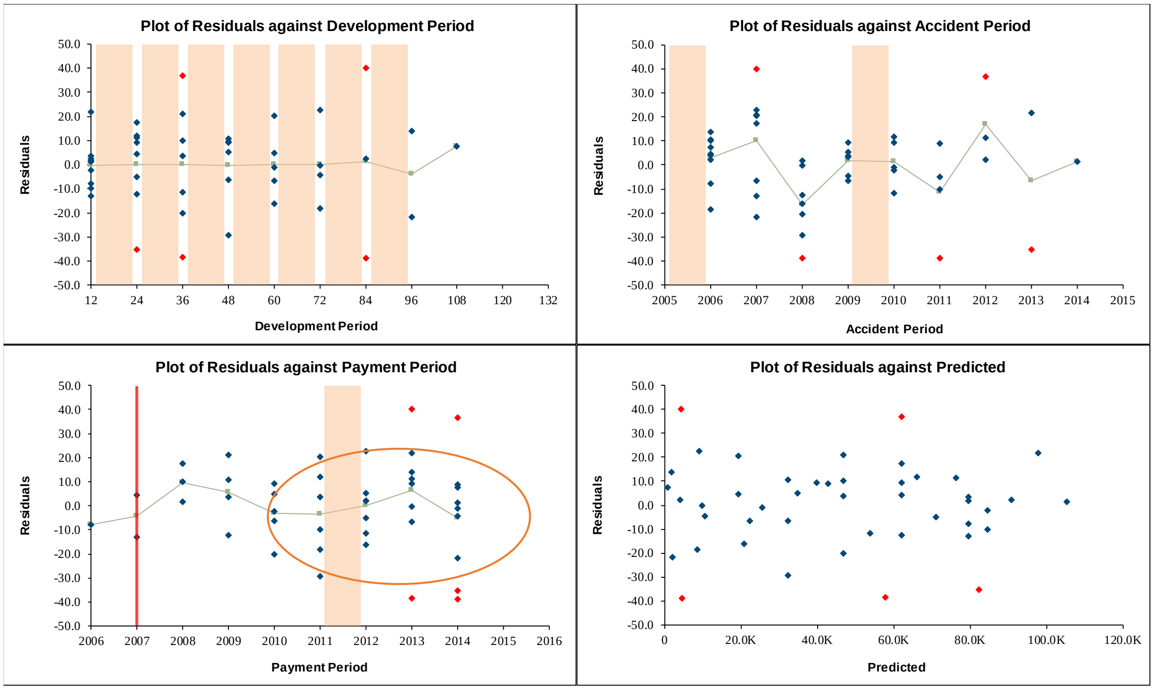


The benefit of hindsight led to an observation that a calendar year trend was evident yet overlooked (see bottom left graph in Figure 5.13). It is important here to pause and contemplate how frequently such trends are observed and disregarded (or considered immaterial). The point here is that the enhanced documentation provides an evidence trail that confirms that the trend was not addressed. With the benefit of hindsight, however, more attention is given to such diagnostics as a material driver of performance.

After identification of this possible explanation, a new model as of the previous valuation date can be calibrated. In this case, the relationship between the ODP bootstrap model and the GLM it is based on became useful. The ODP bootstrap model uses one parameter for every development year and one parameter for every accident year (minus one). Therefore the ODP bootstrap model is unable to add parameters to account for calendar year effects without removing corresponding accident year or development year parameters.

New GLM Bootstrap models based on paid and incurred data were calibrated with calendar year parameters, which was able to model the calendar year effect (see Figure 5.14, where shading refers to the parameters being used). The underlying calendar year trends inherent in the new GLM Bootstrap models imply no trend from 2006 until 2011, but an annual trend of 7.3% for years 2011 and subsequent using the paid data and a trend of 6.4% using the incurred data.

Figure 5.14 Commercial Auto: Plots of Residuals (Paid) for GLM Bootstrap Model



The new GLM Bootstrap models based on paid and incurred data performed better than the prior selected models, as seen in Table 5.13, and many of the model statistics are better.

At first glance Table 5.13 does not appear to be significantly better than Table 5.10. However, a review of Figures 5.15 and 5.16 (for the GLM Bootstrap) reveals that adding the calendar year trend to the models counteracts the upward trend in Figures 5.8 and 5.9 (prior to GLM Bootstrap) to a significant degree (more for paid than incurred) which provides a rationale (or evidence) for the increasing loss ratios over the last few years. This corroborates the earlier back-test of the Bornhuetter-Ferguson a priori loss ratios. The resulting variations in Figures 5.15 and 5.16 also indicates that the variability of the potential outcomes may still be too narrow (e.g., Bornhuetter-Ferguson a priori variance could be larger), but this is just a preliminary review.

Table 5.13 Assessing the Commercial Auto Observations for the GLM Bootstrap Models

Sample Insurance Company Commercial Auto Stochastic Actual vs. Expected as of December 31, 2015							
AY	Age	Actual Paid	Expected Paid	Percentile	Actual Incurred	Expected Incurred	Percentile
2006	120	543	432	69.4%	(47)	228	2.0%
2007	108	2,387	942	96.6%	1,040	516	86.8%
2008	96	1,177	2,117	14.0%	851	1,181	37.9%
2009	84	5,403	5,001	64.1%	2,954	2,665	64.7%
2010	72	14,120	12,100	82.3%	9,035	6,659	89.8%
2011	60	23,636	27,514	11.8%	16,524	13,869	84.2%
2012	48	51,020	46,010	87.6%	36,454	31,896	87.7%
2013	36	75,813	66,910	94.6%	61,541	50,020	98.5%
2014	24	88,832	88,362	54.1%	83,154	78,184	77.8%
2015	12	99,123			178,539		
Totals		362,054			390,045		
AY<CY		262,931	249,388	86.0%	211,506	185,218	98.7%

The ERM process has provided the information to identify the problem segment and the enhanced documentation has allowed quick testing of the prior assumptions to provide an alternative model which can be considered and implemented by the actuarial resources for the current valuation. Additionally, the GLM approach has both identified when the positive calendar year trend begins (i.e., the break point) and quantified the trend rates, which allows the actuary to engage more directly with the claims department, where deeper knowledge may exist to improve the modeling process.

Figure 5.15 Assessing the Incurred AY Observations for Commercial Auto (GLM Bootstrap Model)

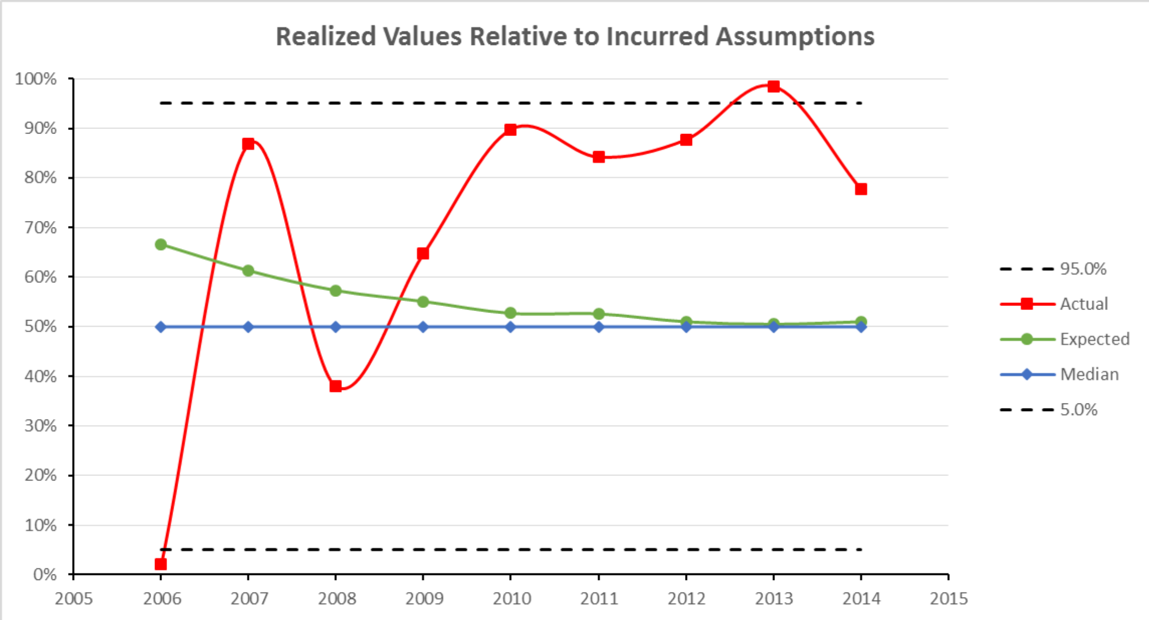
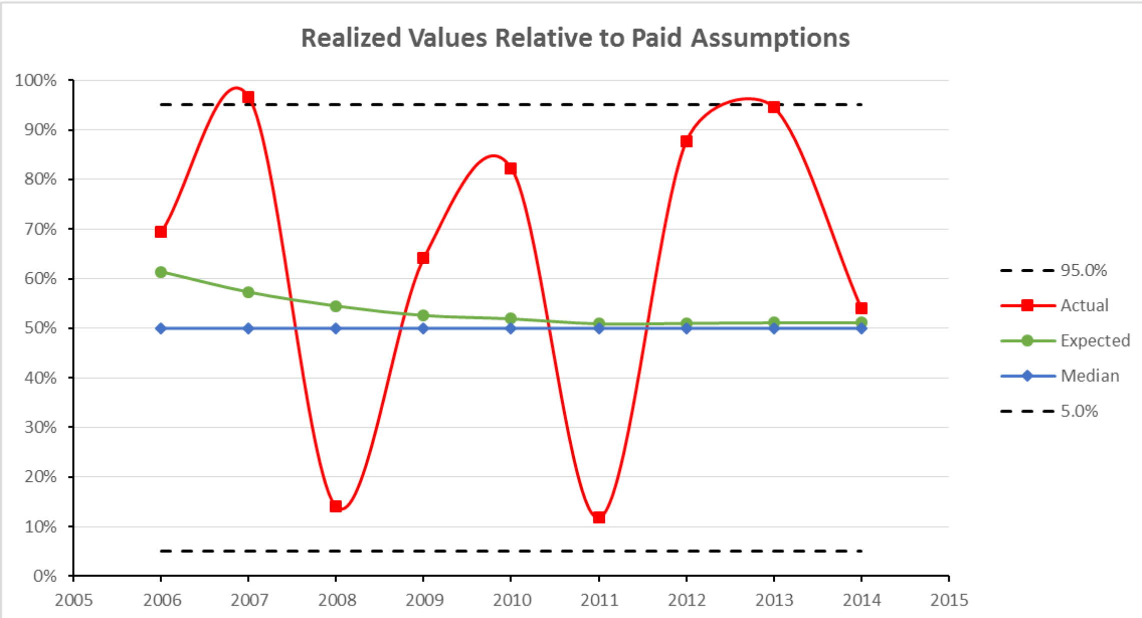
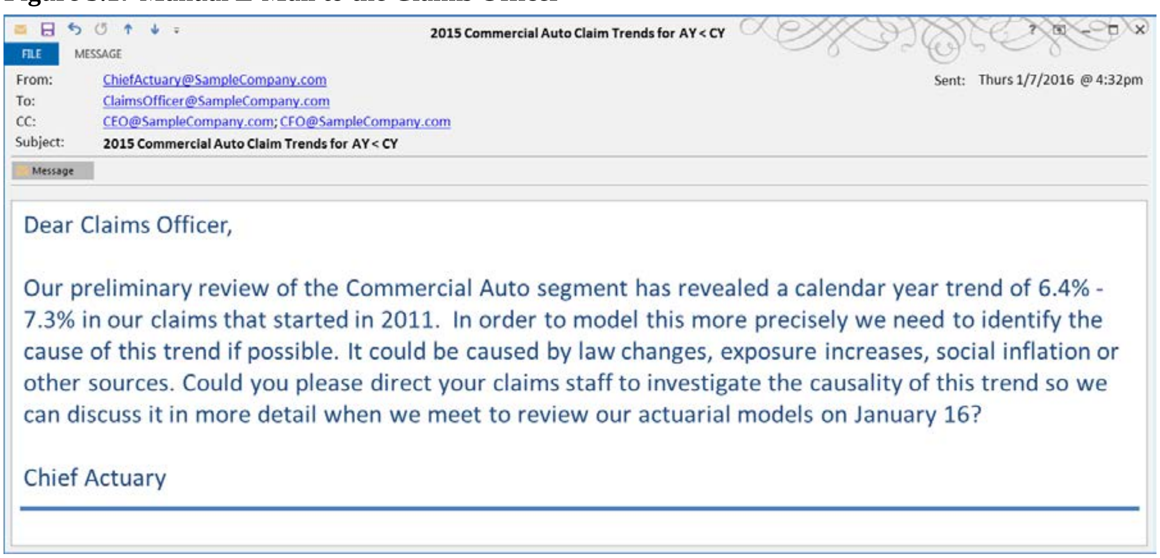


Figure 5.16 Assessing the Paid AY Observations for Commercial Auto (GLM Bootstrap Model)



A direct email from the Chief Actuary to the relevant Claims Officer, as illustrated in Figure 5.17, is the logical next step in the process so that communication around this issue can begin. Note that the process allows the actuary to speak to the claims officer in the language the claims officer understands: no mention of triangles, IBNR, accident years, or any other actuarial concepts that may be unfamiliar.

Figure 5.17 Manual E-Mail to the Claims Officer


2015 Commercial Auto Claim Trends for AY < CY

FILE MESSAGE

From: ChiefActuary@SampleCompany.com Sent: Thurs 1/7/2016 @ 4:32pm
To: ClaimsOfficer@SampleCompany.com
CC: CEO@SampleCompany.com; CFO@SampleCompany.com
Subject: 2015 Commercial Auto Claim Trends for AY < CY

Message

Dear Claims Officer,

Our preliminary review of the Commercial Auto segment has revealed a calendar year trend of 6.4% - 7.3% in our claims that started in 2011. In order to model this more precisely we need to identify the cause of this trend if possible. It could be caused by law changes, exposure increases, social inflation or other sources. Could you please direct your claims staff to investigate the causality of this trend so we can discuss it in more detail when we meet to review our actuarial models on January 16?

Chief Actuary

The value of this active feedback loop on reserving risk within the ERM process can't be overestimated. Not only does it naturally expand the actuarial conversation regarding risk drivers to the entire firm, but it also flows into other risks such as claims management and pricing risk. Indeed, consider the impact that identifying this trend will have on future pricing discussions for Commercial Auto.

6. Conclusions

While the value of including reserve variability estimates as part of the "normal" reserving cycle processes is questioned by some, and perhaps feared by others, the purpose of this paper is to show how making reserve variability estimates a routine part of the analysis can greatly benefit the risk management process. Keeping these estimates in the "back room" or "hidden until needed" does not benefit anyone. If casualty actuaries are going to truly embrace Enterprise Risk Management, then deep discussions of reserving risk must become part of the actuarial lexicon.

Acknowledgments

The authors gratefully acknowledge the many authors listed in the References (and others not listed) that contributed to the foundation of stochastic reserving and Enterprise Risk Management, without which this research would not have been possible. The authors are also grateful to Wayne Blackburn for his thorough review and insightful comments. The authors are also grateful to the participants in various seminars and sessions at GIRO, the CLRS, and the European Actuarial Academy stochastic modeling seminars where the concepts in the paper were first presented and discussed. Finally, the authors are grateful to the CAS Committee on Reserves for their comments which also greatly improved the quality of the paper.

Supplementary Material

There are companion files designed to give the reader a deeper understanding of the concepts discussed in the paper. The files are all in the "Actuary & ERM.zip" file. The files are:

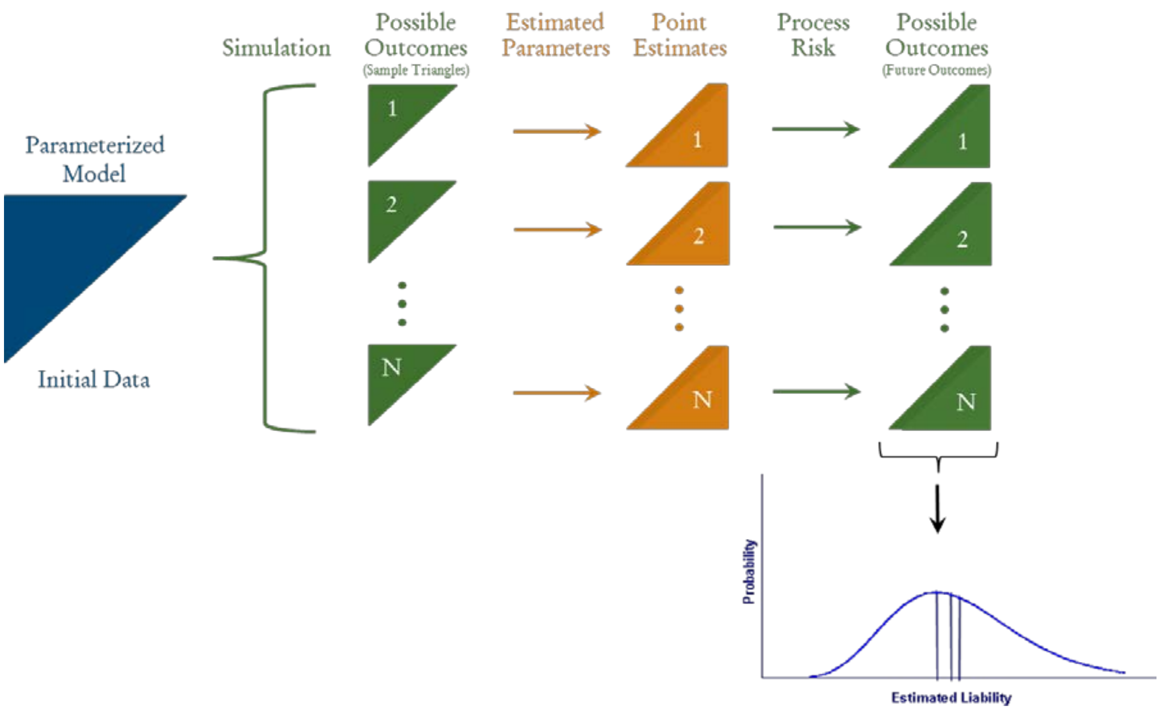
LOB Backtest.xlsm – this file contains the detailed calculations described in this paper for a single segment or line of business. Data can be entered and simulation output can be added for calculating both expected and actual outcomes, along with various statistical measures and results. Deterministic calculations and results are also included for comparison to stochastic results.

AGG Backtest.xlsm – this file can be used to summarize the deterministic and stochastic results from the LOB Backtest.xlsm file (selected results need to be copied to this file) for three lines of business. Aggregate simulation output can be added for calculating both expected and actual outcomes, along with various statistical measures and results.

APPENDICIES

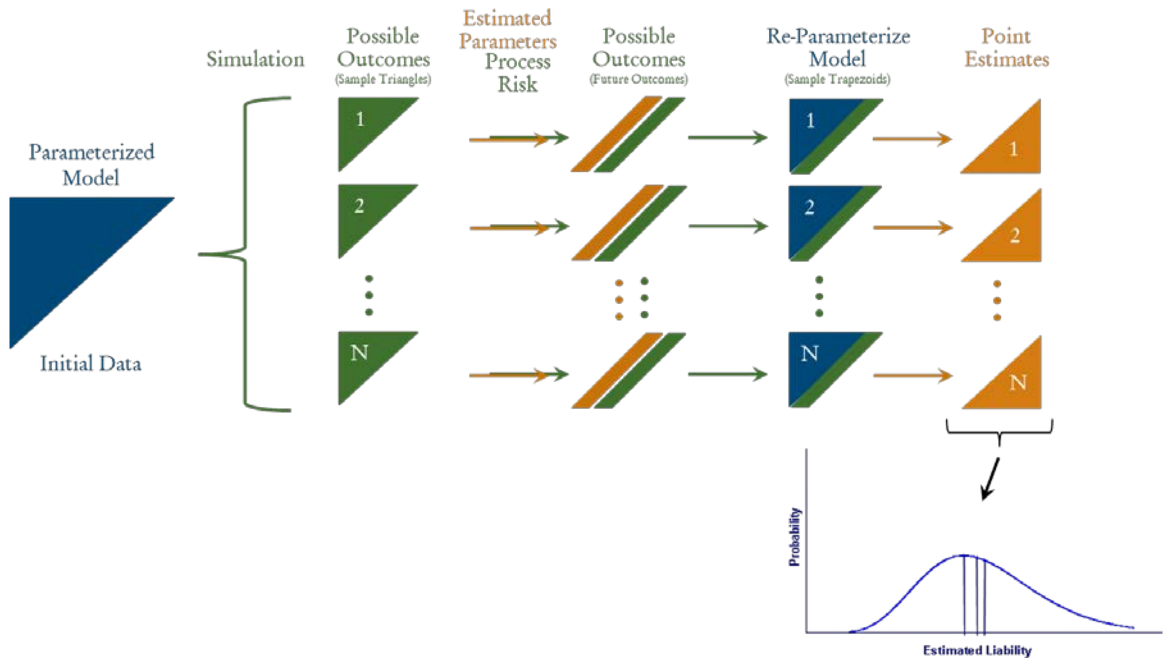
Appendix A – Overview of One-Year Time Horizon

A "standard" ODP bootstrap model can be represented graphically as follows:



- The "standard" model is based on paid data, but incurred data can also be used to reflect information in case reserves and converted to a random payment stream.
- The "standard" model is based on the chain ladder methodology, but other methods such as Bornhuetter-Ferguson and Cape Cod can also be included.
- Multiple models can also be "weighted" and "shifted" to reconcile with the deterministic "best estimate".
- Aggregation of the segment results can be done to derive a consolidated corporate result, even though these graphs are for one segment.

By using the first diagonal of the possible future outcomes and then calculating a point estimate for the remaining unpaid claims, the one-year time horizon can be represented graphically as follows:



- The "one-year" model is based on paid data, but incurred data can also be used to reflect information in case reserves and converted to a random payment stream for the first diagonal and expected payments for the remaining diagonals.
- The "one-year" model is based on the chain ladder methodology, but other methods such as Bornhuetter-Ferguson and Cape Cod can also be included. For internal consistency, all of the assumptions for the "standard" model should apply unchanged for the "one-year" model.
- Multiple models can also be "weighted" and "shifted" to reconcile with the deterministic "best estimate". The weights should be the same as for the "standard" model and "shifting" should be consistent with "standard" model so that the first diagonal after shifting is identical.
- Distributions of conditional point estimates can also be created for each accident year even though the total of all accident years combined is shown in the graphs.
- Aggregation of the segment results can be done to derive a consolidated corporate result, even though these graphs are for one segment.

Appendix B – Reports Attached to Emails

Figure B.1 – Report on 2015 Aggregate Exposures

Stochastic Model Results

2015 Aggregation of All Segments Exposure

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Stochastic Model Detail

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Model Name	2015 Aggregation of All Segments Exposure	**Assumption Owner**	Chief Actuary
Description	Expected Aggregation of All Segments claim payments during 2015 for exposure periods prior to 2015 based on data generated by claims system as of 12/31/2015 relative to the 12/31/2014 actuarial assumptions.	**Reports To**	Chief Executive Officer
Assumption Value	Expected Value	**Assumption Value Date**	12/31/2014
Assumption Minimum	5.0%	**Next Update Due**	12/31/2015
Assumption Maximum	95.0%		

▼ Realized Value

Paid Actual	1,571,872	**Incurred Actual**	847,136
Paid Expected	1,560,637	**Incurred Expected**	858,093
Paid Percentile	61.2%	**Incurred Percentile**	37.6%

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Stochastic Values

New Value

Help

Action	Number	Exposure Period	Age	Paid Actual	Paid Expected	Paid Percentile	Incurred Actual	Incurred Expected	Incurred Percentile
Edit | Del	0001	12/31/2006	120	3,069	4,077	31.8%	1,863	2,115	49.8%
Edit | Del	0002	12/31/2007	108	5,905	6,163	47.9%	3,145	1,819	80.6%
Edit | Del	0003	12/31/2008	96	8,986	10,176	33.6%	3,553	6,026	20.9%
Edit | Del	0004	12/31/2009	84	18,992	20,033	39.0%	9,872	10,399	46.3%
Edit | Del	0005	12/31/2010	72	51,003	48,298	71.6%	25,942	25,562	55.3%
Edit | Del	0006	12/31/2011	60	105,067	104,415	54.3%	52,012	53,101	44.8%
Edit | Del	0007	12/31/2012	48	202,932	196,083	74.2%	106,624	104,075	61.7%
Edit | Del	0008	12/31/2013	36	334,434	331,701	57.1%	189,908	185,173	64.0%
Edit | Del	0009	12/31/2014	24	841,484	839,689	52.8%	454,217	469,822	29.3%
Edit | Del	0010	12/31/2015	12	1,798,138	0		2,528,235	0	

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Figure B.2 – Report on 2015 Private Passenger Auto Exposures

Stochastic Model Results

2015 Private Passenger Auto Exposure

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Stochastic Model Detail

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Model Name	2015 Private Passenger Auto Exposure	Assumption Owner	Reserving Actuary
Description	Expected Private Passenger Auto claim payments during 2015 for exposure periods prior to 2015 based on data generated by claims system as of 12/31/2015 relative to the 12/31/2014 actuarial assumptions.	Reports To	Chief Actuary
Assumption Value	Expected Value	Assumption Value Date	12/31/2014
Assumption Minimum	5.0%	Next Update Due	12/31/2015
Assumption Maximum	95.0%		

▼ Realized Value

Paid Actual	1,071,854	Incurred Actual	571,794
Paid Expected	1,076,388	Incurred Expected	631,511
Paid Percentile	44.9%	Incurred Percentile	0.6%

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Stochastic Values — New Value — Help

Action	Number	Exposure Period	Age	Paid Actual	Paid Expected	Paid Percentile	Incurred Actual	Incurred Expected	Incurred Percentile
Edit | Del	0011	12/31/2006	120	2,500	2,733	48.2%	2,042	2,056	56.7%
Edit | Del	0012	12/31/2007	108	3,485	2,908	69.4%	2,261	1,312	81.0%
Edit | Del	0013	12/31/2008	96	7,582	8,098	43.4%	4,061	5,207	33.2%
Edit | Del	0014	12/31/2009	84	13,765	14,773	37.5%	8,076	8,835	41.7%
Edit | Del	0015	12/31/2010	72	33,083	35,326	30.5%	16,495	20,439	15.6%
Edit | Del	0016	12/31/2011	60	75,969	74,381	61.4%	35,496	40,022	21.2%
Edit | Del	0017	12/31/2012	48	139,715	140,849	45.5%	68,886	74,159	25.6%
Edit | Del	0018	12/31/2013	36	234,781	243,390	26.5%	119,582	128,507	20.2%
Edit | Del	0019	12/31/2014	24	560,974	553,931	62.3%	314,895	350,974	2.9%
Edit | Del	0020	12/31/2015	12	764,210	0		1,205,957	0	

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Figure B.3 – Report on 2015 Commercial Auto Exposures

Stochastic Model Results

2015 Commercial Auto Exposure

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Stochastic Model Detail

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Model Name	2015 Commercial Auto Exposure	Assumption Owner	Reserving Actuary
Description	Expected Commercial Auto claim payments during 2015 for exposure periods prior to 2015 based on data generated by claims system as of 12/31/2015 relative to the 12/31/2014 actuarial assumptions.	Reports To	Chief Actuary
Assumption Value	Expected Value	Assumption Value Date	12/31/2014
Assumption Minimum	5.0%	Next Update Due	12/31/2015
Assumption Maximum	95.0%		

▼ Realized Value

Paid Actual	262,931	Incurred Actual	211,506
Paid Expected	232,199	Incurred Expected	161,054
Paid Percentile	98.9%	Incurred Percentile	100.0%

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Stochastic Values

New Value

Help

Action	Number	Exposure Period	Age	Paid Actual	Paid Expected	Paid Percentile	Incurred Actual	Incurred Expected	Incurred Percentile
Edit | Del	0021	12/31/2006	120	543	571	57.9%	(47)	154	0.0%
Edit | Del	0022	12/31/2007	108	2,387	3,131	21.8%	1,040	448	82.8%
Edit | Del	0023	12/31/2008	96	1,177	1,665	33.5%	851	1,167	44.5%
Edit | Del	0024	12/31/2009	84	5,403	5,044	63.1%	2,954	1,669	86.1%
Edit | Del	0025	12/31/2010	72	14,120	11,061	91.1%	9,035	5,606	94.2%
Edit | Del	0026	12/31/2011	60	23,636	23,276	56.1%	16,524	11,960	93.9%
Edit | Del	0027	12/31/2012	48	51,020	45,272	86.7%	36,454	29,103	92.7%
Edit | Del	0028	12/31/2013	36	75,813	62,481	96.5%	61,541	44,392	99.3%
Edit | Del	0029	12/31/2014	24	88,832	79,698	86.1%	83,154	66,555	97.0%
Edit | Del	0030	12/31/2015	12	99,123	0		178,539	0	

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Figure B.4 – Report on 2015 Homeowners Exposures

Stochastic Model Results

2015 Homeowners Exposure

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Stochastic Model Detail

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Model Name	2015 Homeowners Exposure	Assumption Owner	Reserving Actuary
Description	Expected Homeowners claim payments during 2015 for exposure periods prior to 2015 based on data generated by claims system as of 12/31/2015 relative to the 12/31/2014 actuarial assumptions.	Reports To	Chief Actuary
Assumption Value	Expected Value	Assumption Value Date	12/31/2014
Assumption Minimum	5.0%	Next Update Due	12/31/2015
Assumption Maximum	95.0%		

▼ Realized Value

Paid Actual	237,087	Incurred Actual	63,836
Paid Expected	252,049	Incurred Expected	65,528
Paid Percentile	28.4%	Incurred Percentile	50.2%

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Stochastic Values

New Value

Help

Action	Number	Exposure Period	Age	Paid Actual	Paid Expected	Paid Percentile	Incurred Actual	Incurred Expected	Incurred Percentile
Edit | Del	0031	12/31/2006	120	26	773	13.9%	(132)	(95)	83.5%
Edit | Del	0032	12/31/2007	108	33	125	61.9%	(156)	59	31.4%
Edit | Del	0033	12/31/2008	96	227	414	57.2%	(1,359)	(349)	23.5%
Edit | Del	0034	12/31/2009	84	(176)	217	14.1%	(1,158)	(105)	18.5%
Edit | Del	0035	12/31/2010	72	3,800	1,911	85.6%	412	(482)	67.2%
Edit | Del	0036	12/31/2011	60	5,462	6,758	37.5%	(8)	1,119	12.2%
Edit | Del	0037	12/31/2012	48	12,197	9,961	74.9%	1,284	813	81.4%
Edit | Del	0038	12/31/2013	36	23,840	25,830	40.5%	8,785	12,274	37.9%
Edit | Del	0039	12/31/2014	24	191,678	206,060	28.0%	56,168	52,293	62.7%
Edit | Del	0040	12/31/2015	12	934,805	0		1,143,739	0	

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Figure B.5 – Report on 2015 Conditional Reserves

Stochastic Model Results

2015 Conditional Reserves by Segment

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Model Name	2015 Conditional Reserves by Segment	**Assumption Owner**	Chief Actuary
Description	Expected conditional reserves as of 12/31/2015 for exposure periods prior to 2015 based on data generated by claims system during CY 2015 relative to the 12/31/2014 actuarial assumptions.	**Reports To**	Chief Executive Officer
Assumption Value	Percentile of One-Year Horizon	**Assumption Value Date**	12/31/2014
Output Value	One-Year Reserve Estimate	**Next Update Due**	12/31/2015

▼ Realized Value

Sum of Yrs	(2,154)	**Sum of Yrs**	10,926	**Sum of Yrs**	72,619	**Sum of Yrs**	(35,107)
CY 2015	(2,086)	**CY 2015**	6,115	**CY 2015**	78,376	**CY 2015**	(40,385)
Overall Change:	Aggregation of All Segm	**Overall Change:**	Sum of Segments	**Largest Increase:**	CA	**Largest Decrease:**	PPA

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Stochastic Values New Value Help

Aggregation of All Segments									
Action	**Number**	**Exposure Period**	**Age**	**Original**	**Actual Paid**	**Current**	**Paid Percentile**	**Conditional**	**Change**
Edit | Del	0001	12/31/2006	120	7,410	3,069	4,341	31.8%	2,539	(1,802)
Edit | Del	0002	12/31/2007	108	16,366	5,905	10,461	47.9%	11,349	888
Edit | Del	0003	12/31/2008	96	23,269	8,986	14,283	33.6%	10,961	(3,322)
Edit | Del	0004	12/31/2009	84	44,378	18,992	25,386	39.0%	21,615	(3,771)
Edit | Del	0005	12/31/2010	72	96,042	51,003	45,039	71.6%	49,308	4,269
Edit | Del	0006	12/31/2011	60	202,705	105,067	97,638	54.3%	97,157	(481)
Edit | Del	0007	12/31/2012	48	413,903	202,932	210,971	74.2%	222,250	11,279
Edit | Del	0008	12/31/2013	36	765,488	334,434	431,054	57.1%	427,667	(3,387)
Edit | Del	0009	12/31/2014	24	1,642,982	841,484	801,499	52.8%	795,671	(5,828)
Edit | Del	0010	SUM OF YRS		3,212,543	1,571,872	1,640,671		1,638,516	(2,154)
Edit | Del	0011	CY 2015		3,212,543	1,571,872	1,640,671	61.2%	1,638,584	(2,086)

Sum of All Segments									
Action	**Number**	**Exposure Period**	**Age**	**Original**	**Actual Paid**	**Current**	**Paid Percentile**	**Conditional**	**Change**
Edit | Del	0012	12/31/2006	120	7,410	3,069	4,341	N/A	3,323	(1,018)
Edit | Del	0013	12/31/2007	108	16,366	5,905	10,461	N/A	10,669	208
Edit | Del	0014	12/31/2008	96	23,269	8,986	14,283	N/A	11,697	(2,586)
Edit | Del	0015	12/31/2009	84	44,378	18,992	25,386	N/A	20,075	(5,311)
Edit | Del	0016	12/31/2010	72	96,042	51,003	45,039	N/A	49,263	4,224
Edit | Del	0017	12/31/2011	60	202,705	105,067	97,638	N/A	97,412	(227)
Edit | Del	0018	12/31/2012	48	413,903	202,932	210,971	N/A	222,487	11,516
Edit | Del	0019	12/31/2013	36	765,488	334,434	431,054	N/A	425,174	(5,880)
Edit | Del	0020	12/31/2014	24	1,642,982	841,484	801,499	N/A	811,496	9,997
Edit | Del	0021	SUM OF YRS		3,212,543	1,571,872	1,640,671		1,651,596	10,926
Edit | Del	0022	CY 2015		3,212,543	1,571,872	1,640,671	N/A	1,646,786	6,115

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Figure B.5 – Report on 2015 Conditional Reserves (Cont.)

Stochastic Model Results

2015 Conditional Reserves by Segment

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Model Name	2015 Conditional Reserves by Segment	Assumption Owner	Chief Actuary
Description	Expected conditional reserves as of 12/31/2015 for exposure periods prior to 2015 based on data generated by claims system during CY 2015 relative to the 12/31/2014 actuarial assumptions.	Reports To	Chief Executive Officer
Assumption Value	Percentile of One-Year Horizon	Assumption Value Date	12/31/2014
Output Value	One-Year Reserve Estimate	Next Update Due	12/31/2015

▼ **Realized Value**

Edit Delete Clone

Stochastic Values New Value Help

Private Passenger Auto (PPA)

Action	Number	Exposure Period	Age	Original	Actual Paid	Current	Paid Percentile	Conditional	Change
Edit | Del	0023	12/31/2006	120	5,491	2,500	2,991	48.2%	2,680	(311)
Edit | Del	0024	12/31/2007	108	8,983	3,485	5,498	69.4%	7,248	1,750
Edit | Del	0025	12/31/2008	96	17,643	7,582	10,061	43.4%	8,654	(1,406)
Edit | Del	0026	12/31/2009	84	33,237	13,765	19,472	37.5%	15,635	(3,836)
Edit | Del	0027	12/31/2010	72	71,149	33,083	38,066	30.5%	31,595	(6,470)
Edit | Del	0028	12/31/2011	60	147,271	75,969	71,302	61.4%	73,359	2,057
Edit | Del	0029	12/31/2012	48	295,776	139,715	156,061	45.5%	151,670	(4,390)
Edit | Del	0030	12/31/2013	36	557,593	234,781	322,812	26.5%	292,882	(29,930)
Edit | Del	0031	12/31/2014	24	1,134,993	560,974	574,019	62.3%	581,448	7,430
Edit | Del	0032	SUM OF YRS		2,272,135	1,071,854	1,200,281		1,165,174	(35,107)
Edit | Del	0033	CY 2015		2,272,135	1,071,854	1,200,281	44.9%	1,159,897	(40,385)

Commercial Auto (CA)

Action	Number	Exposure Period	Age	Original	Actual Paid	Current	Paid Percentile	Conditional	Change
Edit | Del	0034	12/31/2006	120	1,146	543	603	57.9%	643	40
Edit | Del	0035	12/31/2007	108	6,629	2,387	4,242	21.8%	3,257	(985)
Edit | Del	0036	12/31/2008	96	3,759	1,177	2,582	33.5%	1,675	(907)
Edit | Del	0037	12/31/2009	84	9,524	5,403	4,121	63.1%	5,593	1,472
Edit | Del	0038	12/31/2010	72	20,752	14,120	6,632	91.1%	13,946	7,313
Edit | Del	0039	12/31/2011	60	43,077	23,636	19,441	56.1%	20,073	632
Edit | Del	0040	12/31/2012	48	96,462	51,020	45,442	86.7%	57,978	12,536
Edit | Del	0041	12/31/2013	36	157,440	75,813	81,627	96.5%	110,701	29,075
Edit | Del	0042	12/31/2014	24	235,978	88,832	147,146	86.1%	170,589	23,442
Edit | Del	0043	SUM OF YRS		574,768	262,931	311,837		384,456	72,619
Edit | Del	0044	CY 2015		574,768	262,931	311,837	98.9%	390,213	78,376

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Figure B.5 – Report on 2015 Conditional Reserves (Cont.)

Stochastic Model Results

2015 Conditional Reserves by Segment

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Stochastic Model Detail

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Model Name	2015 Conditional Reserves by Segment	Assumption Owner	Chief Actuary
Description	Expected conditional reserves as of 12/31/2015 for exposure periods prior to 2015 based on data generated by claims system during CY 2015 relative to the 12/31/2014 actuarial assumptions.	Reports To	Chief Executive Officer
Assumption Value	Percentile of One-Year Horizon	Assumption Value Date	12/31/2014
Output Value	One-Year Reserve Estimate	Next Update Due	12/31/2015

▼ Realized Value

Edit | Delete | Clone

Stochastic Values

New Value

Help

Homeowners (HO)										
Action	Number	Exposure Period	Age	Original	Actual Paid	Current	Paid Percentile	Conditional	Change
Edit | Del	0045	12/31/2006	120	773	26	747	13.9%	0	(747)
Edit | Del	0046	12/31/2007	108	754	33	721	61.9%	164	(557)
Edit | Del	0047	12/31/2008	96	1,867	227	1,640	57.2%	1,367	(272)
Edit | Del	0048	12/31/2009	84	1,617	(176)	1,793	14.1%	(1,153)	(2,946)
Edit | Del	0049	12/31/2010	72	4,140	3,800	340	85.6%	3,722	3,381
Edit | Del	0050	12/31/2011	60	12,356	5,462	6,894	37.5%	3,979	(2,915)
Edit | Del	0051	12/31/2012	48	21,665	12,197	9,468	74.9%	12,839	3,370
Edit | Del	0052	12/31/2013	36	50,455	23,840	26,615	40.5%	21,590	(5,024)
Edit | Del	0053	12/31/2014	24	272,011	191,678	80,333	28.0%	59,458	(20,875)
Edit | Del	0054	SUM OF YRS		365,640	237,087	128,553		101,967	(26,586)
Edit | Del	0055	CY 2015		365,640	237,087	128,553	28.4%	96,676	(31,876)

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Appendix C – Back-Testing Results for Private Passenger Auto

Table C.1 – Calculation of Weighted Ultimate (Deterministic)

Sample Insurance Company
Private Passenger Auto
Calculation of Weighted Ultimate as of December 31, 2014

		Ultimate Values by Method				Weights by Method				Weighted
AY	Age	Paid CL	Inc CL	Paid BF	Inc BF	Paid CL	Inc CL	Paid BF	Inc BF	Ultimate
2006	108	1,218,574	1,218,574	1,218,578	1,218,577	50.0%	50.0%	0.0%	0.0%	1,218,574
2007	96	1,376,278	1,375,860	1,376,284	1,375,866	50.0%	50.0%	0.0%	0.0%	1,376,069
2008	84	1,439,598	1,439,241	1,439,624	1,439,261	50.0%	50.0%	0.0%	0.0%	1,439,420
2009	72	1,561,673	1,558,592	1,561,726	1,558,664	50.0%	50.0%	0.0%	0.0%	1,560,133
2010	60	1,649,696	1,645,907	1,649,700	1,646,004	50.0%	50.0%	0.0%	0.0%	1,647,802
2011	48	1,669,252	1,665,339	1,670,112	1,665,994	50.0%	50.0%	0.0%	0.0%	1,667,295
2012	36	1,746,970	1,739,396	1,750,509	1,741,935	25.0%	25.0%	25.0%	25.0%	1,744,703
2013	24	1,841,516	1,816,296	1,855,755	1,827,462	0.0%	0.0%	50.0%	50.0%	1,841,608
2014	12	1,897,487	1,829,829	1,944,009	1,877,128	0.0%	0.0%	50.0%	50.0%	1,910,569
Totals		14,401,045	14,289,034	14,466,298	14,350,890					14,406,172

Table C.2 – Reconciliation of Total Unpaid (Deterministic)

Sample Insurance Company
Private Passenger Auto
Total Unpaid Reconciliation as of December 31, 2014

AY	Age	Paid to Date	Incurred to Date	Weighted Ultimate	Case Reserve	IBNR	Total Unpaid	Selected Ultimate	Selected IBNR	Total Unpaid
2006	108	1,213,083	1,214,471	1,218,574	1,388	4,103	5,491	1,218,574	4,103	5,491
2007	96	1,367,086	1,369,955	1,376,069	2,869	6,114	8,983	1,376,069	6,114	8,983
2008	84	1,421,777	1,427,920	1,439,420	6,143	11,500	17,643	1,439,420	11,500	17,643
2009	72	1,526,896	1,538,117	1,560,133	11,221	22,016	33,237	1,560,133	22,016	33,237
2010	60	1,576,653	1,604,722	1,647,802	28,069	43,080	71,149	1,647,802	43,080	71,149
2011	48	1,520,024	1,584,626	1,667,295	64,602	82,669	147,271	1,667,295	82,669	147,271
2012	36	1,448,927	1,583,503	1,744,703	134,576	161,200	295,776	1,744,703	161,200	295,776
2013	24	1,284,015	1,535,603	1,841,608	251,588	306,005	557,593	1,841,608	306,005	557,593
2014	12	775,576	1,238,406	1,910,569	462,830	672,163	1,134,993	1,910,569	672,163	1,134,993
Totals		12,134,037	13,097,323	14,406,172	963,286	1,308,849	2,272,135	14,406,172	1,308,849	2,272,135

Table C.3 – Expected Incremental Development – Paid (Deterministic)

Sample Insurance Company
Private Passenger Auto -- Paid Data
Expected Incremental Future Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										2,742	2,749	5,491
2007									2,783	3,097	3,104	8,983
2008								8,029	3,128	3,239	3,247	17,643
2009							13,923	8,893	3,390	3,511	3,519	33,237
2010						34,453	16,297	9,393	3,581	3,708	3,717	71,149
2011					73,449	36,693	16,490	9,504	3,623	3,752	3,761	147,271
2012				139,035	79,111	38,585	17,340	9,994	3,810	3,946	3,955	295,776
2013			237,853	152,195	84,565	41,245	18,536	10,683	4,073	4,218	4,227	557,593
2014		547,018	256,629	157,719	87,634	42,742	19,208	11,071	4,220	4,371	4,381	1,134,993

Table C.4 – Expected Incremental Development – Incurred (Deterministic)

Sample Insurance Company
Private Passenger Auto -- Incurred Data
Expected Incremental Future Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										2,050	2,053	4,103
2007									1,481	2,315	2,319	6,114
2008								5,322	1,331	2,421	2,425	11,500
2009							9,743	5,576	1,443	2,624	2,629	22,016
2010						21,433	8,685	5,890	1,524	2,772	2,776	43,080
2011					40,949	19,818	8,788	5,959	1,542	2,805	2,809	82,669
2012				76,014	41,204	20,892	9,264	6,282	1,626	2,957	2,962	161,200
2013			135,434	78,332	44,616	22,622	10,031	6,802	1,760	3,201	3,207	306,005
2014		361,322	130,571	82,786	47,153	23,908	10,601	7,189	1,860	3,383	3,389	672,163

Table C.5 – Actual vs. Expected Back-test (Deterministic)

Sample Insurance Company
Private Passenger Auto
Deterministic Actual vs. Expected as of December 31, 2015

AY	Age	Actual Paid	Expected Paid	Difference	Actual Incurred	Expected Incurred	Difference
2006	120	2,500	2,742	(242)	2,042	2,050	(8)
2007	108	3,485	2,783	702	2,261	1,481	780
2008	96	7,582	8,029	(447)	4,061	5,322	(1,261)
2009	84	13,765	13,923	(158)	8,076	9,743	(1,667)
2010	72	33,083	34,453	(1,370)	16,495	21,433	(4,938)
2011	60	75,969	73,449	2,520	35,496	40,949	(5,453)
2012	48	139,715	139,035	680	68,886	76,014	(7,128)
2013	36	234,781	237,853	(3,072)	119,582	135,434	(15,852)
2014	24	560,974	547,018	13,956	314,895	361,322	(46,427)
2015	12	764,210			1,205,957		
Totals		1,836,064			1,777,751		
AY<CY		1,071,854	1,059,284	12,569	571,794	653,748	(81,954)

Table C.6 – Actual to Range of Estimates Back-test (Deterministic)

Sample Insurance Company
Private Passenger Auto
Deterministic Actual vs. Method Range as of December 31, 2015

AY	Age	Actual Paid	Paid Minimum	Paid Maximum	Range Percent	Actual Incurred	Incurred Minimum	Incurred Maximum	Difference
2006	120	2,500	2,742	2,744	-12977.0%	2,042	2,050	2,052	-332.1%
2007	108	3,485	2,574	2,993	217.7%	2,261	1,272	1,691	236.3%
2008	96	7,582	7,851	8,218	-73.5%	4,061	5,144	5,515	-291.9%
2009	84	13,765	12,402	15,469	44.5%	8,076	8,215	11,282	-4.5%
2010	72	33,083	32,601	36,307	13.0%	16,495	19,564	23,302	-82.1%
2011	60	75,969	71,579	75,753	105.2%	35,496	39,041	43,372	-81.8%
2012	48	139,715	134,970	143,551	55.3%	68,886	71,591	80,910	-29.0%
2013	36	234,781	222,411	249,543	45.6%	119,582	117,907	148,270	5.5%
2014	24	560,974	500,290	570,167	86.8%	314,895	308,639	389,322	7.8%
2015	12	764,210				1,205,957			
Totals		1,836,064				1,777,751			
AY<CY		1,071,854	987,421	1,104,745	72.0%	571,794	573,423	705,671	-1.2%

Table C.7 – Estimated Unpaid Claims by Accident Year (Stochastic)

Sample Insurance Company
Private Passenger Auto
Stochastic Estimates as of December 31, 2014
Estimated Unpaid Claims by Accident Year

AY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2006	5,491	2,751	50.1%	19	16,929	1,188	3,538	5,318	19	7,256	10,281
2007	8,983	3,423	38.1%	(395)	27,201	3,633	6,557	8,844	13,467	11,195	14,917
2008	17,643	4,155	23.6%	5,353	34,375	11,018	14,771	17,448	14,798	20,330	24,790
2009	33,237	5,245	15.8%	15,269	60,704	24,910	29,619	33,085	32,036	36,639	42,225
2010	71,149	6,902	9.7%	48,314	99,369	60,123	66,324	71,033	72,699	75,783	82,763
2011	147,271	9,088	6.2%	114,275	187,688	132,806	141,043	147,027	142,651	153,290	162,219
2012	295,776	14,568	4.9%	244,570	348,069	272,495	285,945	295,225	281,357	305,146	320,628
2013	557,593	25,394	4.6%	457,369	651,838	516,980	540,414	556,720	552,490	574,475	599,860
2014	1,134,993	46,822	4.1%	973,312	1,337,053	1,062,388	1,102,616	1,132,386	1,181,722	1,165,441	1,216,110
Total	2,272,135	59,102	2.6%	2,064,755	2,479,344	2,177,063	2,231,575	2,270,627	2,295,340	2,311,669	2,371,532

Table C.8 – Estimated Claims Paid by Calendar Year (Stochastic)

Sample Insurance Company Private Passenger Auto Stochastic Estimates as of December 31, 2014 Estimated Paid Claims by Calendar Year											
CY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2015	1,076,388	31,344	2.9%	949,483	1,213,672	1,025,966	1,054,657	1,075,871	1,048,875	1,096,712	1,129,462
2016	551,046	19,390	3.5%	479,596	631,486	519,806	537,516	550,967	553,695	564,102	582,949
2017	311,957	13,916	4.5%	259,341	367,185	289,477	302,543	311,686	316,778	321,297	335,118
2018	163,631	9,937	6.1%	130,776	200,970	147,538	156,774	163,477	162,064	170,225	180,340
2019	80,988	7,270	9.0%	52,760	116,518	69,328	76,043	80,859	84,649	85,870	93,146
2020	40,653	5,645	13.9%	20,217	62,342	31,712	36,714	40,478	39,787	44,381	50,138
2021	22,548	4,548	20.2%	7,784	40,869	15,431	19,416	22,362	21,178	25,499	30,348
2022	12,196	3,877	31.8%	(166)	29,026	6,142	9,531	12,012	8,133	14,672	18,808
2023	8,412	3,700	44.0%	(121)	27,344	2,614	5,876	8,238	(121)	10,742	14,779
2024	4,316	2,311	53.6%	(50)	15,575	764	2,652	4,155	(50)	5,756	8,407
Total	2,272,135	59,102	2.6%	2,064,755	2,479,344	2,177,063	2,231,575	2,270,627	2,295,340	2,311,669	2,371,532

Table C.9 – Mean Future Incremental – Paid (Stochastic)

Sample Insurance Company Private Passenger Auto - Paid Mean Future Incremental as of December 31, 2014												
AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										2,733	2,758	5,491
2007									2,908	3,022	3,053	8,983
2008								8,098	3,080	3,226	3,239	17,643
2009							14,773	8,493	3,216	3,363	3,392	33,237
2010						35,326	15,895	9,164	3,479	3,614	3,670	71,149
2011					74,381	36,251	16,246	9,369	3,594	3,713	3,719	147,271
2012				140,849	78,253	38,124	17,114	9,886	3,733	3,891	3,925	295,776
2013			243,390	149,664	83,084	40,493	18,186	10,534	3,985	4,107	4,150	557,593
2014		553,931	253,630	155,843	86,574	42,317	19,004	10,953	4,164	4,262	4,316	1,134,993

Table C.10 – Standard Deviation of Future Incremental – Paid (Stochastic)

Sample Insurance Company Private Passenger Auto - Paid Standard Deviation Future Incremental as of December 31, 2014												
AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										1,534	1,543	2,751
2007									1,496	1,721	1,722	3,423
2008								2,135	1,567	1,785	1,763	4,155
2009							2,748	2,262	1,679	1,864	1,895	5,245
2010						4,154	2,887	2,321	1,745	1,952	1,988	6,902
2011					5,827	4,105	2,892	2,358	1,770	1,987	2,013	9,088
2012				8,864	6,479	4,403	3,076	2,516	1,860	2,084	2,091	14,568
2013			13,598	9,804	6,879	4,728	3,270	2,652	1,990	2,215	2,225	25,394
2014		25,362	14,095	10,125	7,121	4,866	3,297	2,703	2,032	2,275	2,311	46,822

Table C.11 – Coefficient of Variation of Future Incremental – Paid (Stochastic)

Sample Insurance Company Private Passenger Auto - Paid CoV Future Incremental as of December 31, 2014												
AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										56.1%	55.9%	50.1%
2007									51.4%	57.0%	56.4%	38.1%
2008								26.4%	50.9%	55.3%	54.4%	23.6%
2009							18.6%	26.6%	52.2%	55.4%	55.9%	15.8%
2010						11.8%	18.2%	25.3%	50.2%	54.0%	54.2%	9.7%
2011					7.8%	11.3%	17.8%	25.2%	49.3%	53.5%	54.1%	6.2%
2012				6.3%	8.3%	11.5%	18.0%	25.5%	49.8%	53.5%	53.3%	4.9%
2013			5.6%	6.6%	8.3%	11.7%	18.0%	25.2%	49.9%	53.9%	53.6%	4.6%
2014		4.6%	5.6%	6.5%	8.2%	11.5%	17.3%	24.7%	48.8%	53.4%	53.6%	4.1%

Table C.12 – Estimated Unpaid Claims by Accident Year in 2015 (Stochastic)

Sample Insurance Company
Private Passenger Auto - Paid
Stochastic Estimates as of December 31, 2014
Estimated Unpaid Claims by Accident Year, Calendar Year 2015 Only

AY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2006	2,733	1,534	56.1%	9	9,689	444	1,629	2,563	9	3,697	5,509
2007	2,908	1,496	51.4%	(269)	10,441	750	1,873	2,750	(252)	3,766	5,640
2008	8,098	2,135	26.4%	1,608	20,022	4,867	6,616	7,934	8,649	9,413	11,850
2009	14,773	2,748	18.6%	6,175	26,858	10,506	12,878	14,607	13,421	16,523	19,567
2010	35,326	4,154	11.8%	19,713	52,817	28,828	32,396	35,169	36,788	38,033	42,514
2011	74,381	5,827	7.8%	52,662	98,238	65,082	70,380	74,239	70,540	78,233	84,209
2012	140,849	8,864	6.3%	105,135	178,702	126,665	134,837	140,706	140,360	146,614	155,792
2013	243,390	13,598	5.6%	189,263	302,308	221,056	234,122	243,174	238,506	252,536	266,186
2014	553,931	25,362	4.6%	462,086	667,072	513,991	536,419	553,004	547,742	570,306	597,839
Total	1,076,388	31,344	2.9%	949,483	1,213,672	1,025,966	1,054,657	1,075,871	1,048,875	1,096,712	1,129,462

Table C.13 – Actual vs. Expected Back-test & Conditional Reserve (Stochastic)

Sample Insurance Company
Private Passenger Auto
Stochastic Actual vs. Expected as of December 31, 2015

AY	Age	Actual Paid	Expected Paid	Percentile	Actual Incurred	Expected Incurred	Percentile	Conditional Reserve	Expected Reserve	Change
2006	120	2,500	2,733	48.2%	2,042	2,056	56.7%	2,680	2,991	(311)
2007	108	3,485	2,908	69.4%	2,261	1,312	81.0%	7,248	5,498	1,750
2008	96	7,582	8,098	43.4%	4,061	5,207	33.2%	8,654	10,061	(1,406)
2009	84	13,765	14,773	37.5%	8,076	8,835	41.7%	15,635	19,472	(3,836)
2010	72	33,083	35,326	30.5%	16,495	20,439	15.6%	31,595	38,066	(6,470)
2011	60	75,969	74,381	61.4%	35,496	40,022	21.2%	73,359	71,302	2,057
2012	48	139,715	140,849	45.5%	68,886	74,159	25.6%	151,670	156,061	(4,390)
2013	36	234,781	243,390	26.5%	119,582	128,507	20.2%	292,882	322,812	(29,930)
2014	24	560,974	553,931	62.3%	314,895	350,974	2.9%	581,448	574,019	7,430
2015	12	764,210			1,205,957					
Totals		1,836,064			1,777,751			1,165,174	1,200,281	(35,107)
AY<CY		1,071,854	1,076,388	44.9%	571,794	631,511	0.6%	1,159,897	1,200,281	(40,385)

Figure C.1 – Graph of KPI Thresholds by Accident Year – Paid (Stochastic)

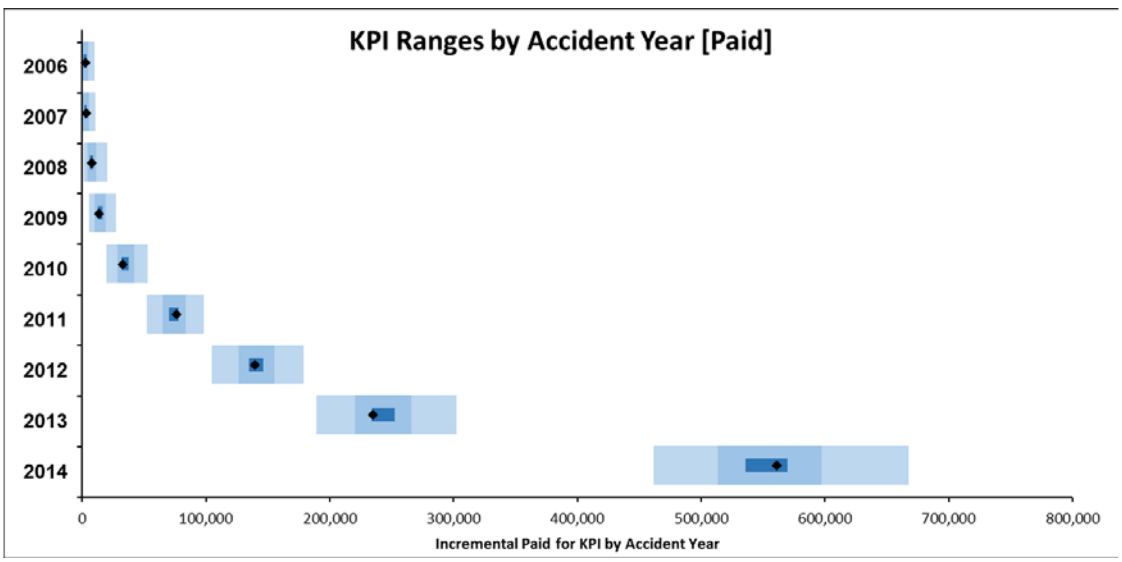


Figure C.2 – Graph of KPI Thresholds by Calendar Year – Paid (Stochastic)

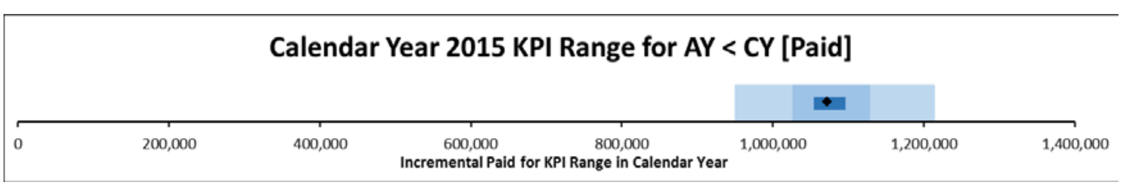


Figure C.3 – Graph of KPI Thresholds by Accident Year – Incurred (Stochastic)

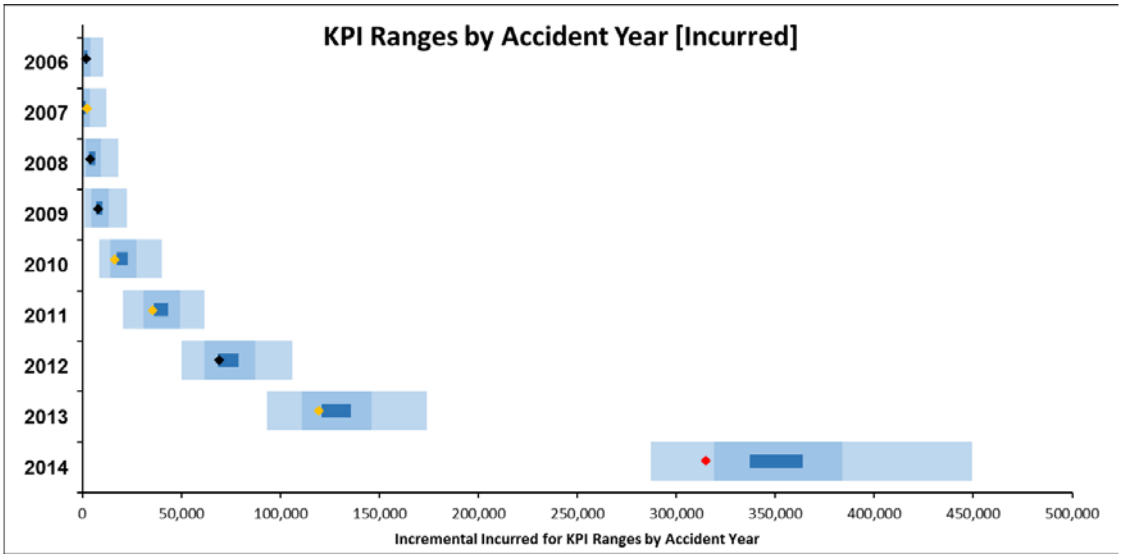


Figure C.4 – Graph of KPI Thresholds by Calendar Year – Incurred (Stochastic)

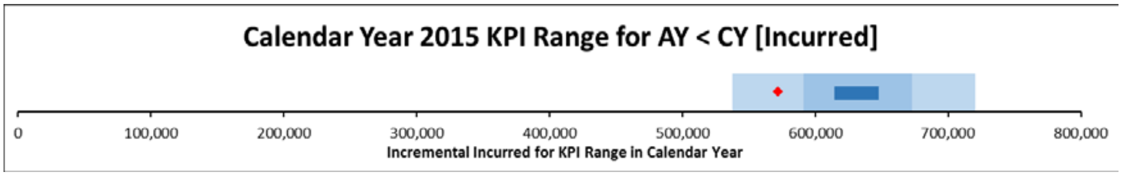


Figure C.5 – Graph of Realized Values vs. Assumptions – Paid (Stochastic)

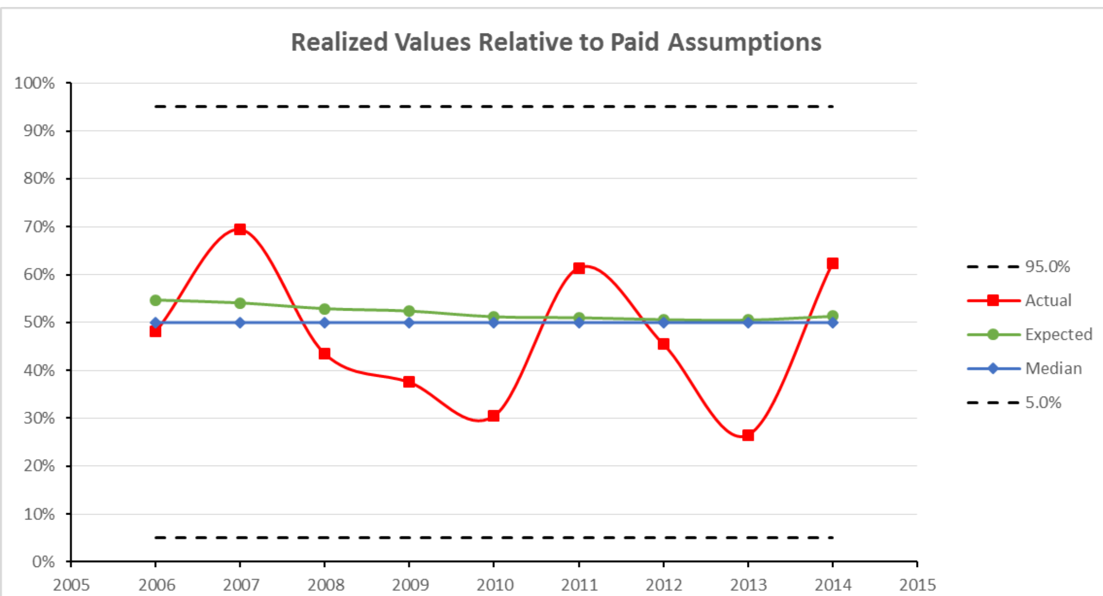
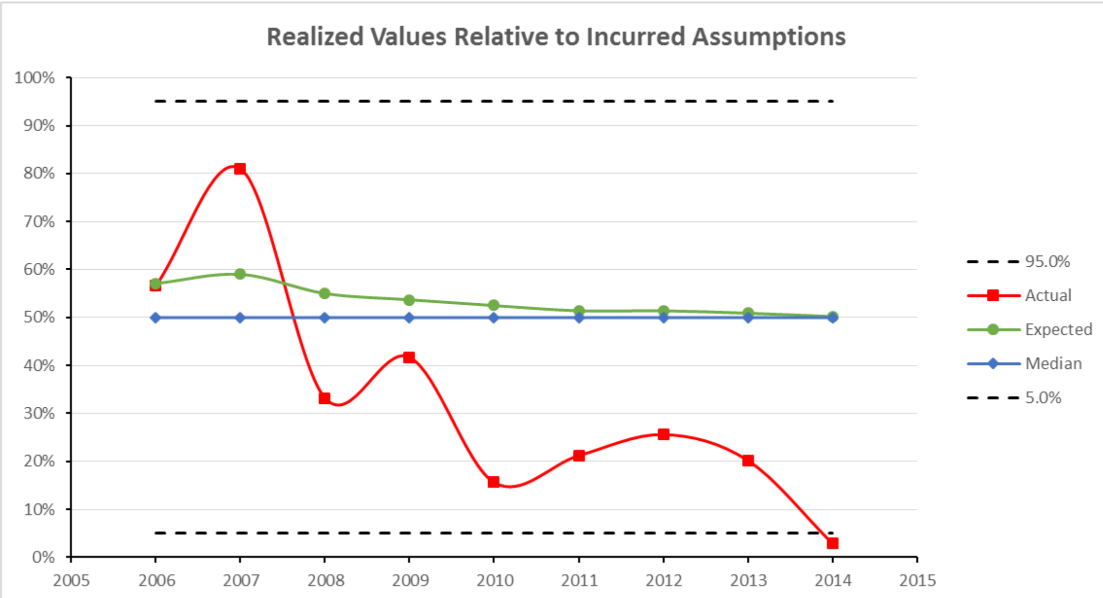


Figure C.6 – Graph of Realized Values vs. Assumptions – Incurred (Stochastic)


Realized Values Relative to Incurred Assumptions
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
2005
2006
2007
2008
2009
2010
2011
2012
2013
2014
2015
95.0%
Actual
Expected
Median
5.0%

Appendix D – Back-Testing Results for Commercial Auto

Table D.1 – Calculation of Weighted Ultimate (Deterministic)

Sample Insurance Company
Commercial Auto
Calculation of Weighted Ultimate as of December 31, 2014

		Ultimate Values by Method				Weights by Method				Weighted
AY	Age	Paid CL	Inc CL	Paid BF	Inc BF	Paid CL	Inc CL	Paid BF	Inc BF	Ultimate
2006	108	258,835	258,835	258,837	258,836	50.0%	50.0%	0.0%	0.0%	258,835
2007	96	267,103	271,591	267,143	271,592	50.0%	50.0%	0.0%	0.0%	269,347
2008	84	243,981	244,137	243,991	244,141	50.0%	50.0%	0.0%	0.0%	244,059
2009	72	267,942	269,784	267,999	269,783	50.0%	50.0%	0.0%	0.0%	268,863
2010	60	290,475	292,079	290,608	292,092	50.0%	50.0%	0.0%	0.0%	291,277
2011	48	288,645	288,592	288,785	288,669	50.0%	50.0%	0.0%	0.0%	288,618
2012	36	335,023	338,775	335,956	338,702	25.0%	25.0%	25.0%	25.0%	337,114
2013	24	333,220	337,698	333,662	336,635	0.0%	0.0%	50.0%	50.0%	335,149
2014	12	357,305	360,286	338,097	344,953	0.0%	0.0%	50.0%	50.0%	341,525
Totals		2,642,529	2,661,779	2,625,078	2,645,402					2,634,788

Table D.2 – Reconciliation of Total Unpaid (Deterministic)

Sample Insurance Company
Commercial Auto
Total Unpaid Reconciliation as of December 31, 2014

AY	Age	Paid to Date	Incurred to Date	Weighted Ultimate	Case Reserve	IBNR	Total Unpaid	Selected Ultimate	Selected IBNR	Total Unpaid
2006	108	257,689	258,524	258,835	835	311	1,146	258,835	311	1,146
2007	96	264,871	270,758	269,347	5,887	(1,411)	4,476	271,500	742	6,629
2008	84	240,300	242,171	244,059	1,871	1,888	3,759	244,059	1,888	3,759
2009	72	259,339	265,496	268,863	6,157	3,367	9,524	268,863	3,367	9,524
2010	60	270,525	281,376	291,277	10,851	9,901	20,752	291,277	9,901	20,752
2011	48	245,541	266,101	288,618	20,560	22,517	43,077	288,618	22,517	43,077
2012	36	240,652	282,394	337,114	41,742	54,720	96,462	337,114	54,720	96,462
2013	24	177,709	235,983	335,149	58,274	99,166	157,440	335,149	99,166	157,440
2014	12	105,547	177,611	341,525	72,064	163,914	235,978	341,525	163,914	235,978
Totals		2,062,173	2,280,414	2,634,788	218,241	354,374	572,615	2,636,941	356,527	574,768

Table D.3 – Expected Incremental Development – Paid (Deterministic)

Sample Insurance Company
Commercial Auto -- Paid Data
Expected Incremental Future Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										572	574	1,146
2007									4,863	882	884	6,629
2008								1,720	959	540	541	3,759
2009							5,468	1,810	1,056	595	596	9,524
2010						11,401	4,957	1,961	1,144	644	646	20,752
2011					23,255	10,556	4,912	1,943	1,134	638	640	43,077
2012				45,941	27,285	12,374	5,758	2,277	1,329	748	750	96,462
2013			62,890	44,425	27,071	12,277	5,712	2,259	1,319	742	744	157,440
2014		80,388	61,679	44,125	26,889	12,194	5,674	2,244	1,310	737	739	235,978

Table D.4 – Expected Incremental Development – Incurred (Deterministic)

Sample Insurance Company
Commercial Auto -- Incurred Data
Expected Incremental Future Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										155	156	311
2007									912	(85)	(85)	742
2008								1,140	455	147	147	1,888
2009							1,202	1,341	502	161	162	3,367
2010						5,271	2,284	1,452	544	175	175	9,901
2011					11,941	5,989	2,263	1,439	539	173	173	22,517
2012				28,462	13,911	6,991	2,642	1,680	629	202	202	54,720
2013			43,797	29,442	13,736	6,903	2,609	1,659	621	200	200	99,166
2014		65,492	44,040	28,917	13,491	6,780	2,562	1,629	610	196	196	163,914

Table D.5 – Actual vs. Expected Back-test (Deterministic)

Sample Insurance Company Commercial Auto Deterministic Actual vs. Expected as of December 31, 2015							
AY	Age	Actual Paid	Expected Paid	Difference	Actual Incurred	Expected Incurred	Difference
2006	120	543	572	(29)	(47)	155	(202)
2007	108	2,387	4,863	(2,476)	1,040	912	128
2008	96	1,177	1,720	(543)	851	1,140	(289)
2009	84	5,403	5,468	(65)	2,954	1,202	1,752
2010	72	14,120	11,401	2,719	9,035	5,271	3,764
2011	60	23,636	23,255	381	16,524	11,941	4,583
2012	48	51,020	45,941	5,079	36,454	28,462	7,992
2013	36	75,813	62,890	12,923	61,541	43,797	17,744
2014	24	88,832	80,388	8,444	83,154	65,492	17,662
2015	12	99,123			178,539		
Totals		362,054			390,045		
AY<CY		262,931	236,497	26,434	211,506	158,372	53,134

Table D.6 – Actual to Range of Estimates Back-test (Deterministic)

Sample Insurance Company Commercial Auto Deterministic Actual vs. Method Range as of December 31, 2015									
AY	Age	Actual Paid	Paid Minimum	Paid Maximum	Range Percent	Actual Incurred	Incurred Minimum	Incurred Maximum	Difference
2006	120	543	572	573	-1947.6%	(47)	155	157	-11482.4%
2007	108	2,387	2,629	7,097	-5.4%	1,040	(1,329)	3,154	52.8%
2008	96	1,177	1,642	1,797	-300.2%	851	1,062	1,220	-133.1%
2009	84	5,403	4,560	6,375	46.4%	2,954	288	2,116	145.9%
2010	72	14,120	10,624	12,177	225.1%	9,035	4,482	6,067	287.2%
2011	60	23,636	23,230	23,355	323.6%	16,524	11,915	12,068	3013.1%
2012	48	51,020	44,341	47,533	209.3%	36,454	26,520	29,980	287.1%
2013	36	75,813	61,648	64,865	440.3%	61,541	41,780	45,513	529.3%
2014	24	88,832	78,521	86,597	127.7%	83,154	63,052	74,156	181.0%
2015	12	99,123				178,539			
Totals		362,054				390,045			
AY<CY		262,931	228,631	250,242	158.7%	211,506	149,974	174,267	253.3%

Table D.7 – Estimated Unpaid Claims by Accident Year (Stochastic)

Sample Insurance Company Commercial Auto Stochastic Estimates as of December 31, 2014 Estimated Unpaid Claims by Accident Year											
AY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2006	1,146	814	71.0%	(10)	5,794	78	535	1,001	(10)	1,614	2,674
2007	6,629	1,224	18.5%	4,226	12,888	4,900	5,718	6,480	5,217	7,369	8,901
2008	3,759	1,453	38.6%	301	11,438	1,635	2,703	3,633	2,931	4,649	6,345
2009	9,524	2,142	22.5%	3,182	20,485	6,275	8,015	9,377	10,379	10,869	13,349
2010	20,752	3,200	15.4%	10,281	35,184	15,708	18,540	20,585	18,785	22,831	26,235
2011	43,077	4,575	10.6%	26,937	64,990	35,935	39,920	42,912	45,008	46,064	50,902
2012	96,462	8,635	9.0%	64,159	131,809	82,929	90,631	96,052	94,959	101,869	111,214
2013	157,440	14,252	9.1%	106,918	218,146	134,900	147,693	157,063	161,109	166,699	181,556
2014	235,978	20,115	8.5%	165,204	320,049	204,296	222,059	235,235	228,038	249,252	269,810
Total	574,768	27,218	4.7%	472,897	687,879	530,792	556,111	574,426	558,264	592,649	620,040

