Credibility for Experience Rating, A Minimum Variance Approach

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Abstract:

This paper attempts to provide a relatively simple, but still mathematically meaningful context for applying Bühlmann credibility to large account experience rating. It further extends this to rating excess layers. It also allows for the inclusion of an additional complement of credibility to the traditional weighting of excess experience and ILF derived indications. Finally this paper gives guidance as to when exposure rating indications should not be used.

Keywords. Credibility, Experience Rating

1. INTRODUCTION

Let's say you are using two, or potentially more, methods to estimate expected losses. It is well known that combining a number of different estimates will generally lead to a better estimate than any single estimate. The question becomes what is the best mix of these different estimators. One very simple answer is to use the weighting that minimizes the estimation error surrounding the composite estimate. The simplest and most common measure of error to minimize is variance, which of course is equivalent to minimizing standard deviation. As it turns out, the optimal weighting of two independent estimates is inversely proportional to their relative variances. In other words, if one estimator has half the variance of the other, than it should be weighted twice as heavily, and therefore has a credibility of two-thirds. This is relationship is true regardless of context, experience rating credibility, class based rating credibility, or even estimations of an utterly non-insurance nature.

This point is implicit in the original Bühlmann[1] formulation of least squares credibility. Boor[2] makes this point much more explicitly and extends it to the case where the errors between the two estimates are correlated. This paper takes this formulation to case of large account excess pricing, and extends it to include the introduction of a third estimate and to cases where some of the estimates may no longer be relevant.

In the context of large account pricing, primary layers are based on two basic estimates: manual rates and experience rates. These estimates are usually based on losses capped at a basic or working layer (here designated WL) limit in order to keep the variance of estimates to a manageable level. Manual rates, also known as class or exposure rates, are class averages which will be more [2]representative for some large risks than for others. Experience rating is based strictly on an account's own historical experience adjusted for development, trends and exposure changes. An estimate based on an account's own history has the advantage of being much more relevant to that account's future experience than the class averages. However, but for all but the largest risks the volatility of year to year experience will be too high for it to be the best estimate on its own.

Pricing excess layers is subject to even greater errors. Historical experience in excess layers is even more volatile, and excess experience rating requires larger development factors and larger trend factors bringing greater estimation errors. The second generally accepted method for estimating excess losses is by applying an increased limits factor to working layer loss estimates. However, by applying ILF's to an already volatile primary loss estimate greatly increases the error in the excess layer. In addition, the ILF Method estimate and the excess experience estimates are correlated, since a single large loss will increase both the working layer and excess loss experience. This correlation will only add to total variance of the estimates and will alter the optimal weighting. If a third estimate of excess losses is available, perhaps a manual excess rate like the Swiss Re or other reinsurance benchmarks or an internal company estimate, it can add accuracy to the overall estimate. This paper attempts to incorporate all these adjustments into an overall optimal weighting scheme.

2. APPLICATION

The theory for this has been in the literature for years. In reality, the tricky part of this calculation is assessing the errors of using each method. In the basic Bühlmann formulation the error associated with using experience rates was the Expected Process Variance, which is how much an account's historical experience is expected to vary year to year. It does not take into account how much a real life company changes year to year (a fact Bühlmann cited as future research), nor does it take into account how much error trend and development estimates bring into the process.

Ignoring these errors in the analysis of large companies can be problematic. Most calculations of the experience rating variance only increase proportionately with exposures, while the variance from manual ratings will increase with the square of exposure. Thus a comparison of relative errors will exaggerate the credibility of experience for large risk. Venter[3] suggests that errors in experience can be modeled with two components, one linear with exposures plus one quadratic. The quadratic factor in effect means process variance is not completely diversifiable. Errors from trending and developing losses, always very significant for excess layers, are certainly not diversifiable and could be approximately quadratic.

Errors from changing operations are an additional significant challenge. Here underwriting knowledge is essential for determining experience rates. The error behind estimates is probably impossible to determine, but are not likely to be significant. The error behind poor experience rating data likewise is probably unmeasurable as well, and can be more important than any other error.

The error associated with using manual rates in the original Bühlmann formula is only how different risks within the same rating class are. This is the Variance of Hypothetical Means (VHM). Techniques are available to estimate this. However, these estimates likely underestimate the errors associated with manual rates. The manual rates themselves may not have a sufficient loss volume to be fully credible. A perhaps even greater problem is that many large risks do not fit well into the class rating schemes. Many large risks are very unique entities, not represented well by any class and conglomerates which include many risk categories. Many of the rates coming from rating bureaus or company rating systems are based only on smaller, single class risks. It is well known that large risks behave differently from small risks and often demonstrate economies of scale when it comes to loss prevention.

Despite these difficulties, assessing these risks is well worthwhile. There will be some arbitrary judgments involved and inevitable there will be factors that cannot be taken into accounts. Breaking down the risks into these smaller components adds insight to the process compared with using more arbitrary measures (just selecting a k in n/n+k) that seem right for a subset of risks. At the very least one can check current credibility measures against this framework for reasonability. Given all these sources of risk, beyond those that can be reasonably estimate, it is always necessary for the actuary to use informed judgment for both designing rating tools and in helping underwriters to interpret and understand the results.

For most situations involving large account pricing a minimum variance approach is to be preferred. Limited fluctuation credibility (the experience has an x% chance of being within y% of the correct answer) implicitly assumes that that the manual rates are appropriate no matter what their relative reliability. So it probably makes little difference whether Minimum Variance or Limited Fluctuation Credibility is used if one cannot assess to any degree the reliability of manual rates. However, if one can come up with a reasonable estimate; say something of the form that manual rates are accurate for any given risk plus or minus z%, then it should be preferable to use a Minimum Variance approach.

True Bayesian approaches to credibility will yield better results when loss distributions are known with certainty, although for many common distributions they yield the same estimates. However, it is very unlikely that the true underlying loss distributions will that closely enough resemble those assumed in Bayesian analysis to yield better results. Thus this minimum variance approach, also known as Bayesian credibility, should yield just as good results without the added complexity and error added by assuming strict distribution forms. For most of this paper we do not need to assume any specific form of distribution, in many cases an empirical distribution will work just as well. We need only work with expected values, variances, covariances and correlations.

2.1 Working Layer Credibility

Let's define

$Exp_{WL} \equiv Experience\ based\ loss\ estimator\ limited\ to\ the\ working\ layer$

Man_{WL}
$\equiv Manual\ Rate, or\ Class$
$/Exposure\ based\ loss\ estimator\ limited\ to\ the\ working\ layer$

$Z_{WL} \equiv Credibility\ weighting\ of\ \ Exp_{WL}$

$Est_{WL} \equiv Z_{WL} \times\ Exp_{WL} + (1 - Z_{WL}) \times Man_{WL}$, this is the credibility weighted expected loss estimator limited to the working layer.

