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The Fall 2010 CAS *E-Forum*

The Fall 2010 Edition of the CAS *E-Forum* is a cooperative effort between the Committee for the CAS *E-Forum* and various other CAS committees.

The CAS Committee on Reserves (CASCOR) presents for discussion 11 papers prepared in response to the 2010 call for papers on reserving topics. This *E-Forum* also includes one additional paper.

This call paper that will be discussed by the authors at the 2010 Casualty Loss Reserve Seminar (CLRS) scheduled for September 20-21, 2010, in Lake Buena Vista, FL.

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Crop Insurance Reserving

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Abstract: The crop insurance industry is a private-public partnership, whereby the private companies issue policies and handle claims for multi-peril crop insurance policies, which are administered by the U.S. Department of Agriculture-Risk Management Agency. The private companies are reinsured by the Federal Crop Insurance Corporation under the terms of the Standard Reinsurance Agreement. Private companies also issue insurance policies not administered by RMA, which provide additional cover, typically referred to as "Crop-Hail."

Crop insurance is a short-tailed line of business; however, significant variation to the ultimate loss ratio exists on an annual basis. Reserving for crop insurance is unique due to the characteristics of the crop insurance policy and catastrophic nature of the risks: weather and price changes. This catastrophic risk is mitigated due to reinsurance from the Federal Crop Insurance Corporation. This paper presents methodology to estimate ultimate losses and reserves for crop insurance.

Keywords: Crop Insurance, Short-Tail, Catastrophe Reserves, Crop Hail

1. INTRODUCTION

Crop insurance is typically viewed as a short-tailed line of business as regards to reserving for ultimate liabilities. Since crop insurance provides coverage for both yield and price risks for a growing season, the annual results will exhibit significant volatility due to the catastrophic nature of weather, as well as price changes. The process for insuring farmers for the revenue risks associated with crops and livestock has been evolving, and the introduction of new policies has changed the calculation of the indemnities to the farmers in the event of loss. This paper addresses the exposures associated with crop insurance and discusses methodologies to estimate ultimate loss ratios and unpaid claim liabilities for these exposures.

The remainder of the paper proceeds as follows. Section 2 will discuss the background of crop insurance from a historic point of view. The underlying exposures such as crops, prices and insurance plans will then be discussed with the implications to forecasting ultimate losses. This will be followed by a discussion of the public-private partnership and the Standard Reinsurance Agreement (SRA). Finally, the accounting treatment of crop insurance is presented. Section 3 will discuss the methodologies and issues of forecasting losses associated with crop insurance. This section will include various pitfalls with traditional loss reserving methods that would apply to crop insurance.

Section 4 will discuss the conclusions of this paper and future areas of additional research.

2. BACKGROUND AND DISCUSSION OF INSURANCE PLANS

2.1 History

Prior to 1938, attempts by commercial insurers to write crop insurance were not successful due to low participation and lack of credible data, as well as the catastrophic nature of the risk. The federal crop insurance program was established in 1938 with the passage of the Federal Crop Insurance Act. The Federal Crop Insurance Corporation (FCIC) was created in 1938 to carry out the program. Initially, the program was limited to major crops in the primary producing areas and was considered mostly experimental. The Federal Crop Insurance Act of 1980 expanded the crop insurance program to many more regions of the country and encouraged more participation by offering a 30% premium subsidy.

While the participation increased during the 1980s, a major drought in 1988 led to an ad hoc disaster assisstance program that was authorized to provide relief to farmers. Additional disaster bills were passed in 1989, 1992, and 1993. The concern that the availability of federal relief in the event of a disaster served to reduce participation in Federal Crop Insurance led to the enactment of the Federal Crop Insurance Reform Act of 1994. This Act made participation in Federal Crop Insurance mandatory for farmers in order to be eligible for deficiency payments under price support programs, certain loans and other federal farm assisstance programs. A policy providing limited coverage was introduced called catastrophic (CAT) coverage, which was available for a nominal charge. Subsidies for additional coverage were increased. The Risk Management Agency (RMA) was created to administer FCIC programs and other non-insurance risk management and educational programs to support agriculture. Policies were introduced that incorporated price risk in addition to yield risk.

The Agricultural Risk Protection Act was passed in 2000, which increased insurance options and subsidies. Participation in revenue policies substantially increased. Partnerships between RMA and private entities were encouraged to develop new, innovative insurance products that covered additional crops, as well as livestock.

2.2 Crop Insurance Plans

A crop insurance plan provides protection to farmers due to loss of yield or revenue from insured perils. The majority of crop insurance plans is administered by RMA and is referred to as multi-peril crop insurance or MPCI. Private insurance plans are typically referred to as "Crop-Hail" and provide additional or gap coverage to MPCI. The following is a brief overview of many of the crop insurance plans. There are significant additional details and regulations for each plan that are beyond the scope of this paper.

2.2.1 Actual Production History

Actual production history (APH) plans were the primary policies issued prior to 2000. APH plans are the basis for most of the other policies so a detailed description of the APH plan will be discussed here and the differences of the other policies will be described later. RMA publishes bulletins and handbooks that should be referred to for any detailed issue regarding MPCI.

APH policies insure farmers against yield losses due to natural causes such as drought, excessive moisture, hail, wind, frost, insects, and disease. The liability of the policy is calculated as:

Acres Insured x Expected Yield x Coverage Level x Price x Share

Expected yield[1] is typically the latest ten-year average of yields. Coverage level (or deductible) is selected by the farmer and can be between 50% and 75% in increments of 5%.[2] Price is established by RMA before the beginning date of the policy based on expected harvest prices. The farmer can also select between 55% and 100% of the price – usually 100% is selected. Share is defined as the percentage of interest in the insured crop as an owner, operator or tenant at the time insurance attaches.[3] An example follows:

The following table displays a farmer's historical yield per acre for a hypothetical crop for an insured unit:[4]

Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10
60	55	64	68	25	72	71	15	78	72

1 The actual methodology to calculate the expected yield (or APH) is very detailed and is beyond the scope of this paper.
2 The maximum coverage level can be greater than 75% in various states for various crops.
3 USDA – RMA Final Agency Determinations "The Definition of 'share'" under 7 C.F.R. &457.8-Definitions.
4 RMA has established regulations how farmers can separate their crop fields into various units of insurance. The details are beyond the scope of this paper.

The average yield (or approved yield) is 58. Assuming the farmer selects a 75% coverage level, insures 100 acres[5] with a 100% share and the price is $4.00, then the liability equals $17,400 or:

Liability = Approved Yield x Price x Coverage Level x Acres x Share

or

$17,400 = 58 x $4.00 x 75% x 100 x 100%

For an APH policy, the trigger of a claim is whether the actual yield[6] is lower than the APH times the coverage level times the acres, or $4,350:

Guaranteed Yield = Approved Yield x Acres x Coverage Level x Share

or

$4,350 = 58 x 100 x 75% x 100%

Let's assume the farmer's yield is 2,250, then the indemnity would be calculated as:

Indemnity = (Guaranteed Yield - Actual Yield) x Price

or

$8,400 = (4,350 – 2,250) x $4.00

MPCI may provide for additional coverage such as replanting or prevented planting as well. Replant provisions cover the anticipated cost of replanting after an initial planting that doesn't produce a stand due to excessive rain or drought. Prevented planting allows coverage in an area where planting is not possible, typically due to wet fields. The farmer could collect an indemnity much smaller than the overall liability and plant a new crop with a lower coverage amount.

2.2.2 Revenue Plans

Participation in revenue plans significantly increased after the passage of the Agricultural Risk Protection Act of 2000. There are currently three plans that are similar to an APH plan, but include a provision for price risk as well. These plans are Crop Revenue Coverage (CRC), Revenue Assurance (RA), and Income Protection (IP). Unlike APH plans, where the same price that is used in determining liability is used in determining indemnity as well, the revenue plans use separate prices. Thus, in addition to yield risk, the revenue product includes an element of price risk. The spring price is established before planting and the fall price is established near harvest time. The basis for the prices is usually a monthly average of the crop's daily settlement value traded on a

5 To avoid adverse selection, a farmer must insure all insurable acreage of a crop within a county.

6 This is referred as "production to count." In the event the insured acreage is not harvested, there are loss adjustment standards published by RMA to measure the yield.

exchange. For example, the December corn futures are traded on the Chicago Board of Trade (CBOT). The following table displays some of these futures and dates for the fall price[7]:

Crop	Insurance Plan	Future	Monthly Average
Corn – 3/15 Close	RA	December	November
Corn – 3/15 Close	CRC	December	October
Soybeans – 3/15 Close	CRC and RA	November	October
Soybeans – 2/28 Close	CRC	September	August

Using the APH example from above, assume that the spring price is also \$4.00, but the fall price increases to \$5.00. The liability is the same as in the APH example, i.e., \$17,400.

Therefore the calculated revenue is the fall price times the production to count or:

\$11,250 = 2,250 x \$5.00

and the indemnity is

\$6,150 = \$17,400 - \$11,250

Beginning with crop year 2011, RMA has combined the APH, RA, CRC and IP policies into a "Combo" policy. The insured would still have the option to exclude the price risk as the original APH plans do. RMA combined these programs to eliminate overlapping policies and reduce administration costs. The RA policy and the Combo policy include an option (for additional premium) where the guarantee (liability) uses the greater of the spring price or the fall price.

Using this option, in the example above, the guarantee (liability) would increase to \$21,750, or:

\$21,750 = 58 x \$5.00 x 75% x 100 x 100%.

The calculated revenue would remain the same at \$11,250, and the indemnity would increase to:

\$10,500 = \$21,750 - \$11,250.

These options have been very popular, so special attention should be paid to years where the fall price exceeds the spring price.

7 Please note these dates can change. Refer to RMA for the actual method to establish Spring and fall prices.

2.2.3 Group Risk Plans

There are currently two insurance plans that use a county index as the basis for determining indemnity: Group Risk Plan (GRP) and Group Risk Income Plan (GRIP). Both plans use the county yield as determined by National Agricultural Statistics Service (NASS). GRP payments are made when the county yield in the crop year falls below the expected county yield for that year. The individual yield for the farmer is not a factor in this plan – other than any impact to the overall county yield. The farmer can only insure as many acres as they plant in the county of the same crop. The coverage for GRP is similar to APH, where the farmer selects a coverage level (up to 90%) of the county average and payments are made when the county yield is lower than the coverage amount.

GRIP includes price in this calculation, as well as yield, and bases indemnity on the expected county revenue versus the actual county revenue. The prices are currently established similar to the CRC plans. The insured can select, for additional premium, the Harvest Revenue Option, where the guarantee is the greater of the spring price or the fall price. The following table displays an example for these policies:

Insurance Plan	(A) Expected Yield	(B) Spring Price	(C) Expected County Revenue = (A)x(B) HRO Max (B) (D)	(D) Fall Price	(E) Actual Yield	(F) Actual Revenue = GRP = (B)x(E) GRIP = (D)x(E)	(G) Indemnity at 90% CL = Max {(C) x 90% - F,0}
GRP	100	$4.00	$400	N/A	75	300	60
GRIP	100	$4.00	$400	$3.00	75	225	135
GRIP-HRO	100	$4.00	$500	$5.00	75	375	75

Currently, the county yields and revenue are not released by RMA until April of the following year for corn and soybeans. Since payments are not made until mid-April, establishing a reserve for this exposure is necessary for year-end reserve analyses. A detailed methodology to establish these reserves is presented in a later section of this paper.

2.2.4 Dollar Plans

Dollar plans were introduced for crops that do not typically have an historical actual yield. There are essentially three different dollar plans currently administered by RMA. The first one is for some vegetable crops. The second plan covers nurseries. The third plan is a dollar tree plan, which insures perennial trees primarily in catastrophic prone areas.

The vegetable dollar plans differ from APH plans in the following ways:

- The historical farmer's yield has no bearing on the guarantee. The guarantee is set by county and is referred to as the maximum reference dollar amount.
- There are a lot of input costs during the season to produce a mature crop. Therefore, stage guarantees limit the amount of insurance coverage from planting to harvest. For example, after 30 days, only 50% of coverage is available.
- There is a significant cost of harvesting the crop, which is deducted from the overall acre guarantee. This is called the allowable cost.
- Causes of loss exclude disease or insect manifestation.

Nursery insurance is a unique coverage that insures the inventory of plants at the nursery. The plant inventory value is a measure of all insurable plants in the nursery. A farmer can insure a percentage of this value. Insured causes of loss include adverse weather conditions, fire, and wildlife.

Fruit tree insurance places a certain dollar amount of insurance on each tree depending on the type and age of the tree. Since many fruit trees are planted in hurricane-prone areas, wind is a major risk for this coverage. The insurance also covers excess moisture and freeze—but not insects, disease, or wildlife.

2.2.5 Rainfall and Vegetation Index

Rainfall Index (RI) and Vegetation Index (VI) are insurance plans introduced in 2007. These plans insure pasture, rangeland, and forage, and are based on rainfall and vegetation indices. A similar plan called Apiculture, subsequently introduced, insures honeybee colonies based on these indices. A farmer can choose to insure acreage used for pasture, rangeland and forage, or honeybee colonies for two or three monthly intervals throughout a year.

An indemnity is paid when the RI or VI is less than the coverage level selected. RI uses data from the National Oceanic and Atmospheric Administration Climate Prediction Center (NOAA CPC). The multiple data sets include weather, satellite, and radar data, and are interpolated and smoothed to 12 by 12 mile grids. The insurance is based off the index within each grid and not individual farms or ranches actual rainfall.

VI uses data from the U.S. Geological Survey Earth Resources Observation and Science data center called the Normalized Difference Vegetation Index (NDVI). The NDVI measures vegetation greenness to estimate plant conditions in approximately 4.8 by 4.8 mile grids. The healthier the plants in the grid, the higher the NDVI will be. Similar to the RI, farmers' own conditions are not considered in the indemnity calculation; only the index is considered.

2.2.6 Livestock Insurance

There are currently two livestock programs currently administered by RMA. Livestock Gross Margin (LGM) provides protection against the gross margin, which is defined as the market value of livestock (or dairy) minus feed costs. Livestock Risk Protection (LRP) provides protection against livestock price declines.

There are three coverages for LGM: Cattle, Dairy and Swine. The gross margin is calculated as the difference between the market price of the livestock (or dairy) and the cost of producing the livestock (or dairy). The insurance period is based on the time it takes to raise the livestock for market and the anticipated cost of feed during this time. Famers can insurer all livestock on a monthly rolling basis. For cattle and dairy, the insurance period is eleven months and for swine, it is six months. This represents the expected time from the beginning of the insurance to the time of selling the livestock. The prices are based on futures and adjusted for state and monthly specific basis. The actual cost of feed or livestock price to the famer is not considered in the indemnity calculation.

LRP is similar to LGM, but it excludes the price of feed in the indemnity calculation. LRP includes lambs, but does not include dairy.

These coverages are similar to GRIP in the sense that indemnity is based off price indices, rather than a farmer's actual revenue loss. However, private insurance can be used to insure the property (the actual livestock) and potential liability caused by their livestock.

2.2.7 Adjusted Gross Revenue

Adjusted Gross Revenue (AGR) insures farmers' overall net income from operations based on filed tax returns. The liability is calculated from previous years' tax returns and is adjusted for any changes in the current operations compared to previous years. Offsets are made for crops that are insured, since any indemnity would be considered revenue to the farmer. These plans are more popular with famers with a variety of operations of which some crops are insurable under MPCI and other crops are not. Examples include vegetable farmers in California or fruit growers in Washington. These plans are generally complicated and can vary significantly by farmer. The plans also pay out later in the following year so this exposure needs to be estimated at year-end for reserving purposes.

2.2.8 Private Crop-Hail

Private crop-hail insurance has been available in various forms since the early twentieth century. This coverage differs from a standard MPCI plan, in that it provides coverage on an acreage basis, rather than a unit basis. In other words, a hail storm could damage part of a field and the crop-hail would provide a payment for the acres that are damaged, whereby MPCI would only pay out if the total unit was damaged enough to lower the yield below the coverage level. Private crop-hail also pays out soon after the occurrence, whereas MPCI will wait until after harvest (unless there is a complete loss).

Private plans also cover wind, transport, and fire damage, both when the crop is in the field and after harvest. MPCI only covers the crop while it is in the field. A farmer can select to exclude hail coverage for the MPCI and receive a reduction in MPCI premium. Crop-hail may also provide replant coverage especially for crops that do not include this coverage in the MPCI policy.

2.3 Crops and Insurance Dates

This section discusses the important dates for MPCI and the implications of these dates. Since the payout of indemnity is typically made quickly after harvest or a major loss that destroys the crop, an understanding of these dates is critical in estimating unpaid claim liabilities. These dates should also be understood in conjunction with the SRA, since additional losses or gains may be primarily ceded to FCIC.

An example of insurance dates is shown for corn in Iowa.[8] These dates are similar for corn-belt crops other than wheat and other specialty crops. In the southern states, the dates are typically a month or so earlier.

Sales Closing Date – March 15: This is the final date that a farmer can sign up to insure crops.

Earliest Planting Date – April 11: This is the first date that a farmer can plant and the crop will be insurable. These dates are based on the climate and may vary by crop.

Final Planting Date – May 31: This is the latest a farmer can plant and still receive all of the coverage. The coverage decreases each day past the final planting date up to a certain date, when no insurance is provided. This is based on the climate and the days to maturity the crop needs before harvest.

Acreage Reporting Date – June 30: At the time of the Sales Closing Date, the farmer may not know what crops will actually be planted and on which fields when they initially sign up to insure crops. This can be based on many issues, including current weather and prices, the availability of land, the cost of seed/fuel/etc. Thus, the acreage reporting date is somewhat later in the year, when all crops have likely been planted. The famer must report what they actually planted (acres and crops) by this date.

Premium Billing Date – October 1: This is the date the farmer is billed for the unsubsidized MPCI premium. While outside the scope of this paper, the cash flows for MPCI differ from traditional property and casualty insurance. An escrow fund is established between the Approved Insurance Provider (AIP) and FCIC, which is used to pay premium, losses, A&O subsidy and the net underwriting gain determined by the SRA.[9]

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[8] 2009 Commodity Insurance Fact Sheet – USDA RMA

[9] Historical Rate of Return Analysis – Milliman, Inc. for USDA - RMA.

End of Insurance – December 10: According to the Iowa Fact Sheet: "Insurance coverage will end at the earliest of: (1) Total destruction of crop, (2) harvest of the unit, (3) final adjustment of a loss, (4) December 10, 2009 or, (5) abandonment of the crop."[10]

These dates are important when establishing ultimate loss ratios for each state/crop. It is also important to note that RMA may issue directives that modify these dates due to unusual circumstances.

2.4 Standard Reinsurance Agreement

The SRA is an agreement between the AIP and FCIC, whereby the AIP provides insurance to farmers and the FCIC reinsures the AIP. The FCIC also pays the AIP a percentage of premiums called A&O subsidy to pay for the administrative and operating expenses of the company. The reinsurance terms of the SRA are calculated on an annual basis for all crops and states. Within thirty days of writing the policy, the AIP assigns each policy to one of the SRA funds[11] for each state. Each fund has a different reinsurance structure in place, where some funds have little risk/reward and other funds have significant risk and reward. This is critical when estimating loss ratios (or underwriting returns) net of the SRA. There is a separate SRA for livestock insurance. The discussions that follow are regarding the MPCI SRA.

The SRA defines the net underwriting gain/loss as the difference between the retained net book premium and the retained ultimate losses. The net book premium excludes A&O subsidy, cancellations, adjustments and administrative fees. The net ultimate loss is defined as any claim paid by the AIP less any recovery or salvage.

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[10] Iowa Fact Sheet.

[11] There is a limit to the amount an AIP can place in the Assigned Risk fund by state. The excess amount automatically gets "spilled over" to the Developmental Fund. The AIP may also cede quota share an amount for each fund/state.

The following table displays each fund and the various retentions of underwriting loss/gain from the AIP's perspective for the 2010 SRA:

Gross Loss Ratio	Assigned Risk	DEVELOPMENTAL			COMMERCIAL		
		CAT	Revenue	Other	CAT	Revenue	Other
0%	2.0%	4.0%	6.0%	6.0%	8.0%	11.0%	11.0%
50%	9.0%	30.0%	50.0%	50.0%	50.0%	70.0%	70.0%
65%	15.0%	45.0%	60.0%	60.0%	75.0%	94.0%	94.0%
100%	5.0%	25.0%	30.0%	25.0%	50.0%	57.0%	50.0%
160%	4.0%	20.0%	22.5%	20.0%	40.0%	43.0%	40.0%
220%	2.0%	11.0%	11.0%	11.0%	17.0%	17.0%	17.0%
500%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

Since the retention of underwriting gain/loss is calculated on an individual state/fund basis, there may be a significant difference between the gross loss ratio and net loss ratio. Due to the sharing of loss/profit by state and fund, it is important to model ultimate losses by state and fund. The following scenarios are shown to highlight the possible differences from the gross loss ratio to the net loss ratio. The following scenarios are provided where the gross loss ratio is the same overall, but differences in state/fund can significantly change the net loss ratio.

State	Gross Premium	Commercial Fund Allocation Percentage	SCENARIO 1		SCENARIO 2		SCENARIO 3	
			Gross Loss Ratio	Net Loss Ratio	Gross Loss Ratio	Net Loss Ratio	Gross Loss Ratio	Net Loss Ratio
IL	150	75%	100.0%	100.0%	13.3%	55.7%	200.0%	147.8%
ND	100	25%	100.0%	100.0%	20.0%	65.6%	20.0%	65.6%
TX	100	25%	100.0%	100.0%	340.0%	157.9%	10.0%	64.7%
WI	50	50%	100.0%	100.0%	40.0%	61.8%	140.0%	119.3%
Total	**400**	**47%**	**100.0%**	**100.0%**	**100.0%**	**74.6%**	**100.0%**	**117.3%**

These examples display why it is important to estimate loss ratios on a state/fund basis, rather than an overall gross and net loss ratio basis.

Beginning for the 2011 crop year, a new SRA was being negotiated between the FCIC and the AIPs. The major changes are:

- The number of funds would be collapsed into two funds: Assigned Risk Fund and Commercial Fund.
- The Commercial Fund would have three different groups (Group 1, 2, and 3) with different retention percentages based on the historical loss experience of the state.
- AIPs would be encouraged to write business in underserved states (Group 3).
- Similar to the prior SRA, there is a limit to the amount an AIP can place in the Assigned Risk Fund.

The current parameters of the SRA for the applicable crop year should always be reviewed when performing an analysis.

2.5 Accounting Issues

Historically, the accounting for MPCI has been treated differently than most property and casualty lines of business. When the NAIC moved towards consistent reporting requirements during codification, they attempted to make MPCI act more like a typical property line of business. As discussed previously, there is an escrow between the AIP and FCIC where premium and losses are placed. While accounting issues are fluid, MPCI is now treated more like typical property and casualty business in reporting to NAIC. There are several considerations that are unique to MPCI:

- The premium is typically earned from sales closing date to the end of the insurance period (or December 31). This allows for little unearned premium reserves at year-end. The unearned premium at year-end is associated with winter wheat and other crops[12] that extend past the end of the year. The winter wheat coverage is placed in the forthcoming SRA year, so the losses associated with this are not in the current year's SRA.
- The NAIC instructions ask that a company describe its method to earn premium throughout the year on the Notes to the Financial Statement. This is asked since the exposure to loss is not uniform throughout the policy period. A review of major companies' Notes indicate that most companies use a uniform earning pattern due to the difficulty in assessing the exposures over the course of a year.

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[12] Usually associated with citrus fruit, trees, nursery, etc.

- The NAIC Statement of Statutory Accounting Principles (SSAP) discusses how to book the amounts associated with MPCI. If the company is at an underwriting gain position, the appropriate amount should be recognized as a write-in asset for a receivable from FCIC. On the other hand, if the company is at a loss position, the company should recognize a write-in liability.
- The SSAP states that the A&O subsidy associated with catastrophic coverage should be recorded as a reduction on loss expenses, whereas the A&O subsidy for other coverages should be recorded as a reduction of underwriting expenses.[13]

The SSAP provides an example of how to calculate the ceded premium and losses after application of the SRA, which is shown below.[14]

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13 The author is unclear as to the basis for the difference in recording A&O subsidy, since the subsidy should be used to pay commissions, general expenses and loss adjusting expenses for all types of policies.

14 For the 2010 SRA, there is an overall 5% quota-share of net underwriting gain (or loss) in addition to the amounts in the table.

FCIC Fund	(1) Retention %	(2) Gross Written Premium	(3) = (1) x (2) Net Retained Premium	(4) Gross Ultimate Losses	(5) = (1) x (4) Net Retained Losses	(6) = (5) / (3) Retained Loss Ratio
Assigned Risk	20%	$20,000	$4,000	$40,000	$8,000	200.0%
Dev-Other	35%	10,000	3,500	16,000	5,600	160.0
Dev-Revenue	35%	5,000	1,750	7,000	2,450	140.0
Dev-CAT	35%	5,000	1,750	4,000	1,400	80.0
Com-Other	100%	100,000	100,000	80,000	80,000	80.0
Com-Revenue	100%	20,000	20,000	18,000	18,000	90.0
Com-CAT	100%	40,000	40,000	22,000	22,000	55.0
Total		**$200,000**	**$171,000**	**$187,000**	**$137,450**	**80.4%**

FCIC Fund	(7) SRA Provisions Underwriting Gain/(Loss)	(8) = (3) - (5) - (7) Stop-Loss Ceded Premium	(9) = (3) - (5) - (7) Stop-Loss Ceded Loss	(10) = (3) - (8) Retained Premium	(11) = (5) - (9) Retained Loss	(12) = (11) / (10) Retained Loss Ratio
Assigned Risk	$(184)	---	$3,816	$4,000	$4,184	104.6%
Dev-Other	(525)	---	1,575	3,500	4,025	115.0
Dev-Revenue	(210)	---	490	1,750	1,960	112.0
Dev-CAT	158	193	---	1,558	1,400	89.9
Com-Other	18,000	1,200	---	98,000	80,000	81.0
Com-Revenue	1,880	120	---	19,800	18,000	90.5
Com-CAT	12,500	5,500	---	34,500	22,000	63.8
Total	**$32,419**	**$7,013**	**$5,881**	**$163,988**	**$131,569**	**80.2%**

These unique characteristics regarding statutory accounting should be noted when providing unpaid claim liability estimates – especially to make sure the entire earned premium is accounted for in the unpaid claim liability estimates.

3. FORECASTING ULTIMATE LOSSES

Forecasting loss ratios for crop insurance are dependent on the available information at the time of the forecast. During the year, more information is available about the success of the current year's crops, as well as the associated prices. Using this information in conjunction with prior year's loss ratios can assist in forecasting loss ratios during the year. Once harvest is completed and claims have been filed, more traditional actuarial methods can be used.

Due to the characteristics of the SRA, one should estimate the loss ratios on a reinsurance year basis so the effect of the SRA can be used to calculate ceded losses. The loss ratios should be projected on a state/fund basis, as well. The following describes various methods to establish estimated ultimate loss ratios for MPCI.

The loss ratio that is being estimated should be consistent with the definition of premium and indemnity provided by the SRA. The target loss ratio currently mandated by RMA is 100%, since expenses are covered by the A&O subsidy. Therefore, the overall rates are set to the expected long-term losses. Defense and Cost and Containment Expenses (DCCE) are minimal or zero for crop insurance. Adjusting and Other Expenses (AOE) is the cost to handle crop insurance claims and is discussed later.

3.1 MPCI Portfolio Review

The first step in the process should be determining the exposures (liabilities and premiums) by state, crop, insurance plan, and SRA fund. The following table and Exhibit 1 display an example portfolio of MPCI premium:

State	Crop	Insurance Plan	Assigned Risk	Commercial Fund	Developmental Fund
IA	Corn	CRC	50	200	0
IL	Corn	GRIP	50	50	50
IA	Corn	APH	20	100	0
IA	Hybrid Corn Seed	APH	10	0	0
TX	Cotton SE	APH	150	50	0
TX	Cotton AO	APH	50	150	0
TX	Peanuts	APH	25	0	25

This MPCI portfolio will be used in the remainder of this paper to determine the overall gross and net loss ratios for a reinsurance year. In practice, a MPCI portfolio will include many more states and crops. Ultimate loss ratios will be estimated for each significant state/crop/insurance plan so that it can be fed through the SRA terms and a net underwriting gain (or loss) can be calculated.

3.2.1 Using Forecasted Yields to Estimate APH Loss Ratios

Because losses will be based on lower than expected yields, an estimate of the ultimate loss ratio can be made by comparing the forecasted yields for the current year to the actual loss ratios and yields of previous years. Forecasted yields for major crops are made available during the year by several institutions. This paper will discuss the yield forecasting performed by NASS, but other forecasted yields could be used as well.

NASS provides crop production estimates for two components: acres to be harvested and yield per acre. Corn and soybean farmers are surveyed in June regarding the planted acres, and crop production estimates are made each month from August through November. NASS uses two survey methods to estimate yields.[15] Agricultural Yield Survey and Objective Yield Survey. Estimates are made for all major crops in major states. Several of the major producing states are further split into about 10 districts each.

An example of this method will be shown using corn data for the State of Iowa. In practice, Iowa may be split into the NASS districts, since different regions may experience better/worse weather during the year. The first step is to obtain the historical Iowa yields from NASS and the historical loss ratios from RMA. A company with credible data may wish to use their own experience, rather than industry data from RMA. Exhibit 2 displays these values for the APH plan.

The first step is to define a relationship of yield and loss ratio using the historical values. The relationship may not be linear, since lower yields will tend to exceed the deductible and increase the losses at a faster rate. Therefore, quadratic or exponential formulas may be used, a formula that fit well was:

Expected loss ratio = a*1/(y^b) + If (y<1, c(1-y), 0)

Where:

a,b,c = regression coefficients – solved by minimizing the squared error

y = yield ratio = current yield / (previous 10-year yield average)

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[15] The detail of NASS methodology can be found in “The Yield Forecasting Program of NASS” issued by the Statistical Methods Branch of NASS.

It should be noted that a 10-year historical average is used as the "expected" yield in the current year. Due to improvements in agricultural practices, as well as the development of crops that are more resistant to adverse weather (particularly drought), crop yields have been increasing. Due to these increasing yield trends, the expected yield in the current year is typically higher than the ten-year average. We can adjust historical yields to "on-level" yields by dividing the current year's yield ratio by the historical yield trend. These results are displayed on Exhibit 3. As shown (by the squared error), the adjustment provides a better fit of the data for this example.

An additional factor was added when yields are significantly low (and loss ratios high) to increase the loss ratios. This is due to both the fact that more policies have claims when yields are lower and the distribution of yields are more diverse in a poor yield year than in a good year. For example, assume that the distribution around the average yield for all corn crops in a county is normally distributed[16] with a mean of one and standard deviation of 0.50. The loss cost for a 65% coverage level APH policy would be 0.0968 per dollar of liability.[17]

The loss cost is calculated using data from many years which have both high yields and low yields. There may also be a difference in the distribution around the mean in a given year; so when yields are high, the distribution around the mean in a given year is low, and when the yields are low, the distribution around the mean in that year is higher. Using the example above, we can compare different scenarios of yields and the deviation around the mean for a 65% coverage level:

Scenario	**Average Yield**	**Variation Around Average**	**Loss Cost Using Scenario Mean and Variation**	**Loss Ratio = Loss Cost / 0.0968**
A	105%	50%	0.0808	83.4%
B	105%	30%	0.0147	15.2%
C	80%	50%	0.1817	187.6%
D	80%	80%	0.2270	286.0%

A review of loss ratios and yield departures indicate that Scenarios B and D are more prevalent than A and C, and this indicates that the yields are more dispersed when yields are low than when they are high.

In the yield and loss ratio regression, a weight is also used in calculating the squared error, which is minimized. This may be used on outlier years or if major changes have been made to the program over the historical experience.

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16 Several studies have been performed to test which statistical distribution best resembles yields. Most conclude that a single distribution does not work well for all crops or for some crops in general. Therefore, most analyses use an empirical fitted formula to calculate rates and coverage level relativities.

17 For ease of example, the yield was set to zero where it is negative.

Using the model described above, we project a 2009 loss ratio of 18%.

A similar approach is used on Exhibit 4 and 5 for Texas cotton. Texas is split into two territories, since there are two distinct growing areas for cotton in Texas: the southeastern coastal bend and the panhandle (or All Other). The model produces loss ratios of:

Southeast	889%
All Other	71%

It should be noted that in years of abnormally low yields, the resulting loss ratio should be compared to the overall liability so that losses do not exceed the liability.

3.2.2 Using Forecasted Yields and Prices to Estimate Revenue Plan Loss Ratios

Revenue plans add an additional parameter in the indemnity calculation; namely, the difference between the spring price and the Harvest Price. There may be several different formulas that can estimate the loss ratio using both the yield and the price component. One method can be to estimate the APH loss ratio and then add a parameter for the revenue risk. However, the popularity of APH plans for corn and soybeans has decreased substantially with the introduction of revenue plans, which reduces the credibility of APH plans loss ratios in recent years. Therefore, in this paper, we show a formula with the combination of yield and price changes.

We can use the same yield departures as in the APH and add the percentage change from the spring price to the fall price, as shown in the following table:

Expected loss ratio = a*1/((y*p)^b) + If (y<1, c*(1-y), 0)

Where:

a,b,c = regression coefficients – solved by minimizing the squared error

y = yield ratio = current yield / (previous ten-year yield average)

p = price change = (fall price - spring price) / spring price

For most crops, the fall price is the average daily settlement value during October. Prior to October, these prices can be estimated using the current futures price or other methods.

An example of a loss ratio estimate for revenue coverage for Iowa corn is shown on Exhibit 6. The price changes are displayed on Exhibit 7. The method results in projected loss ratios of 17% for CRC and 13% for RA/IP.

The major difference in the revenue plans is that the price change affects all policies equally, while, as discussed before, the yield distributions vary by whether it was a high- or low-yield year. A low-price and low-yield year would have a multiplicative effect on the losses, and the opposite is also true. The model may need to reflect the harvest price revenue option where the guarantee is the higher of the spring and fall price.

In summary, the loss ratios can be estimated during the year as the forecasted yields and prices become available. Once the harvest is completed and claims are reported, more traditional methods may be added as well.

3.3 Paid to Case Ratios

At the end of the year, when most crops have been harvested and claims have been reported, a more traditional actuarial method may be used. Using the relationship of prior years' ultimate paid losses compared to the case reserves can be used as an indication. Crops or states can be grouped or separated to gain homogeneous groups of claims.

Attention should be paid to the causes of loss from the current year compared to the prior years. For example, are the remaining open claims at a similar point in closing as they were at the same point in time in prior years? If fall weather was poor and harvest was delayed, there may be a delay in the payout process. Agents may report claims differently; some may report a claim for all policyholders in the case of poor weather or prices. Discussions with claims personnel are also important to understand how case reserves are originally set and how they are handled. Claims management may also know certain intricacies about states or crops that are not obvious by looking at the bulk case reserves.

3.4 Estimating GRIP and GRP Liabilities

GRIP and GRP policies are unique in that they do not use the farmer's actual yield, but rather, a county yield index as the basis for payment. GRIP also includes price changes. Losses will occur when county revenue per acre is less than the trigger revenue. Because county revenue is based on both the county average yield and the harvest price, the year-end loss ratio estimate will require estimates of the county average yield. The harvest price should be known at the end of October or November for most crops with GRIP.

The difficulty in estimating loss ratios for GRIP policies at year-end results from the difficulty in estimating the county yield. Publicly available data from NASS estimates yields by state and by districts for the major agricultural states. These crop-production reports are released for each crop year beginning in August for corn and soybeans.

Because yields do not increase or decrease by uniform amounts by county within a district, an additional review is necessary to determine the difference between expected county yields and actual yields within a district for each available year. The table below presents the data for the 2005 year in Illinois District 10:

District 10 County	Acres	GRIP Expected Yield (bushels per acre)	GRIP Final Yield (bushels Per acre)	Final Yield Deviation (%)	GRIP Expected Revenue ($)	GRIP Final Revenue ($)	Final Revenue Deviation (%)
Bureau	288,000	158.6	134.9	(14.9)	377	260	(31.0)
Carroll	139,000	161.9	160.1	(1.1)	385	309	(19.8)
Henry	236,000	152.8	125.0	(18.2)	364	241	(33.7)
Jo Daviess	85,000	142.8	152.0	6.4	340	293	(13.7)
Lee	259,000	158.4	140.7	(11.2)	357	272	(28.0)
Mercer	138,000	150.1	153.9	2.5	369	297	(16.9)
Ogle	221,000	155.2	135.7	(12.6)	385	262	(29.1)
Putnam	42,000	161.8	136.0	(15.9)	373	262	(31.8)
Rock Island	74,000	156.7	138.9	(11.4)	346	268	(28.1)
Stephenson	158,000	145.4	139.4	(4.1)	360	269	(22.3)
Whiteside	229,000	151.3	124.5	(17.7)	322	240	(33.3)
Winnebago	94,000	135.2	125.7	(7.0)	365	243	(24.6)
Total	**1,963,000**	**153.4**	**137.3**	**(10.5)**	**365**	**265**	**(27.4)**

The estimated Illinois farm yield for District 10 was 140 bushels per acre, which was released in November 14, 2005, in the Illinois Farm Report. It should be noted that the published estimated yield is based on harvested acres, whereas the final NASS yield used in calculating the county revenue uses planted acres. NASS does publish forecasted planted and harvest acres so an adjustment can be made.[18] According to the 2005 Farm Report, there were 1,931,000 acres planted and 1,903,000 acres harvested for grain in Illinois District 10; therefore, the yield per planted acre (comparable with the GRP/GRIP yields) would be 137.3. Larger variations between the planted and harvested yields will occur when yields are low, due to total losses caused by floods or droughts. The spring price declined from $2.38 (expected) to $1.93 (final - harvest), or by 18.9%. Therefore, the combination of yield and price decline led to significant indemnities in 2005.

The loss ratios for GRIP policies would be underestimated if one only considered the difference in district or statewide yields, because the variability in county yields is greater due to a smaller sample and local weather events. In the example above, on a district-wide basis, a 70% coverage-level policy would not incur an indemnity since the loss is 27.4%. Due to variability within county yields, however, indemnities would incur at a 70% coverage level for four of the counties in the district.

The following are two methods for calculating potential losses at year-end. The first is to calculate the difference between the expected district yield and the predicted district yield. This amount can be used as the difference in the county yields and includes a provision for variability by

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18 There are also several counties which the farmer can choose planted or harvested yields. These counties have a lot of corn which is harvested for silage.

county—for example, reduce all yields by 5% and estimate the losses. If more information is available by county from field adjustors or from the other insurance plans, these could also be used.

The following example shows this methodology for Illinois District 10 during 2007. The price fell from $4.06 per bushel to $3.58, or 11.8%, but the yields were much higher than expected. The Illinois Farm Report, released November 13, 2007, estimated a planted yield of 182 bushels per acre for the district. The table below shows the calculation without any variation in the county yield compared to the district's difference.

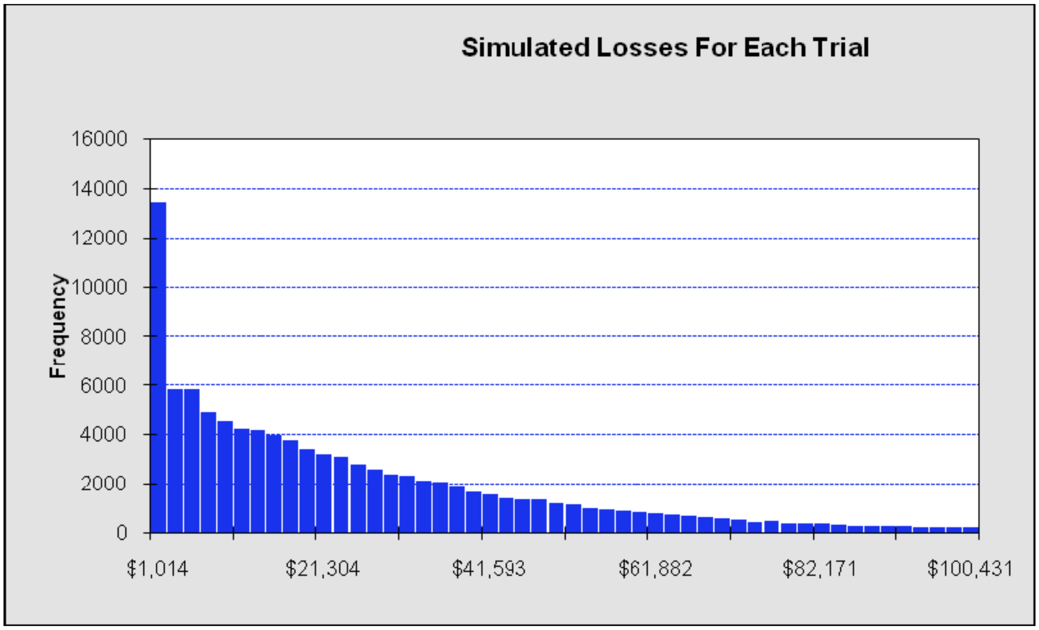
District 10 County	Planted Acres	(A) GRIP Expected Yield (bushels per acre)	(B) GRIP Forecasted Yield (bushels per acre)	(C) = (B)/(A)-1 Final Yield Deviation (%)	(D) = (A)*$4.06 GRIP Expected Revenue ($)	(E) = (B)*$3.58 GRIP Final Revenue ($)	(F)= (E)/(D)-1 Final Revenue Deviation (%)
Bureau	311,000	166	191	14.8%	$674	$682	1.2%
Carroll	155,000	170	195	14.8	689	698	1.2
Henry	257,000	161	185	14.8	654	662	1.2
Jo Daviess	98,000	148	170	14.8	603	610	1.2
Lee	282,000	162	185	14.8	656	664	1.2
Mercer	156,000	164	188	14.8	667	675	1.2
Ogle	244,000	154	177	14.8	626	634	1.2
Putnam	46,000	166	191	14.8	674	683	1.2
Rock Island	79,000	166	191	14.8	674	683	1.2
Stephenson	180,000	152	174	14.8	616	624	1.2
Whiteside	253,000	151	173	14.8	611	619	1.2
Winnebago	102,000	141	162	14.8	572	579	1.2
Total	**2,163,000**	**159**	**182**	**14.8%**	**$644**	**$652**	**1.2%**

The table above assumed that the deviation from expected yields was uniform for each county. The table below shows what estimated final revenues would be for each county based on the yield deviation each county experienced for the 2005 year and applied to 2007:

District 10 County	Planted Acres	(A) GRIP Expected Yield (bushels per acre)	(B) GRIP Forecasted Yield (bushels per acre)	(C) = (B)/(A)-1 Final Yield Deviation (%)	(D) = (A)*$4.06 GRIP Expected Revenue ($)	(E) = (B)*$3.58 GRIP Final Revenue ($)	(F)= (E)/(D)-1 Final Revenue Deviation (%)
Bureau	311,000	166	181	9.1%	$674	$648	(3.8)%
Carroll	155,000	170	215	26.8	689	771	11.8
Henry	257,000	161	169	4.9	654	605	(7.5)
Jo Daviess	98,000	148	203	36.5	603	725	20.4
Lee	282,000	162	184	13.9	656	659	0.5
Mercer	156,000	164	216	31.5	667	773	15.9
Ogle	244,000	154	173	12.1	626	619	(1.1)
Putnam	46,000	166	179	7.8	674	641	(4.9)
Rock Island	79,000	166	189	13.7	674	676	0.2
Stephenson	180,000	152	187	23.0	616	668	8.4
Whiteside	253,000	151	159	5.5	611	569	(6.9)
Winnebago	102,000	141	168	19.2	572	602	5.1
Total	**2,163,000**	**159**	**182**	**14.8%**	**$644**	**$652**	**1.2%**

The second method is to set up a Monte Carlo simulation for actual county yields based on the difference between the expected district yields and the predicted district yields.

A simulation model could vary the GRIP forecasted yields by county based on the historical difference between the yields by county. The following graph shows the results of 100,000 trials for the district based on the 2007 estimated yields for a hypothetical portfolio of GRIP policies. A normal distribution with a standard deviation of 12.5% was used to model the difference from the forecasted yield as a district as a whole compared to each individual county yield.



While the graph above displays the results for only Illinois District 10, the simulation could capture losses for each state or crop with significant GRIP (and GRP) liabilities. The results of this model could then be fed into the overall SRA model to estimate the net underwriting gain (or loss) for the reinsurance year.

3.5 Minor State, Crops, and Plans

Because of credibility considerations, the methods described above are only suitable for the largest states and crops in the MPCI portfolio. The remaining liabilities need to be accounted for in estimating the overall loss ratios by state and SRA fund. There are several methods to estimate the losses for minor states and crops. These crops are typically minor field crops, fruits and vegetables for which NASS does not provide a forecasted yield.

As a first step, we can calculate the historical loss ratios for these crops and a comparative crop using the RMA data. Using the comparative crop's loss ratio for the current year, we can adjust the historical loss ratio to the current year. This is shown in the table below:

(1) State / Crop / Plan	(2) Historical Loss Ratio	(3) Comparative Historical Loss Ratio[1]	(4) 2009 Comparative Loss Ratio[2]	(5) 2009 Estimated Loss Ratio[3]
IA, Hybrid Corn Seed, APH	51%	54%	18%	17%
TX, Peanuts	50%	93%	71%	38%

[1] Iowa Corn APH and Texas Cotton All Other Counties APH

[2] From previous analysis.

[3] = (2)/(3) x (4)

For hurricane-prone areas, we may adjust the historical loss ratios, based on the time of the year and whether a hurricane occurred or not, using a Bornhuetter-Ferguson approach. At year-end, it may be possible to use a paid to case method to estimate these losses.

3.6 Timeline of Indications

The process for estimating ultimate loss ratios for crop insurance is similar to other property lines of business where one starts with an expected loss ratio and changes the expectation due to events during the year.

The following table displays a possible timeline for various crops and the different actuarial methods that may be used. The timelines are shown for a given reinsurance year (consistent with the SRA).

	Oct - April	May - July	Aug - Nov	Dec - March	April - June
Winter Wheat	Expected Loss Ratio	Paid-to-Case	Paid-to-Case	Paid-to-Case	Paid-to-Case
Traditional Row Crops	N/A	Expected Loss Ratio	Forecasted Loss Ratio	Forecasted LR Paid-to-Case Expected Paid Method	Paid-to-Case
Citrus	N/A	Expected Loss Ratio	Expected Paid Method	Expected Paid Method	Paid-to-Case
GRP/GRIP	N/A	Expected Loss Ratio	Forecasted Loss Ratio	Forecasted Loss Ratio	Actual Results

3.7 Summarizing data into the SRA

After the ultimate loss ratios are estimated, they can be summarized into state/fund to apply the reinsurance of the SRA. As shown on Exhibit 8, the overall gross loss ratio is 207%. This gross loss ratio needs to be applied to the SRA to produce a net underwriting gain. In this example, the gross loss ratio of 207% equates to a 9.2% gain after the SRA parameters are applied. This is because most of the losses occurred in one state (Texas) and the majority of these policies were placed in the assigned risk fund.

3.8 Crop-Hail

The major difference in most crop-hail plans compared to MPCI is the occurrence of a claim is on a certain date and payment is made shortly afterwards. Therefore, more traditional actuarial methods may be appropriate. At the end of the year, most crop-hail claims should be reported and many of these are settled. Therefore, more traditional actuarial methods may be appropriate. During the year, an expected paid (or reported) method may be used with an expected loss ratio and a payment (or reported) pattern. States could be grouped where the hail exposure is similar. For example, more hail storms occur earlier in the year the further south the area is.

3.9 Reserve Ranges

There are many ways to measure reasonable reserve ranges in property and casualty insurance. A key issue with MPCI is the SRA, which limits the net underwriting gain or loss by state and SRA fund. For example, a state with a very high or low loss ratio may not significantly change the overall underwriting gain or loss by using an even higher or lower loss ratio.

A few examples to produce reserve ranges are discussed here. The regression methods using forecasted yields would produce a standard error for each regression which can be used in selecting the ranges. The loss ratios for nearby states are most likely not independent and this should be considered in the range. The GRIP/GRP simulation can create distributions, but the overall range would need to account for the dependency between these policies and other policies. It would also depend on the time of year the range is calculated.

3.10 Issues with Traditional Actuarial Methods

There are several reasons why traditional actuarial methods may not be appropriate for crop insurance. The structure of the SRA, which limits underwriting gain or loss by state and fund, requires the projection of losses by state/fund. As shown previously, net losses can be significantly different than gross losses due to the distribution of losses (and placement of policies) between funds. Unique characteristics of some policies such as GRIP make loss development type methods inappropriate. The payout of claims throughout the year is not consistent between years. For example, a flood in the spring may bring many early payments, but the harvest may turn out well and have few losses. The change in price, which is a significant function of many policies, is not known until the end of October for most crops. The harvesting of crops may be delayed in the fall, which may delay the reporting of claims and the settlement of these claims.

These and other reasons should be accounted for when making actuarial projections. The time of the year when the evaluation is taking place should be a key consideration in the appropriate actuarial methods to use.

3.11 Adjusting and Other Expenses

Adjusting and Other Expense (AOE) liabilities may also need to be estimated for crop insurance. Both MPCI and crop-hail are similar to property insurance, where the more claims there are, the more adjusting costs will be. Therefore, traditional actuarial methods may be used to estimate AOE liabilities. Since some of the policies are based on indices (GRIP, RI, VI, livestock, etc.) where considerable less claim handling involved, an adjustment to an overall paid-to-paid type approach may be warranted for these policies.

3.12 Areas for Further Research

While this paper presents several methodologies to estimate ultimate loss ratios, there are other methods that could be used. A "ground-up" method where the indemnity of all policies would be calculated with an expected yield compared to approved yields. These yields could vary based on yield distributions. In other words, instead of all yields being 10% below approved yields, one could make a distribution of yield deviations from the approved yield. As discussed previously, when yields are low, the distribution tends to be greater. Prices would also have to be estimated as well.

4. CONCLUSIONS

Crop insurance is unique to the property/casualty insurance industry. The short-tailed catastrophic exposure and the terms of the SRA need to be recognized when estimating ultimate loss ratios and unpaid claim liabilities. This paper outlines several methods of estimating the ultimate loss ratios for different policy types. New and unique insurance products are being introduced and will be introduced over time. Changes in farming practices will impact future yields and which crops are grown. The process for estimating ultimate loss ratios should be adaptable to the current policies and conditions.

Acknowledgment

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Abbreviations and Notations

A&O Subsidy - Administrative and Operational Expenses
AGR - Adjusted Gross Revenue
AIP - Approved Insurance Provider
AOE - Adjusting and Other Expenses
APH - Actual Production History
CAT - Catastrophic Coverage
CBOT - Chicago Board of Trade
CRC - Crop Revenue Coverage
DCCE - Defense and Cost Containment Expenses
FCIC - Federal Crop Insurance Corporation
GRIP - Group Risk Income Plan
GRP - Group Risk Plan
IP - Income Protection
LGM - Livestock Gross Margin
LRP - Livestock Risk Protection
MPCI - Multi-peril Crop Insurance
NAIC - National Association of Insurance Commissioners
NASS - United States Department of Agriculture - National Agricultural Statistics Service
NDVI - Normalized Difference Vegetation Index
NOAA-CPC - National Oceanic and Atmospheric Administration - Climate Prediction Center
RA - Revenue Assurance
RI - Rainfall Index
RMA - Risk Management Agency of the United States Department of Agriculture
SRA - Standard Reinsurance Agreement
USDA - United States Department of Agriculture
VI - Vegetation Index

Biography of the Author

Carl Xavier Ashenbrenner is a Principal and Consulting Actuary at Milliman, Inc. in Brookfield, Wisconsin. He specializes in reserving and ratemaking for property/casualty insurance with an emphasis on non-traditional lines of business. He holds a bachelor of Business Administration in Actuarial Science and Risk Management and Insurance from the University of Wisconsin–Madison. He is a Fellow of the CAS and a Member of the American Academy of Actuaries. He is a volunteer of the CAS Program Planning Committee, and is a frequent speaker and moderator at CAS meetings and other industry events.

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Exhibit 1

Hypothetical MPCI Portfolio

			Premium			
State	Crop	Insurance Plan	Assigned Risk	C-Fund	D-Fund	Total
IA	Corn	CRC	50	200	0	250
IL	Corn	GRIP	50	50	50	150
IA	Corn	APH	20	100	0	120
IA	Hybrid Corn Seed	APH	10	0	0	10
TX	Cotton SE	APH	150	50	0	200
TX	Cotton AO	APH	50	150	0	200
TX	Peanuts	APH	25	0	25	50
		Total	**355**	**550**	**75**	**980**

Exhibit 2

Iowa Corn
Crop Year 2009
Loss Ratio Projection
APH Plan

(1) Year	(2) Nass Yield	(3) 10yr Average	(4) Yield Ratio	(5) Low Yield Indicator	(6) Actual Loss Ratio	(7) Fitted Loss Ratio	(8) Squared Error	(9) Weight
1980	105							
1981	120							
1982	115							
1983	82							
1984	108							
1985	123							
1986	132							
1987	127							
1988	80							
1989	115							
1990	122	111	110%	0.00	30%	35%	0.00	1
1991	114	112	102%	0.00	72%	52%	0.04	1
1992	144	112	129%	0.00	17%	16%	0.00	1
1993	73	115	64%	0.36	496%	498%	0.00	1
1994	148	114	130%	0.00	5%	15%	0.01	1
1995	120	118	102%	0.00	98%	52%	0.21	1
1996	135	118	115%	0.00	24%	29%	0.00	1
1997	135	118	114%	0.00	7%	30%	0.05	1
1998	142	119	119%	0.00	41%	24%	0.03	1
1999	145	125	116%	0.00	20%	27%	0.00	1
2000	140	128	110%	0.00	11%	36%	0.06	1
2001	142	130	110%	0.00	43%	36%	0.00	1
2002	158	132	120%	0.00	11%	24%	0.02	1
2003	152	134	113%	0.00	15%	31%	0.03	1
2004	177	142	125%	0.00	9%	19%	0.01	1
2005	169	145	117%	0.00	22%	26%	0.00	1
2006	163	149	109%	0.00	20%	37%	0.03	1
2007	167	152	110%	0.00	13%	36%	0.05	1
2008	165	156	106%	0.00	76%	43%	0.11	1
					54%	56%		
		A	B	C	D			
		0.565	4.873	-	1.000		66	<-Minimize
2009	178	158	113%	0.00		31%		

Fitted Loss Ratio = A * [1 / (Yield Ratio^B)] + Low Yield Indicator * C

(2)	NASS: Production / Planted Acres	(6)	From RMA
(3)	Previous 10-year average of (2)	(7)	Fitted Loss Ratio
(4)	= (2) / (3)	(8)	=[(6) - (7)] ^2
(5)	If (4) < 1.00 then [1-(4)]	(9)	Judgment

Exhibit 3

Iowa Corn
Crop Year 2009
Loss Ratio Projection
APH Plan

(1) Year	(2) Nass Yield	(3) 10yr Average	(4) Yield Ratio	(5) Adjusted Yield Ratio	(6) Low Yield Indicator	(7) Actual Loss Ratio	(8) Fitted Loss Ratio	(9) Squared Error	(10) Weight
1980	105								
1981	120								
1982	115								
1983	82								
1984	108								
1985	123								
1986	132								
1987	127								
1988	80								
1989	115								
1990	122	111	110%	99%	0.01	30%	27%	0.00	1
1991	114	112	102%	91%	0.09	72%	87%	0.02	1
1992	144	112	129%	116%	0.00	17%	10%	0.01	1
1993	73	115	64%	57%	0.43	496%	496%	0.00	1
1994	148	114	130%	117%	0.00	5%	9%	0.00	1
1995	120	118	102%	91%	0.09	98%	87%	0.01	1
1996	135	118	115%	103%	0.00	24%	17%	0.00	1
1997	135	118	114%	102%	0.00	7%	17%	0.01	1
1998	142	119	119%	107%	0.00	41%	14%	0.07	1
1999	145	125	116%	104%	0.00	20%	16%	0.00	1
2000	140	128	110%	98%	0.02	11%	30%	0.04	1
2001	142	130	110%	98%	0.02	43%	32%	0.01	1
2002	158	132	120%	107%	0.00	11%	14%	0.00	1
2003	152	134	113%	102%	0.00	15%	17%	0.00	1
2004	177	142	125%	112%	0.00	9%	12%	0.00	1
2005	169	145	117%	105%	0.00	22%	15%	0.00	1
2006	163	149	109%	98%	0.02	20%	37%	0.03	1
2007	167	152	110%	98%	0.02	13%	30%	0.03	1
2008	165	156	106%	95%	0.05	76%	58%	0.03	1
	Average Yield Trend:		112%			54%	54%		
			A	B	C	D			
			0.188	4.374	6.605	1.000		28	<-Minimize
2009	178	158	113%	101%	0.00		18%		

Fitted Loss Ratio = A * [1 / (Adjusted Yield Ratio^B)] + Low Yield Indicator * C

(2)	NASS: Production / Planted Acres	(7)	From RMA
(3)	Previous 10-year average of (2)	(8)	Fitted Loss Ratio
(4)	= (2) / (3)	(9)	=[(6) - (7)] ^2
(5)	= (4) / Average Yield Trend	(10)	Judgment
(6)	If (4) < 1.00 then [1-(4)]		

Exhibit 4

Texas Cotton Southeast
Crop Year 2009
Loss Ratio Projection
APH Plan

(1) Year	(2) Nass Yield	(3) 10yr Average	(4) Yield Ratio	(5) Adjusted Yield Ratio	(6) Low Yield Indicator	(7) Actual Loss Ratio	(8) Fitted Loss Ratio	(9) Squared Error	(10) Weight
1980	338								
1981	482								
1982	524								
1983	576								
1984	635								
1985	751								
1986	649								
1987	663								
1988	622								
1989	510								
1990	541	575	94%	94%	0.06	246%	130%	1.34	1
1991	663	595	111%	112%	0.00	104%	94%	0.01	1
1992	506	614	82%	83%	0.17	244%	175%	0.47	1
1993	520	612	85%	85%	0.15	140%	164%	0.06	1
1994	638	606	105%	106%	0.00	58%	102%	0.20	1
1995	433	606	71%	72%	0.28	245%	229%	0.02	1
1996	345	575	60%	60%	0.40	249%	303%	0.30	1
1997	512	544	94%	94%	0.06	95%	130%	0.13	1
1998	385	529	73%	73%	0.27	164%	222%	0.33	1
1999	659	505	130%	131%	0.00	75%	74%	0.00	1
2000	627	520	121%	121%	0.00	73%	83%	0.01	1
2001	503	529	95%	96%	0.04	143%	127%	0.03	1
2002	493	513	96%	97%	0.03	172%	123%	0.24	1
2003	665	512	130%	131%	0.00	53%	74%	0.04	1
2004	750	526	143%	143%	0.00	22%	64%	0.18	1
2005	594	537	111%	111%	0.00	51%	95%	0.19	1
2006	400	553	72%	73%	0.27	291%	224%	0.45	1
2007	798	559	143%	143%	0.00	26%	64%	0.14	1
2008	442	588	75%	76%	0.24	205%	209%	0.00	1
	Average Yield Trend:		100%			140%	141%		
			A	B	C	D			
			1.116	1.539	1.508	1.000		415	<-Minimize
2009	167	593	28%	28%	0.72		889%		

Fitted Loss Ratio = A * [1 / (Adjusted Yield Ratio^B)] + Low Yield Indicator * C

(2) NASS: Production / Planted Acres
(3) Previous 10-year average of (2)
(4) = (2) / (3)
(5) = (4) / Average Yield Trend
(6) If (4) < 1.00 then [1-(4)]
(7) From RMA
(8) Fitted Loss Ratio
(9) =[(6) - (7)] ^2
(10) Judgment

Exhibit 5

Texas Cotton All Other
Crop Year 2009
Loss Ratio Projection
APH Plan

(1) Year	(2) Nass Yield	(3) 10yr Average	(4) Yield Ratio	(5) Adjusted Yield Ratio	(6) Low Yield Indicator	(7) Actual Loss Ratio	(8) Fitted Loss Ratio	(9) Squared Error	(10) Weight
1980	192								
1981	358								
1982	203								
1983	269								
1984	307								
1985	338								
1986	215								
1987	461								
1988	462								
1989	610								
1990	475	341	139%	126%	0.00	89%	62%	0.07	1
1991	332	370	90%	81%	0.19	238%	158%	0.65	1
1992	276	367	75%	68%	0.32	408%	220%	3.55	1
1993	459	374	122%	111%	0.00	77%	69%	0.01	1
1994	418	393	106%	96%	0.04	65%	92%	0.07	1
1995	324	405	80%	73%	0.27	103%	199%	0.92	1
1996	381	403	95%	86%	0.14	145%	138%	0.01	1
1997	452	420	108%	98%	0.02	46%	86%	0.16	1
1998	377	419	90%	82%	0.18	197%	156%	0.17	1
1999	370	410	90%	82%	0.18	120%	156%	0.13	1
2000	249	386	64%	58%	0.42	168%	271%	1.05	1
2001	321	364	88%	80%	0.20	151%	164%	0.02	1
2002	433	363	119%	108%	0.00	78%	71%	0.01	1
2003	327	378	86%	78%	0.22	161%	171%	0.01	1
2004	621	365	170%	154%	0.00	26%	52%	0.07	1
2005	707	385	184%	166%	0.00	34%	49%	0.02	1
2006	451	424	106%	96%	0.04	130%	91%	0.15	1
2007	821	431	191%	173%	0.00	20%	48%	0.08	1
2008	433	468	93%	84%	0.16	133%	146%	0.02	1
	Average Yield Trend:		110%			126%	126%		
			A	B	C	D			
			0.755	0.847	3.630	1.000		715	<-Minimize
2009	566	473	120%	108%	0.00		71%		

Fitted Loss Ratio = A * [1 / (Adjusted Yield Ratio^B)] + Low Yield Indicator * C

(2) NASS: Production / Planted Acres
(3) Previous 10-year average of (2)
(4) = (2) / (3)
(5) = (4) / Average Yield Trend
(6) If (4) < 1.00 then [1-(4)]
(7) From RMA
(8) Fitted Loss Ratio
(9) =[(6) - (7)] ^2
(10) Judgment

Exhibit 6

Iowa Corn
Crop Year 2009
Loss Ratio Projection
Revenue Plans

(1) Year	(2) Ins Plan	(3) Nass Yield	(4) 10yr Average	(5) Yield Ratio	(6) Adjusted Yield Ratio	(7) Low Yield Indicator	(8) Price Change	(9) Price Times Yield	(10) Actual Loss Ratio	(11) Fitted Loss Ratio	(12) Squared Error	(13) Wght
2000	CRC	140	128	110%	97%	0.03	-0.16	0.82	41%	45%	0.00	1
2001	CRC	142	130	110%	97%	0.03	-0.17	0.81	70%	48%	0.05	1
2002	CRC	158	132	120%	106%	0.00	0.09	1.15	17%	5%	0.01	1
2003	CRC	152	134	113%	100%	0.00	-0.07	0.94	19%	15%	0.00	1
2004	CRC	177	142	125%	110%	0.00	-0.28	0.80	18%	31%	0.02	1
2005	CRC	169	145	117%	103%	0.00	-0.13	0.90	23%	18%	0.00	1
2006	CRC	163	149	109%	96%	0.04	0.17	1.13	24%	28%	0.00	1
2007	CRC	167	152	110%	97%	0.03	-0.12	0.86	21%	39%	0.03	1
2008	CRC	165	156	106%	93%	0.07	-0.24	0.71	93%	91%	0.00	1
2000	RA/IP	140	128	110%	97%	0.03	-0.16	0.82	32%	45%	0.02	1
2001	RA/IP	142	130	110%	97%	0.03	-0.17	0.81	88%	48%	0.16	1
2002	RA/IP	158	132	120%	106%	0.00	0.05	1.11	24%	7%	0.03	1
2003	RA/IP	152	134	113%	100%	0.00	-0.02	0.98	18%	12%	0.00	1
2004	RA/IP	177	142	125%	110%	0.00	-0.30	0.77	23%	36%	0.02	1
2005	RA/IP	169	145	117%	103%	0.00	-0.17	0.86	34%	22%	0.01	1
2006	RA/IP	163	149	109%	96%	0.04	0.37	1.32	23%	25%	0.00	1
2007	RA/IP	167	152	110%	97%	0.03	-0.06	0.91	15%	33%	0.03	1
2008	RA/IP	165	156	106%	93%	0.07	-0.31	0.65	118%	124%	0.00	1
	Average Yield Trend:			113%					39%	37%		
					A	B	C	D				
					0.106	4.800	5.840	1.000			12	<-Min
2009	CRC	178	158	113%	100%	0.00	-0.08	0.92		16%		
2009	RA/IP	178	158	113%	100%	0.00	-0.03	0.96		13%		

Fitted Loss Ratio = A * [1 / ([Adjusted Yield Ratio*(1-Price Change)]^B)] + Low Yield Indicator * C

(3)	NASS: Production / Planted Acres	(8)	From RMA
(4)	Previous 10-year average of (2)	(9)	= (6) X [1 + (8)]
(5)	= (3) / (4)	(10)	From RMA
(6)	= (5) / Average Yield Trend	(11)	Fitted Loss Ratio
(7)	If (6) < 1.00 then [1-(6)]	(12)	=[(10) - (11)] ^2
		(13)	Judgment

Exhibit 7

Corn
Price Changes
March 15th Sales Closing

Crop Year	RA and IP Base	RA and IP Harvest	RA and IP Change	CRC - HRO Base	CRC - HRO Harvest	CRC - HRO Change
2000	$ 2.51	$ 2.11	-16%	$ 2.51	$ 2.11	-16%
2001	$ 2.46	$ 2.05	-17%	$ 2.46	$ 2.05	-17%
2002	$ 2.32	$ 2.43	5%	$ 2.32	$ 2.52	9%
2003	$ 2.42	$ 2.37	-2%	$ 2.42	$ 2.26	-7%
2004	$ 2.83	$ 1.99	-30%	$ 2.83	$ 2.05	-28%
2005	$ 2.32	$ 1.93	-17%	$ 2.32	$ 2.02	-13%
2006	$ 2.59	$ 3.56	37%	$ 2.59	$ 3.03	17%
2007	$ 4.06	$ 3.82	-6%	$ 4.06	$ 3.58	-12%
2008	$ 5.40	$ 3.74	-31%	$ 5.40	$ 4.13	-24%
2009	$ 4.04	$ 3.90	-3%	$ 4.04	$ 3.72	-8%

Exhibit 8

Hypothetical MPCI Portfolio

			Premium			
State	Crop	Insurance Plan	Assigned Risk	C-Fund	D-Fund	Total
IA	Corn	CRC	50	200		250
IL	Corn	GRIP	50	50	50	150
IA	Corn	APH	20	100		120
IA	Hybrid Corn Seed	APH	10			10
TX	Cotton SE	APH	150	50		200
TX	Cotton AO	APH	50	150		200
TX	Peanuts	APH	25		25	50
		Total	**355**	**550**	**75**	**980**

			Gross Loss Ratio			
State	Crop	Insurance Plan	Assigned Risk	C-Fund	D-Fund	Total
IA	Corn	CRC	16%	16%	16%	16%
IL	Corn	GRIP	17%	17%	17%	17%
IA	Corn	APH	18%	18%		18%
IA	Hybrid Corn Seed	APH	17%	17%		17%
TX	Cotton SE	APH	889%	889%		889%
TX	Cotton AO	APH	71%	71%		71%
TX	Peanuts	APH	38%		38%	38%
		Total	**394%**	**111%**	**24%**	**207%**

			Gross Losses			
State	Crop	Insurance Plan	Assigned Risk	C-Fund	D-Fund	Total
IA	Corn	CRC	8	32	-	40
IL	Corn	GRIP	9	9	9	26
IA	Corn	APH	4	18	-	21
IA	Hybrid Corn Seed	APH	2	-	-	2
TX	Cotton SE	APH	1,333	444	-	1,777
TX	Cotton AO	APH	35	106	-	141
TX	Peanuts	APH	9	-	9	19
		Total	**1,399**	**608**	**18**	**2,026**

On the Accuracy of Loss Reserving Methodology

Tapio Boles and Andy Staudt

Abstract

We evaluate the performance of various loss reserving methods and their associated parameterizations under a number of environments (e.g., changes in case reserve adequacy). We simulate proxy loss development data for each environment, which enables us to measure the accuracy of various actuarial projection methods. Then, based on our results, we offer a roadmap the reserving actuary may use in order to select appropriate methodologies and parameterizations given the current, past, and expected future environmental conditions affecting the reserving process.

Keywords: suitability testing; loss reserving; reserving methods; loss development; management best estimate; simulation.

1. INTRODUCTION

We evaluate the accuracy of various loss reserving methods and their associated parameterizations for several lines of business under a variety of common environmental conditions.[1] Based on our results, we offer a roadmap to guide the actuary in evaluating the appropriateness of these methodologies under different circumstances, understanding the differences in projections between various methods and supporting the choices an actuary makes given the past, present, and expected future conditions.

1.1 Use of Simulated Data

Most similar research falls into one of three categories: hindsight testing,[2] mathematical proof, or simulation.[3] In theory, all are viable options, but limitations in the former two make them unsuitable for our current purposes. Essentially, we are interested in how various loss reserving methods will perform given our reasonable expectations as to the future in a real-world setting. With hindsight testing, we are only able to evaluate the performance of methods under one set of environmental conditions—namely the past; and even then, we are only able to make these evaluations many years after the fact. Furthermore, as the future presents entirely new, unknown environmental conditions that reasonably can be expected to differ from the past, we cannot extrapolate results from hindsight

[1] By environmental conditions we refer to characteristics such as inflation, changes in case reserve adequacy, changes in rate adequacy, changes in claim settlement practices, and changes in the mix-of-business.

[2] Hindsight testing is the process by which past predictions are compared with current results; this occurs some years after the predictions are made and gauges the effectiveness of the methods (and/or actuary) that produced those predictions. See Mahon [12] and Jing, Lebens, and Lowe [10].

[3] Stanard [22], Pentikäinen and Rantala [17], Rollins [19] and Narayan and Warthen [15].

history. And most mathematical proofs, while elegant, can be complex and difficult to apply to a diverse array of methods and environments.

For these reasons, we chose to simulate proxy loss development data. We simulate the historical triangle (what the actuary sees) as well as future periods, thus enabling us to evaluate the accuracy of various methods at ultimate. We are also able to isolate environmental conditions, in order to determine how the accuracy of methods is affected by various environmental changes. Finally, by adding noise to the simulated data, we are able to evaluate how susceptible the accuracy of each method is to random volatility.

1.2 Outline

Section 2 describes the various aspects of our approach. Section 3 discusses possible biases in our approach. Section 4 presents the application of our approach to specific examples.

We have included several appendices to help the reader understand the specifics underlying many of the concepts. In Appendix A, we describe numerous loss reserving methodologies and our implementation of them. In Appendix B, we classify these methods into families based on various common characteristics. In Appendix C, we provide more detailed descriptions of the environmental scenarios evaluated. In Appendix D, we describe how to read the graph that we use to present many of the results. In Appendix E, we rank the methods by their accuracy in various environments. In Appendix F, we show in what direction, if any, the methods were biased in various environments. In Appendix G, we give a complete list of the abbreviations and notations used throughout this paper.

2. BACKGROUND AND METHODS

The following briefly introduces the various dimensions of our work including how we simulated data, how we programmed various loss reserving methodologies and how we evaluated their performance.

2.1 Simulation Method

To create proxy data, we used the following general process. When reviewing this section, it may be helpful to refer to Appendix C, to learn basic properties of the proxy data and how the environments were constructed, and Appendix D, to understand how the tests were applied. Each of the data components that we created are italicized in the section below.

First, we deconstructed the loss process into basic component parts and a system of mathematical functions to describe the relationships between the parts. Essentially, we used *exposure* and *frequency* vectors as a starting point to produce ultimate claim counts. We then applied *incremental-reported-on-unreported claim count patterns* to derive reported claim counts at various evaluations. We then applied *incremental-closed-on-incremental-reported claim count patterns* and *incremental-closed-on-open claim count patterns* to get closed and open claim counts at various evaluations. To estimate incremental loss payments, we applied *incremental-paid-on-closed severities* and *incremental-paid-on-open severities* to the incremental closed and open claim counts, respectively. To estimate case reserves, we applied *case-reserve-per-open severities* to the open claim counts.[4] Both the paid severities and reserve severities differ by evaluation period.

We parameterized each proxy data component mentioned above with real world data to produce a deterministic data set with a stable environment. At this stage, the development patterns for each accident period are identical. To assure that the result was realistic, we evaluated various aggregate diagnostics of the proxy data, such as cumulative loss development patterns and loss severities by age of development and compared the results to diagnostics of the underlying data.

To build each of the environments, we adjusted the basic deterministic components based on the unique characteristics of the environment (as described in Appendix C). For example, in environment 4, the change in case reserve adequacy affects reserved severities but often has no impact on claim counts or paid severities. Each of the environments contains an identical stable history of loss development data (i.e., the upper left portion of the triangle), prior to the first testing period (as defined in Appendix D). The first environmental change (whether applied on an accident-year or calendar-year basis) coincides with the first testing period. We do not apply any tests to accident years prior to the start of the first environmental change (although we can reasonably assume that some of the environmental changes affect the ultimate losses of these older accident years).

The process above results in deterministic data sets for each environment. In our last step, we produced stochastic data sets by applying noise multipliers (by accident year and evaluation period) to each of the basic deterministic components mentioned above (except for *exposure* and *frequency*).

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[4] Because simulated paid losses are not directly based on simulated case reserves, a change in case reserve adequacy does not affect paid losses. However, as mentioned later, the simulated noise multipliers are correlated between data components, so that random changes in paid losses are not independent from random changes in case reserves.

Each of the noise multipliers is a normal random variable with a mean of 1.0. We estimated coefficients of variation based on the underlying real world data that we used to parameterize the proxy data. We also correlated the noise multipliers applied to each data components based on correlations that we observed in the real-world data (i.e., higher incremental paid loss on open claims is correlated with higher incremental paid loss on closed claims).

2.2 Loss Reserving Methods

Previous work regarding suitability testing of loss reserving methods focuses on a few methods. To provide a more complete picture, we included techniques beyond the traditional actuarial methods. We focused on methods that can be automated and that do not require optimization routines or knowledge of advanced mathematics or computer science to implement. Where possible, we contacted the authors for exact implementations of their methods to ensure accuracy; we are grateful to those who responded. While our paper is not a complete survey of loss reserving methods, we have included many of the methods that are commonly used by practicing actuaries when developing losses to ultimate. See Schmidt [20] for an excellent bibliography of more recent literature, and Skurnick [21] for descriptions of earlier methods.

2.3 Tests, Criteria, and Statistics

Estimating ultimate loss is vital for major actuarial functions, including loss reserving and ratemaking. While the reserving actuary is interested in projecting ultimate loss for all immature years, the ratemaking actuary may be interested only in projecting ultimate loss for the latest few years. To simplify the presentation of results, we focused on evaluating methods on how well they project loss from earliest evaluation (i.e., 12 months) to ultimate.

There are a variety of criteria that an actuary can use to evaluate the performance of a loss reserving method: accuracy, bias, stability, responsiveness, robustness, consistency, independence, etc. However, we focused on those we believe most important for the practicing actuary—accuracy and bias. We chose accuracy for the obvious reason and bias because it is often helpful to know which methods err in opposite directions, in order to provide upper and lower bounds around the actuary's estimate.

To measure the accuracy of a method, we used the mean absolute percentage error statistic.[5] We

[5] We use the mean absolute percentage error for three reasons. First, accuracy is not dependent on whether a method misses high or low, but rather how close the method is to the true value. Second, we chose the absolute value, rather than the commonly used squared error, as the latter implicitly is a function of the standard deviation and as such does

defined error as the projected ultimate loss minus the actual ultimate loss and the percentage error as the ratio of the error to the actual ultimate loss. We also used the mean percentage error statistic, which represents a combination of accuracy (magnitude of error) and bias (direction of error).

We did not attempt to measure stability or responsiveness, but we use these terms qualitatively. If we describe a method as stable, we mean that its estimates are "sticky" or relatively unaffected by noise or environmental changes. If we describe a method as responsive, we mean that this method corrects itself to produce accurate results shortly after an environmental change. Responsive methods, however, often suffer from a temporary period of inaccuracy during a period of change.

Throughout this paper, we generally refer to results within the context of accuracy. If we say that a method is distorted by or susceptible to a change, we mean that the accuracy is reduced.

3. CAVEATS

Prior to discussing results, we should highlight some of the possible biases in our simulation model and caution the actuary against blindly applying the results, without serious consideration of the differences in situation.

3.1 Specific Books of Business

The most obvious bias in our work is that it necessarily reflects the data we used to parameterize the simulations. The underlying line of business is the medical component of workers compensation. To parameterize the proxy data, we relied on publicly available California industry data, as summarized by the Workers' Compensation Insurance Rating Bureau of California (WCIRB). If we had chosen data from a different region, for example, the errors would be different but the main conclusions would likely be similar. Also, workers compensation is characterized by partial payments on open claims. If we substitute workers compensation with a long-tailed liability line of business, in which there are very few payments prior to claim settlement, there may be differences in some of the conclusions, particularly with methods that separate loss between frequency and severity components. Application of similar testing to other sets of data and other lines of business represents an opportunity for future research.

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not purely assess accuracy. Finally, we use "percentage" error so as not to give disproportionate weight to tests with large dollar values.

3.2 Number of Evaluation Periods

For presentation purposes, we used 11 evaluation periods for each accident period, where the first 10 evaluation periods represent normal development and the eleventh evaluation period is the tail period (i.e., 120 months to ultimate). However, we also tested results using loss triangles with 30 evaluation periods (i.e., with a tail from 348 months to ultimate) and found that the results were more or less invariant to the number of evaluation periods as long as there was a sufficient volume of data at later evaluation ages. The accuracy of methods based on cumulative data (such as the chain ladder method) was relatively unaffected when switching between 30 evaluation periods and 11 periods. However, a method that is dependent on open claim counts or incremental payments may break down if there are no open claim counts or incremental payments in later evaluations. The practicing actuary should consider the credibility of data in the tail before applying the observations in this paper. Alternatively, an actuary may want to combine different methods for different evaluation periods based on the volume of data available as well as the relevant environmental effects by maturity level.

3.3 Structure of Simulated Data

When designing the building blocks of the proxy data, we chose a structure that we believe is realistic (i.e., it maintains appropriate relationships between data types) without being overly complex. If we had chosen a different underlying structure, it likely would have impacted our testing results somewhat. As mentioned previously, our simulated losses are based on the product of simulated claim counts and simulated severities. If instead we had simulated claim count data independently of loss data, then it is likely that methods that exploit the relationship between claim counts and loss severities (such as the Adler-Kline method) would perform poorly.

3.4 Distribution of Noise

As mentioned previously, in order to produce stochastic data, we simulated noise multipliers based on normal[6] random variables. We then multiplied these random variables by the basic components of our deterministic data, such as paid loss severities. Without knowing the underlying frequency and severity distributions, we used the simplifying assumption that a normal distribution would adequately approximate the shape of aggregate noise affecting the development triangles. To test this assumption, we also considered gamma and lognormal noise multipliers. The results were

[6] Although in theory it is possible for normal random variables to produce negative values, this was not an issue in our simulation, because the standard deviation was small enough so that the probability of a negative value was infinitesimal.

approximately the same.

3.5 Amount of Noise

The results of our testing are based on levels of noise consistent with that observed in the data used to parameterize the proxy data. Our aim was to reproduce the level of noise that would be typical for a large insurance carrier, with a fairly consistent history of homogeneous exposures. In order to assess the sensitivity of our results to various levels of noise, we increased and decreased the coefficient of variation of the noise multipliers. We found that if we had chosen a higher level of noise consistent with a smaller, less credible set of exposures, it is likely that methods based on more granular data (e.g., incremental payments, claim counts, incremental paid severities) would suffer due to the leveraged effect of noise on the basic components of such methods.

3.6 Methods

The results of any loss reserving method are influenced by the way in which the method is parameterized. As noted in a Section 4.2.6, methods whose parameters are based on short-term observations are more responsive and less stable than those based on long-term observations. Similarly, methods that rely on estimates of loss trend, such as the incremental additive method, can be significantly affected by choosing a short-term or long-term trend rate. More complicated methods, such as the Berquist-Sherman adjustments, are dependent on how the methods are constructed.

As much as possible, we have attempted to construct our methods using the same rules as described in the original literature referenced in Appendix A. The actuary should consider how the structure or parameterization of a method may impact its accuracy in various situations.

3.7 Environments

Our results are very much tied to the specific environments we tested and to the parameterizations we chose to describe and define those environments. These are described in detail in Appendix C. For each of the environments, our aim was to model a significant change in the development data, large enough to show a measurable distortion in projection methods, while being reasonably likely to occur. If we had chosen environments with smaller or larger changes, our testing results would have been muted or exaggerated, respectively.

Another element to consider is time. For environment 2, for example, we modeled three years of elevated inflation. If we had increased the number of years of elevated inflation, responsive methods (such as the incremental multiplicative) would have been more accurate and unresponsive methods (such as exposure-based methods) would have been less accurate. The inverse is also true.

Conclusions about which methods perform well or poorly under various conditions (such as "the Berquist-Sherman adjustment for case reserve adequacy is accurate during a period of changing case reserves") would not change if we had chosen different values for the environmental parameters: only the relative difference in accuracy between the methods would change.

3.8 Test Statistics

In some respects, our results directly depend on the tests we used to evaluate the various methods: the mean percentage error and the mean absolute percentage error for the 12-month to ultimate projection. However, we also considered various other statistics such as the mean squared error to measure the volatility and responsiveness with regard to overall accuracy of these methods. We found that using these other statistics did not noticeably change our conclusions regarding the *relative* accuracy of the methods.

3.9 Limitations of Our Recommendations

The recommendations in this paper should not be used in place of the actuary's due diligence and appropriate judgment. Our findings presuppose that the actuary is able to review relevant diagnostics or leading indicators to evaluate the characteristics of the current environment and make assumptions about future conditions. It is outside the scope of this paper to provide a list of diagnostics or to determine how easy or difficult it may be to determine the current environment based on these diagnostics. In some instances, it may be difficult for an actuary to ascertain the precise nature of the underlying environment affecting the loss development data. However, we believe the environments reviewed in this paper are broad enough so that they could be identified with diagnostics or other available information, such as a law change or economic data.

4. RESULTS AND DISCUSSION

The following is our roadmap to help the actuary in evaluating various loss reserving methodologies. In Section 4.1 we present several of our high-level findings and general recommendations. In Section 4.2, we comment on several basic components of loss reserving methods that are not unique to any one method. In Section 4.3, we present findings as they relate to families of projection methods, where the families are defined in Appendix B. In Section 4.4, we present findings by environment.

4.1 General Findings and Recommendations

4.1.1 Stable versus responsive methods in periods of consistent conditions

We found that stable methods outperform methods that are more responsive in environments where, although there is still random noise, conditions are consistent over time (or more generally, when the actuary cannot discern the cause of variability). Stable methods tend to rely on a longer history by design or through selection of parameters, and therefore are better at avoiding distortions due to variability that does not reflect environmental changes. In environments that remain consistent over time, methods using parameters based on longer-term averages outperform those whose parameters reflect shorter histories; exposure-based methods (like the Bornhuetter-Ferguson) beat methods that rely on loss only; and cumulative methods outperform incremental methods. Other methods that incorporate a longer history by construction are the Berquist-Sherman adjusted methods (which perform better than the corresponding unadjusted methods), as well as several more complex methods that rely on the entire triangle to project ultimate loss.

4.1.2 The importance of environmental changes

In practice, it is unlikely that conditions would be consistent over a long period. Loss triangles are constantly subject to forces that can distort loss reserving methodologies. The workers compensation system is subject to forces that cause shifts in loss development data, whether slow and subtle (e.g., a change in the mix of claim types due to a shift away from manufacturing) or sudden and dramatic (such as legislative benefit reform). Therefore, while we found that stable methods would theoretically outperform responsive methods in environments where conditions are consistent, in practice the actuary may not often encounter such environments. More likely, there will be subtle shifts in a manner that is either unknown or at least not yet quantifiable.

4.1.3 Adjusting the data during periods of significant upheaval

During periods of significant upheaval, the mechanical application of any loss reserving method to raw data is unlikely to yield reasonably accurate projections. In fact, we found that all methods tested perform quite poorly under such circumstances. This result highlights the importance of not relying blindly on loss reserving methods when the underlying data has been significantly distorted by environmental changes. In these situations, alternatives include making data adjustments (e.g., restating history to current cost level or current claim mix) or otherwise correcting for environmental changes before applying loss reserving methodologies. If the nature of the distortion is understood, it is also useful for the actuary to identify which method's performance will be most affected.

4.1.4 Responding after periods of severe environmental change

If the assumption is that the system has reached a plateau after major disruptions, our findings would point the actuary toward more responsive methods. Our work confirms that methods with short or no *memory* (such as methods where parameters are selected based on recent observations only or incremental methods) fare better than those that use a longer history (e.g., longer-term averages or cumulative methods) under these circumstances.

4.1.5 Type of change versus direction of change

In general, how a method performs under each environment is defined by elements of the environment that change, rather than by the *direction* of the change. For example, we found that methods that work well when claim settlements slow down prove to also work well when those settlements accelerate.

4.1.6 Accident year vs. calendar year effects

We tested the impact of various environmental changes, including accident year shifts (such as an increase in frequency from one accident year to the next) and calendar year shifts (such as a change in inflation, which affects all accident years simultaneously). Our analysis showed that:

(i) Calendar year changes (e.g., inflation that simultaneously impacts all accident years) always affect the accuracy of loss reserving methodologies, since they always distort development patterns.[7] As a result, consideration should be given to adjusting the data underlying the development projection.

(ii) Accident year changes (e.g., change in frequency) do not affect accuracy of methods based on loss unless the shift also causes a change in loss development patterns (e.g., a change in the mix of claim types).

(iii) While an accident or calendar year shift will distort most methods for many years after

[7] Even methods that do not rely on development patterns are affected, because calendar year changes like those described in this paper affect actual unpaid losses, resulting in prediction errors.

the change, incremental methods, by their nature, are able to respond immediately after the change to calendar year shifts.

(iv) Accident year shifts may distort even incremental methods for many years after the change, as the projection for the latest accident year is dependent upon observations from older accident years.

The following figures compare the incremental multiplicative (IM) method with the chain ladder (CL) method as an illustration. Both methods are parameterized equivalently save that the former is applied to incremental paid loss, and the latter is applied to cumulative paid loss.

Figure 1[8] compares the mean error of the IM and CL methods[9] in an environment affected by a calendar year shift in medical inflation.[10] In the first three testing periods, inflation is higher than normal (15%) and in the fourth and subsequent periods, inflation is consistently at its historical average rate (5%). Both methods are distorted; however, the incremental method immediately corrects itself after the change. The cumulative CL method does not—and it will not produce unbiased estimates until the distortion disappears from the data the actuary is using.

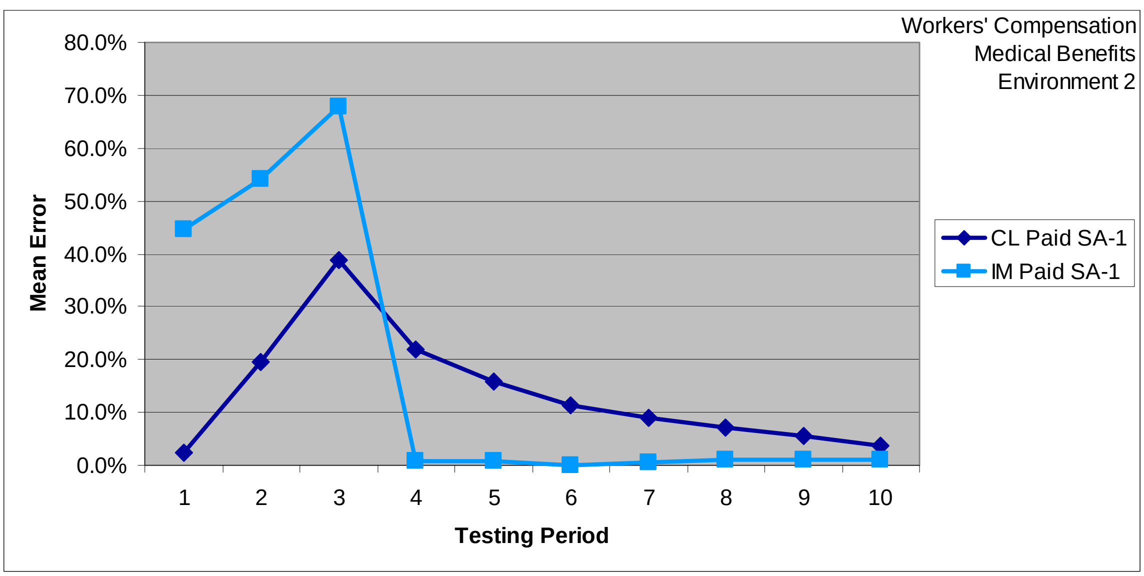

Figure 1: Comparison of incremental and cumulative methods during a calendar year shift.

[8] Please refer to Appendix D, which describes in detail how to interpret the graphs in the paper.
[9] Appendix A provides information about each of the methods and how they were programmed.
[10] Appendix C describes each of the environments in detail and may be useful in understanding logic underlying the conclusions presented.

Figure 2 compares the IM and CL methods during a permanent accident year shift in the frequency of serious injuries, which distorts the development patterns. Both methods are affected, but the IM method corrects itself more quickly after the change.

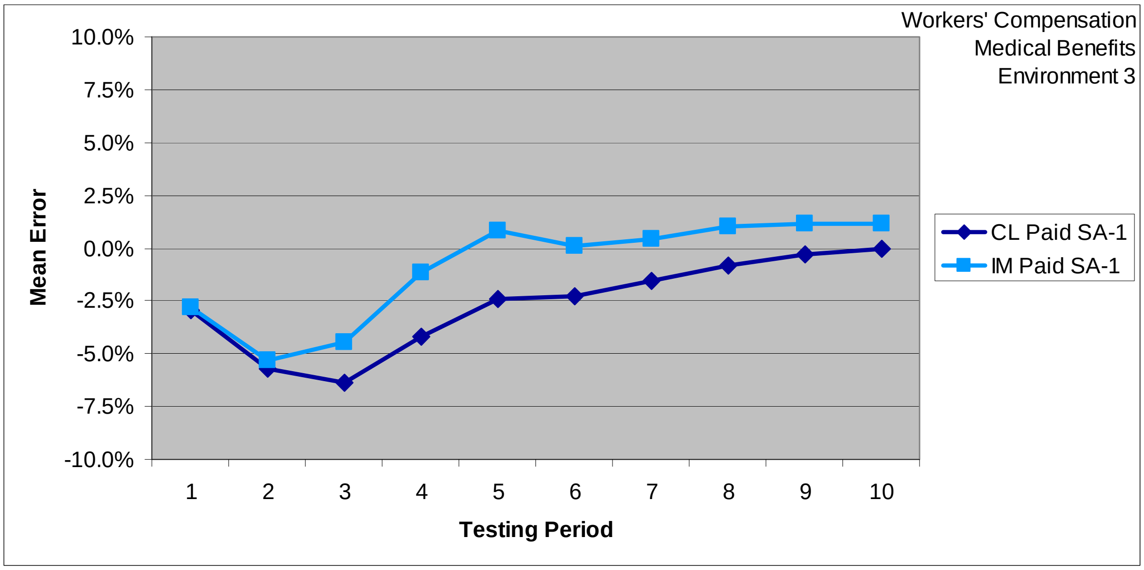

Figure 2: Comparison of incremental and cumulative methods during an accident year shift that distorts development.

