Casualty Actuarial Society E-Forum, Summer 2014-Volume 2

The CAS *E-Forum*, Summer 2014-Volume 2

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This *E-Forum* contains five reserves call papers, created in response to a call issued by the CAS Committee on Reserves (CASCOR). Some of the Reserves Call Papers will be presented at the 2014 Casualty Loss Reserve Seminar (CLRS) on September 15-17, 2014, in San Diego, CA.

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Reserves Call Papers

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Applying Credibility Concepts to Develop Weights for Ultimate Claim Estimators

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June 19, 2014 *

*LaTeXed on July 19, 2014

In estimating ultimate claim and claim expense amounts, actuaries often rely on estimates developed using multiple actuarial methods. Combining these estimates is often left to the actuary's professional judgment. That judgment generally reflects informed but subjective opinion on the relative stability and responsiveness of various methods and the reasonableness of the results of those methods.

This paper proposes a more quantitative approach. The approach is based on credibility concepts which are often used in ratemaking contexts but have yet to find their way into this particular aspect of estimating unpaid claims and claim expenses. As with the ratemaking context, credibility is based on the variance of estimators. However the application to unpaid claim estimates requires a different approach. That approach is the subject of this paper.

Keywords Credibility, Reserving

1. Introduction

1.1. Research Context

The author was not able to locate any prior research on this specific topic. During the course of preparing this paper, the author identified Rehman & Klugman[RK10] as having some similarity in underlying concepts - though that paper has a much different application.

The CAS Taxonomy for this paper is as follows:

- Actuarial Applications and Methodologies > Reserving >Management Best Estimate
- Actuarial Applications and Methodologies > Reserving >Reserving Methods

1.2. Objective

Typically[1], actuarial estimates of ultimate claims are based on a review of multiple actuarial indications (such as those based on the chain-ladder and Bornhuetter-Ferguson methods) and the actuary's professional judgment. We can think of the final (selected) actuarial estimate as a weighted average of actuarial indications and expert opinion and/or prior knowledge. The mathematical description of that process would likely be similar to the following:

$$\hat{C} = \hat{\boldsymbol{I}} \times \boldsymbol{Z} + P \times (1 - Z) \tag{1}$$

which we can recognize as being similar to credibility-weighted averages commonly used in ratemaking contexts.

In the context of estimating ultimate claims[2]:

$C =$ ultimate claims

$\hat{C} =$ an estimator of ultimate claims

$\hat{\boldsymbol{I}} =$ a vector of actuarial projections (indications) of C

$P =$ the actuary's prior estimate of ultimate claims; possibly equal to $\hat{C}_{t-1}$

$\boldsymbol{Z} =$ a vector of credibility factors

$Z =$ the sum of the elements of $\boldsymbol{Z}$

There are several important considerations with respect to these variables:

[1] This section is based strictly on the author's observation of common practices and does not imply that all actuaries use the approach described.

[2] In this paper, the term "claim" is used rather than "loss" to be consistent with Actuarial Standard of Practice No. 43, *Property/Casualty Unpaid Claim Estimates*

- In Equation (1), we assume that $\hat{I}$, P and, by consequence, $\hat{C}$, are all unbiased[3] estimators of C. The author is aware that certain papers challenge this assumption. Measuring and correcting for bias of actuarial methods is outside the scope of this paper.
- "Prior" in this paper is not used in the rigorous statistical sense of that term. Rather, the term refers to an estimate that management or the actuary may be "targeting" and is based on the author's observation that, in practice, reasonableness of an estimate is often evaluated relative to some other (benchmark) estimate[4]. For example, the implied loss ratios of estimates based on the first evaluation of an experience period are assessed through reconciliation of those estimates to the loss ratio used in establishing the premium.

In the usual judgment-based model, "credibility" for method i is then usually assigned based on the distance between $\hat{I}_i$ and P. That is:

$$Z_i \propto \frac{1}{d(\hat{I}_i, P)} \tag{2}$$

$$Z_i = f_j\left(\frac{1}{d(\hat{I}_i, P)}\right) \tag{3}$$

where d represents a generic difference function such as absolute or squared difference. As commonly applied, f_j might be termed the "actuarial judgment function." In Figure 1, we present a visualization of that "actuarial judgment function."

The objective of this paper is to offer an approach to *calculate* Z rather than use an "actuarial judgment function."

1.3. Outline

The remainder of the paper proceeds as follows:

- Section 2 is a discussion of approaches that are commonly used and the approach proposed by this paper.
- Section 3 provides the theory and practice of the credibility weighted approach. A workbook accompanies this report to supplement the reader's understanding of the application in practice.
- Section 4 provides a generalized model and summarizes the findings of this paper.

[3] Or, more precisely, that we are aware of and can adjust for any biases in $\hat{I}$ or P.

[4] As with all generalizations, this of course is not universally true. For example, if there were a known shock loss, the benchmark may be disregarded.

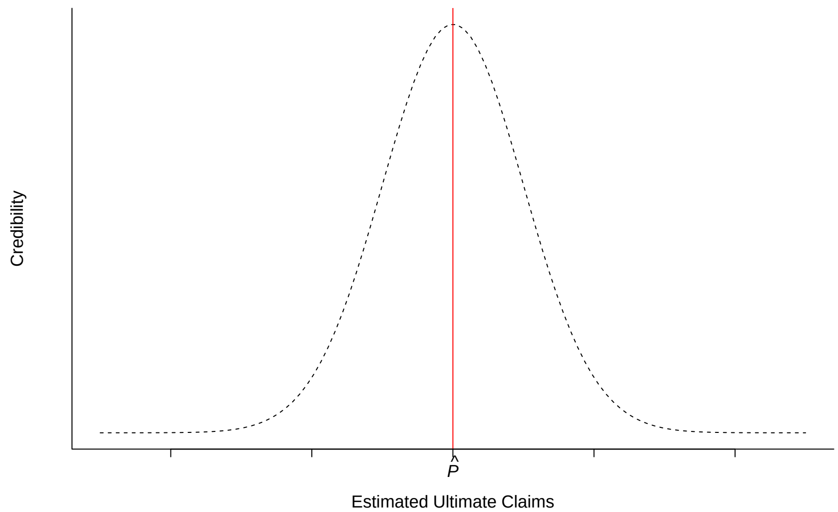

Figure 1: The Actuarial Judgment Function

2. Background and Methods

2.1. Credibility in a Ratemaking Context

Actuaries are familiar with the use of credibility in a ratemaking context. In that context, the goal is to assign predictive value to the experience of a class relative to the predictive value of the experience of an aggregation of classes. Although there are various models to estimate this predictive value, credibility, generally, is proportional to the variance between classes (also referred to as the *variance of hypothetical means, VHM*) and inversely proportional to the average variance within classes (commonly referred to as the *expected value of process variance* ($EVPV$))[5].

2.2. Credibility in a Reserving Context

The extension of credibility to a reserving context may not be immediately clear until we consider the *general* definition of the term "credibility." That is, we need to consider credibility as a measure of the predictive value, possibly measured on a relative basis, of an estimator. We are not referring to credibility as calculated under a *specific* model. In the approach presented in this paper, credibility for each available actuarial method is

[5]Though these terms are used in several papers, to the best of the author's knowledge, they were first used (or at least popularized) by Philbrick [Phi81].

developed based on the variance of that method *relative to the variance of each of the other methods.*

For the moment, assume that we have two competing estimates, I_1 and I_2, for C. Then, the credibility model becomes:

$$\hat{C} = \hat{I}_1 \times Z_1 + \hat{I}_2 \times Z_2 \tag{4}$$

There is an important difference between Equation (1) and Equation (4). The latter does not include the *prior* estimate. In ratemaking, the prior estimate is considered to the extent that we cannot assign credibility to an indication(s). In unpaid claim estimation, we have models (such as the loss ratio or Bornhuetter-Ferguson models) that allow for the consideration of a "prior" estimate so we need not consider the estimate separately and explicitly. This paper presents an approach where we assign 100% of the credibility to available estimates based on the *relative* variance.

In evaluating variance, we consider the residual, or the difference between the observed prediction and the "best" prediction. Furthermore, rather than consider the variance of I_i, we consider the variance of the underlying actuarial method or model i at maturity j. We denote the observed indication $\hat{M}_{i,j}$.

Then, the credibility weighted average of estimators can be written as:

$$M_{1,j} \times Z_{1,j} + \ldots + M_{n,j} \times Z_{n,j} \tag{5}$$

Finally, we define credibility in this context as the probability that the error (residual) of $M_{i,j}$ is smaller than that of $n-1$ competing estimates $M_{1,j} \ldots M_{i-1,j}$, $M_{i+1,j} \ldots M_{n,j}$. where $\sum Z_{1,j} \ldots Z_{n,j} = 1$.

2.3. Residual Errors

The proposed approach is based on an analysis of the distribution of residuals (ε) of each method, M_i, at a particular maturity j. As an illustration of the concept of the variance of residuals, we consider chain-ladder estimates based on an analysis of a development triangle. In that situation, we would have a series of observed predictions at 12 months which are the product of the 12-month claim development factor and previously observed claims at 12 months for prior experience periods. After normalizing the triangle for differences between experience periods[6], we would then calculate the residuals as the difference between those predictions and the "best" values. This is only one method for developing an estimate of the variance of the residuals. We explore the issue further in Section 3.3.1.

[6] For example, such difference may include those attributable to changes in costs level or exposure volume.

3. Results and Discussion

Rather than separating theory and practice/example, I have elected to present the theoretical underpinnings of the proposed credibility model in the context of a minimally specified example. I intend for this presentation to demonstrate the practicality of this approach. This presentation also allows for easier identification of assumptions underlying the credibility model.

3.1. Proposed Credibility Model

For our minimally specified example, we will assume the following:

- We have two competing methods, the paid chain-ladder (Method 1, M_1) and the reported incurred chain-ladder (Method 2, M_2). We are concerned with the estimate at 12 months maturity. (The maturity is unimportant to this example but it is helpful to define the context.)
- Assume that ultimate claims (C) are 1,000. This estimate provides a sense of scale though it is not necessary for our minimally specified example as we are provided the distribution of residuals. (In a different circumstance, we may have the coefficient of variation of the residuals. In this case, the estimate of ultimate claims would be necessary.)
- We assume that the methods are unbiased. Therefore the means of the residuals for both models are assumed to be 0 ($\mu(\varepsilon_1) = \mu(\varepsilon_2) = 0$).
- In our example, the paid chain-ladder has more variability in predictions than the reported incurred chain-ladder. The residual errors are assumed to be normally distributed with standard deviations of 200 and 300 for the reported incurred chain-ladder and paid chain-ladder methods, respectively, ($s(\varepsilon_1) = 300; s(\varepsilon_2) = 200; E[s] = \sigma$) as presented in Figure 2.

Under our definition, the credibility of the reported incurred chain-ladder is the probability that the error of M_2 (random variable denoted X_2) is less than or equal to the error of M_1 (random variable denoted X_1).

So for any $X_2 = x_2$ (where x_2 is an observation of X_2), we have the following possibilities:

1. $|X_1| < |x_2|$ (Credibility to Method 1)
2. $|X_1| > |x_2|$ (Credibility to Method 2)

Given the symmetric distribution centered around 0, for simplicity, we use only the positive domain of x and consider both tails of the distribution of x_1. We use F and f to represent the distribution and density functions, respectively, of the residuals. We then have the

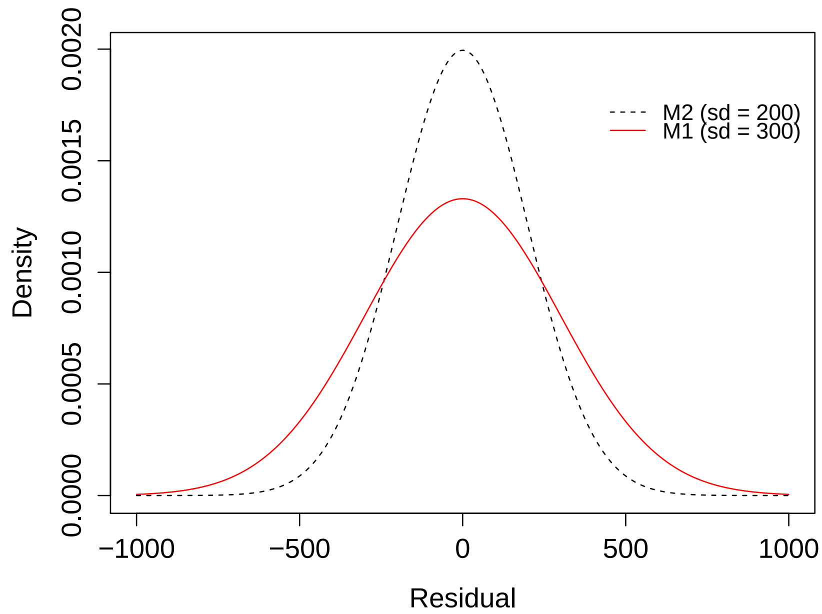

Figure 2: Distribution of Residual Errors

following credibility calculation:

$$Z_2/2 = \int_0^\infty 2\,[1 - F_1(x)]\, f_2(x)\, dx \tag{6}$$

In words, Equation (6) states:

> "Over the domain of positive values of x (i.e. $\int_0^\infty$), the credibility assigned to Method 2 (i.e. Z_2) is the probability that the error of Method 1 is greater than x (i.e. $(1 - F_1(x))$) or less than $-x$ (which by symmetry is also equal to $1 - F_1(x)$) given that $X_2 = x$ (i.e. $f_2(x)\,dx$)."
>
> The *2* inside the integral provides consideration for both values of X_1 less than $-x_2$ and greater than $+x_2$. For example, if $x_2 = 100$, we would assign credibility to Method 2 for X_1 probabilistically greater than 100 or X_1 probabilistically less than -100.
>
> The *2* on the left-side of Equation (6) is necessary as our limits of integration only consider one-half the domain of possible x values.

We should also recognize that if we were evaluating over the domain of negative x values, we would replace $1 - F_1(x)$ with $F_1(x)$. This is due to the property of symmetric distributions centered at 0 where:

$$F(-x) = (1 - F(x)) \qquad x > 0$$
$$F(x) = (1 - F(-x)) \qquad x < 0.$$

This is further demonstrated in Appendix A, where we provide Equation (6) with separate terms for values of x that are less than 0 and those greater than 0.

We can then expand Equation (6) as follows:

$$Z_2/2 = 2\int_0^\infty f_2(x)\,dx - 2\int_0^\infty F_1(x)f_2(x)\,dx$$
$$Z_2/2 = 1 - 2\int_0^\infty F_1(x)f_2(x)\,dx \tag{7}$$
$$Z_2 = 2 - 4\int_0^\infty F_1(x)f_2(x)\,dx \tag{8}$$

Equations (7) and (8) are intuitively appealing as they state that credibility is lost when a competing estimate has a lower error. The constants of 2 and 4 in Equation (8) may seem disconcerting at first but we should recognize that they are twice what they would be had we integrated over both positive and negative values of x.

Furthermore, if we consider the limiting case where Method 1 has no error (i.e. it is a perfect indicator of C) then:

1. $F_1(0) = 1$
2. $\therefore \int_0^\infty F_1(x)f_2(x)\,dx = 1/2$
3. $\therefore Z_2 = 0$

We can use the trapezoidal rule to numerically integrate Equation (8). If we apply the trapezoidal rule over 1,000 evenly-spaced (unit) intervals between 0 and 1,000, we can calculate the value of the integral to be 0.342. The resulting credibility to Method 2 is 0.630.

We can also reverse the subscripts and calculate the credibility of Method 1 using Equation (8).

$$Z_1 = 2 - 4\int_0^\infty F_2(x)f_1(x)\,dx$$

The resulting credibility of Method 1 is 0.379. The sum of these credibilities don't quite equal 1 but that is simply the result of the approximation of the numerical integration. We can address this issue through normalization as presented in Table 1.

	Raw Credibility	Normalized Credibility
Method 1	0.379	0.376
Method 2	0.630	0.624
Total	1.009	1.000

Table 1: Two Method Example

3.2. Simulation

We can also use simulation if we want to avoid the effort of numerical integration. We present the R[R C13] code to estimate credibilities via simulation.

```
> set.seed(12345)
> trials <- 1000
> pd.dev.errors <- abs(rnorm(n = trials, mean = 0, sd = 300))
> rptd.dev.errors <- abs(rnorm(n = trials, mean = 0, sd = 200))
> pd.dev.cred <- length(which(pd.dev.errors < rptd.dev.errors)) / trials
> rptd.dev.cred <- length(which(rptd.dev.errors < pd.dev.errors)) / trials
> pd.dev.cred
[1] 0.375
> rptd.dev.cred
[1] 0.625
```

Alternatively, one could use the `integrate` function with the following code[7].

```
> f <- function(x) pnorm(x, 0, 300) * dnorm(x, 0, 200)
> integral <- integrate(f, 0, Inf)$value
> rptd.dev.cred <- 2 - 4 * integral
> rptd.dev.cred
[1] 0.6256659
```

We note that the results of the simulation are quite close to those calculated from numerical integration.

3.3. Assumptions

The minimally specified model includes several simplifying assumptions that we explore in this section.

[7]This code was contributed by Mark Mordechai Goldburd. Mr. Goldburd reviewed this paper for the CAS 2014 Fall *E-Forum*

- We have assumed that the residual errors are normally distributed. Is this reasonable?
- We were provided with the standard errors in the examples in Section 3.1. What approaches can we use to develop the estimated standard error?
- Can we consider the management's recorded estimate within this model?

3.3.1. Residual Standard Error

Modeling the distribution of residuals is a complex topic that is outside the scope of this paper. We should keep in mind that we are more focused on *relative errors* than *absolute errors*. Furthermore, even when we cannot calculate that uncertainty, we should be able to assign uncertainty based on judgment. That judgment (e.g. the uncertainty / volatility of the reported loss development method at 12 months is ± 200) is more "testable" than the implicit assignments of credibility to methods based on the "actuarial judgment function."

Generally, we would expect that the variance for any method would decrease over time as paid claims are a greater percentage of ultimate claims. In this model, we calculate credibility separately for each maturity. As such, shifts in credibility weights between methods will occur due to *differences in the rate of decrease* in the variance as a function of maturity.

Further, we would *expect* that methods that emphasize stability will have lower variances at early maturities than those that emphasize responsiveness. However, this will depend on the deviations between the *a priori* expected ultimate claims and the current best estimates.

Below, we present one example approach to developing error estimates using the `auto$PersonalAutoIncurred` data included in the `ChainLadder` [GMZ13] package for `R`.

1. In Table 2, we present the triangle of paid claims and volume-weighted development factors.
2. In Table 3, we present the indications of ultimate claims based on the paid development method and the current valuation of paid claims.
3. In Table 4, we present the indications of ultimate claims based on paid development factors and historical valuation of paid claims from Table 2.
4. In Table 5, we present the triangle of residuals. Those residuals are calculated as the difference between the ultimates in Table 4 and those in Table 3.

Origin	Maturity 1	2	3	4	5	6	7	8	9	10
1	101125	209921	266618	305107	327850	340669	348430	351193	353353	353584
2	102541	203213	260677	303182	328932	340948	347333	349813	350523	
3	114932	227704	298120	345542	367760	377999	383611	385224		
4	114452	227761	301072	340669	359979	369248	373325			
5	115597	243611	315215	354490	372376	382738				
6	127760	259416	326975	365780	386725					
7	135616	262294	327086	367357						
8	127177	244249	317972							
9	128631	246803								
10	126288									
Incremental DF	1.990	1.285	1.137	1.064	1.031	1.017	1.006	1.004	1.001	1.000
Cumulative DF	3.278	1.647	1.282	1.128	1.060	1.028	1.011	1.005	1.001	1.000

Table 2: auto$PersonalAutoPaid{ChainLadder}

Origin	Current Claims	Cumulative Development Factor	Ultimates
1	353584	1.000	353584
2	350523	1.001	350874
3	385224	1.005	387150
4	373325	1.011	377432
5	382738	1.028	393455
6	386725	1.060	409928
7	367357	1.128	414379
8	317972	1.282	407640
9	246803	1.647	406485
10	126288	3.278	413972

Table 3: Ultimate Claim Estimates

In Table 5, we also present the standard deviations of the residuals[8] for each age that may be used as a basis to select the standard error of the paid development method at each of the respective ages. Those standard deviations may be slightly understated in an absolute sense as our triangle includes the current valuation - which will have a deviation of 0 under this approach. However, we elected to include that diagonal to generalize the approach and allow for other estimates of the "best" estimate of ultimate claims.

This is only one example algorithm to estimate the standard errors for results of an actuarial method at a particular age. We note that the `ChainLadder` package includes many useful functions for developing these estimates and recommend that readers review the vignette accompanying that package.

3.3.2. Distribution of Residuals

The credibility model presented does not require the use of the normal model to describe the distribution of errors. Identification of the appropriate model is a complex topic that is outside the scope of this paper. However, it would seem reasonable to use a model that

[8] Mr. Goldburd also noted that the standard deviations presented in Table 5 are calculated as sample standard deviations from an estimated mean. If we assume residuals are centered on zero, it may be more appropriate to calculate standard deviations as root mean squared distance from 0, with n as the denominator rather than $n-1$, since a degree of freedom is not lost to an estimated mean. This would not apply to generalizations of the model (such as those presented in the Appendix) and the calculation of residual variance is not the focus of the paper.

	1	2	3	4	5	6	7	8	9	10
1	331488	345740	341804	344161	347521	350208	352263	352949	353706	353584
2	336129	334692	334188	341989	348668	350495	351154	351562	350874	
3	376747	375028	382190	389771	389826	388583	387831	387150		
4	375174	375122	385974	384275	381578	379587	377432			
5	378927	401227	404106	399865	394719	393455				
6	418797	427258	419182	412600	409929					
7	444549	431998	419324	414379						
8	416886	402278	407640							
9	421652	406485								
10	413972									

Table 4: Retrospective Ultimates

	1	2	3	4	5	6	7	8	9	10
1	-22096	-7844	-11780	-9423	-6063	-3376	-1321	-635	122	0
2	-14745	-16182	-16686	-8885	-2206	-379	280	688	-0	
3	-10403	-12122	-4960	2621	2676	1433	681	0		
4	-2258	-2310	8542	6843	4146	2155	-0			
5	-14528	7772	10651	6410	1264	-0				
6	8869	17330	9254	2672	0					
7	30170	17619	4945	-0						
8	9246	-5362	0							
9	15167	-0								
10	0									
sd	16105	12122	10270	6704	3676	2135	867	662	87	

Table 5: Residuals

is symmetric with a zero mode and mean. Rehman & Klugman [RK10] includes discussion related to the use of a normal distribution to describe reserve variability.

3.3.3. Credibility Assigned to Management Estimates

We can consider the recorded management estimate to simply be another indication. If we are able to compile a triangle of prior recorded estimates, we can apply a model similar to that presented above for the paid chain-ladder method. Additionally, review of Rehman & Klugman [RK10] may be useful in determining a model for describing such errors.

4. Conclusion

In this section, we summarize the findings of the research presented in this paper.

4.1. Principal Finding

In this paper, we have proposed a method for weighting methods that is based on the uncertainty of the estimate. We recognize that developing measures of that uncertainty is not a trivial matter.

Equation (6) is the primary finding of working through the minimally specified example. Equation (6) may be generalized for n methods. The initial generalization is presented below as Equation (10). That is, the credibility of method n relative to methods $1 \ldots n-1$ may be calculated as follows.

$$Z_n/2 = \int_0^\infty 2^{n-1}\{[1-F_1(x)]\ldots[1-F_{n-1}(x)]\}f_n(x) \tag{9}$$

$$Z_n = \int_0^\infty 2^{n}\{[1-F_1(x)]\ldots[1-F_{n-1}(x)]\}f_n(x) \tag{10}$$

Equation (11) presents the final generalization and the primary finding of this paper. Specifically that the credibility of method i relative to methods $1 \ldots i-1, i+1 \ldots n$ is calculated as:

$$Z_i = \int_0^\infty 2^{n}\{[1-F_1(x)]\ldots[1-F_{i-1}(x)] [1-F_{i+1}(x)]\ldots[1-F_{n}(x)]\}f_i(x) \tag{11}$$

4.2. Distribution of Residuals

We acknowledge that determination of the distribution of errors is not trivial. However, it would seem that assuming a normal distribution would be reasonable. In addition, so long as we have a consistent approach to determining errors from the various methods, we should be able to apply our model. Those "consistent" approaches may include an approach that assigns uncertainty based on professional judgment.

4.3. Simulation as an Alternative

In practice, using numerical integration to calculate credibility under the proposed model is not overly difficult. It may also be appealing as setting up the model requires that we think through issues of estimation uncertainty. In the companion workbook, we present the calculation for four methods with standard errors 100, 200, 400, and 600. Of course, as n increases, using simulation to estimate credibilities becomes a more attractive option.

5. Supplemental Information

5.1. Acknowledgment

The author thanks Marc Pearl, Nina Gau, Jennifer Wu and John Alltop for their review of this paper as part of the Casualty Actuarial Society (CAS) 2014 Reserves Call Paper Program. I also thank the reviewers for the CAS 2014 Fall *E-Forum*. Finally, I thank my Philadelphia-based Oliver Wyman colleagues (Jason Shook, Alexandra Taggart and Evelyn Shen) for their reviews. Any errors that remain are the sole responsibility of the author.

5.2. Additional `R` Packages

In addition to those previously cited, the `R` packages listed below were also used to develop the model theory and associated documentation.

- reshape2 [Wic07]
- xtable [Dah14]

5.3. Author Biography

Rajesh Sahasrabuddhe is currently a consulting actuary with Oliver Wyman. He has a Bachelor of Science degree in Mathematics / Actuarial Science from the University of Connecticut. He is a Fellow of the CAS and a Member of the American Academy of Actuaries. He has previously served as Chairperson of the CAS Syllabus Committee.

5.4. Further Research

I welcome the opportunity to collaborate with others on improving and furthering the research presented in this paper. I can be reached at rajesh1004@gmail.com. My current professional contact information is available to CAS members on the CAS website. Non-members may obtain contact information through the Online Directory of Actuarial Memberships at https://actuarialdirectory.org.

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Appendix A Expanded Credibility Model

Equation (6) presents a model that is simplified by assuming a symmetric distribution centered at 0. The model prior to simplification (with the assumption maintained) is as follows:

$$Z_2 = \int_{-\infty}^{0} 2F_1(x)f_2(x)\,dx + \int_{0}^{\infty} 2\left[1 - F_1(x)\right] f_2(x)\,dx \tag{12}$$

We could further relax the assumption of symmetry centered at 0. Doing so would produce the following:

$$\begin{aligned} Z_2 = &\int_{-\infty}^{0} F_1(x)f_2(x)\,dx + \int_{-\infty}^{0} [1 - F_1(-x)]f_2(x)\,dx + \\ &\int_{0}^{\infty} [1 - F_1(x)]\,f_2(x)\,dx + \int_{0}^{\infty} F_1(-x)f_2(x)\,dx \end{aligned} \tag{13}$$

The Analysis of "All-Prior" Data

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Abstract

Motivation. Some data sources, such as the NAIC Annual Statement – Schedule P as an example, contain a row of all-prior data within the triangle. While the CAS literature has a wealth of papers that have developed various methods for estimating tail factors, and the CAS Tail Factor Working Party recently published a report on tail factor methods, tail factors are not *directly* applicable to all-prior data.[1] Moreover, the author is not aware of any papers dealing directly with the analysis of all-prior data. Absent a defined methodology, it seems to be common practice for an analysis of data triangles that include an all-prior row to either exclude the all-prior data or to make the explicit assumption that the case reserves, or case plus IBNR reserves, for these claims are adequate. This may be reasonable in certain situations but given the potential materiality of this part of the reserve it would be a useful addition to the actuary's toolkit to develop some methods for analyzing the all-prior data or for testing the reasonability of assuming the case reserves, or case plus IBNR reserves, are adequate.

Method. The process followed in this paper is to both graphically and formulaically illustrate the data issues and analysis, then apply the concepts of a well-known method with three different data sets. While only a deterministic point estimate method is illustrated in this paper, the framework should be quite easily adaptable to other deterministic methods or stochastic models. The paper also illustrates the calculations for this method and examples in a companion Excel spreadsheet.

Conclusions. The methods used for any standard analysis can be adapted to accommodate all-prior data whenever it is present. Even in cases where the all-prior reserves prove adequate, the process of analyzing the all-prior data will help calibrate the tail factor used for all years by validating the selected tail factor using actual data.

Availability. The Excel spreadsheets created for this paper "All Prior Analysis.xlsm" and "Creating All Prior Data.xls" are available at http://www.casact.org/pubs/forum/14fforum/.

Keywords. Reserving (Reserving Methods); Reserving (Data Organization); Reserving (Reserve Variability); Reserving (Tail Factors).

1. INTRODUCTION

From our training in the art and science of actuarial practice, familiarity with basic data triangles and a wide variety of methods and models[2] for extrapolating that data to its ultimate value is a way of life for casualty actuaries. Recently, a significant portion of published CAS papers and research has been devoted to the analysis and quantification of the distribution of future payments[3] and tail factors[4] in order to greatly enhance the usefulness of a "standard" unpaid claim estimate analysis. However, the author is unaware of any research or papers related to the estimation of unpaid claims for the all-prior data found in some triangles.

1 While it may be tempting to simply apply the tail factor to the all-prior data, we will see that this is not a sound practice.

2 Keeping with the definitions of methods and models in [4], the primary feature that distinguishes a model from a method is that a model is used to calculate a "distribution of possible outcomes" whereas a method will only produce a single point estimate.

3 See for example [4], which includes a large number of research papers in the Reference section.

4 See for example [5].

Estimating future payments for unpaid claims is often referred to as "squaring" the triangle when there is no claim development beyond the end of the triangle. Development beyond the end of the triangle, or the calculation of tail factors, can be thought of as the analysis of what's beyond the end of or "to the right of" the square. Similarly, ratemaking and pricing can be thought of as the analysis of what comes after or "below" the triangle. The purpose of this paper is to introduce the analysis of what's before or "above" the triangle.

As we will see, the analyses "to the right of" and "above" the triangle are related, so this paper will build a bridge from the analysis and application of tail factors to the analysis of all-prior data. Once this bridge is built, it should be possible to adapt this framework to other deterministic methods and to stochastic models for estimating distributions of possible outcomes for the all-prior data.

1.1 Research Context

From a research perspective, this paper deals mainly with unpaid claim estimate analysis and presents a new method for a subset of the data in a typical analysis. Along the way, the paper will also review data organization related to unpaid claim estimates and then show its applicability for this new method. While not specifically addressed in this paper, other methods for calculating point estimates and models used for unpaid claim variability and the calculation of uncertainty and distributions could also be adapted to use the all-prior data in a similar fashion, although within the specific frameworks of those methods and models.

1.2 Objective

The two primary goals of this paper are to provide the practicing actuary with some new tools for the analysis of all-prior data and to develop the foundation for further research in this area.

1.3 Outline

In order to achieve these goals, Section 2 will start by reviewing and slightly expanding the notation used by recent CAS research Working Parties for describing unpaid claim estimation methods and models. Section 3 will then review the basic data structure of all-prior data and show, both graphically and formulaically, how the calculation of tail factors can be extended to include all-prior data. Section 4 will apply this basic methodology to the chain ladder method to illustrate that estimates of all-prior data are not only possible but a very useful extension of existing techniques. Finally, some possible areas for future research will be suggested in Section 5 and conclusions will be discussed in Section 6.

2. NOTATION

For the sake of uniform notation, we will use the notation from the CAS Working Party on Quantifying Variability in Reserve Estimates Summary Report [2] and expanded by the CAS Tail Factor Working Party [5], since it was intended to serve as a basis for further research. Many models visualize loss statistics as a two dimensional array. The row dimension is the period[5] by which the loss information is subtotaled, most commonly an accident period.[6] For each accident period, w, the (w,d) element of the array is the total of the loss information as of development age d.[7] For this discussion, we assume that the loss information available is an "upper triangular" subset of the two-dimensional array for rows $w=1,2,\ldots,n$. For each row, w, the information is available for development ages 1 through $n-w+1$. If we think of period n as not only the most recent accident period, but also the latest accounting period for which loss information is available, the triangle represents the loss information as of accounting dates 1 through n. The "diagonal" for which $w+d$ equals a constant, k, represents the loss information for each accident period w as of accounting period k.[8]

In general, the two-dimensional array will extend to columns $d=1,2,\ldots,n$.[9] For purposes of calculating tail factors, we are interested in understanding the development beyond the observed data for periods $d=n+1,n+2,\ldots,u$, where u is the ultimate time period for which any claim activity occurs – i.e., u is the period in which all claims are final and paid in full. As an aide to any reader not familiar with this notation, a graphical representation of each item is contained in Appendix F. [10]

The paper uses the following notation for certain important loss statistics:

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5 Most commonly the periods are annual (years), but as most methods can accommodate periods other than annual we will use the more generic term "period" to represent year, half-year, quarter, month, etc. unless noted otherwise.

6 Other exposure period types, such as policy period and report period, also utilize tail factor methods. For ease of description, we will use the generic term "accident" period to mean all types of exposure periods, unless otherwise noted.

7 Depending on the context, the (w,d) cell can represent the cumulative loss statistic as of development age d or the incremental amount occurring during the d^{th} development period.

8 For a more complete explanation of this two-dimensional view of the loss information see the *Foundations of Casualty Actuarial Science* [7], Chapter 5, particularly pages 210-226.