Let's determine the weighting w, that minimizes the variance of combination of the estimators Exp_{WL} and Man_{WL}. This quantity would be $Var(Est_{WL}) = Var(w \times Exp_{WL} + (1 - w) \times Man_{WL})$. To find the minimum variance estimator we set the derivative of $Var(Est_{WL})$ with respect to w equal to zero.

$$\frac{dVar(Est_{WL})}{dw} = \frac{dVar(w \times Exp_{WL} + (1 - w) \times Man_{WL})}{dw}$$
$$= 2w\sigma^2_{ExpXS} + 2(1 - 2w)\rho_{ExpWL,ManWL}\sigma_{ExpWL}\sigma_{ManWL} - 2(1 - w)\sigma^2_{ManWL} = 0$$

The correlation between manual rates and experience rates will generally be zero as they are separate calculations. While the experience of a given account is potentially in the manual rating database, in most cases this effect should be very small. In the next section we will analyze the situation where the correlation is unequal to zero. Assuming independence, the above equation simplifies to:

$$\frac{dVar(Est_{WL})}{dw} = 2w\sigma^2_{ExpWL} - 2(1 - w)\sigma^2_{ManWL} = 0$$

Rearranging terms and solving for w yields,

$w = \frac{\sigma^2_{ManWL}}{\left(\sigma^2_{ManWL}+\sigma^2_{ExpWL}\right)}$ and

$$1 - w = \frac{\sigma^2_{ExpWL}}{\left(\sigma^2_{ManWL} + \sigma^2_{ExpWL}\right)}$$

What this implies is that the minimum variance of the estimator Est_{WL} is achieved when the weight assigned to the experience rating is equal to the relative size of the variance of manual rate. For instance if the variance of the manual rate is twice variance of the experience rate, then the weighting assigned to the experience rate is twice that of the manual rate.

How do we estimate the variance of our manual rates and experience rating? In Bühlmann's original formulation, σ^2_{ManWL} was the Variance of Hypothetical Means. That is to say how different is the class mean or exposure rate from the true underlying mean of a given risk. We should amend that view to include error coming from less than perfectly credible manual rates and questions about the applicability of manual rates based generally on small risks to large risks. The variance in manual rates can come from the original class rating statistics, or an examination of the distance between manual rates and similar rating classes. This will generally require a decent amount of subjective judgment.

Estimation of σ^2_{ExpWL} also requires judgment. In the original Bühlmann formulation, this was the Expected Process Variance (EPV) of a single risk or period divided by the number of risks or periods observed. For large account rating n will generally be the number of years a risk is observed. Like in our ordinary calculation of sample variance, increasing the number of years observed decreases the variance of our estimate of the mean loss cost. Because the EPV is presumed to be known, we do not need to divide the EPV by (n-1), but can divide by n instead.

Calculation of σ^2_{ExpWL} could come from looking at the variance of annual loss costs in the history, adjusted for trends, development and exposure changes. In practice this estimate is too unstable. A better estimate is to use the manual rates for the risk. The implied severity distributions coming from ILF's can provide an estimate for both mean and variance of limited severities. Dividing the manual rates by the expected average limited severity provides an estimate of annual frequency. Making the usual Poisson assumptions about claim frequencies, we can set the frequency variance equal to the mean claim frequency. We could of course assume some contagion factor in our claims and use a negative binomial distribution. We can then estimate σ^2_{ExpWL} from the usual variance formula.

$$\sigma^2_{ExpWL} = \mu_{frequency}\,\sigma^2_{severity} + \mu^2_{severity}\sigma^2_{frequency}$$

Going back to the original Bühlmann equation we have, and substituting in VHM for σ^2_{ManWL} and EPV/n for σ^2_{ExpWL}

$$w = \frac{\sigma^2_{ManWL}}{\left(\sigma^2_{ManWL}+\sigma^2_{ExpWL}\right)} = \frac{VHM}{(VHM+EPV/n)} = \frac{VHM\times(\frac{n}{VHM})}{(VHM+EPV/n)\times(\frac{n}{VHM})} = \frac{n}{(n+EPV/VHM)} = \frac{n}{n+k} = Z_{WL}$$

So we can now call, w, our weight, the credibility factor

$$Z_{WL} = \frac{\sigma^2_{ManWL}}{\left(\sigma^2_{ManWL}+\sigma^2_{ExpWL}\right)} \; This\ is\ Equation\ 1.$$

This makes the total variance of our estimate:

$$\begin{aligned}\sigma^2_{EstWL} &\equiv Var(Est_{WL}) = Var(Z_{WL}\times\ Exp_{WL} + (1-Z_{WL})\times Man_{WL}) \\ &= \left[\frac{\sigma^2_{ManWL}}{\left(\sigma^2_{ManWL}+\sigma^2_{ExpWL}\right)}\right]^2\times\sigma^2_{ExpWL} + \left[\frac{\sigma^2_{ExpWL}}{\left(\sigma^2_{ManWL}+\sigma^2_{ExpWL}\right)}\right]^2\times\sigma^2_{ManWL} \\ &= \frac{\sigma^2_{ExpWL}\sigma^2_{ManWL}}{\left(\sigma^2_{ManWL}+\sigma^2_{ExpWL}\right)} \; This\ is\ Equation\ 2.\end{aligned}$$

Inspection of this formula shows that the variance of the combined estimate is lower than either of the individual variances. Thus we have shown that we have a working layer estimate with the minimum error.

We now move to a similar formula for excess rates. Instead of weighting primary experience rates and primary manual rates, we will now wish to weight excess experience rates and rates generated by multiplying an ILF to our working layer loss estimate. This becomes more complicated because use of ILF's adds error to our working layer estimate with its own error. It is also more complicated because the ILF Method estimate and the Experience Rating estimate are correlated, as both will be affected by random large losses. This correlation will alter our optimal proportions away from our inverse variance rule, towards a greater weighting to the lower variance estimate. This shift will be seen when we derive Equation 5 later in this essay, which is an expansion of Equation 1 to include correlation.

2.2 Variance of ILF Method for Estimating Excess Loss Rates

ILF (Increased Limits Factor) is generally defined as the relation between total limits pure premium and basic limits pure premium, and has also been shown to the be the ratio of the expected severity limited to total limits to the expected severity limited to basic limits. For the purposes of this paper, E(ILF) will be defined as the ratio between an expected Excess Layer severity (Sev_{XS}) and an expected Working Layer severity (Sev_{WL}). We are not defining it here, for these purposes, as the ratio of the random variables representing either total loss or average severities in the two layers.

$$ILF = E(Sev_{XS})/E(Sev_{WL})$$

Let $\widehat{ILF}$ represent our estimator of the ILF, perhaps as represented in company or ISO ILF tables. Therefore the ILF Method for determining excess losses will be defined as

$$ILF\ Method\ Expected\ Loss \equiv \widehat{ILF} \times Estimated\ Working\ Layer\ Expected\ Loss$$

As has already been shown the credibility weighted estimate of the working layer is

$$Est_{WL} = Z_{WL} \times Exp_{WL} + (1 - Z_{WL}) \times Man_{WL}$$

Therefore the ILF Method Expected Loss estimate is

$$\widehat{ILF} \times Est_{WL} = \widehat{ILF} \times [Z_{WL} \times Exp_{WL} + (1 - Z_{WL}) \times Man_{WL}]$$

The difference between the ILF and the $\widehat{ILF}$ will merely be the estimation error in measuring an ILF for a given account. For this paper we are not defining the ILF as the ratio of excess and working layer losses but instead the ratio of expected losses in the excess layer to expected losses in the working layer.