Finally, Figure 3 illustrates that in the event of an accident year shift that does not distort development patterns (such as exposure growth), both incremental and cumulative methods are unaffected.

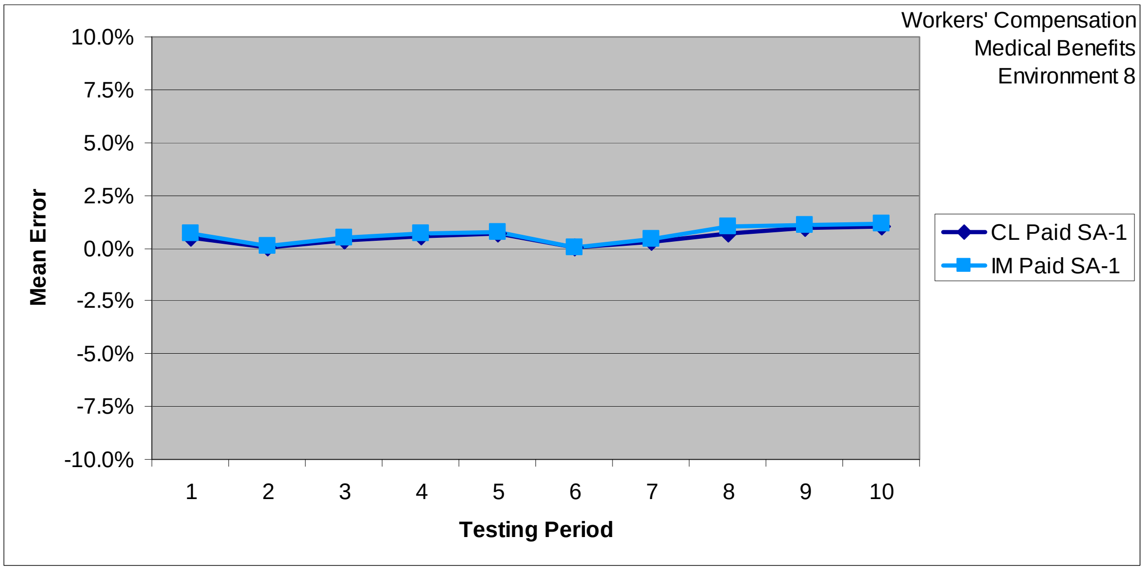

Figure 3: Comparison of incremental and cumulative methods during an accident year shift that does not distort development.

4.1.7 Deterministic versus stochastic analysis

The results of tests that do not take into account residual noise (e.g., tests based on well-behaved, deterministic data) may lead to conclusions that are not appropriate for application in the real world.[11] While enlightening and useful for assessing the reasonability of an approach, we believe that tests performed on deterministic data tend to over-recommend responsive methods, which are susceptible to noise, and under-recommend less responsive methods. When reasonable levels of noise are added, the accuracy of responsive methods is more adversely affected than that of stable methods. This conclusion may caution actuaries against evaluating methods using simplistic examples, which ignore the real-world noise dimension.

[11] Note that this is the only section in the paper where we show test results based on deterministic data. All other tests, shown before and after, are applied to stochastic data.

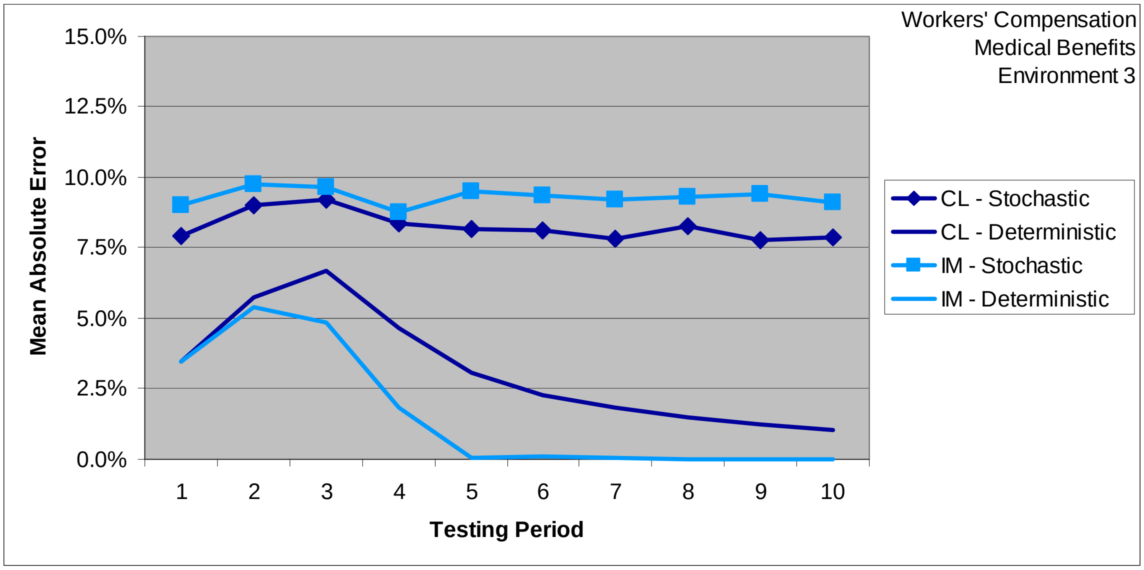

Figure 4: Comparison of the cumulative and incremental methods during both the stochastic and deterministic variants during an increase to a new plateau in the frequency of serious injuries.

Figure 4 shows an example of this phenomenon. Without any residual noise, the incremental method is more accurate and responds more quickly to the change than the cumulative method. However, after we add residual noise to the simulated data, the incremental method is more affected than the cumulative method. Furthermore, the noise in this particular situation is more important to the accuracy of the methods than the environmental change. This analysis invariably depends on the level of noise inherent in the data. When the data is noisy, such as in lines of business characterized by low frequency and high severity, approaches involving more stable methodologies and parameter selection are preferable.

4.1.8 Independence and bias

In cases where two independent methods are biased in opposite directions and produce similar magnitudes of error, a combined method based on the average of those two methods often outperforms either method individually, as the positive and negative errors offset. Additionally, it is helpful to know which methods are biased in opposite directions (and in which environments), as the best estimate is likely to fall between such methods.

4.1.9 Limitations of hindsight testing

Our tests of accuracy are designed to measure errors in the projected ultimate value that, for long-tailed lines, are not capable of being observed in practice. In practice, actuaries typically evaluate success by reviewing changes in *estimates* of ultimate loss over a shorter period (e.g., less than five years). Our analysis showed that such commonly used tests may lead the actuary to discard a good method, which, while it may appear to significantly over- or under-predict in the short term, in actuality performs quite well in predicting the ultimate value.

4.2 The Components of a Loss Reserving Function

Most loss reserving methods are built from the same basic component parts. For example, methods that rely on loss development factors (LDFs) use some type of average of recently observed factors. However, each loss reserving method has some unique aspects that makes it different from other loss reserving methodologies. Since we intend to study these unique aspects, we made sure to implement all methods as consistently as possible, so that the difference in test results directly represents the difference in the unique aspects of the method (e.g., when comparing the chain ladder method with the incremental multiplicative method, we parameterized the loss development factors the same way so that the comparison would only differentiate between structure of the methods).

However, during this process, we noticed that many seemingly different methods are, in practice, identical. The following sections present these results as well as results about characteristics of the various shared components.

4.2.1 The equivalence of Fisher-Lange and Adler-Kline

An excellent example of two distinct methods generating essentially identical results is that offered by the Fisher-Lange (FL) and Adler-Kline (AK) claims closure models. Although the authors describe different methods of computing future severities, the claims component is identical.

The FL method, as described in Fisher and Lange [6], is a frequency-severity approach that operates on report-year data. A key advantage of using report-year data is that the ultimate number of claims is fixed at the end of each report year. The only development is on loss amounts and future claims closure. However, in the absence of report year data, ultimate claim counts can be projected, and the FL method is equally applicable. This is the approach we took in our analysis. After making this modification, however, Fisher and Lange's closure ratios produce identical incremental closed claim counts as Adler and Kline's disposal ratios.[12] This can be shown algebraically, but Figure 13 and Figure 14 provide illustrations of this phenomenon.

4.2.2 The equivalence of the cumulative frequency-severity method with the chain ladder method

The cumulative frequency-severity approach (FS), as described by Friedland [7], projects ultimate loss by applying the chain ladder method separately to claim counts and claim severities. If this approach is parameterized using the latest set of development factors, it is algebraically equivalent to the chain ladder approach on cumulative loss. This relationship holds true whether or not the definition of claim counts is internally consistent and homogeneous. Furthermore, any other parameterization, if applied consistently to both the cumulative frequency-severity approach and the chain ladder approach, will produce results that are virtually identical.

This is not to say that the cumulative frequency-severity method is without purpose. Frequency, in particular, is often impacted by external factors that may not be reflected in the underlying data (e.g., changes in economic conditions and legislative changes). Often there is an advantage to incorporating information exogenous to triangle data when selecting future severities and closure ratios, especially if future frequency and severity are expected to differ from historical frequency and severity. Friedland [7] notes that "[frequency-severity methods] can be particularly valuable when an organization is undergoing changes in operations, philosophy or management."

4.2.3 The equivalence of the incremental additive method, Bühlmann's complementary loss ratio method and the chain ladder method

Both the incremental additive (IA) method and Bühlmann's complementary loss ratio (CLR) method project future incremental loss as a means of estimating the outstanding liability. The IA method computes these amounts based on the relationship of historical incremental loss to on-level exposure, and the CLR method trends forward historical incremental loss. These methods are algebraically equivalent to the chain ladder if parameters are based on the latest observation and loss trend is estimated using a link-ratio approach. By this, we mean that one trend factor is computed for each set of accident years and that these trend factors are calculated as the ratio of the cumulative loss at the latest period to the ratio of the cumulative loss at the earlier period. While this is not the only way (or the best way) to compute trend, it does indicate that both the incremental

[12] This equality only holds when parameterizing the methods using simple averages of one year. Longer-term averages and other types of averages will produce results that are slightly different.

additive method and Bühlmann's complementary loss ratio approach are to some extent intrinsically linked to the chain ladder methodology, and thus it may not offer an independent estimate of loss.

4.2.4 Selecting a projection base (paid loss, reported loss, case reserves, or exposure)

Each loss reserving method reviewed is based on one or more projection bases: paid loss, reported loss, case reserves, or exposures. Each of these projection bases has its own unique advantages and disadvantages that can be good predictors of a method's performance in various environments. For example, methods based on paid loss are immune to changes in case reserve adequacy. However, paid methods appear to be more susceptible to residual noise than methods based on reported loss, because paid methods lack the useful information provided by case reserves, and there is greater prediction error in the paid LDFs due to the larger magnitude of the factors.[13] Methods based on reported loss are quite susceptible to distortions in the reporting pattern caused by changes in case reserve adequacy or claim settlement rates. Methods based on case reserves can be even more distorted than reported methods during changing conditions, as they lack the stability provided by adding paid loss. However, methods based on case reserves are often the most responsive after a period of changing conditions, as they contain information about future loss amounts that is not distorted by volatility in historical amounts. Unlike loss-based methods, methods that rely solely on exposures (such as the budgeted loss method) are completely unresponsive to movements in loss amounts as they represent an a priori estimate of ultimate loss rather than a current estimate of future remaining payments. Generally, exposure-based methods produce stable estimates during changing conditions, but they can err wildly when there are significant changes in loss costs that are not reflected in the underlying exposures. However, as exposure-based methods are often independent of the other loss reserving methods, they are good candidates for establishing bounds within which ultimate loss is likely to be.

4.2.5 Incorporating loss trend

Most of the methods reviewed in this paper incorporate the concept of loss development (i.e., measuring changes in an accident year's losses from one evaluation period to the next). Some of the methods reviewed also incorporate the concept of loss trend (i.e., measuring changes in losses from one accident period to the next).

[13] This observation may depend on the evaluation age and the line of business. Also, in the real world, reported losses may be distorted by small undetected changes in case reserve adequacy, which may increase error in projections based on reported loss.

Several of the methods, including Bühlmann's complementary loss ratio method, Adler-Kline, Fisher-Lange, and the two Ghezzi methods, rely on trending forward historical loss severities from the loss triangle. For these methods, a trend rate is calculated for each evaluation age separately, by fitting a line to the log of the severity amounts. This has the advantage of capturing different trend rates by evaluation age, to the extent that they exist. However, separate trend rates are more susceptible to residual noise than a single trend rate for the entire triangle, especially when the trending period is limited to relatively few data points.

To parameterize the modified Bornhuetter-Ferguson method, which has a self-correcting loss ratio, we employed a three-year trend (i.e., four data points) on an accident year basis. For the incremental additive method, we chose a three-year trend measured on a calendar year basis (i.e., the trend observed based on calendar-year payments). The incremental additive method with a three-year trend often produces very similar results to the incremental multiplicative method based on the latest three years of observations. By contrast, using a long-term trend produces more stable results.

A short-term trend benefits from responsiveness following the end of an environmental change (i.e., when a new period of normalcy is reached), but it may result in wildly inaccurate results during a period of upheaval. A long-term trend, similar to long-term averages of development factors, produces more stable results but is slow to react to emerging conditions.

4.2.6 Comparison of short-term vs. long-term parameterizations

Short-term averages are more responsive than longer-term averages, which are more stable. Figure 5 shows Marker and Mohl's method parameterized using a one-year simple average and a three-year simple average during a three-year bubble in medical inflation.[14] Henceforth, we will generally show only one type of parameterization and work under the assumption that the observed errors will either be muted (and delayed) or intensified depending on whether a longer-term or shorter-term average is used, respectively.

[14] As mentioned previously, all environmental changes begin in the first testing period. Each environment is described in greater detail in Appendix C.

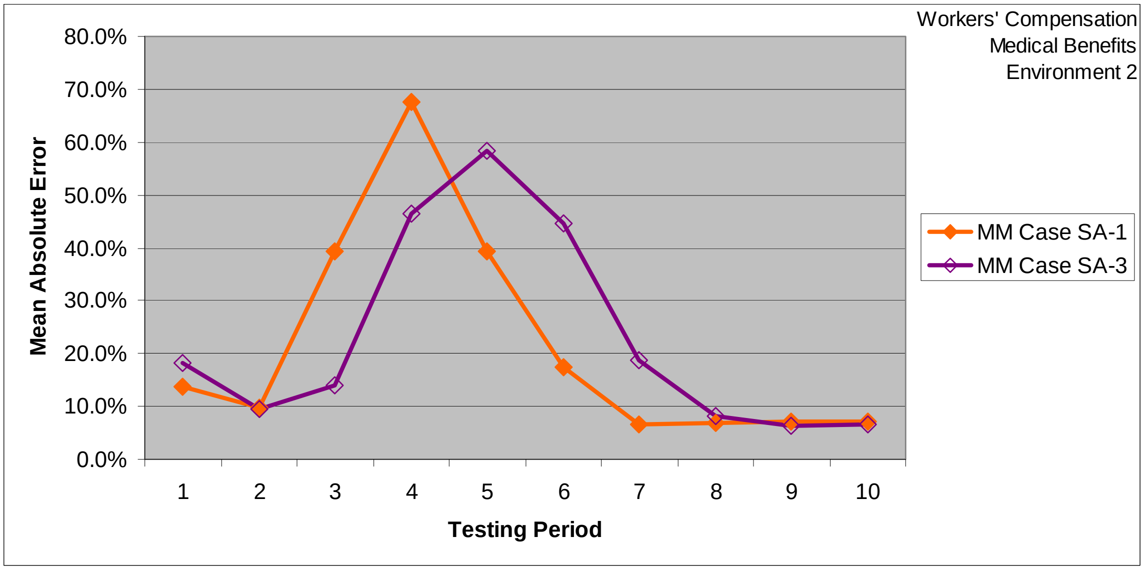

Figure 5: Comparison of a long-term parameterization to a short-term parameterization during a bubble in medical inflation.

4.2.7 Comparison of simple and volume-weighted averages

In our proxy data set, the volume of exposures is stable over time; because of this, there is little difference in accuracy by choosing development factors based on simple averages or volume-weighted averages. See Figure 6 for a comparison of methods based on simple and volume-weighted averages. For simplicity, for the remainder of the paper, we focus on methods that rely on simple averages (or more complex parameterization methods such as regression). However, in the real world, the actuary should be aware that simple averages can be distorted by individual accident years with a small volume of exposures, which are more volatile as a result.

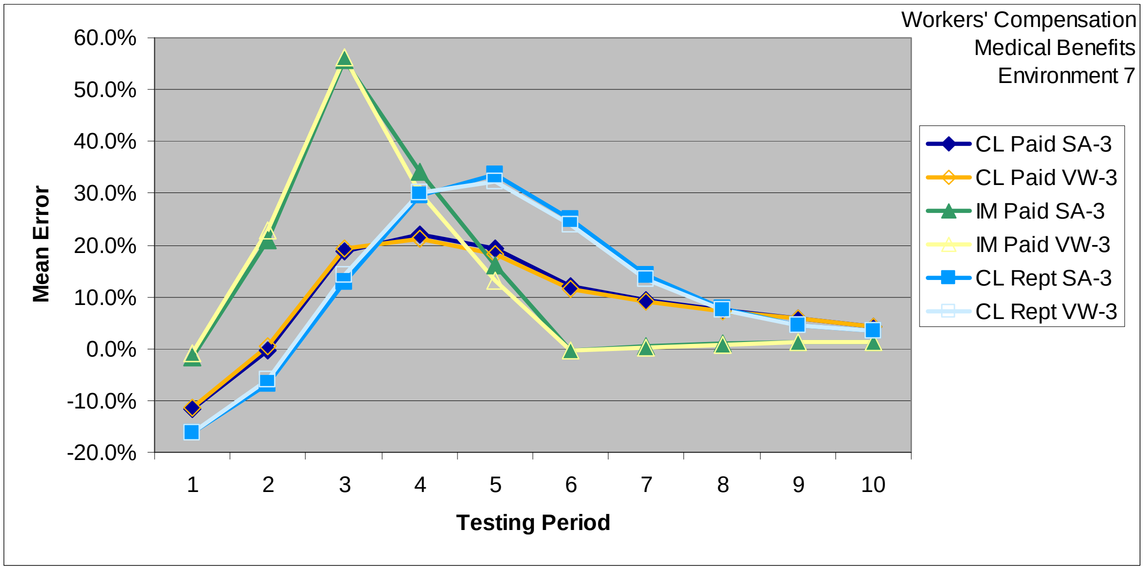

Figure 6: Comparison of volume-weighted and simple averages for the chain ladder method on paid loss, the chain ladder method on reported loss and the incremental multiplicative paid loss. This environment consists of a bubble in the rate of medical inflation coupled with an increase in the frequency of serious injuries.

4.3 Findings by Family of Methods

We grouped the various methods into several families based on certain characteristics (e.g., exposure-based methods, frequency-severity methods, incremental methods, regression methods, etc.). These classifications can be found in Appendix B. The following conclusions all pertain to one family or another and are meant to identify differences between methods within a family. By reviewing these results, we can draw conclusions that tie our understanding of how the methods are constructed to how accurately they perform in various environments.

4.3.1 Exposure-based methods

Figure 7 shows the mean error of methods based on exposures and/or paid loss in environment 2, a temporary three-year period of high calendar-year inflation. In the first year after the onset of inflation, all methods underestimate because they are unaware of the higher-than-expected inflation. Soon after the start of high inflation, the paid method overestimates the ultimate loss because it expects the higher inflation to continue indefinitely. The budgeted loss method (BL) never recognizes the change and therefore always underestimates. The Bornhuetter-Ferguson (BF) and Benktander (BT) methods, meanwhile, lie between the extremes. They still underestimate, but not to the same degree as the budgeted loss method. The modified Bornhuetter-Ferguson (MBF) closely follows the chain ladder method, because the trend underlying the MBFs expected loss ratio is based on projections of ultimate loss produced by the chain ladder method. The incremental additive method has a self-correcting trend rate, so that after inflation reverts to historical norms, the IA produces unbiased estimates.

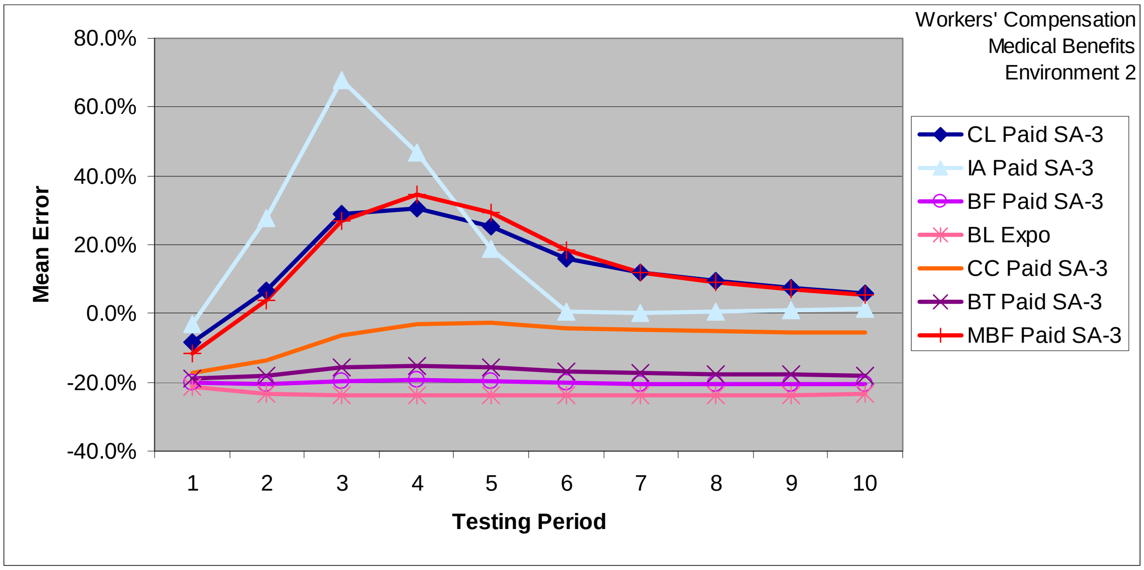

Figure 7: Comparison of various exposure-based methods on paid loss during a bubble in medical inflation.

Figure 8 shows the same methods based on reported loss instead of paid loss. Because the expected percentage reported at 12 months is higher than the expected percentage paid, the BF and BT on reported loss methods are relatively more responsive.

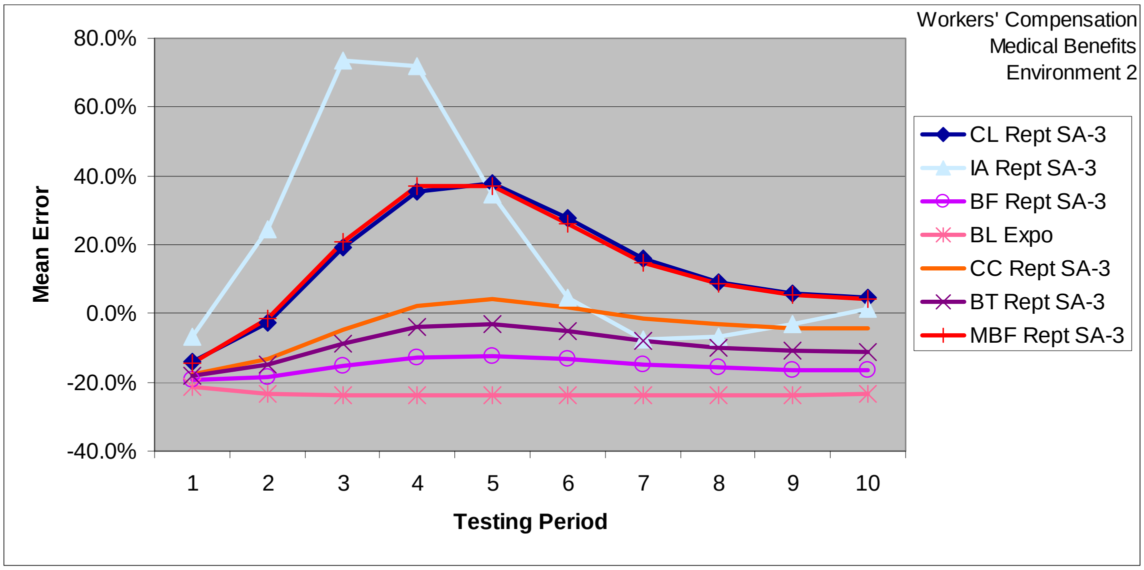

Figure 8: Comparison of various exposure-based methods during a bubble in medical inflation. Where Figure 7 shows the exposure-based methods on paid loss, this figure shows the exposure-based methods on reported loss.

Figure 9 shows the exposure-based methods when subject to an acceleration in claim settlement rates. All methods initially overestimate the ultimate loss, because in this environment, the faster claim closures result in lower ultimate losses. The BF and BT methods, by their nature, fall somewhere between the results of the BL and CL methods. Similar to the previous example, the IA method is the most responsive after the change.

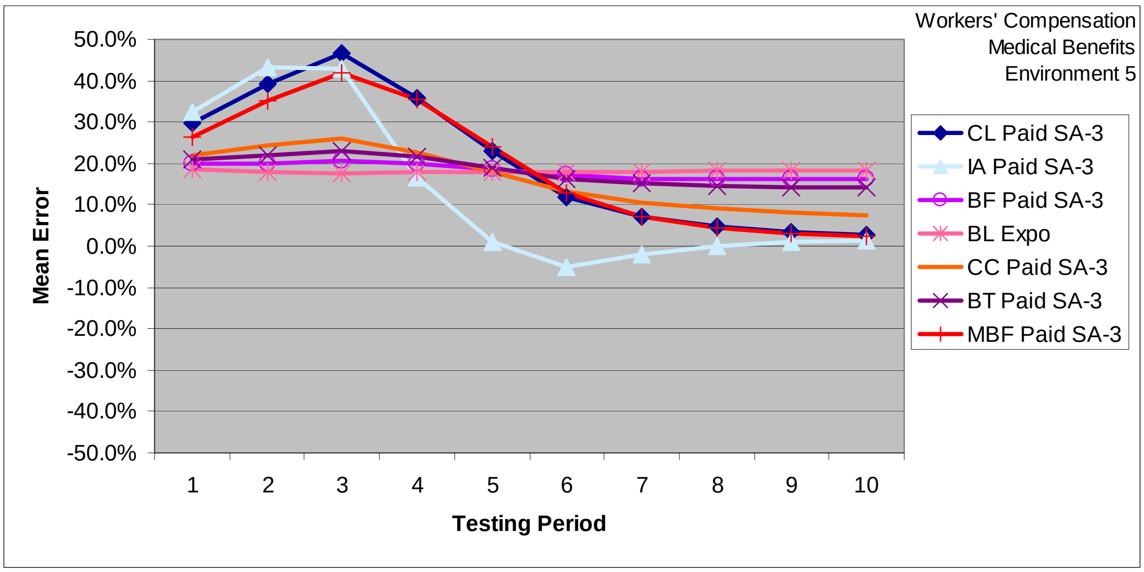

Figure 9: Comparison of various exposure-based methods during a permanent acceleration in claim settlement rates.

4.3.2 Regression-based methods

Regression-based loss reserving methods appear frequently in actuarial literature. The following are some general considerations regarding members of the regression family.

First, consider Brosius's least squares development (LS) method since it serves to highlight some advantages and disadvantages of regression methods. Figure 10 and Figure 11 make it obvious that LS is by far the worst method *during* the period of change, but that it responds much faster than any other method *after* the change, quickly becoming one of the most accurate methods. To understand this, consider how LS works. The LS method begins with the oldest accident years and uses data at the $n-1$ evaluation period to project data at the n^{th} evaluation period (or ultimate) by fitting a line through least squares. Subsequently, this projection is added to the vector of n^{th} evaluation (or ultimate) values and with the addition of another accident year, data at the $n-2$ evaluation period is used to project ultimate loss at the n^{th} evaluation period, and so forth. This approach is reasonably accurate as long as future actual observations are within the range of historical observations. But when future values fall outside of the range of history, the model must extrapolate and errors are increased. This is true of most regression methods; however, the problem is exacerbated by the LS method as the predictions are iteratively fed back into the model in such a way that the error propagates itself.

However, immediately after the change, when conditions stabilize, the LS and other regression methods correct themselves with varying degrees of responsiveness. This is an example of the stability/responsiveness trade-off as determined by the number of parameters in the model. In general (but not always), methods with more parameters are unstable during changing conditions (i.e., they are greatly affected by the changing conditions and produce inaccurate results), but very responsive after the conditions stabilize. As the number of parameters increases, the amount of variability in the dependent variable understood and explained increases as each successive parameter can mine for the residual relationship. However, when conditions are changing, these types of regression models overfit to the historical data (as described above) and produce more inaccurate results. Figure 10, in particular, provides an example of this. The LS method (2 parameters) and Murphy's least squares linear (Mur-LSL) parameterization (3 parameters) are less stable during the changing conditions and more responsive after than Murphy's least squares multiplicative (Mur-LSM) parameterization (1 parameter), and the chain ladder method based on a simple average of all observations (CL SA-All).

Consider now the multivariate (MV) method. This method actually performs very well during an increase in case reserve adequacy coupled with an acceleration in claim settlement rates as shown in Figure 10. However, the MV method performs rather poorly during a bubble in medical inflation as shown in Figure 11. In the simpler environment, the MV method overfits, however, in the more complex environment, the MV is able to combine disjoint pieces of information and perform relatively well.

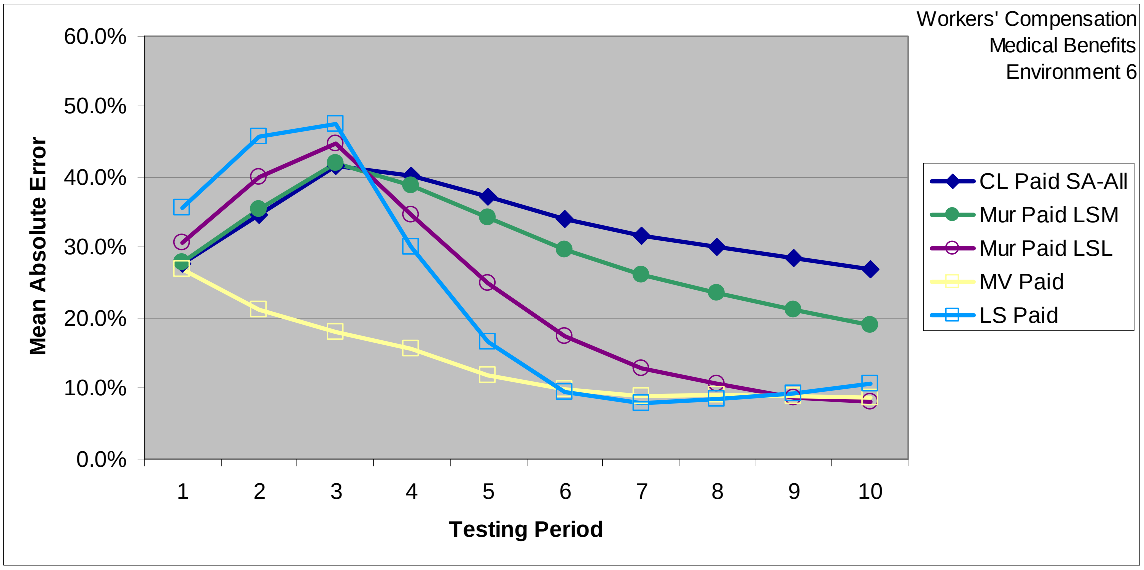

Figure 10: Comparison of various regression methods during a permanent acceleration in claim settlement rates coupled with a permanent increase in case reserve adequacy.

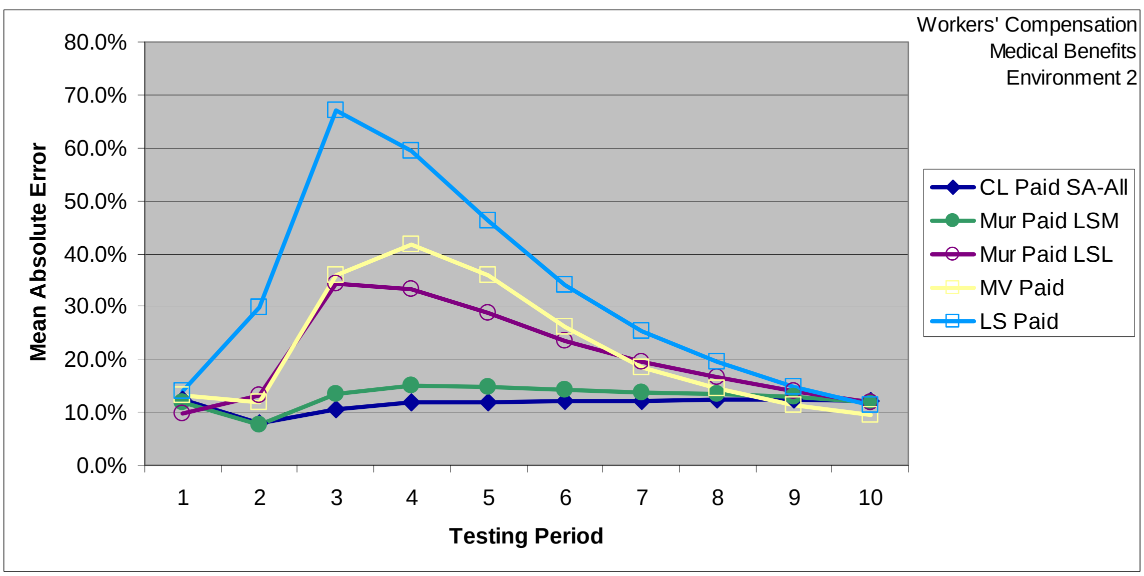

Figure 11: Comparison of several regression-based methods during a bubble in medical inflation.

If the actuary is able to find a regression method that does a good job of describing the loss process, then it may produce accurate results. However, regression methods just as often "overfit" historical data without providing a good prediction of future observations. As a cautionary note, Figure 12 compares each of Verrall's three log-linear models, as described in Narayan and Warthen [15], with the CL method in environment 7 (bubble in the rate of medical inflation coupled with an increase in the frequency of serious injuries). The first model (LL1) has parameters that vary freely by accident year and evaluation period. The second model (LL2) is restricted so that parameters vary only by evaluation period. The third model (LL3) is restricted further so that its parameters do not vary by accident period or evaluation period.

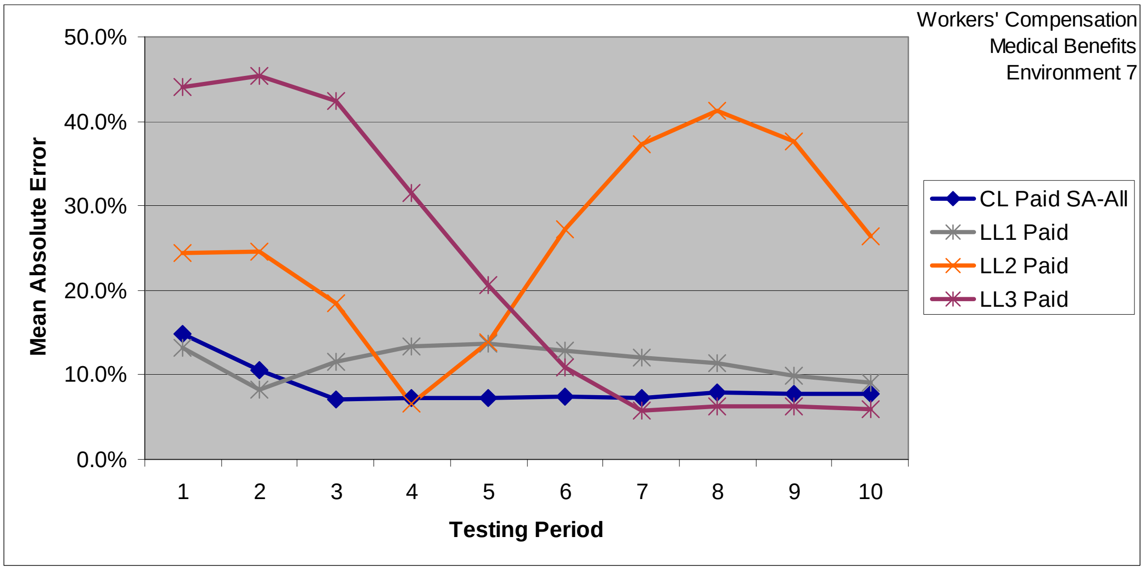

Figure 12: Comparison of Verrall's log-linear models during a bubble in medical inflation coupled with a permanent increase in the frequency of serious injuries.

What is immediately obvious about Figure 12 is the wide variation in results. This highlights the idea that while there are an infinite number of elegant regression models that adhere to theoretically desirable loss development properties, when these models are applied in practice to data that does not mimic those properties the results will be less than desirable. LL3, the simplest of these models, is unable to capture the complex interactions of the calendar year inflation with accident year increase in the frequency of serious claims and errs significantly until these changes work themselves out of the data. LL2 produces large and seemingly unpredictable errors, first underestimating the ultimate loss and then overestimating in later periods. LL1, the most complex of these models, is actually the most accurate during the change, however, it overestimates after the data have stabilized. At the end of the day, the actuary would have been better off using the CL method.

This section further highlights how important it is that the actuary gather both qualitative and quantitative insights from underwriters, claims administrators, and other data sources to improve understanding as to what disturbances underlie the data and, consequently, which methods and parameterizations are likely to over-, under- or correctly estimate future unpaid loss amounts.

4.3.3 Frequency-severity methods

In environment 2, a three-year bubble in medical inflation (see Figure 13), the FS method is distorted in a similar manner as the chain ladder method (not shown). The FL and AK methods start off well, but produce less accurate results a few years after the onset of higher inflation. This is mainly because these methods project future severities using exponential growth curves fit at each evaluation age, which are distorted by the kink in growth caused by a bubble in medical inflation.

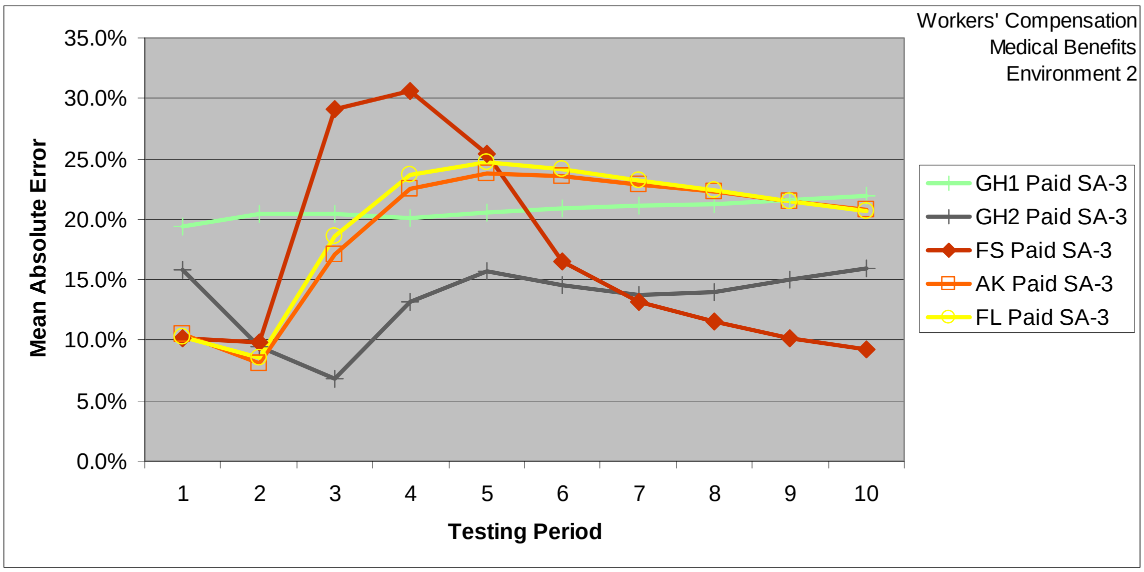

Figure 13: Comparison of various frequency-severity methods during a bubble in medical inflation.

In environment 6, a permanent change in case reserve adequacy combined with a permanent acceleration of claim settlement rates (see Figure 14), each of Ghezzi's methods (GH1 and GH2) shows its merit. This is because these methods are especially effective when data undergo a change that has little or no effect on actual ultimate loss, but the change serves to confuse and distort more traditional loss reserving methodologies. As mentioned previously, the FS method performs similarly to the chain ladder and does not offer any advantage over the AK and FL methods.

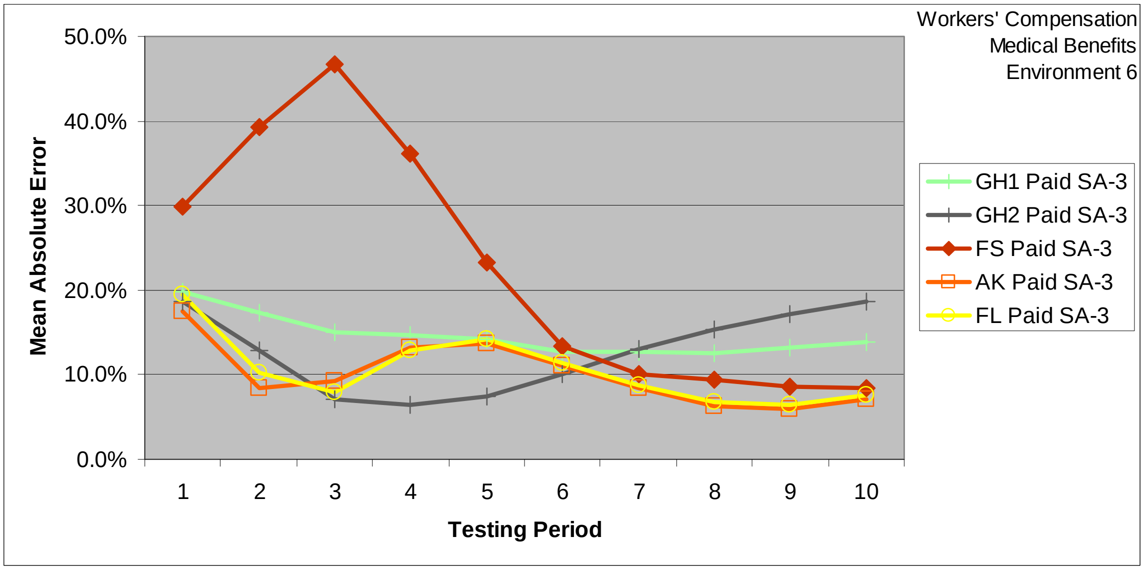

Figure 14: Comparison of various frequency-severities methodologies during a permanent increase in case reserve adequacy coupled with a permanent acceleration in claim settlement rates.

4.3.4 Berquist-Sherman adjustments

We use the phrase "Berquist-Sherman adjustments" to refer to the family of methods that adjust historical triangles prior to projecting. These methods are particularly accurate in environments where the historical change is similar to the one for which the adjustment corrects. Of course, if the emerging environmental change is different from the historical adjustment, then the accuracy of these methods may suffer.

The Berquist-Sherman adjustment for changes in case reserve adequacy (BSRA) method adjusts very well for changes in case reserve adequacy (see Figure 15). Furthermore, this method will also perform reasonably well during an acceleration in claim settlement rates (see Figure 16), where although there is no change in case reserve adequacy per se, there is a change in average outstanding case reserves. This may happen because the BSRA is a more stable (by construction) method and the adjustment for reserve adequacy somewhat dampens the high development factors that distort the chain ladder method on reported loss.

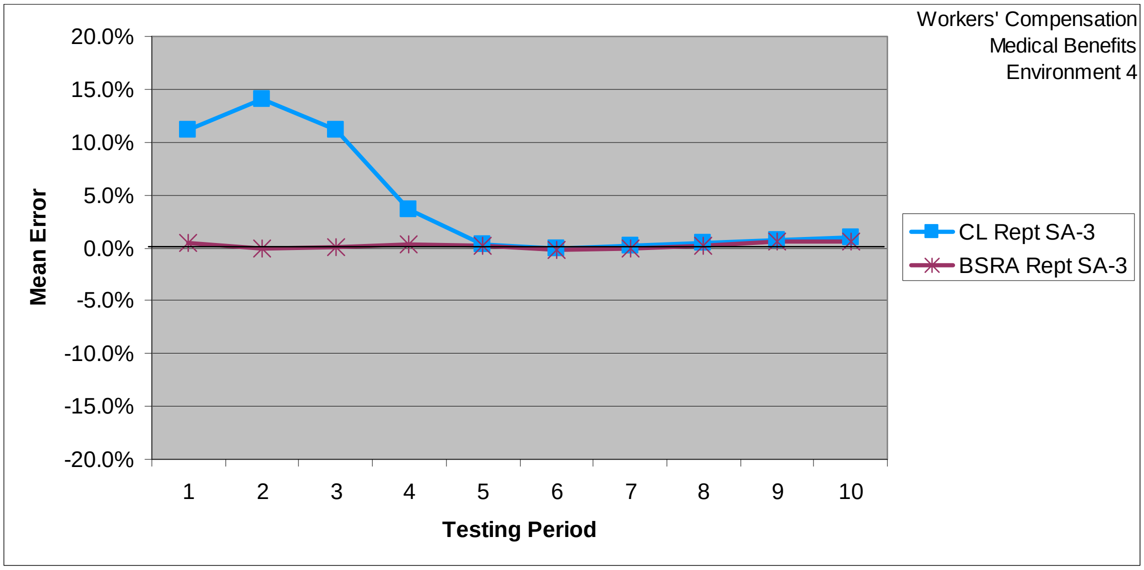

Figure 15: Comparison of the CL and BSRA method during a permanent change in case reserve adequacy.

The Berquist-Sherman adjustment for changes in claim settlement rates (BSCS) and the Fleming-Mayer adjustment for changes in claim settlement rates (FMCS) perform better than the chain ladder on paid loss when there are changes in settlement rates (see Figure 16). The BSCS and FMCS somewhat overreact to the change and underestimate the ultimate loss after the first testing period; neither method perfectly corrects for the change. This is perhaps evidence that in the presence of a change in the rate of claim settlement, the best estimate of ultimate loss lies somewhere between the BSCS and FMCS methods and the traditional CL method, which often in these situations are biased in opposite directions. Also shown is the BSRA, which beats the chain ladder but still overestimates the ultimate loss.

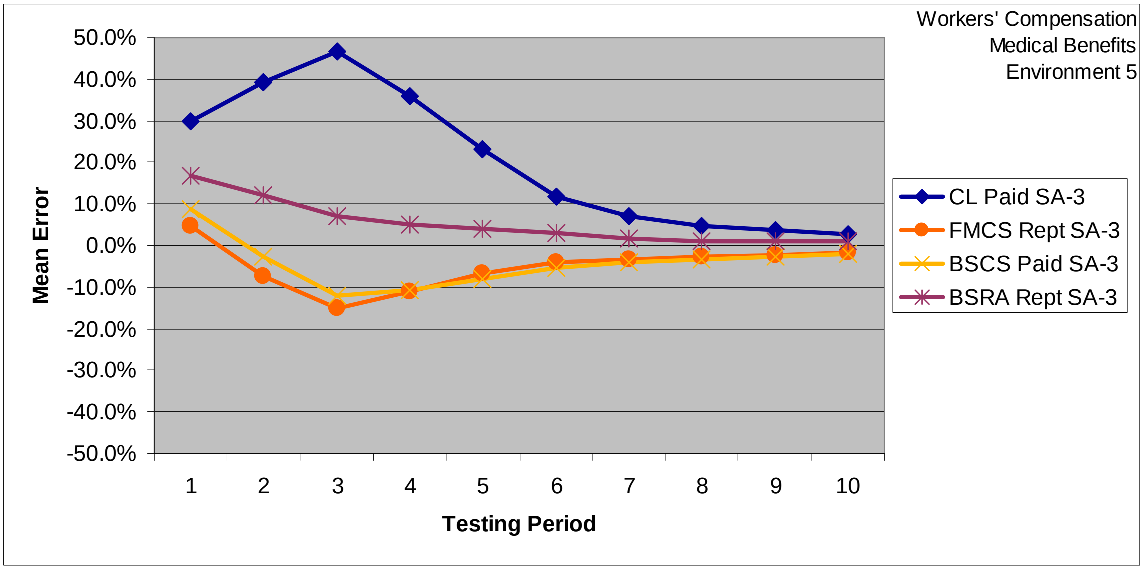

Figure 16: Comparison of the CL method with the Berquist-Sherman adjustments during a permanent acceleration in claim settlement rates to a higher plateau.

Figure 17 compares these methods during an increase in the frequency of serious injuries (environment 3). This change in the mix of claim types results in a change in claim settlement rates as well as a change in case reserve adequacy, although these changes manifest from one accident year to the next, and do not affect historical accident years. Similar to the previous example, the adjusted methods perform better than the chain ladder as they are able to correct somewhat for the environmental change. This is an interesting result, because it suggests that it may be worthwhile to incorporate methods that use Berquist-Sherman adjustments even if the actuary does not have a strong reason to believe that there has been a significant change in case reserve adequacy or claim settlement rates.[15]

[15] This conclusion warrants further investigation, because it is also possible that these methods may overreact to noise in the data that is not indicative of changing case reserve adequacy of claim settlement rates.

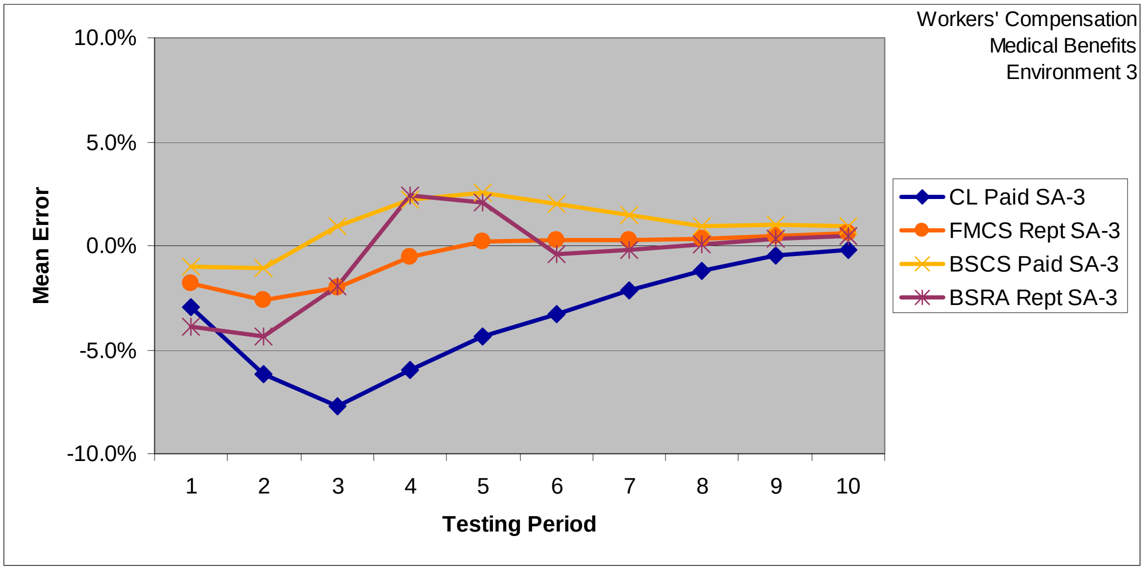

Figure 17: Comparison of the CL method with the Berquist-Sherman adjustments during a permanent increase in the frequency of serious injuries.

4.3.5 Case reserve methods

As would be expected, case reserve methods are adversely affected during a change in case reserve adequacy. Figure 18 compares Atkinson's case development (CD) method with the modified case development (MCD) method and Marker and Mohl's backwards recursive case development (MM) method during a permanent increase in case reserve adequacy. Methods that use case reserves as the projection basis are more adversely affected than the CL method on reported loss because they lack the ballast provided by paid amounts, which are unaffected by changes in case reserves. The MM method is the most adversely affected because the distortion in case reserves not only distorts future predictions of case reserves, but it also distorts future predictions of payments based on projected case reserves (i.e., the error is compounded in the iterative projections of paid and case loss).

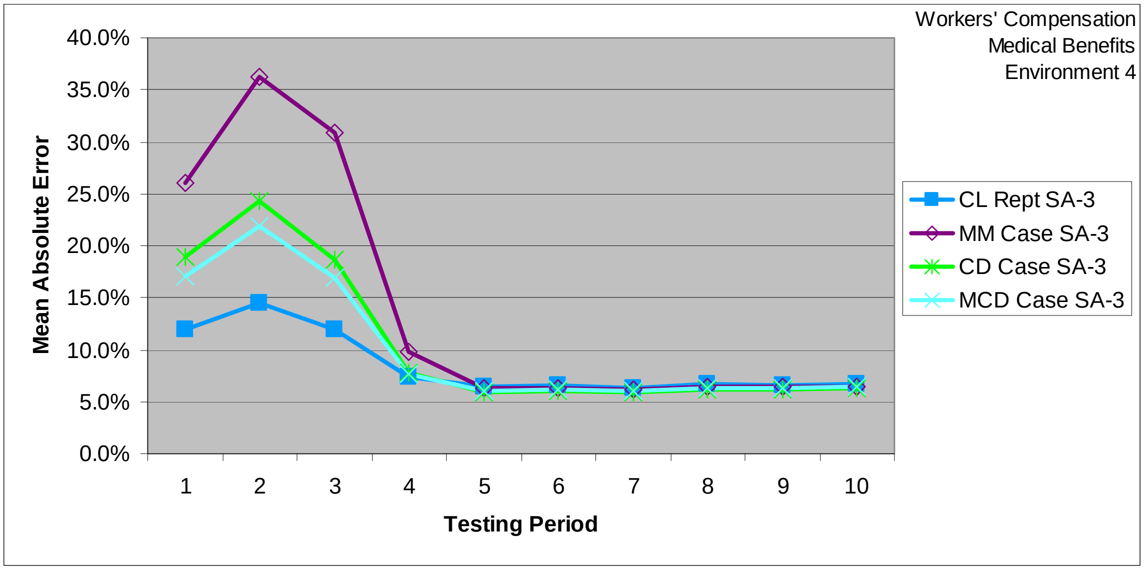

Figure 18: Comparison of case reserve methods during a permanent increase in case reserve adequacy.

What is not as obvious is that case reserve methods perform exceptionally well when there are similar distortions in both paid and reported triangles. Consider Figure 19, which compares the case reserve methods during a permanent increase in the frequency of serious injuries (environment 3). The CD and MCD methods project case reserves (which are unaffected in this environment) based on a function of the reported loss pattern and paid loss pattern. Both these patterns are lengthened due to the increase in frequency of serious injuries. However, the case development method is really only interested in the relative difference between the paid and reported pattern, not the nominal patterns. And since this difference is relatively unchanged in this environment, the CD and MCD methods are relatively unaffected.

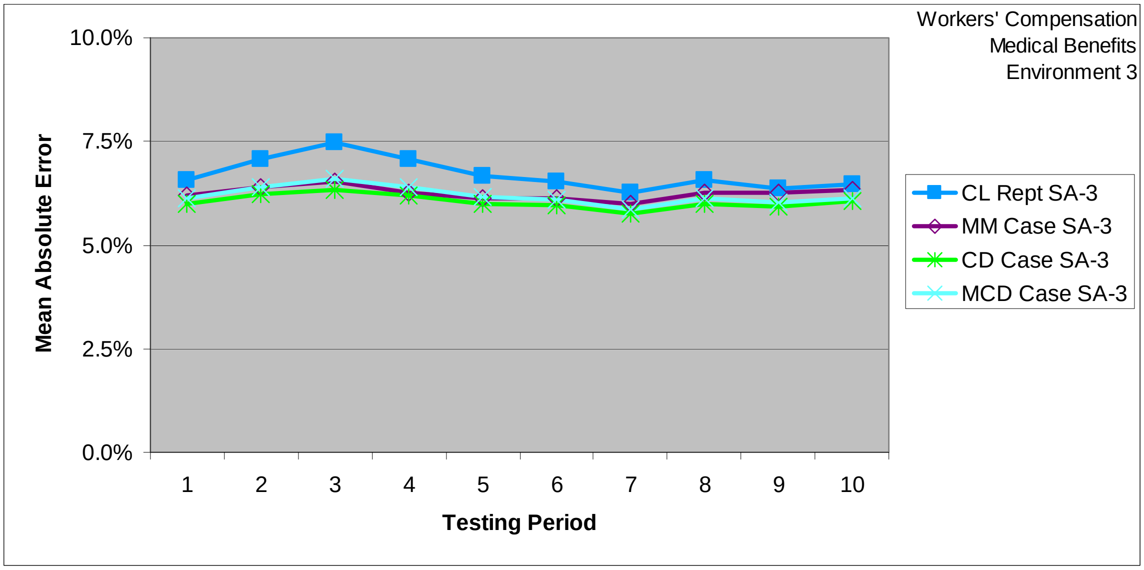

Figure 19: Comparison of case reserve methods during a permanent increase in the frequency of serious injuries.

The MM method is also relatively unaffected because an increase in the frequency of serious injuries drives up both the paid and case incremental severities as serious claims are more costly than average. However, because MM successively applies paid-on-prior case and case-on-prior case ratios, it is not as distorted since both numerator and denominator decrease at reasonably similar rates.[16] And the projection base, case reserves, adjusts to post-change levels more quickly than paid loss.

However, if reported loss patterns are significantly more distorted than paid loss patterns (or vice versa), then the case reserve methods will be distorted. Figure 20 illustrates this phenomenon by comparing the case reserve methods during an acceleration in claim settlement rates. Coupled with this acceleration in claim settlement rates is an increase in the average case reserve as the claims that remain open are the larger, more complex cases.

[16] This is an aspect of this environment, and may not apply in all situations with an increase in the frequency of large claims.

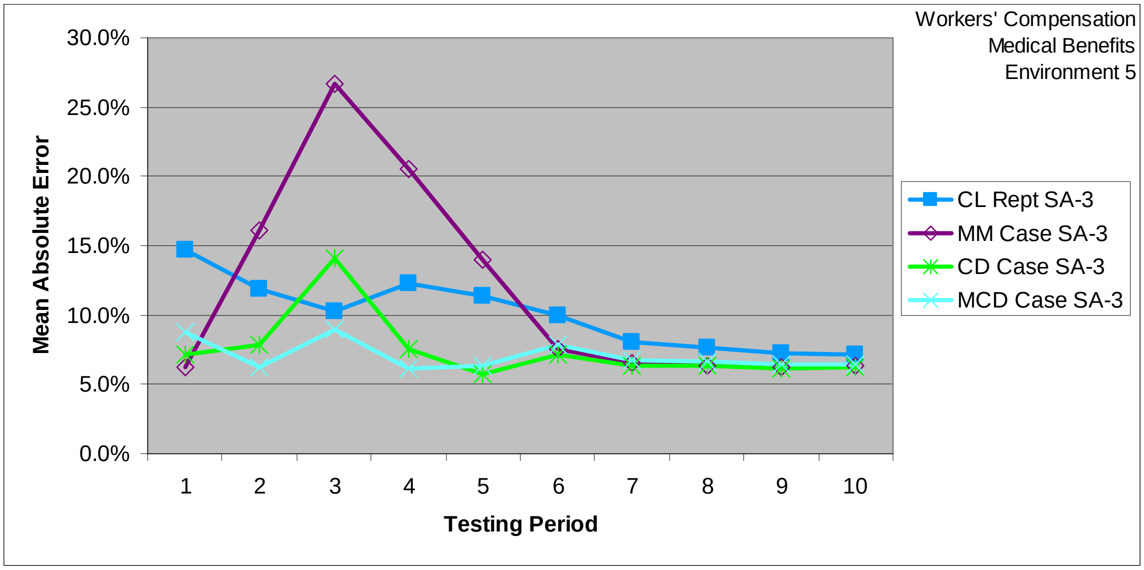

Figure 20: Comparison of case reserve methods during a permanent acceleration in claim settlement rates.

Finally, note that in Figure 18, Figure 19, and Figure 20 the MCD method performs marginally better than the CD method over all testing periods. This is because the MCD method incorporates information about the amount of loss paid to date, which the CD method ignores.

4.3.6 Joint paid-reported methods

The Munich chain ladder (MCL) method produces indications of ultimate loss that are nearly identical whether they are based on reported or paid loss amounts (see Figure 21, Figure 22, and Figure 23). However, those indications are often significantly less accurate than either the CL on paid loss or the CL on reported loss. There are several reasons for this phenomenon. The most obvious reason is that any distortion in loss development, whether it affects paid development (i.e., change in claim settlement rates) or reported development (i.e., change in case reserve adequacy) will always be captured as the MCL models paid and reported amounts simultaneously. For example, consider Figure 21, which compares the MCL and CL methods during a permanent increase in case reserve adequacy.

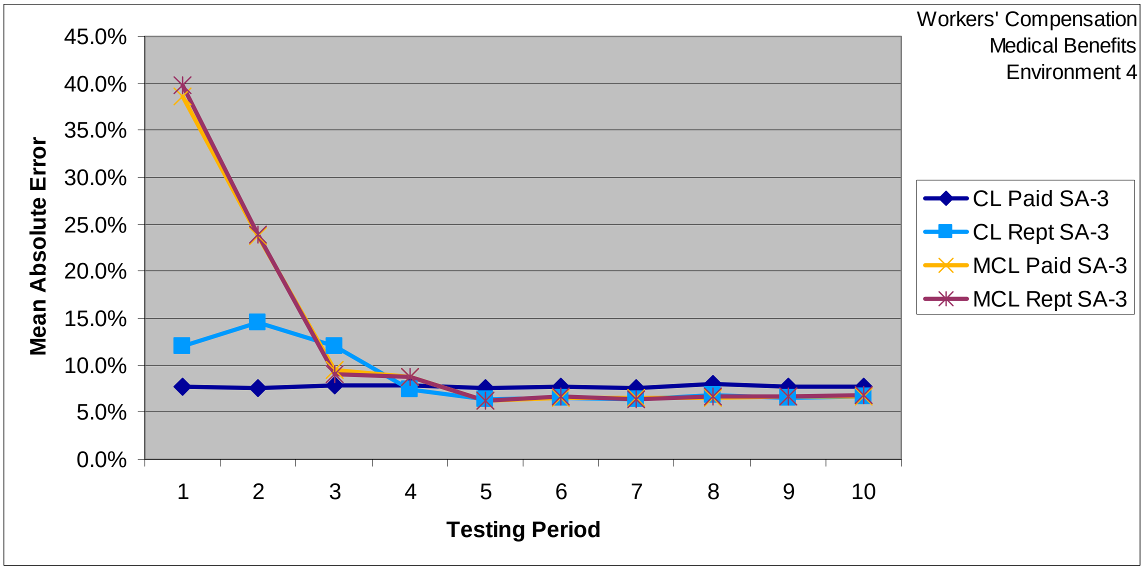

Figure 21: Comparison of the MCL and CL methods during a permanent increase in case reserve adequacy.

Furthermore, the Munich chain ladder appears to magnify distortions to paid development, as any distortion in paid development is implicitly reflected in reported development. To see this, consider Figure 22 where the error in the MCL is close to the combined error of the individual CL methods.

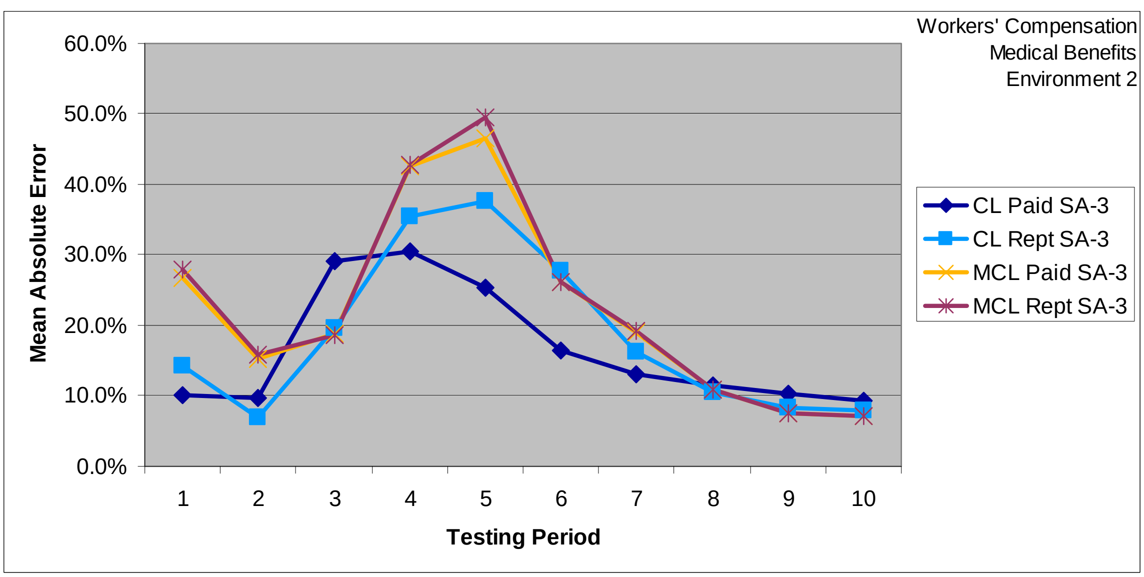

Figure 22: Comparison of the MCL and CL methods during a bubble in the rate of medical inflation.

Finally, note that in situations where there are no severe environmental distortions, although there is still residual noise, the MCL will produce more accurate paid and reported projections. Generally speaking, the MCL is useful to smooth out small distortions in paid and/or reported development that are not expected to be indicative of a larger shift in development. Figure 23 illustrates this phenomenon during a permanent acceleration in claim settlement rates. After the initial shock, the MCL responds much quicker than the CL on paid loss and is the most accurate method for testing periods 4 and subsequent (i.e., periods where there is no significant environmental distortion).

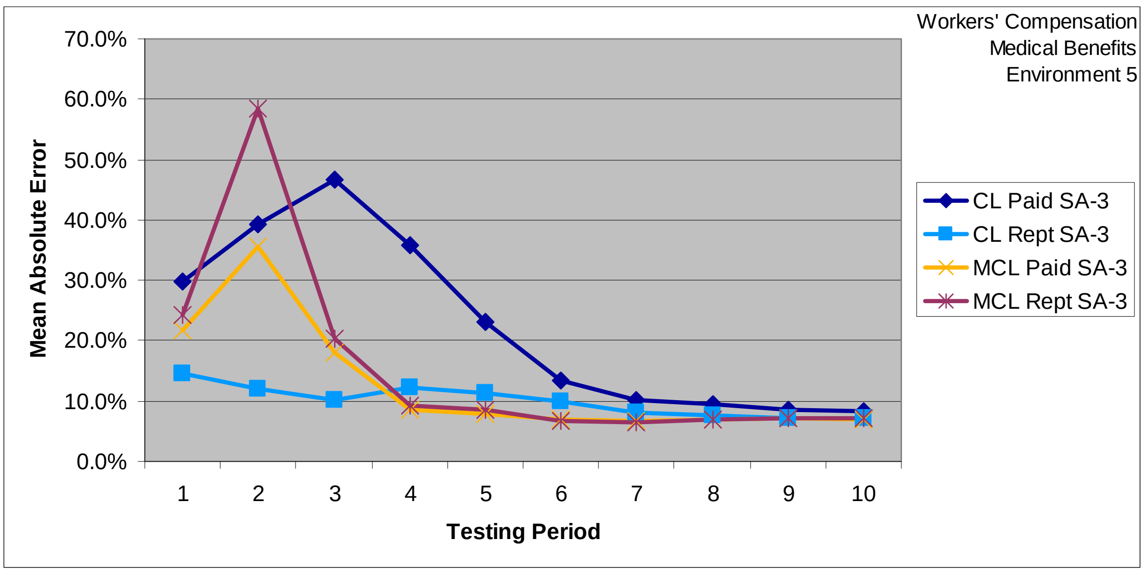

Figure 23: Comparison of the MCL and CL methods during a permanent acceleration in claim settlement rates.

4.4 Findings by Environment

4.4.1 Base environment (environment 1)

In the base environment, both the historical and future loss development patterns are stable, and the only source of error is residual noise. All methods perform similarly well (i.e., minimal mean errors), but no method is completely accurate, because of noise. The extent to which they differ shows their susceptibility to noise. For example, the chain ladder applied to paid loss is generally less accurate than the chain ladder applied to reported loss. This is not surprising, because paid loss development factors from age 12 to ultimate are significantly greater than reported LDFs, and larger factors leave more room for residual noise to distort development patterns. Not shown here are the exposure-based BL and BF methods. These, not surprisingly, are among the most accurate (close to

5% mean absolute error) because they are stabilized by giving weight to an unbiased a priori estimate of ultimate loss. Figure 24 shows these results.

The test results from the base environment illustrate the lowest potential level of mean absolute error possible for these methods under the residual noise levels assumed by our proxy data. Thus, for the rest of the environments, we should not expect to see levels of mean absolute error lower than we see here.

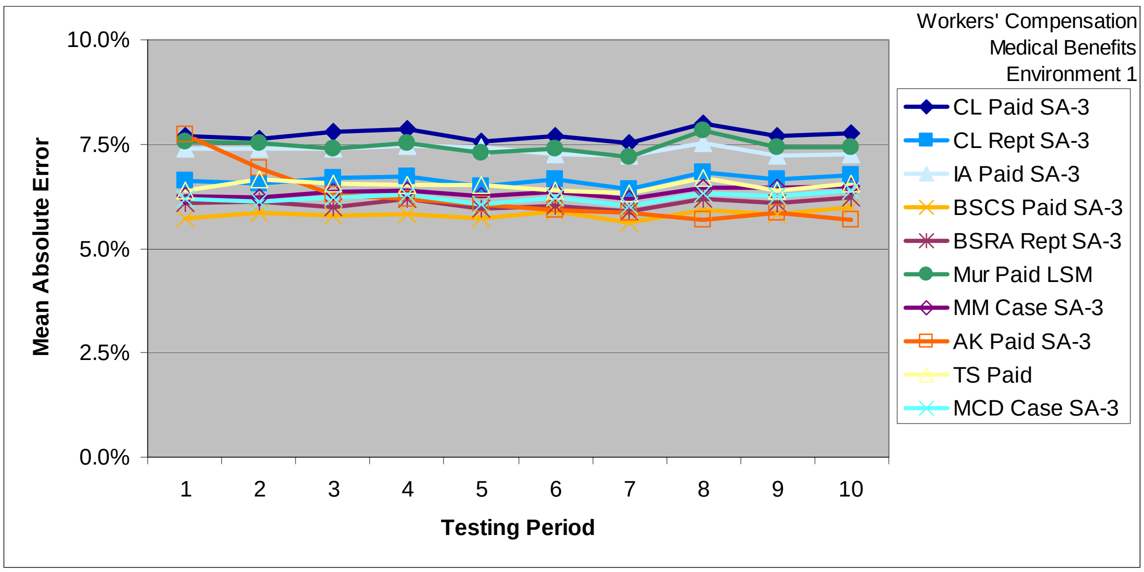

Figure 24: Comparison of various methods in the base environment.

4.4.2 Bubble in the rate of medical inflation (environment 2)

Figure 25 shows a comparison of various methods during a three-year bubble in medical inflation. Here paid severities are immediately affected; however, case severities respond more slowly as claims adjusters account for this new information. What is apparent is that methods based on case reserves or reported loss, such as the CL, MM, and MCD methods, show a significant delayed distortion, due to the lagged effect of higher inflation on case reserves that was assumed in the proxy data. Methods that perform well during the bubble include Murphy's least squares multiplicative (Mur-LSM) and Taylor's separation method (TS), possibly because these methods rely on more data points and are therefore more stable. The IA method performs poorly during the bubble, but it responds quickly after the rate of inflation reverts to its historical level, because the method relies on incremental payments and a short-term trend rate. If the actuary believes that recent inflation rates are likely to continue, then the IA with a short-term trend may be a good method. If the actuary believes that future inflation will be similar to historical long-term inflation, then the IA with a long-term trend may be preferable. Note that this environment is one of many where the data do not reveal the true nature of the change (i.e., specifically that it is only temporary rather than permanent) and other sources should be used to make an informed actuarial judgment.

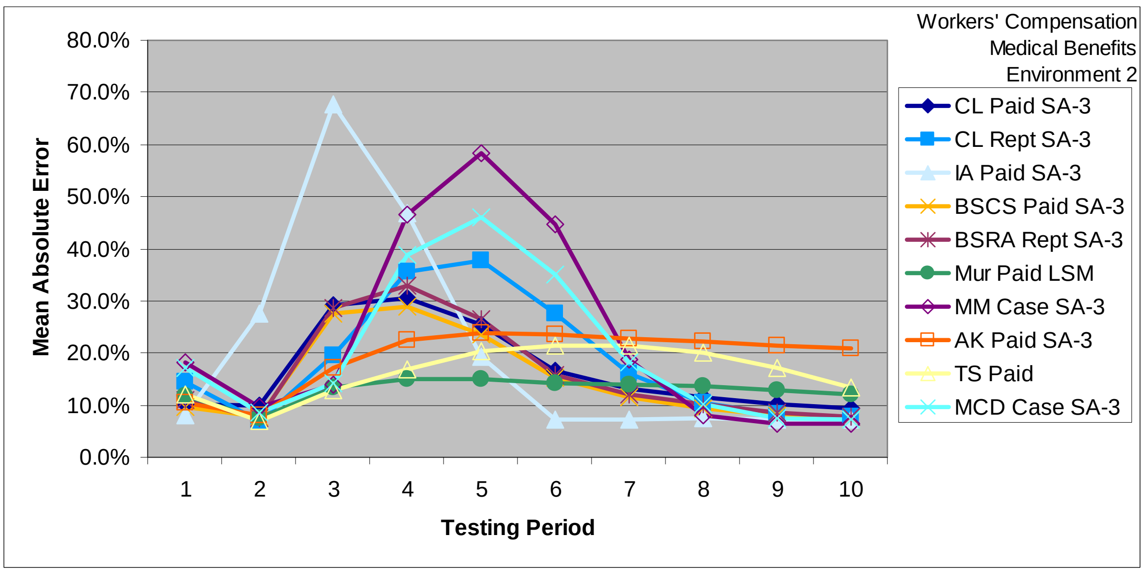

Figure 25: Comparison of various methods during a bubble in medical inflation.

4.4.3 Increase in the frequency of serious injuries (environment 3)

Environment 3 consists of a permanent increase in the frequency of serious injuries. Although claim severities are unchanged within each injury type, there are increases in ultimate claim counts, ultimate average severities, and ultimate losses.

Figure 26 isolates the mean error of the Adler-Kline claims closure model (AK) and the chain ladder method (CL). Both methods underestimate ultimate loss with the error gradually shrinking back to 0% error in the years after the change is complete. The CL underestimates because the increase in the frequency of serious injuries distorts the payment pattern, as serious claims report and close much slower than typical claims in our simulated data.

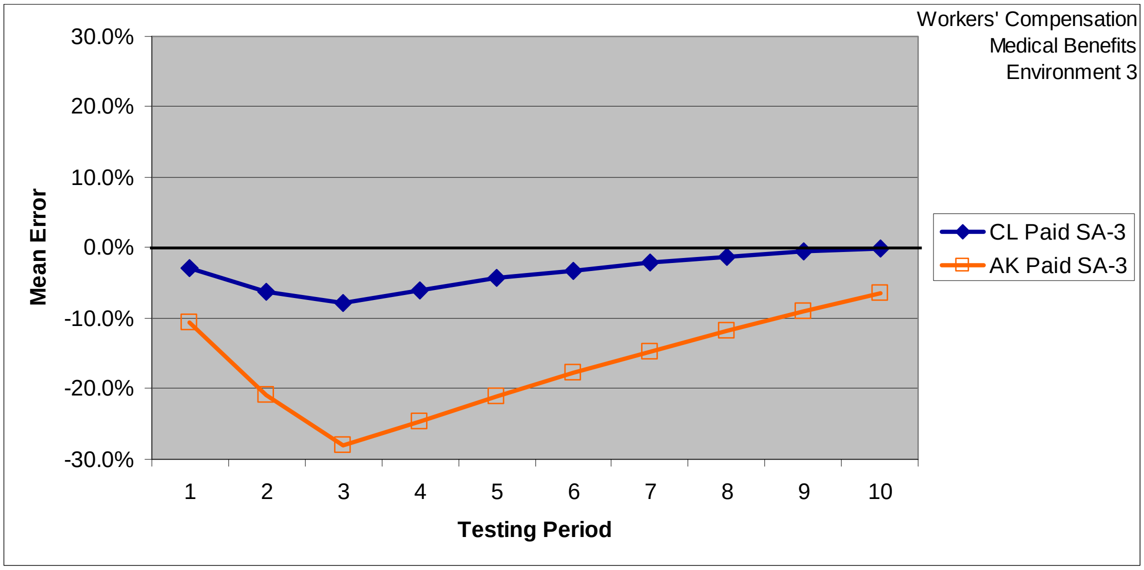

Figure 26: Comparison using the mean error statistic of the Adler-Kline claims closure model with the chain ladder method during and after an increase in the frequency of serious injuries.

To understand the additional error in the Adler-Kline method, note first that it relies on projecting incremental closed claim counts based on estimates of ultimate claim counts. Therefore, the underestimation in ultimate claim counts leads to an underestimation of future incremental closed claim counts. This problem is further exacerbated by applying the slowdown in claims closure pattern. The AK method interprets this slowdown in claims closure incorrectly, and allocates too many of the projected ultimate claims to earlier evaluation periods and too few to mature evaluation ages. And since loss severities are generally smaller at earlier periods and larger at later evaluation periods, the AK method further underestimates ultimate loss.

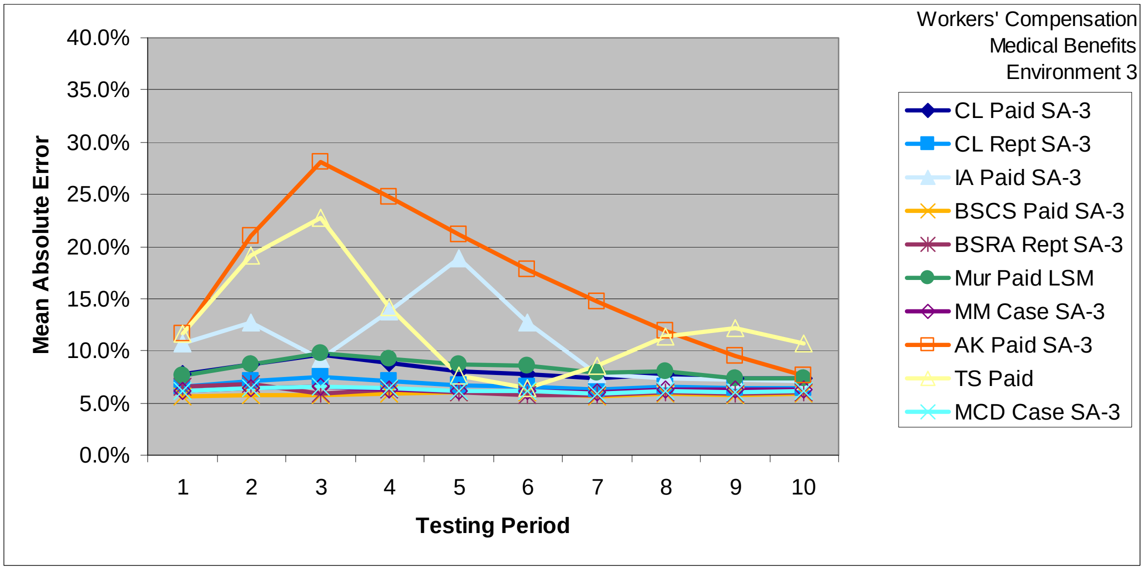

Figure 27: Comparison of various methods during a permanent increase in the frequency of serious injuries.

In Figure 27, we can see that the best performers in this scenario are the MM method, the MCD method, the BSCS and the BSRA method. As mentioned previously, the methods based on case reserves (MM and MCD) perform well in this environment, because although the loss payment and loss reporting patterns are distorted, there is less of a distortion in case reserve development.

It is interesting that both Berquist-Sherman adjustments, although intended to adjust for calendar year effects, perform well in this environment, in which there is a change in the mix of claim types from one accident year to the next. The BSRA method performs well as the increase in the frequency of serious injuries effectively mimics a change in case reserve adequacy, at least for the most recent accident years, and the BSRA method is able to immediately adjust to this new level of reserving. The BSCS method also performs well as the increase in the frequency of serious injuries effectively mimics a change in claim settlement rate that the BSCS method is able to model.

4.4.4 Increase in case reserve adequacy (environment 4)

Figure 28 shows the results for environment 4, in which case reserve adequacy permanently increases to a higher plateau, although paid and ultimate losses remain unchanged. Because only case reserves are affected in this environment, methods that do not incorporate case reserves are unaffected, and so we have excluded most of them from the graph. However, the best-performing method is the BSRA method, which, although it relies on case reserves, is able to correct for this misleading change by restating the historical triangle at the latest year of case reserve adequacy.

Furthermore, as the BSRA method smoothes the historical data, it adds an additional layer of stability minimizing the long-run error relative to other methods such as the chain-ladder approach.

Here the worst-performing method is the MM method since the distortion in case reserves not only distorts future predictions of case reserves, but it distorts future predictions of payments based on projected case reserves (i.e., the error is compounded in the iterative projections of paid and case loss).

The CL method on reported loss and the MCD method are also affected. The MCD method is more adversely affected since it applies the computed development factors to case reserves in isolation rather than to reported loss, which is somewhat stabilized by the paid component of reported loss.

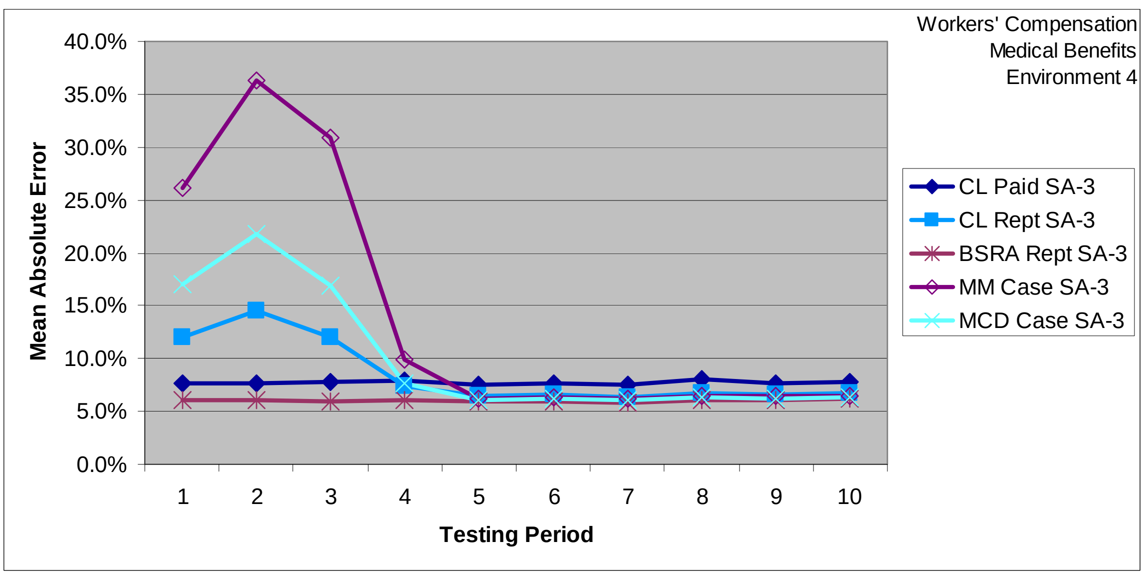

Figure 28: Comparison of various methods based on reported loss and/or case reserves during an environment of permanent increasing case reserve adequacy without an associated change in loss payments. The chain ladder on paid loss method is included as a base.

4.4.5 Acceleration in claim settlement rates (environment 5)

In this environment (see Figure 29), although there is a permanent acceleration in claim settlement rates, there is no change in the ultimate frequency or severity of claims. The FS method, which does not recognize this acceleration, overestimates the ultimate claim severity. In contrast, the AK method actually does quite well in this environment: since it models future incremental closed claims as a function of both ultimate counts and prior closed claims, it adequately responds to the acceleration in claim settlement rates.

In addition to the permanent acceleration in claim settlement rates, there is also a permanent increase in the average outstanding amounts (i.e., the claims that remain open are the more costly cases). This somewhat distorts those methods based on case reserve including the MM method and the CL method on reported loss. The MM method is more seriously distorted as it iteratively projects both case and paid amounts with paid amounts being very much affected in this environment (the case reserve development pattern is expected to change in this scenario). However, the MCD method is actually more accurate than the CL method on reported loss as it is able to adjust to the changing levels of case reserves. In addition to the MCD method, among the most accurate methods are the two Berquist-Sherman methods that are able to adjust for the several distortions in this environment.

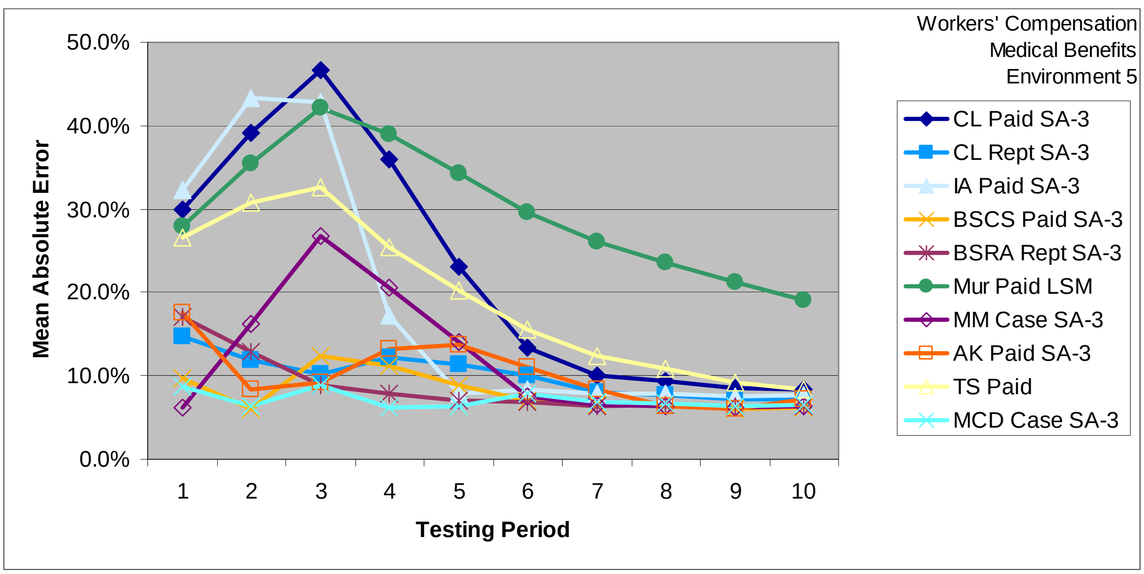

Figure 29: Comparison of various methods during a permanent acceleration in claim settlement rates.

4.4.6 Increase in case reserve adequacy with an acceleration in claim settlement rates (environment 6)

This environment is a combination of environment 4, an increase in case reserve adequacy, and environment 5, an acceleration in claim settlement rates.

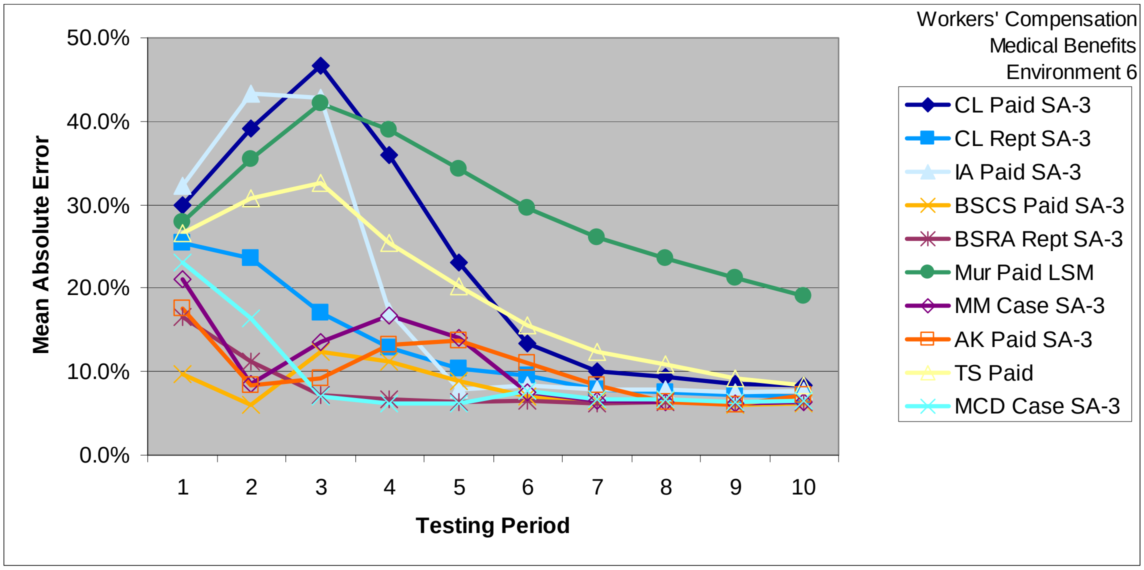

Figure 30: Comparison of various methods during a period of increasing case reserve adequacy, which plateaus at a permanently higher level coupled with a permanent acceleration in claim settlement rates.

As Figure 30 shows, the permanent change in claim settlement rates is the more significant of the two distortions, and so the results of environment 6 are similar to those of environment 5. As a result, the least accurate methods are those based on unadjusted paid loss. The Berquist-Sherman adjustments and the AK method are among the most accurate methods. Interestingly, the MM method is more accurate here than in either environment 4 or 5, because the biases created by the two environments offset each other. As shown in Figure 31, the higher average case reserves of environment 4 cause MM to overestimate, and the faster claim settlement rate of environment 5 cause MM to underestimate.

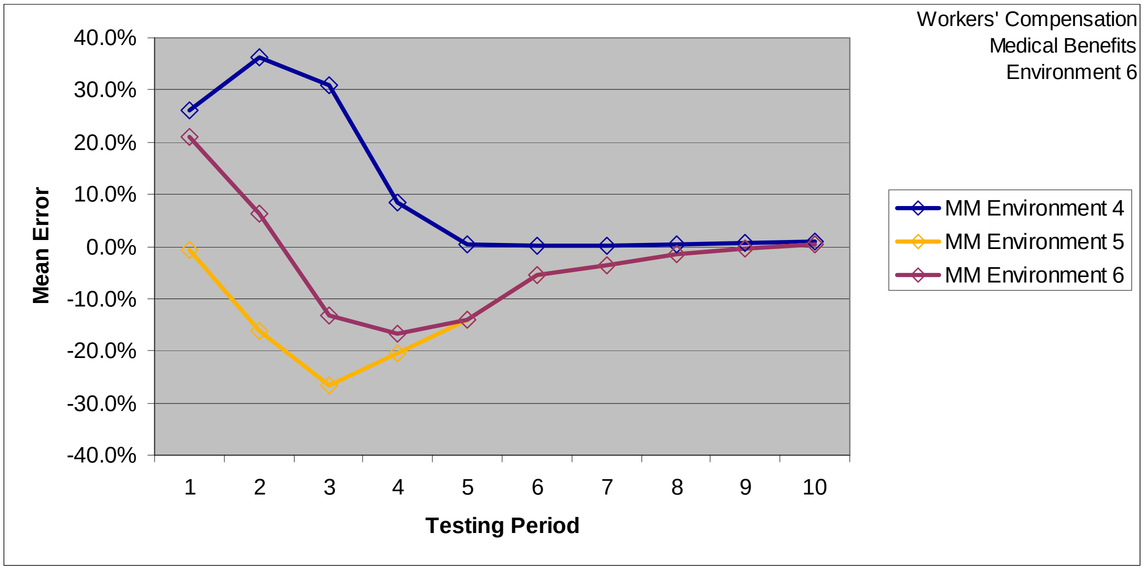

Figure 31: Comparison of various methods during a period of increasing case reserve adequacy, which plateaus at a permanent higher level coupled with a permanent acceleration in claim settlement rates.