Table D.8 – Estimated Claims Paid by Calendar Year (Stochastic)

Sample Insurance Company Commercial Auto Stochastic Estimates as of December 31, 2014 Estimated Paid Claims by Calendar Year											
CY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2015	232,199	12,743	5.5%	186,133	286,448	211,733	223,345	231,854	239,707	240,793	253,653
2016	155,214	10,078	6.5%	123,220	202,461	138,975	148,466	154,950	152,408	161,829	172,239
2017	94,488	7,627	8.1%	67,914	124,583	82,240	89,213	94,253	97,115	99,485	107,381
2018	49,452	5,311	10.7%	33,520	73,129	40,823	45,820	49,320	49,423	52,929	58,355
2019	22,776	3,557	15.6%	10,658	37,548	17,087	20,273	22,624	21,106	25,137	28,853
2020	10,624	2,554	24.0%	2,401	21,272	6,697	8,827	10,460	11,167	12,231	15,060
2021	4,974	1,804	36.3%	522	13,768	2,328	3,680	4,783	5,419	6,057	8,218
2022	2,823	1,412	50.0%	(123)	11,759	872	1,773	2,649	2,360	3,651	5,416
2023	1,476	950	64.4%	8	7,844	222	771	1,325	8	2,002	3,244
2024	741	621	83.8%	4	4,737	28	275	596	4	1,045	1,956
Total	574,768	27,218	4.7%	472,897	687,879	530,792	556,111	574,426	558,264	592,649	620,040

Table D.9 – Mean Future Incremental – Paid (Stochastic)

Sample Insurance Company Commercial Auto - Paid Mean Future Incremental as of December 31, 2014												
AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										571	575	1,146
2007									3,131	1,735	1,763	6,629
2008								1,665	983	557	555	3,759
2009							5,044	1,988	1,170	657	666	9,524
2010						11,061	5,146	2,028	1,189	658	672	20,752
2011					23,276	10,564	4,895	1,925	1,135	636	646	43,077
2012				45,272	27,668	12,508	5,837	2,304	1,348	757	768	96,462
2013			62,481	44,600	27,194	12,354	5,746	2,265	1,308	744	746	157,440
2014		79,698	61,955	44,373	26,936	12,267	5,703	2,264	1,311	730	741	235,978

Table D.10 – Standard Deviation of Future Incremental – Paid (Stochastic)

Sample Insurance Company Commercial Auto - Paid Standard Deviation Future Incremental as of December 31, 2014												
AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										515	519	814
2007									881	534	538	1,224
2008								908	826	500	500	1,453
2009							1,465	990	879	523	533	2,142
2010						2,208	1,565	1,042	912	547	559	3,200
2011					3,189	2,197	1,559	1,027	908	563	556	4,575
2012				5,203	3,869	2,573	1,795	1,181	1,062	626	625	8,635
2013			7,006	5,566	4,081	2,625	1,792	1,197	1,056	629	634	14,252
2014		8,276	6,947	5,516	4,013	2,599	1,783	1,182	1,064	623	621	20,115

Table D.11 – Coefficient of Variation of Future Incremental – Paid (Stochastic)

Sample Insurance Company Commercial Auto - Paid CoV Future Incremental as of December 31, 2014												
AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										90.1%	90.2%	71.0%
2007									28.2%	30.8%	30.5%	18.5%
2008								54.6%	84.0%	89.8%	90.1%	38.6%
2009							29.0%	49.8%	75.2%	79.6%	80.0%	22.5%
2010						20.0%	30.4%	51.4%	76.7%	83.2%	83.2%	15.4%
2011					13.7%	20.8%	31.8%	53.4%	80.0%	88.5%	86.1%	10.6%
2012				11.5%	14.0%	20.6%	30.7%	51.3%	78.8%	82.7%	81.3%	9.0%
2013			11.2%	12.5%	15.0%	21.2%	31.2%	52.8%	80.8%	84.5%	84.9%	9.1%
2014		10.4%	11.2%	12.4%	14.9%	21.2%	31.3%	52.2%	81.2%	85.4%	83.8%	8.5%

Table D.12 – Estimated Unpaid Claims by Accident Year in 2015 (Stochastic)

Sample Insurance Company
Commercial Auto - Paid
Stochastic Estimates as of December 31, 2014
Estimated Unpaid Claims by Accident Year, Calendar Year 2015 Only

AY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2006	571	515	90.1%	(5)	4,550	7	182	441	(5)	813	1,573
2007	3,131	881	28.2%	1,923	8,619	2,052	2,457	2,966	2,052	3,634	4,804
2008	1,665	908	54.6%	47	6,639	440	990	1,522	1,421	2,191	3,355
2009	5,044	1,465	29.0%	1,265	11,797	2,893	3,975	4,902	5,069	5,945	7,666
2010	11,061	2,208	20.0%	4,960	20,538	7,667	9,509	10,915	10,312	12,486	14,886
2011	23,276	3,189	13.7%	13,209	37,472	18,316	21,040	23,131	21,086	25,331	28,725
2012	45,272	5,203	11.5%	28,879	68,025	37,212	41,731	44,991	42,206	48,538	54,277
2013	62,481	7,006	11.2%	36,066	90,980	51,265	57,668	62,265	61,583	67,022	74,418
2014	79,698	8,276	10.4%	49,321	113,281	66,688	74,012	79,329	73,977	85,090	93,641
Total	232,199	12,743	5.5%	186,133	286,448	211,733	223,345	231,854	239,707	240,793	253,653

Table D.13 – Actual vs. Expected Back-test & Conditional Reserve (Stochastic)

Sample Insurance Company
Commercial Auto
Stochastic Actual vs. Expected as of December 31, 2015

AY	Age	Actual Paid	Expected Paid	Percentile	Actual Incurred	Expected Incurred	Percentile	Conditional Reserve	Expected Reserve	Change
2006	120	543	571	57.9%	(47)	154	0.0%	643	603	40
2007	108	2,387	3,131	21.8%	1,040	448	82.8%	3,257	4,242	(985)
2008	96	1,177	1,665	33.5%	851	1,167	44.5%	1,675	2,582	(907)
2009	84	5,403	5,044	63.1%	2,954	1,669	86.1%	5,593	4,121	1,472
2010	72	14,120	11,061	91.1%	9,035	5,606	94.2%	13,946	6,632	7,313
2011	60	23,636	23,276	56.1%	16,524	11,960	93.9%	20,073	19,441	632
2012	48	51,020	45,272	86.7%	36,454	29,103	92.7%	57,978	45,442	12,536
2013	36	75,813	62,481	96.5%	61,541	44,392	99.3%	110,701	81,627	29,075
2014	24	88,832	79,698	86.1%	83,154	66,555	97.0%	170,589	147,146	23,442
2015	12	99,123			178,539					
Totals		362,054			390,045			384,456	311,837	72,619
AY<CY		262,931	232,199	98.9%	211,506	161,054	100.0%	390,213	311,837	78,376

Figure D.1 – Graph of KPI Thresholds by Accident Year – Paid (Stochastic)

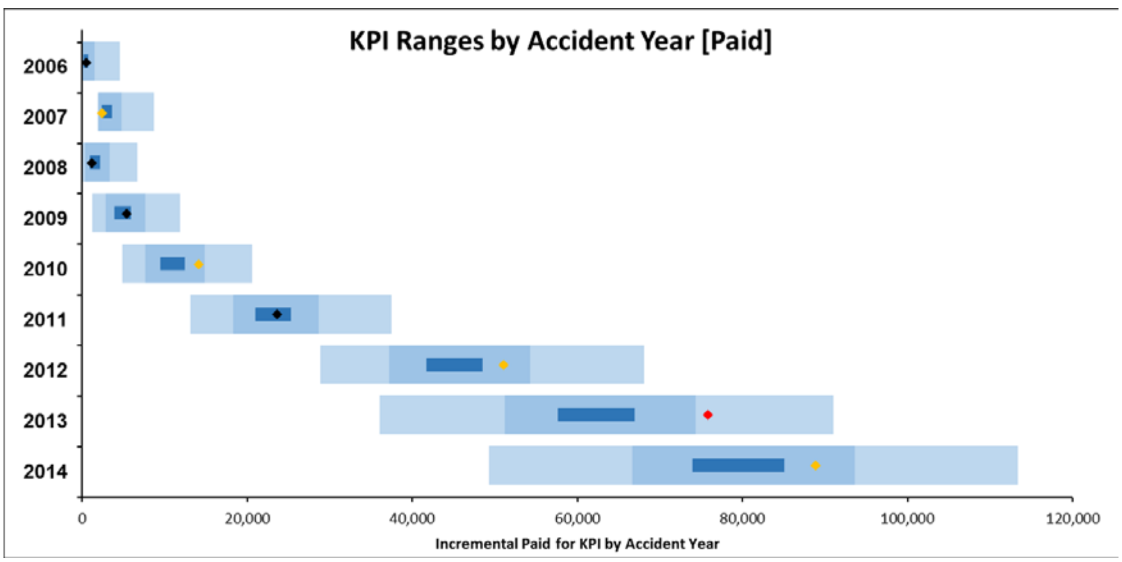


Figure D.2 – Graph of KPI Thresholds by Calendar Year – Paid (Stochastic)

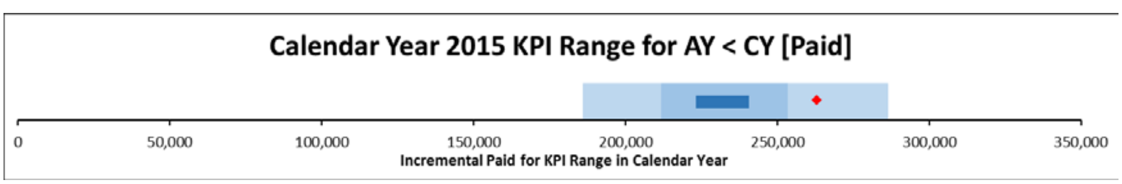


Figure D.3 – Graph of KPI Thresholds by Accident Year – Incurred (Stochastic)

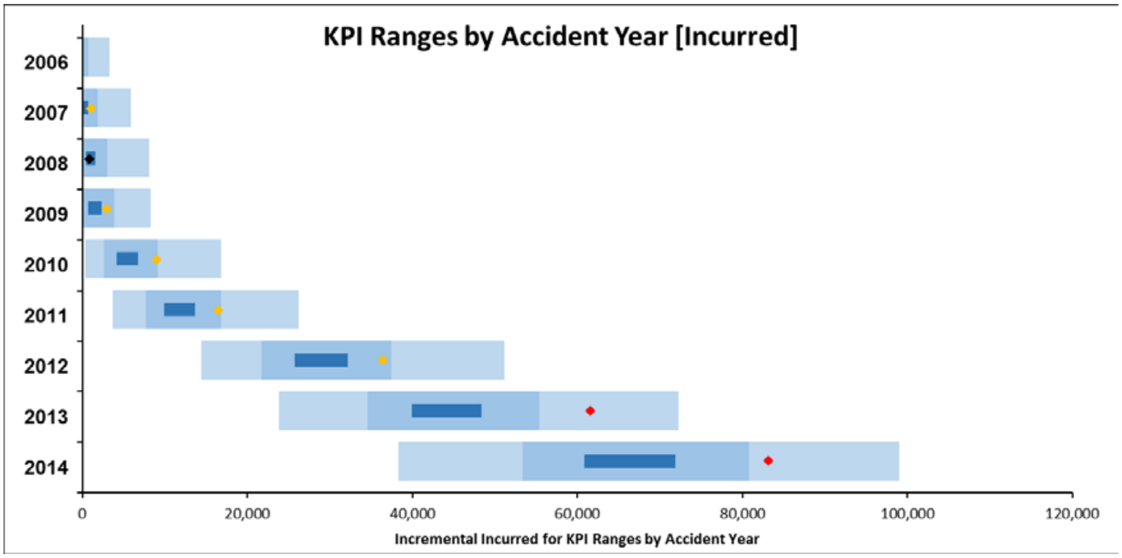


Figure D.4 – Graph of KPI Thresholds by Calendar Year – Incurred (Stochastic)

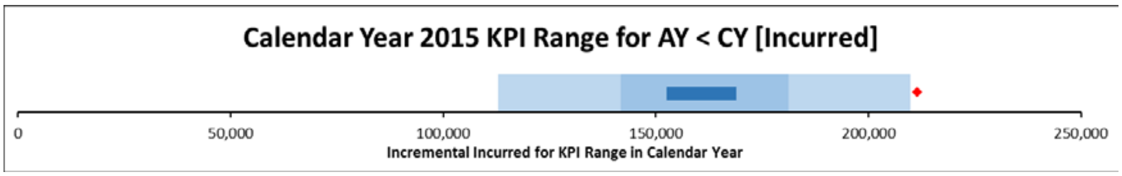


Figure D.5 – Graph of Realized Values vs. Assumptions – Paid (Stochastic)

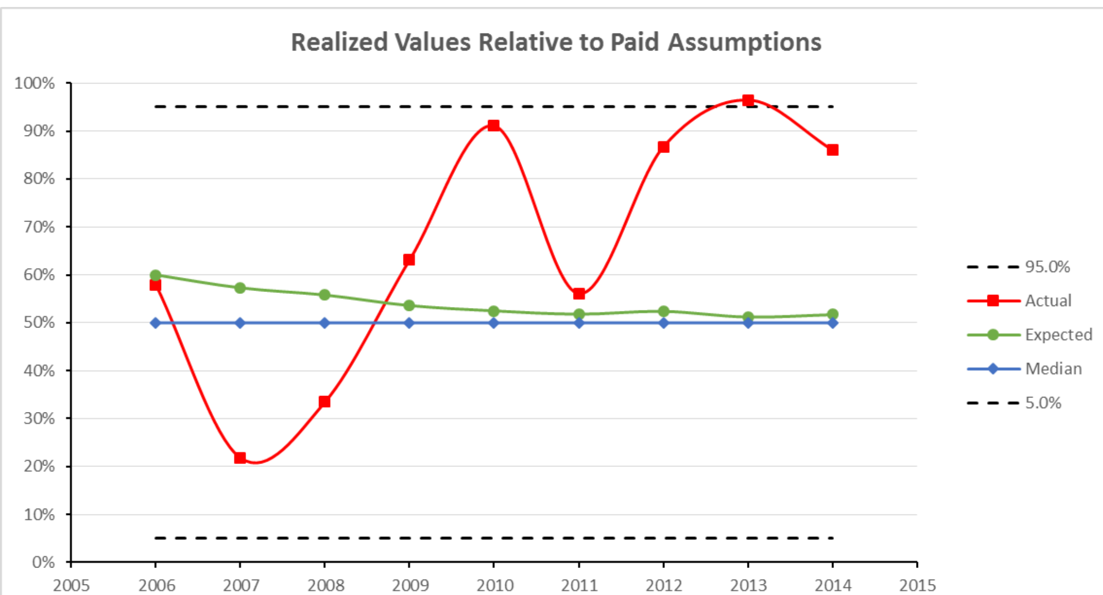
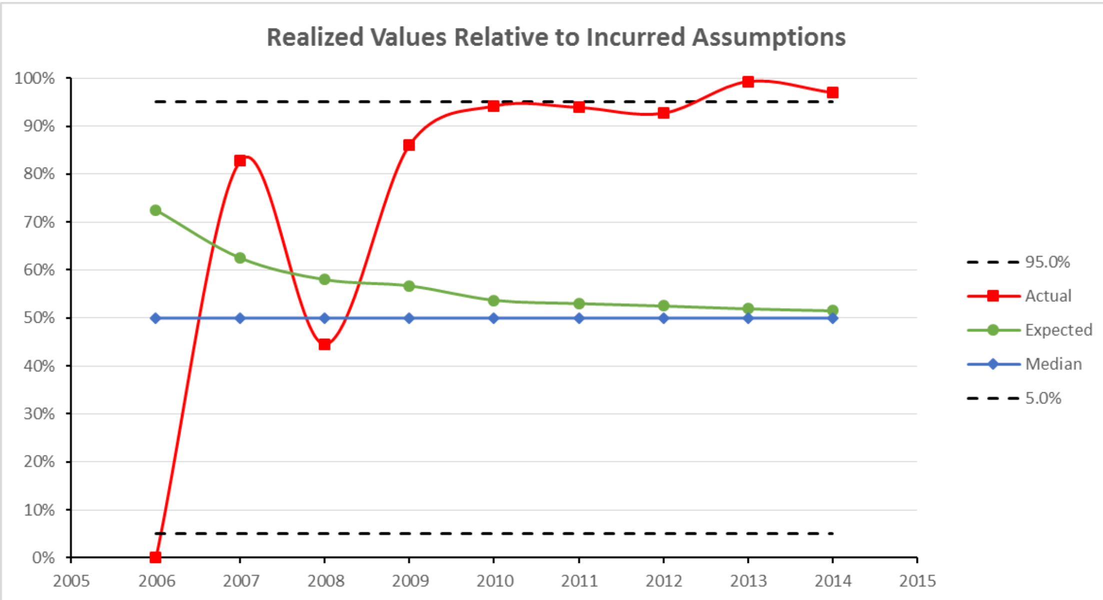


Figure D.2 – Graph of Realized Values vs. Assumptions – Incurred (Stochastic)



Appendix E – Back-Testing Results for Homeowners

Table E.1 – Calculation of Weighted Ultimate (Deterministic)

Sample Insurance Company
Homeowners
Calculation of Weighted Ultimate as of December 31, 2014

		Ultimate Values by Method				Weights by Method				Weighted
AY	Age	Paid CL	Inc CL	Paid BF	Inc BF	Paid CL	Inc CL	Paid BF	Inc BF	Ultimate
2006	108	328,806	328,806	328,806	328,806	50.0%	50.0%	0.0%	0.0%	328,806
2007	96	423,382	422,484	423,380	422,484	50.0%	50.0%	0.0%	0.0%	422,933
2008	84	542,749	542,575	542,751	542,574	50.0%	50.0%	0.0%	0.0%	542,662
2009	72	551,124	549,747	551,123	549,745	50.0%	50.0%	0.0%	0.0%	550,435
2010	60	680,803	678,422	680,808	678,412	50.0%	50.0%	0.0%	0.0%	679,612
2011	48	758,487	757,002	758,506	756,997	50.0%	50.0%	0.0%	0.0%	757,744
2012	36	702,481	700,796	702,653	700,788	25.0%	25.0%	25.0%	25.0%	701,679
2013	24	801,498	797,111	801,473	797,161	0.0%	0.0%	50.0%	50.0%	799,317
2014	12	992,257	996,379	993,794	996,481	0.0%	0.0%	50.0%	50.0%	995,137
Totals		5,781,585	5,773,322	5,783,294	5,773,446					5,778,327

Table E.2 – Reconciliation of Total Unpaid (Deterministic)

Sample Insurance Company
Homeowners
Total Unpaid Reconciliation as of December 31, 2014

AY	Age	Paid to Date	Incurred to Date	Weighted Ultimate	Case Reserve	IBNR	Total Unpaid	Selected Ultimate	Selected IBNR	Total Unpaid
2006	108	328,033	328,901	328,806	868	(95)	773	328,806	(95)	773
2007	96	422,179	422,654	422,933	475	279	754	422,933	279	754
2008	84	540,795	543,199	542,662	2,404	(537)	1,867	542,662	(537)	1,867
2009	72	548,818	550,729	550,435	1,911	(294)	1,617	550,435	(294)	1,617
2010	60	675,472	680,658	679,612	5,186	(1,046)	4,140	679,612	(1,046)	4,140
2011	48	745,388	758,597	757,744	13,209	(853)	12,356	757,744	(853)	12,356
2012	36	680,014	701,622	701,679	21,608	57	21,665	701,679	57	21,665
2013	24	748,862	787,351	799,317	38,489	11,966	50,455	799,317	11,966	50,455
2014	12	723,126	930,676	995,137	207,550	64,461	272,011	995,137	64,461	272,011
Totals		5,412,687	5,704,387	5,778,327	291,700	73,940	365,640	5,778,327	73,940	365,640

Table E.3 – Expected Incremental Development – Paid (Deterministic)

Sample Insurance Company
Homeowners -- Paid Data
Expected Incremental Future Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										386	387	773
2007									(240)	497	497	754
2008								325	266	638	638	1,867
2009							(364)	418	270	647	647	1,617
2010						1,297	397	516	333	798	799	4,140
2011					6,423	2,763	443	575	371	890	891	12,356
2012				9,503	6,648	2,568	412	535	345	827	828	21,665
2013			24,902	11,755	7,541	2,913	467	607	391	939	940	50,455
2014		206,388	33,665	14,702	9,432	3,643	584	759	489	1,174	1,175	272,011

Table E.4 – Expected Incremental Development – Incurred (Deterministic)

Sample Insurance Company
Homeowners -- Incurred Data
Expected Incremental Future Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										(48)	(47)	(95)
2007									401	(61)	(61)	279
2008								(319)	(61)	(78)	(78)	(537)
2009							340	(412)	(62)	(80)	(80)	(294)
2010						169	(432)	(509)	(76)	(98)	(98)	(1,046)
2011					1,645	(1,143)	(482)	(568)	(85)	(109)	(109)	(853)
2012				1,543	839	(1,064)	(449)	(528)	(79)	(102)	(102)	57
2013			12,913	745	955	(1,212)	(511)	(602)	(90)	(116)	(116)	11,966
2014		52,259	13,378	925	1,185	(1,504)	(634)	(747)	(112)	(144)	(144)	64,461

Table E.5 – Actual vs. Expected Back-test (Deterministic)

Sample Insurance Company Homeowners Deterministic Actual vs. Expected as of December 31, 2015							
AY	Age	Actual Paid	Expected Paid	Difference	Actual Incurred	Expected Incurred	Difference
2006	120	26	386	(360)	(132)	(48)	(84)
2007	108	33	(240)	273	(156)	401	(557)
2008	96	227	325	(98)	(1,359)	(319)	(1,040)
2009	84	(176)	(364)	188	(1,158)	340	(1,498)
2010	72	3,800	1,297	2,503	412	169	243
2011	60	5,462	6,423	(961)	(8)	1,645	(1,653)
2012	48	12,197	9,503	2,694	1,284	1,543	(259)
2013	36	23,840	24,902	(1,062)	8,785	12,913	(4,128)
2014	24	191,678	206,388	(14,710)	56,168	52,259	3,909
2015	12	934,805			1,143,739		
Totals		1,171,892			1,207,575		
AY<CY		237,087	248,619	(11,532)	63,836	68,902	(5,066)

Table E.6 – Actual to Range of Estimates Back-test (Deterministic)

Sample Insurance Company Homeowners Deterministic Actual vs. Method Range as of December 31, 2015									
AY	Age	Actual Paid	Paid Minimum	Paid Maximum	Range Percent	Actual Incurred	Incurred Minimum	Incurred Maximum	Difference
2006	120	26	386	386	-143771.0%	(132)	(48)	(47)	-33682.3%
2007	108	33	(688)	207	80.5%	(156)	(48)	850	-12.1%
2008	96	227	235	413	-4.6%	(1,359)	(407)	(229)	-534.5%
2009	84	(176)	(1,051)	322	63.7%	(1,158)	(350)	1,030	-58.5%
2010	72	3,800	99	2,485	155.1%	412	(1,028)	1,372	60.0%
2011	60	5,462	5,673	7,170	-14.1%	(8)	900	2,417	-59.9%
2012	48	12,197	8,582	10,415	197.2%	1,284	650	2,526	33.8%
2013	36	23,840	22,756	27,002	25.5%	8,785	10,700	15,091	-43.6%
2014	24	191,678	203,968	207,819	-319.1%	56,168	49,431	53,586	162.1%
2015	12	934,805				1,143,739			
Totals		1,171,892				1,207,575			
AY<CY		237,087	243,694	253,519	-67.2%	63,836	63,878	73,919	-0.4%

Table E.7 – Estimated Unpaid Claims by Accident Year (Stochastic)

Sample Insurance Company Homeowners Stochastic Estimates as of December 31, 2014 Estimated Unpaid Claims by Accident Year											
AY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2006	773	920	119.1%	(18)	7,510	(16)	121	459	(18)	1,101	2,668
2007	754	1,334	176.9%	(2,345)	11,715	(831)	(164)	445	(446)	1,359	3,384
2008	1,867	1,847	98.9%	(2,791)	15,138	(541)	573	1,534	1,422	2,847	5,402
2009	1,617	1,975	122.1%	(4,363)	14,310	(989)	206	1,315	921	2,700	5,238
2010	4,140	2,932	70.8%	(4,812)	24,814	9	2,020	3,791	1,561	5,885	9,480
2011	12,356	4,435	35.9%	404	35,123	5,775	9,158	11,996	12,056	15,160	20,191
2012	21,665	5,686	26.2%	5,673	46,724	13,069	17,642	21,254	23,445	25,267	31,717
2013	50,455	9,708	19.2%	23,208	98,051	35,582	43,515	49,808	41,265	56,737	67,307
2014	272,011	30,285	11.1%	176,947	402,593	224,048	250,890	271,241	293,093	291,855	323,755
Total	365,640	33,369	9.1%	247,985	505,728	312,138	342,419	364,523	360,985	387,991	421,695

Table E.8 – Estimated Claims Paid by Calendar Year (Stochastic)

Sample Insurance Company
Homeowners
Stochastic Estimates as of December 31, 2014
Estimated Paid Claims by Calendar Year

CY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2015	252,049	25,430	10.1%	171,900	348,486	211,598	234,404	251,252	261,859	269,070	294,959
2016	55,570	9,158	16.5%	29,368	103,028	41,386	49,232	55,076	52,236	61,369	71,445
2017	26,772	6,387	23.9%	7,593	56,696	17,092	22,144	26,470	27,827	30,888	37,890
2018	14,401	4,923	34.2%	333	38,744	7,102	10,932	13,965	13,221	17,409	23,173
2019	6,241	3,422	54.8%	(2,952)	24,140	1,334	3,813	5,881	5,630	8,306	12,436
2020	3,212	2,583	80.4%	(4,367)	18,449	(318)	1,383	2,867	2,281	4,693	7,986
2021	2,735	2,471	90.3%	(5,722)	17,438	(656)	1,006	2,423	770	4,070	7,339
2022	2,318	2,271	98.0%	(3,834)	15,984	(819)	769	1,965	1,163	3,562	6,552
2023	2,340	1,852	79.1%	0	18,642	155	940	1,938	-	3,281	5,981
Total	365,640	33,369	9.1%	247,985	505,728	312,138	342,419	364,523	360,985	387,991	421,695

Table E.9 – Mean Future Incremental – Paid (Stochastic)

Sample Insurance Company
Homeowners - Paid
Mean Future Incremental as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	Total
2006										773	773
2007									125	629	754
2008								414	237	1,215	1,867
2009							217	293	205	903	1,617
2010						1,911	319	403	259	1,248	4,140
2011					6,758	2,604	416	545	348	1,685	12,356
2012				9,961	6,391	2,487	402	503	333	1,588	21,665
2013			25,830	11,299	7,304	2,814	459	585	373	1,792	50,455
2014		206,060	33,797	14,743	9,478	3,682	608	775	527	2,340	272,011

Table E.10 – Standard Deviation of Future Incremental – Paid (Stochastic)

Sample Insurance Company
Homeowners - Paid
Standard Deviation Future Incremental as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	Total
2006										920	920
2007									831	1,054	1,334
2008								952	995	1,243	1,847
2009							704	934	1,030	1,236	1,975
2010						1,805	844	1,062	1,187	1,397	2,932
2011					3,045	1,966	892	1,170	1,287	1,508	4,435
2012				3,658	2,927	1,919	867	1,092	1,236	1,419	5,686
2013			6,340	4,080	3,298	2,086	951	1,234	1,378	1,574	9,708
2014		24,137	7,203	4,746	3,852	2,459	1,138	1,508	1,636	1,852	30,285

Table E.11 – Coefficient of Variation of Future Incremental – Paid (Stochastic)

Sample Insurance Company
Homeowners - Paid
CoV Future Incremental as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	Total
2006										119.1%	119.1%
2007									665.2%	167.5%	176.9%
2008								229.9%	419.4%	102.3%	98.9%
2009							324.5%	318.6%	503.5%	136.9%	122.1%
2010						94.4%	264.4%	263.5%	458.1%	112.0%	70.8%
2011					45.1%	75.5%	214.7%	214.7%	369.8%	89.5%	35.9%
2012				36.7%	45.8%	77.2%	215.6%	217.1%	370.6%	89.4%	26.2%
2013			24.5%	36.1%	45.2%	74.1%	207.1%	210.9%	370.0%	87.9%	19.2%
2014		11.7%	21.3%	32.2%	40.6%	66.8%	187.1%	194.6%	310.6%	79.1%	11.1%

Table E.12 – Estimated Unpaid Claims by Accident Year in 2015 (Stochastic)

Sample Insurance Company
Homeowners - Paid
Stochastic Estimates as of December 31, 2014
Estimated Unpaid Claims by Accident Year, Calendar Year 2015 Only

AY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2006	773	920	119.1%	(18)	7,510	(16)	121	459	(18)	1,101	2,668
2007	125	831	665.2%	(1,973)	6,958	(1,083)	(157)	(63)	(74)	294	1,701
2008	414	952	229.9%	(2,175)	9,496	(742)	(26)	118	(26)	693	2,285
2009	217	704	324.5%	(1,892)	9,688	(523)	(96)	(27)	(96)	360	1,645
2010	1,911	1,805	94.4%	(2,885)	14,491	(317)	565	1,550	(564)	2,884	5,331
2011	6,758	3,045	45.1%	47	22,789	2,482	4,544	6,378	4,282	8,579	12,327
2012	9,961	3,658	36.7%	1,207	28,737	4,701	7,304	9,587	9,740	12,199	16,585
2013	25,830	6,340	24.5%	8,694	52,980	16,319	21,257	25,371	19,688	29,857	37,189
2014	206,060	24,137	11.7%	132,533	295,967	167,429	189,609	205,307	200,574	221,714	247,353
Total	252,049	25,430	10.1%	171,900	348,486	211,598	234,404	251,252	261,859	269,070	294,959

Table E.13 – Actual vs. Expected Back-test & Conditional Reserve (Stochastic)

Sample Insurance Company
Homeowners
Stochastic Actual vs. Expected as of December 31, 2015

AY	Age	Actual Paid	Expected Paid	Percentile	Actual Incurred	Expected Incurred	Percentile	Conditional Reserve	Expected Reserve	Change
2006	120	26	773	13.9%	(132)	(95)	83.5%	-	747	(747)
2007	108	33	125	61.9%	(156)	59	31.4%	164	721	(557)
2008	96	227	414	57.2%	(1,359)	(349)	23.5%	1,367	1,640	(272)
2009	84	(176)	217	14.1%	(1,158)	(105)	18.5%	(1,153)	1,793	(2,946)
2010	72	3,800	1,911	85.6%	412	(482)	67.2%	3,722	340	3,381
2011	60	5,462	6,758	37.5%	(8)	1,119	12.2%	3,979	6,894	(2,915)
2012	48	12,197	9,961	74.9%	1,284	813	81.4%	12,839	9,468	3,370
2013	36	23,840	25,830	40.5%	8,785	12,274	37.9%	21,590	26,615	(5,024)
2014	24	191,678	206,060	28.0%	56,168	52,293	62.7%	59,458	80,333	(20,875)
2015	12	934,805			1,143,739					
Totals		1,171,892			1,207,575			101,967	128,553	(26,586)
AY<CY		237,087	252,049	28.4%	63,836	65,528	50.2%	96,676	128,553	(31,876)

Figure E.1 – Graph of KPI Thresholds by Accident Year – Paid (Stochastic)

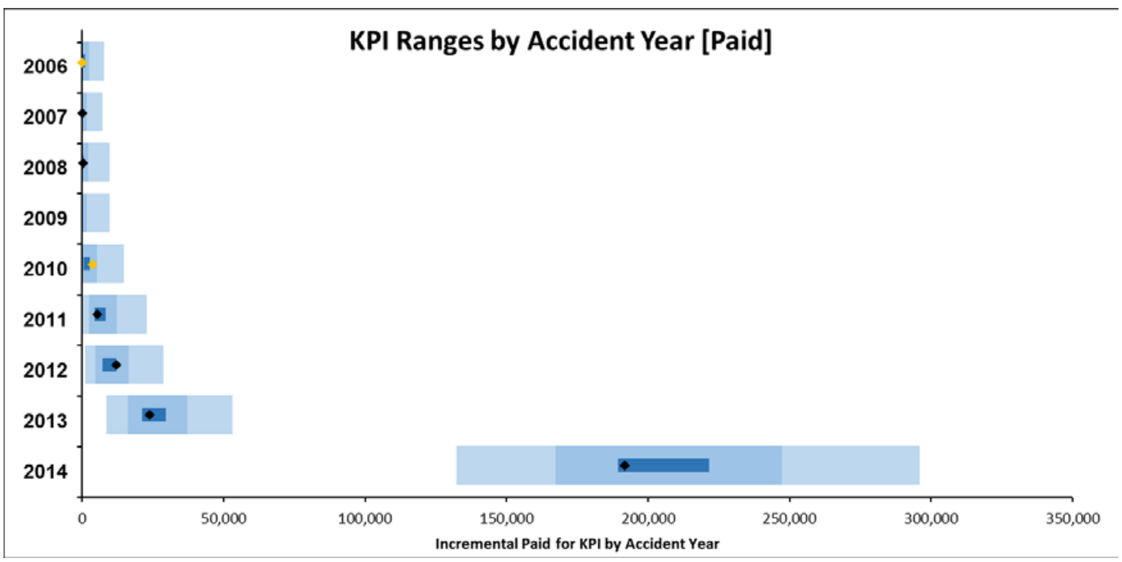


Figure E.2 – Graph of KPI Thresholds by Calendar Year – Paid (Stochastic)

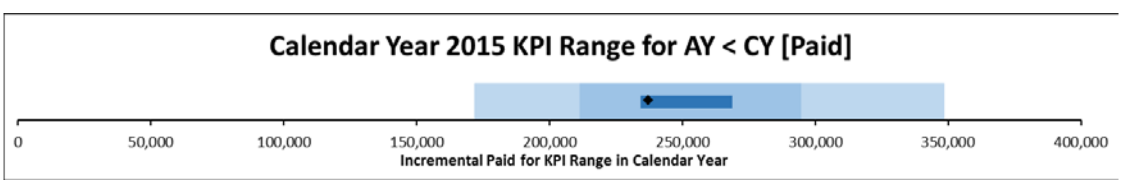


Figure E.3 – Graph of KPI Thresholds by Accident Year – Incurred (Stochastic)

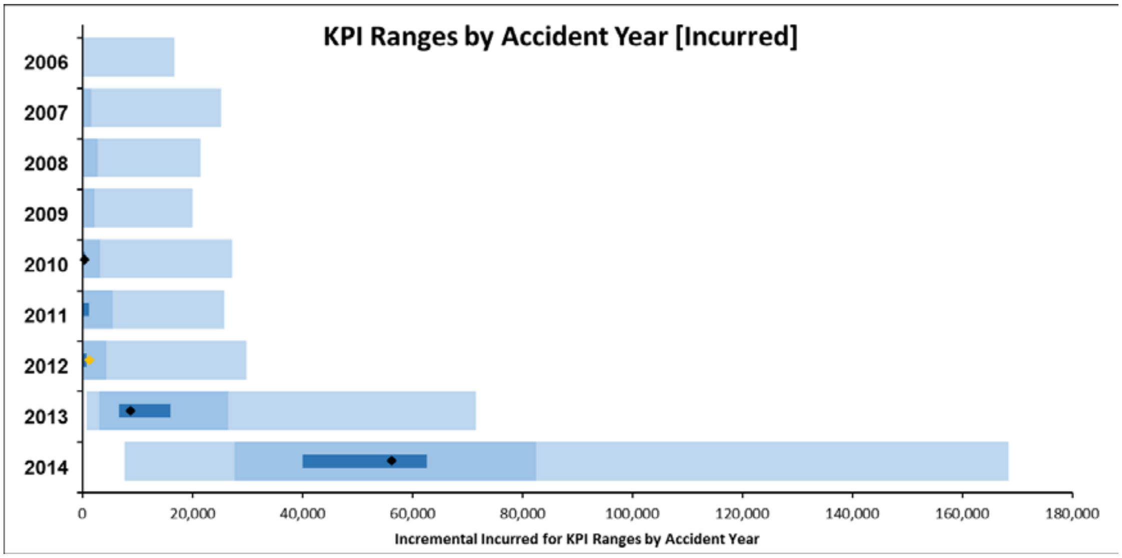


Figure E.4 – Graph of KPI Thresholds by Calendar Year – Incurred (Stochastic)

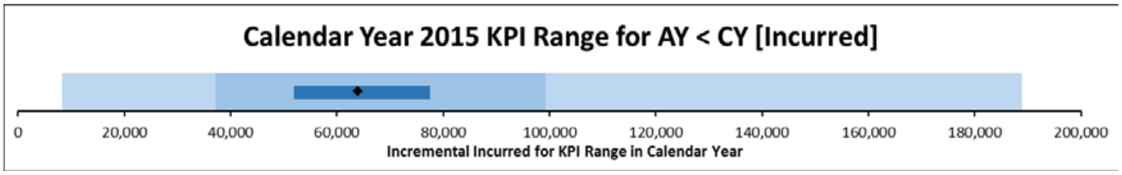


Figure E.5 – Graph of Realized Values vs. Assumptions – Paid (Stochastic)

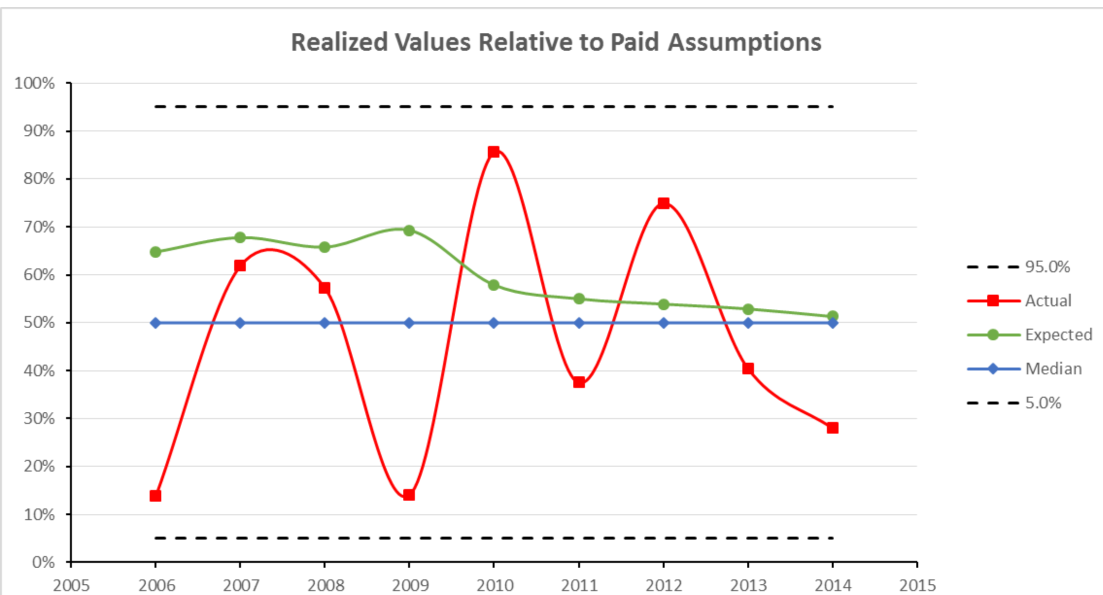
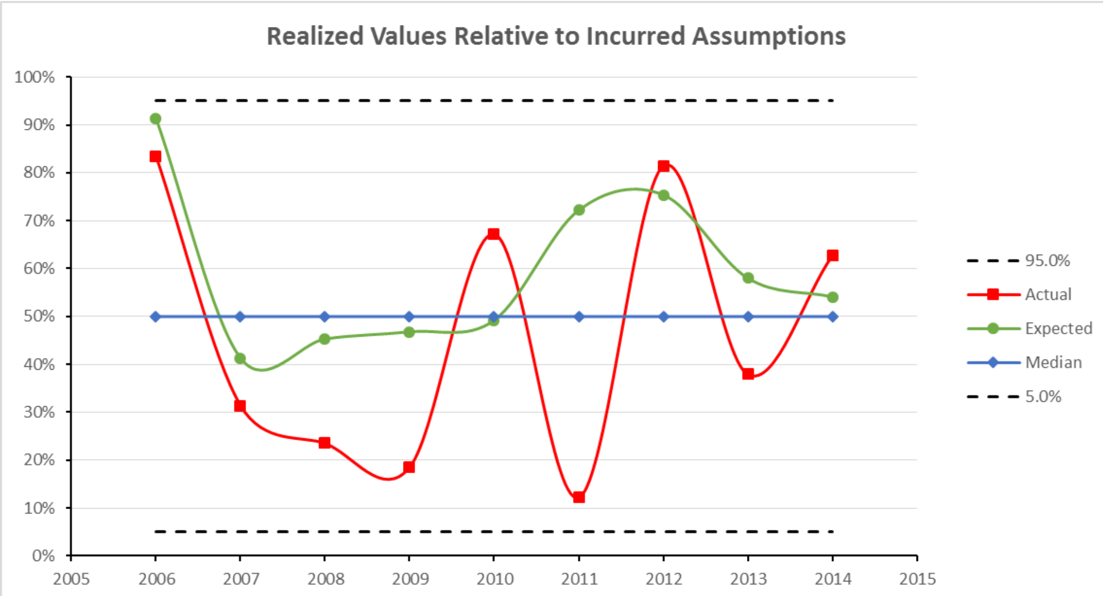


Figure E.6 – Graph of Realized Values vs. Assumptions – Incurred (Stochastic)



Appendix F – Back-Testing Aggregate Results

Table F.1 – Reconciliation of Total Unpaid (Deterministic)

Sample Insurance Company
Consolidation of All Segments
Total Unpaid Reconciliation as of December 31, 2014

AY	Age	Paid to Date	Incurred to Date	Weighted Ultimate	Case Reserve	IBNR	Total Unpaid	Selected Ultimate	Selected IBNR	Total Unpaid
2006	108	1,798,805	1,801,896	1,806,215	3,091	4,319	7,410	1,806,215	4,319	7,410
2007	96	2,054,136	2,063,367	2,068,349	9,231	4,982	14,213	2,070,502	7,135	16,366
2008	84	2,202,872	2,213,290	2,226,141	10,418	12,851	23,269	2,226,141	12,851	23,269
2009	72	2,335,053	2,354,342	2,379,431	19,289	25,089	44,378	2,379,431	25,089	44,378
2010	60	2,522,650	2,566,756	2,618,692	44,106	51,936	96,042	2,618,692	51,936	96,042
2011	48	2,510,953	2,609,324	2,713,658	98,371	104,334	202,705	2,713,658	104,334	202,705
2012	36	2,369,593	2,567,519	2,783,496	197,926	215,977	413,903	2,783,496	215,977	413,903
2013	24	2,210,586	2,558,937	2,976,074	348,351	417,137	765,488	2,976,074	417,137	765,488
2014	12	1,604,249	2,346,693	3,247,231	742,444	900,538	1,642,982	3,247,231	900,538	1,642,982
Totals		19,608,897	21,082,124	22,819,287	1,473,227	1,737,163	3,210,390	22,821,440	1,739,316	3,212,543

Table F.2 – Expected Incremental Development – Paid (Deterministic)

Sample Insurance Company
Consolidation of All Segments -- Paid Data
Expected Incremental Future Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										3,701	3,709	7,410
2007									7,405	4,476	4,485	16,366
2008								10,073	4,353	4,417	4,426	23,269
2009							19,027	11,120	4,716	4,752	4,762	44,378
2010						47,151	21,651	11,869	5,058	5,151	5,162	96,042
2011					103,127	50,012	21,845	12,022	5,128	5,281	5,292	202,705
2012				194,479	113,044	53,527	23,509	12,806	5,484	5,521	5,533	413,903
2013			325,644	208,375	119,178	56,435	24,715	13,549	5,783	5,899	5,911	765,488
2014		833,793	351,973	216,546	123,955	58,580	25,466	14,073	6,020	6,282	6,295	1,642,982

Table F.3 – Expected Incremental Development – Incurred (Deterministic)

Sample Insurance Company
Consolidation of All Segments -- Incurred Data
Expected Incremental Future Development as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										2,158	2,161	4,319
2007									2,794	2,169	2,172	7,135
2008								6,142	1,726	2,489	2,494	12,851
2009							11,285	6,504	1,883	2,706	2,711	25,089
2010						26,873	10,537	6,833	1,991	2,849	2,853	51,936
2011					54,534	24,663	10,569	6,831	1,995	2,868	2,873	104,334
2012				106,020	55,954	26,819	11,457	7,434	2,175	3,057	3,062	215,977
2013			192,143	108,519	59,307	28,313	12,129	7,859	2,291	3,285	3,291	417,137
2014		479,073	187,988	112,628	61,829	29,184	12,530	8,072	2,358	3,436	3,441	900,538

Table F.4 – Actual vs. Expected Back-test (Deterministic)

Sample Insurance Company
Consolidation of All Segments
Deterministic Actual vs. Expected as of December 31, 2015

AY	Age	Actual Paid	Expected Paid	Difference	Actual Incurred	Expected Incurred	Difference
2006	120	3,069	3,701	(632)	1,863	2,158	(295)
2007	108	5,905	7,405	(1,500)	3,145	2,794	351
2008	96	8,986	10,073	(1,087)	3,553	6,142	(2,589)
2009	84	18,992	19,027	(35)	9,872	11,285	(1,413)
2010	72	51,003	47,151	3,852	25,942	26,873	(931)
2011	60	105,067	103,127	1,940	52,012	54,534	(2,522)
2012	48	202,932	194,479	8,453	106,624	106,020	604
2013	36	334,434	325,644	8,790	189,908	192,143	(2,235)
2014	24	841,484	833,793	7,691	454,217	479,073	(24,856)
2015	12	1,798,138			2,528,235		
Totals		3,370,010			3,375,371		
AY<CY		1,571,872	1,544,400	27,471	847,136	881,022	(33,886)

Table F.5 – Actual to Range of Estimates Back-test (Deterministic)

Sample Insurance Company
Consolidation of All Segments
Deterministic Actual vs. Method Range as of December 31, 2015

AY	Age	Actual Paid	Paid Minimum	Paid Maximum	Range Percent	Actual Incurred	Incurred Minimum	Incurred Maximum	Difference
2006	120	3,069	3,701	3,704	-21075.4%	1,863	2,158	2,162	-6790.5%
2007	108	5,905	5,827	8,983	2.5%	3,145	1,210	4,380	61.0%
2008	96	8,986	9,887	10,277	-230.8%	3,553	5,955	6,356	-599.0%
2009	84	18,992	17,726	20,381	47.7%	9,872	9,981	12,657	-4.1%
2010	72	51,003	44,889	49,487	133.0%	25,942	24,600	29,236	28.9%
2011	60	105,067	100,495	106,278	79.1%	52,012	51,856	57,857	2.6%
2012	48	202,932	191,183	198,745	155.4%	106,624	102,222	110,845	51.1%
2013	36	334,434	310,031	338,355	86.2%	189,908	174,120	205,898	49.7%
2014	24	841,484	794,706	853,821	79.1%	454,217	436,298	503,306	26.7%
2015	12	1,798,138				2,528,235			
Totals		3,370,010				3,375,371			
AY<CY		1,571,872	1,481,602	1,586,896	85.7%	847,136	811,568	929,564	30.1%

Table F.6 – Estimated Unpaid Claims by Accident Year (Stochastic)

Sample Insurance Company
Aggregation of All Segments
Stochastic Estimates as of December 31, 2014
Estimated Unpaid Claims by Accident Year

AY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2006	7,410	3,000	40.5%	209	20,930	2,762	5,258	7,230	7,126	9,376	12,584
2007	16,366	3,857	23.6%	4,326	35,971	10,293	13,681	16,160	13,955	18,874	23,025
2008	23,269	4,798	20.6%	7,340	41,630	15,697	19,961	23,038	24,448	26,387	31,552
2009	44,378	6,012	13.5%	23,290	73,490	34,774	40,249	44,172	43,645	48,324	54,552
2010	96,042	8,137	8.5%	68,354	129,130	82,986	90,380	95,868	97,281	101,523	109,899
2011	202,705	11,141	5.5%	162,433	245,913	184,872	195,065	202,429	213,672	210,093	221,392
2012	413,903	18,019	4.4%	348,396	495,863	385,145	401,826	413,324	431,386	425,535	444,597
2013	765,488	31,256	4.1%	643,540	893,747	714,958	744,538	764,726	758,282	786,020	818,610
2014	1,642,982	62,139	3.8%	1,378,415	1,972,517	1,544,716	1,602,194	1,641,001	1,633,958	1,682,508	1,746,787
Total	3,212,543	79,355	2.5%	2,811,937	3,596,084	3,084,602	3,161,789	3,211,505	3,295,980	3,261,725	3,343,252

Table F.7 – Estimated Claims Paid by Calendar Year (Stochastic)

Sample Insurance Company
Aggregation of All Segments
Stochastic Estimates as of December 31, 2014
Estimated Unpaid Claims by Calendar Year

CY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2015	1,560,637	43,888	2.8%	1,326,487	1,761,442	1,490,151	1,531,594	1,560,068	1,569,675	1,589,323	1,634,164
2016	761,830	24,692	3.2%	671,495	861,974	721,379	745,435	761,974	778,026	778,144	802,553
2017	433,217	17,767	4.1%	368,636	499,640	404,462	420,952	433,003	430,492	445,020	463,153
2018	227,484	12,686	5.6%	180,708	277,701	206,908	218,837	227,342	231,979	235,833	248,870
2019	110,005	8,936	8.1%	81,148	145,658	95,506	104,003	109,870	108,106	115,810	124,858
2020	54,489	6,783	12.4%	30,217	81,348	43,677	49,928	54,233	53,345	58,990	65,976
2021	30,258	5,508	18.2%	11,536	54,292	21,555	26,490	30,113	31,602	33,792	39,599
2022	17,338	4,694	27.1%	1,748	38,761	9,925	14,127	17,132	15,736	20,273	25,447
2023	12,228	4,234	34.6%	351	31,873	5,612	9,261	12,025	15,750	14,892	19,631
2024	5,057	2,388	47.2%	(46)	15,791	1,427	3,333	4,900	4,363	6,546	9,313
Total	3,212,543	79,355	2.5%	2,811,937	3,596,084	3,084,602	3,161,789	3,211,505	3,295,980	3,261,725	3,343,252

Table F.8 – Mean Future Incremental – Paid (Stochastic)

Sample Insurance Company
Aggregation of All Segments - Paid
Mean Future Incremental as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										4,077	3,333	7,410
2007									6,163	5,387	4,816	16,366
2008								10,176	4,300	4,998	3,794	23,269
2009							20,033	10,774	4,591	4,922	4,058	44,378
2010						48,298	21,360	11,595	4,927	5,520	4,342	96,042
2011					104,415	49,419	21,556	11,839	5,077	6,033	4,365	202,705
2012				196,083	112,311	53,119	23,353	12,692	5,415	6,236	4,693	413,903
2013			331,701	205,564	117,582	55,662	24,391	13,384	5,665	6,643	4,896	765,488
2014		839,689	349,382	214,959	122,988	58,266	25,315	13,992	6,001	7,332	5,057	1,642,982

Table F.9 – Standard Deviation of Future Incremental – Paid (Stochastic)

Sample Insurance Company
Aggregation of All Segments - Paid
Standard Deviation Future Incremental as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										1,851	1,623	3,000
2007									1,927	2,080	1,809	3,857
2008								2,494	2,030	2,244	1,833	4,798
2009							3,202	2,660	2,162	2,280	1,974	6,012
2010						5,017	3,331	2,742	2,331	2,477	2,065	8,137
2011					7,305	5,065	3,417	2,795	2,369	2,568	2,101	11,141
2012				10,921	8,101	5,518	3,644	3,008	2,443	2,580	2,185	18,019
2013			16,733	12,067	8,683	5,833	3,853	3,164	2,615	2,786	2,312	31,256
2014		36,658	17,799	12,858	9,241	6,087	3,943	3,330	2,814	2,992	2,388	62,139

Table F.10 – Coefficient of Variation of Future Incremental – Paid (Stochastic)

Sample Insurance Company
Aggregation of All Segments - Paid
CoV Future Incremental as of December 31, 2014

AY	12	24	36	48	60	72	84	96	108	120	132	Total
2006										45.4%	48.7%	40.5%
2007									31.3%	38.6%	37.6%	23.6%
2008								24.5%	47.2%	44.9%	48.3%	20.6%
2009							16.0%	24.7%	47.1%	46.3%	48.6%	13.5%
2010						10.4%	15.6%	23.6%	47.3%	44.9%	47.6%	8.5%
2011					7.0%	10.2%	15.8%	23.6%	46.7%	42.6%	48.1%	5.5%
2012				5.6%	7.2%	10.4%	15.6%	23.7%	45.1%	41.4%	46.6%	4.4%
2013			5.0%	5.9%	7.4%	10.5%	15.8%	23.6%	46.2%	41.9%	47.2%	4.1%
2014		4.4%	5.1%	6.0%	7.5%	10.4%	15.6%	23.8%	46.9%	40.8%	47.2%	3.8%

Table F.11 – Estimated Unpaid Claims by Accident Year in 2015 (Stochastic)

Sample Insurance Company
Aggregation of All Segments - Paid
Stochastic Estimates as of December 31, 2014
Estimated Unpaid Claims by Accident Year, Calendar Year 2015 Only

AY	Mean	Std Dev	CoV	Min	Max	5%	25%	Median	Mode	75%	95%
2006	4,077	1,851	45.4%	4	12,459	1,386	2,758	3,891	3,545	5,211	7,424
2007	6,163	1,927	31.3%	92	14,962	3,317	4,823	5,994	6,136	7,317	9,584
2008	10,176	2,494	24.5%	2,955	24,018	6,391	8,444	9,987	8,710	11,747	14,546
2009	20,033	3,202	16.0%	9,752	35,160	15,071	17,795	19,882	19,530	22,094	25,607
2010	48,298	5,017	10.4%	27,691	69,353	40,292	44,825	48,117	49,900	51,560	56,893
2011	104,415	7,305	7.0%	76,379	135,132	92,822	99,305	104,299	105,433	109,283	116,607
2012	196,083	10,921	5.6%	157,181	242,812	178,556	188,588	195,828	193,134	203,222	214,311
2013	331,701	16,733	5.0%	257,765	396,823	304,516	320,387	331,465	315,168	342,845	359,464
2014	839,689	36,658	4.4%	679,077	1,011,508	781,489	815,305	839,033	862,142	862,844	900,811
Total	1,560,637	43,888	2.8%	1,326,487	1,761,442	1,490,151	1,531,594	1,560,068	1,569,675	1,589,323	1,634,164

Table F.12 – Actual vs. Expected Back-test & Conditional Reserve (Stochastic)

Sample Insurance Company
Aggregation of All Segments
Stochastic Actual vs. Expected as of December 31, 2015

AY	Age	Actual Paid	Expected Paid	Percentile	Actual Incurred	Expected Incurred	Percentile	Conditional Reserve	Expected Reserve	Change
2006	120	3,069	4,077	31.8%	1,863	2,115	49.8%	2,539	4,341	(1,802)
2007	108	5,905	6,163	47.9%	3,145	1,819	80.6%	11,349	10,461	888
2008	96	8,986	10,176	33.6%	3,553	6,026	20.9%	10,961	14,283	(3,322)
2009	84	18,992	20,033	39.0%	9,872	10,399	46.3%	21,615	25,386	(3,771)
2010	72	51,003	48,298	71.6%	25,942	25,562	55.3%	49,308	45,039	4,269
2011	60	105,067	104,415	54.3%	52,012	53,101	44.8%	97,157	97,638	(481)
2012	48	202,932	196,083	74.2%	106,624	104,075	61.7%	222,250	210,971	11,279
2013	36	334,434	331,701	57.1%	189,908	185,173	64.0%	427,667	431,054	(3,387)
2014	24	841,484	839,689	52.8%	454,217	469,822	29.3%	795,671	801,499	(5,828)
2015	12	1,798,138			2,528,235					
Totals		3,370,010			3,375,371			1,638,516	1,640,671	(2,154)
AY<CY		1,571,872	1,560,637	61.2%	847,136	858,093	37.6%	1,638,584	1,640,671	(2,086)

Figure F.1 – Graph of KPI Thresholds by Accident Year – Paid (Stochastic)

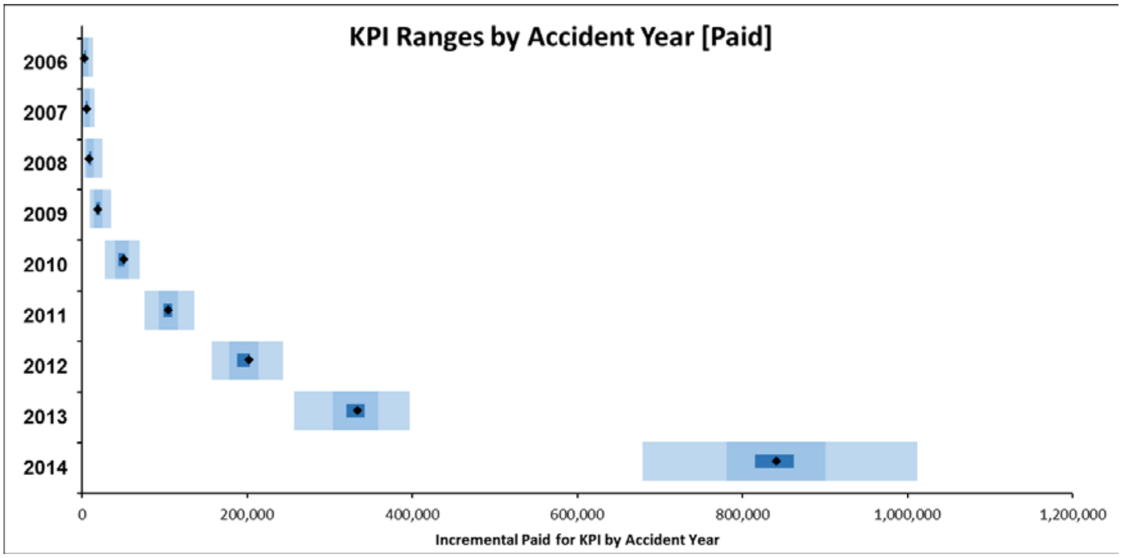


Figure F.2 – Graph of KPI Thresholds by Calendar Year – Paid (Stochastic)

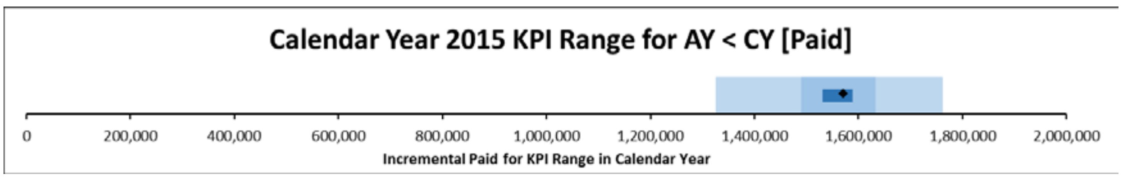


Figure F.3 – Graph of KPI Thresholds by Accident Year – Incurred (Stochastic)

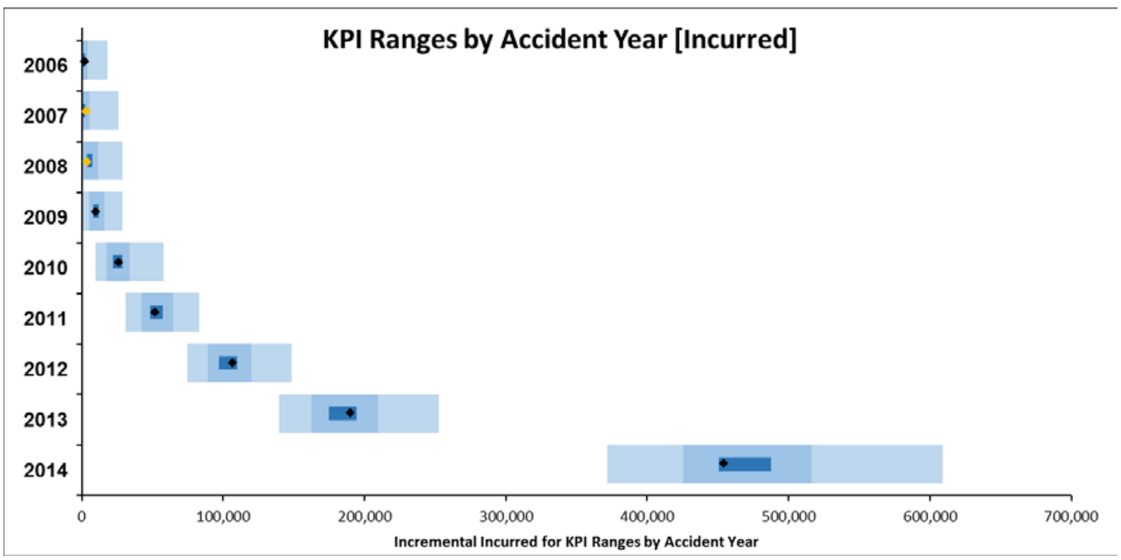


Figure F.4 – Graph of KPI Thresholds by Calendar Year – Incurred (Stochastic)

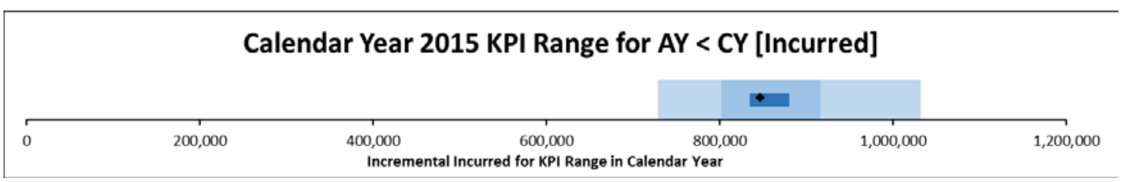


Figure F.5 – Graph of Realized Values vs. Assumptions – Paid (Stochastic)

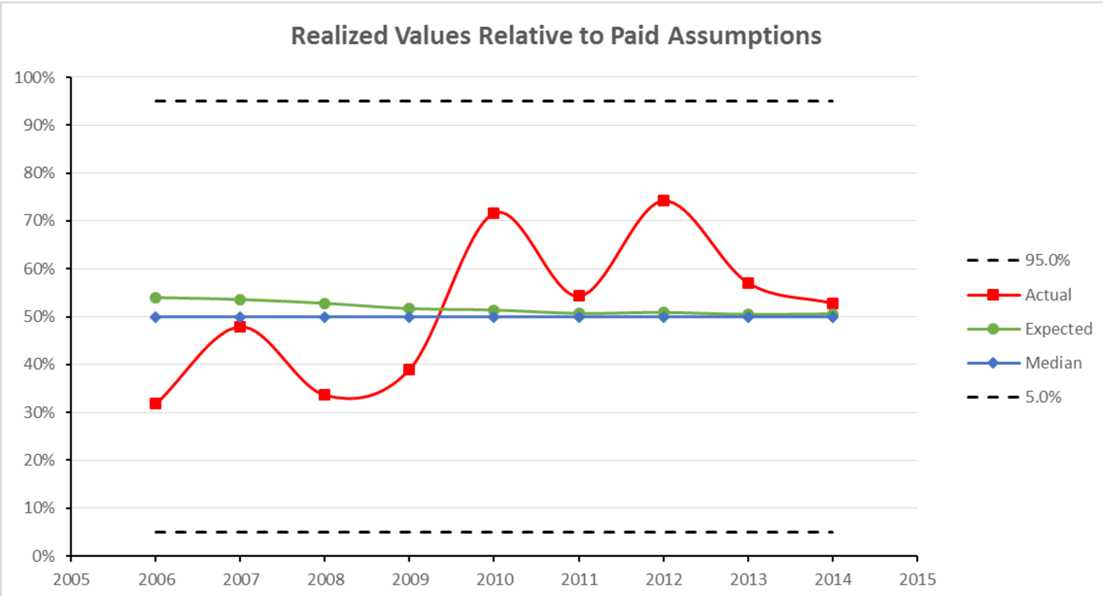


Figure F.6 – Graph of Realized Values vs. Assumptions – Paid (Stochastic)

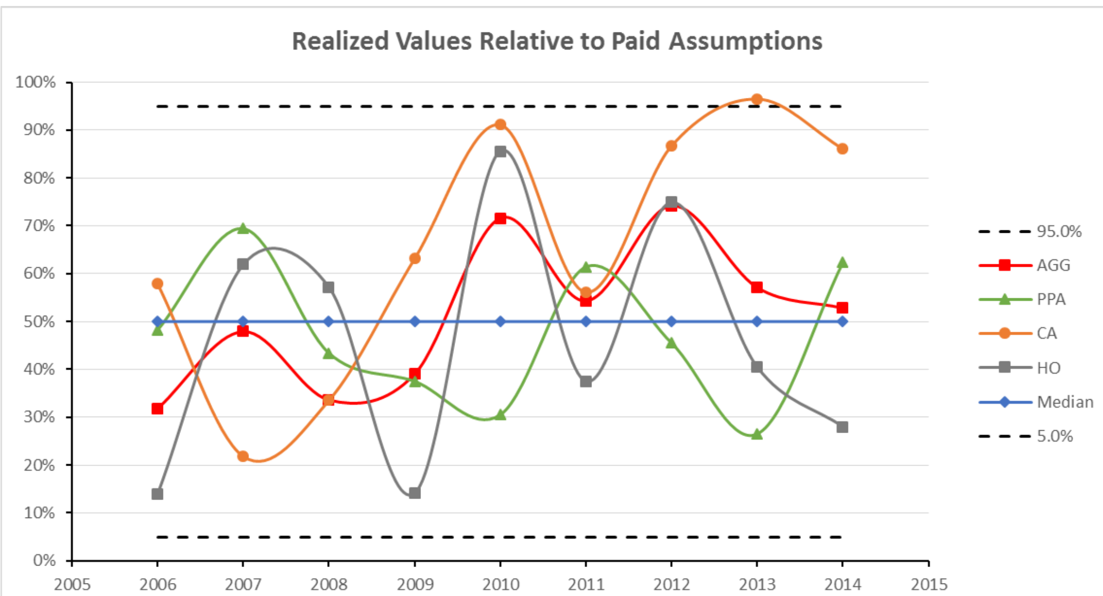


Figure F.7 – Graph of Realized Values vs. Assumptions – Incurred (Stochastic)

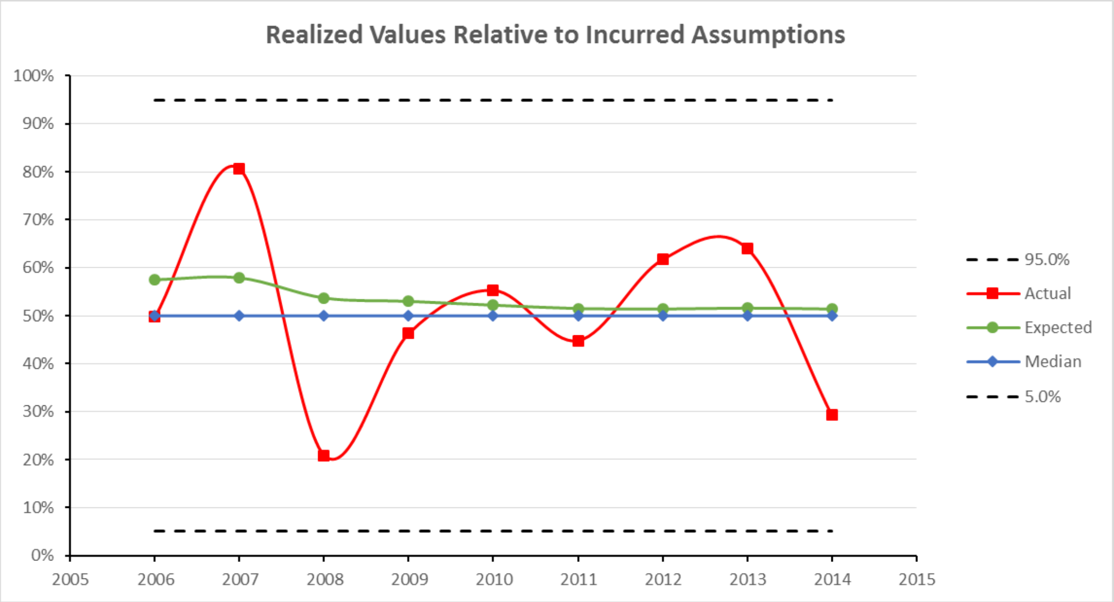
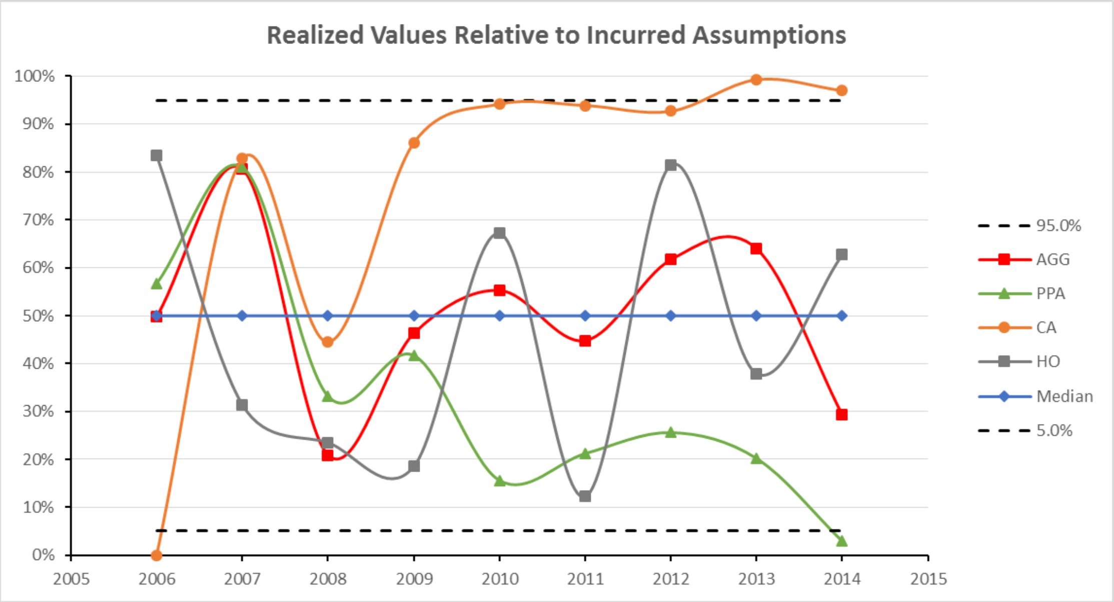


Figure F.8 – Graph of Realized Values vs. Assumptions – Incurred (Stochastic)



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Abbreviations and notations

The following abbreviations and notations are used in the paper.

AY, Accident Year	CY, Calendar Year
AY = CY, the latest AY for which there is no comparable expectation based on the prior annual reserve analysis	AY < CY, all AYs except the latest AY for which there is a comparable expectation based on the prior annual reserve analysis
AYLWA, All Year Loss Weighted Average	IELR, Initial Expected Loss Ratio
BF, Bornhuetter-Ferguson	Inc BF, Incurred Bornhuetter-Ferguson Method
CA, Commercial Automobile	Inc CL, Incurred Chain Ladder Method
CEO, Chief Executive Officer	KPI, Key Performance Indicator
CL, Chain Ladder	LDF, Loss Development Factor
CoV, Coefficient of Variation	MLE, Maximum Likelihood Estimation
ENID, Events Not In the Data	ODP, over-dispersed Poisson
ERM, Enterprise Risk Management	Pd BF, Paid Bornhuetter-Ferguson Method
FD, Framework Directive	Pd CL, Paid Chain Ladder Method
GLM, Generalized Linear Models	PPA, Private Passenger Automobile
HO, Homeowners	TAS-M, Technical Actuarial Standard: Modelling

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Supplements for The Actuary and Enterprise Risk Management: Integrating Reserve Variability

http://www.casact.org/pubs/forum/16sforum/Shapland-Courchene-AGG.xlsm
http://www.casact.org/pubs/forum/16sforum/Shapland-Courchene-LOB.xlsm

Using the Hayne MLE Models: A Practitioner's Guide

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Abstract

Motivation. The Hayne MLE family of models are quite elegant in their application, but like most models in order to address the needs of the practicing actuary the modeling framework needs to allow for the flexibility to deal with many different practical issues. While actuaries are accustomed to making practical adjustments to their algorithms, there is motivation to stay as close to the theoretical underpinnings of the models as possible in order to maintain a sound foundation. Whenever the paper strays a bit from the theory, those departures are noted so practitioners can adequately judge their impact.

Method. This paper starts by reviewing the Hayne MLE modeling framework using a standard notation. Then it covers a number of practical data issues and addresses the diagnostic testing of the model assumptions. Next it will explore a variety of enhancements to the basic framework to allow the models to address other issues related to reserving and pricing risk. Finally, since no single model is perfect, ways to combine or credibility weight the Hayne MLE model results with various other models are explored in order to arrive at a "best estimate" of the distribution. This is similar to how a deterministic best estimate is generally derived in practice, so ways for the practitioner to correlate models by segment in order to simulate aggregate results are discussed.

Results. The paper will illustrate the practical implementation of the Hayne MLE modeling framework as a powerful tool for estimating a distribution of unpaid claims.

Conclusions. The paper outlines the full versatility of the Hayne MLE models for the practicing actuary.

Availability. In lieu of technical appendices, several companion Excel workbooks are included that illustrate the calculations described in this paper. The companion materials are summarized in the Supplementary Materials section and are available at [CAS to fill in location].

Keywords. Maximum Likelihood Estimate, Reserve Variability, Reserve Range, Distribution of Possible Outcomes, Generalized Linear Model, Best Estimate.

Table of Contents

1. Introduction

With the introduction of the Hayne [8] MLE family of models the CAS membership has gained a very powerful and useful new toolset for estimating unpaid claim distributions from a data triangle. The growing need for stochastic models for use as part of Enterprise Risk Management and the changing regulatory landscape makes these new stochastic models all the more important. However, like most papers on stochastic modeling, the Hayne [8] paper focuses primarily on the theory and development of the basic modeling framework, which of course is the critical first step. This paper is an attempt to build and expand upon the foundation of these models by exploring different aspects of their use on a regular basis so that the practicing actuary has a more complete toolset for solving a wider variety of actuarial problems.

