# The Lognormal Random Multivariate

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**Abstract**

For decades the lognormal random variable has been widely used by actuaries to analyze heavy-tailed insurance losses. More recently, especially since ERM and Solvency II, actuaries have had to solve problems involving the interworking of many heavy-tailed risks. Solutions to some of these problems may involve the relatively unknown extension of the lognormal into the multivariate realm. The purpose of this paper is present the basic theory of the lognormal random multivariate.

**Keywords:** lognormal, multivariate, moment generating function, positive-definite

## 1. INTRODUCTION

The lognormal random variable $Y = e^{X \sim N(\mu, \sigma)}$ is familiar to casualty actuaries, especially to those in reinsurance. It vies with the Pareto for the description of heavy-tailed and catastrophic losses. However, unlike the Pareto, all its moments are finite. Moreover, the formula for the lognormal moments is rather simple: $E[Y^n] = e^{n\mu + n^2\sigma^2/2}$. So its first two moments are $E[Y] = e^{\mu+\sigma^2/2}$ and $E[Y^2] = e^{2\mu+2\sigma^2} = E[Y]^2 e^{\sigma^2}$. Hence, its variance is $Var[Y] = E[Y]^2\left(e^{\sigma^2} - 1\right)$, a formula so well known that actuaries commonly refer to $e^{\sigma^2} - 1$ as the "CV squared" of the lognormal. But in recent years, with the rise of ERM and capital modeling, actuaries have needed to model many interrelated random variables. If these random variables are heavy-tailed, it may be apt to model them with the lognormal random multivariate, which we will now present.[1]

[1] The standard reference for the lognormal distribution is Klugman [1998, Appendix A.4.1.1]. On the subject of heavy-tailed distributions, see Klugman [1998, §2.7.2] and Halliwell [2013].

## 2. MOMENT GENERATION AND THE LOGNORMAL MULTIVARIATE

The lognormal random multivariate is $\mathbf{y} = e^{\mathbf{x}}$, where $\mathbf{x} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ is an $n\times 1$ normal multivariate with $n\times 1$ mean μ and $n\times n$ variance Σ. As a realistic variance, Σ must be positive-definite, hence invertible.[2]

The probability density function of the normal random vector **x** with mean μ and variance Σ is:[3]

$$f_{\mathbf{x}}(\mathrm{x}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}(\mathrm{x}-\mu)'\Sigma^{-1}(\mathrm{x}-\mu)}$$

Therefore, $\int_{\mathrm{x}\in\Re^n} f_{\mathbf{x}}(\mathrm{x})dV = 1$. The single integral over $\Re^n$ represents an $n$-multiple integral over each $x_j$ from $-\infty$ to $+\infty$; $dV = dx_1 \ldots dx_n$. The moment generating function of **x** is $M_{\mathbf{x}}(\mathrm{t}) = E\left[e^{\mathrm{t}'\mathbf{x}}\right] = E\left[e^{\sum_{j=1}^{n} t_j X_j}\right]$, where t is an $n\times 1$ vector. Partial derivatives of the moment generating function evaluated at $\mathrm{t} = 0_{n\times 1}$ equal moments of ***x***, since:

$$\left.\frac{\partial^{k_1+\ldots k_n} M_{\mathbf{x}}(\mathrm{t})}{\partial^{k_1} x_1 \ldots \partial^{k_n} x_n}\right|_{\mathrm{t}=0} = E\left[X_1^{k_1} \ldots X_n^{k_n} e^{\mathrm{t}'\mathbf{x}}\right]\Big|_{\mathrm{t}=0} = E\left[X_1^{k_1} \ldots X_n^{k_n}\right]$$

The lognormal moments come directly from the normal moment generating function. For example, if $\mathrm{t} = \mathrm{e}_j$, the $j^{\text{th}}$ unit vector, then $M_{\mathbf{x}}(\mathrm{e}_j) = E\left[e^{\mathrm{e}_j'\mathbf{x}}\right] = E\left[e^{X_j}\right] = E\left[Y_j\right]$. Likewise,

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[2] For a review of positive-definite matrices see Judge [1988, Appendix A.14].
[3] See Johnson and Wichern [1992, Chapter 4] and Judge [1988, §2.5.7].

$M_{\mathbf{x}}(\mathrm{e}_j + \mathrm{e}_k) = E\left[e^{X_j} e^{X_k}\right] = E\left[Y_j Y_k\right]$. So the normal moment generating function is the key to the lognormal moments.