4.4.7 Bubble in the rate of medical inflation with an increase in the frequency of serious injuries (environment 7)

This environment is a combination of environments 2 and 3. Here the IA and AK methods are distorted by the change in mix of claim types, and most other methods are distorted by the inflation bubble. One method that performs well during the period of change is Murphy's least squares multiplicative model. The BSCS and BSRA methods perform well in the first two testing periods, prior to being distorted by the inflation bubble.

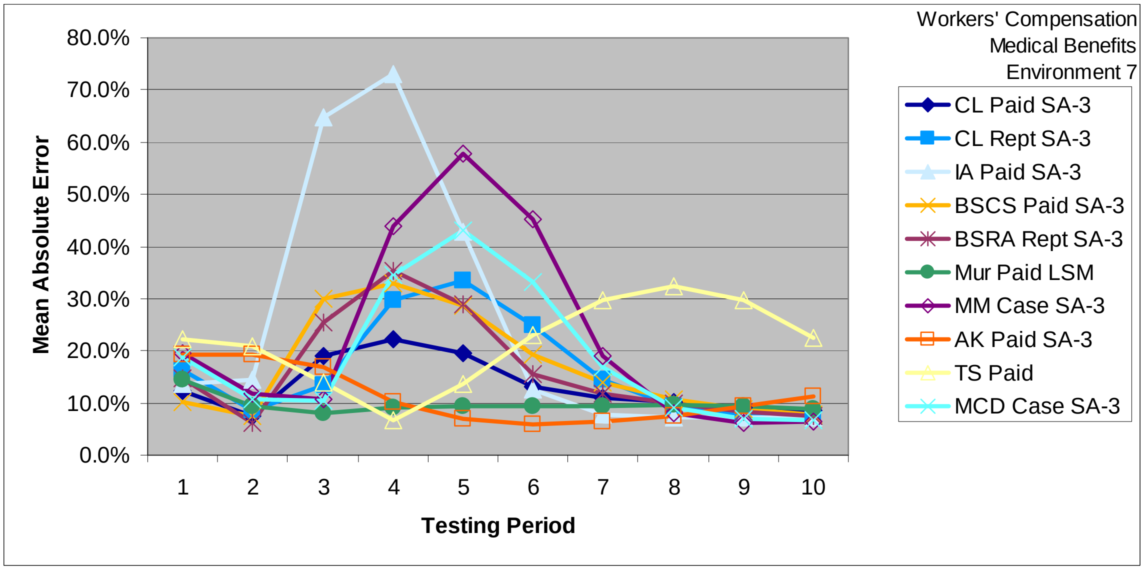

Figure 32: Comparison of various methods during a bubble in the rate of medical inflation coupled with an increase in the frequency of serious injuries.

4.4.8 Change in loss ratio (environment 8)

Figure 33 shows the mean error of various methods during and following a sharp, permanent change in the accident year loss ratio with no change in the mix of claims or the claims reporting, closing, payment, or reporting patterns. This simple environment, where loss doubles relative to exposures/premiums, provides insight into many of the traditional actuarial techniques. Note that the CL method is unaffected as this accident year shift has no effect on loss development patterns. The BL method is the worst-performing as it completely ignores loss information and relies entirely on the a priori estimate. The BF method performs slightly better than the BL method as it incorporates current loss amounts that reflect the higher level of loss. The BT method does slightly better as it gives twice as much weight to current loss amounts.

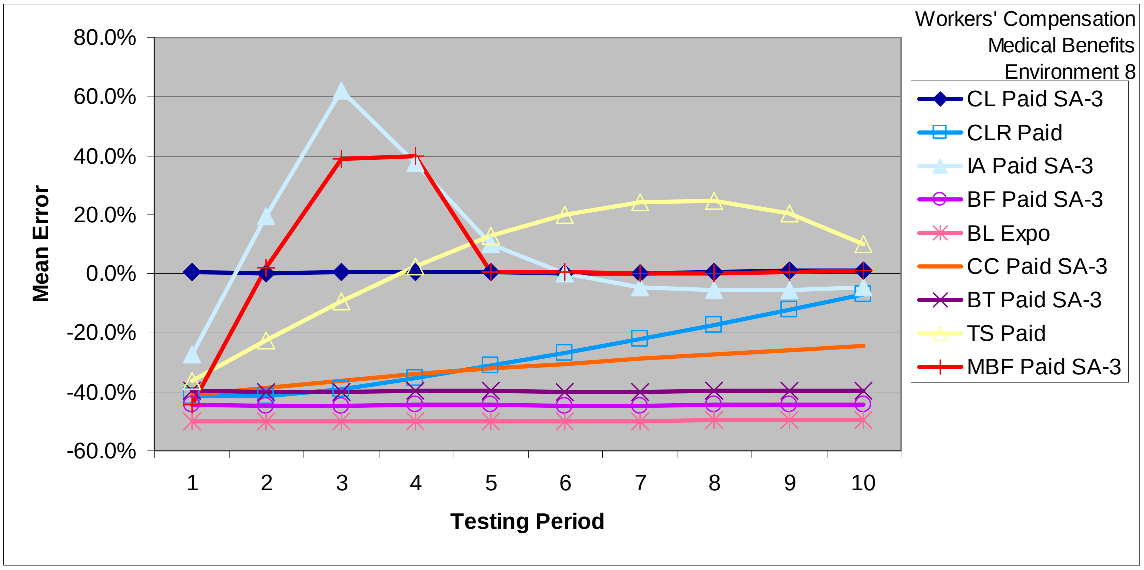

Figure 33: Comparison of various methods following a permanent shift in loss ratio.

The Cape Cod (CC) method is also based on exposures, and as a result it underestimates ultimate loss. Unlike the BL, BF, and BT methods, it does respond to changing conditions by re-evaluating the expected loss ratio. The response rate is gradual, because the expected loss ratio has been parameterized based on a long-term average.

The other methods shown (CLR, IA, TS, and MBF) incorporate loss trend, but in different ways. The modified Bornhuetter-Ferguson method initially matches the BF method, but in subsequent periods the expected loss ratio in the MBF is adjusted based on the observed loss trend in the latest four accident years. Once the accident year loss trend stabilizes, the MBF uses an accurate expected loss ratio and produces unbiased results. In this environment, a shorter-term trend rate would respond even more quickly. The incremental additive method relies on the loss trend observed based on payments in the latest four calendar years, and so the results are a bit worse than the MBF, because this environment is characterized by an accident year change. The CLR projects losses forward by fitting a trend line to columns of historical incremental paid losses, with trend rates calculated separately by evaluation age. Because the CLR uses long-term trend rates, it responds more gradually than the MBF or IA. Finally, Taylor's separation (TS) method uses a blended trend (based on both calendar year and accident year components), and as a result it responds more quickly than the CLR but more slowly than the MBF or IA.

5. CONCLUSION

Our observations and recommendations are intended to guide the actuary in evaluating the strengths and weaknesses of available loss reserving methods. While it is impossible to produce a fool-proof instruction guide for selecting actuarial methodologies, we envision that the practicing actuary will consider our recommendations within the context of the actuarial control cycle.

At the start of this control cycle, the actuary reviews a carefully selected set of diagnostics and leading indicators compiled to assist with detection and interpretation of trends in the system. The review of diagnostics and leading indicators should be accompanied by insights from other sources such as claims administrators, underwriters, and advisory organizations, in order to guide the identification of characteristics of the current and expected future environment. Although it may not be possible to pinpoint the environment, the actuary may be able to narrow down the possibilities and assess the volatility or level of noise in the underlying data.

After identifying characteristics of the environment and volatility of the data, the actuary can use general or specific observations from our analysis or similar work—either directly or by extrapolation of the conclusions contained within these sources—to identify suitable loss reserving methodologies. While we expect that the actuary's focus will first be on the expected accuracy of the various methods in the environment at the time (the priority of this paper), it is also important the actuary consider the relative importance of other criteria, such as bias, stability, and responsiveness.

Finally, the actuary should review the projections both in the short term using actual versus expected comparisons and in the longer term using hindsight testing. Actual versus expected analyses will allow the actuary to make minor corrections to optimize performance of the selected method (or methods). Hindsight testing allows the actuary to identify and subsequently correct for any systematic biases present in the data, the loss reserving methods considered, or the actuary's assumptions.

We hope that using the results of this paper, in the greater framework of the actuarial control cycle, will lead not only to more accurate projections of ultimate loss, but also help increase credibility of the actuarial profession by increasing documentation and arguments for selection of one methodology over another.

Acknowledgment

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Appendix A – Loss Reserving Methods

The following contains descriptions of various loss reserving methodologies considered and comments about their overall performance. The two-to-three letter parenthetical abbreviations are identifiers used to reference that specific method in our analysis.

A.1 Adler-Kline Claims Closure Method (AK)

The Adler-Kline claims closure method is a frequency-and-severity model where projected incremental closed claim counts are multiplied by projected incremental paid on incremental closed claim count severities. First, incremental closed claim counts are computed using disposal ratios; the disposal ratio is defined as the ratio of incremental closed claims to open claim counts. Open claim counts are developed by projecting reported claim counts to ultimate using the chain ladder method and then subtracting cumulative closed claim counts. Incremental closed claim counts are then projected by iteratively multiplying open claims by the disposal ratio and then updating the number of claims still open. Incremental paid on closed claim count severities are trended forward at each evaluation period. See Adler & Kline [1]. This method is typically applied in lines of business for which claims are reported rather quickly, such as coverages written on a claims-made basis.

A.2 Bornhuetter-Ferguson Method (BF)

The Bornhuetter-Ferguson method computes the outstanding loss as the product of the percentage of loss outstanding and an initial expected loss estimate. It sums this amount with the current cumulative loss amount to produce an estimate of ultimate loss. The initial expected loss estimate is the product of the a priori loss ratio with exposures. In our parameterization, the a priori loss ratio is equal to the loss ratio observed prior to the start of first environmental change. See Bornhuetter and Ferguson [4].

A.3 Budgeted Loss Method (BL)

The budgeted loss method computes ultimate loss as the product of an a priori loss ratio with exposures. In our parameterization, the a priori loss ratio is equal to the loss ratio observed prior to the start of first environmental change. See Brosius [5].

A.4 Berquist-Sherman Adjustment for Change in Claim Settlement Rate (BSCS)

This method adjusts actual experience to a common level of claim settlement speed first by computing claims closure ratios, i.e., the ratios of closed claims to ultimate claims (computed by developing reported claim counts to ultimate). Then, the latest diagonal of claims closure ratios is assumed for all historical diagonals. Adjusted paid loss on closed claim count severities are computed using log-linear interpolation between the actual paid loss on closed claim count severities and the actual claims closure ratio. This is to find the implied paid loss severity on closed claims associated with the new claims closure ratio. See Berquist and Sherman [3].

A.5 Berquist-Sherman Adjustment for Change in Case Reserve Adequacy (BSRA)

This method adjusts actual historical case reserves to a common level of reserve adequacy by de-trending the latest diagonal of the triangle of average case reserves per open claim by the trend in the average severity of paid loss on closed claims. See Berquist and Sherman [3].

A.6 Benktander Method (BT)

The Benktander method is a variant of the Bornhuetter-Ferguson method where instead of using the budgeted loss method as a prior, a credibility-weighted sum of the budgeted loss method with the chain ladder method is used. See Mack [11].

A.7 Stanard-Bühlmann or Cape-Cod Method (CC)

The Cape-Cod method is a variant of the Bornhuetter-Ferguson method where the a priori loss ratio is computed as the simple average of the historical loss ratios. The historical loss ratios are computed as the ratio of the latest diagonal of loss to used-up exposure. Used-up exposure is the product of exposure and the percent of loss developed for the year. See Friedland [7].

A.8 Atkinson Case Reserve Development (CD)

The chain ladder on case reserves method, as described by Atkinson [2], works by first selecting reported and paid loss development factors. Then, using the relationship between case reserves and paid and reported loss, the actuary derives case development factors from the paid and reported development factors. These case development factors are then applied to case reserves to project ultimate loss.

A.9 Chain Ladder Method (CL)

The chain ladder method we used is the traditional loss development method, whereby the change in cumulative loss from age to age is used to project the latest diagonal of the cumulative loss triangle.

A.10 Bühlmann's Complementary Loss Ratio Method (CLR)

Bühlmann's complementary loss ratio method computes incremental payments at each evaluation period by trending forward historical incremental payments at similar evaluation periods. These are then summed to provide an estimate of cumulative ultimate loss. See Pentikäinen and Rantala [17].

A.11 Fisher-Lange Claims Closure Method (FL)

Fisher and Lange's claims closure model is very similar to the Adler-Kline claims closure model. However, Fisher and Lange project incremental closed claims by using closure ratios that are the ratio of incremental closed claims to ultimate claims, where ultimate claims are computed by applying the chain ladder method to reported claim counts (as opposed to Adler and Kline who use disposal ratios that are the ratio of incremental closed claims to open claims). Incremental paid on incremental closed severities are trended forward at each exposure period. The product of the projected incremental closed claim counts with the projected incremental paid on incremental closed claim count severities then produces the reserve estimate. See Fisher and Lange [6]. Fisher and Lange advocate using report-year data because there is no development on reported claim counts in this situation, obviating the need to project ultimate claim counts. However, the method can equally apply to accident-year data by developing reported claim counts to ultimate.

A.12 Fleming-Mayer Adjustment for Change in Claim Settlement Rate (FMCS)

The Fleming-Mayer Adjustment for change in claim settlement rates (FMCS) is similar to the Berquist-Sherman adjustment for claim settlement rate (BSCS) in that it adjusts for changes in claim settlement rate. However, the FMCS applies to reported loss rather than paid loss. The paid component in the reported loss amounts are adjusted as they are in the BSCS; however, the case component is also adjusted, in a similar way as the paid component, to reflect the fact that changes in claim settlement often have a ripple effect onto outstanding case amounts. See Fleming and Mayer [7].

A.13 Cumulative Frequency-Severity Method (FS)

The basic frequency-severity approach included here works by projecting claim counts (frequency) to ultimate and projecting loss on claim count (severity) to ultimate using the chain ladder method. The product of these two then produces an estimate of ultimate loss. We have included both the "reported claim count/reported loss on reported claim count" variant as well as the "closed claim count/paid loss on closed claim count" variant. We choose this cumulative frequency-and-severity approach to contrast it with the various other incremental approaches included in our analysis including Fisher-Lange and Adler-Kline.

A.14 Ghezzi's Incremental Closed Claim Severity Method (GH1)

Future incremental closed claim counts are projected by applying the percentage of claims closed pattern to open claims. To compute open claims, ultimate claim counts are projected by applying the chain ladder method to reported claim counts. Future incremental paid loss on incremental closed claim count severities are computed by trending forward historical incremental paid on incremental closed claim count severities using exponential growth. To produce an estimate of outstanding loss, the actuary multiplies the vector of projected incremental paid on incremental closed claim count severities with the vector of projected incremental closed claims. The dot-product of these amounts produces an estimate of reserves. The key to this method (as well as Ghezzi's Ultimate Unclosed Claim Severity Method) is that only ratios prior to the significant environmental change are considered. See Ghezzi [9].

A.15 Ghezzi's Ultimate Unclosed Claim Severity Method (GH2)

Unpaid loss amounts are computed by estimating preliminary ultimate loss amounts using the chain ladder method on either paid or reported loss amounts (we used reported loss); and subtracting paid amounts. Similarly, unclosed claims are computed by estimating ultimate claim counts using the chain ladder method on either closed or reported claim counts (we used reported claim counts). Ghezzi's ultimate unclosed claim severity method then works by trending the ratios of unpaid loss to unclosed claims using exponential growth. The loss reserve is then computed by multiplying these trended ratios by current unclosed claim counts. The key to this method (as well as Ghezzi's incremental closed claim severity method above) is that only ratios prior to the significant environmental change are considered. See Ghezzi [9].

A.16 Incremental Additive Method (IA)

The incremental additive method uses both the triangle of incremental losses and the exposure vector for each accident year as a base. Incremental additive ratios are computed by taking the ratio of incremental loss to the exposure (which has been adjusted for the measurable effect of inflation), for each accident year. This gives the amount of incremental loss in each year and at each age expressed as a percentage of exposure, which we then use to square the triangle.

A.17 Incremental Multiplicative Method (IM)

The incremental multiplicative method uses incremental loss to compute incremental loss development factors, sometimes referred to as "decay ratios," which are defined to be the ratio of incremental loss at a later age to the incremental loss at an earlier age. With these loss development factors it is possible to extrapolate future incremental payments as means of squaring the triangle.

A.18 Verrall's Log-Linear Methods (LL1, LL2, LL3)

Pentikäinen and Rantala [17] include three log-linear regression models with varying numbers of parameters in their analysis. For comparison purposes, we have chosen to include these models, as described on page 184 of the above. For greater detail, we refer the reader to Verrall [25]. We refer to these three models as LL1, LL2 and LL3, respectively, based on the order of presentation in Pentikäinen and Rantala [17].

A.19 Brosius's Least Squares Development (LS)

Least squares development as described by Brosius iteratively regressed ultimate loss on cumulative loss at successive maturity starting with the maturity one evaluation period prior to ultimate. Each successive regression produces one more estimate of ultimate loss that is used in the next regression. See Brosius [5].

A.20 Modified Bornhuetter-Ferguson (MBF)

The modified Bornhuetter-Ferguson method is identical to the Bornhuetter-Ferguson method except that the initial expected loss estimate is the average of the prior years' ultimates (rather than the product of an a priori loss ratio with exposures). We have trended the initial expected loss estimate to adjust for growth in exposures. See Pentikäinen and Rantala [17].

A.21 Munich Chain Ladder (MCL)

The Munich chain ladder method we implemented is identical to the basic method described in Quarg and Mack [18]. However, we allowed the initial selection of parameters to be based on simple and volume-weighted averages of less than all years (e.g., volume-weighted average of latest 3 sets of observed data points).

A.22 Marker-Mohl Backwards Recursive Case Development Method (MM)

The backwards recursive case development method works by first computing the percentage of loss paid (to case reserves in the previous period) and the percentage of case reserves (to case reserves in the previous period) at each age. These are then iteratively applied to case reserves so as to produce case reserves and paid losses at each age. See Marker and Mohl [13]. This method is typically applied in lines of business for which claims are reported rather quickly, such as coverages written on a claims-made basis.

A.23 Murphy's Family of Chain Ladder Parameterizations (Mur)

As part of his 1994 work, Daniel Murphy outlines five possible parameterizations of the chain ladder method. We abbreviate them as follows: (1) LSL – least squares linear, (2) LSM – least squares multiplicative, (3) SA – simple-average development, (4) VW – weighted-average development, and (5) GA – geometric-average development. See Murphy [14] for exact solutions of each parameterization. The last three of these are equivalent to a chain ladder method using various types of averages of all historical observations.

A.24 Multivariate Regression Development Method (MV)

A great variety of multivariate regression models is suggested in the literature. While it would be extremely difficult to include all varying combinations of dependent and independent variables, we have, as a proxy, included the multivariate regression model. In this model, cumulative loss is regressed on case reserves and cumulative payments in the prior period to develop an estimate of loss development factors. These are then applied successively to the latest diagonal of loss (a la the chain ladder method) in order to project ultimate loss.

A.25 Taylor's Separation Method (TS)

Taylor's separation method (TS) attempts to "separate" the calendar year inflation effect from the evaluation period development effect. Our implementation is similar to that in Taylor [23].

A.26 Weller's Algebraic Method (WA)

Weller's algebraic method (WA) describes the claims reserve triangle as a system of linear equations involving various unknown parameters represented the percentage paid or reported at various evaluations. These linear equations can be iteratively solved to establish development factors that can be used to project ultimate losses. See Weller [26].

Appendix B – Loss Reserving Method Families

LOSS RESERVING METHODS CLASSIFIED BY FAMILY

Family	**Methods**
Exposure-Based Methods	(1) Budgeted Loss Method (BL) (2) Bornhuetter-Ferguson Method (BF) (3) Modified Bornhuetter-Ferguson Method (MBF) (4) Benktander Method (BT) (5) Cape-Cod Method (CC)
Regression Methods	(1) Least Squares Development (LS) (2) Murphy's *Least Squares Linear* Parameterization (Mur-LSL) (3) Murphy's *Least Squares Multiplicative* Parameterization (Mur-LSM) (4) Multivariate Regression (MV) (5) Verrall's Log-Linear Models (LL1, LL2, LL3)
Frequency-Severity Methods	(1) Adler-Kline Claims Closure Model (AK) (2) Fisher-Lange Claims Closure Model (FL) (3) Ghezzi's Incremental Closed Claim Severity Method (GH1) (4) Ghezzi's Ultimate Unclosed Claim Severity Method (GH2) (5) Cumulative Frequency-Severity Method (FS)
Case-Reserve Methods	(1) Marker-Mohl Backwards Recursive Case Development (MM) (2) Atkinson Case Development (CD) (3) Modified Atkinson Case Development (MCD)
Incremental Methods	(1) Incremental Multiplicative Method (IM) (2) Incremental Additive Method (IA) (3) Bühlmann's Complementary Loss Ratio Method (CLR)
Joint Paid-Reported Models	(1) Munich Chain Ladder (MCL)
Berquist-Sherman Adjustments	(1) Berquist-Sherman adjustment for case reserve adequacy (BSRA) (2) Berquist-Sherman adjustment for claim settlement rate (BSCS) (3) Fleming-Mayer adjustment for claim settlement rate (FMCS)
Miscellaneous	(1) Taylor's Separation Method (TS) (2) Weller's Algebraic Method (WA)

Appendix C – Environments

The following appendix contains descriptions of the various environments we considered. Each of these environments describes one or two changes which are common to the workers compensation line of business (as well as many other lines). However, the reader should note that these constructed environments are to some degree simplifications of the real world, which would include a combination of many of these changes in tandem.

C.1 Base environment (environment 1)

In the base scenario, exposures grow gradually at 1% from one accident year to the next, and claim frequency is constant. Claim reporting speed, claim payment speed, and claim closure speed are each consistent over time. Claim payments increase gradually with inflation, with case reserves moving in tandem (cost inflation is assumed at 5% per annum for medical on a calendar-year basis). The resulting claims and loss development patterns generally align with recent workers' compensation medical loss development patterns in California.

C.2 Bubble in the rate of medical inflation (environment 2)

In this environment, calendar-year medical inflation is 15% in the first three testing periods (as compared to 5% historically). For the fourth and subsequent testing periods, calendar-year medical inflation returns to the original 5% level. The changes apply consistently to all medical costs, independent of injury type. A practical example of this scenario would be that of runaway medical costs that are subsequently tamed by the implementation of treatment guidelines. Paid losses immediately reflect the increase and subsequent drop in medical inflation rates, whereas case reserves respond more gradually, lagging by three periods. Claim reporting speed and claim closure speed are unchanged from the base environment.

C.3 Increase in the frequency of serious injuries (environment 3)

In relation to the base environment, this environment features an approximate 75% increase in the frequency of serious claims, which occurs evenly over the course of three accident years (corresponding with the first three testing periods). Thereafter, the claim frequency remains at this elevated level. Severities for each injury type, patterns by injury type and the frequency of other injury types are unaffected.

C.4 Increase in case reserve adequacy (environment 4)

In this scenario, case reserve adequacy increases relative to the base scenario. The change occurs over the course of two calendar years (corresponding with the first two testing periods), after which case reserve adequacy remains at the higher level. This change applies consistently to all injury types. It does not affect any ultimate levels or the rate of payments or claim closures.

C.5 Acceleration in claim settlement rate (environment 5)

In this environment, the speed at which claims are settled increases. This change causes claims to be paid and closed earlier than in the base environment. The earlier closure of claims results in fewer payments at later ages, resulting in reduced ultimate losses compared to the base environment. The change applies to all injury types. In addition to speedier claim settlements, there is an increase in the incremental paid amounts per claim at each stage of development. Also, as the claims being settled are likely to be the less serious of the open claims, there are also small increases in the observed average case reserve per open claim (without any change in case reserving adequacy). The acceleration in claim settlement rates and the corresponding changes in severities take place over three years (corresponding with the first three testing periods). In the fourth year, claim settlement rates stabilize at rates observed in the third year (but still higher than historical norms). Similarly, paid severities and average case reserves on open severities stabilize at levels observed in the third year.

C.6 Increase in case reserve adequacy with an acceleration in claim settlement rates (environment 6)

This environment is a combination of the above fourth environment (permanent increase in case reserve adequacy) with the fifth environment (permanent acceleration in claim settlement rates). In addition to more claims being settled sooner with a higher average outstanding case severities (i.e., environment 5), case adjusters overreact to this shift and permanently over-reserve on the few large claims that remain open.

C.7 Bubble in the rate of medical inflation with an increase in the frequency of serious injuries (environment 7)

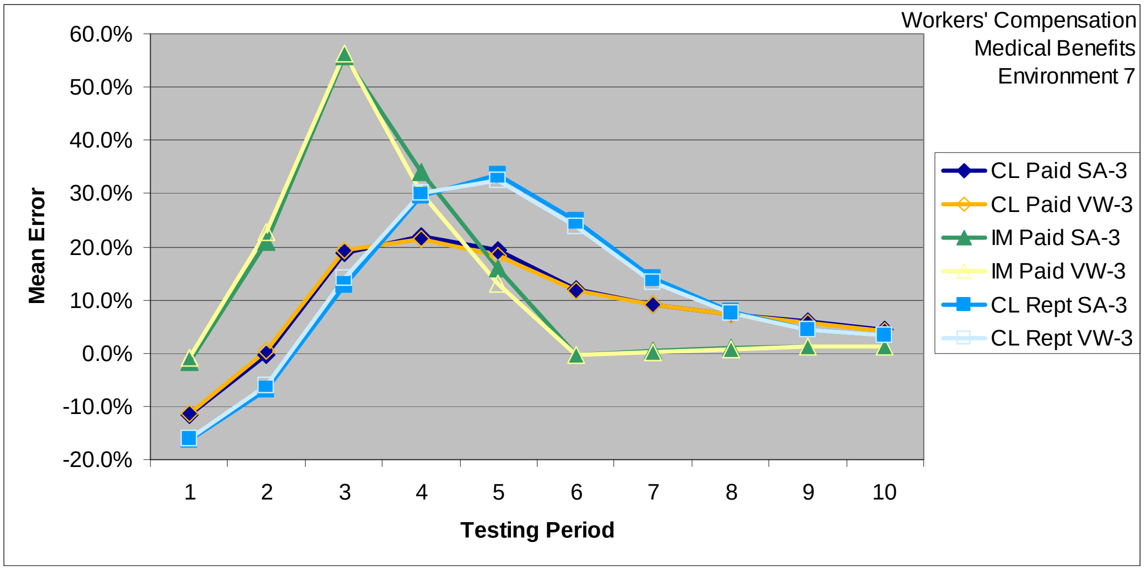
This environment is a combination of the above second environment (bubble in the rate of medical inflation) with the third environment (permanent increase in the frequency of serious injuries).

C.8 Change in loss ratio (environment 8)

In this environment, claim counts and losses increase relative to premium/exposures yet severities remain unchanged. This change occurs suddenly in the first testing period, and losses remain at the elevated level in subsequent testing periods. This simple environment models a change in loss ratio (i.e., as is present in loss ratio cycles) and is used to highlight (i) the effect of accident year changes that do not affect loss development patterns and (ii) the how exposure-based methods are adversely affected in the absence of correct a priori loss estimates.

Appendix D – Results graph interpretation

For the most part, we have used the same graph to present results. The following describes the various components of this graph and how it should be read.



(1) Y-axis: this will reflect either the "Mean Error" (i.e., measuring bias and accuracy) or "Mean Absolute Error" (i.e., measuring accuracy). Error is defined as the projected ultimate loss for the latest accident year (i.e., from age 12 to ultimate) minus the actual ultimate loss, expressed as a percentage of the actual ultimate loss. If a method overestimates an ultimate loss of 100 by 10, then the error would be 10%. Values close to 0 are optimal; values far from 0 indicate distortions.

(2) X-axis: testing period 1 represents the latest accident year one year after the start of the environmental change; testing period 2 represents the latest accident year two years after the start of the environmental change, and so on. So, if testing period 1 is accident year XX at 12/31/XX (at which point it is 12-months-old), then testing period 2 is accident year XX+1 at 12/31/XX+1 (at which point it is also 12-months-old). Most environmental changes take place over the first four testing periods with the data stabilizing in the fifth and subsequent periods. Each testing period measures the same test statistic, which is the error from age 12 to ultimate (i.e., the latest accident year).

(3) Right header: the right header provides three pieces of information. The first line indicates the line of business (i.e., workers compensation). The second line indicates the type of data (i.e., medical benefits). The third line indicates the environment being tested.

(4) Legend: the legend maps the lines with each method. Each method is described by four pieces of information. The first two-to-four letters indicate the loss reserving method (i.e., CL indicates the Chain Ladder method and IM is the incremental multiplicative method). The next word indicates to what data the method applies (i.e., paid, reported, case reserves, or exposure data). The next few letters describe the parameterization (i.e., SA indicates a simple average, VW indicates a volume-weighted average, and the number identifies the number of calendar years of loss development factors used to parameterize the loss reserving model.

(5) Reading the graph: ideally, we are looking for methods that perform well during the change (i.e., small errors in testing periods 1-3), but also important are responsive methods that quickly self-correct after the change (i.e., sharply sloped lines in testing periods 4-7), methods that are relatively stable throughout the testing period (i.e., flat lines in testing periods 1-10), and methods that are biased in opposite directions (i.e., lines that show mirror image mean errors above and below 0).

Appendix E – Accuracy Report Card

Grading of the methods' average accuracy over the second, third, and fourth testing periods in various environments based on a bell curve, with grade thresholds selected judgmentally (A is best; F is worst).

				Environment							
Family	Abbr.	Data	Param.	1	2	3	4	5	6	7	8
Chain-Ladder	CL	Paid	SA-3	D	C	C	C	F	F	B	B
Chain-Ladder	CL	Reported	SA-3	C	C	B	C	C	C	B	A
Exposure	BL	Exposure	N/A	A	C	F	A	C	C	F	F
Exposure	BF	Paid	SA-3	A	C	F	A	C	C	D	F
Exposure	BT	Paid	SA-3	A	B	D	A	C	C	D	F
Exposure	CC	Paid	SA-3	A	A	D	A	C	C	D	F
Exposure	MBF	Paid	SA-3	C	C	C	B	F	F	C	D
Exposure	BF	Reported	SA-3	A	B	D	A	C	C	D	F
Exposure	BT	Reported	SA-3	A	A	D	B	C	C	C	D
Exposure	CC	Reported	SA-3	A	A	D	B	C	C	C	D
Exposure	MBF	Reported	SA-3	B	C	C	C	C	C	C	D
Frequency-Severity	AK	Paid	SA-3	C	B	D	B	B	B	B	A
Frequency-Severity	FL	Paid	SA-3	C	B	D	B	B	B	B	A
Frequency-Severity	GH1	Paid	SA-3	F	C	D	D	C	C	D	C
Frequency-Severity	GH2	Paid	SA-3	B	A	D	B	A	A	C	A
Frequency-Severity	FS	Paid	SA-3	D	C	C	C	F	F	B	B
Frequency-Severity	FS	Reported	SA-3	C	C	B	C	C	C	B	A
Berquist-Sherman	BSCS	Paid	SA-3	B	C	A	A	B	B	C	A
Berquist-Sherman	BSRA	Reported	SA-3	B	C	A	A	B	A	C	A
Berquist-Sherman	FMCS	Reported	SA-3	B	C	A	C	C	B	C	A
Case	CD	Case	SA-3	B	C	A	F	B	A	C	A
Case	MCD	Case	SA-3	B	C	A	F	A	B	C	A
Case	MM	Case	SA-3	C	C	A	F	C	B	C	A
Incremental	CLR	Paid	N/A	F	C	D	C	D	D	C	F
Incremental	IA	Paid	SA-3	C	F	C	C	D	D	F	F
Incremental	IM	Paid	SA-3	D	F	C	C	D	D	D	B
Incremental	CLR	Reported	N/A	F	D	D	C	C	C	B	D
Incremental	IA	Reported	SA-3	C	F	C	D	B	B	F	F
Incremental	IM	Reported	SA-3	C	F	B	D	B	B	F	B
Joint Paid-Reported	MCL	Paid	SA-3	C	D	B	D	C	F	C	A
Joint Paid-Reported	MCL	Reported	SA-3	C	D	B	D	D	F	C	A
Regression	LL1	Paid	LS-All	D	B	C	C	F	F	A	B
Regression	LL2	Paid	LS-All	C	B	D	B	D	D	B	D
Regression	LL3	Paid	LS-All	F	C	F	F	A	A	D	F
Regression	LS	Paid	LS-All	D	F	C	C	F	F	F	C
Regression	Mur	Paid	LSL-All	D	D	C	C	F	F	C	B
Regression	Mur	Paid	LSM-All	C	B	C	C	F	F	A	B
Regression	MV	Paid	LS-All	C	D	B	C	C	C	C	B
Regression	LL1	Reported	LS-All	C	B	B	C	B	C	A	A
Regression	LL2	Reported	LS-All	B	C	C	C	B	B	B	C
Regression	LL3	Reported	LS-All	F	B	F	D	C	B	C	D
Regression	LS	Reported	LS-All	D	F	C	D	C	C	F	C
Regression	Mur	Reported	LSL-All	C	D	B	C	C	C	C	B
Regression	Mur	Reported	LSM-All	C	A	B	C	C	C	A	A
Regression	MV	Reported	LS-All	C	D	A	D	B	B	C	A
Miscellaneous	WA	Paid	All	D	C	C	C	D	D	D	D
Miscellaneous	TS	Paid	All	C	B	C	B	D	D	B	C
Miscellaneous	WA	Reported	All	D	D	D	F	C	D	F	F
Miscellaneous	TS	Reported	All	B	B	C	C	C	C	A	C

Appendix F – Bias Report Card

Methods' average bias categorized based on mean error over the second, third' and fourth testing periods (H+: 15% or greater, H: between 5% and 15%, U: between -5% and 5%, L: between -15% and -5%, L-: below -15%).

				Environment							
Family	Abbr.	Data	Param.	1	2	3	4	5	6	7	8
Chain-Ladder	CL	Paid	SA-3	U	H+	L	U	H+	H+	H	U
Chain-Ladder	CL	Reported	SA-3	U	H+	U	H	H	H+	H	U
Exposure	BL	Exposure	N/A	U	L-	L-	U	H+	H+	L-	L-
Exposure	BF	Paid	SA-3	U	L-	L-	U	H+	H+	L-	L-
Exposure	BT	Paid	SA-3	U	L-	L-	U	H+	H+	L-	L-
Exposure	CC	Paid	SA-3	U	L	L-	U	H+	H+	L-	L-
Exposure	MBF	Paid	SA-3	U	H+	L	U	H+	H+	H	H+
Exposure	BF	Reported	SA-3	U	L-	L-	U	H+	H+	L-	L-
Exposure	BT	Reported	SA-3	U	L	L-	U	H	H+	L-	L-
Exposure	CC	Reported	SA-3	U	L	L-	U	H	H+	L-	L-
Exposure	MBF	Reported	SA-3	U	H+	U	H	H	H+	H	H+
Frequency-Severity	AK	Paid	SA-3	U	H	L-	U	L	L	L-	U
Frequency-Severity	FL	Paid	SA-3	U	H+	L-	U	U	U	L	U
Frequency-Severity	GH1	Paid	SA-3	U	L-	L-	U	H	H	L-	U
Frequency-Severity	GH2	Paid	SA-3	U	U	L-	U	U	H	L-	U
Frequency-Severity	FS	Paid	SA-3	U	H+	L	U	H+	H+	H	U
Frequency-Severity	FS	Reported	SA-3	U	H+	U	H	H	H+	H	U
Berquist-Sherman	BSCS	Paid	SA-3	U	H+	U	U	L	L	H+	U
Berquist-Sherman	BSRA	Reported	SA-3	U	H+	U	U	H	H	H+	U
Berquist-Sherman	FMCS	Reported	SA-3	U	H+	U	H	L	L	H+	U
Case	CD	Case	SA-3	U	H	U	H+	L	U	H	U
Case	MCD	Case	SA-3	U	H+	U	H	U	H	H	U
Case	MM	Case	SA-3	U	H+	U	H+	L-	L	H	U
Incremental	CLR	Paid	N/A	U	H+	L-	U	H+	H+	L-	L-
Incremental	IA	Paid	SA-3	U	H+	U	U	H+	H+	H+	H+
Incremental	IM	Paid	SA-3	U	H+	L	U	H+	H+	H+	U
Incremental	CLR	Reported	N/A	U	H+	L-	H	H+	H+	L	L-
Incremental	IA	Reported	SA-3	U	H+	H	H	H	H	H+	H+
Incremental	IM	Reported	SA-3	U	H+	U	U	U	U	H+	U
Joint Paid-Reported	MCL	Paid	SA-3	U	H+	U	H	L	L-	H	U
Joint Paid-Reported	MCL	Reported	SA-3	U	H+	U	H	L-	L-	H	U
Regression	LL1	Paid	LS-All	U	H	L	U	H+	H+	H	U
Regression	LL2	Paid	LS-All	U	H	L-	U	H+	H+	L-	L-
Regression	LL3	Paid	LS-All	L-	L-	L-	L-	U	U	L-	L-
Regression	LS	Paid	LS-All	U	H+	L	U	H+	H+	H+	U
Regression	Mur	Paid	LSL-All	U	H+	L	U	H+	H+	H+	U
Regression	Mur	Paid	LSM-All	U	H	L	U	H+	H+	U	U
Regression	MV	Paid	LS-All	U	H+	U	H	H+	H+	H+	U
Regression	LL1	Reported	LS-All	U	H	L	H	H	H+	U	U
Regression	LL2	Reported	LS-All	U	H+	L-	H	H	H	L	L-
Regression	LL3	Reported	LS-All	L-	L	L-	L	L-	L	L-	L-
Regression	LS	Reported	LS-All	U	H+	U	H	H	H+	H+	U
Regression	Mur	Reported	LSL-All	U	H+	U	H	H	H+	H+	U
Regression	Mur	Reported	LSM-All	U	U	U	H	H	H+	U	U
Regression	MV	Reported	LS-All	U	H+	U	H	U	U	H+	U
Miscellaneous	WA	Paid	All	U	L-	L-	U	H+	H+	L-	L-
Miscellaneous	TS	Paid	All	U	H	L-	U	H+	H+	L	L
Miscellaneous	WA	Reported	All	U	L-	L-	H	H	H+	L-	L-
Miscellaneous	TS	Reported	All	U	H	L	H	H	H+	L	L

Appendix G – Abbreviations and notation

Miscellanea:
LDF, Loss development factor
SA, Simple average
SA-3, Simple average of the latest three observations
SA-All, Simple average of all historical observations
VW, Volume-weighted average
VW-3, Volume-weighted average of the latest three observations
WCIRB, Workers' Compensation Insurance Rating Bureau of California

Loss Reserving Methods:
AK, Adler-Kline claims closure method
AM, Weller's algebraic reserving method
BF, Bornhuetter-Ferguson method
BLM, budgeted loss method
BT, Benktander method
CC, Stanard-Bühlmann or Cape Cod method
CD, Atkinson chain ladder on case reserves method
CL, chain ladder method
CLR, Bühlmann's complementary loss ratio method
BSCS, Berquist-Sherman adjustment for the change in claim settlement rate method
BSRA, Berquist-Sherman's adjustment for reserve adequacy method
FL, Fisher-Lange claims closure method
FMCS, Fleming-Mayer adjustment for change in claim settlement rate
FS, cumulative frequency-severity method
GH1, Ghezzi's incremental closed claim severity method
GH2, Ghezzi's ultimate unclosed claim severity method
IA, incremental additive method
IM, incremental multiplicative method
LL1, Verrall's log-linear method #1
LL2, Verrall's log-linear method #2
LL3, Verrall's log-linear method #3
LS, Brosius's least squares development method
MBF, modified Bornhuetter-Ferguson method
MCD, modified Atkinson chain ladder on case reserves method
MCL, Munich chain ladder
MM, Marker & Mohl's backwards recursive case development method
MV, multivariate regression method
Mur-LSL, Murphy's family of parameterizations (least squares linear) method
Mur-LSM, Murphy's family of parameterizations (least squares multiplicative) method
TS, Taylor's separation method
WA, Weller's algebraic reserving method

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Reserving for Extended Reporting Endorsement Coverage, Including the Death, Disability, and Retirement Policy Provision

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Abstract. Writers of physicians professional liability (PPL) claims-made coverage typically offer a death, disability and retirement (DDR) provision within their policy language, stating that, in the event of one of these three described events, an extended reporting endorsement (ERE) will be provided to the insured without additional premium charge. The current methodology for deriving an indicated DDR reserve is time-consuming, leveraged, and uncertain as a result of its reliance on calendar period projections up to 50 years beyond the evaluation date. This monograph proposes a fundamentally different methodology for developing an indicated DDR reserve that addresses these concerns.

A related methodology for pricing the DDR policy provision will also be presented.

In addition, two methodologies that may be used to develop an indicated loss and loss adjustment expense (LAE) reserve associated with issued EREs are presented.

Keywords. Medical Malpractice—Claims-Made; reserving; reserving methods; Statutory Accounting Principles; unearned premium reserves.

1. INTRODUCTION

Medical professional liability (MPL) insurers and other carriers that write claims-made coverage typically also provide ERE coverage upon termination of an insured's claims-made policy. The ERE (commonly referred to as a "tail policy") provides coverage for a claim reported after the expiration date of the insured's last claims-made policy provided that the event giving rise to the claim occurred subsequent to the retroactive date of the insured's claims-made coverage and prior to the non-renewal of that coverage.

The cost of an ERE can be several times that of a mature claims-made policy, and consequently a significant expense. For this reason, most insurers also offer DDR coverage for their physician insureds. DDR coverage provides that the insured will receive an ERE without additional premium charge if the claims-made policy is terminated due to the insured's death, disability, or retirement.[1]

Statutory accounting provides that a loss and loss adjustment expense (LAE) reserve be held for any ERE of unlimited duration (EREs of fixed duration require an unearned premium reserve prior to expiration of the endorsement, and a loss and LAE reserve for these EREs is held only for

[1] Certain restrictions are often in place in the event of retirement, such as a minimum age and a minimum number of years that the physician must be continuously insured prior to qualification.

reported claims).[2] In addition, the National Association of Insurance Commissioners (NAIC) requires insurers that offer DDR coverage to carry a reserve for yet-to-be-issued DDR EREs on in-force claims-made policies (referred to here as the "DDR reserve"). The NAIC has entitled this the "extended reporting endorsement policy reserve" and requires that this reserve be "classified as a component part of the unearned premium reserve."[3]

Although included in the unearned premium reserve, the DDR reserve is within the scope of the Statement of Actuarial Opinion.[4] Consequently, derivation of an indicated DDR reserve[5] is of particular importance for appointed actuaries of insurers with this reserve component.

To the best of the author's knowledge, there has been no actuarial literature to date documenting a methodology for reserving for EREs. There is limited actuarial literature discussing the DDR reserve, and what does exist provides an unnecessarily leveraged and complex methodology. This monograph will address these two deficiencies in the current actuarial research.

The remainder of the paper proceeds as follows. Section 2 outlines two methodologies that can be used to develop an indicated loss and LAE reserve for issued EREs. Section 3 discusses the source of the liability for the DDR policy provision and ways in which this liability can be viewed and consequently evaluated. Section 4 outlines a proposed methodology for developing an indicated DDR reserve. Section 5 compares the proposed methodology from Section 4 to the methodology commonly in place today for developing an indicated DDR reserve. Section 6 provides an application of the idea underlying the proposed DDR reserving methodology to pricing the DDR policy provision. Lastly, Section 7 summarizes the key points of the monograph.

2. INDICATED LOSS AND LAE RESERVE FOR ISSUED ERES

Two methods that may be used to develop an indicated loss and LAE reserve for issued EREs are described briefly as follows:

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2 Statement of Statutory Accounting Principles 65-7.
3 Statement of Statutory Accounting Principles 65-8.
4 See the American Academy of Actuaries' Committee on Property and Liability Financial Reporting's Practice Note on Statements of Actuarial Opinion on Property and Casualty Loss Reserves as of December 31, 2009 (in particular, pages 38, 53 and 54).
5 As is common in actuarial literature, the term "indicated reserve" will be used throughout this monograph to refer to indicated unpaid loss and LAE or to indicated unearned premium. The term "reserve" should not be understood to refer to the reserve carried on the financial statements unless explicitly identified as such.

(1) Pure Premium Methodology – Develop an ERE pure premium to which a claim reporting pattern can be applied to allocate the yet-to-be-reported portion of this pure premium for each issued ERE. Claims reported to date would need to be reserved for separately, typically by inclusion within the claims-made portion of the actuarial analysis; and

(2) Triangular Methodology – Include the ERE claims within the occurrence analysis (i.e., within the occurrence triangle) on a policy year basis (i.e., the year in which the endorsement is issued). If the company writes no occurrence business, or if the EREs on their own are of sufficient volume, ERE claims could be aggregated within their own triangle, again on a policy-year basis.

The above two methodologies will be discussed further in the following two sections, respectively.

2.1 Pure Premium Methodology

Exhibit 1 outlines the pure premium methodology that may be used to derive an indicated loss and LAE[6] reserve for unreported claims on EREs. The fundamental idea of this methodology is to derive an a priori ultimate loss and LAE associated with each policy year (for EREs only), and from this, allocate a portion estimated to be unreported as of the evaluation date. The details of Exhibit 1 by column are as follows:

(1) These are the number of issued EREs by policy year, adjusted by the classification of each physician insured to be base-class equivalent (i.e., each physician is counted according to the pricing relativity of his or her classification relative to the base class). The adjustment to a base-class equivalent basis is necessary, as this is the basis on which the pure premiums shown in Column (2) are developed. Note that they are not adjusted to be mature claims-made (MCM) equivalent, another standard adjustment typically included in measuring exposure in MPL reserving. This is because each ERE is assumed to have the same level of exposure regardless of the retroactive date of the claims-made policy to which it attaches (this assumption, and possible deviation from it,

[6] If the actuary intends to develop a provision for unallocated loss adjustment expense (ULAE) costs associated with issued ERE policies elsewhere, the methodology can easily be modified to exclude ULAE from the pure premium and develop solely an indicated loss and allocated loss adjustment expense (ALAE) reserve.

is discussed further toward the end of this section). It would also be appropriate to adjust each issued DDR ERE count by a factor intended to account for the reduction in exposure (due to reduced practice hours, lessened acuity of patients, etc.) that typically occurs prior to a physician's retirement (80% is a commonly used adjustment factor for the exposure associated with DDR EREs relative to purchased EREs).

(2) The indicated loss and LAE base-class ERE pure premium, developed for the most recent policy year on Exhibit 3, is typically based on the insurer's claims-made book of business, along with its indicated or filed ERE factors. To avoid overstating the pure premium, the actuary should take care to exclude claims reported on EREs from the indications. Note that the pure premium for each of the older policy years is developed by de-trending the indication for the most recent year at an assumed trend rate of 5.0% per annum.

(3) The multiplication of Columns (1) and (2) produces an a priori ultimate loss and LAE for each policy year. Note that this ultimate loss and LAE is likely to be different from the ultimate loss and LAE that would be derived based on an analysis of ERE claims reported to date. However, in this context, the liability associated with reported claims is unimportant, as we intend to use this indication of ultimate loss and LAE solely to derive a subsequent indication of loss and LAE on unreported claims alone.

(4) The portion of ultimate loss and LAE estimated to stem from claims unreported at the current evaluation date is based on the trended claim reporting pattern given on Exhibit 5, which is itself based on the untrended reporting pattern on Exhibit 4. Note that it is important to rely on a claim reporting pattern rather than a loss reporting pattern for these indications, as it is not the incurred but not reported (IBNR) reserve itself that we are deriving through this methodology, but solely the reserve associated with IBNR claims. In other words, this provision should exclude the bulk reserve for any indicated deficiency in currently held case reserves. The IBNR reserve itself could be either less than or greater than the reserve associated with IBNR claims, depending on the magnitude of the bulk reserve indicated for claims reported to date.

The use of a trended reporting pattern reflects the assumption that the calendar year of claim payment will determine the cost level of the claim. This is discussed further in [1]. However, these calculations otherwise assume that no severity differential exists by report lag. The actuary should consider whether this assumption is reasonable for the book of business under review, as in many cases, larger claims may take longer to reach a verdict or settlement.

(5) The multiplication of Columns (3) and (4) derives, by policy year, an indicated loss and LAE reserve for unreported claims on EREs on a gross of reinsurance basis.

The above methodology could be adjusted to a net of reinsurance basis by use of an indicated pure premium net of reinsurance or by adjusting the indicated unpaid loss and LAE by a net-to-gross ratio. One would expect in most cases that these ratios would vary by accident year, depending on the reinsurance in effect at the given time.

Note that the ERE pure premium used within this methodology is essentially an "average" ERE pure premium. If the book of business being reviewed is in a steady state, this is likely a reasonable assumption. However, if the book of business is expanding or undergoing other changes, it is possible that the expected pure premium associated with each policy year may be changing significantly as well.

This could be the case, for example, for an insurer that only began writing PPL policies several years ago. If the retroactive date for each of the issued policies was coincident with the initial effective date, the exposure associated with each issued ERE would be growing significantly over time, as the average length of time between the ERE effective date and the retroactive date grows. In this case, a different average ERE-to-claims-made factor should be derived by policy year, and it might be prudent to take into account the retroactive dates on each ERE in estimating exposure.

A second potential pitfall that should be avoided is failing to account for issued DDR EREs in addition to purchased EREs. This could result, for example, if a loss ratio methodology rather than a pure premium methodology were used to develop the a priori ultimate loss and LAE by policy year. Unless the premium used was adjusted to reflect DDR EREs written (which can be half or more of issued EREs), such a methodology could significantly understate the reserve associated with unreported claims on EREs. For the same reason, the actuary should also take care that issued ERE counts reflect issued DDR EREs, in addition to purchased EREs.

As mentioned above, under the pure premium methodology, a separate reserve indication will need to be derived for reported claims on EREs. This is typically done by including claims reported to date on EREs in the analysis of the reserve for claims-made policies. Both claims reported on claims-made policies, as well as claims reported on EREs, would be included in the analysis on a report-year basis.

2.2 Triangular Methodology

If a sufficient volume of ERE claims is available (or if the company has an occurrence book of business with which the ERE claims can be combined), a standard actuarial analysis can be performed on the reported ERE claims to develop an indicated liability for both reported and unreported claims on EREs. Generally, including ERE claims in a triangle separate from the occurrence business is considered preferable, since the development patterns exhibited by each of these policy types can be materially different. For many companies, the proportion of EREs written relative to occurrence policies could vary over time, possibly having a significant impact on the analysis. However, for many companies, there is an insufficient volume of ERE claims to analyze EREs on their own, and including these claims with the occurrence business (or opting for the pure premium methodology described above) is necessary.

In performing a triangular analysis of the liability associated with ERE claims, it is important to organize the ERE claims reported to date on a policy year basis. Organizing the claims on an accident-year basis (as is sometimes done in error, possibly out of confusion due to including these claims with the occurrence business), would effectively develop a reserve for all claims to be reported on EREs that have occurred as of the evaluation date, regardless of whether an ERE to cover such a claim has been written. This would result in a possibly significant overstatement of the indicated reserve.[7]

3. THE LIABILITY FOR THE DDR POLICY PROVISION

The methodology proposed within this monograph for derivation of an indicated DDR reserve is based on a different perception of the source of the liability for the DDR policy provision than the current methodology. This merits further discussion before proceeding to the details of the proposed methodology.

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[7] Organization of ERE claims by policy year also has the benefit of being consistent with the NAIC's Annual Statement instructions for Schedule P, which require ERE premium and claims to be included on a policy year basis within the MPL-Occurrence section (see the Annual Statement instructions, under the heading Schedule P-Parts 1A through 1T).

The following chart demonstrates four categories in which unreported claims on an in-force claims-made book of business may fall:

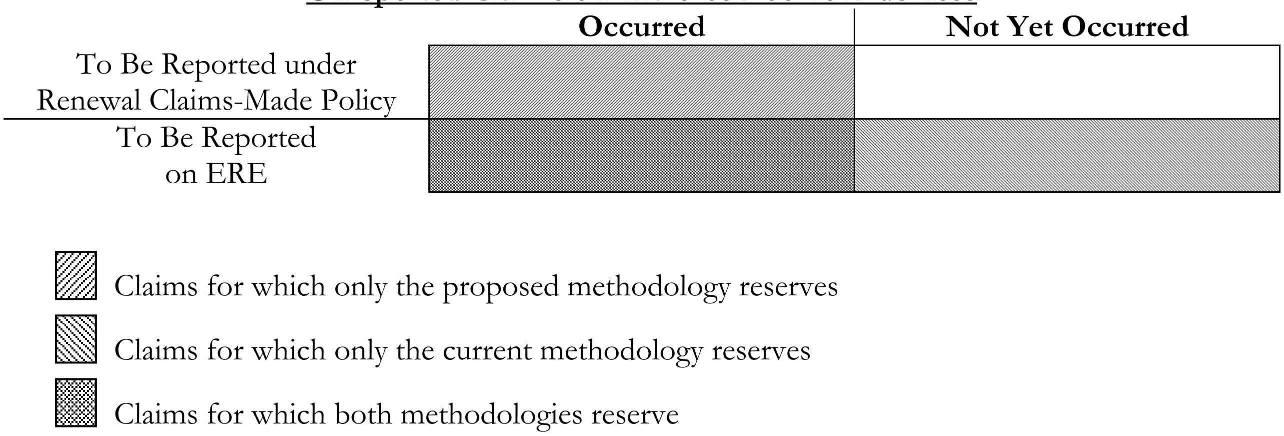
Unreported Claims on In-Force Claims-Made Book of Business

	Occurred	**Not Yet Occurred**
To Be Reported under Renewal Claims-Made Policy		
To Be Reported on ERE		

As shown by this chart, unreported claims may stem from an event that has either occurred as of a given evaluation date or has not yet occurred. Once reported, the claim will either be reported on a renewal of the in-force claims-made policy (or on the unexpired portion of the claims-made policy in-force), or on an ERE issued at the time the insured's last claims-made policy is non-renewed (assuming such an ERE is issued; otherwise, the insurer would have no liability for a claim reported after the date of non-renewal).

Next, we overlay on this chart a visual representation of two ways of viewing the DDR liability:

<u>Unreported Claims on In-Force Book of Business</u>



As shown above, the methodology proposed in this monograph (to be discussed in detail in the following section) assumes that the liability associated with the DDR policy provision stems only from claims that have occurred as of the given evaluate date, regardless of whether such a claim will be reported on the renewal of a claims-made policy or on an ERE issued at the time of the claims-made policy's non-renewal. This is in contrast to the current methodology for development of the DDR reserve, which views the liability as stemming only from yet-to-be-issued DDR EREs on the in-force book, regardless of whether the loss costs associated with those EREs stem from claims that have occurred as of the evaluation date (see Section 5 and Appendix A for further information on the current methodology).

This is easier to discuss and comprehend if we consider, rather than the DDR policy provision, a policy provision in which the insurer contractually agrees to provide EREs to all of its claims-made insureds at the time of their non-renewal. Such a construct is not entirely theoretical, as there are several MPL writers offering such a policy in the current market. For ease of discussion, let us refer to such a policy form as the Enhanced Claims-Made policy form.

As shown in the chart above, under the Enhanced Claims-Made policy form, there are two ways that the liability can be viewed. The first is to view the liability as essentially that of an occurrence policy and reserve for it as such (i.e., reserve for claims that have occurred as of the evaluation date, regardless of whether such a claim is expected to be reported on an ERE or on the renewal of an in-force claims-made policy). The second is to view the liability as that of a claims-made policy, plus the liability for EREs unissued as of the evaluation date for the in-force book. Each of these viewpoints deserves further discussion:

(1) Occurrence Liability — If one considers that an ERE is intended to cover the gap in coverage between a claims-made and an occurrence policy, the notion that an Enhanced Claims-Made policy should be reserved for as an occurrence policy seems like the most natural course. Consider, for example, a claim that has occurred but remains unreported under the Enhanced Claims-Made policy form. Given the contractual language of this policy form stating that an ERE will be issued upon non-renewal of the policy, it is clear that coverage will be provided for such a claim and that a reserve should be held (although it remains unknown whether this claim will be reported under a renewal of the claims-made policy or under the ERE to be issued upon non-renewal). However, whether such a reserve should be held as a loss reserve or an unearned premium reserve would be a matter of debate.

(2) Claims-Made Plus Tail Liability — Under this viewpoint, the insurer would reserve for a claim if it had been reported as of the given evaluation date (this would be in common with the Occurrence Liability viewpoint discussed above, as shown in the chart preceding this discussion), or if the claim was expected to be reported on an ERE to be written upon expiration of the insured's last claims-made policy, regardless of whether such a claim had occurred as of the given evaluation date. On the surface, this viewpoint may seem technically consistent with the contractual policy language (which explicitly refers to coverage for claims reported during the policy period and to the offer of a pre-funded ERE at policy termination). However, including a reserve for claims that have not yet occurred may be inconsistent with the claims whose liability the insurer has in fact assumed at the given evaluation date, and seems counterintuitive relative to all other property & casualty reserving practices.

As discussed above, the writer of an Enhanced Claims-Made policy will have liability for any claim that has occurred subsequent to the retroactive date of a given Enhanced Claims-Made policy, regardless of whether the claim has been reported as of the evaluation date. As is always the case, the insurer retains the right to cancel the policy at any time (although an ERE would be issued without additional premium charge upon such a cancellation), and so can be considered to have no liability for any claim that has not yet occurred. This is similar to other aspects of property & casualty reserving, in which a reserve is carried only for claims that have occurred and for which the insurer is contractually liable (excluding, in particular, any claim to be incurred on the unearned portion of a policy).

Most fundamentally, the principal difference between the current and proposed methodologies is that the proposed methodology is based on actuarial methods common to property & casualty coverage, while the current methodology is characteristic of methodologies associated with life insurance. Reserving for fixed-premium life insurance policies evolved to its current status because of the insurer's contractual agreement as to the fixed-premium amount. Consequently, the life insurance reserving methodology requisitely reflects the possible difference between expected premium and expected payments over the remaining life of the policy. However, there is no contractual agreement on the part of the MPL insurer to continue to provide coverage at the current level of pricing. Consequently, a methodology more akin to typical property & casualty reserving (where a reserve is developed only for those claims that have already occurred) seems appropriate.

The methodology proposed in the following section assumes the first of the viewpoints discussed above; that is, the liability associated with the DDR policy provision stems from claims that have occurred as of the given evaluation date, and in particular, the portion of the loss and LAE on these claims that will be reported on DDR EREs.

4. AN INDICATED RESERVE FOR DDR EXPOSURE

As discussed above, the proposed methodology used to derive an indicated reserve for DDR exposure is based on the observation that a claims-made policy with a DDR provision offers coverage that is effectively between a claims-made policy without this provision and an occurrence policy. Thus, the DDR reserve can be thought of as a subset of the difference between the reserve that would exist for a claims-made book of business, if the business had been written on an occurrence basis, and the reserve that exists for the business as it was written (on a claims-made basis). The key is recognizing that the difference between these two reserve indications (the claims-made and the occurrence) can be grouped into the following categories:

(1) Claims that will be reported on claims-made policies (either unissued – i.e., non-renewed – as of the evaluation date or, if issued, on the unexpired portion of such a policy).

(2) Claims that will be reported after an insured's last claims-made policy has terminated, which themselves can be allocated to three subcategories:

   (a) Claims that will be reported on a yet-to-be-issued DDR ERE.

   (b) Claims that will be reported on a yet-to-be-issued purchased ERE.

   (c) Claims for which the insurer will have no liability, as the insured was not eligible for a DDR ERE at termination of the claims-made policy and the insured chose not to purchase an ERE. (In a case such as this, the insured may have coverage from another insurer for such a claim, if the other insurer agreed to provide the insured with a retroactive date preceding the newly purchased policy's initial effective date. This is oftentimes referred to as "prior acts" coverage within the MPL industry.)

At any given evaluation date, the insurer of a claims-made book has no current liability for claims in category (1), (2b) or (2c). The DDR reserve can be thought of as a provision for claims in category (2a).

Note that the above classification pertains to reserve indications derived for the in-force claims-made book only. In other words, the liability associated with issued EREs is not included above. Deriving an indicated reserve for this liability was discussed in Section 2.

Exhibit 2 outlines the proposed methodology to derive an indicated reserve for the DDR exposure. This methodology is based on the categorization of unreported claims discussed above for an in-force book of claims-made policies. In brief, the methodology derives an indicated a priori ultimate loss and LAE indication for the in-force claims-made book on an accident-year (i.e., occurrence) basis. The portion of this indication assumed to stem from unreported claims is then estimated, and from this, the estimated portion associated with DDR claims (i.e., claims that are projected to be reported under yet-to-be-issued DDR EREs) is allocated. The resulting value is the indicated DDR reserve.

The details of Exhibit 2 by column are as follows:

(1) These are the number of earned exposures by accident year for insureds remaining in-force on claims-made policies only (i.e., excluding insureds for whom an ERE has been issued). Consistent with standard exposure calculations for PPL reserving, the exposures are adjusted to a base-class equivalent basis by the classification of each physician insured. However, they are not adjusted to be mature claims-made (MCM) equivalent. This is because, on an occurrence basis, as the indicated ultimate loss and LAE will be measured here, there is no reduction in liability for an insured holding a claims-made policy that is less than fully mature.

Note furthermore that the exposures are calculated on an accident-year basis, as opposed to the report-year basis on which claims-made exposures would normally be calculated. In other words, the exposures are determined based on the retroactive date of the policies, as opposed to their initial effective dates. If retroactive dates preceding the initial policy effective date have been provided, the accident-year exposures could be very different in magnitude from the report-year exposures. This will be the case if insureds were offered "full prior acts" coverage upon their initial purchase of a claims-made policy.

Lastly, note that the exposures in any given accident year are effectively a subset of the exposures in any subsequent accident year. This is because the exposures in a given accident year represent that portion of the in-force exposures with retroactive dates in or preceding this accident year. This observation can be helpful in understanding the methodology that follows.

(2) The indicated loss and LAE base-class occurrence pure premium is developed for the most recent accident year on Exhibit 3. Note that the underlying data is consistent with the data used in the derivation of the ERE pure premium used in Section 2.1 above. While the methodology in Section 2.1 serves to develop an indicated reserve for issued ERE policies, and hence, relies on an ERE pure premium applied to counts of these policies, the methodology under discussion for the DDR reserve relies on an occurrence pure premium. This is because the methodology is based on exposures measured on an accident-year basis and develops an indicated reserve for as yet unissued DDR EREs. The pure premium for each of the older accident years is

developed by de-trending the indication for the most recent year at an assumed trend rate of 5.0% per annum.

(3) The multiplication of Columns (1) and (2) produces an a priori ultimate loss and LAE for each accident year. As mentioned previously, in the context of the pure premium methodology for unreported ERE claims, this a priori ultimate loss and LAE could be very different from the ultimate loss and LAE that would be derived based on an analysis of claims reported to date. In this context, the liability associated with reported claims is unimportant, as we intend to use this indication of ultimate loss and LAE solely to derive a subsequent indication of loss and LAE on unreported claims alone.

(4) The portion of ultimate loss and LAE estimated to stem from claims unreported at the current evaluation date is based on the trended claim reporting pattern given on Exhibit 5. As was the case with the pure premium methodology for unreported ERE claims, it is important to rely on a claim reporting pattern rather than a loss reporting pattern for these indications, as it is solely the indicated reserve associated with IBNR claims that we are deriving through this methodology (as opposed to the IBNR reserve in its totality).

(5) The multiplication of Columns (3) and (4) derives, by accident year, an indicated reserve for the loss and LAE expected to stem from unreported claims on in-force claims-made policies, which have occurred as of the evaluation date of the analysis.

(6) The portion of the indicated reserve of interest is the portion expected to be reported on yet-to-be-issued DDR EREs. To segregate this portion of the indicated reserve, it is necessary to estimate, by accident year, the portion of loss and LAE on unreported claims that is expected to be reported on DDR EREs. This is done on Exhibit 6 (which is, in turn, based on the selected per annum retention and DDR rates from Exhibit 7).

On Exhibit 6, the average portion of the in-force book of insureds remaining in-force during subsequent calendar periods (cumulative retention) is estimated based on the selected per annum retention ratio. The cumulative retention ratios are then used to estimate the expected portion of insureds to experience DDR in each future calendar year (relative to the insureds in-force as of the current evaluation date). The incremental portion expected to DDR in any year is equal to the selected per annum

DDR rate times the portion remaining in-force. The result is referred to as the "incremental DDR portion." The cumulative portion of insureds expected to have obtained a DDR ERE at any given evaluation date (cumulative DDR portion) is the sum of these incremental DDR portions.

It is possible that the prospective portion of loss and LAE expected to be reported on DDR EREs is biased low for older accident years under the above methodology. This is because the physicians whose exposures are contemplated in the older accident years will, on average, be older than the physicians whose exposures are included in the more recent accident years (recall that the exposures of an older accident year are a subset of the exposures in any more recent accident year; in particular, they are the subset with retroactive dates in or preceding the given older accident year, and are consequently more likely to consist of older physicians). The likelihood of a claim being reported on a DDR ERE can be expected to increase as a physician ages (although this does not necessarily imply that the weighted average portion of claims expected to be reported on DDR EREs will increase for older accident years, as this is influenced by other factors, such as the reporting pattern). The actuary may wish to consider an adjustment to the methodology for this aging phenomenon, although in doing so, the actuary should observe that the portion of the indicated DDR reserve stemming from older accident years is usually quite small, and consequently, the effect of such an adjustment may be immaterial.

This observation may also hold for the later report periods associated with the more recent accident years, in which the physicians will have aged relative to the evaluation date of the analysis. Consequently, their DDR rate may have increased relative to the in-force book from which it was projected. However, their retention rate can also be expected to have decreased (as a result of the increase in the DDR rate), and the effect of these two on the prospective incremental DDR portions may be offsetting. As was noted in the prior paragraph concerning the older accident years, the loss and LAE associated with these later report periods is minimal, and consequently, any attempt to adjust for this phenomenon may be immaterial. However, the actuary should consider the appropriateness and possible effect of the underlying assumptions for the book of business under review.

Two factors that may affect the portion of insureds to DDR on an historical basis that should be considered in selecting a prospective portion to DDR are the economic cycle and the overall age of the claims-made book. Physicians may choose to postpone retirement during an economic recession, so the portion of insureds that are observed to experience a DDR event during such a time period may be lower than during times of economic growth.

In addition, an insurer that has provided claims-made coverage for a relatively short period of time will have experienced few DDR occurrences. This is in part due to what may be a younger book of insureds than will be experienced as the book ages, but also due to stipulations that may exist in the DDR ERE provision, such as a frequent requirement that a physician maintain a claims-made policy in-force for a minimum of typically five years in order to qualify for the retirement benefit. The large majority of DDR policy issuances stem from a physician's retirement, and consequently, the number of such issuances can be very small for a relatively new PPL insurer.

Changes may also occur over time in the particular policy language of the DDR provision. As mentioned in the previous paragraph, physician insureds are frequently required to maintain a claims-made policy in-force for five years in order to qualify for a pre-funded ERE in the event of retirement. This requirement is often relaxed or eliminated during a soft market, and in some cases, insurers may also eliminate the age requirement from the policy language. Such a change can, of course, affect the portion of insureds to earn a DDR ERE over time and should be considered in a prospective selection.

The retention ratio can also be expected to vary over time, and is largely a function of market factors. The retention ratio will vary depending on the state in which the insurer provides coverage and can also vary as a result of the insurance cycle. Consequently, it is prudent to consider multi-year time periods in measuring indications of this ratio. The claims for which this methodology derives an indicated reserve are expected to be reported over several years, and a multi-year average is consequently appropriate.

(7) The product of the indicated reserve in Column (5) (for all unreported claims) and the weighted average portion of claims to be reported on DDR EREs in Column (6) produces an indicated reserve for claims expected to be reported on DDR EREs.

(8) The last adjustment included in this methodology is for the assumed reduction in exposure associated with DDR EREs relative to purchased EREs. This assumed adjustment is due to a reduction in a physician's exposure preceding retirement. While such an adjustment would be largely judgmental for some insurers, insurers with larger books of business may be able to compare the frequency on DDR EREs with the frequency on purchased EREs to develop an indication for this adjustment.

(9) The indicated DDR reserve is the product of the total from Column (7) with Row (8).

Note that, unlike the pure premium methodology for issued EREs discussed in Section 2.1 above, the analysis described above is performed solely on a gross of reinsurance basis, with no separate reduction to net liability. This is typically the manner in which the DDR reserve is carried within an insurer's financial statements, and results from the observation that the DDR ERE remains unissued as of the evaluation date of these statements. Consequently, there is no reinsurance treaty in-force to cover the ERE.[8] The lack of a ceded DDR reserve can also be considered the interpretation of Statement of Statutory Accounting Principles (SSAP) 65-8, which states that "The amount of the reserve should be adequate to pay for all future claims arising from these coverage features, after recognition of future premiums to be paid by current insureds for these benefits."

The requirement of SSAP 65-8 that the indicated DDR reserve include an offset for "recognition of future premiums to be paid by current insureds for these benefits" merits further discussion. Clearly, the methodology proposed above includes no such offset. However, the lack of such an offset seems reasonable, as the only claims for which a reserve is projected are those that have already occurred, as opposed to "all future claims arising from these coverage features." The methodology also seems consistent with the NAIC's original intent in requiring the DDR reserve, which was "to assure that amounts collected by insurers to pay for these benefits are not earned prematurely and that an insurer with an aging book of business will not show adverse operating

[8] Reinsurance treaties for MPL coverage typically apply either on a claims-made or policies-issued basis. The language of a reinsurance treaty on a policies-issued basis is usually such that an ERE would attach upon the effective date of the endorsement itself, and not at the effective date of the claims-made policy that it endorses.

results simply because an increasing portion of insureds is earning the benefits for which it has paid." [9]

5. COMPARISON TO CURRENT METHODOLOGY FOR DDR RESERVING

There are various forms of the current methodology for developing an indicated DDR reserve commonly in use today. One of these is described by Walker and Skrodenis [3], as well as by Walling [4]. Fundamentally similar methodology has been employed by various MPL writers, as well as various consulting firms providing actuarial services to MPL writers. While each user has incorporated (or chosen not to incorporate) various adjustments into the methodology, and has also organized the presentation of the methodology differently, the fundamental concept underlying each version of the methodology remains the same. Appendix A presents a version of the methodology commonly employed, and will be the focus of the discussion here.

5.1 Overview of the Current Methodology

The current methodology requires a number of assumptions, including:

- Death, disability and retirement rates by age;
- Policy renewal rates, also typically analyzed by age;
- Average pure premium and collected premium per exposure;
- Prospective pricing provision for DDR coverage;
- Age demographics for the in-force book of claims-made insureds; and
- Interest rate and trend assumptions.

These are used to project the number of physicians to die, become disabled, retire, or to lapse his or her policy over each of the next 50 or more calendar years, from among the in-force book of claims-made insureds. The estimated loss cost associated with physicians who DDR is then determined, and offset on a discounted basis by the premium collected during these same calendar years associated with DDR coverage (based on the pricing provision for DDR within the claims-made policy).

[9] NAIC Proceedings - 1991 Vol. IIB (also, NAIC Accounting Practices & Procedures Manual, Issue Paper 65, Section 41).

This method is complex and highly leveraged on its underlying assumptions (including the discount rate, trend rate, the death and disability rates, and, in particular, the retirement rates, which are highly uncertain). Given the magnitude of the DDR reserve relative to the loss and LAE reserve, a more streamlined methodology seems appropriate. Perhaps most significantly, the current method projects losses to be reported at some future date, but which have not yet occurred. These losses contribute significantly to the indicated reserve, yet given that no reserve for these claims would be required even if an occurrence policy had been written, their inclusion within the methodology seems intuitively suspect.

A more complete description of one version of the current methodology can be found in Appendix A.

5.2 Actuarial Research on the Current Methodology

Prior to the 1980s, MPL policies were largely written on an occurrence basis, consistent with the rest of the property & casualty industry. However, this changed during the early 1980s, largely in response to pressure from reinsurers who wanted to limit the uncertainty associated with the coverage they were providing. MPL insurers introduced claims-made policy forms, and some eliminated occurrence coverage entirely (although others have continued to offer occurrence coverage, sometimes under certain limitations, such as only for particular specialties or only up to particular policy limits that would be below the level of loss ceded to reinsurers). Not long after this, the DDR policy provision was introduced, although the liability associated with this provision seems not to have been immediately understood.

McClenahan [2] may have been the first to consider the need for an accrued liability related to yet-to-be-issued DDR EREs, and much of his paper, authored in 1988, is devoted to arguing for this accrual. However, his conclusion is that most insureds will remain in-force until death, disability or retirement, and consequently, the insurer should carry the difference between the indicated occurrence reserve and the claims-made reserve as an accrual for the DDR liability.

Subsequently, other actuaries observed that the portion of in-force insureds who will DDR (as opposed to canceling coverage for another reason and purchasing an ERE tail) may in fact be much smaller than the book as a whole. In addition, much of the difference between the occurrence liability and the claims-made liability will be covered by claims-made policies that have not yet been renewed, and an accrual for DDR liability should be offset by these future premiums.

Walker and Skrodenis [3] recognized both of these offsets to the DDR liability. The focus of Walker and Skrodenis's work was on pricing for DDR coverage within a claims-made policy, but the three techniques presented are also discussed by the authors as applicable to establishing the DDR reserve. Each of these techniques is based on estimating rates of mortality, disability, retirement and policy lapse by age.

When applied to the in-force group of physicians, estimates of the number of insureds to die, become disabled, retire, or lapse their policy can be obtained by calendar year. Together with associated estimates of each physician's earned premium and pure premium during these calendar years, an estimate of the DDR reserve is derived. Note that this model effectively makes projections over the next fifty years or more (the length of time that a relatively young physician may be continuously insured).

Walling [4] presents two methods for estimating the DDR reserve. The first is fundamentally identical to Walker and Skrodenis's model, but adds modifications for items such as the waiting period for eligibility and trends in mortality. The second method is a stochastic approach in which interest rates, inflation rates, and mortality are simulated, but is otherwise similar to the model presented by Walker and Skrodenis.

The modifications proposed by Walling within his first method can be considered improvements over the Walker and Skrodenis model. However, the disadvantages associated with the model itself (discussed further below) remain. It is not clear whether the second method proposed by Walling (in which the parameters are stochastically simulated) represents an improvement in methodology, or rather, a difference in methodology. The appeal of stochastically varying the underlying assumptions seems to lie in the recognition that these assumptions are highly uncertain, and stochastically varying the assumptions allows the actuary to incorporate a wider range of parameter values into the indicated reserve. However, as Walling notes, "significant parameter risk still exists and may actually be increased by using a stochastic model."[10]

5.3 Comparison of the Current and Proposed Methodologies

There are several advantages of the proposed methodology over the current. In particular:

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[10] [4], page 10.

(1) It avoids the projection of claims from accidents that have not occurred as of the evaluation date.

(2) It avoids the projection of future earned premiums on policies that have not renewed as of the evaluation date.

(3) It avoids the discounting of the above projections for the time value of money and the need to select a discount rate.

(4) The current methodology has the potential to be highly inaccurate, given its reliance on unknown parameters such as mortality, disability, and retirement rates by age (which may differ from the general population).

(5) The proposed methodology is significantly less leveraged than the current methodology, which has the potential to produce a wide range of reserves based on seemingly small variations in the underlying assumptions.

(6) The time requirement for the actuary of the proposed methodology is appropriate to the relative magnitude of the reserve.

The following table highlights the leverage of the current methodology by providing the indicated increase in reserve under various changes in parameter assumptions:

Indicated Change in Reserve Under Various Parameter Assumptions Current Methodology[1]			
Parameter	**Initial Value**	**Revised Value**	**Indicated Change in Reserve**
Retention	91.0%	89.0%	(24.4)%
Pure Premium[2]	$11,171	$10,054	(17.1)%
DDR Provision	3.0%	4.0%	(23.8)%
Per Annum Trend[3]	5.0%	6.0%	22.6%
Per Annum Discount Rate	3.0%	4.0%	(20.9)%
Retirement Rates	Additive Increase of 1.0% from Age 55 to 79		16.6%

1 Under an assumed 3.0% per annum discount rate for the time value of money.

2 A 10% reduction in revised value relative to initial value.

3 Using same selected pure premium and changing prospective selected trend only.

Other changes in assumptions not outlined here can also have a significant impact on the resulting indicated reserve. For example, varying the DDR rates (perhaps under the assumption that physicians may have slightly longer life expectancies than the general population) or the projected average premium (for long-term pricing assumptions) can also have significant impacts on the analysis.

For comparison, the following table provides the effect of revisions consistent with the above on the indicated reserve under the proposed methodology:

Indicated Change in Reserve Under Various Parameter Assumptions Proposed Methodology			
Parameter	**Initial Value**	**Revised Value**	**Indicated Change in Reserve**
Retention	91.0%	89.0%	(1.3)%
Per Annum Trend[1]	5.0%	6.0%	1.3%
Pure Premium[2]	$7,680	$6,912	(10.0)%

1 Using same selected pure premium and changing factor used to de-trend selected pure premium and derive trended reporting pattern only.

2 A 10% reduction in revised value relative to initial value.

For most other parameters on which the indicated DDR reserve depends under the proposed methodology, similar to the pure premium parameter, the effect on the indicated reserve of a change in the parameter is also proportional. This is the case, for example, for the exposure adjustment for DDR EREs relative to purchased EREs.

6. PRICING THE DDR POLICY PROVISION

The idea behind the methodology proposed above for developing a DDR reserve also provides a simplified methodology to price the DDR provision within the claims-made policy. Recall that a claims-made policy with a DDR policy provision can be thought of as providing coverage between a claims-made policy without this provision and an occurrence policy. Thus the premium charge for a claims-made policy with the DDR provision should also be between the premium charges for these two policy types.

In MPL policies, a typical occurrence factor is 1.10 (i.e., it is typical to charge 10% more for an occurrence policy than for a comparable claims-made policy). This factor is typically derived actuarially by use of a claims reporting pattern and a selected trend rate. An offset for investment income may also be included, depending on the pricing targets for the claims-made and occurrence books of business.

If an insurer experiences a very high retention rate, perhaps so high that the only insureds non-renewing their coverage are those experiencing a DDR event, then that insured is effectively providing occurrence coverage. This is because for every issued claims-made policy, an insurer can expect to provide coverage for all claims occurring during the policy period (whether reported during this same period, on a subsequent renewal of the claims-made policy, or on the DDR ERE eventually issued). Consequently, the charge by such an insurer for a claims-made policy with a DDR provision should be equal to the charge for an occurrence policy (i.e., a DDR provision of 10%).

Conversely, consider an insurer with an abnormally low retention rate. For purposes of this theoretical argument, suppose that the retention rate is so low that effectively no insureds qualify for DDR EREs (recall that the number of in-force insureds experiencing death or disability is very small, and there is typically a vesting period required to qualify for the retirement provision of DDR coverage). In this admittedly theoretical case, there is no cost to the insurer of the DDR provision, and a pricing provision of 0% would be indicated.

Lastly, consider a more realistic insurer, whose retention falls between these two theoretical examples. Suppose that the insurer experiences a per annum retention of 91.0%, and a per annum DDR rate (i.e., the portion of in-force insureds to experience a DDR event) of 3.5%, each measured as a portion of the in-force claims-made book. Thus approximately 39%[11] of the in-force claims-made insureds of this insurer can be expected to obtain a DDR ERE upon non-renewal. The insurer is thus effectively providing occurrence coverage for this 39% of its in-force claims-made insureds, and claims-made coverage only for its remaining insureds. Assuming an occurrence factor of 1.10 for this insurer implies a DDR factor of approximately 1.04[12] within the ratemaking process.

The DDR factor incorporated into the indicated rate level could also vary judgmentally from this indication in the event of changes to the DDR policy language (e.g., a restriction or expansion of the retirement qualification) or in the event of expected changes, such as increased retention or aging of the book of business. A judgmental adjustment to the DDR factor could also be made to reflect the reduction in exposure preceding retirement discussed previously. However, in applying such an adjustment factor, the actuary would want to take into account that this reduction occurs only

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11 Calculated as 3.5% / [100.0% - 91.0%].
12 Calculated as [1.10 – 1.00] x 39% + 1.00.

during the three to four years preceding retirement, and not to the full exposure associated with these insureds.

The current process commonly used for deriving an indicated rate load for the DDR policy provision is discussed in [3]. Similar to the comparison of the current and proposed reserving methodologies, the proposed methodology for developing an indicated pricing provision for DDR coverage is much less time-consuming and more stable than the methodology currently in place.

7. CONCLUSION

Methodologies have been presented for deriving an indicated loss and LAE reserve on issued EREs, a reserve associated with yet-to-be-issued DDR EREs, and for a DDR factor to be incorporated within the ratemaking process. Each of these methodologies has been shown to be an improvement over the methodologies currently in place for deriving each of these indications. The author believes these methodologies would constitute a significant improvement in the techniques employed by actuaries within the MPL industry.

Acknowledgment

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Appendix A

One version of the most common methodology currently in use for developing an indicated DDR reserve is provided on Exhibits A1 through A18. The following paragraphs describe this methodology in more detail, taking care also to highlight the most significant differences that exist between this and the proposed methodology. Note that where assumptions of this methodology overlap with those of the proposed methodology (such as for the selected pure premium), the assumptions have been made consistently. While both sets of exhibits consist of manufactured data, an attempt has been made to present reasonable parameter selections and to maintain consistency between the method assumptions, in order to facilitate comparison.

The results of the current methodology are given on Exhibit A1. The indicated reserve is the discounted loss and LAE paid on yet-to-be-issued DDR EREs, less the discounted DDR premium yet to be earned (i.e., the portion of premium included within the ratemaking analysis for DDR exposure), both for the in-force book of claims-made insureds as of the evaluation date. While SSAP 65 does not explicitly allow or disallow discounting within the DDR reserve, a provision for the time value of money is usually included so as to develop an intuitively reasonable reserve. In addition, other NAIC accounting guidance characterizes the time value of money as one of the factors that "should be considered" in estimating the DDR reserve.[13]

The first step in the development of the indicated reserve is the selection of per annum retention ratios (typically done by age) as well as per annum rates of death, disability, and retirement. The DDR rates assumed are given on Exhibit A2. These can be based on information from the Census Bureau (for the death rates), or other public sources, as well as information from the insurer, which will typically be of limited credibility. In selecting the DDR rates, the actuary should take care that they are consistent with the selected retention rates by age (i.e., that the sum of all the rates is less than or equal to 100.0%, and presumably that this sum increases with age).

Exhibit A3 provides the selected overall per annum retention ratio. Due to the long-term nature of the methodology, the selected retention rate will typically be based on a longer-term indication, unless the actuary has reason to expect a difference in retention rates prospectively. The retention rate can also be selected to vary by age, as shown on Exhibit A4. While indicated retention rates for most age groups will be substantially similar until a typical retirement age (e.g., 65), retention rates can be expected to decrease somewhat for older physicians.

The selected retention rates are used on Exhibits A5 through A8 to project the number of in-force insureds to remain insured at future evaluation points. Note that projections are made over a 50-year time period, as a small portion of the insureds age 30 or less at the current evaluation is expected to remain continuously insured up to that point.

Similar projections are made on Exhibits A9 through A12 for the number of in-force insureds expected to die, become disabled, or retire during the next 50 calendar years. This information, together with the projections of physicians expected to remain in-force, is summarized on Exhibit A13. As a check of reasonability, comparisons can be made here of the number of insureds expected to lapse (whether due to DDR or other reasons) during the next several calendar years to the number known to have lapsed for the claims-made book in the calendar years preceding the evaluation date, recognizing that the projected values will likely be less due to the run-off nature of the methodology.

Exhibit A14 provides the selected average ERE pure premium to be applied to the projected DDR ERE issuances. Its derivation is analogous to the ERE pure premium derivation for the issued ERE reserve methodology discussed in Section 2.1, and shown on Exhibit 3. Note that Exhibit A14 includes an adjustment to the indicated pure premium for an assumed reduction in exposure for DDR EREs due to a physician's reduced practice hours preceding retirement. This is not reflected on Exhibit 3 for the proposed methodology, but is reflected instead on Exhibit 2, as a final adjustment to the indicated DDR reserve.

The pure premium is combined with the projected number of DDR ERE issuances on Exhibit A15. The projected total loss and LAE to be incurred on these EREs is allocated to calendar period by a selected payment pattern, given on Exhibit A18. The projected loss and LAE

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13 NAIC Accounting Practices & Procedures Manual, Issue Paper 65, Section 41.

to be paid by calendar period is discounted for the time value of money at various rates of return on Exhibit A16.

Note that the projection of DDR EREs and associated loss and LAE reflects all DDR EREs to be issued at any future point in time for claims-made insureds in-force as of the current evaluation date. Consequently, these DDR EREs include exposure associated with claims that will have occurred subsequent to the evaluation date of the analysis, but prior to the issuance of the DDR ERE. This is a disadvantage of the current methodology, as a reserve based on these claims would not be required even if occurrence coverage had been written.

Exhibit A17 provides the projection of yet-to-be-earned DDR premium on the renewal of the in-force claims-made policies, based on the current DDR provision included within the ratemaking process for these policies. This projected DDR premium is then discounted at the same per annum rates of return used for the loss and LAE payments on Exhibit A16. It is the total discounted projections from Exhibits A16 and A17 that are used on Exhibit A1 to derive the resulting indicated DDR reserve.

8. REFERENCES

[1] Marker, J.O., and F.J. Mohl, "Rating Claims-Made Insurance Policies," *CAS Discussion Paper Program*, May 1980, 365-304.
[2] McClenahan, C.L., "Liabilities for Extended Reporting Endorsement Guarantees Under Claims-Made Policies," *CAS Discussion Paper Program*, May 1988, 345-364.
[3] Walker, C.P., and D.P. Skrodenis, "Death, Disability and Retirement Coverage: Pricing the 'Free' Claims-Made Tail," *CAS Forum*, Winter 1996, 317-346.
[4] Walling, R.J., III, "A Dynamic Approach to Modeling Free Tail Coverage," *CAS Forum*, Fall 1999.

Abbreviations and notations

ALAE, allocated loss adjustment expense
BCE, base-class equivalent
DDR, death, disability and retirement
ERE, extended reporting endorsement
IBNR, incurred but not reported
LAE, loss adjustment expense
MCM, mature claims-made
MPL, medical professional liability
NAIC, National Association of Insurance Commissioners
PPL, physicians professional liability
ULAE, unallocated loss adjustment expense
SSAP, Statement of Statutory Accounting Principles

Biography of the Author

Susan J. Forray is a Consulting Actuary in the Milwaukee office of Milliman. She is a Fellow of the Casualty Actuarial Society and a Member of the American Academy of Actuaries. She has provided actuarial assistance to a spectrum of risk-bearing entities in both the private and public sectors, ranging from large multi-line commercial insurance companies to self-insured programs. She has also been active in the Casualty Actuarial Society, having served on the Examination Committee, the editorial board of the journal *Variance*, and the committee on professionalism education. She is a frequent presenter at industry symposia, and her work has been published in *Best's Review*, *The Physician Insurer*, and *Contingencies*. She can be reached at susan.forray@milliman.com.

Analysis of Unreported Tail Claims Loss and LAE Reserve And DDR Reserve, as of December 31, 2009

Indicated Unreported Tail Claims Loss and LAE Reserve

Policy Year	(1) Issued ERE Base Class Equivalent Exposures (Not MCM Equivalent)	(2) Indicated ERE Policy Loss & LAE Pure Premium[1]	(3) = (1) x (2) Indicated A Priori Ultimate Loss & LAE On Issued ERE Policies	(4) A Priori Portion of Loss and LAE on Claims Unreported as of 12/31/09 [2]	(5) = (3) x (4) *Gross of Reinsurance* Indicated Loss & LAE on Claims Unreported, On Issued ERE Policies as of 12/31/09
1997	9	7,776	67,687	0.0%	0
1998	10	8,164	85,547	0.0%	0
1999	13	8,573	110,301	0.2%	174
2000	14	9,001	126,632	0.5%	580
2001	16	9,451	148,859	1.3%	1,951
2002	24	9,924	235,961	2.3%	5,312
2003	57	10,420	589,595	2.9%	17,028
2004	74	10,941	814,874	4.2%	34,287
2005	90	11,488	1,038,188	6.7%	69,217
2006	106	12,063	1,280,700	13.7%	175,779
2007	122	12,666	1,542,150	21.8%	335,896
2008	128	13,299	1,702,278	45.8%	779,120
2009	134	13,964	1,871,176	73.4%	1,373,810
Total	797		9,613,948	29.1%	2,793,155

[1] From Exhibit 3, detrended by a per annum trend of 5.0%.

[2] From Exhibit 5, and assumed a twelve-month lag between accident year and ERE policy year claim reporting patterns.

Analysis of Unreported Tail Claims Loss and LAE Reserve And DDR Reserve, as of December 31, 2009

Indicated DDR Reserve

Accident Year	(1) In Force as of 12/31/09 Base Class Equivalent Exposures (Not MCM Equivalent)	(2) Indicated Occurrence Loss & LAE Pure Premium[1]	(3) = (1) x (2) Indicated A Priori Ultimate Loss & LAE On an Occurrence Basis	(4) A Priori Portion of Loss and LAE on Claims Unreported as of 12/31/09 [2]	(5) = (3) x (4) Indicated Loss & LAE on Claims Unreported, But Having Occurred as of 12/31/09	(6) Weighted Average Portion of Claims To be Reported On DDR Policies[3]	(7) = (5) x (6) Indicated Unreported Loss & LAE to be Reported on DDR Policies
1997	167	4,277	712,798	0.0%	0	1.8%	0
1998	201	4,490	900,871	0.2%	1,423	1.8%	25
1999	246	4,715	1,161,550	0.5%	5,321	2.9%	156
2000	269	4,951	1,333,536	1.3%	17,478	3.3%	584
2001	302	5,198	1,567,601	2.3%	35,292	4.6%	1,632
2002	455	5,458	2,484,844	2.9%	71,764	6.6%	4,706
2003	1,083	5,731	6,208,885	4.2%	261,249	7.2%	18,940
2004	1,426	6,017	8,581,241	6.7%	572,120	7.2%	41,329
2005	1,730	6,318	10,932,914	13.7%	1,500,569	6.0%	89,302
2006	2,033	6,634	13,486,747	21.8%	2,937,548	6.4%	189,005
2007	2,331	6,966	16,240,017	45.8%	7,432,933	5.5%	409,327
2008	2,451	7,314	17,926,284	73.4%	13,161,410	6.1%	805,404
2009	2,762	7,680	21,213,994	96.9%	20,559,196	7.5%	1,542,578
Total	15,457		102,751,282		46,556,302		3,102,987

Assumed Reduction in DDR Liability Due to Reduced Exposure Prior to Retirement (8)	80.0%
Indicated DDR Reserve; (7) Total x (8) = (9)	2,482,390

[1] From Exhibit 3, detrended by a per annum trend of 5.0%.
[2] From Exhibit 5.
[3] Weighted average portion to DDR from Exhibit 6.