1.1 Objectives

One objective of this paper is to review the theoretical foundation of Hayne MLE models to better understand the assumptions and parameters. If model assumptions and parameters do not fit the statistical features found in the data then the results of a simulation may not be a very good estimate of the distribution of possible outcomes. Thus, the modeling framework must be able to adapt or "fit" the model to the data so this point will be elaborated on in later sections.

Another objective of this paper is to show how the Hayne MLE modeling framework can be used in practice to help the wider adoption of unpaid claim distributions. Most of the papers describing stochastic models, including the Hayne [8] paper, tend to focus primarily on the theoretical aspects of the model while ignoring the data issues that commonly arise in practice. As a result the models can be quite elegantly implemented yet suffer from practical limitations such as only being useful for complete triangles or only for positive incremental values. Thus, while keeping as close to the theoretical foundation as possible, this objective is to illustrate how practical adjustments can be made to accommodate common data issues and allow the model to "fit" the data. As a practical matter, it is also possible that the model does not fit the data very well, or less well than other models, so the process of diagnosing the reasonability of the assumptions will inform the actuary's judgment when considering adjustments to the parameters or how much weight, if any, to give the model in relation to other models.

A related issue seems to be the notion that actuaries are still searching for the perfect model to describe "the" distribution of unpaid claims, as if imperfections in a model remove it from all consideration since it can't be "the one." This notion can also manifest itself when an actuary settles for a model that seems to work the best or is the easiest to use, or with the idea that each model must be used in its entirety or not at all. Interestingly, this notion was dispelled long ago with respect to deterministic point estimates as actuaries commonly use many different methods, which range from easy to complex, and judgmentally weight the results to arrive at their best estimate.

Model risk – the risk that the model you have chosen is not the same as the one that generates future losses – is very real. Weighting or combining multiple estimates is a very practical way of addressing model risk. Thus, another objective of this paper is to show how stochastic reserving can be similar to deterministic reserving when it comes to analyzing and using the best parts of multiple models by illustrating how the results from a Hayne MLE model can be weighted together with other models. More importantly, the paper hopes to illustrate the advantage of using a more complete set of risk estimation tools (which can include both stochastic models and deterministic methods) to arrive at an actuarial best estimate of the distribution of possible outcomes, rather than to focus on deterministic methods to select the "mean" and then simply "add on" a simple approximation or use only a favorite model to turn that selected mean into a distribution.

2. Notation

Rather than use the notation in the Hayne [8] paper, the notation from the CAS Working Party on Quantifying Variability in Reserve Estimates Summary Report [4] will be used since it is intended to serve as a "standard notation" for further research.

Many models visualize loss data as a two-dimensional array, (w,d) with accident period or policy period w, and development age d (think w = "when" and d = "delay"). For this discussion, it is assumed that the loss information available is an "upper triangular" subset for rows $w=1, 2, \ldots, n$ and for development ages $d=1, 2, \ldots, n-w+1$. The "diagonal" for which $w+d$ equals the constant, k, represents the loss information for each accident period w as of accounting period k.[1]

For purposes of including tail factors, the development beyond the observed data for periods $d=n+1, n+2, \ldots, u$, where u is the ultimate time period for which any claim activity occurs, or the period in which all claims are final and paid in full, must also be considered.

The paper uses the following notation for certain important loss statistics:

[1] For a more complete explanation of this two-dimensional view of the loss information, see the *Foundations of Casualty Actuarial Science* [6], Chapter 5, particularly pages 210-226.

$c(w,d)$: cumulative loss from accident[2] year w as of age d.

$q(w,d)$: incremental loss for accident year w from d - 1 to d.

$c(w,n) = U(w)$: total loss from accident year w when claims are at ultimate values at time n[3], or

$c(w,u) = U(w)$: total loss from accident year w when claims are at ultimate values at time u.

$R(w)$: future development after age d for accident year w, i.e., = $U(w) - c(w,d)$.

$f(d)$: parameter or factor applied to $c(w,d)$ to estimate $q(w,d+1)$ or can be used more generally to indicate any parameter or factor relating to age d.

$F(d)$: parameter or factor applied to $c(w,d)$ to estimate $c(w,d+1)$ or $c(w,n)$ or can be used more generally to indicate any cumulative parameter or factor relating to age d.

$T = T(n)$: ultimate tail factor at end of triangle data, which is applied to the estimated $c(w,n)$ to estimate $c(w,u)$.

$G(w)$: parameter or factor relating to accident year w – capitalized to designate ultimate loss level.

$h(k)$: parameter or factor relating to the diagonal k along which $w+d$ is constant.[4]

$M(w,d)$: matrix factors relating to both accident year w and development year d parameters.

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2 The use of accident year is used for ease of discussion. All of the discussion and formulas that follow could also apply to underwriting year, policy year, report year, etc. Similarly, year could also be half-year, quarter or month.

3 This would imply that claims reach their ultimate value without any tail factor. This is generalized by changing n to $n+t=u$, where t is the number of periods in the tail.

4 Some authors define $d = 0,1,\ldots,n-1$ which intuitively allows $k = w$ along the diagonals, but in this case the triangle size is $n \times n-1$ which is not intuitive. With $d = 1,2,\ldots,n$ defined as in this paper, the triangle size $n \times n$ is intuitive, but then $k = w+1$ along the diagonals is not as intuitive. A way to think about this which helps tie everything together is to assume the w variables are the beginning of the accident periods and the d variables are at the end of the development periods. Thus, if years are used then cell $c(n,1)$ represents accident year n evaluated at 12/31/*n*, or essentially 1/1/*n+1*.

$e(w,d)$:	a random fluctuation, or error, which occurs at the w,d cell.
$b(w,d)$:	cumulative claim count from accident year w as of age d.
$p(w,d)$:	incremental claim count for accident year w from d - 1 to d.
$N(w)$:	the exposures for accident year w.
$A(w,d)$:	the incremental average for accident year w from d - 1 to d.
$E[x]$:	the expectation of the random variable x.
$Var[x]$:	the variance of the random variable x.
κ, ρ:	variance parameters.

What are called factors here could also be summands, but if factors and summands are both used, some other notation for the additive terms would be needed. The notation does not distinguish paid vs. incurred, but if this is necessary, capitalized subscripts P and I could be used.

3. The Hayne MLE Models

The Hayne MLE models[5] are based on a triangular array of incremental values:

		d 1	2	3	…	n-1	n
w	**1**	q(1,1)	q(1,2)	q(1,3)	…	q(1,n-1)	q(1,n)
	2	q(2,1)	q(2,2)	q(2,3)	…	q(2,n-1)	
	3	q(3,1)	q(3,2)	q(3,3)	…		
	…	…	…				
	n-1	q(n-1,1)	q(n-1,2)				
	n	q(n,1)					

By incorporating an exposure adjustment the variety of methods available for analysis is widened, as the focus shifts to the incremental averages:

$$A(w,d) = \frac{q(w,d)}{N(w)}. \tag{3.1}$$

Hayne [8] notes that the exposure adjustment for average incremental values (3.1) can be based on exposure counts or premium amounts, which would commonly be referred to as an average pure premium or burning cost. In addition, the exposure adjustment can be based on

[5] While condensed for ease of exposition, significant portions of Section 3 are based on Hayne [8].

an estimate of ultimate claim counts, which would be commonly referred to as an average claim severity:

$$A(w,d) = \frac{q(w,d)}{b(w,u)}. \tag{3.2}$$

In the case of the average claim severity, the ultimate claim counts are often only estimates and as such could be treated as random variables, which will be addressed in Section 4.

The Hayne MLE models are then based on a generalized framework that expresses each underlying method as a matrix-valued function of a parameter vector $\boldsymbol{\theta}$:

$$A(w,d) = M(\boldsymbol{\theta}). \tag{3.3}$$

In order to turn this general framework into a stochastic model two key assumptions are made. First, the variance of each incremental value is assumed to be proportional to a power of the square of the mean. It is quite common to assume the variance is proportional to a power of the expected values, but the square of the mean is used to allow incremental values to be negative. Also, the constant of proportionality is exponential allowing the parameter to take on any value while assuring positive values for the variance. Second, as the variance of an average of a sample with a finite variance will be inversely proportional to the number of items in the sample, the constant of proportionality is assumed to vary inversely to the number of exposures.

The stochastic model is then expressed as follows:

$$E[A(w,d)] = \mu \tag{3.4}$$

$$Var[A(w,d)] = \frac{e^{\kappa}(\mu^2)^{\rho}}{N(w)} = e^{\kappa - \ln[N(w)]}(\mu^2)^{\rho}. \tag{3.5}$$

Hayne [8] notes that this model includes an implicit structural heteroscedasticity and that both the expected values and variances differ by accident and development year. The two variance parameters, κ and ρ, provide a mechanism to approximate the variance structure of the data without over-parameterizing the model. However, the formulae can be modified to allow κ to vary by development period if additional control over the heteroscedasticity is desired.

Hayne [8] eloquently describes additional assumptions and processes for estimating the parameters for the stochastic model expressed in (3.4) and (3.5), including R code in the appendix. As this can't be improved upon here, it is left to the reader to review the Hayne [8] paper for further details, but the focus will turn to the five different implementations of this general framework before moving on to various practical implementation issues. For anyone not familiar with R, the implementation of the process of estimating model parameters in R is replicated in Excel in the companion "Hayne MLE Models.xlsm" file. Note, however, that while the Solver algorithm in Excel should estimate parameters which are very close to those estimated in R there can be differences and in some cases constraints may need to be added to the Excel Solver algorithm.

3.1 Berquist-Sherman Model

Berquist and Sherman [2] developed methods to recognize that incremental severities can have different "levels" by accident year as well as different trends by development year. Hayne [8] simplifies this approach by assuming a uniform trend from one accident year to the next which replaces different levels with uniform changes in level, which also indirectly impact the development for each year.

$$E[A(w,d)] = f(d) \times e^{wG} \tag{3.6}$$

In the Hayne Berquist-Sherman model, the $f(d)$ parameters represent an average incremental by development period. The G parameter is a constant accident year trend where $w = 1,2,3,...,n$. Using the data from Hayne [8], the companion Excel file summarizes the Berquist-Sherman model parameters as in Table 3.1.

Table 3.1. Summary of Berquist-Sherman Parameters

	Development Period Parameters (Average Incremental)									
	12	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**
Mean	620.95	760.66	708.15	553.57	349.99	181.39	70.96	43.88	11.08	15.21
Std Dev	40.50	46.55	43.00	35.49	26.17	17.66	10.39	8.74	4.22	7.34
Decay Ratios:		122.5%	93.1%	78.2%	63.2%	51.8%	39.1%	61.8%	25.2%	137.3%
CoV:	*6.5%*	*6.1%*	*6.1%*	*6.4%*	*7.5%*	*9.7%*	*14.6%*	*19.9%*	*38.1%*	*48.3%*
	Accident Year Trend	**K**	**p**	**AIC**	**BIC**				**Parameters**	
Mean	0.045	11.216	0.654	643.4	669.5			Acc Period	0	
Std Dev	0.009	1.037	0.085					Dev Period	10	
CoV:	*18.9%*	*9.2%*	*12.9%*					Trend	1	
									11	

In addition to the mean and standard deviation of each parameter, which are nearly identical to those in Hayne [8], the Coefficient of Variation ("CoV") row is added so that the heteroscedastistic variance by parameter is more apparent. The Decay Ratios row is simply the mean of the development parameter divided by the mean of the prior development parameter, which will be used in later discussions about tail extrapolation.

Table 3.2. Expected Incremental Mean Values for Berquist-Sherman Model

Predicted Incremental Mean [Model Fitted] (Paid [÷ Ultimate Claims])

Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**Future Totals**
2006	649.69	795.86	740.93	579.17	366.20	189.78	74.25	45.91	11.59	15.92	0.00
2007	679.73	832.66	775.18	605.95	383.13	198.56	77.69	48.03	12.13	16.65	16.65
2008	711.16	871.16	811.03	633.96	400.84	207.74	81.28	50.25	12.69	17.42	30.11
2009	744.04	911.43	848.52	663.28	419.37	217.34	85.04	52.57	13.27	18.23	84.07
2010	778.44	953.57	887.75	693.94	438.76	227.39	88.97	55.00	13.89	19.07	176.93
2011	814.43	997.66	928.80	726.03	459.05	237.90	93.08	57.55	14.53	19.95	423.01
2012	852.08	1,043.79	971.74	759.59	480.27	248.90	97.38	60.21	15.20	20.88	922.84
2013	891.48	1,092.05	1,016.67	794.71	502.48	260.41	101.89	62.99	15.90	21.84	1,760.22
2014	932.70	1,142.54	1,063.67	831.46	525.71	272.45	106.60	65.90	16.64	22.85	2,905.28
2015	975.82	1,195.36	1,112.85	869.90	550.02	285.05	111.53	68.95	17.41	23.91	4,234.97
											10,554.09

Using formulas (3.6) and (3.5) to calculate the expected mean and standard deviation the results for each incremental value are shown in Tables 3.2 and 3.3, respectively.

Table 3.3. Incremental Standard Deviation Values for Berquist-Sherman Model

Predicted Incremental Standard Deviation [Model Fitted] (Paid [÷ Ultimate Claims])

Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**Future Totals**
2006	95.13	108.63	103.66	88.24	65.39	42.55	23.04	16.82	6.84	8.42	0.00
2007	98.60	112.59	107.44	91.46	67.77	44.10	23.88	17.43	7.09	8.72	8.72
2008	97.68	111.54	106.45	90.61	67.14	43.69	23.65	17.27	7.02	8.64	11.14
2009	100.06	114.26	109.04	92.82	68.78	44.75	24.23	17.69	7.19	8.85	21.05
2010	104.03	118.79	113.36	96.50	71.51	46.53	25.19	18.39	7.48	9.20	33.37
2011	108.82	124.26	118.59	100.95	74.80	48.67	26.35	19.24	7.82	9.63	59.90
2012	107.65	122.92	117.31	99.86	74.00	48.15	26.07	19.03	7.74	9.52	94.79
2013	112.81	128.81	122.93	104.64	77.54	50.45	27.32	19.95	8.11	9.98	144.29
2014	114.36	130.58	124.62	106.08	78.61	51.15	27.69	20.22	8.22	10.12	192.16
2015	110.40	126.07	120.31	102.41	75.89	49.38	26.73	19.52	7.94	9.77	224.29
											349.83

Reviewing Table 3.2 you can see how the expected mean values for each development period relate to the model parameters for $f(d)$ in Table 3.1 by looking at each column. Also, comparing rows allow you to see how the trend parameter G impacts each accident year.

3.2 Cape Cod Model

Hayne [8] notes that the traditional Bornhuetter-Ferguson [3] method estimates future losses by accident year as a percent of an a priori estimate of the ultimate losses for that year.

In contrast, a feature of the Cape Cod method is that it derives the a priori estimates directly from the data. Hayne [8] essentially combines these methods by assuming that the incremental average amounts are the product of an accident year factor and lag factor, which are usually taken as ultimate loss for the year and the percentage of losses emerging that year.

$$E[A(w,d)] = \begin{cases} G(1,1), & w=1, d=1 \\ G(1,1)\times G(w), & w>1, d=1 \\ G(1,1)\times f(d), & w=1, d>1 \\ G(1,1)\times G(w)\times f(d), & w>1, d>1 \end{cases} \tag{3.7}$$

In the Hayne Cape Cod model, the $G(1,1)$ parameter, or scale, is a constant from which all other parameters are based. The $G(w)$ parameters are factors multiplied times the constant which essentially adjust the base for average exposure changes by accident year. The $f(d)$ parameters are factors multiplied times the constant, or constant adjusted by the $G(w)$ parameters, which essentially adjust the base (by accident year) for average incremental changes by development year. Using the data from Hayne [8], the companion Excel file summarizes the Cape Cod model parameters as in Table 3.4.

Table 3.4. Summary of Cape Cod Parameters

	Accident Period Parameters									
	Scale	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**
Mean	620.067	1.160	1.123	1.322	1.376	1.521	1.533	1.580	1.169	1.164
Std Dev	30.048	0.066	0.064	0.072	0.075	0.082	0.084	0.091	0.082	0.105
CoV	*4.8%*	*5.7%*	*5.7%*	*5.4%*	*5.4%*	*5.4%*	*5.5%*	*5.8%*	*7.1%*	*9.0%*
	Development Period Parameters (Average Incremental)									
		24	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**
Mean		1.181	1.063	0.838	0.534	0.284	0.111	0.067	0.015	0.024
Std Dev		0.041	0.040	0.036	0.029	0.023	0.016	0.016	0.009	0.017
Decay Ratios			*90.0%*	*78.8%*	*63.7%*	*53.2%*	*39.0%*	*60.7%*	*22.8%*	*158.0%*
CoV		*3.5%*	*3.8%*	*4.3%*	*5.5%*	*8.1%*	*14.8%*	*23.1%*	*61.1%*	*70.4%*

	K	**p**	**AIC**	**BIC**
Mean	13.104	0.435	619.3	661.5
Std Dev	1.010	0.083		
CoV	*7.7%*	*19.0%*		

	Parameters
Acc Period	9
Dev Period	9
Scale	1
	19

Using formulas (3.7) and (3.5) to calculate the expected mean and standard deviation the results for each incremental value are shown in Tables 3.5 and 3.6, respectively.

Table 3.5. Expected Incremental Mean Values for Cape Cod Model

Predicted Incremental Mean [Model Fitted] (Paid [÷ Ultimate Claims])

Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**Future Totals**
2006	620.07	732.01	659.04	519.38	330.98	176.23	68.79	41.73	9.53	15.05	0.00
2007	719.49	849.38	764.70	602.65	384.04	204.48	79.82	48.43	11.05	17.47	17.47
2008	696.47	822.21	740.24	583.37	371.76	197.94	77.27	46.88	10.70	16.91	27.61
2009	819.84	967.86	871.37	686.71	437.61	233.01	90.95	55.18	12.60	19.90	87.68
2010	853.00	1,006.99	906.61	714.48	455.31	242.43	94.63	57.41	13.11	20.71	185.86
2011	943.01	1,113.26	1,002.28	789.88	503.36	268.01	104.62	63.47	14.49	22.89	473.48
2012	950.77	1,122.42	1,010.52	796.38	507.50	270.22	105.48	63.99	14.61	23.08	984.87
2013	979.71	1,156.58	1,041.28	820.62	522.95	278.44	108.69	65.94	15.05	23.78	1,835.47
2014	725.16	856.08	770.74	607.41	387.08	206.10	80.45	48.81	11.14	17.60	2,129.33
2015	721.47	851.72	766.81	604.31	385.10	205.05	80.04	48.56	11.08	17.52	2,970.19
											8,711.96

Table 3.6. Incremental Standard Deviation Values for Cape Cod Model

Predicted Incremental Standard Deviation [Model Fitted] (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	58.08	62.43	59.64	53.77	44.20	33.60	22.31	17.95	9.44	11.52	0.00
2007	62.35	67.02	64.03	57.73	47.45	36.07	23.96	19.28	10.14	12.37	12.37
2008	59.13	63.56	60.72	54.75	45.00	34.21	22.72	18.28	9.61	11.73	15.17
2009	63.13	67.86	64.83	58.45	48.04	36.52	24.26	19.52	10.26	12.52	25.36
2010	64.83	69.69	66.58	60.02	49.34	37.51	24.91	20.04	10.54	12.86	36.04
2011	68.78	73.94	70.63	63.68	52.35	39.79	26.43	21.26	11.18	13.64	55.18
2012	66.30	71.26	68.08	61.38	50.45	38.35	25.47	20.49	10.78	13.15	73.31
2013	68.34	73.45	70.17	63.27	52.01	39.53	26.26	21.12	11.11	13.56	98.55
2014	59.01	63.43	60.59	54.63	44.90	34.14	22.67	18.24	9.59	11.70	104.47
2015	55.18	59.32	56.67	51.09	42.00	31.92	21.20	17.06	8.97	10.95	114.30
											210.79

Reviewing Table 3.5 you can see that the scale, or constant, is the value for 2006 at 12 months of development. The $G(w)$, or accident year, parameters are used to adjust the scale in the 12 month column and then the $f(d)$, or development year, parameters are used to adjust the scale, or scale adjusted by accident year, for each development column.

3.3 Chain Ladder Model

For the traditional Chain Ladder method, average development factors are multiplied by the cumulative amounts by accident year to estimate the expected future incremental values. Hayne [8] also uses the cumulative amounts by accident year, but instead derives parameters which represent the proportion of the incremental value in each development year. The parameters are constrained so that the incremental values sum to the cumulative values. In addition, $n-1$ parameters are used with the last development year parameter derived so that the sum of all parameters is 100%.

$$E[A(w,d)] = \begin{cases} G(w) \times f(d), & w=1, d<n \\ G(w) \times \left[1 - \sum_{d=1}^{d=n-1} f(d)\right], & w=1, d=n \\ \dfrac{G(w) \times f(d)}{\sum_{k=1}^{k=n+1-w} f(k)}, & w>1, d<n \\ \dfrac{G(w) \times \left[1 - \sum_{d=1}^{d=n-1} f(d)\right]}{\sum_{k=1}^{k=n+1-w} f(k)}, & w>1, d=n \end{cases} \quad (3.8)$$

In the Hayne Chain Ladder model, the $G(w)$ parameters are the cumulative values for each accident year. The $f(d)$ parameters are factors multiplied times the cumulative values to derive the expected incremental values by development year. Only $n-1$ parameters are derived and the "parameter" for the last development period is one minus the sum of the $n-1$ parameters. In order to constrain the sum of the expected incremental values to equal the cumulative values, the $f(d)$ parameters are divided by the sum of the parameters for that accident year so that the proportional factors for that accident year up to the diagonal sum to 100%. Using the data from Hayne [8], the companion Excel file summarizes the Chain Ladder model parameters as in Table 3.7.

Table 3.7. Summary of Chain Ladder Parameters

	Development Period Parameters (Average Incremental)									
	12	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**
Mean	0.195	0.231	0.208	0.164	0.104	0.056	0.022	0.013	0.003	0.005
Std Dev	0.005	0.005	0.005	0.005	0.005	0.004	0.003	0.003	0.002	0.003
Decay Ratios:		118.1%	90.0%	78.8%	63.7%	53.2%	39.0%	60.8%	22.9%	157.7%
CoV:	*2.5%*	*2.3%*	*2.5%*	*3.1%*	*4.5%*	*7.3%*	*14.3%*	*22.6%*	*60.4%*	*69.6%*

	K	p	AIC	BIC		Parameters
Mean	13.074	0.438	619.4	661.5	Acc Period	10
Std Dev	1.007	0.082			Dev Period	9
CoV:	*7.7%*	*18.8%*			Trend	0
						19

The parameter for 120 months is greyed since it is derived by subtracting the sum of the other parameters from one. Using formulas (3.8) and (3.5) to calculate the expected mean and standard deviation the results for each incremental value are shown in Tables 3.8 and 3.9, respectively.

Table 3.8. Expected Incremental Mean Values for Chain Ladder Model

Predicted Incremental Mean [Model Fitted] (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	617.57	729.07	656.33	517.19	329.57	175.45	68.44	41.59	9.51	15.00	0.00
2007	715.93	845.18	760.86	599.56	382.05	203.39	79.34	48.21	11.03	17.39	17.39
2008	695.14	820.64	738.76	582.16	370.96	197.49	77.04	46.81	10.71	16.89	27.60
2009	823.53	972.20	875.21	689.67	439.47	233.96	91.26	55.46	12.69	20.01	88.15
2010	854.54	1,008.81	908.16	715.64	456.02	242.77	94.70	57.55	13.16	20.76	186.17
2011	943.04	1,113.29	1,002.22	789.76	503.25	267.91	104.51	63.51	14.53	22.91	473.37
2012	951.15	1,122.87	1,010.84	796.55	507.58	270.22	105.41	64.06	14.65	23.11	985.02
2013	981.03	1,158.13	1,042.59	821.57	523.52	278.70	108.72	66.07	15.11	23.83	1,837.53
2014	726.85	858.06	772.46	608.70	387.88	206.49	80.55	48.95	11.20	17.66	2,133.89
2015	723.30	853.88	768.69	605.74	385.99	205.49	80.16	48.71	11.14	17.57	2,977.37
											8,726.49

Table 3.9. Incremental Standard Deviation Values for Chain Ladder Model

Predicted Incremental Standard Deviation [Model Fitted] (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	58.10	62.48	59.67	53.76	44.14	33.49	22.18	17.84	9.35	11.41	0.00
2007	62.38	67.08	64.06	57.72	47.38	35.96	23.81	19.15	10.04	12.25	12.25
2008	59.23	63.69	60.83	54.80	44.99	34.14	22.61	18.18	9.53	11.63	15.04
2009	63.44	68.22	65.15	58.70	48.19	36.57	24.22	19.47	10.21	12.46	25.27
2010	65.08	69.98	66.84	60.22	49.44	37.51	24.84	19.98	10.47	12.78	35.91
2011	69.01	74.21	70.87	63.86	52.42	39.78	26.34	21.18	11.11	13.56	55.07
2012	66.53	71.54	68.32	61.56	50.54	38.35	25.40	20.42	10.71	13.07	73.29
2013	68.61	73.78	70.46	63.48	52.12	39.55	26.19	21.06	11.04	13.48	98.71
2014	59.22	63.68	60.82	54.79	44.98	34.14	22.61	18.18	9.53	11.63	104.68
2015	55.39	59.56	56.88	51.25	42.07	31.93	21.14	17.00	8.91	10.88	114.60
											211.05

Reviewing Table 3.8 it is not as obvious how the parameters relate to the incremental values compared to the Berquist-Sherman or Cape Cod models. However, if you sum the incremental values up to the diagonal for each accident year, you will discover that they sum to the cumulative value for each accident year. Thus, the $f(d)$ parameters can be seen as representing an average proportion of the incremental values compared to the cumulative values.

3.4 Hoerl Curve Model

The Hoerl Curve is a three parameter exponential model which uses the development lag for all three parameters; i.e., number of periods, number of periods squared and the natural log of the number of periods. Hayne [8] combines these three parameters with a constant level parameter and an accident year trend factor.

$$E[A(w,d)] = e^{G(1)+d\times f(1)+d^2\times f(2)+\ln(d)\times f(3)+w\times G(2)} \qquad (3.9)$$

In the Hayne Hoerl Curve model, the $G(1)$ parameter is the constant level on a log scale. The $G(2)$ parameter is a constant trend which adjusts the level by accident year. The $f(1)$, $f(2)$, and $f(3)$ parameters are factors multiplied times the development lags; i.e., by d, d^2, and $\ln(d)$, respectfully. Using the data from Hayne [8], the companion Excel file summarizes the Hoerl Curve model parameters as in Table 3.10.

Table 3.10. Summary of Hoerl Curve Parameters

	Parameters (Average Incremental)				
	Level	d	d^2	ln(d)	Trend
Mean	6.496	0.005	(0.065)	0.596	0.043
Std Dev	0.220	0.240	0.019	0.323	0.008
CoV:	*3.4%*	*4687.1%*	*-28.4%*	*54.2%*	*19.5%*

	K	p	AIC	BIC		Parameters
Mean	13.147	0.506	639.7	653.8	Level	1
Std Dev	1.014	0.083			Development	3
CoV:	*7.7%*	*16.3%*			Trend	1
						5

Using formulas (3.9) and (3.5) to calculate the expected mean and standard deviation the results for each incremental value are shown in Tables 3.11 and 3.12, respectively.

Table 3.11. Expected Incremental Mean Values for Hoerl Curve Model

Predicted Incremental Mean [Model Fitted] (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	651.30	813.57	751.30	567.59	362.10	197.86	93.30	38.14	13.55	4.20	0.00
2007	679.90	849.29	784.29	592.51	378.00	206.55	97.40	39.81	14.15	4.38	4.38
2008	709.75	886.58	818.73	618.53	394.60	215.62	101.67	41.56	14.77	4.57	19.34
2009	740.92	925.51	854.68	645.68	411.93	225.08	106.14	43.38	15.42	4.77	63.58
2010	773.45	966.15	892.20	674.04	430.01	234.97	110.80	45.29	16.09	4.98	177.16
2011	807.41	1,008.57	931.38	703.63	448.90	245.28	115.66	47.28	16.80	5.20	430.22
2012	842.86	1,052.85	972.28	734.53	468.61	256.05	120.74	49.35	17.54	5.43	917.72
2013	879.87	1,099.08	1,014.97	766.78	489.18	267.30	126.04	51.52	18.31	5.67	1,724.80
2014	918.50	1,147.34	1,059.53	800.45	510.66	279.03	131.58	53.78	19.11	5.92	2,860.07
2015	958.83	1,197.72	1,106.06	835.59	533.08	291.28	137.35	56.14	19.95	6.18	4,183.37
											10,380.64

Table 3.12. Incremental Standard Deviation Values for Hoerl Curve Model

Predicted Incremental Standard Deviation [Model Fitted] (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	95.72	107.11	102.88	89.29	71.14	52.41	35.84	22.80	13.51	7.47	0.00
2007	98.43	110.15	105.81	91.82	73.16	53.89	36.85	23.45	13.90	7.68	7.68
2008	96.76	108.28	104.00	90.26	71.91	52.98	36.23	23.05	13.66	7.55	15.61
2009	98.34	110.05	105.71	91.73	73.09	53.84	36.82	23.42	13.88	7.67	28.29
2010	101.45	113.52	109.04	94.63	75.39	55.54	37.98	24.16	14.32	7.92	47.90
2011	105.29	117.83	113.18	98.22	78.25	57.65	39.42	25.08	14.86	8.22	76.12
2012	103.35	115.65	111.08	96.40	76.81	56.58	38.69	24.61	14.59	8.06	107.15
2013	107.45	120.25	115.50	100.23	79.86	58.83	40.23	25.59	15.17	8.38	149.87
2014	108.08	120.95	116.18	100.82	80.33	59.18	40.47	25.74	15.26	8.43	190.32
2015	103.53	115.85	111.28	96.57	76.94	56.68	38.76	24.66	14.62	8.08	216.00
											354.98

Reviewing Table 3.11, the link to the parameters must be viewed on a log scale. Starting with the first development column, the beginning "levels" for each accident year on a log scale is the $G(1)$ parameter plus the trend times the number of years, plus one of the $f(1)$ and $f(2)$ parameters. Moving from left to right across the development years, the combination of the three development parameters acts to first increase the incremental values, then to decrease the incremental values in a smooth curve.

3.5 Wright Model

The Wright model also uses the three parameter Hoerl curve, but instead of a constant level and trend parameters, individual parameters for each accident year "level" are used.

$$E[A(w,d)] = e^{G(w)+d\times f(1)+d^2\times f(2)+\ln(d)\times f(3)} \tag{3.10}$$

In the Hayne Wright model, the $G(w)$ parameters are the individual levels for each accident year. Similar to the Hoerl Curve model, the $f(1)$, $f(2)$, and $f(3)$ parameters are factors multiplied times the development lags; i.e., by d, d^2, and $\ln(d)$, respectfully. Using the data from Hayne [8], the companion Excel file summarizes the Wright model parameters as in Table 3.13.

Table 3.13. Summary of Wright Parameters

Accident Period Parameters

	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Mean	6.312	6.472	6.436	6.587	6.636	6.738	6.742	6.771	6.475	6.468
Std Dev	0.168	0.167	0.167	0.166	0.167	0.167	0.166	0.164	0.166	0.184
CoV	*2.7%*	*2.6%*	*2.6%*	*2.5%*	*2.5%*	*2.5%*	*2.5%*	*2.4%*	*2.6%*	*2.8%*

Development Period Parameters (Average Incremental)

	d	d²	ln(d)
Mean	0.192	(0.078)	0.290
Std Dev	0.183	0.015	0.232
CoV	*95.4%*	*-19.5%*	*80.0%*

	K	p	AIC	BIC		Parameters
Mean	14.592	0.319	612.3	642.4	Acc Period	10
Std Dev	0.909	0.075			Dev Period	3
						13
CoV	*6.2%*	*23.4%*				

Using formulas (3.10) and (3.5) to calculate the expected mean and standard deviation the results for each incremental value are shown in Tables 3.14 and 3.15, respectively.

Table 3.14. Expected Incremental Mean Values for Wright Model

Predicted Incremental Mean [Model Fitted] (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	617.75	724.24	668.24	509.80	326.60	176.91	81.32	31.79	10.59	3.01	0.00
2007	724.55	849.44	783.76	597.94	383.06	207.49	95.38	37.29	12.42	3.52	3.52
2008	698.92	819.39	756.04	576.79	369.51	200.15	92.00	35.97	11.98	3.40	15.38
2009	813.22	953.39	879.68	671.11	429.93	232.88	107.05	41.85	13.94	3.96	59.74
2010	854.17	1,001.41	923.98	704.91	451.59	244.61	112.44	43.96	14.64	4.16	175.19
2011	945.66	1,108.66	1,022.94	780.41	499.95	270.81	124.48	48.67	16.21	4.60	464.77
2012	949.61	1,113.29	1,027.21	783.67	502.04	271.94	125.00	48.87	16.27	4.62	968.75
2013	977.65	1,146.17	1,057.55	806.81	516.87	279.97	128.70	50.31	16.75	4.76	1,804.17
2014	726.83	852.12	786.23	599.82	384.26	208.14	95.68	37.41	12.46	3.54	2,127.54
2015	721.95	846.40	780.95	595.80	381.68	206.75	95.04	37.16	12.37	3.51	2,959.65
											8,578.71

Table 3.15. Incremental Standard Deviation Values for Wright Model

Predicted Incremental Standard Deviation [Model Fitted] (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	57.98	61.00	59.45	54.53	47.30	38.89	30.35	22.48	15.83	10.59	0.00
2007	61.39	64.59	62.95	57.74	50.09	41.18	32.13	23.81	16.76	11.21	11.21
2008	58.37	61.41	59.86	54.90	47.63	39.16	30.55	22.64	15.93	10.66	19.17
2009	60.93	64.10	62.48	57.31	49.71	40.87	31.89	23.63	16.63	11.13	30.96
2010	62.47	65.73	64.06	58.76	50.97	41.91	32.70	24.23	17.05	11.41	45.57
2011	65.55	68.96	67.21	61.65	53.48	43.97	34.31	25.42	17.89	11.97	64.96
2012	63.03	66.32	64.64	59.29	51.43	42.28	32.99	24.44	17.21	11.51	80.91
2013	64.73	68.10	66.38	60.88	52.81	43.42	33.88	25.10	17.67	11.82	103.01
2014	57.96	60.98	59.43	54.51	47.28	38.88	30.33	22.47	15.82	10.58	109.72
2015	54.21	57.03	55.58	50.98	44.22	36.36	28.37	21.02	14.80	9.90	117.40
											225.23

Reviewing Table 3.14 you can see the similarities to Table 3.11. Starting with the first development column, the beginning "levels" for each accident year on a log scale is the $G(w)$ parameter plus one of the $f(1)$ and $f(2)$ parameters. Moving from left to right across the development years, the combination of the three development parameters acts to first increase the incremental values, then to decrease the incremental values in a smooth curve.

3.6 The Simulation Process

For each of the Hayne MLE models, using the parameters to calculate the expected mean and standard deviation for each incremental cell is only the starting point. Additional outputs for each model are the standard deviations for each parameter (shown in Tables 3.1, 3.4, 3.7, 3.10, and 3.13) and the variance-covariance matrix of all the parameters (not shown). Using the means and variance-covariance matrix, the simulation process starts by sampling a random set of new parameters using the multi-variate Normal distribution. For example, a sample iteration for the Berquist-Sherman model could look like Table 3.16.

Table 3.16. Sample of Berquist-Sherman Parameters

Berquist-Sherman:

Development Period Parameters (Average Incremental)

12	24	36	48	60	72	84	96	108	120
668.32	704.13	645.21	559.41	380.69	165.37	84.01	33.80	26.55	15.75

Trend	K	p
0.047	11.268	0.661

Using the sample parameters, the next step in the simulation process is to calculate the mean and standard deviation for each cell as in Tables 3.17 and 3.18.

Table 3.17. Sampled Incremental Mean Values for Berquist-Sherman

Generate Incremental Mean from Random Parameters (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	700.40	737.93	676.18	586.26	398.96	173.31	88.05	35.42	27.82	16.50	0.00
2007	734.02	773.35	708.64	614.40	418.11	181.63	92.27	37.12	29.16	17.29	17.29
2008	769.26	810.47	742.66	643.89	438.18	190.35	96.70	38.90	30.56	18.12	48.68
2009	806.18	849.37	778.30	674.79	459.21	199.48	101.34	40.77	32.02	18.99	91.78
2010	844.88	890.14	815.66	707.18	481.25	209.06	106.21	42.73	33.56	19.91	202.40
2011	885.43	932.87	854.81	741.13	504.35	219.09	111.31	44.78	35.17	20.86	431.21
2012	927.93	977.65	895.84	776.70	528.56	229.61	116.65	46.93	36.86	21.86	980.47
2013	972.47	1,024.57	938.84	813.98	553.93	240.63	122.25	49.18	38.63	22.91	1,841.51
2014	1,019.15	1,073.75	983.91	853.06	580.52	252.18	128.11	51.54	40.48	24.01	2,913.81
2015	1,068.07	1,125.29	1,031.13	894.00	608.39	264.29	134.26	54.01	42.43	25.16	4,178.97
											10,706.12

Table 3.18. Sampled Incremental Std. Dev. Values for Berquist-Sherman

Generate Incremental Standard Deviation from Random Parameters (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	107.53	111.30	105.06	95.60	74.12	42.71	27.30	14.95	12.74	9.02	0.00
2007	111.61	115.53	109.04	99.23	76.93	44.33	28.33	15.52	13.23	9.37	9.37
2008	110.73	114.62	108.19	98.45	76.33	43.98	28.11	15.40	13.12	9.29	16.08
2009	113.59	117.58	110.98	100.99	78.30	45.12	28.84	15.79	13.46	9.53	22.84
2010	118.27	122.42	115.55	105.15	81.52	46.98	30.02	16.44	14.02	9.92	38.30
2011	123.90	128.25	121.05	110.15	85.40	49.21	31.45	17.23	14.69	10.40	63.50
2012	122.74	127.05	119.92	109.12	84.60	48.75	31.16	17.07	14.55	10.30	105.43
2013	128.81	133.33	125.84	114.51	88.79	51.16	32.70	17.91	15.27	10.81	159.23
2014	130.77	135.36	127.76	116.26	90.14	51.94	33.20	18.18	15.50	10.97	206.04
2015	126.42	130.86	123.52	112.40	87.14	50.22	32.09	17.58	14.98	10.61	238.34
											376.95

Next, using the sampled mean and standard deviation for each incremental cell process variance is added by randomly generating an observation for each cell using the Normal distribution and the sampled mean and standard deviation for that cell. Continuing the example, U(0,1) random values are shown in Table 3.19 and the random observations based on the means and standard deviations by cell in Tables 3.17 and 3.18, respectively, are shown in Table 3.20.

Table 3.19. Random Values

Simulated Random Values [Correlated] (Paid)

Year	12	24	36	48	60	72	84	96	108	120
2006	0.4009	0.4189	0.9459	0.3101	0.3192	0.1740	0.4005	0.0364	0.1201	0.0822
2007	0.3078	0.7144	0.5731	0.1989	0.4034	0.4817	0.3595	0.8254	0.8173	0.6103
2008	0.3334	0.8134	0.5619	0.9379	0.3830	0.0163	0.1479	0.8463	0.9088	0.9352
2009	0.9491	0.2084	0.7126	0.2911	0.4702	0.6269	0.7621	0.4779	0.1540	0.0921
2010	0.7837	0.4402	0.1229	0.8062	0.4995	0.3770	0.3096	0.5040	0.8820	0.0521
2011	0.1960	0.2693	0.0002	0.3931	0.1450	0.0349	0.1155	0.0600	0.3554	0.0203
2012	0.7020	0.0977	0.2878	0.7736	0.5855	0.0297	0.9950	0.3926	0.7570	0.6794
2013	0.5225	0.0925	0.9975	0.3746	0.1550	0.5164	0.0112	0.7273	0.1654	0.5295
2014	0.4272	0.7301	0.3417	0.6337	0.3146	0.7889	0.2524	0.8902	0.8295	0.6409
2015	0.0630	0.4542	0.8377	0.4535	0.9946	0.1432	0.5699	0.1098	0.7175	0.1494

Table 3.20. Sample Observations for Berquist-Sherman

Generate Random Observation from Sampled Incremental Mean & Variance (Paid [÷ Ultimate Claims])

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	667.37	709.01	844.47	532.84	359.51	130.00	79.63	7.10	11.80	3.16	0.00
2007	670.92	834.93	723.93	523.28	394.97	177.35	80.40	51.29	40.82	19.53	19.53
2008	714.78	909.75	754.66	794.60	411.07	91.53	65.10	54.30	47.90	32.15	80.05
2009	991.65	745.44	836.84	612.69	449.33	212.28	121.06	39.09	17.25	5.51	61.86
2010	934.48	865.17	672.08	795.40	477.14	191.62	89.39	42.09	49.95	2.84	184.27
2011	770.31	845.46	403.96	704.98	407.24	124.96	71.03	16.39	28.85	(1.54)	239.69
2012	988.80	802.30	820.93	855.55	543.17	132.74	197.56	41.30	46.56	26.29	987.63
2013	973.59	836.35	1,296.12	770.69	456.90	240.28	43.88	59.43	22.61	23.20	1,617.00
2014	988.06	1,152.40	924.07	888.17	531.34	292.49	103.72	73.59	54.89	27.54	2,895.81
2015	862.99	1,103.37	1,150.12	874.98	832.28	206.77	138.49	30.96	50.55	13.31	4,400.84
											10,486.67

Since the model is typically based on average severities, the final step is to multiply the random observations times the ultimate claim counts[6] by year to convert the sample to total claim values, as in Table 3.21.

Table 3.21. Conversion to Total Value for Berquist-Sherman

Convert Incremental Severity (Paid [÷ Ultimate Claims]) to Total Incremental Value (in 000's)

Year	12	24	36	48	60	72	84	96	108	120	Future Totals
2006	26,135	27,766	33,070	20,866	14,079	5,091	3,118	278	462	124	0
2007	25,946	32,289	27,996	20,236	15,275	6,859	3,109	1,983	1,578	755	755
2008	29,878	38,029	31,546	33,215	17,183	3,826	2,721	2,270	2,002	1,344	3,346
2009	41,910	31,505	35,368	25,894	18,990	8,972	5,116	1,652	729	233	2,614
2010	38,763	35,888	27,879	32,994	19,792	7,948	3,708	1,746	2,072	118	7,643
2011	30,978	34,000	16,245	28,350	16,377	5,025	2,856	659	1,160	(62)	9,639
2012	43,110	34,979	35,791	37,301	23,682	5,787	8,613	1,801	2,030	1,146	43,059
2013	41,006	35,226	54,590	32,460	19,244	10,120	1,848	2,503	952	977	68,105
2014	42,960	50,106	40,178	38,617	23,103	12,717	4,510	3,200	2,387	1,197	125,908
2015	42,712	54,608	56,922	43,305	41,192	10,234	6,854	1,532	2,502	659	217,808
											478,879

Repeating these steps a large number of times, the results for all iterations can be saved and summarized by accident year, calendar year, and a variety of other ways. The output will be discussed in more detail in Sections 5 and 6.

4. Practical Issues

Now that the basic Hayne MLE framework has been described, a variety of practical issues needed for addressing many common problems can be addressed. In order to distinguish whether the underlying model has parameters associated with individual development period, the underlying models can be categorized into two families. The first family has parameters tied to individual development age — Berquist Sherman, Cape Cod, and Chain Ladder models fall into this family. The other family has no definite parameters on individual development period and the parameters are more comparable to coefficient of regression on development age (operational time) — Hoerl Curve and Wright models belong to this family.

4.1. Negative Incremental Values

In general for the Hayne MLE framework, no special care is required in modeling triangles with a few negative entries. When the total incremental values for a given development period is significantly lower than zero, models from the first family have no problem dealing with this type of triangle. Calibrated development period parameters, most likely, will turn out to be negative to reflect negative expected incremental values for the period. For models from the second family, incremental means are exponential and hence are always positive so negative incremental values in the triangle are difficult to model, which typically implies

[6] This step depends on the original exposure basis used to parameterize the model. For example, if the model is based on pure premiums then the last step is to multiply times exposures by year.

inappropriateness of the model and resulting in a bad fit to the data. However, negative numbers can still be simulated due to the process variance during simulation so a close fit may still work.

4.2. Standardized Residuals

As the Hayne MLE framework is based on an assumed distribution, i.e., the normal distribution for incremental values, this implies that the standardized residuals should be normally distributed with mean of zero and a standard deviation of one. If the average of all the residuals is significantly different than zero, then the fit of the model should be questioned. The goodness of fit to a standard normal distribution of standardized residuals, to some degree, implies the appropriateness of the chosen model. Unlike the ODP Bootstrap model, however, the standardized residuals are not used during the simulation process.

While the residuals are not sampled, the mean and standard deviation of the residuals can be used to adjust the process variance simulations. For the mean, an average of the residuals greater than zero implies that the mean of the parameters are "low" compared to means that would result in an average of zero. Thus, the adjustment for the mean is to increase the mean for each cell by the standard deviation for that cell times the average of the residuals. Similarly, a standard deviation of the residuals greater than one implies "less" variability than would be "normal" so the standard deviation for each cell can be increased by multiplying it times the standard deviation of the residuals.

Another way of thinking about this adjustment is to remember that the process variance in the simulations is based on $N(0, 1)$, so if the residuals exhibit a mean and standard deviation which differ from zero and one, respectively, then this adjustment allows the process variance to more closely match the residuals. In the "Hayne MLE Models.xlsm" file, the "Include Residual Adjustment" option on the Inputs sheet allows the user to use this adjustment or not as this will move away from the calculated Hayne MLE parameters but it could be a way fitting the model to the data.

4.3. Using an *N*-Year Average

It is quite common for actuaries to use averages that are less than all years in their chain-ladder and related methods. Similarly, the Hayne MLE models can be adjusted to only consider the data in the most recent diagonals. For the Hayne MLE framework, only the most recent $L+1$ diagonals (since an L-year average uses $L+1$ diagonals) could be used to parameterize the model. The shape of the data to be modeled essentially becomes a trapezoid instead of a triangle and the excluded diagonals are given zero weight in the models. When running the simulations the entire triangle can still be used since the parameterization of the model has already been constrained by the number of diagonals.

The companion "Hayne MLE Models.xlsm" file has not been specifically designed to select an L-year model, but that can easily be accomplished by using the outlier table to give zero weight to the prior diagonals.

4.4. Missing Values

Sometimes the loss triangle will have missing values. For example, values may be missing from the middle of the triangle, or a triangle may be missing the oldest diagonals, if loss data was not kept in the early years of the book of business.

If values are missing, then the following calculations will be affected:

- Fitted parameters
- Variance-Covariance Matrix
- Fitted triangle
- Residuals
- Degrees of freedom

There are several solutions. The missing value may be estimated using the surrounding values. Or, the parameterization of the model can exclude the missing values as long as the missing value is not compromising the surrounding incremental values, or for the Chain Ladder model the cumulative values. In any case, zero weights are applied to corresponding entries in maximizing log-likelihood functions. The mean and standard deviation of the incremental corresponding to the missing value can be derived from simulated parameters.

If the missing value lies on the most recent diagonal, parameters can be calibrated without any issue except for the Chain Ladder model, which relies on paid-to-date losses to estimate average incremental values. A solution is to use the value in the second most recent diagonal to fit the triangle and the average incremental formula should be adjusted to be divided by the sum of the first $n-w$ parameters rather than $n-w+1$ parameters. Of course for other MLE models, simply using the outliers to apply zero weight to the corresponding cell will allow the model to be parameterized without disturbing the overall framework.

4.5. Outliers

There may be a few extreme or incorrect values in the original triangle dataset that could be considered outliers. These may not be representative of the variability of the dataset in the future and, if so, the modeler may want to remove their impact from the model. These values could be removed, and dealt with in the same manner as missing values by applying zero weight to corresponding incremental.

If there are a significant number of outliers, then this could be an indication that the model is not a good fit to the data. Outliers should always be removed only after careful consideration of the underlying data to make sure it is truly an unusual event.

4.6. Heteroscedasticity

As noted earlier, the Hayne MLE models include variance parameters which adjust the variance for each cell instead of assuming a constant variance throughout. In essence, the modeling framework assumes heteroscedasticity. However, since the variance for the incremental value is only specified using two parameters, it is still possible that the modeled heteroscedasticity does not match up well with the variances in the data. In this case, additional variance parameters can be specified as described in Hayne [8], but that is outside the scope of this paper.

4.7. Heteroecthesious Data

The basic Hayne MLE framework assumes both a symmetrical shape (e.g., annual by annual, quarterly by quarterly, etc. triangles) and homoecthesious data (i.e., similar exposures).[7] Other non-symmetrical shapes (e.g., annual x quarterly data) can also be modeled with the Hayne MLE framework as assumptions are independent from triangle shapes.

Most often, the actuary will encounter heteroecthesious data (i.e., incomplete or uneven exposures) at interim evaluation dates, with the two most common data triangles being either a partial first development period or a partial last calendar period. For example, with annual data evaluated as of June 30, partial first development period data would have all development periods ending at 6, 18, 30, etc. months, while partial last calendar period data would have development periods as of 12, 24, 36, etc. months for all of the data in the triangle except the

[7] The terms *homoecthesious* and *heteroecthesious* are a combination of the Greek *homos* (or ὁμός) meaning the same or *hetero* (or ἑτερο) meaning different and the Greek *ekthesē* (or ἐκθεση) meaning exposure. They were introduced in Shapland [15].

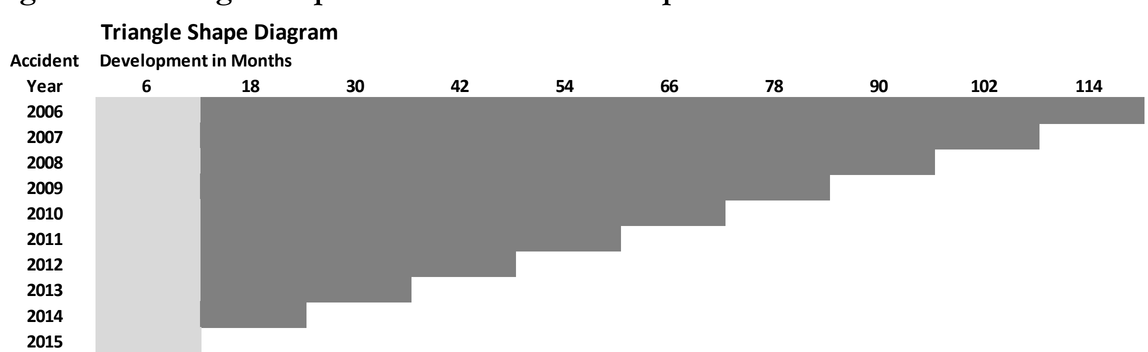
last diagonal, which would have development periods as of 6, 18, 30, etc. months. In either case, not all of the data in the triangle has full annual exposures – i.e., it is heteroecthesious data.

4.7.1. Partial first development period data

For partial first development period data, the first development column has a different exposure period than the rest of the columns (e.g., in the earlier example the first column has six months of development exposure while the rest have 12), as illustrated in Figure 4.1. In models such as Berquist Sherman, Cape Cod and Chain Ladder, where a parameter is specified for each development period, it is not an issue in the parameterization process. Likewise, for the Hoerl Curve or Wright models, development age or operational time is embedded in the model so the development age component should reflect this partial first development period and no further adjustment is required when fitting the model.

After simulation, an additional adjustment for this type of heteroecthesious data is applied in the projection of future incremental values. In a deterministic analysis, the most recent accident year needs to be adjusted to remove exposures beyond the evaluation date. For example, continuing the previous example the development periods at 18 months and later are all for an entire year of exposure whereas the six month column is only for six months of exposure. Thus, the 18 month incremental values will effectively extrapolate the first six months of exposure in the latest accident year to a full accident year's exposure. Accordingly, it is common practice to reduce the projected future payments by half to remove the exposure from June 30 to December 31.[8]

Figure 4.1. Triangle Shape for Partial First Development Period



[8] Reduction by half is actually an approximation since it would also make sense to account for the differences in development between the first and second half years.

The simulation process for Hayne MLE models can be adjusted similarly to the way a deterministic analysis would be adjusted. After simulated parameters are used to project the future incremental values the last accident year's values can be reduced (in the previous example by 50%) to remove the future exposure and then process variance can be simulated as before. Alternatively, the future incremental values can be reduced after the process variance step. For example, Table 4.1 can be compared to Table 3.21 to see the reduction in the future exposures for the last accident year.

Table 4.1 Total Values Adjusted to Remove Future Exposures

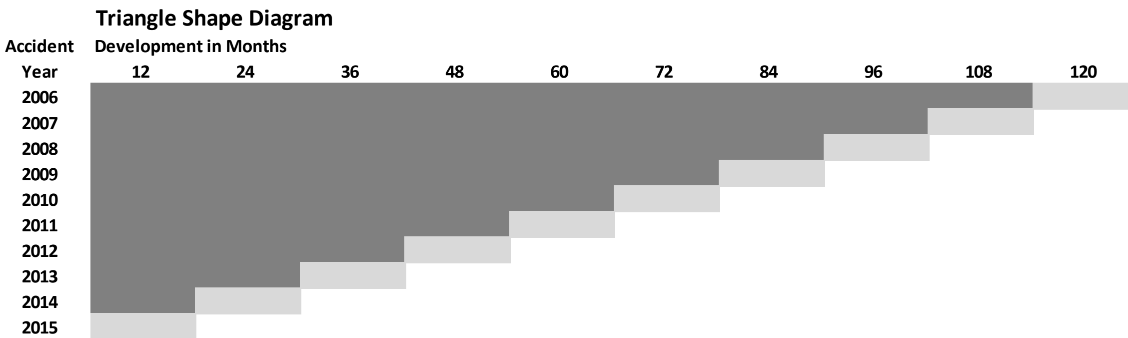
Adjust Total Incremental Value to Remove Future Exposures (Paid)

Year	6	18	30	42	54	66	78	90	102	114	Acc Yr Unpaid
2006	28,857,379	34,633,541	34,647,465	21,990,975	15,245,558	7,118,499	4,118,037	1,554,858	909,294	1,036,639	0
2007	26,990,356	36,365,617	37,107,448	20,096,463	16,226,379	4,625,682	4,294,384	742,007	1,685,216	437,669	437,669
2008	27,339,334	38,824,698	41,206,538	27,457,777	19,874,669	8,614,451	4,357,317	2,305,030	1,936,202	875,195	2,811,397
2009	33,025,357	44,270,589	34,259,044	30,153,257	16,962,781	7,431,261	5,249,446	2,650,407	852,779	1,794,671	5,297,857
2010	22,528,035	48,022,691	34,464,620	26,371,971	17,854,923	9,339,285	4,970,790	1,779,924	1,169,219	1,112,603	9,032,536
2011	30,981,966	42,055,231	39,432,800	30,503,214	18,950,981	4,885,726	5,591,433	3,801,018	2,533,830	512,219	17,324,226
2012	41,326,250	54,286,380	40,110,104	34,320,894	20,507,632	9,781,702	5,854,535	3,771,950	2,580,919	2,407,282	44,904,020
2013	38,369,807	48,715,364	48,470,486	35,752,741	15,959,829	10,458,698	5,220,660	4,310,765	1,937,228	599,141	74,239,063
2014	48,530,079	61,040,684	52,480,950	38,089,201	22,880,897	7,109,981	3,763,221	4,049,779	864,925	1,953,303	131,192,259
2015	56,887,997	***33,868,120***	***29,884,232***	***23,102,485***	***17,305,781***	***7,240,564***	***2,881,388***	***1,974,971***	***573,449***	***47,687***	116,878,677
											402,117,704

4.7.2. Partial last calendar period data

For a partial last calendar period data, most of the data in the triangle has annual exposures and annual development periods, except for the last diagonal which, continuing the example, only has a six-month development period as illustrated in Figure 4.2. A simple approach is to adjust the raw data incremental values along the diagonal to a full development period to make them consistent with the rest of the triangle. The parameterization process can then be done with the adjusted incremental values.

Figure 4.2. Triangle Shape for Partial Last Calendar Period



During the Hayne MLE simulation process, incremental means and standard deviations can be calculated from the fully annualized sample parameters and used to simulate incremental values. Then, the last diagonal from the sample triangle can be adjusted to de-annualize the incremental values in the last diagonal – i.e., reversing the annualization of the original last diagonal – as illustrated in Table 4.2. Finally, the future incremental values for the last accident year must be reduced (in the previous example by 50%) to remove the future exposure, as illustrated in Table 4.3.[9]

Table 4.2 Total Values Adjusted to De-Annualize Incremental Values

Adjust Total Incremental Value for Exposures (Paid)

Year	12	24	36	48	60	72	84	96	108	120	132	Future Totals
2006	28,857,379	34,633,541	34,647,465	21,990,975	15,245,558	7,118,499	4,118,037	1,554,858	909,294	518,319	518,319	518,319
2007	26,990,356	36,365,617	37,107,448	20,096,463	16,226,379	4,625,682	4,294,384	742,007	842,608	1,061,442	218,834	1,280,277
2008	27,339,334	38,824,698	41,206,538	27,457,777	19,874,669	8,614,451	4,357,317	1,152,515	2,120,616	1,405,699	437,598	3,963,913
2009	33,025,357	44,270,589	34,259,044	30,153,257	16,962,781	7,431,261	2,624,723	3,949,926	1,751,593	1,323,725	897,336	7,922,580
2010	22,528,035	48,022,691	34,464,620	26,371,971	17,854,923	4,669,643	7,155,037	3,375,357	1,474,571	1,140,911	556,302	13,702,178
2011	30,981,966	42,055,231	39,432,800	30,503,214	9,475,491	11,918,353	5,238,579	4,696,226	3,167,424	1,523,025	256,110	26,799,716
2012	41,326,250	54,286,380	40,110,104	17,160,447	27,414,263	15,144,667	7,818,119	4,813,243	3,176,434	2,494,100	1,203,641	62,064,467
2013	38,369,807	48,715,364	24,235,243	42,111,613	25,856,285	13,209,264	7,839,679	4,765,713	3,123,997	1,268,185	299,571	98,474,306
2014	48,530,079	30,520,342	56,760,817	45,285,076	30,485,049	14,995,439	5,436,601	3,906,500	2,457,352	1,409,114	976,651	161,712,601
2015	14,221,999	76,534,118	63,752,353	52,986,717	40,408,266	24,546,345	10,121,952	4,856,358	2,548,419	621,136	47,687	276,423,351
												652,861,709

Table 4.3 Total Values Adjusted to Remove Future Exposures

Adjust Total Incremental Value to Remove Future Exposures (Paid)

Year	12	24	36	48	60	72	84	96	108	120	132	Acc Yr Unpaid
2006	28,857,379	34,633,541	34,647,465	21,990,975	15,245,558	7,118,499	4,118,037	1,554,858	909,294	518,319	518,319	518,319
2007	26,990,356	36,365,617	37,107,448	20,096,463	16,226,379	4,625,682	4,294,384	742,007	842,608	1,061,442	218,834	1,280,277
2008	27,339,334	38,824,698	41,206,538	27,457,777	19,874,669	8,614,451	4,357,317	1,152,515	2,120,616	1,405,699	437,598	3,963,913
2009	33,025,357	44,270,589	34,259,044	30,153,257	16,962,781	7,431,261	2,624,723	3,949,926	1,751,593	1,323,725	897,336	7,922,580
2010	22,528,035	48,022,691	34,464,620	26,371,971	17,854,923	4,669,643	7,155,037	3,375,357	1,474,571	1,140,911	556,302	13,702,178
2011	30,981,966	42,055,231	39,432,800	30,503,214	9,475,491	11,918,353	5,238,579	4,696,226	3,167,424	1,523,025	256,110	26,799,716
2012	41,326,250	54,286,380	40,110,104	17,160,447	27,414,263	15,144,667	7,818,119	4,813,243	3,176,434	2,494,100	1,203,641	62,064,467
2013	38,369,807	48,715,364	24,235,243	42,111,613	25,856,285	13,209,264	7,839,679	4,765,713	3,123,997	1,268,185	299,571	98,474,306
2014	48,530,079	30,520,342	56,760,817	45,285,076	30,485,049	14,995,439	5,436,601	3,906,500	2,457,352	1,409,114	976,651	161,712,601
2015	14,221,999	38,267,059	31,876,176	26,493,358	20,204,133	12,273,173	5,060,976	2,428,179	1,274,210	310,568	23,844	138,211,676
												514,650,033

4.8. Parameter Adjustments

The Hayne MLE framework will find the optimal parameters for the specified model. Like

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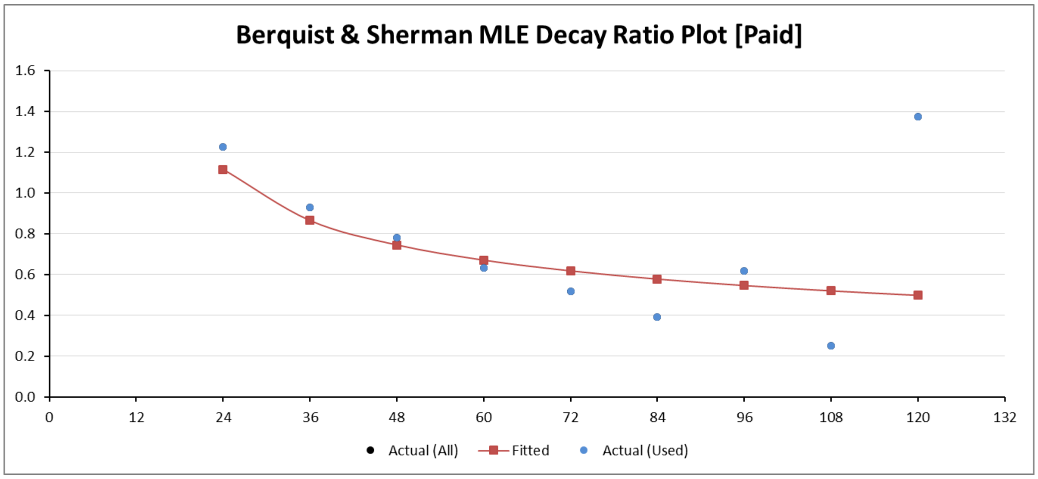
[9] These heteroecthesious data issues can be addressed in the "Hayne MLE Models.xlsm" file.

all models, this also means that there will be times that the noise in the data will lead to "distortions" in the parameters. This is akin to the need to select age-to-age factors to smooth the development pattern. The ability to judgmentally adjust some of the parameters is also possible with the Hayne MLE models. For example, consider the plot of the decay ratios for the Berquist-Sherman model in Figure 4.3.

Figure 4.3. Decay Ratios for Berquist-Sherman Model



In Figure 4.3, notice the "outlier" for the 120 month development period. This is actually an indication that the fitted or modeled parameter for 108 months may be lower than would have been expected. Reviewing the development year parameters, the choice for the modeler boils down to deciding whether to accept the parameters as reasonable or adjusting them to smooth out some of the noise in the data. For this Berquist-Sherman model example, the manual adjustment in Table 4.4 can be compared to the parameters in Table 3.1.[10]

Table 4.4. User Selected Parameters for Berquist-Sherman

User Selected Parameters:	12	24	36	48	60	72	84	96	108	120
Mean	620.96	760.67	708.16	553.57	350.00	181.39	70.97	43.88	**26.00**	15.21
Std Dev	40.50	46.55	43.00	35.49	26.17	17.66	10.40	8.75	7.60	7.36
Decay Ratios:		122.5%	93.1%	78.2%	63.2%	51.8%	39.1%	61.8%	59.3%	58.5%
CoV:	*6.5%*	*6.1%*	*6.1%*	*6.4%*	*7.5%*	*9.7%*	*14.7%*	*19.9%*	*29.2%*	*48.4%*

	Accident Year Trend	K	p	AIC	BIC
Mean	0.045	11.216	0.654	647.9	674.0
Std Dev	0.009	1.094	0.089		
CoV:	[illegible]	*9.8%*	*13.6%*		

To adjust the mean for 108 months, the decay ratios were reviewed and the original mean of 11.08 was seen to be low compared to the surrounding parameters due to the low decay ratio for 108 months and high decay ratio for 120 months. The parameter of 26.00 was selected by smoothing the decay ratios for the last three development periods. Notice that only the

[10] Similar manual adjustments for each of the models are illustrated in Appendix A.

mean parameters need to be adjusted since the MLE framework allows the variance-covariance parameters to be recalculated based on the selected parameter, we are essentially assuming the expected incremental losses are derived from selected parameters, or the true parameters for the data. Also, the diagnostics will give an indication of the significance of the change to the model parameters. Finally, while user selected parameters will tend to move the statistics away from optimal, the goal is to reasonably replicate the statistical features of the data and other adjustments, like the residual adjustment discussed in section 4.2, can also be made if the impact on the residuals is significant.

4.9. Tail Extrapolation

One of the most common data issues is that claim development is not complete within the loss triangle and tail factors are commonly used to extrapolate beyond the end of the data triangle. There are many common methods for calculating tail factors and a useful reference in this regard is the CAS Tail Factor Working Party Report [5]. However, for the Hayne MLE models a different approach is required in order to extrapolate the parameters so that a multi-variant normal distribution can continue to be used. Once extrapolation is used to extend the parameters, incremental values can all be extended to include development periods beyond the end of the triangle – i.e., the tail periods.

For the first family of models (i.e., Berquist-Sherman, Cape Cod, and Chain Ladder) the decay ratios shown in Tables 3.1, 3.4, and 3.7 can be used as a mean of extrapolating the development parameters for each model similarly to how a tail factor might be calculated for a deterministic method. In the "Hayne MLE Models.xlsm" file, five different regression models (i.e., average, linear, logarithmic, power, and polynomial) can be used to extrapolate decay ratios for up to 5 years from either the modeled or user selected parameters. For example, Table 4.5 illustrates the extrapolation for the Berquist-Sherman model, which is based on the user selected parameters in Table 4.4 so the graph in Table 4.5 can be compared to Figure 4.3.

Table 4.5. Berquist-Sherman Model Tail Extrapolation

Decay Ratio Analysis:

Parameters: User **Curve Type:** Power 3

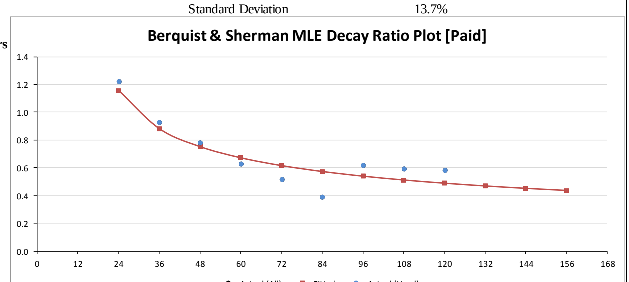
Least Squares Regression Coefficients:

x^a	-0.3916
coefficient	1.1559

Goodness of Fit Statistics:

R^2 Statistic	0.710
Regression Deviation	13.5%
Suggested Decay Parameters:	
Mean	45.3%
Standard Deviation	13.7%

Periods	Decay Ratio	Outliers	Selected Age	Selected Decay Ratio	Incremental Fitted Factors
12-24	1.225	0	1	1.225	1.156
24-36	0.931	0	2	0.931	0.881
36-48	0.782	0	3	0.782	0.752
48-60	0.632	0	4	0.632	0.672
60-72	0.518	0	5	0.518	0.615
72-84	0.391	0	6	0.391	0.573
84-96	0.618	0	7	0.618	0.539
96-108	0.593	0	8	0.593	0.512
108-120	0.585	0	9	0.585	0.489
120-132					0.469
132-144					0.452
144-156					0.437
156-168					0.423
168-180					0.411



From these regression models, the implied tail decay mean is the fitted decay ratio from the regression and the decay standard deviation is the average deviation for the actual decay ratios from the regression curve. The length of the tail period can then be determined by reviewing the means of the incremental periods beyond the triangle and then including enough periods such that the means in the final development column are close to zero. Using the decay ratio statistics and selected number of periods in the tail, the Hayne MLE framework will also extend the variance-covariance matrix to include the tail periods. Continuing the Berquist-Sherman example, the extended parameters for 3 years are illustrated in Table 4.6, which can be compared to Table 4.4.[11]

Table 4.6. Extended Parameters for Berquist-Sherman Model

	User Selected Parameters:												
	12	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**
Mean	620.96	760.67	708.16	553.57	350.00	181.39	70.97	43.88	26.00	15.21	*6.89*	*3.12*	*1.41*
Std Dev	40.50	46.55	43.00	35.49	26.17	17.66	10.40	8.75	7.60	7.36	*4.05*	*2.13*	*1.09*
Decay Ratios:		122.5%	93.1%	78.2%	63.2%	51.8%	39.1%	61.8%	59.3%	58.5%			
CoV:	*6.5%*	*6.1%*	*6.1%*	*6.4%*	*7.5%*	*9.7%*	*14.7%*	*19.9%*	*29.2%*	*48.4%*	*58.8%*	*68.4%*	*77.6%*
	Accident Year					**Tail Extrapolation**			**Implied Tail Factor**				
	Trend	**K**	**p**	**AIC**	**BIC**	**Decay Ratio**	**Periods**	**Distribution**	**Adjusted**	**Actual**	**Tail Sampling Option**		
Mean	0.045	11.216	0.654	647.9	674.0	45.3%	3	Gamma	1.0034	1.0034	Conditional Variance		
Std Dev	0.009	1.094	0.089			13.7%							
CoV:	*18.9%*	*9.8%*	*13.6%*										

One of the interesting features of this extrapolation process is that Coefficients of Variation in the tail parameters are increasing which is a statistical feature you would expect to find. The implied tail factor is also shown in the table in order to better compare with other models and traditional methods.[12] Finally, two different "Tail Sampling Options" are included for use in the simulation process. For the "Conditional Variance" option, the parameters in the tail are sampled using the multi-variate normal along with all the other parameters. For the "Sampling" option, a decay ratio is sampled using the mean and standard deviation from the regression and the selected distribution (i.e., Gamma, Normal, or Lognormal can be selected).

For the second family of models (i.e., Hoerl Curve and Wright), there are no parameters tied specifically to development age, so it is a simple matter to extend the "development" ages. The length of the tail period can be determined by reviewing the means of the incremental periods beyond the triangle and then including enough periods such that the means in the final

[11] The modeled parameters are also extended in the companion file, but they are not illustrated in the paper.

[12] The "adjusted" tail factor would be for annualized data if there were exposure issues as discussed in Section 4.7, whereas the "actual" tail factor would be for the data as is.

development column are close to zero.

A key ingredient for all of these considerations is to verify that the simulations in the tail are reasonable. For example, the tail period represents the extension of development parameters and using just a single period may not produce appropriate incremental results.

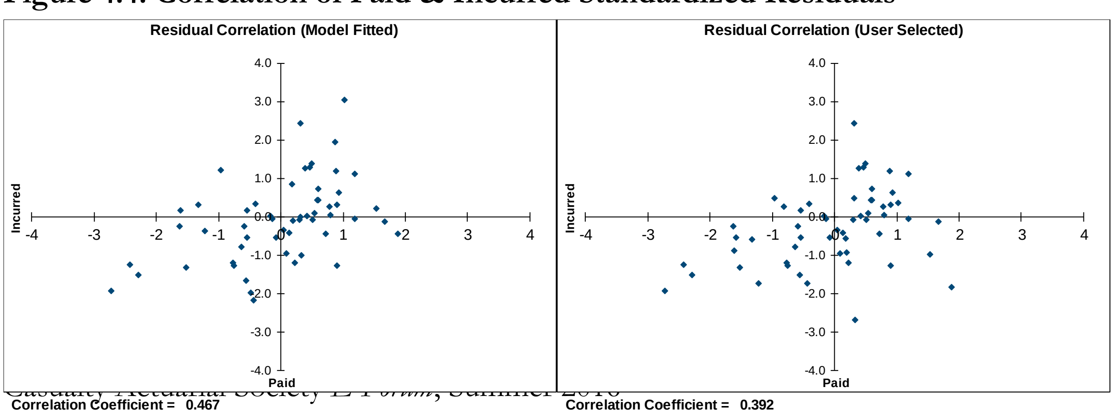
4.10. Incurred Data

The Hayne MLE models can be used to model both paid and incurred loss data. Using incurred data incorporates case reserves, thus perhaps improving the ultimate estimates. However, the resulting distribution from using incurred data will be possible outcomes of the IBNR, not a distribution of the unpaid. There are two possible approaches for modeling an unpaid loss distribution using incurred loss data: modeling incurred data and convert the ultimate values to a payment pattern, or, modeling paid and case reserves separately.

Using the first approach, a convenient way of converting the results of an incurred data model to a payment stream is to run the paid data model in parallel with the incurred data model, and use the random payment pattern from each iteration from the paid data model to convert the ultimate values from each corresponding iteration from the incurred data to a payment pattern for each iteration (for each accident year individually). The "Hayne MLE Models.xlsm" file illustrates this concept. It is worth noting, however, that this process allows the "added value" of using the case reserves to help predict the ultimate results to work its way into the calculations, thus perhaps improving the ultimate estimates, while still focusing on the payment stream for measuring risk. In effect, it allows a distribution of IBNR to become a distribution of IBNR and case reserves.

This process can also be made more sophisticated by correlating the multi-variate normal simulation of the paid and incurred models (e.g., the model parameters and/or process variance). In order to specify a correlation coefficient between the paid and incurred models, the correlation of the standardized residuals can be measured as, for example, in Figure 4.4 for the Berquist-Sherman model.

Figure 4.4. Correlation of Paid & Incurred Standardized Residuals



From Figure 4.4 observe that there is a positive correlation between the paid and incurred standardized residuals for the Berquist-Sherman model. This is not surprising as incurred data includes paid data, but using this to correlate the paid and incurred simulations is a way of including this statistical feature of the data in the model. In the "Hayne MLE Models.xlsm" file the correlation assumption is specified in the Inputs sheet and it will only be used to correlate the process variance portion of the paid and incurred data models.

The second approach could be accomplished by applying the Hayne MLE models to the case reserve triangle and then "combining" the case reserve and paid claim simulations. This has the advantage over the first approach of not modeling the paid losses twice, but it would also require specifying the correlation of the paid and outstanding losses. This second approach is beyond the scope of this paper.