9 Some authors define $d=0,1,\ldots,n-1$ which intuitively allows $k=w$ along the diagonals, but in this case the triangle size is $n \times n-1$ is not intuitive. With $d=1,2,\ldots,n$ defined as in this paper, the triangle size $n \times n$ is intuitive but then $k=w+1$ along the diagonals is not as intuitive. A way to think about this which helps tie everything together is to assume the w variables are the beginning of the accident periods and the d variables are at the end of the development periods. Thus, if we are using years then cell $c(n,1)$ represents accident year n evaluated at $12/31/n$, or essentially $1/1/n+1$.

10 Readers familiar with this notation could skip ahead to section 3.2. Even if you are not familiar with the notation, it is recommended to focus on the concepts in section 3.1 which should be familiar and not get bogged down in the notation. The Notation sheet in the "All Prior Analysis.xlsm" companion file should also be useful for gaining an understanding of the notation.

$c(w,d)$: cumulative loss from accident period w as of age d. Think "when" and "delay."

$q(w,d)$: incremental loss for accident period w during the development age from d - 1 to d. Note that $q(w,d) = c(w,d) - c(w,d-1)$.

$c(w,u) = U(w)$: total loss from accident period w when at the end of ultimate development u.

$R(w)$: future development after age $d = n - w + 1$ for accident period w, *i.e.*, = $U(w) - c(w, n - w + 1)$.

$D(k)$: future development after age $d = n - w + 1$ during calendar period k, *i.e.*, for all $q(w,d)$ where $w + d = k$ and $w + d > n + 1$.

$A(d)$: all-prior data by development age d.

$f(d) = 1 + v(d)$: factor applied to $c(w,d)$ to estimate $c(w,d+1)$ or more generally any factor relating to age d. This is commonly referred to as a link ratio. $v(d)$ is referred to as the 'development portion' of the link ratio, which is used to estimate $q(w,d+1)$. The other portion, the number one, is referred to as the 'unity portion' of the link ratio.

$F(d)$: ultimate development factor relating to development age d. The factor applied to $c(w,d)$ to estimate $c(w,u)$ or more generally any cumulative development factor relating to development age d. The capital indicates that the factor produces the ultimate loss level. As with link ratios, $V(d)$ denotes the 'development portion' of the loss development factor, the number one is the 'unity portion' of the loss development factor.

$T = T(n)$: ultimate tail factor at end of triangle data, which is applied to the estimated $c(w,n)$ to estimate $c(w,u)$.

$\hat{x}$ an estimate of any value or parameter x.

What are called factors here could also be summands, but if factors and summands are both used, some other notation for the additive terms would be needed. The notation does not distinguish paid *vs.* incurred, but if this is necessary, capitalized subscripts P and I could be used.

3. ALL-PRIOR ANALYSIS OVERVIEW

In order to analyze the all-prior data, we must start by understanding the make-up of this data and how it is related to the main triangle data as it is commonly understood. But before we delve into the all-prior data, we will start with a triangle array of cumulative data, illustrated in Table 3.1, and a typical method for estimating unpaid claims excluding any all-prior data.

Table 3.1 – Loss Triangle Data

		d 1	2	3	…	n-1	n
w	1	**c**(1,1)	**c**(1,2)	**c**(1,3)	…	**c**(1,n-1)	**c**(1,n)
	2	**c**(2,1)	**c**(2,2)	**c**(2,3)	…	**c**(2,n-1)	
	3	**c**(3,1)	**c**(3,2)	**c**(3,3)			
	…	…	…				
	n-1	**c**(n-1,1)	**c**(n-1,2)				
	n	**c**(n,1)					

3.1 A Typical Unpaid Claim Estimate

As an example, a typical deterministic analysis of this data will start with an array of link ratios or development factors:

$$f(w,d) = \frac{c(w,d+1)}{c(w,d)}. \tag{3.1}$$

Then two key assumptions are made in order to make a projection of the known elements to their respective ultimate values. **First**, it is typically assumed that each accident period has the same development factor. Equivalently, for each $w = 1, 2, \dots, n-d$:

$$f(w,d) = f(d).$$

Under this first assumption, one of the more popular estimators for the development factor is the weighted average:[11]

$$\hat{f}(d) = \frac{\sum_{w=1}^{n-d} c(w,d+1)}{\sum_{w=1}^{n-d} c(w,d)}. \tag{3.2}$$

Certainly there are other popular estimators in use, but they are beyond our scope at this stage and nothing is gained by exploring other estimators. Suffice it to say that many methods and their corresponding estimators are still consistent with our first assumption that each accident period has the same factor. There are, of course, methods that do not rely on this assumption that all accident periods use the same development factor,[12] but they are beyond the scope of this paper so that we can focus on a basic understanding of the analysis process.

Assuming there is no claim development beyond the end of the triangle, projections of the ultimate values, $\hat{c}(w,u)$ [or $\hat{c}(w,n)$ since $u = n$ in this case], for $w = 2, 3, \dots, n$, are then computed using:

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[11] The popularity of this estimator may stem from it being unbiased as shown by Mack [8] and others.

[12] For example methods that trend the data can directly or indirectly result in different factors for each accident period.

$$\hat{c}(w,n) = c(w,d)\prod_{i=d}^{n-1}\hat{f}(i) \text{, for all } d = n-w+1. \tag{3.3}$$

For completeness, carrying out the calculations for formula (3.3) sequentially for each $\hat{f}(i)$ is often done to estimate each future $\hat{c}(w,d)$, and then by subtraction each future $\hat{q}(w,d)$ is used to estimate cash flows (for paid data). Alternatively, ultimate development factors can be calculated as:

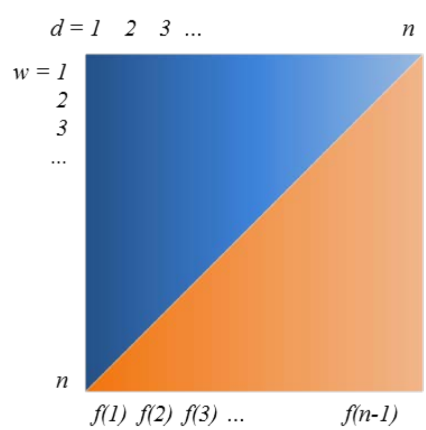
$$\hat{F}(d) = \prod_{i=d}^{n-1}\hat{f}(i) \text{, for each } d = 1,2,\ldots,n-1. \tag{3.4}$$

And then formula (3.3) simplifies to:

$$\hat{c}(w,n) = c(w,d)\times\hat{F}(d) \text{, for all } d = n-w+1. \tag{3.5}$$

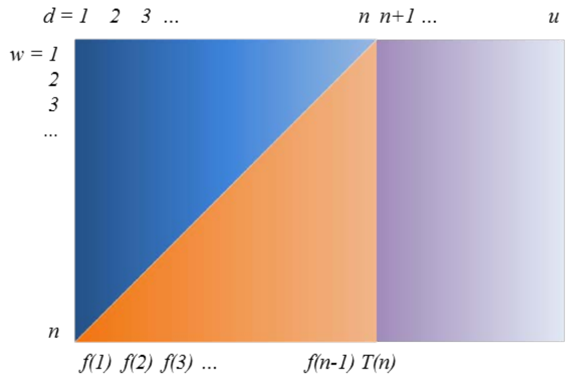
This part of the claim projection algorithm relies explicitly on the **second** assumption, namely that each accident period has a parameter representing its relative level. These level parameters are the current cumulative values for each accident period, or $c(w,n-w+1)$. Of course variations on this second assumption are also common, but the point is that every method has explicit assumptions that are an integral part of understanding the quality of that method. Graphically, our estimation model looks like Graph 3.1, where the blue triangle is the data we know and the orange triangle is estimated.

Graph 3.1 – Loss Estimation without a Tail



If the assumption of no claim development past the end of the triangle is true, then as we will see the analysis needs no further extensions as the all-prior data would similarly need no extrapolation beyond the end of the triangle. On the other hand, it is quite common to expect development beyond the end of the triangle, in which case a tail factor is generally used to extrapolate to the end of the expected development or the ultimate period, u. We can illustrate this graphically by expanding Graph 3.1 to include tail development, as shown in Graph 3.2, where the rectangle in purple is the tail extrapolation.

Graph 3.2 – Loss Estimation with a Tail



There are a variety of methods for estimating a tail factor, $T(n)$, but we will only use one of the common methods, namely, the exponential decay method.[13] The method utilizes link ratios, $f(d)=1+v(d)$, and assumes that the $v(d)$s decay at a constant rate, r, i.e., $v(d_{i+1})=v(d_i)\times r$. The process consists of first fitting an exponential curve to the $v(d)$s, which can be accomplished by using a regression with the natural logarithms (natural log) of the $v(d)$s. Next, the decay constant r can be estimated as the inverse natural log of the slope of the fitted curve. The remaining development, from a given development age d, can be estimated as:

$$T(d)=\prod_{i=1}^{\infty}(1+v(d)\times r^i)\text{, for } d\geq n. \tag{3.6}$$

While formula (3.6) is infinite in theory, in practice the incremental factors in this formula, $\hat{f}(d)=1+v(d)\times r^i$, will get close enough to one[14] such that no new development is expected or the development is small enough to stop. Thus, one of the decision points for a typical tail factor selection is determining the ultimate number of periods or u. The goal of this analysis is to complete the "rectangle" and estimate the future cumulative values, as illustrated in Table 3.2.

Table 3.2 – Cumulative Loss Triangle Data with Estimated Ultimate Projections

		d							
		1	**2**	**3**	**…**	**n-1**	**n**	**…**	**u**
w	**1**	**c**(1,1)	**c**(1,2)	**c**(1,3)	…	**c**(1,n-1)	**c**(1,n)	…	*ĉ(1,u)*
	2	**c**(2,1)	**c**(2,2)	**c**(2,3)	…	**c**(2,n-1)	*ĉ(2,n)*	…	*ĉ(2,u)*
	3	**c**(3,1)	**c**(3,2)	**c**(3,3)	…	*ĉ(3,n-1)*	*ĉ(3,n)*	…	*ĉ(3,u)*
	…	…	…	…	…	…	…	…	…
	n-1	**c**(n-1,1)	**c**(n-1,2)	*ĉ(n-1,3)*	…	*ĉ(n-1,n-1)*	*ĉ(n-1,n)*	…	*ĉ(n-1,u)*
	n	**c**(n,1)	*ĉ(n,2)*	*ĉ(n,3)*	…	*ĉ(n,n-1)*	*ĉ(n,n)*	…	*ĉ(n,u)*

Of course for an analysis using cumulative data it is a simple step to subtract the last known value

[13] For a more complete discussion of tail factor methods see [5]. The exponential decay method is shown in the "Tail Factors" sheet in the "All Prior Analysis.xlsm" file.

[14] Under certain circumstances the regression can result in increasing factors with could become infinite, but when this happens the method is normally discarded as being unreasonable.

for each accident period from the estimated ultimate value to arrive at the estimated unpaid for each accident period w using formula (3.7).

$$\hat{R}_{(w)} = \hat{c}(w,u) - c(w,n-w+1) \tag{3.7}$$

For our purposes, we will also take the additional step of converting the cumulative values to incremental values, as illustrated in Table 3.3.

Table 3.3 – Incremental Loss Triangle Data with Estimated Ultimate Projections

w \ d	1	2	3	…	n-1	n	…	u
1	**q**(1,1)	**q**(1,2)	**q**(1,3)	…	**q**(1,n-1)	**q**(1,n)	…	$\hat{q}(1,u)$
2	**q**(2,1)	**q**(2,2)	**q**(2,3)	…	**q**(2,n-1)	$\hat{q}(2,n)$	…	$\hat{q}(2,u)$
3	**q**(3,1)	**q**(3,2)	**q**(3,3)	…	$\hat{q}(3,n-1)$	$\hat{q}(3,n)$	…	$\hat{q}(3,u)$
…	…	…	…	…	…	…	…	…
n-1	**q**(n-1,1)	**q**(n-1,2)	$\hat{q}(n-1,3)$	…	$\hat{q}(n-1,n-1)$	$\hat{q}(n-1,n)$	…	$\hat{q}(n-1,u)$
n	**q**(n,1)	$\hat{q}(n,2)$	$\hat{q}(n,3)$	…	$\hat{q}(n,n-1)$	$\hat{q}(n,n)$	…	$\hat{q}(n,u)$

From the estimated incremental values we have an estimate of the unpaid claims for each accident period w using formula (3.8) to sum the estimated incremental values.

$$\hat{R}_{(w)} = \sum_{d=n-w+2}^{d=u} \hat{q}(w,d) \tag{3.8}$$

Also, adding the estimates for each accident period, we can derive a formula for the total estimated unpaid as shown in formula (3.9).

$$\hat{R}_{(T)} = \sum_{w=1}^{w=n} \hat{R}_{(w)} = \sum_{w=1}^{w=n} \sum_{d=n-w+2}^{d=u} \hat{q}(w,d) \tag{3.9}$$

Using the estimated incremental values we can also create an estimate of the future cash flows by calendar period k using formula (3.10) to sum the estimated incremental values along the diagonal instead of by row.

$$\begin{aligned} \hat{D}_{(k)} &= \sum_{w=1}^{w=n} \hat{q}(w,k-w), \text{ for } n+2 \le k \le u+1 \\ \hat{D}_{(k)} &= \sum_{w=k-u}^{w=n} \hat{q}(w,k-w), \text{ for } u+2 \le k \le u+n \end{aligned} \tag{3.10}$$

For the formulas in (3.10), the first one is for complete diagonals (all rows) as k increases from $n+2$ to $u+1$, while in the second formula the diagonals are shrinking each period as k goes from $u+2$ to $u+n$.[15] Similarly, adding the estimates for each calendar period we can derive a formula for the total estimated unpaid as shown in formula (3.11).

$$\hat{R}_{(T)} = \sum_{k=n+2}^{k=n+u} \hat{D}_{(k)} = \sum_{k=n+2}^{k=u+1} \sum_{w=1}^{w=n} \hat{q}(w,k-w) + \sum_{k=u+2}^{k=n+u} \sum_{w=k-u}^{w=n} \hat{q}(w,k-w) \tag{3.11}$$

[15] Keep in mind that $k = w + d$ and the last row is contained in each diagonal sum, so the incremental values from $\hat{q}(n,2)$ to $\hat{q}(n,u)$ are part of the details in formulas (3.10) and (3.11).

3.2 The All-Prior Data

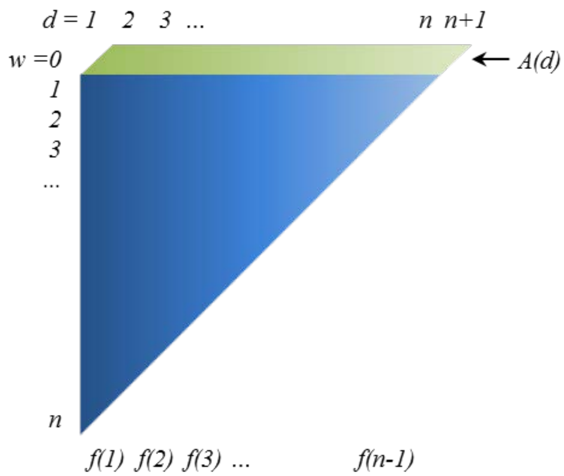
With this brief review complete, we can now expand the analysis by examining the all-prior data. First, the basic loss development triangle will include the extra row as shown in Table 3.4.

Table 3.4 – Loss Triangle Data with All-Prior Row

		d						
		1	2	3	…	n-1	n	n+1
w	0		**A**(2)	**A**(3)	…	**A**(n-1)	**A**(n)	**A**(n+1)
	1	**c**(1,1)	**c**(1,2)	**c**(1,3)	…	**c**(1,n-1)	**c**(1,n)	
	2	**c**(2,1)	**c**(2,2)	**c**(2,3)	…	**c**(2,n-1)		
	3	**c**(3,1)	**c**(3,2)	**c**(3,3)				
	…	…	…					
	n-1	**c**(n-1,1)	**c**(n-1,2)					
	n	**c**(n,1)						

Graphically the addition of all-prior data can be illustrated in Graph 3.3, with the all-prior data shown in green.

Graph 3.3 – Loss Triangle with All-Prior Data



The color and shape for the all-prior data is significant for three reasons. First, while the main triangle can be either cumulative or incremental values, the all-prior data could be either[16] but, more importantly, it is a combination of multiple periods and as such we need to introduce new notation, $A(d)$, for the cells in the all-prior row. Second, the addition of this extra row does not always include

[16] Technically, it is possible to use either incremental or cumulative data in the underlying data used to calculate the all-prior row. In addition, all development periods for $d = 1, 2, ..., u$ could be included or only the periods beyond the end of the triangle or $d = n+1, ..., u$. For purposes of this paper we will assume the underlying data is incremental and use all development periods.

any value in the first column(s)[17] so the overall shape is no longer strictly triangular. And third, because the data includes multiple periods at different stages of development we can't ***directly*** apply the factors from our typical analysis to extend it for the analysis of the all-prior row.

The all-prior data is included in accounting statements so that a triangle large enough to show all development can be truncated by collapsing the triangle down to a specific maximum size, while still including all of the relevant claim information for reconciliation with the balance sheet. Thus, the all-prior row is actually a summary of the claim activity for all claims that occurred prior to the first accident period ($w=1$) in the triangle as of the date of the financial statement.

As there can be different ways of compiling the all-prior data, the key to any analysis is to first understand exactly what is in the data or how it was created. As a common source of all-prior data is the NAIC Annual Statement Schedule P (for companies operating in the United States), we will use those rules here which result in each all-prior cell being the calendar period (i.e., diagonal) sum of all prior accident periods.[18] Rather than spending time and space here dissecting the NAIC rules [10], we direct the interested reader to the "Creating All Prior Data.xls" companion file, which uses one data set to walk through the rules for compiling Schedule P and then reconciles this with a more direct calculation. To illustrate this we can restate Table 3.4 as Table 3.5.

Table 3.5 – Loss Triangle Data with All-Prior Row Details

w \ d	1	2	3	…	n	n+1	n+2	n+3	…	u
-u+2										**q**(-u+2,u)
-u+3									…	**q**(-u+3,u)
…						…	…	…	…	
-2				…	**q**(-2,n)	**q**(-2,n+1)	**q**(-2,n+2)	**q**(-2,n+3)		
-1			**q**(-1,3)	…	**q**(-1,n)	**q**(-1,n+1)	**q**(-1,n+2)			
0		**q**(0,2)	**q**(0,3)	…	**q**(0,n)	**q**(0,n+1)				
1	**c**(1,1)	**c**(1,2)	**c**(1,3)	…	**c**(1,n)					
2	**c**(2,1)	**c**(2,2)	**c**(2,3)	…						
3	**c**(3,1)	**c**(3,2)	**c**(3,3)	…						
…	…	…								
n-1	**c**(n-1,1)	**c**(n-1,2)								
n	**c**(n,1)									

As we are assuming the all-prior data starts with $A(2)$, the first diagonal will include all incremental cells were $k=w+d=2$, so the earliest accident period with data should be $-u+2$ and the earliest accident period with data in development period u should be $-u+n+1$. Graphically, we can illustrate this as shown in Graph 3.4.

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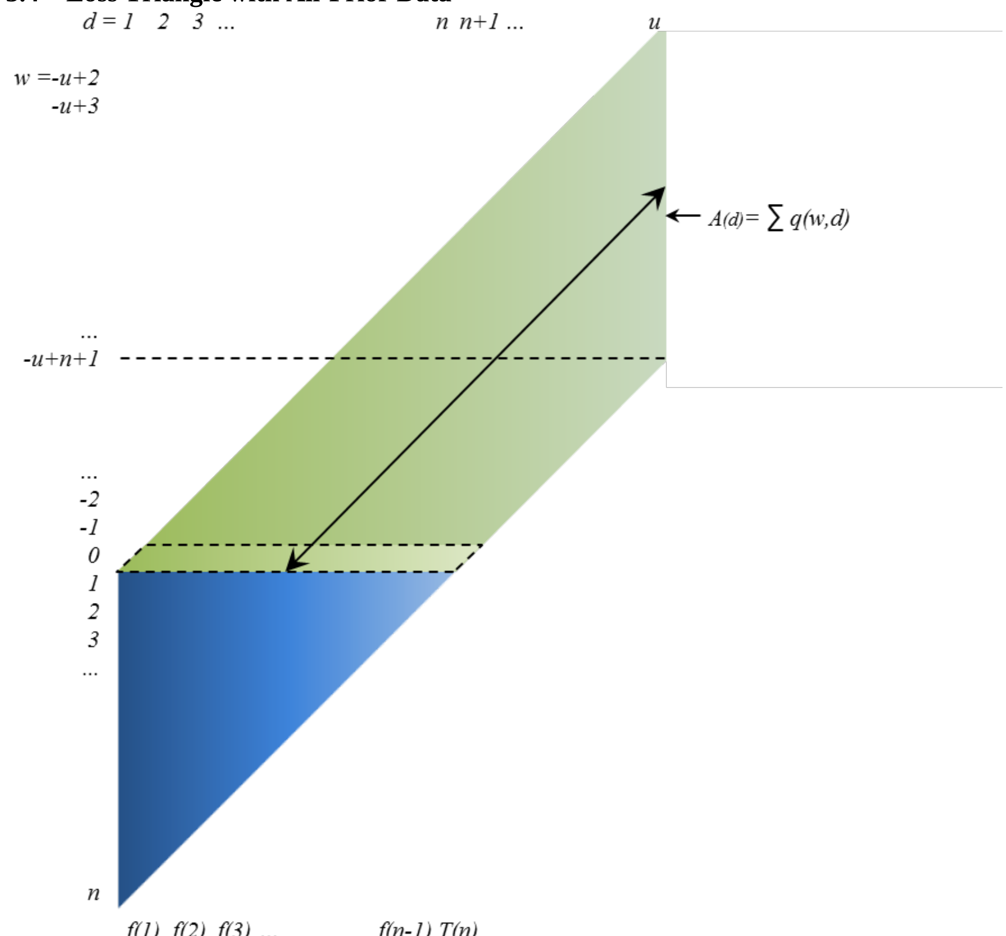
[17] Of course none of the columns need to be missing or blank, but for purposes of this paper we will assume the first column $A(1)$ is blank and include data in columns $A(2)$ and later to be consistent with the NAIC Schedule P. In Schedule P the paid data for $A(2)$ is zero, but for incurred data it only contains reserves and no payments.

[18] Two useful references for understanding the all-prior data in the NAIC Schedule P are [6] and [10].

Graph 3.4 – Loss Triangle with All-Prior Data



Now we can more precisely define each cell in the all-prior row of data using formula (3.12), which is the diagonal sum of the claim activity in those periods.[19, 20]

$$A_{(k)} = \sum_{w=-u+k}^{w=0} q(w, k-w)\text{, for } k = 2, 3, \ldots, n+1.^{21} \qquad (3.12)$$

It is not a coincidence that the diagonal sum of the all-prior row stretches out for the same number of periods, u, as we will expect for the tail factor. Indeed, if we can get the incremental data that was used to create the all-prior row then we can use this to calibrate the length of the tail factors.

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[19] Technically, $A(2)$ could be the sum of all diagonals prior to $A(3)$, thus the first cell in the graph would be a different color and Graphs 3.4 and 3.5 could be extended even further, but our focus will be on the incremental changes in the $A(k)$, so we can ignore this technicality.

[20] Of course if the company did not start writing business that long ago, then claims for these older accident years would not exist at all and any estimates of the all-prior unpaid claims would need to be adjusted accordingly. For purposes of this paper we will assume business was written at least as early as is implied by the ultimate tail extrapolation.

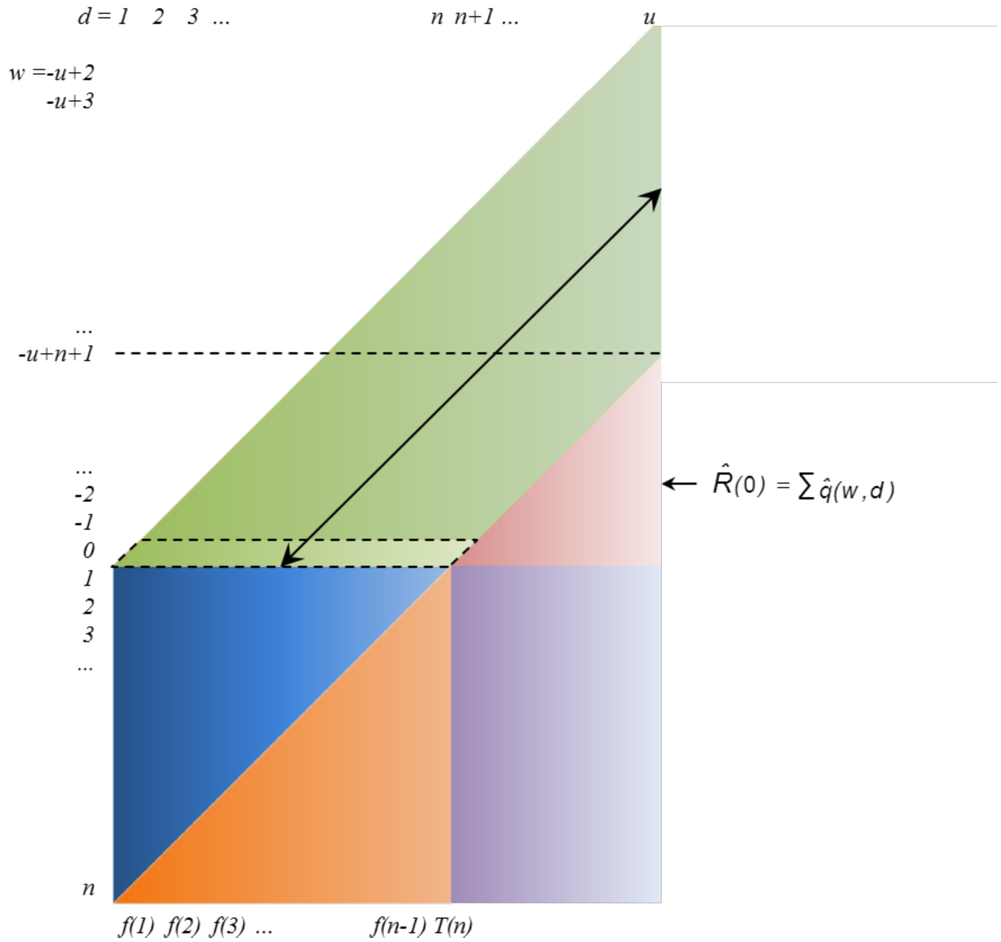
[21] In Graphs 3.3 and 3.4, we used d with our notation for the all-prior row, $A(d)$, since it is used in those contexts consistent with development columns. In formula (3.12) and beyond we switch to using k in our notation for the all-prior row, $A(k)$, since we are illustrating how this is a diagonal sum of the incremental values. For the all-prior row $d = k$, so they can be used interchangeably.

The last step in examining the all-prior row is to define the unpaid claims we need to estimate as the sum of the future all-prior diagonals. Graphically, we can combine Graph 3.2 with Graph 3.4 and illustrate the unpaid claim estimate we are working toward in red in Graph 3.5.

Completing the description for our all-prior estimate, we need to develop methods to solve for the future incremental cells for the all-prior data that will allow us to use formula (3.13) to estimate the total unpaid claims for the all-prior data.

$$\hat{R}_{(0)} = \sum_{k=n+2}^{k=u} \hat{A}_{(k)} = \sum_{k=n+2}^{k=u} \sum_{w=-u+k}^{w=0} \hat{q}(w, k-w) \tag{3.13}$$

Graph 3.5 – Loss Estimation with All-Prior Data and a Tail



3.3 All-Prior Analysis

Even though we have more clearly delineated the problem, we can't just apply the tail factors we would use for the rest of the analysis because those factors are based on cumulative values and, even if we have the incremental details for the all-prior row, we can't calculate the appropriate cumulative values unless we have all of the claim data, not just the data used to calculate the all-prior row. In effect, to use a normal tail factor we would need the entire triangle for all periods – i.e., a $u \times u$ triangle[22] instead of an $n \times n$ triangle. If we had all of the data for the $u \times u$ triangle, then we could use formula (3.6) (or something similar) to successively apply a different factor $T(d)$ to each accident period for each $d > n$. Then again, if we have that data we would not need to calculate tail factors or use all-prior data.

[22] In keeping with the notation in Graph 3.5, the rows for the u x u triangle would run from $-u+n+1$ to n. Renumbering by adding $u-n$ to each row, the rows would then run from 1 to u.

Whenever we don't have complete cumulative data for every accident period that is part of the all-prior data, we will need to make some assumptions about the history prior to our data triangle in order to use our normal tail factors. For example, we could use the Bornhuetter-Ferguson [3] algorithm which uses an a priori estimate of the total losses and the loss development pattern to derive an estimate. With premium and/or exposure data prior to the data triangle, we can apply the Bornhuetter-Ferguson algorithm to estimate the cumulative values for the prior periods.

4. ALL-PRIOR METHODS

In order to illustrate the calculations for, and the usefulness of, the analysis of all-prior data within a typical deterministic analysis, three data sets were simulated, each with all of the historical data needed to estimate the all-prior unpaid claims.[23] While the data is simulated, it was done in a way to make it look real and tested using methods such as those suggested in Venter [12] and other sources to make sure it has realistic statistical properties. The three data sets approximate companies with three different case reserving philosophies, "medium" case reserves, "low" case reserves and "high" case reserves, respectively, as well as different exposures and development patterns. Within the body of the paper, we will only review and primarily discuss the "medium" scenario, but the analysis and results for the other two are contained in the Appendices.[24]

In addition to having simulated claim triangles for 10 years with an all-prior row, we are also assuming that we have 11 years of earned premium and expected loss ratios for the years in the all-prior row to approximate what you might find in practice (i.e., for the 11 years prior to the oldest year in the triangle). For the older periods where this information is unavailable (i.e., prior to those 11 years), we derive estimates for premium and expected loss ratios as you would need to do in practice. The paid data for the "medium" scenario is shown in Table 4.1.

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[23] The simulated data is for complete 30 x 30 rectangles, with different development, exposure growth, parameters, etc., but all of the simulated data is fully developed prior to 30 periods. This size was chosen to be consistent with the limits of flexibility set up in the companion Excel file. Each data set was then collapsed into 10 x 10 triangles, with an all-prior row, to illustrate the analysis. In addition, the prior 11 years of premiums and "ultimate" loss ratios are included to approximate the information you could obtain from the oldest accident years in the 11 Annual Statements prior to the current year.

[24] The complete details for all three scenarios are also included in the "All Prior Analysis.xlsm" file. The interested reader can select a different data set in cell "V1" on the Data sheet and recalculate the sheet to see the calculations for any of the scenarios.

Table 4.1 – "Medium" Paid Loss Triangle with All-Prior Data

	12	24	36	48	60	72	84	96	108	120	132
A-P		-	124,151	196,502	234,850	256,775	269,143	276,080	279,086	281,182	282,390
2004	74,998	189,335	252,351	284,850	301,895	311,600	317,040	319,748	321,762	322,784	
2005	92,015	216,237	283,370	316,672	335,600	346,804	352,535	356,275	357,748		
2006	90,909	191,270	262,856	289,054	310,018	319,763	325,725	328,463			
2007	100,503	215,220	271,927	315,048	333,808	343,553	348,988				
2008	94,647	225,979	295,390	330,250	349,553	359,694					
2009	99,464	204,539	271,740	308,343	329,792						
2010	83,463	200,265	274,434	309,186							
2011	76,140	184,681	255,177								
2012	112,865	243,840									
2013	100,689										

Extending the chain ladder method for a triangle of data that includes an all-prior row, the steps to our analysis can be summed up as follows:

1) Calculate the age-to-age factors excluding the all-prior row,
2) Extrapolate the age-to-age factors and select a tail factor,
3) Estimate the cumulative data for each prior accident period which is part of the all-prior row,
4) Estimate the incremental data for each prior accident period (from Step 3) and sum the diagonals to estimate the values in the all-prior row,
5) Use comparisons of the estimated all-prior row data to the actual all-prior row data to evaluate and calibrate the selected factors,
6) Re-select, re-estimate and re-calibrate (repeat Steps 2 through 5) as needed, and
7) Sum all future diagonals for each prior accident period to estimate the all-prior row reserves.

4.1 Calculate Age-to-Age Factors

The first step is to calculate the age-to-age factors or link ratios for the data triangle. Using formula (3.2), and excluding the all-prior (A-P) row, the weighted average age-to-age factors for this data are shown in Table 4.2. [25]

[25] Note that if you are trying to reproduce the calculated values in the Tables in this paper, the actual values are generally unrounded in Excel so you may encounter rounding differences.