Analysis of Unreported Tail Claims Loss and LAE Reserve And DDR Reserve, as of December 31, 2009

Indicated Pure Premiums For Occurrence and Tail Coverage

Report Year	Ultimate Loss & ALAE Limited to Policy Limits[1]	Mature Claims-Made Base Class Equivalent Exposures	Ultimate Loss & ALAE Pure Premium	Trended[2] Ultimate Loss & ALAE Pure Premium
1998	5,424,163	1,359	3,992	6,828
1999	2,396,646	1,427	1,679	2,736
2000	9,206,638	1,574	5,847	9,071
2001	4,793,956	1,777	2,698	3,986
2002	7,511,520	1,825	4,117	5,793
2003	8,367,549	1,774	4,717	6,321
2004	17,946,284	2,526	7,106	9,069
2005	19,199,929	2,736	7,017	8,529
2006	14,844,834	2,844	5,220	6,043
2007	10,900,615	2,500	4,360	4,807
2008	13,165,910	2,414	5,455	5,728
2009	16,783,983	2,674	6,277	6,277
2003 - 2009				6,713
2005 - 2009				6,315

(1) Selected Base Class Claims-Made Loss & ALAE Pure Premium at Total Limits	6,525
(2) ULAE Load[1]	7.0%
(3) Selected Base Class Claims-Made Loss & LAE Pure Premium at Total Limits; (1) x [1 + (2)]	6,982
(4) Mature Claims-Made to Occurrence Factor[3]	1.100
(5) Selected Base Class Loss & LAE Occurrence Pure Premium; (3) x (4)	**7,680**
(6) Mature Claims-Made to Average ERE Factor[3]	2.000
(7) Selected Base Class Loss & LAE Tail Pure Premium; (3) x (6)	**13,964**

[1] Based on claims-made reserve analysis
[2] Trended at 5.0% per annum to average report date of July 1, 2009
[3] Based on actuarial analysis or currently filed occurrence and ERE factors

Analysis of Unreported Tail Claims Loss and LAE Reserve And DDR Reserve, as of December 31, 2009

Indicated Portion Unreported by Accident Year

Accident Year	Reported Claim Counts 12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
1995	6	48	119	161	163	168	171	173	174	175	176	176	176	176	176
1996	4	32	72	96	101	105	107	109	109	110	111	111	111	111	
1997	6	48	114	158	166	177	183	184	186	187	188	189	189		
1998	5	57	119	170	181	195	200	202	203	204	205	205			
1999	6	60	122	164	178	191	195	197	197	199	200				
2000	5	59	149	222	237	253	258	261	262	263					
2001	5	60	140	207	225	241	247	250	251						
2002	9	72	139	195	209	223	228	231							
2003	10	81	158	213	236	255	261								
2004	7	64	133	198	219	234									
2005	10	70	141	194	212										
2006	6	67	136	195											
2007	12	82	152												
2008	10	71													
2009	14														

Accident Year	Reported Claim Counts Development Factors 12 - 24	24 - 36	36 - 48	48 - 60	60 - 72	72 - 84	84 - 96	96 - 108	108 - 120	120 - 132	132 - 144	144 - 156	156 - 168	168 - 180	180 - ult
1995	8.000	2.479	1.353	1.012	1.031	1.018	1.012	1.006	1.006	1.006	1.000	1.000	1.000	1.000	
1996	8.000	2.250	1.333	1.052	1.040	1.019	1.019	1.000	1.009	1.009	1.000	1.000	1.000		
1997	8.000	2.375	1.386	1.051	1.066	1.034	1.005	1.011	1.005	1.005	1.005	1.000			
1998	11.400	2.088	1.429	1.065	1.077	1.026	1.010	1.005	1.005	1.005	1.000				
1999	10.000	2.033	1.344	1.085	1.073	1.021	1.010	1.000	1.010	1.005					
2000	11.800	2.525	1.490	1.068	1.068	1.020	1.012	1.004	1.004						
2001	12.000	2.333	1.479	1.087	1.071	1.025	1.012	1.004							
2002	8.000	1.931	1.403	1.072	1.067	1.022	1.013								
2003	8.100	1.951	1.348	1.108	1.081	1.024									
2004	9.143	2.078	1.489	1.106	1.068										
2005	7.000	2.014	1.376	1.093											
2006	11.167	2.030	1.434												
2007	6.833	1.854													
2008	7.100														
Average	9.039	2.149	1.405	1.073	1.064	1.023	1.012	1.004	1.007	1.006	1.001	1.000	1.000	1.000	
Wtd Average	8.624	2.118	1.409	1.075	1.066	1.023	1.011	1.004	1.006	1.006	1.001	1.000	1.000	1.000	
Avg L5	8.249	1.985	1.410	1.093	1.071	1.022	1.011	1.005	1.007	1.006					
Avg L3	8.367	1.966	1.433	1.102	1.072	1.024	1.012	1.003	1.006	1.005	1.002	1.000			
Avg L5 x H/L	7.748	1.998	1.404	1.095	1.069	1.022	1.011	1.004	1.006	1.005					
															Tail
Select	8.249	1.985	1.410	1.093	1.071	1.022	1.011	1.005	1.007	1.006	1.002	1.001	1.000	1.000	1.000
Cumulative	28.516	3.457	1.742	1.235	1.130	1.055	1.032	1.021	1.016	1.009	1.003	1.001	1.000	1.000	1.000

Implicit Portion of Claims Unreported at Given Month of Development

12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
96.5%	71.1%	42.6%	19.0%	11.5%	5.2%	3.1%	2.1%	1.6%	0.9%	0.3%	0.1%	0.0%	0.0%	0.0%

Analysis of Unreported Tail Claims Loss and LAE Reserve And DDR Reserve, as of December 31, 2009

Indicated Portion Unreported by Accident Year -- Adjusted for Trend in Payments

(1) **Implicit Portion of Claims Unreported at Given Month of Development**[1]

12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
96.5%	71.1%	42.6%	19.0%	11.5%	5.2%	3.1%	2.1%	1.6%	0.9%	0.3%	0.1%	0.0%	0.0%	0.0%

(2) **Incremental Portion of Claims Reported Between Given Months of Development**[2]

0 - 12	12 - 24	24 - 36	36 - 48	48 - 60	60 - 72	72 - 84	84 - 96	96 - 108	108 - 120	120 - 132	132 - 144	144 - 156	156 - 168	168 - 180	Total
3.5%	25.4%	28.5%	23.5%	7.5%	6.3%	2.1%	1.1%	0.5%	0.7%	0.6%	0.2%	0.1%	0.0%	0.0%	100.0%

(3) **Trend Factor at 5.0% per annum Relative to 0 - 12 Reporting Period**

0 - 12	12 - 24	24 - 36	36 - 48	48 - 60	60 - 72	72 - 84	84 - 96	96 - 108	108 - 120	120 - 132	132 - 144	144 - 156	156 - 168	168 - 180
1.000	1.050	1.103	1.158	1.216	1.276	1.340	1.407	1.477	1.551	1.629	1.710	1.796	1.886	1.980

(4) = (2) x (3) **Trended Incremental Portion of Claims Reported Between Given Months of Development**

0 - 12	12 - 24	24 - 36	36 - 48	48 - 60	60 - 72	72 - 84	84 - 96	96 - 108	108 - 120	120 - 132	132 - 144	144 - 156	156 - 168	168 - 180	Total
3.5%	26.7%	31.4%	27.3%	9.2%	8.0%	2.8%	1.5%	0.7%	1.1%	1.0%	0.3%	0.2%	0.0%	0.0%	113.6%

(5) = (4) / (4) Total **Normalized and Trended Incremental Portion of Claims Reported Between Given Months of Development**

0 - 12	12 - 24	24 - 36	36 - 48	48 - 60	60 - 72	72 - 84	84 - 96	96 - 108	108 - 120	120 - 132	132 - 144	144 - 156	156 - 168	168 - 180	Total
3.1%	23.5%	27.7%	24.0%	8.1%	7.1%	2.5%	1.3%	0.6%	0.9%	0.9%	0.3%	0.2%	0.0%	0.0%	100.0%

(6) **Normalized and Trended Portion of Claims Unreported at Given Month of Development (i.e., Portion of Loss and LAE on Unreported Claims)**[3]

12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
96.9%	73.4%	45.8%	21.8%	13.7%	6.7%	4.2%	2.9%	2.3%	1.3%	0.5%	0.2%	0.0%	0.0%	0.0%

[1] From Exhibit 4.

[2] Incremental differences of the portion of claims unreported at the given evaluations.

[3] 100% less the cumulative sum of the portion reported in each time interval preceding the given evaluation.

Analysis of Unreported Tail Claims Loss and LAE Reserve And
DDR Reserve, as of December 31, 2009

Indicated Portion of Claims by Accident Year Yet to be Reported on Claims-Made Policies

Accident Year	Selected Percentage of Loss and LAE on Claims Unreported @ 12/31/09	Percentage of Loss and LAE on Unreported Claims to be Reported in the Twelve Months Preceding [1]													Weighted Average Portion of Loss and LAE on Unreported Claims to be Reported on DDR Policies[2]
		12/31/10	12/31/11	12/31/12	12/31/13	12/31/14	12/31/15	12/31/16	12/31/17	12/31/18	12/31/19	12/31/20	12/31/21	12/31/22	
1997	0.0%	100.0%													1.8%
1998	0.2%	100.0%	0.0%												1.8%
1999	0.5%	65.5%	34.5%	0.0%											2.9%
2000	1.3%	65.0%	22.9%	12.0%	0.0%										3.3%
2001	2.3%	41.8%	37.9%	13.3%	7.0%	0.0%									4.6%
2002	2.9%	22.0%	32.6%	29.5%	10.4%	5.5%	0.0%								6.6%
2003	4.2%	31.4%	15.1%	22.4%	20.3%	7.1%	3.8%	0.0%							7.2%
2004	6.7%	36.9%	19.8%	9.6%	14.1%	12.8%	4.5%	2.4%	0.0%						7.2%
2005	13.7%	51.4%	17.9%	9.6%	4.6%	6.9%	6.2%	2.2%	1.2%	0.0%					6.0%
2006	21.8%	37.0%	32.4%	11.3%	6.1%	2.9%	4.3%	3.9%	1.4%	0.7%	0.0%				6.4%
2007	45.8%	52.4%	17.6%	15.4%	5.4%	2.9%	1.4%	2.1%	1.9%	0.7%	0.3%	0.0%			5.5%
2008	73.4%	37.7%	32.7%	11.0%	9.6%	3.3%	1.8%	0.9%	1.3%	1.2%	0.4%	0.2%	0.0%		6.1%
2009	96.9%	24.2%	28.5%	24.8%	8.3%	7.3%	2.5%	1.4%	0.7%	1.0%	0.9%	0.3%	0.2%	0.0%	7.5%
Cumulative Retention[3]		95.5%	91.2%	87.1%	83.2%	79.4%	75.9%	72.4%	69.2%	66.1%	63.1%	60.3%	57.5%	55.0%	
Incremental DDR Portion[4]		3.5%	3.3%	3.2%	3.0%	2.9%	2.8%	2.7%	2.5%	2.4%	2.3%	2.2%	2.1%	2.0%	
Cumulative DDR Portion[5]		1.8%	5.2%	8.4%	11.6%	14.5%	17.4%	20.1%	22.7%	25.2%	27.5%	29.8%	32.0%	34.0%	

[1] Percentage of ultimate loss and LAE expected to be reported in the given interval divided by the percentage unreported as of 12/31/2009.
[2] Weighted average portion of exposures to have experienced DDR ("Cumulative DDR Portion"), where the weights are proportional to the percentage of loss and LAE to be reported in the corresponding interval.
[3] Annual retention (from Exhibit 7) compounded over time; retention within first calendar period is adjusted to be the average of the annual retention and 100%.
[4] Selected per annum DDR rate (from Exhibit 7) times portion remaining in-force ("cumulative retention") from prior column.
[5] Cumulation of incremental DDR portions; adjusted to reflect the average portion expected to have experienced DDR during the calendar year of the given column.

Analysis of Unreported Tail Claims Loss and LAE Reserve And DDR Reserve, as of December 31, 2009

Selected Per Annum Retention

(1) Policy Year	(2) Number of Base Class Insureds	(3) Number of Base Class Insureds To Renew	(4) = (3) / (2) Indicated Retention	(5) Number of Base Class Insureds to DDR	(6) = (5) / (2) Indicated Portion to DDR
1996	1,359	1,182	87.0%	58	4.3%
1997	1,393	1,227	88.1%	53	3.8%
1998	1,501	1,339	89.2%	62	4.1%
1999	1,676	1,455	86.8%	60	3.6%
2000	1,801	1,516	84.2%	56	3.1%
2001	1,799	1,556	86.5%	62	3.4%
2002	2,150	1,903	88.5%	81	3.8%
2003	2,631	2,344	89.1%	86	3.3%
2004	2,790	2,558	91.7%	115	4.1%
2005	2,672	2,413	90.3%	96	3.6%
2006	2,457	2,290	93.2%	97	3.9%
2007	2,544	2,320	91.2%	87	3.4%
2008	2,674	2,423	90.6%	83	3.1%
Total	27,446	24,525	89.4%	994	3.6%
1998 - 2006	19,476	17,374	89.2%	714	3.7%
2004 - 2008	13,137	12,004	91.4%	478	3.6%
Select			91.0%		3.5%

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Summary of Indicated DDR Reserve

	(1)	(2)	(3) = (1) - (2)
Discount rate	Present Value of Benefits[1]	Present Value of Future DDR Premiums[2]	Indicated Reserve
Undiscounted	$33,074,314	$10,488,515	$22,585,799
3.0%	$18,471,516	$7,692,363	$10,779,153
4.0%	$15,568,357	$7,039,737	$8,528,620
5.0%	$13,253,623	$6,482,270	$6,771,353
6.0%	$11,386,908	$6,002,160	$5,384,748

[1] From Exhibit A16.
[2] From Exhibit A17.

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Selected Per Annum Rates of DDR by Age

	Selected DDR Rates			
Age	Death	Disability	Retirement	Total
22				
23				
24				
25	0.118%	0.090%	0.000%	0.208%
26	0.118%	0.100%	0.000%	0.218%
27	0.118%	0.100%	0.000%	0.218%
28	0.118%	0.110%	0.000%	0.228%
29	0.118%	0.120%	0.000%	0.238%
30	0.130%	0.130%	0.000%	0.260%
31	0.130%	0.140%	0.000%	0.270%
32	0.130%	0.150%	0.000%	0.280%
33	0.130%	0.160%	0.000%	0.290%
34	0.130%	0.170%	0.000%	0.300%
35	0.169%	0.180%	0.000%	0.349%
36	0.169%	0.190%	0.000%	0.359%
37	0.169%	0.200%	0.000%	0.369%
38	0.169%	0.210%	0.000%	0.379%
39	0.169%	0.230%	0.000%	0.399%
40	0.263%	0.240%	0.000%	0.503%
41	0.263%	0.260%	0.000%	0.523%
42	0.263%	0.280%	0.000%	0.543%
43	0.263%	0.300%	0.000%	0.563%
44	0.263%	0.320%	0.000%	0.583%
45	0.399%	0.350%	0.000%	0.749%
46	0.399%	0.380%	0.000%	0.779%
47	0.399%	0.410%	0.000%	0.809%
48	0.399%	0.450%	0.000%	0.849%
49	0.399%	0.490%	0.000%	0.889%
50	0.595%	0.530%	0.000%	1.125%
51	0.595%	0.580%	0.000%	1.175%
52	0.595%	0.640%	0.000%	1.235%
53	0.595%	0.690%	0.000%	1.285%
54	0.595%	0.750%	0.000%	1.345%
55	0.833%	0.820%	4.000%	5.653%
56	0.833%	0.890%	4.000%	5.723%
57	0.833%	0.960%	4.000%	5.793%
58	0.833%	1.040%	4.000%	5.873%
59	0.833%	1.120%	4.000%	5.953%
60	1.279%	1.210%	4.000%	6.489%
61	1.279%	1.300%	5.000%	7.579%
62	1.279%	1.400%	5.000%	7.679%
63	1.279%	1.490%	5.000%	7.769%
64	1.279%	1.590%	5.000%	7.869%
65	1.904%	1.690%	6.000%	9.594%
66	1.904%	0.000%	6.500%	8.404%
67	1.904%	0.000%	6.500%	8.404%
68	1.904%	0.000%	6.500%	8.404%
69	1.904%	0.000%	6.500%	8.404%
70	2.991%	0.000%	6.500%	9.491%
71	2.991%	0.000%	6.500%	9.491%
72	2.991%	0.000%	6.500%	9.491%
73	2.991%	0.000%	6.500%	9.491%
74	2.991%	0.000%	6.500%	9.491%
75	4.694%	0.000%	6.500%	11.194%
76	4.694%	0.000%	6.500%	11.194%
77	4.694%	0.000%	6.500%	11.194%
78	4.694%	0.000%	6.500%	11.194%
79	4.694%	0.000%	6.500%	11.194%
80	7.566%	0.000%	92.434%	100.000%
81	7.566%	0.000%	92.434%	100.000%
82	8.932%	0.000%	91.068%	100.000%
83	9.753%	0.000%	90.248%	100.000%

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Selected Per Annum Retention

Policy Year	Number of Base Class Insureds	Number of Base Class Insureds To Renew	Indicated Retention
1996	1,359	1,182	87.0%
1997	1,393	1,227	88.1%
1998	1,501	1,339	89.2%
1999	1,676	1,455	86.8%
2000	1,801	1,516	84.2%
2001	1,799	1,556	86.5%
2002	2,150	1,903	88.5%
2003	2,631	2,344	89.1%
2004	2,790	2,558	91.7%
2005	2,672	2,413	90.3%
2006	2,457	2,290	93.2%
2007	2,544	2,320	91.2%
2008	2,674	2,423	90.6%
Total	27,446	24,525	89.4%
1998 - 2006	19,476	17,374	89.2%
2004 - 2008	13,137	12,004	91.4%
Select			91.0%

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Selected Retention Pattern by Age

Age Group	Number of Base Class Insureds	Number of Base Class Insureds To Renew	Indicated Retention	Selected Retention[1]
<35	2,295	1,944	84.7%	86.0%
35 - 39	4,013	3,539	88.2%	90.0%
40 - 44	4,610	4,159	90.2%	92.0%
45 - 49	4,941	4,535	91.8%	93.0%
50 - 54	3,916	3,501	89.4%	91.0%
55 - 59	3,171	2,857	90.1%	92.0%
60 - 64	2,329	2,107	90.5%	92.0%
65 - 69	1,196	1,066	89.2%	91.0%
70 - 74	530	456	85.9%	87.0%
75 - 79	323	266	82.4%	84.0%
80 +	123	105	84.9%	0.0%
Total	27,446	24,535	89.4%	

[1] Selection normalized to balance to overall selected retention of 91%

Exhibit A5

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Persistency Schedule by Calendar Year

			Number of Equivalent Physicians Remaining as of December 31, xxxx														
Age	Selected Retention Factors[1]	Claims-Made In-Force BCE[2] Physicians As Of 12/31/09	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024
25	86.0%	0.3	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0
26	86.0%	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
27	86.0%	0.4	0.4	0.3	0.3	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
28	86.0%	1.0	0.8	0.7	0.6	0.5	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.2
29	86.0%	1.1	1.0	0.8	0.7	0.6	0.5	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.2	0.2
30	86.0%	3.7	3.2	2.7	2.4	2.0	1.7	1.6	1.4	1.3	1.1	1.0	0.9	0.9	0.8	0.7	0.7
31	86.0%	8.0	6.9	5.9	5.1	4.4	4.0	3.6	3.2	2.9	2.6	2.4	2.2	2.0	1.9	1.7	1.6
32	86.0%	12.7	10.9	9.4	8.0	7.2	6.5	5.9	5.3	4.8	4.4	4.0	3.7	3.4	3.1	2.9	2.7
33	86.0%	15.9	13.6	11.7	10.6	9.5	8.6	7.7	6.9	6.4	5.9	5.4	5.0	4.6	4.2	4.0	3.7
34	86.0%	30.9	26.6	23.9	21.6	19.4	17.5	15.7	14.5	13.3	12.2	11.3	10.4	9.6	9.0	8.3	7.7
35	90.0%	32.4	29.1	26.2	23.6	21.2	19.1	17.6	16.2	14.9	13.7	12.6	11.7	10.9	10.1	9.4	8.8
36	90.0%	60.7	54.6	49.1	44.2	39.8	36.6	33.7	31.0	28.5	26.2	24.4	22.7	21.1	19.6	18.3	16.6
37	90.0%	47.5	42.8	38.5	34.6	31.9	29.3	27.0	24.8	22.8	21.2	19.7	18.4	17.1	15.9	14.4	13.1
38	90.0%	75.6	68.0	61.2	56.3	51.8	47.7	43.9	40.4	37.5	34.9	32.5	30.2	28.1	25.6	23.3	21.2
39	90.0%	77.1	69.4	63.9	58.8	54.1	49.7	45.8	42.6	39.6	36.8	34.2	31.8	29.0	26.4	24.0	21.8
40	92.0%	80.3	73.9	68.0	62.6	57.6	52.9	49.2	45.8	42.6	39.6	36.8	33.5	30.5	27.8	25.3	23.0
41	92.0%	79.4	73.0	67.2	61.8	56.8	52.9	49.2	45.7	42.5	39.5	36.0	32.8	29.8	27.1	24.7	22.7
42	92.0%	98.4	90.5	83.3	76.6	71.3	66.3	61.6	57.3	53.3	48.5	44.1	40.2	36.6	33.3	30.6	28.2
43	92.0%	88.9	81.8	75.2	70.0	65.1	60.5	56.3	52.3	47.6	43.4	39.4	35.9	32.7	30.1	27.6	25.4
44	92.0%	67.9	62.5	58.1	54.1	50.3	46.7	43.5	39.6	36.0	32.8	29.8	27.1	25.0	23.0	21.1	19.4
45	93.0%	84.9	79.0	73.5	68.3	63.5	59.1	53.8	48.9	44.5	40.5	36.9	33.9	31.2	28.7	26.4	24.3
46	93.0%	75.9	70.6	65.6	61.0	56.8	51.7	47.0	42.8	38.9	35.4	32.6	30.0	27.6	25.4	23.3	21.5
47	93.0%	69.8	64.9	60.4	56.2	51.1	46.5	42.3	38.5	35.0	32.2	29.7	27.3	25.1	23.1	21.2	19.5
48	93.0%	76.8	71.4	66.4	60.5	55.0	50.1	45.6	41.5	38.1	35.1	32.3	29.7	27.3	25.1	23.1	21.3
49	93.0%	69.1	64.2	58.4	53.2	48.4	44.0	40.1	36.9	33.9	31.2	28.7	26.4	24.3	22.4	20.6	18.9
50	91.0%	64.0	58.2	53.0	48.2	43.9	39.9	36.7	33.8	31.1	28.6	26.3	24.2	22.3	20.5	18.9	17.3
51	91.0%	65.1	59.3	53.9	49.1	44.7	41.1	37.8	34.8	32.0	29.4	27.1	24.9	22.9	21.1	19.4	17.7
52	91.0%	86.0	78.3	71.2	64.8	59.6	54.9	50.5	46.4	42.7	39.3	36.2	33.3	30.6	28.2	25.6	23.3
53	91.0%	118.1	107.4	97.8	89.9	82.8	76.1	70.0	64.4	59.3	54.5	50.2	46.2	42.5	38.6	35.2	32.0
54	91.0%	54.4	49.5	45.6	41.9	38.6	35.5	32.6	30.0	27.6	25.4	23.4	21.5	19.6	17.8	16.2	14.8
55	92.0%	59.6	54.8	50.4	46.4	42.7	39.2	36.1	33.2	30.6	28.1	25.9	23.5	21.4	19.5	17.7	16.1
56	92.0%	97.3	89.5	82.4	75.8	69.7	64.1	59.0	54.3	49.9	46.0	41.8	38.1	34.6	31.5	28.7	24.9
57	92.0%	80.9	74.4	68.5	63.0	58.0	53.3	49.1	45.1	41.5	37.8	34.4	31.3	28.5	25.9	22.5	19.6
58	92.0%	57.0	52.5	48.3	44.4	40.9	37.6	34.6	31.8	29.0	26.4	24.0	21.8	19.9	17.3	15.0	13.1
59	92.0%	80.7	74.2	68.3	62.8	57.8	53.2	48.9	44.5	40.5	36.9	33.6	30.5	26.6	23.1	20.1	17.5
60	92.0%	70.4	64.8	59.6	54.8	50.4	46.4	42.2	38.4	35.0	31.8	29.0	25.2	21.9	19.1	16.6	14.4
61	92.0%	65.6	60.3	55.5	51.1	47.0	42.8	38.9	35.4	32.2	29.3	25.5	22.2	19.3	16.8	14.6	12.3
62	92.0%	82.7	76.1	70.0	64.4	58.6	53.3	48.5	44.1	40.2	35.0	30.4	26.5	23.0	20.0	16.8	14.1
63	92.0%	69.4	63.9	58.8	53.5	48.7	44.3	40.3	36.7	31.9	27.8	24.2	21.0	18.3	15.4	12.9	10.8
64	92.0%	57.5	52.9	48.2	43.8	39.9	36.3	33.0	28.7	25.0	21.7	18.9	16.5	13.8	11.6	9.8	8.2
65	91.0%	45.5	41.4	37.7	34.3	31.2	28.4	24.7	21.5	18.7	16.3	14.1	11.9	10.0	8.4	7.0	5.9
66	91.0%	36.8	33.5	30.5	27.8	25.3	22.0	19.1	16.6	14.5	12.6	10.6	8.9	7.5	6.3	5.3	0.0
67	91.0%	26.6	24.2	22.0	20.0	17.4	15.2	13.2	11.5	10.0	8.4	7.0	5.9	5.0	4.2	0.0	0.0
68	91.0%	38.7	35.2	32.1	27.9	24.3	21.1	18.4	16.0	13.4	11.3	9.5	8.0	6.7	0.0	0.0	0.0
69	91.0%	36.7	33.4	29.1	25.3	22.0	19.1	16.6	14.0	11.7	9.9	8.3	7.0	0.0	0.0	0.0	0.0
70	87.0%	37.7	32.8	28.5	24.8	21.6	18.8	15.8	13.2	11.1	9.3	7.8	0.0	0.0	0.0	0.0	
71	87.0%	11.9	10.3	9.0	7.8	6.8	5.7	4.8	4.0	3.4	2.8	0.0	0.0	0.0	0.0		
72	87.0%	17.6	15.3	13.3	11.6	9.8	8.2	6.9	5.8	4.9	0.0	0.0	0.0	0.0			
73	87.0%	27.3	23.8	20.7	17.4	14.6	12.3	10.3	8.6	0.0	0.0	0.0	0.0				
74	87.0%	13.5	11.7	9.9	8.3	7.0	5.8	4.9	0.0	0.0	0.0	0.0					
75	84.0%	7.8	6.6	5.5	4.6	3.9	3.3	0.0	0.0	0.0	0.0						
76	84.0%	8.1	6.8	5.7	4.8	4.0	0.0	0.0	0.0	0.0							
77	84.0%	10.4	8.8	7.4	6.2	0.0	0.0	0.0	0.0								
78	84.0%	6.9	5.8	4.9	0.0	0.0	0.0	0.0									
79	84.0%	17.3	14.5	0.0	0.0	0.0	0.0										
80	0.0%	13.4	0.0	0.0	0.0	0.0											
81	0.0%	6.2	0.0	0.0	0.0												
82	0.0%	1.5	0.0	0.0													
83	0.0%	17.5	0.0														
Total		2,649.0	2,379.9	2,158.5	1,965.9	1,789.6	1,630.7	1,485.1	1,349.6	1,221.7	1,106.7	1,002.7	902.6	811.5	728.1	653.9	584.8

[1] From Exhibit A4.
[2] Base Class, Mature Claims-Made, Full-Time Equivalent Physicians

Exhibit A6

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Persistency Schedule by Calendar Year

Age	Selected Retention Factors[1]	Claims-Made In-Force BCE[2] Physicians As Of 12/31/09	Number of Equivalent Physicians Remaining as of December 31, xxxx 2025	2026	2027	2028	2029	2030	2031	2032	2033	2034	2035	2036	2037	2038	2039
25	86.0%	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	86.0%	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
27	86.0%	0.4	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
28	86.0%	1.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0
29	86.0%	1.1	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
30	86.0%	3.7	0.6	0.6	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.2	0.2
31	86.0%	8.0	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.7	0.6	0.6	0.5	0.5	0.4
32	86.0%	12.7	2.5	2.3	2.2	2.0	1.8	1.6	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.8
33	86.0%	15.9	3.4	3.2	2.9	2.6	2.4	2.2	2.0	1.8	1.7	1.5	1.4	1.3	1.2	1.1	1.0
34	86.0%	30.9	7.2	6.6	6.0	5.4	4.9	4.5	4.1	3.8	3.5	3.2	3.0	2.7	2.5	2.3	2.1
35	90.0%	32.4	8.0	7.3	6.6	6.0	5.5	5.0	4.6	4.3	3.9	3.6	3.3	3.1	2.8	2.6	2.4
36	90.0%	60.7	15.1	13.8	12.5	11.4	10.5	9.6	8.9	8.2	7.5	6.9	6.4	5.8	5.4	4.9	4.5
37	90.0%	47.5	12.0	10.9	9.9	9.1	8.4	7.7	7.1	6.5	6.0	5.5	5.1	4.7	4.3	3.9	3.6
38	90.0%	75.6	19.3	17.5	16.1	14.8	13.6	12.6	11.5	10.6	9.8	9.0	8.3	7.6	6.9	6.3	5.7
39	90.0%	77.1	19.9	18.3	16.8	15.5	14.2	13.1	12.0	11.1	10.2	9.4	8.6	7.9	7.1	6.5	5.9
40	92.0%	80.3	21.1	19.5	17.9	16.5	15.2	13.9	12.8	11.8	10.9	10.0	9.1	8.3	7.5	6.8	6.2
41	92.0%	79.4	20.9	19.2	17.7	16.3	15.0	13.8	12.7	11.7	10.7	9.8	8.9	8.1	7.4	6.7	5.8
42	92.0%	98.4	25.9	23.8	21.9	20.2	18.6	17.1	15.7	14.5	13.1	12.0	10.9	9.9	9.0	7.8	6.8
43	92.0%	88.9	23.4	21.5	19.8	18.2	16.8	15.4	14.2	12.9	11.8	10.7	9.7	8.9	7.7	6.7	5.8
44	92.0%	67.9	17.9	16.5	15.1	13.9	12.8	11.8	10.7	9.8	8.9	8.1	7.4	6.4	5.6	4.8	4.2
45	93.0%	84.9	22.4	20.6	18.9	17.4	16.0	14.6	13.3	12.1	11.0	10.0	8.7	7.6	6.6	5.7	5.0
46	93.0%	75.9	19.8	18.2	16.7	15.4	14.0	12.7	11.6	10.6	9.6	8.4	7.3	6.3	5.5	4.8	4.0
47	93.0%	69.8	18.0	16.5	15.2	13.9	12.6	11.5	10.4	9.5	8.3	7.2	6.3	5.4	4.7	4.0	3.3
48	93.0%	76.8	19.6	18.0	16.4	14.9	13.6	12.4	11.2	9.8	8.5	7.4	6.4	5.6	4.7	4.0	3.3
49	93.0%	69.1	17.4	15.8	14.4	13.1	11.9	10.9	9.5	8.2	7.2	6.2	5.4	4.5	3.8	3.2	2.7
50	91.0%	64.0	15.8	14.4	13.1	11.9	10.8	9.4	8.2	7.1	6.2	5.4	4.5	3.8	3.2	2.7	2.3
51	91.0%	65.1	16.1	14.6	13.3	12.1	10.5	9.2	8.0	6.9	6.0	5.1	4.3	3.6	3.0	2.5	0.0
52	91.0%	86.0	21.2	19.3	17.6	15.3	13.3	11.6	10.1	8.8	7.4	6.2	5.2	4.4	3.7	0.0	0.0
53	91.0%	118.1	29.1	26.5	23.1	20.1	17.5	15.2	13.2	11.1	9.3	7.8	6.6	5.5	0.0	0.0	0.0
54	91.0%	54.4	13.4	11.7	10.2	8.8	7.7	6.7	5.6	4.7	4.0	3.3	2.8	0.0	0.0	0.0	0.0
55	92.0%	59.6	14.0	12.2	10.6	9.2	8.0	6.8	5.7	4.8	4.0	3.4	0.0	0.0	0.0	0.0	
56	92.0%	97.3	21.7	18.9	16.4	14.3	12.0	10.1	8.5	7.1	6.0	0.0	0.0	0.0	0.0		
57	92.0%	80.9	17.1	14.8	12.9	10.8	9.1	7.7	6.4	5.4	0.0	0.0	0.0	0.0			
58	92.0%	57.0	11.4	9.9	8.3	7.0	5.9	4.9	4.1	0.0	0.0	0.0	0.0				
59	92.0%	80.7	15.2	12.8	10.7	9.0	7.6	6.4	0.0	0.0	0.0	0.0					
60	92.0%	70.4	12.1	10.2	8.6	7.2	6.0	0.0	0.0	0.0	0.0						
61	92.0%	65.6	10.3	8.7	7.3	6.1	0.0	0.0	0.0	0.0							
62	92.0%	82.7	11.9	10.0	8.4	0.0	0.0	0.0	0.0								
63	92.0%	69.4	9.1	7.6	0.0	0.0	0.0	0.0									
64	92.0%	57.5	6.9	0.0	0.0	0.0	0.0										
65	91.0%	45.5	0.0	0.0	0.0	0.0											
66	91.0%	36.8	0.0	0.0	0.0												
67	91.0%	26.6	0.0	0.0													
68	91.0%	38.7	0.0														
69	91.0%	36.7															
70	87.0%	37.7															
71	87.0%	11.9															
72	87.0%	17.6															
73	87.0%	27.3															
74	87.0%	13.5															
75	84.0%	7.8															
76	84.0%	8.1															
77	84.0%	10.4															
78	84.0%	6.9															
79	84.0%	17.3															
80	0.0%	13.4															
81	0.0%	6.2															
82	0.0%	1.5															
83	0.0%	17.5															
Total		2,649.0	521.5	463.3	409.7	360.5	318.0	279.9	245.2	215.7	187.9	162.3	141.6	123.3	104.5	89.1	76.3

[1] From Exhibit A4.

[2] Base Class, Mature Claims-Made, Full-Time Equivalent Physicians

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Persistency Schedule by Calendar Year

Age	Selected Retention Factors[1]	Claims-Made In-Force BCE[2] Physicians As Of 12/31/09	Number of Equivalent Physicians Remaining as of December 31, xxxx													
			2040	2041	2042	2043	2044	2045	2046	2047	2048	2049	2050	2051	2052	2053
25	86.0%	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	86.0%	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
27	86.0%	0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
28	86.0%	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
29	86.0%	1.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
30	86.0%	3.7	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0
31	86.0%	8.0	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1
32	86.0%	12.7	0.7	0.6	0.6	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.2	0.2
33	86.0%	15.9	0.9	0.9	0.8	0.7	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.3	0.2	0.2
34	86.0%	30.9	2.0	1.8	1.6	1.5	1.3	1.2	1.1	0.9	0.8	0.7	0.6	0.5	0.4	0.4
35	90.0%	32.4	2.2	2.0	1.8	1.6	1.5	1.3	1.1	1.0	0.8	0.7	0.6	0.5	0.4	0.4
36	90.0%	60.7	4.1	3.7	3.4	3.1	2.7	2.3	2.0	1.8	1.5	1.3	1.1	0.9	0.8	0.6
37	90.0%	47.5	3.2	3.0	2.7	2.3	2.0	1.8	1.5	1.3	1.1	0.9	0.8	0.7	0.6	0.0
38	90.0%	75.6	5.2	4.7	4.1	3.6	3.1	2.7	2.4	2.0	1.7	1.4	1.2	1.0	0.0	0.0
39	90.0%	77.1	5.4	4.7	4.1	3.5	3.1	2.7	2.3	1.9	1.6	1.3	1.1	0.0	0.0	0.0
40	92.0%	80.3	5.4	4.7	4.1	3.6	3.1	2.6	2.2	1.8	1.5	1.3	0.0	0.0	0.0	0.0
41	92.0%	79.4	5.1	4.4	3.8	3.3	2.8	2.4	2.0	1.7	1.4	0.0	0.0	0.0	0.0	
42	92.0%	98.4	5.9	5.2	4.5	3.8	3.2	2.7	2.2	1.9	0.0	0.0	0.0	0.0		
43	92.0%	88.9	5.1	4.4	3.7	3.1	2.6	2.2	1.8	0.0	0.0	0.0	0.0			
44	92.0%	67.9	3.7	3.1	2.6	2.2	1.8	1.5	0.0	0.0	0.0	0.0				
45	93.0%	84.9	4.2	3.5	3.0	2.5	2.1	0.0	0.0	0.0	0.0					
46	93.0%	75.9	3.4	2.8	2.4	2.0	0.0	0.0	0.0	0.0						
47	93.0%	69.8	2.8	2.4	2.0	0.0	0.0	0.0	0.0							
48	93.0%	76.8	2.8	2.3	0.0	0.0	0.0	0.0								
49	93.0%	69.1	2.3	0.0	0.0	0.0	0.0									
50	91.0%	64.0	0.0	0.0	0.0	0.0										
51	91.0%	65.1	0.0	0.0	0.0											
52	91.0%	86.0	0.0	0.0												
53	91.0%	118.1	0.0													
54	91.0%	54.4														
55	92.0%	59.6														
56	92.0%	97.3														
57	92.0%	80.9														
58	92.0%	57.0														
59	92.0%	80.7														
60	92.0%	70.4														
61	92.0%	65.6														
62	92.0%	82.7														
63	92.0%	69.4														
64	92.0%	57.5														
65	91.0%	45.5														
66	91.0%	36.8														
67	91.0%	26.6														
68	91.0%	38.7														
69	91.0%	36.7														
70	87.0%	37.7														
71	87.0%	11.9														
72	87.0%	17.6														
73	87.0%	27.3														
74	87.0%	13.5														
75	84.0%	7.8														
76	84.0%	8.1														
77	84.0%	10.4														
78	84.0%	6.9														
79	84.0%	17.3														
80	0.0%	13.4														
81	0.0%	6.2														
82	0.0%	1.5														
83	0.0%	17.5														
Total		2,649.0	65.0	54.9	45.7	37.9	31.0	24.9	20.0	15.5	11.6	8.7	6.2	4.3	2.8	1.9

[1] From Exhibit A4.
[2] Base Class, Mature Claims-Made, Full-Time Equivalent Physicians

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Persistency Schedule by Calendar Year

Age	Selected Retention Factors[1]	Claims-Made In-Force BCE[2] Physicians As Of 12/31/09	Number of Equivalent Physicians Remaining as of December 31, xxxx													
			2054	2055	2056	2057	2058	2059	2060	2061	2062	2063	2064	2065	2066	2067
25	86.0%	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	86.0%	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
27	86.0%	0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
28	86.0%	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		
29	86.0%	1.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0			
30	86.0%	3.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0				
31	86.0%	8.0	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0					
32	86.0%	12.7	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0						
33	86.0%	15.9	0.2	0.1	0.1	0.0	0.0	0.0	0.0							
34	86.0%	30.9	0.3	0.3	0.0	0.0	0.0	0.0								
35	90.0%	32.4	0.3	0.0	0.0	0.0	0.0									
36	90.0%	60.7	0.0	0.0	0.0	0.0										
37	90.0%	47.5	0.0	0.0	0.0											
38	90.0%	75.6	0.0	0.0												
39	90.0%	77.1	0.0													
40	92.0%	80.3														
41	92.0%	79.4														
42	92.0%	98.4														
43	92.0%	88.9														
44	92.0%	67.9														
45	93.0%	84.9														
46	93.0%	75.9														
47	93.0%	69.8														
48	93.0%	76.8														
49	93.0%	69.1														
50	91.0%	64.0														
51	91.0%	65.1														
52	91.0%	86.0														
53	91.0%	118.1														
54	91.0%	54.4														
55	92.0%	59.6														
56	92.0%	97.3														
57	92.0%	80.9														
58	92.0%	57.0														
59	92.0%	80.7														
60	92.0%	70.4														
61	92.0%	65.6														
62	92.0%	82.7														
63	92.0%	69.4														
64	92.0%	57.5														
65	91.0%	45.5														
66	91.0%	36.8														
67	91.0%	26.6														
68	91.0%	38.7														
69	91.0%	36.7														
70	87.0%	37.7														
71	87.0%	11.9														
72	87.0%	17.6														
73	87.0%	27.3														
74	87.0%	13.5														
75	84.0%	7.8														
76	84.0%	8.1														
77	84.0%	10.4														
78	84.0%	6.9														
79	84.0%	17.3														
80	0.0%	13.4														
81	0.0%	6.2														
82	0.0%	1.5														
83	0.0%	17.5														
Total		2,649.0	1.1	0.6	0.3	0.2	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

[1] From Exhibit A4.

[2] Base Class, Mature Claims-Made, Full-Time Equivalent Physicians

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

DDR Lapse Schedule by Calendar Year

	Percent of In-force:[1]			Number of Equivalent Physicians to DDR During Period[3]														
Age	Death	Disability	Retirement	1/10 - 12/10	1/11 - 12/11	1/12 - 12/12	1/13 - 12/13	1/14 - 12/14	1/15 - 12/15	1/16 - 12/16	1/17 - 12/17	1/18 - 12/18	1/19 - 12/19	1/20 - 12/20	1/21 - 12/21	1/22 - 12/22	1/23 - 12/23	1/24 - 12/24
25	0.118%	0.090%	0.000%	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
26	0.118%	0.100%	0.000%	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
27	0.118%	0.100%	0.000%	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
28	0.118%	0.110%	0.000%	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
29	0.118%	0.120%	0.000%	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
30	0.130%	0.130%	0.000%	0.010	0.009	0.008	0.007	0.006	0.006	0.006	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.004
31	0.130%	0.140%	0.000%	0.022	0.019	0.017	0.015	0.015	0.014	0.013	0.012	0.012	0.013	0.012	0.012	0.011	0.011	0.013
32	0.130%	0.150%	0.000%	0.035	0.032	0.028	0.028	0.026	0.024	0.022	0.021	0.024	0.023	0.022	0.021	0.020	0.023	0.023
33	0.130%	0.160%	0.000%	0.046	0.041	0.041	0.038	0.035	0.032	0.031	0.035	0.033	0.032	0.030	0.029	0.034	0.033	0.032
34	0.130%	0.170%	0.000%	0.093	0.093	0.086	0.080	0.074	0.070	0.079	0.076	0.072	0.069	0.066	0.078	0.075	0.072	0.071
35	0.169%	0.180%	0.000%	0.113	0.105	0.097	0.090	0.085	0.096	0.092	0.088	0.084	0.080	0.094	0.091	0.088	0.086	0.084
36	0.169%	0.190%	0.000%	0.218	0.202	0.186	0.177	0.200	0.192	0.183	0.175	0.166	0.196	0.190	0.184	0.179	0.174	0.205
37	0.169%	0.200%	0.000%	0.175	0.162	0.154	0.174	0.167	0.159	0.152	0.145	0.171	0.165	0.160	0.156	0.152	0.179	0.170
38	0.169%	0.210%	0.000%	0.287	0.272	0.308	0.295	0.282	0.269	0.256	0.302	0.292	0.282	0.276	0.268	0.316	0.300	0.287
39	0.169%	0.230%	0.000%	0.308	0.349	0.334	0.319	0.304	0.290	0.343	0.331	0.320	0.312	0.304	0.358	0.340	0.326	0.308
40	0.263%	0.240%	0.000%	0.404	0.387	0.369	0.352	0.336	0.397	0.384	0.370	0.362	0.352	0.414	0.394	0.377	0.357	0.340
41	0.263%	0.260%	0.000%	0.415	0.397	0.378	0.360	0.426	0.412	0.398	0.388	0.378	0.445	0.423	0.404	0.383	0.365	1.395
42	0.263%	0.280%	0.000%	0.535	0.510	0.486	0.574	0.555	0.536	0.523	0.510	0.600	0.570	0.545	0.516	0.492	1.880	1.751
43	0.263%	0.300%	0.000%	0.501	0.477	0.564	0.545	0.526	0.514	0.500	0.589	0.560	0.535	0.507	0.483	1.847	1.720	1.602
44	0.263%	0.320%	0.000%	0.396	0.468	0.453	0.437	0.427	0.416	0.489	0.465	0.445	0.421	0.401	1.534	1.428	1.330	1.241
45	0.399%	0.350%	0.000%	0.636	0.615	0.594	0.580	0.565	0.665	0.632	0.604	0.572	0.545	2.084	1.941	1.808	1.686	1.572
46	0.399%	0.380%	0.000%	0.591	0.571	0.557	0.543	0.639	0.607	0.581	0.550	0.524	2.002	1.865	1.737	1.620	1.511	1.515
47	0.399%	0.410%	0.000%	0.565	0.551	0.537	0.632	0.600	0.574	0.544	0.518	1.981	1.845	1.718	1.603	1.494	1.499	1.611
48	0.399%	0.450%	0.000%	0.652	0.635	0.747	0.710	0.679	0.643	0.613	2.344	2.183	2.033	1.896	1.768	1.773	1.906	1.776
49	0.399%	0.490%	0.000%	0.614	0.722	0.687	0.657	0.622	0.592	2.265	2.110	1.965	1.833	1.709	1.714	1.842	1.717	1.598
50	0.595%	0.530%	0.000%	0.720	0.684	0.654	0.620	0.590	2.257	2.103	1.958	1.826	1.703	1.708	1.835	1.711	1.592	1.484
51	0.595%	0.580%	0.000%	0.765	0.732	0.693	0.660	2.525	2.352	2.190	2.043	1.905	1.911	2.053	1.914	1.781	1.660	1.862
52	0.595%	0.640%	0.000%	1.062	1.006	0.958	3.376	3.414	3.179	2.965	2.765	2.773	2.980	2.778	2.585	2.409	2.702	2.154
53	0.595%	0.690%	0.000%	1.517	1.445	5.007	4.799	4.794	4.471	4.170	4.182	4.493	4.188	3.899	3.633	4.075	3.248	2.956
54	0.595%	0.750%	0.000%	0.732	2.344	2.447	2.353	2.265	2.112	2.118	2.276	2.122	1.975	1.840	2.064	1.645	1.497	1.362
55	0.833%	0.820%	4.000%	2.443	2.730	2.771	2.704	2.540	2.547	2.737	2.551	2.375	2.213	2.482	1.978	1.800	1.638	1.491
56	0.833%	0.890%	4.000%	3.913	4.444	4.614	4.481	4.525	4.862	4.532	4.218	3.931	4.409	3.514	3.198	2.910	2.648	2.722
57	0.833%	0.960%	4.000%	2.947	3.080	3.609	4.015	4.393	4.095	3.811	3.552	3.984	3.176	2.890	2.630	2.393	2.459	2.140
58	0.833%	1.040%	4.000%	2.753	2.774	3.050	3.313	3.138	2.921	2.722	3.053	2.434	2.215	2.015	1.834	1.885	1.640	1.427
59	0.833%	1.120%	4.000%	3.831	4.026	4.714	4.776	4.491	4.185	4.694	3.742	3.405	3.099	2.820	2.898	2.521	2.193	1.908
60	1.279%	1.210%	4.000%	3.736	4.289	4.432	4.235	3.970	4.452	3.549	3.230	2.939	2.675	2.749	2.391	2.080	1.810	1.575
61	1.279%	1.300%	5.000%	4.134	4.058	4.031	4.019	4.507	3.593	3.270	2.975	2.708	2.783	2.421	2.106	1.832	1.594	1.636
62	1.279%	1.400%	5.000%	5.274	5.120	5.317	6.093	4.924	4.480	4.077	3.710	3.813	3.317	2.886	2.511	2.184	2.241	1.883
63	1.279%	1.490%	5.000%	4.267	4.194	5.248	4.309	4.091	3.723	3.388	3.481	3.029	2.635	2.293	1.995	2.046	1.719	1.444
64	1.279%	1.590%	5.000%	3.591	4.468	3.853	3.554	3.352	3.050	3.135	2.727	2.373	2.064	1.796	1.843	1.548	1.300	1.092
65	1.904%	1.690%	6.000%	3.243	2.815	2.993	2.849	2.622	2.694	2.344	2.039	1.774	1.543	1.584	1.330	1.117	0.939	0.788
66	1.904%	0.000%	6.500%	2.001	2.400	2.257	2.302	2.398	2.087	1.815	1.579	1.374	1.410	1.184	0.995	0.836	0.702	5.267
67	1.904%	0.000%	6.500%	1.607	1.769	1.687	1.800	1.654	1.439	1.252	1.089	1.117	0.939	0.788	0.662	0.556	4.175	0.000
68	1.904%	0.000%	6.500%	2.883	2.730	2.950	2.637	2.303	2.004	1.743	1.789	1.503	1.262	1.060	0.891	6.684	0.000	0.000
69	1.904%	0.000%	6.500%	2.792	2.904	2.758	2.400	2.088	1.816	1.864	1.566	1.315	1.105	0.928	6.963	0.000	0.000	0.000
70	2.991%	0.000%	6.500%	2.248	2.483	2.239	2.200	2.048	2.101	1.765	1.482	1.245	1.046	7.850	0.000	0.000	0.000	
71	2.991%	0.000%	6.500%	1.033	0.898	0.852	0.741	0.761	0.639	0.537	0.451	0.379	2.842	0.000	0.000	0.000		
72	2.991%	0.000%	6.500%	1.100	1.213	1.267	1.300	1.092	0.917	0.770	0.647	4.855	0.000	0.000	0.000			
73	2.991%	0.000%	6.500%	1.766	1.652	2.276	1.944	1.633	1.372	1.152	8.647	0.000	0.000	0.000				
74	2.991%	0.000%	6.500%	1.192	1.314	1.104	0.927	0.779	0.654	4.910	0.000	0.000	0.000					
75	4.694%	0.000%	6.500%	0.762	0.708	0.618	0.519	0.436	3.275	0.000	0.000	0.000						
76	4.694%	0.000%	6.500%	0.679	0.664	0.636	0.535	4.011	0.000	0.000	0.000							
77	4.694%	0.000%	6.500%	1.007	0.899	0.755	6.191	0.000	0.000	0.000								
78	4.694%	0.000%	6.500%	0.665	0.559	4.875	0.000	0.000	0.000									
79	4.694%	0.000%	6.500%	1.857	13.605	0.000	0.000	0.000										
80	7.566%	0.000%	92.434%	10.770	0.000	0.000	0.000											
81	7.566%	0.000%	92.434%	3.616	0.000	0.000												
82	8.932%	0.000%	91.068%	1.507	0.000													
83	9.753%	0.000%	90.248%	13.710														
Total				102.7	88.6	85.3	86.3	81.9	77.8	75.7	75.4	68.4	63.2	64.3	59.6	56.3	50.9	48.8

[1] From Exhibit A2.

[2] Base Class, Mature Claims-Made, Full-Time Equivalent Physicians

[3] Lapses due to retirement are excluded if the provider has been continuously insured less than five years

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

DDR Lapse Schedule by Calendar Year

	Percent of In-force:[1]			Number of Equivalent Physicians to DDR During Period														
Age	Death	Disability	Retirement	1/25 - 12/25	1/26 - 12/26	1/27 - 12/27	1/28 - 12/28	1/29 - 12/29	1/30 - 12/30	1/31 - 12/31	1/32 - 12/32	1/33 - 12/33	1/34 - 12/34	1/35 - 12/35	1/36 - 12/36	1/37 - 12/37	1/38 - 12/38	1/39 - 12/39
25	0.118%	0.090%	0.000%	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
26	0.118%	0.100%	0.000%	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
27	0.118%	0.100%	0.000%	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001
28	0.118%	0.110%	0.000%	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.003	0.003	0.003
29	0.118%	0.120%	0.000%	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.004	0.004	0.004	0.003
30	0.130%	0.130%	0.000%	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.004	0.017	0.015	0.014	0.013	0.013
31	0.130%	0.140%	0.000%	0.012	0.012	0.012	0.011	0.013	0.013	0.012	0.012	0.011	0.042	0.039	0.036	0.034	0.032	0.032
32	0.130%	0.150%	0.000%	0.022	0.021	0.021	0.025	0.023	0.022	0.021	0.020	0.077	0.072	0.067	0.062	0.058	0.058	0.062
33	0.130%	0.160%	0.000%	0.031	0.030	0.036	0.034	0.032	0.031	0.029	0.112	0.104	0.097	0.091	0.085	0.085	0.091	0.085
34	0.130%	0.170%	0.000%	0.069	0.081	0.077	0.074	0.070	0.066	0.254	0.237	0.220	0.206	0.192	0.192	0.207	0.193	0.179
35	0.169%	0.180%	0.000%	0.099	0.094	0.090	0.085	0.081	0.309	0.288	0.268	0.250	0.233	0.234	0.251	0.234	0.218	0.203
36	0.169%	0.190%	0.000%	0.195	0.187	0.177	0.168	0.644	0.600	0.558	0.521	0.486	0.487	0.523	0.488	0.454	0.423	0.475
37	0.169%	0.200%	0.000%	0.162	0.154	0.146	0.560	0.522	0.486	0.453	0.423	0.424	0.455	0.424	0.395	0.368	0.413	0.329
38	0.169%	0.210%	0.000%	0.272	0.259	0.990	0.923	0.859	0.801	0.747	0.749	0.805	0.751	0.699	0.651	0.730	0.582	0.530
39	0.169%	0.230%	0.000%	0.294	1.123	1.046	0.974	0.908	0.847	0.850	0.913	0.851	0.792	0.738	0.828	0.660	0.601	0.546
40	0.263%	0.240%	0.000%	1.299	1.210	1.127	1.051	0.980	0.983	1.056	0.985	0.917	0.854	0.958	0.764	0.695	0.632	0.575
41	0.263%	0.260%	0.000%	1.299	1.210	1.129	1.052	1.056	1.134	1.057	0.984	0.917	1.029	0.820	0.746	0.679	0.618	0.635
42	0.263%	0.280%	0.000%	1.631	1.521	1.419	1.423	1.529	1.425	1.326	1.236	1.386	1.105	1.006	0.915	0.833	0.856	0.745
43	0.263%	0.300%	0.000%	1.494	1.393	1.397	1.501	1.399	1.303	1.214	1.361	1.085	0.988	0.899	0.818	0.840	0.731	0.636
44	0.263%	0.320%	0.000%	1.157	1.160	1.247	1.162	1.082	1.008	1.131	0.901	0.820	0.746	0.679	0.698	0.607	0.528	0.460
45	0.399%	0.350%	0.000%	1.577	1.694	1.579	1.470	1.370	1.537	1.225	1.115	1.014	0.923	0.949	0.825	0.718	0.625	0.543
46	0.399%	0.380%	0.000%	1.628	1.518	1.413	1.316	1.476	1.177	1.071	0.975	0.887	0.911	0.793	0.690	0.600	0.522	0.536
47	0.399%	0.410%	0.000%	1.501	1.397	1.302	1.460	1.164	1.059	0.964	0.877	0.902	0.784	0.682	0.594	0.517	0.530	0.445
48	0.399%	0.450%	0.000%	1.653	1.541	1.728	1.377	1.254	1.141	1.038	1.067	0.928	0.807	0.702	0.611	0.627	0.527	0.442
49	0.399%	0.490%	0.000%	1.489	1.670	1.331	1.211	1.102	1.003	1.031	0.897	0.780	0.679	0.591	0.606	0.509	0.428	0.359
50	0.595%	0.530%	0.000%	1.664	1.327	1.207	1.099	1.000	1.027	0.894	0.778	0.677	0.589	0.604	0.507	0.426	0.358	0.301
51	0.595%	0.580%	0.000%	1.484	1.351	1.229	1.118	1.149	1.000	0.870	0.757	0.658	0.676	0.568	0.477	0.400	0.336	2.524
52	0.595%	0.640%	0.000%	1.960	1.784	1.623	1.668	1.451	1.263	1.098	0.956	0.981	0.824	0.692	0.581	0.488	3.663	0.000
53	0.595%	0.690%	0.000%	2.690	2.448	2.515	2.188	1.904	1.656	1.441	1.479	1.242	1.043	0.876	0.736	5.524	0.000	0.000
54	0.595%	0.750%	0.000%	1.240	1.274	1.109	0.964	0.839	0.730	0.749	0.629	0.528	0.444	0.373	2.798	0.000	0.000	0.000
55	0.833%	0.820%	4.000%	1.532	1.333	1.160	1.009	0.878	0.901	0.757	0.635	0.534	0.448	3.365	0.000	0.000	0.000	
56	0.833%	0.890%	4.000%	2.368	2.060	1.792	1.559	1.600	1.344	1.129	0.948	0.797	5.977	0.000	0.000	0.000		
57	0.833%	0.960%	4.000%	1.861	1.619	1.409	1.446	1.214	1.020	0.857	0.720	5.401	0.000	0.000	0.000			
58	0.833%	1.040%	4.000%	1.241	1.080	1.108	0.931	0.782	0.657	0.552	4.139	0.000	0.000	0.000				
59	0.833%	1.120%	4.000%	1.660	1.703	1.431	1.202	1.010	0.848	6.364	0.000	0.000	0.000					
60	1.279%	1.210%	4.000%	1.616	1.357	1.140	0.958	0.804	6.036	0.000	0.000	0.000						
61	1.279%	1.300%	5.000%	1.374	1.154	0.969	0.814	6.111	0.000	0.000	0.000							
62	1.279%	1.400%	5.000%	1.581	1.328	1.116	8.374	0.000	0.000	0.000								
63	1.279%	1.490%	5.000%	1.213	1.019	7.646	0.000	0.000	0.000									
64	1.279%	1.590%	5.000%	0.917	6.884	0.000	0.000	0.000										
65	1.904%	1.690%	6.000%	5.917	0.000	0.000	0.000											
66	1.904%	0.000%	6.500%	0.000	0.000	0.000												
67	1.904%	0.000%	6.500%	0.000	0.000													
68	1.904%	0.000%	6.500%	0.000														
69	1.904%	0.000%	6.500%															
70	2.991%	0.000%	6.500%															
71	2.991%	0.000%	6.500%															
72	2.991%	0.000%	6.500%															
73	2.991%	0.000%	6.500%															
74	2.991%	0.000%	6.500%															
75	4.694%	0.000%	6.500%															
76	4.694%	0.000%	6.500%															
77	4.694%	0.000%	6.500%															
78	4.694%	0.000%	6.500%															
79	4.694%	0.000%	6.500%															
80	7.566%	0.000%	92.434%															
81	7.566%	0.000%	92.434%															
82	8.932%	0.000%	91.068%															
83	9.753%	0.000%	90.248%															
Total				46.2	44.0	41.7	39.2	34.3	31.4	29.0	24.7	23.7	22.0	17.6	15.4	16.3	13.0	10.7

[1] From Exhibit A2.

[2] Base Class, Mature Claims-Made,Full-Time Equivalent Physicians

Exhibit A11

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

DDR Lapse Schedule by Calendar Year

	Percent of In-force:[1]			Number of Equivalent Physicians to DDR During Period													
Age	Death	Disability	Retirement	1/40 - 12/40	1/41 - 12/41	1/42 - 12/42	1/43 - 12/43	1/44 - 12/44	1/45 - 12/45	1/46 - 12/46	1/47 - 12/47	1/48 - 12/48	1/49 - 12/49	1/50 - 12/50	1/51 - 12/51	1/52 - 12/52	1/53 - 12/53
25	0.118%	0.090%	0.000%	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
26	0.118%	0.100%	0.000%	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
27	0.118%	0.100%	0.000%	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
28	0.118%	0.110%	0.000%	0.003	0.002	0.002	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001
29	0.118%	0.120%	0.000%	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.002	0.002	0.002	0.002	0.002	0.001
30	0.130%	0.130%	0.000%	0.013	0.014	0.013	0.012	0.011	0.012	0.010	0.009	0.008	0.007	0.008	0.007	0.006	0.005
31	0.130%	0.140%	0.000%	0.034	0.032	0.030	0.028	0.031	0.025	0.022	0.020	0.019	0.019	0.017	0.014	0.013	0.011
32	0.130%	0.150%	0.000%	0.058	0.054	0.051	0.057	0.045	0.041	0.037	0.034	0.035	0.030	0.026	0.023	0.020	0.021
33	0.130%	0.160%	0.000%	0.079	0.074	0.083	0.066	0.060	0.055	0.050	0.051	0.044	0.039	0.034	0.029	0.030	0.025
34	0.130%	0.170%	0.000%	0.167	0.187	0.149	0.136	0.124	0.113	0.116	0.101	0.088	0.076	0.066	0.068	0.057	0.048
35	0.169%	0.180%	0.000%	0.228	0.182	0.165	0.150	0.137	0.141	0.122	0.107	0.093	0.081	0.083	0.069	0.058	0.049
36	0.169%	0.190%	0.000%	0.378	0.344	0.313	0.285	0.293	0.255	0.222	0.193	0.168	0.172	0.145	0.122	0.102	0.086
37	0.169%	0.200%	0.000%	0.300	0.273	0.248	0.255	0.222	0.193	0.168	0.146	0.150	0.126	0.106	0.089	0.075	0.560
38	0.169%	0.210%	0.000%	0.482	0.439	0.451	0.392	0.341	0.297	0.258	0.265	0.223	0.187	0.157	0.132	0.990	0.000
39	0.169%	0.230%	0.000%	0.497	0.511	0.445	0.387	0.337	0.293	0.300	0.252	0.212	0.178	0.150	1.122	0.000	0.000
40	0.263%	0.240%	0.000%	0.591	0.515	0.448	0.389	0.339	0.348	0.292	0.245	0.206	0.173	1.299	0.000	0.000	0.000
41	0.263%	0.260%	0.000%	0.552	0.481	0.418	0.364	0.373	0.314	0.263	0.221	0.186	1.394	0.000	0.000	0.000	
42	0.263%	0.280%	0.000%	0.648	0.564	0.490	0.503	0.423	0.355	0.298	0.250	1.880	0.000	0.000	0.000		
43	0.263%	0.300%	0.000%	0.553	0.482	0.494	0.415	0.349	0.293	0.246	1.846	0.000	0.000	0.000			
44	0.263%	0.320%	0.000%	0.400	0.410	0.345	0.290	0.243	0.204	1.533	0.000	0.000	0.000				
45	0.399%	0.350%	0.000%	0.558	0.468	0.393	0.330	0.278	2.083	0.000	0.000	0.000					
46	0.399%	0.380%	0.000%	0.450	0.378	0.318	0.267	2.002	0.000	0.000	0.000						
47	0.399%	0.410%	0.000%	0.374	0.314	0.264	1.980	0.000	0.000	0.000							
48	0.399%	0.450%	0.000%	0.372	0.312	2.343	0.000	0.000	0.000								
49	0.399%	0.490%	0.000%	0.302	2.264	0.000	0.000	0.000									
50	0.595%	0.530%	0.000%	2.256	0.000	0.000	0.000										
51	0.595%	0.580%	0.000%	0.000	0.000	0.000											
52	0.595%	0.640%	0.000%	0.000	0.000												
53	0.595%	0.690%	0.000%	0.000													
54	0.595%	0.750%	0.000%														
55	0.833%	0.820%	4.000%														
56	0.833%	0.890%	4.000%														
57	0.833%	0.960%	4.000%														
58	0.833%	1.040%	4.000%														
59	0.833%	1.120%	4.000%														
60	1.279%	1.210%	4.000%														
61	1.279%	1.300%	5.000%														
62	1.279%	1.400%	5.000%														
63	1.279%	1.490%	5.000%														
64	1.279%	1.590%	5.000%														
65	1.904%	1.690%	6.000%														
66	1.904%	0.000%	6.500%														
67	1.904%	0.000%	6.500%														
68	1.904%	0.000%	6.500%														
69	1.904%	0.000%	6.500%														
70	2.991%	0.000%	6.500%														
71	2.991%	0.000%	6.500%														
72	2.991%	0.000%	6.500%														
73	2.991%	0.000%	6.500%														
74	2.991%	0.000%	6.500%														
75	4.694%	0.000%	6.500%														
76	4.694%	0.000%	6.500%														
77	4.694%	0.000%	6.500%														
78	4.694%	0.000%	6.500%														
79	4.694%	0.000%	6.500%														
80	7.566%	0.000%	92.434%														
81	7.566%	0.000%	92.434%														
82	8.932%	0.000%	91.068%														
83	9.753%	0.000%	90.248%														
Total				9.3	8.3	7.5	6.3	5.6	5.0	3.9	3.7	3.3	2.5	2.1	1.7	1.4	0.8

[1] From Exhibit A2.

[2] Base Class, Mature Claims-Made,Full-Time Equivalent Physicians

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

DDR Lapse Schedule by Calendar Year

	Percent of In-force:[1]			Number of Equivalent Physicians to DDR During Period													
Age	Death	Disability	Retirement	1/54 - 12/54	1/55 - 12/55	1/56 - 12/56	1/57 - 12/57	1/58 - 12/58	1/59 - 12/59	1/60 - 12/60	1/61 - 12/61	1/62 - 12/62	1/63 - 12/63	1/64 - 12/64	1/65 - 12/65	1/66 - 12/66	1/67 - 12/67
25	0.118%	0.090%	0.000%	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000
26	0.118%	0.100%	0.000%	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
27	0.118%	0.100%	0.000%	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	
28	0.118%	0.110%	0.000%	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.003	0.000	0.000	0.000		
29	0.118%	0.120%	0.000%	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.004	0.000	0.000	0.000			
30	0.130%	0.130%	0.000%	0.004	0.004	0.004	0.003	0.003	0.002	0.017	0.000	0.000	0.000				
31	0.130%	0.140%	0.000%	0.011	0.009	0.008	0.007	0.006	0.042	0.000	0.000	0.000					
32	0.130%	0.150%	0.000%	0.017	0.015	0.012	0.010	0.077	0.000	0.000	0.000						
33	0.130%	0.160%	0.000%	0.021	0.018	0.015	0.112	0.000	0.000	0.000							
34	0.130%	0.170%	0.000%	0.040	0.034	0.254	0.000	0.000	0.000								
35	0.169%	0.180%	0.000%	0.041	0.309	0.000	0.000	0.000									
36	0.169%	0.190%	0.000%	0.644	0.000	0.000	0.000										
37	0.169%	0.200%	0.000%	0.000	0.000	0.000											
38	0.169%	0.210%	0.000%	0.000	0.000												
39	0.169%	0.230%	0.000%	0.000													
40	0.263%	0.240%	0.000%														
41	0.263%	0.260%	0.000%														
42	0.263%	0.280%	0.000%														
43	0.263%	0.300%	0.000%														
44	0.263%	0.320%	0.000%														
45	0.399%	0.350%	0.000%														
46	0.399%	0.380%	0.000%														
47	0.399%	0.410%	0.000%														
48	0.399%	0.450%	0.000%														
49	0.399%	0.490%	0.000%														
50	0.595%	0.530%	0.000%														
51	0.595%	0.580%	0.000%														
52	0.595%	0.640%	0.000%														
53	0.595%	0.690%	0.000%														
54	0.595%	0.750%	0.000%														
55	0.833%	0.820%	4.000%														
56	0.833%	0.890%	4.000%														
57	0.833%	0.960%	4.000%														
58	0.833%	1.040%	4.000%														
59	0.833%	1.120%	4.000%														
60	1.279%	1.210%	4.000%														
61	1.279%	1.300%	5.000%														
62	1.279%	1.400%	5.000%														
63	1.279%	1.490%	5.000%														
64	1.279%	1.590%	5.000%														
65	1.904%	1.690%	6.000%														
66	1.904%	0.000%	6.500%														
67	1.904%	0.000%	6.500%														
68	1.904%	0.000%	6.500%														
69	1.904%	0.000%	6.500%														
70	2.991%	0.000%	6.500%														
71	2.991%	0.000%	6.500%														
72	2.991%	0.000%	6.500%														
73	2.991%	0.000%	6.500%														
74	2.991%	0.000%	6.500%														
75	4.694%	0.000%	6.500%														
76	4.694%	0.000%	6.500%														
77	4.694%	0.000%	6.500%														
78	4.694%	0.000%	6.500%														
79	4.694%	0.000%	6.500%														
80	7.566%	0.000%	92.434%														
81	7.566%	0.000%	92.434%														
82	8.932%	0.000%	91.068%														
83	9.753%	0.000%	90.248%														
Total				0.8	0.4	0.3	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

[1] From Exhibit A2.

[2] Base Class, Mature Claims-Made,Full-Time Equivalent Physicians

Exhibit A13

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Attrition Pattern by Calendar Year

	(1)	(2)	(3) = (4) - (1) - (2)	(4)
	Number of Base Class/Mature CM/Full-Time Equivalents:			
Calendar Period	In-Force at Beginning Of Period[1]	Lapsing Due to DDR[2]	Lapsing *Other Than* Due to DDR	In-Force at End Of Period[1]
1/10 - 12/10	2,649.0	102.7	166.4	2,379.9
1/11 - 12/11	2,379.9	88.6	132.8	2,158.5
1/12 - 12/12	2,158.5	85.3	107.3	1,965.9
1/13 - 12/13	1,965.9	86.3	90.1	1,789.6
1/14 - 12/14	1,789.6	81.9	77.0	1,630.7
1/15 - 12/15	1,630.7	77.8	67.8	1,485.1
1/16 - 12/16	1,485.1	75.7	59.7	1,349.6
1/17 - 12/17	1,349.6	75.4	52.6	1,221.7
1/18 - 12/18	1,221.7	68.4	46.5	1,106.7
1/19 - 12/19	1,106.7	63.2	40.8	1,002.7
1/20 - 12/20	1,002.7	64.3	35.8	902.6
1/21 - 12/21	902.6	59.6	31.6	811.5
1/22 - 12/22	811.5	56.3	27.0	728.1
1/23 - 12/23	728.1	50.9	23.3	653.9
1/24 - 12/24	653.9	48.8	20.3	584.8
1/25 - 12/25	584.8	46.2	17.1	521.5
1/26 - 12/26	521.5	44.0	14.1	463.3
1/27 - 12/27	463.3	41.7	11.9	409.7
1/28 - 12/28	409.7	39.2	10.0	360.5
1/29 - 12/29	360.5	34.3	8.1	318.0
1/30 - 12/30	318.0	31.4	6.7	279.9
1/31 - 12/31	279.9	29.0	5.6	245.2
1/32 - 12/32	245.2	24.7	4.9	215.7
1/33 - 12/33	215.7	23.7	4.1	187.9
1/34 - 12/34	187.9	22.0	3.6	162.3
1/35 - 12/35	162.3	17.6	3.2	141.6
1/36 - 12/36	141.6	15.4	2.9	123.3
1/37 - 12/37	123.3	16.3	2.5	104.5
1/38 - 12/38	104.5	13.0	2.3	89.1
1/39 - 12/39	89.1	10.7	2.2	76.3
1/40 - 12/40	76.3	9.3	2.0	65.0
1/41 - 12/41	65.0	8.3	1.8	54.9
1/42 - 12/42	54.9	7.5	1.7	45.7
1/43 - 12/43	45.7	6.3	1.5	37.9
1/44 - 12/44	37.9	5.6	1.3	31.0
1/45 - 12/45	31.0	5.0	1.1	24.9
1/46 - 12/46	24.9	3.9	0.9	20.0
1/47 - 12/47	20.0	3.7	0.8	15.5
1/48 - 12/48	15.5	3.3	0.6	11.6
1/49 - 12/49	11.6	2.5	0.5	8.7
1/50 - 12/50	8.7	2.1	0.3	6.2
1/51 - 12/51	6.2	1.7	0.2	4.3
1/52 - 12/52	4.3	1.4	0.2	2.8
1/53 - 12/53	2.8	0.8	0.1	1.9
1/54 - 12/54	1.9	0.8	0.1	1.1
1/55 - 12/55	1.1	0.4	0.0	0.6
1/56 - 12/56	0.6	0.3	0.0	0.3
1/57 - 12/57	0.3	0.1	0.0	0.2
1/58 - 12/58	0.2	0.1	0.0	0.1
1/59 - 12/59	0.1	0.0	0.0	0.0
1/60 - 12/60	0.0	0.0	0.0	0.0
1/61 - 12/61	0.0	0.0	0.0	0.0
1/62 - 12/62	0.0	0.0	0.0	0.0
1/63 - 12/63	0.0	0.0	0.0	0.0
1/64 - 12/64	0.0	0.0	0.0	0.0
1/65 - 12/65	0.0	0.0	0.0	0.0
1/66 - 12/66	0.0	0.0	0.0	0.0
1/67 - 12/67	0.0	0.0	0.0	0.0
Total	26,451.6	1,557.7	1,091.3	23,802.6

[1] From Exhibits A5 through A8.
[2] From Exhibit A9 through A12.

Reserving for Extended Reporting Endorsement Coverage, Including the Death, Disability, and Retirement Policy Provisioin

Analysis of DDR Unearned Premium Reserve

Under Current Methodology, as of December 31, 2009

Indicated Pure Premiums For ERE Coverage

(1)	(2)	(3)	(4) = (2) / (3)	(5)
Report Year	Ultimate Loss & ALAE Limited to Policy Limits[1]	Mature Claims-Made Base Class Equivalent Exposures	Ultimate Loss & ALAE Pure Premium	Trended[2] Ultimate Loss & ALAE Pure Premium
1998	5,424,163	1,359	3,992	6,828
1999	2,396,646	1,427	1,679	2,736
2000	9,206,638	1,574	5,847	9,071
2001	4,793,956	1,777	2,698	3,986
2002	7,511,520	1,825	4,117	5,793
2003	8,367,549	1,774	4,717	6,321
2004	17,946,284	2,526	7,106	9,069
2005	19,199,929	2,736	7,017	8,529
2006	14,844,834	2,844	5,220	6,043
2007	10,900,615	2,500	4,360	4,807
2008	13,165,910	2,414	5,455	5,728
2009	16,783,983	2,674	6,277	6,277
2003 - 2009				6,713
2005 - 2009				6,315

(6)	Selected Base Class Claims-Made Loss & ALAE Pure Premium at Total Limits	6,525
(7)	ULAE Load[1]	7.0%
(8)	Selected Base Class Claims-Made Loss & LAE Pure Premium at Total Limits; (6) x [1 + (7)]	6,982
(9)	Mature Claims-Made to Average ERE Factor[3]	2.000
(10)	Assumed Reduction in DDR Liability Due to Reduced Exposure Prior to Retirement	80.0%
(11)	**Selected Base Class Loss & LAE ERE Pure Premium; (8) x (9) x (10)**	**11,171**

[1] Based on claims-made reserve analysis

[2] Trended at 5.0% per annum to average report date of July 1, 2009

[3] Based on actuarial analysis or currently filed tail factor

Exhibit A15

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Projections of Loss and LAE on DDR Policies

Assumptions	
Per Annum Trend:	5.00%

Calendar Period	Projected Number to DDR[1]	Projected Pure Premium[2]	Loss & LAE Incurred[3]	Loss & LAE Paid[4] 0.5% 1st Year	2.5% 2nd Year	2.0% 3rd Year	25.0% 4th Year	20.0% 5th Year	15.0% 6th Year	10.0% 7th Year	5.0% 8th Year	5.0% 9th Year	5.0% 10th Year	5.0% 11th Year	2.5% 12th Year	2.5% 13th Year	Total
1/10 - 12/10	102.7	11,171	1,147,766	5,739													5,739
1/11 - 12/11	88.6	11,730	1,039,567	5,198	28,694												33,892
1/12 - 12/12	85.3	12,316	1,050,612	5,253	25,989	22,955											54,198
1/13 - 12/13	86.3	12,932	1,115,615	5,578	26,265	20,791	286,941										339,576
1/14 - 12/14	81.9	13,578	1,112,278	5,561	27,890	21,012	259,892	229,553									543,909
1/15 - 12/15	77.8	14,257	1,109,064	5,545	27,807	22,312	262,653	207,913	172,165								698,396
1/16 - 12/16	75.7	14,970	1,133,551	5,668	27,727	22,246	278,904	210,122	155,935	114,777							815,378
1/17 - 12/17	75.4	15,719	1,185,086	5,925	28,339	22,181	278,069	223,123	157,592	103,957	57,388						876,575
1/18 - 12/18	68.4	16,505	1,129,272	5,646	29,627	22,671	277,266	222,456	167,342	105,061	51,978	57,388					939,436
1/19 - 12/19	63.2	17,330	1,096,102	5,481	28,232	23,702	283,388	221,813	166,842	111,561	52,531	51,978	57,388				1,002,915
1/20 - 12/20	64.3	18,196	1,169,327	5,847	27,403	22,585	296,272	226,710	166,360	111,228	55,781	52,531	51,978	57,388			1,074,082
1/21 - 12/21	59.6	19,106	1,137,841	5,689	29,233	21,922	282,318	237,017	170,033	110,906	55,614	55,781	52,531	51,978	28,694		1,101,716
1/22 - 12/22	56.3	20,062	1,129,504	5,648	28,446	23,387	274,026	225,854	177,763	113,355	55,453	55,614	55,781	52,531	25,989	28,694	1,122,540
1/23 - 12/23	50.9	21,065	1,073,047	5,365	28,238	22,757	292,332	219,220	169,391	118,509	56,678	55,453	55,614	55,781	26,265	25,989	1,131,591
1/24 - 12/24	48.8	22,118	1,079,123	5,396	26,826	22,590	284,460	233,865	164,415	112,927	59,254	56,678	55,453	55,614	27,890	26,265	1,131,635
1/25 - 12/25	46.2	23,224	1,073,217	5,366	26,978	21,461	282,376	227,568	175,399	109,610	56,464	59,254	56,678	55,453	27,807	27,890	1,132,304
1/26 - 12/26	44.0	24,385	1,073,050	5,365	26,830	21,582	268,262	225,901	170,676	116,933	54,805	56,464	59,254	56,678	27,727	27,807	1,118,284
1/27 - 12/27	41.7	25,604	1,068,412	5,342	26,826	21,464	269,781	214,609	169,426	113,784	58,466	54,805	56,464	59,254	28,339	27,727	1,106,287
1/28 - 12/28	39.2	26,884	1,054,322	5,272	26,710	21,461	268,304	215,825	160,957	112,950	56,892	58,466	54,805	56,464	29,627	28,339	1,096,072
1/29 - 12/29	34.3	28,229	968,651	4,843	26,358	21,368	268,262	214,643	161,868	107,305	56,475	56,892	58,466	54,805	28,232	29,627	1,089,146
1/30 - 12/30	31.4	29,640	931,744	4,659	24,216	21,086	267,103	214,610	160,983	107,912	53,652	56,475	56,892	58,466	27,403	28,232	1,081,690
1/31 - 12/31	29.0	31,122	903,937	4,520	23,294	19,373	263,581	213,682	160,957	107,322	53,956	53,652	56,475	56,892	29,233	27,403	1,070,340
1/32 - 12/32	24.7	32,678	807,177	4,036	22,598	18,635	242,163	210,864	160,262	107,305	53,661	53,956	53,652	56,475	28,446	29,233	1,041,287
1/33 - 12/33	23.7	34,312	812,854	4,064	20,179	18,079	232,936	193,730	158,148	106,841	53,652	53,661	53,956	53,652	28,238	28,446	1,005,584
1/34 - 12/34	22.0	36,028	791,523	3,958	20,321	16,144	225,984	186,349	145,298	105,432	53,421	53,652	53,661	53,956	26,826	28,238	973,239
1/35 - 12/35	17.6	37,829	665,111	3,326	19,788	16,257	201,794	180,787	139,762	96,865	52,716	53,421	53,652	53,661	26,978	26,826	925,833
1/36 - 12/36	15.4	39,720	610,757	3,054	16,628	15,830	203,214	161,435	135,591	93,174	48,433	52,716	53,421	53,652	26,830	26,978	890,956
1/37 - 12/37	16.3	41,706	680,517	3,403	15,269	13,302	197,881	162,571	121,077	90,394	46,587	48,433	52,716	53,421	26,826	26,830	858,709
1/38 - 12/38	13.0	43,792	568,706	2,844	17,013	12,215	166,278	158,305	121,928	80,718	45,197	46,587	48,433	52,716	26,710	26,826	805,769
1/39 - 12/39	10.7	45,981	490,337	2,452	14,218	13,610	152,689	133,022	118,728	81,285	40,359	45,197	46,587	48,433	26,358	26,710	749,649
1/40 - 12/40	9.3	48,280	449,029	2,245	12,258	11,374	170,129	122,151	99,767	79,152	40,643	40,359	45,197	46,587	24,216	26,358	720,437
1/41 - 12/41	8.3	50,694	420,964	2,105	11,226	9,807	142,176	136,103	91,614	66,511	39,576	40,643	40,359	45,197	23,294	24,216	672,826
1/42 - 12/42	7.5	53,229	397,460	1,987	10,524	8,981	122,584	113,741	102,078	61,076	33,256	39,576	40,643	40,359	22,598	23,294	620,696
1/43 - 12/43	6.3	55,891	352,834	1,764	9,936	8,419	112,257	98,067	85,306	68,052	30,538	33,256	39,576	40,643	20,179	22,598	570,592
1/44 - 12/44	5.6	58,685	329,396	1,647	8,821	7,949	105,241	89,806	73,550	56,871	34,026	30,538	33,256	39,576	20,321	20,179	521,781
1/45 - 12/45	5.0	61,619	309,735	1,549	8,235	7,057	99,365	84,193	67,354	49,034	28,435	34,026	30,538	33,256	19,788	20,321	483,150
1/46 - 12/46	3.9	64,700	255,247	1,276	7,743	6,588	88,208	79,492	63,145	44,903	24,517	28,435	34,026	30,538	16,628	19,788	445,287
1/47 - 12/47	3.7	67,935	254,557	1,273	6,381	6,195	82,349	70,567	59,619	42,096	22,451	24,517	28,435	34,026	15,269	16,628	409,806
1/48 - 12/48	3.3	71,332	236,522	1,183	6,364	5,105	77,434	65,879	52,925	39,746	21,048	22,451	24,517	28,435	17,013	15,269	377,369
1/49 - 12/49	2.5	74,899	186,359	932	5,913	5,091	63,812	61,947	49,409	35,283	19,873	21,048	22,451	24,517	14,218	17,013	341,507
1/50 - 12/50	2.1	78,644	164,669	823	4,659	4,730	63,639	51,049	46,460	32,940	17,642	19,873	21,048	22,451	12,258	14,218	311,791
1/51 - 12/51	1.7	82,576	138,720	694	4,117	3,727	59,131	50,911	38,287	30,973	16,470	17,642	19,873	21,048	11,226	12,258	286,357
1/52 - 12/52	1.4	86,705	117,458	587	3,468	3,293	46,590	47,304	38,184	25,525	15,487	16,470	17,642	19,873	10,524	11,226	256,172
1/53 - 12/53	0.8	91,040	73,564	368	2,936	2,774	41,167	37,272	35,478	25,456	12,762	15,487	16,470	17,642	9,936	10,524	228,273
1/54 - 12/54	0.8	95,592	74,784	374	1,839	2,349	34,680	32,934	27,954	23,652	12,728	12,762	15,487	16,470	8,821	9,936	199,986
1/55 - 12/55	0.4	100,372	39,341	197	1,870	1,471	29,365	27,744	24,700	18,636	11,826	12,728	12,762	15,487	8,235	8,821	173,841
1/56 - 12/56	0.3	105,390	31,140	156	984	1,496	18,391	23,492	20,808	16,467	9,318	11,826	12,728	12,762	7,743	8,235	144,405
1/57 - 12/57	0.1	110,660	14,882	74	778	787	18,696	14,713	17,619	13,872	8,233	9,318	11,826	12,728	6,381	7,743	122,769
1/58 - 12/58	0.1	116,193	10,127	51	372	623	9,835	14,957	11,035	11,746	6,936	8,233	9,318	11,826	6,364	6,381	97,676
1/59 - 12/59	0.0	122,002	5,603	28	253	298	7,785	7,868	11,218	7,356	5,873	6,936	8,233	9,318	5,913	6,364	77,443
1/60 - 12/60	0.0	128,102	2,323	12	140	203	3,721	6,228	5,901	7,478	3,678	5,873	6,936	8,233	4,659	5,913	58,975
1/61 - 12/61	0.0	134,507	686	3	58	112	2,532	2,976	4,671	3,934	3,739	3,678	5,873	6,936	4,117	4,659	43,289
1/62 - 12/62	0.0	141,233	491	2	17	46	1,401	2,025	2,232	3,114	1,967	3,739	3,678	5,873	3,468	4,117	31,680
1/63 - 12/63	0.0	148,294	208	1	12	14	581	1,121	1,519	1,488	1,557	1,967	3,739	3,678	2,936	3,468	22,081
1/64 - 12/64	0.0	155,709	52	0	5	10	171	465	840	1,013	744	1,557	1,967	3,739	1,839	2,936	15,287
1/65 - 12/65	0.0	163,495	96	0	1	4	123	137	348	560	506	744	1,557	1,967	1,870	1,839	9,658
1/66 - 12/66	0.0	171,669	0	0	2	1	52	98	103	232	280	506	744	1,557	984	1,870	6,429
1/67 - 12/67	0.0	180,253	0	0	0	2	13	42	74	69	116	280	506	744	778	984	3,608
12/67 - 11/68				0	0	0	24	10	31	49	34	116	280	506	372	778	2,202
12/68 - 11/69				0	0	0	0	19	8	21	25	34	116	280	253	372	1,128
12/69 - 11/70				0	0	0	0	0	14	5	10	25	34	116	140	253	598
12/70 - 11/71				0	0	0	0	0	0	10	3	10	25	34	58	140	280
12/71 - 11/72				0	0	0	0	0	0	0	5	3	10	25	17	58	118
12/72 - 11/73				0	0	0	0	0	0	0	0	5	3	10	12	17	47
12/73 - 11/74				0	0	0	0	0	0	0	0	0	5	3	5	12	25
12/74 - 11/75				0	0	0	0	0	0	0	0	0	0	5	1	5	11
12/75 - 11/76				0	0	0	0	0	0	0	0	0	0	0	2	1	4
12/76 - 11/77				0	0	0	0	0	0	0	0	0	0	0	0	2	2
12/77 - 11/78				0	0	0	0	0	0	0	0	0	0	0	0	0	0
Total	1,557.7		33,074,314	165,372	826,858	661,486	8,268,578	6,614,863	4,961,147	3,307,431	1,653,716	1,653,716	1,653,716	1,653,716	826,858	826,858	33,074,314

[1] From Exhibit A13.
[2] First calendar period value is from Exhibit A14; subsequent values are trended at the per annum trend rate selected above.
[3] Product of the projected number of insureds to DDR and the projected pure premium.
[4] From Exhibit A18.

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Discounted Projections of Loss & LAE

Calendar	Total Paid	Paid Loss & LAE Discounted at a Per Annum Rate of			
Period	Loss & LAE[1]	3.0%	4.0%	5.0%	6.0%
1/10 - 12/10	5,739	5,655	5,627	5,601	5,574
1/11 - 12/11	33,892	32,422	31,956	31,500	31,055
1/12 - 12/12	54,198	50,337	49,136	47,974	46,851
1/13 - 12/13	339,576	306,201	296,020	286,269	276,928
1/14 - 12/14	543,909	476,166	455,907	436,691	418,456
1/15 - 12/15	698,396	593,604	562,883	534,024	506,897
1/16 - 12/16	815,378	672,848	631,891	593,784	558,304
1/17 - 12/17	876,575	702,279	653,189	607,952	566,233
1/18 - 12/18	939,436	730,720	673,107	620,523	572,489
1/19 - 12/19	1,002,915	757,374	690,951	630,908	576,578
1/20 - 12/20	1,074,082	787,493	711,520	643,502	582,540
1/21 - 12/21	1,101,716	784,227	701,756	628,627	563,706
1/22 - 12/22	1,122,540	775,776	687,520	610,008	541,849
1/23 - 12/23	1,131,591	759,254	666,407	585,644	515,300
1/24 - 12/24	1,131,635	737,168	640,801	557,778	486,151
1/25 - 12/25	1,132,304	716,121	616,519	531,532	458,905
1/26 - 12/26	1,118,284	686,654	585,466	499,952	427,568
1/27 - 12/27	1,106,287	659,502	556,909	471,037	399,039
1/28 - 12/28	1,096,072	634,382	530,545	444,465	372,976
1/29 - 12/29	1,089,146	612,012	506,916	420,625	349,641
1/30 - 12/30	1,081,690	590,119	484,082	397,852	327,591
1/31 - 12/31	1,070,340	566,919	460,580	374,931	305,806
1/32 - 12/32	1,041,287	535,467	430,844	347,385	280,665
1/33 - 12/33	1,005,584	502,046	400,069	319,499	255,700
1/34 - 12/34	973,239	471,745	372,308	294,498	233,467
1/35 - 12/35	925,833	435,696	340,551	266,812	209,524
1/36 - 12/36	890,956	407,071	315,118	244,534	190,218
1/37 - 12/37	858,709	380,910	292,031	224,461	172,956
1/38 - 12/38	805,769	347,016	263,488	200,593	153,106
1/39 - 12/39	749,649	313,444	235,708	177,735	134,380
1/40 - 12/40	720,437	292,456	217,811	162,676	121,834
1/41 - 12/41	672,826	265,174	195,593	144,691	107,342
1/42 - 12/42	620,696	237,503	173,498	127,124	93,420
1/43 - 12/43	570,592	211,972	153,359	111,297	81,018
1/44 - 12/44	521,781	188,193	134,846	96,930	69,893
1/45 - 12/45	483,150	169,184	120,060	85,479	61,055
1/46 - 12/46	445,287	151,384	106,395	75,029	53,086
1/47 - 12/47	409,806	135,264	94,152	65,763	46,090
1/48 - 12/48	377,369	120,930	83,365	57,674	40,040
1/49 - 12/49	341,507	106,250	72,541	49,708	34,184
1/50 - 12/50	311,791	94,180	63,682	43,221	29,443
1/51 - 12/51	286,357	83,977	56,237	37,805	25,510
1/52 - 12/52	256,172	72,937	48,374	32,210	21,529
1/53 - 12/53	228,273	63,101	41,448	27,335	18,099
1/54 - 12/54	199,986	53,671	34,915	22,807	14,959
1/55 - 12/55	173,841	45,296	29,183	18,882	12,267
1/56 - 12/56	144,405	36,530	23,309	14,938	9,613
1/57 - 12/57	122,769	30,152	19,055	12,095	7,710
1/58 - 12/58	97,676	23,291	14,577	9,165	5,787
1/59 - 12/59	77,443	17,928	11,113	6,920	4,329
1/60 - 12/60	58,975	13,255	8,137	5,019	3,110
1/61 - 12/61	43,289	9,446	5,743	3,509	2,153
1/62 - 12/62	31,680	6,712	4,041	2,445	1,487
1/63 - 12/63	22,081	4,542	2,709	1,623	978
1/64 - 12/64	15,287	3,053	1,803	1,070	638
1/65 - 12/65	9,658	1,872	1,095	644	381
1/66 - 12/66	6,429	1,210	701	408	239
1/67 - 12/67	3,608	659	378	218	127
12/67 - 11/68	2,202	391	222	127	73
12/68 - 11/69	1,128	194	109	62	35
12/69 - 11/70	598	100	56	31	18
12/70 - 11/71	280	45	25	14	8
12/71 - 11/72	118	19	10	6	3
12/72 - 11/73	47	7	4	2	1
12/73 - 11/74	25	4	2	1	1
12/74 - 11/75	11	2	1	0	0
12/75 - 11/76	4	1	0	0	0
12/76 - 11/77	2	0	0	0	0
12/77 - 11/78	0	0	0	0	0
Total	$33,074,314	$18,471,516	$15,568,357	$13,253,623	$11,386,908

[1] From Exhibit A15.