The moment generating function of the normal random vector **x** is:

$$\begin{aligned}
M_{\mathbf{x}}(\mathrm{t}) &= E\left[e^{\mathrm{t}'\mathbf{x}}\right] \\
&= \int_{\mathrm{x}\in\Re^n} \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}(\mathrm{x}-\mu)'\Sigma^{-1}(\mathrm{x}-\mu)} e^{\mathrm{t}'\mathrm{x}} dV \\
&= \int_{\mathrm{x}\in\Re^n} \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}\left\{(\mathrm{x}-\mu)'\Sigma^{-1}(\mathrm{x}-\mu)-2\mathrm{t}'\mathrm{x}\right\}} dV
\end{aligned}$$

A multivariate "completion of the square" results in the identity:

$$(\mathrm{x}-\mu)'\Sigma^{-1}(\mathrm{x}-\mu) - 2\mathrm{t}'\mathrm{x} = (\mathrm{x}-[\mu+\Sigma\mathrm{t}])'\Sigma^{-1}(\mathrm{x}-[\mu+\Sigma\mathrm{t}]) - 2\mathrm{t}'\mu - \mathrm{t}'\Sigma\mathrm{t}$$

We leave it for the reader to verify the identity. By substitution, we have:

$$\begin{aligned}
M_{\mathbf{x}}(\mathrm{t}) &= \int_{\mathrm{x}\in\Re^n} \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}\left\{(\mathrm{x}-\mu)'\Sigma^{-1}(\mathrm{x}-\mu)-2\mathrm{t}'\mathrm{x}\right\}} dV \\
&= \int_{\mathrm{x}\in\Re^n} \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}\left\{(\mathrm{x}-[\mu+\Sigma\mathrm{t}])'\Sigma^{-1}(\mathrm{x}-[\mu+\Sigma\mathrm{t}])-2\mathrm{t}'\mu-\mathrm{t}'\Sigma\mathrm{t}\right\}} dV \\
&= \int_{\mathrm{x}\in\Re^n} \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}(\mathrm{x}-[\mu+\Sigma\mathrm{t}])'\Sigma^{-1}(\mathrm{x}-[\mu+\Sigma\mathrm{t}])} dV \cdot e^{\mathrm{t}'\mu+\mathrm{t}'\Sigma\mathrm{t}/2} \\
&= 1 \cdot e^{\mathrm{t}'\mu+\mathrm{t}'\Sigma\mathrm{t}/2} \\
&= e^{\mathrm{t}'\mu+\mathrm{t}'\Sigma\mathrm{t}/2}
\end{aligned}$$

The reduction of the integral to unity in the second last line is due to the fact that $\frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}(\mathrm{x}-[\mu+\Sigma\mathrm{t}])'\Sigma^{-1}(\mathrm{x}-[\mu+\Sigma\mathrm{t}])}$ is the probability density function of the normal random vector with mean $\mu+\Sigma\mathrm{t}$ and variance $\Sigma$.

So the moment generating function of the normal multivariate $\mathbf{x} \sim N(\mu, \Sigma)$ is $M_{\mathbf{x}}(\mathrm{t}) = e^{\mathrm{t}'\mu + \mathrm{t}'\Sigma\mathrm{t}/2}$. As a check:[4]

$$\frac{\partial M_{\mathbf{x}}(\mathrm{t})}{\partial \mathrm{t}} = (\mu + \Sigma\mathrm{t})e^{\mathrm{t}'\mu + \mathrm{t}'\Sigma\mathrm{t}/2} \Rightarrow E[\mathbf{x}] = \left.\frac{\partial M_{\mathbf{x}}(\mathrm{t})}{\partial \mathrm{t}}\right|_{\mathrm{t}=0} = \mu$$

And for the second derivative:

$$\begin{aligned}
\frac{\partial^2 M_{\mathbf{x}}(\mathrm{t})}{\partial \mathrm{t}\partial \mathrm{t}'} &= \frac{\partial(\mu + \Sigma\mathrm{t})e^{\mathrm{t}'\mu + \mathrm{t}'\Sigma\mathrm{t}/2}}{\partial \mathrm{t}'} \\
&= \left(\Sigma + (\mu + \Sigma\mathrm{t})(\mu + \Sigma\mathrm{t})'\right)e^{\mathrm{t}'\mu + \mathrm{t}'\Sigma\mathrm{t}/2} \\
&\Rightarrow E[\mathbf{x}\mathbf{x}'] = \left.\frac{\partial^2 M_{\mathbf{x}}(\mathrm{t})}{\partial \mathrm{t}\partial \mathrm{t}'}\right|_{\mathbf{t}=0} = \Sigma + \mu\mu' \\
&\Rightarrow Var[\mathbf{x}] = E[\mathbf{x}\mathbf{x}'] - \mu\mu' = \Sigma
\end{aligned}$$