5. Diagnostics

The quality of any model depends on the quality of the underlying assumptions. When a model fails to "fit" the data, it is unlikely to produce a good estimate of the distribution of possible outcomes.[13] However, a balance must be considered between parsimony of parameters and the goodness-of-fit. Over-parameterization may cause the model to be less predictive of future losses. On the other hand, no model will perfectly "fit" the data, so the best you can hope for with any model is that it reasonably represents the data and your understanding of the processes that impact the data. Therefore, diagnostically evaluating the assumptions underlying a model is important for evaluating whether it will produce reasonable results or not and whether it should stay in your selected group of reasonable models.

The CAS Working Party [4], in the third section of their report on quantifying variability in reserve estimates, identified 20 criteria or diagnostic tools for gauging the quality of a stochastic model. The Working Party also noted that, in trying to determine the optimal fit of a model, or indeed an optimal model, no single diagnostic tool or group of tools can be considered definitive. Depending on the statistical features found in the data, a variety of diagnostic tools are necessary to best judge the quality of the model assumptions and to adjust the parameters

[13] While the examples are different, significant portions of sections 5 and 6 are based on IAA [10] and Milliman [13].

of the model. This paper will discuss some of these tools in detail as they relate to the Hayne MLE models.

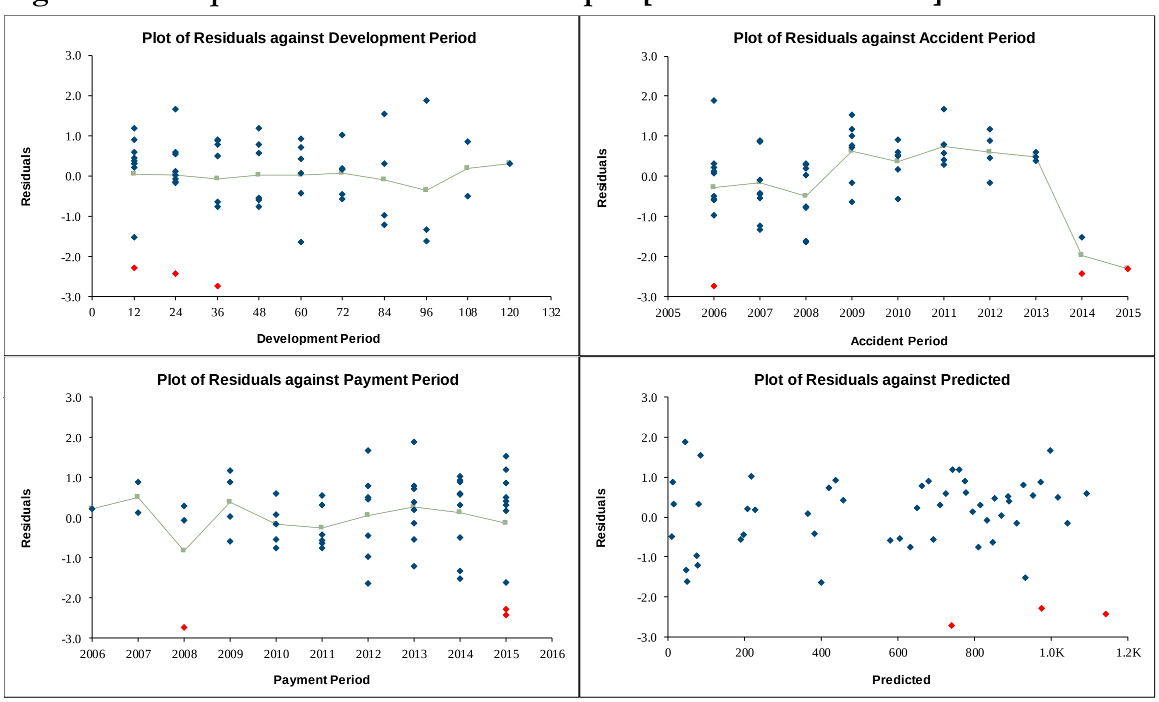
The key diagnostic tests are designed for three purposes: to test various assumptions in the model, to gauge the quality of the model fit to the data, and to help guide the adjustment of model parameters, if needed. Some tests are relative in nature, enabling results from one set of model parameters to be compared to those of another, for a specific model, allowing a modeler to improve the fit of the model. For the most part, however, the tests can't be used to compare different models. The objective, consistent with the goals of a deterministic analysis, is ***not*** to find the one best model, but rather a set of reasonable models.

Some diagnostic measures include statistical tests, providing a pass/fail determination for some aspects of the model assumptions. This can be useful even though a "fail" does not necessarily invalidate an entire model; it only points to areas where improvements can be made to the model or its parameterization. The goal is to find the sets of models and parameters that will yield the most realistic, most consistent simulations, based on statistical features found in the data.[14]

5.1 Residual Graphs

As noted earlier, the Hayne MLE models rely on the normal distribution assumption for incremental values and the standardized residuals are independent and identically distributed about the standard normal distribution conditional on parameters. Graphing residuals is a good way to check this. Consider the residual graphs for the Berquist-Sherman model in Figure 5.1 for the modeled parameters.

Figure 5.1. Berquist-Sherman Residual Graphs [Modeled Parameters]



For each model, going clock-wise, and starting from the lower-left-hand corner, the graphs in Figure 5.1 show the residuals (blue and red dots[15]) by calendar period, development period, and accident period and against the fitted incremental value (in the lower-right-hand corner). In addition, the graphs include a trend line (in green) that highlights the averages for each period.

Most residuals from the Berquist-Sherman model appear reasonably random and the averages do not deviate significantly from zero by development periods and payment periods. The averages by development period are not surprising since there is a parameter for each development period, but the lack of a trend by payment year is more useful since without a calendar year trend parameter this would be problematic for the Berquist-Sherman model. The averages by accident period appear significantly different from zero, which may indicate that a single trend component is not enough to model the level of incremental values by exposure periods.

Also of interest are the three large negative residuals in early development period, which are indicated in red as outliers. This could indicate the need to adjust those development period parameters although adjustments to remove outliers is typically a last resort compared to other options.

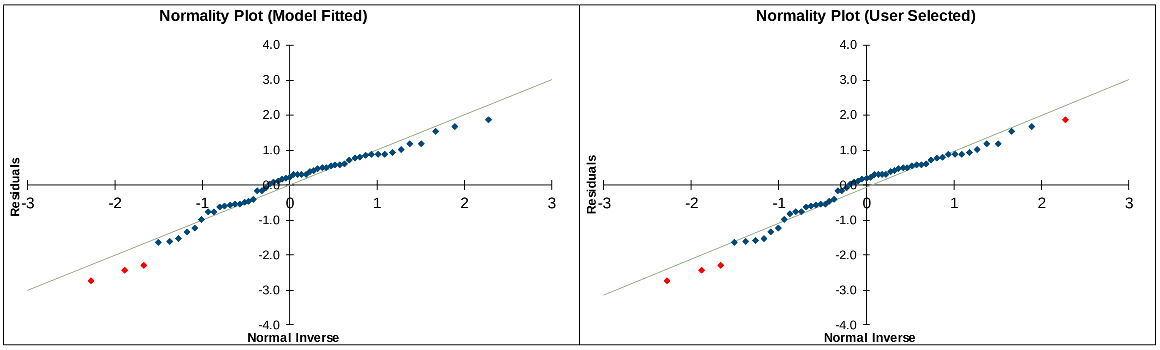
5.2 Normality Test

To see whether the standardized residuals are normally distributed, tests comparing the residuals against a normal distribution are useful. This also enables a comparison of the modeled parameters to the user selected parameter sets and gauging the skewness of the residuals in order to further validate the suitability of the chosen model. For example, Figure 5.2 shows the normality tests for the Berquist-Sherman model comparing the modeled and

[15] In the graphs that follow, the red dots are outliers as identified in Figure 5.3.

user selected parameters.

Figure 5.2. Normality Plots for Berquist-Sherman



N = 55 P-Value = 3.9% R^2 = 95.5%
Normal: MU = 0.00, Sigma = 1.01 AIC = 643.4, BIC = 669.5

N = 55 P-Value = 8.0% R^2 = 96.3%
Normal: MU = -0.05, Sigma = 1.03 AIC = 647.9, BIC = 674.0

The residual plots appear close to normally distributed, with the data points tightly distributed around the diagonal line. While there is an additional outlier for the user selected parameters, the *p*-value, a statistical pass-fail test for normality, improved from 3.9% to 8.0%, and the R^2 improved from 95.5% to 96.3%. The *p*-value is generally considered a "passing" score of the normality test when it is greater than 5.0%.[16] The graphs in Figure 5.2 also show N (the number of data points).

While the *p*-value and R^2 tests assess the goodness of fit of the model to the data, they do not penalize for added parameters. Adding more parameters will almost always improve the fit of the model to the data, but the goal is to have a good fit with as few parameters as possible. Two other tests, the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC), address this limitation, using the difference between each residual and its normal counterpart from the normality plot to calculate the Residual Sum Squared (RSS) and include a penalty for additional parameters, as shown in (5.1) and (5.2), respectively.[17]

$$AIC = 2 \times p + n \times \left[\ln\left(\frac{2 \times \pi \times RSS}{n}\right) + 1 \right] \tag{5.1}$$

$$BIC = n \times \ln\left(\frac{RSS}{n}\right) + p \times \ln(n) \tag{5.2}$$

A smaller value for the AIC and BIC tests indicate an improvement, especially with respect to overcoming the penalty of adding a parameter. For the Berquist-Sherman model test in

[16] Note that this doesn't indicate whether the Hayne MLE model itself passes or fails, it only tests whether the residuals can be judged to be normally distributed.

[17] There are different versions of the AIC and BIC formula from various authors and sources, but the general idea of each version is consistent. Other similar formulas could also be used.

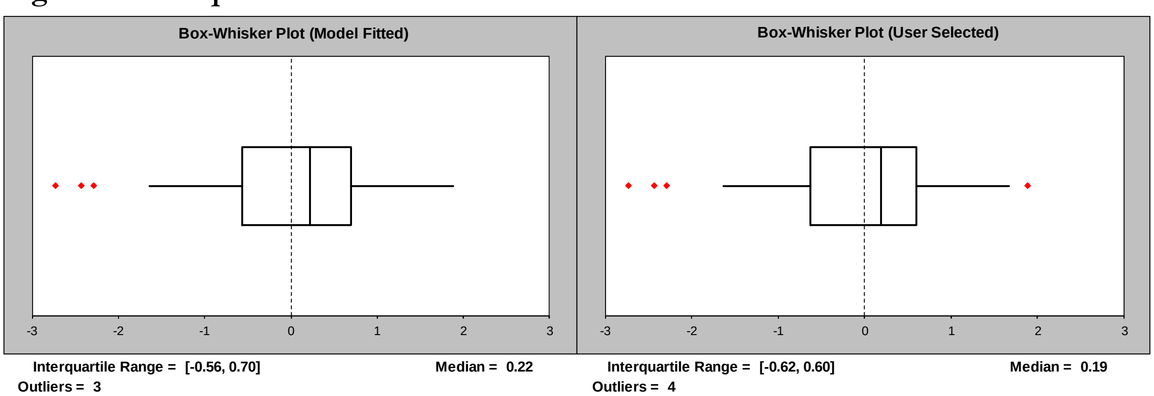
Figure 5.2, there were no parameters added but the values increased a little which is expected since the user selected parameters are not the optimal parameters. It is important to remember that the AIC and BIC tests are model specific in the sense that they are not well suited for comparing different model, but rather different parameterizations of the same model.

5.3 Outliers

Identifying outliers in the data provides another useful test in determining model fit. Outliers can be represented graphically in a box-whisker plot, which shows the inter-quartile range (the 25th to 75th percentiles) and the median (50th percentile) of the residuals—the so-called box. The whiskers then extend to the largest values within three times this inter-quartile range.[18] Values beyond the whiskers may generally be considered outliers and are identified individually with a point. For example, the Box-Whisker plots in Figure 5.3 compare the modeled and user selected parameters for the Berquist-Sherman model.

If the data in those outlier cells genuinely represent events that cannot be expected to happen again, the outlier(s) may be removed from the model (by giving it/them zero weight). But extreme caution should be taken even when the removal of outliers seems warranted. The possibility always remains that apparent outliers may actually represent realistic extreme values, which, of course, are critically important to include as part of any sound analysis.

Figure 5.3. Berquist-Sherman Box-Whisker Plots



Additionally, when residuals are not normally distributed a significant number of outliers tend to result – i.e., the distributional shape of the residuals may be skewed or otherwise not

[18] Various authors and textbooks use widths for the whiskers which tend to span from 1.5 to 3 times the inter-quartile range. Changing the multiplier will therefore make the Box-Whisker plot more or less sensitive to outliers. It is also possible to illustrate "mild" outliers with a multiplier of 1.5 and the more "extreme" outliers with a multiplier of 3 using different colors and/or symbols in the graphs. Of course the actual multipliers can be adjusted based on personal preference.

normal. In this case, it is impossible for the Hayne MLE simulation to capture this shape as it relies on the normality assumption, although adjusting the parameters may help "restore" normality. Finally, a significant number of residuals can also mean the underlying model is not a good fit to the data so other models should be used or this model given less weight (see Section 6).

While the three diagnostic tests shown above demonstrate techniques commonly used with most types of models, they are not the only tests available.[19] Next, we'll take a look at the flexibility of the Hayne MLE framework and some of the diagnostic elements of the simulation results. For a more extensive list of other tests available, see the report, CAS Working Party on Quantifying Variability in Reserve Estimates [4].

5.4. Model Results

Once the parameter diagnostics have been reviewed, simulations should be run for each model.[20] These simulation results provide an additional diagnostic tool to aid in evaluation of the model, as described in section 3 of CAS Working Party [4]. As an example, the results for the Berquist-Sherman Hayne MLE model will be reviewed. The estimated-unpaid results shown in Table 5.1 were simulated using 10,000 iterations with the parameters from Table 4.6.

5.4.1. Estimated-Unpaid Results

It's recommended to start a diagnostic review of the estimated unpaid results with the standard error (standard deviation) and coefficient of variation (standard error divided by the mean), shown in Table 5.1. Keep in mind that for books of business with relatively stable volume the standard error should increase when moving from the oldest years to the most recent years, as the standard errors (value scale) should follow the magnitude of the mean of unpaid estimates. In Table 5.1, the standard errors conform to this pattern. At the same time, the standard error for the total of all years should be larger than any individual year.

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[19] For example, see Venter [17].

[20] Throughout the paper, all simulations include both parameter uncertainty and process uncertainty as illustrated in Tables 3.16 through 3.21.

Table 5.1. Estimated Unpaid Model Results for Berquist-Sherman

Sample Insurance Company
Hayne Paper Data
Accident Year Unpaid (in 000's)
Paid Berquist & Sherman Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	123,738	441	573	129.9%	(1,475)	2,372	391	823	1,420	1,881
2007	140,983	1,083	825	76.2%	(1,675)	4,401	1,048	1,611	2,466	3,113
2008	147,516	2,459	1,168	47.5%	(1,527)	6,082	2,417	3,252	4,462	5,274
2009	174,349	4,793	1,595	33.3%	(172)	11,597	4,758	5,809	7,391	8,954
2010	173,637	8,629	1,992	23.1%	1,588	16,582	8,542	9,810	11,955	13,951
2011	174,996	18,214	3,136	17.2%	7,989	30,302	18,135	20,292	23,509	25,381
2012	169,224	41,402	5,008	12.1%	25,322	59,952	41,302	44,862	49,756	53,216
2013	134,010	75,281	7,480	9.9%	53,427	105,936	74,961	80,194	87,930	93,542
2014	68,911	127,141	11,108	8.7%	93,649	164,080	127,078	134,809	144,791	152,998
2015	35,798	210,599	16,205	7.7%	159,908	275,851	210,505	221,397	236,756	253,297
Totals	1,343,162	490,041	31,334	6.4%	405,127	622,322	488,329	510,471	542,250	566,151

Also, the coefficients of variation should generally decrease when moving from the oldest year to the more recent years and the coefficient of variation for all years combined should be less than for any individual year.

The main reason for the decrease in the coefficient of variation has to do with the independence in the incremental claim-payment stream. Because the oldest accident year typically has only a few incremental payments remaining, or even just one, the variability is nearly all reflected in the coefficient. For more current accident years, random variations in the future incremental payment stream may tend to offset one another, thereby reducing the variability of the total unpaid loss.[21]

While the coefficients of variation should go down, they could also start to rise again in the most recent years. Such reversals are from a couple of issues:

- With an increasing number of parameters used in the model, the parameter uncertainty tends to increase when moving from the oldest years to the more recent years, particularly for models with accident year parameters, where uncertainty could increase in more recent accident years.
- In the most recent years, parameter uncertainty can grow to overpower process uncertainty, which may cause the coefficient of variation to start rising again. At a minimum, increasing parameter uncertainty will slow the rate of decrease in the coefficient of variation.

[21] To visualize this reducing Coefficient of Variation, recall that the standard deviation for the total of several independent variables is equal to the square root of the sum of the squares.

The model may be overestimating the uncertainty in recent accident years if the increase is significant. In that case, another model may need to be used. Keep in mind also that the standard error or coefficient of variation for the total of all accident years will be less than the sum of the standard error or coefficient of variation for the individual years. This is because the model assumes that the random process generating the process uncertainty in each accident year is independent.

Minimum and maximum results are the next diagnostic element in the analysis of the estimated unpaid claims in Table 5.1, representing the smallest and largest values from all iterations of the simulation. These values will need to be reviewed in order to determine their veracity. If any of them seem implausible, the model assumptions would need to be reviewed. Their effects could materially alter the mean indication.

5.4.2. Mean, Standard Deviation and CoV of Incremental Values

The mean, standard deviation and coefficients of variation for every incremental value from the simulation process can also provide useful diagnostic results, enabling a deeper review into potential coefficient of variation issues that may be found in the estimated unpaid results. Consider, for example, the mean, standard deviation and coefficient of variation results shown in Tables 5.2, 5.3, and 5.4, respectively.

Table 5.2. Mean of Incremental Values for Berquist-Sherman

Sample Insurance Company
Hayne Paper Data
Accident Year Incremental Values by Development Period
Paid Berquist & Sherman Model

Accident	Mean Values (in 000's)												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156
2006	25,064	31,145	28,656	22,440	14,281	7,309	2,843	1,814	1,079	613	269	116	56
2007	25,835	32,119	29,703	23,223	14,691	7,552	2,961	1,878	1,113	617	278	134	54
2008	29,579	36,189	33,544	26,237	16,817	8,639	3,384	2,100	1,250	695	309	138	66
2009	31,088	38,234	35,446	27,737	17,569	9,087	3,546	2,182	1,318	747	329	139	78
2010	31,976	39,197	36,545	28,680	18,113	9,362	3,640	2,270	1,354	789	336	162	80
2011	32,175	39,680	36,868	29,088	18,350	9,495	3,691	2,294	1,384	767	343	162	78
2012	36,809	45,089	42,259	32,820	20,700	10,715	4,251	2,642	1,571	883	374	184	82
2013	36,915	45,693	42,709	33,487	20,936	10,886	4,241	2,615	1,582	860	396	192	85
2014	40,158	49,481	45,856	36,060	22,785	11,583	4,600	2,882	1,699	972	425	189	88
2015	47,924	58,862	54,790	43,026	27,063	13,895	5,533	3,402	2,037	1,139	498	234	118

Table 5.3. Standard Deviation of Incremental Values for Berquist-Sherman

Sample Insurance Company
Hayne Paper Data
Accident Year Incremental Values by Development Period
Paid Berquist & Sherman Model

Accident	Standard Error Values (in 000's)												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156
2006	4,010	4,911	4,418	3,910	2,782	1,895	1,037	776	619	498	365	238	162
2007	4,203	5,015	4,679	3,993	2,819	1,920	1,079	791	626	465	365	254	171
2008	4,524	5,085	4,684	4,094	3,163	1,992	1,185	906	688	533	407	285	191
2009	4,337	5,232	5,277	4,218	3,313	2,126	1,228	929	743	544	439	286	190
2010	4,665	5,270	5,114	4,576	3,282	2,213	1,243	911	708	563	420	301	199
2011	4,639	5,546	5,234	4,562	3,410	2,243	1,240	955	759	589	441	303	196
2012	5,184	6,314	5,718	4,887	3,589	2,420	1,337	996	800	655	473	324	220
2013	5,169	6,197	6,028	5,178	3,800	2,491	1,415	1,022	788	625	479	337	226
2014	5,652	6,619	6,140	5,421	4,013	2,649	1,467	1,142	881	676	548	353	239
2015	6,057	7,346	7,284	6,284	4,601	3,062	1,707	1,244	982	789	605	416	288

Table 5.4. Coefficient of Variation of Incremental Values for Berquist-Sherman

Sample Insurance Company
Hayne Paper Data
Accident Year Incremental Values by Development Period
Paid Berquist & Sherman Model

Accident	Coefficient of Variation Values												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156
2006	16.0%	15.8%	15.4%	17.4%	19.5%	25.9%	36.5%	42.8%	57.3%	81.3%	135.7%	204.6%	290.1%
2007	16.3%	15.6%	15.8%	17.2%	19.2%	25.4%	36.4%	42.1%	56.2%	75.4%	131.3%	189.1%	319.2%
2008	15.3%	14.1%	14.0%	15.6%	18.8%	23.1%	35.0%	43.2%	55.1%	76.6%	131.6%	206.4%	291.3%
2009	14.0%	13.7%	14.9%	15.2%	18.9%	23.4%	34.6%	42.6%	56.4%	72.9%	133.3%	206.4%	241.8%
2010	14.6%	13.4%	14.0%	16.0%	18.1%	23.6%	34.1%	40.2%	52.3%	71.4%	125.1%	186.2%	248.6%
2011	14.4%	14.0%	14.2%	15.7%	18.6%	23.6%	33.6%	41.7%	54.8%	76.8%	128.4%	187.1%	249.9%
2012	14.1%	14.0%	13.5%	14.9%	17.3%	22.6%	31.4%	37.7%	50.9%	74.2%	126.3%	175.8%	267.6%
2013	14.0%	13.6%	14.1%	15.5%	18.1%	22.9%	33.4%	39.1%	49.8%	72.7%	121.0%	175.4%	265.3%
2014	14.1%	13.4%	13.4%	15.0%	17.6%	22.9%	31.9%	39.6%	51.9%	69.6%	128.8%	187.2%	271.3%
2015	12.6%	12.5%	13.3%	14.6%	17.0%	22.0%	30.8%	36.6%	48.2%	69.3%	121.4%	177.3%	243.1%

The mean values in Table 5.2 appear consistent throughout and support the increases in estimated unpaid by accident year that are shown in Table 5.1. In fact, the future mean values, which lay beyond the stepped diagonal line in Table 5.2, sum to the results in Table 5.1. The standard deviation values in Table 5.3 also appear consistent, but the standard deviations can't be added because the standard deviations in Table 5.1 represent those for aggregated incremental values by accident year, which are less than perfectly correlated. The coefficient of variation values in Table 5.4 help the user efficiently review both the incremental mean and standard deviation values in Tables 5.2 and 5.3 as inconsistencies in a column will highlight issues with either the means or standard deviations or both. The coefficients by column in Table 5.4 all appear consistent, so the other main use of this table is to review the progression of CoVs by development period which should increase over time as they do in Table 5.4 indicating that the final incremental payments in the tail tend to be the most uncertain.

6. Using Multiple Models

So far the focus has only been on one model. In practice, multiple stochastic models should be used in the same way that multiple methods should be used in a deterministic analysis. First the results for each model must be reviewed and finalized, after an iterative process of diagnostic testing and reviewing model output to make sure the model "fits" the data, has reasonable assumptions and produces reasonable results. Then these results can be combined by assigning a weight to the results of each model.

Two primary methods exist for combining the results for multiple models:

- **Run models with the same random variables.** For this algorithm, every model uses the exact same random variables. In the "Hayne MLE Models.xlsm" file, the random values are simulated before they are used to simulate results, which means that this algorithm may be accomplished by reusing the same set of random variables for each model. At the end, the incremental values for each model, for each iteration by accident year (that have a partial weight), can be weighted together.
- **Run models with independent random variables.** For this algorithm, every model is run with its own random variables. In the "Hayne MLE Models.xlsm" file the random values are simulated before they are used to simulate results, which means that this algorithm may be accomplished by simulating a new set of random variables for each model.[22] At the end, the weights are used to randomly select a model for each iteration by accident year so that the result is a weighted "mixture" of models.

Both algorithms are similar to the process of weighting the results of different deterministic methods to arrive at an actuarial best estimate. The process of weighting the results of different stochastic models produces an actuarial best estimate of a distribution. In practice it is also common to further "adjust" or "shift" the weighted results by year after considering case reserves and the calculated IBNR. For example, in an older year the weighted value could result in a negative IBNR which offsets case reserves and a reasonable adjustment could be to accept the case reserves by "shifting" the IBNR to zero. This "shifting" can also be done for weighted distributions, either additively to maintain the exact shape and width of the distribution by year or multiplicatively to maintain the exact shape of the distribution but adjusting the width of the distribution.

Table 6.1. Model weights by accident year

Accident	Model Weights by Accident Year										
Year	Paid BS	Incd BS	Paid CC	Incd CC	Paid CL	Incd CL	Paid HC	Incd HC	Paid WR	Incd WR	TOTAL
2006	25.0%	25.0%	25.0%	25.0%							100.0%
2007	25.0%	25.0%	25.0%	25.0%							100.0%
2008	25.0%	25.0%	25.0%	25.0%							100.0%
2009	25.0%	25.0%	25.0%	25.0%							100.0%
2010	25.0%	25.0%	25.0%	25.0%							100.0%
2011	16.7%	16.7%	16.7%	16.7%	16.7%	16.7%					100.0%
2012	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%			100.0%
2013	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%			100.0%
2014	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%			100.0%
2015	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%	12.5%			100.0%

[22] In general, in order to simulate new random values a new seed value must be selected (or a seed value of zero can be used), otherwise the same random values will be simulated. In the "Hayne MLE Models.xlsm" file the seed value is incremented for each model and data type so that different seed values are being used as long as new random numbers are generated for each model and data type.

By comparing the results for all ten models (or fewer, depending on how many are used)[23] a qualitative assessment of the relative merits of each model may be determined. Bayesian methods can be used to determine weighting based on the quality of each model's forecasts.[24] The weights can be determined separately for each year. The table in Table 6.1 shows an example of weights for the Hayne MLE data.[25] The weighted results are displayed in the "Best Estimate" column of Table 6.2. As a parallel to a deterministic analysis, the means from the eight models given some weight could be used to derive a reasonable range from the modeled results (i.e., from $395,563 to $490,041) as shown in Table 6.3. Alternatively, if only results by accident year which are given some weight when deriving the best estimate are considered, then the "weighted range" may be a more representative view of the uncertainty of the actuarial central estimate.[26]

Table 6.2. Summary of mean results by model

Sample Insurance Company
Hayne Paper Data
Summary of Results by Model (in 000's)

	Mean Estimated Unpaid										
Accident	Berquist & Sherman		Cape Cod		Chain Ladder		Hoerl Curve		Wright		Best Est.
Year	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	(Weighted)
2006	441	528	485	488	168	177	86	91	64	65	471
2007	1,083	1,164	1,201	1,228	477	507	269	281	218	218	1,148
2008	2,459	2,494	2,355	2,427	1,281	1,389	919	937	694	718	2,453
2009	4,793	4,812	5,172	5,182	3,975	4,278	2,872	2,861	2,715	2,769	4,945
2010	8,629	8,400	9,239	8,940	8,073	8,721	7,681	7,516	7,597	7,429	8,642
2011	18,214	17,179	20,571	20,421	19,370	20,588	17,664	16,874	19,119	19,046	19,280
2012	41,402	38,115	44,568	42,079	43,332	44,793	40,416	37,923	42,804	40,657	41,487
2013	75,281	66,959	78,842	74,018	77,959	80,697	73,354	67,037	76,810	72,994	74,398
2014	127,141	110,465	93,698	93,653	93,147	101,410	125,089	112,174	93,415	94,782	107,115
2015	210,599	178,646	147,763	150,595	147,782	162,612	207,924	182,932	147,450	151,814	173,575
Totals	490,041	428,763	403,895	399,031	395,563	425,172	476,274	428,627	390,884	390,491	433,516

23 Other models in addition to the Hayne MLE models could also be included in the weighting process as long as the simulated results are in the form of random incremental payment streams.

24 Quality of the forecast could be defined in a number of ways, but the essential idea is to measure the relative predictive power of competing models.

25 For simplicity, the weights are only illustrative and not derived using Bayesian methods.

26 The "modeled range" in Figure 6.3 is derived using each model that is given at least some weight for any accident year – i.e., if the model is used. In contrast, the "weighted range" is derived using only the models given weight for each accident year, which are highlighted in grey in Figure 6.2 and 6.4.

Table 6.3. Summary of ranges by accident year

Sample Insurance Company
Hayne Paper Data
Summary of Results by Model (in 000's)

Accident Year	Best Est. (Weighted)	Ranges: Weighted Minimum	Weighted Maximum	Modeled Mininum	Modeled Maximum
2006	471	441	528	86	528
2007	1,148	1,083	1,228	269	1,228
2008	2,453	2,355	2,494	919	2,494
2009	4,945	4,793	5,182	2,861	5,182
2010	8,642	8,400	9,239	7,516	9,239
2011	19,280	17,179	20,588	16,874	20,588
2012	41,487	37,923	44,793	37,923	44,793
2013	74,398	66,959	80,697	66,959	80,697
2014	107,115	93,147	127,141	93,147	127,141
2015	173,575	147,763	210,599	147,763	210,599
Totals	433,516	380,045	502,488	395,563	490,041

When selecting weights for stochastic models, the standard deviations should also be considered in addition to the means by model since the weighted best estimate should reflect the actuary's judgments about the entire distribution not just a central estimate. Thus, coefficients of variation by model can be used for this purpose as illustrated in Table 6.4.

Table 6.4. Summary of CoV results by model

Sample Insurance Company
Hayne Paper Data
Summary of Results by Model (in 000's)

Accident Year	Coefficient of Variation: Berquist & Sherman Paid	Berquist & Sherman Incurred	Cape Cod Paid	Cape Cod Incurred	Chain Ladder Paid	Chain Ladder Incurred	Hoerl Curve Paid	Hoerl Curve Incurred	Wright Paid	Wright Incurred
2006	129.9%	118.0%	131.4%	131.3%	239.3%	254.9%	279.2%	281.7%	632.5%	639.9%
2007	76.2%	78.5%	97.7%	98.2%	156.0%	163.8%	166.4%	168.8%	303.7%	311.0%
2008	47.5%	48.6%	64.5%	64.5%	78.5%	82.7%	92.4%	93.5%	146.0%	147.4%
2009	33.3%	33.7%	38.6%	38.3%	39.7%	45.7%	51.1%	51.4%	58.0%	58.2%
2010	23.1%	25.1%	27.2%	27.1%	25.0%	32.5%	32.2%	32.7%	31.1%	30.8%
2011	17.2%	17.0%	15.6%	15.0%	14.1%	24.0%	20.8%	20.6%	17.1%	16.3%
2012	12.1%	13.5%	10.0%	9.8%	9.3%	22.4%	13.4%	13.9%	10.7%	10.1%
2013	9.9%	10.6%	7.7%	7.0%	6.4%	20.8%	10.2%	10.5%	7.7%	6.7%
2014	8.7%	9.4%	8.5%	7.0%	5.9%	24.0%	8.5%	9.0%	8.2%	6.5%
2015	7.7%	8.4%	9.4%	5.8%	5.2%	22.0%	7.2%	7.8%	9.4%	5.4%
Totals	6.4%	6.1%	5.9%	4.6%	4.1%	11.8%	6.0%	5.7%	5.5%	3.9%

With a focus on the entire distribution, the weights by year were used to randomly sample the specified percentage of iterations from each model. A more complete set of the results for the "weighted" iterations can be created similar to the tables shown in section 5. The companion "Best Estimate.xlsm" file can be used to weight ten different models together in order to calculate a weighted best estimate. An example is shown in the table in Table 6.5 for the Hayne [8] data.

Table 6.5. Estimated unpaid model results (weighted)

Sample Insurance Company
Hayne Paper Data
Accident Year Unpaid (in 000's)
Best Estimate (Weighted)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	123,738	471	644	136.7%	(2,545)	4,909	405	829	1,576	2,333
2007	140,983	1,148	1,049	91.4%	(3,520)	6,314	1,092	1,780	2,957	3,957
2008	147,516	2,453	1,357	55.3%	(3,302)	10,083	2,408	3,290	4,714	5,954
2009	174,349	4,945	1,789	36.2%	(4,448)	12,718	4,898	6,102	7,933	9,502
2010	173,637	8,642	2,208	25.5%	(1,331)	19,227	8,604	10,106	12,319	14,029
2011	174,996	19,280	3,656	19.0%	4,625	39,886	19,143	21,530	25,382	28,830
2012	169,224	41,487	6,136	14.8%	16,382	75,478	41,413	45,225	51,128	58,189
2013	134,010	74,398	9,887	13.3%	25,947	157,876	74,300	79,822	90,176	104,245
2014	68,911	107,115	17,580	16.4%	28,733	187,403	104,724	120,254	137,020	148,299
2015	35,798	173,575	30,419	17.5%	9,842	285,509	170,237	197,558	224,280	240,117
Totals	1,343,162	433,516	38,243	8.8%	254,901	599,252	432,354	460,201	497,529	524,069
Normal Dist.		433,516	38,243	8.8%			433,516	459,310	496,420	522,483
logNormal Dist.		433,522	38,456	8.9%			431,826	458,398	499,520	530,586
Gamma Dist.		433,516	38,243	8.8%			432,392	458,661	498,279	527,410
TVaR							464,489	483,287	513,512	536,643
Normal TVaR							464,029	482,127	512,401	535,442
logNormal TVaR							464,105	483,728	518,630	546,956
Gamma TVaR							463,996	483,043	516,174	542,419

As one final check of the weighted results it would be common to review the implied IBNR to make sure there are no issues as shown in Table 6.6. By reviewing this reconciliation, and perhaps also comparing it to deterministic results, additional adjustments could be made to various assumptions. For example, from year 2006 in Table 6.6 it may be more realistic to revisit the tail factor assumptions or the weights by model so that the unpaid estimate is more consistent with the case reserves. Finally, after the interactive process of reviewing results and adjusting assumptions is complete, it may still be prudent to make adjustments to the best estimate of the unpaid by shifting the results as noted earlier in this section.

Table 6.6. Reconciliation of total results (weighted)

Sample Insurance Company
Hayne Paper Data
Reconciliation of Total Results (in 000's)
Best Estimate (Weighted)

Accident Year	Paid To Date	Incurred To Date	Case Reserves	IBNR	Estimate of Ultimate	Estimate of Unpaid
2006	123,738	124,486	748	(277)	124,209	471
2007	140,983	141,488	505	643	142,131	1,148
2008	147,516	150,057	2,541	(88)	149,969	2,453
2009	174,349	180,737	6,388	(1,443)	179,294	4,945
2010	173,637	182,952	9,315	(673)	182,279	8,642
2011	174,996	193,196	18,200	1,080	194,276	19,280
2012	169,224	199,879	30,655	10,832	210,711	41,487
2013	134,010	189,518	55,508	18,890	208,408	74,398
2014	68,911	132,561	63,650	43,465	176,026	107,115
2015	35,798	110,269	74,471	99,104	209,373	173,575
Totals	1,343,162	1,605,143	261,981	171,535	1,776,678	433,516

6.1 Additional Output

Three rows of percentile numbers for the normal, lognormal, and gamma distributions, which have been fitted to the total unpaid-claim distribution, may be seen at the bottom of the table in Table 6.5. The fitted mean, standard deviation, and selected percentiles are in their respective columns; the smoothed results can be used to assess the quality of fit, parameterize a dynamic financial analysis ("DFA") model, or used to smooth the estimate of extreme values,[27] among other applications.

Four rows of numbers indicating the Tail Value at Risk ("TVaR"), defined as the average of all of the simulated values greater than or equal to the percentile value, may also be seen at the bottom of Table 6.5. For example, in this table, the 99th percentile value for the total unpaid claims for all accident years combined is $524,069, while the average of all simulated values that are greater than or equal to is $536,643. The Normal TVaR, Lognormal TVaR, and Gamma TVaR rows are calculated similarly, except that they use the respective fitted distributions in the calculations rather than actual simulated values from the model.

An analysis of the TVaR values is likely to help clarify a critical issue: if the actual outcome exceeds the X percentile value, by how much will it exceed that value on average? This type of assessment can have important implications related to risk-based capital calculations and other technical aspects of enterprise risk management. But it is worth noting that the purpose of the normal, lognormal, and gamma TVaR numbers is to provide "smoothed" values—that is, that some of the random statistical noise is essentially prevented from distorting the calculations.

6.2. Estimated Cash Flow Results

A model's output may also be reviewed by calendar year (or by future diagonal), as shown in the table in Table 6.7. A comparison of the values in Tables 6.5 and 6.7 indicates that the total rows are identical, because summing the future payments horizontally or diagonally will produce the same total. Similar diagnostic issues (as discussed in Section 5) may be reviewed in the table in Table 6.7, with the exception of the relative values of the standard errors and coefficients of variation moving in opposite directions for calendar years compared to accident

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[27] A random instance of an extreme percentile can be quite erratic compared to the same percentile of a distribution fitted to the simulated distribution. This random noise for extreme percentiles could be cause for increasing the number of iterations, but if the same percentiles for the fitted distributions are stable perhaps they can be used in lieu of more iterations. Of course the use of the extreme values assumes that the models are reliable.

years. This phenomenon makes sense on an intuitive level when one considers that "final" payments, projected to the furthest point in the future, should actually be the smallest, yet relatively most uncertain.

Table 6.7. Estimated Cash Flow (weighted)

Sample Insurance Company
Hayne Paper Data
Calendar Year Unpaid (in 000's)
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	160,184	14,166	8.8%	109,684	222,966	159,583	169,553	184,716	195,263
2017	116,073	12,102	10.4%	72,833	166,146	115,235	124,202	136,915	145,439
2018	75,084	8,938	11.9%	34,373	111,295	74,509	80,772	90,836	97,566
2019	42,212	6,021	14.3%	16,605	71,311	41,859	46,173	52,711	57,524
2020	21,143	3,889	18.4%	8,545	37,308	20,894	23,666	27,935	30,994
2021	9,680	2,613	27.0%	(212)	20,773	9,541	11,348	14,156	16,596
2022	4,960	1,802	36.3%	(2,713)	13,036	4,900	6,101	8,021	9,492
2023	2,371	1,338	56.4%	(3,187)	8,932	2,299	3,229	4,684	5,783
2024	1,102	992	90.0%	(2,827)	6,547	1,003	1,691	2,847	3,857
2025	462	632	136.8%	(3,435)	4,443	376	790	1,644	2,350
2026	182	383	210.8%	(2,728)	2,866	122	357	865	1,365
2027	61	221	363.4%	(1,545)	1,829	24	130	460	799
Totals	433,516	38,243	8.8%	254,901	599,252	432,354	460,201	497,529	524,069

6.3. Estimated Ultimate Loss Ratio Results

Another output table, Table 6.8, shows the estimated ultimate loss ratios by accident year. Similar to the estimated unpaid and estimated cash-flow tables, the values in this table are calculated using all simulated values, not just the values beyond the end of the historical triangle. Because the simulated sample triangles represent additional possibilities of what could have happened in the past, even as the "squaring of the triangle" and process variance represent what could happen as those same past values are played out into the future, there is sufficient information to enable estimation of the variability in the loss ratio from day one until all claims are completely paid and settled for each accident year.[28]

Table 6.8. Estimated loss ratio (weighted)

Sample Insurance Company
Hayne Paper Data
Accident Year Ultimate Loss Ratios (in 000's)
Best Estimate (Weighted)

Accident Year	Earned Premium	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	184,450	71.9%	7.2%	10.0%	48.2%	105.4%	70.9%	76.1%	85.2%	91.9%
2007	237,093	60.5%	4.5%	7.5%	38.0%	84.3%	60.4%	63.3%	67.9%	71.8%
2008	297,807	52.3%	3.9%	7.5%	37.0%	71.5%	52.1%	54.7%	59.0%	62.6%
2009	349,324	49.3%	3.6%	7.3%	28.3%	61.3%	49.6%	51.8%	54.8%	56.9%
2010	361,198	48.1%	3.4%	7.1%	32.3%	61.8%	48.3%	50.5%	53.3%	55.2%
2011	374,921	50.0%	6.7%	13.4%	14.0%	100.2%	50.3%	52.8%	60.8%	73.3%
2012	370,904	54.2%	6.3%	11.6%	20.5%	102.4%	54.3%	57.3%	62.8%	77.2%
2013	345,267	58.3%	7.0%	12.0%	21.2%	125.7%	58.3%	61.6%	68.9%	82.2%
2014	301,114	61.4%	9.5%	15.5%	16.3%	104.4%	59.9%	68.8%	76.9%	82.8%
2015	277,987	77.0%	13.1%	17.0%	4.4%	129.2%	75.3%	87.5%	98.8%	105.2%
Totals	3,100,065	57.0%	2.2%	3.9%	48.8%	66.6%	56.9%	58.4%	60.7%	62.5%

[28] If one is only interested in the "remaining" volatility in the loss ratio, then the values in the estimated unpaid table (Figure 6.5) can be added to the cumulative paid values by year and divided by the premiums.

Reviewing the simulated values indicates that the standard errors in Table 6.8 should be proportionate to the means, while the coefficients of variation should be relatively constant by accident year. In terms of diagnostics, any increases in standard error and coefficient of variation for the most recent years would be consistent with the reasons previously cited in Section 5.4 for the estimated unpaid tables. Risk management-wise, the loss ratio distributions have important implications for projecting pricing risk – the mean loss ratios can be used to view any underwriting cycles and help inform the projected mean for the next few years, while the coefficients of variation can be used to select a standard deviation for the next few years.[29]

6.4. Estimated Unpaid Claim Runoff Results

Table 6.9, shows the runoff of the total unpaid claim distribution by future calendar year. Like the estimated unpaid and estimated cash-flow tables, the values in this table are calculated using only future simulated values, except that future diagonal results are sequentially removed so that only the unpaid claims at the end of each future calendar period are remaining. These results are quite useful for calculating the runoff of the unpaid claim distribution when calculating risk margins using the cost of capital method.

Table 6.9. Estimated unpaid claim runoff (weighted)

Sample Insurance Company
Hayne Paper Data
Calendar Year Unpaid Claim Runoff (in 000's)
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	433,516	38,243	8.8%	254,901	599,252	432,354	460,201	497,529	524,069
2016	273,331	27,072	9.9%	142,347	383,736	272,131	292,254	319,167	337,111
2017	157,258	17,542	11.2%	67,252	220,814	156,499	169,104	187,514	200,341
2018	82,174	10,939	13.3%	32,880	123,782	81,702	89,376	100,832	108,855
2019	39,962	6,966	17.4%	14,345	69,981	39,632	44,447	52,029	57,298
2020	18,819	4,746	25.2%	1,463	41,958	18,626	21,805	27,058	30,892
2021	9,139	3,442	37.7%	(4,763)	26,381	8,926	11,285	15,161	18,408
2022	4,178	2,466	59.0%	(5,361)	15,768	3,933	5,672	8,598	11,114
2023	1,807	1,647	91.2%	(7,328)	10,335	1,565	2,713	4,837	6,709
2024	704	938	133.2%	(4,654)	6,189	539	1,172	2,474	3,628
2025	243	491	202.5%	(3,442)	3,710	152	455	1,135	1,876
2026	61	221	363.4%	(1,545)	1,829	24	130	460	799

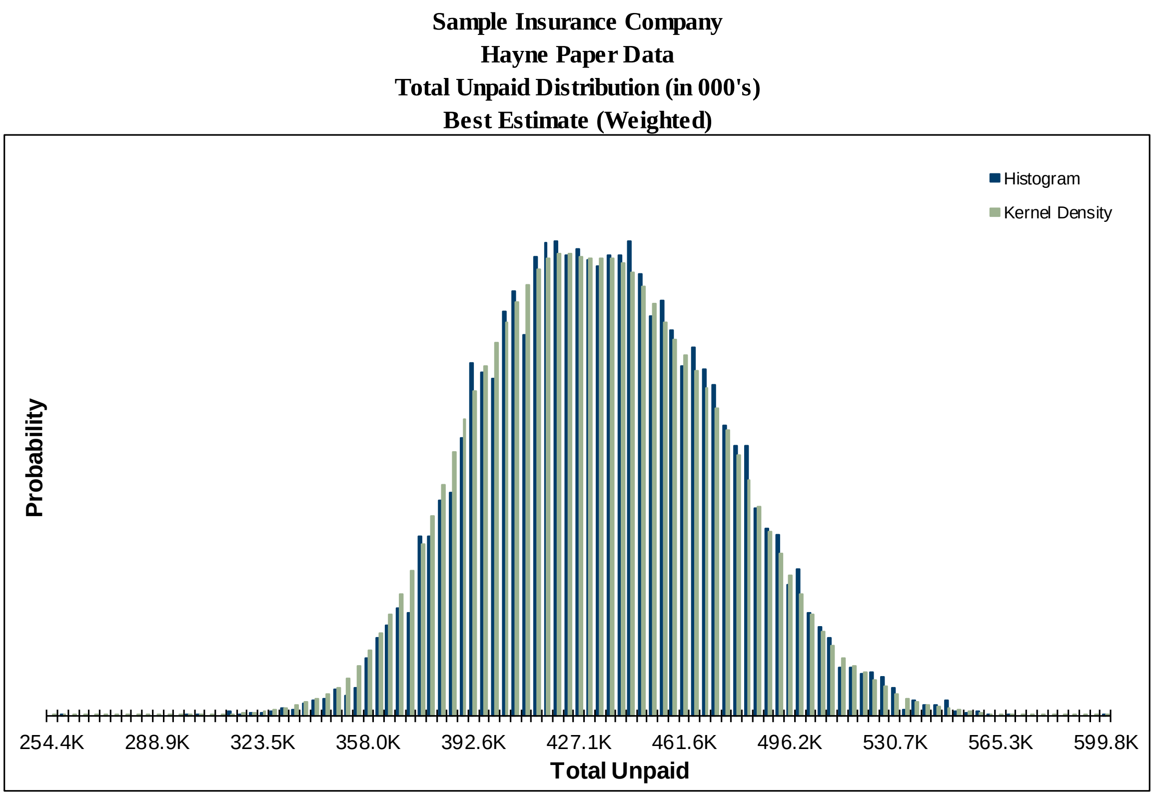
[29] The coefficients of variation measure the variability of the loss ratios, given the movements by year. Without this information, it is common to base the future standard deviation on the standard deviation of the historical mean loss ratios, but this is not ideal since the variability of the mean loss ratios is not the same as the possible variation in the actual outcomes given movements in the means.

6.3 Distribution Graphs

A final model output to consider is a histogram of the estimated unpaid amounts for the total of all accident years combined, as shown in the graph in Figure 6.1. The histogram is created by counting the number of outcomes within each of 100 "buckets" of equal size spread between the minimum and maximum outcome. To smooth the histogram a kernel density function[30] is often used, which is the green bars in Figure 6.1.

Another useful strategy for graphing the total unpaid distribution may be accomplished by creating a summary of the ten model distributions used to determine the weighted "best estimate" and distribution. An example of this graph using the kernel density functions is shown in Figure 6.2 and dots for the mean estimates, which would represent a traditional range[31], are also included.

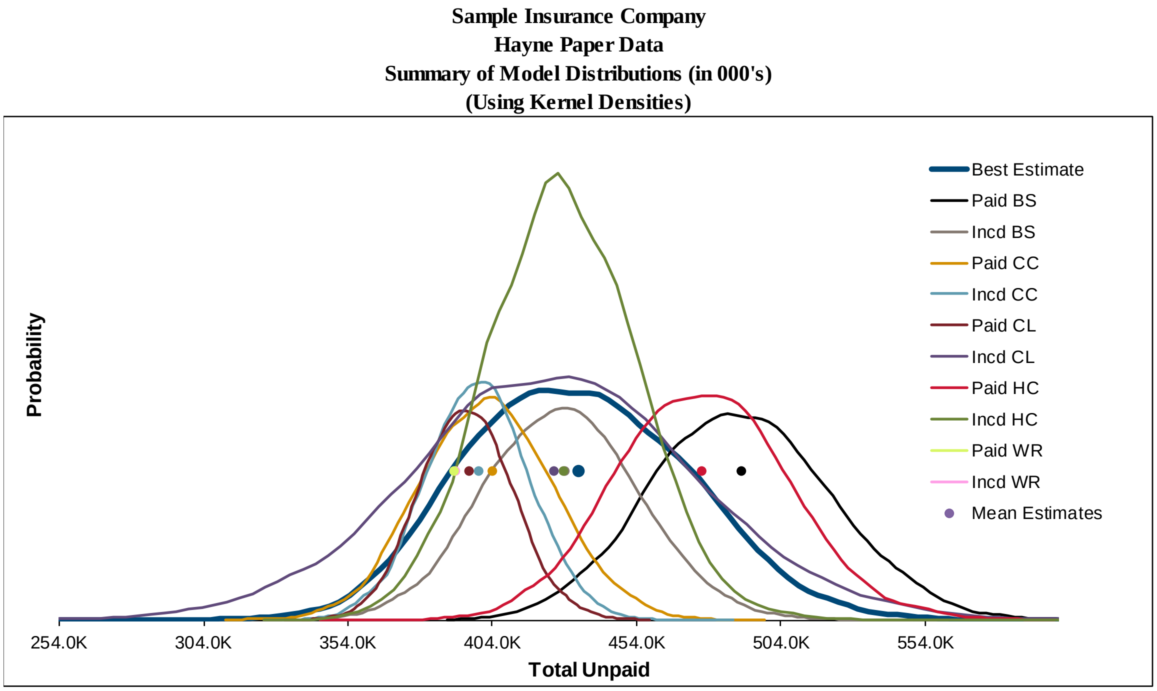
Figure 6.1. Total Unpaid Claims Distribution



[30] A kernel density function uses weighed values of the surrounding values, with decreasing weight the further from the value in question, in order to smooth the values.

[31] A traditional range would use deterministic point estimates instead of means of the distributions, but the intent is consistent. While the points would technically have an infinitesimal probability and should therefore sit on the x-axis, they are elevated above the zero probability level purely for illustration purposes.

Figure 6.2. Summary of model distributions



6.4 Correlation & Aggregation

Results for an entire business unit can be estimated, after each business segment has been analyzed and weighted into best estimates, using aggregation. This represents another area where caution is warranted. The procedure is not a simple matter of adding up the distributions for each segment. In order to estimate the distribution of possible outcomes for a company as a whole, a correlation of results among segments must be used.[32] To illustrate aggregation, data from the "Industry Data.xls" file for Parts A, B, and C are used. The various model tables and graphs for the Part A, Part B, and Part C results are shown in Appendices B, C, and D, respectively.

Simulating correlated variables is commonly accomplished with a multi-variate distribution whose parameters and correlations have been previously specified. This type of simulation is most easily applied when distributions are uniformly identical and known in advance (for example, all derived from a multi-variate normal distribution). Unlike the ODP bootstrap framework, in which the characteristics of the overall distribution are unknown in advance, the multi-variate normal distribution assumption in the Hayne MLE framework could allow

[32] This section assumes the reader is familiar with correlation.

model correlation for multiple business segments. However, the correlation among parameters from each segment has to be defined before consolidating the variance-covariance matrices to simulate parameters for all segments. Thus, a fair amount of parameters are needed for correlation and it is difficult to visualize the gigantic aggregated variance-covariance matrix, so it is beyond the scope of this paper.

Alternatively, two useful correlation processes for the Hayne MLE model are synchronized parameter simulation and re-sorting.[33]

With synchronized parameter simulation, in each iteration, independent normal random values are simulated for each parameter and each segment, then correlation is applied to adjust the simulated random numbers for the second segment and beyond, and modified random numbers are used for multi-variate normal distribution sampling.

The synchronized simulation process can be implemented in Excel once a correlation matrix has been estimated. There are, however, two potential drawbacks to this process. First, since multiple LOB/segments are being simulated simultaneously either the size of the workbook needs to increase to accommodate all of the segments or the random number streams need to be correlated in a separate process. Second, when the multiple models are weighted to get a "best estimate" for each segment the coordination of multiple models and segments is even more complex.

The second correlation process, re-sorting, can be accomplished with algorithms such as Iman-Conover[34] or Copulas, among others. The primary advantages of re-sorting include:

- The correlation is a combination of parameter uncertainty and process variance,
- Different correlation assumptions may be employed, and
- Different correlation algorithms may also have other beneficial impacts on the aggregate distribution.

For example, using a *t*-distribution Copula with low degrees of freedom rather than a normal-distribution Copula, will effectively "strengthen" the focus of the correlation in the tail of the distribution, all else being equal. This type of consideration is important for risk-

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[33] For a useful reference see Kirschner, et al. [11]. The Kirschner paper is about correlation for the ODP Bootstrap model, but the two processes can be used with other models.

[34] For a useful reference see Iman and Conover [9] or Mildenhall [12]. In the "Aggregate Estimate.xlsm" file the Iman-Conover algorithm is used to "Generate Rank Values" on the Inputs sheet.

based capital and other risk modeling issues.

To induce correlation among different segments in the "Aggregation.xlsm" file, a correlation matrix can be calculated using Spearman's Rank Order for each data / model type combination in order to select a correlation assumption. Using the selected correlation, re-sorting based on the ranks of the total unpaid claims for all accident years combined can be done. The calculated correlations for Parts A, B, and C based on the paid residuals for Berquist-Sherman may be seen in the first part of Table 6.10. A second part of Table 6.10 are the *p*-values for each correlation coefficient, which are an indication of whether a correlation coefficient is significantly different than zero as the *p*-value gets close to zero.[35]

Table 6.10. Estimated Correlation and P-values

Rank Correlation of Residuals Paid BS Model - [Modeled]

LOB	HO	PPA	CA
HO	1.00	0.26	0.22
PPA	0.26	1.00	0.15
CA	0.22	0.15	1.00

P-Value of Rank Correlation of Residuals Paid BS Model - [Modeled]

LOB	HO	PPA	CA
HO	0.00	0.06	0.11
PPA	0.06	0.00	0.29
CA	0.11	0.29	0.00

By reviewing the correlation coefficients for each "pair" of segments, along with the *p*-values, from different sets of correlations matrices (e.g., from paid or incurred data for each model) judgment can be used to select a correlation matrix assumption. As noted above, caution is warranted as these calculated correlation matrices are limited to the data used in the calculation and the impact of other systemic issues, such as contagion, may also need to be considered.

[35] While judgment is clearly appropriate, the typical threshold is a *p*-value of 5% – i.e., a *p*-value of 5% or less indicates the correlation is significantly different than zero, while a *p*-value greater than 5% indicates the correlation is not significantly different than zero.

Table 6.11. Aggregate estimated unpaid

Sample Insurance Company
Aggregate Three Lines of Business
Accident Year Unpaid (in 000's)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	18,613	146	1,002	688.1%	(2,013)	74,778	37	55	421	2,422
2007	20,618	198	993	500.3%	(1,523)	37,034	70	94	503	3,069
2008	22,866	246	927	377.4%	(5,763)	54,447	128	162	542	3,227
2009	22,842	367	1,286	350.7%	(2,918)	90,399	230	268	695	3,778
2010	22,351	535	1,359	254.3%	(1,875)	69,139	406	452	860	3,458
2011	22,422	869	1,266	145.7%	(3,632)	68,690	760	826	1,253	4,003
2012	24,350	1,589	939	59.1%	(4,107)	27,387	1,518	1,633	2,198	4,927
2013	19,973	2,814	1,424	50.6%	(8,046)	80,667	2,785	2,963	3,667	6,153
2014	18,919	5,418	4,384	80.9%	(8,120)	407,319	5,420	5,768	6,863	9,408
2015	15,961	13,369	3,352	25.1%	(11,431)	98,644	13,319	14,627	17,722	21,777
Totals	208,915	25,550	9,304	36.4%	(815)	476,278	24,635	26,612	32,642	55,933
Normal Dist.		25,550	9,304	36.4%			25,550	31,826	40,854	47,195
logNormal Dist.		25,528	6,217	24.4%			24,803	29,163	36,812	43,354
Gamma Dist.		25,550	9,304	36.4%			24,430	31,065	42,526	52,000
TVaR							28,995	32,475	48,429	89,074
Normal TVaR							32,974	37,377	44,742	50,348
logNormal TVaR							30,371	33,900	40,865	47,165
Gamma TVaR							32,838	38,140	48,373	57,295

Using these correlation coefficients, the "Aggregate Estimate.xlsm" file, and the simulation data for Parts A, B, and C, the aggregate results for the three lines of business were calculated and summarized in Table 6.11. A more complete set of tables for the aggregate results is shown in Appendix E.

Note that using residuals to correlate the lines of business (or other segments), as in the synchronized simulation method, and measuring the correlation between residuals, as in the re-sorting method, both tend to create correlations that are close to zero. For reserve risk, the correlation that is desired is between the total unpaid amounts for two segments. The correlation that is being measured is the correlation between each incremental future loss amount, given the underlying model describing the overall trends in the data. This may or may not be a reasonable approximation.

While not the direct measure being sought, keep in mind that some level of implied correlation between lines of business will naturally occur due to correlations between the model parameters – e.g., similarities in development parameters, so correlation based on the correlation between the remaining random movements in the incremental values given the model parameters (i.e., residuals) may be reasonable. However, an example of an issue not particularly well suited to measurement via residual correlation is contagion between lines of business – i.e., single events that result in claims in multiple lines of business. To account for this, and to add a bit of conservatism, the correlation assumption can be easily changed based on actuarial judgment.

Correlation is often thought of as being much stronger than "close to zero", but in this case the correlation being considered is typically the loss ratio movements by line of business. For pricing risk, the correlation that is desired is between the loss ratio movements by accident year between two segments. This correlation is not as likely to be close to zero, so correlation of loss ratios (e.g., for the data in Table 6.8) is often done with a different correlation assumption compared to reserving risk.

7. Future Research

While common use of the Hayne MLE models may be in its infancy, the hope is that this paper will spur more widespread use of the models. Nevertheless, there are many area where further research can add value, but only a few key areas are offered up here.

- **Use of Other Distributions** – The key assumption which allows the framework for the Hayne MLE is the Normal distribution. Other distribution assumptions, while more complex mathematically, may provide useful alternatives;
- **Simulating Frequency and Severity** – Instead of simply basing the Hayne MLE on the estimate ultimate claim count, the claim count could also be generated stochastically, with correlation between frequency and severity outputs, and thus simulating both at the same time;
- **A Flexible Model** – Similar to the GLM bootstrap or incremental log models it may be possible to develop a model using the Hayne MLE framework where the user can specify the place for parameters and include a diagonal parameter;
- **Time Horizon Models** – As other models have been adapted for calculation of the one-year time horizon for Solvency II purposes, the Hayne MLE models could also be so adapted;
- **MCMC Models** – It is possible that Markov Chain Monte Carlo (MCMC) models could be used to induce additional correlation into the Hayne MLE models; and
- **Pricing Models** – In order to expand the usefulness of the models, they could be extrapolated into future underwriting periods.

8. Conclusions

While this paper endeavored to show how the Hayne MLE models can be used in a variety of practical ways, and to illustrate the diagnostic tools the actuary needs to assess whether the model is working well, it should not be assumed that a given Hayne MLE model is well suited for every data set. However, it is hoped that the Hayne MLE "toolsets" can become an integral part of the actuary's regular estimation of unpaid claim liabilities, rather than just a "black box" to be used only if necessary or after the deterministic methods have been used to select a point estimate. Finally, the modeling framework allows the actuary to "adjust" the model parameters to smooth anomalies in the data instead of simply accepting the model as is and essentially forcing the data to "fit" the model.

Acknowledgment

The authors acknowledge the foundational research done by Roger Hayne and the many other authors listed in the References (and others not listed) that contributed to the foundation of the stochastic modeling, without which this research would not have been possible. The authors would like to thank the peer reviewers, Roger Hayne, Steve Finch, and Blair Manktelow, who helped to improve the quality of the paper in a variety of ways. Finally, the authors are also grateful to the CAS Committee on Reserves for their comments which greatly improved the quality of the paper.

Supplementary Material

There are several companion files designed to give the reader a deeper understanding of the concepts discussed in the paper. The files are all in the "Hayne MLE Practitioners Guide.zip" file. The files are:

Model Instructions.pdf – this file contains a written description of how to use the primary Hayne MLE modeling files.

Primary modeling files:

Industry Data.xls – this file contains Schedule P data by line of business for the entire U.S. industry and five of the top 50 companies, for each LOB that has 10 years of data.

Hayne MLE Models.xlsm – this file contains the detailed model steps described in this paper as well as various modeling options and diagnostic tests. Data can be entered and simulations run and saved for use in calculating a weighted best estimate.

Best Estimate.xlsm – this file can be used to weight the results from ten different models to get a "best estimate" of the distribution of possible outcomes.

Aggregate Estimate.xlsm – this file can be used to correlate the best estimate results from 3 LOBs/segments.

Correlation Ranks.xlsm – this file contains examples of ranks used to correlate results by LOB/segment.

Appendix A – User Selected Parameters & Diagnostics

In this appendix, the selected parameters and diagnostics are shown for paid data for each model.

Figure A.1. User Selected Parameters for Berquist-Sherman

	12	24	36	48	60	72	84	96	108	120	132	144	156
Mean	620.96	760.67	708.16	553.57	350.00	181.39	70.97	43.88	26.00	15.21	*7.05*	*3.27*	*1.51*
Std Dev	40.50	46.55	43.00	35.49	26.17	17.66	10.40	8.75	7.60	7.36	*6.47*	*4.32*	*2.67*
Decay Ratios:		122.5%	93.1%	78.2%	63.2%	51.8%	39.1%	61.8%	59.3%	58.5%			
CoV:	*6.5%*	*6.1%*	*6.1%*	*6.4%*	*7.5%*	*9.7%*	*14.7%*	*19.9%*	*29.2%*	*48.4%*	*91.8%*	*132.3%*	*176.2%*

	Accident Year					Tail Extrapolation			Implied Tail Factor		
	Trend	**K**	**p**	**AIC**	**BIC**	**Decay Ratio**	**Periods**	**Distribution**	**Adjusted**	**Actual**	**Tail Sampling Option**
Mean	0.045	11.216	0.654	647.9	674.0	46.3%	3	Gamma	1.0036	1.0036	Conditional Variance
Std Dev	0.009	1.094	0.089			32.5%					
CoV:	*18.9%*	*9.8%*	*13.6%*								

Decay Ratio Analysis:

Parameters: Model **Curve Type:** Power 3

Least Squares Regression Coefficients:

x^a	-0.3673
coefficient	1.1166

Goodness of Fit Statistics:

R^2 Statistic	0.240
Regression Deviation	32.5%
Suggested Decay Parameters:	
Mean	46.3%
Standard Deviation	32.5%

Periods	Decay Ratio	Outliers	Selected Age	Selected Decay Ratio	Incremental Fitted Factors
12-24	1.225	0	1	1.225	1.117
24-36	0.931	0	2	0.931	0.866
36-48	0.782	0	3	0.782	0.746
48-60	0.632	0	4	0.632	0.671
60-72	0.518	0	5	0.518	0.618
72-84	0.391	0	6	0.391	0.578
84-96	0.618	0	7	0.618	0.546
96-108	0.252	0	8	0.252	0.520
108-120	1.373	0	9	1.373	0.498
120-132					0.479
132-144					0.463
144-156					0.448
156-168					0.435
168-180					0.424

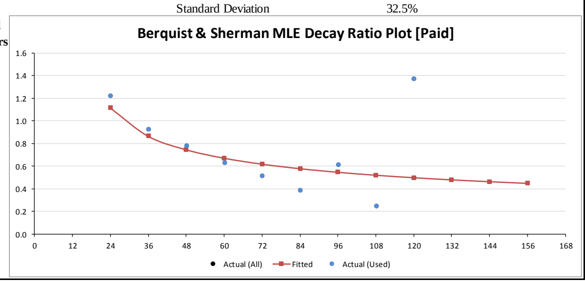


Figure A.2. Residual Graphs for Berquist-Sherman [Modeled Parameters]

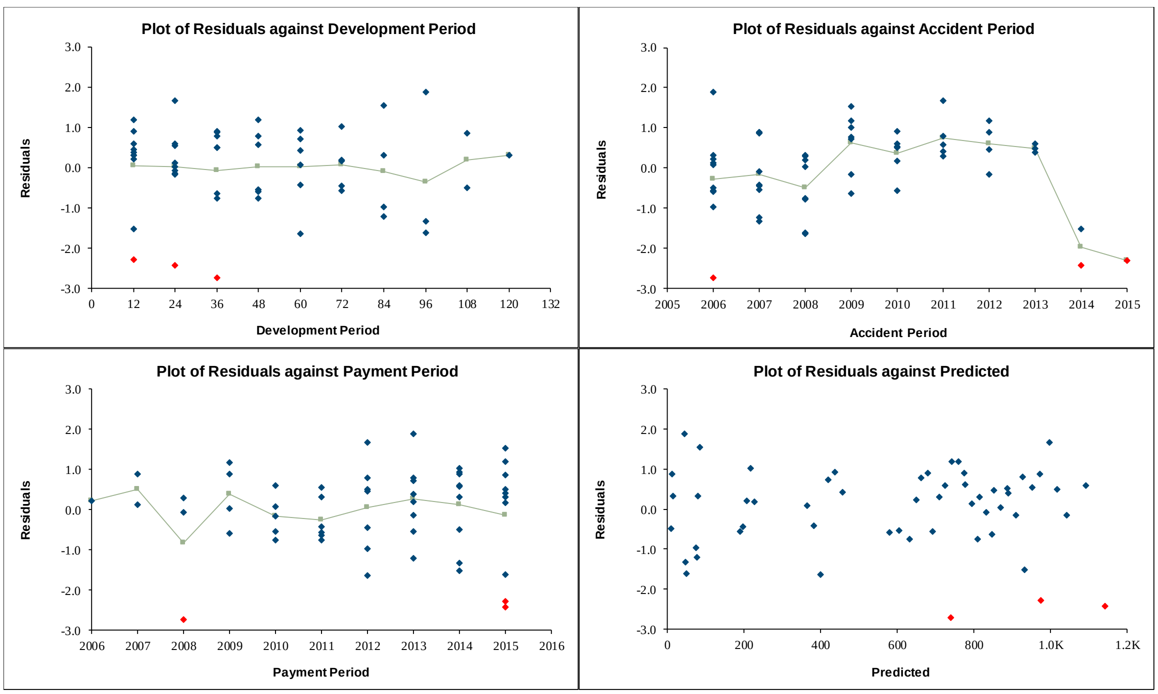


Figure A.3. Residual Graphs for Berquist-Sherman [Selected Parameters]

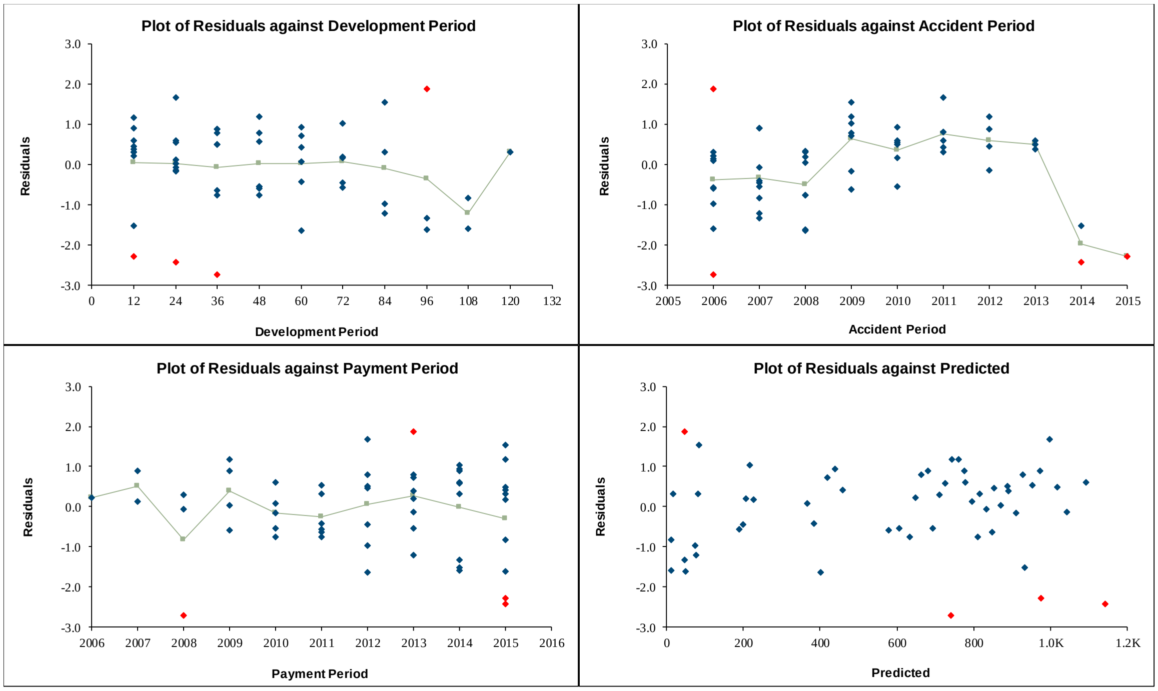
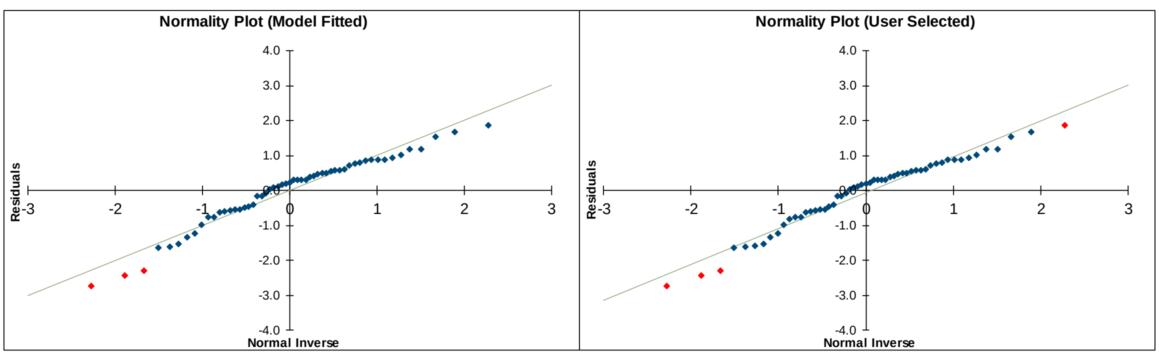


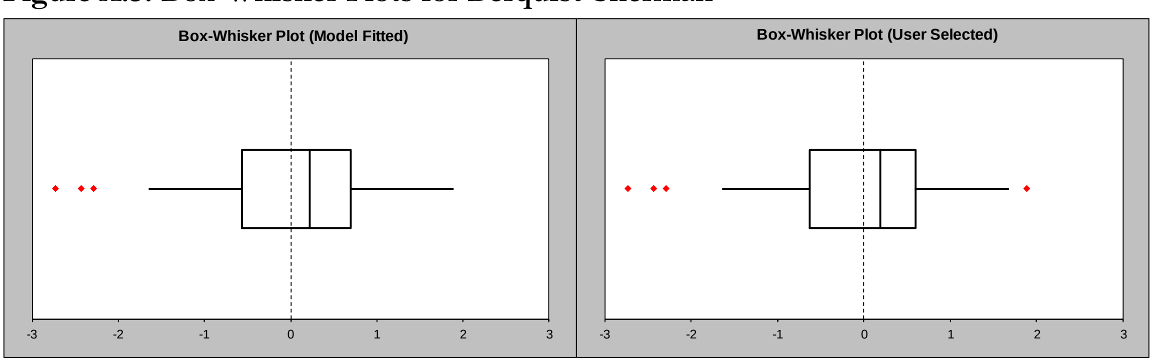
Figure A.4. Normality Plots for Berquist-Sherman



N = 55 P-Value = 3.9% R^2 = 95.5%
Normal: MU = 0.00, Sigma = 1.01 AIC = 643.4, BIC = 669.5

N = 55 P-Value = 8.0% R^2 = 96.3%
Normal: MU = -0.05, Sigma = 1.03 AIC = 647.9, BIC = 674.0

Figure A.5. Box-Whisker Plots for Berquist-Sherman



Interquartile Range = [-0.56, 0.70] Median = 0.22
Outliers = 3

Interquartile Range = [-0.62, 0.60] Median = 0.19
Outliers = 4

Figure A.6. Model Structure Graphs for Berquist-Sherman

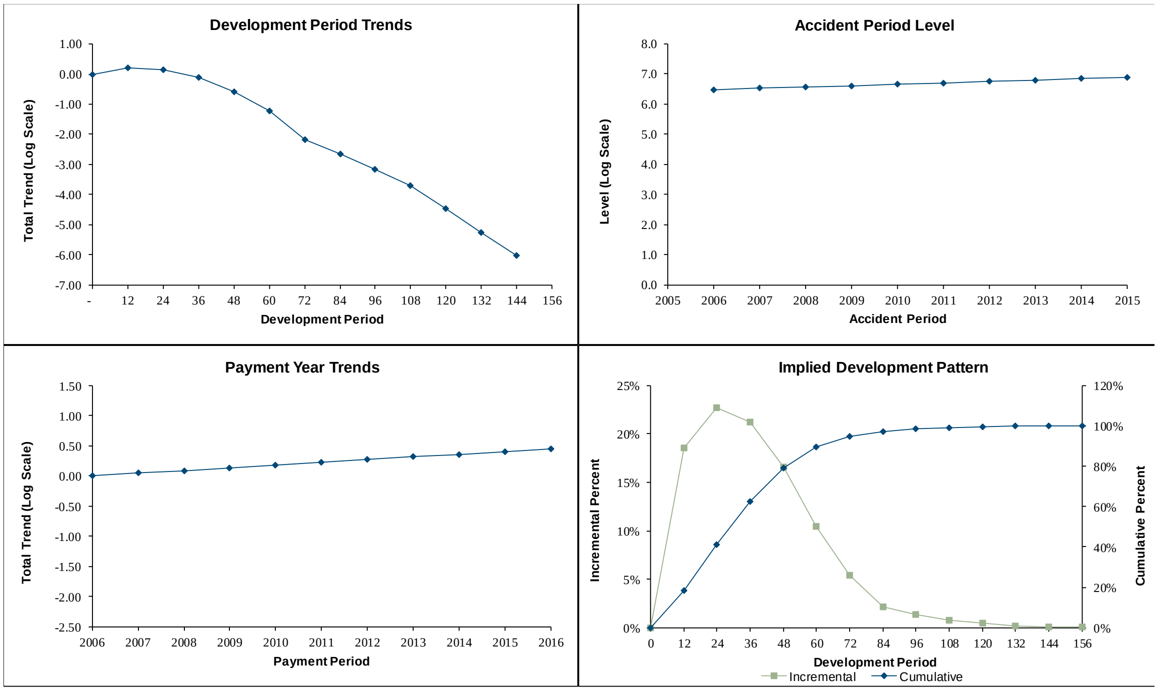


Figure A.7. User Selected Parameters for Cape Cod

User Selected Parameters:

	Scale	2007	2008	2009	2010	2011	2012	2013	2014	2015
Mean	620.067	1.160	1.123	1.322	1.376	1.521	1.533	1.580	1.169	1.164
Std Dev	30.027	0.066	0.064	0.072	0.075	0.082	0.084	0.091	0.082	0.105
CoV	*4.8%*	*5.7%*	*5.7%*	*5.4%*	*5.4%*	*5.4%*	*5.5%*	*5.8%*	*7.0%*	*9.0%*

Development Period Parameters (Average Incremental)

	24	36	48	60	72	84	96	108	120	132	144	156
Mean	1.181	1.063	0.838	0.534	0.284	0.111	0.067	0.040	0.024	*0.011*	*0.005*	*0.002*
Std Dev	0.041	0.040	0.036	0.029	0.023	0.016	0.016	0.015	0.017	0.009	0.004	0.002
Decay Ratios		*90.0%*	*78.8%*	*63.7%*	*53.2%*	*39.0%*	*60.7%*	*59.4%*	*60.7%*			
CoV	*3.5%*	*3.8%*	*4.3%*	*5.5%*	*8.1%*	*14.9%*	*23.1%*	*37.8%*	*70.6%*	*77.1%*	*83.4%*	*89.7%*

					Tail Extrapolation			Implied Tail Factor		
	K	**p**	**AIC**	**BIC**	**Decay Ratio**	**Periods**	**Distribution**	**Adjusted**	**Actual**	**Tail Sampling Option**
Mean	13.104	0.435	663.9	706.0	46.4%	3	Gamma	1.0037	1.0037	Sampling
Std Dev	1.061	0.087			11.8%					
CoV	*8.1%*	*19.9%*								

Decay Ratio Analysis:

Parameters: User **Curve Type:** Power 3

Least Squares Regression Coefficients:

x^a	-0.3195
coefficient	1.0261

Goodness of Fit Statistics:

R^2 Statistic	0.435
Regression Deviation	11.6%
Suggested Decay Parameters:	
Mean	46.4%
Standard Deviation	11.8%

Periods	Decay Ratio	Outliers	Selected Age	Selected Decay Ratio	Incremental Fitted Factors
24-36	0.900	0	2	0.900	0.822
36-48	0.788	0	3	0.788	0.722
48-60	0.637	0	4	0.637	0.659
60-72	0.532	0	5	0.532	0.614
72-84	0.390	0	6	0.390	0.579
84-96	0.607	0	7	0.607	0.551
96-108	0.594	0	8	0.594	0.528
108-120	0.607	0	9	0.607	0.509
120-132					0.492
132-144					0.477
144-156					0.464
156-168					0.452
168-180					0.442

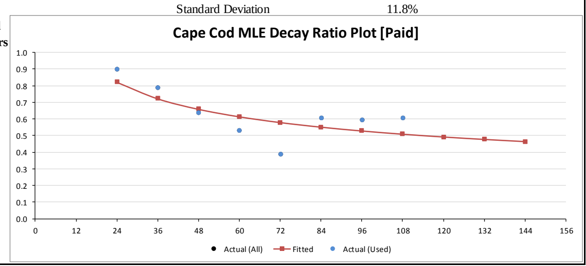


Figure A.8. Residual Graphs for Cape Cod [Modeled Parameters]

Residual Graphs for Cape Cod Hayne MLE Paid Model [Model Fitted]

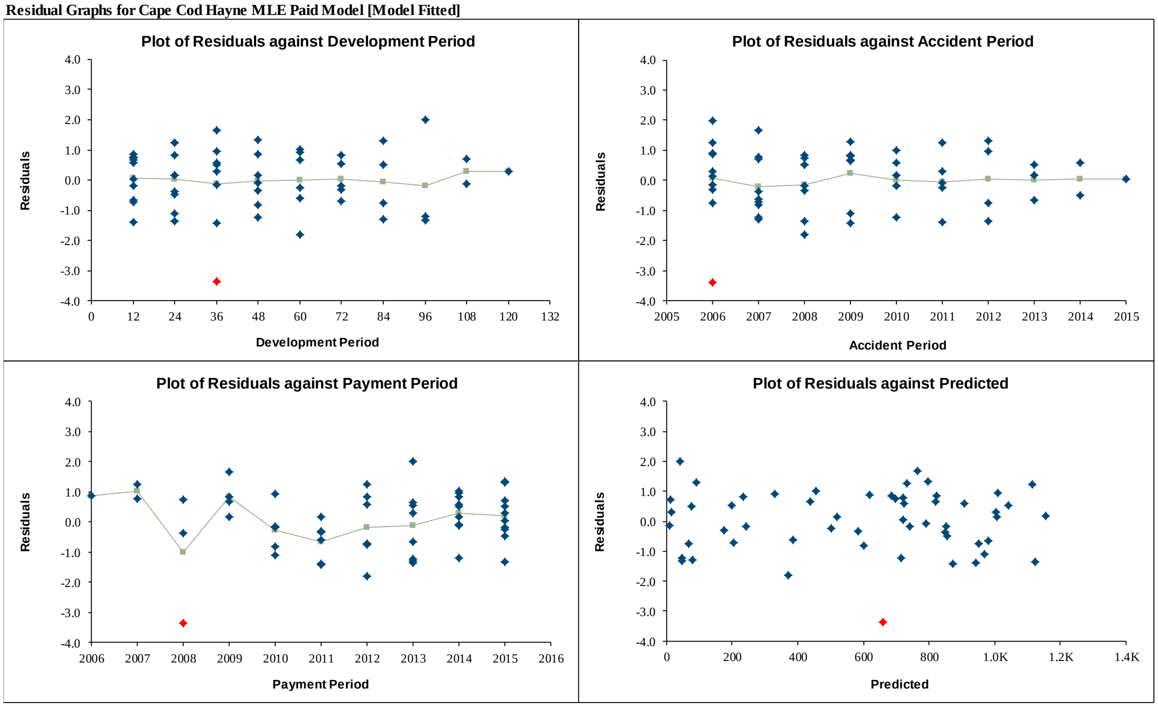


Figure A.9. Residual Graphs for Cape Cod [Selected Parameters]

Residual Graphs for Cape Cod Hayne MLE Paid Model [User Selected]

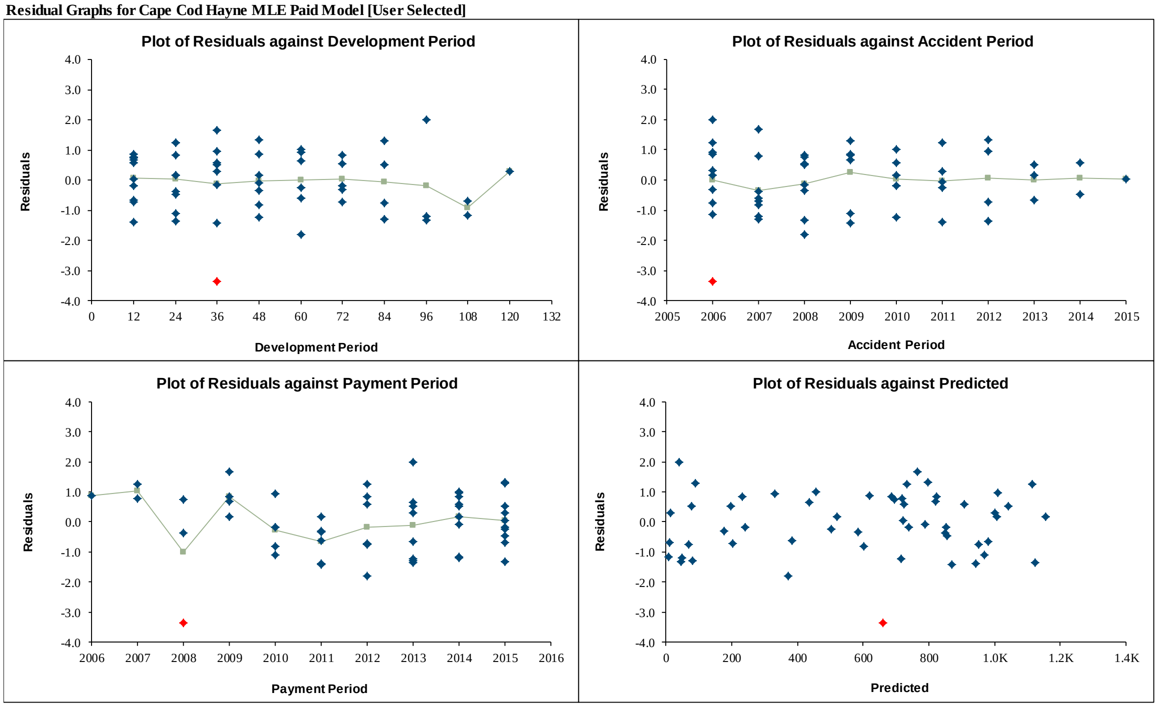


Figure A.10. Normality Plots for Cape Cod

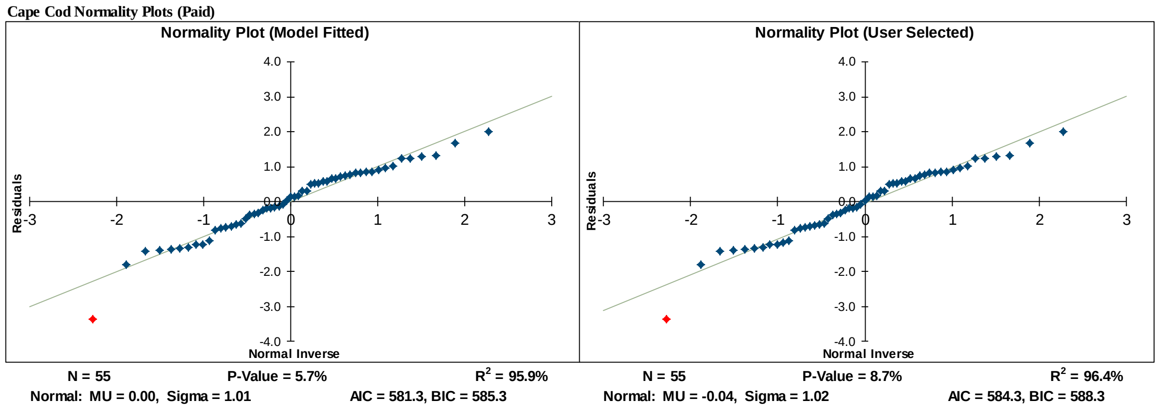


Figure A.11. Box-Whisker Plots for Cape Cod

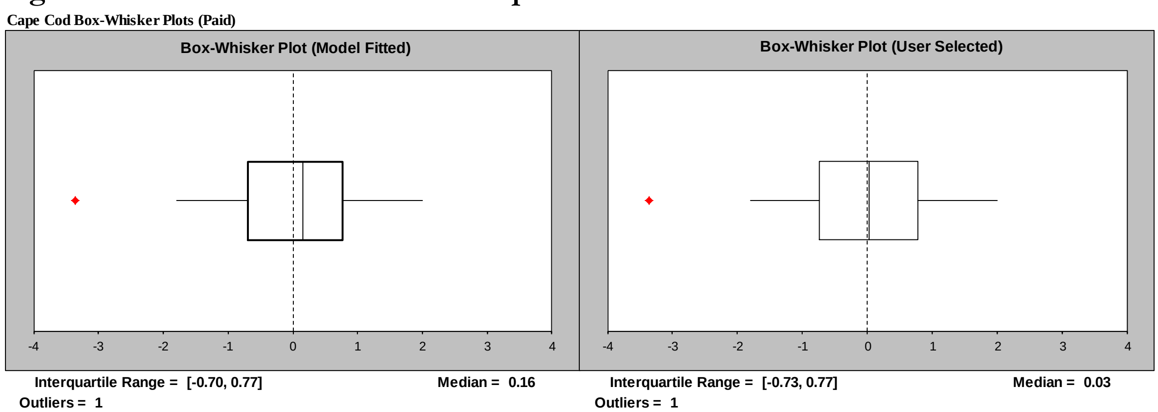


Figure A.12. Model Structure Graphs for Cape Cod

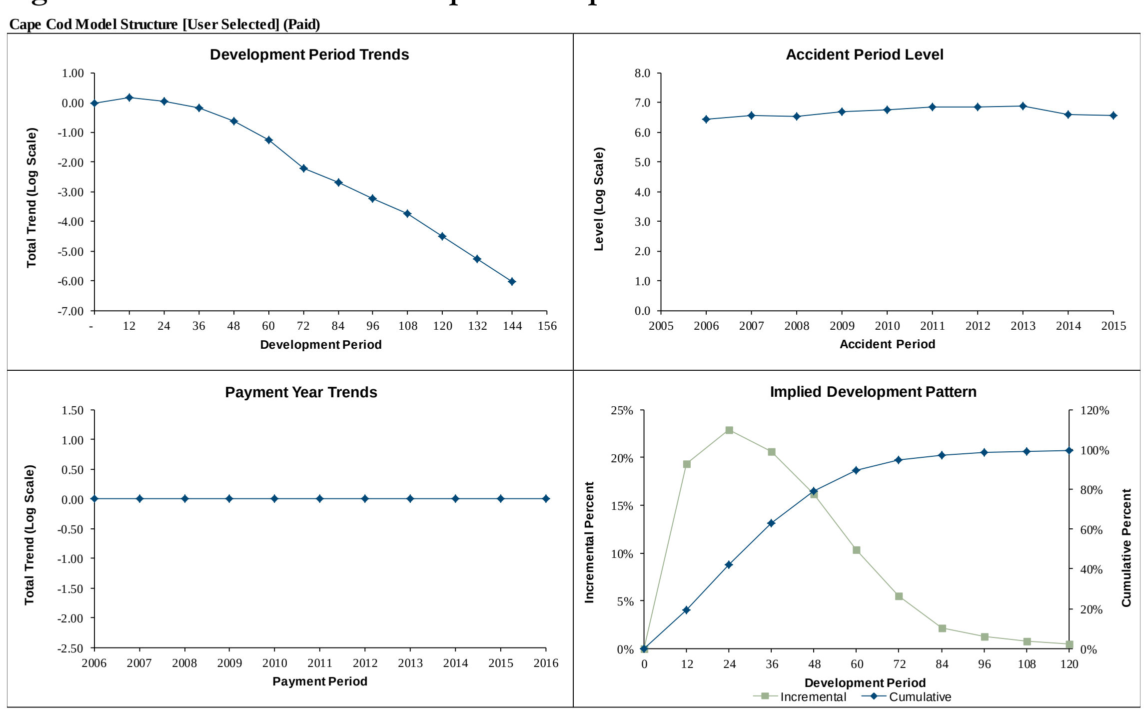


Figure A.13. User Selected Parameters for Chain Ladder

User Selected Parameters:

	12	24	36	48	60	72	84	96	108	120	132	144	156
Mean	0.195	0.231	0.208	0.164	0.104	0.056	0.022	0.013	0.006	0.002	*0.001*	*0.000*	*0.000*
Std Dev	0.005	0.005	0.005	0.005	0.005	0.004	0.003	0.003	0.002	0.002	*0.001*	*0.000*	*0.000*
Decay Ratios:		118.1%	90.0%	78.8%	63.7%	53.2%	39.0%	60.8%	41.8%	41.1%			
CoV:	*2.5%*	*2.3%*	*2.5%*	*3.1%*	*4.5%*	*7.3%*	*14.3%*	*22.6%*	*45.1%*	*96.4%*	*103.0%*	*109.6%*	*116.3%*

	K	p	AIC	BIC	Tail Extrapolation Decay Ratio	Tail Extrapolation Periods	Distribution	Implied Tail Factor Adjusted	Implied Tail Factor Actual	Tail Sampling Option
Mean	13.074	0.438	621.6	663.8	37.6%	3	Gamma	1.0013	1.0013	Sampling
Std Dev	1.018	0.083			9.8%					
CoV:	*7.8%*	*19.0%*								

Decay Ratio Analysis:

Parameters: User **Curve Type:** Power 3

Least Squares Regression Coefficients:

x^a	-0.4965
coefficient	1.2345

Goodness of Fit Statistics:

R2 Statistic	0.870
Regression Deviation	9.5%
Suggested Decay Parameters:	
Mean	37.6%
Standard Deviation	9.8%

Periods	Decay Ratio	Outliers	Selected Age	Selected Decay Ratio	Incremental Fitted Factors
12-24	1.181	0	1	1.181	1.235
24-36	0.900	0	2	0.900	0.875
36-48	0.788	0	3	0.788	0.715
48-60	0.637	0	4	0.637	0.620
60-72	0.532	0	5	0.532	0.555
72-84	0.390	0	6	0.390	0.507
84-96	0.608	0	7	0.608	0.470
96-108	0.418	0	8	0.418	0.440
108-120	0.411	0	9	0.411	0.415
120-132					0.394
132-144					0.375
144-156					0.359
156-168					0.345

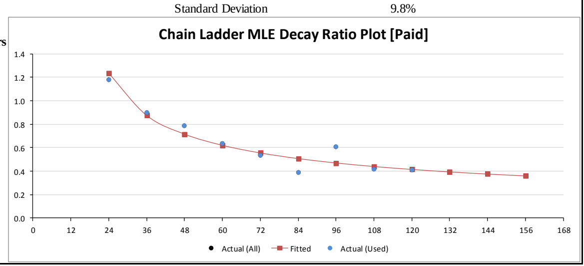


Figure A.14. Residual Graphs for Chain Ladder [Modeled Parameters]

Residual Graphs for Chain Ladder Hayne MLE Paid Model [Model Fitted]

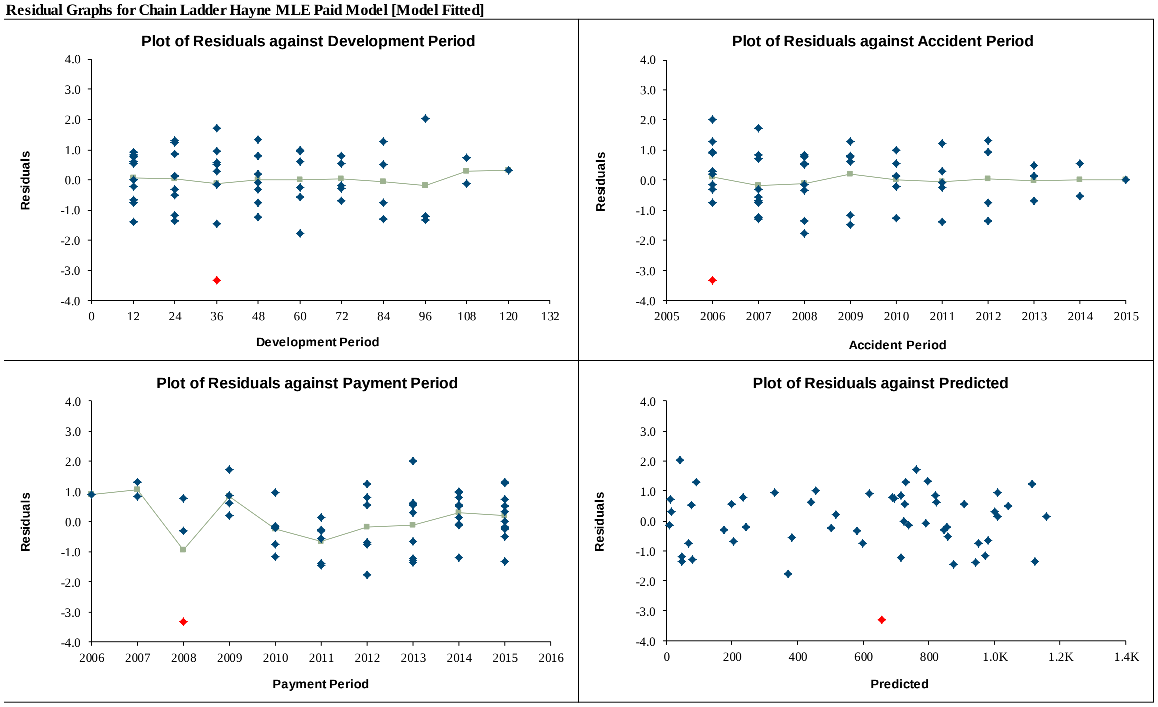


Figure A.15. Residual Graphs for Chain Ladder [Selected Parameters]

Residual Graphs for Chain Ladder Hayne MLE Paid Model [User Selected]

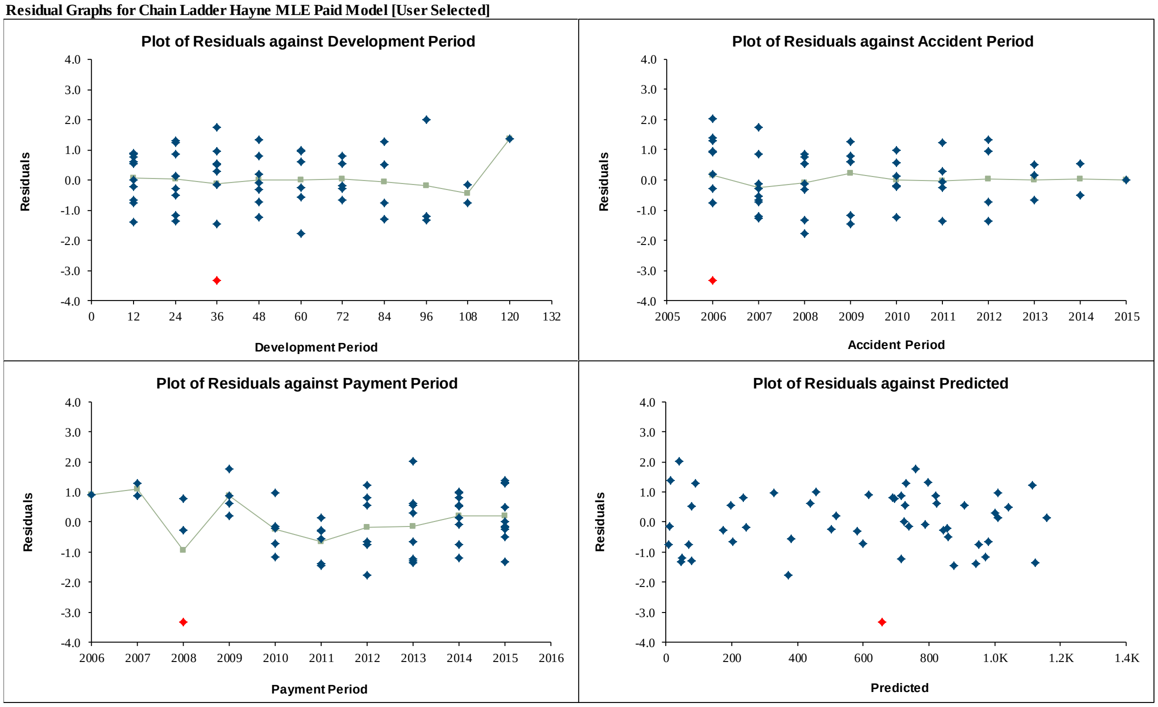
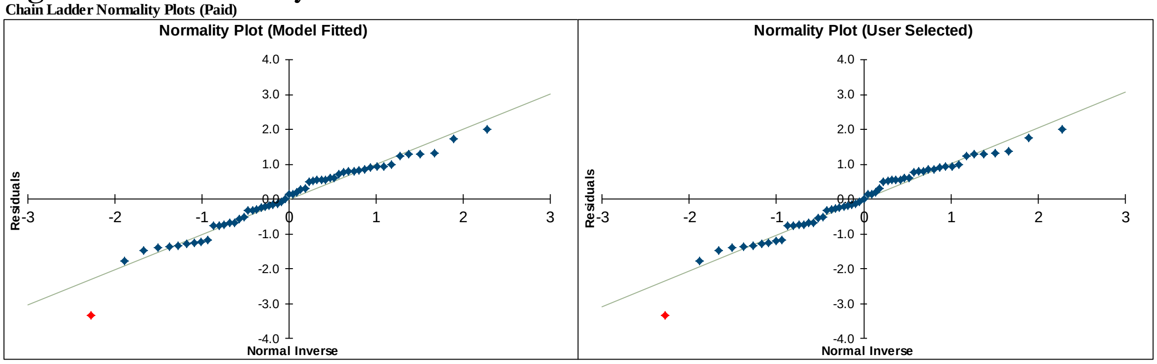


Figure A.16. Normality Plots for Chain Ladder

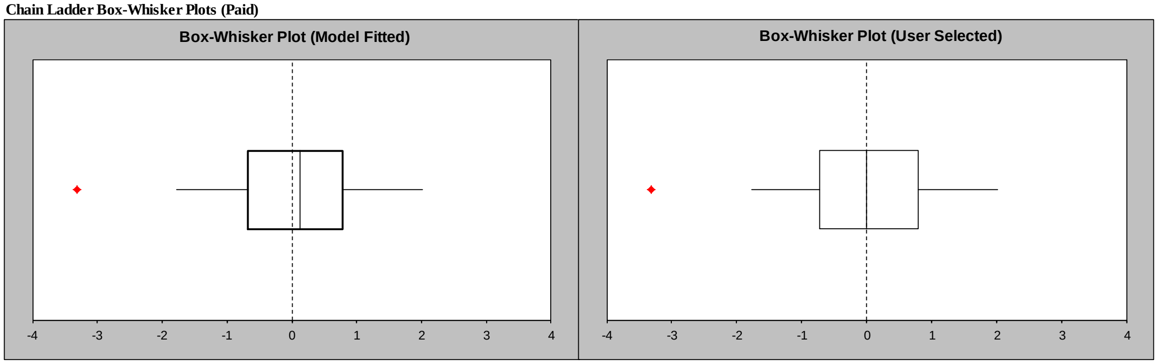
Chain Ladder Normality Plots (Paid)



N = 55	P-Value = 7.5%	R^2 = 96.2%	N = 55	P-Value = 11.3%	R^2 = 96.6%
Normal: MU = 0.01, Sigma = 1.01		AIC = 581.4, BIC = 585.4	Normal: MU = 0.00, Sigma = 1.03		AIC = 583.6, BIC = 587.7

Figure A.17. Box-Whisker Plots for Chain Ladder

Chain Ladder Box-Whisker Plots (Paid)



Interquartile Range = [-0.68, 0.79]	Median = 0.13	Interquartile Range = [-0.72, 0.80]	Median = 0.00
Outliers = 1		Outliers = 1	

Figure A.18. Model Structure Graphs for Chain Ladder

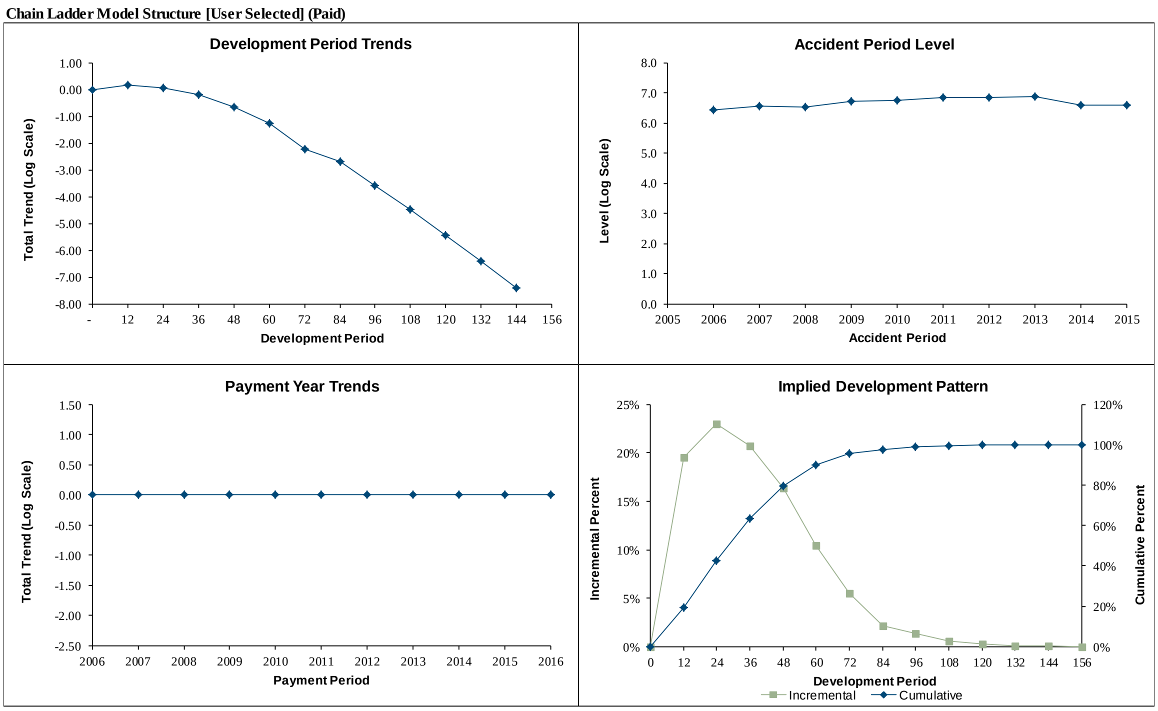


Figure A.19. User Selected Parameters for Hoerl Curve

User Selected Parameters:

	Level	d	d²	ln(d)	Trend
Mean	6.496	0.005	(0.065)	0.596	0.043
Std Dev	0.220	0.240	0.019	0.323	0.008
CoV:	*3.4%*	*4687.1%*	*-28.4%*	*54.2%*	*19.5%*

	K	p	AIC	BIC	Tail Extrapolation Periods	Implied Tail Factor Adjusted	Implied Tail Factor Actual
Mean	13.147	0.506	635.9	649.9	3	1.0004	1.0004
Std Dev	1.014	0.083					
CoV:	*7.7%*	*16.3%*					

Figure A.20. Residual Graphs for Hoerl Curve [Modeled Parameters]

Residual Graphs for Hoerl Curve Hayne MLE Paid Model [Model Fitted]

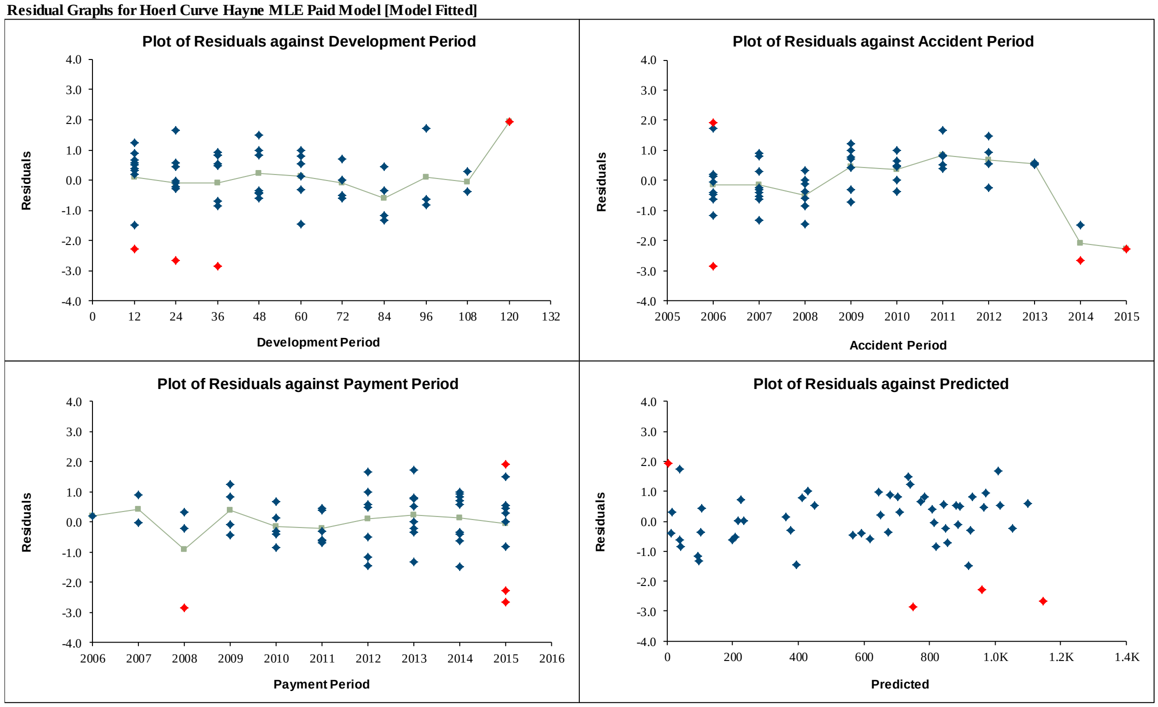


Figure A.21. Residual Graphs for Hoerl Curve [Selected Parameters]

Residual Graphs for Hoerl Curve Hayne MLE Paid Model [User Selected]

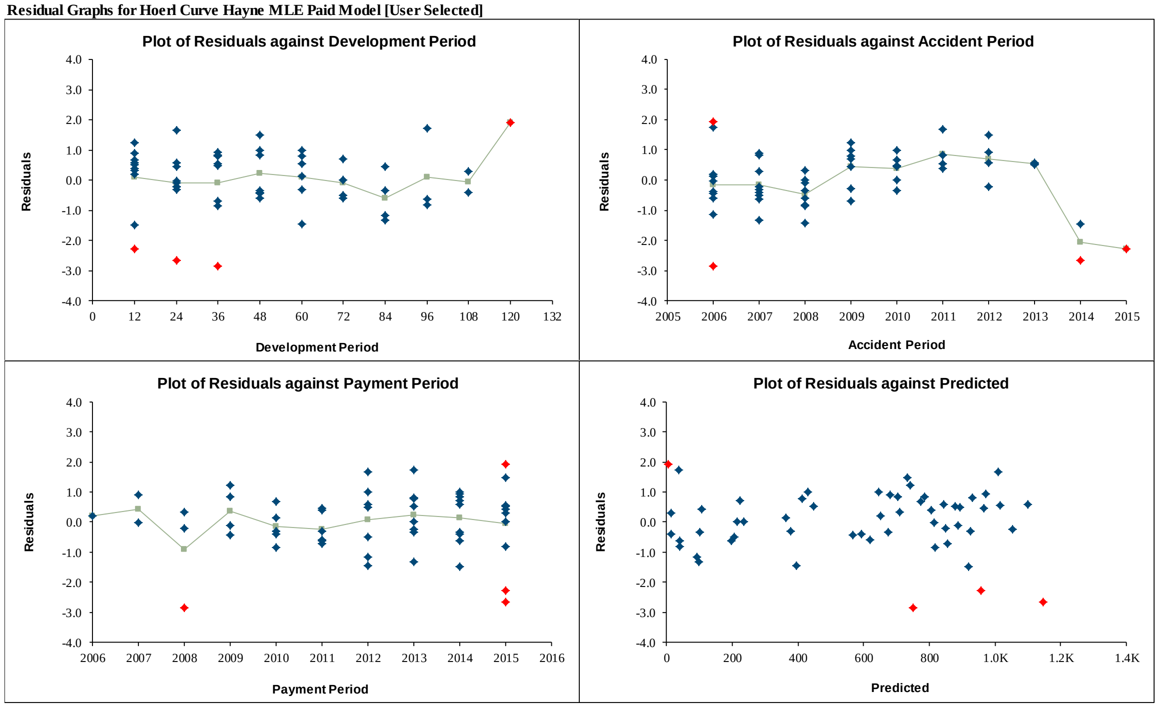


Figure A.22. Normality Plots for Hoerl Curve

Hoerl Curve Normality Plots (Paid)

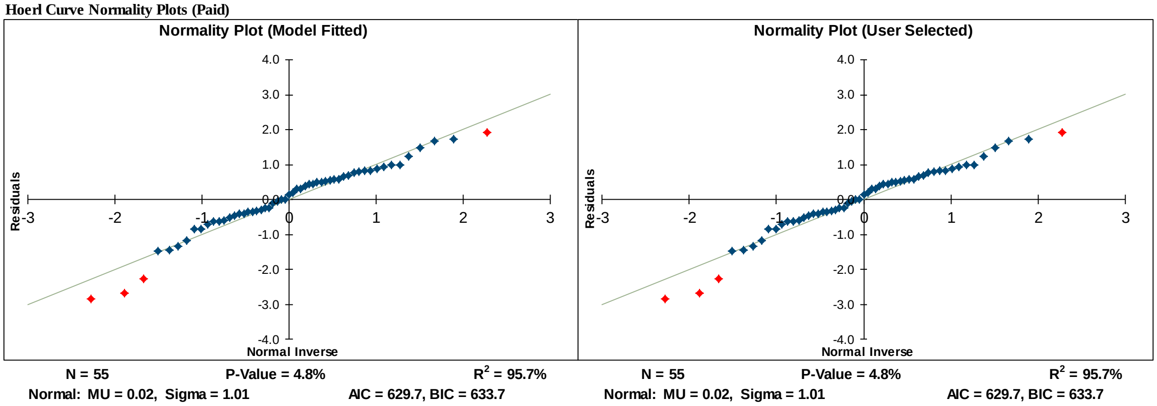


Figure A.23. Box-Whisker Plots for Hoerl Curve

Hoerl Curve Box-Whisker Plots (Paid)

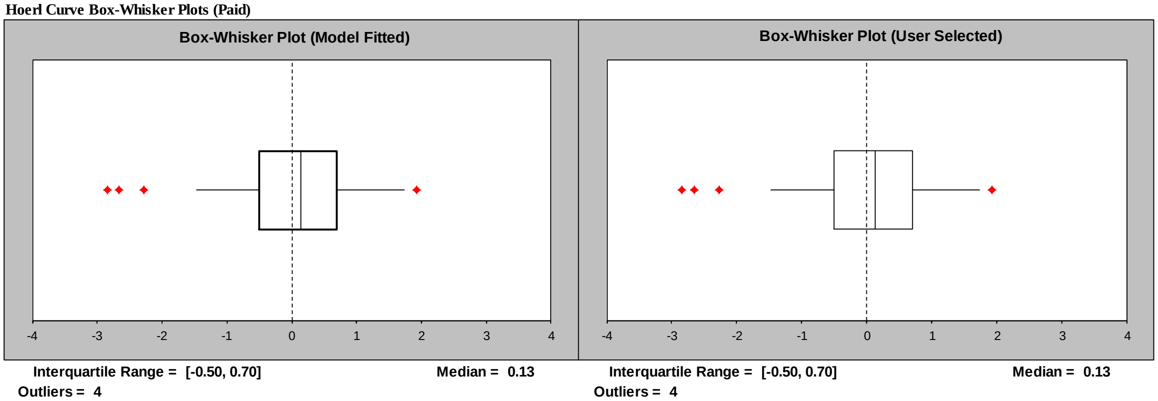


Figure A.24. Model Structure Graphs for Hoerl Curve

Hoerl Curve Model Structure [User Selected] (Paid)

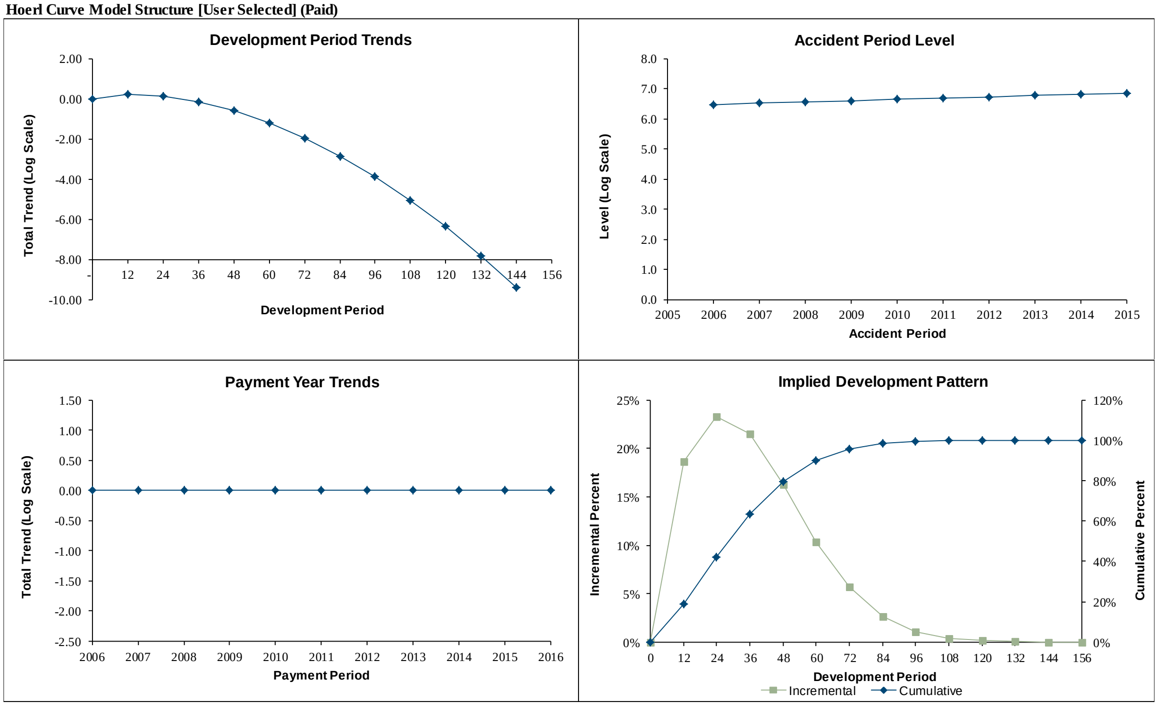


Figure A.25. User Selected Parameters for Wright

	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
User Selected Parameters:										
Mean	6.312	6.472	6.436	6.587	6.636	6.738	6.742	6.771	6.475	6.468
Std Dev	0.168	0.167	0.167	0.166	0.167	0.167	0.166	0.164	0.166	0.184
CoV	*2.7%*	*2.6%*	*2.6%*	*2.5%*	*2.5%*	*2.5%*	*2.5%*	*2.4%*	*2.6%*	*2.8%*

Development Period Parameters (Average Incremental)

	d	d²	ln(d)
Mean	0.192	(0.078)	0.290
Std Dev	0.183	0.015	0.232
CoV	*95.4%*	*-19.5%*	*80.0%*

	K	p	AIC	BIC	Tail Extrapolation Periods	Implied Tail Factor Adjusted	Implied Tail Factor Actual
Mean	14.592	0.319	612.3	642.4	3	1.0003	1.0003
Std Dev	0.909	0.075					
CoV	*6.2%*	*23.4%*					

Figure A.26. Residual Graphs for Wright [Modeled Parameters]

Residual Graphs for Wright Hayne MLE Paid Model [Model Fitted]

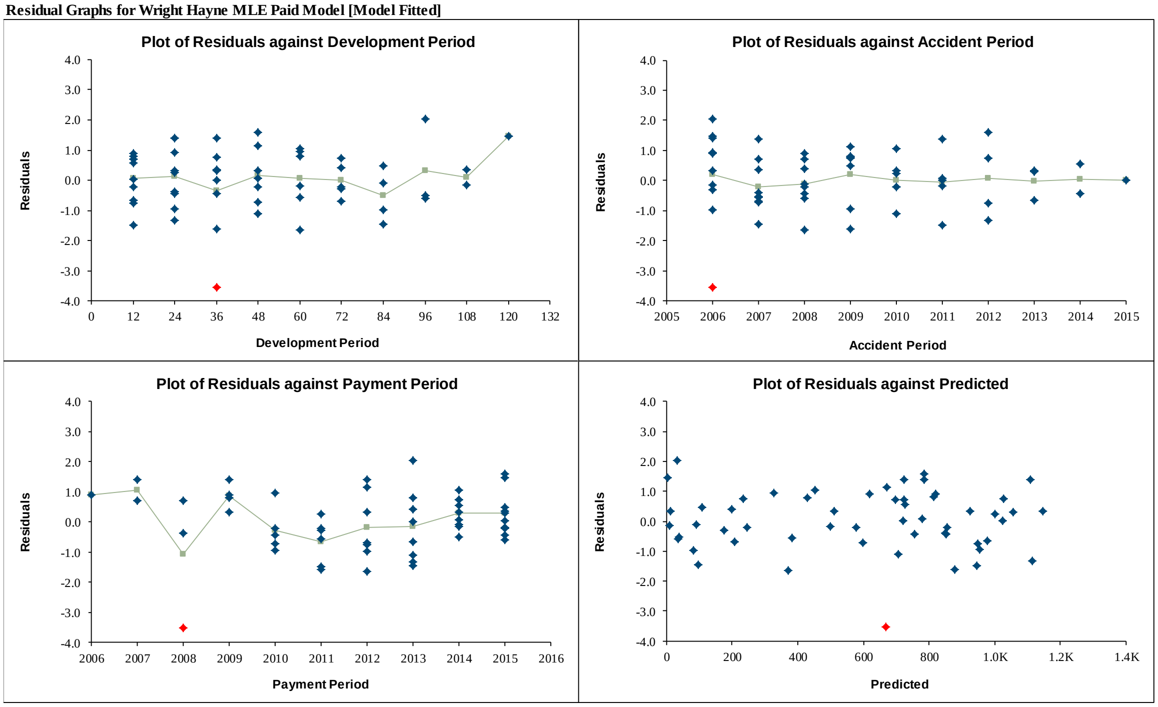


Figure A.27. Residual Graphs for Wright [Selected Parameters]

Residual Graphs for Wright Hayne MLE Paid Model [User Selected]

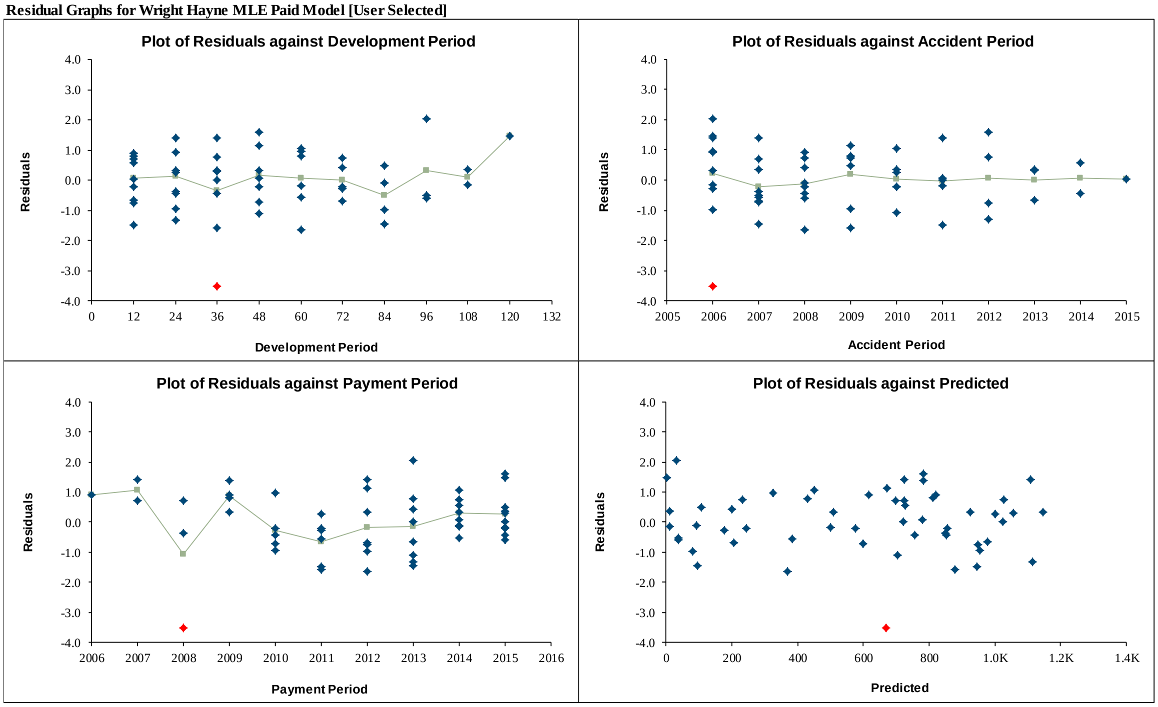
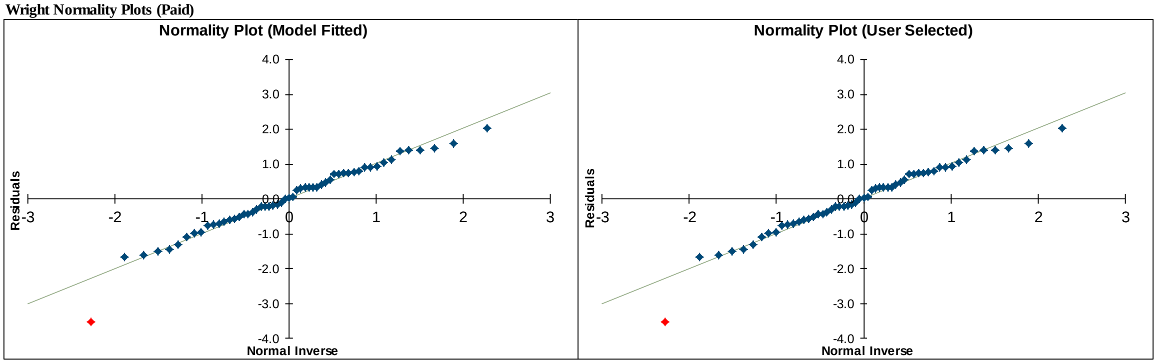


Figure A.28. Normality Plots for Wright

Wright Normality Plots (Paid)

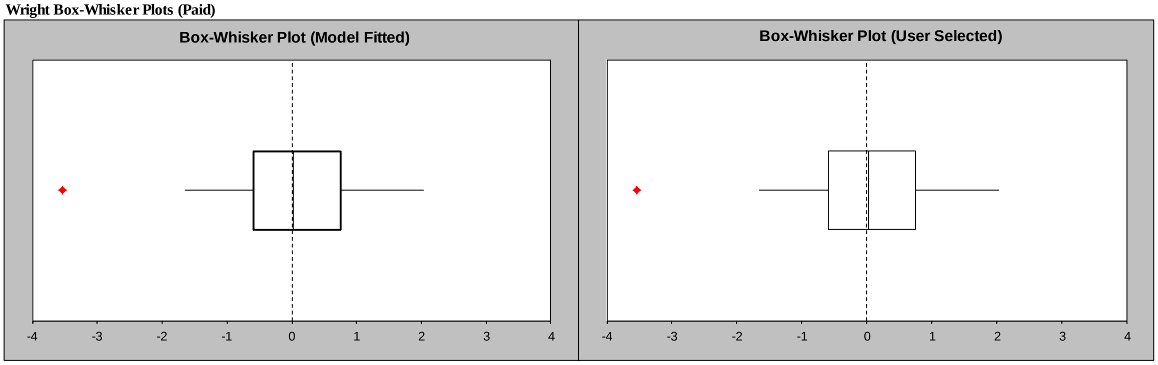


N = 55 P-Value = 6.5% R^2 = 96.1%
Normal: MU = 0.02, Sigma = 1.01 AIC = 586.3, BIC = 590.3

N = 55 P-Value = 6.5% R^2 = 96.1%
Normal: MU = 0.02, Sigma = 1.01 AIC = 586.3, BIC = 590.3

Figure A.29. Box-Whisker Plots for Wright

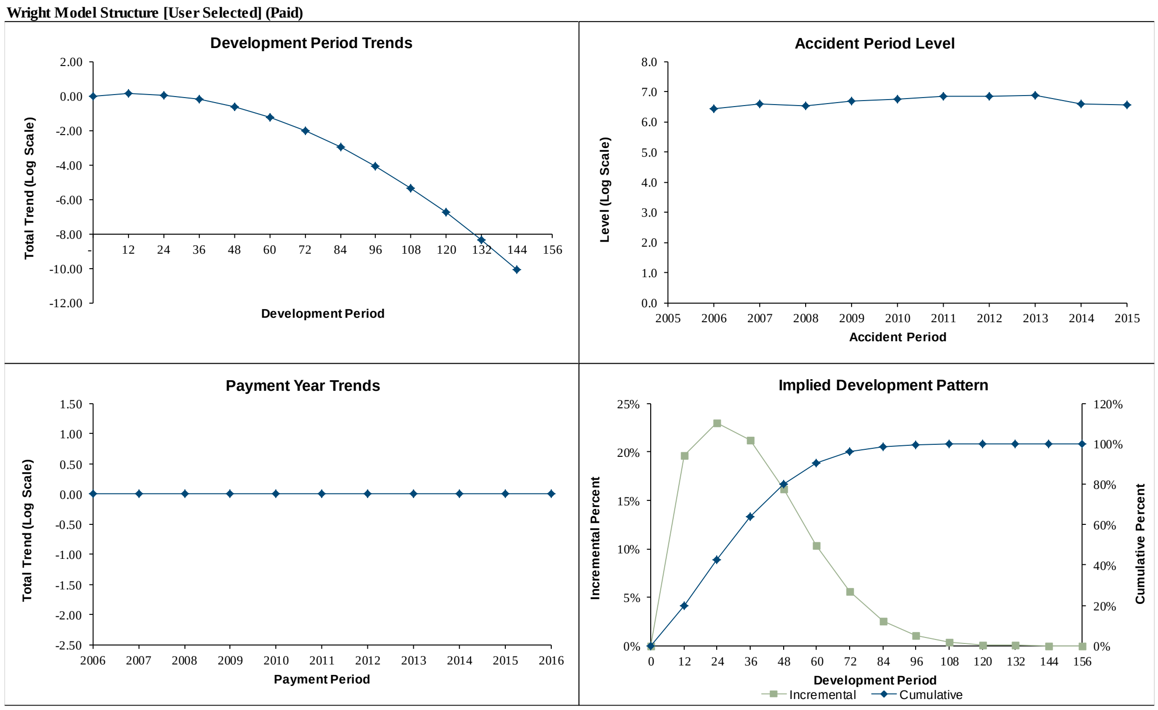
Wright Box-Whisker Plots (Paid)



Interquartile Range = [-0.59, 0.75] Median = 0.02
Outliers = 1

Interquartile Range = [-0.59, 0.75] Median = 0.02
Outliers = 1

Figure A.30. Model Structure Graphs for Wright



Appendix B – Schedule P, Part A Results

In this appendix the results for Schedule P, Part A (Homeowners / Farmowners) are shown.

Figure B.1. Estimated unpaid model results (Paid Berquist-Sherman)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Paid Berquist & Sherman Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	1	2	180.3%	(13)	12	1	3	6	9
2007	6,470	3	5	145.9%	(12)	25	3	6	11	19
2008	7,848	9	11	119.8%	(26)	45	8	15	27	36
2009	7,020	18	19	106.5%	(47)	103	18	30	50	65
2010	7,291	38	33	88.7%	(84)	218	38	59	94	118
2011	8,134	80	60	75.4%	(120)	263	79	120	177	219
2012	10,800	181	113	62.5%	(211)	575	181	253	362	478
2013	7,522	342	207	60.6%	(274)	1,106	343	470	707	810
2014	7,968	789	427	54.2%	(727)	2,126	800	1,062	1,461	1,789
2015	9,309	4,880	1,850	37.9%	(2,872)	11,865	4,846	6,061	7,993	9,246
Totals	77,596	6,340	1,916	30.2%	(896)	13,657	6,355	7,623	9,484	10,650
Normal Dist.		6,340	1,916	30.2%			6,340	7,632	9,491	10,797
logNormal Dist.		6,791	3,793	55.9%			5,929	8,426	13,971	19,927
Gamma Dist.		6,340	1,916	30.2%			6,149	7,507	9,785	11,624

Figure B.2. Total unpaid claims distribution (Paid Berquist-Sherman)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Total Unpaid Distribution (in 000's)
Paid Berquist & Sherman Model

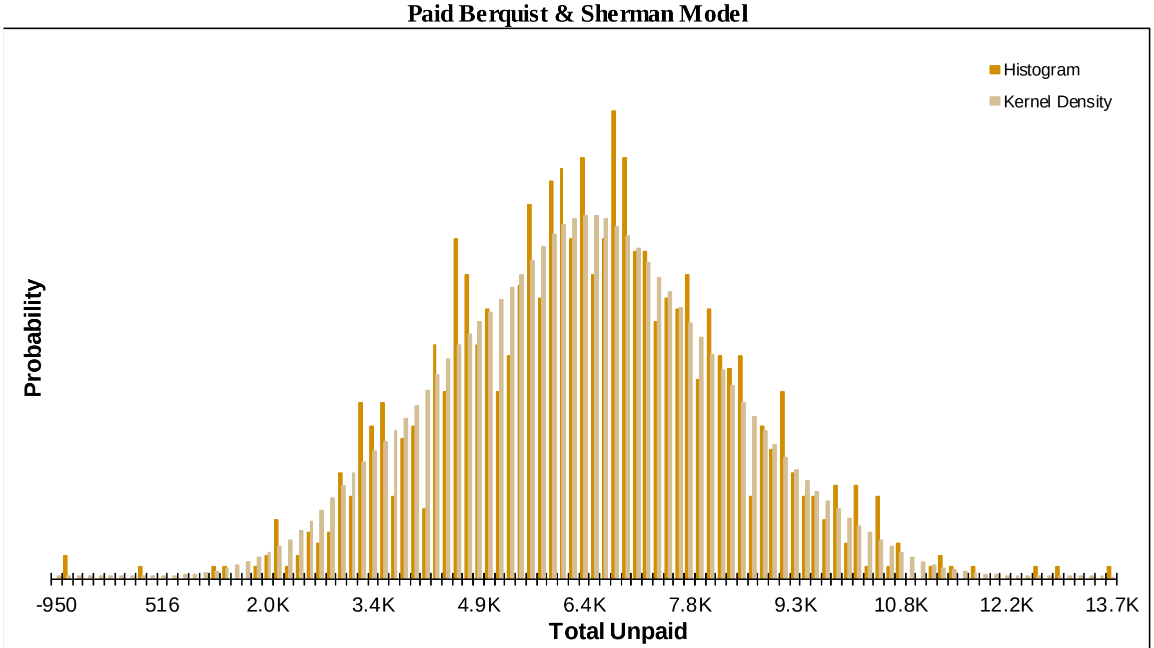


Figure B.3. Estimated unpaid model results (Incurred Berquist-Sherman)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Incurred Berquist & Sherman Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	1	4	296.4%	(54)	50	1	3	7	13
2007	6,470	3	9	267.7%	(172)	109	3	6	15	23
2008	7,848	10	35	354.9%	(735)	675	8	16	31	50
2009	7,020	21	41	189.4%	(106)	1,032	18	31	60	96
2010	7,291	44	138	311.7%	(155)	3,281	32	56	107	251
2011	8,134	82	105	129.2%	(1,215)	1,430	70	114	218	400
2012	10,800	181	289	159.6%	(5,037)	5,874	159	252	419	713
2013	7,522	339	684	201.7%	(12,497)	9,046	282	453	902	1,762
2014	7,968	794	2,795	351.9%	(63,725)	50,307	656	965	1,816	3,496
2015	9,309	4,260	2,334	54.8%	(695)	46,021	4,081	5,048	7,206	11,774
Totals	77,596	5,736	3,744	65.3%	(56,400)	54,796	5,441	6,633	9,385	14,683
Normal Dist.		5,736	3,744	65.3%			5,736	8,262	11,895	14,446
logNormal Dist.		6,881	6,211	90.3%			5,108	8,597	18,185	30,775
Gamma Dist.		5,736	3,744	65.3%			4,945	7,637	12,945	17,771

Figure B.4. Total unpaid claims distribution (Incurred Berquist-Sherman)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Total Unpaid Distribution (in 000's)
Incurred Berquist & Sherman Model

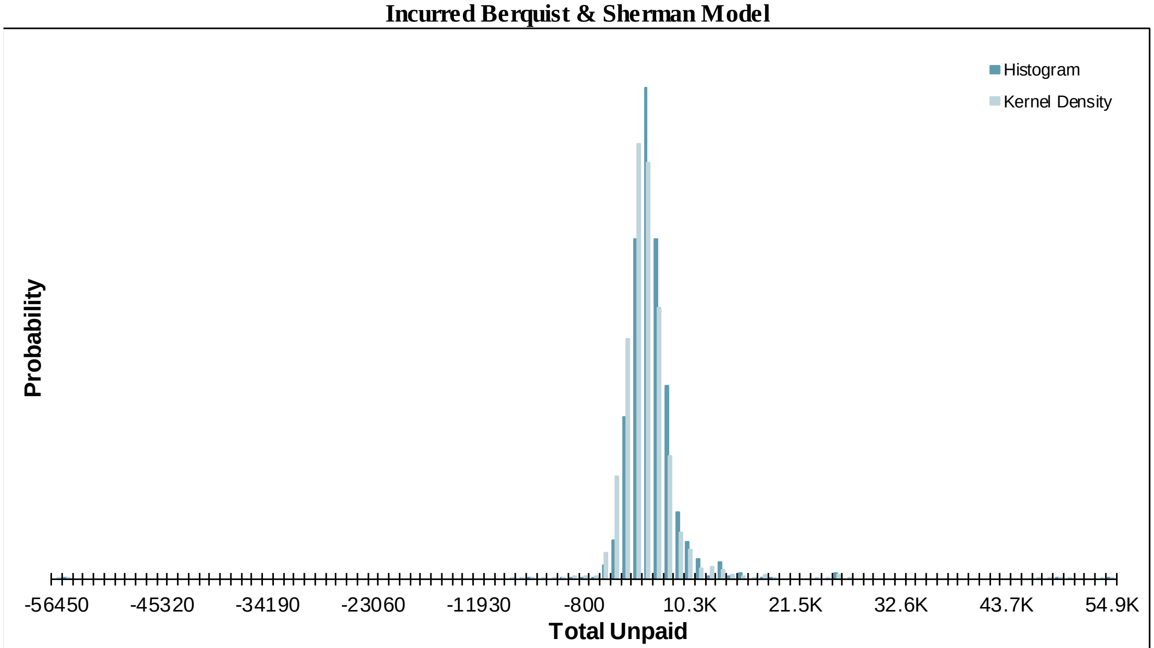


Figure B.5. Estimated unpaid model results (Paid Cape Cod)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Paid Cape Cod Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	2	2	145.7%	(7)	13	1	3	6	9
2007	6,470	4	4	112.8%	(8)	29	3	5	10	17
2008	7,848	10	9	88.7%	(19)	40	10	16	26	34
2009	7,020	21	16	76.6%	(32)	76	22	32	47	62
2010	7,291	40	28	70.0%	(50)	130	40	60	84	106
2011	8,134	81	48	59.7%	(88)	286	81	112	156	199
2012	10,800	240	119	49.6%	(124)	659	240	323	441	501
2013	7,522	298	157	52.7%	(398)	969	301	395	553	677
2014	7,968	717	336	46.8%	(322)	1,894	711	933	1,276	1,577
2015	9,309	3,937	1,416	36.0%	(6)	8,153	3,904	4,835	6,319	7,312
Totals	77,596	5,350	1,478	27.6%	1,256	10,155	5,392	6,272	7,856	8,680
Normal Dist.		5,350	1,478	27.6%			5,350	6,347	7,781	8,788
logNormal Dist.		5,374	1,707	31.8%			5,121	6,313	8,529	10,535
Gamma Dist.		5,350	1,478	27.6%			5,214	6,259	7,990	9,374

Figure B.6. Total unpaid claims distribution (Paid Cape Cod)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Total Unpaid Distribution (in 000's)
Paid Cape Cod Model

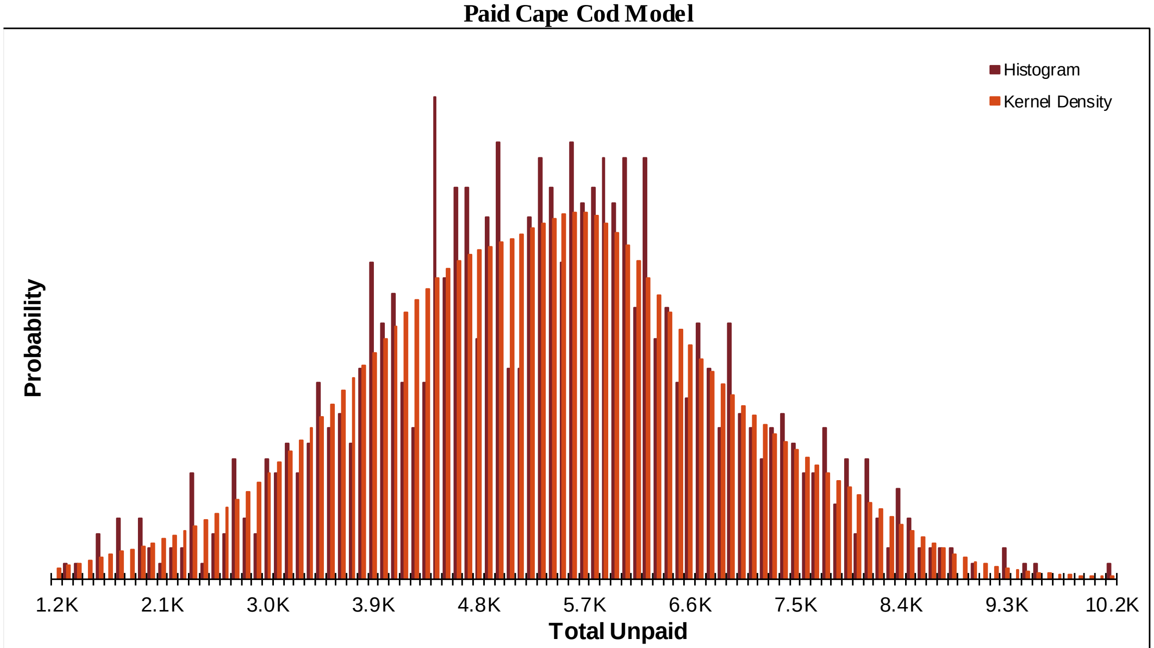


Figure B.7. Estimated unpaid model results (Incurred Cape Cod)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Incurred Cape Cod Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	1	3	185.0%	(28)	20	1	2	6	10
2007	6,470	3	10	283.2%	(235)	59	3	6	11	25
2008	7,848	11	13	120.2%	(160)	154	10	17	30	49
2009	7,020	26	28	110.4%	(136)	428	23	36	65	111
2010	7,291	50	114	226.4%	(72)	2,555	40	63	113	204
2011	8,134	92	99	107.8%	(1,066)	1,254	79	122	214	385
2012	10,800	211	164	78.0%	(72)	3,242	189	270	452	668
2013	7,522	297	698	234.9%	(16,494)	5,425	272	404	768	1,308
2014	7,968	1,315	15,993	1216.4%	(9,454)	498,887	652	944	1,711	2,637
2015	9,309	3,884	1,745	44.9%	(5)	21,243	3,736	4,672	6,505	9,280
Totals	77,596	5,890	16,156	274.3%	(12,718)	504,979	5,150	6,196	8,438	11,426
Normal Dist.		5,890	16,156	274.3%			5,890	16,788	32,465	43,476
logNormal Dist.		5,943	3,903	65.7%			4,967	7,439	13,302	20,006
Gamma Dist.		5,890	16,156	274.3%			150	3,384	33,123	80,833

Figure B.8. Total unpaid claims distribution (Incurred Cape Cod)

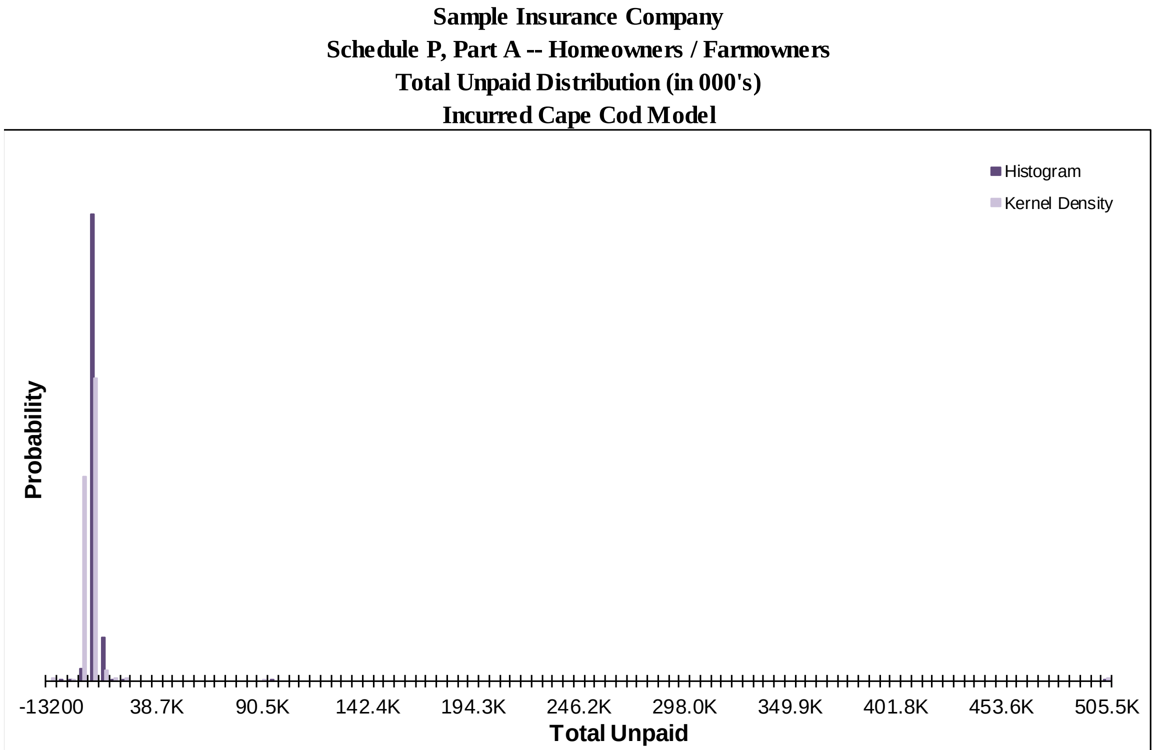


Figure B.9. Estimated unpaid model results (Paid Chain Ladder)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Paid Chain Ladder Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	12	11	93.8%	(16)	66	10	17	32	45
2007	6,470	23	17	73.8%	(21)	98	22	33	53	71
2008	7,848	44	23	52.8%	(18)	131	42	59	86	104
2009	7,020	53	25	47.1%	(13)	165	51	70	95	116
2010	7,291	75	29	38.3%	(4)	188	74	95	125	146
2011	8,134	125	36	28.9%	(6)	259	125	149	183	215
2012	10,800	244	57	23.3%	60	413	245	282	339	372
2013	7,522	311	68	21.7%	55	506	311	358	417	451
2014	7,968	698	113	16.1%	355	1,036	694	771	881	969
2015	9,309	3,841	364	9.5%	2,667	4,806	3,832	4,091	4,437	4,673
Totals	77,596	5,425	443	8.2%	3,925	6,815	5,422	5,731	6,159	6,411
Normal Dist.		5,425	443	8.2%			5,425	5,724	6,154	6,456
logNormal Dist.		5,425	449	8.3%			5,407	5,717	6,194	6,552
Gamma Dist.		5,425	443	8.2%			5,413	5,717	6,174	6,509

Figure B.10. Total unpaid claims distribution (Paid Chain Ladder)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Total Unpaid Distribution (in 000's)
Paid Chain Ladder Model

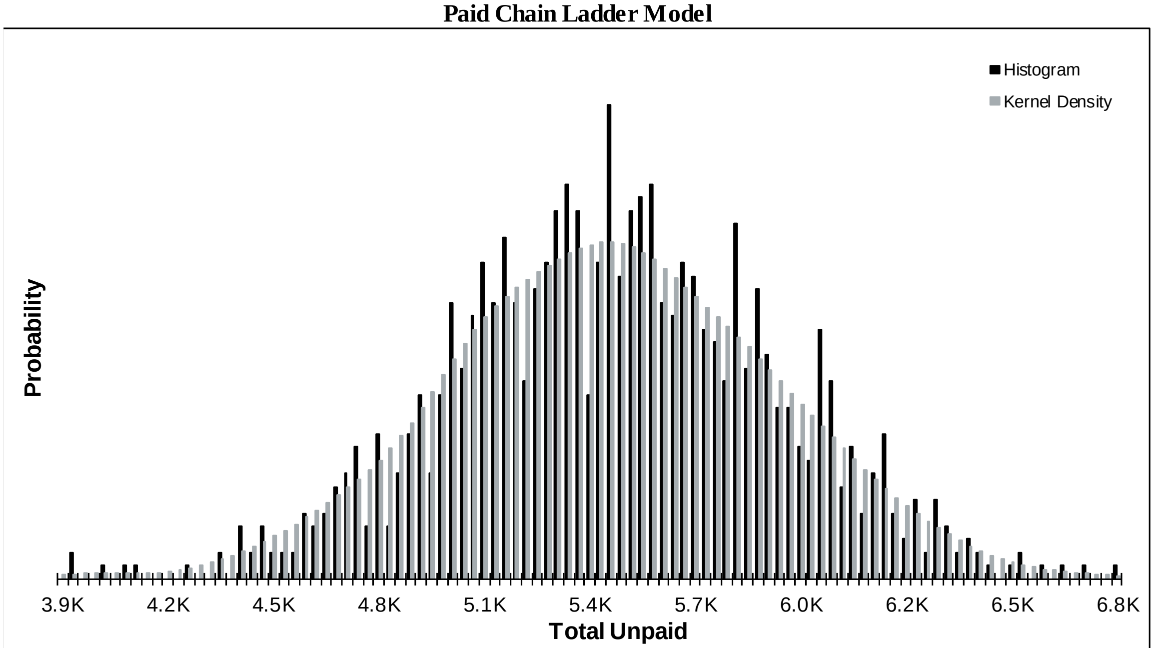


Figure B.11. Estimated unpaid model results (Incurred Chain Ladder)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Incurred Chain Ladder Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	12	11	97.4%	(17)	86	10	17	31	45
2007	6,470	23	18	77.9%	(14)	126	21	33	56	71
2008	7,848	43	24	55.5%	(17)	129	41	59	87	109
2009	7,020	52	28	53.3%	(16)	178	49	68	99	135
2010	7,291	74	32	43.2%	(3)	240	71	94	131	161
2011	8,134	124	42	34.4%	(6)	259	122	150	198	237
2012	10,800	243	68	28.0%	45	470	242	287	362	402
2013	7,522	304	87	28.5%	42	633	300	357	459	528
2014	7,968	704	157	22.4%	240	1,476	697	793	969	1,130
2015	9,309	3,701	596	16.1%	1,605	5,935	3,684	4,060	4,695	5,169
Totals	77,596	5,279	651	12.3%	3,057	7,701	5,258	5,697	6,332	6,838
Normal Dist.		5,279	651	12.3%			5,279	5,718	6,349	6,793
logNormal Dist.		5,279	664	12.6%			5,238	5,700	6,437	7,010
Gamma Dist.		5,279	651	12.3%			5,252	5,702	6,393	6,910