Analysis of DDR Unearned Premium Reserve
Under Current Methodology, as of December 31, 2009

Discounted Projections of DDR Premiums

Assumptions	
Annual Avg. Rate Change:	5.00%
DDR Provision (% of premium)	3.00%

Calendar Period	Average Base Class/ Mature CM Full-Time Equivalent Premium[1]	Projected Number of Equivalents Renewing During Period[2]	Undiscounted DDR Premium[3]	DDR Premium Discounted at a Per Annum Rate of 3.0%	4.0%	5.0%	6.0%
1/10 - 12/10	9,302	2,379.9	664,128	654,384	651,231	648,122	645,058
1/11 - 12/11	9,767	2,158.5	632,464	605,034	596,329	587,830	579,532
1/12 - 12/12	10,255	1,965.9	604,842	561,758	548,351	535,388	522,850
1/13 - 12/13	10,768	1,789.6	578,124	521,304	503,970	487,370	471,466
1/14 - 12/14	11,307	1,630.7	553,121	484,231	463,629	444,087	425,543
1/15 - 12/15	11,872	1,485.1	528,916	449,554	426,288	404,432	383,888
1/16 - 12/16	12,466	1,349.6	504,724	416,497	391,144	367,556	345,593
1/17 - 12/17	13,089	1,221.7	479,700	384,318	357,454	332,698	309,867
1/18 - 12/18	13,743	1,106.7	456,301	354,924	326,940	301,400	278,069
1/19 - 12/19	14,430	1,002.7	434,067	327,796	299,048	273,060	249,546
1/20 - 12/20	15,152	902.6	410,299	300,822	271,801	245,818	222,530
1/21 - 12/21	15,910	811.5	387,308	275,695	246,702	220,994	198,171
1/22 - 12/22	16,705	728.1	364,909	252,185	223,495	198,298	176,141
1/23 - 12/23	17,540	653.9	344,080	230,865	202,633	178,076	156,686
1/24 - 12/24	18,417	584.8	323,091	210,468	182,954	159,250	138,800
1/25 - 12/25	19,338	521.5	302,534	191,337	164,724	142,017	122,612
1/26 - 12/26	20,305	463.3	282,238	173,301	147,763	126,181	107,912
1/27 - 12/27	21,320	409.7	262,055	156,222	131,920	111,578	94,524
1/28 - 12/28	22,386	360.5	242,115	140,130	117,194	98,179	82,388
1/29 - 12/29	23,506	318.0	224,277	126,026	104,384	86,615	71,998
1/30 - 12/30	24,681	279.9	207,220	113,050	92,736	76,217	62,757
1/31 - 12/31	25,915	245.2	190,662	100,987	82,044	66,787	54,474
1/32 - 12/32	27,211	215.7	176,067	90,540	72,850	58,738	47,456
1/33 - 12/33	28,571	187.9	161,016	80,389	64,060	51,159	40,943
1/34 - 12/34	30,000	162.3	146,091	70,813	55,886	44,206	35,045
1/35 - 12/35	31,500	141.6	133,767	62,951	49,204	38,550	30,273
1/36 - 12/36	33,075	123.3	122,346	55,899	43,272	33,580	26,121
1/37 - 12/37	34,729	104.5	108,830	48,275	37,011	28,447	21,920
1/38 - 12/38	36,465	89.1	97,495	41,988	31,881	24,271	18,525
1/39 - 12/39	38,288	76.3	87,648	36,648	27,559	20,781	15,712
1/40 - 12/40	40,203	65.0	78,383	31,819	23,698	17,699	13,255
1/41 - 12/41	42,213	54.9	69,462	27,376	20,193	14,938	11,082
1/42 - 12/42	44,323	45.7	60,784	23,258	16,990	12,449	9,148
1/43 - 12/43	46,540	37.9	52,944	19,668	14,230	10,327	7,517
1/44 - 12/44	48,867	31.0	45,448	16,392	11,745	8,443	6,088
1/45 - 12/45	51,310	24.9	38,288	13,407	9,514	6,774	4,838
1/46 - 12/46	53,875	20.0	32,312	10,985	7,720	5,444	3,852
1/47 - 12/47	56,569	15.5	26,287	8,676	6,039	4,218	2,956
1/48 - 12/48	59,398	11.6	20,660	6,621	4,564	3,157	2,192
1/49 - 12/49	62,368	8.7	16,195	5,039	3,440	2,357	1,621
1/50 - 12/50	65,486	6.2	12,241	3,698	2,500	1,697	1,156
1/51 - 12/51	68,760	4.3	8,906	2,612	1,749	1,176	793
1/52 - 12/52	72,198	2.8	6,084	1,732	1,149	765	511
1/53 - 12/53	75,808	1.9	4,311	1,192	783	516	342
1/54 - 12/54	79,599	1.1	2,517	676	439	287	188
1/55 - 12/55	83,579	0.6	1,571	409	264	171	111
1/56 - 12/56	87,757	0.3	825	209	133	85	55
1/57 - 12/57	92,145	0.2	468	115	73	46	29
1/58 - 12/58	96,753	0.1	226	54	34	21	13
1/59 - 12/59	101,590	0.0	92	21	13	8	5
1/60 - 12/60	106,670	0.0	37	8	5	3	2
1/61 - 12/61	112,003	0.0	20	4	3	2	1
1/62 - 12/62	117,603	0.0	9	2	1	1	0
1/63 - 12/63	123,484	0.0	4	1	0	0	0
1/64 - 12/64	129,658	0.0	2	0	0	0	0
1/65 - 12/65	136,141	0.0	0	0	0	0	0
1/66 - 12/66	142,948	0.0	0	0	0	0	0
1/67 - 12/67	150,095	0.0	0	0	0	0	0
Total			$10,488,515	$7,692,363	$7,039,737	$6,482,270	$6,002,160

[1] First calendar period based on most recent rate-making analysis; subsequent calendar periods based on selected per annum trend rate given above.

[2] From Exhibit A13.

[3] Product of the preceding two columns with the selected DDR premium provision given above.

Exhibit A18

Reserving for Extended Reporting Endorsement Coverage, Including the Death, Disability, and Retirement Policy Provisioin

Analysis of DDR Unearned Premium Reserve

Under Current Methodology, as of December 31, 2009

Indicated Payment Pattern

Report Year	**Paid and ALAE Limited to Total Limits** 6	18	30	42	54	66	78	90	102	114	126	138	Ultimate Loss & ALAE [1]
1998	25,164	89,752	1,130,799	2,542,591	2,967,665	2,995,798	3,437,037	3,559,235	3,801,522	4,491,207	5,169,227	5,424,163	5,424,163
1999	6,521	105,758	1,003,750	1,031,549	1,473,937	1,941,283	2,214,501	2,377,473	2,396,646	2,396,646	2,396,646		2,396,646
2000	40,354	454,320	714,843	2,533,727	5,365,433	7,885,766	8,493,316	8,775,703	9,206,638	9,206,638			9,206,638
2001	3,676	81,617	1,970,348	2,133,686	2,784,400	3,241,764	3,339,004	3,789,487	4,067,321				4,793,956
2002	30,553	173,218	1,894,060	2,377,270	2,591,217	4,738,023	4,844,448	5,411,389					7,511,520
2003	19,591	255,093	1,730,200	5,041,606	7,464,685	8,363,029	8,367,549						8,367,549
2004	11,365	64,274	997,369	7,616,690	12,400,882	12,098,981							17,946,284
2005	31,269	923,539	7,581,324	12,268,755	13,911,166								19,199,929
2006	53,595	1,172,375	5,880,446	7,986,521									14,844,834
2007	48,480	990,712	3,243,150										10,900,615
2008	57,096	261,824											13,165,910
2009	24,899												16,783,983

Report Year	**Months of Development** 6	18	30	42	54	66	78	90	102	114	126	138
1998	0.5%	1.7%	20.8%	46.9%	54.7%	55.2%	63.4%	65.6%	70.1%	82.8%	95.3%	100.0%
1999	0.3%	4.4%	41.9%	43.0%	61.5%	81.0%	92.4%	99.2%	100.0%	100.0%	100.0%	
2000	0.4%	4.9%	7.8%	27.5%	58.3%	85.7%	92.3%	95.3%	100.0%	100.0%		
2001	0.1%	1.7%	41.1%	44.5%	58.1%	67.6%	69.7%	79.0%	84.8%			
2002	0.4%	2.3%	25.2%	31.6%	34.5%	63.1%	64.5%	72.0%				
2003	0.2%	3.0%	20.7%	60.3%	89.2%	99.9%	100.0%					
2004	0.1%	0.4%	5.6%	42.4%	69.1%	67.4%						
2005	0.2%	4.8%	39.5%	63.9%	72.5%							
2006	0.4%	7.9%	39.6%	53.8%								
2007	0.4%	9.1%	29.8%									
2008	0.4%	2.0%										
2009	0.1%											
Average	0.3%	3.8%	27.2%	46.0%	62.2%	74.3%	80.4%	82.2%	88.7%	94.3%	97.7%	100.0%
Weighted Average	0.3%	4.0%	26.0%	48.5%	65.4%	74.2%	81.4%	81.5%	89.2%	94.5%	96.7%	100.0%
Average L5	0.3%	4.8%	27.0%	50.4%	64.7%	76.7%	83.8%	82.2%				
Average L3	0.3%	6.3%	36.3%	53.4%	76.9%	76.8%	78.0%	82.1%	94.9%	94.3%		
Selected	6 - Ult	18 - Ult	30 - Ult	42 - Ult	54 - Ult	66 - Ult	78 - Ult	90 - Ult	102 - Ult	114 - Ult	126 - Ult	138 - Ult
Cumulative	0.3%	5.0%	30.0%	50.0%	65.0%	75.0%	80.0%	85.0%	90.0%	95.0%	97.5%	100.0%

Tail Year Payment Pattern

	12 - Ult	24 - Ult	36 - Ult	48 - Ult	60 - Ult	72 - Ult	84 - Ult	96 - Ult	108 - Ult	120 - Ult	132 - Ult	144 - Ult	156 - Ult
Cumulative[2]	0.5%	3.0%	5.0%	30.0%	50.0%	65.0%	75.0%	80.0%	85.0%	90.0%	95.0%	97.5%	100.0%
Incremental	**0.5%**	**2.5%**	**2.0%**	**25.0%**	**20.0%**	**15.0%**	**10.0%**	**5.0%**	**5.0%**	**5.0%**	**5.0%**	**2.5%**	**2.5%**

[1] Based on claims-made reserve analysis.

[2] In order to determine a "tail year" payment pattern, we assume an 18 month lag between a report year pattern and a tail year pattern (12 and 24 month factors are judgmentally selected)

Bootstrapping Generalized Linear Models for Development Triangles Using Deviance Residuals

Thomas Hartl, ACAS

Abstract: This paper presents a practical study of how to bootstrap a development triangle using a generalized linear model (GLM) and deviance residuals. We also point out some limitations inherent in bootstrapping approaches. (Interested readers can contact the author and request a copy of an MS Excel application to further explore the concepts discussed in this paper.) First we demonstrate how Pearson residual bootstrapping can fail when applied to GLMs because of their linear rescaling properties. Next we describe an algorithm for rescaling deviance residuals based on the identity variance function. We continue with an example where Pearson residual bootstrapping fails, while deviance residuals bootstrapping works. We then present bootstrap simulation results for two GLMs: one where both approaches work and the original example where only deviance residuals can be applied. Subsequently we prove that deviance residuals based on the identity variance function are bounded below for any given data point with the lower bound depending on the fitted value for the data point. We then give an example of a GLM where deviance residual bootstrapping fails because of this property. The paper concludes with a discussion of "distribution-free" versus parametric resampling.

Keywords. Bootstrapping and Resampling Methods, Generalized Linear Modeling, Reserve Variability, Reserving Methods, Nonparametric Methods.

1. INTRODUCTION

In the context of stochastic reserving, several authors (e.g., [4], [7] and [9]) have stressed the need of casting the task of projecting reserves in a rigorous way as a regression problem. Several authors (e.g., [4], [7] and [9]) have also pointed out that performing an all-years, volume weighted, link-ratio estimate leads to the same result as fitting a GLM with the logarithmic link function and the identity variance function. Some papers (e.g., [4]) and many practitioners have exploited this equivalence to implement spreadsheet based applications for deriving a distribution of possible reserve outcomes based on bootstrap simulations by repeated resampling and application of the link-ratio estimate. While suitable to illustrate the concept of bootstrapping, these applications are typically not flexible enough to deal with practical judgments reserving analysts have to make about which cells of the triangle are deemed to be representative of future development (e.g., use data from last *n* diagonals or exclude obvious abnormalities). In [5] the author of this paper presented a practical account of how to rigorously translate such judgments into a well-defined regression problem using the apparatus of GLM theory. This paper builds on the framework established in [5] and presents a case study to give a practical demonstration of some limitations inherent in non-parametric bootstrapping approaches based on Pearson or deviance residuals.

1.1 Research Context

General accounts of how to apply bootstrapping methods to a GLM for a incremental development triangle have been provided in [4] and [7]. Details on how to apply bootstrapping to a GLM can also be found in chapter 7.2 of [3]. This paper demonstrates that the common Boostrapping approach based on Pearson residuals does break down when the linear rescaling of residuals leads to negative incremental values in the resampling distribution for some triangle cells. In [3], [4], and [7] the authors do mention that there are also alternative ways of defining residuals. We illustrate that a bootstrapping approach based on deviance residuals (using the identity variance function) may succeed where bootstrapping based on Pearson residuals fails. As we demonstrate, this alternative approach also has practical limitations. At least for the identity variance function, deviance residuals are technically not identically distributed, and it may not be possible to rescale all residuals for resampling purposes. Throughout this paper we use the GLM framework for incomplete development triangles established in [5]. Our case study suggests that there may be good practical reasons to prefer parametric resampling over nonparametric resampling.

1.2 Objective

Bootstrapping has become a popular method for deriving distributions of reserve outcomes based on development triangles. This paper provides a case study to demonstrate some inherent limitations of applying versions of this method to a GLM for an incremental development triangle. These practical limitations point to the need for further research into alternative resampling schemes. We also hope that the case study and the discussion of the issues encountered will provide readers with a better understanding of what bootstrapping really is.

1.3 Outline

The remainder of the paper proceeds as follows. Section 2 explains the difference between linear and non-linear rescaling of residuals, introduces a version of a Newton-Raphson algorithm for rescaling deviance residuals based on the identity variance function, and presents an example of a GLM for an incremental development triangle for which bootstrapping with deviance residuals is possible while bootstrapping with Pearson residuals fails. In section 3 we analyze a limitation of bootstrapping with deviance residuals based on the identity variance function. We demonstrate that the theoretical distribution of deviance residuals for a given triangle cell is bounded below, and that the bound varies with the square root of the expected mean. This means that negative deviance residuals for triangle cells with larger expected means may be "out of bounds" for some triangle cells with smaller expected means. We provide an example of a GLM for an incremental development triangle where bootstrapping with deviance residuals is not possible for this reason. In the summary and discussion section we review our results provide and invite the reader to explore the concepts presented with the accompanying MS Excel file. In the conclusion we reflect on the "distribution free" or "non-parametric" attributes of bootstrapping approaches. We suggest that in the context of GLMs (or other stochastic models) for development triangles "parametric" resampling may be just as useful while avoiding some of the limitations demonstrated in this paper.

2 RESCALING OF RESIDUALS FOR RESAMPLING

In general bootstrapping deals with the heteroscedasticity of the underlying stochastic model by rescaling the residuals obtained from a specific data point so they can be applied to the expected means of other data points. The procedure for rescaling depends on the definition of residual used. While various residuals are mentioned in [4] and [7], the examples presented in these papers use Pearson residuals. Here we demonstrate a limitation imposed by the linear rescaling of Pearson residuals and contrast this with the non-linear rescaling properties of deviance residuals.

2.1 Linear vs. non-linear rescaling

We assume that we have a collection of suitably standardized residuals that were obtained by fitting a GLM to an incremental development triangle using pseudo-likelihood with the identity variance function and the natural logarithm as the link function. Suitably standardized means that the standardized residuals for various triangle cells can be considered as being approximately independent identically distributed (iid). We will return to the question of whether deviance residuals can be considered iid in section 3 of this paper. For the time being we simply follow the standardization suggested by equations 12.4 and 12.5 on page 397 in [6]. So, in continuing we assume that we a have vector of standardized residuals denoted by **s**. Note that technically this vector will be defined differently for Pearson and deviance residuals, but we will not distinguish this in our notation. For any given data point we can think of the residuals as a measure of how much an actual observation differs from the expected value for that observation. If the expected value, $\hat{y}$, for the observation is considered a constant parameter the residual, r, is a function of the actual observation, y, alone:

$$r = \mathrm{H}_{\hat{y}}(y). \tag{2.1}$$

In abstract terms the rescaling of the standardized residuals, **s**, is accomplished by applying the functional inverse of $\mathrm{H}_{\hat{y}}$ to the elements of **s**. So the vector of resampling values **y*** can be defined by:

$$\mathbf{y}^* = \mathrm{H}_{\hat{y}}^{-1}(\mathbf{s}). \tag{2.2}$$

Noting that $\mathrm{V}(\hat{y})$ stands for the variance function of the expected mean of a data point, equation (2.1) for Pearson residuals becomes:

$$r_{\mathrm{P}} = \frac{y - \hat{y}}{\sqrt{\mathrm{V}(\hat{y})}}. \tag{2.3}$$

As one can seem this is a linear function of y, and equation (2.2) therefore also takes linear form:

$$\mathbf{y}_{\mathrm{P}}^* = \hat{y} + \sqrt{\mathrm{V}(\hat{y})} \cdot \mathbf{s}. \tag{2.4}$$

So the distribution of resampling values is a linear transformation of the standardized residuals, with the same rescaling factor being applied to all standardized residuals.

Since the standardized residuals have a mean of zero,[1] this also means that for sufficiently small expected values, $\hat{y}$, there will always be negative values in the resampling distribution. For a GLM with a logarithmic link function, this represents a violation of the fundamental model assumption of positive incremental values and the MLE algorithm cannot be applied since we cannot take the logarithm of a negative number.

In the context of quasi-likelihood estimation, equation (2.1) for deviance residuals can generally be expressed as follows (see equation 9.4 on page 327 in [6]):

$$r_{\mathrm{D}} = \mathrm{sign}(y - \hat{y}) \cdot \sqrt{2\int_{\hat{y}}^{y} \frac{y - t}{\mathrm{V}(t)}\, dt}. \tag{2.5}$$

Needless to say that dealing with this sort of expression is mathematically more complex than the relatively simple functional form of equation (2.3). In particular it is not possible to give a general expression for the functional inverse analogous to equation (2.4). We will explain how to approach this task numerically for the identity variance function in the next subsection. We conclude this subsection by noting that while highly non-linear, deviance residuals are perfectly well-behaved, and provided that $\mathrm{V}(t)$ goes to 0 sufficiently fast as t goes to 0, we are guaranteed that inverting the function will not result in negative resampling values.

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[1] If necessary we will make a centering adjustment to guarantee that this is the case.

2.2 Rescaling deviance residuals based on the identity variance function

With the identity variance function (i.e., $V(\hat{y}) = \hat{y}$) the integral in equation (2.6) can be given in closed form and we get the following expression (see expression for Poisson distribution on page 39 in [6]) for the deviance residuals:

$$r_D = \text{sign}(y - \hat{y}) \cdot \sqrt{2\left(y \cdot \log\left(y / \hat{y}\right) - y + \hat{y}\right)}. \tag{2.6}$$

We are still not in a position to give a closed-form expression for the functional inverse, but we can numerically solve this using a variant of the Newton-Raphson algorithm (combined with the bisection method) based on the code provided on pages 366/7 in [8].[2] Since this is a well documented standard algorithm we will not go into all implementation details here. We will, however, give some details on the modifications we have made to tweak this to the concrete task at hand. Firstly, to simplify treatment of the sign of the residual and to get rid of the square root, we actually just invert the right-hand side of the following equation:

$$\left(r_D\right)^2 = 2 \cdot \hat{y} \cdot \left(y / \hat{y} \cdot \log\left(y / \hat{y}\right) - y / \hat{y} + 1\right). \tag{2.7}$$

Note that the right-hand side of this equation does not define a one-on-one function, so we need to choose an appropriate domain (i.e., upper and lower bound for y) based on the sign of the residual in question. To further simplify, we substitute $x = y/\hat{y}$ and $w = (r_D)^2 / (2 \cdot \hat{y}) - 1$, and thus arrive at:

$$w = x \cdot \log(x) - x. \tag{2.8}$$

In order to numerically solve this for x, we need upper and lower bounds depending on the value of w. With negative residuals, it suffices to restrict x to (0,1). For positive residuals the lower bound for x is clearly 1, but our algorithm uses more refined initial estimates as detailed in appendix 1. With this set-up, we solve for x in terms of w by using the Newton-Raphson algorithm to find the zero of the function $f(x)$ defined by:

$$f(x) = x \cdot \log(x) - x - w. \tag{2.9}$$

For those readers not familiar with the Newton-Raphson method, the algorithm proceeds by iterating over x until convergence is achieved, using the following formula:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{2.10}$$

[2] A pdf of the cited section is freely available at http://www.nrbook.com/a/bookcpdf/c9-4.pdf.

For completeness, in our case the derivative of $f(x)$ is given by:

$$f'(x) = \log(x). \tag{2.11}$$

As indicated above the actual algorithm implemented is mixing the Newton-Raphson method with the bisection method to prevent x from jumping out of bounds or to improve the speed of convergence when $f(x) << f'(x)$.

The interested reader can contact the author and request a copy of the companion MS Excel application to study and explore the source code of user defined function VB_PoissonDevianceResidual_Inverse. Also note that, as implemented here, the algorithm assumes that residuals have not been adjusted (i.e., normalized) for the dispersion factor.

Now that we know how to compute resampling distributions using both Pearson and deviance residuals, we can apply this apparatus to real-life data.

2.3 Example

We are using a data set that the authors of [7] attribute to Taylor and Ashe (1983). Here is the data in incremental form:

~~357,848~~	~~766,940~~	~~610,542~~	~~482,940~~	~~527,326~~	~~574,398~~	146,342	139,950	227,229	67,948
~~352,118~~	~~884,021~~	~~933,894~~	~~1,183,289~~	445,745	320,996	527,804	266,172	425,046	
~~290,507~~	~~1,001,799~~	~~926,219~~	1,016,654	750,816	~~146,923~~	495,992	280,405		
~~310,608~~	~~1,108,250~~	776,189	~~1,562,400~~	272,482	352,053	206,286			
~~443,160~~	693,190	991,983	769,488	504,851	470,639				
396,132	937,085	847,498	805,037	705,960					
440,832	847,631	1,131,398	1,063,269						
359,480	1,061,648	1,443,370							
376,686	986,608								
344,014									

To avoid visual clutter we have omitted row and column labels. There are no non-positive or missing data points, but we have chosen only to include the latest five diagonals of incremental values and to exclude three further triangle cells that were identified as outliers in preliminary analysis. Excluded data points from the original data set have been indicated by "~~strikethrough~~" formatting.

Fitting a GLM with the logarithm as link function and the identity variance function results in the following fitted values:

140,801	*338,807*	*431,201*	*358,694*	*242,579*	*197,553*	**185,516**	**116,383**	**211,622**	**67,948**
293,186	*705,487*	*897,876*	*746,898*	**505,115**	**411,359**	**386,295**	**242,341**	**440,653**	*141,486*
396,579	*954,279*	*1,214,515*	**1,010,295**	**683,246**	*556,426*	**522,523**	**327,803**	*596,051*	*191,382*
214,098	*515,178*	**655,669**	*545,419*	**368,858**	**300,393**	**282,090**	176,968	321,785	103,319
307,853	**740,778**	**942,791**	**784,261**	**530,383**	**431,937**	*405,619*	*254,464*	*462,697*	*148,564*
343,763	**827,188**	**1,052,766**	**875,744**	**592,251**	*482,321*	*452,933*	*284,146*	*516,669*	*165,893*
386,316	**929,583**	**1,183,083**	**984,148**	*665,564*	*542,025*	*509,000*	*319,320*	*580,625*	*186,429*
442,821	**1,065,549**	**1,356,128**	*1,128,096*	*762,913*	*621,305*	*583,450*	*366,025*	*665,551*	*213,697*
400,230	**963,064**	*1,225,695*	*1,019,595*	*689,536*	*561,548*	*527,333*	*330,821*	*601,538*	*193,143*
344,014	*827,792*	*1,053,534*	*876,383*	*592,684*	*482,673*	*453,264*	*284,354*	*517,046*	*166,014*

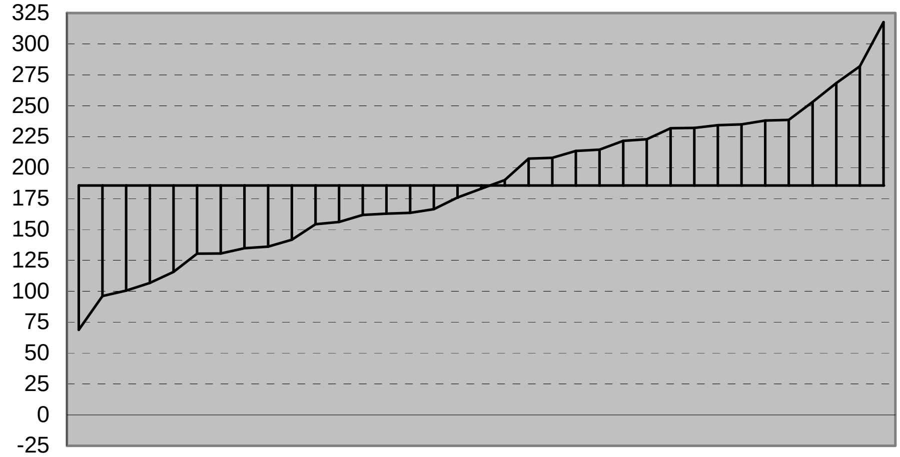
As the author of this paper has demonstrated in [5], this type of GLM does not only project expected values for future triangle cells but also extrapolates the expected values for all past triangle cells that were excluded from the analysis. In the above table, we show all fitted values that correspond to included data points in bold letters. All values in italics correspond to projections/extrapolations based on the fitted parameters for the model. Since we have chosen the identity variance function the reader can also verify that fitted values in bold preserve the row and column sums of the original data points for the included triangles cells.

To bootstrap this GLM, we repeatedly generate pseudo-data for each of the included triangle cells (see bold-face fitted values in the above table). During each iteration step we re-estimate the GLM based on the pseudo-data generated. For a stochastic reserving application we also calculate the estimated reserve and save the total by accident year or in aggregate to get a simulated distribution of reserve estimates to evaluate the inherent parameter error. Typically one would similarly simulate future development amounts to account for the process error.

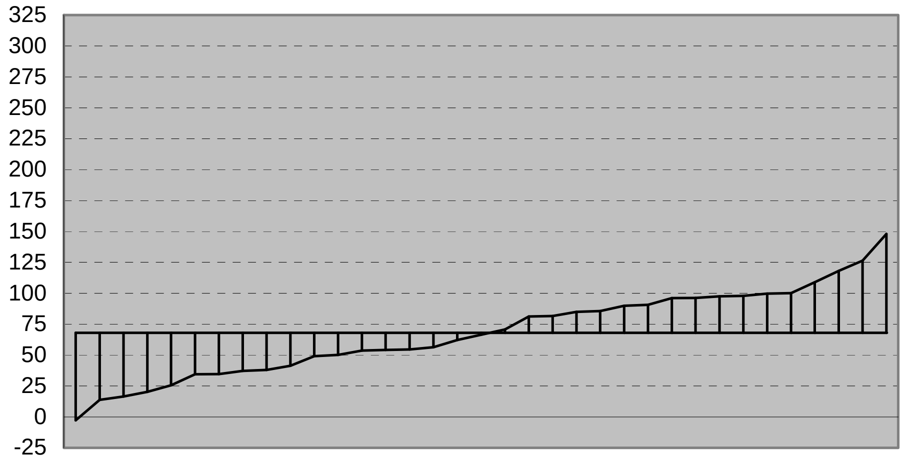
Effectively this resampling process defines a resampling distribution for each data point, which is obtained by rescaling all available standardized residuals and applying them to the fitted values as described in the previous two subsections. These resampling distributions can be pre-computed and stored both to save execution time during each bootstrap iteration, and to evaluate them for consistency with the underlying model assumptions. Here our main concern is with negative incremental values, since these prevent us from fitting the model with the MLE algorithm. On the following two pages we graph the resampling distributions for two of the triangle cells. Figure 1 shows the results obtained based on Pearson residuals and Figure 2 shows the corresponding results using deviance residuals.

Resampling Distributions with Pearson Residuals

Triangle Cell (1,7)—Fitted Mean = 185,516



Triangle Cell (1,10)—Fitted Mean = 67,948



Triangle Cell (1,10)—Fitted Mean = 67,948—Values Below Mean

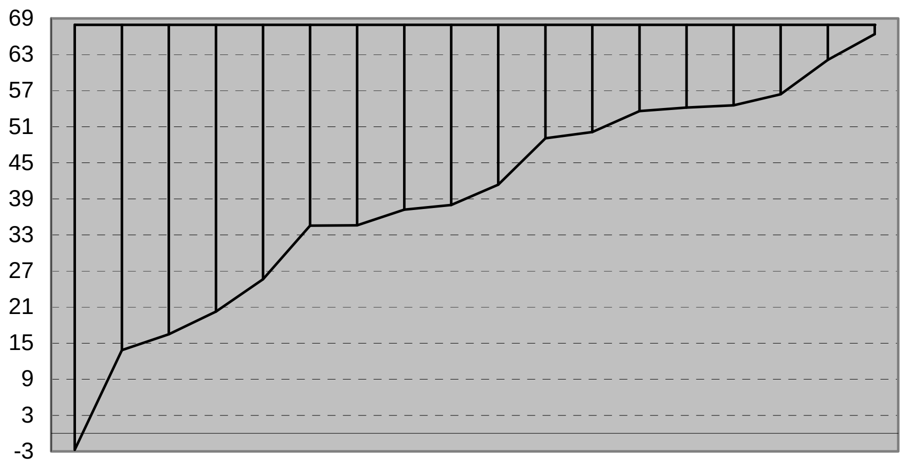
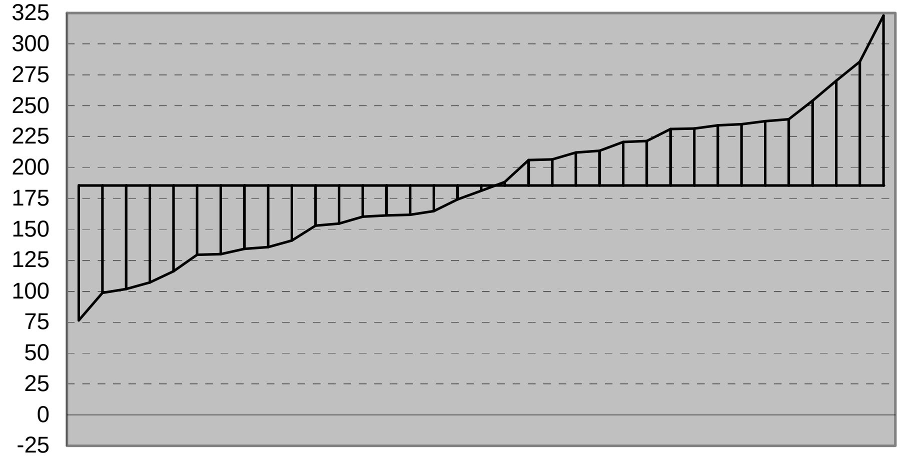


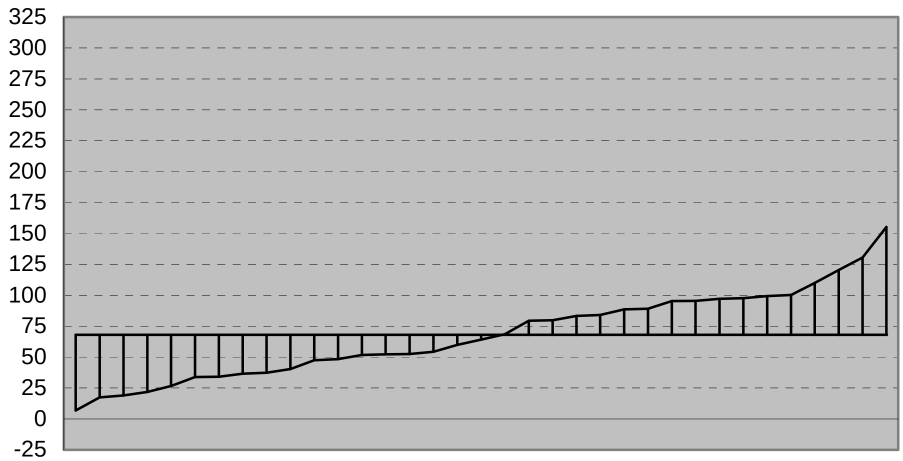
Figure 1

Resampling Distributions with Deviance Residuals

Triangle Cell (1,7)—Fitted Mean = 185,516



Triangle Cell (1,10)—Fitted Mean = 67,948



Triangle Cell (1,10)—Fitted Mean = 67,948—Values Below Mean

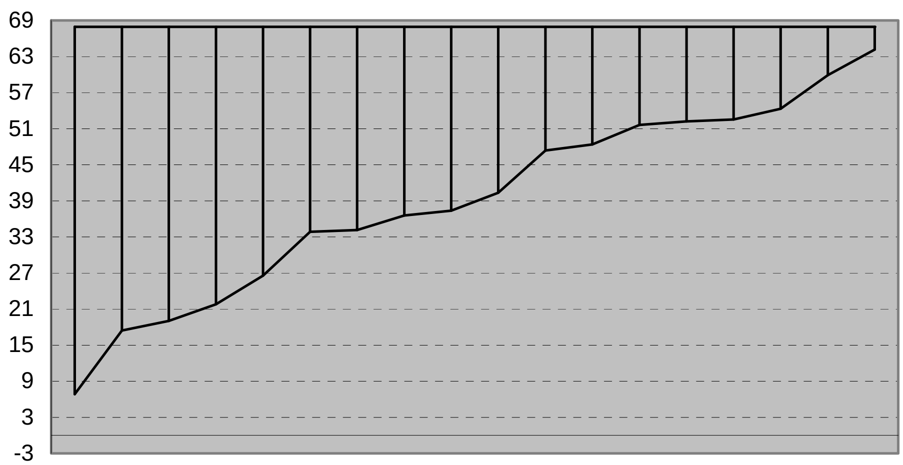


Figure 2

Overall the two sets of resampling distributions are very similar. Comparing the top graphs for triangle cell (1,7) one can see that the smallest value obtained by Pearson residual resampling is below that obtained by deviance residual resampling. The same holds true for the largest value. The fact that both distributions nevertheless have the same mean of 185,516 (and approximately the same variance) is the first indication that deviance residuals are subtly different.

Comparing the middle graphs for triangle cell (1,10) to the corresponding top graphs for cell (1,7) one notices how the spread from minimum to maximum resampling value is smaller for triangle cell (1,7) than for triangle cell (1,10). This is the effect of the rescaling to ensure that each resampling distribution has the appropriate variance as defined by the variance function assumed in the GLM specification.

Since our main concern here is with negative incremental values we have included slightly larger scale graphs of the lower part of the resampling distribution for triangle cell (1,10) at the bottom of figure 1 and 2. As we can see, Pearson resampling leads to one negative resampling value, while deviance resampling stays positive for the corresponding residual.

In this particular example the negative resampling value for cell (1,10) is actually the only negative value resulting from Pearson resampling. Nevertheless it does mean that the GLM as specified cannot be bootstrapped using Pearson residuals. One alternative would be to exclude cell (1,10) from the model specification. In this particular case this leads to a model that projects a reserve of versus a reserve of for the original model including cell (1,10). Noting that excluding cell (1,10) amounts to assuming zero development for the 10^{th} development period, it is not surprising that the projected reserve is smaller.

To demonstrate the differences in outcomes, we conclude this section with output obtained by bootstrapping the model excluding cell (1,10) both with Pearson and with deviance residuals. We also include the bootstrapping results for the model including cell (1,10). The "Modeled Reserve" column of each output table shows the reserve projection representing the expected future development amounts that result from fitting the GLM. The "Bootstrap Projection" column shows the mean of the reserve projections based on simulated data. Note that the "Bootstrap Projection" distribution defines the parameter error. The "Simulated Future Development" column shows the mean of the simulated future development amounts used to incorporate process error. Comparing the "Modeled Reserve" to "Simulated Future Development" also allows one to gauge whether we have sufficiently many bootstrap iterations to keep bias resulting from sampling error to an acceptable level. The "Standard Prediction Error" is the root of the mean square error of simulated reserve outcomes (i.e., projected reserve based on pseudo-data less simulated future development). All the way to the right we also show a confidence interval based on empirical percentiles of simulated reserve outcomes. Note that a positive number represents a reserve projection above the simulated future development amount.

Bootstrapping Results with 10,000 Iterations

Excluding Cell (1,10)—Pearson Residuals

Accident Period	Modeled Reserve	Bootstrap Projection	Sim. Future Development	Standard Pred. Error	5%-ile Sim. Outcome	95%-ile Sim. Outcome
1	-	-	-	-	-	-
2	-	-	-	-	-	-
3	596,051	603,398	595,127	166,522	(254,940)	288,038
4	498,753	504,064	498,789	135,273	(214,047)	231,751
5	1,122,779	1,134,746	1,125,780	224,917	(345,901)	394,656
6	1,736,070	1,751,181	1,734,825	302,852	(467,485)	522,686
7	2,616,534	2,640,194	2,612,849	407,758	(613,245)	724,636
8	4,127,340	4,164,901	4,132,367	586,633	(892,087)	1,040,074
9	4,956,065	4,990,267	4,959,138	801,618	(1,232,452)	1,417,929
10	5,087,731	5,161,854	5,082,052	1,393,141	(2,030,612)	2,510,119
Total	**20,741,324**	**20,950,606**	**20,740,927**	**2,504,915**	**(3,645,668)**	**4,603,584**

Excluding Cell (1,10)—Deviance Residuals

Accident Period	Modeled Reserve	Bootstrap Projection	Sim. Future Development	Standard Pred. Error	5%-ile Sim. Outcome	95%-ile Sim. Outcome
1	-	-	-	-	-	-
2	-	-	-	-	-	-
3	596,051	601,425	595,682	165,133	(254,824)	283,543
4	498,753	502,686	497,483	135,937	(213,619)	233,681
5	1,122,779	1,130,897	1,122,776	225,374	(348,761)	388,058
6	1,736,070	1,748,560	1,735,691	300,235	(460,344)	514,240
7	2,616,534	2,636,940	2,619,302	409,205	(630,112)	709,919
8	4,127,340	4,156,304	4,128,423	582,196	(885,958)	1,016,114
9	4,956,065	5,002,022	4,962,549	802,422	(1,215,009)	1,405,855
10	5,087,731	5,169,300	5,088,560	1,404,841	(2,048,259)	2,501,894
Total	**20,741,324**	**20,948,135**	**20,750,465**	**2,530,813**	**(3,764,693)**	**4,599,894**

Including Cell (1,10)—Deviance Residuals

Accident Period	Modeled Reserve	Bootstrap Projection	Sim. Future Development	Standard Pred. Error	5%-ile Sim. Outcome	95%-ile Sim. Outcome
1	-	-	-	-	-	-
2	141,486	148,558	141,810	99,435	(142,027)	181,427
3	787,433	802,512	786,345	227,758	(332,132)	415,547
4	602,073	612,774	600,459	168,556	(252,157)	302,197
5	1,271,343	1,290,547	1,271,004	266,900	(394,291)	476,089
6	1,901,963	1,926,750	1,906,391	343,783	(513,444)	607,984
7	2,802,963	2,834,990	2,804,315	448,446	(679,871)	795,858
8	4,341,037	4,384,089	4,338,730	639,559	(958,332)	1,144,621
9	5,149,209	5,209,231	5,145,549	844,468	(1,259,637)	1,509,566
10	5,253,745	5,354,869	5,249,988	1,444,013	(2,074,331)	2,567,191
Total	**22,251,251**	**22,564,319**	**22,244,592**	**2,868,629**	**(4,054,094)**	**5,235,817**

Figure 3

3 A PRACTICAL LIMIT OF DEVIANCE RESIDUALS

In the previous section we demonstrated that the non-linear rescaling properties of deviance residuals allow us to bootstrap a GLM in some instances where Pearson residuals lead to negative values in the resampling distribution for some data points. This does not mean that any GLM for an incomplete development triangle can be bootstrapped using deviance residuals. In this brief section we explore the mathematical reason for why this is the case for deviance residuals based on the identity variance function. We also give an example of a GLM based on the same data set used in the previous section where deviance residual resampling cannot be applied.

3.1 Taking the limit

For convenience we repeat the definition of deviance residuals based on the identity variance function (i.e., equation 2.):

$$r_{\mathrm{D}} = \mathrm{sign}(y - \hat{y}) \cdot \sqrt{2\left(y \cdot \log\left(y / \hat{y}\right) - y + \hat{y}\right)}\,. \tag{3.1}$$

Here we are interested in the lower limit as $y \to 0$, hence we can substitute -1 for sign(y-ŷ). After some rearranging we obtain:

$$\lim_{y\to 0} r_{\mathrm{D}} = -\sqrt{2 \cdot \lim_{y\to 0}\left(\frac{\log\left(y/\hat{y}\right)}{y^{-1}} - y + \hat{y}\right)}\,. \tag{3.2}$$

Dealing with the easy parts we get:

$$\lim_{y\to 0} r_{\mathrm{D}} = -\sqrt{2} \cdot \sqrt{\lim_{y\to 0}\left(\frac{\log\left(y/\hat{y}\right)}{y^{-1}}\right) + \hat{y}}\,. \tag{3.3}$$

The remaining part goes to -∞/∞ as $y \to 0$, so we can use l'Hôpital's rule to evaluate it, leading to:

$$\lim_{y\to 0} r_{\mathrm{D}} = -\sqrt{2} \cdot \sqrt{\lim_{y\to 0}\left(\frac{\left(y/\hat{y}\right)^{-1}}{-y^{-2}}\right) + \hat{y}}\,. \tag{3.4}$$

This simplifies to:

$$\lim_{y\to 0} r_{\mathrm{D}} = -\sqrt{2} \cdot \sqrt{\lim_{y\to 0}\left(-y \cdot \hat{y}\right) + \hat{y}} = -\sqrt{2\hat{y}}\,. \tag{3.5}$$

So we can see that for any given triangle cell the smallest theoretical value for the deviance residual is $-(2\hat{y})^{.5}$. Obviously this result is dependent on the particular functional form of the deviance residual, which in the case of the identity variance function is given by equation 3.1.

This does raise the question of whether deviance residuals can be considered approximately iid, which is a fundamental underlying assumption of resampling methods. Theory aside, we are left with a practical issue when trying to use deviance residuals for bootstrapping: a deviance residual obtained from a data point with a larger expected mean may be below the lower bound for deviance residuals for a data point with a smaller expected mean. If this happens we cannot rescale the deviance residual in question for the purpose of resampling. So, if $\hat{y}_{min}$ is the smallest fitted value for a particular GLM with the identity variance function, we can only use deviance residual bootstrapping, if for all data points included in the model we have

$$r_D > -\sqrt{2 \cdot \hat{y}_{min}} \, . \tag{3.6}$$

We will now demonstrate that it is not difficult to come up with an example where this relationship does not hold for all data points.

3.2 Example

The GLM used as an example here is not very different from the one introduced in section 2.3. We include cells (1,6) and (3,6) which were previously excluded:

~~357,848~~	~~766,940~~	~~610,542~~	~~482,940~~	~~527,326~~	574,398	146,342	139,950	227,229	67,948
~~352,118~~	~~884,021~~	~~933,894~~	~~1,183,289~~	445,745	320,996	527,804	266,172	425,046	
~~290,507~~	~~1,001,799~~	~~926,219~~	1,016,654	750,816	146,923	495,992	280,405		
~~310,608~~	~~1,108,250~~	776,189	~~1,562,400~~	272,482	352,053	206,286			
~~443,160~~	693,190	991,983	769,488	504,851	470,639				
396,132	937,085	847,498	805,037	705,960					
440,832	847,631	1,131,398	1,063,269						
359,480	1,061,648	1,443,370							
376,686	986,608								
344,014									

As before we fit a GLM with a logarithmic link function and the identity variance function. This results in the following fitted values:

254,672	*611,704*	*774,193*	*665,389*	*434,726*	**320,588**	**299,529**	**184,715**	**283,087**	**67,948**
332,131	*797,756*	*1,009,667*	*867,770*	**566,950**	**418,096**	**390,632**	**240,897**	**369,188**	*88,615*
359,730	*864,049*	*1,093,569*	**939,880**	**614,062**	**452,839**	**423,093**	**260,915**	*399,867*	*95,978*
223,757	*537,449*	**680,214**	*584,618*	**381,955**	**281,672**	**263,169**	*162,293*	*248,723*	*59,700*
311,253	**747,608**	**946,198**	**813,221**	**531,310**	**391,814**	*366,076*	*225,754*	*345,981*	*83,044*
343,043	**823,968**	**1,042,841**	**896,282**	**585,577**	*431,833*	*403,467*	*248,812*	*381,319*	*91,526*
384,679	**923,974**	**1,169,412**	**1,005,065**	*656,650*	*484,245*	*452,436*	*279,011*	*427,600*	*102,635*
444,666	**1,068,059**	**1,351,772**	*1,161,796*	*759,049*	*559,759*	*522,990*	*322,520*	*494,280*	*118,640*
400,741	**962,553**	*1,218,240*	*1,047,030*	*684,067*	*504,464*	*471,327*	*290,660*	*445,454*	*106,920*
344,014	*826,299*	*1,045,792*	*898,818*	*587,234*	*433,055*	*404,608*	*249,516*	*382,397*	*91,785*

We can see that if $\hat{y}_{\min}$ = 67,948 from cell (1,10). For bootstrapping purposes this results in a lower bound of -368.64 for (unscaled) deviance residuals. Applying equation 3.1 to cell (3,6) with y = 146,923 and $\hat{y}$ = 452,839 we get the following (unscaled) deviance residual:

$$r_{\mathrm{D}} = -530.16\,. \tag{3.7}$$

Hence we can see that it is not possible to bootstrap this GLM using deviance residuals. Finally, our discussion has focused on the resampling of included data points. If the residuals are also used for simulating the process error, the allowable minimum would also depend on the smallest expected value for future development periods.

4. RESULTS AND DISCUSSION

In section 2.1 we showed how resampling with Pearson residuals can lead to negative incremental values in the resampling distribution. This in turn means that we cannot apply the MLE algorithm to for fitting the GLM during bootstrap iterations. We also presented the concept of deviance residuals and explained how these hold the promise of avoiding the issue of negative incremental values.

Section 2.2 provided details on how to compute the inverse of the deviance residual function based on the identity variance function. The algorithm is based on a variation of the Newton-Raphson method.

In section 2.3 an example of a GLM was presented, where bootstrapping with Pearson residuals is not possible, but bootstrapping with deviance residuals works. The crucial difference in the resampling distributions resulting from the different techniques was graphically illustrated. We also showed bootstrapping output for a slightly modified model where both approaches work and for the main example where only the deviance residual approval can be applied. It should be noted that deviance residuals therefore can broaden the scope of bootstrapping approaches.

In section 3.1 we proved that for any given data point, the deviance residuals for log-link GLMs with the identity variance function are bounded below $-(2\hat{y})^{.5}$, where $\hat{y}$ is the fitted value for that data point. This result means that Boostrapping with deviance residuals can only work if all standardized deviance residuals exceed the lower bound for the smallest fitted value. In section 3.2 we presented another GLM (again a slightly modified version of the example in section 2.3) where this condition is indeed violated.

The material in this paper is based on standard GLM theory and standard numerical methods. We hope this paper contributes to making more actuaries aware of how these powerful methods can practically be applied in the context of stochastic reserving. Given the popularity of bootstrapping approaches, we also feel that it is also important to draw attention to some of their inherent limitations.

Interested readers are encouraged to contact the author and request a copy of the companion MS Excel application to further explore the concepts and algorithms presented in this paper. Interacting with this application should prove a useful aid to gaining a deeper understanding of what regression models can accomplish in the context of development triangles. As far as the bootstrapping functionality is concerned, the application follows the approach outlined in [7]. In particular we use "procedure 2" as outlined in Figure 2 of that paper. As pointed out in [7], this is also the approach described in [3].

We want to conclude this section reflecting on "distribution-free" (or "non-parametric") versus parametric approaches to bootstrapping. Before bootstrapping was applied in an actuarial context, it was introduced as stochastic modeling techniques for detecting bias for estimators or to derive confidence intervals for parameter estimates in cases where standard regression may not work well because the underlying error structure is not normally distributed (or does not follow other known error distributions, for which specialized regression techniques are available). Especially in cases where there is a decent number of observations which all can be assumed to come from the same underlying distribution, bootstrapping can provide results that are superior to those obtained by applying standard regression techniques, based on assumptions not satisfied by the data at hand. This is the context in which "distribution-free" approaches shine.

The question to consider here is "What makes bootstrapping attractive in the context of stochastic reserving?" One advantage is that bootstrapping can derive a distribution for just about any function one may want to calculate based on an observed sample of data points. Note that the estimated reserves technically are not the fitted model itself, but a projection of expected amounts for future (i.e., out of sample) development periods. Hence bootstrapping is useful, because we can just recalculate the reserve based on the pseudo data generated for each iteration, and thus we simulate an empirical distribution of reserve estimates. There is no need to theoretically understand how the errors in the parameter estimates of the underlying model (and their correlations) might affect the distribution of reserve estimates.

The "distribution free" aspect of bootstrapping does not appear to be particularly important in the context of stochastic reserving based on development triangles. Often the data sample (number of triangle cells) is not particularly large in relation to the number of parameters we are trying to estimate. Furthermore, it is not obvious that the error structure for incremental development in the first year, for example, is in anyway systematically related to the error structure in the fifth year. Variation in the first year may be due to subtle variation in the mix of claims for subcoverages, while fifth-year development might be caused by sporadic late reporting claims, or an unexpected judicial decision for a single open claim. Non-parametric bootstrapping is based on the assumption that (after standardization and dealing with heteroscedasticity) the residuals are the best available approximation for the error structure driving the underling stochastic process.

So, can we harvest the power of bootstrapping as a simulation technique for deriving a distribution of reserve estimates while not implicitly relying on treating the residuals as our best approximation to the "true" error structure? The answer is "yes." If we can make educated guesses about how the error structure for various development periods should look like (preferably in the shape of assumed parametric distributions), we can generate pseudo-data based on these educated guesses and then continue with calculating the resulting reserve and thus build up an empirical distribution of reserve estimates Monte Carlo style. This type of approach also avoids the practical limitations of Pearson or deviance residual bootstrapping we demonstrated in this paper.

5. CONCLUSIONS

Our case study of GLM-based bootstrapping for incremental development triangles reveals a serious limitation of the standard approach based on Pearson residuals: the possible existence of negative resampling values due to the linear nature of the rescaling procedure. We demonstrated that the obstacle of negative resampling value can be overcome by using deviance residuals that rely on non-liner rescaling. We also proved that deviance residuals based on the identity variance function cannot be used for resampling under all circumstances. A practical example of this was provided. In our discussion, we suggested that in a stochastic reserving context the main advantage of bootstrapping is that it can generate a distribution of reserve estimates that accounts for parameter correlations imposed by the estimation process. To use this advantage, we can also employ resampling schemes that are based on assumed parametric distributions and thus circumvent the limitations of non-parametric resampling revealed by this case study. Further research may also reveal more robust non-parametric resampling schemes.

Acknowledgment

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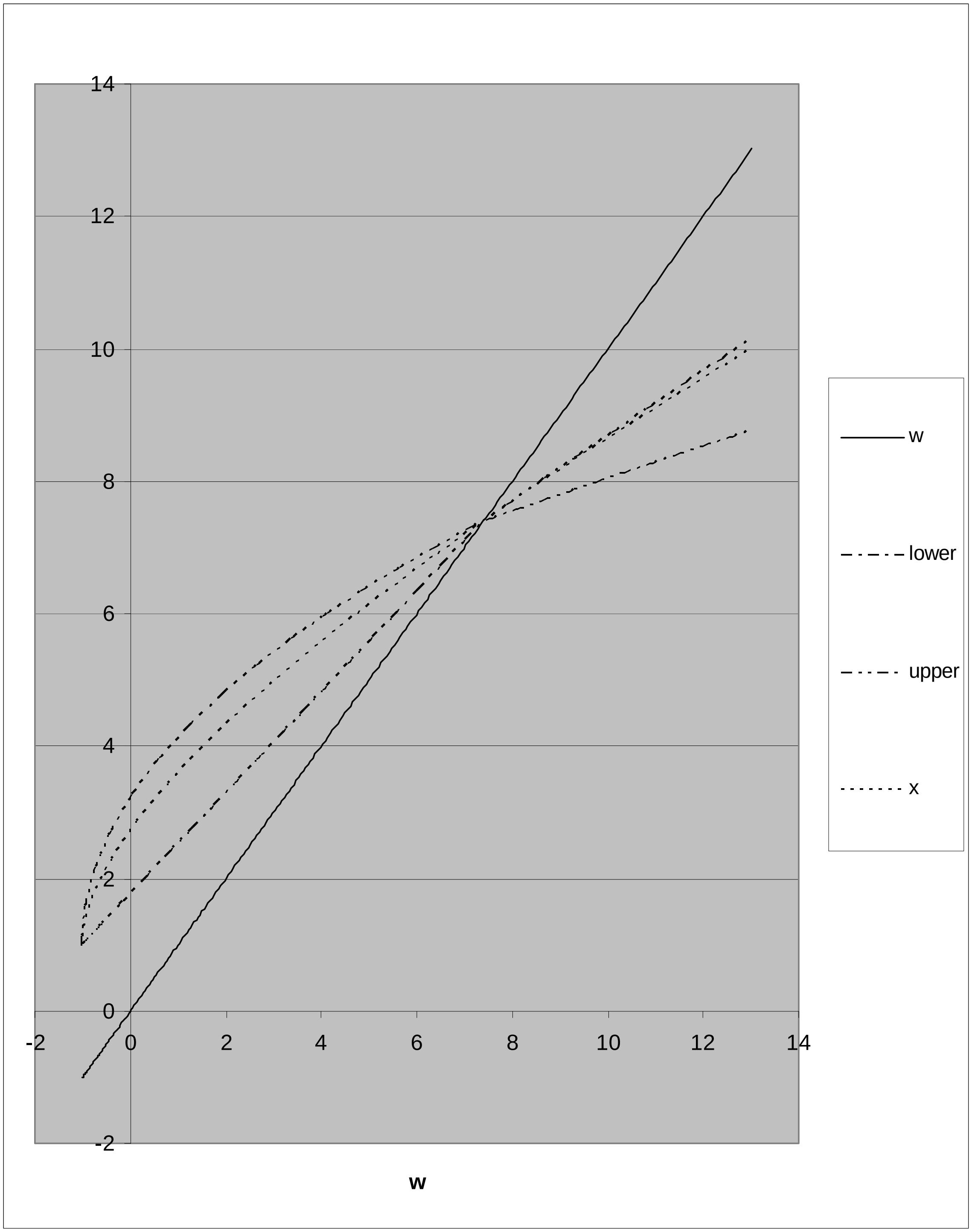
APPENDIX A

In the case of positive deviance residuals we use the following upper and lower bounds to initialize the inversion algorithm:

w	Lower Bound	Upper Bound
-1	1	1
$\left(-1, e^2\right)$	$1+\dfrac{(w+1)\cdot\left(e^2-1\right)}{\left(e^2+1\right)}$	$1+\dfrac{\sqrt{w+1}\cdot\left(e^2-1\right)}{\sqrt{e^2+1}}$
e^2	e^2	e^2
$> e^2$	$\dfrac{w}{\ln(w)}+\dfrac{e^2}{2}$	$\dfrac{2\cdot w}{\ln(w)}$

We do not provide a formal proof that these bounds are valid, but *x*, *w*, and the lower and upper bounds can easily be plotted to visually demonstrate that this is case (see next page).


14
12
10
8
6
4
2
0
-2
-2
0
2
4
6
8
10
12
14
w
w
lower
upper
x

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Abbreviations and notations

GLM, generalized linear model
LSQ, least squares
MLE, maximum likelihood estimator (or estimation)
iid, independent identically distributed

Biography of the Author

Thomas Hartl is a Manager within PricewaterhouseCoopers' Actuarial Insurance Management Solutions (AIMS) practice. He provides consulting services to insurance companies, reinsurers, and regulators. His responsibilities include the design, validation, and implementation of simulation models supporting statistical analysis for ERM, litigation support, predictive modeling, and stochastic reserving. Thomas Hartl is an Associate of the Casualty Actuarial Society, a Member of the American Academy of Actuaries, and holds a Ph.D. in mathematics from the University of Glasgow, Scotland.

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Fitting a GLM to Incomplete Development Triangles

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Abstract: When fitting a generalized liner model (GLM) to a development triangle is discussed in the existing actuarial literature, reference is usually made to statistical packages for accomplishing this task. This paper presents a practical discussion of how to use Visual Basic to fit a GLM to a triangle with special emphasis on how to deal with incomplete data. Interested readers can contact the author to request a copy of an MS Excel application that implements the algorithms discussed in this paper. The application of GLMs to incomplete development triangles is motivated by translating judgments of practicing actuaries (e.g., use last *n* diagonals) into a rigorous regression framework. The key original contribution of this paper is the discussion of how graph theory can be used to analyze the topology of an arbitrary selection of triangle cells, and how to use the information gained to set up a regression model that is suitable for projecting future development. Once properly specified, fitting a GLM using maximum likelihood estimation (MLE) is straight forward, and we describe how this can be accomplished from a practical point of view in Visual Basic. To round off our discussion of model fitting, we briefly describe the standardization of residuals, and how to plot them for graphically evaluating goodness of fit. Finally we briefly discuss how the described class of GLMs for development triangles compares to some other stochastic models proposed in the actuarial literature.

Keywords. Generalized Linear Modeling, Reserving Methods, Regression, Data Diagnostics, Data Visualization, Bootstrapping, and Resampling Methods.

1. INTRODUCTION

In the context of stochastic reserving, several authors (e.g., [4], [7], and [9]) have stressed the need of casting the task of projecting reserves in a rigorous way as a regression problem. These authors have also pointed out that performing an all years volume weighted link-ratio estimate leads to the same result as fitting a GLM with the logarithmic link function and the identity variance function. Many practitioners have exploited this equivalence to implement spreadsheet-based applications for deriving a distribution of possible reserve outcomes based on bootstrap simulations by repeated resampling and application of the link-ratio estimate. While suitable to illustrate the concept of bootstrapping, these applications are typically not flexible enough to deal with practical judgments reserving analysts have to make about which cells of the triangle are deemed to be representative of future development (e.g., use data from last *n* diagonals or exclude obvious abnormalities). At other times practicing actuaries are also faced with data that are simply incomplete to start with. The important question here is how incomplete can a triangle ultimately be, while still providing information that is useful for the purpose of projecting future development? The key result presented in this paper is that concepts from an area of mathematics know as graph theory can be used to answer this question. Once we have analyzed some key aspects of the graph topology of a set of triangle cells, we can easily set up a well-defined regression problem and gain further insights into what information about the variability of the underlying stochastic process can be gained from the resulting model.

1.1 Research Context

Several papers (e.g., [4], [7], and [9]) in the actuarial literature do describe in abstract terms how to apply GLM theory to fitting a model to an actuarial development triangle. The actual algorithm for fitting a GLM follows the description in McCullagh and Nelder's classic *Generalized Linear Models* (2nd Edition) [6]. When fitting a GLM to an incomplete development triangle, however, the question of what constitutes a valid regression model specification naturally arises. We discuss how to use graph theory to algorithmically deal with this issue. By fitting a GLM to incomplete development triangles we furthermore extend the scope of traditional triangle-based reserving techniques: often a reserve projection can be made even if we only have partial information about the past development history. While this paper deals with the fitting of a regression model, the graph topology of the selected set of triangle cells also determines what information about the variability of the underlying stochastic process can be gleaned from the data. As it turns out, even when a data set supports projections for all development periods, different regions of an incomplete triangle may split into areas that are effectively fit without any influence from other areas. So, we can have multiple weakly connected regression models, rather than one comprehensive model for all selected data points. We use some of this information in our description of how to standardize residuals and how to plot them for diagnostic purposes. This insight also has implications for the scope and applicability of bootstrapping methods.

1.2 Objective

The iterative weighted LSQ algorithm for fitting a GLM is described in [6] and [9], but these textbooks generally assume that the reader is already familiar with the algorithms for performing regression fits. This paper seeks to explain at a practical level how to fit a GLM to a triangle of incremental development amounts. In particular, we address the issue of what happens if we either do not have complete information about the development history or want to exercise actuarial judgment about what data to include in our model. In addition, while there may be many advantages to using a fully fledged statistical package, we hope that interested readers who contact the author to request a copy of the companion MS Excel application will be able to explore the issues discussed and thus deepen their understanding of the process of fitting a regression model to a development triangle.

1.3 Outline

The remainder of the paper proceeds as follows. The second section starts with a visual description of the structure of a regression model for an incomplete development triangle. We then briefly discuss how we can link the discussed features of the model structure to aspects of the graph topology of an incomplete development triangle. Next we introduce an algorithm known as "breadth first search" which can be used to identify the situations previously described. We conclude by indicating how the information gathered can be used to specify the regression problem and deal with data points that require special attention. We believe that the application of graph theory to specifying a regression problem has not been previously discussed in the actuarial literature. The third section provides an overview of how to use Visual Basic to implement a maximum likelihood estimator based on iterative weighted least squares. The core algorithm here follows standard textbook treatment, but we make use of the graph topology of the incomplete development triangle to piece together the overall regression model from its subcomponents, if applicable. The fourth section deals with the standardization of residuals and plotting them for graphically evaluating goodness of fit. This section also walks through a number of the diagnostic exhibits using a concrete data set to demonstrate how an analyst may use them in practice. Finally we briefly discuss how the class of GLMs described in this paper compares to some other stochastic models for development triangles that are discussed in the actuarial literature.

2 SETTING UP THE MODEL SPECIFICATION

In this section we go into the details of how to set up the model specification that formally describes the regression problem corresponding to a multiplicative model for an incremental development triangle with separate parameters for rows and columns.

2.1 Notes on the structure of the regression model

To visualize what our set-up algorithm is trying to accomplish, we use the example of a five-by-five triangle. We have dispensed with row or column labels to reduce clutter. We follow the convention that rows denote exposure periods and columns development periods. With this said, a multiplicative model for expected incremental amounts looks something like this:

$$\begin{array}{ccccc}
c & b_2c & b_3c & b_4c & b_5c \\
a_2c & a_2b_2c & a_2b_3c & a_2b_4c & \\
a_3c & a_3b_2c & a_2b_3c & & \\
a_4c & a_4b_2c & & & \\
a_5c & & & &
\end{array} \tag{2.1}$$

This parameterization corresponds to a common method for dealing with extrinsic aliasing for factorial models: drop one level from each factor and replace them by one offset parameter common to all observations. For (2.1) we have dropped the first exposure and the first development period parameter and replaced them with an offset parameter. In this parameterization the offset parameter c denotes the value of a base (or reference) cell and the a_i and b_j parameters are relativities for exposure and development periods, respectively. Also note that the choice of reference cell generally does not affect the fitted values produced by the model. We could have equally chosen the following parameterization:

$$\begin{array}{ccccc}
a_1b_1c & a_1b_2c & a_1c & a_1b_4c & a_1b_5c \\
a_2b_1c & a_2b_2c & a_2c & a_2b_4c & \\
b_1c & b_2c & c & & \\
a_4b_1c & a_4b_2c & & & \\
a_5b_1c & & & &
\end{array} \tag{2.2}$$

For a complete triangle either of the above parameterizations can straightforwardly be translated into a standard regression problem and the fitted incremental amounts will be identical. When we start excluding data points from the analysis, we may encounter a number of issues that force us to pay closer attention the structure and parameterization of our regression model. The algorithm presented here deals with four specific issues relating to ensuring we are dealing with a well-defined regression problem, and to identifying triangle cells requiring special treatment in our subsequent goodness of fit analysis.

2.1.1 Not enough data points to estimate some parameters

$$\begin{array}{ccccc}
c & b_2c & b_3c & \times & b_5c \\
a_2c & a_2b_2c & a_2b_3c & \times & \\
a_3c & a_3b_2c & a_2b_3c & & \\
a_4c & a_4b_2c & & & \\
a_5c & & & &
\end{array} \tag{2.3}$$

The cross symbol here denotes data points that are missing or excluded by the analyst (e.g., truncated triangles or want to use last n diagonals). Clearly we have no information on the b_4

parameter and it therefore has to be dropped from the model. Note that despite the "gap" at development period 4, there is no issue with relating the top right corner (development period 5) with the rest of the triangle—we can compare this value to the first 3 incremental values for exposure period 1.

2.1.2 Choice of reference cell does matters after all

Assume we are trying to use our cell (3,1) as our reference cell for the following data set:

$$\begin{array}{ccccc}
a_1b_1c & a_1b_2c & a_1c & a_1b_4c & a_1b_5c \\
a_2b_1c & a_2b_2c & a_2c & a_2b_4c & \\
\times & \times & \times & & \\
a_4b_1c & a_4b_2c & & & \\
a_5b_1c & & & &
\end{array} \tag{2.4}$$

Clearly we have no information on a_3, and this situation could be remedied by dropping this parameter from the model. In this case, however, we cannot do this because a_3 has already been replaced by the common offset parameter c. Actually, our algorithm circumvents this problem altogether by first analyzing which rows and columns are part of the connected component of triangle cells for which fit a model before attempting to assign parameters to rows and columns.

2.1.3 Data splits into unrelated regions

$$\begin{array}{ccccc}
\times & \times & b_3c & b_4c & b_5c \\
\times & \times & a_2b_3c & a_2b_4c & \\
\times & \times & a_2b_3c & & \\
a_4c & a_4b_2c & & & \\
a_5c & & & &
\end{array} \tag{2.5}$$

In this situation, there is no information on how to relate the upper right sub-triangle to the lower triangle. This is an issue that cannot be fixed in a meaningful way, as far as predicting future values for all exposure and development periods is concerned. We handle this issue by using graph theory, noting that two triangle cells can be regarded as connected if they are either in the same row or in the same column. We determine what is called the maximal connected components of the triangle viewed as a graph. Further information on graph theory will be provided in section 2.2, below. One could fit a separate regression model for each of the connected components, but this is not useful for projecting future development amounts. Generally we hope that there is only one connected component. If not, we continue with the connected component that has the maximum number of triangle cells. If the number of triangle cells does not uniquely determine which component to pick, we take the component with the left-most column.

2.1.4 Exact fit cells

$$\begin{array}{ccccc}
\times & \times & b_3c & b_4c & \underline{b_5c} \\
\times & \times & a_2b_3c & a_2b_4c & \\
a_3c & a_3b_2c & \underline{\underline{a_3b_3c}} & & \\
a_4c & a_4b_2c & & & \\
\underline{a_5c} & & & &
\end{array} \tag{2.6}$$

One of the general goals in stochastic model fitting is to assess goodness-of-fit and measure the variability inherent in the observed process. To this end residuals (actual value less fitted value) need to be analyzed. This analysis can be distorted by triangle cells where the fitted value will always be exactly the same as the actual value. For a complete triangle this will always be the case for the top right and the bottom left corner, but when there are missing data points the same may be true for other cells. In the above example parameters a_5 and b_5 each appear in exactly one triangle cell, so they will always take values that ensure a perfect fit. What may be less obvious is that there will always be an exact fit for the cell in row 3, column 3. The reason for this is that removing this cell would split the incomplete triangle into two unrelated regions (as discussed under 2.1.2 above). Our algorithm for analyzing the model structure identifies exact fit cells by looping over all cells in the selected connected component and checking for each cell whether the removing the cell from the model changes to model structure by either dropping a row or column, or by splitting the model into two unrelated regions.

2.1.5 Further remarks on the model structure

Until now our discussion on the model structure preserved the shape of the triangle because the distinction between exposure and development periods is meaningful to us as P&C actuaries. Algorithms for fitting a simple GLM model as described above, however, are indifferent to how we perceive the various triangle cells as data points that are somehow ordered by exposure and development periods. Consider the following sparse data set for a hypothetical ratemaking problem with a multiplicative model with two classification dimensions, namely group and territory:

$$\begin{array}{ccccc} t_1 g_1 b & t_1 g_2 b & b & t_1 g_4 b & t_1 g_5 b \\ \times & \times & t_2 b & \times & \times \\ \times & t_3 g_2 b & t_3 b & \times & t_3 g_5 b \\ \times & \times & t_4 b & \times & t_4 g_5 b \\ t_5 g_1 b & t_5 g_2 b & t_5 b & \times & t_5 g_5 b \end{array} \tag{2.7}$$

Group 3 in territory 1 corresponds to the base rate b, while the g_i and t_i parameters represent group and territory relativities. This may not look like a development triangle, but this data set is structurally identical to multiplicative model for a complete triangle of incremental development amounts (see our original example at the beginning of section 2).

Hence, when using a model based on distinct parameters for each exposure and development period, a triangle is just an unordered list of data points, and the only relationships between data points are defined by certain parameters simultaneously affecting the fitted values of multiple triangle cells. For the purpose of implementing a concrete algorithm, however, we do need to find a way of listing all triangle cells. To keep things simple our default is to loop over rows, then columns, resulting in the following order of processing cells:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & \times & 8 & \\ \times & 9 & 10 & & \\ 11 & 12 & & & \\ 13 & & & & \end{array} \tag{2.8}$$

Note how excluded cells are skipped. There will be occasions when this order changes, so when interpreting the Visual Basic code, the reader should generally not rely on triangle cells being ordered in this way.

Another point, that is worth understanding in translating between triangles and the common representation of regression problems (e.g., following McCullagh and Nelder), is that all explanatory variables are on the same footing after aliasing has been taking care of—the regression algorithm does not distinguish between exposure or development period parameters (or the offset parameter).

2.1.6 Moving to GLMs—taking the log transform

The above discussion on the model parameterization is generic in the sense that it applies to any multiplicative regression model for an incremental development triangle that is restricted to distinct, unordered parameters for exposure and development periods. This paper is more specifically about fitting a GLM to a development triangle. To linearize the multiplicative model we need to choose the logarithm as a link function. This results in the following additive model structure for the logarithms of the expected incremental amounts:

$$\begin{array}{ccccc} \gamma & \beta_2+\gamma & \beta_3+\gamma & \beta_4+\gamma & \beta_5+\gamma \\ \alpha_2+\gamma & \alpha_2+\beta_2+\gamma & \alpha_2+\beta_3+\gamma & \alpha_2+\beta_4+\gamma & \\ \alpha_3+\gamma & \alpha_3+\beta_2+\gamma & \alpha_2+\beta_3+\gamma & & \\ \alpha_4+\gamma & \alpha_4+\beta_2+\gamma & & & \\ \alpha_5+\gamma & & & & \end{array} \tag{2.9}$$

We are emphasizing this step here for two reasons. Firstly, the use of a logarithmic link function restricts the model to positive incremental values. Secondly, understanding the connection between this additive model and the more generic multiplicative model is crucial to interpreting the output from GLM packages.

2.2 Graph topology of an incomplete development triangle

We noted above the cells of an incomplete development triangle can be thought of as forming a mathematical structure know as a graph. Generally a graph is collection of nodes (or vertices) that are connected to each other by edges. Two nodes are considered neighbors if there is an edge that directly joins them. We can also define an equivalence relationship among nodes called connectedness. Two nodes are connected if there is a path (or sequence) of neighboring nodes that leads from one node to the other. Two nodes are disconnected if there is no way of getting from one to the other by passing from neighbor to neighbor. This equivalence relationship of connectedness defines equivalence classes of nodes that are called maximal connected components. Note that in this paper we often refer to maximal connected components as connected components since repeating "maximal" becomes cumbersome.

If we think of the triangle cells as nodes and define two triangle cells as being neighbors if they are in the same row or column of the triangle, the collection of cells from an incomplete development triangle can be seen to form a graph. We will now briefly outline how graph theory to can be used to handle the issues regarding model parameterization and exact fit cells identified above.

2.2.1 What parameters are needed for the model?

We use an algorithm known as "breadth first search" (described in detail in section 2.4) to first identify the maximal connected components of the incomplete triangle. As explained above in section 2.1.3 we can only "complete the triangle" for projection purposes if the given triangle cells form a connected component. If there is more than one connected component our algorithm proceeds by picking the largest (most triangle cells) connected component. If there is more than one largest connected component the algorithm chooses the component that has the left most column. Once we have identified the connected component for which we will fit a regression model we analyze which rows and columns are covered by the connected component. We parameterize our regression model by choosing the cell corresponding to the top row and left most column as our reference cell, with separate parameters for each other row and column. This takes care of the issues identified in section 2.1.1 and 2.1.2.

2.2.2 Which cells are exact fit cells?

As indicated above there are two circumstances under which the fitted value for a particular triangle cell will always match the given data point. Both situations can be identified by eliminating a particular cell from the regression model and seeing how the elimination affects the structure of the model. Technically we do this by looping over all cells in the selected connected component, remove each cell in turn and then run the "breath first search" algorithm also used in section 2.2.1 to analyze the structure of the remaining model. This allows us to identify three different types of cells:

Single parameter cells: when this cell is eliminated from the model, we lose one row or column and the corresponding parameter. Since this is therefore the only data point for that parameter, the parameter will always take a value that produces an exact fit for this cell.

Critical connector cells: eliminating this cell from the model, splits it into exactly two disconnected components (proof left as an easy exercise for the reader[1]). The issue of aliasing now affects both disconnected components separately, so we lose a parameter. The details of how this parameter disappears are more subtle than for single parameter cells, but the bottom line is that there is some "slack" in the parameterization and we always get an exact fit for a critical connector cell.

Regression cells: when eliminating this cell, the model structure is not affected in a significant way (same number of rows and columns covered, same number of parameters needed to parameterize the model).

Both types of exact-fit cells need to be excluded when analyzing standardized residuals and measuring the inherent uncertainty of the underlying stochastic process. The impact of the critical connector cells is more far reaching. Despite having a valid regression model for the entire connected component, the critical connector cells (if they exist) split the incomplete triangle into

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[1] Hint: if another cell, C, is connected to the critical connector cell, there has to be at least one cell that is connected to C, which either shares a row or column with the critical connector cell.

regions for which the regression fit is performed without any influence from the other regions. Our algorithm for fitting the GLM powerfully demonstrates this feature by literally applying the iterated weighted least square procedure to these separate regression components. To get the final parameterization for the overall model we then perform a single-weighted least square fit based on the fitted values separately obtained for regression regions and the actual data points for the identified exact fit cells. Note that it is not necessary to perform the fit separately for the regression regions, but when it comes to actual computations, it is usually more efficient to split a larger problem into separate smaller problems, especially when the computational cost scales non-linearly.

2.3 Formal set-up

To formally establish the relationship between the data for which we want to find a model (i.e., the cells of the triangle) and the explanatory variables (i.e., the exposure and development periods) we set up a data structure known as the model matrix, **X**, the columns of which represent the explanatory variables, and a corresponding column vector, **y**, which represents the data. Interested readers are referred to the CAS Practitioners' Guide to GLMs ([1]) for a general introduction to setting up information matrices. Here we will simply show an example of how a complete 5 x 5 triangle can be represented:

	Parameters **P**	γ	β_2	α_2	β_3	α_3	β_4	α_4	β_5	α_5	
	Data					*Model matrix*					*Fitted Values*
Unit #	**Y**					**X**					Exp(**X.p**)
1	inc(1,1)	1	0	0	0	0	0	0	0	0	Exp(γ)
2	inc(1,2)	1	1	0	0	0	0	0	0	0	Exp($\beta_2+\gamma$)
3	inc(2,1)	1	0	1	0	0	0	0	0	0	Exp($\alpha_2+\gamma$)
4	inc(2,2)	1	1	1	0	0	0	0	0	0	Exp($\alpha_2+\beta_2+\gamma$)
5	inc(1,3)	1	0	0	1	0	0	0	0	0	Exp($\beta_3+\gamma$)
6	inc(3,1)	1	0	0	0	1	0	0	0	0	Exp($\alpha_3+\gamma$)
7	inc(2,3)	1	0	1	1	0	0	0	0	0	Exp($\alpha_2+\beta_3+\gamma$)
8	inc(3,2)	1	1	0	0	1	0	0	0	0	Exp($\alpha_3+\beta_2+\gamma$)
9	inc(1,4)	1	0	0	0	0	1	0	0	0	Exp($\beta_4+\gamma$)
10	inc(4,1)	1	0	0	0	0	0	1	0	0	Exp($\alpha_4+\gamma$)
11	inc(3,3)	1	0	0	1	1	0	0	0	0	Exp($\alpha_3+\beta_3+\gamma$)
12	inc(2,4)	1	0	1	0	0	1	0	0	0	Exp($\alpha_2+\beta_4+\gamma$)
13	inc(4,2)	1	1	0	0	0	0	1	0	0	Exp($\alpha_4+\beta_2+\gamma$)
14	inc(1,5)	1	0	0	0	0	0	0	1	0	Exp($\beta_5+\gamma$)
15	inc(5,1)	1	0	0	0	0	0	0	0	1	Exp($\alpha_5+\gamma$)

Note that the unit #s are simply for referencing values stored in arrays and that inc(i,j) denotes the incremental amount for accident period i and development period j. The model matrix, **X**, has one row for each triangle cell and one column for each parameter of the model. All entries of **X** are either 0 or 1. A value of 1 simply means that the parameter corresponding to the respective column contributes to the fitted value for the triangle cell corresponding to the respective row; a value of 0 implies the converse. In the above table we have introduced the parameter vector, **p**. If $\hat{\mathbf{y}}$ denotes the vector of fitted values, the systematic part of our GLM model can neatly be summarized by the following equation:

$$log(\hat{\mathbf{y}}) = \mathbf{X.p} \tag{2.10}$$

2.4 The algorithm for setting up the model specification

Setting up the actual model matrix is straightforward. The real challenge here is to algorithmically analyze the incomplete development triangle to come up with a valid parameterization and to identify the exact fit cells. Before we introduce the pseudo-code for the "breadth first search" algorithm that is at the core of this undertaking, we briefly describe some auxiliary data elements. Readers in a hurry can jump right to section 2.4.1 where the pseudo-code is presented.

In the following we use some name ranges and variable names specific to the MS Excel application available from the author at request. We hope that their mention here does provide the reader with an idea of what is involved from a practical implementation point of view. We start with a complete triangle in a range called data_incremental. To exclude triangle cells from the analysis, we use a second triangle of 0s and 1s in a range called data_excluded to mask the corresponding cells in the triangle of incremental amounts.

For determining the maximal connected components of the selected triangle cells viewed as a graph, we use a "breadth first search"-type algorithm (see [5]) adapted to the fact that our graph edges (the links between cells) come in two "flavors:" shared row or shared column. To this end we need to maintain four lists of cells: (connected component) assigned, rows tested, columns tested and untested. To facilitate moving cells from and to these lists we use an array called UnitIndex that has a row for each selected triangle cell and columns for storing the cells predecessor and successor in its current list. In addition, for each list we maintain a special pointer to the first element in the list. This data structure allows us to store all four lists in parallel in the same array. The array UnitIndex also has additional columns for identifying a cell's row and column in the triangle. At times, we use a separate triangle of unit numbers to efficiently locate triangle cells in the UnitIndex array based on their position in the triangle.

There are also a number of arrays to facilitate moving from the model matrix to the actual triangle and vice versa:

UnitIndex() has fields for mapping a row of the model matrix to the corresponding triangle cell

Data_Selected() or Data_Regression() are used to either mark some specific triangle cells for further processing or to store pointers from the triangle cells to the corresponding row in UnitIndex()

GLM_Par_To_Triangle() maps each column of the model matrix (or the parameter row vector) to the corresponding row or columns of the triangle

ExporsurePeriod_To_GLM_Paramter() maps rows of the triangle to columns of the model matrix

DevelopmentPeriod_To_GLM_Paramter() maps columns of the triangle to columns of the model matrix

2.4.1 Determining connected components of the incomplete development triangles

Below we outline pseudo-code for the "breadth first search"-type algorithm for an incomplete development triangle operates as follows. Here is how the general "breadth first search" algorithm works. It is called "breadth first" because while we are trying to find all nodes connected a particular untested node we first mark all its immediate neighbors before trying to find neighbors of neighbors. Once all the immediate neighbors of a node have been identified we are done with that node. What is left to do is to loop over all the immediate neighbors previously identified and check whether they have any neighbors we have not looked at, yet. For each node we are done once we have marked all of its immediate neighbors. This process continues until we cannot find any further new neighbors. We have now identified the maximal connected component of the original untested node. If there are untested nodes left, we know that our graph has a further connected component and we start the process all over to find all the nodes belonging to the next connected component.

For a development triangle the algorithm generally proceeds exactly the same way except that we are now done with a particular cell when we have identified all its row and all its column neighbors. In order to efficiently do this, we introduce two lists of cells with an intermediate testing status: all row neighbors identified (i.e., still need to check column neighbors) and all column neighbors identified (i.e., still need to check row neighbors). Hence there are two possible processing paths for a particular triangle cell: untested to column tested to component assigned, or untested to row tested to component assigned.

Here is the pseudo-code:

Input: DataSelected.....................two-dimensional array indicating which triangle cells are included in model

UnitIndex..........................array for storing information about data points and maintaining list structures utilized by algorithm

ConnectedComponent...array for keeping track of maximal connected components and some key properties

Goal: Assign component sequence number to each selected triangle cell and gather summary information about connected components

BreadthFirst 1) Initialize UnitIndex and associated data structures needed for determining maximal connected components of selected triangle cells. In particular list of untested cells contains all cells included by user, while the list of assigned, row tested, or column tested cells are empty.

BreadthFirst 2) Beginning of outer loop - keep going while the list of untested cells is empty.

BreadthFirst 3) Increment component counter; get column of last untested cell; loop over untested rows in this column; assign component counter to untested cells found and move them from the untested list to the column tested list.

BreadthFirst 4) Beginning of inner loop - keep going while there are newly identified neighbors for the current connected component.

BreadthFirst 5) Loop over cells in column tested list; get row of current column tested cell; move current cell from column tested list to assigned list; loop over untested columns in current row; assign component counter to untested cells found and move them from the untested list to the row tested list.

BreadthFirst 6) Loop over cells in row tested list; get column of current row tested cell; move current cell from row tested list to assigned list; loop over untested rows in current column; assign component counter to untested cells found and move them from the untested list to the column tested list.

BreadthFirst 7) End of inner loop - if list of column tested cells is not empty, execution will continue with BreadthFirst 5).

BreadthFirst 8) End of outer loop - if list of untested cells in non-empty, execution will continue with BreadthFirst 3).

2.4.2 Selecting connected component for the subsequent model fit

While it would be possible to fit a regression model for each separate connected component, there is no way of using these disconnected models for projecting future development for all periods. Hence our algorithm proceeds by selecting the component with the most triangle cells for further processing. If there is more than one largest connected component the algorithm simply picks the component with the left most column. Note that in the companion spreadsheet, which is available from the author at request, there is an exhibit that shows all connected components. If the user does not like the default choice imposed by the algorithm, they can change the selected data points to only include the connected component that they are interested in.

2.4.3 Determining the cell types for the selected connected component

As described in section 2.2.2, above, there are three types of cells within each connected component: single parameter cells, critical connector cells, and regression cells. For reference let NoUnits stand for the number of cells in the selected connected component. We test for the cell types by looping over all the cells in the selected connected component and remove each cell in turn from the selected component. After removing the cell we analyze the structure of the remaining cells. In order to do so it is sufficient to start with any of the remaining cells and then run the inner loop of the "breadth first search" algorithm described above. If the maximal connected component associated with that cell has less than NoUnits - 1 cells, we know that the removed cell must be a critical connector cell. Otherwise we need to check whether we lost a row or column by removing the cell. If we did lose a row or column, then the removed cell is a single parameter cell. If the latter is not the case, we know by elimination that the removed cell is a regression cell. The above-mentioned exhibit for the connected components also visualized the cell types for cells in the selected components by formatting; single parameter cells have a border, critical connector cells are crossed out, and regression cells have a grey fill.

2.4.4 Determining the regression components within the selected connected component

We now run the full "breadth first search" algorithm again, but only on the regression cells within the selected connected component. This allows us to identify decoupled areas of connected regression cells if they exist. Gathering this information allows us to split the fitting of the overall model into smaller pieces, each of which purely consists of regression cells.

2.4.5 Setting up the model matrix and associated data vector

Having done all the preparatory work of analyzing the graph topology of the incomplete triangle the task of setting up the model matrix and data vector is trivial. We set up separate model matrices and data vectors for each of the regression regions identified above. We also set up one overall model matrix and data vector. Note however, that technically we will only run the full-fitting algorithm on the model matrices for the regression regions. The model matrix for the overall model will only be used to for a single iteration weighted least square fit based on the fitted values obtained for the regression regions and the actual data points for the exact fit cells. This last step is only needed to obtain a convenient parameterization for the overall model that can be used for projection purposes. Note, however, that the model matrix for the overall model is perfectly valid and that feeding it into a GLM fitting algorithm should produce the same parameter values (give or take some rounding) as the approach we are taking. Keeping track of the exact fit cells and regression regions, however, is computationally advantageous and provides useful information for the subsequent residual analysis.

3 MAXIMUM LIKELIHOOD ESTIMATION USING ITERATED WEIGHTED LEAST SQUARES

In section 2 we went into considerable detail of how our algorithm for setting up the model matrix works, because we are not aware of such a step-by-step description in the actuarial literature. Algorithms for fitting a generalized linear model (GLM) by using a maximum likelihood estimator, however, are described in many text books (e.g., chapter 2.5 in [6] or chapter 6.4.2 in [9]). We will therefore concentrate on how to implement such an approach relying on standard linear algebra routines available in open source form—we have used code from the ALGLIB project available for download at www.alglib.net. Other than demonstrating that Visual Basic for MS Excel is well capable of fitting a GLM to triangles of considerable size, we also want to show that in the process of performing the calculations we can also extract useful information[2] other than the fitted parameter values.

3.1 Notation

Before continuing we need to introduce some standard notation for describing a GLM model. Often a GLM model is specified by assuming a specific distribution from the exponential family for the observations. In practice the algorithm for fitting a GLM works just as well under the weaker assumption that the second moment of the distribution is a function of the expected mean (see chapter 9 on pseudo-likelihood functions in [6]). Hence we regard a GLM type problem to be fully specified by the following elements:

- Model matrix, $\mathbf{X}$
- Data vector, $\mathbf{y}$
- Fitted values vector, $\hat{\mathbf{y}}$
- Link function, g—note that here we only will use the logarithm
- Variance function, V

The following notation is also useful in describing the algorithm and computations:

- Parameter vector, $\mathbf{p}$
- Linear estimator, $\boldsymbol{\eta}$
- Vector of quadratic weights for weighted LSQ regression, $\mathbf{w}$
- Vector of linearized data, $\mathbf{z}$
- Vector of expected variance for data points, $\mathbf{v}$

The algorithm described here follows the approach outlined in chapter 2.5 of [6]—we perform iterated weighted least square regressions. This procedure can be understood as an adaption of the multi-dimensional Newton-Raphson method to the problem of solving the maximum likelihood equations for a GLM. In effect we are repeatedly solving a linearized regression problem until the successive solutions have converged to a sufficient degree. In general such numerical procedures can be sensitive to the starting point chosen. Here we can use the actually observed values for the data points (triangle cells), which makes the practical implementation easy.

3.2 Pseudo-code for MLE algorithm

Deferring our discussion of how to perform the weighted least squares (LSQ) regression, here is

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[2] Specifically we are referring to the diagonal elements of the hat matrix, which can be used to standardize residuals. This will be discussed in more detail in section 4.

how the MLE algorithm implemented here works:

Step 0) Based on the actual data points, initialize the fitted data points, the linear estimator, and the expected variances: $\hat{\mathbf{y}}_0 = \mathbf{y}$, $\boldsymbol{\eta}_0 = g(\mathbf{y}) = log(\mathbf{y})$, and $\mathbf{v}_0 = V(\mathbf{y})$.

Step 1) Based on a current estimator, $\boldsymbol{\eta}_i$, determine the vector of linearized data using the following formula:

$$\mathbf{z_i} = \boldsymbol{\eta}_\mathbf{i} + (\mathbf{y} - \hat{\mathbf{y}}_\mathbf{i})\left(\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}\hat{\mathbf{y}}}\right). \tag{3.1}$$

Step 2) Based on a current estimator, $\boldsymbol{\eta}_i$, the expected variances, $\mathbf{v}_i$, calculate the vector of weights for weighted LSQ regression:

$$\mathbf{w_i} = \left(\left(\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}\hat{\mathbf{y}}}\right)^2 \mathbf{v_i}\right)^{-1}. \tag{3.2}$$

Step 3) Perform weighted LSQ regression of $\mathbf{z}_i$ on $\mathbf{X}$ subject to $\mathbf{w}_i$ to obtain new set of parameters, $\mathbf{p}_{i+1}$, leading to a new liner estimator, $\boldsymbol{\eta}_{i+1}$, and fitted values, $\hat{\mathbf{y}}_{i+1}$.

Step 4) Compare $\mathbf{p}_i$ with $\mathbf{p}_{i+1}$ to determine whether convergence is satisfactory. If not, repeat from Step 1).

Step 5) Extract diagonal elements of hat matrix, $\mathbf{H}$, calculate deleveraged residuals, estimate dispersion parameter, and calculate standardized residuals.

Step 6) Loop over exact-fit cells and recursively determine exact-fit parameters based on parameters obtained with MLE algorithm.

Note that with the logarithm as our link function we get $\boldsymbol{\eta} = g(\hat{\mathbf{y}}) = log(\hat{\mathbf{y}})$, and $\mathrm{d}\boldsymbol{\eta}/\mathrm{d}\hat{\mathbf{y}} = 1/\hat{\mathbf{y}}$. Readers who are interested in why this procedure (specifically steps 0 to 4) works are referred to chapter 2.5.1 in [6]. Step 5) will be discussed in detail in section 3. Step 6) utilizes the information gathered by the algorithm for setting up the model matrix, $\mathbf{X}$, as described in section 2. While this also reduces the computational cost, the main purpose is to keep track for which data points we can calculate standardized residuals.