Estimating an excess layer pure premium with an ILF adds an additional source of variance to the overall estimate.

Therefore, $Var\left(\widehat{ILF} \times Est_{WL}\right) > \widehat{ILF}^2 \times Var(Est_{WL})$.

It is generally reasonable to assume that the working layer expected losses and $\widehat{ILF}$ are independent:

$$
\begin{aligned}
Var\left(\widehat{ILF} \times Est_{WL}\right) &= E\left(\widehat{ILF}^2 \times {Est_{WL}}^2\right) - E^2\left(\widehat{ILF}\right) \times E^2(Est_{WL}) \\
&= E\left(\widehat{ILF}^2\right) \times E\left({Est_{WL}}^2\right) - E^2\left(\widehat{ILF}\right) \times E^2(Est_{WL}) \\
&= \left[Var\left(\widehat{ILF}\right) + E^2\left(\widehat{ILF}\right)\right] \times \left[Var(Est_{WL}) + E^2(Est_{WL})\right] - E^2\left(\widehat{ILF}\right) \times E^2(Est_{WL}) \\
&= \left(Var\left(\widehat{ILF}\right) \times Var(Est_{WL})\right) + \left(\widehat{ILF}^2 \times Var(Est_{WL})\right)\left(Est_{WL}^2 \times Var\left(\widehat{ILF}\right)\right)
\end{aligned}
$$

Equivalently, we can write:

$$\sigma^2_{\mathrm{ILF \times WL}} = \left[(\sigma^2_{\mathrm{ILF}} \times \sigma^2_{\mathrm{EstWL}}) + \left(\widehat{ILF}^2 \times \sigma^2_{\mathrm{EstWL}}\right) + (Est_{WL}^2 \times \sigma^2_{\mathrm{ILF}})\right], \qquad \text{(Equation 3)}$$

.

This is also our first major equation for determining excess layer credibilities.

Estimating the variability of $\widehat{ILF}$ is naturally a tricky proposition. One way to estimate these is too look at different ILF tables. One could estimate the standard deviation as half the distance between the current ILF and the next higher ILF table. Thus for a certain combination of limits the ISO Table A ILF is 1.4 and Table B is 1.6. A fair estimate of standard deviation might be 0.1. Thus for the majority of Table A risks the appropriate ILF to use would be between 1.3 and 1.5. Remember the error we are interested in is the appropriateness of using the expected ILF's found in our ILF tables. $Var\left(\widehat{ILF}\right)$ is analogous to the Variance of Hypothetical means in our manual rates, where we (unlike Bühlmann's most basic formulation) do not have to assume they are perfectly accurate for a class as a whole. In fact, a very similar technique can be used for estimating the VHM of our manual rates.

2.3 Correlation between Excess Experience Rating and the ILF Method

Let us define:

Exp_{XS}; Experience based loss estimate of the excess layer, trended, developed and exposure adjusted.

The correlation between the excess experience rating and the ILF method is clearly greater than zero. Large losses will increase both the working layer experience rating and the excess layer experience rating. Assume we already have the correlation between aggregate working layer and excess losses ($\rho_{ExpXS,ExpWL}$). This can be estimated either empirically from a sample of loss projections or calculated explicitly from frequency and severity distributions. This is calculated in the Appendix. The rest will need to be broken down into its components.

$$\rho(Exp_{XS}, \widehat{ILF} \times Est_{WL})) = \rho(Exp_{XS}, \widehat{ILF} \times (Z_{WL} \times Exp_{WL} + (1 - Z_{WL}) \times Man_{WL}))$$

Examining the covariance between the excess experience and the ILF Method we have

$$
\begin{aligned}
&Cov(Exp_{XS}, \widehat{ILF} \times (Z_{WL} \times Exp_{WL} + (1 - Z_{WL}) \times Man_{WL})) \\
&\qquad = Cov\left(Z_{WL} \times \widehat{ILF} \times Exp_{WL}, Exp_{XS}\right) + Cov((1 - Z_{WL}) \times \widehat{ILF} \times Man_{WL}, Exp_{XS})
\end{aligned}
$$

There is no apparent dependence between $\widehat{ILF}$, Z_{WL}, the manual rate or excess experience so we can set $Cov((1 - Z_{WL}) \times \widehat{ILF} \times Man, Exp_{XS})$ to zero. This yields:

$$
\begin{aligned}
&Cov(Exp_{XS}, \widehat{ILF} \times (Z_{WL} \times Exp_{WL} + (1 - Z_{WL}) \times Man_{WL})) \\
&\qquad = Cov\left(Z_{WL} \times \widehat{ILF} \times Exp_{WL}, Exp_{XS}\right)
\end{aligned}
$$

Since $\widehat{ILF}$ and Z_{WL} are estimators rather than the original random variables they can be treated as independent of our actual experience turning our equation into

$$Cov(Exp_{XS}, \widehat{ILF} \times (Z_{WL} \times Exp_{WL} + (1 - Z_{WL}) \times Man_{WL})) = Z_{WL} \times \widehat{ILF} \times Cov(Exp_{WL}, Exp_{XS}) = Z_{WL} \times \widehat{ILF} \times \rho_{ExpXS,ExpWL} \times \sigma_{ExpXS} \times \sigma_{ExpWL}$$

This implies that

$$\rho(Exp_{XS}, \widehat{ILF} \times (Z_{WL} \times Exp_{WL} + (1 - Z_{WL}) \times Man_{WL})) = \frac{Z_{WL} \times \widehat{ILF} \times \rho_{ExpXS,ExpWL}\sigma_{ExpXS}\sigma_{ExpWL}}{\sigma_{ExpXS}\sigma_{ILF((ZExpWL+(1-Z)Man)}} = \frac{Z_{WL} \times \widehat{ILF} \times \rho_{ExpXS,ExpWL}\sigma_{ExpWL}}{\sigma_{ILF\times \text{EstWL}}}$$

Or

$$\rho(Exp_{XS}, \widehat{ILF} \times Est_{WL})) \equiv \rho_{ExpXS,ILF\times WL} = \frac{Z_{WL} \times \widehat{ILF} \times \rho_{ExpXS,ExpWL}\sigma_{ExpWL}}{\sigma_{ILF\times \text{EstWL}}} \quad (Equation\ 4)$$

This is our second main equation to determine the credibility of our excess experience. We can see that it is a function of the credibility of the working layer pick. If the experience is little used in the working layer pick then there is little correlation between the excess experience and the working layer pick, in other words there is little correlation between the excess experience and the ILF times a manual rate.