Figure B.12. Total unpaid claims distribution (Incurred Chain Ladder)

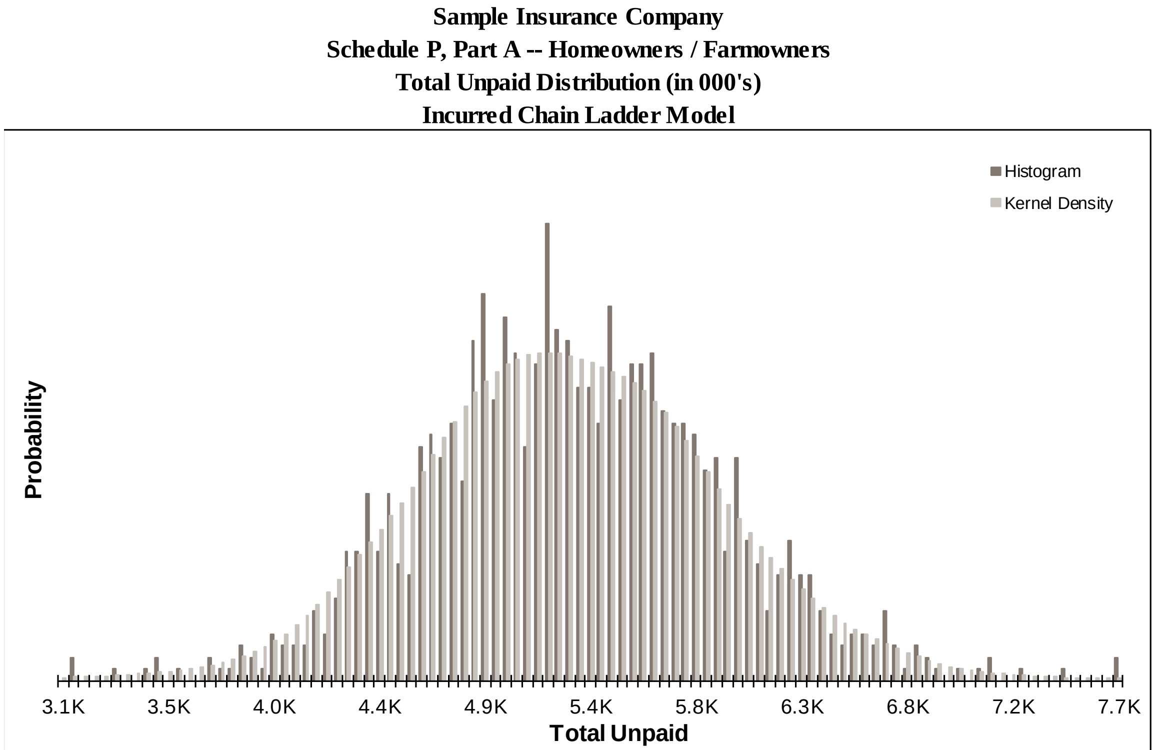


Figure B.13. Estimated unpaid model results (Paid Hoerl Curve)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Paid Hoerl Curve Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	-	-		-	-	-	-	-	-
2007	6,470	47	29	63.3%	(37)	170	44	63	100	129
2008	7,848	79	42	52.3%	(47)	262	75	102	154	208
2009	7,020	97	45	46.3%	(42)	291	93	124	173	224
2010	7,291	111	46	41.7%	(34)	329	106	140	193	232
2011	8,134	148	56	38.0%	2	396	142	180	250	317
2012	10,800	236	71	30.0%	35	523	233	277	361	422
2013	7,522	320	78	24.5%	21	613	318	368	452	502
2014	7,968	798	137	17.2%	345	1,259	796	888	1,028	1,127
2015	9,309	4,428	451	10.2%	3,062	5,849	4,422	4,724	5,179	5,546
Totals	77,596	6,264	616	9.8%	4,469	8,520	6,225	6,665	7,308	7,868
Normal Dist.		6,264	616	9.8%			6,264	6,680	7,278	7,698
logNormal Dist.		6,264	618	9.9%			6,234	6,662	7,329	7,837
Gamma Dist.		6,264	616	9.8%			6,244	6,668	7,311	7,786

Figure B.14. Total unpaid claims distribution (Paid Hoerl Curve)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Total Unpaid Distribution (in 000's)
Paid Hoerl Curve Model

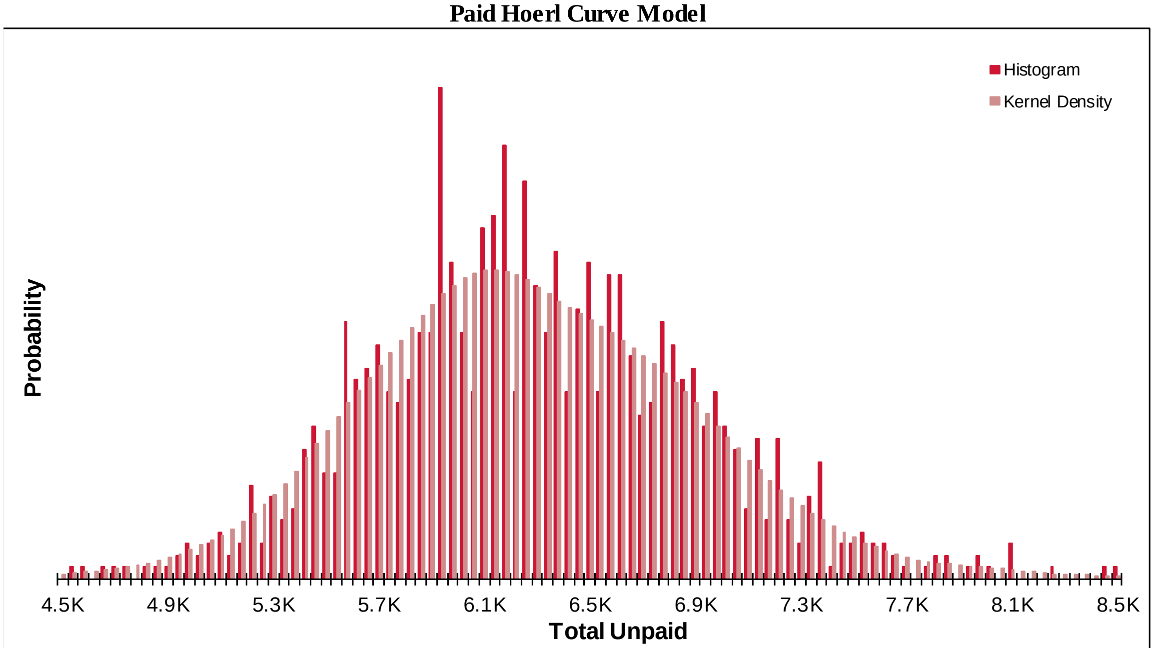


Figure B.15. Estimated unpaid model results (Incurred Hoerl Curve)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Incurred Hoerl Curve Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	-	-		-	-	-	-	-	-
2007	6,470	47	31	64.7%	(45)	177	44	64	101	137
2008	7,848	80	42	52.7%	(50)	278	76	103	155	205
2009	7,020	96	45	47.2%	(37)	317	91	124	175	220
2010	7,291	110	47	42.5%	(31)	340	105	136	191	237
2011	8,134	145	56	38.6%	2	423	140	177	243	307
2012	10,800	229	69	30.3%	36	532	225	269	348	405
2013	7,522	305	78	25.6%	22	574	302	353	441	497
2014	7,968	759	137	18.1%	349	1,500	756	844	991	1,102
2015	9,309	4,140	424	10.2%	3,012	5,953	4,134	4,401	4,861	5,250
Totals	77,596	5,911	554	9.4%	4,278	8,496	5,876	6,251	6,842	7,399
Normal Dist.		5,911	554	9.4%			5,911	6,285	6,822	7,199
logNormal Dist.		5,911	553	9.4%			5,885	6,268	6,862	7,313
Gamma Dist.		5,911	554	9.4%			5,894	6,275	6,850	7,275

Figure B.16. Total unpaid claims distribution (Incurred Hoerl Curve)

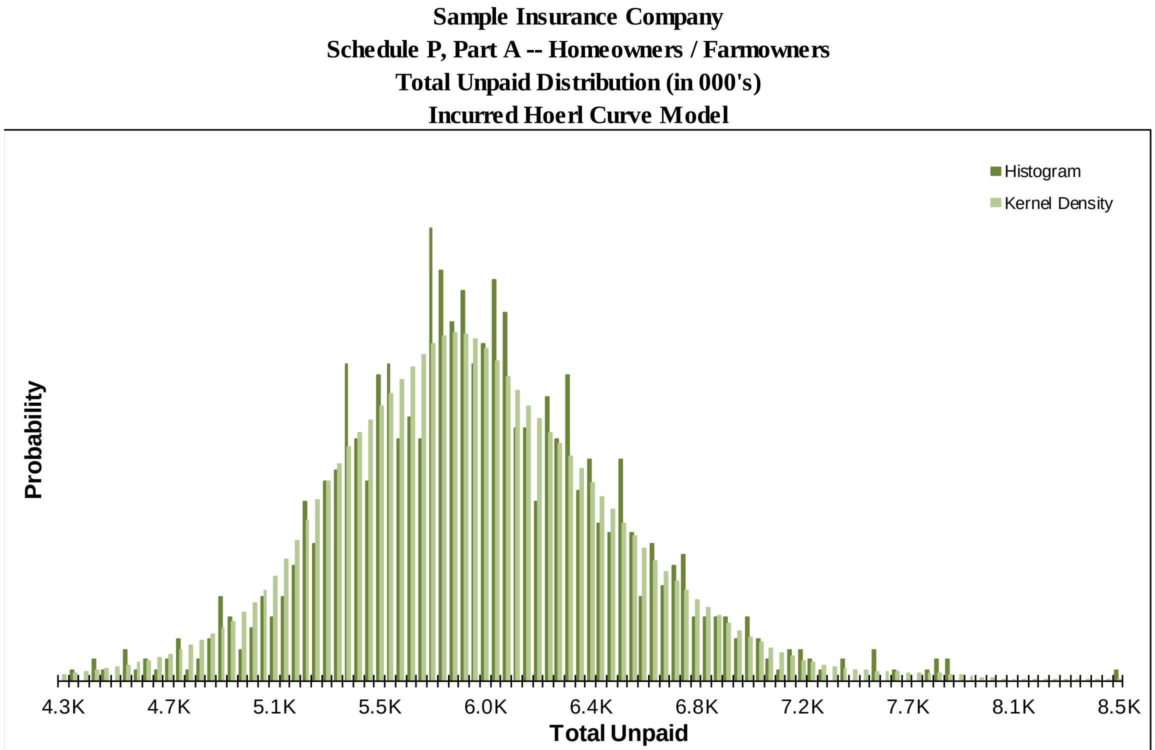


Figure B.17. Estimated unpaid model results (Paid Wright)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Paid Wright Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	-	-		-	-	-	-	-	-
2007	6,470	47	31	65.1%	(31)	194	44	65	105	138
2008	7,848	83	44	52.6%	(51)	281	81	107	161	210
2009	7,020	93	45	48.5%	(30)	262	86	118	176	212
2010	7,291	111	49	43.9%	(5)	346	106	143	195	236
2011	8,134	150	58	38.9%	(17)	399	147	185	252	304
2012	10,800	265	81	30.6%	56	615	257	312	411	480
2013	7,522	304	75	24.8%	41	603	300	353	430	487
2014	7,968	791	124	15.7%	373	1,197	788	868	993	1,077
2015	9,309	3,905	343	8.8%	2,992	4,995	3,902	4,135	4,465	4,703
Totals	77,596	5,750	514	8.9%	4,193	7,586	5,711	6,056	6,678	7,075
Normal Dist.		5,750	514	8.9%			5,750	6,096	6,595	6,946
logNormal Dist.		5,750	514	8.9%			5,727	6,082	6,632	7,048
Gamma Dist.		5,750	514	8.9%			5,734	6,088	6,621	7,013

Figure B.18. Total unpaid claims distribution (Paid Wright)

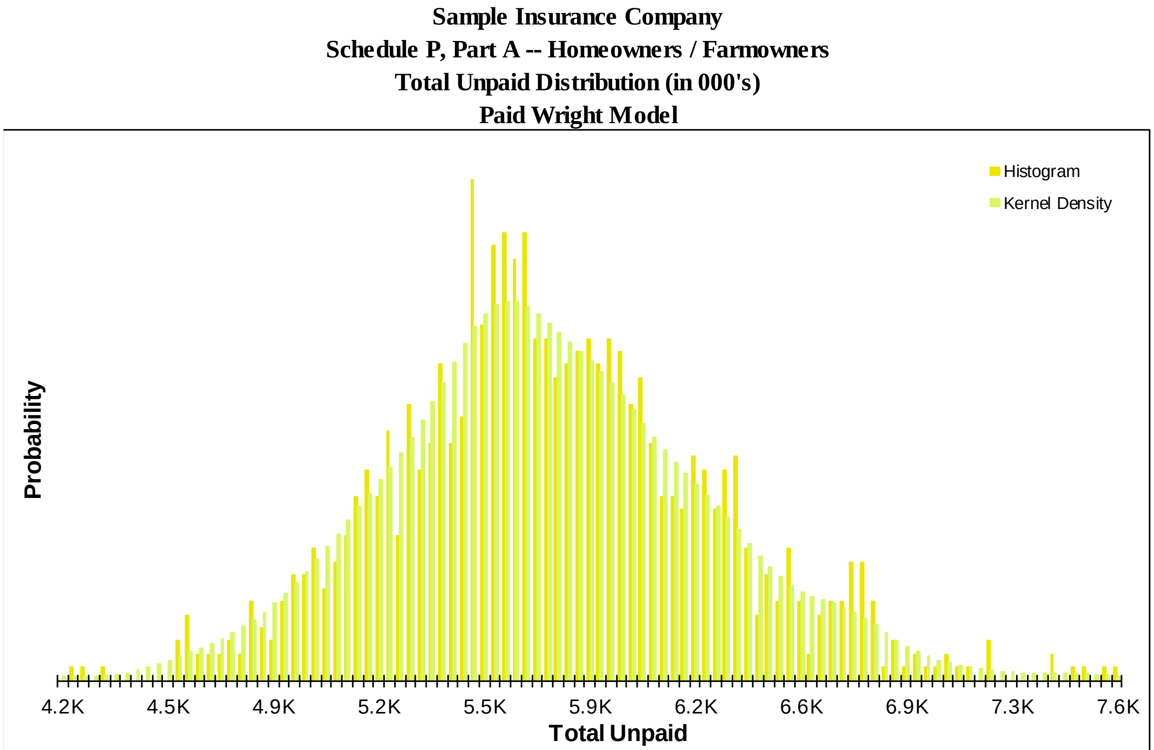


Figure B.19. Estimated unpaid model results (Incurred Wright)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Incurred Wright Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	-	-		-	-	-	-	-	-
2007	6,470	44	28	63.3%	(39)	192	41	61	93	122
2008	7,848	81	43	52.5%	(39)	283	77	104	160	205
2009	7,020	94	47	50.6%	(46)	358	88	118	180	236
2010	7,291	115	51	44.0%	(14)	392	111	146	204	256
2011	8,134	154	58	37.5%	(8)	497	150	185	253	322
2012	10,800	251	72	28.7%	55	530	248	296	383	442
2013	7,522	297	73	24.4%	86	540	295	346	416	477
2014	7,968	777	114	14.7%	412	1,187	775	854	971	1,053
2015	9,309	3,812	266	7.0%	3,013	4,733	3,808	3,988	4,264	4,448
Totals	77,596	5,625	440	7.8%	4,333	7,210	5,605	5,887	6,392	6,829
Normal Dist.		5,625	440	7.8%			5,625	5,923	6,350	6,650
logNormal Dist.		5,625	438	7.8%			5,608	5,911	6,374	6,721
Gamma Dist.		5,625	440	7.8%			5,614	5,916	6,369	6,701

Figure B.20. Total unpaid claims distribution (Incurred Wright)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Total Unpaid Distribution (in 000's)
Incurred Wright Model

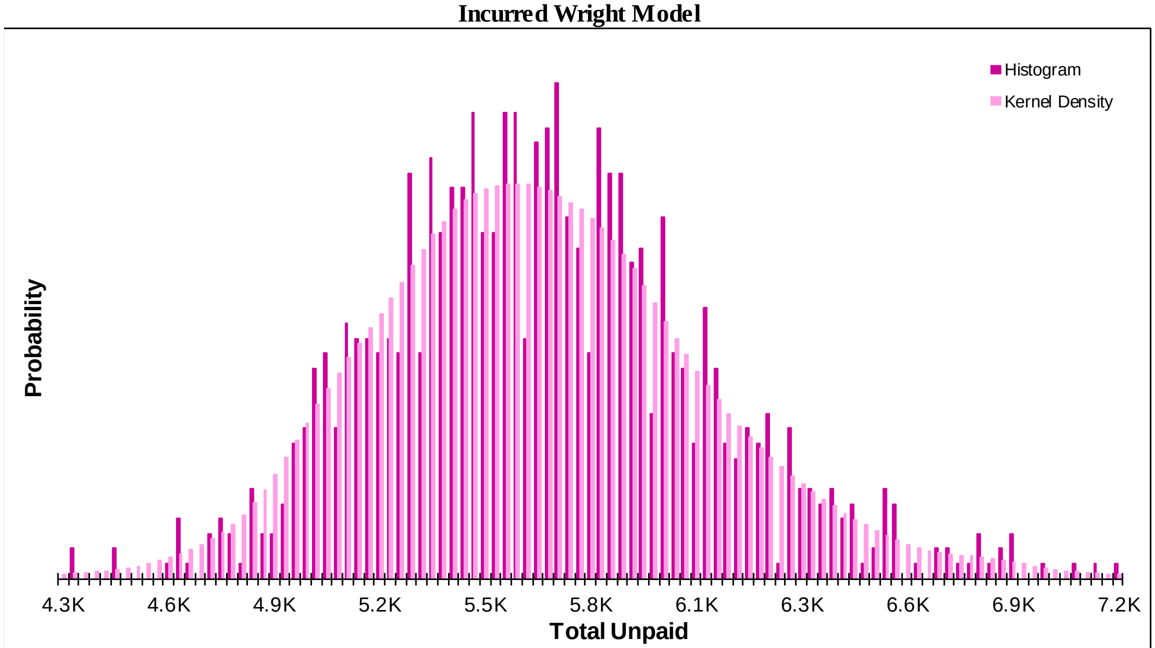


Figure B.21. Model weights by accident year

Accident Year	Model Weights by Accident Year										
	Paid BS	Incd BS	Paid CC	Incd CC	Paid CL	Incd CL	Paid HC	Incd HC	Paid WR	Incd WR	TOTAL
2006	40.0%		30.0%		30.0%						100.0%
2007	40.0%		30.0%		30.0%						100.0%
2008	40.0%		30.0%		30.0%						100.0%
2009	40.0%		30.0%		30.0%						100.0%
2010	40.0%		30.0%		30.0%						100.0%
2011	40.0%		30.0%		30.0%						100.0%
2012	40.0%		30.0%		30.0%						100.0%
2013	40.0%		30.0%		30.0%						100.0%
2014	40.0%		30.0%		30.0%						100.0%
2015	40.0%		30.0%		30.0%						100.0%

Figure B.22. Estimated mean unpaid by model

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Summary of Results by Model (in 000's)

Accident	Mean Estimated Unpaid										
	Berquist & Sherman		Cape Cod		Chain Ladder		Hoerl Curve		Wright		Best Est.
Year	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	(Weighted)
2006	1	1	2	1	12	12	-	-	-	-	5
2007	3	3	4	3	23	23	47	47	47	44	9
2008	9	10	10	11	44	43	79	80	83	81	20
2009	18	21	21	26	53	52	97	96	93	94	29
2010	38	44	40	50	75	74	111	110	111	115	49
2011	80	82	81	92	125	124	148	145	150	154	94
2012	181	181	240	211	244	243	236	229	265	251	217
2013	342	339	298	297	311	304	320	305	304	297	318
2014	789	794	717	1,315	698	704	798	759	791	777	739
2015	4,880	4,260	3,937	3,884	3,841	3,701	4,428	4,140	3,905	3,812	4,312
Totals	6,340	5,736	5,350	5,890	5,425	5,279	6,264	5,911	5,750	5,625	5,792

Figure B.23. Estimated ranges

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Summary of Results by Model (in 000's)

Accident	Best Est.	Ranges			
		Weighted		Modeled	
Year	(Weighted)	Minimum	Maximum	Mininum	Maximum
2006	5	1	12	1	12
2007	9	3	23	3	23
2008	20	9	44	9	44
2009	29	18	53	18	53
2010	49	38	75	38	75
2011	94	80	125	80	125
2012	217	181	244	181	244
2013	318	298	342	298	342
2014	739	698	789	698	789
2015	4,312	3,841	4,880	3,841	4,880
Totals	5,792	5,166	6,587	5,350	6,340

Figure B.24. Reconciliation of total results (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Reconciliation of Total Results (in 000's)
Best Estimate (Weighted)

Accident Year	Paid To Date	Incurred To Date	Case Reserves	IBNR	Estimate of Ultimate	Estimate of Unpaid
2006	5,234	5,237	3	2	5,239	5
2007	6,470	6,479	9	1	6,480	9
2008	7,848	7,867	19	1	7,868	20
2009	7,020	7,046	26	3	7,050	29
2010	7,291	7,341	50	(1)	7,340	49
2011	8,134	8,225	91	3	8,228	94
2012	10,800	11,085	285	(68)	11,017	217
2013	7,522	7,810	288	30	7,840	318
2014	7,968	8,703	735	4	8,707	739
2015	9,309	12,788	3,478	834	13,621	4,312
Totals	77,596	82,580	4,984	808	83,388	5,792

Figure B.25. Estimated unpaid model results (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Unpaid (in 000's)
Best Estimate (Weighted)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5,234	5	8	169.9%	(37)	63	2	6	21	35
2007	6,470	9	14	148.1%	(30)	103	4	11	40	59
2008	7,848	20	22	110.9%	(38)	156	13	28	65	92
2009	7,020	29	25	85.5%	(71)	227	26	43	76	99
2010	7,291	49	35	70.7%	(90)	210	49	72	107	133
2011	8,134	94	55	58.3%	(132)	318	96	130	180	219
2012	10,800	217	106	49.0%	(281)	659	222	284	385	478
2013	7,522	318	162	51.0%	(438)	1,177	314	400	600	759
2014	7,968	739	335	45.3%	(1,016)	2,588	719	903	1,341	1,678
2015	9,309	4,312	1,512	35.1%	(2,872)	12,591	4,060	5,087	7,090	8,710
Totals	77,596	5,792	1,571	27.1%	(800)	14,273	5,568	6,609	8,652	10,410
Normal Dist.		5,792	1,571	27.1%			5,792	6,851	8,375	9,446
logNormal Dist.		5,846	1,890	32.3%			5,562	6,880	9,343	11,583
Gamma Dist.		5,792	1,571	27.1%			5,651	6,760	8,594	10,056

Figure B.26. Estimated cash flow (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Calendar Year Unpaid (in 000's)
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	3,871	1,443	37.3%	(3,836)	11,833	3,690	4,610	6,468	7,923
2017	959	413	43.0%	(878)	3,461	923	1,183	1,709	2,114
2018	446	221	49.6%	(643)	1,488	429	563	848	1,073
2019	214	113	52.7%	(371)	721	208	279	408	514
2020	124	69	55.9%	(183)	529	122	165	240	304
2021	72	44	61.1%	(103)	286	71	99	146	185
2022	44	29	66.8%	(83)	180	43	62	94	120
2023	28	22	78.5%	(51)	167	26	40	66	88
2024	16	16	104.0%	(38)	132	12	23	47	68
2025	10	12	124.9%	(26)	125	7	14	33	51
2026	6	9	155.5%	(34)	87	3	9	24	39
2027	3	7	220.4%	(23)	100	1	3	16	29
Totals	5,792	1,571	27.1%	(800)	14,273	5,568	6,609	8,652	10,410

Figure B.27. Estimated loss ratio (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Ultimate Loss Ratios (in 000's)
Best Estimate (Weighted)

Accident Year	Earned Premium	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	7,878	69.8%	21.6%	31.0%	-26.9%	168.0%	67.7%	80.4%	108.4%	129.3%
2007	8,257	79.7%	22.8%	28.6%	-29.4%	192.7%	78.8%	90.5%	120.1%	141.0%
2008	8,812	89.6%	24.5%	27.4%	-11.8%	254.9%	89.0%	100.9%	132.4%	155.1%
2009	9,823	75.4%	22.6%	29.9%	-57.7%	189.6%	73.1%	86.1%	116.6%	138.5%
2010	11,499	66.1%	20.0%	30.3%	-44.6%	173.3%	64.5%	75.4%	101.7%	122.0%
2011	12,965	65.2%	19.1%	29.4%	-37.5%	169.8%	64.0%	74.1%	99.1%	117.4%
2012	13,875	84.3%	25.1%	29.8%	-32.7%	231.7%	80.7%	96.3%	130.5%	157.9%
2013	14,493	57.6%	18.6%	32.3%	-36.6%	160.7%	55.3%	66.5%	91.7%	109.1%
2014	15,202	60.4%	19.3%	32.0%	-17.0%	175.8%	58.1%	70.0%	95.2%	114.0%
2015	15,148	96.2%	26.7%	27.7%	-16.9%	235.7%	90.5%	110.0%	146.5%	172.7%
Totals	117,952	74.0%	7.2%	9.7%	47.5%	109.0%	73.9%	78.6%	85.9%	91.8%

Figure B.28. Estimated unpaid claim runoff (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Calendar Year Unpaid Claim Runoff (in 000's)
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	5,792	1,571	27.1%	(800)	14,273	5,568	6,609	8,652	10,410
2016	1,920	505	26.3%	(102)	4,329	1,876	2,206	2,823	3,308
2017	961	275	28.6%	(206)	2,428	950	1,116	1,441	1,713
2018	515	159	30.8%	(97)	1,203	515	616	779	913
2019	301	110	36.5%	(101)	828	299	371	485	574
2020	178	80	45.3%	(135)	674	171	225	321	402
2021	106	62	58.8%	(132)	515	98	139	221	297
2022	62	48	77.6%	(74)	417	51	84	153	217
2023	34	35	102.2%	(96)	305	24	47	103	156
2024	18	23	123.2%	(58)	243	11	26	63	100
2025	9	14	154.2%	(38)	162	4	12	37	61
2026	3	7	220.4%	(23)	100	1	3	16	29

Figure B.29. Mean of incremental values (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Mean Values (in 000's)												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156
2006	3,865	1,191	233	103	43	25	14	8	6	3	2	1	1
2007	4,622	1,425	285	123	53	30	17	10	7	3	2	2	1
2008	5,563	1,705	335	148	61	35	20	12	9	4	3	2	2
2009	5,203	1,608	317	138	59	33	18	11	8	4	3	2	2
2010	5,342	1,647	323	144	61	34	19	11	8	4	3	2	2
2011	5,969	1,800	359	159	67	38	22	13	9	4	3	2	2
2012	8,260	2,509	495	217	91	51	29	18	12	6	4	3	2
2013	5,857	1,818	356	160	67	37	21	13	9	4	3	2	2
2014	6,467	1,975	393	172	73	41	24	14	10	5	3	2	2
2015	10,266	3,145	620	276	114	64	37	22	15	8	5	4	3

Figure B.30. Standard deviation of incremental values (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Standard Error Values (in 000's)												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156
2006	1,557	610	166	88	43	27	17	11	8	5	4	3	3
2007	1,742	676	187	99	49	30	19	12	9	6	5	4	4
2008	2,010	765	209	111	55	34	22	14	11	7	6	5	4
2009	2,042	785	212	111	55	34	21	14	10	7	5	4	4
2010	2,106	807	223	116	59	35	22	14	11	7	5	4	4
2011	2,300	869	240	127	63	38	24	15	12	8	6	5	4
2012	3,134	1,159	309	158	80	48	30	20	14	9	8	6	5
2013	2,476	953	263	139	68	41	26	16	12	8	6	5	4
2014	2,723	1,050	284	148	74	44	28	18	13	8	6	5	5
2015	3,609	1,403	374	200	97	59	36	23	17	12	9	8	7

Figure B.31. Coefficient of variation of incremental values (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Coefficients of Variation												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156
2006	40.3%	51.2%	71.1%	85.1%	100.7%	107.7%	120.5%	128.4%	136.7%	174.4%	200.4%	226.6%	249.8%
2007	37.7%	47.5%	65.4%	80.9%	92.4%	99.3%	112.8%	119.4%	129.6%	172.6%	195.5%	218.5%	244.5%
2008	36.1%	44.9%	62.3%	75.2%	89.6%	95.6%	107.4%	114.5%	123.2%	165.8%	185.0%	206.7%	235.0%
2009	39.2%	48.8%	66.8%	80.7%	93.0%	101.2%	114.9%	121.9%	127.6%	172.6%	189.1%	214.2%	239.5%
2010	39.4%	49.0%	68.8%	80.7%	97.1%	104.3%	115.3%	123.1%	130.5%	172.1%	192.4%	210.9%	237.6%
2011	38.5%	48.3%	66.8%	79.8%	94.1%	99.3%	112.5%	120.7%	128.3%	168.0%	185.7%	209.3%	236.7%
2012	37.9%	46.2%	62.3%	72.7%	88.2%	94.8%	102.4%	109.4%	117.7%	159.1%	176.0%	198.6%	228.9%
2013	42.3%	52.4%	73.9%	86.7%	101.8%	110.0%	119.8%	129.4%	136.3%	170.6%	190.1%	213.5%	244.3%
2014	42.1%	53.2%	72.3%	85.8%	101.0%	107.9%	117.3%	125.6%	133.1%	169.4%	191.4%	212.9%	232.3%
2015	35.2%	44.6%	60.4%	72.2%	85.1%	90.8%	99.1%	104.9%	115.8%	153.8%	171.3%	193.9%	220.4%

Figure B.32. Total unpaid claims distribution (weighted)

Sample Insurance Company
Schedule P, Part A -- Homeowners / Farmowners
Total Unpaid Distribution (in 000's)
Best Estimate (Weighted)

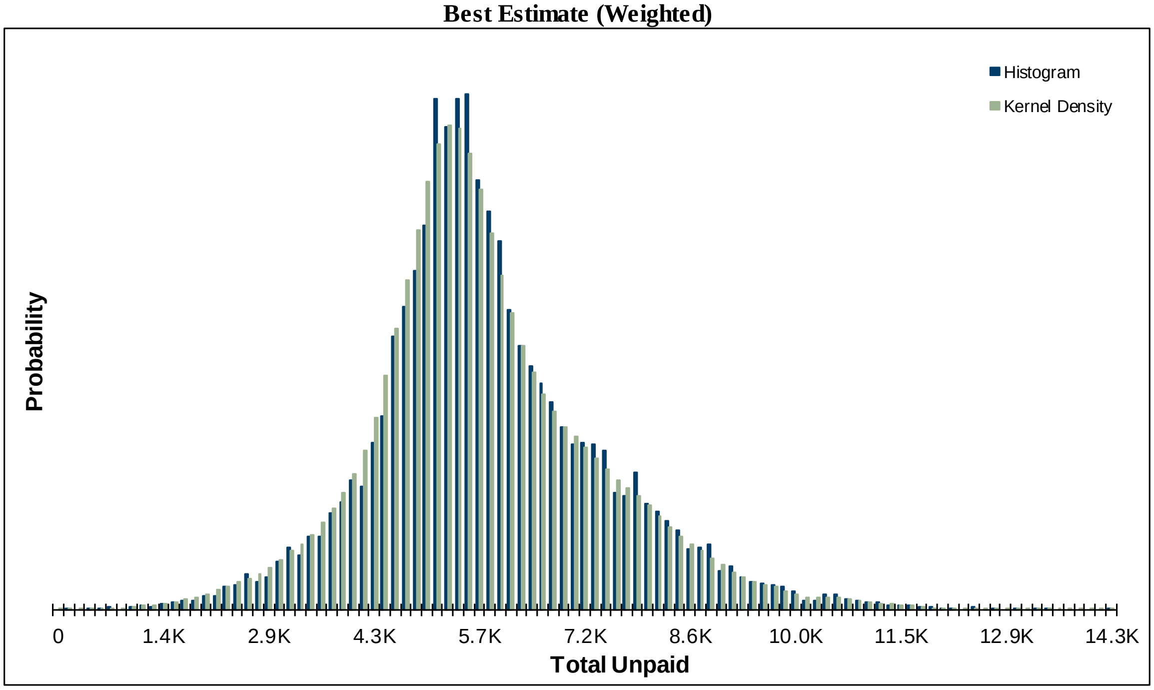
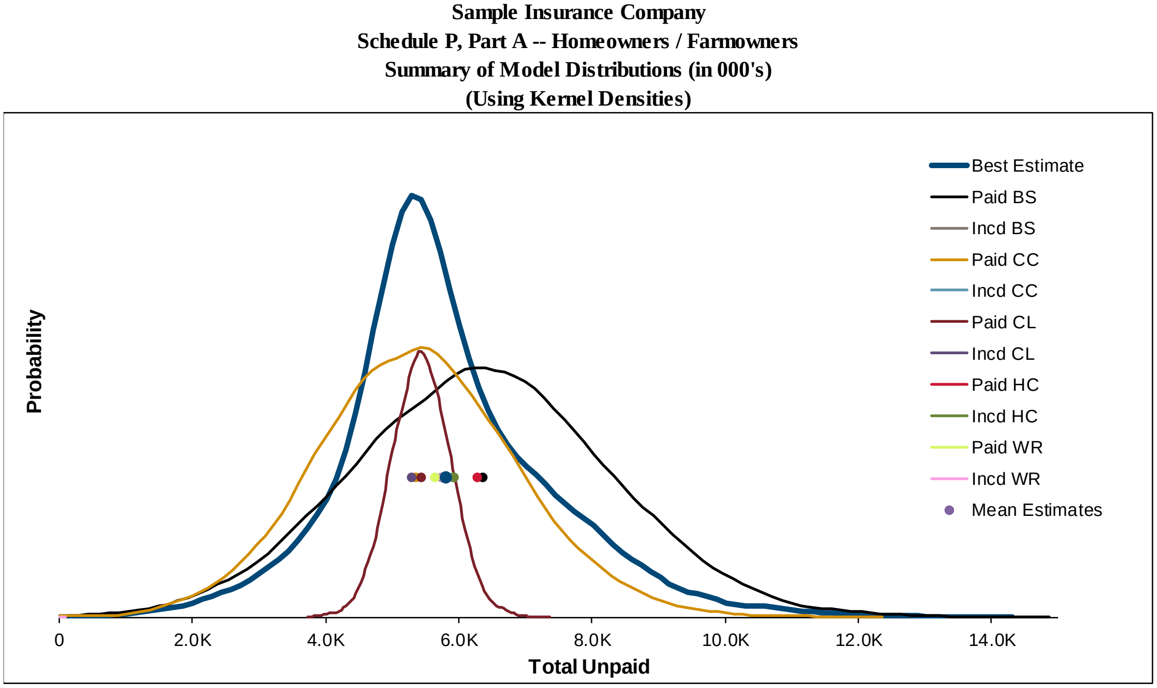


Figure B.33. Summary of model distributions



Appendix C – Schedule P, Part B Results

In this appendix the results for Schedule P, Part B (Private Passenger Auto Liability) are shown.

Figure C.1. Estimated unpaid model results (Paid Berquist-Sherman)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Paid Berquist & Sherman Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	39	8	20.8%	13	72	39	45	53	59
2007	12,679	68	9	13.6%	36	103	68	74	83	89
2008	13,631	108	10	9.4%	75	144	108	114	124	132
2009	14,472	184	11	6.2%	151	224	185	192	204	212
2010	13,717	311	14	4.4%	263	369	312	320	333	343
2011	13,090	571	18	3.2%	510	627	572	583	602	618
2012	12,490	1,107	29	2.6%	1,025	1,215	1,108	1,128	1,154	1,171
2013	11,598	2,110	48	2.3%	1,964	2,276	2,112	2,140	2,192	2,223
2014	10,306	3,964	87	2.2%	3,680	4,247	3,962	4,021	4,109	4,167
2015	6,357	8,078	173	2.1%	7,523	8,628	8,074	8,192	8,369	8,484
Totals	120,157	16,541	271	1.6%	15,759	17,433	16,553	16,724	16,991	17,159
Normal Dist.		16,541	271	1.6%			16,541	16,724	16,987	17,172
logNormal Dist.		16,541	271	1.6%			16,538	16,722	16,991	17,182
Gamma Dist.		16,541	271	1.6%			16,539	16,723	16,989	17,178

Figure C.2. Total unpaid claims distribution (Paid Berquist-Sherman)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Total Unpaid Distribution (in 000's)
Paid Berquist & Sherman Model

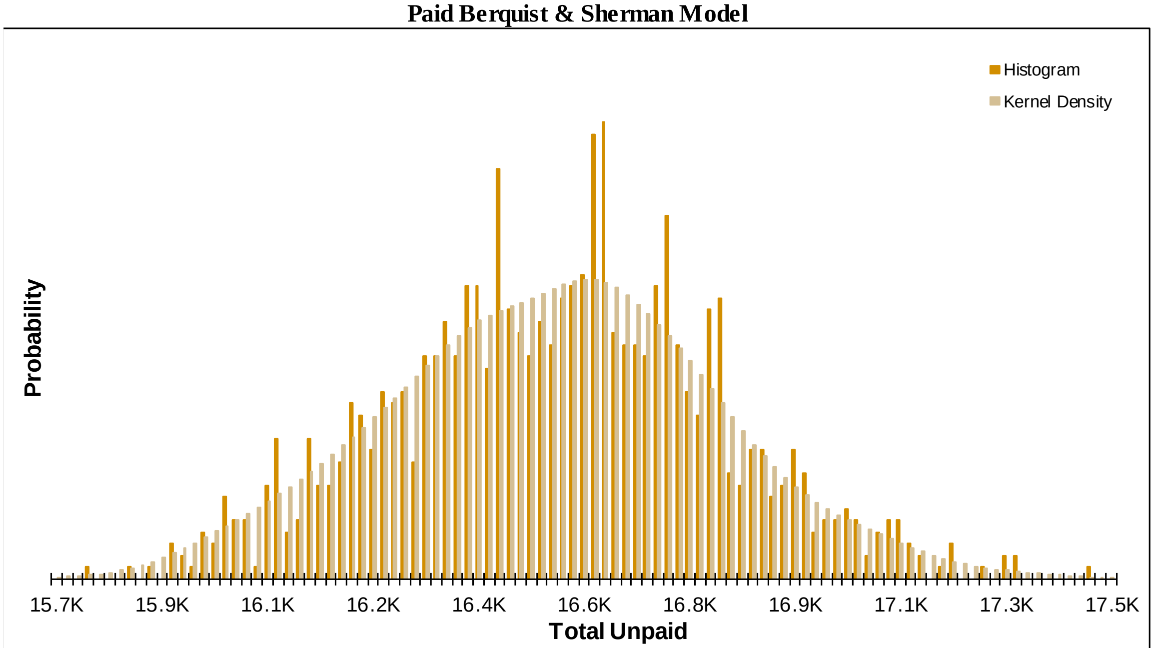


Figure C.3. Estimated unpaid model results (Incurred Berquist-Sherman)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Incurred Berquist & Sherman Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	41	9	21.3%	13	77	41	46	55	62
2007	12,679	69	10	14.6%	39	107	69	76	87	93
2008	13,631	110	11	10.5%	74	156	109	117	129	137
2009	14,472	187	14	7.5%	144	240	187	196	211	222
2010	13,717	315	19	6.0%	261	391	315	328	347	363
2011	13,090	576	30	5.3%	468	707	576	596	623	646
2012	12,490	1,113	54	4.8%	845	1,264	1,116	1,147	1,199	1,243
2013	11,598	2,109	96	4.6%	1,787	2,498	2,111	2,177	2,266	2,330
2014	10,306	3,950	178	4.5%	3,393	4,637	3,952	4,066	4,239	4,355
2015	6,357	8,041	366	4.6%	6,334	9,228	8,026	8,288	8,650	8,948
Totals	120,157	16,511	492	3.0%	14,729	18,489	16,495	16,835	17,297	17,794
Normal Dist.		16,511	492	3.0%			16,511	16,843	17,320	17,655
logNormal Dist.		16,511	492	3.0%			16,504	16,838	17,332	17,687
Gamma Dist.		16,511	492	3.0%			16,506	16,840	17,328	17,676

Figure C.4. Total unpaid claims distribution (Incurred Berquist-Sherman)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Total Unpaid Distribution (in 000's)
Incurred Berquist & Sherman Model

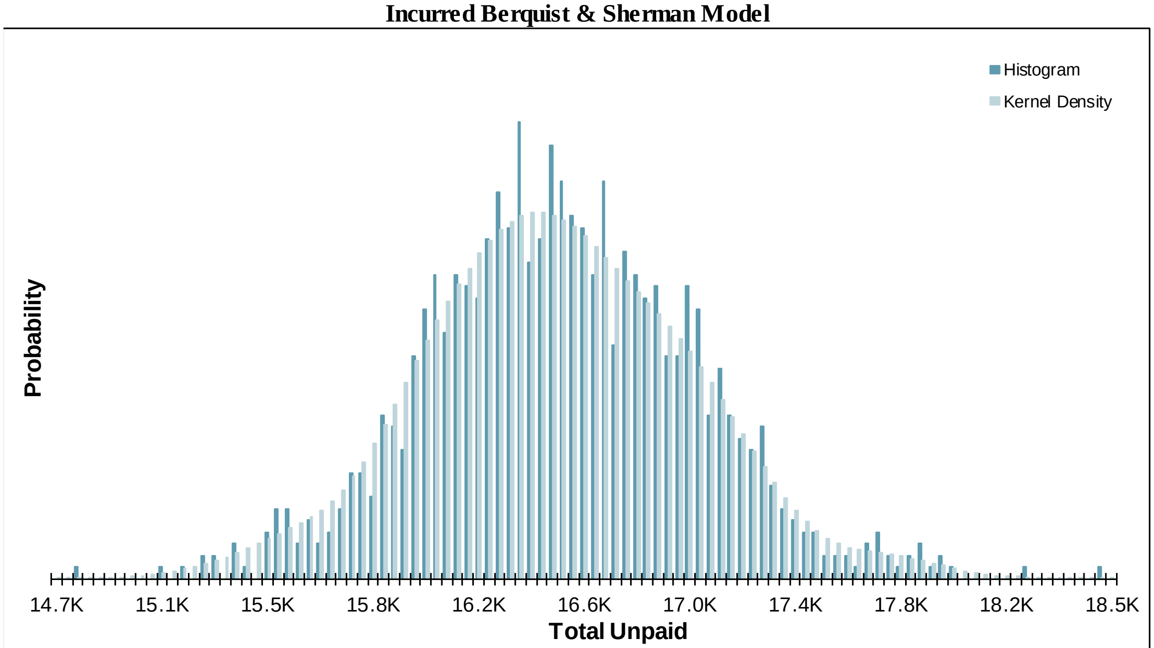


Figure C.5. Estimated unpaid model results (Paid Cape Cod)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Paid Cape Cod Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	305	1,958	641.5%	0	36,642	16	69	956	4,630
2007	12,679	351	2,087	595.3%	22	39,584	43	100	1,033	4,939
2008	13,631	413	2,214	535.9%	59	41,095	84	146	1,180	5,464
2009	14,472	511	2,371	463.5%	130	44,495	161	226	1,273	5,749
2010	13,717	633	2,322	366.8%	249	45,048	294	355	1,426	5,889
2011	13,090	884	2,272	256.9%	484	43,956	555	611	1,665	5,851
2012	12,490	1,401	2,230	159.2%	976	44,387	1,082	1,142	2,089	6,281
2013	11,598	2,374	2,201	92.7%	1,858	43,272	2,069	2,130	3,085	7,659
2014	10,306	4,212	2,322	55.1%	3,532	48,773	3,897	3,997	4,965	9,352
2015	6,357	8,351	2,347	28.1%	7,248	52,155	8,056	8,265	9,150	13,969
Totals	120,157	19,435	22,304	114.8%	15,233	439,407	16,251	16,796	26,583	69,103
Normal Dist.		19,435	22,304	114.8%			19,435	34,479	56,121	71,321
logNormal Dist.		18,404	5,673	30.8%			17,587	21,550	28,867	35,446
Gamma Dist.		19,435	22,304	114.8%			11,848	26,812	64,241	103,101

Figure C.6. Total unpaid claims distribution (Paid Cape Cod)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Total Unpaid Distribution (in 000's)
Paid Cape Cod Model

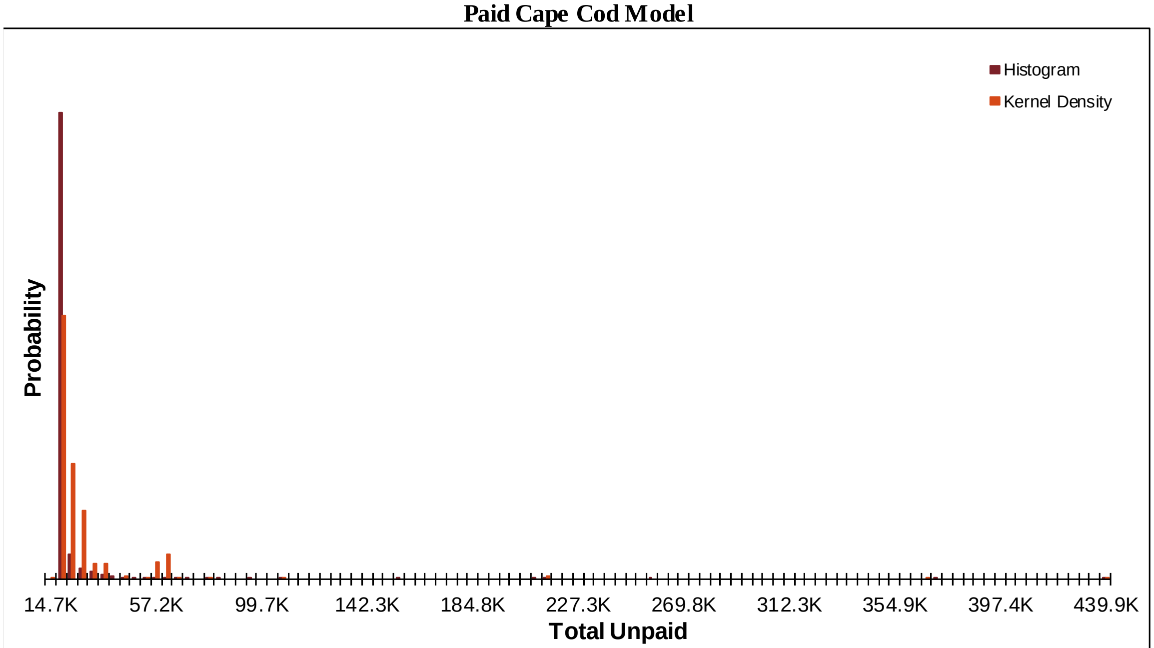


Figure C.7. Estimated unpaid model results (Incurred Cape Cod)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Incurred Cape Cod Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	190	746	391.9%	0	9,272	16	69	877	3,473
2007	12,679	223	768	344.3%	22	9,932	43	100	942	3,562
2008	13,631	286	852	297.8%	55	10,795	86	146	1,087	3,948
2009	14,472	384	914	238.4%	128	11,267	169	234	1,251	4,318
2010	13,717	508	878	172.8%	253	11,161	304	365	1,316	4,388
2011	13,090	742	826	111.3%	469	10,301	557	612	1,488	4,300
2012	12,490	1,255	778	62.0%	885	10,899	1,091	1,149	1,966	4,634
2013	11,598	2,195	726	33.1%	1,762	10,855	2,059	2,139	2,836	5,297
2014	10,306	4,034	675	16.7%	3,364	11,716	3,923	4,062	4,582	6,785
2015	6,357	8,415	515	6.1%	7,434	14,114	8,371	8,612	9,068	10,103
Totals	120,157	18,232	7,536	41.3%	15,207	109,195	16,630	17,144	25,086	50,824
Normal Dist.		18,232	7,536	41.3%			18,232	23,315	30,627	35,763
logNormal Dist.		18,025	3,936	21.8%			17,610	20,370	25,116	29,095
Gamma Dist.		18,232	7,536	41.3%			17,205	22,599	32,131	40,152

Figure C.8. Total unpaid claims distribution (Incurred Cape Cod)

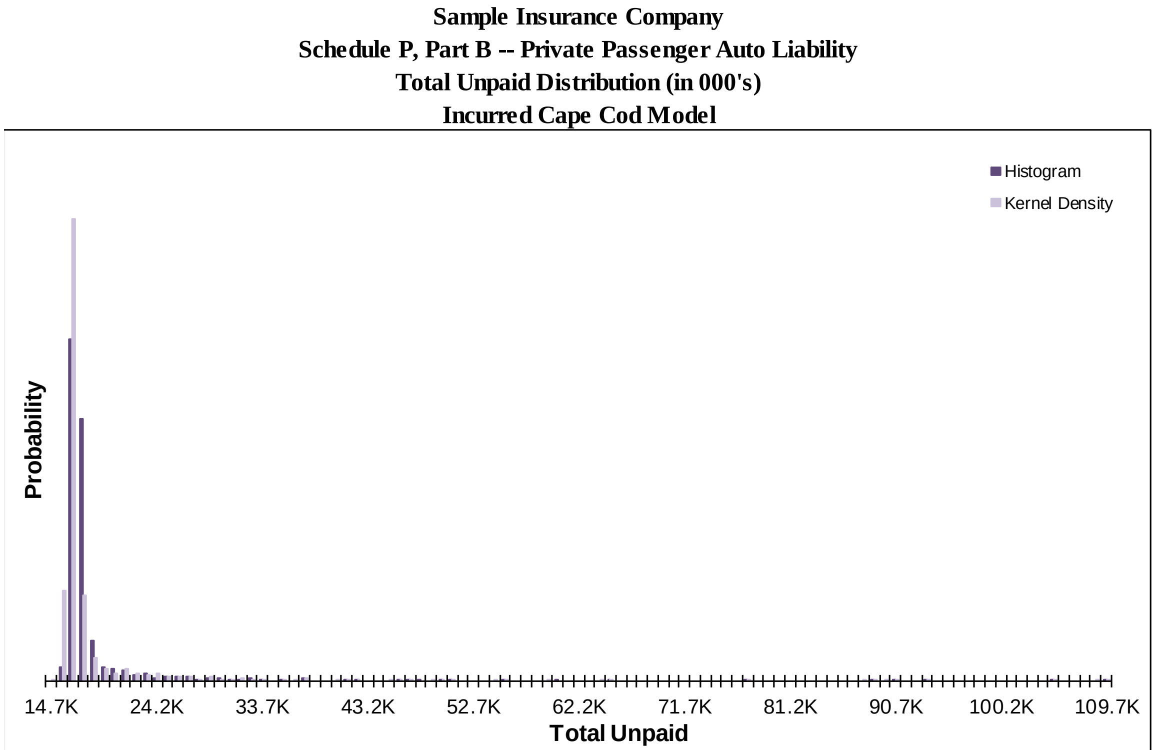


Figure C.9. Estimated unpaid model results (Paid Chain Ladder)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Paid Chain Ladder Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	536	3,745	698.6%	0	70,078	19	90	1,532	8,689
2007	12,679	602	4,030	669.2%	20	75,408	48	126	1,704	9,201
2008	13,631	681	4,276	628.2%	57	79,197	88	170	1,892	10,040
2009	14,472	798	4,564	572.0%	129	85,847	165	259	2,008	10,661
2010	13,717	901	4,410	489.2%	243	84,539	294	379	2,129	10,702
2011	13,090	1,135	4,285	377.7%	474	82,537	547	628	2,315	10,327
2012	12,490	1,649	4,245	257.4%	953	81,846	1,072	1,149	2,810	10,613
2013	11,598	2,636	4,296	163.0%	1,896	82,205	2,061	2,142	3,817	12,255
2014	10,306	4,493	4,528	100.8%	3,577	89,297	3,897	3,996	5,690	14,021
2015	6,357	8,629	4,501	52.2%	7,540	92,918	8,051	8,195	9,809	18,554
Totals	120,157	22,060	42,872	194.3%	15,256	823,872	16,189	16,962	34,096	115,071
Normal Dist.		22,060	42,872	194.3%			22,060	50,977	92,579	121,796
logNormal Dist.		19,588	7,896	40.3%			18,168	23,603	34,394	44,805
Gamma Dist.		22,060	42,872	194.3%			4,314	23,741	104,947	208,038

Figure C.10. Total unpaid claims distribution (Paid Chain Ladder)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Total Unpaid Distribution (in 000's)
Paid Chain Ladder Model

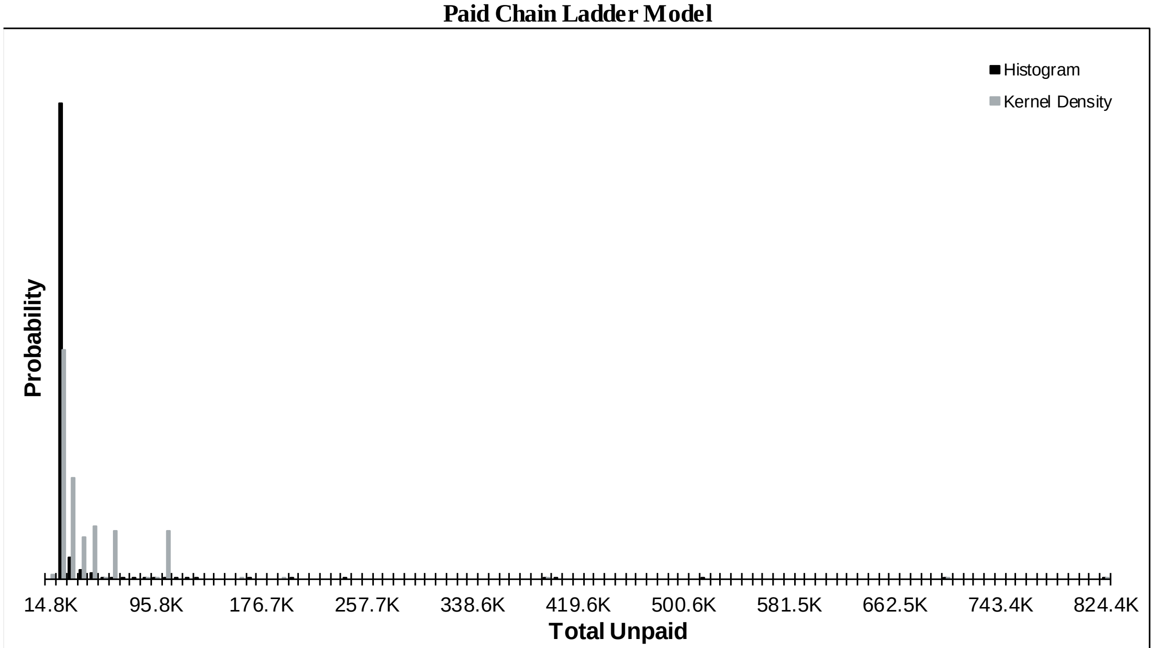


Figure C.11. Estimated unpaid model results (Incurred Chain Ladder)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Incurred Chain Ladder Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	212	994	469.1%	(1,028)	16,639	10	55	904	4,948
2007	12,679	226	1,009	446.6%	(5,368)	16,438	38	85	1,047	3,770
2008	13,631	243	867	357.1%	(4,064)	10,852	70	135	1,029	4,617
2009	14,472	375	1,083	288.8%	(601)	14,903	151	227	1,236	5,709
2010	13,717	445	1,049	235.9%	(3,824)	15,213	255	399	1,511	4,644
2011	13,090	598	1,083	181.0%	(2,561)	13,647	441	674	1,523	5,014
2012	12,490	990	1,092	110.3%	(2,239)	15,426	897	1,315	2,329	4,854
2013	11,598	1,704	1,577	92.6%	(3,079)	13,891	1,641	2,364	3,908	7,614
2014	10,306	3,106	2,388	76.9%	(5,810)	20,138	3,137	4,479	6,789	8,088
2015	6,357	6,652	4,635	69.7%	(14,979)	22,158	6,703	9,505	14,259	16,821
Totals	120,157	14,551	9,189	63.1%	(13,211)	115,434	13,628	17,226	25,849	49,616
Normal Dist.		14,551	9,189	63.1%			14,551	20,749	29,666	35,928
logNormal Dist.		25,561	52,784	206.5%			11,140	26,572	92,799	223,349
Gamma Dist.		14,551	9,189	63.1%			12,669	19,278	32,188	43,849

Figure C.12. Total unpaid claims distribution (Incurred Chain Ladder)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Total Unpaid Distribution (in 000's)
Incurred Chain Ladder Model

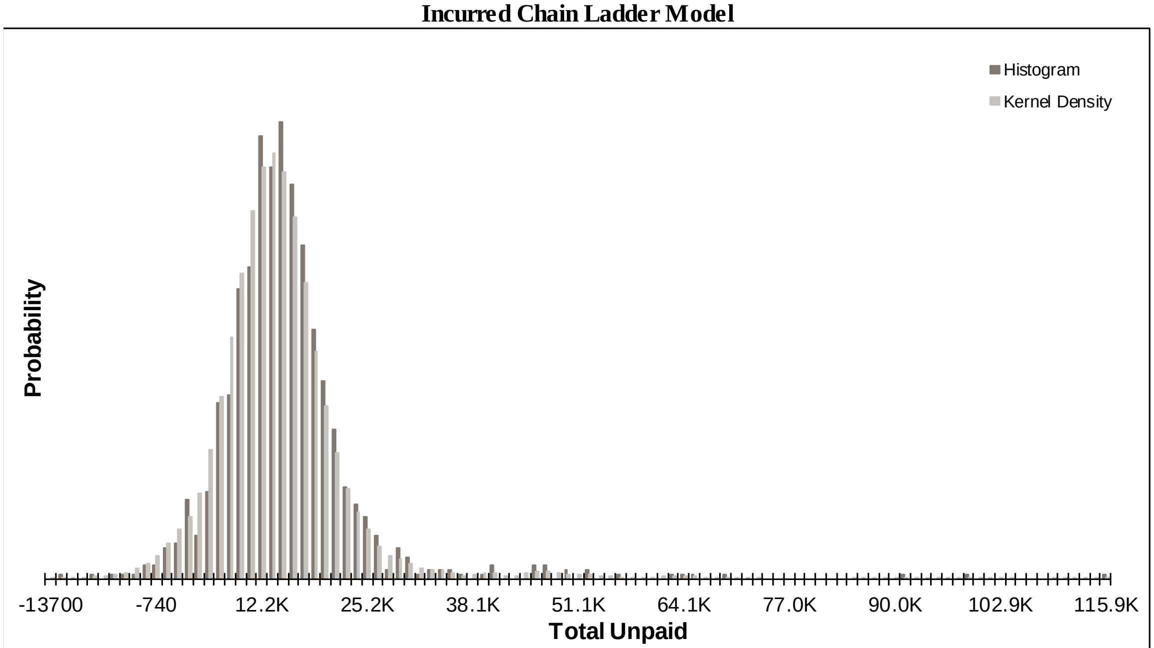


Figure C.13. Estimated unpaid model results (Paid Hoerl Curve)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Paid Hoerl Curve Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	31	6	19.3%	15	55	31	35	42	47
2007	12,679	55	8	15.0%	35	90	54	60	69	78
2008	13,631	98	11	11.3%	66	137	97	105	118	127
2009	14,472	180	15	8.6%	140	234	179	190	206	220
2010	13,717	309	22	7.0%	246	384	309	323	345	356
2011	13,090	561	34	6.1%	459	688	560	583	618	641
2012	12,490	1,057	57	5.4%	880	1,275	1,058	1,093	1,154	1,188
2013	11,598	2,052	101	4.9%	1,716	2,381	2,052	2,114	2,222	2,283
2014	10,306	4,145	201	4.9%	3,441	4,783	4,154	4,273	4,458	4,653
2015	6,357	8,030	386	4.8%	6,852	9,105	8,032	8,286	8,689	8,951
Totals	120,157	16,517	562	3.4%	14,682	18,267	16,519	16,894	17,410	17,867
Normal Dist.		16,517	562	3.4%			16,517	16,896	17,442	17,825
logNormal Dist.		16,517	563	3.4%			16,507	16,891	17,459	17,869
Gamma Dist.		16,517	562	3.4%			16,511	16,893	17,453	17,853

Figure C.14. Total unpaid claims distribution (Paid Hoerl Curve)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Total Unpaid Distribution (in 000's)
Paid Hoerl Curve Model

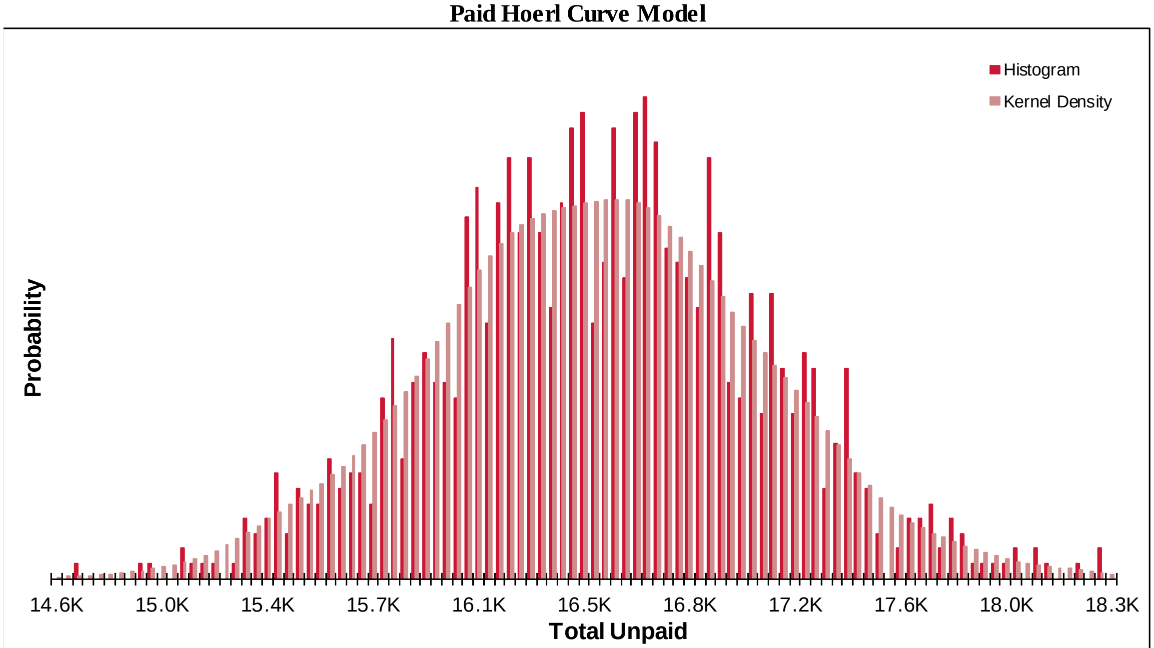


Figure C.15. Estimated unpaid model results (Incurred Hoerl Curve)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Incurred Hoerl Curve Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	32	6	20.1%	15	55	31	36	43	50
2007	12,679	56	9	15.9%	34	97	55	61	72	79
2008	13,631	100	12	12.4%	66	143	99	107	120	131
2009	14,472	183	17	9.5%	132	238	182	194	214	227
2010	13,717	314	26	8.2%	212	400	314	332	357	375
2011	13,090	570	43	7.5%	426	716	569	597	645	679
2012	12,490	1,074	73	6.8%	835	1,329	1,074	1,122	1,196	1,251
2013	11,598	2,082	132	6.3%	1,641	2,530	2,084	2,169	2,300	2,401
2014	10,306	4,203	241	5.7%	3,414	5,065	4,196	4,363	4,600	4,783
2015	6,357	8,167	443	5.4%	6,639	10,105	8,175	8,444	8,904	9,166
Totals	120,157	16,781	549	3.3%	14,964	19,012	16,788	17,130	17,678	18,120
Normal Dist.		16,781	549	3.3%			16,781	17,151	17,684	18,059
logNormal Dist.		16,781	549	3.3%			16,772	17,146	17,698	18,097
Gamma Dist.		16,781	549	3.3%			16,775	17,148	17,695	18,085

Figure C.16. Total unpaid claims distribution (Incurred Hoerl Curve)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Total Unpaid Distribution (in 000's)
Incurred Hoerl Curve Model

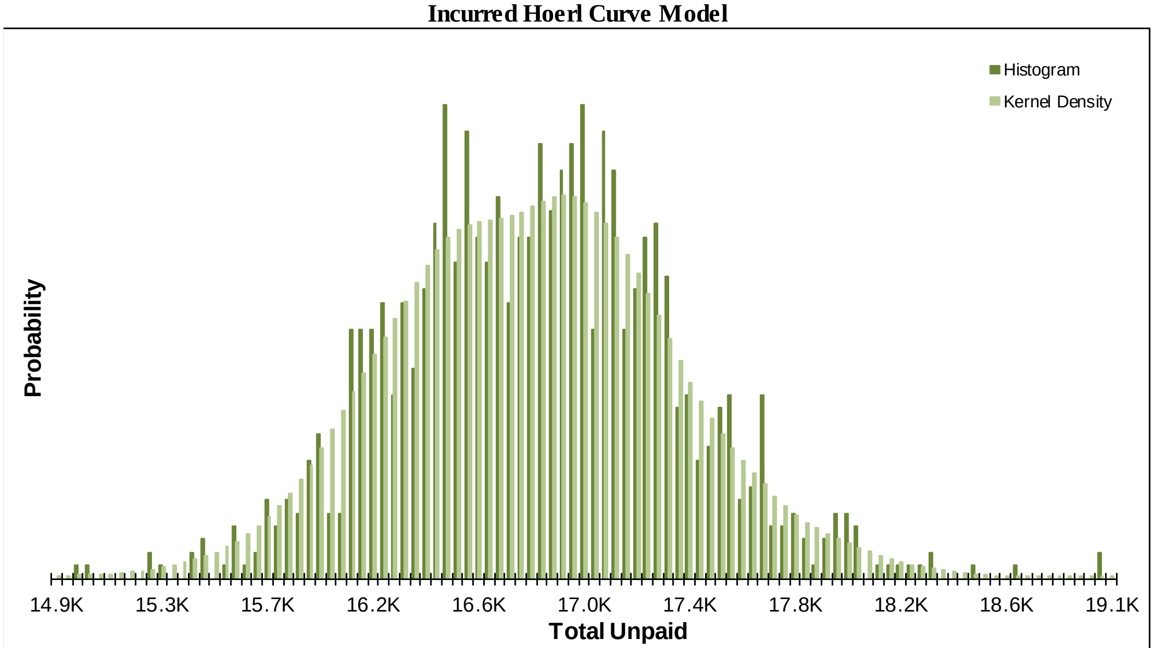


Figure C.17. Estimated unpaid model results (Paid Wright)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Paid Wright Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	33	6	18.0%	19	52	32	36	44	49
2007	12,679	57	8	14.2%	35	87	56	62	72	77
2008	13,631	103	12	11.2%	69	149	102	110	124	131
2009	14,472	188	16	8.5%	132	238	188	198	215	227
2010	13,717	322	23	7.2%	252	392	321	337	361	375
2011	13,090	572	36	6.4%	471	719	572	597	635	656
2012	12,490	1,039	63	6.0%	832	1,219	1,042	1,082	1,140	1,185
2013	11,598	1,982	116	5.8%	1,603	2,345	1,983	2,057	2,174	2,245
2014	10,306	4,172	259	6.2%	3,263	5,107	4,178	4,339	4,619	4,755
2015	6,357	7,932	596	7.5%	6,151	10,392	7,894	8,315	8,943	9,467
Totals	120,157	16,399	712	4.3%	14,387	18,935	16,364	16,858	17,619	18,179
Normal Dist.		16,399	712	4.3%			16,399	16,879	17,570	18,055
logNormal Dist.		16,399	710	4.3%			16,383	16,869	17,593	18,119
Gamma Dist.		16,399	712	4.3%			16,389	16,873	17,587	18,100

Figure C.18. Total unpaid claims distribution (Paid Wright)

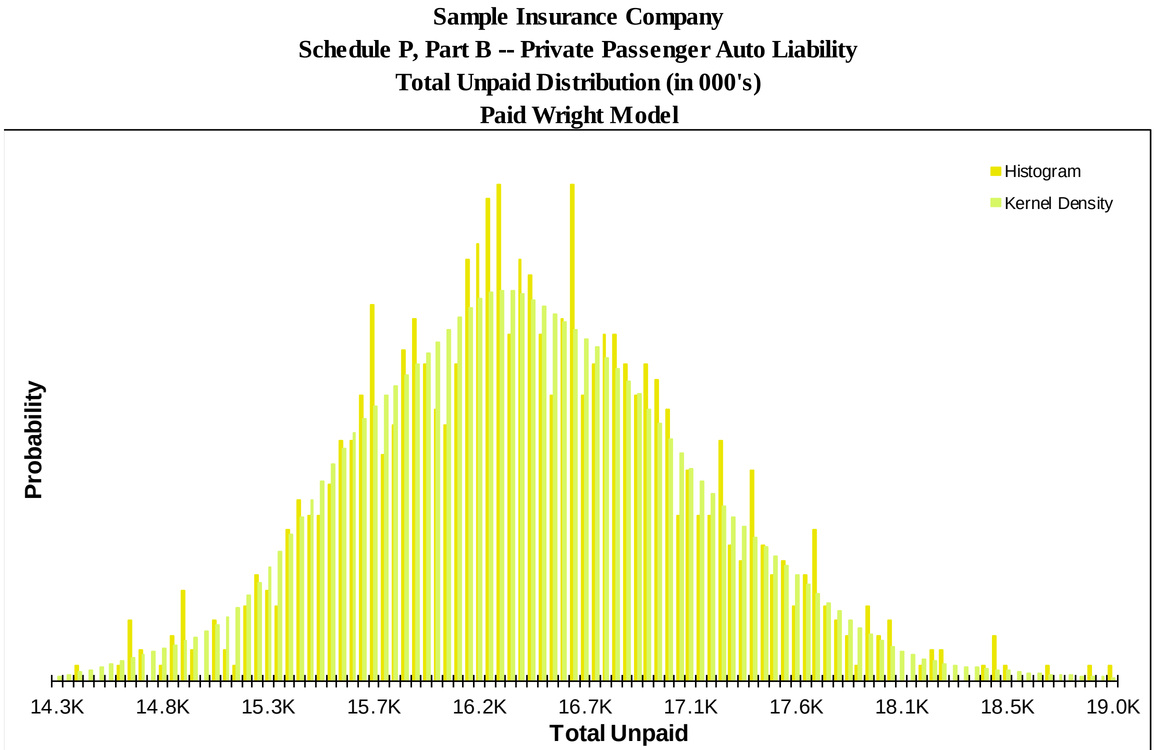


Figure C.19. Estimated unpaid model results (Incurred Wright)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Incurred Wright Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	3,966,020	89,251,382	2250.4%	11	2,562,613,734	32	37	47	182
2007	12,679	6,836,660	151,785,937	2220.2%	35	4,285,467,202	57	63	75	397
2008	13,631	11,394,086	249,064,047	2185.9%	70	6,902,830,083	105	114	130	631
2009	14,472	22,062,894	473,328,887	2145.4%	(19)	12,454,255,734	197	209	231	1,248
2010	13,717	33,137,459	697,879,848	2106.0%	175	17,782,233,386	332	349	382	2,198
2011	13,090	57,164,640	1,225,268,490	2143.4%	307	33,399,406,687	577	606	652	3,679
2012	12,490	112,525,697	2,407,249,389	2139.3%	479	64,465,798,848	1,066	1,105	1,196	6,437
2013	11,598	217,196,589	4,589,378,234	2113.0%	876	115,499,405,106	2,020	2,103	2,279	12,504
2014	10,306	395,484,469	8,302,943,031	2099.4%	(546)	208,307,514,180	4,137	4,285	4,552	24,508
2015	6,357	854,159,749	18,202,471,420	2131.0%	2,861	478,840,892,606	8,366	8,599	9,075	50,131
Totals	120,157	1,713,928,261	36,360,742,828	2121.5%	6,088	944,500,417,566	16,866	17,188	17,873	101,915
Normal Dist.		1,713,928,261	36,360,742,828	2121.5%			1,713,928,261	26,238,876,608	61,522,027,980	86,301,665,037
logNormal Dist.		42,038	82,461	196.2%			19,093	44,556	150,791	355,000
Gamma Dist.		1,713,928,261	36,360,742,828	2121.5%			0	0	41	4,737,757,328

Figure C.20. Total unpaid claims distribution (Incurred Wright)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Total Unpaid Distribution (in 000's)
Incurred Wright Model