3.3 Notes on implementation issues

Implementing this general algorithm in Visual Basic is pretty straight forward. To make the implementation flexible regarding specific choices of variance and link functions we are using a number of generic functions that such as Calc_dL (for $\mathrm{d}\boldsymbol{\eta}/\mathrm{d}\hat{\mathbf{y}}$), Calc_Variance (for $\mathbf{v}$), and Calc_Weights (for $\mathbf{w}$), that are simply passing through a symbolic parameter indicating the current choice of link and variance function to lower level functions that compute those values for specific data points (Link_Scalar, LinkInv_Scalar, Var_Scalar, dL_Scalar). Note that despite this flexibility in design, the template does only implement the logarithmic link function.

For those readers who want to go over the Visual Basic code in detail, please note that we are making extensive use of passing data by reference to provide functions with input and to return the results. The formal function results are mainly used for debugging and error handling. Passing by reference means that subroutines are given direct access to data structures (variables and/or arrays) defined at a higher level rather than having their own copy of the data passed in. So the main point of a statement like

```
If Not Calc_dL(dL, Y_, LinkFunc) Then Stop
```

is not to stop execution if the function Calc_dL returns FALSE. Rather we want to calculate the vector of derivatives for the link function, which is accomplished as a side effect to the function call by updating the values of the vector (1-d array) dL that is passed to the function by reference. The "If … Then Stop" statement is simply a way of pointing the developer to the right position in the code in case something were to go wrong.

Performing the weighted LSQ regression is mainly an exercise in linear algebra. Built-in Excel functions only provide limited support for matrix operations and the matrices the built in functions can deal with is limited. We are therefore providing some auxiliary routines for simple matrix operations. A list with short descriptions follows:

- MClearUpper ..clears the upper right triangle of a square matrix **A**; needed because the Cholesky transform routine from ALGLIB assumes that symmetric matrix is represented in triangular format
- MTrans............populates matrix transpose $\mathbf{A}^\mathbf{T}$ of matrix **A**
- MMult ....populates matrix **R** with the product of matrices **A** and **B**
- MDiagRMult ...left multiplies matrix **A** with a diagonal matrix that is represented as a vector **diag**
- MDiagLMult ...right multiplies matrix **A** with a diagonal matrix that is represented as a vector **diag**
- MSet ....populates matrix **R** by copying entries of matrix **A**
- MSwapCol ....swap column c1 and c2 of matrix **A**
- MSwapRow .....swap row r1 and r2 of matrix **A**
- MQuickMult....populates vector **r** with the product of matrix **A** and vector **b**
- VectorSqrt ....populate vector **sqrt_w** with square root of elements of vector **w**

As indicated above, we are also using code from the open source ALGLIB project (www.alglib.net). The following functions are Visual Basic implementations of LAPACK routines (www.netlib.org/lapack/):

- SPDMatrixCholesky. perform Cholesky decomposition of matrix A which is assumed to be a symmetric matrix stored in triangular format.
- RMatrixTRInverse.... perform matrix inversion of triangular matrix A.

We will not explain the algorithms in detail, but we do want to briefly outline why these routines are useful for our purposes. The key computational step in performing the weighted LSQ regression is the inversion of the matrix $\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{X}$, where $\mathbf{W}$ is the diagonal matrix of weights represented by the vector $\mathbf{w}$. If the regression problem is well defined, this matrix is symmetric and positive-definite. Now, if you have a symmetric, positive-definite matrix $\mathbf{A}$, then there is a lower triangular matrix $\mathbf{L}$ such that $\mathbf{A}=\mathbf{L}.\mathbf{L}^\mathbf{T}$. This representation of $\mathbf{A}$ is called its Cholesky decomposition. It follows from basic matrix algebra that $\mathbf{A}^{-1}=(\mathbf{L}^{-1})^{\mathbf{T}}.(\mathbf{L}^{-1})$. Hence we can see how the inversion of $\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{X}$. can be accomplished by first finding its Cholesky decomposition and then inverting the resulting lower triangular matrix.

In symbolic form the weighted LSQ regression estimate for the parameters based on the linearized data is given by:

$$\mathbf{p} = (\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{X})^{-1}.\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{z}. \tag{3.3}$$

3.4 Pseudo-code for weighted LSQ regression

We now present an outline of the code for performing the weighted LSQ regression. Note that in essence we are building up three matrices: $\mathbf{X}^\mathbf{T}.\mathbf{W}$, $(\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{X})^{-1}$, and $(\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{X})^{-1}.\mathbf{X}^\mathbf{T}.\mathbf{W}$. The corresponding variables are called XtW, XtWXinv, and XtWXinvXtW:

wLSQ 1) MTrans_C(XtW, X)—i.e., XtW = $\mathbf{X}^\mathbf{T}$

wLSQ 2) MDiagRMult_C(XtW, w)—now XtW = $\mathbf{X}^\mathbf{T}.\mathbf{W}$

wLSQ 3) MMult_C(XtWXinv, XtW, X)—i.e., XtWXinv = $\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{X}$

wLSQ 4) MSet_C(M1, XtWXinv)—i.e., M1 = $\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{X}$

wLSQ 5) MClearUpper_C(M1)—prepare M1 for Cholesky routine

wLSQ 6) MCholesky_C(M1)—calculate the $\mathbf{L}$ for $\mathbf{X}^\mathbf{T}.\mathbf{W}.\mathbf{X}$

wLSQ 7) MTriInv_C(M1, t)—i.e., M1 = $\mathbf{L}^{-1}$

wLSQ 8) MTrans_C(M2, tmpM1)—i.e., M2 = $(\mathbf{L}^{-1})^{\mathbf{T}}$

wLSQ 9) MMult_C(XtWXinv, M2, M1)—i.e., XtWXinv = $(\mathbf{L}^{-1})^{\mathbf{T}}.(\mathbf{L}^{-1})$

wLSQ 10) MMult_C(XtWXInvXtW, XtWXinv, XtW)—used for calculating $\mathbf{h}$

The rest of the tasks associated with Step 3) of the <u>iterated</u> weighted LSQ algorithm is accomplished with the following statements:

wLSQ 11) MQuickMult(p, XtWXInvXtW, z)—i.e., $\mathbf{p_1} = (\mathbf{X^T}.\mathbf{W}.\mathbf{X})^{-1}.\mathbf{X^T}.\mathbf{W}.\mathbf{z_0}$

wLSQ 12) MQuickMult(z_, X, p)—i.e., $\boldsymbol{\eta}_1 = \mathbf{X}.\ \mathbf{p_1}$

wLSQ 13) LinkInv(y_, z_, LinkFunc)—i.e., $\hat{\mathbf{y}}_1 = g^{-1}(\boldsymbol{\eta}_1)$

3.5 Concluding remarks

We encourage interested readers to contact the author and request a copy of the accompanying MS Excel application, so they can study the commented source code for further implantation details.

One note on performance: the computational cost of many sub-tasks except for the matrix operations varies linearly with the number of triangle cells included in the analysis. For fitting a single model the performance of the Visual Basic code seems satisfactory, even for larger triangles (the author tested up to 40 by 40). For repeated applications (such as bootstrapping) execution time can become an issue. Without changing the logic, significant gains in performance can be gained from porting the weighted LSQ routines to C++ and compiling them into a dll, which is then loaded by the Visual Basic code. In this paper we do not discuss the issues of interfacing Visual Basic with a dll and/or how to write code in C++ that can be compiled into a dll. For readers interested in those issues we recommend Steve Dalton's *Financial Applications using Excel Add-in Development in C/C++* (different versions exist for MS Excel 2003 and MS Excel 2007).

4 STANDARDIZED RESIDUALS AND THEIR APPLICATION

In this section, we introduce standardized residuals and how to compute them. After briefly describing how to create diagnostic plots based on these standardized residuals we present a practical example (based on a data set also used in a number of other papers on triangle-based stochastic reserving) of how these plots can be used to assess goodness-of-fit and make decisions about the model structure.

4.1 Standardized Residuals

Residuals are simply the difference between the actual data points and their fitted values based on a concrete model specification. Suitably standardized, so comparisons for residuals corresponding to different data points can be made, residuals can be a powerful tool for visually analyzing the goodness of fit of a model. Standardizing residuals is also a crucial step in bootstrapping a GLM. The approach here follows chapter 12.5 and 12.7 in [6], but similar descriptions of the standardization procedure can also be found in [7] and in chapter 7.2 of [3].

For a GLM, there are two adjustments to residuals that need to be made in order get residuals that are approximately identically distributed. Firstly we need to adjust for the differences in expected variances based on the relationship imposed by the variance function. This motivates the following definition of Pearson residuals:

$$\overline{\mathbf{r}} = \frac{\mathbf{y} - \hat{\mathbf{y}}}{\sqrt{V(\hat{\mathbf{y}})}}. \tag{4.1}$$

In addition we also need to adjust for the leverage that individual data points exert on their corresponding fitted value. At the extreme there are the exact-fit cells where the fitted value will always identically match the observed data point. For other points the observed value will still exert a pull on the fitted value that biases the residuals as a measure of variability inherent in the data. For a detailed discussion of the concept of leverage, we refer the reader to chapter 12.7 in [6]. A useful measure of leverage can be obtained by the diagonal values of the hat matrix, H, which for a GLM is defined as (equation 12.3 in [6]):

$$\mathbf{H} = \mathbf{W}^{½}.\mathbf{X}.(\mathbf{X}^{\mathbf{T}}.\mathbf{W}.\mathbf{X})^{-1}.\mathbf{X}^{\mathbf{T}}.\mathbf{W}^{½}. \tag{4.2}$$

Note that since we are only interested in the vector **h** of diagonal elements we can also use the following formula:

$$\mathbf{h} = \text{diag}(\mathbf{X}.(\mathbf{X}^{\mathbf{T}}.\mathbf{W}.\mathbf{X})^{-1}.\mathbf{X}^{\mathbf{T}}.\mathbf{W}). \tag{4.3}$$

With this we can now introduce the following definition of deleveraged Pearson residuals:

$$\tilde{\mathbf{r}} = \frac{\mathbf{y} - \hat{\mathbf{y}}}{\sqrt{V(\hat{\mathbf{y}}) \cdot (1 - \mathbf{h})}}. \tag{4.4}$$

Note that the deleveraged Pearson residuals are not defined for exact fit cells. Also note that this definition is still missing the estimated dispersion factor as a normalizing constant. The reason for this is that we are proposing the following estimator for the dispersion factor:

$$\hat{\phi} = \mathbf{Var}(\tilde{\mathbf{r}}) \tag{4.5}$$

Compare this to the conventional estimator (unnumbered formula on page 328 in [6]), based on an after-the-fact degree of freedom adjustment:

$$\hat{\phi} = \mathbf{Var}(\overline{\mathbf{r}}) \cdot \frac{n}{n - p} \tag{4.6}$$

where n is the number of data points and p is the number of estimated parameters. Both estimators for the dispersion parameter are ad hoc and are trying to adjust for the leverage effect. Equation 4.5 relies on the bias correction applied to each individual residual. Equation 4.6 does not distinguish between the difference in leverage for individual data points. Further research is required for assessing the relative performance (e.g., in terms of bias or standard error) of these two estimators. For the purposes of assessing goodness of fit and for bootstrapping applications, deleveraged residuals are the better choice since the assumption that they are approximately identically distributed is more likely to be true. For this reason, we continue with using equation 4.5, but the accompanying excel application displays both versions of the estimate of the dispersion factor.

With this we derive at our final definition of standardized Pearson residuals:

$$\mathbf{r}^* = \frac{\mathbf{y} - \hat{\mathbf{y}}}{\sqrt{\hat{\phi} \cdot V(\hat{\mathbf{y}}) \cdot (1 - \mathbf{h})}} . \tag{4.7}$$

This concludes our discussion of the subtasks summarized as step 5 in MLE algorithm outlined in section 2. For convenience, we are repeating the step:

Step 5) Extract diagonal elements of hat matrix, **H**, calculate deleveraged residuals, estimate dispersion parameter, and calculate standardized residuals.

4.2 Graphical Representation

Now that we have defined standardized residuals and know how to calculate them, the task of plotting them in MS Excel is straightforward. The main trick for getting pretty plots is to keep track of how many residuals we are actually trying to plot. We address this issue by having the MLE algorithm automatically update named ranges, which are used to specify the data source for the plots. We are graphing standardized residuals against exposure period, development period, calendar period, and size of fitted value. As an aid in the visual inspection we are adding a trend line to each of the plots. For the period-based plots this trend line simply is the average of all residuals for that period. For the plot against size of fitted value, we perform a linear regression of the residuals on the fitted values to plot the trend line.

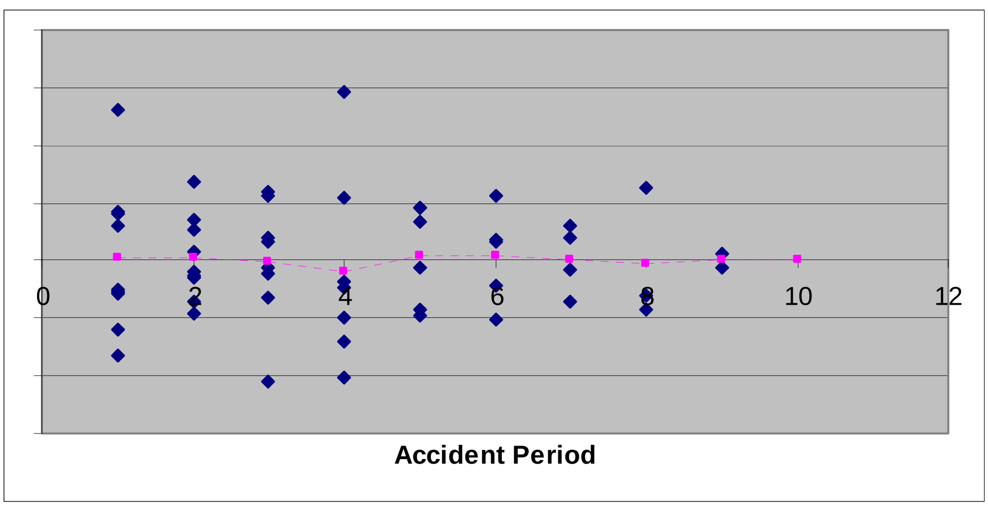
Plots of standardized residuals against various axis of interest are a standard tool for assessing goodness of fit in stochastic modeling. Barnett and Zenwirth's paper on "Best Estimates for Reserves" ([2]) has popularized the concept in the context of analyzing actuarial development triangles and they are now a staple of stochastic reserving packages. The idea is that standardized residuals should be randomly and identically distributed. If there are any obvious systematic trends visible in the plots, then the residuals are not random after all. In addition one can also detect extreme outliers and an experienced analyst may also infer other information that can be used to find an optimal model.

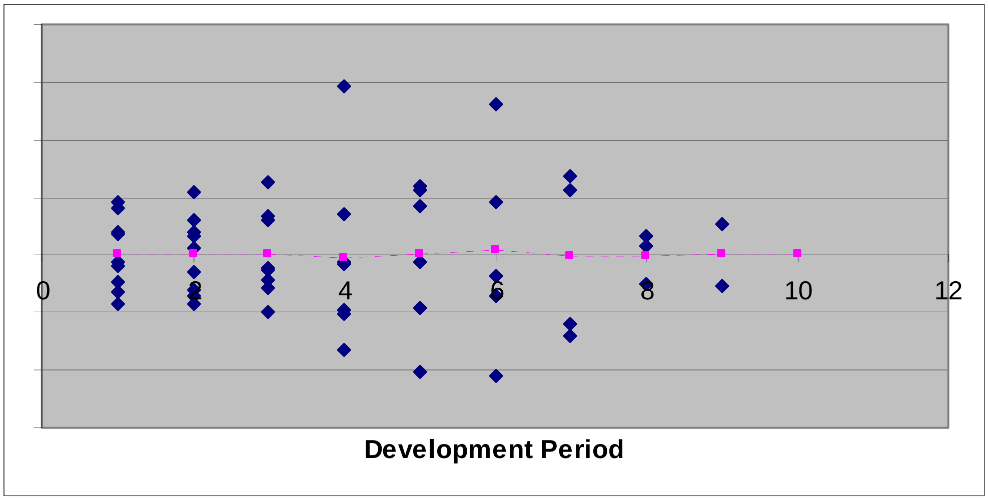
4.2 Example

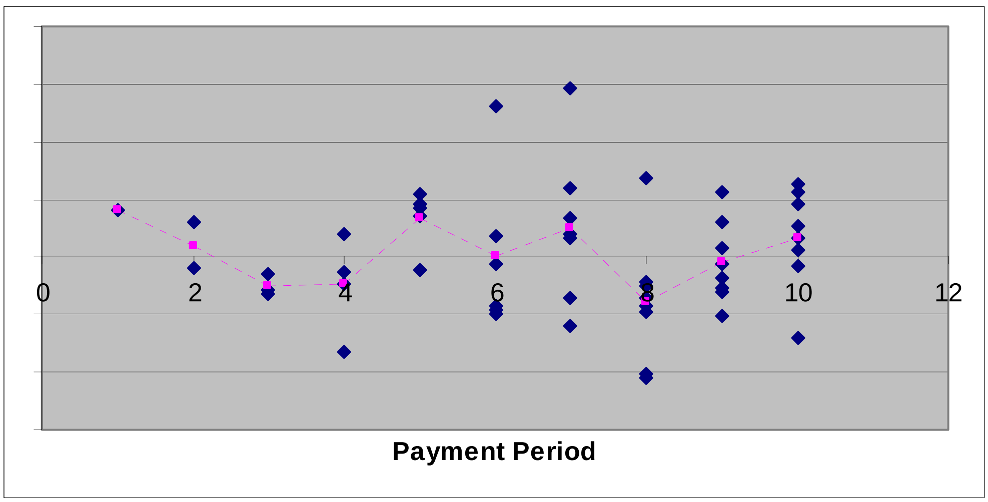
We are using a data set that the authors of [7] attribute to Taylor and Ashe (1983). Here is the data in incremental form:

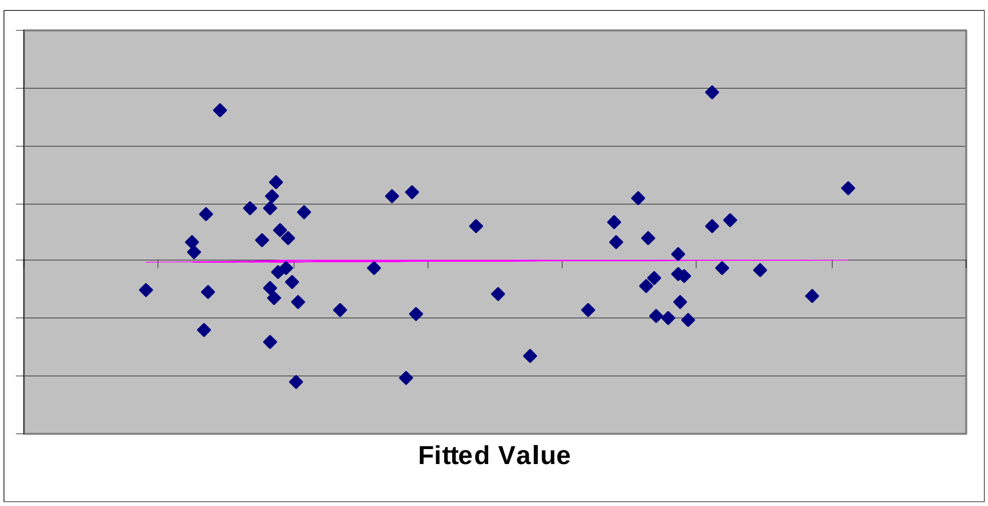
357,848	766,940	610,542	482,940	527,326	574,398	146,342	139,950	227,229	67,948
352,118	884,021	933,894	1,183,289	445,745	320,996	527,804	266,172	425,046	
290,507	1,001,799	926,219	1,016,654	750,816	146,923	495,992	280,405		
310,608	1,108,250	776,189	1,562,400	272,482	352,053	206,286			
443,160	693,190	991,983	769,488	504,851	470,639				
396,132	937,085	847,498	805,037	705,960					
440,832	847,631	1,131,398	1,063,269						
359,480	1,061,648	1,443,370							
376,686	986,608								
344,014									

There are no non-positive or missing data points, so we can go ahead and fit a model based on the full triangle. To start with we chose the identity variance function, which produces a model that preserves the row and column sums of the triangle and is equivalent to the model obtained by developing the triangle using the all-year, volume-weighted link ratios. This results in the following residual plots:

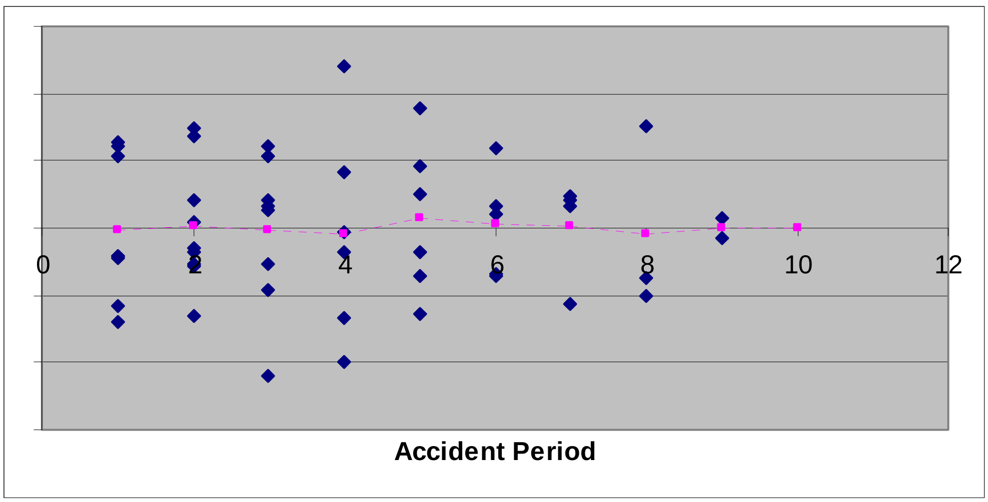


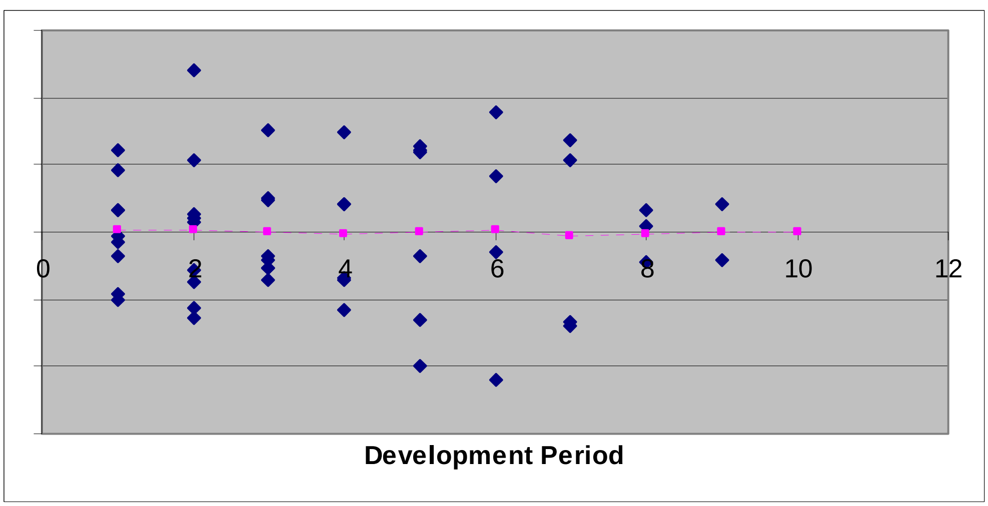

0
2
4
6
8
10
12
Development Period


0
2
4
6
8
10
12
Payment Period

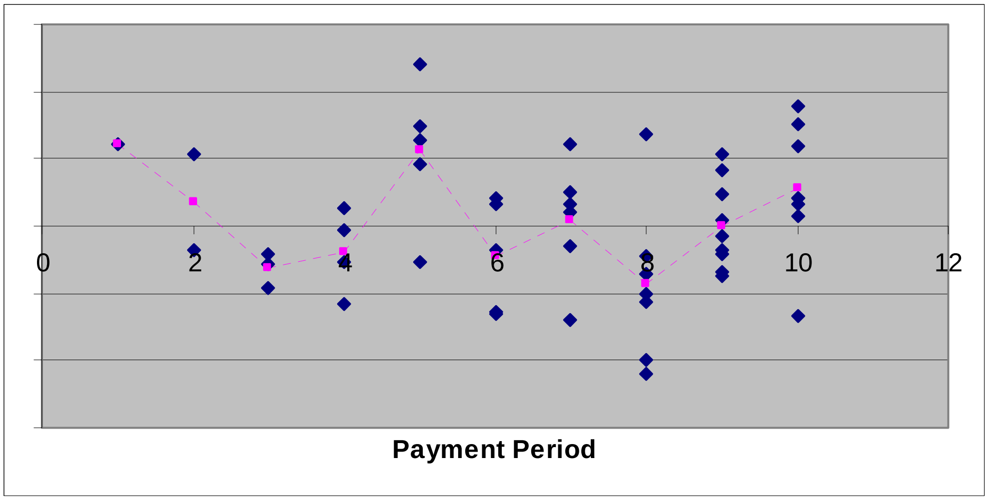

Fitted Value

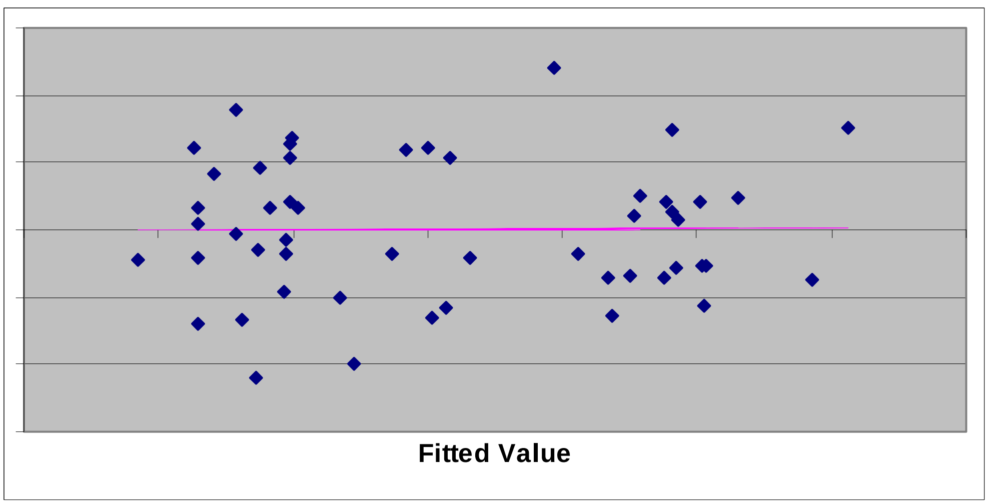
From the jagged trend line for calendar period plot we can see that there are calendar year effects that our model does not capture. From the exposure period and development period plots, we can see that there are two relatively large residuals for cells (4,4) and (1,6)—looking at the data set above we can see that the corresponding values are 1,562,400 and 574,398, respectively. Inspection of the corresponding columns in the triangle does confirm that these values appear unusually high. For demonstration purposes, we assume that theses values represent abnormal circumstances that should not be included in our model to predict future development. Hence we exclude these data points. Technically we have now parameterized a new model and the new residual plots look as follows:





Before looking at the remaining plots on the next page, please note that these plots should mainly be used for assessing the internal consistency of the current model. We emphasize that by excluding data points we end up with different models that cannot be directly compared. With that said, we note that while there are still ups and downs in the exposure and development period plots, the biggest and smallest residuals now do not look way out.



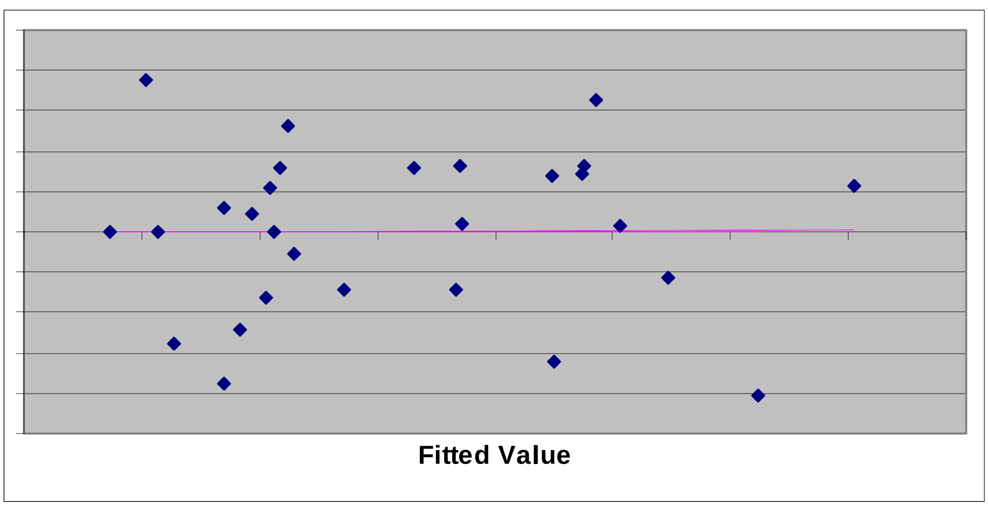


The calendar period plot looks no better than before. Since our model only has exposure and development period parameters, we have limited options for responding to these calendar period effects. For demonstration purposes we will try a model that only includes the latest five diagonals except for the diagonal for calendar period 8, which looks out of line, and cells (4,4) and (1,6), which we previously excluded.

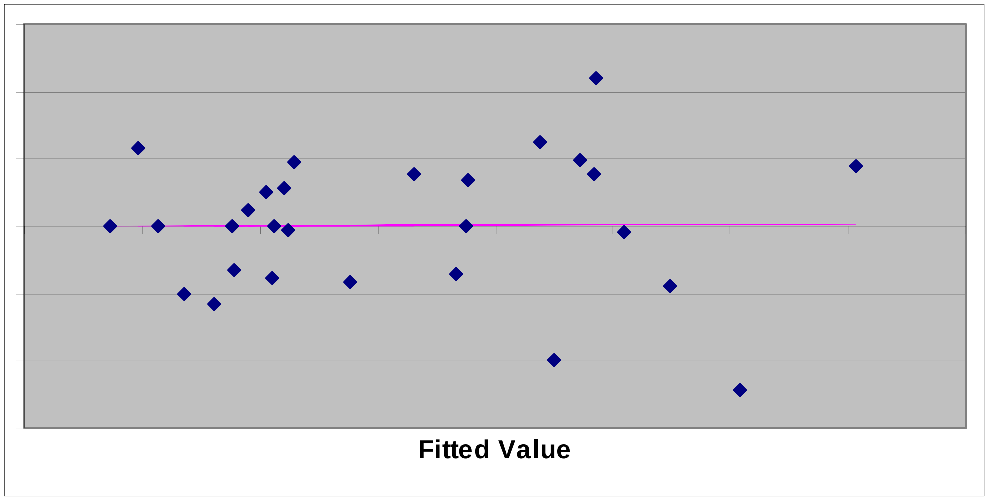
At this point, let us have a look at the output produced by our model. One feature of this type of model is that we cannot only project expected values for future triangle cells, but we can also extrapolate what the expected values for all past triangle cells that were excluded from the analysis. In the table on the following page we show all fitted values that correspond to included data points in bold letters. All values in italics correspond to projections/extrapolations based on the fitted parameters for the model. Since we have chosen the identity variance function the reader can also verify that fitted values in bold preserve the row and column sums of the original data points for the included triangles cells along the last three diagonals.

142,392	*330,441*	*425,664*	*331,922*	*244,123*	*196,001*	**146,600**	*110,970*	**226,971**	**67,948**
266,817	*619,189*	*797,621*	*621,963*	**457,443**	**367,273**	*274,703*	**207,939**	**425,304**	*127,323*
434,523	*1,008,377*	*1,298,962*	**1,012,895**	**744,967**	*598,120*	**447,366**	**338,638**	*692,627*	*207,351*
247,343	*573,998*	**739,407**	*576,569*	*424,057*	**340,467**	**254,654**	*192,763*	*394,263*	*118,030*
316,708	**734,969**	**946,766**	*738,262*	**542,980**	**435,948**	*326,069*	*246,821*	*504,831*	*151,130*
386,120	**896,049**	*1,154,264*	**900,064**	**661,982**	*531,493*	*397,532*	*300,915*	*615,472*	*184,253*
416,980	*967,665*	**1,246,518**	**972,001**	*714,890*	*573,972*	*429,304*	*324,966*	*664,663*	*198,979*
471,751	**1,094,769**	**1,410,249**	*1,099,674*	*808,792*	*649,363*	*485,694*	*367,650*	*751,967*	*225,115*
410,550	**952,744**	*1,227,297*	*957,013*	*703,867*	*565,121*	*422,684*	*319,955*	*654,414*	*195,911*
344,014	*798,336*	*1,028,394*	*801,913*	*589,794*	*473,534*	*354,182*	*268,101*	*548,356*	*164,160*

There is one last feature of the MLE algorithm implemented here that we want to demonstrate: the user can choose from a number of pre-defined variance functions (identity, unity, square root, power 2, and power family with a specifiable positive exponent). To see the effect this may have on the model, consider the residuals versus fitted size plot for our latest model with the identity variance function:



For comparison we will also be fitting a model with the same data points but using the unity variance function. With the unity variance function all data points are assumed to have the same expected variance. Hence we would expect that, relative to the identity variance function, smaller fitted values are given more weight and therefore should have smaller residuals (relative to the residuals for larger fitted values). The reader can inspect the residual plot on the following page to see whether this expectation holds up.



This concludes our walk through on how to use the MLE template. We encourage readers to download the accompanying MS Excel template and put it through its paces with data sets of your choice.

5. RESULTS AND DISCUSSION

In section 2.1, we visually presented the issues that need to be addressed when setting up a model matrix for a development triangle-assuming a multiplicative model which has distinct unordered parameters for exposure and development periods.

The following section 2.2 more formally introduces some concepts from graph theory that allow us to analyze the graph topology of an incomplete development triangle. This enables us to generate a valid model specification and to gather information that is important for interpreting the output of the fitting procedure. In particular we can identify exact fit cells and sub-regions of the incomplete development triangle for which the regression fit is performed without any influence from other regions.

Section 2.4 provided details for an original algorithm based on graph theory that generates a valid model specification for a regression model that can be used to project future development for an incomplete development triangle. If necessary the algorithm will restrict the model to a maximal connected component of the selected incomplete triangle. Note that with minor modifications the algorithm would also work for a rectangle of data points (still assuming a two-dimensional factorial model).

An outline of how to use iterated weighted least squares to implement a maximum likelihood estimator for a GLM was given in section 3.2. The algorithm (and its implementation in Visual Basic, which can be found in the accompanying MS Excel application) is generic and should work for any valid model matrix. An interested reader could easily extend the functionality by adding their own code for other link functions, the derivative of the link function, and/or code for other variance functions.

After providing further guidance on implementation issues in section 3.3, we explained the concrete steps required for performing a weighted least squares regression in section 3.4. This code utilized some simple routines for basic matrix operations tailored to our specific application and standard LAPACK routines implemented in Visual Basic from the open source ALGLIB project. We note in section 3.5 that performance of the MLE algorithm can be significantly improved by porting the matrix routines to C++ and compiling them into a dll that then can be accessed from Visual Basic.

In section 4.1 we presented an intuitive account of the concept of leverage and its role in the computation of standardized residuals that can be used for visual analysis of goodness of fit and for bootstrapping applications. We also proposed an alternative estimator for the dispersion parameter which is based on deleveraged residuals that have been individually adjusted for the bias introduced by leverage. The conventional estimator uses a degree of freedom adjustment that in effect is uniformly applied to all Pearson residuals.

After providing a brief outline on how to create plots of standardized residuals we presented a "walk through" of how these plots can be used in assessing goodness of fit and making informed choices about which data points should be included for a concrete triangle that has also been used in other papers on stochastic reserving. Note that the intention of the walk through is not to provide an optimal model for the specific data set, but to showcase the kind of judgments an analyst can make in the context of the type of GLM model presented here.

The material in this paper is based on standard GLM theory and standard numerical methods. We hope, that by presenting a complete open-source implementation, we can contribute to making more actuaries aware of how these powerful methods can be used in the context of stochastic reserving, and that interacting with the accompanying MS Excel template will prove a useful aid to gaining a deeper understanding of what regression models can accomplish in the context of development triangles.

Interested readers are encouraged to contact the author and request a copy of the companion MS

Excel application to further explore the concepts and algorithms presented in this paper.

We want to conclude this section by comparing the model presented in this paper to some other stochastic models for incremental development triangles discussed in the actuarial literature. The type of model described in this paper (multiplicative effects with discrete parameters for exposure and development periods) is the same as the model described in [7], except that we provide the full apparatus that allows an analyst to exclude arbitrary triangle cells from the analysis. By allowing for excluded data points and different choices of variance functions we somewhat extend the simple bootstrapping models discussed in [4] and [7]. As noted also in these papers a GLM with the identity variance function produces the same estimated reserve as the all-year, volume-weighted link ratio method, when applied to a complete development triangle. But, as observed in [9], such a GLM is more like the traditional BF or Cape Cod method than the link ratio method, in the sense that generally we derive a development pattern and an estimate of exposure by exposure period and then calculate future development by multiplying the two.

Models such as those proposed in [2] and [8] differ from the model presented here in at least two important ways. The first difference is that the models in [2] and [8] effectively group development (or even exposure) periods together and parameterize them using either parametric curves or forms non-parametric smoothing (also discussed in [4]). The second difference is that these models add the calendar period as an additional dimension to the analysis. The model proposed in [8] and some models discussed in [4] furthermore use a Bayesian framework utilizing Monte Carlo Markov Chain simulation techniques.

One straight-forward extension for the model presented here is to allow for arbitrary prior weights for the various triangle cells. Some grouping of development periods (or accident periods) based on parametric curves should not be too difficult to implement either.

A persistent nuisance of most stochastic reserving models for development triangles is that they do not work for negative incremental values. As we have noted in the case of our model in section 2, this seems to be due to the seemingly inevitable choice of a logarithmic link function or a similar transformation involving taking a logarithm. Given how often we encounter triangles with negative incremental values as practicing U.S. P&C reserving actuaries, one would hope that a solution to this problem is found soon.

We conclude our discussion by reiterating that the model and material presented here is intended to introduce a wider audience of P&C actuaries to regression analysis for development triangles. The paper should aid practitioners in deepening their understanding of regression analysis, in general, and GLM analysis, in particular. We also hope that practitioners will start appreciating that development triangles represent a rather condensed form of data and that even the most sophisticated stochastic models cannot recover the information that was destroyed in the process of aggregating individual claims data.

6. CONCLUSIONS

We have demonstrated that an incomplete development history is no obstacle to projecting future development. Our analysis of the graph topology of an incomplete development triangle precisely describes to what extent such projections are possible based on the data points given. Understanding the nature and implications of exact fit cells and critical connector cells is crucial for assessing the goodness of fit of the model and for bootstrapping applications. To our knowledge the application of graph theory in this context has not previously been discussed in the actuarial literature. The companion MS Excel application, which is available from the author at request, demonstrates that performing a GLM-based regression fit to a development triangle is a tool within easy reach of any practitioner with access to a personal computer.

Acknowledgment

The companion MS Excel application to this paper is derived from a stochastic reserving model internally developed by PricewaterhouseCoopers' Actuarial and Insurance Management Solutions (AIMS) practice. The author thanks all PwC staff members who helped him with this development project. PwC AIMS also supported the author of this paper by providing him with writing time. Any errors, inaccuracies or views expressed in the paper are the author's responsibility alone. PwC's support for this project in no way represents an endorsement by PwC or PwC AIMS of any of the views expressed by the author or the methods presented in the paper.

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Abbreviations and notations

GLM, generalized linear model
LSQ, least squares
MLE, maximum likelihood estimator (or estimation)

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Thomas Hartl is a Manager within PricewaterhouseCoopers' Actuarial Insurance Management Solutions (AIMS) practice. He provides consulting services to insurance companies, reinsurers, and regulators. His responsibilities include the design, validation and implementation of simulation models supporting statistical analysis for ERM, litigation support, predictive modeling and stochastic reserving. Thomas Hartl is an Associate of the Casualty Actuarial Society, a Member of the American Academy of Actuaries, and holds a PhD in Mathematics from the University of Glasgow, Scotland.

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On Small Samples and the Use of Robust Estimators in Loss Reserving

Hou-wen Jeng*

Abstract

This paper explores the use of robust location estimators such as Average-Excluding-High-And-Low and Huber's M-estimators in loss reserving. Standard order statistics results are used to investigate the finite-sample properties of Average-Excluding-High-And-Low for positively skewed distributions including bias and efficiency, based on the criterion of mean squared error. The paper concludes that Averages-Excluding-High-And-Low, although biased with respect to the population mean for positively skewed distributions, is more efficient than the sample average in small samples. The paper also shows that the use of Huber's M-estimators can enhance the consistency in loss development factor selections by identifying the implied risk preference.

Keywords: Robust Estimators; Order Statistics; Averages-Excluding-High-And-Low; Huber's M-Estimators; Loss Reserving.

1 Introduction

In practice, actuarial data are usually plagued by two problems: heterogeneity and small sample sizes. Heterogeneity refers to the fact that the underlying exposures consist of policies with vastly different statistical properties, either within a rating period or between different periods. For example, losses from separate policies may follow different probability distributions, or follow the same type of distribution but with different parameters. Actuaries try hard to adjust the data by using trend factors, rate change history, and other cross-section and time series factors. After these adjustments, in many instances, doubts may still linger as to whether more adjustments are needed to make the data homogeneous.

*Thanks are due to David Homer for discussion, and to Elizabeth Smith of the CAS for editorial review. The usual disclaimer applies.

Heterogeneity, which usually renders the data from older years obsolete, may exacerbate the problem of small sample sizes. A typical example is that the insurer changes its underwriting focus and the current policy mix becomes drastically different from those just a few years before. As a result, when it comes to estimating loss development factors or loss ratios, one rarely can have more than a dozen quality data points. That poses difficult problems in parameter estimation, confidence interval calculation, and hypothesis testing.

A recent paper by Blumsohn and Laufer [2] describes in great detail such dilemmas faced by casualty actuaries. The authors asked a group of actuaries to select loss development factors for an umbrella incurred loss triangle. The methods used by the participants were tabulated and the resulting estimated reserves compared. They found that, due largely to the instability of the loss development, the number of approaches and the selected factors varied widely. They concluded that (1) actuaries should keep an open mind and to approach unstable triangles from a variety of perspectives, and (2) if the selected factors or the fitted model differ significantly from the sample average, one must be sure there is a good reason for the discrepancy.

Blumsohn and Laufer also noted that the majority of the participants were using a variety of averaging methods such as loss weighted averages and Average-Excluding-High-And-Low ($\bar{A}_{xHL}$), which calculates the sample average after discarding the sample maximum and sample minimum. Notice that these methods are equivalent to either down-weighting or rejecting outliers. In the case of $\bar{A}_{xHL}$, the sample maximum and minimum are automatically identified as outliers and excluded. $\bar{A}_{xHL}$ is widely used by practicing actuaries in the estimation of loss development factors and loss ratios despite the potential downward bias pointed out by Wu [12], who argues that if the data exhibit a long-tailed property as they do in most of the insurance loss distributions, the use of $\bar{A}_{xHL}$ would lead to downward bias when compared to the sample average.

Wu's argument seems to be consistent with most of the current actuarial methodologies, which focus mainly on estimating the population means of the underlying distributions, with a clear preference for unbiased estimators. Naturally, the most frequent choice is the sample average due to its simplicity and unbiasedness. However, from a modern robust statistics point of view, the sample average is probably the worst estimator for the population mean. The sample average is not robust in the sense that it takes only one outlier to make the sample average arbitrarily large or small. Thus it is not difficult to understand why Averages-Excluding-High-And-Low are popular with actuaries since in many instances (particularly when the sample sizes are small), the necessity of eliminating extreme outliers seems to outweigh the consequences of possible downward bias. But, is $\bar{A}_{xHL}$ just a convenient escape route for actuaries when facing selection dilemmas? Or are there instances where one can justifiably select $\bar{A}_{xHL}$ over unbiased estimators such as the sample average?

Robust statistics studies the construction of statistical methods and estimators that can produce reliable parameter estimates and that are less sensitive to sample outliers (see Maronna et al. [10]). The idea of robust statistics also stems from the fact that the underlying distribution may not always be correctly specified and the existence of outliers may be the result of contaminated data. In these circumstances, robust estimators can often perform better and are more efficient in terms of variance or mean squared error than, say, the sample average.

This paper tries to argue that in the case of small samples and skewed distributions the use of robust estimators is even more valuable and can help the analyst make difficult selections. The goal here is to rationalize the use of $\bar{A}_{xHL}$ and Huber's M-estimator in loss reserving by providing evidence from the statistics literature on theoretical grounds, and constructing examples to show its relative efficiency in the context of small samples and skewed distributions. The main reasons of using Huber's M-estimator are its relative simplicity and ease of calculation. In addition, the analyst's selection of the critical value in Huber's M-estimator may also reveal his or her risk preference in identifying outliers.

Section 2 explores the implications of small sample sizes, while the standard results from order statistics are used in Section 3 to investigate the finite-sample properties of $\bar{A}_{xHL}$. The means and variances of $\bar{A}_{xHL}$ are calculated and compared with those of the sample average for four positively skewed distributions, namely exponential, Weibull, lognormal, and Pareto. It shows by example that the mean squared error of $\bar{A}_{xHL}$ can be smaller than that of the sample average for positively skewed distributions, and thus more efficient than the sample average despite its downward bias with respect to the population mean. Section 4 discusses the general properties of Huber's M-estimator and its use in loss development factor selections. Section 5 uses the incurred loss triangle from Blumsohn and Laufer [2] to illustrate the merit of $\bar{A}_{xHL}$ and Huber's M-estimator when volatility is the main issue. Section 6 provides a summary of the publicly available softwares in Excel and R that calculate Huber's M-estimators. The concluding remarks are in Section 7.

2 Outliers and Small Samples

Since the sample average can be significantly altered by outliers, the positive skewness of the underlying distribution can exacerbate the outlier problem as outliers may be coming further from the right tail. In the case of small samples, the potential influence of outliers on the sample average is even greater than those in large samples as the weight of each observation is larger. One might think that the impact of the outliers from both tails of the distribution on the sample average may cancel out each other. This may be true for symmetric distributions in a relatively large sample. But most of the actuarial applications considered here involve small samples that presumably are drawn from positively skewed, heavy-tailed distributions with non-negative support, such as lognormal or Pareto distributions. Thus outliers from the right tail, if present in the sample, tend to be larger and their effect on the sample average more significant.

Table 1 shows the probability, by sample size, of having at least one outlier from the right tail of an independent sample when outliers are defined as data points greater than either the 95th percentile or the 90th percentile of the underlying distribution.[1] Given the measurable chance for outliers in small samples, the sample average may not be a reliable estimator for the population mean if the underlying distribution is heavy-tailed.

Table 1 : Probability of At Least One Outlier from the Right Tail in A Sample of Size n

Sample Size	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
Outlier defined as $\geq$ 95th percentile	22.6%	26.5%	30.2%	33.7%	37.0%	40.1%
Outlier defined as $\geq$ 90th percentile	41.0%	46.9%	52.2%	57.0%	61.3%	65.1%

There are other problems associated with small samples from skewed distributions. For example, Fleming [5] warns that the sample average of a small sample from a positively skewed distribution is most likely smaller than the population mean. In other words, the skewness of the parent distribution can be carried over to the sampling distribution of $\bar{X}$. The statistics literature provided an elegant explanation of this phenomenon nearly 70 years ago through the Berry-Esseen theorem, [2] [3] which says that the largest difference between the sampling distribution function of $\bar{X}$ and the standard normal distribution (its limiting distribution) is bounded by a ratio of the skewness of the underlying distribution to the square root of the sample size. In short, it simply means that the larger the skewness, the slower the speed of convergence to normality. Thus in order to achieve a certain level of sampling precision, the sample average may require a considerably large

[1]The probability is $(1-(0.95)^n)$ or $(1-(0.90)^n)$.

[2]David Homer pointed out this fact to me.

[3]Formally, let $X_1, ..., X_n$ be i.i.d. with $E(X_1)=\mu$, $Var(X_1)=\sigma^2$, and $\beta_3 = E(|X_1-\mu|^3) < \infty$. Then there exists a constant C, independent of n or the distribution of the X_i, such that

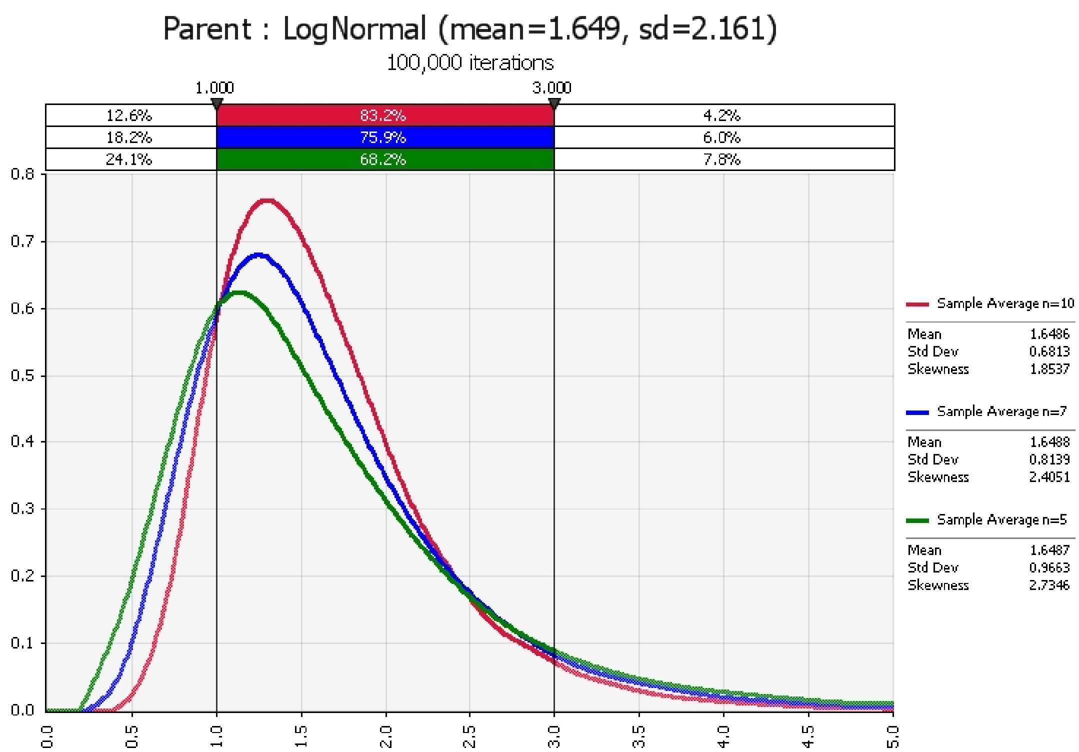
$$\sup_X \left| P(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq x) - \Phi(x) \right| \leq \frac{C\beta_3/\sigma^3}{\sqrt{n}},$$

where β_3/σ^3 is the skewness and $C \leq 0.7655$.

sample size to compensate for the skewness of the parent distribution. This issue is of a different nature from the outlier problem. The solution seems to either get a larger sample or use certain transformation methods to get around the skewness problem. From a robust statistics point of view, the outlier problem exists in samples of all sizes. For small samples, however, the choice of the estimator (either robust or non-robust) may have a more significant impact on the final results.

The following graph shows the simulated results for the sampling distributions of $\bar{X}$ from a lognormal parent distribution (mean = 1.649, sd = 2.161, or $\mu = 0$, $\sigma = 1$) with different sample sizes (n=10, 7, and 5). Notice the gradual increase in skewness (thicker tail) when the sample size decreases from 10 to 5.

Graph 1 : Sampling Distributions of $\bar{X}$ (n = 10, 7, and 5)



3 $\bar{A}_{xHL}$: A Robust Estimator

Trimmed means, which are considered robust estimators for location parameters, calculate the sample average after discarding a fixed *number* or a fixed *percentage* of the observations from both ends of an ordered sample. Trimmed means are less sensitive to outliers compared to the sample average $\bar{X}$. Trimmed means come in many varieties, and their statistical properties as well as asymptotic behavior are studied extensively in the statistics literature (see Maronna et al. [10] and Wilcox [11]). The use of Average-Excluding-High-And-Low $\bar{A}_{xHL}$ in actuarial practice is a classical example of trimmed means. It is obvious that $\bar{A}_{xHL}$ is just a special case of trimmed means, where only the sample maximum and minimum are discarded.

3.1 Finite-Sample Statistics of $\bar{A}_{xHL}$

In this section we calculate the finite-sample mean and variance of $\bar{A}_{xHL}$ while the asymptotic properties of $\bar{A}_{xHL}$ are explored in Section 3.3. Let $(X_1, \ldots, X_n)$ be a sample of n independent and identically distributed random variables. Denote the cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$ with mean μ and variance σ^2 (subject to existence).

Let $X_{(i)}$ be the ith order statistic of $(X_1, \ldots, X_n)$. Thus $X_{(1)} = min(X_1, \ldots, X_n)$, $X_{(n)} = max(X_1, \ldots, X_n)$, and $X_{(1)} \leq \ldots \leq X_{(i)} \leq \ldots \leq X_{(n)}$ for $1 \leq i \leq n$. Average-Excluding-High-And-Low $\bar{A}_{xHL}$ is defined as

$$\bar{A}_{xHL} = \frac{\sum_{i=1}^{n} X_i - X_{(1)} - X_{(n)}}{n-2}.$$

The mean and the variance of $\bar{A}_{xHL}$ when the sample size is n are

$$E(\bar{A}_{xHL}) = \frac{n\mu - E(X_{(1)}) - E(X_{(n)})}{n-2} \tag{1}$$

and

$$Var(\bar{A}_{xHL}) = Var\left\{\frac{\sum_{i=2}^{n-1} X_{(i)}}{n-2}\right\} = \frac{\sum_{i=2}^{n-1}\sum_{j=2}^{n-1} Cov(X_{(i)}, X_{(j)})}{(n-2)^2}, \tag{2}$$

respectively. Note that although all observations are i.i.d., the order statistics (i.e., $X_{(i)}$) of an independent sample are correlated with one another.

3.2 Bias and Relative Efficiency of $\bar{A}_{xHL}$

Next, define $Bias$ of $\bar{A}_{xHL}$ with respect to the population mean μ as

$$Bias(\bar{A}_{xHL}; \mu) = \frac{E(\bar{A}_{xHL}) - \mu}{\mu} = \frac{2\mu - E(X_{(1)}) - E(X_{(n)})}{(n-2)\mu}.$$

It can be shown that $2\mu = (E(X_{(1)}) + E(X_{(n)}))$ for symmetric distributions and $2\mu < (E(X_{(1)}) + E(X_{(n)}))$ for positively skewed distributions. For the latter case, it implies $Bias(\bar{A}_{xHL}) < 0$.

Define MSE with respect to the population mean μ as

$$\begin{aligned} MSE(\bar{A}_{xHL};\mu) &= Var(\bar{A}_{xHL}) + \left\{Bias(\bar{A}_{xHL};\mu) * \mu\right\}^2 \\ &= Var(\bar{A}_{xHL}) + \left\{\frac{2\mu - E(X_{(1)}) - E(X_{(n)})}{(n-2)}\right\}^2 \end{aligned}$$

and the relative efficiency between $\bar{X}$ and $\bar{A}_{xHL}$ with respect to the population mean μ as

$$REff(\bar{X}, \bar{A}_{xHL};\mu) = \frac{MSE(\bar{X};\mu)}{MSE(\bar{A}_{xHL};\mu)} = \frac{Var(\bar{X})}{Var(\bar{A}_{xHL}) + \left\{\frac{2\mu - E(X_{(1)}) - E(X_{(n)})}{(n-2)}\right\}^2}.$$

$Bias$ is a common way to quantify the distance between an estimator and a parameter while MSE is a widely accepted measure of accuracy for estimators with respect to a parameter. Traditionally, the efficiency measure is a ratio between the Cramér-Rao lower bound and the variance of an unbiased estimator. Here, however, $REff(\bar{X}, \bar{A}_{xHL};\mu)$ is narrowly defined to compare the MSEs of the sample average $\bar{X}$ and $\bar{A}_{xHL}$. Note that if the underlying distribution is skewed, $\bar{A}_{xHL}$ is always biased with respect to the population mean. As such, MSE may be a more appropriate measure in comparing $\bar{X}$ and $\bar{A}_{xHL}$ as it penalizes the estimator for its deviation from the parameter μ.

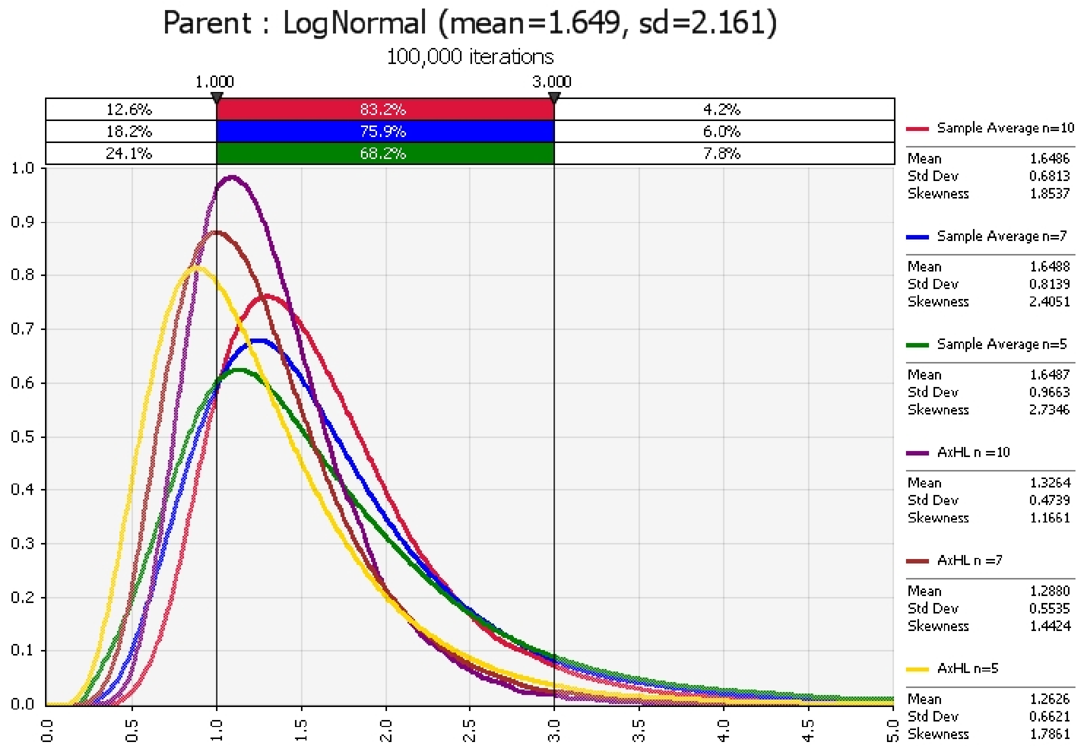
In the appendix, the mean, variance, $Bias$, MSE, $Asym$, and $REff$ of $\bar{A}_{xHL}$ from five distributions are calculated for sample sizes from five to ten as shown in Table 2. The distributions range from symmetric (standard normal), light-tailed (exponential) to positively skewed and heavy-tailed (lognormal, Pareto) distribution. The selections of the parameter values are subjective as the goals are to illustrate the influence of sample size on $E(\bar{A}_{xHL})$ and $Var(\bar{A}_{xHL})$ and to contrast their differences with $E(\bar{X})$ and $Var(\bar{X})$, respectively.

Table 2 : Means, Variances, Coefficients of Variation and Skewness of Selected Distributions

Distribution	Mean	Variance	Coeff. Vari.	Skewness
Standard Normal	0	1	N.A.	0
Exponential ($\theta = 1$)	1	1	100%	2
Pareto ($\theta = 1$, $\alpha = 4$)	1.333	0.222	35%	7.071
LogNormal ($\mu = 0, \sigma^2 = 1$)	1.649	4.671	131%	6.185
Weibull (θ=1, τ=0.5)	2	20	224%	6.618

Overlaying on Graph 1, Graph 2 shows the simulated results for the sampling distributions of $\bar{A}_{xHL}$ from a lognormal parent distribution with sample sizes of 10, 7, and 5. Note the differences in skewness (thicker tail) and standard deviation between the corresponding distributions of $\bar{X}$ and $\bar{A}_{xHL}$ with the same sample size.

Graph 2 : Sampling Distributions of $\bar{X}$ and $\bar{A}_{xHL}$ (n = 10, 7, and 5)



As indicated earlier, $\bar{A}_{xHL}$ are biased downward in positively skewed distributions. The degree of the bias depends on the shape parameter and the sample size. The larger the sample size, the smaller the bias. Table 3 summarizes the results from the calculations using order statistics in the appendix. Note that $\bar{A}_{xHL}$ are unbiased if the underlying distribution is symmetric.

Table 3 : $Bias(\bar{A}_{xHL}; \mu)$ by Sample Size n

Sample Size	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
Standard Normal	0 %	0%	0%	0%	0%	0%
Exponential ($\theta = 1$)	-16%	-15%	-15%	-14%	-13%	-13%
Pareto ($\theta = 1$, $\alpha = 4$)	-6%	-6%	-6%	-6%		
LogNormal ($\mu = 0, \sigma^2 = 1$)	-23%	-23%	-22%	-21%	-20%	-20%
Weibull (θ=1, τ=0.5)	-46%	-44%	-43%	-41%	-40%	-38%

While $\bar{A}_{xHL}$ is biased with respect to the population mean for positively skewed distributions, they are more efficient than the sample average in terms of mean squared error. The efficiency advantage is consistent across the sample sizes as shown in Table 4, which summarizes the results from the appendix. Note that given a normal distribution, the sample average is universally more efficient than

$\bar{A}_{xHL}$ regardless of the sample size. For an exponential distribution, $\bar{A}_{xHL}$ is almost as efficient as the sample average. However, for the Pareto, LogNormal, and Weibull distributions, $\bar{A}_{xHL}$ are much more efficient than $\bar{X}$.

Table 4 : $REff(\bar{X}, \bar{A}_{xHL}; \mu)$ **by Sample Size** n

Sample Size	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
Standard Normal	88%	91%	92%	93%	94%	95%
Exponential ($\theta = 1$)	97%	97%	97%	97%	97%	97%
Pareto ($\theta = 1$, $\alpha = 4$)	151%	148%	145%	142%		
LogNormal ($\mu = 0, \sigma^2 = 1$)	159%	159%	159%	159%	159%	159%
Weibull (θ=1, τ=0.5)	185%	185%	185%	185%	185%	185%

3.3 Asymptotic Properties of $\bar{A}_{xHL}$

The asymptotic properties of trimmed means depend on the nature of trimming in relation to the sample size n. If the number of the trimmed observations is fixed, the trimming is considered *light*, such as $\bar{A}_{xHL}$. All other cases are considered either *intermediate* or *heavy* trimming, where the number of the trimmed observations may be infinite as n goes to infinity. For example a 25% trimmed mean is calculated by trimming 25% of the observations from both ends of the ordered sample regardless of the sample size.

Light Trimming - Note that the value of $\bar{A}_{xHL}$ approaches $\bar{X}$ as n becomes large. Kesten [8] also shows that the convergence in distribution of *lightly* trimmed means and sample average are equivalent. In other words, both $\bar{A}_{xHL}$ and the sample average are asymptotically normal with the same normalizing factors (i.e., the asymptotic mean and standard deviation). Thus the asymptotic mean of $\bar{A}_{xHL}$ is the population mean μ and it is in this sense that $\bar{A}_{xHL}$ is asymptotically unbiased. However, as shown in Section 3.2 and the appendix, depending on the type of the distribution, the finite-sample properties of $\bar{A}_{xHL}$ and the sample average can be very different.

Heavy Trimming - In the case of heavy trimming, where a fixed *percentage* of the sample points are trimmed from both ends of the ordered sample, Csörgő et al. [3] have shown that a normalized trimmed mean so defined converges in distribution to a standard normal random variable, and the asymptotic mean is the expected value of a truncated parent distribution with the upper and lower truncation points at the same fixed percentiles as in the sample. For example, if a trimmed mean is obtained by trimming 20% of the sample from both ends, the support of the truncated distribution is from the 20th percentile to the 80th percentile of the parent distribution.

Wu [12] indicates that $\bar{A}_{xHL}$ would underestimate the population mean of a positively skewed distribution. He first defines the asymptotic means of $\bar{A}_{xHL}$ ([12]

p. 717, Exhibit 1) as

$$Asym(\bar{A}_{xHL}) = \frac{1}{1-2/n} \int_{F^{-1}(\frac{1}{n})}^{F^{-1}(1-\frac{1}{n})} xf(x)dx, \tag{3}$$

which[4] is equivalent to the asymptotic mean for *heavily* trimmed means when the trimming percentage is fixed at $1/n$. For example, if the sample size is five, $Asym(\bar{A}_{xHL})$ is the expected value of a truncated parent distribution with the upper and lower truncation points at the 80th and 20th percentiles of the parent distribution, respectively. As such, $F^{-1}(1-\frac{1}{n}) = F^{-1}(0.8)$, $F^{-1}(\frac{1}{n}) = F^{-1}(0.2)$, and

$$Asym(\bar{A}_{xHL}) = \frac{1}{1-2/5} \int_{F^{-1}(0.2)}^{F^{-1}(0.8)} xf(x)dx.$$

The magnitude of the truncation is based on the size of the sample. As a result, $Asym(\bar{A}_{xHL})$ can be different when the sample size varies. Wu [12] then estimates the bias of $\bar{A}_{xHL}$ by comparing $Asym(\bar{A}_{xHL})$ with the population mean, and argues that the sampling bias can be corrected by using a ratio of the population mean and $Asym(\bar{A}_{xHL})$.

Wu's approach to the problem raises two issues. First, we know through the statistics literature (e.g., Kesten [8]), when the trimming is light, such as $\bar{A}_{xHL}$, the asymptotic mean is the same as the underlying population mean regardless of the sample size. Second, although $\bar{A}_{xHL}$ and $\bar{X}$ have the same asymptotic mean, the finite-sample expected values for $\bar{A}_{xHL}$ can be very different, depending on the sample size. The sample sizes under consideration in actuarial practice are usually quite small. Despite the fact that the exact distribution of $\bar{A}_{xHL}$ is often intractable, the means, variances, and covariances of $\bar{A}_{xHL}$ for small samples can often be derived explicitly or numerically approximated. Therefore, it is not necessary to use the asymptotic mean to calculate the theoretical bias in small samples since doing so would actually overstate the size of the bias.

Table 5 : $Bias(Asym(\bar{A}_{xHL}); \mu)$ **by Sample Size** n

Sample Size	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
Standard Normal	0 %	0%	0%	0%	0%	0%
Exponential ($\theta = 1$)	-24%	-22%	-21%	-19%	-18%	-17%
Pareto ($\theta = 1$, $\alpha = 4$)	-11%	-10%	-10%	-10%		
LogNormal ($\mu = 0, \sigma^2 = 1$)	-33%	-31%	-29%	-27%	-26%	-25%
Weibull (θ=1, τ=0.5)	-64%	-60%	-57%	-54%	-52%	-50%

[4]Since we are only interested in continuous distributions, $F^{-1}(u)$ here is assumed to be uniquely determined for each u in $[0, 1]$.

The appendix compares the estimated bias resulting from using $Asym(\bar{A}_{xHL})$ instead of $E(\bar{A}_{xHL})$ for the five distributions. Table 5 summarizes the results by sample size and indicates that the **overstatement** exists across the sample sizes of five to ten and can be as much as 50% for some positively skewed distributions, compared to $Bias(\bar{A}_{xHL}; \mu)$ in Table 3.

One may also argue on philosophical grounds that the correction, for either small or large samples, is not necessary. That is, from a robust statistics point of view, the examination and treatment of outliers are of fundamental importance[5] while the unbiasedness with respect to the population mean is never an objective nor a concern. In general, the goal of robust location estimators is to measure the **central tendency** of the distribution, *not* the population mean. Thus the question is not whether the outliers should be eliminated or not, but how to lessen their impact if outliers exert undue influence on the estimation.

Unbiasedness seems to have been fully embraced in the casualty literature as the most important property for an estimator, but in practice unbiased estimators, such as the sample average are rarely used as selections. Instead, it is always some type of modified average depending on the circumstance, the data, and the analyst. Moving away from the "first moment only" mentality can help us achieve a shorter confidence interval and gain efficiency in terms of mean squared error, which considers both the first and second moments. Here it should be emphasized that we are not advocating abandoning the sample average as an estimator. Rather, we suggest that efficient robust estimators should always be considered along with other unbiased estimators.

4 Huber's M-Estimators

To calculate $\bar{A}_{xHL}$, automatically trimmed are the sample maximum and sample minimum, which may or may not be outliers relative to the rest of the sample. Thus it makes sense if the trimming can be limited to the outliers identified during the calculating process. Huber's M-estimator does exactly that. The theory of Huber (See Huber and Ronchetti [7]) is to solve the following problem given n i.i.d. observations $(x_1, \ldots, x_n)$:

$$\min_t \left(\sum_{i=1}^{n} \xi(x_i - t)\right)$$

[5] The usual benchmarks for robustness measurement are breakdown point and influence function (see Maronna et al. [10]).

with ξ a suitable function. Or equivalently, $\sum_{i=1}^{n} \Psi(x_i - t) = 0$ where Ψ is the derivative of ξ. Specifically, Huber's Ψ is defined as follows:

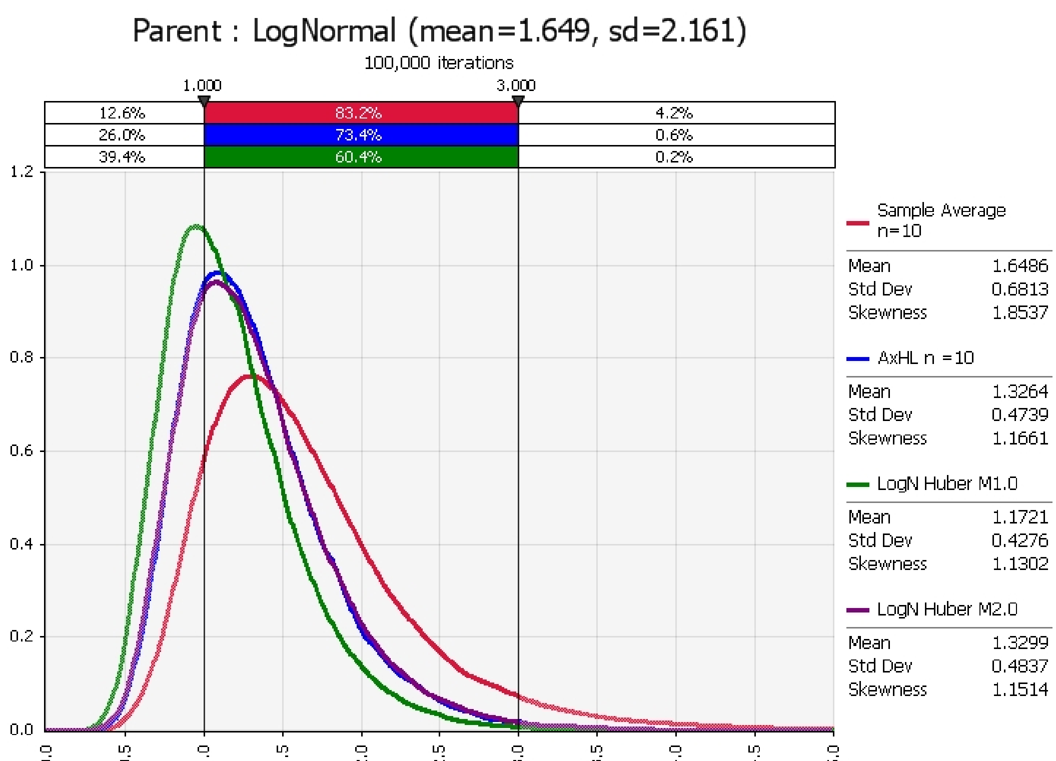
$$\Psi(x) = \begin{cases} K & \text{if } x > K, \\ x & \text{if } |x| \leq K, \\ -K & \text{if } x < -K, \end{cases}$$

where $K > 0$ is a factor selected by the analyst. In practice, the following form of Ψ is used:

$$\sum_{i=1}^{n} \Psi\left(\frac{x_i - t}{\tau}\right) = 0 \tag{4}$$

where τ is a scale measure added to ensure that the resulting solution $t = M$ is scale equivariant. The intuition here is that instead of trimming a fixed number or percentage of observations, only those observations with the adjusted values of $(x - M)/\tau$ outside the range of $[-K, K]$ are replaced by either $(M - \tau K)$ or $(M + \tau K)$. Note that the presumed outliers are not trimmed but replaced.

Graph 3 : Sampling Distributions of $\bar{X}, \bar{A}_{xHL}$, and Huber's M-Estimators (n=10)



If $K = \infty$, $\sum_{i=1}^{n}(\Psi(x_i - t)) = \sum_{i=1}^{n}(x_i - t) = 0$ and the solution to the optimization is the sample average $\bar{X}$. And if $K = 0$, the sample median is the solution. If K is between 0 and infinity, no closed-form solutions exist and a numerical approximation using the Newton-Raphson algorithm is usually employed to derive the solution. Note that when K is between 0 and infinity, the solution is not necessarily between the median and $\bar{X}$ due to the non-linearity of the problem.

The finite-sample properties of Huber's M-estimator can be obtained through simulation. Graph 3 shows the simulation results for the sampling distributions of $\bar{X}$, $\bar{A}_{xHL}$, and Huber's M-estimators with $K = 1.0$ and 2.0 from a lognormal parent distribution when the sample size is 10. The distribution of $\bar{A}_{xHL}$ is almost indistinguishable from that of Huber's M-estimators with $K = 2.0$ while $\bar{X}$ has a thicker tail and a larger standard deviation than the two robust estimators. Note that Huber's M-estimator with $K = 1.0$ has a smaller standard deviation but a larger bias than Huber's M-estimator with $K = 2.0$.

In theory, the selection of the K value is to balance between efficiency (asymptotic variance at the normal distribution) and robustness (resistance against outliers from heavy-tailed distributions). For example, compared with Huber's M-estimator with $K = 1.0$, Huber's M-estimator with $K = 2.0$ has a lower asymptotic variance at the normal distribution. Huber's M-estimator with $K = 1.0$, on the other hand, is more robust in terms of guarding against the impact of outliers.

Using the standard normal approximation may provide another perspective on what the K value implies in the calculation of Huber's M-estimator. Given 1.64 is the 95th percentile of the standard normal distribution, a range of [-1.64, 1.64] for the adjusted value $(x - t)/\tau$ may be interpreted as covering 90% of the underlying distribution.[6] With a higher K value, the range for admissible observations is getting larger and thus fewer observations are classified as outliers. If we define risk as the influence of outliers on the measure of the distribution center, then the selection of the K value may have an added benefit in reflecting the risk preference of the analyst. In other words, the more risk averse the analyst is, the lower the K value may be selected.

5 An LDF Example Using Robust Estimators

In this section, we illustrate the calculation of $\bar{A}_{xHL}$ and Huber's M-estimators using the data from Blumsohn and Laufer [2]. For completeness, the age-to-age development factors of the incurred loss triangle from Blumsohn and Laufer [2] (p. 22) are reproduced in Table 6 below along with the medians, $\bar{A}_{xHL}$, and means of the respective age-to-age factors.[7]

[6] No references can be found for this interpretation, which may be regarded as speculative.

[7] Here we assume that the age-to-age factors in each column are i.i.d.

Table 6: Age-to-Age Loss Development Factors

Year	2-1	3-2	4-3	5-4	6-5	7-6	8-7	9-8	10-9	11-10	12-11
1991	1.68	2.31	1.47	1.22	1.14	0.97	1.05	0.97	1.04	1.01	0.99
1992	6.54	1.26	1.62	1.57	0.87	1.11	1.03	1.01	0.99	0.99	
1993	1.75	2.78	1.32	1.24	1.08	1.01	0.98	0.99	1.01		
1994	3.88	1.83	0.86	0.96	1.20	1.01	1.05	1.01			
1995	2.69	1.81	0.91	1.19	1.00	1.57	1.00				
1996	1.11	1.42	1.12	1.14	1.23	1.01					
1997	1.98	1.41	0.96	1.17	1.02						
1998	3.91	1.10	1.53	1.02							
1999	1.45	0.97	1.44								
2000	1.44	1.13									
2001	1.23										
Med.	1.75	1.41	1.32	1.18	1.08	1.01	1.03	1.00	1.01	1.00	0.99
$\bar{A}_{xHL}$	2.23	1.53	1.25	1.16	1.09	1.03	1.03	1.00	1.01	1.00	0.99
Avg	2.52	1.60	1.25	1.19	1.08	1.11	1.02	0.99	1.01	1.00	0.99

Table 7: Implied Loss Reserves and M-Estimates of LDF for Various K Values

K	Prob	Implied Reserve	2-1	3-2	4-3	5-4	6-5	7-6	8-7	9-8	10-9
0.06	5%	22,017k	1.75	1.41	1.32	1.18	1.08	1.01	1.03	1.00	1.01
0.13	10%	22,089k	1.76	1.41	1.32	1.18	1.08	1.01	1.03	1.00	1.01
0.25	20%	22,382k	1.80	1.41	1.32	1.18	1.08	1.01	1.03	1.00	1.01
0.39	30%	22,741k	1.80	1.43	1.32	1.18	1.08	1.02	1.03	1.00	1.01
0.52	40%	22,673k	1.82	1.46	1.30	1.18	1.08	1.02	1.03	1.00	1.01
0.67	50%	22,854k	1.87	1.48	1.28	1.18	1.08	1.02	1.03	1.00	1.01
0.84	60%	23,415k	1.92	1.49	1.27	1.18	1.09	1.02	1.03	1.00	1.01
1.04	70%	23,503k	1.97	1.51	1.25	1.18	1.09	1.02	1.02	1.00	1.01
1.15	75%	23,650k	2.00	1.52	1.25	1.17	1.09	1.03	1.02	1.00	1.01
1.28	80%	23,758k	2.04	1.54	1.25	1.17	1.09	1.03	1.02	1.00	1.01
1.64	90%	24,227k	2.14	1.57	1.25	1.17	1.08	1.03	1.02	1.00	1.01
1.96	95%	24,908k	2.23	1.59	1.25	1.16	1.08	1.04	1.02	0.99	1.01
2.58	99%	25,799k	2.31	1.60	1.25	1.16	1.08	1.04	1.02	0.99	1.01
Med.		22,017k	1.75	1.41	1.32	1.18	1.08	1.01	1.03	1.00	1.01
$\bar{A}_{xHL}$		25,502k	2.23	1.53	1.25	1.16	1.09	1.03	1.03	1.00	1.01
Avg		33,349k	2.52	1.60	1.25	1.19	1.08	1.11	1.02	0.99	1.01

Using the data from Table 6, Table 7 shows the resulting estimated loss reserves with various K values in Huber's M-estimators. For each K value, the implied loss reserve is calculated by assuming that the same K value is used for each of the columns in the age-to-age factor selection. The K value ranges from 0.06 to 2.58, which correspond to the 5% and 99% pseudo-probability ranges, respectively (i.e., as if approximated by a standard normal distribution). Using a K value of 0.06 implies that the analyst classifies any observation x as an outlier if its adjusted value $(x - M)/\tau$ is outside the range of [-0.06, 0.06].

In the example, at the 5% pseudo-probability level, all observations are deemed outliers and the Huber's M-estimate is the sample median for all ages. With higher K values, the M-estimates change gradually from the sample median to the sample average. Finally at the 99% pseudo-probability level, only a handful of observations are deemed outliers and the Huber's M-estimate is the sample average for most of the age-to-age factors. The range of the implied reserves is between 22.0 million and 25.8 million, corresponding to K=0.06 and K=2.58, respectively.

Table 8 below shows the data points in the 2-1 age-to-age factors that are deemed outliers for various K values in the calculation of Huber's M-estimates. At K = 0.06, all points are outliers except 1.75, which happens to be the sample median. As K becomes larger, fewer data points are declared outliers. At K = 2.58, the only outlier is 6.54.

Table 8: Implied Outliers For 2-1 Age-To-Age Factors By K Value

K	Out1	Out2	Out3	Out4	Out5	Out6	Out7	Out8	Out9	Out10
0.06	6.54	3.91	3.88	2.69	1.98	1.68	1.45	1.44	1.23	1.11
0.13	6.54	3.91	3.88	2.69	1.98		1.45	1.44	1.23	1.11
0.25	6.54	3.91	3.88	2.69			1.45	1.44	1.23	1.11
0.39	6.54	3.91	3.88	2.69			1.45	1.44	1.23	1.11
0.52	6.54	3.91	3.88	2.69					1.23	1.11
0.67	6.54	3.91	3.88	2.69					1.23	1.11
0.84	6.54	3.91	3.88	2.69					1.23	1.11
1.04	6.54	3.91	3.88							1.11
1.28	6.54	3.91	3.88							
1.64	6.54	3.91	3.88							
1.96	6.54	3.91	3.88							
2.58	6.54									

A few comments on the age-to-age selection methods may be in order:

- One potential flaw or inconsistency of $\bar{A}_{xHL}$ when applied to the setting of age-to-age factor calculation is that $\bar{A}_{xHL}$ trims a different percentage of data for each of the columns. For example, for the 2-1 factor, two out of

11 observations or 18.2% of the data are trimmed while for the 8-7 factors, 40% of the data (two out of five) are trimmed. The inconsistency stems from the fact that $\bar{A}_{xHL}$ trims less percentage of the data when the data are more volatile (e.g., the 2-1 factors) and more percentage of the data when the data are relatively stable (e.g., the 8-7 factors). As indicated in Section 3.2, the finite-sample properties of $\bar{A}_{xHL}$ are dependent on the sample size and can be very different between trimming 18.2% and 40% of the data. On the other hand, using Huber's M-estimators and selecting "appropriate" K values by age may avoid this problem and maintain some level of consistency in the age-to-age factor calculation.

- The loss reserve estimates based on the M-estimates are in the middle-to-lower range of the reserves estimated by the participants in the Blumsohn and Laufer study. The primary reason is that many participants downweight or ignore the negative development in the age-to-age factor selection. For the earlier development ages, their age-to-age factor selections seem to largely fall within the range of the M-estimates with the K values between 0.06 and 2.58, except for the age 7-6 factors, where 1.566 is a prominent outlier and causes a great deal of variations in the participants' selection.

- One interesting observation regarding the age-to-age factor selections by the participants of the Blumsohn and Laufer study is that the implied K values across ages are not consistent. For example, one may select 1.75 for the 2-1 factor with an implied K value of 0.06 while selecting 1.60 for the 3-2 factor with an implied K value of 2.58 (see Table 7). This lack of consistency in terms of the K value may be due to the fact that different averaging methods were used for different ages in selections while the statistical implications of the methods are not obvious.

- When Huber's M-estimator is used with a specific K value, the confidence interval for the loss reserves can be obtained by bootstrapping individual age-to-age factors. One potential problem of this approach is that the τ value can easily become zero in equation (4) for the bootstrap samples when the sample size is small. Note that Huber's M-estimator is not well-defined when $\tau = 0$.

6 Software Implementation

Software in Excel VBA -

- Written by this author and included in the appendix are two Excel/VBA functions (*HuberM* and *MADN*) for calculating Huber's M-estimators. To implement the functions, copy the code for both functions into a Visual Basic module of the desired Excel file. The first required input for *HuberM* is a numeric range/vector while the second required input is the selected K value. Note that *HuberM* is not well-defined when τ from Equation (4) is zero. When this occurs, Excel will exhibit a warning message.

- The Royal Society of Chemistry has made available an Excel Add-in for Huber's M-estimator, *RobStat.xla*.[8] All the installation instructions are in the *ReadMe.txt* file, as well as in the full help system. The Add-in has two Excel functions, *A15_MEAN* and *H15_MEAN*, which calculate two types of Huber's M-estimators. The difference is that the former uses a fixed MADN for τ from Equation (4) in the iteration process while the latter continues to update the τ in each iteration.

  Despite the Add-in's strength in error handling and help system, this author was not able to reconcile the calculation results from *A15_MEAN* (or *H15_MEAN*) with the results from any **R**-based functions including *huberM* in the **R** package and *mest* in Wilcox's collection.

- The function *TRIMMEAN(array,*α*%)* supplied by Excel calculates the α% trimmed mean for the array specified in the first argument of the function. For example, the 20% trimmed mean *TRIMMEAN(array,20%)* for a sample size five is the same as $\bar{A}_{xHL}$.

Software in **R** -

- Two **R** packages ("**robust**" and "**robustbase**") are available on the **R** website to calculate a variety of robust estimators. The function *huberM* in "**robustbase**" calculates Huber's M-estimator, which requires a numeric vector and a K value as inputs.
- Wilcox [11] maintains a significant collection of **R** functions in robust statistics.[9] *mest* is the function that calculates Huber's M-estimator.
- Interested readers can also find other collections of related **R** or **S-Plus** functions in *http://www.statistik.tuwien.ac.at/rsr/index.html.*

[8] http://www.rsc.org/Membership/Networking/InterestGroups/Analytical/AMC/Software /RobustStatistics.asp

[9] http://www-rcf.usc.edu/ rwilcox/

7 Conclusion

Modern robust statistics has made it well-known that outliers can have unbounded influence on classical estimators such as the sample average, resulting in: (1) inaccurate parameter estimates/inference, (2) large standard errors, and (3) wide confidence intervals.

The purpose of this paper is to provide some theoretical facts and examples regarding average-excluding-high-and-low and more broadly, some robust estimators, which may not have been given proper credit in our literature. We have shown by example that $\bar{A}_{xHL}$ is more efficient than the sample average. It also shows that using Huber's M-estimators with selected K values may have some more benefit than using $\bar{A}_{xHL}$.

Although these two estimators only represent a tiny portion of the large number of robust estimators in the statistics literature, one of the major advantages of $\bar{A}_{xHL}$ and Huber's M-estimators is that they can be easily implemented through simple software (see Section 6). The famed Princeton Study on robust estimators (see Andrews et al. [1]) also shows that (1) some trimmed means (similar to $\bar{A}_{xHL}$) and Huber's M-estimators behave rather well under many scenarios in comparison with other robust estimators, and (2) no single robust estimator is more efficient for all distributions.

John Tukey, an early pioneer of the modern robust statistics, once said "*ust which robust/resistant methods you use is not important – what is important is that you use some.*" It is this author's belief that the use of $\bar{A}_{xHL}$ and Huber's M-estimators may be beneficial to actuaries in tackling the day-to-day selection problems.

Appendix

A.0 Basic Formulas in Order Statistics

Let $(X_1, \ldots, X_n)$ be a sample of n independent and identically distributed random variables. Denote the cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$ with mean μ and variance σ^2 (subject to existence).

Let $X_{(i)}$ be the ith order statistic of $(X_1, \ldots, X_n)$. Thus $X_{(1)} = \min(X_1, \ldots, X_n)$, $X_{(n)} = \max(X_1, \ldots, X_n)$, and $X_{(1)} \leq \ldots \leq X_{(i)} \leq \ldots \leq X_{(n)}$ for $1 \leq i \leq n$. If $F(x)$ is absolutely continuous, the expected value and the variance of $X_{(i)}$, and the expected value of $X_{(i)}$ and $X_{(j)}$ for $1 \leq i < j \leq n$ can be expressed as (see David and Nagaraja [4])

$$E(X_{(i)}) = \binom{n}{i} \int_{-\infty}^{\infty} x f(x)[F(x)]^{i-1}[1 - F(x)]^{n-i} dx$$

$$Var(X_{(i)}) = \binom{n}{i} \int_{-\infty}^{\infty} x^2 f(x)[F(x)]^{i-1}[1 - F(x)]^{n-i} dx - [E(X_{(i)})]^2$$

and

$$\begin{aligned} E(X_{(i)}, X_{(j)}) &= Cov(X_{(i)}, X_{(j)}) + E(X_{(i)})E(X_{(j)}) \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times \\ &\quad \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy f(x) f(y)[F(x)]^{i-1}[F(y) - F(x)]^{j-i-1}[1 - F(y)]^{n-j} dx dy, \end{aligned}$$

respectively.

The closed-form solutions to $E(X_{(i)}), Var(X_{(i)})$, and $Cov(X_{(i)}, X_{(j)})$ can be derived explicitly for the exponential, Weibull, and Pareto distributions. For the lognormal distribution, numerical approximation is needed to calculate these statistics. In the order statistics literature, extensive studies (see David and Nagraja [4]) were performed in the 1950s and 1960s on the calculations of the moments of order statistics for various distributions by sample size. Harter and Balakrishnan [6] have summarized and tabulated the numerical results of those studies in their 1996 Handbook.

In this section, we calculate and tabulate the numerical values of the means and variances of $\bar{A}_{xHL}$ for the standard normal distribution and four distributions with nonnegative supports, namely the exponential, lognormal, Pareto and Weibull distributions.[10] In other words, We assume the underlying distribution is known and there is no model misspecification or data contamination. We then employ these results to calculate the exact values of the bias and the relative efficiency of the sample average and $\bar{A}_{xHL}$ for the sample sizes between five and ten. Although $\bar{A}_{xHL}$ may be significantly downward biased for a positively skewed distribution, $MSE(\bar{A}_{xHL})$ usually is smaller than $MSE(\bar{X})$, which is just $Var(\bar{X}) = \sigma^2/n$. That is, even considering the penalty for bias, the average distance as defined by MSE between $\bar{A}_{xHL}$ and μ may still be shorter than that between $\bar{X}$ and μ.

A.1 The Standard Normal Distribution

Since the standard normal is symmetric, $\bar{A}_{xHL}$ is unbiased. Note that $\bar{X}$ is more efficient than $\bar{A}_{xHL}$ as $Var(\bar{X}) \leq Var(\bar{A}_{xHL})$ for all sample sizes. In fact, the standard normal distribution has the rare property that $\bar{X}$ is more efficient than most robust location estimators.

Table A.1 : Efficiency of $\bar{A}_{xHL}$ for Standard Normal

Sample Size	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$E(\bar{X})$	0	0	0	0	0	0
$E(\bar{A}_{xHL})$	0	0	0	0	0	0
$Bias(\bar{A}_{xHL};\mu)$	0%	0%	0%	0%	0%	0%
$Asym(\bar{A}_{xHL})$	0	0	0	0	0	0
$Bias(Asym(\bar{A}_{xHL});\mu)$	0%	0%	0%	0%	0%	0%
$Var(\bar{X})$	0.20000	0.16667	0.14286	0.12500	0.11111	0.10000
$Var(\bar{A}_{xHL})$	0.22706	0.18403	0.15494	0.13387	0.11790	0.10535
$MSE(\bar{A}_{xHL};\mu)$	0.22706	0.18403	0.15494	0.13387	0.11790	0.10535
$REff(\bar{X},\bar{A}_{xHL};\mu)$	88%	91%	92%	93%	94%	95%

A.2 The Exponential Distribution

The pdf and cdf of the exponential distribution with scale parameter θ are

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \qquad x \geq 0, \theta > 0,$$

$$F(x;\theta) = 1 - e^{-x/\theta},$$

respectively. Given that the rth moment of X is $E(x^r) = \theta^r r!$, the exponential distribution has a fixed skewness of 2, independent of θ as shown below

$$Skewness(x) = \frac{E(x^3) - 3\theta E(x^2) + 2\theta^3}{\theta^3} = 2.$$

[10]The distribution forms and the corresponding statistics are based on Klugman et al. [9].