Table 4.2 – "Medium" Paid Loss Development Factors

	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	Tail
2004	2.525	1.333	1.129	1.060	1.032	1.017	1.009	1.006	1.003	
2005	2.350	1.310	1.118	1.060	1.033	1.017	1.011	1.004		
2006	2.104	1.374	1.100	1.073	1.031	1.019	1.008			
2007	2.141	1.263	1.159	1.060	1.029	1.016				
2008	2.388	1.307	1.118	1.058	1.029					
2009	2.056	1.329	1.135	1.070						
2010	2.399	1.370	1.127							
2011	2.426	1.382								
2012	2.160									
VWA	2.268	1.332	1.126	1.063	1.031	1.017	1.009	1.005	1.003	
5-Yr VWA	2.270	1.328	1.128	1.064	1.031	1.017	1.009	1.005	1.003	
3-Yr VWA	2.308	1.359	1.126	1.062	1.030	1.017	1.009	1.005	1.003	
TF Fitted	1.395	1.213	1.115	1.062	1.034	1.018	1.010	1.005	1.003	1.003
User	2.250									
Selected	2.250	1.332	1.126	1.063	1.034	1.018	1.010	1.005	1.003	1.0015
Ultimate	3.856	1.714	1.287	1.143	1.075	1.040	1.021	1.012	1.006	1.0033
% Paid	25.9%	58.4%	77.7%	87.5%	93.0%	96.2%	97.9%	98.9%	99.4%	99.7%
% Unpaid	74.1%	41.6%	22.3%	12.5%	7.0%	3.8%	2.1%	1.1%	0.6%	0.3%

In addition to the volume weighted average (VWA) factors from formula (3.2), other averages are shown in Table 4.2 to mimic a more typical process in practice where the actuary would compare different averages to select their age-to-age factors. A user entered row is also included and the selected factors by development period are outlined.

4.2 Select a Tail Factor

Using formula (3.6), we can also estimate a tail factor, including the incremental age-to-age factors that comprise the tail factor, which by itself is a factor to ultimate. The tail factor calculation for the paid data is illustrated in Table 4.3. Note that while the incremental factors that make up the tail factor could be ignored in an analysis without an all-prior row, they are a necessary part of this analysis since we need to estimate the incremental values that sum to the all-prior row data and we will need tail factors for $d > n$ in order to estimate the all-prior unpaid claims. Note also that age-to-age and tail factors can often be rounded to 3 decimal places in practice, but in order to calibrate the incremental tail factors with the ultimate development length of the data, u, more than 3 decimal places may be needed to help identify more precisely how many periods to include in the tail.

Table 4.3 – "Medium" Paid Tail Factor Calculation

				All Prior		
Tail Years:	12	Actual		282,390	Decay	0.540
Tail Factor:	1.0033	Estimated		303,022	Intercept	0.732
		Error %		7.3%		

Period	Factor	Dev	Log	Excl	Period	Log	Fitted	Selected	ATA	ATU
1	2.26832	1.26832	0.238	Y			1.395339		1.395339	2.155306
2	1.33162	0.33162	(1.104)	Y			1.213371		1.213371	1.544647
3	1.12622	0.12622	(2.070)		3	(2.070)	1.115159		1.115159	1.273022
4	1.06314	0.06314	(2.762)		4	(2.762)	1.062153		1.062153	1.141560
5	1.03099	0.03099	(3.474)		5	(3.474)	1.033545		1.033545	1.074760
6	1.01707	0.01707	(4.070)		6	(4.070)	1.018105		1.018105	1.039878
7	1.00923	0.00923	(4.685)		7	(4.685)	1.009771		1.009771	1.021386
8	1.00516	0.00516	(5.267)		8	(5.267)	1.005274		1.005274	1.011502
9	1.00318	0.00318	(5.752)		9	(5.752)	1.002846		1.002846	1.006195
10							1.001536		1.001536	1.003339
11							1.000829		1.000829	1.001800
12							1.000447		1.000447	1.000970
13							1.000242		1.000242	1.000523
14							1.000130		1.000130	1.000281
15							1.000070		1.000070	1.000151
16							1.000038		1.000038	1.000080
17							1.000020		1.000020	1.000042
18							1.000011		1.000011	1.000022
19							1.000006		1.000006	1.000011
20							1.000003		1.000003	1.000005
21							1.000002		1.000002	1.000002

4.3 Estimate Prior Cumulative Values

With the development factors and tail factor calculated it is a simple matter to "rectangle"[26] the triangle, so that will not be illustrated here.[27] Instead we will examine a process for estimating the incremental values that comprise the all-prior row of data shown in Table 4.1. To do this we can use the prior earned premiums, estimated ultimate loss ratios, estimated percent paid (from Table 4.2), and Bornhuetter-Ferguson methodology to estimate the cumulative paid for each prior year, as illustrated in Table 4.4.

For example, from the simulated data we know that the premium for 2003 is 468,659 and the estimated ultimate loss ratio is 71.6%.[28] Combining this with the estimated percent paid at 24 months from Table 4.2 of 54.8% we can estimate the cumulative losses for 2003 as 468,659 x .716 x .548 = 195,823. The estimated values for all years shown in Table 4.4, for development periods from 24 to 120 months were calculated using the same methodology. Using these estimated cumulative values at 120 months for each prior accident year, we can then use the incremental (age-to-age) tail factors from Table 4.3 to estimate the remaining cumulative values to ultimate.

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[26] Technically, it is more precise to say we are "rectangling" the triangle when we have a tail, but as a square is a type of rectangle, some may prefer to think of "squaring" in more general terms meaning turning the triangle into either a square or rectangle.

[27] While some calculations are skipped (or knowledge of the calculations is assumed) in the body of the paper, they are all contained in the companion Excel file "All Prior Analysis.xlsm" for easy reference.

[28] See the Data sheet in the "All Prior Analysis.xlsm" file.

Table 4.4 – "Medium" Paid All-Prior Projection (Cumulative)

	Premium	Loss Ratio	24	36	48	60	72	84	96	108	120	132
1984	*402,171*	*70.0%*	*164,287*	*218,768*	*246,380*	*261,937*	*270,724*	*275,625*	*278,319*	*279,786*	*280,583*	281,014
1985	*406,193*	*70.0%*	*165,930*	*220,956*	*248,844*	*264,557*	*273,431*	*278,382*	*281,102*	*282,584*	*283,389*	283,824
1986	*410,255*	*70.0%*	*167,589*	*223,165*	*251,332*	*267,202*	*276,165*	*281,165*	*283,913*	*285,410*	*286,222*	286,662
1987	*414,357*	*70.0%*	*169,265*	*225,397*	*253,846*	*269,874*	*278,927*	*283,977*	*286,752*	*288,264*	*289,085*	289,529
1988	*418,501*	*70.0%*	*170,958*	*227,651*	*256,384*	*272,573*	*281,716*	*286,817*	*289,619*	*291,147*	*291,975*	292,424
1989	*422,686*	*70.0%*	*172,667*	*229,927*	*258,948*	*275,299*	*284,534*	*289,685*	*292,516*	*294,058*	*294,895*	295,348
1990	*426,913*	*70.0%*	*174,394*	*232,226*	*261,537*	*278,052*	*287,379*	*292,582*	*295,441*	*296,999*	*297,844*	298,302
1991	*431,182*	*70.0%*	*176,138*	*234,549*	*264,153*	*280,832*	*290,253*	*295,508*	*298,395*	*299,969*	*300,823*	301,285
1992	*435,494*	*70.0%*	*177,899*	*236,894*	*266,794*	*283,640*	*293,155*	*298,463*	*301,379*	*302,968*	*303,831*	304,298
1993	439,848	69.1%	*177,368*	*236,187*	*265,998*	*282,794*	*292,280*	*297,572*	*300,479*	*302,064*	*302,924*	303,389
1994	472,929	64.9%	*179,117*	*238,515*	*268,620*	*285,581*	*295,161*	*300,505*	*303,441*	*305,041*	*305,910*	306,380
1995	412,911	75.1%	*180,964*	*240,975*	*271,390*	*288,526*	*298,205*	*303,604*	*306,570*	*308,187*	*309,064*	309,539
1996	460,127	68.0%	*182,592*	*243,143*	*273,831*	*291,122*	*300,888*	*306,335*	*309,328*	*310,960*	*311,845*	312,324
1997	471,803	67.0%	*184,472*	*245,646*	*276,651*	*294,120*	*303,986*	*309,490*	*312,514*	*314,162*	*315,056*	315,540
1998	443,804	71.9%	*186,215*	*247,968*	*279,265*	*296,899*	*306,858*	*312,414*	*315,467*	*317,130*	*318,033*	318,522
1999	448,454	71.9%	*188,166*	*250,565*	*282,191*	*300,009*	*310,073*	*315,687*	*318,772*	*320,453*	*321,365*	321,859
2000	439,491	74.1%	*190,048*	*253,071*	*285,013*	*303,010*	*313,174*	*318,844*	*321,960*	*323,658*	*324,579*	325,078
2001	499,204	65.9%	*191,981*	*255,646*	*287,912*	*306,092*	*316,360*	*322,088*	*325,235*	*326,950*	*327,881*	328,384
2002	447,766	74.2%	*193,888*	*258,184*	*290,772*	*309,132*	*319,502*	*325,286*	*328,465*	*330,197*	*331,137*	331,646
2003	468,659	71.6%	*195,823*	*260,762*	*293,675*	*312,218*	*322,691*	*328,534*	*331,744*	*333,493*	*334,443*	334,956

	Growth	Loss Ratio
Prior to 1993	*1.0%*	*70.0%*

Note that the cumulative projections in Table 4.4 extend 12 periods beyond 120 months to match the number of periods used for the tail factor selection in Table 4.3,[29] but we have included a total of 20 pre-2004 accident years since that's how many periods of all-prior data we will need to estimate the all-prior row in the next steps. Thus, in addition to the 11 years of prior earned premiums and estimated ultimate loss ratios we have, we need to make some additional assumptions for years prior to the those 11, namely a 1% growth rate and an expected loss ratio of 70% were assumed. Of course whether you have any premium and loss ratio data prior to the start of the triangle or not, the materiality of these assumptions can be stronger than the tail factor assumption when "calibrating" these assumptions by estimating the actual all-prior data.

4.4 Estimate Prior Incremental Values

After estimating the projected cumulative values, the projected incremental values are estimated by a simple subtraction, as illustrated in Table 4.5. With the incremental values, we can also sum along the diagonal using formula (3.11) to compare these estimated values with the actual incremental values from the data in Table 4.1.

[29] To keep Table 4.4 from becoming unreadable only projections to 132 months are shown, but all projections can be seen in the companion "All Prior Analysis.xlsm" file.

Table 4.5 – "Medium" Paid All-Prior Projection (Incremental)

	12	24	36	48	60	72	84	96	108	120	132	144
1994												254
1995											475	257
1996										885	479	259
1997									1,648	894	484	262
1998								3,053	1,664	903	489	264
1999							5,614	3,085	1,681	912	494	267
2000						10,164	5,670	3,116	1,698	921	499	270
2001					18,180	10,268	5,728	3,147	1,715	931	504	272
2002				32,587	18,360	10,370	5,785	3,179	1,732	940	509	275
2003			64,939	32,913	18,543	10,473	5,842	3,210	1,750	949	514	278
Totals:	**(144+)**	**(36-132)**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**
Estimated	1,309	303,022	138,094	73,886	41,383	23,068	12,720	6,947	3,774	2,044	1,106	*598*
Actual		282,390	124,151	72,351	38,348	21,925	12,368	6,937	3,006	2,096	1,208	
Differences		20,632	13,943	1,535	3,035	1,143	352	10	768	(52)	(102)	
Cumulative Percent Difference			7.3%	4.2%	6.0%	4.5%	3.8%	4.7%	9.7%	-4.6%	-8.4%	
Weights			0.25	0.50	1.00	2.00	3.00	4.00	5.00	6.00	7.00	
Weighted Average			0.4%									

4.5 Compare to Actual & Calibrate

Comparing the estimates to the actual all-prior data we can see in Table 4.5 that the differences are not too far off.[30] The totals for both the actual and estimated all-prior row are also included in Table 4.3, which shows the estimates are 7.3% higher than the actual values. While the cumulative percentage difference of 7.3% is useful for gauging all of the assumptions for the all-prior row, it tends to be heavily influenced by the early development periods and is, thus, not usually responsive to changes in the tail factor assumptions. To calibrate the tail factor assumptions, it is much better to focus on the cumulative percent differences close to the end of the triangle, or use a weighted average of all cumulative differences with much more weight given to later development periods, which shows a difference of 0.4%, as illustrated in Table 4.5.

The process of using the all-prior estimates to help "calibrate" the tail factor assumptions (i.e., what are reasonable for $v(d)$ and u) can be quite useful in practice. For example, if we had used only 3 decimal places in the tail factors in Table 4.3, and thus only 2 years appear to be needed in the tail,[31] the weighted average of the cumulative percentage differences changes to -14.9% instead of +0.4%. Of course either $v(d)$ or u, or both, can be adjusted to see whether changing the tail factor assumption improves the fit of the estimated all-prior data to the actual data, thus validating the tail factor

[30] Again for readability values beyond 144 months of development are excluded from Table 4.5 so the diagonal values will not sum to the values in the Incremental row without referencing all of the values in the companion Excel file.

[31] Since all fitted factors beyond the 11th period in Table 4.3 would round to 1.000.

assumption with actual data in the all-prior row.[32]

To illustrate a more complete validation process, Table 4.6 summarizes key results when changing the number of years in the tail estimation from 1 to 14 years. Of course the actual validation process in practice can include other assumptions and methods for calculating the tail, but in the end judgment is required for making the final selections.

Table 4.6 – "Medium" Paid Tail Calibration Summary

		All-Prior Projection					Change in IBNR	
Tail Years	(*u*) Ultimate	Total Difference	Cumulative Percent	Weighted Percent	IBNR	Total IBNR	All-Prior	Total
1	11	16,039	5.7%	-28.1%	(1,323)	176,381		
2	12	18,173	6.4%	-14.9%	(1,045)	179,629	278	3,248
3	13	19,311	6.8%	-7.8%	(746)	181,532	299	1,903
4	14	19,920	7.1%	-4.0%	(506)	182,639	241	1,107
5	15	20,245	7.2%	-2.0%	(334)	183,279	172	640
6	16	20,419	7.2%	-0.9%	(218)	183,647	116	368
7	17	20,512	7.3%	-0.4%	(143)	183,857	75	211
8	18	20,562	7.3%	0.0%	(97)	183,978	47	120
9	19	20,588	7.3%	0.1%	(68)	184,046	29	68
10	20	20,602	7.3%	0.2%	(51)	184,085	17	39
11	21	20,619	7.3%	0.3%	(31)	184,116	20	31
12	22	20,632	7.3%	0.4%	(14)	184,139	17	23
13	23	20,642	7.3%	0.4%	(2)	184,155	13	16
14	24	20,648	7.3%	0.5%	7	184,166	9	11

4.6 Estimate All-Prior Reserves

Finally, summing all of the diagonals below the diagonal line in Table 4.5, using formula (3.13), allows us to derive an independent estimate of the unpaid claims for all-prior years, as shown in Table 4.5.[33] Using this estimate of all-prior unpaid claims, we can complete the typical summary of our chain ladder estimates, as illustrated in Table 4.7.[34]

[32] While calibrating and validating could be used somewhat interchangeably, I think it is more useful to think of them as different yet related processes. In this case, calibration is the process of adjusting the parameters used to estimate a tail factor and validation is the process of checking the tail factor against the actual data in the all-prior row.

[33] As Table 4.5 is truncated beyond 144 months for readability, the interested reader can refer to the Excel file for the details beyond 144 months of development which sum to derive the all-prior row estimate.

[34] Note that the columns in Table 4.8 are a continuation of Table 4.7, so the column (7) referenced in Table 4.7 can be found in Table 4.8.

Table 4.7 – "Medium" Paid Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Paid Data
***All-Prior Estimate in Separate Exhibit**

	(1) Paid to Date	(2) Paid CDF	(3) (1) x (2) Ultimate	(4) (3) - (1) Estimated Unpaid	(5) (7) - (1) Case Reserve	(6) (4) - (5) Estimated IBNR
A-P*	282,390	*1.0046*	*283,699*	*1,309*	1,323	(14)
2004	322,784	1.0033	323,862	1,078	1,132	(54)
2005	357,748	1.0062	359,964	2,216	2,030	186
2006	328,463	1.0115	332,241	3,778	3,473	305
2007	348,988	1.0214	356,451	7,463	6,054	1,409
2008	359,694	1.0399	374,038	14,344	11,865	2,479
2009	329,792	1.0748	354,447	24,655	19,049	5,607
2010	309,186	1.1426	353,283	44,097	34,772	9,326
2011	255,177	1.2868	328,373	73,196	61,512	11,684
2012	243,840	1.7136	417,840	174,000	118,332	55,669
2013	100,689	3.8556	388,215	287,525	189,983	97,542
				633,661	449,522	184,139

The all-prior (A-P) row in Table 4.7 is highlighted to signify that it was not calculated the same as the remaining rows. For the all-prior row, the estimated unpaid amount is the sum of the future diagonals from Table 4.5, the ultimate is (1) plus (4) and the Paid CDF is (3) divided by (1), which is only included for comparison purposes with the other CDFs in column (2). Note that simply using the tail factor for the all-prior row (1.0033 instead of 1.0046) would have misestimated the all-prior unpaid claims, perhaps significantly in some cases.

The analysis in Tables 4.1 to 4.7 used paid data. Analogous work using incurred data is included in Appendix A as Tables A.1 to A.7, respectively. For ease of comparison, the summary of results for the incurred data (Table A.7) is repeated here as Table 4.8.

Table 4.8 – "Medium" Incurred Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Incurred Data
***All-Prior Estimate in Separate Exhibit**

	(7) Incurred to Date	(8) Incurred CDF	(9) (7) x (8) Ultimate	(10) (11) + (12) Estimated Unpaid	(11) (7) - (1) Case Reserve	(12) (9) - (7) Estimated IBNR
A-P*	283,713	*1.0001*	*283,735*	1,344	1,323	*21*
2004	323,915	1.0001	323,948	1,164	1,132	33
2005	359,778	1.0002	359,866	2,118	2,030	88
2006	331,936	1.0006	332,131	3,668	3,473	195
2007	355,042	1.0014	355,543	6,555	6,054	501
2008	371,559	1.0039	373,025	13,331	11,865	1,466
2009	348,841	1.0093	352,096	22,304	19,049	3,255
2010	343,957	1.0226	351,733	42,548	34,772	7,776
2011	316,689	1.0525	333,326	78,149	61,512	16,637
2012	362,172	1.1214	406,131	162,291	118,332	43,959
2013	290,672	1.2840	373,216	272,527	189,983	82,544
				605,997	449,522	156,475

Comparing the results in Tables 4.7 and 4.8, it seems fair to conclude that the case reserves for the all-prior years are adequate and that an IBNR reserve near zero for these years would be reasonable.[35]

Appendices B and C include analyses for the "low" case reserve simulated data for paid and incurred data, respectively. For ease of comparison, Tables B.7 and C.7 are repeated here as Tables 4.9 and 4.10, respectively.

Table 4.9 – "Low" Paid Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Paid Data
***All-Prior Estimate in Separate Exhibit**

	(1)	(2)	(3) (1) x (2)	(4) (3) - (1)	(5) (7) - (1)	(6) (4) - (5)
	Paid to Date	**Paid CDF**	**Ultimate**	**Estimated Unpaid**	**Case Reserve**	**Estimated IBNR**
A-P*	546,393	*1.0122*	*553,046*	*6,653*	6,075	578
2004	386,452	1.0114	390,872	4,420	3,476	944
2005	434,642	1.0185	442,661	8,020	5,946	2,074
2006	407,012	1.0306	419,475	12,463	7,684	4,779
2007	457,165	1.0518	480,866	23,701	16,130	7,571
2008	398,617	1.0892	434,190	35,574	23,671	11,903
2009	431,152	1.1550	497,975	66,823	33,566	33,257
2010	400,155	1.2794	511,940	111,786	63,349	48,437
2011	304,450	1.5237	463,877	159,427	94,442	64,985
2012	231,388	2.2836	528,388	297,000	159,371	137,629
2013	105,488	5.0838	536,281	430,793	206,653	224,140
				1,156,658	620,362	536,296

Table 4.10 – "Low" Incurred Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Incurred Data
***All-Prior Estimate in Separate Exhibit**

	(7)	(8)	(9) (7) x (8)	(10) (11) + (12)	(11) (7) - (1)	(12) (9) - (7)
	Incurred to Date	**Incurred CDF**	**Ultimate**	**Estimated Unpaid**	**Case Reserve**	**Estimated IBNR**
A-P*	552,468	*1.0019*	*553,494*	7,101	6,075	*1,026*
2004	389,928	1.0025	390,883	4,432	3,476	955
2005	440,588	1.0045	442,586	7,944	5,946	1,998
2006	414,696	1.0084	418,178	11,166	7,684	3,482
2007	473,295	1.0164	481,067	23,902	16,130	7,772
2008	422,287	1.0298	434,869	36,252	23,671	12,581
2009	464,718	1.0551	490,328	59,176	33,566	25,610
2010	463,503	1.1028	511,172	111,017	63,349	47,669
2011	398,892	1.1871	473,531	169,080	94,442	74,639
2012	390,758	1.3800	539,250	307,862	159,371	148,491
2013	312,141	1.7137	534,926	429,438	206,653	222,785
				1,167,370	620,362	547,007

Comparing the results in Tables 4.9 and 4.10, we have evidence that the case reserves for the all-prior years are inadequate, so we have the ability to compare our estimates to any held IBNR to see if it is sufficient.

[35] Some tables in the Appendices have also been reduced for readability, so the reader is directed to the companion Excel file for all of the details.

Appendices D and E include the analysis for the "high" case reserve simulated data for paid and incurred data, respectively.[36] For ease of comparison, Tables D.7 and E.7 are repeated here as Tables 4.11 and 4.12, respectively.

Table 4.11 – "High" Paid Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Paid Data
***All-Prior Estimate in Separate Exhibit**

	(1) Paid to Date	(2) Paid CDF	(3) (1) x (2) Ultimate	(4) (3) - (1) Estimated Unpaid	(5) (7) - (1) Case Reserve	(6) (4) - (5) Estimated IBNR
A-P*	2,028,756	*1.0040*	*2,036,779*	*8,024*	13,009	(4,985)
2004	962,203	1.0093	971,173	8,969	11,874	(2,904)
2005	898,591	1.0184	915,098	16,508	21,878	(5,370)
2006	907,581	1.0363	940,536	32,955	42,994	(10,040)
2007	977,881	1.0722	1,048,462	70,581	83,430	(12,849)
2008	1,040,208	1.1459	1,191,977	151,769	140,745	11,025
2009	914,456	1.2918	1,181,321	266,865	257,107	9,758
2010	732,524	1.7372	1,272,516	539,993	528,128	11,865
2011	496,043	2.6041	1,291,769	795,726	696,830	98,896
2012	271,729	5.2619	1,429,810	1,158,081	933,516	224,565
2013	99,365	14.9591	1,486,405	1,387,040	1,129,608	257,432
				4,436,510	3,859,117	577,393

Table 4.12 – "High" Incurred Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Incurred Data
***All-Prior Estimate in Separate Exhibit**

	(7) Incurred to Date	(8) Incurred CDF	(9) (7) x (8) Ultimate	(10) (11) + (12) Estimated Unpaid	(11) (7) - (1) Case Reserve	(12) (9) - (7) Estimated IBNR
A-P*	2,041,764	*0.9996*	*2,040,912*	12,156	13,009	*(853)*
2004	974,077	0.9989	973,045	10,841	11,874	(1,032)
2005	920,468	0.9981	918,726	20,135	21,878	(1,742)
2006	950,576	0.9972	947,946	40,364	42,994	(2,630)
2007	1,061,310	0.9942	1,055,132	77,251	83,430	(6,179)
2008	1,180,953	0.9933	1,173,030	132,822	140,745	(7,923)
2009	1,171,563	0.9942	1,164,732	250,275	257,107	(6,832)
2010	1,260,651	1.0042	1,265,965	533,442	528,128	5,314
2011	1,192,873	1.0589	1,263,124	767,081	696,830	70,252
2012	1,205,245	1.1466	1,381,967	1,110,238	933,516	176,722
2013	1,228,972	1.2667	1,556,760	1,457,395	1,129,608	327,787
				4,412,001	3,859,117	552,884

Comparing the results in Tables 4.11 and 4.12, we have evidence that the case reserves for the all-prior years are more than adequate, and again we have the ability to assess any held IBNR.

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[36] Note that the exponential decay method (3.6) of estimating tail factors is not well suited to fitting development factors less than 1.000. Thus, the selected tail factor in Table E.3 needed to be estimated using a different method.

5. FUTURE RESEARCH

As this is the first paper outlining a process for estimating unpaid claims for all-prior data, there is much that can be done to expand this in various ways. Only a few suggestions for such future research are offered here.

- The historical estimation process could also incorporate assumptions from other estimation methods such as Berquist and Sherman [3].
- Closed-form estimates for the standard deviation as in Mack [8] or alternative assumptions for age-to-age factors as in Murphy [9] may be adaptable to all-prior data.
- The Over-Dispersed Poisson (ODP) Bootstrap models such as those discussed in Shapland and Leong [11] could incorporate the all-prior data analysis to simulate a distribution for the all-prior claims.
- The incremental log models in Barnett and Zehnwirth [1] or Zehnwirth [13] can be extended backwards to simulate a distribution for the all-prior claims.

6. CONCLUSIONS

Whenever data being used to estimate unpaid claims includes an all-prior row and a tail factor is needed, the starting point to analyzing the all-prior data is understanding the data (i.e., how was it created and what is included). Once the data is understood, the methods introduced in this paper can be used to analyze the all-prior row. Regardless of whether the unpaid claims in the all-prior row are a significant portion of the total unpaid claims or not, the value of the methodology in helping to calibrate the tail factor should not be underestimated. Indeed, the process of calibrating the tail factor and validating it by comparing estimates of the all-prior data to the actual all-prior data may reveal that the tail factor is different than otherwise expected, which will have an impact on estimates for all accident periods.

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Supplementary Material

A more complete review of the notation, data and examples used in this paper are contained in the companion Excel file "All Prior Analysis.xlsm". An example of how all-prior data is compiled for the NAIC Schedule P is contained in the "Creating All Prior Data.xls" file.

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Biography of the Author

Mark R. Shapland is a Senior Consulting Actuary in Milliman's Dubai office. He is responsible for various reserving, pricing and risk modeling projects for a wide variety of clients. He has a B.S. degree in Integrated Studies (Actuarial Science) from the University of Nebraska-Lincoln. He is a Fellow of the Casualty Actuarial Society, a Fellow of the Society of Actuaries and a Member of the American Academy of Actuaries. He has previously served the CAS as a chair of the CAS Committee on Reserves, a chair of the Dynamic Risk Modeling Committee, a co-chair of the CAS Loss Simulation Model Working Party and a co-chair of the CAS Tail Factor Working Party. He is a co-creator and co-presenter for the CAS Reserve Variability Limited Attendance Seminar and a frequent speaker on reserve variability at actuarial meetings in the United States and many other countries.

Appendix A – Incurred Analysis for "Medium" Case Reserve Data

Table A.1 – "Medium" Incurred Loss Triangle with All-Prior Data

	12	24	36	48	60	72	84	96	108	120	132
A-P		226,614	253,212	272,185	278,519	281,496	283,003	283,520	283,663	283,741	283,713
2004	250,529	286,453	307,218	317,077	321,489	322,467	323,628	323,685	323,858	323,915	
2005	277,084	325,918	342,040	353,268	356,648	358,593	359,498	359,761	359,778		
2006	271,418	298,981	316,852	323,994	328,877	330,662	331,705	331,936			
2007	284,989	320,743	335,916	347,257	352,265	354,693	355,042				
2008	297,906	334,537	353,299	365,298	369,420	371,559					
2009	277,237	307,715	333,225	343,673	348,841						
2010	270,103	313,682	337,891	343,957							
2011	255,515	292,838	316,689								
2012	323,902	362,172									
2013	290,672										

Table A.2 – "Medium" Incurred Loss Development Factors

	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	Tail
2004	1.143	1.072	1.032	1.014	1.003	1.004	1.000	1.001	1.000	
2005	1.176	1.049	1.033	1.010	1.005	1.003	1.001	1.000		
2006	1.102	1.060	1.023	1.015	1.005	1.003	1.001			
2007	1.125	1.047	1.034	1.014	1.007	1.001				
2008	1.123	1.056	1.034	1.011	1.006					
2009	1.110	1.083	1.031	1.015						
2010	1.161	1.077	1.018							
2011	1.146	1.081								
2012	1.118									
VWA	1.133	1.065	1.029	1.013	1.005	1.003	1.001	1.000	1.000	
5-Yr VWA	1.131	1.068	1.028	1.013	1.005	1.003	1.001	1.000	1.000	
3-Yr VWA	1.140	1.080	1.028	1.014	1.006	1.002	1.001	1.000	1.000	
TF Fitted	1.157	1.066	1.027	1.011	1.005	1.002	1.001	1.000	1.000	1.000
User	1.145									
Selected	1.145	1.065	1.029	1.013	1.005	1.003	1.001	1.000	1.000	1.0001
Ultimate	1.284	1.121	1.053	1.023	1.009	1.004	1.001	1.001	1.000	1.0001
% Reported	0.779	0.892	0.950	0.978	0.991	0.996	0.999	0.999	1.000	1.000
% Unrptd	0.221	0.108	0.050	0.022	0.009	0.004	0.001	0.001	0.000	0.000

Table A.3 – "Medium" Incurred Tail Factor Calculation

Tail Years:	5	Actual	57,099	Decay	0.417
Tail Factor:	1.0001	Estimated	63,910	Intercept	0.377
		Error %	11.9%		

Period	Factor	Dev	Log	Excl	Period	Log	Fitted	Selected	ATA	Ultimate
1	1.13328	0.13328	(2.015)		1	(2.015)	1.157232		1.157232	1.291574
2	1.06541	0.06541	(2.727)		2	(2.727)	1.065522		1.065522	1.116089
3	1.02927	0.02927	(3.531)		3	(3.531)	1.027304		1.027304	1.047457
4	1.01315	0.01315	(4.331)		4	(4.331)	1.011378		1.011378	1.019618
5	1.00537	0.00537	(5.228)		5	(5.228)	1.004741		1.004741	1.008147
6	1.00253	0.00253	(5.979)		6	(5.979)	1.001976		1.001976	1.003390
7	1.00054	0.00054	(7.521)		7	(7.521)	1.000823		1.000823	1.001411
8	1.00028	0.00028	(8.184)		8	(8.184)	1.000343		1.000343	1.000587
9	1.00018	0.00018	(8.646)		9	(8.646)	1.000143		1.000143	1.000244
10							1.000060		1.000060	1.000101
11							1.000025		1.000025	1.000041
12							1.000010		1.000010	1.000016
13							1.000004		1.000004	1.000006
14							1.000002		1.000002	1.000002

Table A.4 – "Medium" Incurred All-Prior Projection (Cumulative)

	Premium	Loss Ratio	24	36	48	60	72	84	96	108	120	132
1991	*431,182*	*70.0%*	*269,158*	*286,762*	*295,154*	*299,037*	*300,641*	*301,402*	*301,650*	*301,754*	*301,797*	301,815
1992	*435,494*	*70.0%*	*271,849*	*289,630*	*298,106*	*302,027*	*303,648*	*304,416*	*304,667*	*304,771*	*304,815*	304,833
1993	*439,848*	*69.1%*	*271,038*	*288,765*	*297,216*	*301,125*	*302,741*	*303,507*	*303,757*	*303,861*	*303,905*	303,923
1994	*472,929*	*64.9%*	*273,709*	*291,611*	*300,146*	*304,094*	*305,725*	*306,499*	*306,751*	*306,856*	*306,900*	306,919
1995	*412,911*	*75.1%*	*276,532*	*294,619*	*303,241*	*307,230*	*308,878*	*309,660*	*309,915*	*310,021*	*310,065*	310,084
1996	*460,127*	*68.0%*	*279,020*	*297,269*	*305,969*	*309,993*	*311,657*	*312,445*	*312,703*	*312,810*	*312,855*	312,873
1997	*471,803*	*67.0%*	*281,893*	*300,330*	*309,120*	*313,186*	*314,866*	*315,663*	*315,923*	*316,031*	*316,076*	316,095
1998	*443,804*	*71.9%*	*284,557*	*303,168*	*312,040*	*316,145*	*317,841*	*318,645*	*318,908*	*319,017*	*319,063*	319,082
1999	*448,454*	*71.9%*	*287,538*	*306,344*	*315,310*	*319,457*	*321,171*	*321,984*	*322,249*	*322,360*	*322,406*	322,425
2000	*439,491*	*74.1%*	*290,414*	*309,408*	*318,463*	*322,652*	*324,383*	*325,204*	*325,472*	*325,583*	*325,630*	325,649
2001	*499,204*	*65.9%*	*293,368*	*312,556*	*321,703*	*325,934*	*327,683*	*328,512*	*328,783*	*328,895*	*328,942*	328,962
2002	*447,766*	*74.2%*	*296,281*	*315,660*	*324,897*	*329,171*	*330,937*	*331,775*	*332,048*	*332,162*	*332,209*	332,229
2003	*468,659*	*71.6%*	*299,239*	*318,811*	*328,141*	*332,457*	*334,241*	*335,087*	*335,363*	*335,478*	*335,526*	335,546

Table A.5 – "Medium" Incurred All-Prior Projection (Incremental)