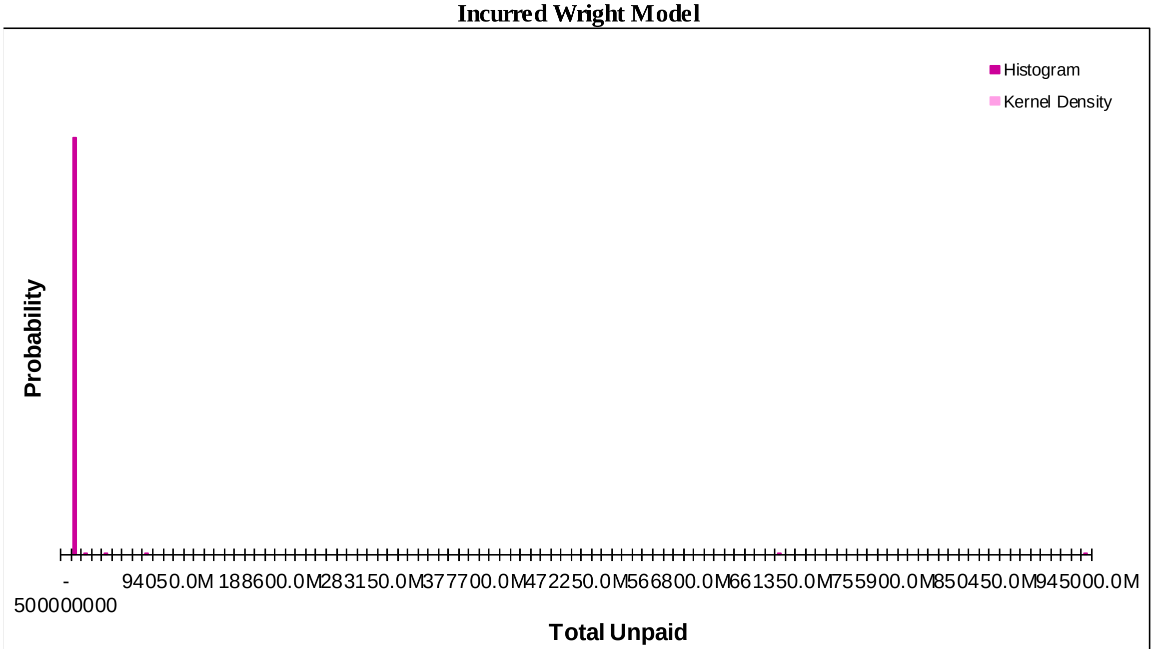


Figure C.21. Model weights by accident year

Accident Year	Model Weights by Accident Year									
	Paid BS	Incd BS	Paid CC	Incd CC	Paid CL	Incd CL	Paid HC	Paid WR	[redacted]	TOTAL
2006	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2007	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2008	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2009	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2010	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2011	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2012	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2013	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2014	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%
2015	10.0%	20.0%	10.0%	20.0%	10.0%	20.0%	5.0%	5.0%		100.0%

Figure C.22. Estimated mean unpaid by model

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Summary of Results by Model (in 000's)

Accident Year	Mean Estimated Unpaid								
	Berquist & Sherman		Cape Cod		Chain Ladder		Hoerl Curve		Best Est.
	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	(Weighted)
2006	39	41	305	190	536	212	31	33	140
2007	68	69	351	223	602	226	55	57	186
2008	108	110	413	286	681	243	98	103	215
2009	184	187	511	384	798	375	180	188	314
2010	311	315	633	508	901	445	309	322	439
2011	571	576	884	742	1,135	598	561	572	677
2012	1,107	1,113	1,401	1,255	1,649	990	1,057	1,039	1,165
2013	2,110	2,109	2,374	2,195	2,636	1,704	2,052	1,982	2,093
2014	3,964	3,950	4,212	4,034	4,493	3,106	4,145	4,172	3,923
2015	8,078	8,041	8,351	8,415	8,629	6,652	8,030	7,932	7,928
Totals	16,541	16,511	19,435	18,232	22,060	14,551	16,517	16,399	17,079

Figure C.23. Estimated ranges

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Summary of Results by Model (in 000's)

Accident Year	Best Est. (Weighted)	Ranges			
		Weighted		Modeled	
		Minimum	Maximum	Mininum	Maximum
2006	140	31	536	31	536
2007	186	55	602	55	602
2008	215	98	681	98	681
2009	314	180	798	180	798
2010	439	309	901	309	901
2011	677	561	1,135	561	1,135
2012	1,165	990	1,649	990	1,649
2013	2,093	1,704	2,636	1,704	2,636
2014	3,923	3,106	4,493	3,106	4,493
2015	7,928	6,652	8,629	6,652	8,629
Totals	17,079	13,687	22,060	14,551	22,060

Figure C.24. Reconciliation of total results (weighted)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Reconciliation of Total Results (in 000's)
Best Estimate (Weighted)

Accident Year	Paid To Date	Incurred To Date	Case Reserves	IBNR	Estimate of Ultimate	Estimate of Unpaid
2006	11,816	11,863	47	92	11,956	140
2007	12,679	12,752	72	113	12,865	186
2008	13,631	13,743	112	103	13,846	215
2009	14,472	14,687	216	99	14,786	314
2010	13,717	14,079	362	77	14,156	439
2011	13,090	13,691	600	76	13,767	677
2012	12,490	13,683	1,193	(28)	13,655	1,165
2013	11,598	13,912	2,313	(221)	13,691	2,093
2014	10,306	14,625	4,319	(396)	14,229	3,923
2015	6,357	15,188	8,830	(902)	14,285	7,928
Totals	120,157	138,223	18,066	(987)	137,236	17,079

Figure C.25. Estimated unpaid model results (weighted)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Unpaid (in 000's)
Best Estimate (Weighted)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11,816	140	1,002	717.3%	(2,012)	74,732	33	46	409	2,417
2007	12,679	186	992	534.2%	(1,534)	37,021	59	75	490	3,024
2008	13,631	215	926	431.1%	(5,790)	54,408	101	118	513	3,196
2009	14,472	314	1,285	408.8%	(2,963)	90,358	179	200	646	3,686
2010	13,717	439	1,359	309.7%	(1,945)	69,048	308	333	765	3,336
2011	13,090	677	1,264	186.9%	(3,824)	68,442	562	598	1,051	3,798
2012	12,490	1,165	928	79.7%	(4,552)	27,150	1,088	1,144	1,762	4,562
2013	11,598	2,093	1,405	67.1%	(8,529)	79,999	2,066	2,153	2,880	5,341
2014	10,306	3,923	4,359	111.1%	(9,679)	405,947	3,935	4,095	5,126	7,619
2015	6,357	7,928	2,727	34.4%	(16,198)	92,918	8,087	8,384	10,346	14,962
Totals	120,157	17,079	8,888	52.0%	(8,740)	467,516	16,313	17,135	22,900	45,682
Normal Dist.		17,079	8,888	52.0%			17,079	23,074	31,698	37,755
logNormal Dist.		17,362	6,693	38.5%			16,200	20,823	29,882	38,509
Gamma Dist.		17,079	8,888	52.0%			15,565	21,956	33,823	44,176

Figure C.26. Estimated cash flow (weighted)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Calendar Year Unpaid (in 000's)
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	7,556	1,252	16.6%	(4,926)	16,377	7,784	8,077	8,947	10,799
2017	3,729	587	15.7%	(1,542)	7,781	3,832	3,979	4,440	5,204
2018	2,056	329	16.0%	(1,017)	4,299	2,111	2,196	2,449	2,901
2019	1,120	275	24.5%	(599)	12,854	1,116	1,175	1,441	1,965
2020	672	853	126.9%	(349)	60,793	576	625	1,032	2,954
2021	426	833	195.4%	(435)	33,332	306	346	791	2,860
2022	293	815	277.8%	(956)	44,837	170	205	692	2,742
2023	244	1,086	445.1%	(1,312)	70,895	101	132	643	3,052
2024	209	1,139	544.5%	(744)	60,995	63	93	574	3,034
2025	180	1,049	584.4%	(1,512)	53,125	34	64	578	2,993
2026	159	825	519.5%	(1,570)	23,121	19	43	550	3,195
2027	156	1,260	805.5%	(3,523)	74,981	11	25	478	3,172
2028	169	3,600	2134.8%	(7,667)	342,488	6	12	412	2,742
2029	110	1,288	1168.6%	(1,140)	66,389	2	4	196	2,239
Totals	17,079	8,888	52.0%	(8,740)	467,516	16,313	17,135	22,900	45,682

Figure C.27. Estimated loss ratio (weighted)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Ultimate Loss Ratios (in 000's)
Best Estimate (Weighted)

Accident Year	Earned Premium	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	15,679	72.5%	22.1%	30.5%	-127.2%	554.0%	76.2%	78.6%	87.9%	125.3%
2007	15,510	77.8%	22.5%	28.9%	-120.8%	321.8%	81.5%	83.9%	94.7%	134.4%
2008	16,428	79.2%	22.7%	28.7%	-80.0%	418.0%	83.0%	85.4%	94.4%	135.3%
2009	18,432	76.0%	20.6%	27.1%	-121.9%	568.8%	78.9%	81.6%	90.2%	128.0%
2010	20,376	66.2%	18.8%	28.5%	-71.4%	405.5%	68.9%	71.2%	79.5%	112.5%
2011	20,821	63.2%	18.2%	28.8%	-133.5%	391.5%	65.9%	67.8%	76.8%	108.6%
2012	20,445	64.2%	18.6%	29.0%	-112.3%	193.2%	66.9%	68.8%	78.9%	114.7%
2013	20,724	63.2%	19.0%	30.1%	-110.9%	441.9%	66.1%	67.9%	76.3%	110.3%
2014	20,414	67.4%	27.9%	41.3%	-151.2%	2038.6%	69.9%	72.0%	81.6%	118.0%
2015	20,467	68.8%	20.5%	29.8%	-139.5%	485.5%	70.7%	73.2%	87.2%	128.7%
Totals	189,295	69.3%	7.0%	10.1%	30.3%	277.8%	70.1%	73.0%	78.1%	84.1%

Figure C.28. Estimated unpaid claim runoff (weighted)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Calendar Year Unpaid Claim Runoff (in 000's)
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	17,079	8,888	52.0%	(8,740)	467,516	16,313	17,135	22,900	45,682
2016	9,523	8,743	91.8%	(3,814)	460,913	8,426	9,009	14,012	40,239
2017	5,794	8,725	150.6%	(2,272)	457,482	4,538	4,972	10,156	37,168
2018	3,738	8,694	232.6%	(1,255)	455,334	2,400	2,757	8,053	35,056
2019	2,619	8,557	326.8%	(656)	452,390	1,281	1,572	6,762	33,341
2020	1,946	8,054	413.8%	(307)	438,625	705	938	5,628	29,716
2021	1,520	7,587	499.1%	(156)	432,875	400	597	4,727	25,537
2022	1,227	7,152	582.9%	(101)	427,826	231	388	3,904	21,800
2023	983	6,647	676.3%	(9,318)	427,376	133	257	3,207	18,459
2024	774	6,071	784.7%	(9,365)	424,965	72	160	2,498	14,866
2025	594	5,472	921.1%	(9,345)	411,264	39	93	1,874	11,321
2026	435	4,936	1134.2%	(9,102)	393,463	19	46	1,261	7,898
2027	279	3,988	1430.4%	(7,664)	342,491	8	17	690	5,233
2028	110	1,288	1168.6%	(1,140)	66,389	2	4	196	2,239

Figure C.29. Mean of incremental values (weighted)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Mean Values (in 000's)														
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
2006	4,987	3,176	1,405	801	432	214	108	56	31	23	14	12	16	29	68
2007	5,286	3,367	1,491	850	458	226	114	59	32	25	15	14	18	34	80
2008	5,706	3,639	1,611	918	495	244	123	63	35	26	16	14	18	32	73
2009	6,134	3,912	1,729	987	532	263	133	68	38	28	17	15	20	38	90
2010	5,909	3,764	1,667	950	512	253	128	66	36	27	16	15	20	38	92
2011	5,763	3,671	1,625	927	499	247	125	64	35	27	16	15	20	37	91
2012	5,748	3,660	1,619	924	498	246	124	64	35	27	16	15	20	36	84
2013	5,728	3,651	1,617	921	497	245	124	64	35	27	16	15	20	38	92
2014	6,012	3,829	1,695	967	521	258	130	67	37	28	17	15	21	43	125
2015	6,161	3,925	1,737	991	534	264	133	69	38	29	17	16	22	43	110

Figure C.30. Standard deviation of incremental values (weighted)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Standard Error Values (in 000's)														
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
2006	1,504	955	425	241	131	65	33	17	9	7	11	22	58	187	752
2007	1,503	957	426	241	131	65	33	17	9	7	12	25	68	210	696
2008	1,629	1,033	460	261	142	70	35	18	10	8	12	25	65	192	656
2009	1,622	1,031	458	260	141	70	35	18	10	8	13	28	76	249	951
2010	1,621	1,027	459	260	141	70	35	18	10	8	13	28	79	269	994
2011	1,612	1,022	458	258	140	69	35	18	10	8	13	28	77	250	917
2012	1,673	1,061	472	268	145	72	36	19	10	8	13	27	67	192	618
2013	1,670	1,063	473	268	145	72	36	19	10	8	13	28	78	254	950
2014	1,709	1,083	485	275	149	74	37	19	11	8	13	31	108	572	3,565
2015	1,730	1,096	486	277	150	74	38	20	11	9	14	31	92	328	1,288

Figure C.31. Coefficient of variation of incremental values (weighted)

Sample Insurance Company
Schedule P, Part B -- Private Passenger Auto Liability
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Coefficients of Variation														
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
2006	30.1%	30.1%	30.2%	30.1%	30.3%	30.4%	30.3%	30.6%	30.9%	31.7%	77.8%	175.4%	359.0%	643.8%	1099.6%
2007	28.4%	28.4%	28.5%	28.4%	28.6%	28.6%	28.7%	28.9%	29.2%	30.0%	78.7%	186.6%	373.9%	612.0%	865.2%
2008	28.5%	28.4%	28.6%	28.4%	28.7%	28.6%	28.7%	28.9%	29.3%	30.2%	78.3%	178.7%	355.4%	591.5%	898.4%
2009	26.4%	26.4%	26.5%	26.4%	26.6%	26.6%	26.6%	26.8%	27.3%	28.2%	77.7%	181.3%	372.8%	661.7%	1057.6%
2010	27.4%	27.3%	27.5%	27.4%	27.5%	27.6%	27.6%	27.9%	28.3%	29.0%	78.5%	188.1%	394.4%	708.1%	1079.9%
2011	28.0%	27.8%	28.1%	27.8%	28.1%	28.0%	28.2%	28.5%	28.9%	29.6%	79.8%	189.0%	390.0%	668.2%	1007.6%
2012	29.1%	29.0%	29.1%	29.0%	29.2%	29.1%	29.1%	29.4%	29.8%	30.8%	79.4%	180.6%	340.5%	529.0%	733.9%
2013	29.2%	29.1%	29.3%	29.1%	29.3%	29.3%	29.4%	29.5%	29.8%	30.6%	80.1%	190.3%	391.1%	673.6%	1037.3%
2014	28.4%	28.3%	28.6%	28.4%	28.6%	28.5%	28.7%	28.8%	29.3%	29.9%	78.3%	202.5%	513.3%	1333.7%	2845.1%
2015	28.1%	27.9%	28.0%	27.9%	28.2%	28.1%	28.2%	28.6%	28.8%	29.9%	80.0%	196.5%	420.2%	757.1%	1168.6%

Figure C.32. Total unpaid claims distribution (weighted)

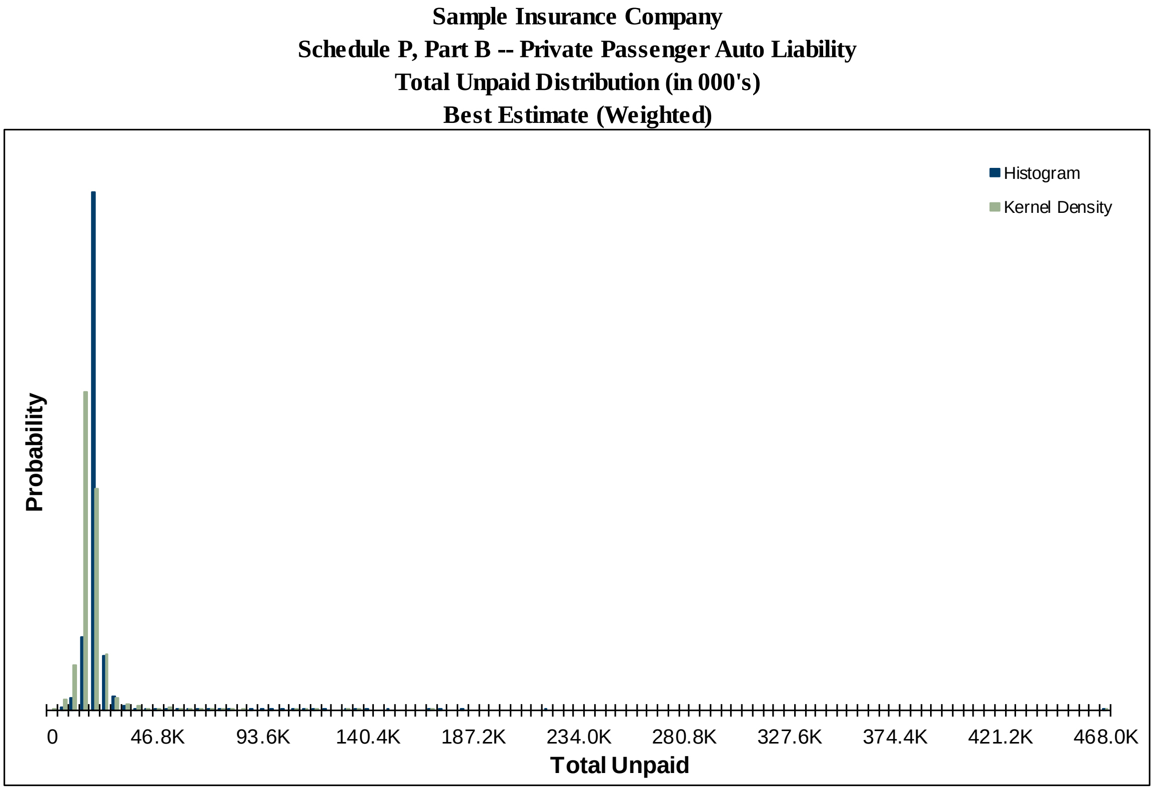
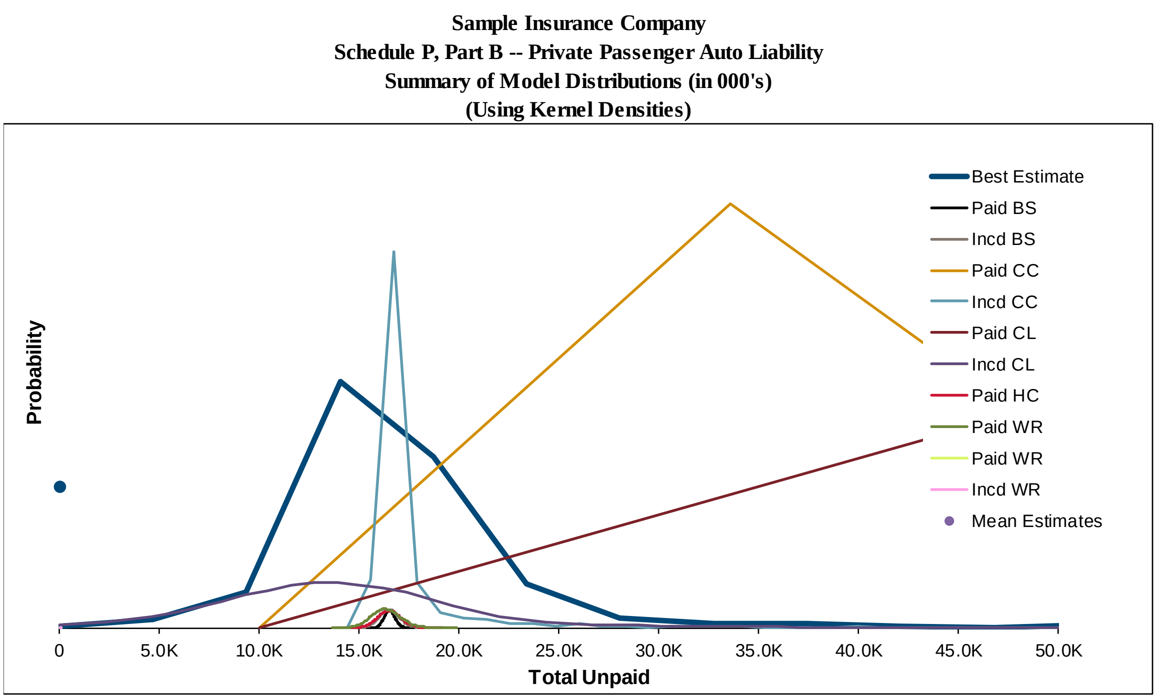


Figure C.33. Summary of model distributions



Appendix D – Schedule P, Part C Results

In this appendix the results for Schedule P, Part C (Commercial Auto Liability) are shown.

Figure D.1. Estimated unpaid model results (Paid Berquist-Sherman)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Paid Berquist & Sherman Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	1	3	251.9%	(20)	14	1	3	7	10
2007	1,469	4	5	149.6%	(23)	22	3	7	13	18
2008	1,387	14	8	54.9%	(21)	43	14	19	28	34
2009	1,350	28	10	35.4%	(9)	62	27	34	44	51
2010	1,342	50	13	25.4%	5	91	50	58	71	81
2011	1,198	103	16	15.6%	55	157	103	114	128	140
2012	1,061	209	22	10.3%	110	303	210	224	243	256
2013	853	402	28	6.9%	308	490	401	421	448	467
2014	645	742	40	5.4%	619	888	741	768	809	842
2015	294	1,176	51	4.4%	1,026	1,341	1,173	1,212	1,260	1,299
Totals	11,162	2,729	111	4.1%	2,361	3,140	2,725	2,798	2,918	2,983
Normal Dist.		2,729	111	4.1%			2,729	2,804	2,911	2,986
logNormal Dist.		2,729	111	4.1%			2,727	2,802	2,915	2,996
Gamma Dist.		2,729	111	4.1%			2,727	2,803	2,913	2,993

Figure D.2. Total unpaid claims distribution (Paid Berquist-Sherman)

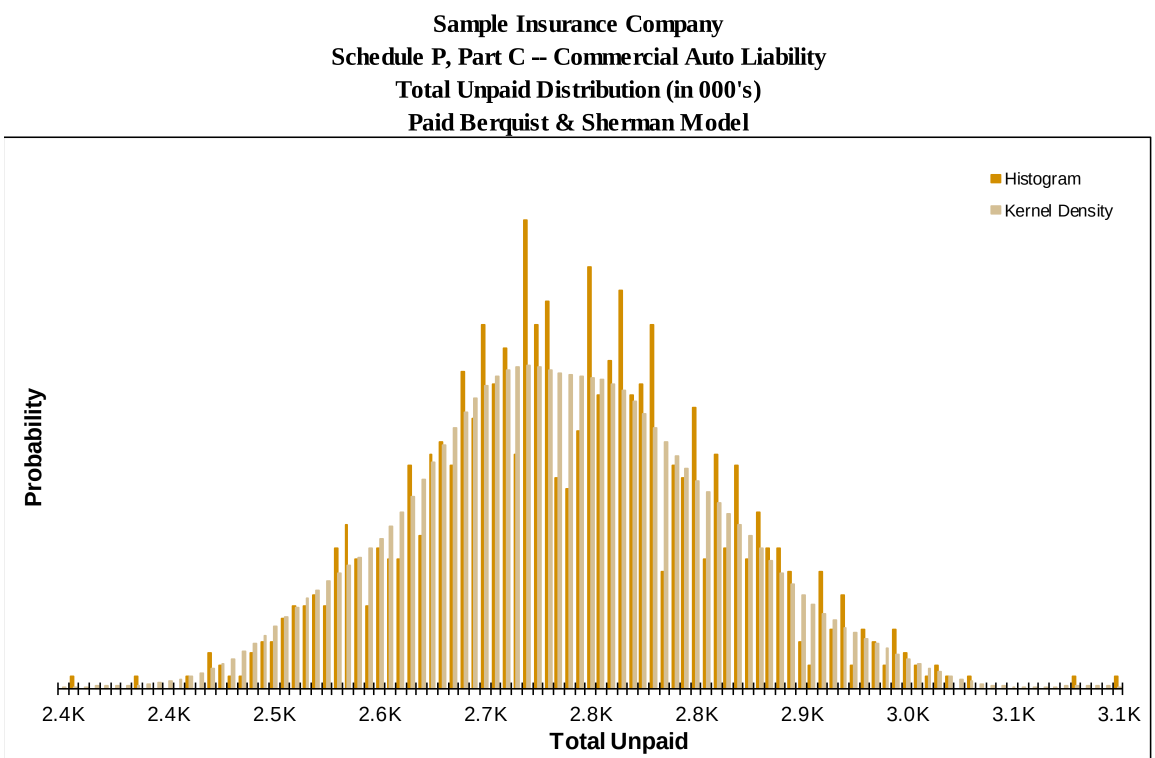


Figure D.3. Estimated unpaid model results (Incurred Berquist-Sherman)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Incurred Berquist & Sherman Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	1	3	258.7%	(25)	13	1	3	6	9
2007	1,469	3	5	152.9%	(27)	24	3	6	12	17
2008	1,387	14	8	56.8%	(23)	45	13	18	27	33
2009	1,350	27	10	38.6%	(7)	76	26	33	45	53
2010	1,342	49	15	30.0%	4	107	49	59	75	88
2011	1,198	103	23	22.1%	46	186	102	117	142	162
2012	1,061	213	39	18.1%	107	334	212	239	275	304
2013	853	418	67	16.0%	160	655	418	463	531	577
2014	645	786	122	15.6%	363	1,249	783	864	981	1,074
2015	294	1,271	182	14.3%	655	1,879	1,274	1,387	1,570	1,699
Totals	11,162	2,885	276	9.6%	1,805	3,901	2,887	3,072	3,340	3,553
Normal Dist.		2,885	276	9.6%			2,885	3,072	3,340	3,528
logNormal Dist.		2,885	281	9.8%			2,872	3,066	3,370	3,601
Gamma Dist.		2,885	276	9.6%			2,876	3,066	3,354	3,567

Figure D.4. Total unpaid claims distribution (Incurred Berquist-Sherman)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Total Unpaid Distribution (in 000's)
Incurred Berquist & Sherman Model

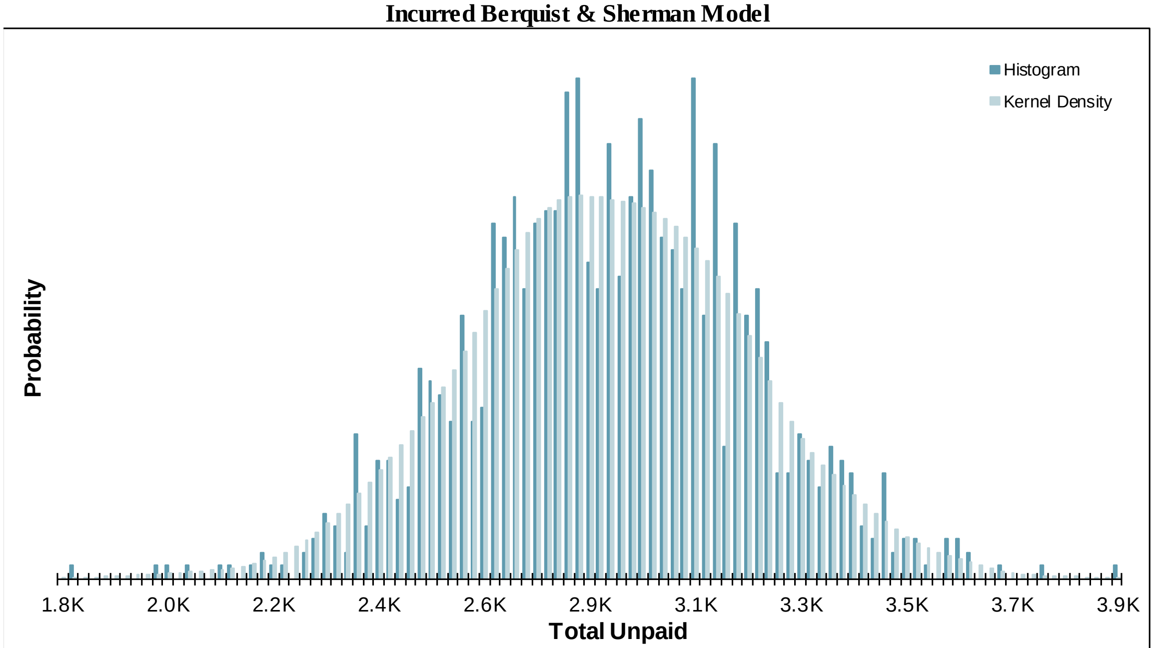


Figure D.5. Estimated unpaid model results (Paid Cape Cod)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Paid Cape Cod Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	2	7	352.9%	(28)	39	2	5	13	21
2007	1,469	5	10	202.5%	(35)	45	4	11	22	33
2008	1,387	15	12	80.5%	(24)	71	14	22	34	45
2009	1,350	29	14	49.5%	(25)	97	28	38	52	63
2010	1,342	53	17	32.0%	2	103	53	65	82	92
2011	1,198	101	19	18.9%	29	165	101	114	133	149
2012	1,061	212	24	11.3%	131	303	211	228	250	266
2013	853	407	30	7.4%	296	509	406	426	455	477
2014	645	767	46	6.0%	619	932	766	796	845	882
2015	294	1,093	73	6.7%	826	1,319	1,093	1,142	1,216	1,265
Totals	11,162	2,684	130	4.8%	2,267	3,103	2,680	2,772	2,895	2,998
Normal Dist.		2,684	130	4.8%			2,684	2,772	2,898	2,987
logNormal Dist.		2,684	131	4.9%			2,681	2,770	2,904	3,002
Gamma Dist.		2,684	130	4.8%			2,682	2,770	2,901	2,996

Figure D.6. Total unpaid claims distribution (Paid Cape Cod)

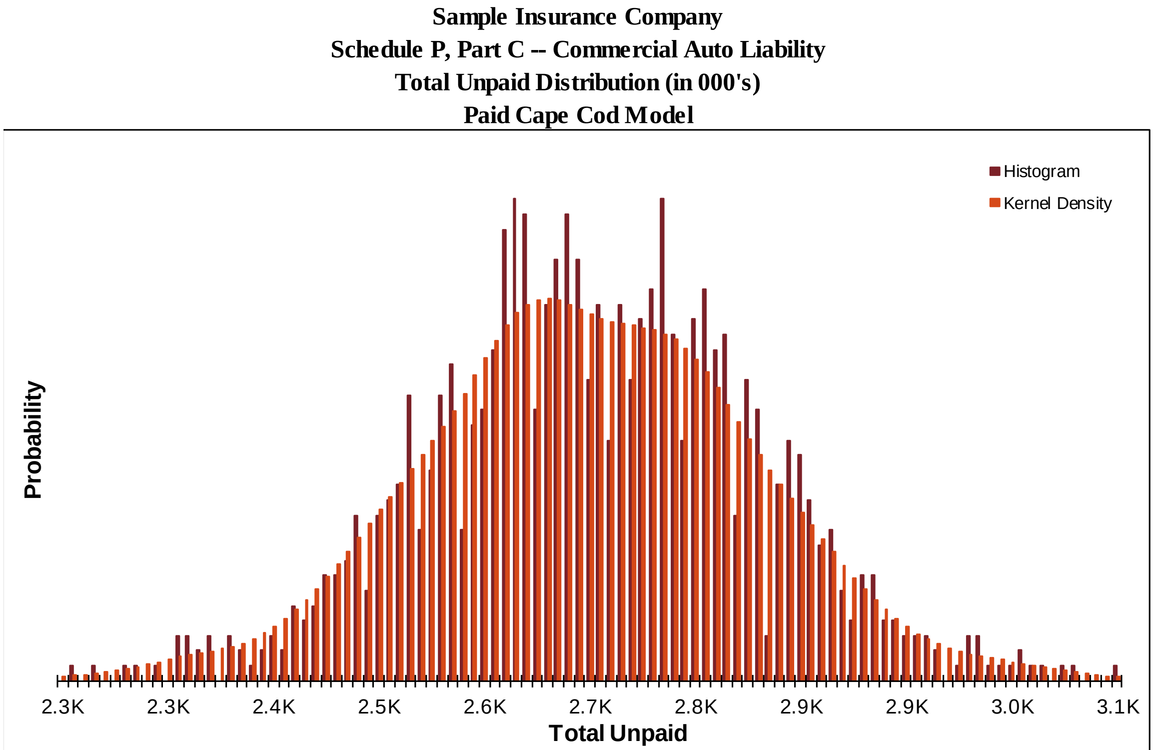


Figure D.7. Estimated unpaid model results (Incurred Cape Cod)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Incurred Cape Cod Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	(34)	1,036	-3051.6%	(32,754)	310	0	3	10	25
2007	1,469	(35)	1,169	-3319.0%	(36,931)	820	2	7	20	56
2008	1,387	42	790	1886.5%	(2,137)	18,514	10	19	40	110
2009	1,350	11	643	5723.4%	(18,141)	8,214	25	37	62	107
2010	1,342	(19)	1,741	-9140.2%	(53,588)	3,106	53	70	102	194
2011	1,198	112	749	666.5%	(13,123)	18,673	113	133	182	506
2012	1,061	459	8,951	1951.6%	(36,649)	279,556	223	255	353	979
2013	853	533	4,998	937.7%	(46,566)	125,917	417	464	613	2,008
2014	645	1,277	16,899	1323.6%	(21,260)	524,102	732	793	1,051	3,347
2015	294	402	10,559	2627.1%	(306,693)	24,226	1,047	1,143	1,540	3,321
Totals	11,162	2,747	23,026	838.1%	(130,300)	711,166	2,611	2,788	3,131	4,659
Normal Dist.		2,747	23,026	838.1%			2,747	18,278	40,621	56,313
logNormal Dist.		7,743	34,578	446.6%			1,692	5,487	29,805	97,831
Gamma Dist.		2,747	23,026	838.1%			0	0	3,034	#NUM!

Figure D.8. Total unpaid claims distribution (Incurred Cape Cod)

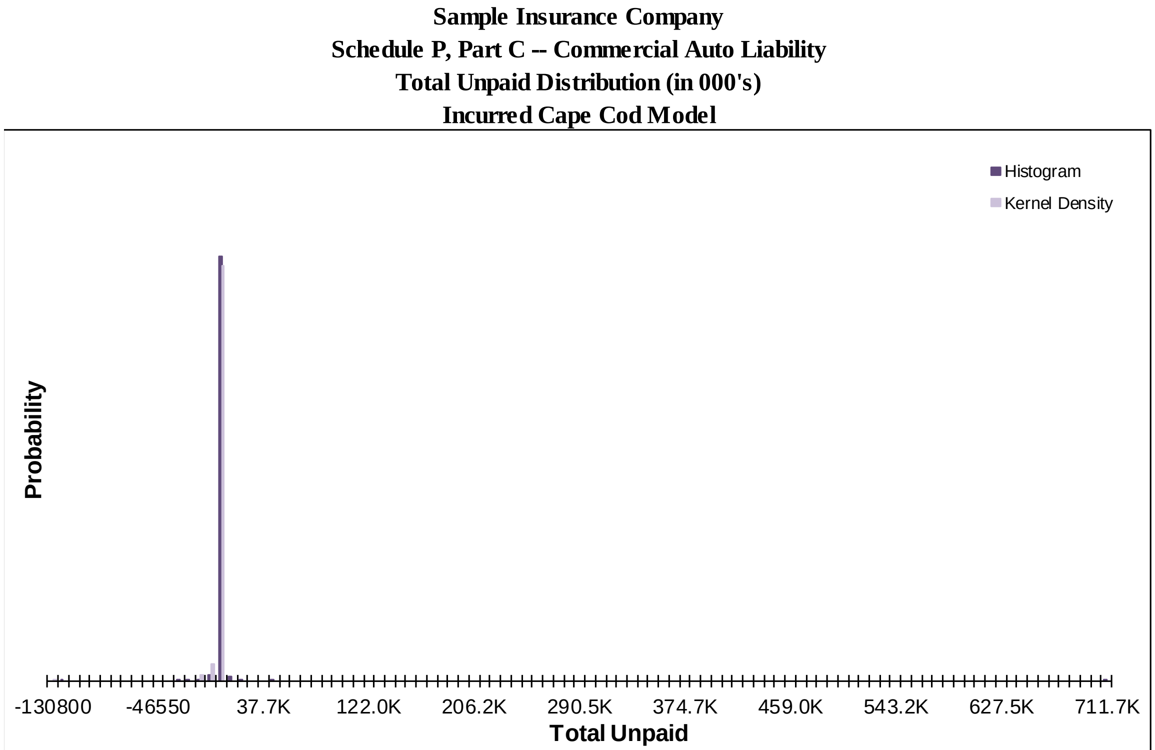


Figure D.9. Estimated unpaid model results (Paid Chain Ladder)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Paid Chain Ladder Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	1	5	494.6%	(26)	28	1	3	9	15
2007	1,469	3	7	245.1%	(38)	30	2	7	15	22
2008	1,387	13	10	75.7%	(27)	46	12	19	28	37
2009	1,350	26	12	46.8%	(29)	71	26	34	46	54
2010	1,342	51	15	29.9%	(13)	103	51	62	76	85
2011	1,198	101	17	17.2%	29	159	101	113	128	142
2012	1,061	211	21	10.2%	121	313	211	225	245	256
2013	853	406	26	6.5%	330	493	405	423	449	470
2014	645	766	34	4.4%	669	882	766	790	822	848
2015	294	1,096	43	3.9%	974	1,249	1,097	1,123	1,168	1,204
Totals	11,162	2,675	95	3.5%	2,374	3,000	2,675	2,739	2,829	2,912
Normal Dist.		2,675	95	3.5%			2,675	2,739	2,831	2,896
logNormal Dist.		2,675	95	3.6%			2,673	2,738	2,834	2,903
Gamma Dist.		2,675	95	3.5%			2,674	2,738	2,833	2,901

Figure D.10. Total unpaid claims distribution (Paid Chain Ladder)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Total Unpaid Distribution (in 000's)
Paid Chain Ladder Model

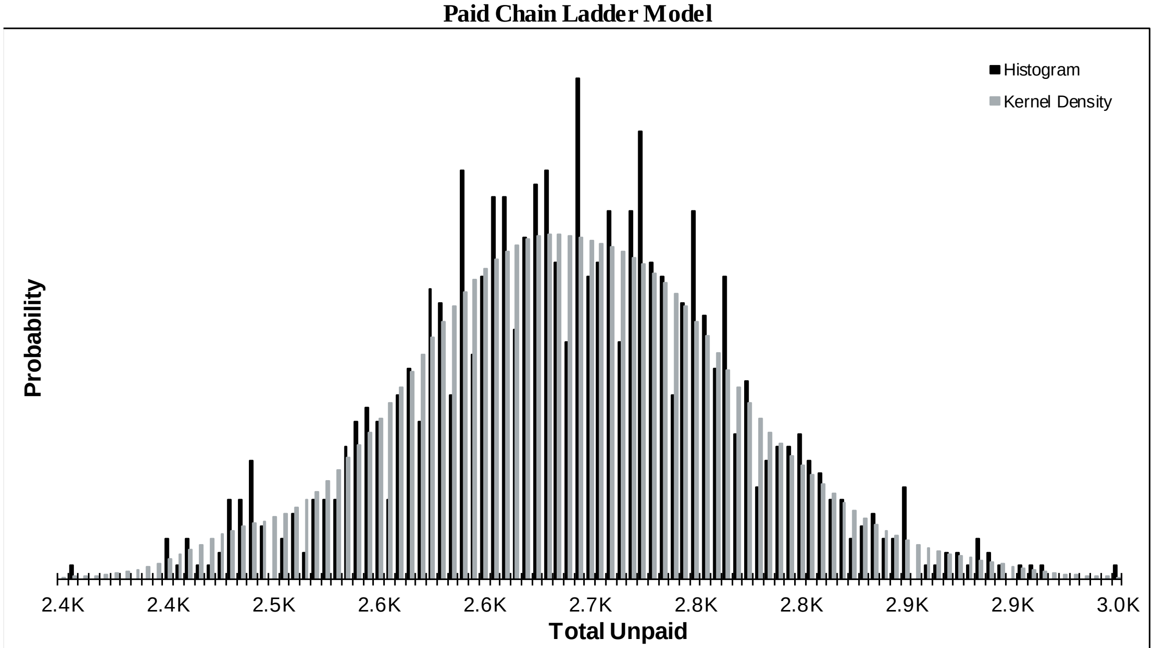


Figure D.11. Estimated unpaid model results (Incurred Chain Ladder)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Incurred Chain Ladder Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	1	6	693.6%	(33)	36	0	3	9	19
2007	1,469	3	9	357.0%	(89)	58	1	6	18	30
2008	1,387	12	16	127.8%	(44)	85	10	20	40	60
2009	1,350	25	26	100.8%	(54)	148	22	38	70	102
2010	1,342	47	47	100.1%	(147)	214	44	74	131	180
2011	1,198	97	89	92.1%	(191)	493	95	149	252	343
2012	1,061	200	196	98.1%	(548)	955	200	317	527	709
2013	853	384	366	95.2%	(1,275)	1,789	381	616	961	1,279
2014	645	718	638	88.8%	(1,751)	3,360	724	1,122	1,759	2,244
2015	294	1,071	907	84.7%	(3,060)	4,244	1,136	1,645	2,559	3,298
Totals	11,162	2,557	1,221	47.7%	(4,786)	7,504	2,541	3,329	4,603	5,312
Normal Dist.		2,557	1,221	47.7%			2,557	3,381	4,566	5,398
logNormal Dist.		4,521	9,406	208.0%			1,959	4,686	16,441	39,696
Gamma Dist.		2,557	1,221	47.7%			2,366	3,243	4,839	6,213

Figure D.12. Total unpaid claims distribution (Incurred Chain Ladder)

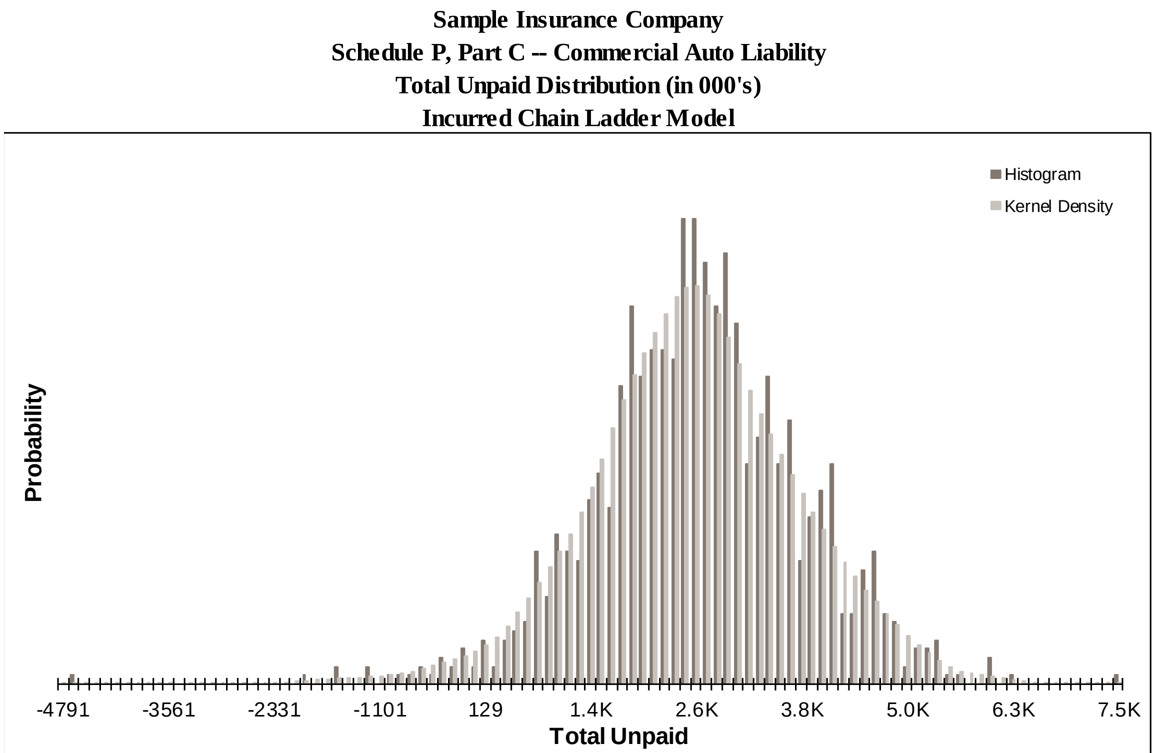


Figure D.13. Estimated unpaid model results (Paid Hoerl Curve)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Paid Hoerl Curve Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	1	8	738.6%	(34)	39	1	5	13	25
2007	1,469	2	10	562.6%	(43)	55	2	7	16	27
2008	1,387	6	12	210.0%	(38)	56	5	12	25	39
2009	1,350	14	15	104.5%	(46)	70	13	23	39	50
2010	1,342	36	19	51.6%	(20)	99	36	48	68	89
2011	1,198	91	23	25.0%	(13)	167	91	106	129	143
2012	1,061	203	27	13.4%	116	294	203	220	247	267
2013	853	395	31	7.9%	306	505	395	415	448	467
2014	645	730	40	5.5%	616	867	729	757	798	824
2015	294	1,164	50	4.3%	1,013	1,342	1,165	1,196	1,247	1,284
Totals	11,162	2,641	110	4.2%	2,328	2,997	2,641	2,719	2,822	2,896
Normal Dist.		2,641	110	4.2%			2,641	2,715	2,822	2,896
logNormal Dist.		2,641	110	4.2%			2,639	2,714	2,826	2,907
Gamma Dist.		2,641	110	4.2%			2,640	2,714	2,824	2,903

Figure D.14. Total unpaid claims distribution (Paid Hoerl Curve)

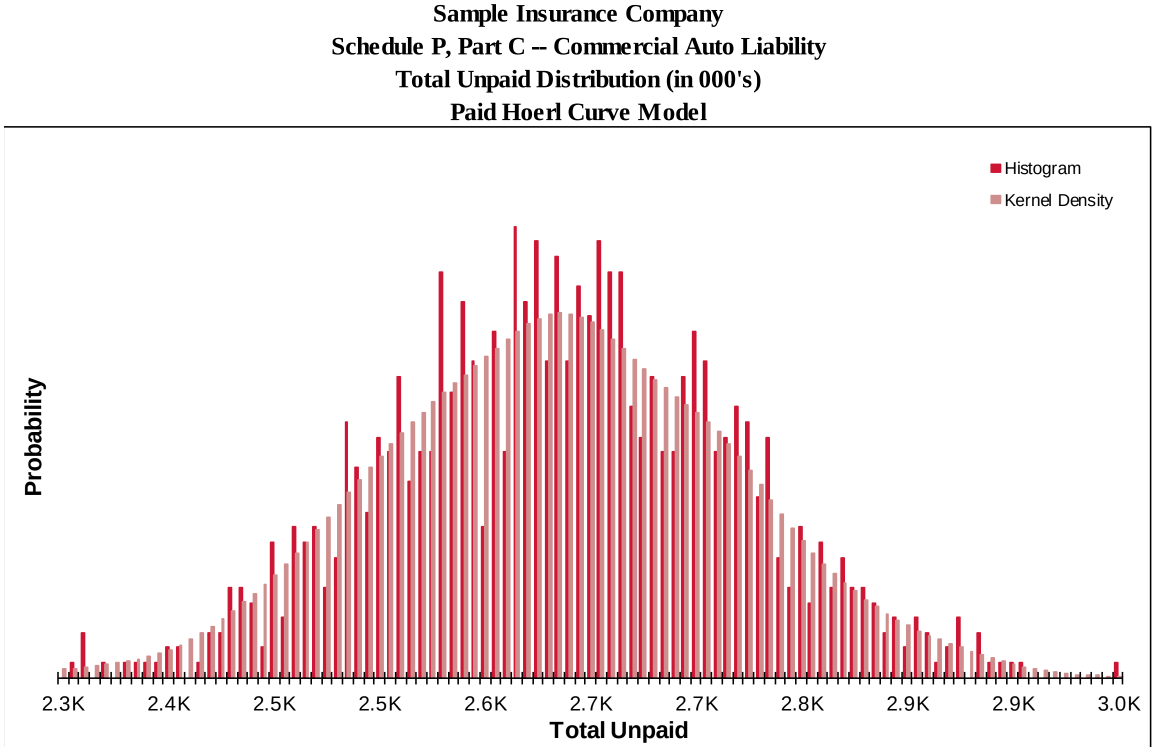


Figure D.15. Estimated unpaid model results (Incurred Hoerl Curve)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Incurred Hoerl Curve Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	###########	###########	-3162.3%	###########	###########	1	4	17	12,332,021
2007	1,469	###########	###########	-3162.3%	###########	###########	2	7	27	37,702,476
2008	1,387	###########	###########	3162.3%	###########	###########	5	12	35	###########
2009	1,350	###########	###########	3162.3%	###########	###########	14	24	182	###########
2010	1,342	###########	###########	3162.3%	(51,418,906)	###########	36	51	4,042	###########
2011	1,198	###########	###########	3162.3%	(487,667)	###########	93	113	17,391	###########
2012	1,061	###########	###########	3162.3%	(5,299,244)	###########	210	244	36,869	###########
2013	853	###########	###########	3162.3%	(2,019,237)	###########	420	475	66,618	###########
2014	645	###########	###########	3162.3%	(54,398,452)	###########	791	877	124,983	###########
2015	294	###########	###########	3162.3%	(157,144)	###########	1,290	1,410	211,790	###########
Totals	11,162	###########	###########	3162.3%	(27,682,830)	###########	2,840	3,023	469,045	###########
Normal Dist.		###########	###########	3162.3%			###########	###########	###########	###########
logNormal Dist.		###########	###########	240623181.5%			7,145	276,650	53,267,753	###########
Gamma Dist.		###########	###########	3162.3%			0	0	###########	###########

Figure D.16. Total unpaid claims distribution (Incurred Hoerl Curve)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Total Unpaid Distribution (in 000's)
Incurred Hoerl Curve Model

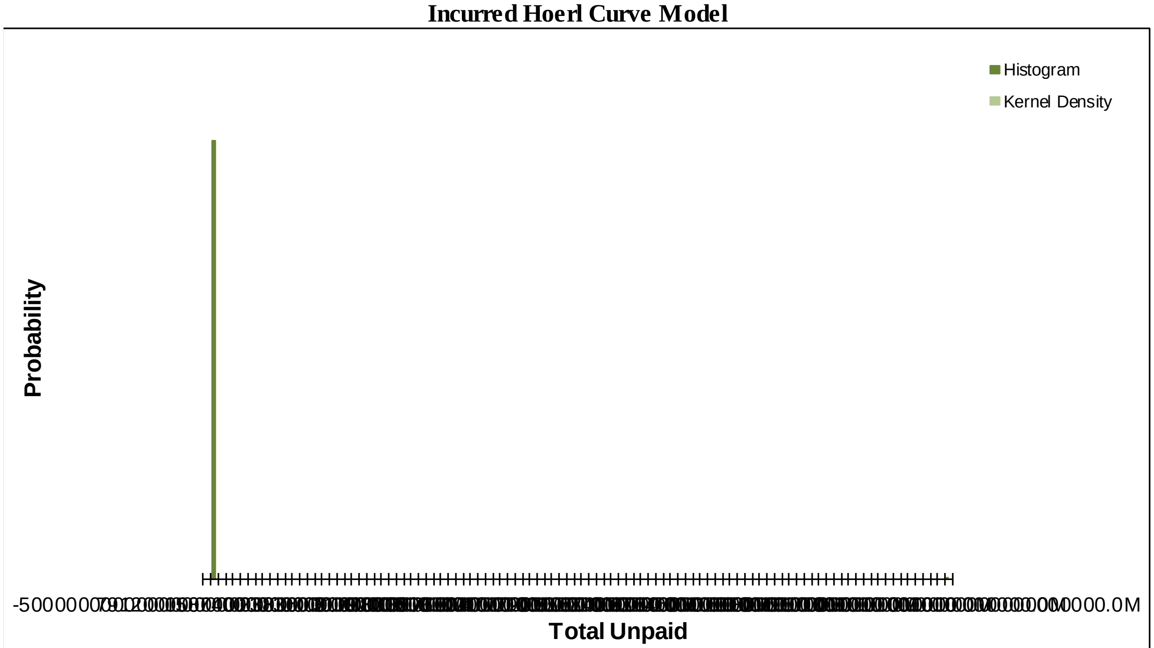


Figure D.17. Estimated unpaid model results (Paid Wright)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Paid Wright Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	1	14	1071.2%	(101)	93	1	8	21	36
2007	1,469	2	16	642.8%	(69)	100	3	10	27	44
2008	1,387	6	17	301.3%	(61)	106	5	14	34	56
2009	1,350	14	20	145.3%	(73)	145	12	25	45	67
2010	1,342	35	23	67.2%	(68)	141	35	49	73	96
2011	1,198	87	25	29.2%	7	205	86	102	128	157
2012	1,061	202	29	14.6%	88	323	202	220	249	270
2013	853	398	34	8.4%	263	509	399	420	454	476
2014	645	754	45	5.9%	628	942	754	783	824	866
2015	294	1,088	69	6.3%	853	1,350	1,086	1,131	1,204	1,255
Totals	11,162	2,587	121	4.7%	2,151	3,124	2,585	2,664	2,786	2,907
Normal Dist.		2,587	121	4.7%			2,587	2,668	2,786	2,868
logNormal Dist.		2,587	121	4.7%			2,584	2,667	2,790	2,881
Gamma Dist.		2,587	121	4.7%			2,585	2,667	2,789	2,876

Figure D.18. Total unpaid claims distribution (Paid Wright)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Total Unpaid Distribution (in 000's)
Paid Wright Model

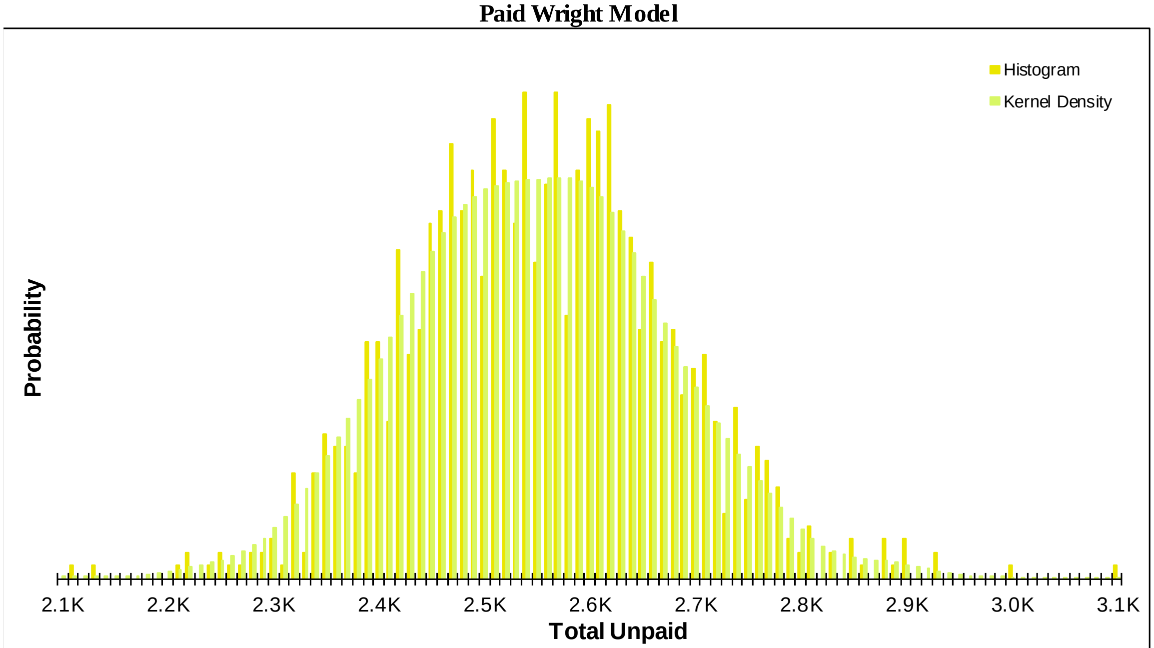


Figure D.19. Estimated unpaid model results (Incurred Wright)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Incurred Wright Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	###########	###########	-3155.7%	###########	###########	1	7	18	37
2007	1,469	###########	###########	3107.7%	(10,943,017)	###########	2	9	26	2,259
2008	1,387	###########	###########	2219.7%	(151,422,813)	###########	5	14	37	2,113
2009	1,350	###########	###########	3289.5%	###########	###########	14	29	54	620
2010	1,342	###########	###########	-3134.5%	###########	###########	39	56	89	14,339
2011	1,198	###########	###########	3109.8%	6	###########	103	124	167	392,998
2012	1,061	###########	###########	3105.4%	65	###########	226	251	293	658,553
2013	853	###########	###########	3107.7%	(2,533)	###########	435	466	516	1,573,045
2014	645	###########	###########	3111.5%	(1,508)	###########	763	806	890	2,689,124
2015	294	###########	###########	3108.7%	338	###########	1,103	1,166	1,283	3,789,386
Totals	11,162	###########	###########	3108.9%	(1,002)	###########	2,684	2,796	2,961	9,258,513
Normal Dist.		###########	###########	3108.9%			###########	###########	###########	###########
logNormal Dist.		18,838	106,398	564.8%			3,284	11,586	71,059	253,986
Gamma Dist.		###########	###########	3108.9%			0	0	0	###########

Figure D.20. Total unpaid claims distribution (Incurred Wright)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Total Unpaid Distribution (in 000's)
Incurred Wright Model

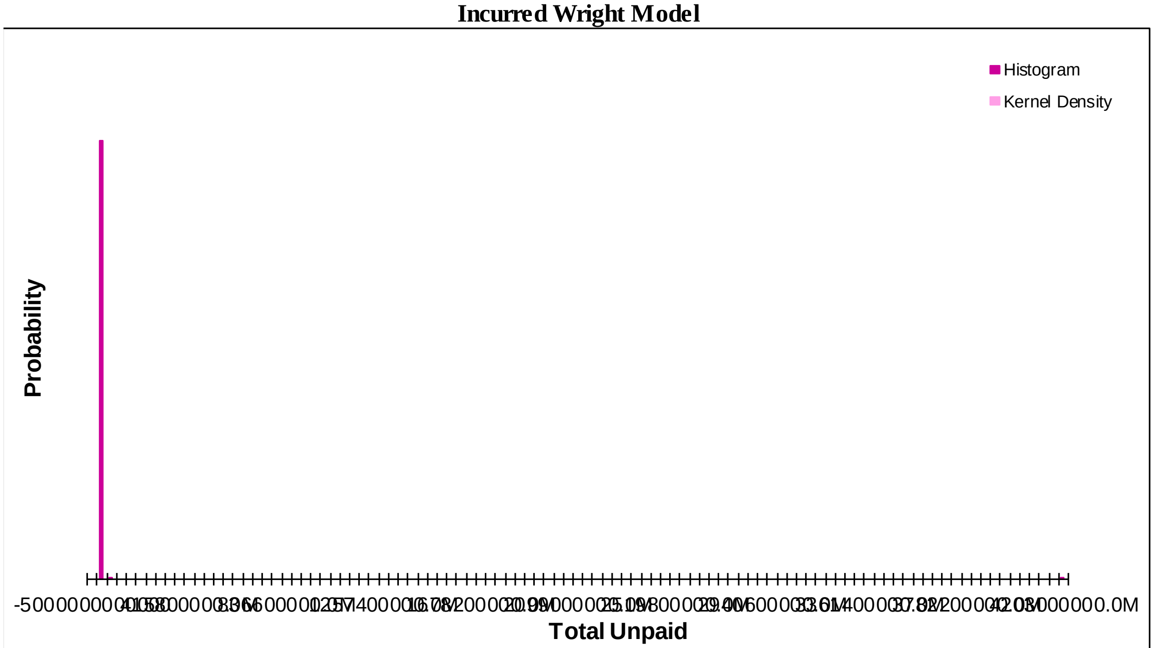


Figure D.21. Model weights by accident year

Accident Year	Model Weights by Accident Year								
	Paid BS	Incd BS	Paid CC	Paid CL	Incd CL	Paid HC	Paid WR	[redacted]	TOTAL
2006	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2007	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2008	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2009	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2010	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2011	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2012	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2013	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2014	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%
2015	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%		100.0%

Figure D.22. Estimated mean unpaid by model

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Summary of Results by Model (in 000's)

Accident	Mean Estimated Unpaid							
	Berquist & Sherman		Cape Cod		Chain Ladder		Hoerl Curve	Best Est.
Year	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	(Weighted)
2006	1	1	2	1	1	1	1	1
2007	4	3	5	3	3	2	2	3
2008	14	14	15	13	12	6	6	11
2009	28	27	29	26	25	14	14	23
2010	50	49	53	51	47	36	35	47
2011	103	103	101	101	97	91	87	99
2012	209	213	212	211	200	203	202	207
2013	402	418	407	406	384	395	398	403
2014	742	786	767	766	718	730	754	756
2015	1,176	1,271	1,093	1,096	1,071	1,164	1,088	1,130
Totals	2,729	2,885	2,684	2,675	2,557	2,641	2,587	2,679

Figure D.23. Estimated ranges

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Summary of Results by Model (in 000's)

Accident		Ranges			
	Best Est.	Weighted		Modeled	
Year	(Weighted)	Minimum	Maximum	Mininum	Maximum
2006	1	1	2	1	2
2007	3	2	5	2	5
2008	11	6	15	6	15
2009	23	14	29	14	29
2010	47	35	53	35	53
2011	99	87	103	87	103
2012	207	200	213	200	213
2013	403	384	418	384	418
2014	756	718	786	718	786
2015	1,130	1,071	1,271	1,071	1,271
Totals	2,679	2,517	2,894	2,557	2,885

Figure D.24. Reconciliation of total results (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Reconciliation of Total Results (in 000's)
Best Estimate (Weighted)

Accident Year	Paid To Date	Incurred To Date	Case Reserves	IBNR	Estimate of Ultimate	Estimate of Unpaid
2006	1,563	1,577	14	(12)	1,565	1
2007	1,469	1,505	36	(33)	1,472	3
2008	1,387	1,436	49	(38)	1,398	11
2009	1,350	1,417	67	(44)	1,373	23
2010	1,342	1,445	102	(56)	1,389	47
2011	1,198	1,345	147	(48)	1,297	99
2012	1,061	1,339	278	(71)	1,267	207
2013	853	1,327	474	(71)	1,256	403
2014	645	1,442	797	(41)	1,401	756
2015	294	1,422	1,128	1	1,424	1,130
Totals	11,162	14,255	3,093	(413)	13,841	2,679

Figure D.25. Estimated unpaid model results (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Unpaid (in 000's)
Best Estimate (Weighted)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,563	1	7	526.2%	(101)	95	1	4	12	23
2007	1,469	3	9	274.3%	(89)	100	3	8	18	30
2008	1,387	11	13	119.2%	(71)	115	11	18	31	46
2009	1,350	23	17	75.1%	(80)	216	24	33	50	68
2010	1,342	47	24	52.3%	(111)	272	47	59	82	118
2011	1,198	99	39	39.1%	(235)	571	100	115	147	224
2012	1,061	207	80	38.8%	(880)	891	208	228	290	490
2013	853	403	139	34.6%	(974)	1,841	401	428	551	920
2014	645	756	244	32.3%	(1,740)	3,765	755	793	1,014	1,635
2015	294	1,130	370	32.8%	(2,262)	4,783	1,133	1,199	1,533	2,406
Totals	11,162	2,679	474	17.7%	(500)	6,591	2,683	2,837	3,362	4,119
Normal Dist.		2,679	474	17.7%			2,679	2,999	3,458	3,781
logNormal Dist.		2,749	897	32.6%			2,614	3,239	4,411	5,478
Gamma Dist.		2,679	474	17.7%			2,651	2,982	3,503	3,903

Figure D.26. Estimated cash flow (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Calendar Year Unpaid (in 000's)
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	1,069	176	16.4%	38	2,391	1,072	1,135	1,337	1,599
2017	744	136	18.3%	(166)	1,830	745	794	945	1,154
2018	443	90	20.2%	(178)	1,298	444	478	568	714
2019	229	52	22.7%	(149)	661	229	252	301	385
2020	106	29	27.5%	(90)	313	106	122	150	182
2021	48	19	40.3%	(55)	176	47	60	80	99
2022	23	15	63.7%	(37)	139	23	32	47	61
2023	11	12	104.9%	(49)	94	11	18	30	42
2024	3	8	248.7%	(60)	55	3	8	16	25
2025	1	6	454.7%	(86)	61	1	4	11	18
2026	1	4	777.7%	(46)	59	0	2	7	14
2027	0	3	1416.3%	(27)	40	0	1	4	9
Totals	2,679	474	17.7%	(500)	6,591	2,683	2,837	3,362	4,119

Figure D.27. Estimated loss ratio (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Ultimate Loss Ratios (in 000's)
Best Estimate (Weighted)

Accident Year	Earned Premium	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	1,748	86.6%	27.2%	31.4%	-264.5%	340.5%	88.5%	91.5%	108.2%	180.7%
2007	1,810	80.6%	24.5%	30.5%	-193.1%	335.3%	81.8%	84.5%	101.8%	166.6%
2008	1,915	73.5%	22.2%	30.3%	-139.9%	287.7%	74.4%	77.5%	93.9%	149.7%
2009	2,275	60.3%	19.4%	32.2%	-126.9%	245.5%	60.6%	62.7%	78.5%	128.9%
2010	2,524	53.4%	18.0%	33.6%	-107.9%	232.9%	54.0%	56.1%	70.8%	116.8%
2011	2,445	53.0%	17.4%	32.9%	-132.3%	248.4%	53.2%	55.0%	69.9%	116.6%
2012	2,543	49.5%	18.2%	36.9%	-190.8%	224.8%	49.7%	51.4%	67.2%	117.4%
2013	2,461	51.1%	17.5%	34.2%	-117.8%	240.1%	50.9%	52.7%	69.3%	116.5%
2014	2,485	56.2%	18.1%	32.2%	-132.1%	278.3%	56.1%	58.3%	75.8%	123.2%
2015	2,383	60.2%	19.8%	32.9%	-118.2%	257.3%	60.3%	63.7%	81.5%	130.0%
Totals	22,588	60.8%	6.2%	10.2%	20.4%	93.6%	61.1%	63.8%	70.9%	77.6%

Figure D.28. Estimated unpaid claim runoff (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Calendar Year Unpaid Claim Runoff (in 000's)
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	2,679	474	17.7%	(500)	6,591	2,683	2,837	3,362	4,119
2016	1,610	308	19.1%	(590)	4,201	1,612	1,714	2,049	2,557
2017	865	179	20.7%	(445)	2,522	866	933	1,114	1,419
2018	422	97	23.0%	(268)	1,247	422	465	561	715
2019	193	54	28.1%	(119)	634	193	222	277	342
2020	88	35	39.9%	(86)	339	87	108	143	179
2021	40	25	62.0%	(65)	204	39	54	80	105
2022	16	17	105.8%	(80)	113	16	26	45	64
2023	5	12	222.4%	(77)	93	5	11	25	38
2024	2	8	387.7%	(83)	81	2	6	15	28
2025	1	5	694.5%	(49)	62	0	3	9	18
2026	0	3	1416.3%	(27)	40	0	1	4	9

Figure D.29. Mean of incremental values (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Mean Values (in 000's)												
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**
2006	321	373	334	236	133	63	27	13	8	2	1	0	0
2007	309	359	322	227	128	61	26	13	8	2	1	0	0
2008	299	347	311	219	124	59	25	13	8	2	1	0	0
2009	291	338	303	213	121	57	25	12	8	2	1	0	0
2010	286	332	298	209	119	56	24	12	7	2	1	0	0
2011	275	320	286	202	114	54	23	11	7	2	1	0	0
2012	267	310	278	196	110	53	23	11	7	2	1	0	0
2013	267	310	278	196	111	53	23	11	7	2	1	0	0
2014	297	345	309	218	123	58	25	12	8	2	1	0	0
2015	305	353	317	223	126	60	26	12	8	2	1	0	0

Figure D.30. Standard deviation of incremental values (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Standard Error Values (in 000's)												
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**
2006	104	117	106	75	44	23	14	10	9	6	5	4	3
2007	98	110	99	71	41	22	13	10	9	6	4	4	3
2008	94	105	96	67	40	21	13	10	9	6	4	4	3
2009	98	109	99	70	41	22	12	10	8	6	4	4	3
2010	100	112	101	72	42	22	12	10	8	6	4	3	3
2011	94	105	95	68	39	21	12	9	8	5	4	3	3
2012	101	114	103	73	43	22	12	9	8	5	4	3	3
2013	95	106	96	69	39	20	12	9	8	5	4	3	3
2014	99	111	100	71	41	22	12	10	8	5	4	3	3
2015	103	117	106	75	43	23	13	10	8	5	4	3	3

Figure D.31. Coefficient of variation of incremental values (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Coefficients of Variation												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156
2006	32.4%	31.4%	31.7%	32.0%	33.1%	36.5%	49.8%	75.4%	108.1%	300.4%	599.5%	982.0%	1807.7%
2007	31.6%	30.6%	30.7%	31.2%	31.9%	36.4%	49.4%	75.6%	108.5%	278.9%	601.2%	1156.2%	1193.4%
2008	31.5%	30.4%	30.7%	30.7%	32.1%	35.9%	50.0%	76.0%	110.9%	297.0%	666.9%	1051.0%	1366.6%
2009	33.5%	32.3%	32.5%	32.9%	33.9%	37.9%	49.9%	77.7%	108.6%	292.6%	586.7%	1061.1%	1535.5%
2010	34.8%	33.7%	34.0%	34.3%	35.3%	38.5%	51.0%	80.1%	109.0%	300.1%	572.5%	934.5%	2346.4%
2011	34.0%	32.9%	33.2%	33.6%	34.5%	38.4%	51.9%	78.3%	114.1%	297.8%	581.3%	1007.1%	1454.7%
2012	37.9%	36.9%	37.1%	37.5%	38.5%	42.1%	54.5%	81.5%	117.4%	298.5%	536.2%	1189.7%	1266.2%
2013	35.4%	34.2%	34.5%	35.0%	35.4%	38.9%	51.9%	79.6%	113.7%	294.5%	575.8%	963.2%	1305.7%
2014	33.4%	32.2%	32.5%	32.8%	33.6%	37.7%	48.8%	77.3%	106.5%	278.9%	578.7%	999.3%	1598.3%
2015	33.9%	33.0%	33.3%	33.6%	34.0%	37.5%	49.6%	76.4%	106.0%	271.3%	531.0%	860.7%	1416.3%

Figure D.32. Total unpaid claims distribution (weighted)

Sample Insurance Company
Schedule P, Part C -- Commercial Auto Liability
Total Unpaid Distribution (in 000's)
Best Estimate (Weighted)

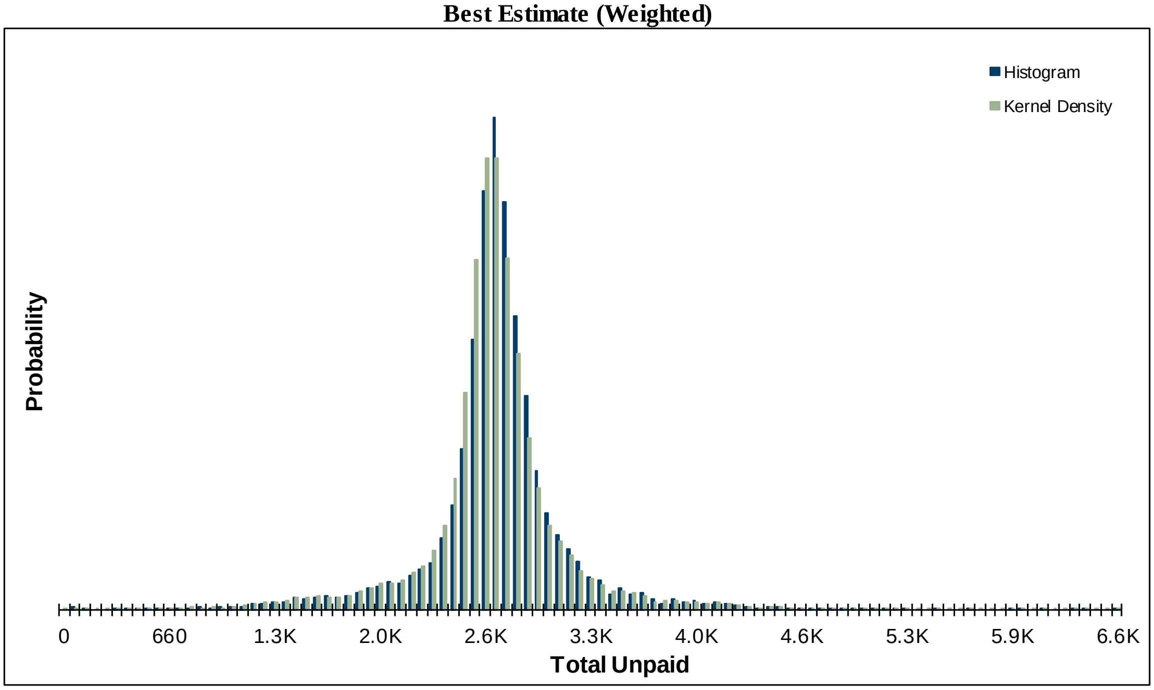
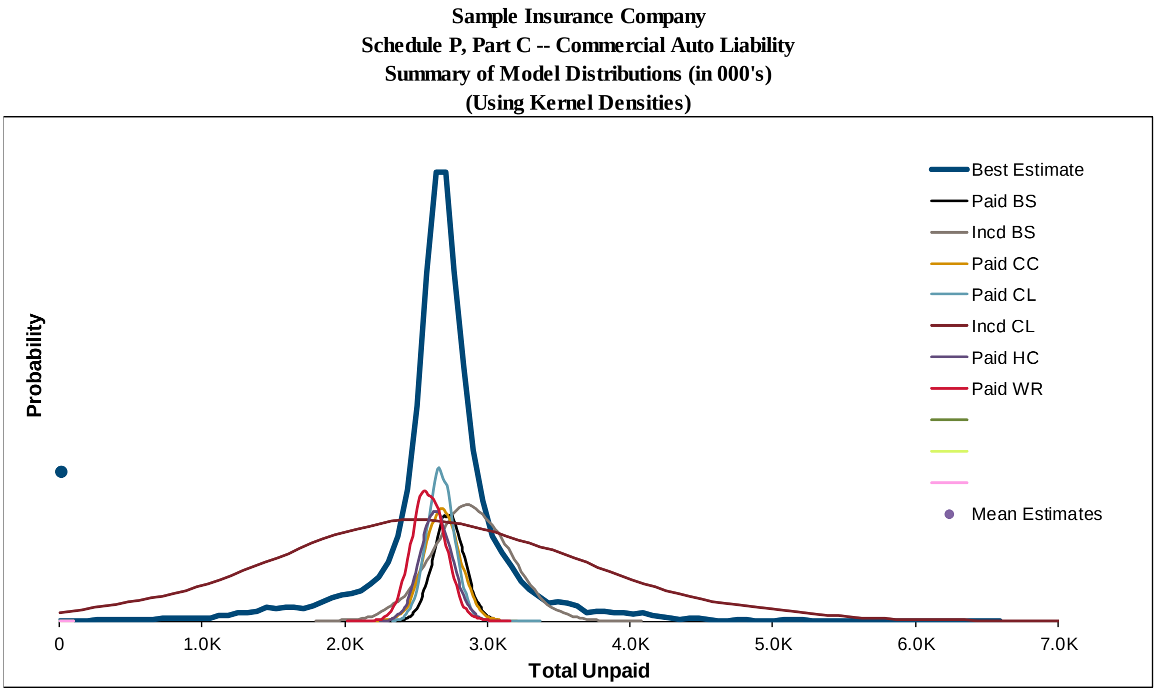


Figure D.33. Summary of model distributions



Appendix E – Aggregate Results

In this appendix the results for the correlated aggregate of the three Schedule P lines of business (Parts A, B, and C) are shown, using the correlation calculated from the paid data for the Berquist-Sherman model.