2.4 Excess Credibility – ILF Method and Excess Experience

Let our Excess Rate, Est_{XS}, be defined as the weighted average of our Excess Experience and ILF Method Estimates. Let w be the weight assigned to excess experience rate.

$$Est_{XS} = \text{w} \times Exp_{xs} + (1 - w) \times (\widehat{ILF} \times Est_{WL})$$

$$Var(Est_{XS}) = Var\left(\text{w} \times Exp_{xs} + (1 - w) \times (\widehat{ILF} \times Est_{WL})\right) = w^2\sigma^2_{ExpXS} + 2w(1 - w)\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILFxWL} + (1 - w)^2\sigma^2_{ILF\times WL}$$

Note that for brevity when we want to represent the ILF Method as a subscript we use ILF x WL, rather than the more precise $\widehat{ILF}$ x EstWL.

Minimize the variance of the above expression by taking its derivative with respect to w and setting it equal to zero:

$$\frac{dVar(Est_{XS})}{dw} = 2w\sigma^2_{ExpXS} + 2(1 - 2w)\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL} - 2(1 - w)\sigma^2_{ILF\times WL} = 0$$

Regrouping terms yields:

$$w\left(\sigma^2_{ExpXS} - 2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL} + \sigma^2_{ILF\times WL}\right) + \rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL} - \sigma^2_{ILF\times WL} = 0$$

Solving for w, we have

$$w = \frac{(\sigma^2_{ILFxWL} - \rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL})}{\left(\sigma^2_{ExpXS} - 2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times EstWL} + \sigma^2_{ILF\times WL}\right)}$$

So w is the weight that minimizes the error when combining the excess experience and ILF method. We can redefine w as our excess credibility σ^2_{ExpXS} factor. This equation, in a more general context has previously been derived by Boor (op. cit.) and others.

$$Z_{xs} = \frac{\sigma^2_{ILF\times EstWL} - \rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}}{\left(\sigma^2_{ExpXS} - 2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL} + \sigma^2_{ILF\times WL}\right)}, \qquad \text{(Equation 5)}$$

From the above equation we need to be able to calculate the variance of the ILF method $\sigma^2_{ILF\times WL}$, the variance of the excess experience rating σ^2_{ExpXS}, and the correlation between the two $\rho_{ExpXS,ILF\times WL}$. Estimation of σ^2_{ExpXS} is very similar to the estimation of working layer experience variance. Here too we can break up the ILF Method expected excess into frequency and severity components. For the other quantities we have already derived Equations 3 and 4.

2.5 Three way credibility: Adding in excess manual rates

Additional reduction in the variance of excess can obtained if you have a third source of estimates, beyond excess experience and the ILF approach. In Europe there are a number of industry benchmarks, such as the Swiss Re curves in casualty. Alternatively companies may have their own excess rate exposure estimates.

Let's call this excess rate our excess manual rate Man_{XS}.

Our new excess estimate will be

$$Est_{XS} = \mathrm{w}_1 \times Exp_{xs} + w_2 \times (\mu_{ILF} \times Est_{WL}) + (1 - \mathrm{w}_1 - \mathrm{w}_2) \times Man_{XS}$$

Taking the variance of this estimate we get:

$$\begin{aligned} Var(Est_{XS}) &= Var\left(\mathrm{w}_1 \times Exp_{xs} + w_2 \times \left(\widehat{ILF} \times Est_{WL}\right) + (1 - \mathrm{w}_1 - \mathrm{w}_2) \times Man_{XS}\right) \\ &= w_1^2\sigma^2_{ExpXS} + w_2^2\sigma^2_{ILF\times WL} + (1 - \mathrm{w}_1 - \mathrm{w}_2)^2\sigma^2_{ManXS} \\ &\quad + 2\mathrm{w}_1\mathrm{w}_2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL} + 2\mathrm{w}_1(1 - \mathrm{w}_1 - \mathrm{w}_2)\rho_{ExpXS,ManXS}\sigma_{ExpXS}\sigma_{ManXS} \\ &\quad + 2\mathrm{w}_2(1 - \mathrm{w}_1 - \mathrm{w}_2)\rho_{ILFxWL,ManXS}\sigma_{ILFxWL}\sigma_{ManXS} \end{aligned}$$

Assuming that there is no correlation between the excess manual rates and either the excess experience and the ILF method, this simplifies to

$$Var(Est_{XS}) = w_1^2\sigma^2_{ExpXS} + w_2^2\sigma^2_{ILF\times WL} + (1 - \mathrm{w}_1 - \mathrm{w}_2)^2\sigma^2_{ManXS} + 2\mathrm{w}_1\mathrm{w}_2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}$$

To minimize this quantity we take its derivative with respect to both w_1 and w_2 and set the equations equal to zero.

$$\frac{dVar(Est_{XS})}{dw_1} = 2w_1\sigma^2_{ExpXS} - 2(1 - \mathrm{w}_1 - \mathrm{w}_2)\sigma^2_{ManXS} + 2\mathrm{w}_2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL} = 0$$

$$\frac{dVar(Est_{XS})}{dw_2} = 2w_2\sigma^2_{ILF\times WL} - 2(1 - \mathrm{w}_1 - \mathrm{w}_2)\sigma^2_{ManXS} + 2\mathrm{w}_1\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL} = 0$$

This yields two equations in two unknowns. Regrouping, and dividing by 2, we get:

$$w_1\left(\sigma^2_{ExpXS}+\sigma^2_{ManXS}\right)+w_2\left(\sigma^2_{ManXS}+\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\right)=\sigma^2_{ManXS}$$

$$w_1\left(\sigma^2_{ManXS}+\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\right)+w_2\left(\sigma^2_{ILF\times WL}+\sigma^2_{ManXS}\right)=\sigma^2_{ManXS}$$