	12	24	36	48	60	72	84	96	108	120	132	144
1994												8
1995											18	8
1996										44	18	8
1997									106	44	18	8
1998								257	107	45	19	8
1999							797	260	108	45	19	8
2000						1,696	804	262	109	46	19	8
2001					4,147	1,714	813	265	111	46	19	8
2002				9,055	4,189	1,731	821	268	112	47	19	8
2003			19,188	9,147	4,232	1,749	829	270	113	47	20	8
Totals:	**(144+)**	**(36-132)**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**
Estimated	21	63,910	35,321	16,297	7,222	3,021	1,285	460	192	80	33	*14*
Actual		57,099	26,597	18,973	6,334	2,976	1,507	517	143	77	(28)	
Differences		6,811	8,724	(2,677)	888	44	(222)	(57)	49	2	61	
Cumulative Percent Difference			11.9%	-6.3%	6.6%	-2.4%	-7.6%	7.6%	57.7%	126.1%	219.1%	
Weights			0.25	0.50	1.00	2.00	3.00	4.00	5.00	6.00	7.00	
Weighted Average			90.0%									

Table A.6 – "Medium" Incurred Tail Calibration Summary

		All-Prior Projection					Change in IBNR	
Tail	(*u*)	Total	Cumulative	Weighted		Total		
Years	Ultimate	Difference	Percent	Percent	IBNR	IBNR	All-Prior	Total
1	11	6,698	11.7%	62.1%	-	156,306		
2	12	6,766	11.9%	79.0%	8	156,403	8	97
3	13	6,795	11.9%	86.0%	15	156,447	7	44
4	14	6,806	11.9%	88.8%	19	156,466	4	20
5	15	6,811	11.9%	90.0%	21	156,475	2	9
6	16	6,813	11.9%	90.5%	23	156,479	1	4
7	17	6,814	11.9%	90.7%	23	156,480	1	2
8	18	6,814	11.9%	90.8%	24	156,481	0	1
9	19	6,815	11.9%	90.8%	24	156,482	0	0
10	20	6,815	11.9%	90.9%	24	156,482	0	0

Table A.7 – "Medium" Incurred Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Incurred Data
***All-Prior Estimate in Separate Exhibit**

	(7)	(8)	(9)	(10)	(11)	(12)
			(7) x (8)	(11) + (12)	(7) - (1)	(9) - (7)
	Incurred to Date	Incurred CDF	Ultimate	Estimated Unpaid	Case Reserve	Estimated IBNR
A-P*	283,713	*1.0001*	*283,735*	1,344	1,323	*21*
2004	323,915	1.0001	323,948	1,164	1,132	33
2005	359,778	1.0002	359,866	2,118	2,030	88
2006	331,936	1.0006	332,131	3,668	3,473	195
2007	355,042	1.0014	355,543	6,555	6,054	501
2008	371,559	1.0039	373,025	13,331	11,865	1,466
2009	348,841	1.0093	352,096	22,304	19,049	3,255
2010	343,957	1.0226	351,733	42,548	34,772	7,776
2011	316,689	1.0525	333,326	78,149	61,512	16,637
2012	362,172	1.1214	406,131	162,291	118,332	43,959
2013	290,672	1.2840	373,216	272,527	189,983	82,544
				605,997	449,522	156,475

Appendix B – Paid Analysis for "Low" Case Reserve Data

Table B.1 – "Low" Paid Loss Triangle with All-Prior Data

	12	24	36	48	60	72	84	96	108	120	132
A-P		-	224,096	349,441	428,145	476,471	506,620	525,072	535,370	541,985	546,393
2004	59,477	172,635	254,266	309,215	335,168	355,021	372,113	378,908	383,860	386,452	
2005	95,293	190,721	287,897	338,580	382,595	407,187	421,132	429,650	434,642		
2006	73,884	165,497	266,958	318,469	366,483	387,022	397,578	407,012			
2007	81,811	222,270	329,320	389,660	419,385	442,175	457,165				
2008	119,772	205,222	277,631	333,442	373,116	398,617					
2009	111,735	225,388	329,885	394,175	431,152						
2010	89,494	212,010	339,510	400,155							
2011	73,009	200,877	304,450								
2012	115,736	231,388									
2013	105,488										

Table B.2 – "Low" Paid Loss Development Factors

	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	Tail
2004	2.903	1.473	1.216	1.084	1.059	1.048	1.018	1.013	1.007	
2005	2.001	1.510	1.176	1.130	1.064	1.034	1.020	1.012		
2006	2.240	1.613	1.193	1.151	1.056	1.027	1.024			
2007	2.717	1.482	1.183	1.076	1.054	1.034				
2008	1.713	1.353	1.201	1.119	1.068					
2009	2.017	1.464	1.195	1.094						
2010	2.369	1.601	1.179							
2011	2.751	1.516								
2012	1.999									
VWA	2.226	1.499	1.191	1.108	1.060	1.036	1.021	1.012	1.007	
5-Yr VWA	2.109	1.483	1.190	1.112	1.060	1.036	1.021	1.012	1.007	
3-Yr VWA	2.316	1.526	1.191	1.095	1.059	1.032	1.021	1.012	1.007	
TF Fitted	1.539	1.313	1.182	1.105	1.061	1.035	1.021	1.012	1.007	1.011
User										
Selected	2.226	1.499	1.191	1.108	1.060	1.036	1.021	1.012	1.007	1.0044
Ultimate	5.084	2.284	1.524	1.279	1.155	1.089	1.052	1.031	1.018	1.0114
% Paid	19.7%	43.8%	65.6%	78.2%	86.6%	91.8%	95.1%	97.0%	98.2%	98.9%
% Unpaid	80.3%	56.2%	34.4%	21.8%	13.4%	8.2%	4.9%	3.0%	1.8%	1.1%

Table B.3 – "Low" Paid Tail Factor Calculation

Paid Tail Factor Analysis

All Prior

Tail Years:	15	Actual	546,393	Decay	0.580
Tail Factor:	1.0114	Estimated	548,874	Intercept	0.930
		Error %	0.5%		

Period	Factor	Dev	Log	Excl	Period	Log	Fitted	Selected	ATA	ATU
1	2.22626	1.22626	0.204	Y			1.539467		1.539467	3.051586
2	1.49874	0.49874	(0.696)	Y			1.313031		1.313031	1.982236
3	1.19095	0.19095	(1.656)	Y			1.181640		1.181640	1.509664
4	1.10768	0.10768	(2.229)	Y			1.105398		1.105398	1.277601
5	1.06036	0.06036	(2.807)		5	(2.807)	1.061159		1.061159	1.155783
6	1.03556	0.03556	(3.337)		6	(3.337)	1.035488		1.035488	1.089171
7	1.02078	0.02078	(3.874)		7	(3.874)	1.020592		1.020592	1.051843
8	1.01230	0.01230	(4.398)		8	(4.398)	1.011949		1.011949	1.030621
9	1.00675	0.00675	(4.998)		9	(4.998)	1.006933		1.006933	1.018451
10							1.004023	1.004440	1.004440	1.011439
11							1.002335	1.002640	1.002640	1.006968
12							1.001355	1.001940	1.001940	1.004316
13							1.000786	1.000940	1.000940	1.002372
14							1.000456	1.000640	1.000640	1.001430
15							1.000265	1.000340	1.000340	1.000790
16							1.000154	1.000240	1.000240	1.000450
17							1.000089		1.000089	1.000210
18							1.000052		1.000052	1.000120
19							1.000030		1.000030	1.000069
20							1.000017		1.000017	1.000039
21							1.000010		1.000010	1.000021
22							1.000006		1.000006	1.000011
23							1.000003		1.000003	1.000005
24							1.000002		1.000002	1.000002

Table B.4 – "Low" Paid All-Prior Projection (Cumulative)

	Premium	Loss Ratio	24	36	48	60	72	84	96	108	120	132
1994	408,252	74.4%	*133,011*	*199,349*	*237,416*	*262,981*	*278,854*	*288,769*	*294,715*	*298,237*	*300,305*	301,638
1995	421,696	74.2%	*137,022*	*205,361*	*244,575*	*270,911*	*287,263*	*297,476*	*303,602*	*307,230*	*309,360*	310,734
1996	426,540	75.5%	*141,024*	*211,359*	*251,718*	*278,824*	*295,653*	*306,165*	*312,470*	*316,203*	*318,396*	319,809
1997	435,782	76.2%	*145,416*	*217,941*	*259,557*	*287,507*	*304,860*	*315,699*	*322,200*	*326,050*	*328,311*	329,768
1998	445,319	76.8%	*149,768*	*224,463*	*267,326*	*296,112*	*313,984*	*325,148*	*331,843*	*335,809*	*338,137*	339,638
1999	479,330	73.5%	*154,280*	*231,225*	*275,379*	*305,032*	*323,443*	*334,943*	*341,840*	*345,925*	*348,323*	349,870
2000	482,332	75.2%	*158,837*	*238,055*	*283,513*	*314,042*	*332,996*	*344,836*	*351,937*	*356,142*	*358,612*	360,204
2001	508,950	73.4%	*163,591*	*245,180*	*291,998*	*323,440*	*342,963*	*355,157*	*362,470*	*366,801*	*369,345*	370,984
2002	499,443	77.0%	*168,409*	*252,401*	*300,598*	*332,966*	*353,063*	*365,617*	*373,146*	*377,604*	*380,222*	381,910
2003	552,073	71.8%	*173,584*	*260,156*	*309,834*	*343,198*	*363,912*	*376,851*	*384,612*	*389,207*	*391,906*	393,646

Table B.5 – "Low" Paid All-Prior Projection (Incremental)

	12	24	36	48	60	72	84	96	108	120	132	144
1994												796
1995											1,374	820
1996										2,192	1,414	844
1997									3,850	2,261	1,458	871
1998								6,696	3,965	2,328	1,501	897
1999							11,500	6,897	4,085	2,398	1,547	924
2000						18,955	11,840	7,101	4,205	2,469	1,592	951
2001					31,443	19,522	12,194	7,313	4,331	2,543	1,640	979
2002				48,197	32,369	20,097	12,553	7,529	4,459	2,618	1,688	1,008
2003			86,573	49,678	33,363	20,715	12,939	7,760	4,596	2,699	1,740	1,039
Totals:	**(144+)**	**(36-132)**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**
Estimated	6,653	548,874	212,814	130,042	82,786	50,912	31,107	18,716	11,286	6,892	4,319	*2,657*
Actual		546,393	224,096	125,345	78,704	48,327	30,149	18,451	10,298	6,615	4,409	
Differences		2,480	(11,282)	4,697	4,082	2,585	958	264	987	277	(89)	
Cumulative Percent Difference			0.5%	4.3%	4.6%	4.2%	3.4%	3.6%	5.5%	1.7%	-2.0%	
Weights			0.25	0.50	1.00	2.00	3.00	4.00	5.00	6.00	7.00	
Weighted Average			2.2%									

Table B.6 – "Low" Paid Tail Calibration Summary

		All-Prior Projection					Change in IBNR	
Tail Years	***(u)* Ultimate**	**Total Difference**	**Cumulative Percent**	**Weighted Percent**	**IBNR**	**Total IBNR**	**All-Prior**	**Total**
1	11	(14,527)	-2.7%	-33.9%	(6,075)	497,077		
2	12	(7,802)	-1.4%	-19.7%	(5,031)	510,460	1,044	13,383
3	13	(3,044)	-0.6%	-9.6%	(3,521)	521,061	1,510	10,602
4	14	(822)	-0.2%	-4.9%	(2,439)	526,557	1,082	5,496
5	15	635	0.1%	-1.7%	(1,471)	530,533	968	3,976
6	16	1,380	0.3%	-0.1%	(837)	532,766	634	2,233
7	17	1,887	0.3%	1.0%	(308)	534,424	529	1,658
8	18	2,068	0.4%	1.4%	(82)	535,070	226	645
9	19	2,170	0.4%	1.6%	66	535,461	148	391
10	20	2,226	0.4%	1.7%	161	535,697	95	236
11	21	2,300	0.4%	1.9%	277	535,895	116	198
12	22	2,366	0.4%	2.0%	383	536,048	105	153
13	23	2,417	0.4%	2.1%	467	536,161	85	113
14	24	2,455	0.4%	2.2%	532	536,241	64	80
15	25	2,480	0.5%	2.2%	578	536,296	46	56
16	26	2,498	0.5%	2.2%	610	536,334	32	38
17	27	2,509	0.5%	2.3%	632	536,359	22	25
18	28	2,516	0.5%	2.3%	647	536,376	15	17
19	29	2,521	0.5%	2.3%	657	536,387	10	11
20	30	2,524	0.5%	2.3%	664	536,394	7	7

Table B.7 – "Low" Paid Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Paid Data
***All-Prior Estimate in Separate Exhibit**

	(1) Paid to Date	(2) Paid CDF	(3) (1) x (2) Ultimate	(4) (3) - (1) Estimated Unpaid	(5) (7) - (1) Case Reserve	(6) (4) - (5) Estimated IBNR
A-P*	546,393	*1.0122*	*553,046*	*6,653*	6,075	578
2004	386,452	1.0114	390,872	4,420	3,476	944
2005	434,642	1.0185	442,661	8,020	5,946	2,074
2006	407,012	1.0306	419,475	12,463	7,684	4,779
2007	457,165	1.0518	480,866	23,701	16,130	7,571
2008	398,617	1.0892	434,190	35,574	23,671	11,903
2009	431,152	1.1550	497,975	66,823	33,566	33,257
2010	400,155	1.2794	511,940	111,786	63,349	48,437
2011	304,450	1.5237	463,877	159,427	94,442	64,985
2012	231,388	2.2836	528,388	297,000	159,371	137,629
2013	105,488	5.0838	536,281	430,793	206,653	224,140
				1,156,658	620,362	536,296

Appendix C – Incurred Analysis for "Low" Case Reserve Data

Table C.1 – "Low" Incurred Loss Triangle with All-Prior Data

	12	24	36	48	60	72	84	96	108	120	132
A-P		313,964	419,793	474,098	509,975	528,993	540,336	546,327	549,438	551,508	552,468
2004	229,846	286,253	326,645	356,188	367,977	378,068	384,592	387,096	389,206	389,928	
2005	272,625	317,769	373,881	395,845	419,735	430,657	435,569	439,389	440,588		
2006	239,240	296,287	343,883	375,203	398,283	406,375	411,221	414,696			
2007	273,614	361,153	416,886	443,360	456,786	467,440	473,295				
2008	280,215	326,745	365,787	389,743	411,549	422,287					
2009	299,423	361,656	410,220	449,671	464,718						
2010	301,843	364,457	432,227	463,503							
2011	263,437	341,716	398,892								
2012	318,040	390,758									
2013	312,141										

Table C.2 – "Low" Incurred Loss Development Factors

	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	Tail
2004	1.245	1.141	1.090	1.033	1.027	1.017	1.007	1.005	1.002	
2005	1.166	1.177	1.059	1.060	1.026	1.011	1.009	1.003		
2006	1.238	1.161	1.091	1.062	1.020	1.012	1.008			
2007	1.320	1.154	1.064	1.030	1.023	1.013				
2008	1.166	1.119	1.065	1.056	1.026					
2009	1.208	1.134	1.096	1.033						
2010	1.207	1.186	1.072							
2011	1.297	1.167								
2012	1.229									
VWA	1.229	1.155	1.076	1.045	1.025	1.013	1.008	1.004	1.002	
5-Yr VWA	1.220	1.153	1.077	1.047	1.025	1.013	1.008	1.004	1.002	
3-Yr VWA	1.242	1.162	1.078	1.039	1.023	1.012	1.008	1.004	1.002	
TF Fitted	1.283	1.153	1.083	1.045	1.024	1.013	1.007	1.004	1.002	1.002
User										
Selected	1.242	1.162	1.076	1.045	1.025	1.013	1.008	1.004	1.002	1.0011
Ultimate	1.714	1.380	1.187	1.103	1.055	1.030	1.016	1.008	1.005	1.0025
% Reported	0.584	0.725	0.842	0.907	0.948	0.971	0.984	0.992	0.995	0.998
% Unrptd	0.416	0.275	0.158	0.093	0.052	0.029	0.016	0.008	0.005	0.002

Table C.3 – "Low" Incurred Tail Factor Calculation

Incurred Tail Factor Analysis

			All Prior		
Tail Years:	10	Actual	238,504	Decay	0.541
Tail Factor:	1.0025	Estimated	230,023	Intercept	0.522
		Error %	-3.6%		

Period	Factor	Dev	Log	Excl	Period	Log	Fitted	Selected	ATA	Ultimate
1	1.22940	0.22940	(1.472)	Y			1.282709		1.282709	1.763247
2	1.15526	0.15526	(1.863)		2	(1.863)	1.152996		1.152996	1.374628
3	1.07641	0.07641	(2.572)		3	(2.572)	1.082798		1.082798	1.192222
4	1.04524	0.04524	(3.096)		4	(3.096)	1.044809		1.044809	1.101057
5	1.02458	0.02458	(3.706)		5	(3.706)	1.024250		1.024250	1.053836
6	1.01316	0.01316	(4.331)		6	(4.331)	1.013123		1.013123	1.028886
7	1.00796	0.00796	(4.834)		7	(4.834)	1.007102		1.007102	1.015558
8	1.00400	0.00400	(5.521)		8	(5.521)	1.003844		1.003844	1.008396
9	1.00185	0.00185	(6.290)		9	(6.290)	1.002080		1.002080	1.004535
10							1.001126		1.001126	1.002450
11							1.000609		1.000609	1.001323
12							1.000330		1.000330	1.000713
13							1.000178		1.000178	1.000384
14							1.000097		1.000097	1.000205
15							1.000052		1.000052	1.000109
16							1.000028		1.000028	1.000056
17							1.000015		1.000015	1.000028
18							1.000008		1.000008	1.000013
19							1.000004		1.000004	1.000004

Table C.4 – "Low" Incurred All-Prior Projection (Cumulative)

	Premium	Loss Ratio	24	36	48	60	72	84	96	108	120	132
1994	*408,252*	*74.4%*	*220,100*	*255,864*	*275,415*	*287,875*	*294,952*	*298,833*	*301,211*	*302,368*	*302,997*	303,338
1995	*421,696*	*74.2%*	*226,737*	*263,579*	*283,720*	*296,556*	*303,846*	*307,844*	*310,293*	*311,486*	*312,134*	312,485
1996	*426,540*	*75.5%*	*233,359*	*271,278*	*292,006*	*305,218*	*312,721*	*316,835*	*319,356*	*320,584*	*321,250*	321,612
1997	*435,782*	*76.2%*	*240,626*	*279,725*	*301,100*	*314,722*	*322,459*	*326,702*	*329,301*	*330,567*	*331,255*	331,627
1998	*445,319*	*76.8%*	*247,828*	*288,097*	*310,112*	*324,142*	*332,110*	*336,480*	*339,157*	*340,461*	*341,169*	341,553
1999	*479,330*	*73.5%*	*255,294*	*296,776*	*319,454*	*333,907*	*342,115*	*346,616*	*349,374*	*350,717*	*351,446*	351,842
2000	*482,332*	*75.2%*	*262,834*	*305,542*	*328,889*	*343,769*	*352,220*	*356,854*	*359,694*	*361,076*	*361,827*	362,234
2001	*508,950*	*73.4%*	*270,701*	*314,687*	*338,733*	*354,058*	*362,761*	*367,534*	*370,459*	*371,883*	*372,656*	373,076
2002	*499,443*	*77.0%*	*278,673*	*323,955*	*348,709*	*364,485*	*373,445*	*378,359*	*381,369*	*382,835*	*383,632*	384,063
2003	*552,073*	*71.8%*	*287,236*	*333,909*	*359,424*	*375,685*	*384,920*	*389,985*	*393,088*	*394,599*	*395,420*	395,865

Table C.5 – "Low" Incurred All-Prior Projection (Incremental)

	12	24	36	48	60	72	84	96	108	120	132	144
1994												174
1995											331	179
1996										629	341	185
1997									1,193	648	351	190
1998								2,521	1,227	667	362	196
1999							4,243	2,600	1,266	688	373	202
2000						7,968	4,369	2,678	1,304	708	384	208
2001					14,453	8,208	4,501	2,758	1,343	730	396	214
2002				23,347	14,880	8,451	4,634	2,840	1,382	751	407	221
2003			43,986	24,046	15,325	8,704	4,773	2,925	1,424	774	419	227
Totals:	(144+)	(36-132)	36	48	60	72	84	96	108	120	132	144
Estimated	1,026	230,023	99,036	56,696	33,627	18,849	10,448	5,845	3,008	1,631	883	*478*
Actual		238,504	105,829	54,304	35,877	19,019	11,343	5,991	3,110	2,071	960	
Differences		(8,481)	(6,793)	2,392	(2,251)	(170)	(894)	(146)	(102)	(439)	(77)	
Cumulative Percent Difference			-3.6%	-1.3%	-5.2%	-4.3%	-7.1%	-6.3%	-10.1%	-17.0%	-8.0%	
Weights			0.25	0.50	1.00	2.00	3.00	4.00	5.00	6.00	7.00	
Weighted Average			-9.4%									

Table C.6 – "Low" Incurred Tail Calibration Summary

		All-Prior Projection					Change in IBNR	
Tail	(*u*)	Total	Cumulative	Weighted		Total		
Years	Ultimate	Difference	Percent	Percent	IBNR	IBNR	All-Prior	Total
1	11	(11,900)	-5.0%	-34.8%	-	539,749		
2	12	(10,267)	-4.3%	-22.7%	227	542,847	227	3,097
3	13	(9,411)	-3.9%	-16.4%	470	544,643	243	1,797
4	14	(8,963)	-3.8%	-13.0%	664	545,678	194	1,035
5	15	(8,729)	-3.7%	-11.3%	802	546,272	138	593
6	16	(8,606)	-3.6%	-10.3%	894	546,610	92	339
7	17	(8,541)	-3.6%	-9.9%	953	546,803	59	192
8	18	(8,508)	-3.6%	-9.6%	990	546,911	37	109
9	19	(8,490)	-3.6%	-9.5%	1,012	546,973	22	61
10	20	(8,481)	-3.6%	-9.4%	1,026	547,007	13	35
11	21	(8,470)	-3.6%	-9.3%	1,041	547,034	15	27
12	22	(8,462)	-3.5%	-9.3%	1,054	547,053	13	19
13	23	(8,456)	-3.5%	-9.2%	1,063	547,066	10	13
14	24	(8,452)	-3.5%	-9.2%	1,070	547,075	7	9
15	25	(8,449)	-3.5%	-9.2%	1,075	547,081	5	6
16	26	(8,447)	-3.5%	-9.2%	1,078	547,084	3	4
17	27	(8,446)	-3.5%	-9.2%	1,080	547,086	2	2
18	28	(8,434)	-3.5%	-9.2%	1,081	547,088	1	1
19	29	(8,421)	-3.5%	-9.2%	1,082	547,088	1	1
20	30	(8,407)	-3.5%	-9.2%	1,082	547,089	0	1

Table C.7 – "Low" Incurred Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Incurred Data
***All-Prior Estimate in Separate Exhibit**

	(7)	(8)	(9)	(10)	(11)	(12)
			(7) x (8)	(11) + (12)	(7) - (1)	(9) - (7)
	Incurred to Date	**Incurred CDF**	**Ultimate**	**Estimated Unpaid**	**Case Reserve**	**Estimated IBNR**
A-P*	552,468	*1.0019*	*553,494*	7,101	6,075	*1,026*
2004	389,928	1.0025	390,883	4,432	3,476	955
2005	440,588	1.0045	442,586	7,944	5,946	1,998
2006	414,696	1.0084	418,178	11,166	7,684	3,482
2007	473,295	1.0164	481,067	23,902	16,130	7,772
2008	422,287	1.0298	434,869	36,252	23,671	12,581
2009	464,718	1.0551	490,328	59,176	33,566	25,610
2010	463,503	1.1028	511,172	111,017	63,349	47,669
2011	398,892	1.1871	473,531	169,080	94,442	74,639
2012	390,758	1.3800	539,250	307,862	159,371	148,491
2013	312,141	1.7137	534,926	429,438	206,653	222,785
				1,167,370	620,362	547,007

Appendix D – Paid Analysis for "High" Case Reserve Data

Table D.1 – "High" Paid Loss Triangle with All-Prior Data

	12	24	36	48	60	72	84	96	108	120	132
A-P		-	694,326	1,233,322	1,605,148	1,798,756	1,911,906	1,969,504	2,002,311	2,019,120	2,028,756
2004	79,078	195,201	376,363	563,604	760,099	854,132	909,879	940,170	953,400	962,203	
2005	55,011	166,607	338,389	508,834	706,763	803,987	853,722	883,714	898,591		
2006	62,645	195,873	369,571	541,058	719,526	811,071	874,968	907,581			
2007	75,825	190,645	413,211	587,344	815,442	914,584	977,881				
2008	81,654	244,999	466,821	694,938	922,414	1,040,208					
2009	81,003	235,834	436,030	702,479	914,456						
2010	100,835	239,091	488,580	732,524							
2011	74,250	228,057	496,043								
2012	91,294	271,729									
2013	99,365										

Table D.2 – "High" Paid Loss Development Factors

	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	Tail
2004	2.468	1.928	1.497	1.349	1.124	1.065	1.033	1.014	1.009	
2005	3.029	2.031	1.504	1.389	1.138	1.062	1.035	1.017		
2006	3.127	1.887	1.464	1.330	1.127	1.079	1.037			
2007	2.514	2.167	1.421	1.388	1.122	1.069				
2008	3.000	1.905	1.489	1.327	1.128					
2009	2.911	1.849	1.611	1.302						
2010	2.371	2.043	1.499							
2011	3.071	2.175								
2012	2.976									
VWA	2.805	1.996	1.499	1.345	1.127	1.069	1.035	1.015	1.009	
5-Yr VWA	2.843	2.021	1.499	1.344	1.127	1.069	1.035	1.015	1.009	
3-Yr VWA	2.774	2.021	1.531	1.336	1.126	1.070	1.035	1.015	1.009	
TF Fitted	2.991	2.013	1.516	1.262	1.134	1.068	1.035	1.018	1.009	1.009
User										
Selected	2.843	2.021	1.499	1.345	1.127	1.069	1.035	1.018	1.009	1.0046
Ultimate	14.959	5.262	2.604	1.737	1.292	1.146	1.072	1.036	1.018	1.0093
% Paid	6.7%	19.0%	38.4%	57.6%	77.4%	87.3%	93.3%	96.5%	98.2%	99.1%
% Unpaid	93.3%	81.0%	61.6%	42.4%	22.6%	12.7%	6.7%	3.5%	1.8%	0.9%

Table D.3 – "High" Paid Tail Factor Calculation

Paid Tail Factor Analysis

			All Prior		
Tail Years:	13	Actual	2,028,756	Decay	0.509
Tail Factor:	1.0093	Estimated	1,885,275	Intercept	3.912
		Error %	-7.1%		

Period	Factor	Dev	Log	Excl	Period	Log	Fitted	Selected	ATA	ATU
1	2.80509	1.80509	0.591		1	0.591	2.990863		2.990863	14.957482
2	1.99552	0.99552	(0.004)		2	(0.004)	2.013295		2.013295	5.001059
3	1.49908	0.49908	(0.695)		3	(0.695)	1.515739		1.515739	2.484017
4	1.34473	0.34473	(1.065)		4	(1.065)	1.262497		1.262497	1.638816
5	1.12735	0.12735	(2.061)		5	(2.061)	1.133604		1.133604	1.298075
6	1.06876	0.06876	(2.677)		6	(2.677)	1.068001		1.068001	1.145086
7	1.03521	0.03521	(3.347)		7	(3.347)	1.034611		1.034611	1.072178
8	1.01541	0.01541	(4.173)		8	(4.173)	1.017616		1.017616	1.036310
9	1.00923	0.00923	(4.685)		9	(4.685)	1.008966		1.008966	1.018371
10							1.004563		1.004563	1.009321
11							1.002323		1.002323	1.004736
12							1.001182		1.001182	1.002408
13							1.000602		1.000602	1.001224
14							1.000306		1.000306	1.000622
15							1.000156		1.000156	1.000316
16							1.000079		1.000079	1.000160
17							1.000040		1.000040	1.000081
18							1.000021		1.000021	1.000040
19							1.000010		1.000010	1.000020
20							1.000005		1.000005	1.000009
21							1.000003		1.000003	1.000004
22							1.000001		1.000001	1.000001

Table D.4 – "High" Paid All-Prior Projection (Cumulative)

	Premium	Loss Ratio	24	36	48	60	72	84	96	108	120	132
1994	669,311	83.4%	*106,085*	*214,352*	*321,331*	*432,104*	*487,131*	*520,628*	*538,647*	*548,135*	*553,050*	555,574
1995	715,259	82.0%	*111,464*	*225,223*	*337,626*	*454,017*	*511,834*	*547,029*	*565,962*	*575,932*	*581,096*	583,748
1996	758,317	81.2%	*117,021*	*236,451*	*354,458*	*476,652*	*537,352*	*574,302*	*594,178*	*604,645*	*610,067*	612,851
1997	811,833	79.6%	*122,811*	*248,150*	*371,996*	*500,235*	*563,938*	*602,716*	*623,576*	*634,561*	*640,251*	643,172
1998	853,244	79.5%	*128,914*	*260,480*	*390,480*	*525,092*	*591,960*	*632,665*	*654,562*	*666,092*	*672,064*	675,131
1999	890,376	80.0%	*135,370*	*273,526*	*410,036*	*551,390*	*621,607*	*664,350*	*687,344*	*699,452*	*705,723*	708,943
2000	986,176	75.9%	*142,251*	*287,429*	*430,878*	*579,417*	*653,203*	*698,119*	*722,281*	*735,005*	*741,595*	744,979
2001	984,188	79.8%	*149,259*	*301,589*	*452,105*	*607,961*	*685,383*	*732,511*	*757,864*	*771,214*	*778,129*	781,680
2002	984,698	83.8%	*156,821*	*316,870*	*475,013*	*638,766*	*720,110*	*769,627*	*796,264*	*810,291*	*817,556*	821,287
2003	1,041,477	83.2%	*164,676*	*332,742*	*498,806*	*670,761*	*756,180*	*808,177*	*836,148*	*850,878*	*858,507*	862,424

	Growth	Loss Ratio
Prior to 1993	*5.0%*	*80.0%*

Table D.5 – "High" Paid All-Prior Projection (Incremental)

	12	24	36	48	60	72	84	96	108	120	132	144
1994												1,290
1995											2,652	1,356
1996										5,421	2,784	1,423
1997									10,985	5,689	2,922	1,494
1998								21,897	11,531	5,972	3,067	1,568
1999							42,743	22,994	12,108	6,271	3,221	1,647
2000						73,787	44,916	24,162	12,724	6,590	3,384	1,730
2001					155,856	77,422	47,128	25,353	13,350	6,915	3,551	1,816
2002				158,142	163,753	81,344	49,516	26,637	14,027	7,265	3,731	1,908
2003			168,066	166,064	171,956	85,419	51,997	27,971	14,729	7,629	3,918	2,003
Totals:	**(144+)**	**(36-132)**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**
Estimated	8,024	1,885,275	642,062	497,792	348,365	185,260	104,849	55,504	28,914	14,896	7,632	*3,901*
Actual		2,028,756	694,326	538,996	371,826	193,608	113,149	57,599	32,807	16,809	9,635	
Differences		(143,480)	(52,263)	(41,204)	(23,461)	(8,349)	(8,300)	(2,094)	(3,892)	(1,913)	(2,003)	
Cumulative Percent Difference			-7.1%	-6.8%	-6.3%	-6.3%	-7.9%	-8.5%	-13.2%	-14.8%	-20.8%	
Weights			0.25	0.50	1.00	2.00	3.00	4.00	5.00	6.00	7.00	
Weighted Average			-13.3%									

Table D.6 – "High" Paid Tail Calibration Summary

		All-Prior Projection					Change in IBNR	
Tail	**(*u*)**	**Total**	**Cumulative**	**Weighted**		**Total**		
Years	**Ultimate**	**Difference**	**Percent**	**Percent**	**IBNR**	**IBNR**	**All-Prior**	**Total**
1	11	(162,400)	-8.0%	-34.1%	(13,009)	514,079		
2	12	(152,459)	-7.5%	-23.4%	(11,001)	543,201	2,008	29,122
3	13	(147,751)	-7.3%	-18.2%	(9,006)	559,028	1,995	15,828
4	14	(145,524)	-7.2%	-15.7%	(7,519)	567,564	1,487	8,536
5	15	(144,473)	-7.1%	-14.5%	(6,533)	572,140	986	4,576
6	16	(143,977)	-7.1%	-13.9%	(5,920)	574,580	613	2,440
7	17	(143,744)	-7.1%	-13.6%	(5,554)	575,876	366	1,296
8	18	(143,634)	-7.1%	-13.5%	(5,342)	576,562	212	686
9	19	(143,583)	-7.1%	-13.4%	(5,222)	576,923	121	362
10	20	(143,559)	-7.1%	-13.4%	(5,154)	577,114	68	190
11	21	(143,526)	-7.1%	-13.3%	(5,082)	577,248	72	134
12	22	(143,499)	-7.1%	-13.3%	(5,025)	577,337	57	89
13	23	(143,480)	-7.1%	-13.3%	(4,985)	577,393	40	56
14	24	(143,468)	-7.1%	-13.3%	(4,959)	577,428	26	35
15	25	(143,461)	-7.1%	-13.3%	(4,942)	577,449	17	21
16	26	(143,457)	-7.1%	-13.3%	(4,932)	577,461	10	12
17	27	(143,454)	-7.1%	-13.3%	(4,926)	577,468	6	7
18	28	(143,453)	-7.1%	-13.3%	(4,922)	577,472	4	4
19	29	(143,452)	-7.1%	-13.3%	(4,920)	577,475	2	2
20	30	(143,452)	-7.1%	-13.3%	(4,919)	577,476	1	1