The closed-form solutions exist for the mean and variance of $X_{(i)}$, which are

$$E(X_{(i)};\theta) = \theta \sum_{j=1}^{i} \frac{1}{n-j+1},$$

and

$$Var(X_{(i)};\theta) = \theta^2 \sum_{j=1}^{i} \frac{1}{(n-j+1)^2},$$

respectively. For $i < j$, the covariance of $X_{(i)}$ and $X_{(j)}$ is the same as $Var(X_{(i)};\theta)$. For a sample of five,

$$Bias(\bar{A}_{xHL}) = \frac{5\theta - \theta(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}) - \theta(\frac{1}{5})}{(5-2)\theta} - 1 = -16.1\%.$$

Given the fixed skewness of the exponential distribution, it is not surprising that $Bias(\bar{A}_{xHL})$ is dependent on the sample size n only and independent of the parameter θ.

The calculation of $Asym(\bar{A}_{xHL})$ depends on the sample size n and θ.

$$Asym(\bar{A}_{xHL}) = \frac{\theta}{1-2/n}\left\{\Gamma(2;-ln(\frac{1}{n})) - \Gamma(2;-ln(1-\frac{1}{n}))\right\}$$

where $\Gamma(.,.)$ is the incomplete Gamma function. The following table shows the statistics for the exponential distribution with $\theta = 1$. Note that $E(x) = 1$ and $V(x) = 1$ when $\theta = 1$. As expected, when the sample size gets larger the bias is getting smaller. On the other hand, $\bar{A}_{xHL}$ is almost as efficient as $\bar{X}$ for sample sizes 5 to 10.

Table A.2 : Bias and Efficiency of $\bar{A}_{xHL}$ for Exponential (θ=1)

Sample Size	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$E(\bar{X})$	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
$E(\bar{A}_{xHL})$	0.83889	0.84584	0.85286	0.85952	0.86570	0.87138
$Bias(\bar{A}_{xHL};\mu)$	-16.1%	-15.4%	-14.7%	-14.0%	-13.4%	-12.9%
$Asym(\bar{A}_{xHL})$	0.76085	0.77970	0.79547	0.80914	0.82076	0.83058
$Bias(Asym(\bar{A}_{xHL});\mu)$	-23.9%	-22.0%	-20.5%	-19.1%	-17.9%	-16.9%
$Var(\bar{X})$	0.20000	0.16667	0.14286	0.12500	0.11111	0.10000
$Var(\bar{A}_{xHL})$	0.17966	0.14634	0.12407	0.10801	0.09585	0.08628
$MSE(\bar{A}_{xHL};\mu)$	0.20562	0.17011	0.14572	0.12775	0.11389	0.10282
$REff(\bar{X},\bar{A}_{xHL};\mu)$	97%	97%	97%	97%	97%	97%

A.3 The Pareto Distribution

The pdf and cdf of the Pareto distribution with scale parameter θ and shape parameter α are

$$f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \qquad F(x) = 1 - (\frac{\theta}{x})^\alpha, \qquad x \geq \theta, \alpha > 0,$$

respectively. The mean, variance, and skewness are

$$E(x) = \frac{\alpha\theta}{\alpha - 1}, \qquad Var(x) = \frac{\theta^2\alpha}{(\alpha-2)(\alpha-1)^2},$$

and

$$Skewness(x) = \frac{2(1+\alpha)}{\alpha - 3}\sqrt{\frac{\alpha-2}{\alpha}},$$

respectively. The calculation of $Asym(\bar{A}_{xHL})$ depends on the sample size n,

$$Asym(\bar{A}_{xHL}) = \frac{1}{(1-2/n)}\frac{\alpha\theta}{(\alpha-1)}\left\{(1-\frac{1}{n})^{1-1/\alpha} - (\frac{1}{n})^{1-1/\alpha}\right\}.$$

Using Tables C13.1 and C13.2 in Harter and Balakrishnan [6] and Eqs. (1)-(2) in Section 3.1, the means and variances of $\bar{A}_{xHL}$ with the underlying Pareto $(\alpha = 4, \theta = 1)$ are shown in the following table. Note that $E(x) = 1.33333$, $Var(x) = 0.22225$, and $Skewness(x) = 7.07106$ for the Pareto distribution with $\alpha = 4$ and $\theta = 1$.

Table A.3 : Bias and Efficiency of $\bar{A}_{xHL}$ for Pareto (θ=1, α=4)

Sample Size	$n = 5$	$n = 6$	$n = 7$	$n = 8$
$E(\bar{X})$	1.33334	1.33332	1.33333	1.33333
$E(\bar{A}_{xHL})$	1.24920	1.25220	1.25530	1.25823
$Bias(\bar{A}_{xHL};\mu)$	-6.3%	-6.1%	-5.9%	-5.6%
$Asym(\bar{A}_{xHL})$	1.18859	1.19496	1.20037	1.20502
$Bias(Asym(\bar{A}_{xHL});\mu)$	-10.9%	-10.4%	-10.0%	-9.6%
$Var(\bar{X})$	0.04445	0.03703	0.03174	0.02777
$Var(\bar{A}_{xHL})$	0.02234	0.01839	0.01578	0.01391
$MSE(\bar{A}_{xHL};\mu)$	0.02942	0.02497	0.02187	0.01954
$REff(\bar{X},\bar{A}_{xHL};\mu)$	151%	148%	145%	142%

A.4 The LogNormal Distribution

The pdf of the standard lognormal distribution with location parameter[11] μ and shape parameter σ^2 is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(ln(x)-\mu)^2/2\sigma^2} \qquad\qquad 0 < x < \infty.$$

The mean, variance, and skewness are

$$E(x) = e^{\mu+\sigma^2/2}, \qquad Var(x) = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2},$$

and

$$Skewness(x) = (e^{\sigma^2} + 2)\sqrt{e^{\sigma^2} - 1},$$

respectively.

No closed-form solutions exist for the cdf. So numerical approximation has to be performed for the means and variance of $\bar{X}$, $\bar{A}_{xHL}$. The calculation of $Asym(\bar{A}_{xHL})$ depends on the sample size n, μ, and σ^2.

$$Asym(\bar{A}_{xHL}) = \frac{e^{\mu+\sigma^2/2}}{1-2/n}\left\{\theta(\theta^{-1}(1-\frac{1}{n}) - \sigma) - \theta(\theta^{-1}(\frac{1}{n}) - \sigma)\right\},$$

where $\theta()$ is the cdf of the standard normal distribution. Using Tables C6.1 and C6.2 in Harter and Balakrishnan [6] and Eqs. (1)-(2) in Section 3.1, the means and variances of $\bar{A}_{xHL}$ with the underlying lognormal ($\mu = 0, \sigma = 1$) are shown in the following table. Note that $E(x) = 1.64872$, $Var(x) = 4.67075$, and $Skewness(x) = 6.1849$ for the lognormal distribution with μ=0 and σ^2=1.

Table A.4 : Bias and Efficiency of $\bar{A}_{xHL}$ for LogNormal (μ=0, σ^2=1)

Sample Size	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$E(\bar{X})$	1.64872	1.64872	1.64873	1.64872	1.64872	1.64872
$E(\bar{A}_{xHL})$	1.26269	1.27623	1.29000	1.30314	1.31542	1.32679
$Bias(\bar{A}_{xHL}; \mu)$	-23.4%	-22.6%	-21.8%	-21.0%	-20.2%	-19.5%
$Asym(\bar{A}_{xHL})$	1.11100	1.14365	1.17164	1.19585	1.21702	1.23571
$Bias(Asym(\bar{A}_{xHL}); \mu)$	-32.6%	-30.6%	-28.9%	-27.5%	-26.2%	-25.1%
$Var(\bar{X})$	0.93415	0.77846	0.66725	0.58385	0.51898	0.46708
$Var(\bar{A}_{xHL})$	0.43857	0.36178	0.31139	0.27534	0.24803	0.22646
$MSE(\bar{A}_{xHL}; \mu)$	0.58759	0.50053	0.44008	0.39477	0.35912	0.33011
$REff(\bar{X}, \bar{A}_{xHL}; \mu)$	159%	159%	159%	159%	159%	159%

[11] This is not the same μ as in $Bias(\bar{A}_{xHL}; \mu)$

A.5 The Weibull Distribution

The pdf and cdf of the two-parameter Weibull distribution with scale parameter θ and shape parameter τ are

$$f(x;\theta,\tau) = \frac{\tau}{\theta}\left\{\frac{x}{\theta}\right\}^{\tau-1} e^{-(x/\theta)^{\tau}} \qquad x \geq 0, \theta > 0, \tau > 0$$

and

$$F(x;\theta,\tau) = 1 - e^{-(x/\theta)^{\tau}},$$

respectively. The mean, variance, and Skewness are

$$E(x) = \theta\Gamma(1+\frac{1}{\tau}), \qquad Var(x) = \theta^2\Gamma(1+\frac{2}{\tau}) - (\theta\Gamma(1+\frac{1}{\tau}))^2,$$

and

$$Skewness(x) = \frac{\Gamma(1+\frac{3}{\tau}) - 3\Gamma(1+\frac{2}{\tau})\Gamma(1+\frac{1}{\tau}) + 2[\Gamma(1+\frac{1}{\tau})]^3}{[\Gamma(1+\frac{2}{\tau}) - [\Gamma(1+\frac{1}{\tau})]^2]^{3/2}},$$

respectively.

Various closed-form solutions exist for the means and variances for $X_{(i)}$ (see Harter and Balakrishnan [6]). The calculation of $Asym(\bar{A}_{xHL})$ depends on the sample size n and parameters θ and τ,

$$\begin{aligned} Asym(\bar{A}_{xHL}) &= \frac{\theta\Gamma(1+\frac{1}{\tau})}{1-2/n}\left\{\Gamma(1+\frac{1}{\tau};[\frac{F^{-1}(1-\frac{1}{n})}{\theta}]^{\tau}) - \Gamma(1+\frac{1}{\tau};[\frac{F^{-1}(\frac{1}{n})}{\theta}]^{\tau})\right\} \\ &= \frac{\theta\Gamma(1+\frac{1}{\tau})}{1-2/n}\left\{\Gamma(1+\frac{1}{\tau};-ln(\frac{1}{n})) - \Gamma(1+\frac{1}{\tau};-ln(1-\frac{1}{n}))\right\}. \end{aligned}$$

Note that $E(x) = 2$, $Var(x) = 20$, and $Skewness(x) = 6.618$ for the Weibull distribution with $\tau = 0.5$ and $\theta = 1$. Using Tables C3.1 and C3.2 in Harter and Balakrishnan [6] and Eqs. (1)-(2) in Section 3.1, the means and variances of $\bar{A}_{xHL}$ with the underlying Weibull distribution (θ=1,τ=0.5) are shown in the following table.

Table A.5 : Bias and Efficiency of $\bar{A}_{xHL}$ for Weibull (θ=1, τ=0.5)

Sample Size	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$E(\bar{X})$	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000
$E(\bar{A}_{xHL})$	1.08093	1.11264	1.14489	1.17576	1.20464	1.23142
$Bias(\bar{A}_{xHL};\mu)$	-46.0%	-44.4%	-42.8%	-41.2%	-39.8%	-38.4%
$Asym(\bar{A}_{xHL})$	0.72468	0.79828	0.86211	0.91824	0.96747	1.01078
$Bias(Asym(\bar{A}_{xHL});\mu)$	-63.8%	-60.1%	-56.9%	-54.1%	-51.6%	-49.5%
$Var(\bar{X})$	4.00000	3.33333	2.85714	2.50000	2.22222	2.00000
$Var(\bar{A}_{xHL})$	1.31714	1.09927	0.96081	0.86327	0.78969	0.73149
$MSE(\bar{A}_{xHL};\mu)$	2.16184	1.88668	1.69202	1.54263	1.42229	1.32221
$REff(\bar{X},\bar{A}_{xHL};\mu)$	185%	185%	185%	185%	185%	185%

A.6 Excel VBA Functions for Huber's M-Estimator

```
Function HuberM(x As Range, KValue As Double) As Double
 Dim vRange1 As Variant
 Dim dTemp, dHuberSum, dTempHuberM, dMADN As Double
 Dim h, i, j, iRowCount, iColumnCount, iHuberCount As Integer
 iRowCount = x.Rows.Count
 iColumnCount = x.Columns.Count
 vRange1 = x.Cells.Value
 dMADN = MADN(x)
 dTempHuberM = WorksheetFunction.Median(x)
 dTemp = 0
 For h = 1 To 20     '20 is arbitrary
    dHuberSum = 0
    iHuberCount = 0
    For i = 1 To iRowCount
       For j = 1 To iColumnCount
            dTemp = (vRange1(i, j) - dTempHuberM) / dMADN
            If Abs(dTemp) < KValue Then
              dHuberSum = dHuberSum + dTemp
              iHuberCount = iHuberCount + 1
            ElseIf dTemp > KValue Then
              dHuberSum = dHuberSum + KValue
            Else
               dHuberSum = dHuberSum - KValue
            End If
       Next j
    Next i
    If iHuberCount = 0 Then
        dTemp = dTempHuberM
    Else
        dTemp = dTempHuberM + dMADN * dHuberSum / iHuberCount
    End If
    If Abs(dTemp - dTempHuberM) < 0.0001 Then
        HuberM = dTemp
        Exit Function
    Else
        dTempHuberM = dTemp
    End If
Next h
End Function
```

```
Function MADN(x As Range) As Double
 Dim vRange1 As Variant
 Dim dMedian As Double
 Dim i, j, iRowCount, iColumnCount As Integer
 iRowCount = x.Rows.Count
 iColumnCount = x.Columns.Count
 vRange1 = x.Cells.Value
 dMedian = WorksheetFunction.Median(x)
 For i = 1 To iRowCount
      For j = 1 To iColumnCount
        vRange1(i, j) = Abs(vRange1(i, j) - dMedian)
      Next j
 Next i
 MADN = WorksheetFunction.Median(vRange1) / 0.6745
End Function
```

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Gauss–Markov Loss Prediction in a Linear Model

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Abstract

In a linear model for loss reserving, Gauss–Markov prediction is the natural principle of prediction: It minimizes the mean squared error of prediction over the class of all unbiased linear predictors, and it provides exact formulas for predictors and their mean squared error of prediction. Another advantage of Gauss–Markov prediction is in the fact that the Gauss–Markov predictor of a sum is just the sum of the Gauss–Markov predictors of the single terms of that sum such that essentially only the most elementary quantities have to be predicted.

The use of Gauss–Markov prediction in loss reserving is not new. For example, the additive (or incremental loss ratio) method and the Panning method are based on Gauss–Markov prediction in an appropriate linear model. Here we propose a systematic study of Gauss–Markov prediction in these and several related models. This leads to a variety of new methods of loss reserving, and for each of these models and methods we obtain straightforward estimators of the mean squared error of prediction.

To complete the discussion, we also explain certain limitations of the Gauss–Markov principle in connection with the chain–ladder method.

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1 Introduction

For at least six decades, loss reserving was determined by a variety of heuristic methods, among which the most popular ones are the chain–ladder method described by Tarbell [1934] and the Bornhuetter–Ferguson method proposed by Bornhuetter and Ferguson [1972].

The first stochastic model for loss reserving is probably that of Hachemeister and Stanard [1975]. In their model, the incremental losses are independent and Poisson distributed with a multiplicative structure of the expectations, and it turns out that maximum–likelihood estimations leads to the chain–ladder predictors. Their model thus provides a first justification of the chain–ladder method, but because of the Poisson assumption it applies to claim numbers rather than claim amounts.[1]

About two decades later, a couple of papers appeared which considerably advanced the use of stochastic models in loss reserving. In one of these papers, Mack [1991] proposed a model in which the incremental losses are uncorrelated with a multiplicative structure of the expectations and variances and in which least squares estimation leads to the additive (or incremental loss ratio) method. Subsequently, Mack [1993] proposed another but similar model in which least squares estimation leads to the chain–ladder method.[2] In both of these papers, however, emphasis is on parameter estimation and not on prediction of future losses.

It is easy to see that the additive model of Mack [1991] is a linear model, and it follows from Schmidt and Schnaus [1996] that the chain–ladder model of Mack [1993] is a sequential linear model.[3] But this was certainly not the usual way of looking at these models at the time when they were published, and it is the merit of Halliwell [1996] of having pointed out that linear models are most useful in loss reserving since the Gauss–Markov principle provides not only estimators of parameters but also predictors of future losses.

About another decade later, linear models turned out to be a driving force for the development of new methods of loss reserving: Inspired by Braun [2004], Pröhl and Schmidt [2005] proposed a sequential linear model in which Gauss–Markov prediction leads to a multivariate version of the chain–ladder method[4] and Hess, Schmidt and Zocher [2006] proposed a linear model in which Gauss–Markov prediction leads to a multivariate version of the additive method. Both methods are of interest for

[1]Extensions of the model of Hachemeister and Stanard [1975], which allow for dependence within the accident years and in which maximum–likelihood estimation still produces the chain–ladder predictors of the ultimate cumulative losses were proposed by Schmidt and Wünsche [1998] and by Schmidt and Zocher [2005].

[2]It is remarkable that the assumptions of the model of Hachemeister and Standard [1972] and those of the model of Mack [1993] cannot be fulfilled simultaneously; see Hess and Schmidt [2002] for a comparison of a variety of models for the chain–ladder method.

[3]See Schmidt [2003] and Radtke and Schmidt [2004].

[4]Another paper which is in the spirit of Pröhl and Schmidt [2005] is that of Kremer [2005].

simultaneous prediction for dependent lines of business.[5] At the same time, Panning [2006] proposed a linear model which in a certain sense is intermediate between the linear model for the additive method and the sequential linear model for the chain–ladder method. More recently, Kloberdanz and Schmidt [2009] used a bivariate version of the additive model to approach the paid & incurred problem which was first studied by Halliwell [1997] and later by Quarg and Mack [2004, 2008].

At this point, it is useful to briefly review some basic aspects of linear and general linear models and of Gauss–Markov estimation and prediction in such models; a more precise discussion will be given in Section 3.

A *linear model* (or *regression model*) essentially consists in the assumption that the unknown expectations of certain random variables $X_1, \dots, X_s$ can be expressed as linear functions of certain unknown parameters $\beta_1, \dots, \beta_r$ with $r < s$. This means that, for every $i \in \{1, \dots, s\}$, there exist known coefficients $a_{i,1}, \dots, a_{i,r}$ such that

$$E[X_i] \;=\; \sum_{k=1}^{r} a_{i,k}\,\beta_k$$

The point is that in a linear model the s unknown expectations are explained by r unknown parameters such that the problem of estimating s expectations is reduced to that of estimating only $r < s$ parameters. A general principle for estimating the parameters in a linear model is *Gauss–Markov estimation* which consists in the computation of the *Gauss–Markov estimators* β_k^{GM} minimizing the *mean squared error of estimation*

$$E[(\widehat{\beta}_k - \beta_k)^2]\,.$$

over all estimators $\widehat{\beta}_k$ which are linear in $X_1, \dots, X_s$ and unbiased for β_k. Thus, with respect to the mean squared error of estimation, the Gauss–Markov estimator β_k^{GM} is the *best linear unbiased estimator* of β_k.

In a *general linear model*, only the first $s_1 < s$ random variables are observable while the remaining $s_2 := s - s_1$ random variables are non–observable. In this case, Gauss–Markov estimation of the parameters is still possible by replacing s with s_1 in the previous identities, but the real problem is *Gauss–Markov prediction* of the non–observable random variables which consists in the computation of the *Gauss–Markov predictors* X_j^{GM} with $j \in \{s_1+1, \dots, s_1+s_2\}$ minimizing the *mean squared error of prediction*

$$E[(\widehat{X}_j - X_j)^2]$$

over all predictors $\widehat{X}_j$ which are linear and unbiased for X_j in the sense that $E[\widehat{X}_j] = E[X_j]$. Thus, with respect to the mean squared error of prediction, the Gauss–Markov predictor X_j^{GM} is the *best linear unbiased predictor* of X_j.

Under mild conditions on the coefficients and the variances and covariances of the random variables, Gauss–Markov estimators and predictors exist and are unique. To

[5] See Schmidt [2006b] for a survey of the results of these papers.

determine Gauss–Markov estimators and predictors, the variances and covariances of the random variables must be known or have to be estimated but no further assumptions on their joint distribution have to be made.[6] Moreover, since Gauss–Markov estimators and predictors are linear and unbiased, it is evident that also the mean squared errors of estimation and prediction are determined by the variances and covariances.

Since loss reserving aims at the prediction of future losses from those observed in the past, every stochastic model for loss reserving typically has to consist of observable and non–observable random variables representing past and future losses. Therefore, general linear models provide a wide class of stochastic models which meet the basic requirement on every stochastic model for loss reserving.

Whenever it is judged to be appropriate, the use of general linear models in loss reserving is strongly recommendable since

- explicit formulas can be given for Gauss–Markov predictors of reserves and for their mean squared error of prediction, and
- estimators of the mean squared errors of prediction can be obtained by simply replacing unknown variances and covariances with appropriate estimators.

Of course, the choice of a particular stochastic model for loss reserving should not be determined by such technical advantages but rather by statistical analysis and actuarial judgement. In many cases, however, such considerations will not end up with a single model and the choice of a general linear model could be reasonable.

In the present paper we propose Gauss–Markov prediction in a general linear model as a common approach to the additive method, the Panning method and a new method which is a combination of both and could be extended further. We thus extend results of Ludwig, Schmeisser, and Thänert [2009].

This paper is organized as follows: We first present the typical data structure in loss reserving (Section 2) and discuss Gauss–Markov prediction in the general linear model (Section 3). We then apply the general results on Gauss–Markov prediction to the additive model (Section 4), the Panning model (Section 5), and the combined model (Section 6). For the sake of comparison, we also consider the Mack model for the chain–ladder method (Section 7), which because of its sequential structure presents certain difficulties with regard to the estimation of the mean squared errors of prediction for reserves.[7] Finally, we present a numerical example (Section 9) and we conclude with some remarks (Section 8).

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[6]In particular, it is not necessary to assume that the random variables are jointly normally distributed. The popularity of the normal assumption is probably due to the fact that, if it holds, then the Gauss–Markov estimators agree with the maximum–likelihood estimators. While the normal assumption is inessential for Gauss–Markov estimation and prediction, it is of interest for the construction of confidence intervals or prediction intervals; these topics, however, will not be dealt with in the present paper.

[7]The use of plug–in estimators for estimating the mean squared errors of prediction is not possible in the Mack model; instead, certain approximations seem to be unavoidable in the construction of estimators or the mean squared errors of prediction and it appears to be difficult to quantify the approximation errors.

2 Data Structure

In the present paper, we consider a portfolio of risks and we assume that each claim of the portfolio is settled either in the accident year or in finitely many subsequent development years.

To model such a portfolio, we consider a family of square integrable random variables

$$\{Z_{i,k}\}_{i \in \{-m,\dots,n\}, k \in \{0,\dots,n\}}$$

and we interpret the random variable $Z_{i,k}$ as the loss of *accident year* i which is settled with a delay of k years and hence in *development year* k and in *calendar year* $i+k$. We refer to $Z_{i,k}$ as the *incremental loss* of accident year i and development year k.

We assume that the incremental losses $Z_{i,k}$ are *observable* for *calendar years* $i+k \leq n$ and that they are *non–observable* for calendar years $i+k \geq n+1$. The observable incremental losses are represented by the following *run–off trapezoid*:

Accident	**Development Year**									
Year	0	1	$\dots$	k	$\dots$	$n-i$	$\dots$	$n-1$	n
$-m$	$Z_{-m,0}$	$Z_{-m,1}$	$\dots$	$Z_{-m,k}$	$\dots$	$Z_{-m,n-i}$	$\dots$	$Z_{-m,n-1}$	$Z_{-m,n}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
0	$Z_{0,0}$	$Z_{0,1}$	$\dots$	$Z_{0,k}$	$\dots$	$Z_{0,n-i}$	$\dots$	$Z_{0,n-1}$	$Z_{0,n}$
1	$Z_{1,0}$	$Z_{1,1}$	$\dots$	$Z_{1,k}$	$\dots$	$Z_{1,n-i}$	$\dots$	$Z_{1,n-1}$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			
i	$Z_{i,0}$	$Z_{i,1}$	$\dots$	$Z_{i,k}$	$\dots$	$Z_{i,n-i}$			
$\vdots$	$\vdots$	$\vdots$		$\vdots$					
$n-k$	$Z_{n-k,0}$	$Z_{n-k,1}$	$\dots$	$Z_{n-k,k}$					
$\vdots$	$\vdots$	$\vdots$							
$n-1$	$Z_{n-1,0}$	$Z_{n-1,1}$							
n	$Z_{n,0}$								

In the traditional case $m = 0$, the run–off trapezoid reduces to a *run–off triangle*. The case $m \geq 1$ is of interest, since it is always desirable to have more than one completely developed accident year and since this also turns out to be necessary for certain stochastic models which to some extent specify the joint distribution of the family of all incremental losses.

For the stochastic models to be considered in this paper, it is essential to linearize the run–off trapezoid of observable incremental losses and the triangle of non–observable incremental losses. Therefore, we define the random vectors

$$
\mathbf{X}_1 := \begin{pmatrix} Z_{-m,0} \\ \vdots \\ Z_{n,0} \\ \hline \vdots \\ \hline Z_{-m,k} \\ \vdots \\ Z_{n-k,k} \\ \hline \vdots \\ \hline Z_{-m,n} \\ \vdots \\ Z_{0,n} \end{pmatrix} \qquad \text{and} \qquad \mathbf{X}_2 := \begin{pmatrix} Z_{n,1} \\ \hline \vdots \\ \hline Z_{n-k+1,k} \\ \vdots \\ Z_{n,k} \\ \hline \vdots \\ \hline Z_{1,n} \\ \vdots \\ Z_{n,n} \end{pmatrix}
$$

such that $\mathbf{X}_1$ represents the run–off trapezoid of observable incremental losses and $\mathbf{X}_2$ represents the triangle of non–observable incremental losses.

The first problem is to *predict*

(1) the *accident year reserves*

$$
R_i := \sum_{k=n-i+1}^{n} Z_{i,k}
$$

for $i \in \{1, \dots, n\}$,

(2) the *calendar year reserves*

$$
R_{(c)} := \sum_{i=c-n}^{n} Z_{i,c-i}
$$

for $c \in \{n+1, \dots, 2n\}$, and

(3) the *total reserve*

$$
R := \sum_{k=1}^{n} \sum_{i=n-k+1}^{n} Z_{i,k}.
$$

In either case, the problem is to predict $\mathbf{d}'\mathbf{X}_2$ for a suitable vector $\mathbf{d}$.

The second problem is to estimate the mean squared error of prediction for the predictors of these reserves.

3 Gauss–Markov Prediction in the General Linear Model

The stochastic models of loss reserving to be studied in the present paper are special cases of the following general linear model for a random vector

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}$$

consisting of an observable part $\mathbf{X}_1$ and a non–observable part $\mathbf{X}_2$ with square integrable coordinates:

General Linear Model: *There exist known matrices* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *and an unknown parameter vector* $\boldsymbol{\beta}$ *such that*

$$E\left[\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}\right] = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}\boldsymbol{\beta}$$

Moreover, $\mathbf{A}_1$ *has full column rank and* $\mathrm{var}[\mathbf{X}_1]$ *is invertible.*

The general linear model is more general than the traditional linear model since it involves the non–observable part $\mathbf{X}_2$. In particular, the problem is not only to estimate the parameter vector $\boldsymbol{\beta}$ but also to predict the non–observable random vector $\mathbf{X}_2$. The matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are called the *design matrices* of the general linear model.

For the remainder of this section, we assume that the assumptions of the general linear model are fulfilled.

Following an idea of Hamer [1999], the best way to simultaneously estimate the parameter vector $\boldsymbol{\beta}$ and predict the non–observable random vector $\mathbf{X}_2$ is to predict a target quantity of the form

$$\mathbf{T} = \mathbf{C}_0\boldsymbol{\beta} + \mathbf{C}_1\mathbf{X}_1 + \mathbf{C}_2\mathbf{X}_2$$

with matrices $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2$ of suitable dimensions which also allows for the prediction of linear combinations of the coordinates of $\mathbf{X}_1$ and $\mathbf{X}_2$.

Since only $\mathbf{X}_1$ is observable, every random variable $\widehat{\mathbf{T}}$ which is a (measurable) transformation of $\mathbf{X}_1$ is said to be a *predictor* of $\mathbf{T}$.

A predictor $\widehat{\mathbf{T}}$ is said to be an *admissible predictor* of $\mathbf{T}$ if there exists a matrix $\mathbf{Q}$ satisfying

$$\widehat{\mathbf{T}} = \mathbf{Q}\mathbf{X}_1$$

and

$$\mathbf{Q}\mathbf{A}_1 = \mathbf{C}_0 + \mathbf{C}_1\mathbf{A}_1 + \mathbf{C}_2\mathbf{A}_2$$

Because of the first identity, every admissible predictor $\widehat{\mathbf{T}}$ of $\mathbf{T}$ is *linear* (in $\mathbf{X}_0$), and because of the second identity it is also *unbiased* since

$$\begin{aligned} E[\widehat{\mathbf{T}}] &= E[\mathbf{Q}\mathbf{X}_1] \\ &= \mathbf{Q}E[\mathbf{X}_1] \\ &= \mathbf{Q}\mathbf{A}_1\boldsymbol{\beta} \\ &= (\mathbf{C}_0 + \mathbf{C}_1\mathbf{A}_1 + \mathbf{C}_2\mathbf{A}_2)\boldsymbol{\beta} \\ &= \mathbf{C}_0\boldsymbol{\beta} + \mathbf{C}_1\mathbf{A}_1\boldsymbol{\beta} + \mathbf{C}_2\mathbf{A}_2\boldsymbol{\beta} \\ &= \mathbf{C}_0\boldsymbol{\beta} + \mathbf{C}_1 E[\mathbf{X}_1] + \mathbf{C}_2 E[\mathbf{X}_2] \\ &= E[\mathbf{C}_0\boldsymbol{\beta} + \mathbf{C}_1\mathbf{X}_1 + \mathbf{C}_2\mathbf{X}_2] \\ &= E[\mathbf{T}] \end{aligned}$$

An admissible predictor $\widehat{\mathbf{T}}$ of $\mathbf{T}$ is said to be a *Gauss–Markov predictor* of $\mathbf{T}$ if it minimizes the *mean squared error of prediction*

$$E[(\widehat{\mathbf{T}} - \mathbf{T})'(\widehat{\mathbf{T}} - \mathbf{T})]$$

which is sometimes also called the *mean squared error of prediction* of $\widehat{\mathbf{T}}$ and is denoted by m.s.e.p.$[\widehat{\mathbf{T}}]$. Since every admissible predictor $\widehat{\mathbf{T}}$ of $\mathbf{T}$ is unbiased, we have $E[\widehat{\mathbf{T}} - \mathbf{T}] = \mathbf{0}$ and hence

$$\begin{aligned} E[(\widehat{\mathbf{T}} - \mathbf{T})'(\widehat{\mathbf{T}} - \mathbf{T})] &= E[\mathrm{trace}((\widehat{\mathbf{T}} - \mathbf{T})(\widehat{\mathbf{T}} - \mathbf{T})')] \\ &= \mathrm{trace}(E[(\widehat{\mathbf{T}} - \mathbf{T})(\widehat{\mathbf{T}} - \mathbf{T})')] \\ &= \mathrm{trace}(\mathrm{var}[\widehat{\mathbf{T}} - \mathbf{T}] + E[\widehat{\mathbf{T}} - \mathbf{T}]\, E[\widehat{\mathbf{T}} - \mathbf{T}]') \\ &= \mathrm{trace}(\mathrm{var}[\widehat{\mathbf{T}} - \mathbf{T}])\ . \end{aligned}$$

We have the following result:

3.1 Proposition (Gauss–Markov Theorem). *There exists a unique Gauss–Markov predictor* $\mathbf{T}^{\mathrm{GM}}$ *of* $\mathbf{T}$ *and it satisfies*

$$\mathbf{T}^{\mathrm{GM}} = \mathbf{C}\boldsymbol{\beta}^* + \mathbf{C}_1\mathbf{X}_1 + \mathbf{C}_2\mathbf{X}_2^*$$

with

$$\boldsymbol{\beta}^* := (\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1$$

and

$$\mathbf{X}_2^* := \mathbf{A}_2\boldsymbol{\beta}^* + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\big(\mathbf{X}_1 - \mathbf{A}_1\boldsymbol{\beta}^*\big)$$

Moreover,

$$\mathrm{var}[\mathbf{T}^{\mathrm{GM}} - \mathbf{T}] = \mathbf{K}\mathrm{var}[\boldsymbol{\beta}^*]\mathbf{K} + \mathbf{C}_2(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12})\mathbf{C}_2'$$

with $\mathbf{K} := \mathbf{C} + \mathbf{C}_2\mathbf{A}_2 - \mathbf{C}_2\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{C}_2'$ *and* $\mathrm{var}[\boldsymbol{\beta}^*] = (\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}$.

Proposition 3.1 is well–known; see e. g. Rao and Toutenburg [1995], Radtke and Schmidt [2004], Schmidt [2004] and, in particular, Hamer [1999].

The previous result shows that Gauss–Markov prediction of the target quantity $\mathbf{T}$ is based on Gauss–Markov estimation of the parameter $\boldsymbol{\beta}$. Although the following result is a special case of Proposition 3.1, we state it because of its importance and for later reference:

3.2 Corollary. *The Gauss–Markov estimator $\boldsymbol{\beta}^{\mathrm{GM}}$ of $\boldsymbol{\beta}$ satisfies*

$$\boldsymbol{\beta}^{\mathrm{GM}} \;=\; (\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1$$

and

$$\mathrm{var}[\boldsymbol{\beta}^{\mathrm{GM}}] \;=\; (\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}.$$

In the models considered in this paper, we always have $\boldsymbol{\Sigma}_{12} = \mathbf{O}$. In this case, we obtain particularly simple formulas for the Gauss–Markov predictor of $\mathbf{X}_2$ and for the variance of the prediction error:

3.3 Corollary. *Assume that $\boldsymbol{\Sigma}_{12} = \mathbf{O}$. Then the Gauss–Markov predictor $\mathbf{X}_2^{\mathrm{GM}}$ of $\mathbf{X}_2$ satisfies*

$$\mathbf{X}_2^{\mathrm{GM}} \;=\; \mathbf{A}_2\boldsymbol{\beta}^{\mathrm{GM}}$$

and

$$\mathrm{var}[\mathbf{X}_2^{\mathrm{GM}} - \mathbf{X}_2] \;=\; \mathbf{A}_2\mathrm{var}[\boldsymbol{\beta}^{\mathrm{GM}}]\mathbf{A}_2' + \boldsymbol{\Sigma}_{22}\,.$$

Because of the previous result, the mean squared error of prediction

$$E[(\mathbf{X}_2^{\mathrm{GM}} - \mathbf{X}_2)'(\mathbf{X}_2^{\mathrm{GM}} - \mathbf{X}_2)] \;=\; \mathrm{trace}\Big(\mathrm{var}[\mathbf{X}_2^{\mathrm{GM}} - \mathbf{X}_2]\Big)$$

is the sum of the *estimation error* $\mathrm{trace}(\mathbf{A}_2\mathrm{var}[\boldsymbol{\beta}^{\mathrm{GM}}]\mathbf{A}_2')$ and the *random error* $\mathrm{trace}(\boldsymbol{\Sigma}_{22})$.

Finally, the Gauss–Markov predictor of a linear transformation $\mathbf{C}_2\mathbf{X}_2$ of $\mathbf{X}_2$ is easily obtained from the Gauss–Markov predictor of $\mathbf{X}_2$:

3.4 Corollary. *The Gauss–Markov predictor $(\mathbf{C}_2\mathbf{X}_2)^{\mathrm{GM}}$ of $\mathbf{C}_2\mathbf{X}_2$ satisfies*

$$(\mathbf{C}_2\mathbf{X}_2)^{\mathrm{GM}} \;=\; \mathbf{C}_2\mathbf{X}_2^{\mathrm{GM}}$$

and

$$\mathrm{var}[(\mathbf{C}_2\mathbf{X}_2)^{\mathrm{GM}} - \mathbf{C}_2\mathbf{X}_2] \;=\; \mathbf{C}_2\mathrm{var}[\mathbf{X}_2^{\mathrm{GM}} - \mathbf{X}_2]\mathbf{C}_2'\,.$$

Because of the previous result, Gauss–Markov prediction is linear in the sense that the Gauss–Markov predictor of a linear combination of non–observable random variables is the same linear combination of their Gauss–Markov predictors.

We shall also need a conditional version of the general linear model and of the Gauss–Markov Theorem. For a sub–σ–algebra $\mathcal{G} \subseteq \mathcal{F}$, the $\mathcal{G}$–conditional linear model is defined as follows:

$\mathcal{G}$–Conditional General Linear Model: *There exist observable $\mathcal{G}$–measurable random matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ and an unknown parameter vector $\boldsymbol{\beta}$ such that*

$$E^{\mathcal{G}}\left[\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}\right] = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \boldsymbol{\beta} \,.$$

Moreover, $\mathbf{A}_1$ *has full column rank and* $\mathrm{var}^{\mathcal{G}}[\mathbf{X}_1]$ *is invertible.*

Here and in the sequel, $E^{\mathcal{G}}[\mathbf{X}_i]$ and $\mathrm{var}^{\mathcal{G}}[\mathbf{X}_i]$ denote the $\mathcal{G}$–conditional expectation and the $\mathcal{G}$–conditional variance of $\mathbf{X}_i$, respectively; accordingly, $\mathrm{cov}^{\mathcal{G}}[\mathbf{X}_1, \mathbf{X}_2]$ denotes the $\mathcal{G}$–conditional covariance of $\mathbf{X}_1$ and $\mathbf{X}_2$.

The discussion of the $\mathcal{G}$–conditional general linear model is entirely analogous to that of the general linear model: Replace the admissible predictors by the $\mathcal{G}$–conditionally admissible predictors (which are obtained by replacing the matrix $\mathbf{Q}$ by a $\mathcal{G}$–measurable random matrix $\mathbf{Q}$ and which are linear and $\mathcal{G}$–conditionally unbiased in the sense that their $\mathcal{G}$–conditional expectation coincides with that of the target quantity) and replace the first and second order moments by their $\mathcal{G}$–conditional counterparts.

4 Gauss–Markov Loss Prediction in the Extended Additive Model

The extended additive model is defined as follows:

Extended Additive Model: *There exist known parameters* $v_i, w_i \in (0, \infty)$ *with* $i \in \{-m, \ldots, n\}$ *as well as unknown parameters* $\zeta_k \in \mathbb{R}$ *and* $\sigma_k^2 \in (0, \infty)$ *with* $k \in \{0, \ldots, n\}$ *such that the incremental losses satisfy*

$$\begin{aligned} E[Z_{i,k}] &= v_i \, \zeta_k \\ \operatorname{cov}[Z_{i,k}, Z_{j,l}] &= w_i \, \sigma_k^2 \, \delta_{i,j} \, \delta_{k,l} \end{aligned}$$

for all $i, j \in \{-m, \ldots, n\}$ *and* $k, l \in \{0, \ldots, n\}$.

In the extended additive model, the accident year parameter v_i is usually referred to as a *volume measure* of accident year i; for example, the volume measure could be the total premium income or the number of contracts in the accident year. Since the first identity in the extended additive model can be written as

$$E[Z_{i,k}/v_i] = \zeta_k$$

the development year parameter ζ_k is the *expected incremental loss ratio* of development year k (with respect to the volume measures) and is assumed to be independent of the accident year such that the collection of these parameters forms a *development pattern*; see Schmidt and Zocher [2009].

The extended additive model extends the traditional additive model in which it is assumed that $m = 0$ and that $w_i = v_i$ holds for all $i \in \{-m, \ldots, n\}$; see Mack [1991], Radtke and Schmidt [2004], Hess, Schmidt and Zocher [2006], and Schmidt and Zocher [2009]. The reason for considering the extended additive model becomes evident from its comparison with the extended Panning model (Section 5) and with the combination of both models (Section 6).

Assume that the assumptions of the extended additive model are fulfilled. Then the expectation of the random vector $\mathbf{X}_1$ of all observable incremental losses satisfies

$$E\left[\begin{pmatrix} Z_{-m,0} \\ \vdots \\ Z_{n,0} \\ \hline \vdots \\ \hline Z_{-m,k} \\ \vdots \\ Z_{n-k,k} \\ \hline \vdots \\ \hline Z_{-m,n} \\ \vdots \\ Z_{0,n} \end{pmatrix}\right] = \begin{pmatrix} v_{-m} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ v_n & \cdots & 0 & \cdots & 0 \\ \hline \vdots & & \vdots & & \vdots \\ \hline 0 & \cdots & v_{-m} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & v_{n-k} & \cdots & 0 \\ \hline \vdots & & \vdots & & \vdots \\ \hline 0 & \cdots & 0 & \cdots & v_{-m} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & v_0 \end{pmatrix} \begin{pmatrix} \zeta_0 \\ \vdots \\ \zeta_k \\ \vdots \\ \zeta_n \end{pmatrix}$$

such that there exist a design matrix $\mathbf{A}_1$ having full column rank and a parameter vector $\boldsymbol{\beta}$ satisfying $E[\mathbf{X}_1] = \mathbf{A}_1\boldsymbol{\beta}$, and the expectation of the random vector $\mathbf{X}_2$ of all non–observable incremental losses satisfies

$$E\left[\left(\begin{array}{c} Z_{n,1} \\ \hline \vdots \\ \hline Z_{n-k+1,k} \\ \vdots \\ Z_{n,k} \\ \hline \vdots \\ \hline Z_{1,n} \\ \vdots \\ Z_{n,n} \end{array}\right)\right] = \left(\begin{array}{cccccc} 0 & v_n & \cdots & 0 & \cdots & 0 \\ \hline \vdots & \vdots & & \vdots & & \vdots \\ \hline 0 & 0 & \cdots & v_{n-k+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & v_{n-k} & \cdots & 0 \\ \hline \vdots & \vdots & & \vdots & & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & v_1 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & v_n \end{array}\right) \left(\begin{array}{c} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_k \\ \vdots \\ \zeta_n \end{array}\right).$$

Moreover, the variance $\mathbf{\Sigma}_{11}$ of $\mathbf{X}_1$ satisfies

$$\operatorname{var}\left[\left(\begin{array}{c} Z_{-m,0} \\ \vdots \\ Z_{n,0} \\ \hline \vdots \\ \hline Z_{-m,k} \\ \vdots \\ Z_{n-k,k} \\ \hline \vdots \\ \hline Z_{-m,n} \\ \vdots \\ Z_{0,n} \end{array}\right)\right] = \left(\begin{array}{ccc|c|ccc|c|ccc} w_{-m}\sigma_0^2 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & w_n\sigma_0^2 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & w_{-m}\sigma_k^2 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & w_{n-k}\sigma_k^2 & \cdots & 0 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & w_{-m}\sigma_n^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & w_0\sigma_n^2 \end{array}\right)$$

and is thus invertible, and the variance $\mathbf{\Sigma}_{22}$ of $\mathbf{X}_2$ satisfies

$$\operatorname{var}\left[\left(\begin{array}{c} Z_{n,1} \\ \hline \vdots \\ \hline Z_{n-k+1,k} \\ \vdots \\ Z_{n,k} \\ \hline \vdots \\ \hline Z_{1,n} \\ \vdots \\ Z_{n,n} \end{array}\right)\right] = \left(\begin{array}{c|c|ccc|c|ccc} w_n\sigma_1^2 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & \cdots & w_{n-k+1}\sigma_k^2 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & w_n\sigma_k^2 & \cdots & 0 & \cdots & 0 \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & \cdots & 0 & \cdots & 0 & \cdots & w_1\sigma_n^2 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & w_n\sigma_n^2 \end{array}\right).$$

Furthermore, we have $\mathbf{\Sigma}_{12} = \operatorname{cov}[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{O}$. We thus obtain the following result:

4.1 Theorem. *The extended additive model is a linear model.*

In a first step, we compute the Gauss–Markov estimators of the coordinates of the parameter vector and their covariances:

4.2 Lemma (Gauss–Markov estimation of parameters). *In the extended additive model, the Gauss–Markov estimators of the coordinates of the parameter vector satisfy*

$$\zeta_k^{\mathrm{GM}} \;=\; \frac{\sum_{i=-m}^{n-k} v_i Z_{i,k}/w_i}{\sum_{i=-m}^{n-k} v_i^2/w_i}$$

and

$$\mathrm{cov}[\zeta_k^{\mathrm{GM}}, \beta_l^{\mathrm{GM}}] \;=\; \frac{1}{\sum_{i=-m}^{n-k} v_i^2/w_i}\, \sigma_k^2\, \delta_{k,l}$$

for all $k, l \in \{0, 1, \dots, n\}$.

Proof. The coordinates of the random vector $\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1$ satisfy

$$\begin{pmatrix} v_{-m} & \cdots & v_{n-k} \end{pmatrix} \begin{pmatrix} w_{-m}\sigma_k^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_{n-k}\sigma_k^2 \end{pmatrix}^{-1} \begin{pmatrix} Z_{-m,k} \\ \vdots \\ Z_{n-k,k} \end{pmatrix} \;=\; \left(\sum_{i=-m}^{n-k} \frac{v_i Z_{i,k}}{w_i} \right) \frac{1}{\sigma_k^2}\;.$$

Moreover, the matrix $\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1$ is diagonal and its diagonal elements satisfy

$$\begin{pmatrix} v_{-m} & \cdots & v_{n-k} \end{pmatrix} \begin{pmatrix} w_{-m}\sigma_k^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_{n-k}\sigma_k^2 \end{pmatrix}^{-1} \begin{pmatrix} v_{-m} \\ \vdots \\ v_{n-k} \end{pmatrix} \;=\; \left(\sum_{i=-m}^{n-k} \frac{v_i^2}{w_i} \right) \frac{1}{\sigma_k^2}\;.$$

Because of Corollary 3.2, we have $\boldsymbol{\beta}^{\mathrm{GM}} = (\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1$ and hence

$$\zeta_k^{\mathrm{GM}} \;=\; \frac{\sum_{i=-m}^{n-k} v_i Z_{i,k}/w_i}{\sum_{i=-m}^{n-k} v_i^2/w_i}$$

which is the first identity, and we also have $\mathrm{var}[\boldsymbol{\beta}^{\mathrm{GM}}] = (\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}$ and hence

$$\mathrm{cov}[\zeta_k^{\mathrm{GM}}, \zeta_l^{\mathrm{GM}}] \;=\; \frac{1}{\sum_{i=-m}^{n-k} v_i^2/w_i}\, \sigma_k^2\, \delta_{k,l}$$

which is the second identity. □

In a second step, we compute the Gauss–Markov predictors of the non–observable incremental losses and the covariances of their prediction errors:

4.3 Lemma (Gauss–Markov prediction of incremental losses). *In the extended additive model, the Gauss–Markov predictors of the non–observable incremental losses satisfy*

$$Z_{i,k}^{\mathrm{GM}} \;=\; v_i\, \zeta_k^{\mathrm{GM}}$$

and

$$\mathrm{cov}[Z_{i,k}^{\mathrm{GM}} - Z_{i,k}, Z_{j,l}^{\mathrm{GM}} - Z_{j,l}] \;=\; \Bigl(v_i v_j\, \mathrm{var}[\zeta_k^{\mathrm{GM}}] + w_i\, \sigma_k^2\, \delta_{i,j} \Bigr)\, \delta_{k,l}$$

for all $i, j \in \{-m, \dots, n\}$ and $k, l \in \{0, \dots, n\}$ such that $\min\{i+k, j+l\} \geq n+1$.

Proof. Because of Corollary 3.3, we have $\mathbf{X}_2^{\mathrm{GM}} = \mathbf{A}_2 \boldsymbol{\beta}^{\mathrm{GM}}$. For all $i \in \{-m, \ldots, n\}$ and $k \in \{0, \ldots, n\}$ such that $i + k \geq n + 1$, this yields

$$Z_{i,k}^{\mathrm{GM}} \;=\; v_i \, \zeta_k^{\mathrm{GM}}$$

which is the first identity.
Corollary 3.3 also provides an identity for $\mathrm{var}[\mathbf{X}_2^{\mathrm{GM}} - \mathbf{X}_2]$, but in the present model the direct computation of the elements of this matrix seems to be more transparent. Consider $i, j \in \{-m, \ldots, n\}$ and $k, l \in \{0, \ldots, n\}$ such that $\min\{i+k, j+l\} \geq n+1$. Lemma 4.2 yields

$$\begin{aligned}
\mathrm{cov}[Z_{i,k}^{\mathrm{GM}}, Z_{j,l}^{\mathrm{GM}}] &= \mathrm{cov}[v_i \zeta_k^{\mathrm{GM}}, v_j \zeta_l^{\mathrm{GM}}] \\
&= v_i v_j \, \mathrm{cov}[\zeta_k^{\mathrm{GM}}, \zeta_l^{\mathrm{GM}}] \\
&= v_i v_j \, \mathrm{var}[\zeta_k^{\mathrm{GM}}] \, \delta_{k,l}
\end{aligned}$$

and we also have

$$\mathrm{cov}[Z_{i,k}, Z_{j,l}] \;=\; w_i \, \sigma_k^2 \, \delta_{i,j} \, \delta_{k,l} \; .$$

Since $Z_{i,k}^{\mathrm{GM}}$ and $Z_{j,l}^{\mathrm{GM}}$ are linear combinations of observable incremental losses whereas $Z_{i,k}$ and $Z_{j,l}$ are non–observable incremental losses, we have $\mathrm{cov}[Z_{i,k}^{\mathrm{GM}}, Z_{j,l}] = 0 = \mathrm{cov}[Z_{i,k}, Z_{j,l}^{\mathrm{GM}}]$ and hence

$$\begin{aligned}
\mathrm{cov}[Z_{i,k}^{\mathrm{GM}} - Z_{i,k}, Z_{j,l}^{\mathrm{GM}} - Z_{j,l}] &= \mathrm{cov}[Z_{i,k}^{\mathrm{GM}}, Z_{j,l}^{\mathrm{GM}}] + \mathrm{cov}[Z_{i,k}, Z_{j,l}] \\
&= v_i v_j \, \mathrm{var}[\zeta_k^{\mathrm{GM}}] \, \delta_{k,l} + w_i \, \sigma_k^2 \, \delta_{i,j} \, \delta_{k,l},
\end{aligned}$$

which is the second identity. □

In a third step, we compute the Gauss–Markov predictors of reserves and their mean squared errors of prediction:

4.4 Theorem (Gauss–Markov prediction of reserves). *In the extended additive model,*
(1) *the Gauss–Markov predictors of the accident year reserves satisfy*

$$R_i^{\mathrm{GM}} \;=\; v_i \sum_{k=n-k+1}^{n} \zeta_k^{\mathrm{GM}}$$

and

$$\mathrm{cov}[R_i^{\mathrm{GM}} - R_i, R_j^{\mathrm{GM}} - R_j] \;=\; v_i v_j \sum_{k=n-i\wedge j+1}^{n} \mathrm{var}[\zeta_k^{\mathrm{GM}}] + w_i \left(\sum_{k=n-i+1}^{n} \sigma_k^2 \right) \delta_{i,j}$$

for all $i, j \in \{1, \ldots, n\}$; *in particular,*

$$E[(R_i^{\mathrm{GM}} - R_i)^2] \;=\; v_i^2 \sum_{k=n-i+1}^{n} \mathrm{var}[\zeta_k^{\mathrm{GM}}] + w_i \sum_{k=n-i+1}^{n} \sigma_k^2$$

holds for all $i \in \{1, \ldots, n\}$.

(2) *the Gauss–Markov predictors of the calendar year reserves satisfy*

$$R_{(c)}^{\mathrm{GM}} \;=\; \sum_{i=c-n}^{n} v_i \, \zeta_{c-i}^{\mathrm{GM}}$$

and

$$\mathrm{cov}[R_{(c)}^{\mathrm{GM}} - R_{(c)}, R_{(d)}^{\mathrm{GM}} - R_{(d)}] \;=\; \sum_{i=c\vee d-n}^{n} v_i v_{i-|c-d|} \, \mathrm{var}[\zeta_{c\vee d-i}^{\mathrm{GM}}] + \left(\sum_{i=c-n}^{n} w_i \, \sigma_{c-i}^2 \right) \delta_{c,d}$$

for all $c, d \in \{n+1, \dots, 2n\}$; *in particular,*

$$E[(R_{(c)}^{\mathrm{GM}} - R_{(c)})^2] \;=\; \sum_{i=c-n}^{n} v_i^2 \, \mathrm{var}[\zeta_{c-i}^{\mathrm{GM}}] + \sum_{i=c-n}^{n} w_i \, \sigma_{c-i}^2$$

holds for all $c \in \{n+1, \dots, 2n\}$.

(3) *the Gauss–Markov predictor of the total reserve satisfies*

$$R^{\mathrm{GM}} \;=\; \sum_{k=1}^{n} \left(\sum_{i=n-k+1}^{n} v_i \right) \zeta_k^{\mathrm{GM}}$$

and

$$E[(R^{\mathrm{GM}} - R)^2] \;=\; \sum_{k=1}^{n} \left(\sum_{i=n-k+1}^{n} v_i \right)^2 \mathrm{var}[\zeta_k^{\mathrm{GM}}] + \sum_{k=1}^{n} \left(\sum_{i=n-k+1}^{n} w_i \right) \sigma_k^2 \,.$$

Proof. Let us first consider the accident year reserves. We have

$$R_i \;=\; \sum_{k=n-i+1}^{n} Z_{i,k}$$

and, since Gauss–Markov prediction is linear, we obtain

$$R_i^{\mathrm{GM}} \;=\; \sum_{k=n-i+1}^{n} Z_{i,k}^{\mathrm{GM}}$$

which because of Lemma 4.3 gives the first identity. This yields

$$R_i^{\mathrm{GM}} - R_i \;=\; \sum_{k=n-i+1}^{n} (Z_{i,k}^{\mathrm{GM}} - Z_{i,k})$$

and because of Lemma 4.3 we obtain

$$\mathrm{cov}[R_i^{\mathrm{GM}} - R_i, R_j^{\mathrm{GM}} - R_j] \;=\; \mathrm{cov}\left[\sum_{k=n-i+1}^{n} (Z_{i,k}^{\mathrm{GM}} - Z_{i,k}), \sum_{l=n-j+1}^{n} (Z_{j,l}^{\mathrm{GM}} - Z_{j,l}) \right]$$

$$
\begin{aligned}
&= \sum_{k=n-i+1}^{n} \sum_{l=n-j+1}^{n} \operatorname{cov}[Z_{i,k}^{\mathrm{GM}} - Z_{i,k}, Z_{j,l}^{\mathrm{GM}} - Z_{j,l}] \\
&= \sum_{k=n-i+1}^{n} \sum_{l=n-j+1}^{n} \Big(v_i v_j \operatorname{var}[\zeta_k^{\mathrm{GM}}] + w_i\, \sigma_k^2\, \delta_{i,j} \Big)\, \delta_{k,l} \\
&= \sum_{k=n-i\wedge j+1}^{n} \Big(v_i v_j \operatorname{var}[\zeta_k^{\mathrm{GM}}] + w_i\, \sigma_k^2\, \delta_{i,j} \Big) \\
&= v_i v_j \sum_{k=n-i\wedge j+1}^{n} \operatorname{var}[\zeta_k^{\mathrm{GM}}] + w_i \left(\sum_{k=n-i+1}^{n} \sigma_k^2 \right) \delta_{i,j} \,,
\end{aligned}
$$

which is the second identity. Since Gauss–Markov predictors are unbiased, the third identity follows from the second.
Let us now consider the calendar year reserves. We have

$$
R_{(c)} = \sum_{i=c-n}^{n} Z_{i,c-i}
$$

and hence

$$
R_{(c)}^{\mathrm{GM}} = \sum_{i=c-n}^{n} Z_{i,c-i}^{\mathrm{GM}}
$$

which because of Lemma 4.3 gives the first identity. This yields

$$
R_{(c)}^{\mathrm{GM}} - R_{(c)} = \sum_{i=c-n}^{n} (Z_{i,c-i}^{\mathrm{GM}} - Z_{i,c-i})
$$

and because of Lemma 4.3 we obtain

$$
\begin{aligned}
\operatorname{cov}[R_{(c)}^{\mathrm{GM}} - R_{(c)}, R_{(d)}^{\mathrm{GM}} - R_{(d)}] &= \operatorname{cov}\left[\sum_{i=c-n}^{n} (Z_{i,c-i}^{\mathrm{GM}} - Z_{i,c-i}) \,, \sum_{j=d-n}^{n} (Z_{j,d-j}^{\mathrm{GM}} - Z_{j,d-j}) \right] \\
&= \sum_{i=c-n}^{n} \sum_{j=d-n}^{n} \operatorname{cov}[(Z_{i,c-i}^{\mathrm{GM}} - Z_{i,c-i}), (Z_{j,d-j}^{\mathrm{GM}} - Z_{j,d-j})] \Big] \\
&= \sum_{i=c-n}^{n} \sum_{j=d-n}^{n} \Big(v_i v_j \operatorname{var}[\zeta_{c-i}^{\mathrm{GM}}] + w_i\, \sigma_{c-i}^2\, \delta_{i,j} \Big)\, \delta_{c-i,d-j} \\
&= \sum_{i=c\vee d-n}^{n} v_i v_{i-|c-d|} \operatorname{var}[\zeta_{c\vee d-i}^{\mathrm{GM}}] + \left(\sum_{i=c-n}^{n} w_i\, \sigma_{c-i}^2 \right) \delta_{c,d}
\end{aligned}
$$

which is the second identity.
Let us finally consider the total reserve. We have

$$
R = \sum_{k=1}^{n} \sum_{i=k+1}^{n} Z_{i,k}
$$

and, since Gauss–Markov prediction is linear, we obtain

$$R^{\mathrm{GM}} \;=\; \sum_{k=1}^{n} \sum_{i=k+1}^{n} Z_{i,k}^{\mathrm{GM}}$$

which because of Lemma 4.3 gives the first identity. This yields

$$R^{\mathrm{GM}} - R \;=\; \sum_{k=1}^{n} \sum_{i=k+1}^{n} (Z_{i,k}^{\mathrm{GM}} - Z_{i,k})$$

and because of Lemma 4.3 we obtain

$$\begin{aligned}
E[(R^{\mathrm{GM}} - R)^2] &= \mathrm{var}[R^{\mathrm{GM}} - R] \\
&= \mathrm{var}\left[\sum_{k=1}^{n} \sum_{i=n-k+1}^{n} (Z_{i,k}^{\mathrm{GM}} - Z_{i,k})\right] \\
&= \sum_{k=1}^{n} \mathrm{var}\left[\sum_{i=n-k+1}^{n} (Z_{i,k}^{\mathrm{GM}} - Z_{i,k})\right] \\
&= \sum_{k=1}^{n} \sum_{i=n-k+1}^{n} \sum_{j=n-k+1}^{n} \mathrm{cov}[Z_{i,k}^{\mathrm{GM}} - Z_{i,k}, Z_{j,k}^{\mathrm{GM}} - Z_{j,k}] \\
&= \sum_{k=1}^{n} \sum_{i=n-k+1}^{n} \sum_{j=n-k+1}^{n} \Big(v_i v_j \, \mathrm{var}[\zeta_k^{\mathrm{GM}}] + w_i \, \sigma_k^2 \, \delta_{i,j}\Big) \\
&= \sum_{k=1}^{n} \left(\sum_{i=n-k+1}^{n} v_i\right)^2 \mathrm{var}[\zeta_k^{\mathrm{GM}}] + \sum_{k=1}^{n} \left(\sum_{i=n-k+1}^{n} w_i\right) \sigma_k^2
\end{aligned}$$

which is the second identity. □

In the special case where $m = 0$ and $w_i = v_i$ holds for all $i \in \{-m, \ldots, n\}$, the Gauss–Markov predictors of incremental losses and reserves are identical with the predictors used in the traditional additive method of loss reserving. This means that Gauss–Markov prediction in the extended additive model provides simultaneously an extension of the additive method and its justification based on a general statistical principle.

Via Lemma 4.3 and Lemma 4.2, the mean squared errors of prediction depend on unknown variance parameters which may be estimated as follows:

4.5 Theorem (Estimation of variance parameters). *In the extended additive model with $m \geq 1$ and for every $k \in \{0, \ldots, n\}$, the random variable*

$$\widehat{\sigma}_k^2 \;:=\; \frac{1}{m+n-k} \sum_{i=-m}^{n-k} \frac{1}{w_i} \,(Z_{i,k} - v_i \zeta_k^{\mathrm{GM}})^2$$

is an unbiased estimator of σ_k^2.

Proof. Consider $i \in \{-m, \dots, n-k\}$. By Lemma 4.2, we have $E[v_i \zeta_k^{\mathrm{GM}}] = E[Z_{i,k}]$ and thus

$$\begin{aligned} E[(Z_{i,k} - v_i \zeta_k^{\mathrm{GM}})^2] &= \mathrm{var}[Z_{i,k} - v_i \zeta_k^{\mathrm{GM}}] \\ &= \mathrm{var}[Z_{i,k}] - 2\, v_i \,\mathrm{cov}[Z_{i,k}, \zeta_k^{\mathrm{GM}}] + v_i^2 \,\mathrm{var}[\zeta_k^{\mathrm{GM}}] \ . \end{aligned}$$

Recall that

$$\mathrm{var}[Z_{i,k}] = w_i \, \sigma_k^2 \ .$$

Furthermore, using Lemma 4.2 and Lemma 4.3 we obtain

$$\begin{aligned} \mathrm{cov}[Z_{i,k}, \zeta_k^{\mathrm{GM}}] &= \mathrm{cov}\left[Z_{i,k}, \frac{\sum_{j=-m}^{n-k} v_j Z_{j,k}/w_j}{\sum_{j=-m}^{n-k} v_j^2/w_j} \right] \\ &= \sum_{j=-m}^{n-k} \frac{v_j/w_j}{\sum_{h=-m}^{n-k} v_h^2/w_h} \,\mathrm{cov}[Z_{i,k}, Z_{j,k}] \\ &= \sum_{j=-m}^{n-k} \frac{v_j/w_j}{\sum_{h=-m}^{n-k} v_h^2/w_h} \, w_i \, \sigma_k^2 \, \delta_{i,j} \\ &= v_i \, \frac{1}{\sum_{h=-m}^{n-k} v_h^2/w_h} \, \sigma_k^2 \end{aligned}$$

and Lemma 4.2 yields

$$\mathrm{var}[\zeta_k^{\mathrm{GM}}] = \frac{1}{\sum_{h=-m}^{n-k} v_h^2/w_h} \, \sigma_k^2 \ .$$

Therefore, we have

$$\begin{aligned} E[(Z_{i,k} - v_i \zeta_k^{\mathrm{GM}})^2] &= \mathrm{var}[Z_{i,k}] - 2\, v_i \,\mathrm{cov}[Z_{i,k}, \zeta_k^{\mathrm{GM}}] + v_i^2 \,\mathrm{var}[\zeta_k^{\mathrm{GM}}] \\ &= w_i \, \sigma_k^2 - 2\, v_i^2 \, \frac{1}{\sum_{h=-m}^{n-k} v_h^2/w_h} \, \sigma_k^2 + v_i^2 \, \frac{1}{\sum_{h=-m}^{n-k} v_h^2/w_h} \, \sigma_k^2 \\ &= \left(w_i - \frac{v_i^2}{\sum_{h=-m}^{n-k} v_h^2/w_h} \right) \sigma_k^2 \end{aligned}$$

and hence

$$\begin{aligned} \sum_{i=-m}^{n-k} \frac{1}{w_i} \, E[(Z_{i,k} - v_i \zeta_k^{\mathrm{GM}})^2] &= \sum_{i=-m}^{n-k} \frac{1}{w_i} \left(w_i - \frac{v_i^2}{\sum_{h=-m}^{n-k} v_h^2/w_h} \right) \sigma_k^2 \\ &= \Big((m+1+n-k) - 1 \Big) \, \sigma_k^2 \\ &= (m+n-k) \, \sigma_k^2 \end{aligned}$$

which proves the assertion. □

In the case $m = 0$, the assertion of Theorem 4.5 remains valid for $k \in \{0, \dots, n-1\}$. To obtain an estimator of σ_n^2 also in this case, one may choose a parametric class $\{f_c \mid c \in C\}$ of real functions (e.g.,the class $\{f_{(a,b)} : \mathbb{R} \to \mathbb{R} \mid (a,b) \in (0,\infty)^2\}$ with $f_{(a,b)}(x) = a\, e^{-bx}$), determine $\widehat{c} \in C$ satisfying

$$\sum_{k=0}^{n-1} \Bigl(f_{\widehat{c}}(k) - \widehat{\sigma}_k^2\Bigr)^2 \;=\; \inf_{c \in C} \sum_{k=0}^{n-1} \Bigl(f_c(k) - \widehat{\sigma}_k^2\Bigr)^2$$

and define

$$\widehat{\sigma}_n^2 \;:=\; f_{\widehat{c}}(n)$$

If the sequence $\{\widehat{\sigma}_k^2\}_{k \in \{0,\dots,n-1\}}$ is decreasing, one might alternatively define $\widehat{\sigma}_n^2 := \widehat{\sigma}_{n-1}^2$.

Now estimators of the mean squared errors of prediction can be obtained by replacing the variance parameters by their estimators in the formulas for the mean squared errors of prediction.

5 Gauss–Markov Loss Prediction in the Extended Panning Model

In the present section, we denote by $\mathcal{F}_0$ the σ–algebra generated by the family $\{Z_{i,0}\}_{i\in\{-m,\dots,n\}}$ of the losses of development year 0. The extended Panning model is defined as follows:

Extended Panning Model: *There exist known $\mathcal{F}_0$–measurable random parameters w_i with $w_i > 0$ and $i \in \{-m,\dots,n\}$ as well as unknown parameters $\xi_k \in \mathbb{R}$ and $\sigma_k^2 \in (0,\infty)$ with $k \in \{0,\dots,n\}$ such that the incremental losses satisfy*

$$\begin{aligned} E^{\mathcal{F}_0}[Z_{i,k}] &= Z_{i,0}\,\xi_k \\ \mathrm{cov}^{\mathcal{F}_0}[Z_{i,k}, Z_{j,l}] &= w_i\,\sigma_k^2\,\delta_{i,j}\,\delta_{k,l} \end{aligned}$$

for all $i,j \in \{-m,\dots,n\}$ and $k,l \in \{0,\dots,n\}$. Moreover, $Z_{i,0} > 0$ holds for all $i \in \{-m,\dots,n\}$.

In the extended Panning model, the *initial losses* $Z_{i,0}$ replace the volume measures used in the extended additive model. Since the first identity in the extended Panning model implies

$$E[Z_{i,k}/Z_{i,0}] = \xi_k$$

the development year parameter ξ_k is assumed to be independent of the accident year such that the collection of these parameters forms a *development pattern*; see Schmidt and Zocher [2009].

The extended Panning model extends the traditional Panning model in which it is assumed that $m = 0$ and that $w_i = 1$ holds for all $i \in \{-m,\dots,n\}$; see Panning [2006] and Schmidt and Zocher [2009]. The reason for considering the extended Panning model becomes evident from its comparison with the extended additive model (Section 4) and with the combination of both models (Section 6).

The following results are entirely analogous to those for the extended additive model and can be obtained by replacing the volume measures v_i used in the extended additive model by the initial losses $Z_{i,0}$ and by replacing the first and second order moments by their $\mathcal{F}_0$–conditional counterparts.

5.1 Theorem. *The extended Panning model is an $\mathcal{F}_0$–conditional linear model.*

5.2 Lemma (Gauss–Markov estimation of parameters). *In the extended Panning model, the $\mathcal{F}_0$–conditional Gauss–Markov estimators of the coordinates of the parameter vector satisfy*

$$\xi_k^{\mathrm{GM}} = \frac{\sum_{i=-m}^{n-k} Z_{i,0} Z_{i,k}/w_i}{\sum_{i=-m}^{n-k} Z_{i,0}^2/w_i}$$

and

$$\mathrm{cov}^{\mathcal{F}_0}[\xi_k^{\mathrm{GM}}, \xi_l^{\mathrm{GM}}] \;=\; \frac{1}{\sum_{i=-m}^{n-k} Z_{i,0}^2/w_i}\, \sigma_k^2\, \delta_{k,l}$$

for all $k, l \in \{0, 1, \dots, n\}$.

5.3 Lemma (Gauss–Markov prediction of incremental losses). *In the extended Panning model, the* $\mathcal{F}_0$*–conditional Gauss–Markov predictors of the non–observable incremental losses satisfy*

$$Z_{i,k}^{\mathrm{GM}} \;=\; Z_{i,0}\, \xi_k^{\mathrm{GM}}$$

and

$$\mathrm{cov}^{\mathcal{F}_0}[Z_{i,k}^{\mathrm{GM}} - Z_{i,k}, Z_{j,l}^{\mathrm{GM}} - Z_{j,l}] \;=\; \Big(Z_{i,0} Z_{j,0}\, \mathrm{var}^{\mathcal{F}_0}[\xi_k^{\mathrm{GM}}] + w_i\, \sigma_k^2\, \delta_{i,j} \Big)\, \delta_{k,l}$$

for all $i, j \in \{-m, \dots, n\}$ *and* $k, l \in \{0, \dots, n\}$ *such that* $\min\{i+k, j+l\} \geq n+1$.

5.4 Theorem (Gauss–Markov prediction of reserves). *In the extended Panning model,*

(1) *the* $\mathcal{F}_0$*–conditional Gauss–Markov predictors of the accident year reserves satisfy*

$$R_i^{\mathrm{GM}} \;=\; Z_{i,0} \sum_{k=n-k+1}^{n} \xi_k^{\mathrm{GM}}$$

and

$$\mathrm{cov}^{\mathcal{F}_0}[R_i^{\mathrm{GM}} - R_i, R_j^{\mathrm{GM}} - R_j] \;=\; Z_{i,0} Z_{j,0} \sum_{k=n-i\wedge j+1}^{n} \mathrm{var}^{\mathcal{F}_0}[\xi_k^{\mathrm{GM}}] + w_i \left(\sum_{k=n-i+1}^{n} \sigma_k^2 \right) \delta_{i,j}$$

for all $i, j \in \{1, \dots, n\}$; *in particular,*

$$E^{\mathcal{F}_0}[(R_i^{\mathrm{GM}} - R_i)^2] \;=\; Z_{i,0}^2 \sum_{k=n-i+1}^{n} \mathrm{var}^{\mathcal{F}_0}[\xi_k^{\mathrm{GM}}] + w_i \sum_{k=n-i+1}^{n} \sigma_k^2$$

holds for all $i \in \{1, \dots, n\}$.

(2) *the* $\mathcal{F}_0$*–conditional Gauss–Markov predictors of the calendar year reserves satisfy*

$$R_{(c)}^{\mathrm{GM}} \;=\; \sum_{i=c-n}^{n} Z_{i,0}\, \xi_{c-i}^{\mathrm{GM}}$$

and

$$\begin{aligned} &\mathrm{cov}^{\mathcal{F}_0}[R_{(c)}^{\mathrm{GM}} - R_{(c)}, R_{(d)}^{\mathrm{GM}} - R_{(d)}] \\ &\quad = \sum_{i=c\vee d-n}^{n} Z_{i,0} Z_{i-|c-d|,0}\, \mathrm{var}^{\mathcal{F}_0}[\xi_{c\vee d-i}^{\mathrm{GM}}] + \left(\sum_{i=c-n}^{n} w_i\, \sigma_{c-i}^2 \right) \delta_{c,d} \end{aligned}$$

for all $c, d \in \{n{+}1, \ldots, 2n\}$; in particular,

$$E^{\mathcal{F}_0}[(R_{(c)}^{\mathrm{GM}} - R_{(c)})^2] \;=\; \sum_{i=c-n}^{n} Z_{i,0}^2 \,\mathrm{var}^{\mathcal{F}_0}[\xi_{c-i}^{\mathrm{GM}}] + \sum_{i=c-n}^{n} w_i\, \sigma_{c-i}^2$$

holds for all $c \in \{n{+}1, \ldots, 2n\}$.

(3) *the $\mathcal{F}_0$–conditional Gauss–Markov predictor of the total reserve satisfies*

$$R^{\mathrm{GM}} \;=\; \sum_{k=1}^{n} \left(\sum_{i=n-k+1}^{n} Z_{i,0} \right) \xi_k^{\mathrm{GM}}$$

and

$$E^{\mathcal{F}_0}[(R^{\mathrm{GM}} - R)^2] \;=\; \sum_{k=1}^{n} \left(\sum_{i=n-k+1}^{n} Z_{i,0} \right)^2 \mathrm{var}^{\mathcal{F}_0}[\xi_k^{\mathrm{GM}}] + \sum_{k=1}^{n} \left(\sum_{i=n-k+1}^{n} w_i \right) \sigma_k^2 \,.$$

In the special case where $m = 0$ and $w_i = 1$ holds for all $i \in \{-m, \ldots, n\}$, the Gauss–Markov predictors of incremental losses and reserves are identical with the predictors used in the traditional Panning method of loss reserving. This means that Gauss–Markov prediction in the extended Panning model provides simultaneously an extension of the Panning method and its justification based on a general statistical principle.

The unknown variance parameters may be estimated as follows:

5.5 Theorem (Estimation of variance parameters). *In the extended Panning model with $m \geq 1$ and for every $k \in \{0, \ldots, n\}$, the random variable*

$$\widehat{\sigma}_k^2 \;:=\; \frac{1}{m+n-k} \sum_{i=-m}^{n-k} \frac{1}{w_i} (Z_{i,k} - Z_{i,0}\xi_k^{\mathrm{GM}})^2$$

is an $\mathcal{F}_0$–conditionally unbiased estimator of σ_k^2.

The final remarks of Section 4 apply to the extended Panning model as well.

6 Gauss–Markov Loss Prediction in the Combined Model

Because of the similarity of the extended additive model and the extended Panning model, it is natural to consider convex combinations of these models. As in the previous section, we denote by $\mathcal{F}_0$ the σ–algebra generated by the family $\{Z_{i,0}\}_{i\in\{-m,\dots,n\}}$ of the losses of development year 0. The combined model is defined as follows:

> **Combined Model:** *There exist known $\mathcal{F}_0$–measurable random parameters v_i, w_i with $v_i, w_i > 0$ and $i \in \{-m,\dots,n\}$ as well as unknown parameters $\zeta_k, \xi_k \in \mathbb{R}$ and $\sigma_k^2 \in (0,\infty)$ with $k \in \{0,\dots,n\}$ such that the incremental losses satisfy*
>
> $$\begin{aligned} E^{\mathcal{F}_0}[Z_{i,k}] &= v_i\,\zeta_k + Z_{i,0}\,\xi_k \\ \operatorname{cov}^{\mathcal{F}_0}[Z_{i,k}, Z_{j,l}] &= w_i\,\sigma_k^2\,\delta_{i,j}\,\delta_{k,l} \end{aligned}$$
>
> *for all $i,j \in \{-m,\dots,n\}$ and $k,l \in \{0,\dots,n\}$. Moreover, $Z_{i,0} > 0$ holds for all $i \in \{-m,\dots,n\}$ and $v_i Z_{j,0} \neq v_j Z_{i,0}$ holds for some $i,j \in \{-m,\dots,0\}$ with $i \neq j$.*

It is evident that the combined model combines the extended additive model and the extended Panning model: Formally, putting $\xi_k := 0$ yields the extended additive model and putting $\zeta_k := 0$ yields the extended Panning model. However, the analysis of the combined model turns out to be a bit more subtle than the analysis of the extended additive and Panning models.

Assume that the assumptions of the combined model are fulfilled. Then the $\mathcal{F}_0$–conditional expectation of the random vector $\mathbf{X}_1$ of all observable incremental losses satisfies

$$E^{\mathcal{F}_0}\left[\left(\begin{array}{c} Z_{-m,0} \\ \vdots \\ Z_{n,0} \\ \hline \vdots \\ \hline Z_{-m,k} \\ \vdots \\ Z_{n-k,k} \\ \hline \vdots \\ \hline Z_{-m,n} \\ \vdots \\ Z_{0,n} \end{array}\right)\right] = \left(\begin{array}{ccccc|ccccc} v_{-m} & \cdots & 0 & \cdots & 0 & Z_{-m,0} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ v_n & \cdots & 0 & \cdots & 0 & Z_{n,0} & \cdots & 0 & \cdots & 0 \\ \hline \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \hline 0 & \cdots & v_{-m} & \cdots & 0 & 0 & \cdots & Z_{-m,0} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & v_{n-k} & \cdots & 0 & 0 & \cdots & Z_{n-k,0} & \cdots & 0 \\ \hline \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \hline 0 & \cdots & 0 & \cdots & v_{-m} & 0 & \cdots & 0 & \cdots & Z_{-m,0} \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & v_0 & 0 & \cdots & 0 & \cdots & Z_{0,0} \end{array}\right) \left(\begin{array}{c} \zeta_0 \\ \vdots \\ \zeta_k \\ \vdots \\ \zeta_n \\ \hline \xi_0 \\ \vdots \\ \xi_k \\ \vdots \\ \xi_n \end{array}\right)$$

such that there exist an $\mathcal{F}_0$–measurable random design matrix $\mathbf{A}_1$ having full column rank and a parameter vector $\boldsymbol{\beta}$ satisfying $E^{\mathcal{F}_0}[\mathbf{X}_1] = \mathbf{A}_1\boldsymbol{\beta}$. A similar identity holds for the $\mathcal{F}_0$–conditional expectation of the random vector $\mathbf{X}_2$ of all non–observable incremental losses. Moreover, the $\mathcal{F}_0$–conditional variance of the random vector $\mathbf{X}$ is the same as in the extended additive model and the extended Panning model.

6.1 Theorem. *The combined model is an $\mathcal{F}_0$–conditional linear model.*

For a concise and transparent presentation of the result for the combined model, we now introduce some auxiliary random variables. By assumption, we have

$$\sum_{i=-m}^{0} \sum_{j=-m}^{0} (v_i Z_{j,0} - v_j Z_{i,0})^2 \;>\; 0$$

We may thus define, for $k \in \{0, \dots, n\}$ and $r, s \in \{0, 1, 2\}$,

$$Y_k^{(r,s)} \;:=\; 2 \, \frac{\sum_{i=-m}^{n-k} v_i^r Z_{i,0}^s / w_i}{\sum_{i=-m}^{n-k} \sum_{j=-m}^{n-k} (v_i Z_{j,0} - v_j Z_{i,0})^2 / w_i w_j} \, .$$

and straightforward calculation shows that

$$Y_k^{(r,s)} \;=\; \frac{\sum_{i=-m}^{n-k} v_i^r Z_{i,0}^s / w_i}{(\sum_{i=-m}^{n-k} v_i^2 / w_i)(\sum_{i=-m}^{n-k} Z_{i,0}^2 / w_i) - (\sum_{i=-m}^{n-k} v_i Z_{i,0} / w_i)^2} \, .$$

Note that these random variables are $\mathcal{F}_0$–measurable.

6.2 Lemma (Gauss–Markov estimation of parameters). *In the combined model, the $\mathcal{F}_0$–conditional Gauss–Markov estimators of the coordinates of the parameter vector satisfy*

$$\begin{aligned}
\zeta_k^{\mathrm{GM}} &= Y_k^{(0,2)} \sum_{i=-m}^{n-k} \frac{v_i Z_{i,k}}{w_i} - Y_k^{(1,1)} \sum_{i=-m}^{n-k} \frac{Z_{i,0} Z_{i,k}}{w_i} \\
\xi_k^{\mathrm{GM}} &= Y_k^{(2,0)} \sum_{i=-m}^{n-k} \frac{Z_{i,0} Z_{i,k}}{w_i} - Y_k^{(1,1)} \sum_{i=-m}^{n-k} \frac{v_i Z_{i,k}}{w_i}
\end{aligned}$$

as well as

$$\begin{aligned}
\mathrm{cov}^{\mathcal{F}_0}[\zeta_k^{\mathrm{GM}}, \zeta_l^{\mathrm{GM}}] &= Y_k^{(0,2)} \, \sigma_k^2 \, \delta_{k,l} \\
\mathrm{cov}^{\mathcal{F}_0}[\zeta_k^{\mathrm{GM}}, \xi_l^{\mathrm{GM}}] &= -Y_k^{(1,1)} \, \sigma_k^2 \, \delta_{k,l} \\
\mathrm{cov}^{\mathcal{F}_0}[\xi_k^{\mathrm{GM}}, \xi_l^{\mathrm{GM}}] &= Y_k^{(2,0)} \, \sigma_k^2 \, \delta_{k,l}
\end{aligned}$$

and, in particular,

$$\mathrm{cov}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix}, \begin{pmatrix} \zeta_l^{\mathrm{GM}} \\ \xi_l^{\mathrm{GM}} \end{pmatrix} \right] = \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix} \right] \delta_{k,l}$$

for all $k, l \in \{0, 1, \dots, n\}$.

Proof. We have

$$\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1 = \begin{pmatrix} \left(\dfrac{1}{\sigma_k^2}\displaystyle\sum_{i=-m}^{n-k}\dfrac{v_i Z_{i,k}}{w_i}\right)_{k\in\{0,\dots,n\}} \\ \left(\dfrac{1}{\sigma_k^2}\displaystyle\sum_{i=-m}^{n-k}\dfrac{Z_{i,0} Z_{i,k}}{w_i}\right)_{k\in\{0,\dots,n\}} \end{pmatrix}$$

and

$$\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1 = \begin{pmatrix} \operatorname{diag}\left(\dfrac{1}{\sigma_k^2}\displaystyle\sum_{i=-m}^{n-k}\dfrac{v_i^2}{w_i}\right)_{k\in\{0,\dots,n\}} & \operatorname{diag}\left(\dfrac{1}{\sigma_k^2}\displaystyle\sum_{i=-m}^{n-k}\dfrac{v_i Z_{i,0}}{w_i}\right)_{k\in\{0,\dots,n\}} \\ \operatorname{diag}\left(\dfrac{1}{\sigma_k^2}\displaystyle\sum_{i=-m}^{n-k}\dfrac{v_i Z_{i,0}}{w_i}\right)_{k\in\{0,\dots,n\}} & \operatorname{diag}\left(\dfrac{1}{\sigma_k^2}\displaystyle\sum_{i=-m}^{n-k}\dfrac{Z_{i,0}^2}{w_i}\right)_{k\in\{0,\dots,n\}} \end{pmatrix}.$$

Therefore, we have

$$\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1 = \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V} & \mathbf{W} \end{pmatrix}$$

with suitable diagonal matrices $\mathbf{U}, \mathbf{V}, \mathbf{W}$, and we also have

$$\begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V} & \mathbf{W} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{U}-\mathbf{V}\mathbf{W}^{-1}\mathbf{V})^{-1} & -\mathbf{U}^{-1}\mathbf{V}(\mathbf{W}-\mathbf{V}\mathbf{U}^{-1}\mathbf{V})^{-1} \\ -\mathbf{W}^{-1}\mathbf{V}(\mathbf{U}-\mathbf{V}\mathbf{W}^{-1}\mathbf{V})^{-1} & (\mathbf{W}-\mathbf{V}\mathbf{U}^{-1}\mathbf{V})^{-1} \end{pmatrix}.$$

Therefore, straightforward calculation yields

$$(\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1} = \begin{pmatrix} \operatorname{diag}(Y_k^{(0,2)}\sigma_k^2)_{k\in\{0,\dots,n\}} & -\operatorname{diag}(Y_k^{(1,1)}\sigma_k^2)_{k\in\{0,\dots,n\}} \\ -\operatorname{diag}(Y_k^{(1,1)}\sigma_k^2)_{k\in\{0,\dots,n\}} & \operatorname{diag}(Y_k^{(2,0)}\sigma_k^2)_{k\in\{0,\dots,n\}} \end{pmatrix}.$$

We thus obtain

$$(\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}\mathbf{A}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1 = \begin{pmatrix} \left(Y_k^{(0,2)}\displaystyle\sum_{i=-m}^{n-k}\dfrac{v_i Z_{i,k}}{w_i} - Y_k^{(1,1)}\displaystyle\sum_{i=-m}^{n-k}\dfrac{Z_{i,0}Z_{i,k}}{w_i}\right)_{k\in\{0,\dots,n\}} \\ \left(Y_k^{(2,0)}\displaystyle\sum_{i=-m}^{n-k}\dfrac{Z_{i,0}Z_{i,k}}{w_i} - Y_k^{(1,1)}\displaystyle\sum_{i=-m}^{n-k}\dfrac{v_i Z_{i,k}}{w_i}\right)_{k\in\{0,\dots,n\}} \end{pmatrix}$$

and hence

$$\begin{aligned} \zeta_k^{\mathrm{GM}} &= Y_k^{(0,2)}\sum_{i=-m}^{n-k}\frac{v_i Z_{i,k}}{w_i} - Y_k^{(1,1)}\sum_{i=-m}^{n-k}\frac{Z_{i,0}Z_{i,k}}{w_i} \\ \xi_k^{\mathrm{GM}} &= Y_k^{(2,0)}\sum_{i=-m}^{n-k}\frac{Z_{i,0}Z_{i,k}}{w_i} - Y_k^{(1,1)}\sum_{i=-m}^{n-k}\frac{v_i Z_{i,k}}{w_i}\ . \end{aligned}$$

The above identity for $(\mathbf{A}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{A}_1)^{-1}$ also yields

$$\begin{aligned}
\mathrm{cov}^{\mathcal{F}_0}[\zeta_k^{\mathrm{GM}}, \zeta_l^{\mathrm{GM}}] &= \quad Y_k^{(0,2)}\, \sigma_k^2\, \delta_{k,l} \\
\mathrm{cov}^{\mathcal{F}_0}[\zeta_k^{\mathrm{GM}}, \xi_l^{\mathrm{GM}}] &= -\, Y_k^{(1,1)}\, \sigma_k^2\, \delta_{k,l} \\
\mathrm{cov}^{\mathcal{F}_0}[\xi_k^{\mathrm{GM}}, \xi_l^{\mathrm{GM}}] &= \quad Y_k^{(2,0)}\, \sigma_k^2\, \delta_{k,l}
\end{aligned}$$

which completes the proof. □

We can now compute the Gauss–Markov predictors of the non–observable incremental losses and the covariances of their prediction errors:

6.3 Lemma (Gauss–Markov prediction of incremental losses). *In the combined model, the $\mathcal{F}_0$–conditional Gauss–Markov predictors of the non–observable incremental losses satisfy*

$$Z_{i,k}^{\mathrm{GM}} = v_i\, \zeta_k^{\mathrm{GM}} + Z_{i,0}\, \xi_k^{\mathrm{GM}}$$

and

$$\begin{aligned}
&\mathrm{cov}^{\mathcal{F}_0}[Z_{i,k}^{\mathrm{GM}} - Z_{i,k}, Z_{j,l}^{\mathrm{GM}} - Z_{j,l}] \\
&= \left(\begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix} \right] \begin{pmatrix} v_j \\ Z_{j,0} \end{pmatrix} + w_i \sigma_k^2 \delta_{i,j} \right) \delta_{k,l}
\end{aligned}$$

for all $i, j \in \{-m, \ldots, n\}$ and $k, l \in \{0, \ldots, n\}$ such that $\min\{i+k, j+l\} \geq n+1$; *in particular,*

$$E^{\mathcal{F}_0}[(Z_{i,k}^{\mathrm{GM}} - Z_{i,k})^2] = \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix} \right] \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix} + w_i\, \sigma_k^2$$

holds for all $i \in \{-m, \ldots, n\}$ and $k \in \{0, \ldots, n\}$ such that $i + k \geq n + 1$.

Proof. The first identity is evident. Furthermore, Lemma 6.2 yields

$$\begin{aligned}
\mathrm{cov}^{\mathcal{F}_0}[Z_{i,k}^{\mathrm{GM}}, Z_{j,l}^{\mathrm{GM}}] &= \mathrm{cov}^{\mathcal{F}_0}[v_i\, \zeta_k^{\mathrm{GM}} + Z_{i,0} \xi_k^{\mathrm{GM}}, v_j\, \zeta_l^{\mathrm{GM}} + Z_{j,0}\, \xi_l^{\mathrm{GM}}] \\
&= \mathrm{cov}^{\mathcal{F}_0}\left[\begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix}, \begin{pmatrix} v_j \\ Z_{j,0} \end{pmatrix}' \begin{pmatrix} \zeta_l^{\mathrm{GM}} \\ \xi_l^{\mathrm{GM}} \end{pmatrix} \right] \\
&= \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \mathrm{cov}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix}, \begin{pmatrix} \zeta_l^{\mathrm{GM}} \\ \xi_l^{\mathrm{GM}} \end{pmatrix} \right] \begin{pmatrix} v_j \\ Z_{j,0} \end{pmatrix} \\
&= \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix} \right] \begin{pmatrix} v_j \\ Z_{j,0} \end{pmatrix} \delta_{k,l}\ .
\end{aligned}$$

Since $\mathrm{cov}^{\mathcal{F}_0}[Z_{i,k}^{\mathrm{GM}}, Z_{j,l}] = 0 = \mathrm{cov}^{\mathcal{F}_0}[Z_{i,k}, Z_{j,l}^{\mathrm{GM}}]$ and

$$\mathrm{cov}^{\mathcal{F}_0}[Z_{i,k} Z_{j,l}] = w_i\, \sigma_k^2\, \delta_{i,j} \delta_{k,l}$$

we obtain

$$\begin{aligned}
&\mathrm{cov}^{\mathcal{F}_0}[Z_{i,k}^{\mathrm{GM}} - Z_{i,k}, Z_{j,l}^{\mathrm{GM}} - Z_{j,l}] \\
&\quad = \mathrm{cov}^{\mathcal{F}_0}[Z_{i,k}^{\mathrm{GM}}, Z_{j,l}^{\mathrm{GM}}] + \mathrm{cov}^{\mathcal{F}_0}[Z_{i,k} Z_{j,l}] \\
&\quad = \left(\begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix} \right] \begin{pmatrix} v_j \\ Z_{j,0} \end{pmatrix} \delta_{k,l} + w_i \sigma_k^2 \delta_{i,j} \right) \delta_{k,l}
\end{aligned}$$

which is the second identity. □

The following result on the Gauss–Markov predictors of reserves and their expected squared prediction errors is formally identical with the results for the extended additive model and the extended Panning model:

6.4 Theorem (Gauss–Markov prediction of reserves). *In the combined model,*

(1) *the $\mathcal{F}_0$–conditional Gauss–Markov predictors of the accident year reserves satisfy*

$$R_i^{\mathrm{GM}} = v_i \sum_{k=n-k+1}^{n} \zeta_k^{\mathrm{GM}} + Z_{i,0} \sum_{k=n-k+1}^{n} \xi_k^{\mathrm{GM}}$$

and

$$\begin{aligned}
&\mathrm{cov}^{\mathcal{F}_0}[R_i^{\mathrm{GM}} - R_i, R_j^{\mathrm{GM}} - R_j] \\
&\quad = \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \left(\sum_{k=n-i\wedge j+1}^{n} \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix} \right] \right) \begin{pmatrix} v_j \\ Z_{j,0} \end{pmatrix} + w_i \left(\sum_{k=n-i+1} \sigma_k^2 \right) \delta_{i,j}
\end{aligned}$$

for all $i, j \in \{1, \ldots, n\}$; in particular,

$$\begin{aligned}
&E^{\mathcal{F}_0}[(R_i^{\mathrm{GM}} - R_i)^2] \\
&\quad = \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \left(\sum_{k=n-i+1}^{n} \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix} \right] \right) \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix} + w_i \sum_{k=n-i+1} \sigma_k^2
\end{aligned}$$

holds for all $i \in \{1, \ldots, n\}$.