The lognormal moments follow from the moment generating function:

$$E[Y_j] = [e^{X_j}] = E\left[e^{\mathrm{e}_j'\mathbf{x}}\right] = M_{\mathbf{x}}(\mathrm{e}_j) = e^{\mathrm{e}_j'\mu + \mathrm{e}_j'\Sigma\mathrm{e}_j/2} = e^{\mu_j + \Sigma_{jj}/2}$$

The second moments are conveniently expressed in terms of first:

$$\begin{aligned}
E[Y_jY_k] = E\left[e^{X_j}e^{X_k}\right] &= E\left[e^{(\mathrm{e}_j + \mathrm{e}_k)'\mathbf{x}}\right] \\
&= e^{(\mathrm{e}_j + \mathrm{e}_k)'\mu + (\mathrm{e}_j + \mathrm{e}_k)'\Sigma(\mathrm{e}_j + \mathrm{e}_k)/2} \\
&= e^{\mu_j + \mu_k + (\Sigma_{jj} + \Sigma_{jk} + \Sigma_{kj} + \Sigma_{kk})/2} \\
&= e^{\mu_j + \Sigma_{jj}/2 + \mu_k + \Sigma_{kk}/2 + (\Sigma_{jk} + \Sigma_{kj})/2} \\
&= e^{\mu_j + \Sigma_{jj}/2} \cdot e^{\mu_k + \Sigma_{kk}/2} \cdot e^{(\Sigma_{jk} + \Sigma_{kj})/2} \\
&= e^{\mu_j + \Sigma_{jj}/2} \cdot e^{\mu_k + \Sigma_{kk}/2} \cdot e^{(\Sigma_{jk} + \Sigma_{jk})/2} \\
&= E[Y_j]E[Y_k] \cdot e^{\Sigma_{jk}}
\end{aligned}$$

So, $Cov[Y_j, Y_k] = E[Y_jY_k] - E[Y_j]E[Y_k] = E[Y_j]E[Y_k](e^{\Sigma_{jk}} - 1)$, which is the multivariate equivalent of the well-known scalar formula $CV^2[e^X] = e^{\sigma^2} - 1$. The whole variance matrix can be expressed as

[4] The vector formulation of partial differentiation is explained in Judge [1988, Appendix A.17].

$Var[\mathbf{y}] = \left( E[\mathbf{y}]E[\mathbf{y}]' \right) \circ \left( e^{\Sigma} - 1_{n \times n} \right)$, where '∘' represents elementwise multiplication (the Hadamard product). Defining the diagonalization of a vector as $diag(\mathrm{v}_{n \times 1}) = \begin{bmatrix} v_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_n \end{bmatrix}$, we may express the variance in terms of the usual matrix multiplication as $Var[\mathbf{y}] = diag(E[\mathbf{y}])\left(e^{\Sigma} - 1_{n \times n}\right)diag(E[\mathbf{y}])$. Because $diag(E[\mathbf{y}])$ is diagonal in positive elements (hence, symmetric and positive-definite), $Var[\mathbf{x}]$ is positive-definite if and only if $e^{\Sigma} - 1_{n \times n}$ is positive-definite. Although beyond the scope of this paper, it can be proven[5] that if Σ is positive-definite, as stipulated above, then so too is $\mathrm{T} = e^{\Sigma} - 1_{n \times n}$.[6]

## 3. CONCLUSION

The mean and the variance of the lognormal multivariate are straightforward extensions of their scalar equivalents. Simulating lognormal random outcomes is nothing more than exponentiating simulated normal random multivariates. Therefore, one faced with the problem of modeling several heavy-tailed random variables in a mean-variance framework may find an acceptable solution in the lognormal random multivariate.

## REFERENCES


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[3.] Judge, George G., Hill, R. C., et al., *Introduction to the Theory and Practice of Econometrics* (Second Edition), New York, Wiley, 1988.

[4.] Klugman, Stuart A., et al., *Loss Models: From Data to Decisions*, New York, Wiley, 1998.


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[5] A proof involving Shur's Product Theorem forms part of an unpublished paper by the author, "Complex Random Variables," which he will make available for the asking.

[6] The converse is not necessarily true: there exist positive-definite T for which $\Sigma = \ln\left(1_{n \times n} + \mathrm{T}\right)$ is not positive-definite. Lognormal variance is a proper subset of (normal) variance. Hereby one can test whether variance T is realistic for interrelated random variables with heavy tails.