Table D.7 – "High" Paid Chain Ladder Summary, with All-Prior

Estimate of Total Unpaid Claims Using Paid Data

***All-Prior Estimate in Separate Exhibit**

	(1)	(2)	(3)	(4)	(5)	(6)
			(1) x (2)	(3) - (1)	(7) - (1)	(4) - (5)
	Paid to Date	**Paid CDF**	**Ultimate**	**Estimated Unpaid**	**Case Reserve**	**Estimated IBNR**
A-P*	2,028,756	*1.0040*	*2,036,779*	*8,024*	13,009	(4,985)
2004	962,203	1.0093	971,173	8,969	11,874	(2,904)
2005	898,591	1.0184	915,098	16,508	21,878	(5,370)
2006	907,581	1.0363	940,536	32,955	42,994	(10,040)
2007	977,881	1.0722	1,048,462	70,581	83,430	(12,849)
2008	1,040,208	1.1459	1,191,977	151,769	140,745	11,025
2009	914,456	1.2918	1,181,321	266,865	257,107	9,758
2010	732,524	1.7372	1,272,516	539,993	528,128	11,865
2011	496,043	2.6041	1,291,769	795,726	696,830	98,896
2012	271,729	5.2619	1,429,810	1,158,081	933,516	224,565
2013	99,365	14.9591	1,486,405	1,387,040	1,129,608	257,432
				4,436,510	3,859,117	577,393

Appendix E – Incurred Analysis for "High" Case Reserve Data

Table E.1 – "High" Incurred Loss Triangle with All-Prior Data

	12	24	36	48	60	72	84	96	108	120	132
A-P		1,874,645	1,989,030	2,049,323	2,067,607	2,056,452	2,052,137	2,046,479	2,044,469	2,042,713	2,041,764
2004	770,485	871,259	892,079	959,581	981,362	979,974	979,594	975,287	974,890	974,077	
2005	755,139	837,212	871,723	909,541	920,876	927,887	924,599	921,732	920,468		
2006	778,857	837,074	908,267	945,531	951,361	950,469	952,152	950,576			
2007	835,631	969,389	991,007	1,048,260	1,058,442	1,062,825	1,061,310				
2008	980,023	1,039,677	1,099,087	1,178,784	1,185,561	1,180,953					
2009	958,889	1,052,715	1,105,673	1,164,752	1,171,563						
2010	1,007,229	1,087,877	1,213,688	1,260,651							
2011	974,991	1,102,902	1,192,873								
2012	1,091,849	1,205,245									
2013	1,228,972										

Table E.2 – "High" Incurred Loss Development Factors

	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	Tail
2004	1.131	1.024	1.076	1.023	0.999	1.000	0.996	1.000	0.999	
2005	1.109	1.041	1.043	1.012	1.008	0.996	0.997	0.999		
2006	1.075	1.085	1.041	1.006	0.999	1.002	0.998			
2007	1.160	1.022	1.058	1.010	1.004	0.999				
2008	1.061	1.057	1.073	1.006	0.996					
2009	1.098	1.050	1.053	1.006						
2010	1.080	1.116	1.039							
2011	1.131	1.082								
2012	1.104									
VWA	1.104	1.061	1.054	1.010	1.001	0.999	0.997	0.999	0.999	
5-Yr VWA	1.095	1.067	1.053	1.008	1.001	0.999	0.997	0.999	0.999	
3-Yr VWA	1.105	1.083	1.054	1.007	1.000	0.999	0.997	0.999	0.999	
TF Fitted	1.192	1.062	1.020	1.006	1.002	1.001	1.000	1.000	1.000	0.999
User										
Selected	1.105	1.083	1.054	1.010	1.001	0.999	0.997	0.999	0.999	0.9995
Ultimate	1.267	1.147	1.059	1.004	0.994	0.993	0.994	0.997	0.998	0.9989
% Reported	0.789	0.872	0.944	0.996	1.006	1.007	1.006	1.003	1.002	1.001
% Unrptd	0.211	0.128	0.056	0.004	(0.006)	(0.007)	(0.006)	(0.003)	(0.002)	(0.001)

Table E.3 – "High" Incurred Tail Factor Calculation

Incurred Tail Factor Analysis

All Prior

Tail Years:	8	Actual	167,119	Decay	0.322
Tail Factor:	0.9989	Estimated	130,156	Intercept	0.597
		Error %	-22.1%		

Period	Factor	Dev	Log	Excl	Period	Log	Fitted	Selected	ATA	Ultimate
1	1.10429	0.10429	(2.261)		1	(2.261)	1.191966		1.191966	1.301527
2	1.06108	0.06108	(2.796)		2	(2.796)	1.061760		1.061760	1.091916
3	1.05445	0.05445	(2.911)		3	(2.911)	1.019870		1.019870	1.028402
4	1.01011	0.01011	(4.595)		4	(4.595)	1.006393		1.006393	1.008366
5	1.00088	0.00088	(7.031)		5	(7.031)	1.002057		1.002057	1.001961
6	0.99911	(0.00089)	7.022	Y			1.000662		1.000662	0.999905
7	0.99694	(0.00306)	5.788	Y			1.000213		1.000213	0.999243
8	0.99912	(0.00088)	7.041	Y			1.000068		1.000068	0.999031
9	0.99917	(0.00083)	7.089	Y			1.000022		1.000022	0.998962
10							1.000007	0.999460	0.999460	0.998940
11							1.000002	0.999780	0.999780	0.999480
12							1.000001	0.999870	0.999870	0.999700
13							1.000000	0.999910	0.999910	0.999830
14							1.000000	0.999960	0.999960	0.999920
15							1.000000	0.999980	0.999980	0.999960
16							1.000000	0.999990	0.999990	0.999980
17							1.000000	0.999990	0.999990	0.999990

Table E.4 – "High" Incurred All-Prior Projection (Cumulative)

	Premium	Loss Ratio	24	36	48	60	72	84	96	108	120	132
1994	*669,311*	*83.4%*	*486,823*	*527,159*	*555,862*	*561,479*	*561,975*	*561,474*	*559,754*	*559,264*	*558,797*	558,496
1995	*715,259*	*82.0%*	*511,511*	*553,892*	*584,051*	*589,953*	*590,474*	*589,947*	*588,140*	*587,625*	*587,135*	586,817
1996	*758,317*	*81.2%*	*537,012*	*581,507*	*613,169*	*619,365*	*619,912*	*619,359*	*617,462*	*616,921*	*616,406*	616,073
1997	*811,833*	*79.6%*	*563,582*	*610,278*	*643,506*	*650,009*	*650,583*	*650,003*	*648,012*	*647,444*	*646,904*	646,555
1998	*853,244*	*79.5%*	*591,586*	*640,602*	*675,482*	*682,308*	*682,911*	*682,301*	*680,211*	*679,615*	*679,049*	678,682
1999	*890,376*	*80.0%*	*621,214*	*672,685*	*709,311*	*716,479*	*717,112*	*716,472*	*714,277*	*713,652*	*713,057*	712,672
2000	*986,176*	*75.9%*	*652,790*	*706,878*	*745,366*	*752,898*	*753,563*	*752,891*	*750,584*	*749,927*	*749,302*	748,897
2001	*984,188*	*79.8%*	*684,950*	*741,702*	*782,086*	*789,989*	*790,687*	*789,981*	*787,561*	*786,872*	*786,215*	785,791
2002	*984,698*	*83.8%*	*719,655*	*779,283*	*821,713*	*830,017*	*830,750*	*830,009*	*827,466*	*826,742*	*826,052*	825,606
2003	*1,041,477*	*83.2%*	*755,702*	*818,316*	*862,872*	*871,592*	*872,362*	*871,583*	*868,913*	*868,152*	*867,428*	866,960

Table E.5 – "High" Incurred All-Prior Projection (Incremental)

	12	24	36	48	60	72	84	96	108	120	132	144
1994												(111)
1995											(288)	(117)
1996										(467)	(302)	(123)
1997									(515)	(490)	(317)	(129)
1998								(1,897)	(541)	(515)	(333)	(136)
1999							(581)	(1,991)	(567)	(540)	(349)	(142)
2000						603	(609)	(2,090)	(596)	(567)	(367)	(149)
2001					7,168	633	(640)	(2,195)	(625)	(595)	(385)	(157)
2002				38,488	7,532	665	(672)	(2,307)	(657)	(626)	(405)	(165)
2003			56,752	40,384	7,903	698	(706)	(2,420)	(690)	(656)	(425)	(173)
Totals:	**(144+)**	**(36-132)**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**
Estimated	(853)	130,156	99,014	44,355	4,165	(3,926)	(4,857)	(4,357)	(2,034)	(1,411)	(793)	*(386)*
Actual		167,119	114,384	60,293	18,285	(11,156)	(4,315)	(5,658)	(2,010)	(1,756)	(949)	
Differences		(36,963)	(15,370)	(15,938)	(14,120)	7,229	(542)	1,300	(24)	344	156	
Cumulative Percent Difference			-22.1%	-40.9%	-74.8%	32.8%	8.4%	17.1%	10.1%	18.5%	16.5%	
Weights			0.25	0.50	1.00	2.00	3.00	4.00	5.00	6.00	7.00	
Weighted Average			11.7%									

Table E.6 – "High" Incurred Tail Calibration Summary

		All-Prior Projection					Change in IBNR	
Tail Years	**(*u*) Ultimate**	**Total Difference**	**Cumulative Percent**	**Weighted Percent**	**IBNR**	**Total IBNR**	**All-Prior**	**Total**
1	11	(34,285)	-20.5%	35.3%	-	559,823		
2	12	(35,485)	-21.2%	24.7%	(173)	557,075	(173)	(2,748)
3	13	(36,159)	-21.6%	18.8%	(372)	555,354	(199)	(1,721)
4	14	(36,603)	-21.9%	14.9%	(574)	554,098	(202)	(1,255)
5	15	(36,791)	-22.0%	13.2%	(691)	553,513	(117)	(585)
6	16	(36,880)	-22.1%	12.4%	(763)	553,208	(71)	(305)
7	17	(36,923)	-22.1%	12.0%	(805)	553,049	(42)	(159)
8	18	(36,963)	-22.1%	11.7%	(853)	552,884	(48)	(165)
9	19	(36,963)	-22.1%	11.7%	(853)	552,884	0	0
10	20	(36,963)	-22.1%	11.7%	(853)	552,884	0	0
11	21	(36,963)	-22.1%	11.7%	(853)	552,884	0	0
12	22	(36,963)	-22.1%	11.7%	(853)	552,884	0	0
13	23	(36,963)	-22.1%	11.7%	(853)	552,884	0	0

Table E.7 – "High" Incurred Chain Ladder Summary, with All-Prior

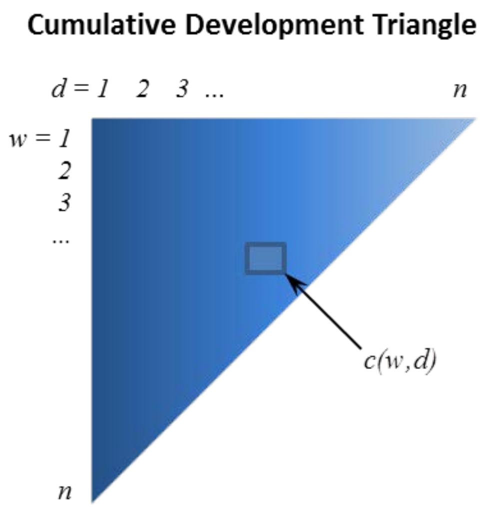
Estimate of Total Unpaid Claims Using Incurred Data
***All-Prior Estimate in Separate Exhibit**

	(7) Incurred to Date	(8) Incurred CDF	(9) (7) x (8) Ultimate	(10) (11) + (12) Estimated Unpaid	(11) (7) - (1) Case Reserve	(12) (9) - (7) Estimated IBNR
A-P*	2,041,764	*0.9996*	*2,040,912*	12,156	13,009	*(853)*
2004	974,077	0.9989	973,045	10,841	11,874	(1,032)
2005	920,468	0.9981	918,726	20,135	21,878	(1,742)
2006	950,576	0.9972	947,946	40,364	42,994	(2,630)
2007	1,061,310	0.9942	1,055,132	77,251	83,430	(6,179)
2008	1,180,953	0.9933	1,173,030	132,822	140,745	(7,923)
2009	1,171,563	0.9942	1,164,732	250,275	257,107	(6,832)
2010	1,260,651	1.0042	1,265,965	533,442	528,128	5,314
2011	1,192,873	1.0589	1,263,124	767,081	696,830	70,252
2012	1,205,245	1.1466	1,381,967	1,110,238	933,516	176,722
2013	1,228,972	1.2667	1,556,760	1,457,395	1,129,608	327,787
				4,412,001	3,859,117	552,884

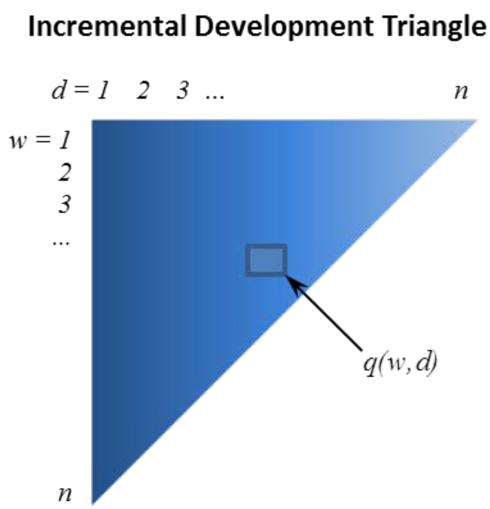
Appendix F – Graphical Representation of Notation

The paper uses the following notation for certain important loss statistics which is also represented graphically:

$c(w,d)$: cumulative loss from accident period w as of age d. Think "when" and "delay."

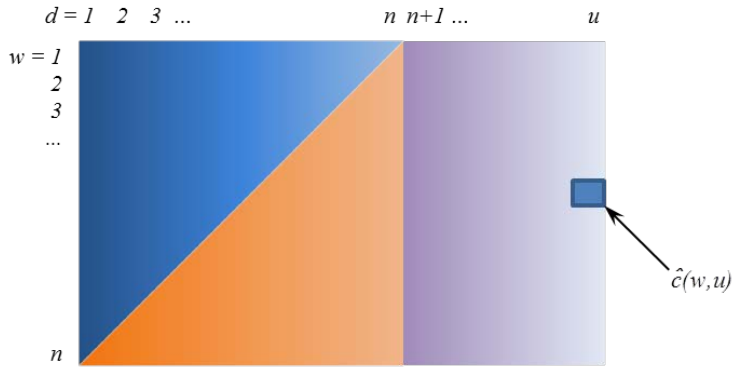


$q(w,d)$: incremental loss for accident period w during the development age from d - 1 to d. Note that $q(w,d) = c(w,d) - c(w,d-1)$.



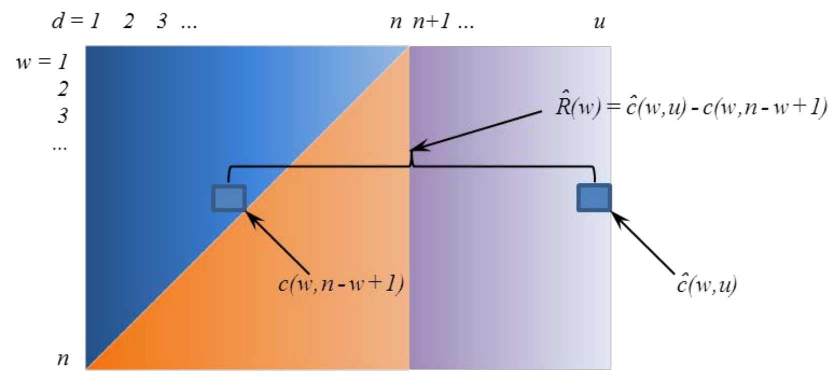
$c(w,u) = U(w)$: total loss from accident period w when at the end of ultimate development u.

Cumulative Development Triangle, estimated to Ultimate

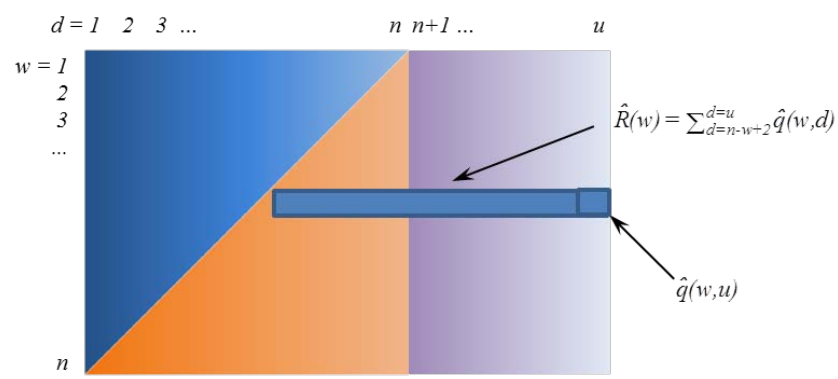


$R(w)$: future development after age $d = n - w + 1$ for accident period w, *i.e.*, $= U(w) - c(w, n - w + 1)$.

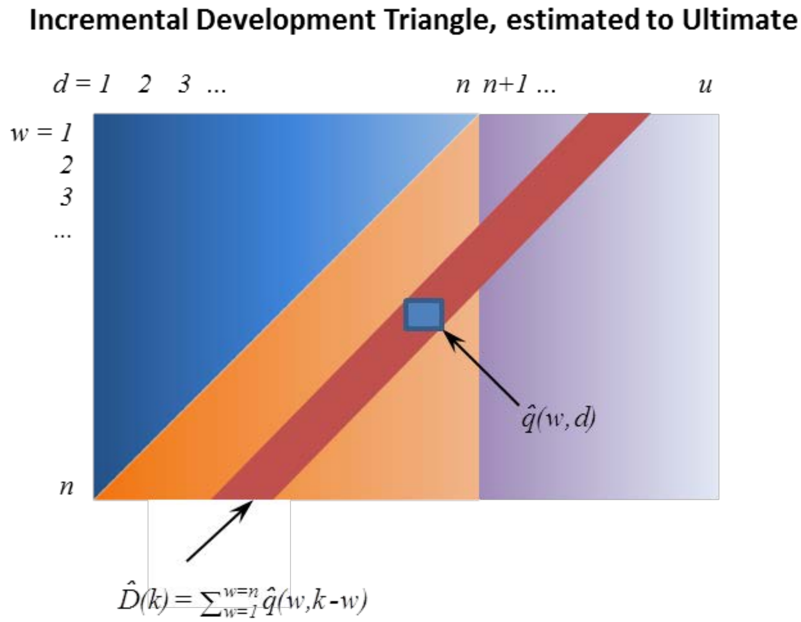
Cumulative Development Triangle, estimated to Ultimate



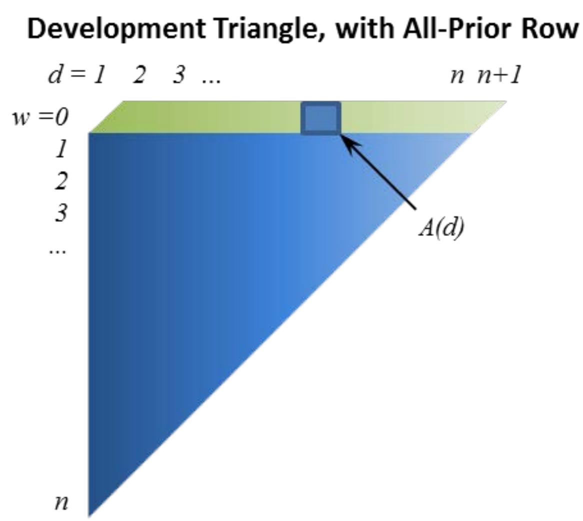
Incremental Development Triangle, estimated to Ultimate



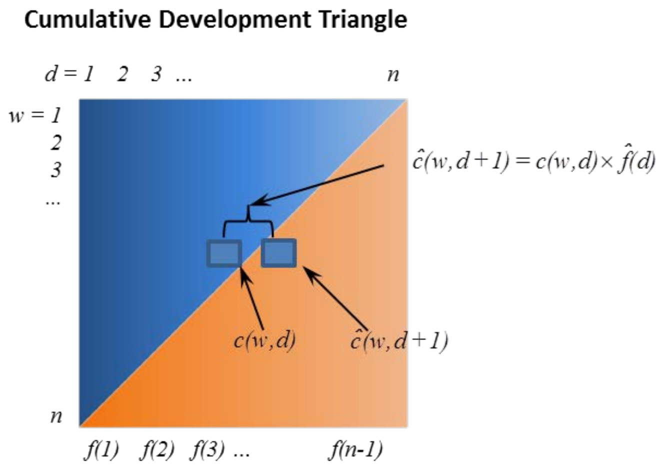
$D(k)$: future development after age $d = n - w + 1$ during calendar period k, *i.e.*, for all $q(w,d)$ where $w + d = k$ and $w + d > n + 1$.



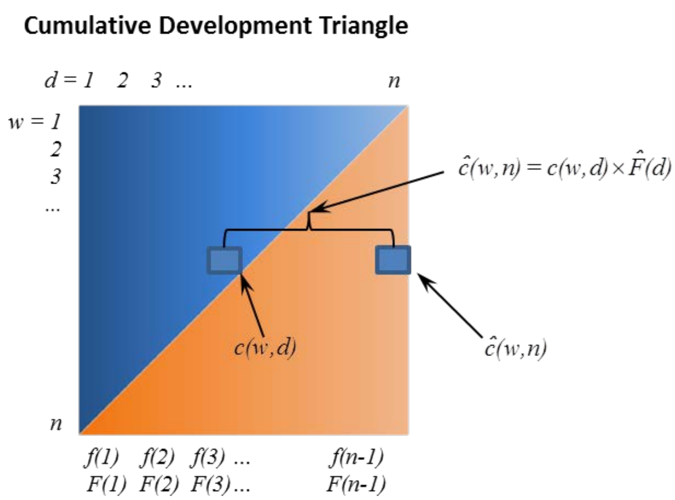
$A(d)$: all-prior data by development age d.



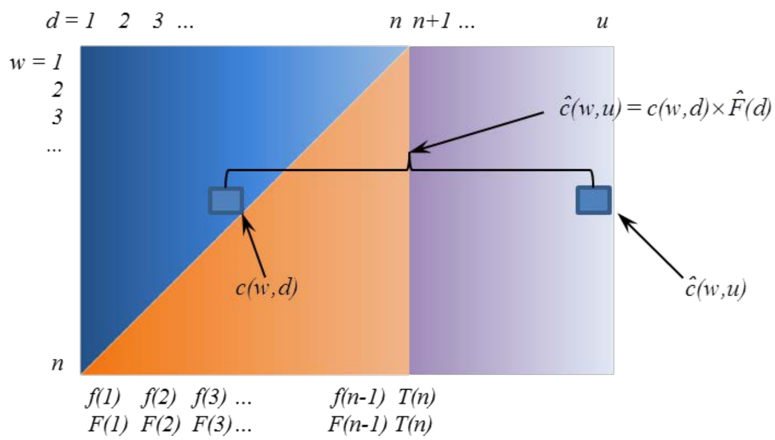
$f(d) = 1 + v(d)$: factor applied to $c(w,d)$ to estimate $c(w,d+1)$ or more generally any factor relating to age d. This is commonly referred to as a link ratio. $v(d)$ is referred to as the 'development portion' of the link ratio, which is used to estimate $q(w,d+1)$. The other portion, the number one, is referred to as the 'unity portion' of the link ratio.



$F(d)$: ultimate development factor relating to development age d. The factor applied to $c(w,d)$ to estimate $c(w,u)$ or more generally any cumulative development factor relating to development age d. The capital indicates that the factor produces the ultimate loss level. As with link ratios, $V(d)$ denotes the 'development portion' of the loss development factor, the number one is the 'unity portion' of the loss development factor.

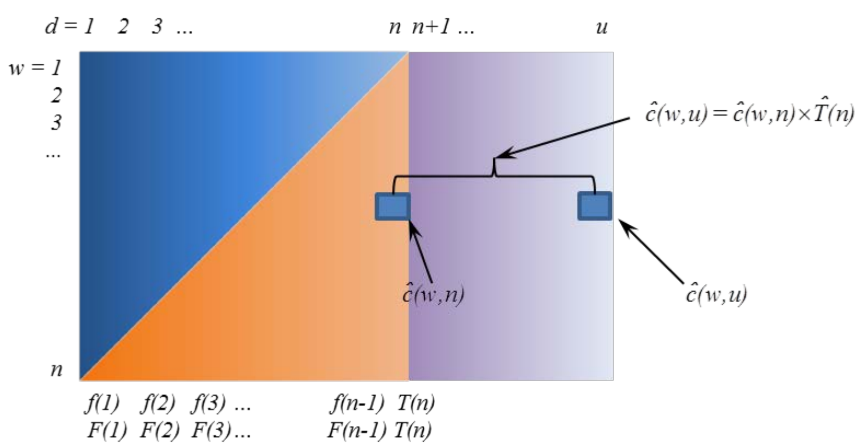


Cumulative Development Triangle, with Tail Factor



$T = T(n)$: ultimate tail factor at end of triangle data, which is applied to the estimated $c(w,n)$ to estimate $c(w,u)$.

Cumulative Development Triangle, with Tail Factor



$\hat{x}$ an estimate of any value or parameter x.

Credibility in Loss Reserving

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Abstract

This article proposes using credibility theory in the context of stochastic claims reserving. We consider the situation where an insurer has access to the claims experience of its peer competitors and has the potential to improve prediction of outstanding liabilities by incorporating information from other insurers. Based on the framework of Bayesian linear models, we show that the development factor in the classical chain-ladder setting has a credibility expression, i.e. a weighted average of the prior mean and the best estimate from the data. In the empirical analysis, we examine loss triangles for the line of commercial auto insurance from a portfolio of insurers in the US. We employ hierarchical model for the specification of prior and show that prediction could be improved through borrowing strength among insurers based on a hold-out sample validation.

Keywords: Bayesian Modeling, Chain-ladder method, Hierarchical model

1 Introduction

General insurance (also known as property-casualty insurance in the U.S. and non-life insurance in other countries) protects a person or business against the losses to its physical property or legal liability through injury property damage. General insurance is a stable cornerstone of and makes significant contributions to any developed economy. In 2011, as the largest insurance market in the world the U.S. underwrote over $0.66 trillion U.S. dollars of premium in property and casualty insurance, which account for about 4.45% of the nation's GDP (*International Insurance Fact Book 2013*). Because of its critical role in the economy, the general insurance industry is usually highly regulated to monitor and ensure its financial health. For example, each insurer is required to provide sufficient technical provisions, also known as loss reserves, to support its potential outstanding liabilities.

Loss reserves represent the best estimate of an insurer's outstanding loss payments. In general insurance potential reporting lags, the settlement process, and potentially reopened claims can all lengthen the time to close a claim. For the purposes of valuation and financial reporting, the insurer predicts the ultimate payment amount for all the claims arising from past exposures. This includes estimates of both incurred but not reported and reported but not settled claims. The loss reserve is then built up based on the best estimate and updated at each valuation.

There is an extensive literature on the prediction of outstanding losses and quantification of associated predictive uncertainty. See, for example, Taylor (2000) and Wüthrich and Merz (2008) for comprehensive reviews. One approach worth mentioning is the chain-ladder method which is the current industry benchmark and is also the building block of the hierarchical model employed in this study. Think of a run-off triangle of cumulative payments, where aggregated paid losses are arranged in a triangular fashion to reflect the occurrence and development over years. The chain-ladder algorithm uses year-to-year development factors to project cumulative payments for each accident year. This simple algorithm is further justified by a variety of statistical models which also provide the foundation to quantify reserving variability. Several commonly used variations include the Mack chain ladder (Mack (1993, 1999)), the Munich chain ladder (Quarg and Mack (2008)), and bootstrap chain ladder (England and Verrall (2002)). Additionally, the chain-ladder model can be easily implemented in the statistical package R (see Sturtz et al. (2005)).

Incorporating the experience of loss payment from peers could add value to the prediction of an insurer's own liabilities. First, an insurer's own claim experience might not be reliable, especially for small insurers. In this case, the insurer might want to give less credibility to its own experience but more to the industry-level information. Second, an insurer could borrow strength in the prediction of its own outstanding claims by combining experiences from other companies that share similar claim payment patterns, often in the same line of business. Third, the claim experience of all insurers are influenced by certain common factors whether macroeconomic or due to a change in regulation, pooling experience from multiple insurers can better capture and measure such factors.

In this work, we develop a formal structure to incorporate claim information from peer insurers with an insurer's own information for reserving purposes. We focus on the chain-ladder approach and using the theory of Bayesian linear models, we show that the development factor in the claim-ladder method has a credibility expression, i.e. a weighted average of prior knowledge and an estimate from data. Furthermore, through hierarchical models we explore the impact of prior specification. The Bayesian approach is a natural choice to blend collateral information with an insurer's own claim experience. Additionally, Bayesian models naturally incorporate parameter uncertainty in the prediction. Bayesian methods have a long history in the loss reserving literature, with the earliest efforts traced back to 1990s (see, for example, Jewell (1989, 1990) and Verrall (1990)). Partly because of the development of the Markov chain Monte Carlo (MCMC) techniques, the loss reserving literature has observed an increasing number of applications from the Bayesian perspective. Some recent examples include Antonio and Beirlant (2008), de Alba and Nieto-Barajas (2008), Peters et al. (2009), Meyers (2009), Merz and Wüthrich (2010), Shi et al. (2012), and Zhang and Dukic (2012) among others.

Apart from the above literature, two recent studies incorporate information from multiple insurers for reserving. Zhang et al. (2012) employed a hierarchical growth curve to predict insurers' outstanding liabilities for a single business line. Extending this idea, Shi (2013) proposed a Bayesian copula regression model for determining reserves for dependent lines of business. Different from these studies, we focus on the classical chain-ladder model and derive a credibility estimate. Note that although credibility is widely used in ratemaking, to the best of our knowledge, it has not been studied in reserving. Furthermore, both Zhang et al. (2012) and Shi (2013) focused on prediction for the portfolio of insurers. In contrast, we emphasize the value of external information for individual insurers.

The rest of the article is structured as follows: Section 2 formulates the Bayesian linear model and presents the credibility results in reserving prediction. Section 3 describes the loss triangle data. Section 4 introduces the hierarchical model and proposes alternative choices for the prior specification. Model inferences are discussed as well. Section 5 demonstrates the prediction using the Bayesian model and compares model performance using out-of-sample validation. Section 6 concludes the paper.

2 Model

Credibility is a technique for incorporating relevant outside data and is widely used in ratemaking. Studies on credibility begin with Mowbray (1914) and Whitney (1918). The theoretical foundation for credibility ratemaking is due to Bühlmann (1967) where traditional credibility formulas are derived in a distribution-free setup using a least-squares criterion. The approach was subsequently extended and popularized by a series of studies (see Bühlmann and Gisler (2005) for a comprehensive review). Despite of its long history in ratemaking, credibility is rarely used in reserving even though the goal is prediction as well.

We investigate credibility in loss reserving based on the framework of Bayesian linear models and show the credibility results for the chain-ladder method. Bayesian credibility was introduced by Bailey (1950) and further extended by Mayerson (1964), Miller and Hickman (Miller and Hickman), and Luo et al. (2004) among others. Our study is unique because instead of focusing on a single insurer we show the credibility results for a group of insurers. We argue that an insurer could borrow predictive strength from the claims experience of peer insurers.

Consider N run-off triangles, each from an individual insurer. Assume all triangles are of the same dimension with I accident years and $J(=I)$ development years. Let $C^n_{i,j}$ denote the cumulative paid loss in the ith ($i = 1, \cdots, I$) accident year and the jth ($j = 0, \cdots, I-1$) development lag of the nth ($n = 1, \cdots, N$) insurer. Define $\mathbf{C}^{(n)}_j = \left(C^{(n)}_{1,j}, \cdots, C^{(n)}_{I,j}\right)'$ for $j = 0, \cdots, I-1$ and $n = 1, \cdots, N$. Denote $\mathbf{C}^{(n)}_{U,j} = \left(C^{(n)}_{1,j}, \cdots, C^{(n)}_{I-j,j}\right)'$ and $\mathbf{C}^{(n)}_{L,j} = \left(C^{(n)}_{I-j+1,j}, \cdots, C^{(n)}_{I,j}\right)'$ as the vector of cumulative payment in the upper triangle (realized loss) and lower triangle (outstanding payment), respectively.

For the purposes of brief presentation, we further define

$$\mathbf{Y}_j = \begin{pmatrix} \mathbf{C}_{U,j}^{(1)} \\ \vdots \\ \mathbf{C}_{U,j}^{(N)} \end{pmatrix}, \ \mathbf{X}_{j-1} = \begin{pmatrix} \mathbf{C}_{U,j-1}^{(1)} & & \\ & \ddots & \\ & & \mathbf{C}_{U,j-1}^{(N)} \end{pmatrix} \tag{1}$$

for $j = 1, \cdots, I-1$. To determine reserve, we follow the spirit of classic chain-ladder method and focus on the year-to-year development factors in the triangle. Specifically we examine the following linear model:

$$\mathrm{E}(\mathbf{Y}_j|\boldsymbol{\beta}_j) = \mathbf{X}_{j-1}\boldsymbol{\beta}_j \tag{2}$$

$$\mathrm{Var}(\mathbf{Y}_j|\boldsymbol{\beta}_j) = \mathbf{R}_j \tag{3}$$

where $\boldsymbol{\beta}_j = (\boldsymbol{\beta}_j^{(1)}, \cdots, \boldsymbol{\beta}_j^{(N)})'$ represents the vector of development factors from lag $j-1$ to j, and $\mathbf{R}_j$ denote the (conditional) covariance matrix for the jth development year.