Solving for w_1 and w_2, we get after some manipulation:

$$Z_{Exp}=w_1=\frac{\sigma^2_{ManXS}\left(\sigma^2_{ILF\times WL}-\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\right)}{\sigma^2_{ExpXS}\sigma^2_{ManXS}+\sigma^2_{ILF\times WL}\sigma^2_{ManXS}-2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\sigma^2_{ManXS}+\sigma^2_{ExpXS}\sigma^2_{ILF\times WL}\left(1-\rho^2_{ExpXS,ILF\times WL}\right)}$$

$$Z_{ILF\times WL}=w_2=\frac{\sigma^2_{ManXS}\left(\sigma^2_{ExpXS}-\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\right)}{\sigma^2_{ExpXS}\sigma^2_{ManXS}+\sigma^2_{ILF\times WL}\sigma^2_{ManXS}-2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\sigma^2_{ManXS}+\sigma^2_{ExpXS}\sigma^2_{ILF\times WL}\left(1-\rho^2_{ExpXS,ILF\times WL}\right)}$$

$$Z_{ManXS}=1-w_1-w_2=\frac{\sigma^2_{ExpXS}\sigma^2_{ILF\times WL}\left(1-\rho^2_{ExpXS,ILF\times WL}\right)}{\sigma^2_{ExpXS}\sigma^2_{ManXS}+\sigma^2_{ILF\times WL}\sigma^2_{ManXS}-2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\sigma^2_{ManXS}+\sigma^2_{ExpXS}\sigma^2_{ILF\times WL}\left(1-\rho^2_{ExpXS,ILF\times WL}\right)}$$

If we assume correlation between the excess manual rates and the other two methods we could expand the above equations even further.

Note that if we could assume no correlation between the excess experience and the ILF Method, then the above equations become simpler. In essence, the credibility assigned to each method would have been proportional to the product of the variances of the other two methods. Thus we can extend our inverse variance rule for two estimates to an inverse product rule for three.

A couple of observations are to be noted. First, the uncorrelated risk gets a greater weighting than the other two risks. For instance if we set all the variances to be equal, the credibility of the manual rate is proportional to $1-\rho^2_{ExpXS,ILF\times WL}$ where the other estimates are proportional to $1-\rho_{ExpXS,ILF\times WL}$. Given a positive but imperfect correlation between excess experience and the ILF method (due to large losses) the excess manual rate gets a greater weighting.

3. A CLASSICAL STATISTICAL TWIST TO BAYESIAN CREDIBILITY

To this point this we have focused strictly on an expansion of the original Bühlmann approach to credibility, which is often called Bayesian Credibility. It is called this because even before Bühlmann the traditional credibility formula (Z x Average Experience + (1-Z) x Class Mean) could give the exact same answer as a formal Bayesian statistical approach, depending on the exact formula for Z. It was recognized that for certain combinations of prior probability distributions (here distribution where the manual rate is the mean) and likelihood functions (the distribution describing the experience) would yield a posterior distribution whose mean was equal to the results of our usual credibility formula. It was later shown that this works for a very broad class of distributions, known as the exponential family.[4]

Bühlmann's insight was that our usual credibility formula was the minimum least-squares estimator for any class, as long as Z= n/n+K, where K= EPV/VHM. Thus this result is pretty robust and does not depend on specifying any distribution. The extension of this method in this paper has the same advantage.

Implicit in this formulation is that the class rating is relevant to the risk we are rating. As was detailed above, for many large risks this will not necessarily be the case. Without adjustment, our credibility approach will always put some weight on our manual rates and our experience will never be considered fully credible. This is because neither the variance of manual rates will approach infinity nor will the variance of the experience rates approach zero. Even if the manual rate is a pretty good estimator for a risk, at some point an account should be sufficiently large that the experience rating alone will be the best estimator of the account's future experience. The NCCI experience rating plans recognize this, but make a somewhat arbitrary adjustment to arrive at a full credibility standard.

An approach which solves both these problems is to subject the manual rating to classical statistical testing. Specifically we will test the null hypothesis that the manual rate is valid, or more specifically that the manual rate is a valid mean for our risk given the actual experience. Our alternative hypothesis that the manual rate is invalid implies that the best estimate of rate for the risk is determined solely by the experience alone. In other words, if the null hypothesis is disproven, the proper credibility of the experience is 100%, and the account should be self-rated.

We chose as our null hypothesis that the manual rating is valid since class rating is the standard for most risks and because we want to temper the volatility of pure experience ratings. As with any hypothesis test a significance level needs to be assumed. The significance level we choose represents the probability that manual rate is either higher or lower than the experience due solely to chance, rather than being inappropriate for the risk being rated. Selection of the specific significance level is to some extent an arbitrary choice dependent on any number of considerations including:

1. Prior beliefs -a very high level of proof is required to disprove long held "facts" or to support a hypothesis that would be considered radical.
2. Consequences – a very high level of proof is required if the practical policy implied by the results is either expensive or risky. For example, there should be a high level of proof that a dangerous drug is effective before it is used to treat an illness.

In this case, we know that most manual rates have issues when applied to large risks, so we have good reason to not make our significance level too exacting. If we set our standards for rejecting the manual rates too low, we are at risk of seriously under-pricing business; however we will explore a way to limit this risk in the next section. Given the circumstances, a relatively commonly selected 5% significance level may not be a bad choice.

So we set up our hypothesis testing, for say the working layer:

$H_0: \mu_{WL} = Man_{WL};$

$H_0: \mu_{WL} < Man_{WL}, for\ the\ case\ where\ the\ manual\ rate\ is\ greater\ than\ the\ experience$

$\alpha = .05$

We could use the standard Student t-test:

$t = (\bar{x} - \mu_{WL})/(s/\sqrt{n})$, with n-1 degrees of freedom, where $\bar{x}$ is the average of your experience rating years and s is the sample standard deviation over n years

Perhaps a better formulation is to look at Man_{WL} and σ_{ManWL} from the prior section. As discussed before there are some better, and particularly more stable, estimates for the variance of the experience rating than just the sample standard deviation across what must be a limited sample of years. With these estimates one can go to either a standard normal test statistic, or better yet to choose an experience distribution that the actuary prefers. For instance with two moments in hand and knowledge that the experience should never be below zero we can do hypothesis testing assuming either a gamma or lognormal distribution for ease. Note that if we do this we will want to test aggregate losses over the experience period. Therefore we will look at $n \times Man_{WL}$ and $\sqrt{n} \times \sigma_{ManWL}$, which represent the first two moments for the aggregate loss distribution given that manual rate is valid. In practice, we will also want to do exposure adjustments to get to a true "As If" future basis. We compare this distribution to the actual, adjusted losses, $n \times Exp_{WL}$.

So again we look at if

$$Prob\left(\sum\nolimits_{n} x = n \times Exp_{WL} \middle| n \times \mu_{WL} = n \times Man_{WL}\right) \leq .05$$

And then we reject the exposure rates.

No explicit distribution is required; we could also use a standard of two or three standard deviations to reject the manual rates. However, using a distribution gives some more flexibility, even if it adds some model specification error.

If we reject the manual rates, the experience rating data becomes the only valid source for expected losses for a given account. At this point the account would generally be self-rated. For smaller accounts, the exposure rating will only be rejected if the exposure rate is dramatically different from experience. For very large accounts the exposure rating may be rejected even if it's reasonably close to the experience. This occurs, of course, because standard deviation of losses grows much more slowly than expected losses as an account gets larger.