Figure E.1. Estimated unpaid model results

Sample Insurance Company
Aggregate Three Lines of Business
Accident Year Unpaid (in 000's)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	18,613	146	1,002	688.1%	(2,013)	74,778	37	55	421	2,422
2007	20,618	198	993	500.3%	(1,523)	37,034	70	94	503	3,069
2008	22,866	246	927	377.4%	(5,763)	54,447	128	162	542	3,227
2009	22,842	367	1,286	350.7%	(2,918)	90,399	230	268	695	3,778
2010	22,351	535	1,359	254.3%	(1,875)	69,139	406	452	860	3,458
2011	22,422	869	1,266	145.7%	(3,632)	68,690	760	826	1,253	4,003
2012	24,350	1,589	939	59.1%	(4,107)	27,387	1,518	1,633	2,198	4,927
2013	19,973	2,814	1,424	50.6%	(8,046)	80,667	2,785	2,963	3,667	6,153
2014	18,919	5,418	4,384	80.9%	(8,120)	407,319	5,420	5,768	6,863	9,408
2015	15,961	13,369	3,352	25.1%	(11,431)	98,644	13,319	14,627	17,722	21,777
Totals	208,915	25,550	9,304	36.4%	(815)	476,278	24,635	26,612	32,642	55,933
Normal Dist.		25,550	9,304	36.4%			25,550	31,826	40,854	47,195
logNormal Dist.		25,528	6,217	24.4%			24,803	29,163	36,812	43,354
Gamma Dist.		25,550	9,304	36.4%			24,430	31,065	42,526	52,000
TVaR							28,995	32,475	48,429	89,074
Normal TVaR							32,974	37,377	44,742	50,348
logNormal TVaR							30,371	33,900	40,865	47,165
Gamma TVaR							32,838	38,140	48,373	57,295

Figure E.2. Estimated cash flow

Sample Insurance Company
Aggregate Three Lines of Business
Calendar Year Unpaid (in 000's)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	12,497	2,095	16.8%	(797)	23,980	12,494	13,633	15,903	17,708
2017	5,432	760	14.0%	175	10,046	5,475	5,859	6,587	7,241
2018	2,945	423	14.4%	(100)	5,390	2,965	3,176	3,586	3,989
2019	1,562	310	19.8%	(95)	13,391	1,553	1,674	1,959	2,463
2020	902	857	95.0%	(102)	60,941	810	893	1,281	3,189
2021	546	835	153.0%	(320)	33,504	431	490	928	3,022
2022	361	817	226.4%	(880)	44,982	242	289	756	2,813
2023	283	1,087	384.4%	(1,221)	70,925	144	183	681	3,090
2024	228	1,139	499.5%	(714)	61,008	84	120	590	3,055
2025	190	1,049	551.1%	(1,481)	53,144	46	79	587	3,006
2026	165	825	499.4%	(1,571)	23,126	27	54	554	3,206
2027	160	1,260	789.6%	(3,531)	74,987	14	33	480	3,172
2028	169	3,600	2134.8%	(7,667)	342,488	6	12	412	2,742
2029	110	1,288	1168.6%	(1,140)	66,389	2	4	196	2,239
Totals	25,550	9,304	36.4%	(815)	476,278	24,635	26,612	32,642	55,933

Figure E.3. Estimated loss ratio

Sample Insurance Company
Aggregate Three Lines of Business
Accident Year Ultimate Loss Ratios (in 000's)

Accident Year	Earned Premium	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	25,305	72.6%	15.3%	21.1%	-52.5%	368.8%	74.2%	79.3%	91.2%	108.2%
2007	25,577	78.6%	15.6%	19.8%	-34.7%	224.3%	80.3%	85.5%	97.6%	114.8%
2008	27,155	82.2%	16.0%	19.5%	-18.6%	276.9%	84.0%	89.3%	102.3%	119.9%
2009	30,529	74.6%	14.5%	19.4%	-59.9%	373.8%	75.7%	80.8%	93.3%	109.0%
2010	34,399	65.2%	13.0%	19.9%	-24.3%	269.8%	66.3%	70.9%	81.9%	94.6%
2011	36,231	63.2%	12.5%	19.8%	-47.2%	251.4%	64.2%	68.8%	79.9%	92.2%
2012	36,863	70.7%	14.1%	20.0%	-37.8%	146.6%	70.7%	77.5%	92.4%	107.1%
2013	37,678	60.2%	12.8%	21.3%	-34.6%	271.2%	60.8%	65.9%	77.7%	89.1%
2014	38,101	63.9%	16.8%	26.2%	-54.2%	1115.7%	64.2%	69.8%	82.1%	95.7%
2015	37,997	79.2%	15.9%	20.1%	-36.7%	313.1%	78.0%	86.8%	104.4%	121.2%
Totals	329,835	70.4%	4.9%	6.9%	47.9%	195.6%	70.5%	73.2%	77.3%	81.3%

Figure E.4. Estimated unpaid claim runoff

Sample Insurance Company
Aggregate Three Lines of Business
Calendar Year Unpaid Claim Runoff (in 000's)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	25,550	9,304	36.4%	(815)	476,278	24,635	26,612	32,642	55,933
2016	13,054	8,804	67.4%	(18)	464,252	11,965	12,832	17,870	43,814
2017	7,621	8,748	114.8%	(193)	459,695	6,389	6,960	12,059	39,214
2018	4,676	8,706	186.2%	(93)	456,649	3,366	3,757	8,953	36,003
2019	3,113	8,561	275.0%	2	452,976	1,799	2,096	7,305	33,834
2020	2,212	8,057	364.3%	67	439,029	986	1,225	5,912	30,057
2021	1,665	7,588	455.6%	21	433,153	557	758	4,879	25,743
2022	1,305	7,152	548.1%	14	427,965	318	480	3,998	21,821
2023	1,022	6,647	650.3%	(9,274)	427,465	177	304	3,245	18,524
2024	794	6,071	764.6%	(9,339)	425,019	94	187	2,507	14,898
2025	604	5,472	906.6%	(9,336)	411,276	49	108	1,873	11,341
2026	438	4,936	1126.0%	(9,105)	393,463	23	52	1,269	7,908
2027	279	3,988	1430.4%	(7,664)	342,491	8	17	690	5,233
2028	110	1,288	1168.6%	(1,140)	66,389	2	4	196	2,239

Figure E.5. Mean of incremental values

Sample Insurance Company
Aggregate Three Lines of Business
Accident Year Incremental Values by Development Period

Accident	Mean Values (in 000's)														
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
2006	9,173	4,740	1,972	1,141	608	302	149	77	45	28	16	14	17	29	68
2007	10,218	5,151	2,099	1,201	639	318	157	82	47	30	18	16	20	34	80
2008	11,568	5,691	2,256	1,285	680	339	169	88	51	33	19	17	20	32	73
2009	11,629	5,858	2,348	1,338	711	353	176	92	53	34	20	18	22	38	90
2010	11,538	5,742	2,289	1,303	691	343	171	89	52	33	20	18	22	38	92
2011	12,007	5,790	2,270	1,288	680	340	170	88	52	33	20	17	22	37	91
2012	14,275	6,479	2,392	1,337	699	350	176	93	54	34	21	18	22	36	84
2013	11,853	5,778	2,251	1,277	674	335	168	87	51	33	20	17	22	38	92
2014	12,776	6,149	2,397	1,357	716	357	178	94	54	35	21	18	23	43	125
2015	16,732	7,423	2,675	1,491	773	388	195	103	60	38	23	20	25	43	110

Figure E.6. Standard deviation of incremental values

Sample Insurance Company
Aggregate Three Lines of Business
Accident Year Incremental Values by Development Period

Accident	Standard Deviation Values (in 000's)														
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
2006	2,168	1,131	467	268	145	74	39	22	15	11	12	22	58	187	752
2007	2,302	1,179	475	272	146	75	40	23	16	11	13	26	68	210	696
2008	2,598	1,299	517	293	156	80	43	25	17	12	14	26	65	192	656
2009	2,610	1,303	513	293	158	81	43	25	16	12	15	29	76	249	951
2010	2,637	1,313	516	292	158	81	44	25	17	12	15	29	79	269	994
2011	2,801	1,342	525	295	158	82	44	26	17	12	15	28	77	250	917
2012	3,566	1,581	577	320	173	90	48	29	20	13	15	28	67	192	618
2013	3,004	1,428	550	312	166	85	47	26	18	12	15	29	78	254	950
2014	3,196	1,509	572	323	171	90	48	28	19	13	15	32	108	572	3,565
2015	4,007	1,903	637	359	187	98	54	32	22	15	17	32	92	328	1,288

Figure E.7. Coefficient of variation of incremental values

Sample Insurance Company
Aggregate Three Lines of Business
Accident Year Incremental Values by Development Period

Accident	Coefficients of Variation														
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
2006	23.6%	23.9%	23.7%	23.5%	23.9%	24.6%	26.1%	28.6%	33.7%	38.3%	74.9%	156.3%	332.5%	643.8%	1099.6%
2007	22.5%	22.9%	22.7%	22.7%	22.8%	23.6%	25.6%	27.7%	33.0%	36.7%	74.4%	164.6%	342.8%	612.0%	865.2%
2008	22.5%	22.8%	22.9%	22.8%	23.0%	23.7%	25.7%	28.1%	33.5%	36.9%	73.2%	155.0%	321.5%	591.5%	898.4%
2009	22.4%	22.2%	21.8%	21.9%	22.2%	22.9%	24.4%	26.9%	31.1%	34.5%	72.3%	160.6%	344.1%	661.7%	1057.6%
2010	22.9%	22.9%	22.6%	22.4%	22.9%	23.5%	25.6%	27.8%	32.6%	35.7%	73.4%	165.2%	363.3%	708.1%	1079.9%
2011	23.3%	23.2%	23.1%	22.9%	23.3%	24.0%	26.0%	28.9%	33.9%	37.1%	73.8%	163.4%	354.9%	668.2%	1007.6%
2012	25.0%	24.4%	24.1%	24.0%	24.7%	25.6%	27.5%	31.1%	36.0%	39.1%	73.3%	151.8%	301.9%	529.0%	733.9%
2013	25.3%	24.7%	24.4%	24.5%	24.7%	25.4%	27.8%	30.1%	34.9%	37.6%	74.0%	164.7%	356.7%	673.6%	1037.3%
2014	25.0%	24.5%	23.9%	23.8%	23.8%	25.1%	26.8%	29.9%	34.4%	37.3%	73.1%	175.0%	466.6%	1333.7%	2845.1%
2015	23.9%	25.6%	23.8%	24.0%	24.1%	25.3%	27.7%	31.1%	36.3%	40.0%	72.2%	160.0%	368.1%	757.1%	1168.6%

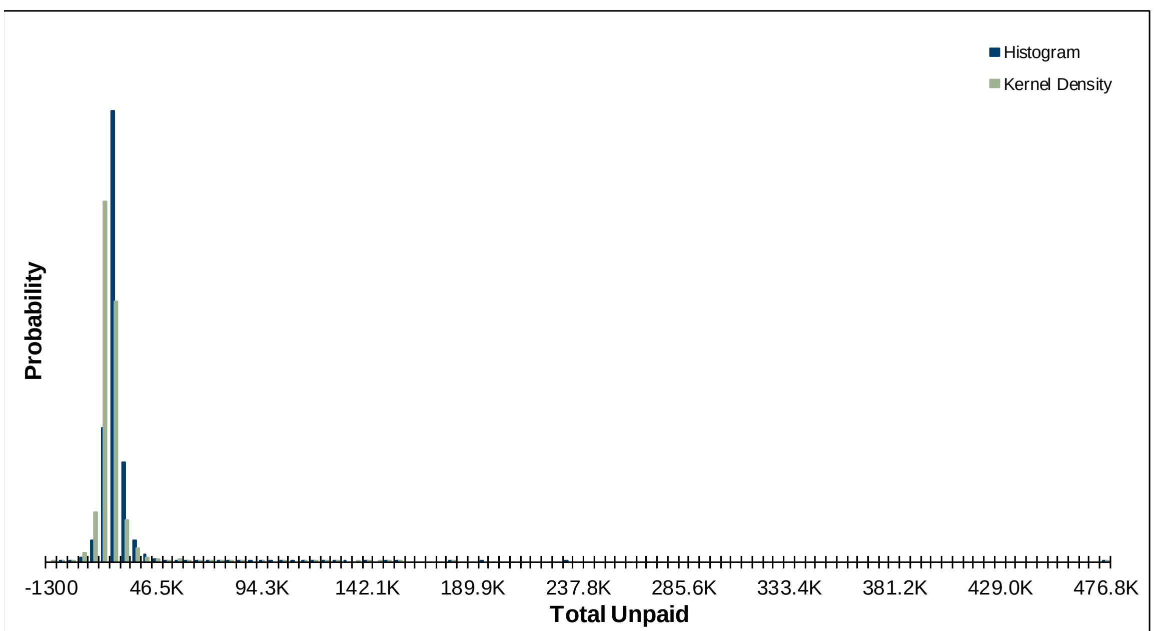
Figure E.8. Calculation of risk based capital

Sample Insurance Company
Aggregate Three Lines of Business
Indicated Unpaid Claim Risk Portion of Required Capital (in 000's)

LOB / Segment	Earned Premium	Mean Unpaid	99.0% Unpaid	Value at Risk Capital	Allocated Capital	Unpaid Ratio	Premium Ratio
Homeowners / Farmowners	15,148	5,792	10,410	4,618	4,048	69.9%	26.7%
Private Passenger Auto Liabilit	20,467	17,079	45,682	28,602	25,072	146.8%	122.5%
Commercial Auto Liability	2,383	2,679	4,119	1,439	1,262	47.1%	52.9%
Total	37,997	25,550	60,210	34,660			
Aggregate	37,997	25,550	55,933	30,382	30,382	118.9%	80.0%

Figure E.9. Total unpaid claims distribution

Sample Insurance Company
Aggregate Three Lines of Business
Total Unpaid Distribution



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Abbreviations and notations

Collect here in alphabetical order all abbreviations and notations used in the paper

AIC, akaiki information criteria	CoV, coefficient of variation
BIC, bayesian information criteria	HC, hoerl curve
BS, berquist-sherman	CC, cape cod
WR, wright	CL, chain ladder
TVaR, Tail Value at Risk	VaR, Value at Risk

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Innovation Fueled by Risk Management

Aaron M. Halpert, ACAS, MAAA

We generally do not tend to think of innovation and risk management as compatible themes. Innovation conjures up images of bold new ideas, thinking outside the box, disrupting established ways of doing things, and breaking new ground. Outdated applications of risk management, on the other hand, focused almost entirely on reducing or transferring risk, primarily by imposing controls, keeping things from getting out of hand, and inside the box. Contemporary enterprise risk management (ERM) programs, which focus not only on the identification, measurement, mitigation, monitoring and communication of risk, but also on capitalizing on risk opportunities, provides a better fit with innovation.

In truth, the two themes are not only compatible, but having an effective ERM protocol actually fuels and enables innovation. This essay makes the case for this compatibility, and offers some ideas from current ERM thinking on how to best use risk management tools to bolster innovation.

MINIMIZING RISKS ASSOCIATED WITH NEW IDEAS

Some have described one of the key pillars of innovation as "never fail to fail."[1] Successful innovation requires a higher tolerance for failure. With failure comes the opportunity to quickly learn from mistakes and inappropriate assumptions about a product or its market, and build a better idea. However, costly failure may not be acceptable to an organization's Board or owners. How do we balance these competing imperatives? Most current approaches to ERM focus on the development of an organization's risk appetite and related risk tolerances. Management and the Board will lay out statements that capture the types and amounts of risk the organization is willing to entertain in conducting its business. Naturally the levels of risk tolerated in different segments of the business will be evaluated in light of the potential returns that can be achieved in each segment. As such, the organization may be willing to tolerate more risk in a new product venture with high potential returns than it would in a mature product producing more limited returns. An articulation of the risk tolerance for innovation products, and how it relates to the organization's overall risk tolerance, will provide an effective framework for pursuing innovation strategies.

Once an idea has been articulated and captured by an organization (usually in a document that summarizes the concept and why it may be attractive to the organization's stakeholders), many effective innovation platforms begin with a process called Minimum Viable Product (MVP) testing. MVP testing accelerates learning by testing a product hypothesis in small scale experimentation for a

[1] "The Eight Pillars of Innovation" by Susan Wojcicki, 2011, http://bit.ly/1qoWpqm.

limited market using minimal resources. As such, MVP testing exemplifies the integration of risk management into the innovation process. By testing on a limited scale and with limited new product features, the risk of undertaking new product ventures that produce disappointing results is minimized.

Looking at this approach more broadly, a small investment in product testing (with relatively high tolerance for limited loss to be experienced in the testing phase) will lead to a refinement of the product/service and marketing plan and also result in a better understanding of the risks associated with product/service implementation. In other words, risk management is not a process that is glued onto an innovation process, but rather is an important component integrated into the approach used to develop new ideas.

AN INNOVATION RISK MANAGEMENT FRAMEWORK

With this in mind, we can sketch out how the risk management function can strengthen the innovation process. The following steps illustrate how risk management specialists can do so:

1. Working with the Board and management to articulate their risk appetite for new innovations and how it relates to the organization's overall risk appetite.

While setting risk tolerance is not an exact science, doing so allows the Board to set risk policy for the organization, and facilitates an ongoing monitoring process for management to demonstrate that risks are maintained within established tolerance levels. As noted earlier, innovation will likely only be successfully achieved through a willingness to take on more risk than an organization would normally consider. Once this is recognized however, the board has an opportunity to set specified higher risk tolerances for innovation efforts with a clear vision of how this fits with the overall risk profile.

2. Reviewing new ideas and their associated product testing plans to develop plausible outcome scenarios, and determine the risks associated with pilot testing efforts. They also provide assurance that the risks associated with these outcomes are within the relatively broad risk tolerances.

Scenario testing and stress testing are important elements in any ERM toolkit. Such an analysis is particularly important in product testing both prior to testing a prototype as well as analyzing results afterwards. At the front end, development of reasonable outcome scenarios provides the innovation team with better insights for developing controls and also helps identify which product features are most susceptible to producing adverse results.

3. Working with the innovation team to review pilot testing results and assist in redefining product attributes and marketing strategies. Concurrently, risks associated with the refined product are

recalibrated, and assurance is provided that the recalibrated risks are within a somewhat narrower risk tolerance level.

Post testing analysis provides clues about whether the risks were fully understood in the first place and also helps to further refine the product/market attributes as well as the controls necessary to keep the full product roll out within defined risk tolerance levels.

4. Assuring that all the relevant controls are in place when the final product launch is recommended to the Board.

This final step brings the pieces together to help launch new product ideas and provide a level of assurance that downside risk is managed at the same time. The more effective the risk management protocols are, the more likely it is that the Board and management will enthusiastically support the organization's efforts.

AN EXAMPLE

Let's consider an example, and see how this process would apply. Suppose an insurer were to assess whether to expand their mechanical breakdown coverage for insured autos to include damage to the auto resulting from engine system hacking. Given the limited knowledge of the exposure (how many systems are susceptible to hacking, how likely are hackers to act, how much damage will result, etc.), the insurer would have to recognize that the financial results attached to this expansion of coverage would be considerably more uncertain than the breakdown coverage to which it is being attached.

By first analyzing a wide range of scenarios that reflect assumptions about who will purchase the coverage, the incidence of hacking, the resulting damage to the auto's systems, etc. the team can develop estimates of the resulting underwriting loss under severely adverse conditions. If the risk is deemed too high relative to the organization's innovation risk tolerance, product features can be scaled back to limit coverage, marketing plans can be limited to certain market segments and geographies, or the project can be deferred to allow more time to better understand the potential incidence of hacking into "the internet of things." Limited prototype versions of the coverage would be tested in limited markets and the results can further inform decisions on product features, pricing, and how broadly the coverage would be marketed for the full roll out.

Finally, risk management professionals can assist in developing controls to assure that the aggregate risk associated with the new coverage remains within pre-determined tolerance levels. In addition to traditional reinsurance considerations, and taking a page from the evolving cyber security world, controls for the program might include industry data on make and model hacking incidents, aggregate exposure monitoring based on available incident frequency, and the likelihood of incidents affecting several insureds simultaneously.

CONCLUSION

While failure provides a critical opportunity for learning and improving innovation efforts, the risk of failure must be actively managed within the innovating organization's ERM protocols. As such, by providing a framework in which innovation efforts are more confidently undertaken, risk management does not inhibit innovation — it actually fuels it.

Innovation in Crop Insurance: The Price-Flex® Story

Michael G. Wacek, FCAS, MAAA, CERA

Starting in 2012 and continuing for several years since, I have been witness to a burst of innovation in the seemingly staid world of crop insurance, which is too often dismissed as an obscure backwater of the U.S. insurance industry. 2012 was the year that Hudson Insurance Group teamed up with leading crop insurance agency Silveus Insurance Group and economic consulting firm Watts and Associates to introduce a new supplemental crop insurance product called Price-Flex®.[1] As chief risk officer of Odyssey Re, Hudson's parent, I was called upon to vet the pricing engine that had been developed by Watts; and thus began a collaboration with Watts, which has expanded to include innovative *loss ratio hedging*, that continues through today.

To understand the nature of the innovation I am going to discuss, it will be helpful to know a little about the Federal Crop Insurance Program, which is administered by the Risk Management Agency (RMA) of the U.S. Department of Agriculture (USDA) on behalf of the Federal Crop Insurance Corporation (FCIC). The RMA has developed a suite of Multiple Peril Crop Insurance (MPCI) policies for sale to American farmers by a limited number of approved insurance providers (AIPs), each of which is eligible for reinsurance protection provided by the FCIC.[2] The RMA has developed both the policy language and the rates for the various MPCI coverages, and under the terms of their agreement with the RMA, the AIPs are not permitted to deviate from either those standard policy terms or rates. In addition, the RMA has imposed restrictions on commissions paid to agents. In other words, the AIPs cannot compete with each other on the basis of coverage or price, nor are they entirely free to compete for agents by paying higher commissions. As a result, competition in the U.S. crop insurance market turns on service, both to agents and farmers, and also on private crop insurance products, which are supplemental coverages outside the RMA's standard MPCI suite (and thus not reinsured by the FCIC). These private supplemental coverages are subject to RMA (as well as state insurance department) approval. While RMA rules prohibit tying the sale of private products to the sale of an MPCI policy, as a practical matter most farmers find it more convenient to buy all of their coverage from a single AIP. As a result, many AIPs find that offering their own suite of private supplemental policies with attractive characteristics in terms of coverage and/or price is an effective way to compete for the pockets of MPCI business they find most attractive.

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[1] Price-Flex is a registered trademark of Watts and Associates, Inc., Billings, MT.

[2] As of January 2016 there were 17 AIPs, including Hudson, eligible to provide MPCI coverage under the Standard Reinsurance Agreement with the FCIC.

The most popular MPCI policies in recent years, comprising about 80% of total premiums, have been those providing *revenue protection*, which the RMA describes as follows:

> Revenue Protection policies insure producers against yield losses due to natural causes such as drought, excessive moisture, hail, wind, frost, insects, and disease, and revenue losses caused by a change in the harvest price from the projected price.[3]

These revenue protection (RP) policies put a floor under a farmer's revenue, defined as yield per acre × crop price × acreage. A farmer can typically buy protection up to 85% of projected revenue (effectively a 15% deductible), where the projected revenue per acre is equal to the yield per acre times the *higher of* the projected price and the harvest price. There is a claim under the revenue policy if, and to the extent that, actual revenue falls below the guaranteed level.

Years ago, in order to facilitate efficient and transparent administration of MPCI policies, the RMA introduced a standardized approach to determining the projected and harvest prices used in establishing coverage and adjusting claims for each crop. Under that approach, the actual price received by a farmer at his local elevator is ignored, and instead, for coverage and claim purposes *all* farmers within specified regions are *deemed* to receive the *same* harvest price for their crops.

This deemed harvest price is the one-month average daily price of a specified futures contract. To illustrate, for policies covering corn in most parts of the Upper Midwest, the harvest price is defined as the October average daily price on the corn futures contract for December delivery; for policies covering soybeans in the same region, the harvest price is equal to the October average daily price on the soybeans futures contract for November delivery. October is said to be the *price discovery period* for determination of the harvest price.

The projected price is established in similar way. For example, for most parts of the Upper Midwest the projected price for corn is the February average price on the December futures contract, and for soybeans it is the February price on the November futures contract. February is said to be the price discovery period for determination of the projected price.

The use of standardized crop prices not only streamlines administration but also simplifies the pricing of the MPCI policies (though that is a task left to the RMA) as well as supplemental private products such as Price-Flex. Rather than having to try to estimate the prices actually received by each farmer, the pricing model can focus on just the relevant futures prices.

As mentioned earlier, under an MPCI revenue protection policy the farmer's revenue guarantee is based on the higher of the projected price and the harvest price, both of which are determined by reference to average futures prices during specified price discovery periods. Price-Flex is supplemental crop insurance that expands the basic coverage provided by MPCI revenue protection

[3] http://www.rma.usda.gov/policies/

and similar policies[4] by allowing the farmer to add one or more additional price discovery periods to the formula for determining the revenue guarantee. The Price-Flex revenue guarantee is based on the highest of the projected price, harvest price and the prices emerging from the additional Price-Flex price discovery periods.

While all MPCI policies for a given crop and region have a common anniversary date, typically a few months before planting begins, Hudson begins selling Price-Flex policies nearly a year earlier. For example, for corn in most parts of the Upper Midwest, the common anniversary date (or sales closing date) for crop year 2016 MPCI policies is March 15, 2016; in contrast, Hudson offered 2016 Price-Flex coverage starting in April 2015.

Because futures prices are used directly in the determination of coverage and claims, Hudson's Price-Flex pricing framework uses the latest available futures price as a rating variable, both in order to rate policies more accurately and, importantly, to avoid the risk of adverse selection associated with using stale prices. As a consequence, the rate paid by a farmer today, with the corn futures price at $3.93 a bushel, would be different from the rate he would have paid yesterday with corn futures at $3.87. Underwriters of most other types of property-casualty insurance rarely have the opportunity to price their policies with such up-to-date information.

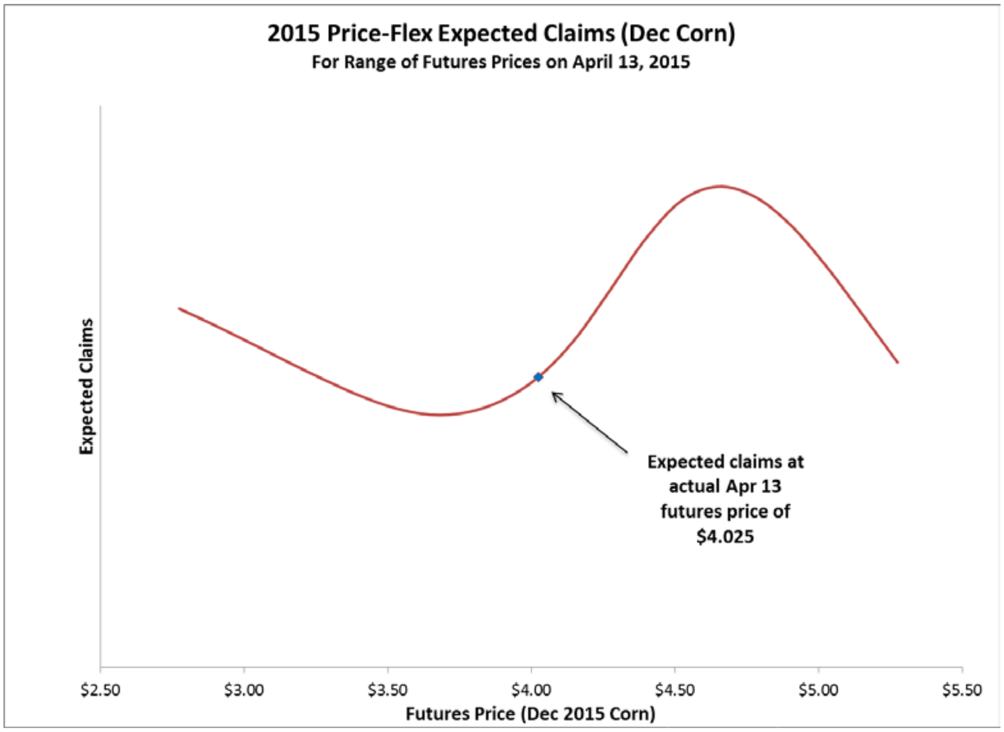
Hudson can use this latest crop price information for pricing Price-Flex because

- The latest futures price is both relevant and observable,
- Regulators have been willing to approve its use as a rating variable, and, critically,
- Through its partnership with Watts and Associates, Hudson can efficiently handle the required quoting, policy issuance and administration.

Thanks to the efficient IT infrastructure and data collection systems, the *entire portfolio* of Price-Flex policies in force can be "repriced" every day using the latest futures price information to provide an updated estimate of the expected portfolio claim costs.

In addition, the pricing model can also be applied to the portfolio on a daily basis to determine the effect of a range of "as-if" futures prices on projected claim costs, as illustrated graphically below for the April 13, 2015 valuation of Hudson's 2015 portfolio of Price-Flex policies linked to the December corn futures contract.

[4] In addition to RP policies protecting a farmer directly, there are similar policies based on county level data.



Since the impact of a futures price movement on expected claims can be predicted using the pricing model, that predictable effect can be offset by taking a suitable position in the underlying futures contract.[5] In other words, that portion of the risk in Price-Flex claims related to crop prices can be hedged, at least over very short periods of time. That last qualification is critical. This is not a static hedge that can be put in place and then forgotten. Instead, the hedge has to be reviewed frequently and adjusted to reflect changes in the portfolio and/or in expected portfolio claims as a function of the current futures price.

From a risk management perspective, the prospect of hedging Hudson's Price-Flex crop price risk was intriguing. Back-testing of the hedging algorithm over the period 1990-2012 revealed that it would have substantially reduced the variability of the Price-Flex loss ratio for policies linked to December corn and November soybeans, bringing the hedged loss ratio in most years much closer to the target loss ratio.[6]

However, as a new and unusual idea for a property-casualty company, there was skepticism at the group level about using derivatives to hedge insurance risk. After all, weren't many supposedly sophisticated players in the financial markets badly burned during the financial crisis by derivatives? How realistic and reliable was the back-testing? What experience did Hudson or Odyssey Re have in

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[5] The size of the position and whether it is long or short is related to the slope of the curve representing expected claims as a function of the futures price.

[6] The effect of the hedging algorithm is generally to reduce loss ratios that would in the absense of hedging be above the target loss ratio and increase those that would otherwise be below the target.

derivatives trading? These questions and others led Hudson to put the hedging idea aside temporarily.

Over the course of more than a year, the ERM team at Odyssey Re modeled the effect of hedging on Hudson's Price-Flex portfolio using as-if "paper trades" instead of real ones. That exercise provided further evidence that the risk mitigation effects of employing the hedging algorithm were real.

The strength of that analysis, together with other supporting material, addressed the roots of the original skepticism, and this time a new proposal to begin hedging Hudson's Price-Flex risk was approved. I am happy to say that the live hedging program using actual trades has proved as effective as the paper exercise had suggested it would be.

In summary, the Price-Flex story at Hudson is one of innovation on several levels. First, unlike most traditional property-casualty ratemaking, the Price-Flex pricing model had to be developed to cope with not only fairly conventional insurance variables like crop yields but also the modeling of futures prices. Second, because Hudson wanted to avoid being confined to a common anniversary date for policy sales, it was necessary to build a pricing model dynamic enough to be able to incorporate daily futures prices, both to maximize pricing accuracy and to avoid adverse selection. Finally, to manage the commodity price risk inherent in the Price-Flex product, the pricing model was extended to devise a portfolio loss ratio hedging algorithm that may be the first of its kind in the property-casualty industry.

Apart from the personal thrill I have experienced in witnessing this innovation, I am also seeing for the first time the outlines of potential disruption of the traditional insurance industry. As a veteran of traditional insurance, I find it difficult to see how an insurance version of *Uber* or *Airbnb* could usurp traditional homeowners or car insurance, much less commercial insurance. However, taking a page from the RMA's book, which decided years ago to standardize the crop prices used in crop insurance by using reference prices instead of the actual ones the farmer receives, what if a new, non-traditional insurer emerged to offer streamlined first party coverages, perhaps including some totally new ones, using observable reference prices to establish coverage amounts and/or to value claims? We all know that insurance industry expense ratios are high, and if there were a way to reduce costs substantially by eliminating underwriting and claims administration costs, it might just take the insurance industry by storm. Food for thought!

Introduction to Bayesian Loss Development

David R. Clark, FCAS

Abstract

This paper provides an introduction to the use of Bayesian methods for blending prior information with a loss development pattern from a triangle. The methods build upon conjugate forms discussed in earlier literature but introduce the Generalized Dirichlet as a prior, which allows for a significant simplification in calculation. The discussion is mainly restricted to the question of blending observed data with prior beliefs and not on the question of reserve ranges.

The paper is aimed at practicing actuaries seeking an introduction to Bayesian ideas for loss development. The methods will work with a single development triangle analyzed in a spreadsheet.

Keywords. Bayesian loss development, conjugate prior, Generalized Dirichlet

1. INTRODUCTION

The selection of loss development patterns is a critical piece of actuarial analysis for casualty insurance business and it arises in both pricing and reserving. The most common data structure for this analysis is the development triangle. The actuary can estimate a pattern from the triangle but would typically impose judgment in selecting the final pattern based on prior knowledge or external data.

The incorporation of expert judgment and external data makes loss development analysis a natural application for the Bayesian framework. The Bayesian framework provides a way to incorporate this prior knowledge in a systematic way. This paper will provide a very basic model to allow the practicing actuary to begin using the Bayesian ideas.

In many non-insurance applications of Bayesian statistics, the observed data overwhelms the prior distribution, making the exact form of the prior irrelevant. This is not so for insurance, where the data is often sparse or volatile; the prior knowledge can have a great influence on the final results.

We will focus on the narrow problem of selecting a pattern from a loss development triangle (no exposure units or loss ratio information), blending the data in the triangle with prior knowledge. For convenience, this will be done using conjugate forms, which make the calculations trivial to perform. Anyone who knows how to calculate an age-to-age factor will be able to begin doing Bayesian analysis right away.

1.1 Research Context

Bayesian ideas have been part of actuarial thinking for many years, often in the context of credibility theory, which has been called the "cornerstone of actuarial science" (Hickman, 1999). Bayesian methods have previously been introduced in the context of reserving, and have gained more attention recently because of advances in computational algorithms such as Markov Chain Monte Carlo (MCMC) techniques.

The Bayesian approach has been noted for three major advantages:

1) It allows the analyst to incorporate prior knowledge or expertise in a logically coherent way.

2) It can incorporate complex, nonlinear relationships to provide a more realistic model than can be done otherwise.

3) It can incorporate uncertainty in all model parameters and therefore produce realistic reasonable ranges around predicted values.

Prior papers such as Meyers (2015) and Zhang, et al (2012) have generally focused on the problem of estimating ranges around reserve estimates. Authors such as Robbin (1986), Mildenhall (2006), Wüthrich (2007), and England, et al (2012) have also used the Bayesian concepts to illuminate the relationships between traditional models such as chain ladder and Bornhuetter-Ferguson.

While many papers acknowledge that "The Bayesian paradigm offers a formal mechanism for incorporating into one's analysis information not contained in the available data" (Zhang, 2012), it is not always clear how this can be done. Diffuse or noninformative priors are used in much of the literature.

1.2 Objective

In this paper, we present a conjugate Bayesian model applied to a standard loss development triangle. We will assume that the goal of the analyst is to estimate a development pattern using this data to update prior beliefs. Our focus will be on how to organize the prior beliefs about the development pattern into an explicit prior distribution for this blending problem.

By staying in the context of the conjugate[1] models, the blending of prior knowledge with new data can be done with very simple calculations. This allows analyst to begin experimenting with these ideas immediately without the need for special software or programming skills. The hope is that this model will help build intuition in the Bayesian framework and become the stepping stone for expanding to more advanced models.

1.3 Outline

Section 2 of this paper will outline the mathematics of the Bayesian conjugate form for the loss development pattern estimation; this will give all of the theory underlying the approach. Section 3 will provide a numerical example showing how the model can be implemented in practice. Section 4 gives a brief sketch of future research and ways to extend the model into more realistic (and more complex) forms.

2. BACKGROUND AND MATHEMATICS

This section provides all of the mathematics needed to derive the conjugate family for Bayesian loss development. Most of this is not critical for the actuary who is only looking to implement the method, and can be skimmed.

2.1 Bayesian Theory in General

Bayesian theory assumes that an analyst working with a loss development triangle does not start as a "blank slate" with no idea of what a development pattern looks like. Instead, it assumes that the analyst comes with some "prior" expectation and is willing to change that prior belief based on what is observed in the new data.

The theory is derived from Bayes' theorem, which calculates the "inverse probability" of a parameter value θ, based on observed data x.

[1] Conjugacy is "the property that the posterior distribution follows the same parametric form as the prior distribution" (Gelman, et al (2013), page 35). This is a technical definition, but the attraction of conjugacy is in the practical implementation and interpretability.

$$f(\theta|x) = \frac{f(x|\theta)\cdot f(\theta)}{f(x)} = \frac{f(x|\theta)\cdot f(\theta)}{\int_\theta f(x|\theta)\cdot f(\theta)d\theta} \quad (1.1)$$

The major challenge for applying Bayes' theorem in practice is that the parameter θ is usually a vector of multiple parameters. This means that we need to specify a multi-dimensional distribution $f(\theta)$ and also be able to evaluate the multi-dimensional integral in the denominator. This presents a computational challenge.

There have been three main strategies for handling the computation challenge:

1) Use of conjugate priors, allowing closed-form solutions for carefully chosen distributional forms.
2) Linear approximations to the formula (e.g., Bühlmann-Straub)
3) Numerical approximations
    a. Quadrature evaluation of the integral
    b. Simulation-based approached (MCMC)

With greater computer speeds and improved algorithms, the simulation-based methods have allowed for Bayesian methods to be used in many fields. These models are especially useful when we need to evaluate complex models.

The conjugate families are much more useful for introductory purposes because they allow the calculations to be done simply and even manually. It is also very useful to include conjugate forms in some of the components of a simulation model ("conditionally conjugate" parameters) to improve efficiency.

For this paper, we will stay in the conjugate world in order to introduce all of the concepts in the loss development application such that any actuary can implement. If you can calculate an age-to-age factor, then you can do Bayesian analysis!

2.2 The Beta-Binomial Conjugate Relationship

Blending patterns is a multivariate problem, but it is easiest to attack the problem by starting with the univariate case. We begin with the univariate Beta-Binomial case, because it will be the main building block for the loss development application.

The Beta distribution works with a continuous random variable, p, that can be any value between 0 and 1. The density function is given below.

$$f(p)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\cdot p^{\alpha-1}\cdot(1-p)^{\beta-1} \tag{2.1}$$

$$E(p)=\frac{\alpha}{\alpha+\beta} \tag{2.2}$$

The Beta distribution is usefully interpreted as the ratio of gamma random variables. The two gamma random variable have different shape parameters, but share a common scale parameter ϕ, which does not affect the Beta random variable.

$$\begin{gathered} Z_1 \sim Gamma(\alpha,\phi) \\ Z_2 \sim Gamma(\beta,\phi) \\ p=\frac{Z_1}{Z_1+Z_2} \end{gathered} \tag{2.3}$$

We can also note that the shape parameters α and β must be positive numbers but they are not restricted to being integer values.

The likelihood function for the observed data x will be assumed to come from a binomial distribution with the probability function below.

$$f(x\mid p)=\binom{n}{x}\cdot p^{x}\cdot(1-p)^{n-x} \tag{2.4}$$

The binomial is often interpreted as the number of "successes" observed in a sample of n trials, given a probability of success p. The maximum likelihood estimator of this probability is calculated easily.

$$\hat{p}=\frac{x}{n} \tag{2.5}$$

While the binomial distribution is strictly speaking restricted to integer values, we will make an approximation in this application that the estimator above can include non-integer values for x and/or n when estimating the proportion p.

If the parameter p has a Beta prior distribution as defined above, then we apply Bayes' theorem to revise our distribution based on the observed data.

$$f\left(p \mid x\right) = \frac{f\left(x \mid p\right) \cdot f\left(p\right)}{f\left(x\right)} = \frac{\Gamma\left(\alpha+\beta+n\right)}{\Gamma\left(\alpha+x\right) \cdot \Gamma\left(\beta+n-x\right)} \cdot p^{\alpha+x-1} \cdot \left(1-p\right)^{\beta+n-x-1} \tag{2.6}$$

The fact that the posterior distribution for p is again a Beta distribution gives us the reason for calling this a "conjugate" form.

The expected value of the proportion can also be written in a linear form.

$$E\left(p \mid x\right) = \frac{\alpha+x}{\alpha+\beta+n} = \left(\frac{x}{n}\right) \cdot \left(\frac{n}{\alpha+\beta+n}\right) + \left(\frac{\alpha}{\alpha+\beta}\right) \cdot \left(\frac{\alpha+\beta}{\alpha+\beta+n}\right) \tag{2.7}$$

Alternatively, we can write the updating of parameters in a simple form:

$$\begin{aligned} \alpha^{(1)} &= \alpha^{(0)} + x \\ \beta^{(1)} &= \beta^{(0)} + n - x \end{aligned} \tag{2.8}$$

With this updating formula, we have a very useful way of interpreting the parameters as being "pseudo-data." That is, the prior parameters $\alpha^{(0)}$ and $\beta^{(0)}$ are combined with the new data as though they were previously observed data points. Our prior knowledge is used as though it had been previously observed data.

Koop, et al (2007, page 19) summarize this concept well:

> "Natural conjugate priors have the desirable feature that prior information can be viewed as 'fictitious sample information' in that it is combined with the sample in exactly the same way that additional sample information would be combined. The only difference is that the prior information is 'observed' in the mind of the researcher, not in the real world."

This interpretability is useful when prior knowledge comes in a subjective form. For example, someone may say "I selected the development pattern based upon my twenty years of experience as an actuary." This is still useful in the Bayesian framework but we need to translate twenty years of experience into equivalent dollars of loss development data.

2.3 The Dirichlet-Multinomial Conjugate Relationship

The Dirichlet distribution is a multivariate generalization of the Beta distribution, which allows for a sequence of proportions, $\{p_1, p_2, \cdots, p_k\}$. The probability density function is similar to the Beta distribution except that the random variable is now a vector of percentages. These are interpreted as the incremental percentages of ultimate loss paid or reported in each identified period.

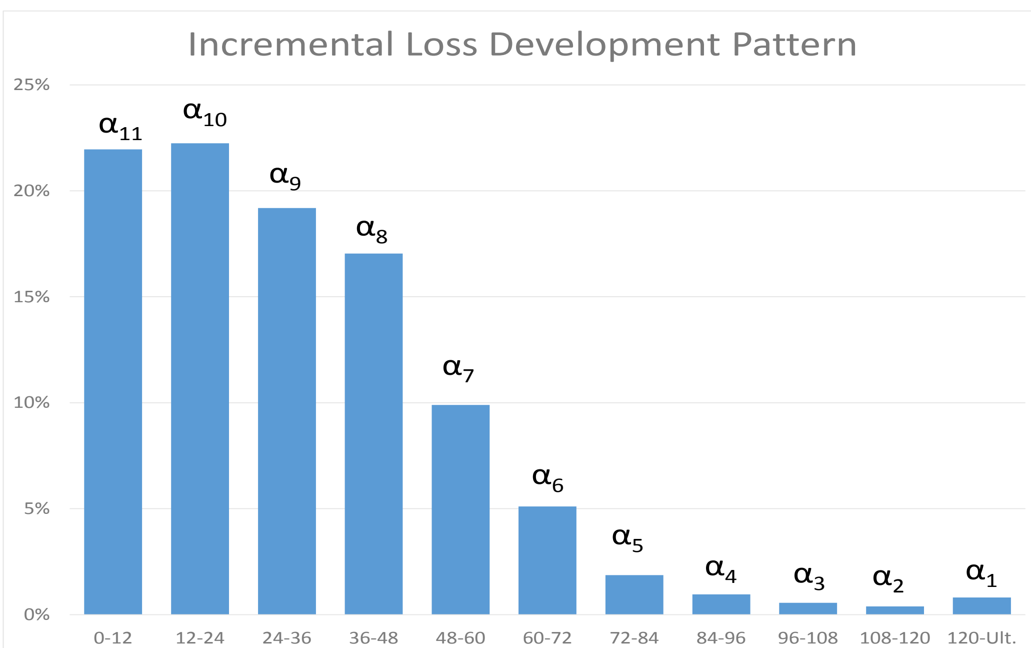
$$f(p) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2) \cdots \Gamma(\alpha_k)} \cdot \prod_{i=1}^{k} p_i^{\alpha_i - 1} \tag{3.1}$$

The expected percent-of-ultimate in each period is proportional to its corresponding alpha.

$$E(p_i) = \frac{\alpha_i}{\alpha_1 + \alpha_2 + \cdots + \alpha_k} \tag{3.2}$$

The sequence of expected percentages produces the expected loss development pattern (either paid or reported). Figure 1 represents the proportion of ultimate loss in each incremental period. This assumption is consistent with Robbin (1986), Hesselager & Witting (1988), de Alba (2002), and Mildenhall (2006).

Figure 1



Similar to the Beta-Binomial model, the Dirichlet is conjugate with a Multinomial distribution, the Multinomial being the multivariate generalization of the Binomial. The parameters are given a similar updating.

$$\begin{aligned} \alpha_1^{(1)} &= \alpha_1^{(0)} + x_1 \\ \alpha_2^{(1)} &= \alpha_2^{(0)} + x_2 \\ &\vdots \\ \alpha_k^{(1)} &= \alpha_k^{(0)} + x_k \end{aligned} \tag{3.3}$$

In this updating formula, the sequence $\{x_1, x_2, \cdots, x_k\}$ is proportional to the observed losses in each development period. It is most convenient to think of these as the shape parameters of gamma random variables, similar to the sequence of $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$. As such, the new data comes as the incremental losses divided by the common scale parameter ϕ.

The scale parameter ϕ is the variance/mean ratio of the loss data. We will assume that this is a fixed and known quantity, though that assumption can be relaxed later in the work.

If an estimate of the variance/mean ratio is needed, it can be approximated from the data just as is done for the dispersion parameter in a GLM[2], where $C_{t,d}$ is the cumulative loss for year t as of development period d. This is approximately a variance/mean ratio.

$$\phi = \frac{Var(C_{t,d})}{E(C_{t,d})} \approx \frac{1}{n - \#param} \cdot \sum_{t,d} \frac{\left(\left(C_{t,d+1} - C_{t,d}\right) - C_{t,ult} \cdot p_{k-d}\right)^2}{C_{t,ult} \cdot p_{k-d}} \tag{3.4}$$

The major difficulty in the Dirichlet-Multinomial model is that we need to have a complete development pattern from the data in order to perform the updating. This is precisely not the case for loss development; we have a triangle of <u>incomplete</u> patterns. Fortunately, this difficulty is solved via the Generalized Dirichlet distribution.

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[2] See for McCullagh & Nelder (1989) as a standard reference.

This is the same concept used in the over-dispersed Poisson (ODP) version of the chain ladder method, as presented in papers such as Renshaw and Verrall (1998). This connection is not accidental, as the binomial model presented here is simply a conditional Poisson model. That is, if X_1 and X_2 are Poisson random variables, then $X_1|X_1 + X_2 = N$ is a binomial random variable. This relationship extends to the over-dispersed and multivariate versions of the distributions.

2.4 The Generalized Dirichlet Distribution

The Generalized Dirichlet distribution was introduced by Connor and Mosimann (1969) in the context of biological science. Wong (1998) further investigated this form and provides the Bayesian updating formulas. Ng, et al (2011) provides more description of this distribution, renaming it the "nested Dirichlet."

Instead of a sequence of model parameters $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$, we have a parameter set with alphas and betas: $\{\alpha_1, \alpha_2, \cdots, \alpha_{k-1}, \beta_1, \beta_2, \cdots, \beta_{k-1}\}$. Just as α_i was seen to be proportional to incremental loss, the β_i parameter is proportional to cumulative loss. This added flexibility means that we can have different weights for each cumulative development age, making it natural for the development triangle data format.

These parameters generalize the Dirichlet distribution given above. But the random variable $p = \{p_1, p_2, \cdots, p_k\}$, is interpreted exactly the same as before.

$$f(p) = p_k^{\beta_{k-1}-1} \cdot \prod_{i=1}^{k-1} \left[\frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i) \cdot \Gamma(\beta_i)} \cdot p_i^{\alpha_i - 1} \cdot \left(\sum_{j=i}^{k} p_j \right)^{\beta_{i-1} - (\alpha_i + \beta_i)} \right] \tag{4.1}$$

The Generalized Dirichlet has independence[3] between p_1 and $p_2/(1-p_1)$ and between subsequent conditional values $p_3/(1-p_1-p_2)$ and so forth. For the loss development application this implies that all of the age-to-age factors are independent. This independence assumption between age-to-age factors is paralleled in the chain ladder method (Mack, 1993).

The expected incremental losses are given as below. Formulas for all of the moments and co-moments are given in Wong (1998).

$$E(p_i) = \frac{\alpha_i}{\alpha_i + \beta_i} \cdot \prod_{j=1}^{i-1} \frac{\beta_j}{\alpha_j + \beta_j} \quad i = 2, \cdots, k \tag{4.2}$$

The expected incremental values are more easily calculated via a recursive formula.

$$E(p_i) = \frac{\alpha_i}{\alpha_i + \beta_i} \cdot E(p_{i-1}) \cdot \left(\frac{\beta_{i-1}}{\alpha_{i-1}} \right) \quad i = 2, \cdots, k \tag{4.3}$$

---

[3] This property is described as "neutrality" by Connor and Mosimann (1969), and it only holds for the Generalized Dirichlet when the variables are ordered. It is for this reason that we use the notation that the first variable is the tail factor, and then move from right to left up to k as the first (usually age 12) factor. In this order the distribution is a perfect model for development triangle data.

The Dirichlet is a special case when $\beta_j = \alpha_{j+1} + \beta_{j+1}$.

The Bayesian updating formulas are also straightforward.

$$\alpha_j^{(1)} = \alpha_j^{(0)} + x_j \qquad (4.4)$$
$$\beta_j^{(1)} = \beta_j^{(0)} + x_{j+1} + x_{j+2} + \cdots + x_k$$

Using the cumulative losses from the triangle, this is written as shown below. For losses in accident year t as of development period d, the cumulative amount is $C_{t,d}$. The values used for updating the parameters remove the scaling parameter: $x = \left(C_{t,d+1} - C_{t,d}\right)/\phi$.

$$\alpha_j^{(1)} = \alpha_j^{(0)} + \frac{1}{\phi}\sum_{t=1}^{k}\left(C_{t,d+1} - C_{t,d}\right) \qquad (4.5)$$
$$\beta_j^{(1)} = \beta_j^{(0)} + \frac{1}{\phi}\sum_{t=1}^{k} C_{t,d}$$

The dispersion parameter ϕ acts as a scaling parameter on the loss data from the triangle.

The great advantage of this Generalized Dirichlet is that we can exclude the first p_1 or the first several points and the remaining points are still a Generalized Dirichlet. Further, the relationship of the first increment to the sum of the remaining increments is always a Beta distribution.

$$E\left(p_1\right) = \frac{\alpha_1}{\alpha_1 + \beta_1} \qquad (4.6)$$

This relationship of one period to all the others is exactly what is needed in the calculation of age-to-age link ratios in the chain ladder method. The notation needs to be reversed: for example, count i=1 for last incremental loss oldest period, and $i = k$ for losses in the first year. The model parameters therefore translate very easily into familiar age-to-age factors.

$$ATA_{12-24} = \frac{\alpha_k + \beta_k}{\beta_k} \qquad ATA_{120-ult} = \frac{\alpha_1 + \beta_1}{\beta_1} \qquad (4.7)$$

The age-to-age factor for development period d is calculated from the triangle as shown below. The weighted average age-to-age (ATA) factor should be familiar to most actuaries.

$$ATA_d = \frac{\sum_{t=1}^{k} C_{t,d+1}}{\sum_{t=1}^{k} C_{t,d}} \tag{4.8}$$

The credibility blended numbers are given in a simple form as in formula (4.9) below.

$$ATA_d = \frac{\phi \cdot \left(\alpha_{k-d} + \beta_{k-d}\right) + \sum_{t=1}^{k} C_{t,d+1}}{\phi \cdot \beta_{k-d} + \sum_{t=1}^{k} C_{t,d}} \tag{4.9}$$

Given the model parameters for the Generalized Dirichlet and the scaling parameter ϕ, this credibility blending can be performed in a spreadsheet cell or even on paper. A numerical example of this calculation is given in Section 3.2.

Part of what has made the conjugate form so easy to implement is the assumption of independence between development ages. Unfortunately, the disadvantage of the independence assumption is that ages with little volume will get little credibility weight. There is no consideration of adjacent points, and no more weight assigned if all ages show consistently better (or worse) development than the benchmark. Most notably, the benchmark tail factor will never change based on the client data.

Most users, however, would want some dependence between ages. For example, if all of the age-to-age factors in the client's triangle are below the benchmark, then the benchmark tail should also be reduced. The next section of our paper will provide a way to include such a dependence structure.

2.5 Mixtures of Generalized Dirichlet Distributions

The model above provides a full conjugate Bayesian model that can be easily implemented by an analyst with knowledge of calculating age-to-age factors. The conjugate family is actually a bit more flexible still and allows for further expansion of the prior distributions.

The principle is that a linear combination of conjugate priors will still be a conjugate prior. If the analyst decides that the prior knowledge includes a library of possible development patterns (perhaps slow/medium/fast), then the prior is defined as a weighted average of these priors. The weights $\{w_1, w_2, w_3\}$ act as a discrete mixture distribution.

$$f(p) = w_1 \cdot GD_1(p) + w_2 \cdot GD_2(p) + w_3 \cdot GD_3(p) \tag{5.1}$$

For each of the individual Generalized Dirichlet distributions $GD_l(p)$, we perform the same updating as outlined in the previous section. In addition, we update the weights in proportion to the likelihood functions for each.

The likelihood functions are the products of the Beta-Binomial functions for each age included in the analysis.

$$f(x) = \int_0^1 f(x \mid p) \cdot f(p) dp = \binom{n}{x} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} \cdot \frac{\Gamma(\alpha+x) \cdot \Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n)} \tag{5.2}$$

For non-integer values of n and x, we can replace $\binom{n}{x}$ with $\frac{\Gamma(n+1)}{\Gamma(n-x+1) \cdot \Gamma(x+1)}$. We may also note that a special case of formula (5.2) is the uniform distribution when $\alpha = \beta = 1$, indicating that all values are equally likely.

The updating of the weights is a straight-forward application of Bayes' theorem.

$$w_j^{(1)} = \frac{w_j^{(0)} \cdot f_j(x)}{w_1^{(0)} \cdot f_1(x) + w_2^{(0)} \cdot f_2(x) + w_3^{(0)} \cdot f_3(x)} \tag{5.3}$$

Section 3.3, below, gives a numerical example illustrating this formula. The ability to adjust the tail factor in the data according to client data is a major practical advantage.

3. NUMERICAL EXAMPLE

3.1 Selecting the Model Parameters

The description of the Bayesian model given in the previous section has flexibility for the analyst to supply a large number of prior parameters. We now discuss how this can be done without making all of these choices arbitrary.

Parodi and Bonche (2010), describe the uncertainty in prior information from two sources:

1. Market heterogeneity – the spread of different risks around some industry average
2. Estimation uncertainty – the industry average, though large, may still be of limited size

We may choose to give the prior distribution more variance depending upon how we evaluate these sources of uncertainty. Nonetheless, we usually have some prior knowledge and are not completely uninformed about external information.

In many application of Bayesian models, the choice of prior is not given much attention because it is assumed that the data will overwhelm the prior assumption anyway. For insurance applications we cannot assume this, and instead want to provide meaningful prior information. The discussion of "noninformative" or "diffuse" priors is therefore just a starting point.

For the Beta or Dirichlet distributions, a standard noninformative prior is to set all of the parameters equal to 1.00. That is, $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$; this is sometimes referred to as the Laplace prior. Even more diffuse is the Jeffreys prior with $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1/k$. In both these cases, the expected pattern would have equal percentages in each period. In the most extreme case, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$; which is an improper prior, sometimes called a Haldane prior, that gives no weight to the prior information and therefore will always result in a posterior expected value equal to the chain ladder calculation.

We would like our prior to have expected values equal to our prior knowledge. In the reserving exercise, this may be equal to the pattern selected in a prior reserve study. In the pricing exercise, the prior pattern may be taken from the expiring pricing or from an average of similar risks.

One approach to setting the sequence of alphas is to make them proportional to the incremental losses in our benchmark pattern. If these are scaled to add up to 1.00 then we have a very wide uncertainty similar to the Jeffreys prior. If the alphas add up to a larger quantity, say 100, then the prior benchmark pattern will be given much more weight. The sequence of betas can be set to make the Generalized Dirichlet equal to a simple Dirichlet: $\beta_j = \alpha_{j+1} + \beta_{j+1}$.

Alternatively, we can set $(\alpha_j + \beta_j)$ as a constant, generally greater than 2, with the α_j and β_j values set to match the ATA factors.

The other key input is the dispersion parameter ϕ, which is defined as the variance/mean of the data in the triangle. A small value of ϕ will result in more weight given to the new data because it implies small process variance.

This dispersion parameter may be estimated empirically from representative triangles, or it can be selected based on other sources for aggregate distributions. For example, in Table M[4] we can approximate the distributions using a Gamma. The expected loss group (ELG) represents the insurance charge at an entry ratio of 1.00. The expected losses for the ELG divided by the Gamma shape parameter is therefore an estimate for the scale ϕ.

Table 1

Theoretical "Table M" (for illustration)

Gamma Shape Parameter	Insurance Charge at Entry=1	Expected Loss Group	Aggregate Loss Size (example)	Implied Variance/Mean
0.5	0.484	48	360,000	720,000
1	0.368	37	1,000,000	1,000,000
1.5	0.308	31	2,000,000	1,333,333
2	0.271	27	3,750,000	1,875,000

For a starting point, we can select a combination of the parameters, such that $\phi \cdot (\alpha_j + \beta_j)$ is constant for all j.

If the prior distribution and scale parameter are calculated from a sample of patterns collected from peer companies, then it may be considered an "empirical Bayes" model. Schmid (2012) and Shi and Hartman (2014) provide models on that basis.

3.2 Numerical Example with One Benchmark Pattern

For an example of the loss development task, we introduce a triangle of cumulative loss payments. This data was taken from a sample of companies collected in the CAS Website, and represents Products Liability loss net of reinsurance. The example is, of course, only for illustration.[5]

The average age-to-age (ATA) factors are calculated as all year weighted averages. The "Col. 1" number represents the sum of losses for a given age, excluding the latest diagonal;

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[4] Table M is an industry tool for excess-of-aggregate charges for Workers' Compensation. The numbers shown here are not from that source, but were created only to illustrate the concept.

[5] For the interested reader, an Excel file including the example that follows can be provided by the author upon request.

the "Col. 2" number represents the sum of the subsequent column.

Table 2

Sample Triangle = Cumulative Products Liability Paid Losses								
	12	24	36	48	60	72	84	96
1990	73	262	469	528	536	591	604	606
1991	148	346	391	502	522	514	567	
1992	99	198	219	394	408	430		
1993	118	255	352	412	581			
1994	275	415	645	803				
1995	261	446	637					
1996	130	471						
1997	148							
Col. 1	1,104	1,922	2,076	1,836	1,466	1,105	604	
Col. 2	2,393	2,713	2,639	2,047	1,535	1,171	606	
Avg ATA	2.168	1.412	1.271	1.115	1.047	1.060	1.003	

The average ATA factors are easily calculated by the actuary and, if desired, could be replaced with the values for only including the latest, say, three diagonals.

The ATA factors in the triangle show considerable volatility, so it is desirable to blend the data with other benchmarks to improve the stability.

Table 3

Age-to-Age Factors							
	12-24	24-36	36-48	48-60	60-72	72-84	84-96
1990	3.589	1.790	1.126	1.015	1.103	1.022	1.003
1991	2.338	1.130	1.284	1.040	0.985	1.103	
1992	2.000	1.106	1.799	1.036	1.054		
1993	2.161	1.380	1.170	1.410			
1994	1.509	1.554	1.245				
1995	1.709	1.428					
1996	3.623						

The table below brings in the prior knowledge. We assume that we know a loss development pattern. This pattern may come from industry sources, peer companies, or prior reserve studies.

We must select the alpha and beta parameters for each age. We can set these such that the expected pattern equals our benchmark: ATA = (Alpha+Beta)/Beta.

The total value of Alpha+Beta is selected to be 4.00 in this example, representing a weakly informative prior. The variance/mean ratio or scale parameter ϕ is selected as 1,000 ($1,000,000 since the original Schedule P units are in thousands). The "Col. 1" and "Col. 2" entries are simply the scale parameter times the Beta and Alpha+Beta parameters of the Generalized Dirichlet.

Table 4

Prior Assumptions

	12	24	36	48	60	72	84	96
LDF	21.950	7.787	3.946	2.512	1.842	1.558	1.415	1.315
ATA	2.819	1.973	1.571	1.364	1.182	1.101	1.076	1.315
Alpha	2.58	1.97	1.45	1.07	0.62	0.37	0.28	0.96
Beta	1.42	2.03	2.55	2.93	3.38	3.63	3.72	3.04
Alpha+Beta	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
Variance/ Mean:		1,000						
Col. 1	1,419	2,027	2,546	2,933	3,383	3,633	3,717	3,042
Col. 2	4,000	4,000	4,000	4,000	4,000	4,000	4,000	4,000

The blended pattern is simply the addition of the Col. 1 and Col. 2 weights from the triangle and the benchmark pattern (scaled by ϕ).

As noted previously, the conjugate form puts the prior knowledge into a form as though it was prior loss development data. The prior knowledge is added to the data from the new triangle as though we actually had more loss in the weighted-average calculation. The table below makes use of formula (4.9) to blend the patterns.

Table 5

Example of Blending Client and Benchmark Patterns

	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-Ult
ATA from Triangle								
Col. 1	1,104	1,922	2,076	1,836	1,466	1,105	604	-
Col. 2	2,393	2,713	2,639	2,047	1,535	1,171	606	-
ATA	2.168	1.412	1.271	1.115	1.047	1.060	1.003	
Benchmark Pattern								
Col. 1	1,419	2,027	2,546	2,933	3,383	3,633	3,717	3,042
Col. 2	4,000	4,000	4,000	4,000	4,000	4,000	4,000	4,000
ATA	2.819	1.973	1.571	1.364	1.182	1.101	1.076	1.315
Blended Pattern								
Col. 1	2,523	3,949	4,622	4,769	4,849	4,738	4,321	3,042
Col. 2	6,393	6,713	6,639	6,047	5,535	5,171	4,606	4,000
ATA	2.534	1.700	1.436	1.268	1.141	1.091	1.066	1.315

This calculation can be easily incorporated into reserving studies or pricing work. The values for the alpha, beta and scale parameters in our example are only for illustration; the actuary can sensitivity test values in real examples in order to gain intuition for setting reasonable values.

One limitation in this implementation is that the "tail" factor will always be equal to the benchmark number. This is because we have assumed independence between all ATA factors. This assumption is relaxed in the next section, as more robust priors are used.

3.3 Numerical Example with Library of Benchmark Patterns

The example in section 3.2 assumes that there is a benchmark development pattern and some level of uncertainty around that pattern. It further assumes independence between the ATA factors for each development age.

We can expand the prior assumptions to instead assume that there is not just a single benchmark pattern but rather a library of such patterns. For example, we may assume that there are fast, medium and slow developing businesses, perhaps differing due to settlement strategies or case reserving practices. Each of these patterns has its own Generalized Dirichlet parameters, and there is some prior belief as to the probability of a given triangle being from any member of the library.

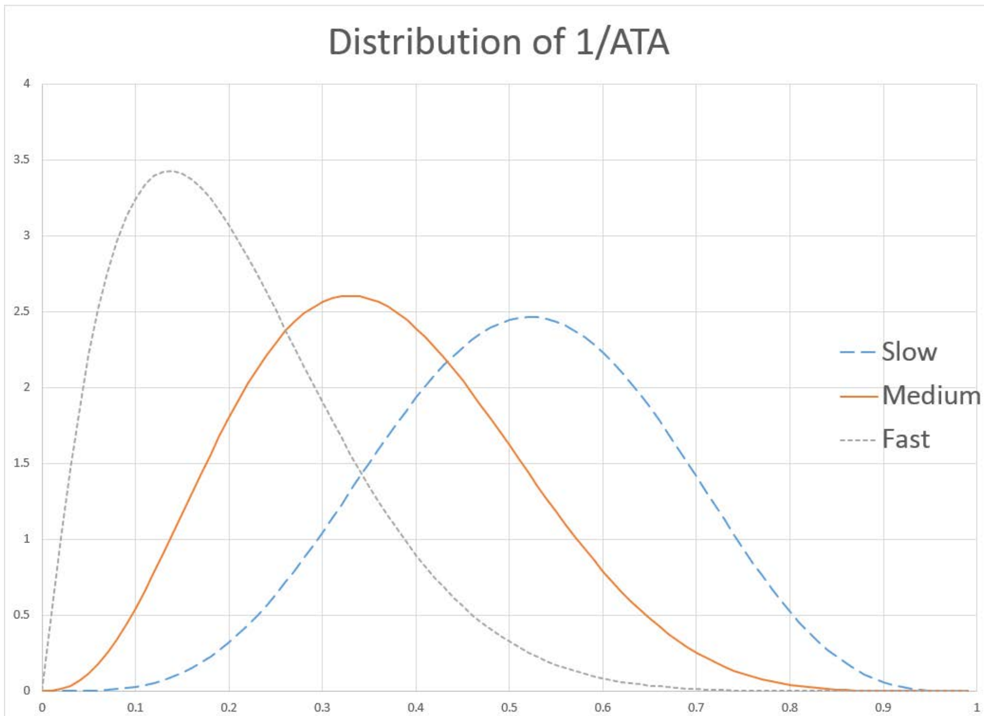
For a reinsurer, this may mean that their client companies' development patterns are naturally clustered into Fast/Medium/Slow groups, but without a perfect way to tell beforehand to which cluster a given client belongs.

Table 6

	Cumulative Loss Development Factors							
	12	24	36	48	60	72	84	96
Fast	14.014	4.930	2.607	1.759	1.406	1.263	1.191	1.155
Medium	21.950	7.787	3.946	2.512	1.842	1.558	1.415	1.315
Slow	49.240	15.860	7.407	4.163	2.706	2.057	1.750	1.567

As we noted earlier, the distribution of 1/ATA always follows a Beta distribution. For each development age, we can make a graph of the density functions for each of the benchmark patterns as a test for reasonableness.

Figure 2



We may have the case that the user has specified three different patterns, with variance within each. The prior mixture weights are assumed to be 1/3 to each of the three benchmark patterns. For Bayesian updating, the same procedure from Section 3.2 is applied for each of these patterns separately.

The mixture weights are then updated using formula (5.3). An example is show for the Fast pattern below, with formula (5.2) calculated as loglikelihood for each development age.

Table 7

Calculaton of Loglikelihood (Fast Pattern)

	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-Ult
Data from Triangle								
Col. 1	1,104	1,922	2,076	1,836	1,466	1,105	604	
Col. 2	2,393	2,713	2,639	2,047	1,535	1,171	606	
Variance/Mean Ratio:		1,000						
N	2.39	2.71	2.64	2.05	1.54	1.17	0.61	
X	1.29	0.79	0.56	0.21	0.07	0.07	0.00	
Benchmark Pattern								
LDF	14.014	4.930	2.607	1.759	1.406	1.263	1.191	1.155
ATA	2.843	1.891	1.482	1.251	1.113	1.060	1.031	1.155
Alpha	6.5	4.7	3.3	2.0	1.0	0.6	0.3	1.3
Beta	3.5	5.3	6.7	8.0	9.0	9.4	9.7	8.7
Alpha+Beta	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0
Loglikelihood	-0.9363	-1.0052	-0.8252	-0.5260	-0.2687	-0.2535	-0.0290	0.0000

This is known as a mixture model and it is still relatively easy to compute because a mixture of conjugate distributions is still a conjugate form. The posterior will again be a discrete mixture of Generalized Dirichlet distributions. Because the data from the triangle generally showed a faster pattern than implied in our benchmark, the weights are revised to shift more weight to the "Fast" curve.

Table 8

Bayesian Updating of Probabilities

	LogLikelihood	Difference in LL	Relative Likelihood	Original Weights	Revised Weights
	A	B=A-Max(A)	C=exp(B)	D	E=C*D/Avg(C)
Slow	-4.61	-0.77	0.464	33.33%	20.41%
Baseline	-4.06	-0.21	0.810	33.33%	35.61%
Fast	-3.84	0.00	1.000	33.33%	43.98%
			0.758	100.00%	100.00%

This use of a mixture of benchmark patterns can be expanded to include as many alternative patterns as desired, though for practical purposes three is sufficient. The major point is simply to illustrate the great flexibility for incorporating prior knowledge.

4. RESULTS AND DISCUSSION

It was the main goal of this paper to provide a Bayesian model that can be implemented quickly for the practicing non-technical actuary. The use of the conjugate form allows that implementation. Once this introductory material has been mastered, it is hoped that actuaries will seek to expand the model to make them more realistic.

4.1 Summary of Conjugate Model

The conjugate model is based on some simplified assumptions for ease of implementation. It is worth remembering some of the assumptions we have required.

1) The variance/mean ratio is assumed to be constant and known (supplied by the analyst)
2) All incremental development should be strictly positive
3) Individual incremental losses are independent

4.2 Extensions of the Model

Some of the ways that we can expand on the simple model are given below. These go beyond the conjugate form and therefore require moving to simulation models. The simple conjugate form may still be a component or special case of these advances.

4.2.1 Parametric versus Nonparametric Models

The use of the Dirichlet or Generalized Dirichlet distribution allows for a pattern with a parameter for each development period. This creates a very flexible shape but requires estimation of many parameters. An alternative is the use of a parametric "growth curve" such as described in Zhang, et al (2012).

A parametric curve creates a much smoother development pattern, which is more constrained because of the fewer parameters. The Dirichlet is sometimes called a nonparametric model because it can follow the data more closely; however, "nonparametric" is a bit of a misnomer because it does not mean "no parameters" but rather potentially "many parameters."

The use of a parametric growth curve can be incorporated in a Bayesian framework, with the prior distribution being on the parameters. This does not fit as neatly into our conjugate form, but can be handled in simulation-based MCMC models.

4.2.2 Including Exposures or Other External Data

The models above assume that the actuary is selecting a loss development pattern from a development triangle, and that the basic assumptions of the chain ladder method apply. For example, that the same pattern is applicable for all accident years.

The Bayesian framework allows us to move beyond this limited data and include other information. We could bring in data such as exposure units (e.g., onlevel premium) or expected loss ratios. This additional information may also have prior distributions reflecting the relative uncertainty in the data. Robbin (1986) and Mildenhall (2006) show that as the relative uncertainties change the results move between familiar methods such as Cape Cod and Bornhuetter-Ferguson.

In addition to exposure or premium information, the model can expand to modify the assumption that all accident years share the same expected development pattern. Meyers (2015) introduces a "changing settlement rate" (CSR) model that includes an interaction term to adjust each accident year.

4.2.3 Calculation of Predictive Distribution

This paper has been focused on getting an estimate of expected ultimate loss that incorporates prior knowledge, and we have not directly discussed the variability around that estimate. However, because all of the analysis presented in this paper has been based on explicit distribution forms, all of the building blocks are in place to calculate ranges around estimated ultimate losses.

The evaluation of variance depends directly upon the scale parameter ϕ, which has been assumed to be fixed and known – in fact supplied by the analyst. For computing ranges we would more generally want this parameter to be considered a random variable with its own prior distribution. The variance should also be considered uncertain in order to evaluate the full uncertainty in the final estimate of ultimate loss.

4. CONCLUSIONS

This paper has presented an introduction to Bayesian loss development and gives an implementation that can be used immediately by any actuary. The use of the Generalized Dirichlet allows simpler computation than presented in earlier papers and allows for calculations that are as direct as the calculation of age-to-age factors. The use of a conjugate form allows an interpretation of prior knowledge in the form of "fictitious" prior loss development. The conjugate form can also be expanded with discrete mixtures to allow greater flexibility in specifying prior knowledge.

It is hoped that this paper will allow more actuaries to experiment with the Bayesian framework and then be comfortable to move to ever more realistic modeling work.

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Supplementary Material

The examples given in this paper are easily implements in spreadsheet format. The author can make the examples available upon request.

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Abbreviations and Mathematical Notation

ATA	Age-to-Age factor , or "link ratio"
LDF	Cumulative Loss Development Factor

p	A variable representing a portion between 0 and 1. It is a parameter (number of successes) in the Binomial distribution or the random variable itself for Beta distribution. In the univariate distributions (Binomial, Beta) it is written without a subscript; in the multivariate cases (Multinomial, Dirichlet) it is written with a subscript.
ϕ	Scale Parameter, or variance-to-mean ratio of aggregate loss
α, β	Shape parameters of Gamma, Beta and Generalized Dirichlet distributions
$C_{t,d}$	Cumulative losses for accident year t as of development age d

Biography of the Author

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