We adopt a Bayesian approach for predicting outstanding payments and quantifying reserve variability. Using a conjugate multivariate normal prior $\boldsymbol{\beta}_j \sim N(\boldsymbol{\mu}_j, \boldsymbol{\Omega}_j)$, we have

$$\begin{pmatrix} \boldsymbol{\beta}_j \\ \mathbf{Y}_j \end{pmatrix} \sim N\left(\begin{pmatrix} \boldsymbol{\mu}_j \\ \mathbf{X}_{j-1}\boldsymbol{\mu}_j \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Omega}_j & \boldsymbol{\Omega}_j\mathbf{X}'_{j-1} \\ \mathbf{X}_{j-1}\boldsymbol{\Omega}_j & \mathbf{R}_j + \mathbf{X}_{j-1}\boldsymbol{\Omega}_j\mathbf{X}'_{j-1} \end{pmatrix} \right) \tag{4}$$

It is straight forward to derive the posterior distribution of $\boldsymbol{\beta}_j$ with

$$\mathrm{E}(\boldsymbol{\beta}_j|\mathbf{Y}_j) = \boldsymbol{\mu}_j + \boldsymbol{\Omega}_j\mathbf{X}'_{j-1}(\mathbf{R}_j + \mathbf{X}_{j-1}\boldsymbol{\Omega}_j\mathbf{X}'_{j-1})^{-1}(\mathbf{Y}_j - \mathbf{X}_{j-1}\boldsymbol{\mu}_j) \tag{5}$$

$$\mathrm{Var}(\boldsymbol{\beta}_j|\mathbf{Y}_j) = \boldsymbol{\Omega}_j - \boldsymbol{\Omega}_j\mathbf{X}'_{j-1}(\mathbf{R}_j + \mathbf{X}_{j-1}\boldsymbol{\Omega}_j\mathbf{X}'_{j-1})^{-1}\mathbf{X}_{j-1}\boldsymbol{\Omega}_j \tag{6}$$

Credibility Result 1: The posterior mean of development factor is a matrix-weighted average of the prior mean and the generalized least squares estimator, i.e. $\mathrm{E}(\boldsymbol{\beta}_j|\mathbf{Y}_j) = (\mathbf{I} - \boldsymbol{\zeta}_\beta)\boldsymbol{\mu}_j + \boldsymbol{\zeta}_\beta\boldsymbol{\beta}_j^{GLS}$, where $\boldsymbol{\zeta}_\beta = (\boldsymbol{\Omega}_j^{-1} + \mathbf{X}'_{j-1}\mathbf{R}_j^{-1}\mathbf{X}_{j-1})^{-1}\mathbf{X}'_{j-1}\mathbf{R}_j^{-1}\mathbf{X}_{j-1}$ and $\boldsymbol{\beta}_j^{GLS} = (\mathbf{X}'_{j-1}\mathbf{R}_j^{-1}\mathbf{X}_{j-1})^{-1}\mathbf{X}'_{j-1}\mathbf{R}_j^{-1}\mathbf{Y}_j$.

Proof.

$$
\begin{aligned}
&\boldsymbol{\Omega}_j\mathbf{X}_{j-1}'(\mathbf{R}_j+\mathbf{X}_{j-1}\boldsymbol{\Omega}_j\mathbf{X}_{j-1}')^{-1}\mathbf{Y}_j\\
=&\{\boldsymbol{\Omega}_j\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}-\boldsymbol{\Omega}_j\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{X}_{j-1}(\boldsymbol{\Omega}_j^{-1}+\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{X}_{j-1})^{-1}\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\}\mathbf{Y}_j\\
=&\boldsymbol{\Omega}_j\{\mathbf{I}-\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{X}_{j-1}(\boldsymbol{\Omega}_j^{-1}+\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{X}_{j-1})^{-1}\}\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{Y}_j\\
=&\boldsymbol{\Omega}_j\{\boldsymbol{\Omega}_j^{-1}\}(\boldsymbol{\Omega}_j^{-1}+\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{X}_{j-1})^{-1}\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{Y}_j\\
=&\boldsymbol{\zeta}_\beta\boldsymbol{\beta}_j^{GLS}\\
&\boldsymbol{\mu}_j-\boldsymbol{\Omega}_j\mathbf{X}_{j-1}'(\mathbf{R}_j+\mathbf{X}_{j-1}\boldsymbol{\Omega}_j\mathbf{X}_{j-1}')^{-1}\mathbf{X}_{j-1}\boldsymbol{\mu}_j\\
=&\boldsymbol{\mu}_j-(\boldsymbol{\Omega}_j^{-1}+\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{X}_{j-1})^{-1}\mathbf{X}_{j-1}'\mathbf{R}_j^{-1}\mathbf{X}_{j-1}\boldsymbol{\mu}_j\\
=&(\mathbf{I}-\boldsymbol{\zeta}_\beta)\boldsymbol{\mu}_j
\end{aligned}
$$

It is straightforward to see that when $\boldsymbol{\Omega}_j\to\infty$ or $\mathbf{0}$, $\boldsymbol{\zeta}_\beta\to\mathbf{I}$ or $\mathbf{0}$, respectively. That is, if one knows the true value of development factors, then zero credibility is given to the data. Otherwise, if one has no prior information on the development factors, full credibility is given to the data.

Credibility Result 2: The predictive mean of cumulative payment in each lower triangle is a weighted average of the prior mean and the best prediction, i.e. $\mathrm{E}(\mathbf{C}_{L,j-1}^{(n)}\boldsymbol{\beta}_j|\mathbf{Y}_j)=(1-\zeta_\beta^{(n)})\mathbf{C}_{L,j-1}^{(n)}\mu_j^{(n)}+\zeta_\beta^{(n)}\mathbf{C}_{L,j-1}^{(n)}\beta_j^{(n)GLS}$, iff

$$
\mathbf{R}_j=\begin{pmatrix}\mathbf{R}_j^{(1)}&&\\&\ddots&\\&&\mathbf{R}_j^{(N)}\end{pmatrix}\text{ and }\boldsymbol{\Omega}_j=\begin{pmatrix}\left(\omega_j^{(1)}\right)^2&&\\&\ddots&\\&&\left(\omega_j^{(N)}\right)^2\end{pmatrix}\tag{7}
$$

where $\zeta_\beta^{(n)}=\frac{\mathbf{C}_{U,j-1}^{(n)\prime}\left(\mathbf{R}_j^{(n)}\right)^{-1}\mathbf{C}_{U,j-1}^{(n)}}{\left(\omega_j^{(n)}\right)^{-2}+\mathbf{C}_{U,j-1}^{(n)\prime}\left(\mathbf{R}_j^{(n)}\right)^{-1}\mathbf{C}_{U,j-1}^{(n)}}$, $\beta_j^{(n)GLS}=\frac{\mathbf{C}_{U,j-1}^{(n)\prime}\left(\mathbf{R}_j^{(n)}\right)^{-1}\mathbf{C}_{U,j}^{(n)}}{\mathbf{C}_{U,j-1}^{(n)\prime}\left(\mathbf{R}_j^{(n)}\right)^{-1}\mathbf{C}_{U,j-1}^{(n)}}$, and $\mu_j^{(n)}$ is the nth element in $\boldsymbol{\mu}_j$. Furthermore, if $\mathbf{R}_j^{(n)}=diag\left(\left(\sigma_j^{(n)}C_{1,j-1}^{(n)}\right)^2,\cdots,\left(\sigma_j^{(n)}C_{I-j,j-1}^{(n)}\right)^2\right)$, the predictive mean of the outstanding payment is a weighted average of the prior mean and the chain-ladder prediction.

Proof. The first part of the result follows from conditional assumption among triangles. The second part of the result is due to $\beta_j^{(n)GLS}=\sum_{i=1}^{I-j}C_{i,j}^{(n)}/\sum_{i=1}^{I-j}C_{i,j-1}^{(n)}$ $(n=1,\cdots,N)$, which is the chain-ladder development factor.

Note that the above results could be derived for each individual triangle. We emphasize that by pooling triangles from multiple insurers we allow an insurer to blend its own claim experience with its peers. The information sharing could be achieved by allowing triangles from different insurers to be correlated with each other. Explicitly, that correlation could be introduced through the sampling distribution. In this study, we focus on an implicit strategy, a hierarchical prior specification (see Section 4). The hierarchical model is more natural and intuitive for this application, allowing an insurer to adjust its priors based on the information borrowed from other insurers.

3 Data

In the empirical analysis, we consider run-off triangles of commercial automobile insurance from a group of property-casualty insurers in the US. The data are from Schedule P of the National Association of Insurance Commissioners (NAIC) database. The triangles are available in terms of both incurred and paid losses. Our analysis uses the 1997 paid losses. Each triangle contains payments for the claims in ten accident years from 1988 to 1997, and for each accident year up to ten development lags. Table 1 illustrates organization of the data. For example, the first row contains payments for claims which occurred in 1988. Because of the reporting and settlement lags, we observe payments from 1988 through the valuation year, 1997. In contrast, for accident year 1997, we only have one year of payments by the valuation year.

Table 1: Run-off triangle from Schedule P of NAIC

Accident Year	0	1	2	3	4	5	6	7	8	9		
1988	×	×	×	×	×	×	×	×	×	×		
1989	×	×	×	×	×	×	×	×	×		←	1998
1990	×	×	×	×	×	×	×	×			←	1999
1991	×	×	×	×	×	×	×				←	2000
1992	×	×	×	×	×	×					←	2001
1993	×	×	×	×	×						←	2002
1994	×	×	×	×							←	2003
1995	×	×	×								←	2004
1996	×	×									←	2005
1997	×										←	2006

The goal of reserving practice is to identify the payment pattern based on realized paid losses and to predict outstanding future payments. Using the example in Table 1, and assuming that all claims will be settled in ten years, we predict the unpaid losses represented by the cells in the highlighted lower triangle. To validate the model, we use a hold-out sample to evaluate the prediction. In our analysis, we will use the data from 1997 to develop the model and use realizations of future payments in lower triangles to examine the predictive performance of alternative models. The validation data are extracted from the Schedule P in the NAIC database of subsequent years 1998-2006. Specifically, the paid losses of accident year 1989 are from the Schedule P of year 1998, the paid losses of accident year 1990 are from the Schedule P of year 1999, and so on. This process is also demonstrated in Table 1 where the last column indicates the year from which the future payments in lower triangles are gathered.

Schedule P contains firm level run-off triangles of aggregated claims for major business lines of U.S. property-casualty insurers. Examples include personal auto liability, commercial auto liability, worker's compensation, general liability, and medical malpractice. The settlement periods for liability insurance could be lengthy due to late reporting, protracted negotiations, or judicial proceedings. However, the triangle data of Schedule P only contains payments for the most recent ten years. Because of this drawback, we focus on commercial auto liability where, compared with other casualty lines, the loss payments have relatively shorter tails and take fewer years to close.

In our analysis, we examine fifteen insurers with large commercial auto liability books. We expect that insurers could borrow more from peers of similar size. In selecting the group of insurers, we also make sure that there is no major merger and acquisition in this particular line of business over the study period. Specifically, the Schedule P of years 1998-2006 contains paid losses in the upper triangles that are already extracted from the Schedule P of year 1997 as well. We use observations in overlapping years to cross-validate the data quality of the selected insurers. To visualize the data, Figure 1 displays the development of cumulative payments for each insurer by accident year. Each curve connects the paid losses over time corresponding to a single accident year. As anticipated, the curve flattens in later development years. In particular, there is no substantial increase in the payment from the eighth to the ninth development lag for accident year 1988, which supports our assumption that it takes about ten years to close all the claims. Notice that the volume of business written varies over years and there is substantive heterogeneity across insurers.

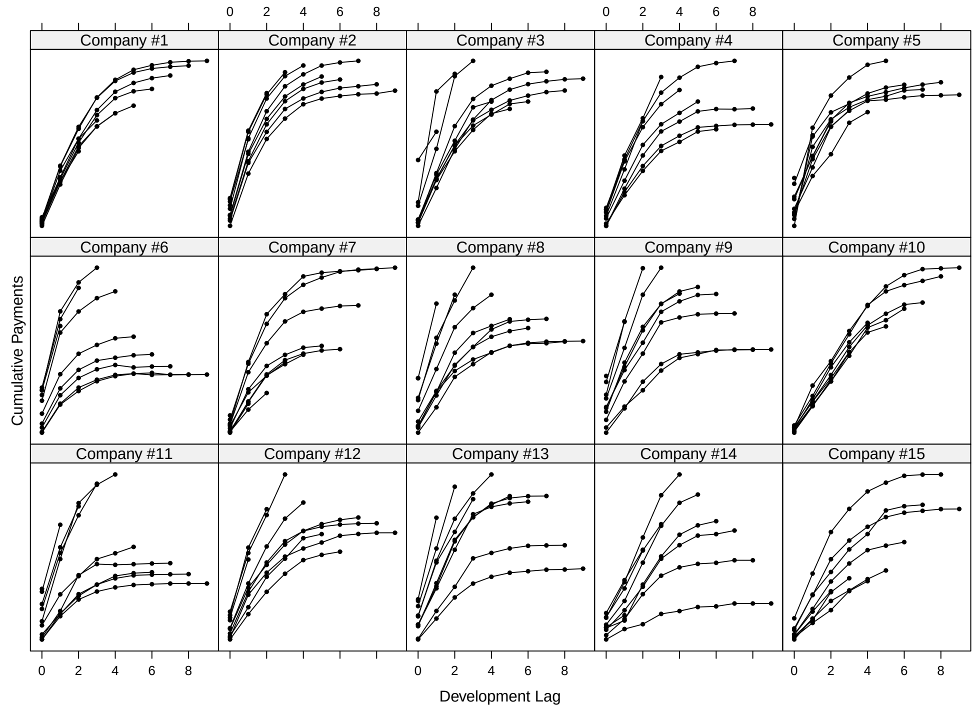

Figure 1: Multiple time series plot of cumulative paid loss

4 Bayesian Inference

In the empirical analysis, we introduce correlation between insurers and thus enable information borrowing simply from the hierarchical specification. Specifically, we start with the model

$$C_{ij}^{(n)} \sim N\left(C_{ij-1}^{(n)}\beta_j^{(n)}, \left(\sigma_j^{(n)}C_{ij-1}^{(n)}\right)^2\right)$$
$$\beta_j^{(n)} \sim N(\mu_j, \theta^2)$$

Here, we assume that the development factors in the jth year, $\beta_j^{(n)}$, have the same prior distribution with mean μ_j. an insurer is expected to incorporate experience of payment development from other insurers into its own experience. The parameter θ^2 is fixed and known. It determines the degrees of shrinkage among multiple insurers in that smaller values will increase the shrinkage and larger values will weaken it. We employ an empirical Bayes estimates for $\sigma_j^{(n)}$ from the classical chain-ladder model. This allows for fair comparison to the chain-ladder prediction and demonstrates the value added by credibility.

There are different ways to specify the prior distribution for hyperparameter μ_j. We discuss two alternatives that are particularly useful in reserving applications. The first and a natural choice is a conjugate prior. We use

$$\mu_j \sim N(a, b^2)$$

where b^2 controls the precision of prior knowledge that one has on μ_j and also create shrinkage in the development factors over years. We use $a = 1$ and impose a diffuse prior $b = +\infty$ assuming that an insurer has no prior knowledge on the development factor and the only way to gather information is to learn through its peers. The diffuse prior also guarantees the heterogeneity in development factors over time, which is desirable because we do not expect shrinkage over time though we anticipate shrinkage across insurers.

Alternatively, we know that as payments develop over time the development factors will tend to one. We can think of it as a change point where at some development time k, the claims are settled and all later factors are one. Specifically, the model is written as follows:

$$\mu_j \sim \begin{cases} N(a, b^2) & \text{if } j < k \\ N(1, 0.0001^2) & \text{if } j \geq k \end{cases}$$

$$k \sim DU(1, 10)$$

Here we assume that there are two states for hyperparameter μ_j. The posterior of parameter k determines the time period that it takes to close all claims such that the development factor is essentially one. Assuming no prior knowledge, we use a discrete uniform prior. Note that it is possible that it takes longer to close all claims than the window period of the triangle. In our application, claims might continue to develop after ten years. In this case, k will be 10. In practice, the domain knowledge of the reserving actuaries will determine the priors. Another choice that serves a similar purpose is to think of the prior of μ_j as a mixture of a normal distribution and 1, then the value of the weight for the normal distribution is the posterior probability that the development factor is significantly different from 1 (π_j):

$$\mu_j \sim \begin{cases} N(a, b^2) & \text{with probability } \pi_j \\ N(1, 0.0001^2) & \text{with probability } 1-\pi_j \end{cases}$$

$$\pi_j \sim Unif(0,1)$$

We estimate the hierarchical model using 50,000 MCMC iterations with the first 40,000 iterations discarded as a burn-in sample. Though not reported here, we generate multiple chains from different initial values, and the convergence for each parameter is confirmed with the Gelman-Rubin statistic. The posterior of k appears to be 8, indicating that the hyperparameter μ_j will transit to the absorbing state (=1) in the eighth development year.

To compare between the conjugate prior and the change point prior, Figure 2 presents the posterior distribution of μ_j. The left and right panel represents the posterior when using the conjugate and change point prior, respectively. Each box-plot corresponds to the prior mean of the development factor in each accident year. As anticipated, we observe relative larger development factors in the early stage and the the rate at which claims develop decreases over time. The two panels display similar patterns in the development factors. The subtle difference is that the development factors in the last two development years are equal to one under the change point process, however, they follow normal distribution under the conjugate prior. It is not surprising to see the little difference because when essentially all the claims are closed in the last two development years as suggested by the change point process, the normal distribution could not pick up much variability in the data.

The Bayesian linear model in Section 2 is based on the normality assumption. We employ residual analysis to validate this assumption. Note that residuals are not well defined in a Bayesian context. We follow the classic definition and calculate residuals as $e_{ij}^{(n)} = \left(C_{ij}^{(n)} - \hat{\beta}_j^{(n)} C_{ij-1}^{(n)}\right) / \left(\hat{\sigma}_j^{(n)} C_{ij-1}^{(n)}\right)$, where $\hat{\beta}_{ij}^{(n)}$ is the the posterior mode and $\hat{\sigma}_j^{(n)}$ is the empirical Bayes estimates. We present the normal qq plot in Figure 3. The agreement with the 45 degree line is consistent with the normality assumption. Also reported in Figure 3 is the plot of residual versus fitted value, where no particular pattern is detected. Note that because there is little difference between the conjugate hyperprior and the change point hyperprior, we only report the results from one of the two models.

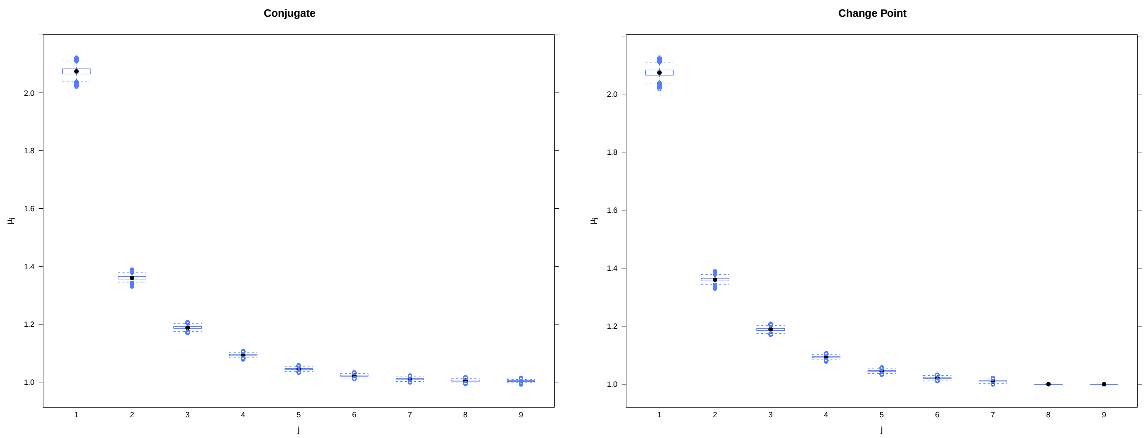

Figure 2: Posterior distribution of μ_j under conjugate and change point priors

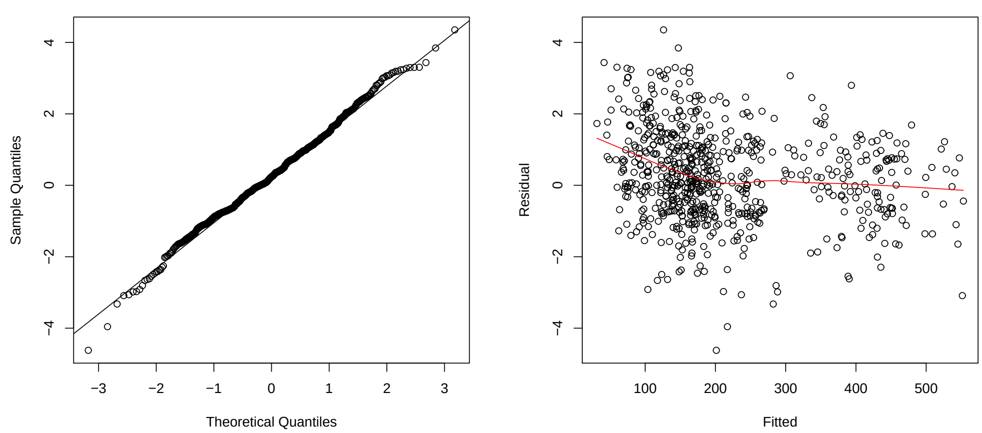

Figure 3: Normal qq plot and residual plot

By pooling multiple triangles, an insurer could gain predictive power by borrowing strength from other insurers. This is reflected by the shrinkage effect on the development factors, which is illustrated in Figure 4. Each panel reports the development factors in a particular year. Recall that parameter θ^2 controls the degrees of shrinkage. We estimate the model at $\theta = 1,\ 0.1,$ and 0.01. Within a panel, each curve connects the development factors estimated at various shrinkage for a single insurer. For comparison, we also report the development factor in the chain ladder model. As anticipated we see that a smaller θ shrinks the development factors of all insurers toward the group average. We also observe a larger shrinkage effect on the development factors in early years but smaller effect for later years. This is explained by the weak heterogeneity across insurers in later development years and the small variability in their posterior mean as shown in Figure 2. In the extreme case, the change point process even suggests that the expected development factors in the most recent two years are equal to one. There is no shrinkage effect in the chain ladder approach. As indicated in Section 2, the Bayesian linear model will reproduce the chain-ladder prediction when diffuse prior is used for inference. Finally, we stress that the degrees of shrinkage, i.e. whether to rely on an insurer's own claim experience or to adjust the prediction toward the industry average, requires the expert knowledge of reserving actuaries.

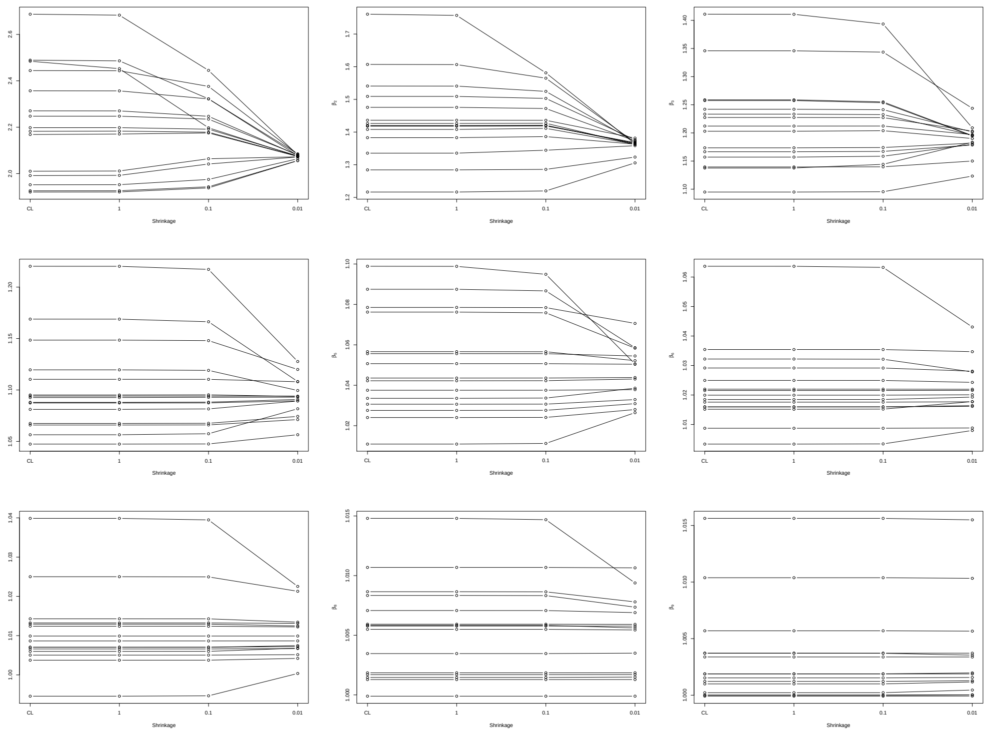

Figure 4: Shrinkage effects on development factors

5 Prediction

In this study we use chain-ladder method as the benchmark from which we employ a hierarchical model to introduce credibility. The inference has focused on the development factor in the chain-ladder framework. Therefore, the prediction follows in a straightforward way. The reserve (outstanding payments) for insurer n, $R^{(n)}$, is estimated by

$$\widehat{R}^{(n)} = \sum_{i=1}^{I} C_{iI-i}^{(n)} \left(\hat{\beta}_{I-i}^{(n)} \cdots \hat{\beta}_{I-1}^{(n)} - 1 \right)$$

where $\hat{\beta}_j^{(n)}$ $(j = 0, \cdots, I-1)$ are the best estimates i.e. the posterior mode from the hierarchical model. Wisely, reserving actuaries are more interested in a credible predictive range than a single point prediction. A commonly used measure of reserve variability is the mean squared prediction error that combines both uncertainty in the stochastic model and the unknown parameters.

Straightforward calculation shows similar decoupling results in a Bayesian context:

$$\text{MSEP}_{R^{(n)}} = \text{E}\left[(R^{(n)} - \widehat{R}^{(n)})^2\right] = \text{E}\left[\text{Var}(R^{(n)}|\boldsymbol{\Theta})\right] + \text{Var}\left[\text{E}(R^{(n)}|\boldsymbol{\Theta})\right]$$

The total variance (TV) is decomposed into average process variance (PV) and estimation error (EE) as in the classical analysis.

Tables 2 and 3 summarize the prediction results using the conjugate hyperprior and the change point hyperprior respectively. We report in each table the best estimate of firm-level reserves and the associated variability with the decoupling components under different degrees of shrinkage. The panel with $\theta = +\infty$ is equivalent to the chain ladder prediction. As θ decreases, the shrinkage effect strengthens. The amount of shrinkage has a pretty significant impact on the reserve, especially when $\theta = 0.01$. Note that the hierarchical specification drives the development factor not necessarily the reserve toward the group average, because the development factor and reserve could be negatively correlated. For example, a small firm might have a larger development factor, thus the shrinkage prediction would lower the reserve prediction. In addition, we also observe the effect of shrinkage on the reserving variability, especially for the estimation uncertainty. This is expected because the uncertainty in hyperpriors will be added to the parameter estimates. The average process variance is small because the conditional process variance is calculated following the chain-ladder approach and the estimation uncertainty is subdued by the averaging process. Consistent with results in Section 4, the predictions under the conjugate hyperprior and the change point hyperprior are quite similar.

In the above analysis, we have used a diffuse prior ($a = 1,\ b = +\infty$) for the hyperparameter μ_j, assuming that no prior knowledge is available at the point of valuation. The Bayesian approach allows expert opinions into the inference process. This could also be viewed as a downside because management could manipulate loss reserves through prior beliefs to manage earnings or hide solvency issues, though this is somewhat true under standard models depending on how the development factors are chosen or which method is used. We perform a prior sensitivity analysis of the reserve predictions to determine the extent of that control. Specifically, we consider the six combinations of $a = 0.5,\ 1,\ 2$ and $b = 0.1,\ 1$. The reserve estimates, total variance, process variance and estimation error are calculated under each specification. Along with the base case,

Table 2: Reserve prediction using conjugate prior

Company	Reserve	$\sqrt{TV}$	$\sqrt{PV}$	$\sqrt{EE}$	Reserve	$\sqrt{TV}$	$\sqrt{PV}$	$\sqrt{EE}$
	$\boldsymbol{\theta = +\infty}$				$\boldsymbol{\theta = 1}$			
1	498,645	28,222	24,953	13,184	498,808	28,244	24,956	13,226
2	410,216	18,174	14,975	10,298	410,150	18,273	14,975	10,472
3	490,000	90,432	80,935	40,341	490,151	90,694	80,958	40,880
4	463,987	30,850	25,902	16,758	463,906	31,072	25,901	17,164
5	157,824	46,527	41,740	20,555	156,413	45,899	41,743	19,085
6	67,497	6,874	4,633	5,078	67,543	6,808	4,634	4,988
7	93,136	10,601	9,447	4,810	93,046	10,538	9,446	4,672
8	145,421	11,175	8,435	7,331	145,446	11,216	8,436	7,391
9	99,618	9,445	7,376	5,899	99,765	9,465	7,380	5,927
10	83,508	7,952	5,818	5,420	83,536	7,979	5,819	5,459
11	84,934	9,971	7,916	6,062	85,222	9,957	7,924	6,029
12	88,281	7,525	5,807	4,785	88,347	7,534	5,809	4,798
13	239,553	21,880	17,005	13,768	239,992	21,913	17,016	13,807
14	82,357	12,395	10,691	6,271	82,179	12,364	10,681	6,228
15	42,301	6,207	5,248	3,316	42,212	6,208	5,243	3,323
	$\boldsymbol{\theta = 0.1}$				$\boldsymbol{\theta = 0.01}$			
1	497,245	28,031	24,938	12,800	450,286	25,672	23,996	9,125
2	413,177	18,227	15,001	10,354	455,377	18,261	15,500	9,656
3	491,968	88,191	81,058	34,747	444,800	79,260	77,503	16,594
4	458,874	30,734	25,808	16,690	379,806	26,294	24,102	10,509
5	146,799	43,454	42,125	10,665	153,487	44,558	44,190	5,717
6	68,806	6,796	4,648	4,958	95,992	6,648	5,024	4,353
7	88,571	10,233	9,386	4,075	80,285	9,472	9,173	2,362
8	146,181	11,180	8,449	7,322	145,011	9,639	8,501	4,544
9	101,446	9,428	7,405	5,835	95,709	7,968	7,224	3,362
10	82,514	7,897	5,789	5,372	52,722	5,954	4,923	3,348
11	87,309	9,791	7,972	5,684	92,342	8,722	8,185	3,013
12	87,964	7,435	5,803	4,648	76,680	6,294	5,548	2,973
13	232,388	21,211	16,845	12,889	179,319	16,735	15,582	6,103
14	75,427	11,404	10,161	5,177	45,127	8,357	7,977	2,491
15	39,045	5,946	5,103	3,051	26,129	4,646	4,474	1,251

Table 3: Reserve prediction using change point prior

Company	Reserve	$\sqrt{TV}$	$\sqrt{PV}$	$\sqrt{EE}$	Reserve	$\sqrt{TV}$	$\sqrt{PV}$	$\sqrt{EE}$
	$\theta = +\infty$				**$\theta = 1$**			
1	498,870	28,317	24,955	13,382	498,910	28,393	24,959	13,535
2	410,435	18,217	14,978	10,369	410,483	18,189	14,979	10,319
3	490,445	91,108	80,969	41,768	489,781	90,693	80,918	40,959
4	464,048	31,019	25,903	17,065	463,516	31,043	25,890	17,127
5	157,670	46,508	41,728	20,537	156,102	46,060	41,725	19,507
6	67,681	6,863	4,635	5,061	67,663	6,818	4,635	5,000
7	93,040	10,576	9,441	4,766	92,919	10,526	9,437	4,662
8	145,140	11,165	8,430	7,320	145,104	11,212	8,430	7,393
9	99,813	9,437	7,381	5,880	99,940	9,427	7,384	5,859
10	83,543	8,011	5,820	5,506	83,692	8,021	5,823	5,517
11	84,922	9,924	7,916	5,986	84,863	9,882	7,914	5,918
12	88,409	7,511	5,810	4,760	88,209	7,551	5,806	4,827
13	239,901	21,956	17,012	13,880	240,141	22,015	17,019	13,964
14	82,378	12,396	10,696	6,266	82,236	12,350	10,685	6,193
15	42,236	6,216	5,245	3,336	42,196	6,235	5,242	3,376
	$\theta = 0.1$				**$\theta = 0.01$**			
1	497,308	28,205	24,936	13,180	450,274	25,654	23,996	9,074
2	413,434	18,341	15,003	10,549	454,381	18,193	15,491	9,540
3	491,147	88,324	80,998	35,220	445,029	79,329	77,528	16,808
4	458,812	30,773	25,806	16,763	379,380	26,257	24,095	10,433
5	147,018	43,513	42,132	10,873	152,649	44,506	44,139	5,700
6	68,799	6,870	4,648	5,059	96,024	6,605	5,024	4,288
7	88,497	10,203	9,385	4,003	80,238	9,468	9,171	2,353
8	146,199	11,119	8,449	7,228	144,272	9,592	8,488	4,468
9	101,371	9,434	7,403	5,848	95,695	7,968	7,224	3,362
10	82,402	7,942	5,787	5,440	51,792	5,837	4,906	3,162
11	87,259	9,824	7,972	5,741	92,359	8,718	8,186	2,998
12	87,938	7,459	5,803	4,687	76,402	6,285	5,543	2,963
13	232,652	21,360	16,853	13,124	179,148	16,699	15,578	6,016
14	75,096	11,400	10,143	5,204	45,057	8,355	7,975	2,491
15	39,236	5,951	5,113	3,046	25,911	4,638	4,467	1,248

we present the reserve estimates and the predictive uncertainty in Figure 5. Each line in the figure represents an individual insurer. The predictions and associated variability are relatively robust to the prior specification, suggesting that data are informative enough for model inference.