Just because we reject the exposure rating, does not mean we have to use the experience rate unmodified. To be specific, we do not have to use the mean of experience, even suitably adjusted. This is particularly the case when the experience rating is less than the exposure rating; by rejecting exposure rates we have now eliminated a measure of conservatism and stability in the loss rating process.

Implicitly this assumes that under-pricing business is more dangerous than over-pricing, as the underwriting loss due to under-pricing is worse than the revenue loss due to over-pricing. This will not in practice always be true. Later we will address the case of when the experience is worse than the exposure rating.

Having eliminated both the stability and conservatism of exposure rates, we are faced with the possibility that the experience rate is still understated. Even if the exposure rating is too high to be valid, it doesn't mean that the good experience isn't partially a product of luck. Let's say that we had determined that based on the exposure rating the experience was in the 1st percentile or lower. We would reject the exposure rate. But to be conservative we could still say the experience was the 5th or 10th percentile, and explicitly make an assumption about how lucky we were.

If we did assume that the experience was the 10th percentile, we would need want to know what the underlying mean was, since this will be our new expected loss pick for the working layer. For many typical distributions we can use numerical techniques to converge upon the proper answer.

Since we will often use two parameter aggregate distributions, we can extend the logic laid out in the Working Layer Credibility section. Let's first assume that the underlying mean of the "true distribution" is, say, twice our selected 10th percentile. We can then imply a variance or standard deviation for that distribution by again assuming that selected ILF's have a reasonable claim severity distribution behind them. From this we can once again back into an implied frequency by dividing out the newly assumed mean and the ILF expected severity. Making the usual Poisson assumption, we assume that the frequency variance equals the mean. From this and our formula for the variance of an aggregate distribution, we can again parameterize the aggregate distribution. With the appropriate aggregate distribution we can then see what percentile our experience rate is. If our experience rating pick is below the 10th percentile, we can use a lower estimate for mean of the aggregate distribution; otherwise we use a higher estimate of the mean. We can through multiple iterations to get to a mean that is close enough.

Unfortunately adding this margin of error over the experience rating mean does mean that we will add some margin of error over the mean for even the largest risks. Surely that margin will be smaller as a percentage of the experience mean for larger risks, but it will always exist. Another solution is to use the newly determined mean as a new complement of credibility to the actual experience mean. The minimum variance credibility formulas of this paper can not be used, since we don't really have two separate variances to minimize, plus as discussed under minimum variance credibility (and all related approaches) you never have full credibility. Here, for this limited case, we can use a full credibility standard derived from a limited fluctuation approach.

Now let's examine the case where the experience rate is so much worse than the exposure rate, and so we reject the exposure rate as being too low. This removes a more liberal element from our calculation as well as a source of stability. We could use the procedure laid out above to come up with an estimate lower than experience, but still above what a credibility calculation including the exposure rate would have yielded. This becomes questionable, because we justified building in the margin of error above based on a principle of conservatism. We may do this if we believe there is some information left in the *direction* of exposure rating, even if we reject the magnitude of exposure rating. If we believe that the exposure rating tells us nothing about a specific account, we can use the experience rating unmodified or even *add* a margin of error to our estimate. This decision becomes a matter of actuarial or underwriting judgment.

4.0 RESULTS AND CONCULUSIONS

In this essay we are looking at the optimal way to combine a number of different estimates in calculating both primary and excess large account losses. At its base we need to compare the relative size of estimation errors in a structured way to come up with the best possible composite estimate. For the simple case of comparing two uncorrelated estimates, the optimal weighting is in inverse proportion to each method's estimation error. The bulk of the paper is expanding that frame work to take into account a third estimate, correlations, and connections between the errors coming from primary and excess ratings.

We have the following main sources of variance:

1. Working Layer Manual Rates – These need to be estimated, either from in informed judgment or from a close examination primary ratemaking techniques. This is more than just the Variance of Hypothetical Means, but need to incorporate additional sources of error including errors in the manual rate estimation as it applies to large accounts.

2. Working Layer Experience Rates – This can be estimated from actual year over year historical experience. However a more stable estimate, which does not bias against risks with better expected experience, will be to look at manual rates and come up with an expected frequency and severity based on ILF's. The Expected Process Variance can be calculated this way. However, as was the case with the Variance of Hypothetical Means, additional sources of error should be contemplated, including trend and development.

3. ILF's – ILF's should not be looked at solely as point estimates, but should be viewed as random variable requiring estimation. Assessing the error in ILF's avoids overweighting the ILF's approach to excess rates.

4. Excess Manual Rates – Like working layer manual rates these will have an uncertainty associated with them, however we would expect these errors to be greater. Excess manual rates may not be based on sufficient data, often have their own sets of implied ILF's, and require trending and developing of data.

5. Excess Experience Rates – Excess experience rates suffer all the same errors as working layer rates, however these too are much greater. A similar approach to determining variance can be estimated for excess rates; however we are even more dependent on imperfect measures to base these upon.

With these estimates, we can use the algebra presented here to come up with appropriate credibility weightings.

1. The Working Layer credibility for experience when combined with a manual rate:

$$Z_{WL} = \frac{\sigma^2_{ManWL}}{\left(\sigma^2_{ManWL} + \sigma^2_{ExpWL}\right)} \quad (Equation\ 1)$$

2. The total variance of the working layer, when using the above weighting to determine the optimal weighting for the experience and manual rates:

$$\sigma^2_{EstWL} = \frac{\sigma^2_{ExpWL}\sigma^2_{ManWL}}{\left(\sigma^2_{ManWL} + \sigma^2_{ExpWL}\right)} \quad (Equation\ 2)$$

3. The credibility assigned to an excess layers experience, when compared to the ILF method for determining excess losses, including recognition of the correlation between the ILF Method and the excess experience due to large losses being part of both estimates.

$$Z_{xs} = \frac{\sigma^2_{ILF\times WL} - \rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}}{\left(\sigma^2_{ExpXS} - 2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL} + \sigma^2_{ILF\times EstWL}\right)} \quad (Equation\ 5)$$

4. Equation 5 above requires an estimated variance for the ILF method, derived from the errors of both the working layer estimate and the error derived from the use of ILF's themselves:

$$\sigma^2_{\mathrm{ILF\times WL}} = (\sigma^2_{\mathrm{ILF}} \times \sigma^2_{\mathrm{EstWL}} + \widehat{ILF}^2 \times \sigma^2_{\mathrm{EstWL}} + Est^2_{WL} \times \sigma^2_{\mathrm{ILF}}) \quad (Equation\ 3)$$

5. Equation 5 above also requires an estimate for the correlation between excess experience and the ILF Method.

$$\rho_{ExpXS,ILF\times WL} = \frac{Z_{WL} \times \widehat{ILF} \times \rho_{ExpXS,ExpWL}\sigma_{ExpWL}}{\sigma_{ILF\times \text{EstWL}}} \quad (Equation\ 4)$$

6. Equation 4 requires a further estimation of $\rho_{ExpXS,ExpWL}$. The best way to estimate this is from severity distributions. From the Appendix we have:

$$\rho_{ExpXS,ExpWL} \cong \rho_{WL,XS} = \frac{(Limit)\mu_{SevXS}}{\sqrt{\sigma^2_{SevWL} + \mu^2_{SevWL}} \times \sqrt{\sigma^2_{SevXS} + \mu^2_{SevXS}}} \quad (Equation\ 6)$$

where the Limit refers to the Working Layer Limit of Liability.