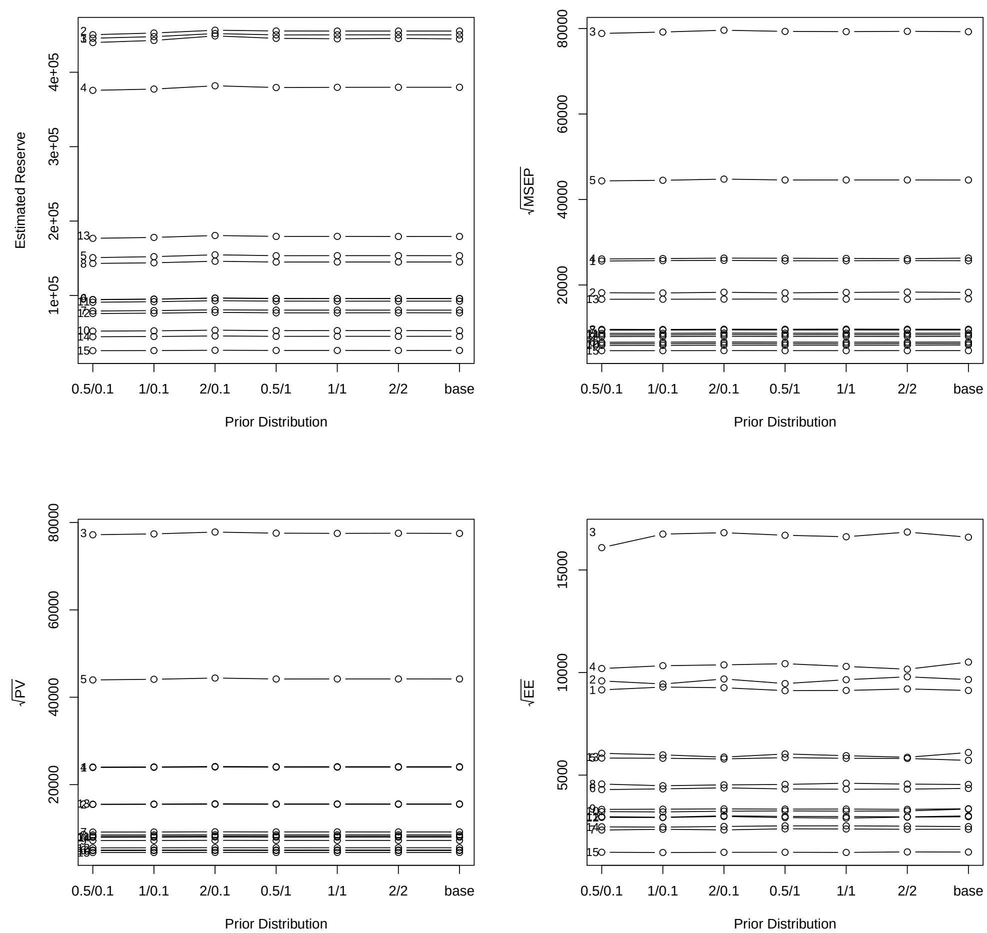

Figure 5: Robust analysis of reserve prediction

We employ out-of-sample validation to examine the value added by the information pooling. Recall that we have access to the Schedule P in years 1998-2006, and thus can calculate the actual amount of future payments (paid losses in the lower triangle) for each insurer, denoted by $Q^{(n)}$. Based on the predictive distribution of reserve $R^{(n)}$, we compute the two-sided p-value $\min\left\{\Pr(R^{(n)} < Q^{(n)}),\ \Pr(R^{(n)} > Q^{(n)})\right\}$. A smaller p-value indicates a more extremal outcome, i.e. the realized outcome is further away from the center of prediction. Because both under and over reserving could be detrimental to the insurer, a small p-value implies poor predictive performance.

The p-value is calculated using both conjugate prior and change point prior and using different degrees of shrinkage for each prior. Results are presented in Table 4 with the largest p-value highlighted. Recall that the diffuse prior $\theta = +\infty$ reproduce the chain ladder predictions. The small p-values for this scenario suggest that using some degrees of shrinkage to borrow information from peer insurers, an insurer is as least as good as the chain ladder method and as the amount of shrinkage increases, the model improves.

Table 4: p-values from out-of-sample validation

	Conjugate Prior				Change Point Prior			
Company	$\theta = +\infty$	$\theta = 1$	$\theta = 0.1$	$\theta = 0.01$	$\theta = +\infty$	$\theta = 1$	$\theta = 0.1$	$\theta = 0.01$
1	0.093	0.093	0.100	**0.333**	0.093	0.093	0.101	**0.332**
2	0.001	**0.001**	0.001	0.000	0.001	**0.001**	0.001	0.000
3	0.025	0.025	0.021	**0.048**	0.025	0.025	0.022	**0.048**
4	0.045	0.046	0.062	**0.113**	0.046	0.048	0.063	**0.109**
5	**0.250**	0.237	0.164	0.211	**0.248**	0.236	0.166	0.205
6	0.015	0.015	**0.023**	0.020	0.016	0.015	**0.024**	0.020
7	**0.000**	0.000	0.000	0.000	**0.000**	0.000	0.000	0.000
8	0.094	**0.094**	0.083	0.069	0.098	**0.099**	0.081	0.078
9	0.151	0.148	0.109	**0.231**	0.146	0.142	0.111	**0.232**
10	0.155	0.155	**0.185**	0.000	0.156	0.152	**0.190**	0.000
11	0.234	0.242	0.310	**0.492**	0.232	0.230	0.308	**0.492**
12	0.003	0.003	0.004	**0.082**	0.003	0.004	0.004	**0.088**
13	0.007	0.006	0.013	**0.362**	0.006	0.006	0.013	**0.358**
14	0.044	0.045	**0.106**	0.027	0.044	0.044	**0.111**	0.027
15	0.291	0.296	**0.489**	0.003	0.295	0.297	**0.476**	0.003
Selected	2	2	4	7	2	2	4	7

6 Conclusion

In this paper, we investigated credibility in reserving. We started with the classical chain-ladder method and, based on Bayesian linear models, we showed credibility results for both development factors and reserve estimates, i.e. a weighted average of prior knowledge and best estimates from the data. Further, we employed a hierarchical model for the prior specification such that an insurer could blend its own experience with claim experience from peer insurers. The hierarchical specification also leads to a shrinkage effect on the information across insurers. We emphasized that the degree of shrinkage used in the prediction is a judgement call of the reserving actuaries, allowing for more flexibility in the model.

In the empirical analysis, we examined a portfolio of fifteen large US property-casualty insurers' commercial auto insurance lines. We explored alternative approaches for prior specification, including conjugate and change point priors. The former is a natural choice for hierarchical model, and the latter is particularly useful if one is more interested in the payment pattern in the tails. We illustrated the advantage of the Bayesian approach to quantify reserve variability. Without loss of interpretability, the total variance can still be decomposed into the process variance and estimation error. Through out-of-sample validation, we showed that prediction for individual insurers can be improved by borrowing strength from peer insurers.

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The Use of GAMLSS in Assessing the Distribution of Unpaid Claims Reserves

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Abstract

Motivation. Regression modeling through generalized linear models (GLM) has known increasing popularity in last decades after milestone papers published in actuarial literature, representing one of the most used tools to assess the variability of unpaid claims reserve. Generalized additive models for location scale and shape (GAMLSS) represent an extension of classical GLM framework allowing not only the location parameters but also shape and scale parameters of a relevant number of distributions to be modeled as function of dependent variable like accident and development years. The paper applies GAMLSS to triangles coming from NAIC loss triangle databases in order to assess the distribution of unpaid loss reserve in term of best estimate as well as distributional form.

The results of GAMLSS are critically compared with those of classical stochastic reserving approach. All the analyses will be performed using R statistical software.

Keywords. Reserving Methods; Reserve Variability; Generalized Linear Models; GAMLSS; R software; NAIC Schedule P database.

1. INTRODUCTION

Regression modeling through generalized linear models (GLM) has been successfully applied in dynamic financial analysis (DFA) to assess the variability of claims reserves. In particular, over-dispersed Poisson models (ODP) have become popular due to the equality of the best estimate (BE) arising from its application to the ones coming from the classical chain ladder (CL). Distributions other than Poisson have been applied for estimating unpaid claims reserves like gamma and negative binomial.

GLMs can be used to obtain an estimate of the variability of outstanding claims reserves, decomposed into the amount due to inherent process variability (process variance) and the amount due to the estimation error (estimation variance). The latter element can be estimated either analytically or numerically thanks to the bootstrap approach (see England & Verrall, 1999) for bootstrap in claims reserve framework and (England, 2002) for process error evaluation).

GLM assumptions regarding the conditional distribution of the dependent variable are quite restrictive, however, since the variance of the outcome variables (that are the triangle cells) is expressed as a function (i.e., the variance function) of the mean of the outcome variables. A new class of statistical models has been introduced, generalized additive models for location, scale and shape (GAMLSS), with the aim to provide a flexible regression framework. In particular, it allows one to use separate regression equations for all parameters of the assumed conditional distribution of the dependent variable. In addition, it provides tools to assess the reasonableness of the regression forms (by means of the functional relationship assumed and variables included) as well as the shape of the conditional distribution.

Rigby & Stasinopoulos (2005) provide a theoretical introduction to GAMLSS, whilst Rigby & Stasinopoulos (2010) show applications of GAMLSS from a practitioners' point of view. At the time of this paper's drafting, no paper applying GAMLSS in the loss reserving context had been found within actuarial literature, making Schewe (2012) and Clemente and Spedicato (2013) the only approaches available. An early introduction of the idea can be found in Spedicato (2012), whilst Schewe (2012) and Clemente & Spedicato (2013) provide more comprehensive expositions. The first paper uses the GAMLSS approach to estimate claims reserves of numerous lines of business by using paid-to-premium ratios and compares the reserve uncertainty to the CL method. The second paper focuses on estimating claims reserve and quantifying reserve risk variability. On the other side, many works on applying GLM and generalized additive models (GAM) exist (see Renshaw & Verrall, 1998 for a general reference). Actuarial applications of GAMLSS are indeed very scarce: Stasinopoulos (2007) and Klein et al. (2014) applied GAMLSS in a ratemaking context, whilst an application to capital modeling has been shown in Spedicato (2011).

The application of GAMLSS for loss reserving is beneficial for two reasons. The first is that the regression assumptions are more flexible. For instance, making the conditional variance a function of external predictors (like the accident, development or calendar years) allows a more flexible modeling of the conditional distribution of triangle cells' outcomes and, therefore, better assesses the process variance. The second reason is applying GAMLSS provides valid tools to assess the shape of the distribution of losses that can be tested against numerous alternative distributions. Loss reserving with GLMs has given little attention to the shape of the conditional distribution of triangle's cells. In general, it can be said that all reserving models based on GLMs are particular cases of those that can be implemented under a GAMLSS framework.

The objective of the paper is twofold: (1) to introduce theoretically GAMLSS as a possible modeling tool for assessing the distribution of loss reserves and (2) to show a practical application on NAIC Schedule P triangles (NAIC DB). The remainder of the paper proceeds as follows: Section 2 discusses the general framework of GAMLSS and the proposed method for claims reserve evaluation, Section 3 describes a practical application on Schedule P databases, Section 4 reports main results and Section 5 drafts conclusions.

2. BACKGROUND AND METHODS

2.1 Introduction to GAMLSS

GLM and GAM proposed to assess loss reserve distribution lead to restrictive modeling for the variance of the response variable since the variance only depends on the mean as expressed within the variance function. Rigby and Stasinopoulos claim that this is true for skewness and kurtosis as well. Thus the authors developed a new model which allows explicit modeling of these moments rather than keeping implicit dependence on the mean. They also relaxed the requirement of a distribution from an exponential family by allowing more general distributions.

GAMLSS is a general class of univariate regression models where the exponential family assumption is relaxed and replaced by a general distribution family. The systematic part of the model allows all the parameters of the conditional distribution of the response variable Y_i ($i = 1,2, \dots, n$) to be modeled as parametric or non-parametric functions of explanatory variables. This means that an actuary can model not only the expected claim payment but also its process variance as a function of accident, development and/or calendar year using a regression expression.

Let $\theta^T = (\theta_1, \theta_2, \dots, \theta_p)$ the p parameters of a probability density function $f_{Y_i}(y_i|\theta_1)$ modeled using an additive model. $\theta_i^T = (\theta_{i,1}, \theta_{i,2}, \dots, \theta_{i,p})$ is a vector of p parameters related to explanatory variables, where the first two parameters $\theta_{i,1}$ and $\theta_{i,2}$ are usually characterized as location μ_i and scale σ_i. The remaining parameters, if any, are characterized as shape parameters. In a reserving context, this framework means that any cell of the triangle can be modeled by any distribution, where the parameters are derived by regression equations of accident and development years. The current R implementation of the software allows distribution up to 4 parameters to be modeled under this framework.

Under this condition, we can derive the following model (when p = 4):

$$\begin{cases} g_1(\mu) = \eta_1 = X_1\beta_1 + \sum_{j=1}^{J_1} Z_{j,1}\,\gamma_{j,1} \\ g_2(\sigma) = \eta_2 = X_2\beta_2 + \sum_{j=1}^{J_2} Z_{j,2}\,\gamma_{j,2} \\ g_3(\nu) = \eta_3 = X_3\beta_3 + \sum_{j=1}^{J_3} Z_{j,3}\,\gamma_{j,3} \\ g_4(\tau) = \eta_4 = X_4\beta_4 + \sum_{j=1}^{J_4} Z_{j,4}\,\gamma_{j,4} \end{cases} \tag{2.1.1}$$

where μ, σ, ν, τ are vectors of length n, X_k are known design matrices, $\beta_k^T = (\beta_{1,k}, \beta_{2,k}, \dots, \beta_{j,k})$ are parameters vector, $Z_{j,k}$ are known design matrices for the random effects and $\gamma_{j,k}$ are random vectors.

In particular, the previous equations imply that the moments of response variable in each cell can be expressed directly as a function of covariates after a convenient parameterization. This since regression equations can be used to model each parameter as a function of covariates and since the moments of any distribution can be expressed as functions of its own parameters. Each linear predictor η_k consists of a parametric component X_k and an additive random component. Instead of random effects, smooth functions may be used as in GAM. Cubic splines, penalized splines, varying coefficients and random effects offer a maximum degree of flexibility since they allow more complex scenarios than GLM (or GAM) to be modeled.

Currently the GAMLSS R package supports more than 60 distributions, non-linear and non-parametric relationships (e.g. cubic splines and non-parametric smoothers) and random effect modeling. See (Rigby & Stasinopoulos, 2010) for more details on the R package. Nevertheless for most real world applications, two-parametric distributions should sufficiently approximate the dependent variable distribution of interest. This means that for the reserving analysis in this paper we will consider only two-parametric distribution families. For example, reserve analysis with up to four parameters can be found. in Schewe (2012).

Applying a GAMLSS model in a reserving exercise involves both selecting the distribution of the dependent variable (for example weibull or a lognormal) and the functional relationship between the parameters of the dependent variable distribution (say μ and σ, if a two-parametric family has been chosen) and the independent variables (say accident and development years). The R package that implements the GAMLSS models provides various instruments to aid the selection of both the functional form and the distribution assumption. (Rigby & Stasinopoulos, Lancaster Booklet, 2010) paper provides an introduction to GAMLSS regression modeling in which the interested reader can find both theoretical details and applied GAMLSS modeling examples. The main instrument to evaluate the reasonableness of GAMLSS is the analysis of normalized quantile residuals (NQR). Normalized randomized quantile residuals (see Dunn & Smyth, 1996) are used to check the adequacy of a GAMLSS model and, in particular, its distribution component. The residuals are given by $\hat{r}_i = \Phi^{-1}(u_i)$, where Φ^{-1} is the inverse of the cumulative distribution function of a standard normal distribution and $u_i = F(y_i|\hat{\theta}_i)$ is derived by applying the estimated cumulative distribution to y_i. If the model is specified correctly the NQR should follow a Gaussian distribution. Apart from model checking, the normality properties have been used for bootstrapping GAMLSS models as shown in forthcoming section.

2.2 GAMLSS Applications to Loss Reserve Analysis

Focusing now on claim reserving analysis, we consider a generic loss development triangle of dimension (I, J) with rows ($i = 1, \dots, I$) representing the claim accident years (AY) and columns (with $j = 0, \dots, J$) describing the development years (DY) for payments. It needs to be emphasized that the number of columns may differ from the number of rows, for example, because of a tail in the payment development.

Following an approach similar to Renshaw & Verrall (1998), we can now define $P_{i,j}$ as the incremental paid claims and identify the incremental paid claims as response variables of the following structure:

$$\begin{cases} E\left[P_{i,j}\right] = g_1^{-1}(\eta_{1,i,j}) \\ \sigma^2\left[P_{i,j}\right] = g_2^{-1}(\eta_{2,i,j}) \end{cases}. \tag{2.2.1}$$

If a model for the distribution of incremental paid claims is found on historical data $P_{i,j}$ $(i + j \leq I)$, the model can be applied to predict future payments $P_{i,j}$ $(i + j > I)$. The key advantage of GAMLSS compared to GLMs is that $\sigma^2\left[P_{i,j}\right]$ can explicitly be modeled within a statistical framework, instead of relying on the GLM variance function assumption.

The ODP model is one of the most used approaches by actuarial practitioners when performing stochastic reserving under a regression framework. Within this framework it is assumed that each triangle cell $P_{i,j}$ follows a Poisson with parameter $\lambda_{i,j}$.

In addition, it is assumed that:

a. $E\left[P_{i,j}\right] = \lambda_{i,j}$ can be modeled using a log-linear regression, for example, as a function of AY and DY dummy indicators: $E\left[\lambda_{i,j}\right] = exp\left(\alpha + \beta_i + \gamma_j\right)$, where α may be parametrized to a baseline accident/development period level.
b. An over-dispersion parameter ϕ exists such that $var\left[P_{i,j}\right] = \phi \cdot E\left[P_{i,j}\right] > E\left[P_{i,j}\right]$ holds for all i, j.

Taking into account the nature of data, however, other distributions may be more appropriate and provide a better fit to the underlying data than a Poisson.

One of the aims of this paper is the investigation of which is the most appropriate distribution for $P_{i,j}$ in a real-world scenario. Triangles from the NAIC DB will be used as the basis of investigation. A GAMLSS with a two-parametric distribution will be fit to the data and the effect of the covariates on the first parameter μ will be examined.

In a second step, we will verify if the second parameter σ can be held constant or can be expressed as a function of either AY or DY within a regression structure similar to the one for μ.

A third step will be to use the GAMLSS to estimate claim reserves and variability of claim reserves. After a suitable conditional distribution and a regression structure for the location and scale parameters has been chosen, the GAMLSS can be applied to the lower part of the triangle to obtain a best estimate of reserves and variability of the estimates, as further detailed in Schewe (2012) and Clemente & Spedicato (2013).

In order to assess the variability of the claims reserves, the following bootstrap-like approach can be used:

1. Fit a GAMLSS model M on an incremental paid claims triangle using a suitable distribution function and development year and accident year as covariates. The functional relationship between the location and scale parameters and their predictors could be modeled using dummy variables or more sophisticated functional relationships such as polynomials or splines. This approach would be similar to a classical ODP modeling approach for development triangles, but here not only a regression for the expected value of the cell but also for its variability would be done. The estimated parameters will be used to derive BE reserves and to model the process variance. Note that the application is not bound to incremental paid claims triangle but incremental incurred claims triangle could be used as well.
2. In order to allow for prediction error, it is proposed to adapt the bootstrap algorithm proposed in Renshaw & Verrall (1998) to GAMLSS model:
    a. Compute the normalized quantile residuals, $\hat{r}_{i,j} = \Phi^{-1}\left(F\left(P_{i,j}|\hat{\theta}_{i,j}\right)\right)$.
    b. Generate N upper triangles of residuals $\hat{r}_{i,j}^{k}$, with $k = 1, \dots, N$ by replacement.
    c. Derive N upper triangles of pseudo incremental payments from the GAMLSS model

by the inverse relation: $\hat{P}_{i,j}^{k} = F^{-1}\left(\Phi\left(\hat{r}_{i,j}^{k} | \hat{\theta}_{i,j}\right)\right)$

d. Refit the GAMLSS model M on N triangles in order to assess model variance

e. For each cell of the lower part of each triangle, simulate the outcome $P_{i,j}$ from the process distribution with mean and variance depending by the fitted GAMLSS

3. The sum of lower triangle part cells values as predicted by the GAMLSS model corresponds to the reserve.

4. The N values derived at step 3 represent the simulated distribution of claims reserve.

5. The main moments, that is, the best estimate and a measure of loss variability, can be estimated by such distribution.

Clemente & Spedicato (2013) applied this approach to the classical Taylor-Ashe triangle (Taylor & Ashe, 1983) finding a Gamma distribution with development year as covariate to best fit the payment pattern within a reasonble set of choices. This paper will apply the outlined approach on generic NAIC loss triangles, using various distributions and shows how to derive with the BE reserve and its variability.

3. RESULTS

3.1 Process variance analysis

The first part of the analysis investigates whether, when performing loss reserving under a regression modeling framework, a statistical distribution may be deemed the most appropriate using a statistical goodness-of-fit criterion. For this purpose, a generalized Akaike information criterion (GAIC) will be used to compare GAMLSS. It is obtained by adding to the fitted global deviance a fixed penalty for each degree of freedom in the model.

Furthermore, to address the question stated above, various GAMLSS models have been fit on NAIC Schedule P loss triangles following the approach outlined as follows:

a) The incremental claim payments are expected to vary both by accident and development year. A second and third structure has been defined, allowing the scale parameter to vary by either accident or development year.

b) Accident and development years enter the GAMLSS regression as dummy variables in all our analyses.

c) The following distributions were tested: Poisson (POI), negative binomial (NBI), gamma (GA), Weibull (WEI), lognormal (LNORM) and inverse Gaussian (IG). Whilst the GAMLSS R package can handle more than 60 different distributions, the relatively limited choice is driven by the authors' aims to introduce the approach and to restrict the analysis to the most used distributions within current actuarial practice. For each distribution, the three regression structures mentioned above were implemented. Note that the two discrete distributions, Poisson and negative binomial, are being used for a continuous random variable for the same reasons outlined in England & Verrall (1999). Recall that it is shown that a GLM reserve estimate under an ODP framework is equal to a chain-ladder reserve estimate.

d) Each combination of regression structure and distribution has been fit to each triangle of the NAIC. For each triangle, we selected the model with the lowest value of GAIC criterion among those for which the GAMLSS algorithm was able to estimate parameters.

e) The conditional distributions and parameter assumptions of the reference models have been tabulated for all the NAIC DB lines of business (product liability, other liability, medical malpractice, workers compensation, commercial auto and private passengers auto).

For the analysis of reserve variability, a triangle was picked at random (group 226, private passengers auto) and the upper and lower parts of the incremental paid loss triangle were arranged. Traditional reserving models were estimated with the aid of Gessman, Zhang, & Murphy (2013) R package as well as various GAMLSS reserving models. The underlying best estimates have been compared with the subsequent payments shown in the lower triangle and released within the NAIC DB package. The models' reserve standard errors have been compared as well.

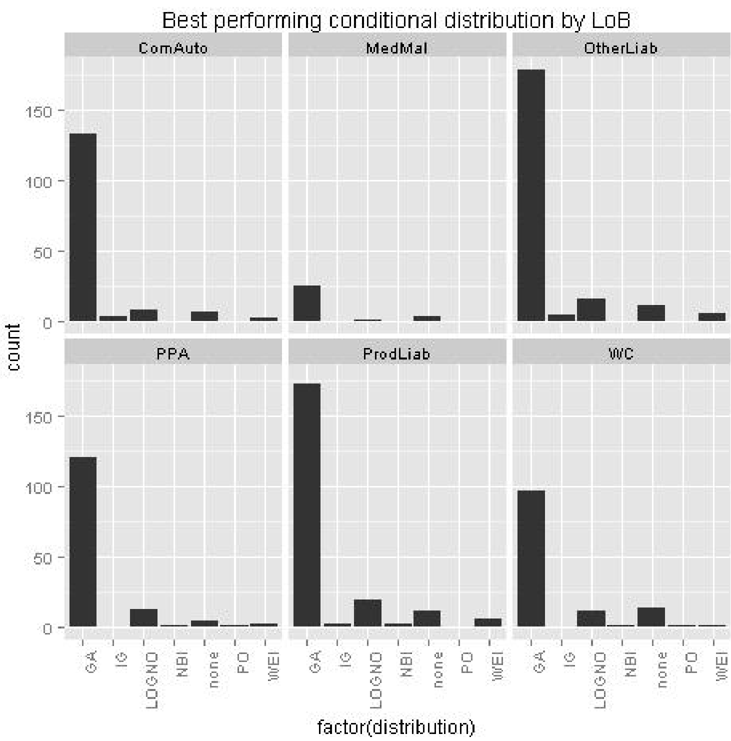

Figure 1: Best performing conditional distribution by line of business (LoB)

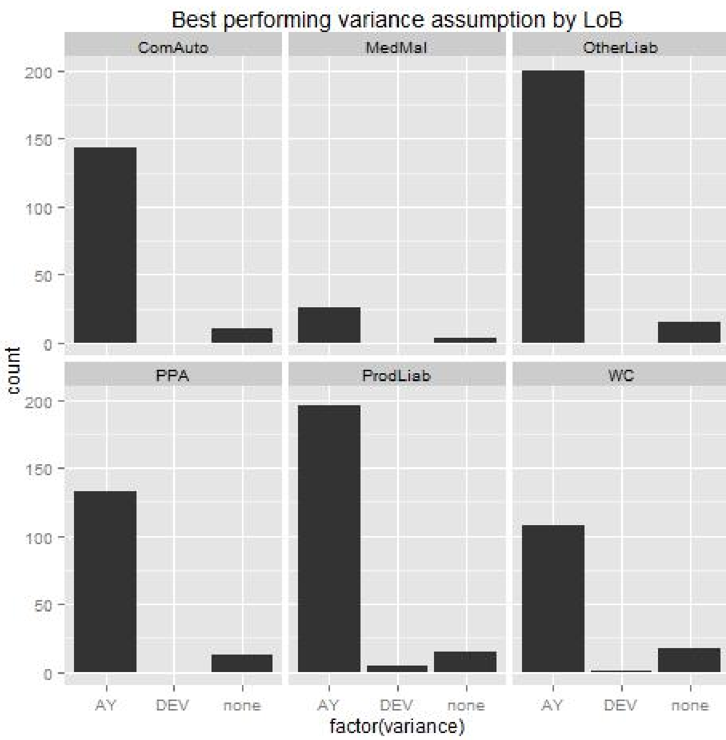

Figure 2: Best performing shape parameter assumption by LoB

Various GAMLSS have been fit on NAIC DB varying the conditional distribution assumption as well as the variance dependency by either accident or development years or neither of them. The best performing model as measured by the goodness of fit has been selected. In other words, for each group's triangle, within each line of business (LoB), a "best" model has been selected. It has been defined by a conditional distribution assumption (tabulated on Figure 1) and a shape parameter assumption (tabulated on Figure 2). The gamma distribution supersedes by far the other distributions as the most appropriate distribution by AY. The lognormal and Weibull distributions are a distant second and third, respectively. Assuming the claim payment follows a discrete distribution (Poisson or negative binomial), as was done in earlier GLM reserving approaches, appears to be not supported by empirical data. Similarly, assuming the scale parameter to vary by accident year appears to improve the model fit in terms of GAIC.

The R programming code that replicates the analysis of this section are the first three files listed in the appendix.

3.2 Full distribution of Unpaid Claim Reserves

The approach outlined in the methodology section has been applied in order to estimate the reserve BE and its variability. The exercise has been carried on a group of 266 triangles for the private passenger LoB (henceforth called example triangle or ET). As pertaining to the NAIC Schedule P triangles set, it shows 10 years of development for AYs 1988–1987. The obtained figures have been compared with the actual incremental payments during calendar year 1998–2006 and with the BE and standard deviation implied with other reserving algorithms applied on the same triangle (Mack formula, bootstrap chain ladder with a gamma process distribution, GLM ODP).

Initially various GAMLSS models have been fit on the ET in order to find an appropriate stochastic model for the claim triangle. The selected model assumes a GAMMA conditional distribution, modeling the expected value to depend on both the accident and development years whilst the variance to vary by development year only. Then the unpaid claim distribution (see Figure 3) has been obtained by estimating both process and parameter uncertainty as described in Section 2.2. The green and red lines in Figure 3 represent observed payments in the lower part of the triangle and GAMLSS BE, respectively.

model	Best Estimate	Standard Deviation
Mack	30.065	2.517
BootstrapCL	32.635	141.905
ODP	30.065	6.695
GAMLSS BASE	31.821	14.354

Table1: BE and standard deviations of various loss development models on private passenger, Group 226

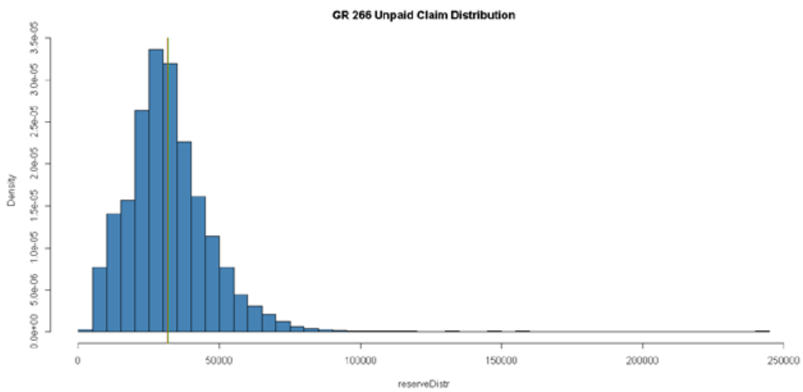

Figure 3: GR 266 Reserve Distribution derived by the GAMLSS model.

The BE and standard deviation of unpaid claims are reported in Table 1. When compared with actual calendar year 1998–2006 payments totaling 31,713, the GAMLSS model BE appears to be the closest to the actual results. However, the standard deviation of the GAMLSS model appears to be quite large (slightly more than double than that given by the ODP and almost seven times than the variability given by the Mack formula, even though far lower than the bootstrapped chain ladder with gamma process variance). It is difficult to explain why the difference in variability is so great when compared with standard models. One possible reason is that when a model that predicts not only the central tendency but also the variability is bootstrapped, the resulting variance is exacerbated.

In addition, the estimated BE appears to be very sensitive to changing conditional distribution assumptions. Varying either the conditional distribution assumption or the variance form assumption can imply large swings in terms of the BE (see Table 2) as well as in the inherent variability.

model	Best Estimate	Standard Deviation
GA, dev	31.821	14.354
GA, ay	26.614	645
GA, none	34.527	18.671
WEI, ay	26.106	521

Table 2: Loss reserve BE and sigma by changing GAMLSS conditional distribution and variance modeling assumptions.

3.3 Comparison of GAMLSS Reserves Estimate with BLUE Chain Ladder

A final exercise was done comparing the accuracy of reserve estimates using mechanic chain ladder and GAMLSS approach with respect to actual lower triangle part payments (since calendar year 1999). The accuracy was measured by means of the root mean-squared error (RMSE); that is, the square root of the average squared difference between an actual outcome (the lower triangle cells' actual payments) and its estimate (the best estimate). The analysis was performed on the full NAIC DB private passenger auto triangle set, excluding those triangles on which a standard GAMLSS model did not converge. In addition, chain ladder link ratios were estimated using the regression through the origin formula, which has been shown to be a best linear unbiased estimator (BLUE) of the development factors (Murphy, 1994).

Model	**RMSE**
BLUE Chain Ladder	97.577
GAMLSS	16.276

Table 3: RMSE comparisons between BLUE Chain Ladder and GAMLSS approach

The analysis shows that the correlation between the GAMLSS and chain ladder estimates is very high. In addition, even if neither GAMLSS nor chain ladder systematically outperforms the other, the GAMLSS RMSE is significantly lower than chain ladder value; thus, suggesting GAMLSS could provide sensible reserves estimates.

The R programming code that replicates this analysis are the last two files listed in the appendix.