7. If we have a third estimator for excess loss we can then expand the credibility formulas to:

$$Z_{ExpXS} = \frac{\sigma^2_{ManXS}(\sigma^2_{ILF\times WL} - \rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL})}{\sigma^2_{ExpXS}\sigma^2_{ManXS} + \sigma^2_{ILF\times WL}\sigma^2_{ManXS} - 2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\sigma^2_{ManXS} + \sigma^2_{ExpXS}\sigma^2_{ILF\times WL}\left(1 - \rho^2_{ExpXS,ILF\times WL}\right)}$$

$$Z_{ILF\times WL} \frac{\sigma^2_{ManXS}(\sigma^2_{ExpXS} - \rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL})}{\sigma^2_{ExpXS}\sigma^2_{ManXS} + \sigma^2_{ILF\times WL}\sigma^2_{ManXS} - 2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\sigma^2_{ManXS} + \sigma^2_{ExpXS}\sigma^2_{ILF\times WL}\left(1 - \rho^2_{ExpXS,ILF\times WL}\right)}$$

$$Z_{ManXS} = \frac{\sigma^2_{ExpXS}\sigma^2_{ILF\times WL}(1 - \rho^2_{ExpXS,ILF\times WL})}{\sigma^2_{ExpXS}\sigma^2_{ManXS} + \sigma^2_{ILF\times WL}\sigma^2_{ManXS} - 2\rho_{ExpXS,ILF\times WL}\sigma_{ExpXS}\sigma_{ILF\times WL}\sigma^2_{ManXS} + \sigma^2_{ExpXS}\sigma^2_{ILF\times WL}\left(1 - \rho^2_{ExpXS,ILF\times WL}\right)}$$

It is well known that rating casualty layers in difficult, especially for excess layers, because of the errors laid out above and more. However, with the framework presented by this paper, plus some more clever solutions to estimating the variances above, a methodical approach to determining credibility can be established. While the mechanics look complicated, most of these factors can be easily be put into a spreadsheet or a computer program. If need be, correlations could be ignored to simplify the calculations.

This method, a generalization of Bühlmann's Bayesian Credibility, does come up with an optimized weighting unlike limited fluctuation credibility, and does not require any kind of distributional assumptions like pure Bayesian analysis. To get answers we need to only estimate variances and correlations, which although tricky, is far easier than estimating appropriate distributions.

We have also examined a procedure for examining the case of when exposure rating should no longer be used at all. We do so by hypothesis testing the exposure rates as the underlying mean of a distribution that yielded our experience. If we reject the exposure rates, then we can use the experience rating mean. Alternatively we can use the experience mean plus a statistically determined margin of error. If desired this new pick can become the complement of credibility in a limited fluctuation calculation.

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Appendix A
Estimating the Correlation between Working Layers and Excess Layer Experience

In order to estimate the correlation between the ILF Method and Excess Experience, $\rho_{ExpXS,ExpWL}$, we need to estimate the correlation between aggregate Working Layer experience and Excess Layer experience. We will approximate this with $\rho_{XS,WL}$ which is ultimately calculated by looking at Working Layer and Excess Severities.

To do this we need to devise a formula for aggregate losses then calculate the correlation between Working Layer severities and Excess severities. Let's look at the covariance between the aggregate losses of in the Working Layer (WL) and Excess Layer (XS). The number of claims is unknown but represented by the random variable N. Lets designate each individual claim $SevWL_i$ for the Severity in the Working Layer of Claim i. We will make the usual assumption that individual claims are independently drawn from the same ground up severity distribution.

$$Cov(WL,XS) = Cov\left(\sum_{i=1}^{N} SevWL_i\,, \sum_{i=1}^{N} SevXS_i\right)$$
$$= E\left(\sum_{i=1}^{N} SevWL_i \sum_{i=1}^{N} XS_i\right) - E\left(\sum_{i=1}^{N} SevWL_i\right) \times E\left(\sum_{i=1}^{N} SevXS_i\right)$$

They key here is to examine the $E(\sum_{i=1}^{N} SevWL_i \sum_{i=1}^{N} SevXS_i)$, by looking at the conditional expectation of that quantity holding N constant, and then take expected value of that quantity with respect to N.

$$E\left(\sum_{i=1}^{N} SevWL_i \sum_{i=1}^{N} SevXS_i\right) = E_N\left(E\left(\sum_{i=1}^{n} SevWL_i \sum_{i=1}^{n} SevXS_i \middle| N = n\right)\right)$$
$$= E_N\left(E\left(\sum_{i=1}^{n} SevWL_i \times SevXS_i + \sum_{i=1}^{n}\sum_{i\neq j}^{n} SevWL_i \times SevXS_j \middle| N = n\right)\right)$$

In the expectation above $E(SevWL_i \times SevXS_i)$ can be broken into $E(SevWL_i) \times E(SevXS_j)$ only when $i \neq j$ since we assume separate claims are independent. When $i = j$, $SevWL_i$ *and* $SevXS_i$ will be highly correlated because they are the limited and excess portion of the same claim.

$$E\left(\sum_{i=1}^{N} SevWL_i \sum_{i=1}^{N} SevXS_i\right)$$
$$= E_N\left(N \times E(SevWL \times SevXS) + \left(N \times (N-1)\right) \times E(SevWL)E(SevXS)\right)$$
$$= E(N)E(SevWL \times SevXS) + E(N^2 - N)E(SevWL)E(SevXS)$$
$$= E(N)E(SevWL \times SevXS) + Var(N)E(SevWL)E(SevXS)$$
$$+ E^2(N)E(SevWL)E(SevXS) - E(N)E(SevWL)E(SevXS),$$

$as\ E(N^2) = Var(N) + E^2(N)$

We also have by similar logic

$$E\left(\sum_{i=1}^{N} SevWL_i\right) = E(N)E(SevWL) \; and \; E\left(\sum_{i=1}^{N} SevXS_i\right) = E(N)E(SevXS)$$

We can then substitute these last three equations into our equation for covariance above.

$$\begin{aligned} Cov(WL, XS) \\ &= E(N)E(SevWL \times SevXS) + Var(N)E(SevWL)E(SevXS) \\ &- E(N)E(SevWL)E(SevXS) \end{aligned}$$