As a general remark, whilst GAMLSS models allow for a great degree of flexibility, they have not yet been studied extensively. In particular, the actual R implementation is not optimized by means of incorporating C code in the computationally most critical part of the estimation process. In addition, model estimation convergence problems may arise, especially when complex regression structures are used or non-standard conditional distributions are chosen. The following measures were taken in order to overcome such drawbacks:

1. The R code was highly parallelized to take advantage of multicore processors when performing the analysis on the whole NAIC database. The aim is running much larger chunks of computations in parallel.

2. Exception was explicitly handled within the R programming code to avoid analysis interruptions.

In addition, the following adjustments to the data were performed in the data preparation part:

1. Incremental paid data were modeled.
2. Negative increments were zeroed.

DISCUSSION

This paper has investigated the use of GAMLSS regression models for P&C loss reserving. On an overall basis, a mixed result can be drawn. For various reasons, these models, appear to be potentially relevant for loss reserving. By allowing an explicit modeling of the variability by either or both accident and development years, GAMLSS overcome the limitations of standard GLMs. Therefore, they would better model the process variance of loss reserves. On the other hand, the additional parameter that is needed to be modeled reduces the degrees of freedom available. This can be a strong limitation of the model when using triangles of similar size to those provided in the NAIC DB , but in case larger triangles (as quarterly based triangles or 15x15 yearly based triangles requested in Solvency II technical reports) are used, this limitation can be overcome.

The preliminary analyses carried out in the paper have shown that actual triangles of the NAIC DB present a source of variability that departs from the variance function assumption on which standard GLMs are based. In addition, a gamma conditional distribution outperforms, in terms of goodness of fit, other distribution like negative binomial and Poisson that are commonly assumed in standard GLM reserving models.

When a GAMLSS approach has been used to assess P&C loss reserves, BE, and variability on an actual triangle, results have been comparable with those of other reserving methods . In addition, the GAMLSS approach systematically applied on the whole NAIC DB private passenger auto triangle set has shown an RMSE lower than the BLUE chain ladder, when predicted payments (i.e., reserves) have been compared with actual payments. On the other hand, unpaid claim distributions arising from bootstrapping GAMLSS models have been shown to be extremely sensitive with respect to changes in marginal distribution assumptions and cell variance. A final limitation, that needs to be stressed, is that convergence problems arise much more frequently than with standard GLMS. This requires a greater effort in data checking and model selection.

Supplementary Material

The full R code to replicate numerical results of the paper is available. In particular:

1. 0-loadNaicTriangles.R loads NAIC CSV files.
2. 1-prepare data set 4 modeling.R performs additional preprocessing.
3. 2-regression models on Naic Triangles.R performs additional analyses.
4. 3-GAMLSS reserve variability.R performs analysis of reserve variability on a real triangle
5. 4-Compare ChainLadder and GAMLSS.R applies ChainLadder and GAMLSS on PAP triangles and compares RMSE.

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Abbreviations and notations

AY, accident year
BE, Best Estimate
BE, best estimate
BLUE, best linear unbiased estimator
CAS, Casualty Actuarial Society
CL, chain ladder
DY, development years
GAIC, generalized Akaike information criterion
GAMLSS, generalized additive models for mean location and shape
GLM, generalized linear models
NAIC DB, NAIC Schedule P triangles data base
NQR, normalized quantile residuals
ODP, over-dispersed Poisson models
OLS, ordinary least squares

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Combining Estimates

Thomas Struppeck, FCAS, ASA, CERA

Abstract:

The problem of combining two or more estimates into a single estimate appears in many applications, such as combining estimates based on paid losses and estimates based on incurred losses, or combining estimates for several accident years or lines of business into a single estimate. A methodology for performing such combinations which allows for correlation is described. An accompanying Excel spreadsheet illustrates the procedure.

Keywords: Reserving, Credibility, Solvency II, Ranges

1. Introduction

Actuaries often are faced with the task of combining two or more estimates into a single estimate. When estimating ultimate losses, there may be one estimate based on paid losses and another estimate based on incurred losses. As a second example, consider estimates for several lines of business that are to be combined into a single estimate. While these two examples are somewhat similar, there is a very important difference between them. In the first example, we have two different estimates for the same quantity. In the second example, we have estimates for different quantities. As we will see, in the first case we will actually be able to improve on both estimates. In the second case, we can only hope to not be much worse than the worst one (at least in terms of the width of our confidence interval).

Patel and Raws [PR] used a simulation technique to study the effects of different methods of combining estimates in the case where the estimates are for different quantities (example 2 above). We review their results in section 3.

The remainder of this paper is structured as follows:

- Combining two or more estimates for the same quantity
    - Precision vs. accuracy
    - An example: two estimators for one parameter
- Combining estimates for multiple components
    - Patel and Raws' simulation work
    - Sums of different component pieces
    - Why normal distributions?
        - Considerations in selecting correlations

- An example
- Concluding remarks

2. Combining Two Or More Estimates For The Same Quantity

First, consider the case where two independent estimates, $\hat{\theta}_A$ and $\hat{\theta}_B$, are available for an unknown parameter, θ . We will follow the usual notational convention of writing carats ("hats") over parameters to denote estimators of those parameters and of using Greek letters to denote unknown parameters. In a reserving context, θ might represent the unpaid claims on a block of business, and $\hat{\theta}_A$ and $\hat{\theta}_B$ might represent two different estimates of θ.

It should be noted that we think of θ as being a fixed (but unknown) number; it is not itself a random variable. On the other hand, $\hat{\theta}_A$ and $\hat{\theta}_B$ are often instances of random variables and as such have sampling distributions. We will follow customary notation and write, for example, E[$\hat{\theta}_A$] for the expected value of (the sampling distribution of) $\hat{\theta}_A$.

2.1 Precision vs. Accuracy

The two terms "precision" and "accuracy" are often used interchangeably; however, there is a slight difference in their definitions. Accuracy refers to the proximity of the expected value of an estimator to the true value of the parameter (the "reference value" in the illustration), whereas precision refers to the spread of estimator values around its expected value. Figure 1, below, illustrates this for a normal density.[1]

[1] Used with permission under the terms of the GNU Free Documentation License, Version 1.2.

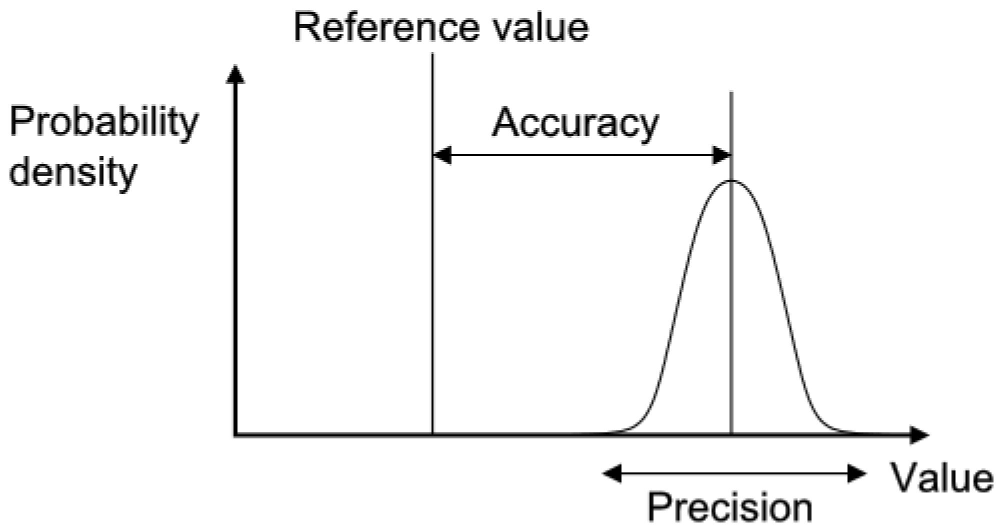

Figure 1: The distinction between accuracy and precision

2.2 An Example: Two Estimators for One Parameter

For our first example, we will attempt to estimate the outstanding loss reserve, θ. We have an unbiased estimator of θ, $\hat{\theta}_A$. For the moment assume that this estimator has a normally distributed sampling distribution with mean \$250 and standard deviation \$30. Then, since $\hat{\theta}_A$ is unbiased, a reasonable choice of point estimate for θ would be the expected value of $\hat{\theta}_A$, namely \$250. Since the sampling distribution is known[2], we can even give an approximate 95% confidence interval, \$250 $\pm$ 2(\$30) = (\$190, \$310)[3].

Now suppose that a second, independent, unbiased estimator of θ, $\hat{\theta}_B$, is available. Further suppose that the sampling distribution of $\hat{\theta}_B$ also is normal but with mean \$275 and standard deviation \$40. This second estimator has less precision than our first estimator (it has a larger standard deviation), and it suggests a different point estimate. Using just the second estimator we obtain another 95% confidence interval, \$275 $\pm$ 2(\$40) = (\$195, \$355).

We would like to create a single estimator that allows these two estimates to work in tandem in a way that maximizes the precision of the resulting estimator. The way to do this is to consider the one-parameter family of estimators obtained by taking weighted averages of $\hat{\theta}_A$ and $\hat{\theta}_B$:

$\hat{\theta}_t = (1-t)\hat{\theta}_A + t\hat{\theta}_B$ where $0 \le t \le 1$.

---

[2] In fact, we do not need to know the entire sampling distribution, only its 2.5th and 97.5th percentiles.
[3] We have used 2.0 instead of 1.96 for ease of exposition; about 95% of the area under a normal is within two standard deviations of the mean. The given interval is actually a 95.45% confidence interval.

Since we have assumed that both $\hat{\theta}_A$ and $\hat{\theta}_B$ are unbiased, each $\hat{\theta}_t$ is unbiased because expectation is a linear operator.

When t is zero, we get the first estimator, and when t is 1, we get the second estimator. In between, we get a family of estimators. Since each of $\hat{\theta}_A$ and $\hat{\theta}_B$ is normal, and they are independent, the weighted average $\hat{\theta}_t$ is normal with mean $(1-t)E[\hat{\theta}_A]+tE[\hat{\theta}_B]$, which in our case is $(1-t)*250+t*275=250+25t$, and variance $(1-t)^2Var[\hat{\theta}_A]+t^2Var[\hat{\theta}_B]$, which in our case is $(1-t)^2*30^2+t^2*40^2$.

We want to maximize the precision, which amounts to minimizing the standard deviation, which is the same as minimizing the variance. This is easily done by taking the derivative with respect to t and setting it to zero:

$$2(1-t)*30^2*(-1)+2*t*40^2=0 \rightarrow (30^2+40^2)t=30^2 \rightarrow t=\left(3/5\right)^2$$

So the minimum occurs when we let t=.36, i.e. we use 64% of estimator A and 36% of estimator B. This produces $\hat{\theta}_{0.36}$ which is normal with mean = 259 and standard deviation = 24. This estimator has the smallest standard deviation of any weighted average of our estimators and suggests a point estimate of $259 with an approximate 95% confidence interval of ($211, $307). This estimate has the most precision of any estimator in this class.

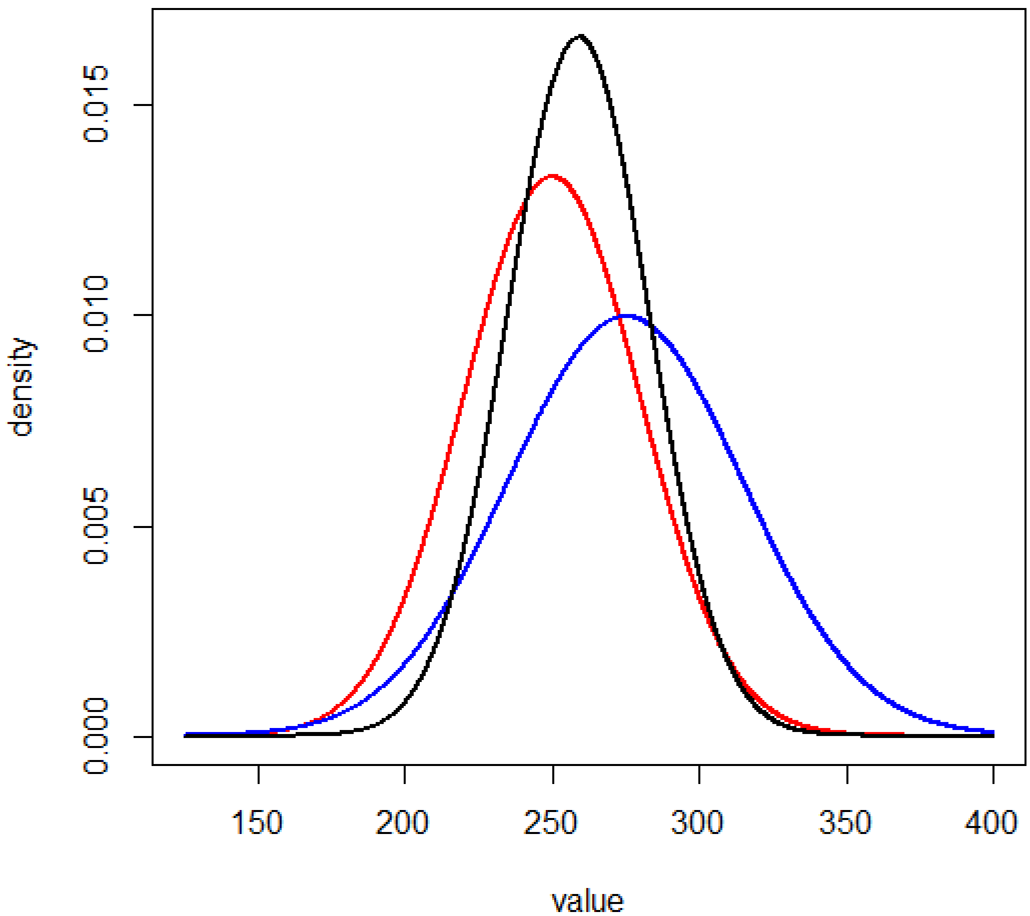

Figure 2: Densities for two estimators and their optimal combination

Figure 2, shown above, illustrates the density function for the two estimators, $\hat{\theta}_A$ (shown in red) and $\hat{\theta}_B$ (shown in blue), and the density function of their optimum weighted average, $\hat{\theta}_{0.36}$ (shown in black). The weighted average, having a more concentrated density, is more precise. Whether it is more accurate is a more delicate question.

Accuracy measures how far the true parameter is from our estimate. Since the true parameter is intrinsically unknowable, we will have to satisfy ourselves with a statement about confidence intervals. Since $\hat{\theta}_{0.36}$ lies between $\hat{\theta}_A$ and $\hat{\theta}_B$, whatever its true value, θ is closer to $\hat{\theta}_{0.36}$ than it is to at least one of $\hat{\theta}_A$ and $\hat{\theta}_B$. In other words, $\hat{\theta}_{0.36}$ is always more accurate than the worst of $\hat{\theta}_A$ and $\hat{\theta}_B$.

In fact, sometimes it is more accurate than either of them. Whenever $\theta > 254.5$, θ will be closer to $\hat{\theta}_{0.36}$ than to $\hat{\theta}_A$. And whenever $\theta < 267$, θ will be closer to $\hat{\theta}_{0.36}$ than to $\hat{\theta}_B$. Using the sampling distribution for $\hat{\theta}_{0.36}$, we see that the first interval is a 57.44% one-sided confidence interval for θ and that the second is a 63.06% one-sided confidence interval. The intersection of these two intervals is the interval $254.5 < \theta < 267$, which is a 20.49% confidence interval for θ.

In the example, we assumed that we had two estimators, that our estimators were normally distributed, and that they were independent. If we have more than two estimators, we can still find the optimum weighting by using multivariate calculus techniques (setting the gradient to zero, etc.). If the estimators are correlated and we have a good estimate of the correlation coefficient(s), we can still compute the standard deviation of the weighted averages and find the optimum weighting. In order to create a confidence interval, we need to know the distribution of the sum. If our summands are bi-normally or multi-normally distributed[4], then the sum will have a normal distribution. In that case, we can create our confidence interval in the usual way, namely by picking the appropriate point from a table of standard normal values (a z-score), multiplying it by the standard deviation and using this as a radius about the point estimate.

In practice, multiple estimates of the same quantity tend to be highly correlated. But, unless the correlation is 100%, some increase in precision will occur when they are combined.

3. Combining Estimates for Multiple Components

3.1 Patel and Raws' Simulation Work

In [PR], Patel and Raws considered the problem of estimating a total reserve from estimates of the component pieces. They used a simulation approach to compare several different possible distributions for the losses in each piece. Among the distributions that they examined were the uniform, triangle, normal, and log-normal distributions. In each iteration of the simulation, they generated losses by line from those distributions, summed them, and repeated this process many times to create the simulated distribution of the sum.

In their simulations, they assumed that the component pieces were independent. That assumption allowed them to select the losses for each line without having to explicitly correlate them. In the text they suggest that correlations between accident years could be adjusted for in the choice of distribution and that correlations between lines of business could be adjusted for similarly.

[4] It is possible for two or more jointly distributed normal random variables to have normal marginal distributions but not be multi-normally distributed.

Capturing the effects of correlation in a simulation is possible, but it can be tricky. If one is not careful, the marginal distributions can fail to be what is expected. Copulae can solve this problem.

A (d-dimensional) copula is a mapping from the d-dimensional unit cube onto the unit interval that is a joint cumulative distribution function with uniform marginals. Copulae are of interest to us because of Sklar's Theorem, which says that any d-dimensional random variable (i.e. d jointly distributed univariate random variables) can be expressed as a composition of its d marginal random variables and a copula that combines them. Furthermore, in the case of continuous random variables, this decomposition is unique. So, utilizing a copula, one can impose any possible correlation structure on a family of random variables, such as having correlation mainly manifest itself in the tails. More in depth discussions of copulae can be found in Mango and Sandor [MS] and Venter [V].

The ranges of estimates that Patel and Raws obtained did not vary greatly by choice of distribution. This suggests an alternative approach: instead of assuming independence and using multiple distributions, only use one distribution, but use one that allows for explicit correlations. One such distribution is the multi-normal distribution.

3.2 Sums of Different Component Pieces

The example in the first section illustrated how multiple estimates for the same parameter can be combined to create an estimate with greater precision than the original individual estimates. Often the quantity that we want to estimate is a sum of several parts, each of which has an associated estimate --- this is the problem that Patel and Raws examined. For instance, we could be interested in the total outstanding losses for a company that writes three lines of business, and we have estimates of the outstanding losses for each line.

We will not be taking a weighted average here, but rather we will just take a sum. The summing and averaging are closely related, but differ in an important way: in a weighted average, each summand gets multiplied by a number between 0 and 1, t and (1-t) in our example:

$$\hat{\theta}_t = (1-t)\hat{\theta}_A + t\hat{\theta}_B \text{ where } 0 \le t \le 1$$

The variances of $\hat{\theta}_A$ and $\hat{\theta}_B$ got multiplied by $(1-t)^2$ and t^2, respectively, and these are strictly less than (1-t) and t (unless t is 0 or 1). <u>This is how we obtained greater precision</u>. There is no similar opportunity when the coefficient is equal to one as it is in a sum.

What this means is that as we add more and more independent pieces to our sum, our confidence intervals will get nominally larger.[5] By nominally larger, we mean that the width of the confidence interval will increase. In relation to the size of the reserve however, the intervals might be getting smaller. One possible measure of the relative size is the coefficient of variation (CV) --- the ratio of the standard deviation to the mean. The square of this, the ratio of the variance to the squared mean, is another measure which is sometimes easier to work with. It is called the squared coefficient of variation or SCV. Both the CV and SCV are dimensionless quantities.

When there is no correlation among the pieces and the number of pieces is large, the sampling distribution of the sum will start to look like a normal distribution--this is essentially the content of the Central Limit Theorem. Intuitively, we expect half of our pieces to be above their respective medians and the other half to be below their medians; when we add them all together the errors tend to cancel.

Often, however, there is reason to believe that there may be some correlation among the pieces. For instance, if we are estimating the total ultimate losses for a book consisting of several lines of long-tail business, we might model each of them separately. Future inflation (or deflation) might simultaneously increase (or decrease) each of those lines effectively creating correlation.

[5] If the pieces had very large negative correlations, it is possible that the confidence intervals could get smaller, but such instances are rare and would be quite unusual.

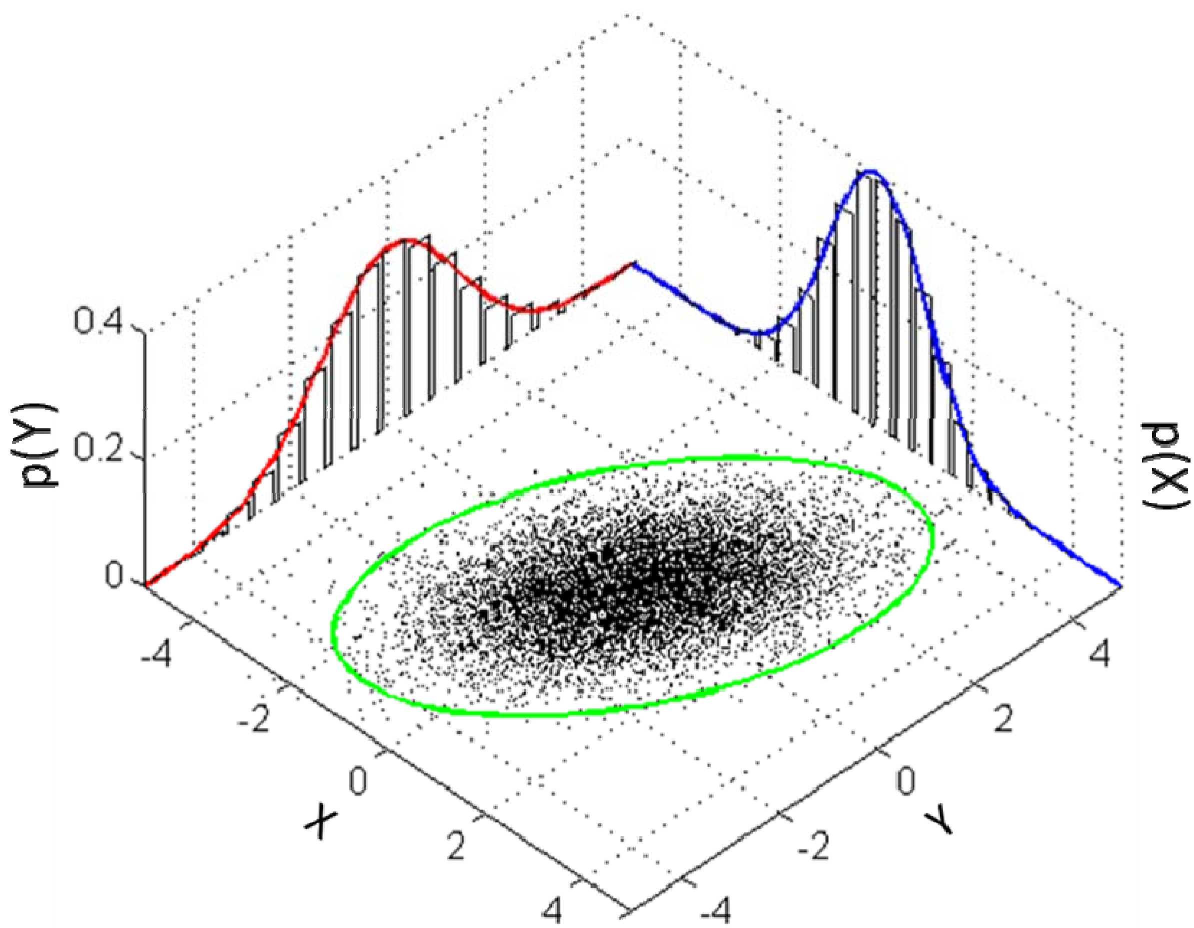

Figure 3: The marginal densities of a bi-normal pair of random variables

Figure 3, above, illustrates the marginal densities (in red and blue) of a bi-normally distributed pair of random variables[6]. The green ellipse represents a level curve of the joint distribution (in this instance, a 3-sigma ellipse[7]). If the correlation is positive, the distribution of the sum corresponds to the major axis of the ellipse (if the correlation is negative, the minor axis). This shows that the precision of the sum is smaller than the precision of the summands (unless the correlation is very negative).

[6] This image is in the public domain. The code to produce this graph can be found here: http://en.wikipedia.org/wiki/File:MultivariateNormal.png

[7] Symmetric confidence intervals for univariate normal variables have the property that the density function is equal at each end of the interval, i.e. the two endpoints of the interval form a level set of the density function. The analog for bivariate or multivariate normal random variables is to use a level set of the density function as the boundary for the analog of a confidence interval--a confidence region. In the case of bivariate normals, these level sets are ellipses. The green one shown corresponds to three standard deviations, i.e. approximately 99.7% of the probability is inside it.

Consider the following example: the company writes three lines of business, A, B, and C. You have estimated the outstanding losses for these three lines of business and selected ultimate losses for each. Your estimated outstanding losses along with some estimated ranges are:

Line of Business	Expected Losses	25^{th}-percentile Losses	75^{th}-percentile Losses
A	100	90	110
B	225	150	300
C	350	200	500

Further suppose that we believe that these estimates of the unpaid losses for lines A and B are 50% correlated, for lines B and C are 60% correlated and for lines A and C are 50% correlated.

Notice that we have selected our range of estimates for each line of business to a symmetric confidence interval (best estimate is in the center). This is because we are going to represent all three lines by a multi-normal random variable with mean vector, $\vec{\mu}$, and variance-covariance matrix, Σ.

First, we select the marginal distribution for line A. It is going to be normal with mean = 100 (the expected value) and some standard deviation, σ_A. We know that the 25^{th}-percentile is 90, which is 10 less than the mean, so using the standard normal table we conclude that σ_A must be 14.82. This technique is just the familiar method-of-moments.

Similarly, for line B we obtain mean = 225 and σ_B = 111.2, and for line C, mean = 350 and σ_C = 222.4.

Since the expected value of a sum is the sum of the expected values, the mean for our total will be 675.

To create the variance-covariance matrix, we start with the correlation matrix and multiply it on both the right and the left by a diagonal matrix with the respective standard deviations down the diagonal.

The correlation matrix: $\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.6 \\ 0.5 & 0.6 & 1 \end{pmatrix}$

The variance-covariance matrix: $$\Sigma = \begin{pmatrix} \sigma_A & 0 & 0 \\ 0 & \sigma_B & 0 \\ 0 & 0 & \sigma_C \end{pmatrix} \begin{pmatrix} 1 & \rho_{BA} & \rho_{CA} \\ \rho_{AB} & 1 & \rho_{CB} \\ \rho_{AC} & \rho_{BC} & 1 \end{pmatrix} \begin{pmatrix} \sigma_A & 0 & 0 \\ 0 & \sigma_B & 0 \\ 0 & 0 & \sigma_C \end{pmatrix}$$

The variance-covariance matrix for our example:

$$\Sigma = \begin{pmatrix} 14.82 & 0 & 0 \\ 0 & 111.2 & 0 \\ 0 & 0 & 222.4 \end{pmatrix} \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.6 \\ 0.5 & 0.6 & 1 \end{pmatrix} \begin{pmatrix} 14.82 & 0 & 0 \\ 0 & 111.2 & 0 \\ 0 & 0 & 222.4 \end{pmatrix} = \begin{pmatrix} 219.6 & 824.0 & 1,648.0 \\ 824.0 & 12,365.4 & 14,838.5 \\ 1,648.0 & 14,838.5 & 49,461.8 \end{pmatrix}$$

The correlation matrix is easier to interpret than the variance-covariance matrix. One reason for this is that the correlation matrix is dimensionless, meaning that it has no units, whereas the variance-covariance matrix has units, in this case square dollars. It is generally easier to work with dimensionless quantities when possible.

Line of Business	Expected losses	25^{th}-percentile losses	75^{th}-percentile losses	St. Dev. (Est.)	Estimated CV
A	100	90	110	14.8	0.148
B	225	150	300	111.2	0.494
C	350	200	500	222.4	0.635
Naïve Total	675	440	910	348.4	0.516
With Covariance Adjustment	675	465.3	884.7	310.9	0.461

The range for the "naïve total" is obtained by summing the endpoints of the intervals for each line of business. This corresponds to comonotonicity[8], which in the case of normal random variables means 100% correlation. See, for example, [S].

The last line labelled "with covariance adjustment" shows the 25^{th}- to 75^{th}-percentiles for the sum using the given correlations. The term covariance adjustment is taken from the US Statutory Risk Based Capital (RBC) calculation.

This calculation is easily reproduced using only Lines A, B, and C in the accompanying Excel spreadsheet, the use of which is described in detail later in this paper.

[8] Incidentally, the copula corresponding to this is the upper Fréchet–Hoeffding bound.

3.3 Why Normal Distributions?

Normal distributions are not the only distributions with the property that they are closed under addition[9]. Gamma distributions and many other families of distribution also have this property, at least when the summands are independent and possibly with some restrictions on the parameters. We choose to use normal distributions to approximate the individual distributions that we are going to combine because:

- Many naturally arising estimators have large-sample normal distributions. In particular, large-sample bootstrap estimators are asymptotically normal.
- The Central Limit Theorem suggests that averages (and hence sums) of independent observations will become normally distributed when the sample sizes get large.
- We can easily incorporate correlation and interpret it.
- It tends to produce results that seem reasonable.

3.3.1 Considerations in Selecting Correlations

Correlations are related to the R^2 statistic that comes from performing a simple regression of one of the two variables on the other. This statistic gives the proportion of the variation in the response variable that is explained by the explanatory variable. The square root of R^2 is an estimator for the correlation between the two variables. So, an R^2 of 49%, which means that about half of the variation on one variable is explained by the other, corresponds to an estimated correlation of 0.70. Higher R^2 values correspond to even larger correlations.

Correlations can be estimated from historic data, if available, or they can be selected judgmentally. Some caution should be exercised when using historic data, as common estimation methods can severely underestimate correlations, especially when the correlation is large. The choice of a value for the correlation coefficient can be influenced by how the result is to be used. If the goal is to obtain a central estimate, correlations based on historic levels may be adequate. On the other hand, if the goal is to obtain estimates in the tails, higher correlation selections may be justified, because correlations tend to be higher in the tails--when it rains, it pours.

4. An Example

The accompanying Excel spreadsheet illustrates how this technique can be used. The spreadsheet accepts up to eight lines-of-business and combines them into a single total. The user

[9] A family of distributions is said to be "closed under addition" if, whenever two members of the family are added together, the resulting sum is in the same family, but possibly with different parameters.

gives two points on a normal curve for each line-of-business. The user also specifies a correlation matrix for the various lines-of-business.

The reserve being estimated is outstanding losses. This purely hypothetical example uses five lines. Lines A, B, and C are taken from the above example[10]. Line D is a large loss, which the claim handler estimates will settle for $250 million, although there is a chance that it could realistically be as little as $150 million or as high as $400 million. Selecting the 20th-percentile to be $150 million and the 90th-percentile to be $400 million produces an expected loss close to $250 million; this is shown on the Output Page. Tweaking the 90th-percentile to be the 89.67th-percentile trues up the expected loss to be $250 million. Line E represents the unpaid losses from a recent catastrophe. They are currently estimated to be $450 million, but are expected to grow to $500 million. There is a small chance that the ultimate losses will turn out to be much worse--we selected a 95th-percentile loss of $1,155 million. We have chosen to enter $500 million as the 50th-percentile (which is the mean for symmetric distributions, such as the normal distribution.) The spreadsheet uses the method of percentile matching (see, for example, [KPW]) and a univariate normal is determined by two parameters (say, the mean and the standard deviation), so we can specify two percentiles for each line.

For the correlations among Lines A, B, and C, we will use the correlations we used above. We believe that Line D will act less like Line A or B than Line C, so we select 25% for the first two and 50% for the third correlation. Finally, we feel that an increase in Line E would come from a general adverse change in insurance loss reserves in general (less friendly courts, unexpected inflation, etc.), so we select 50% for the correlation with each of Lines A, B, C, and D.

The resulting correlation matrix is positive definite, so there is a multi-normal distribution that has our selected correlation matrix and that has our modelled losses for our lines of business as its marginals. The sum of our five lines of business has a normal distribution with mean and standard deviation computed by the spreadsheet. It is now an easy matter to select the mean and the two specified percentiles. These are shown on the Output Page along with the ranges corresponding to no correlation and comonotonicity (“100% correlation”).

The spreadsheet will calculate a value for any given percentile; however, it is designed for estimates somewhat close to the center of the distribution. Solvency II calls for calculations at specified percentiles such as the 99.5th-percentile. Caution should be exercised in estimating such high percentiles using these methods.

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[10] To reproduce the earlier example, simply set the Line D and Line E losses at the 25th- and 75th-percentiles to zero.

5. Concluding Remarks

Estimates for a given quantity often are obtained by combining other estimates. In the case of multiple estimates for the given quantity, we can combine them and obtain an estimate with more precision than any of the individual estimates. On the other hand, if our estimate is for a sum and we have estimates for the summands, we cannot hope to obtain (nominal) precision better than the worst of our summands, and in fact we cannot even do that well. Often, however, it is possible to improve the precision in a relative sense, using a measure such as the coefficient of variation.

Patel and Raws' simulation work showed that the choice of distribution did not matter very much, but they had to assume independence. We give up the choice of distribution, always selecting a normal distribution, but in exchange we recapture the ability to select the correlation structure. Since, in many cases, independence cannot reasonably be assumed, this is a real advantage.

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Supplementary Material

There is an accompanying Excel spreadsheet which contains the examples from this paper. It can be found on the CAS website. The URL is: http://www.casact.org/pubs/forum/14fforum/.

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