Now we know that $Cov(WL, XS) = E(WL \times XS) - E(WL)E(XS)$, so rearranging this and inserting this in to the above equation yields:

$$Cov(WL, XS) = E(N)Cov(SevWL \times SevXS) + Var(N)E(SevWL)E(SevXS)$$

$$Corr(WL, XS) = \frac{Cov(\sum_{i=1}^{N} SevWL_i, \sum_{i=1}^{N} SevXS_i)}{\sqrt{Var(\sum_{i=1}^{N} SevWL_i)Var(\sum_{i=1}^{N} SevXS_i)}}$$

$$= \frac{E(N)Cov(SevWL \times SevXS) + Var(N)E(SevWL)E(SevXS)}{\sqrt{Var(SevWL)E(N) + E^2(SevWL)Var(N)} \times \sqrt{Var(SevXS)E(N) + E^2(SevXS)Var(N)}}$$

The equation within the square root is our usual equation for the variance of a compound process combining frequency and severity. For ease we write the above as

$$\rho_{WL,XS} = \frac{\mu_N \sigma_{SevWL,SevXS} + \sigma_N^2 \mu_{SevWL} \mu_{SevXS}}{\sqrt{\sigma_{SevWL}^2 \mu_N + \mu_{SevWL}^2 \sigma_N^2} \times \sqrt{\sigma_{SevXS}^2 \mu_N + \mu_{SevXS}^2 \sigma_N^2}}$$

Substituting in $\sigma_{SevWL,SevXS} = \rho_{SevWL,SevXS} \sigma_{SevWL} \sigma_{SevWL}$ yields

$$\rho_{WL,XS} = \frac{\mu_N \rho_{SevWL,SevXS} \sigma_{SevWL} \sigma_{SevWL} + \sigma_N^2 \mu_{SevWL} \mu_{SevXS}}{\sqrt{\sigma_{SevWL}^2 \mu_N + \mu_{SevWL}^2 \sigma_N^2} \times \sqrt{\sigma_{SevXS}^2 \mu_N + \mu_{SevXS}^2 \sigma_N^2}}$$

If we assume a Poisson process for claims, with the variance of N equal to mean of N, then all the N's cancel out and we have the following:

$$\rho_{WL,XS} = \frac{\rho_{SevWL,SevXS} \sigma_{SevWL} \sigma_{SevXS} + \mu_{SevWL} \mu_{SevXS}}{\sqrt{\sigma_{SevWL}^2 + \mu_{SevWL}^2} \times \sqrt{\sigma_{SevXS}^2 + \mu_{SevXS}^2}}$$

So now we have an equation for the correlation between aggregate losses in the working and excess layers. Presumably we have estimates for both expected severity and the variance of severity for both our working layer. Thus all we are missing is an estimate for the correlation between the working layer severity and the excess severity.

$$\rho_{SevWL,SevXS} = \frac{\sigma_{SevWL,SevXS}}{\sigma_{SevWL} \sigma_{SevXS}} = \frac{E(SevWL \times SevXS) - \mu_{SevWL} \mu_{SevXS}}{\sigma_{SevWL} \sigma_{SevXS}}$$

$$\begin{aligned} E(SevWL \times SevXS) \\ &= \iint_0^{Policy\ Limit} SevWL \times SevXS \times f(SevWL \times SevXS) \, d(SevWL) d(SevXS) \end{aligned}$$

Let our ground up claims be designated SevGU and let the Working layer limit be designated Limit.

If SevGU < Limit then SevWL = SevGU and SevXS =0

If SevGU >= Limit then SevWL = Limit and SevXS =GU-Limit

We can then rewrite the above equation in terms of SevGU

$$E(SevWL, \text{Sev}XS) = \int_0^{\text{Limit}} \text{SevGU} \times 0 \text{ f(SevGU) d(SevGU)} + \int_{\text{Limit}}^{\text{Policy Limit}} \text{Limit} \times (\text{SevGU} - \text{Limit}) \text{ f(SevGU)d(SevGU)}$$

The first integral is easily valued as zero, the second integral can be identified as Limit times the expected XS Claim severity.

$$E(SevWL \times SevXS) = \text{Limit} \times \text{E(SevXS)}$$

Thus we can rewrite $\rho_{SevWL,SevXS}$ as

$$\rho_{SevWL,SevXS} = \frac{\sigma_{SevWL,SevXS}}{\sigma_{SevWL}\sigma_{SevXS}} = \frac{E(SevWL, SevXS) - \mu_{SevWL}\mu_{SevXS}}{\sigma_{SevWL}\sigma_{SevXS}} = \frac{(Limit)\mu_{SevXS} - \mu_{SevWL}\mu_{SevXS}}{\sigma_{SevWL}\sigma_{SevXS}} = \frac{(Limit - \mu_{SevWL})\mu_{SevXS}}{\sigma_{SevWL}\sigma_{SevXS}}$$

Then we can substitute the above equation into our aggregate equation:

$$\rho_{WL,XS} = \frac{\rho_{SevWL,SevXS}\sigma_{SevWL}\sigma_{SevXS} + \mu_{SevWL}\mu_{SevXS}}{\sqrt{\sigma^2_{SevWL} + \mu^2_{SevWL}} \times \sqrt{\sigma^2_{SevXS} + \mu^2_{SevXS}}} = \frac{\rho_{SevWL,SevXS}\sigma_{SevWL}\sigma_{SevXS} + \mu_{SevWL}\mu_{SevXS}}{\sqrt{\sigma^2_{SevWL} + \mu^2_{SevWL}} \times \sqrt{\sigma^2_{SevXS} + \mu^2_{SevXS}}} = \frac{(Limit - \mu_{SevWL})\mu_{SevXS} + \mu_{SevWL}\mu_{SevXS}}{\sqrt{\sigma^2_{SevWL} + \mu^2_{SevWL}} \times \sqrt{\sigma^2_{SevXS} + \mu^2_{SevXS}}} = \frac{(Limit)\mu_{SevXS}}{\sqrt{\sigma^2_{SevWL} + \mu^2_{SevWL}} \times \sqrt{\sigma^2_{SevXS} + \mu^2_{SevXS}}}$$

This is Equation 6.

5. REFERENCES

[1] Hans Bühlmann, ASTIN Bulletin International Actuarial Association - Brussels, Belgium 1967: 4:3 p.199-207.

[2] Joseph Boor, Proceedings of the Casualty Actuarial Society Casualty Actuarial Society - Arlington, Virginia 1992: LXXIX p.166-185

[3] Venter, Gary G., Casualty Actuarial Society Forum Casualty Actuarial Society - Arlington, Virginia 2003: Winter 621-267

[4] Jewell, William S., ASTIN Bulletin International Actuarial Association - Brussels, Belgium 1974: 8:1, 77-90