(2) *the $\mathcal{F}_0$–conditional Gauss–Markov predictors of the calendar year reserves satisfy*

$$R_{(c)}^{\mathrm{GM}} = \sum_{i=c-n}^{n} \left(v_i \, \zeta_{c-i}^{\mathrm{GM}} + Z_{i,0} \, \xi_{c-i}^{\mathrm{GM}} \right)$$

and

$$\begin{aligned}
&\mathrm{cov}^{\mathcal{F}_0}[R_{(c)}^{\mathrm{GM}} - R_{(c)}, R_{(d)}^{\mathrm{GM}} - R_{(d)}] \\
&\quad = \sum_{i=c\vee d-n}^{n} \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_{c\vee d-i}^{\mathrm{GM}} \\ \xi_{c\vee d-i}^{\mathrm{GM}} \end{pmatrix} \right] \begin{pmatrix} v_{i-|c-d|} \\ Z_{i-|c-d|,0} \end{pmatrix} + \left(\sum_{i=c\vee d-n}^{n} w_i \sigma_{c-i}^2 \right) \delta_{c,d}
\end{aligned}$$

for all $c, d \in \{n{+}1, \dots, 2n\}$; *in particular,*

$$E^{\mathcal{F}_0}[(R_{(c)}^{\mathrm{GM}} - R_{(c)})^2] = \sum_{i=c-n}^{n} \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix}' \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_{c-i}^{\mathrm{GM}} \\ \xi_{c-i}^{\mathrm{GM}} \end{pmatrix}\right] \begin{pmatrix} v_i \\ Z_{i,0} \end{pmatrix} + \sum_{i=c-n}^{n} w_i \sigma_{c-i}^2$$

holds for all $c \in \{n{+}1, \dots, 2n\}$.

(3) *the* $\mathcal{F}_0$*–conditional Gauss–Markov predictor of the total reserve satisfies*

$$R^{\mathrm{GM}} = \sum_{k=1}^{n} \sum_{i=n-k+1}^{n} \Big(v_i \, \zeta_k^{\mathrm{GM}} + Z_{i,0} \, \xi_k^{\mathrm{GM}} \Big)$$

and

$$E^{\mathcal{F}_0}[(R_i^{\mathrm{GM}} - R_i)^2] = \sum_{k=1}^{n} \begin{pmatrix} \displaystyle\sum_{i=n-k+1}^{n} v_i \\ \displaystyle\sum_{i=n-k+1}^{n} Z_{i,0} \end{pmatrix}' \mathrm{var}^{\mathcal{F}_0}\left[\begin{pmatrix} \zeta_k^{\mathrm{GM}} \\ \xi_k^{\mathrm{GM}} \end{pmatrix}\right] \begin{pmatrix} \displaystyle\sum_{i=n-k+1}^{n} v_i \\ \displaystyle\sum_{i=n-k+1}^{n} Z_{i,0} \end{pmatrix} + \sum_{k=1}^{n} \Bigg(\sum_{i=n-k+1}^{n} w_i \Bigg) \sigma_k^2 \, .$$

The proof of Theorem 6.4 is analogous to that of Theorem 4.4 (using Lemma 6.3 instead of Lemma 4.3).

Finally, the unknown variance parameters may be estimated as follows:

6.5 Theorem (Estimation of variance parameters). *In the combined model with* $m \geq 2$ *and for every* $k \in \{0, \dots, n\}$, *the random variable*

$$\widehat{\sigma}_k^2 := \frac{1}{m+n-k-1} \sum_{i=-m}^{n-k} \frac{1}{w_i} \Big(Z_{i,k} - (v_i \zeta_k^{\mathrm{GM}} + Z_{i,0} \xi_k^{\mathrm{GM}}) \Big)^2$$

is an $\mathcal{F}_0$*–conditionally unbiased estimator of* σ_k^2.

The proof of Theorem 6.5 is analogous to that of Theorem 4.5 (using Lemmas 6.2 and 6.3 instead of Lemmas 4.2 and 4.3).

In the case $m = 1$, the assertion of Theorem 6.5 remains valid for $k \in \{0, \dots, n{-}1\}$, and in the case $m = 0$ it remains valid for $k \in \{0, \dots, n{-}2\}$. Thus, the final remarks of Section 4 apply *mutatis mutandis* to the combined model as well.

7 Loss Prediction in the Mack Model

For the sake of comparison, the present section provides a brief discussion of the famous Mack model for the chain–ladder method. In a sense to be made precise below, the Mack model is related to linear models but it is not a linear model as such.

For $k \in \{0,\ldots,n\}$, we denote by $\mathcal{F}_k$ the σ–algebra generated by the family

$$\{S_{j,l}\}_{l\in\{0,\ldots,k\},\, j\in\{-m,\ldots,n-l\}}$$

of all observable cumulative losses up to development year k and, for $i \in \{-m,\ldots,n\}$ and $k \in \{0,\ldots,n\}$, we denote by $\mathcal{F}_{i,k}$ the σ–algebra generated by the family

$$\{S_{i,l}\}_{l\in\{0,\ldots,k\}}$$

of all cumulative losses of accident year i up to development year k; note that the definition of $\mathcal{F}_0$ is in accordance with that used in Sections 5 and 6. The Mack model is defined as follows:

> **Mack Model:** *The accident years are independent* (*in the sense that the family of σ–algebras $\{\mathcal{F}_{i,n}\}_{i\in\{-m,\ldots,n\}}$ is independent*) *and, for every development year $k \in \{1,\ldots,n\}$, there exist unknown parameters $\varphi_k \in \mathbb{R}$ and $\sigma_k^2 \in (0,\infty)$ such that the cumulative losses satisfy*
>
> $$\begin{aligned} E^{\mathcal{F}_{i,k-1}}[S_{i,k}] &= S_{i,k-1}\,\varphi_k \\ \mathrm{var}^{\mathcal{F}_{i,k-1}}[S_{i,k}] &= S_{i,k-1}\,\sigma_k^2 \end{aligned}$$
>
> *for all $i \in \{-m,\ldots,n\}$. Moreover, $S_{i,k} > 0$ holds for all $i \in \{-m,\ldots,n\}$ and $k \in \{0,\ldots,n-1\}$.*

In the Mack model, the cumulative losses $S_{i,k}$ replace the incremental losses used in the models considered before, the cumulative losses $S_{i,k-1}$ replace the volume measures used in the extended additive model and the initial losses used in the extended Panning model, and they also replace the accident year parameters w_i used in each of these models. Since the first identity in the Mack model implies

$$E[S_{i,k}/S_{i,k-1}] \;=\; \varphi_k$$

the development year parameter φ_k is assumed to be independent of the accident year and the collection of these parameters forms a *development pattern*; see Schmidt and Zocher [2009].

The Mack model is due to Mack [1993] who assumed that $m = 0$.

Assume that the assumptions of the Mack model are fulfilled. Then we have, for every $k \in \{1,\ldots,n\}$,

$$E^{\mathcal{F}_{k-1}}\left[\begin{pmatrix} S_{-m,k} \\ \vdots \\ S_{n-k,k} \end{pmatrix}\right] \;=\; \begin{pmatrix} S_{-m,k-1} \\ \vdots \\ S_{n-k,k-1} \end{pmatrix}\varphi_k$$

and

$$E^{\mathcal{F}_{k-1}}[S_{n-k+1,k}] \;=\; S_{n-k+1,k-1}\,\varphi_k$$

as well as

$$\mathrm{var}^{\mathcal{F}_{k-1}}\left[\begin{pmatrix} S_{-m,k} \\ \vdots \\ S_{n-k,k} \end{pmatrix}\right] \;=\; \begin{pmatrix} S_{-m,k-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_{n-k,k-1} \end{pmatrix} \sigma_k^2$$

and

$$\mathrm{var}^{\mathcal{F}_{k-1}}[S_{n-k+1,k}] \;=\; S_{n-k+1,k-1}\,\sigma_k^2$$

and we also have

$$\mathrm{cov}^{\mathcal{F}_{k-1}}[S_{i,k}, S_{j,k}] \;=\; 0$$

for all $i,j \in \{-m,\ldots,n-k+1\}$ such that $i \neq j$; see Schmidt and Schnaus [1996]. We thus obtain the following result:

7.1 Theorem. *For every development year $k \in \{1,\ldots,n\}$, the Mack model provides an $\mathcal{F}_{k-1}$–conditional linear model for the family $\{S_{i,k}\}_{i\in\{-m,\ldots,n-k+1\}}$.*

Because of Theorem 7.1, the Mack model may be called a *sequential linear model.*

Let us first consider Gauss–Markov estimation of the parameter in the conditional linear models provided by the Mack model:

7.2 Lemma (Gauss–Markov estimation of parameters). *In the Mack model and for every development year $k \in \{1,\ldots,n\}$, the $\mathcal{F}_{k-1}$–conditional Gauss–Markov estimator of the parameter φ_k satisfies*

$$\varphi_k^{\mathrm{GM}} \;=\; \frac{\sum_{i=-m}^{n-k} S_{i,k}}{\sum_{i=-m}^{n-k} S_{i,k-1}}$$

and

$$\mathrm{var}^{\mathcal{F}_{k-1}}[\varphi_k^{\mathrm{GM}}] \;=\; \frac{1}{\sum_{i=-m}^{n-k} S_{i,k-1}}\,\sigma_k^2 \ .$$

The linear models for the families $\{S_{i,k}\}_{i\in\{-m,\ldots,n-k+1\}}$ cannot be extended to the families $\{S_{i,k}\}_{i\in\{-m,\ldots,n\}}$ since the cumulative losses $S_{i,k-1}$ with $i \in \{n-k+2,\ldots,n\}$ are non–observable and hence cannot be part of the design matrix of a conditional linear model (in which the design matrix is assumed to be observable); therefore, Gauss–Markov prediction is possible only for the non–observable cumulative losses $S_{n-k+1,k}$ of the first non–observable calendar year $n+1$:

7.3 Lemma (Gauss–Markov prediction of cumulative losses). *In the Mack model and for every development year $k \in \{1,\dots,n\}$, the $\mathcal{F}_{k-1}$–conditional Gauss–Markov predictor of the cumulative loss $S_{n-k+1,k}$ satisfies*

$$S^{\text{GM}}_{n-k+1,k} \;=\; S_{n-k+1,k-1}\varphi^{\text{GM}}.$$

At this point, let us recall that, for every $k \in \{1,\dots,n\}$, the *chain–ladder factor* φ^{CL}_k is defined as

$$\varphi^{\text{CL}}_k \;:=\; \frac{\sum_{i=-m}^{n-k} S_{i,k}}{\sum_{i=-m}^{n-k} S_{i,k-1}}$$

and that, for all $i,k \in \{0,\dots,n\}$ such that $i+k \geq n$, the *chain–ladder predictor* of the cumulative loss $S_{i,k}$ (which is non–observable for $i+k \geq n+1$) is defined as

$$S^{\text{CL}}_{i,k} \;:=\; S_{i,n-i} \prod_{l=n-i+1}^{k} \varphi^{\text{CL}}_l$$

(such that $S^{\text{CL}}_{i,n-i} = S_{i,n-i}$). Thus, Lemmas 7.2 and 7.3 assert that

$$\varphi^{\text{GM}}_k \;=\; \varphi^{\text{CL}}_k$$

and

$$S^{\text{GM}}_{n-k+1,k} \;=\; S^{\text{CL}}_{n-k+1,k}$$

holds for all $k \in \{1,\dots,n\}$. Since Gauss–Markov predictors are unbiased, the previous identity yields

$$E[S^{\text{CL}}_{i,k} - S_{i,k}] \;=\; 0$$

and hence

$$E[(S^{\text{CL}}_{i,k} - S_{i,k})^2] \;=\; \text{var}[S^{\text{CL}}_{i,k} - S_{i,k}]$$

for all $i,k \in \{1,\dots,n\}$ such that $i+k = n+1$, and it can be shown that these identities are also true for all $i,k \in \{1,\dots,n\}$ such that $i+k \geq n+2$.

Following Mack [1993], however, one should consider the $\mathcal{F}_n$–conditional mean squared error of prediction

$$E^{\mathcal{F}_n}[(S^{\text{CL}}_{i,k} - S_{i,k})^2] \;=\; \text{var}^{\mathcal{F}_n}[S^{\text{CL}}_{i,k} - S_{i,k}] + \Big(E^{\mathcal{F}_n}[S^{\text{CL}}_{i,k} - S_{i,k}]\Big)^2$$

instead of the unconditional mean squared error of prediction $E[(S^{\text{CL}}_{i,k} - S_{i,k})^2]$. Since

$$E^{\mathcal{F}_n}[S^{\text{CL}}_{i,k}] \;=\; S_{i,n-i} \prod_{l=n-i+1}^{k} \varphi^{\text{CL}}_l$$

$$E^{\mathcal{F}_n}[S_{i,k}] \;=\; S_{i,n-i} \prod_{l=n-i+1}^{k} \varphi_l$$

we have

$$E^{\mathcal{F}_n}[S_{i,k}^{\mathrm{CL}} - S_{i,k}] \;=\; S_{i,n-i}\left(\prod_{l=n-i+1}^{k} \varphi_l^{\mathrm{CL}} - \prod_{l=n-i+1}^{k} \varphi_l\right)$$

which shows that the chain–ladder predictors fail to be $\mathcal{F}_n$–conditionally unbiased. Thus, the bias does not vanish in the identity for the $\mathcal{F}_n$–conditional mean squared error of prediction, which is most unfortunate since obviously plug–in estimators cannot be used to estimate the bias. By contrast, Mack [1993] has shown that the $\mathcal{F}_n$–conditional variance of the prediction error satisfies

$$\mathrm{var}^{\mathcal{F}_n}[S_{i,k}^{\mathrm{CL}} - S_{i,k}] \;=\; S_{i,n-i} \sum_{l=n-i+1}^{k} \left(\prod_{h=n-i+1}^{l-1} \varphi_h\right) \sigma_l^2 \left(\prod_{h=l+1}^{k} \varphi_h^2\right)$$

(which provides the identity

$$\mathrm{var}^{\mathcal{F}_n}[S_{i,k}] \;=\; S_{i,n-i} \sum_{l=n-i+1}^{k} \left(\prod_{h=n-i+1}^{l-1} \varphi_h\right) \sigma_l^2 \left(\prod_{h=l+1}^{k} \varphi_h^2\right)$$

needed in Theorem 7.4 below). In conclusion, estimation of the bias causes a serious difficulty in the estimation of the $\mathcal{F}_n$–conditional mean squared error of prediction of the chain–ladder predictor of a non–observable cumulative loss.

These observations also apply to the chain–ladder predictors of non–observable incremental losses which are defined as

$$Z_{i,k}^{\mathrm{CL}} \;:=\; S_{i,k}^{\mathrm{CL}} - S_{i,k-1}^{\mathrm{CL}}$$

and, in particular, to the chain–ladder predictors of reserves which are defined as

$$\begin{aligned} R_i^{\mathrm{CL}} &:= \sum_{k=n-i+1}^{n} Z_{i,k}^{\mathrm{CL}} \\ R_{(c)}^{\mathrm{CL}} &:= \sum_{i=c-n}^{n} Z_{i,c-i}^{\mathrm{CL}} \\ R^{\mathrm{CL}} &:= \sum_{k=1}^{n} \sum_{i=n-k+1}^{n} Z_{i,k}^{\mathrm{CL}} \quad . \end{aligned}$$

This can be seen from the following result:

7.4 Theorem (Chain–ladder prediction of reserves). *In the Mack model,*
(1) *the chain–ladder predictors of the accident year reserves satisfy*

$$E^{\mathcal{F}_n}[R_i^{\mathrm{CL}} - R_i] \;=\; S_{i,n-i}\left(\prod_{k=n-i+1}^{n} \varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{n} \varphi_k\right)$$

and

$$E^{\mathcal{F}_n}[(R_i^{\mathrm{CL}}-R_i)^2] = \left(S_{i,n-i}\left(\prod_{k=n-i+1}^{n}\varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{n}\varphi_k\right)\right)^2 + \mathrm{var}^{\mathcal{F}_n}[S_{i,n}]$$

as well as

$$\mathrm{cov}^{\mathcal{F}_n}[R_i^{\mathrm{CL}}-R_i, R_j^{\mathrm{CL}}-R_j] = \mathrm{var}^{\mathcal{F}_n}[S_{i,n}]\,\delta_{i,j}$$

(2) *the chain–ladder predictors of the calendar year reserves satisfy*

$$\begin{aligned} & E^{\mathcal{F}_n}[R_{(c)}^{\mathrm{CL}}-R_{(c)}] \\ & = \sum_{i=c-n}^{n} S_{i,n-i}\left(\left(\prod_{k=n-i+1}^{c-i-1}\varphi_k^{\mathrm{CL}}\right)(\varphi_{c-i}^{\mathrm{CL}}-1) - \left(\prod_{k=n-i+1}^{c-i-1}\varphi_k\right)(\varphi_{c-i}-1)\right) \end{aligned}$$

and

$$\begin{aligned} & E^{\mathcal{F}_n}[(R_{(c)}^{\mathrm{CL}}-R_{(c)})^2] \\ & = \left(\sum_{i=c-n}^{n} S_{i,n-i}\left(\left(\prod_{k=n-i+1}^{c-i-1}\varphi_k^{\mathrm{CL}}\right)(\varphi_{c-i}^{\mathrm{CL}}-1) - \left(\prod_{k=n-i+1}^{c-i-1}\varphi_k\right)(\varphi_{c-i}-1)\right)\right)^2 \\ & \quad + \sum_{i=c-n}^{n}\left(\mathrm{var}^{\mathcal{F}_n}[S_{i,c-i-1}]\,(\varphi_{c-i}-1)^2 + S_{i,n-i}\left(\prod_{k=n-i+1}^{c-i-1}\varphi_k\right)\sigma_{c-i}^2\right) \end{aligned}$$

(3) *the chain–ladder predictor of the total reserve satisfies*

$$E^{\mathcal{F}_n}[R^{\mathrm{CL}}-R] = \sum_{i=1}^{n} S_{i,n-i}\left(\prod_{k=n-i+1}^{n}\varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{n}\varphi_k\right)$$

and

$$E^{\mathcal{F}_n}[(R^{\mathrm{CL}}-R)^2] = \left(\sum_{i=1}^{n} S_{i,n-i}\left(\prod_{k=n-i+1}^{n}\varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{n}\varphi_k\right)\right)^2 + \sum_{i=1}^{n}\mathrm{var}^{\mathcal{F}_n}[S_{i,n}]$$

A proof of Theorem 7.4 will be given in the Appendix.

Theorem 7.4 provides explicit formulas for the $\mathcal{F}_n$–conditional mean squared errors of prediction, but the use of plug–in estimators in these formulas is not recommendable since it would result in wiping out a part of the $\mathcal{F}_n$–conditional mean squared errors of prediction.

7.5 Theorem (Estimation of variance parameters). *In the Mack model with $m \geq 1$ and for every $k \in \{1, \ldots, n\}$, the random variable*

$$\widehat{\sigma}_k^2 := \frac{1}{m+n-k}\sum_{i=-m}^{n-k}\frac{1}{S_{i,k-1}}\,(S_{i,k}-S_{i,k-1}\varphi_k^{\mathrm{CL}})^2$$

is an $\mathcal{F}_{k-1}$*–conditionally unbiased estimator of* σ_k^2.

As noted before, the use of plug–in estimators for the parameters of the development pattern in the formulas provided by Theorem 7.4 is not recommendable. Mack [1993] proposed the estimators

$$\begin{aligned}
&\widehat{E^{\mathcal{F}_n}}[(R_i^{\mathrm{CL}} - R_i)^2] \\
&\quad := (S_{i,n}^{\mathrm{CL}})^2 \sum_{k=n-i+1}^{n} \left(\frac{1}{\sum_{h=-m}^{n-k} S_{h,k}} + \frac{1}{S_{i,k}^{\mathrm{CL}}} \right) \frac{\widehat{\sigma}_k^2}{\varphi_k^{\mathrm{CL}}} \\
&\quad = (S_{i,n}^{\mathrm{CL}})^2 \sum_{k=n-i+1}^{n} \frac{1}{\sum_{h=-m}^{n-k} S_{h,k}} \frac{\widehat{\sigma}_k^2}{\varphi_k^{\mathrm{CL}}} + (S_{i,n}^{\mathrm{CL}})^2 \sum_{k=n-i+1}^{n} \frac{1}{S_{i,k}^{\mathrm{CL}}} \frac{\widehat{\sigma}_k^2}{\varphi_k^{\mathrm{CL}}}
\end{aligned}$$

for the $\mathcal{F}_n$–conditional mean squared errors of prediction of the accident year reserves and

$$\begin{aligned}
&\widehat{E^{\mathcal{F}_n}}[(R^{\mathrm{CL}} - R)^2] \\
&:= \sum_{i=1}^{n} \sum_{j=1}^{n} S_{i,n}^{\mathrm{CL}} S_{j,n}^{\mathrm{CL}} \sum_{k=n-i\wedge j+1}^{n} \frac{1}{\sum_{h=-m}^{n-k} S_{h,k}} \frac{\widehat{\sigma}_k^2}{\varphi_k^{\mathrm{CL}}} + \sum_{i=1}^{n} (S_{i,n}^{\mathrm{CL}})^2 \sum_{k=n-i+1}^{n} \frac{1}{S_{i,k}^{\mathrm{CL}}} \frac{\widehat{\sigma}_k^2}{\varphi_k^{\mathrm{CL}}}
\end{aligned}$$

for the $\mathcal{F}_n$–conditional mean squared error of prediction of the total reserve. The construction of each of these estimators involves certain approximations.

Apparently, no estimators have been proposed in the literature for the conditional mean squared errors of prediction of the calendar year reserves.

8 Remarks

In the Panning model and in the combined model, it would be sufficient to assume $Z_{i,0} > 0$ only for $i \in \{1, \ldots, n\}$, but then the formulas for predictors and mean squared errors of prediction have would have to be modified as to avoid divisions by zero.

The accident year parameters w_i may e. g.,be chosen as follows:
– In the extended additive model, one may choose $w_i := 1$ (corresponding to the traditional Panning model) or $w_i := v_i$ (traditional additive model) or, more generally, $w_i := \alpha + \beta v_i$ with $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$.
– In the extended Panning model, one may choose $w_i := 1$ (traditional Panning model) or $w_i := v_i$ (corresponding to the traditional additive model) or $w_i := Z_{i,0}$ (in analogy with the traditional additive model) or, more generally, $w_i := \alpha + \beta v_i + \gamma Z_{i,0}$ with $\alpha, \beta, \gamma \in [0, 1]$ and $\alpha + \beta + \gamma = 1$.
– In the combined model, one may choose $w_i := 1$ (corresponding to the traditional Panning model) or $w_i := v_i$ (corresponding to the traditional additive model) or $w_i := Z_{i,0}$ or, more generally, $w_i := \alpha + \beta v_i + \gamma Z_{i,0}$ with $\alpha, \beta, \gamma \in [0, 1]$ and $\alpha + \beta + \gamma = 1$.

The combined model uses volume measures and initial losses as regressors and thus provides an example for a broad class of general linear models combining different sources of information on the accident years. As there are several possible choices for the volume measure, like the number of contracts, the premium income, market statistics or even information on a similar portfolio of risks, one might want to use some of them simultaneously; also, as for example in excess–of–loss reinsurance, one might want to use several volume measures but avoid initial losses. In both cases, it is straightforward to construct appropriate modifications of the combined model and the analysis of the resulting models would follow the lines of Section 6.

For the additive method and the Panning method, the principle of Gauss–Markov prediction in an appropriate linear model shows that, under certain assumptions on the first and second order moments of the incremental losses,
– the predictors used in these methods are unbiased and minimize the mean squared error of prediction, and
– the mean squared errors of prediction can be estimated by the simple use of plug–in estimators for the unknown variance parameters.

In addition, the systematic use of Gauss–Markov prediction in a linear model leads to variations and combinations of these methods; see Section 9 below for nine such methods using the available information in a slightly different way. The analysis of results from different but similar methods may be useful to study the sensitivity of result with respect to model variations and to analyze the impact of loss development data and volume measures; see also Schmidt and Zocher [2009] for a similar discussion of another family of models and methods.

Unfortunately, the situation is not that comfortable for the chain–ladder method. While the Mack model was certainly a breakthrough in stochastic modelling for the chain–ladder method and provides a partial justification of that method, it seems that in this model

– the question of whether or not the chain–ladder predictors minimize the mean squared error of prediction cannot be settled and that
– the construction of estimators of the mean squared errors of prediction presents a serious problem and seems to require certain delicate approximations.

This is due to the sequential character of the Mack model, which provides a linear model for every development year but not for the entire loss development.

9 A Numerical Example

In the present section we illustrate the results of this paper by a numerical example. In the example, we consider a portfolio of auto liability and use the incremental losses provided by Braun [2004], truncated at development year 9, and the volume measures proposed by Merz and Wüthrich [2009]. These data are presented in Table 1.

For each of the additive model, the Panning model and the combined model we consider the three cases in which the accident year parameters of the variances are chosen as $w_i = 1$, $w_i = v_i$, and $w_i = Z_{i,0}$, respectively, and we also consider the Mack model in which the corresponding parameters are the cumulative losses $S_{i,k-1}$. The Gauss–Markov estimators of the parameters ζ_k (additive model and combined model), ξ_k (Panning model and combined model), and φ_k (Mack model) are displayed in Tables 2–5.

In the combined model, the signs of the Gauss–Markov estimators given Table 4 show that the volume measures and the initial losses have an opposite effect on the Gauss–Markov predictors of reserves; see Theorem 6.4. The Gauss–Markov predictors of the reserves of accident years 1–9, the total reserves and the reserves of calendar years 10–18 are displayed in Table 6.

The *standard error of prediction* is defined as the square root of the mean squared error of prediction and measures uncertainty in the monetary unit. The estimated standard errors of prediction are displayed in Table 7.

As an alternative measure of uncertainty, one could also consider the *coefficient of variation* which is defined as the ratio between the standard error of prediction and the predictor and is dimension–free. The coefficients of variation are displayed in Table 8.

Of course, the choice of a stochastic model should not be driven by the numerical results which it produces. Nevertheless, model selection should perhaps proceed in steps, starting with the choice of a plausible *class* of models (like the class of general linear models) and subsequently shrinking this class to only a few models or even a single one. In this process a comparative analysis of a family of similar models could help to obtain some insight into some of the characteristics of these models.

For the example considered here, we make the following observations:

- The choice of regressors (volume measures in the additive model, initial losses in the Panning model, and both of them in the combined model) may affect the predictors and the standard errors of prediction. For example, for the Panning model, the predictors of the total reserves are smaller and the standard errors of calendar year 10 are larger than for the additive model and the combined model.

- The choice of the accident year parameters may affect the predictors and the standard errors of prediction. For example, for $w_i = 1$, the total reserves are larger and the standard errors are smaller than for $w_i = v_i$ and $w_i = Z_{i,0}$.
- For the Mack model, the predictors are in the range of those obtained for the other models but the standard errors are larger.

Such considerations combined with actuarial judgement could help to determine estimates of reserves and estimates of standard errors of prediction for the portfolio under consideration.

Nevertheless, such an analysis for a particular portfolio cannot justify a general preference for a particular stochastic model.

Accident Year	Volume Measure	Development Year 0	1	2	3	4	5	6	7	8	9
0	413213	114423	133538	65021	31358	27139	-377	9889	4477	-316	7108
1	537988	152296	152879	71438	41686	22009	25315	7961	4843	-113	1593
2	589145	144325	162919	106365	50432	55224	7951	8234	1409	2061	669
3	523419	145904	161732	79458	46642	29384	15811	3598	5527	-2484	462
4	501498	170333	171168	92601	36227	11872	18760	3180	3538	948	-875
5	598345	189643	171480	85734	61226	18479	13556	7523	1964	88	
6	608376	179022	217202	101080	56183	28362	29791	11244	12568		
7	698993	205908	210139	104397	45277	34888	30193	17563			
8	704129	210951	215478	98618	62846	52435	22824				
9	903557	213426	295796	140211	82259	59209					
10	947326	249508	330502	142126	122023						
11	1134129	258425	427587								
12	1538916	368762	540304								
13	1487234	394997									

Table 1: Incremental Losses

Gauss–Markov Estimators	Development Year 0	1	2	3	4	5	6	7	8	9
$w_i = 1$	0.2605	0.3368	0.1642	0.0934	0.0570	0.0326	0.0158	0.0091	0.0001	0.0030
$w_i = v_i$	0.2680	0.3290	0.1613	0.0905	0.0558	0.0317	0.0155	0.0091	0.0001	0.0035
$w_i = Z_{i,0}$	0.2648	0.3307	0.1626	0.0911	0.0573	0.0311	0.0156	0.0090	0.0001	0.0036

Table 2: Gauss–Markov Estimators in the Additive Model

Gauss–Markov Estimators	Development Year 1	2	3	4	5	6	7	8	9
$w_i = 1$	1.2747	0.6003	0.3308	0.1955	0.1121	0.0535	0.0313	0.0004	0.0100
$w_i = v_i$	1.2021	0.5769	0.3167	0.1890	0.1091	0.0522	0.0312	0.0002	0.0116
$w_i = Z_{i,0}$	1.2258	0.5891	0.3220	0.1964	0.1083	0.0531	0.0313	0.0002	0.0123

Table 3: Gauss–Markov Estimators in the Panning Model

Gauss–Markov Estimators	Development Year 1	2	3	4	5	6	7	8	9
$w_i = 1$	0.4795	0.2686	0.1731	0.1731	-0.0300	0.0305	0.0033	0.0024	0.0148
	-0.5505	-0.2023	-0.4139	-0.4139	0.2140	-0.0504	0.0199	-0.0077	-0.0419
$w_i = v_i$	0.4444	0.2403	0.1421	0.1896	-0.0340	0.0335	0.0047	0.0011	0.0177
	-0.4302	-0.2886	-0.1832	-0.4714	0.2246	-0.0618	0.0150	-0.0035	-0.0502
$w_i = Z_{i,0}$	0.4545	0.2542	0.1393	0.1861	-0.0414	0.0292	0.0003	0.0032	0.0146
	-0.4679	-0.3392	-0.1735	-0.4589	0.2499	-0.0471	0.0303	-0.0108	-0.0392

Table 4: Gauss–Markov Estimators in the Combined Model

Gauss–Markov Estimators	Development Year 1	2	3	4	5	6	7	8	9
	2.2258	1.2694	1.1204	1.0668	1.0354	1.0168	1.0097	1.0001	1.0037

Table 5: Gauss–Markov Estimators in the Mack Model

Year	Additive Model			Panning Model			Combined Model			Mack Model
	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	
1	1792	2089	2165	1890	2195	2336	919	1086	1304	2054
2	1912	2160	2258	1859	2100	2241	1581	1822	1874	2415
3	8567	8842	8896	8756	8833	9026	8232	8485	8588	8762
4	19763	19804	19937	20073	20068	20459	19024	18942	19200	20232
5	54806	54017	53717	44226	43588	43812	47548	47272	44396	52994
6	111440	109465	110578	100490	98103	100217	114045	114790	113047	116698
7	239298	233738	235656	189570	183455	187008	265053	265185	259631	251872
8	577322	565374	569989	491876	474513	484091	619938	613617	610210	562574
9	1058893	1035648	1042712	1030382	983097	1002726	1061093	1052418	1050462	1028283
Total	2073790	2031136	2045907	1888941	1815952	1851916	2137432	2123616	2108712	2045884
10	962268	940978	947253	902762	859493	876786	979515	965487	966517	943140
11	505930	495009	499106	457482	440535	449395	539568	536922	534841	498805
12	288908	281751	284390	253895	245074	250111	302808	302239	298209	285564
13	163703	160341	161950	142379	138618	141695	158496	158669	155306	163089
14	85982	84427	83876	74307	72919	73147	81916	81565	78020	85532
15	40543	40394	40590	35258	35053	35668	42187	42770	41316	40861
16	17173	17583	17656	15079	15360	15628	19610	20461	19767	18026
17	4829	5460	5706	3841	4330	4621	7846	8973	8498	5568
18	4454	5193	5380	3938	4570	4865	5486	6530	6239	5300

Table 6: Gauss–Markov Predictors of Reserves

Year	Additive Model			Panning Model			Combined Model			Mack Model
	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	
1	3672	4260	4458	3823	4428	4619	4627	5255	5661	5507
2	4046	4645	4730	4125	4738	4821	4598	5241	5439	6523
3	5816	6616	6722	5879	6702	6782	6504	7381	7575	7940
4	7213	8122	8252	7456	8420	8505	8012	8963	9244	9688
5	12257	15329	14299	11468	14555	13423	18370	20789	19572	17168
6	18424	22991	22327	20582	25468	24802	18160	21689	20921	28047
7	24595	30909	28394	27585	34861	31879	33790	38016	35710	38512
8	33753	44489	42401	43931	57054	54984	41550	50472	47660	55852
9	43298	56745	56753	72712	90441	91254	40463	54210	53349	129110
Total	86154	101944	100194	109448	130181	129282	113638	126545	121534	176968
10	41519	52118	51402	71084	86744	86557	41168	52158	50498	
11	31861	39778	38650	43229	53290	52786	33925	41131	39429	
12	25884	34347	32733	29612	39777	38020	30784	36890	35322	
13	20602	28982	27921	22623	31995	31032	26262	31509	30190	
14	13984	19671	19057	12946	18666	17902	20091	24483	23357	
15	8860	11802	11264	8567	11801	11172	13289	15601	14868	
16	7334	9780	9340	6859	9445	8940	11274	13729	13186	
17	5899	8354	7987	5439	8027	7642	9457	12073	11676	
18	5318	7602	7437	5183	7540	7375	6467	8997	8806	

Table 7: Standard Errors of Prediction

Year	Additive Model			Panning Model			Combined Model			Mack Model
	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	$w_i = 1$	$w_i = v_i$	$w_i = Z_{i,0}$	
1	204.97%	203.94%	205.95%	202.25%	201.76%	197.77%	503.36%	484.10%	434.04%	268.09%
2	211.56%	215.06%	209.53%	221.84%	225.59%	215.17%	290.77%	287.65%	290.26%	270.14%
3	67.91%	74.82%	75.56%	68.55%	75.88%	75.13%	79.00%	86.99%	88.20%	90.62%
4	36.50%	41.01%	41.39%	37.15%	41.96%	41.57%	42.11%	47.32%	48.14%	47.89%
5	22.36%	28.38%	26.62%	25.93%	33.39%	30.64%	38.63%	43.98%	44.09%	32.40%
6	16.53%	21.00%	20.19%	20.48%	25.96%	24.75%	15.92%	18.89%	18.51%	24.03%
7	10.28%	13.22%	12.05%	14.55%	19.00%	17.05%	12.75%	14.34%	13.75%	15.29%
8	5.85%	7.87%	7.44%	8.93%	12.02%	11.36%	6.70%	8.23%	7.81%	9.93%
9	4.09%	5.48%	5.44%	7.06%	9.20%	9.10%	3.81%	5.15%	5.08%	12.56%
Total	4.15%	5.02%	4.90%	5.79%	7.17%	6.98%	5.32%	5.96%	5.76%	8.65%
10	4.31%	5.54%	5.43%	7.87%	10.09%	9.87%	4.20%	5.40%	5.22%	
11	6.30%	8.04%	7.74%	9.45%	12.10%	11.75%	6.29%	7.66%	7.37%	
12	8.96%	12.19%	11.51%	11.66%	16.23%	15.20%	10.17%	12.21%	11.84%	
13	12.59%	18.07%	17.24%	15.89%	23.08%	21.90%	16.57%	19.86%	19.44%	
14	16.26%	23.30%	22.72%	17.42%	25.60%	24.47%	24.53%	30.02%	29.94%	
15	21.85%	29.22%	27.75%	24.30%	33.67%	31.32%	31.50%	36.48%	35.99%	
16	42.70%	55.62%	52.90%	45.49%	61.49%	57.20%	57.49%	67.10%	66.71%	
17	122.14%	153.00%	139.97%	141.61%	185.37%	165.39%	120.52%	134.54%	137.40%	
18	119.40%	146.41%	138.23%	131.62%	164.95%	151.61%	117.87%	137.78%	141.15%	

Table 8: Coefficients of Variation

Appendix

Here we present a proof of Theorem 7.4:

Proof. We have $R_i^{\mathrm{CL}} - R_i = S_{i,n}^{\mathrm{CL}} - S_{i,n}$ and hence

$$\begin{aligned} E^{\mathcal{F}_n}[R_i^{\mathrm{CL}} - R_i] &= E^{\mathcal{F}_n}[S_{i,n}^{\mathrm{CL}} - S_{i,n}] \\ &= S_{i,n-i}\left(\prod_{k=n-i+1}^{n} \varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{n} \varphi_k\right). \end{aligned}$$

Since the accident years are independent, we also have

$$\begin{aligned} \mathrm{cov}^{\mathcal{F}_n}[R_i^{\mathrm{CL}} - R_i, R_j^{\mathrm{CL}} - R_j] &= \mathrm{cov}^{\mathcal{F}_n}[S_{i,n}^{\mathrm{CL}} - S_{i,n}, S_{j,n}^{\mathrm{CL}} - S_{j,n}] \\ &= \mathrm{cov}^{\mathcal{F}_n}[S_{i,n}, S_{j,n}] \\ &= \mathrm{var}^{\mathcal{F}_n}[S_{i,n}]\, \delta_{i,j} \end{aligned}$$

In particular, we have

$$\begin{aligned} E^{\mathcal{F}_n}[(R_i^{\mathrm{CL}} - R_i)^2] &= \mathrm{var}^{\mathcal{F}_n}[R_i^{\mathrm{CL}} - R_i] + \Big(E^{\mathcal{F}_n}[R_i^{\mathrm{CL}} - R_i]\Big)^2 \\ &= \mathrm{var}^{\mathcal{F}_n}[S_{i,n}] + S_{i,n-i}^2\left(\prod_{k=n-i+1}^{n} \varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{n} \varphi_k\right)^2. \end{aligned}$$

This proves (1).
We have

$$\begin{aligned} R_{(c)}^{\mathrm{CL}} - R_{(c)} &= \sum_{i=c-n}^{n} \left((S_{i,c-i}^{\mathrm{CL}} - S_{i,c-i-1}^{\mathrm{CL}}) - (S_{i,c-i} - S_{i,c-i-1})\right) \\ &= \sum_{i=c-n}^{n} (S_{i,c-i}^{\mathrm{CL}} - S_{i,c-i}) - \sum_{i=c-n}^{n} (S_{i,c-i-1}^{\mathrm{CL}} - S_{i,c-i-1}) \end{aligned}$$

and hence

$$\begin{aligned} & E^{\mathcal{F}_n}[R_{(c)}^{\mathrm{CL}} - R_{(c)}] \\ &= \sum_{i=c-n}^{n} E^{\mathcal{F}_n}[(S_{i,c-i}^{\mathrm{CL}} - S_{i,c-i})] - \sum_{i=c-n}^{n} E^{\mathcal{F}_n}[(S_{i,c-i-1}^{\mathrm{CL}} - S_{i,c-i-1})] \\ &= \sum_{i=c-n}^{n} S_{i,n-i}\left(\prod_{k=n-i+1}^{c-i} \varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{c-i} \varphi_k\right) - \sum_{i=c-n}^{n} S_{i,n-i-1}\left(\prod_{k=n-i+1}^{c-i-1} \varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{c-i-1} \varphi_k\right) \\ &= \sum_{i=c-n}^{n} S_{i,n-i}\left(\left(\prod_{k=n-i+1}^{c-i-1} \varphi_k^{\mathrm{CL}}\right)(\varphi_{c-i}^{\mathrm{CL}} - 1) - \left(\prod_{k=n-i+1}^{c-i-1} \varphi_k\right)(\varphi_{c-i} - 1)\right). \end{aligned}$$

Since the accident years are independent, we also have

$$\mathrm{var}^{\mathcal{F}_n}[R_{(c)}^{\mathrm{CL}} - R_{(c)}] = \mathrm{var}^{\mathcal{F}_n}\left[\sum_{i=c-n}^{n} \left(\left(S_{i,c-i}^{\mathrm{CL}} - S_{i,c-i-1}^{\mathrm{CL}}\right) - \left(S_{i,c-i} - S_{i,c-i-1}\right)\right)\right]$$

$$
\begin{aligned}
&= \operatorname{var}^{\mathcal{F}_n}\left[\sum_{i=c-n}^{n}\Big(S_{i,c-i}-S_{i,c-i-1}\Big)\right] \\
&= \sum_{i=c-n}^{n}\operatorname{var}^{\mathcal{F}_n}[S_{i,c-i}-S_{i,c-i-1}]
\end{aligned}
$$

as well as

$$
\operatorname{var}^{\mathcal{F}_n}[S_{i,c-i}-S_{i,c-i-1}] = \operatorname{var}^{\mathcal{F}_n}[S_{i,c-i-1}]\,(\varphi_{c-i}-1)^2 + S_{i,n-i}\left(\prod_{k=n-i+1}^{c-i-1}\varphi_k\right)\sigma_{c-i}^2
$$

and hence

$$
\operatorname{var}^{\mathcal{F}_n}[R_{(c)}^{\mathrm{CL}}-R_{(c)}] = \sum_{i=c-n}^{n}\left(\operatorname{var}^{\mathcal{F}_n}[S_{i,c-i-1}]\,(\varphi_{c-i}-1)^2 + S_{i,n-i}\left(\prod_{k=n-i+1}^{c-i-1}\varphi_k\right)\sigma_{c-i}^2\right).
$$

In particular, we have

$$
\begin{aligned}
&E^{\mathcal{F}_n}[(R_{(c)}^{\mathrm{CL}}-R_{(c)})^2] \\
&= \operatorname{var}^{\mathcal{F}_n}[R_{(c)}^{\mathrm{CL}}-R_{(c)}] + \Big(E^{\mathcal{F}_n}[R_{(c)}^{\mathrm{CL}}-R_{(c)}]\Big)^2 \\
&= \sum_{i=c-n}^{n}\left(\operatorname{var}^{\mathcal{F}_n}[S_{i,c-i-1}]\,(\varphi_{c-i}-1)^2 + S_{i,n-i}\left(\prod_{k=n-i+1}^{c-i-1}\varphi_k\right)\sigma_{c-i}^2\right) \\
&\quad + \left(\sum_{i=c-n}^{n} S_{i,n-i}\left(\left(\prod_{k=n-i+1}^{c-i-1}\varphi_k^{\mathrm{CL}}\right)(\varphi_{c-i}^{\mathrm{CL}}-1) - \left(\prod_{k=n-i+1}^{c-i-1}\varphi_k\right)(\varphi_{c-i}-1)\right)\right)^2.
\end{aligned}
$$

This proves (2).
We have $R^{\mathrm{CL}} - R = \sum_{i=1}^{n}(R_i^{\mathrm{CL}}-R_i)$ and hence

$$
\begin{aligned}
E^{\mathcal{F}_n}[R^{\mathrm{CL}}-R] &= \sum_{i=1}^{n} E^{\mathcal{F}_n}[R_i^{\mathrm{CL}}-R_i] \\
&= \sum_{i=1}^{n} S_{i,n-i}\left(\prod_{k=n-i+1}^{n}\varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{n}\varphi_k\right)
\end{aligned}
$$

From (1) we obtain

$$
\begin{aligned}
\operatorname{var}^{\mathcal{F}_n}[R^{\mathrm{CL}} - R] &= \sum_{i=1}^{n}\operatorname{var}^{\mathcal{F}_n}[R_i^{\mathrm{CL}}-R_i] \\
&= \sum_{i=1}^{n}\operatorname{var}^{\mathcal{F}_n}[S_{i,n}]\,.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
E^{\mathcal{F}_n}[(R^{\mathrm{CL}}-R)^2] &= \operatorname{var}^{\mathcal{F}_n}[R^{\mathrm{CL}}-R] + \Big(E^{\mathcal{F}_n}[R^{\mathrm{CL}}-R]\Big)^2 \\
&= \sum_{i=1}^{n}\operatorname{var}^{\mathcal{F}_n}[S_{i,n}] + \left(\sum_{i=1}^{n} S_{i,n-i}\left(\prod_{k=n-i+1}^{n}\varphi_k^{\mathrm{CL}} - \prod_{k=n-i+1}^{n}\varphi_k\right)\right)^2.
\end{aligned}
$$

This proves (3). □

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The Technical Provisions in Solvency II
What EU Insurers Could Do if They Had Schedule P

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Abstract: The goal of this paper is to demonstrate how publicly available data can be used to calculate the technical provisions in Solvency II. This is a purely hypothetical exercise, since the publicly available data is in America, and Solvency II applies to the European Union. Using American Schedule P data, this paper:

Develops "prior information" to be used in an empirical Bayesian loss reserving method.

Uses the Metropolis-Hastings algorithm to develop a posterior distribution of parameters for a Bayesian Analysis.

Develops a series of diagnostics to assess the applicability of the Bayesian model.

Uses the results to calculate the best estimate and the risk margin in accordance with the principles underlying Solvency II.

Develops an ongoing process to regularly compare projected results against experience.

The paper includes analyses of the Schedule P data for four American Insurers based on its methodology.

Keywords: Solvency II, reserving methods, reserve variability, uncertainty and ranges, Bayesian estimation

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1. INTRODUCTION

In 2009 the European Parliament passed a new act for regulating insurers known as Solvency II. Its objectives include:

- Increased focus on effective risk management, control, and governance,
- Market consistent valuation of assets and liabilities,
- Increased disclosure and transparency.

This act will become effective on October 31, 2012. Because of the growing international nature of the business of insurance, the development of the provisions in this act has been watched and debated by interested parties worldwide.

This paper focuses on calculating the "technical provisions" specified in this act[1]. The "technical provisions" refer to the insurer's liability for unpaid losses. Specifically:

- "The value of the technical provisions shall be equal to the sum of a best estimate and a risk margin."[2]

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[1]The provisions quoted below are stated in Section 2 of Chapter VI of the act, p 222, http://register.consilium.europa.eu/pdf/en/09/st03/st03643-re01.en09.pdf.

[2] Article 77

- "The best estimate shall correspond to the probability-weighted average of future cash flows, taking account of the time value of money using the relevant risk-free interest rate term structure."[3]
- "The risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance obligations over the lifetime thereof."[4]
- "Insurance undertakings shall segment their insurance obligations into homogeneous risk groups, and as a minimum by lines of business, when calculating the technical provisions."[5]

With regard to technical provisions, the act also requires insurers to have "processes and procedures in place to insure that best estimates, and the assumptions underlying the calculation of the best estimates, are regularly compared against experience.

When the comparison identifies systematic deviations between the experience and the best estimate, the insurer shall make appropriate adjustments to the actuarial methods and/or the assumptions being made."[6]

These provisions of the act implicitly, if not explicitly, call for a stochastic model of the loss development process. Details such as the particular models and the data being used are not specified.

In America, insurers are required to report very detailed data to regulators. Relevant to the topic of technical provisions is Schedule P of the National Association of Insurance Commissioners (NAIC) Annual Statement.[7] This data contains net premiums, along with paid and incurred loss triangles spanning a period of ten accident years. The data is organized into 36 specific lines of insurance such as Personal Auto, Commercial Auto, Homeowners, and Workers' Compensation. Note that all dollar amounts are in thousands.

This paper describes how to use data provided by the NAIC to develop a stochastic model for the loss development process. A feature of this model will be that it draws on the information provided by several insurers to provide "prior information" for use in the Bayesian estimation of the model parameters. The Bayesian methodology will also quantify the uncertainty in the parameters.

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[3] Article 77
[4] Article 77
[5] Article 80
[6] Article 83
[7] One can purchase an electronic copy of the Annual Statements for all American insurers for a nominal price from the NAIC. http://www.naic.org/store_financial_home.htm.

This paper will then show how to use this model to carry out the calculations required for the technical provisions of Solvency II. In watching parts of the debate that led to Solvency II, I have seen reasonable alternatives to its methodology. This paper will explore some of those alternatives.

The data in Schedule P is available to the public for all American insurers and thus the calculations described in this paper can be done by external interested parties. The intent of this paper is not to replace the more detailed analysis that insurers can do internally. Instead its intent is to do a credible analysis with publicly available data.

2. A STOCHASTIC MODEL OF THE LOSS DEVELOPMENT PROCESS

The stochastic model in this paper describes the random incremental paid loss, $X_{AY,Lag}$, for accident year AY, and settlement lag, Lag. The data used to fit the model will consist of a loss triangle of ten accident years of incremental paid net losses and the net earned premium for each accident year. The model can be used to predict the distribution of losses paid in future settlement lags through the tenth year. It can also be used to predict the distribution of sums of losses for any given combination of future settlement lags in the given accident years.

For a given accident year, AY, and settlement lag, Lag, the expected loss is equal to

$$\mu_{AY,Lag} = Premium_{AY} \cdot ELR_{AY} \cdot Dev_{Lag} \cdot t^{AY+Lag-1}, \tag{1}$$

where:

- $Premium_{AY}$ is the accident year premium obtained from the data,
- ELR_{AY} is a parameter representing the expected loss ratio for the accident year,
- Dev_{Lag} is a parameter representing the incremental paid loss development factor for the settlement lag,
- t is a parameter representing the calendar year trend for the claim frequency.

The claim severity, Z, in this model is a random variable with a gamma distribution,

$$f(z) = \frac{z^{\alpha-1} \cdot e^{-z/\theta}}{\Gamma(\alpha) \cdot \theta^{\alpha}}. \tag{2}$$

The claim severity distribution will vary by settlement lag with its mean given by the parameter $\tau_{Lag} = \alpha \cdot \theta_{Lag}$ and a fixed shape parameter, $\alpha = 1/2$. In accordance with the general observation that claim severity increases with the settlement lag, this model sets

$$\tau_{Lag} = sev \cdot \left(1 - \left(1 - \frac{Lag}{10}\right)^3\right) \text{ for } Lag = 1,2 \ldots,10. \tag{3}$$

As was done in Meyers (2007a), the claim count, N, in this model has a distribution with its mean given by $\lambda_{AY,Lag} = \mu_{AY,Lag}/\tau_{Lag}$, and its variance given by

$$Var\left[N\right] = \lambda_{AY,Lag} + c \cdot \lambda^2_{AY,Lag}\,. \tag{4}$$

The model, as described by Equations 1-4, depends upon the unknown parameters

- ELR_{AY}, for $AY = 1,2,\ldots,10$.
- Dev_{Lag}, for $Lag = 1,2,\ldots,10$.
- *sev* (the claim severity for the 10th settlement lag).
- *t* (the calendar year frequency trend factor).
- *c* (the contagion parameter).

My selection of the fixed parameters in the model (i.e. the τ_{Lag} parameters used to describe variation by settlement lag and the α parameter in the gamma claim severity distribution) was based on a combination of prior experience and sensitivity testing.

The expected loss in each (*AY*,*Lag*) cell is given by Equation (1). The variance of the loss in each cell is given by:

$$Var\left[X_{AY,Lag}\right] = \mu_{AY,Lag} \cdot \tau_{Lag} \cdot \left(1 + 1/\alpha\right) + c \cdot \mu^2_{AY,Lag}\,. \tag{5}$$

For each (*AY*,*Lag*) cell, the model will be approximated by a Tweedie distribution with the same mean and variance[8]. The mean and variance of the Tweedie distribution are given by μ and $\phi \cdot \mu^p$, respectively, with $p = (\alpha+2)/(\alpha+1)$. Using the value of p that is implied by the value of α and solving for the ϕ that forces the variances to be equal yields:

$$\phi_{AY,Lag} = \frac{\mu^{1-p}_{AY,Lag} \cdot \tau_{Lag}}{2-p} + c \cdot \mu^{2-p}_{AY,Lag}\,. \tag{6}$$

Note that the approximation is exact if N has a Poisson distribution with (implied) $c = 0$.

3. BAYESIAN ESTIMATION OF THE MODEL PARAMETERS

It is generally regarded as good statistical practice to use models with as few parameters as possible. As illustrated by Meyers (2008), too many parameters can lead to overfitting problems when estimating the parameters by maximum likelihood. Attempts such as Clark (2006) and Meyers (2009) to formulate models for loss reserving, with a small number of parameters have not found

[8] See Meyers (2009) and/or Smyth and Jørgensen (2002) for an introduction to the Tweedie distribution.

general use in the actuarial community.[9]

In the same paper, Meyers (2008) suggests, by way of example, that a Bayesian analysis can overcome the problems associated with overfitting. The paper recommends using a mixture of models over the posterior distribution of parameters. This paper takes a similar Bayesian approach.

Let *Parm* denote the set of unknown parameters $\left\{\left\{ELR_{AY}\right\},\left\{Dev_{Lag}\right\},sev,t,c\right\}$. Let $\mathbf{X}=\left\{x_{AY,Lag},AY=1,\ldots,10,Lag=1,\ldots,11-AY\right\}$ denote the observed incremental paid losses from a 10x10 Schedule P loss development triangle. According to Bayes' Theorem:

$$f\left(Parm|\,\mathbf{X}\right)\propto \ell\left(\mathbf{X}\,|\,Parm\right)\cdot f\left(Parm\right) \tag{7}$$

where:

- $f\left(Parm|\,\mathbf{X}\right)$ is the posterior distribution of *Parm*.
- $\ell\left(\mathbf{X}\,|\,Parm\right)$ is the likelihood function of **X.**
- $f\left(Parm\right)$ is the prior distribution of Parm.

The likelihood function is given by

$$\ell\left(\mathbf{X}\,|\,Parm\right)=\prod_{AY=1}^{10}\prod_{Lag=1}^{11-AY} dtweedie\left(x_{AY,Lag}\,|\,p,\mu_{AY,Lag},\phi_{AY,Lag}\right), \tag{8}$$

where:

- *dtweedie* is the probability density function for the Tweedie distribution.
- p is the power parameter. $p = (\alpha+2)/(\alpha+1) = 1.67$.
- $\mu_{AY,Lag}$ and $\phi_{AY,Lag}$ are calculated from *Parm* and Equations 1 and 6.

Following Meyers (2009) this paper uses the Metropolis-Hastings algorithm to generate a sample of 500 parameter sets that represent the posterior distribution. Appendix A describes how that algorithm was implemented in this paper. That appendix also provides the code (written in the R programming language) used for this paper.

This paper uses a gamma distribution (Equation 2) to represent its prior distributions. Table 1 gives the α and θ parameters of the prior distribution for each parameter in *Parm*.

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[9] I intend no disparagement here. I consider Clark's paper to be a very good introduction to the use of maximum likelihood methods for fitting loss reserve models.

Table 1

			Implied	
Parameter	α	θ	Mean	Std. Dev.
sev	1.3676	136.2478	186.3386	159.34
t	1290.2307	0.0008	0.9931	0.0276
c	0.0740	0.1391	0.0103	0.0379
ELR_1	29.8506	0.0237	0.7073	0.1295
ELR_2	33.8347	0.0227	0.7674	0.1319
ELR_3	35.3338	0.0214	0.7545	0.1269
ELR_4	24.4908	0.0285	0.6981	0.1411
ELR_5	28.6618	0.0254	0.7272	0.1358
ELR_6	25.6341	0.0304	0.7790	0.1539
ELR_7	16.8043	0.0501	0.8417	0.2053
ELR_8	14.3680	0.0602	0.8650	0.2282
ELR_9	9.3053	0.1017	0.9465	0.3103
ELR_{10}	6.3667	0.1609	1.0246	0.4061
Dev_1	15.8100	0.0135	0.2137	0.0537
Dev_2	42.8538	0.0059	0.2517	0.0385
Dev_3	56.4944	0.0036	0.2028	0.0270
Dev_4	30.4528	0.0046	0.1403	0.0254
Dev_5	10.2309	0.0085	0.0870	0.0272
Dev_6	5.8094	0.0083	0.0480	0.0199
Dev_7	3.6954	0.0068	0.0250	0.0130
Dev_8	2.3934	0.0057	0.0135	0.0087
Dev_9	1.3559	0.0066	0.0090	0.0077
Dev_{10}	0.4552	0.0200	0.0091	0.0135

These prior distributions were obtained by the following steps.

1. Obtain the maximum likelihood estimates (MLEs) of the parameters for 50 large active insurers using Schedule P data.
2. Using the MLEs obtained in Step 1 as prior means, run the Metropolis-Hastings algorithm to get a sample of 100 parameter sets.
3. Using the 5,000 parameter sets obtained from the Steps 1 and 2 above, fit the gamma distributions by matching the mean and standard deviation of the gamma distribution with the sample mean and standard deviation for each parameter in the set.

Loss reserving is considered by many to be an art that depends on the data and actuarial judgment. The experience gained from many reserving analyses often forms the basis of such judgments. These steps taken to derive the prior distribution are an attempt to capture the experience needed for such judgments in a repeatable and transparent way. The Bayesian approach taken by this paper merges the data with the "judgment" supplied by the prior distribution.

For a given insurer, the iterations generated by the Metropolis-Hastings algorithm can be thought of as a sample of equally likely parameter sets describing the posterior distribution of their loss development process. Denote the n^{th} parameter set by:

$$Parm_n = \left\{ sev_n, t_n, c_n, \left\{ ELR_{n,AY} \right\}, \left\{ Dev_{n,Lag} \right\} \right\}. \tag{9}$$

Each $Parm_n$ can be used to construct "statistics of interest" that can be either used to describe parameter risk, or be averaged to get an overall expected value. The sections below provide several examples of statistics of interest that involve model diagnostics, prediction intervals, and items in a financial statement, such as a best estimate and a risk margin.

4. EXAMPLES WITH FOUR ILLUSTRATIVE INSURERS

This paper has illustrative analyses with data from four real insurers. The paid loss triangles were taken from the 1997 Schedule P each insurer reported to the NAIC for the commercial auto line of insurance. The data are reported in the form of cumulative paid losses for each accident year. Incremental paid losses were obtained by taking the difference of the cumulative paid losses by settlement lag. Occasionally, the cumulative paid losses decreased with subsequent settlement lag. My understanding of the reporting instructions is that this should not happen, but when it did happen, I removed the negative incremental paid loss from the data, and fit the models without that data point. The data used for fitting the model consisted of the earned net premium, the incremental paid losses indexed by accident year and settlement lag. These data are tabulated in Appendix B. Table 2 gives an indication of the size of each insurer.

Table 2

Insurer	1997 Net Premium
1	73,359
2	24,030
3	99,940
4	241,228

Before selecting the particular insurers to put in this paper, I fit the model to the data from several insurers. I selected these insurers to illustrate the variety of stories that these kinds of data can tell. I would discourage any attempts to draw conclusions about the Commercial Auto line of business or about other insurers not analyzed in this paper.

Let us start by looking at the variability of each parameter in the model. Exhibits 1-3 plot histograms of the s*ev*, *t* and *c* parameters. The top of each exhibit has a histogram of a sample of parameters taken from the prior distribution. This shows how much of the initial uncertainty in the parameters is reduced by each insurer's data. Here are some casual observations about the *sev*, *t,* and *c* parameters

The width of the histograms indicates uncertainty in the parameters. An inspection of the exhibits indicates that there is no apparent relationship between the parameter uncertainty and the size of the insurer.

Exhibit 2 confirms a general industry trend of a slight decrease in claim frequency over time for commercial auto. Given that the trend factor of 1.00 is close to the center of the histograms, one might be tempted to drop the trend parameter but, in light of the industry trend, I chose to keep it in.

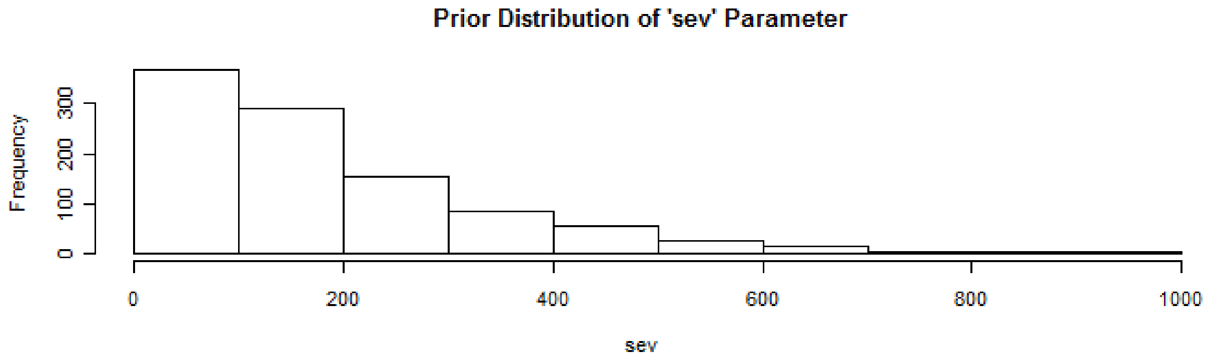
As indicated in Meyers (2007a), a positive c parameter indicates that there is a random external factor that affects all claims at once. The c parameter is the coefficient of variation squared of the external factor. For insurers 2 and 3, the minimal size of the c parameter indicates that the external factor is something usual, such as changing inflation rates. The c parameter for Insurer 4 is enormous. Something is systematically affecting large blocks of claims.

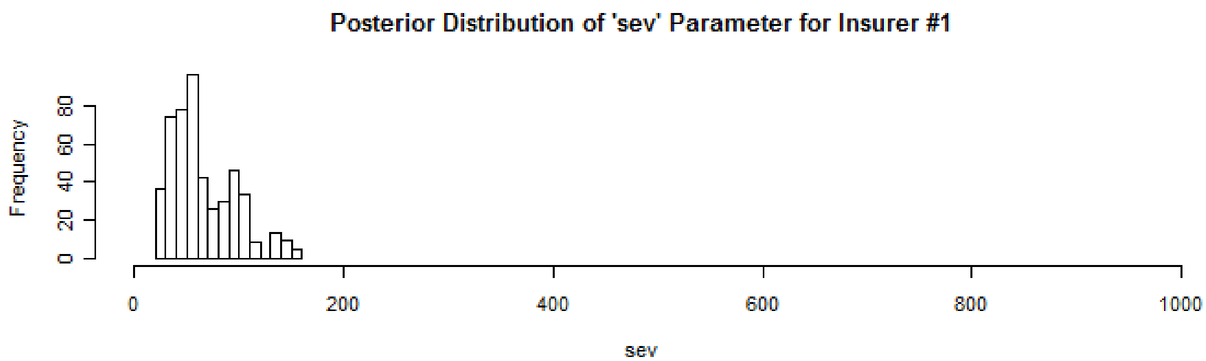
Exhibit 4 shows the $\{ELR\}$ and $\{Dev\}$ parameters expressed as paths over time for both the prior and posterior distributions. One general observation is that the uncertainty in the $\{ELR\}$ parameters decreases as we gain information over time. In other words, we have better information about the loss ratio for earlier years.

It might seem natural to define the "parameter estimates" as the mean of the parameter sets $Parm_n$. But the analyses below do not make any use of such a parameter estimate. Instead they create "statistics of interest" as functions of each parameter set. They then combine them by either:

1. taking an average "statistic of interest" over all the $Parm_n$s;
2. plotting related statistics of interest; or
3. simulating predicted losses derived from a random selection of $Parm_n$s.

Exhibit 1a


Prior Distribution of 'sev' Parameter
Frequency
0
100
200
300
0
200
400
600
800
1000
sev


Posterior Distribution of 'sev' Parameter for Insurer #1
Frequency
0
20
40
60
80
0
200
400
600
800
1000
sev

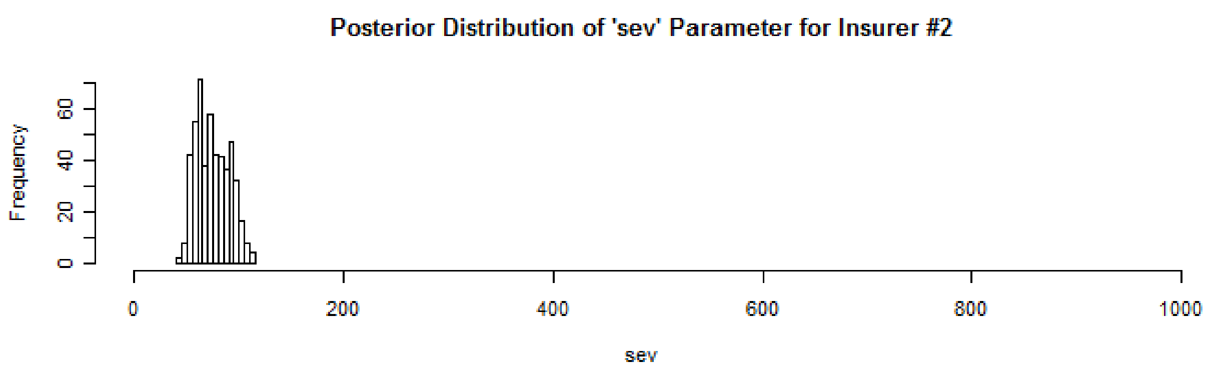

Posterior Distribution of 'sev' Parameter for Insurer #2
Frequency
0
20
40
60
0
200
400
600
800
1000
sev

Exhibit 1b

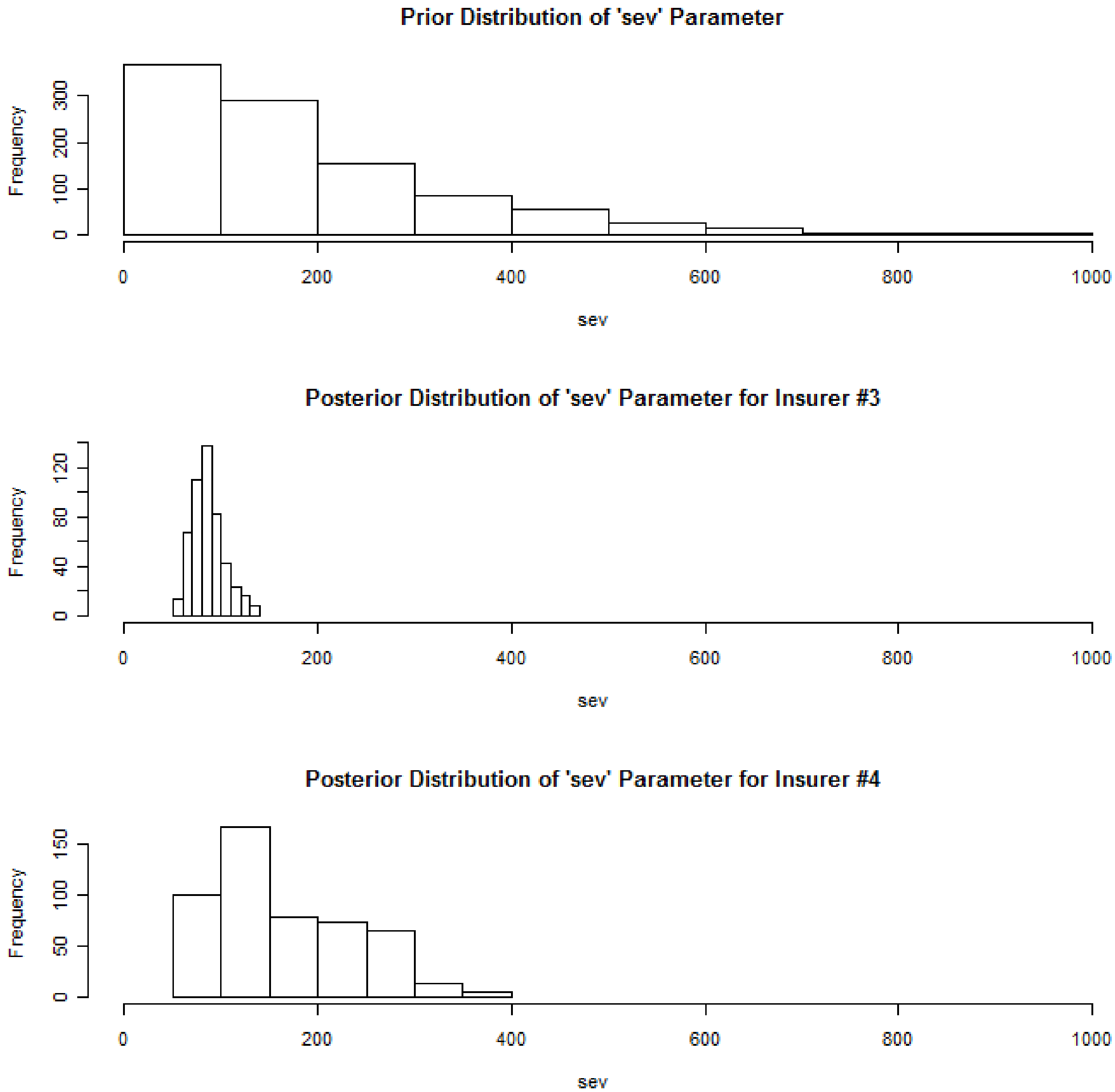
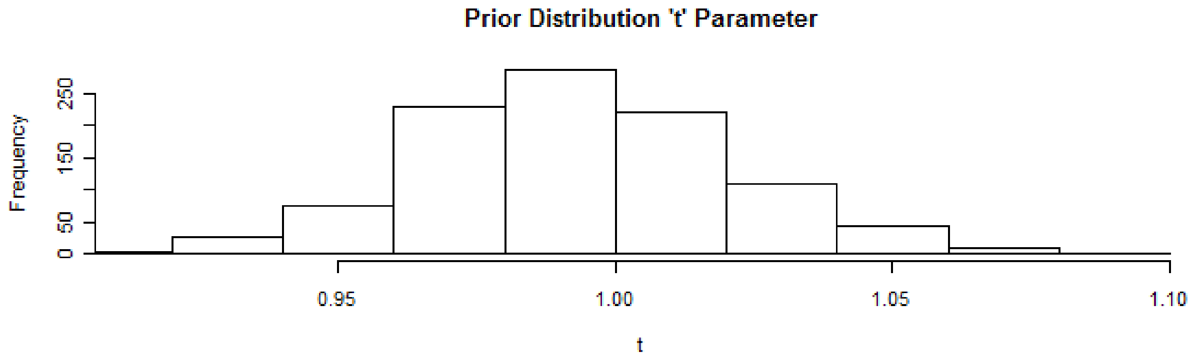
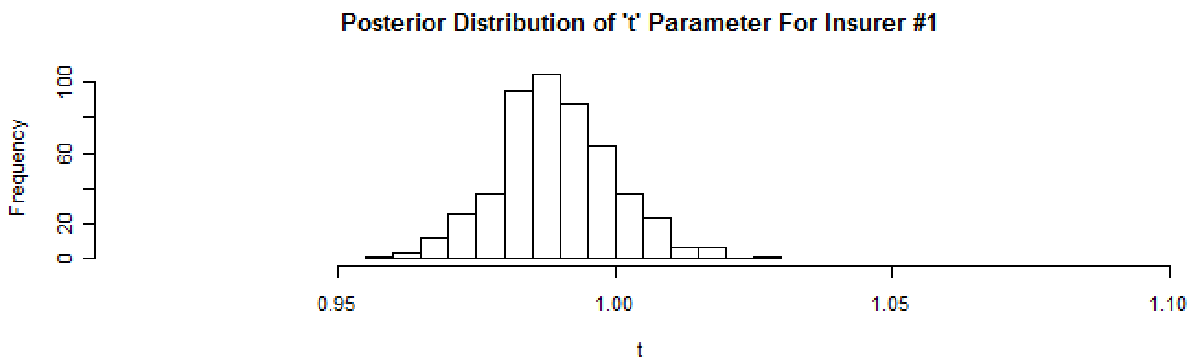
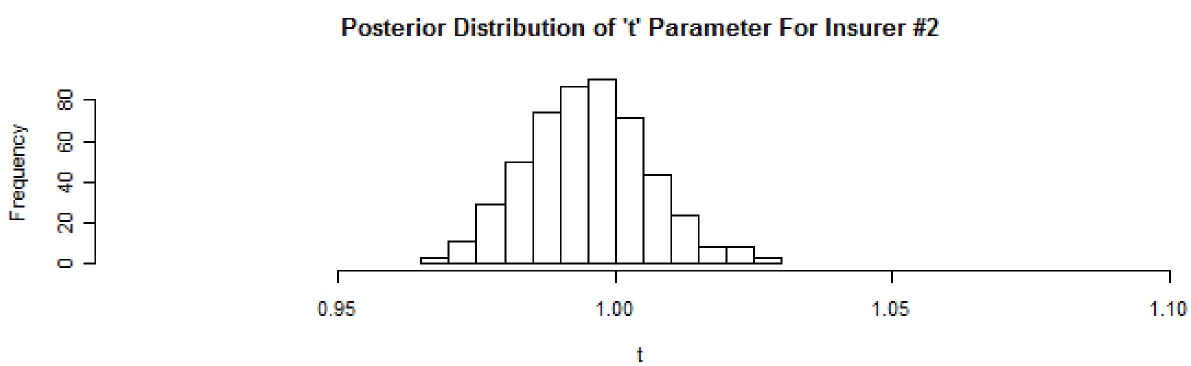

Prior Distribution of 'sev' Parameter
Frequency
0 100 200 300
0 200 400 600 800 1000
sev
Posterior Distribution of 'sev' Parameter for Insurer #3
Frequency
0 40 80 120
0 200 400 600 800 1000
sev
Posterior Distribution of 'sev' Parameter for Insurer #4
Frequency
0 50 100 150
0 200 400 600 800 1000
sev

Exhibit 2a


Prior Distribution 't' Parameter
Frequency
0
50
150
250
0.95
1.00
1.05
1.10
t


Posterior Distribution of 't' Parameter For Insurer #1
Frequency
0
20
60
100
0.95
1.00
1.05
1.10
t


Posterior Distribution of 't' Parameter For Insurer #2
Frequency
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20
40
60
80
0.95
1.00
1.05
1.10
t

Exhibit 2b

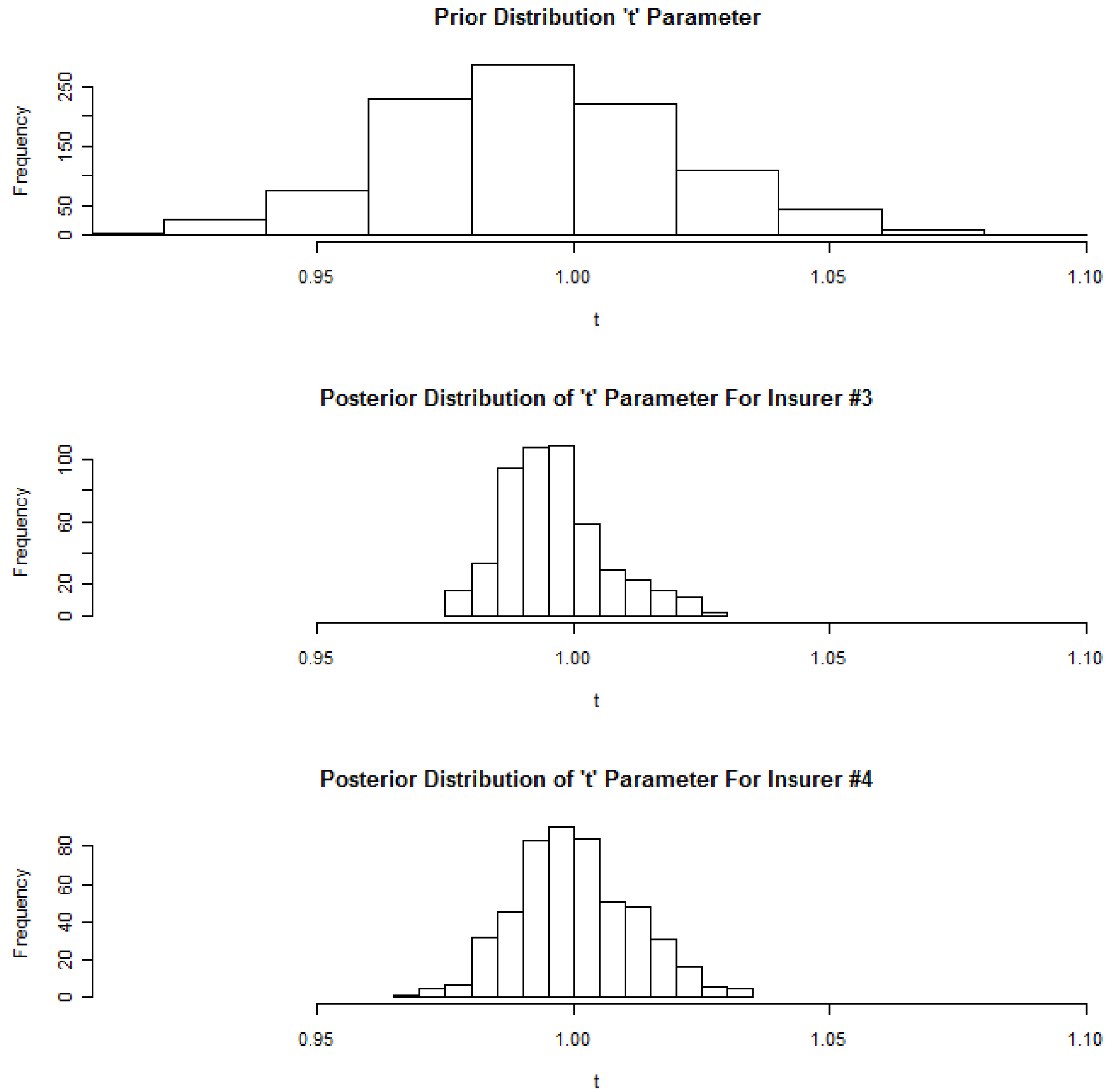

Prior Distribution 't' Parameter
Frequency
0
50
150
250
0.95
1.00
1.05
1.10
t
Posterior Distribution of 't' Parameter For Insurer #3
Frequency
0
20
60
100
0.95
1.00
1.05
1.10
t
Posterior Distribution of 't' Parameter For Insurer #4
Frequency
0
20
40
60
80
0.95
1.00
1.05
1.10
t

Exhibit 3a

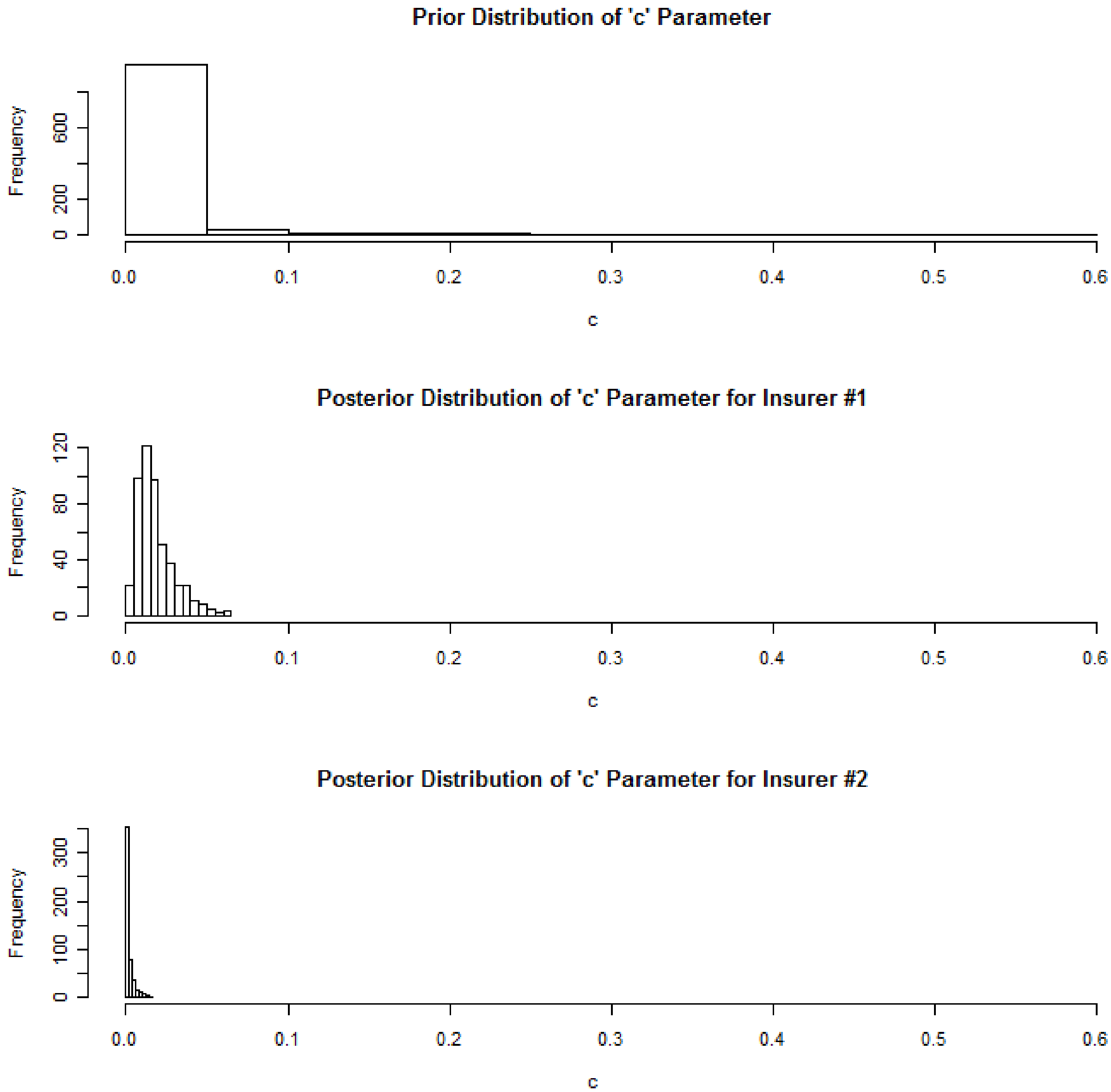

Prior Distribution of 'c' Parameter
Frequency
0
200
600
0.0
0.1
0.2
0.3
0.4
0.5
0.6
c
Posterior Distribution of 'c' Parameter for Insurer #1
Frequency
0
40
80
120
0.0
0.1
0.2
0.3
0.4
0.5
0.6
c
Posterior Distribution of 'c' Parameter for Insurer #2
Frequency
0
100
200
300
0.0
0.1
0.2
0.3
0.4
0.5
0.6
c

Exhibit 3b

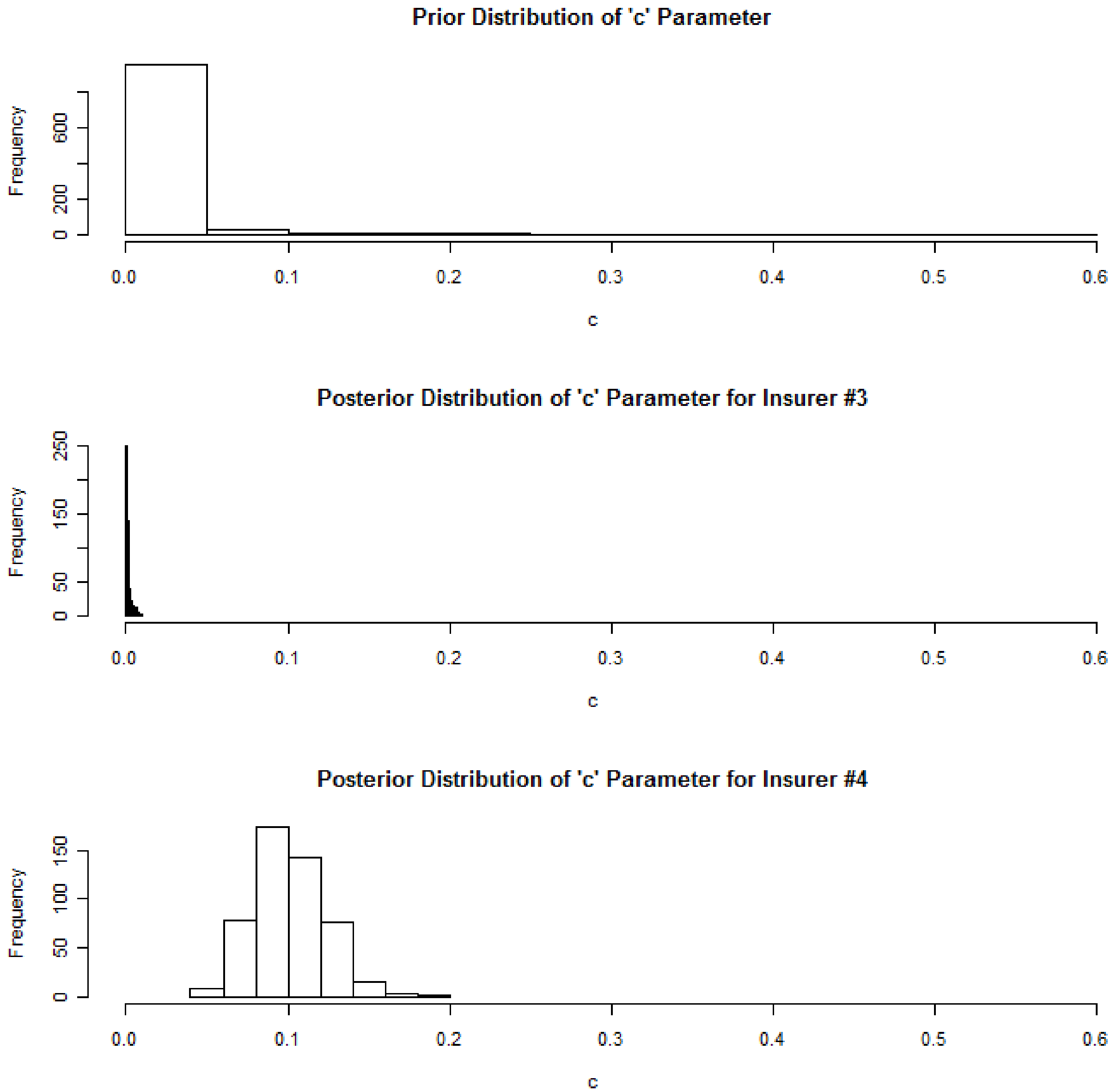
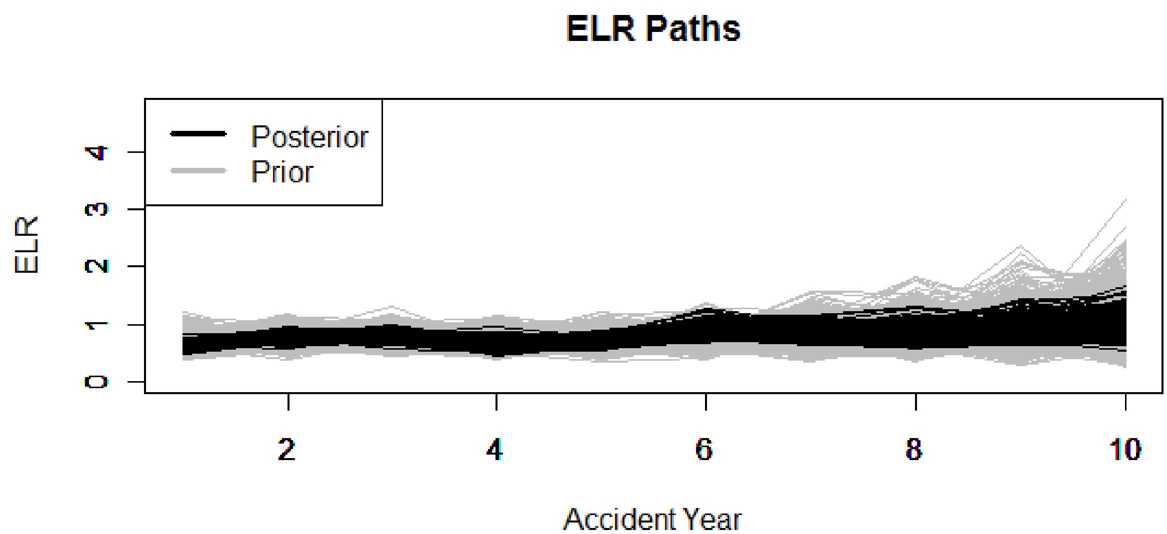

Prior Distribution of 'c' Parameter
Frequency
0
200
600
0.0
0.1
0.2
0.3
0.4
0.5
0.6
c
Posterior Distribution of 'c' Parameter for Insurer #3
Frequency
0
50
150
250
0.0
0.1
0.2
0.3
0.4
0.5
0.6
c
Posterior Distribution of 'c' Parameter for Insurer #4
Frequency
0
50
100
150
0.0
0.1
0.2
0.3
0.4
0.5
0.6
c

Exhibit 4a
ELR and Dev Paths for Insurer #1



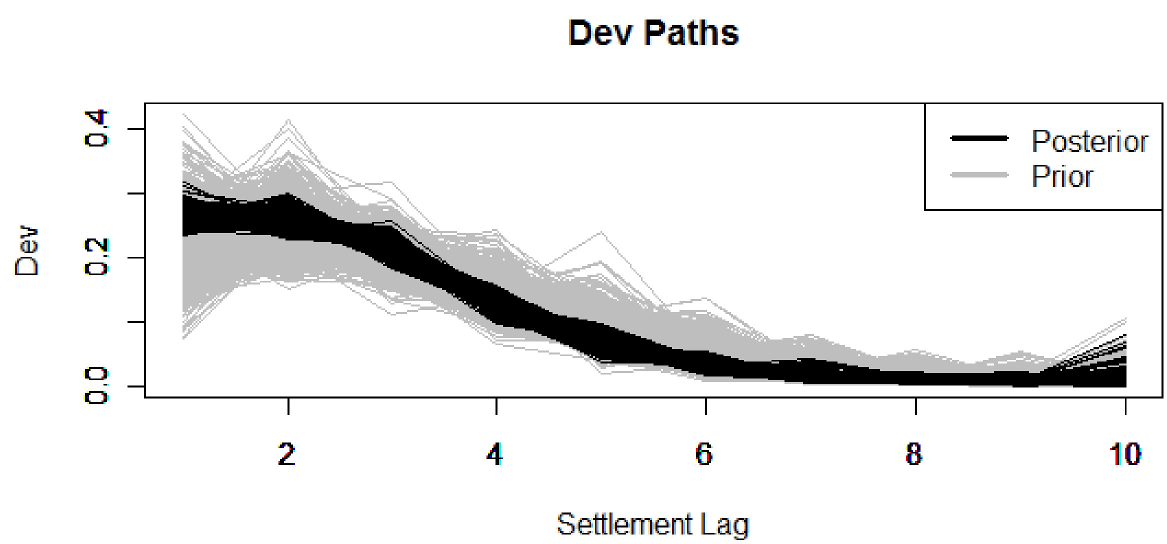
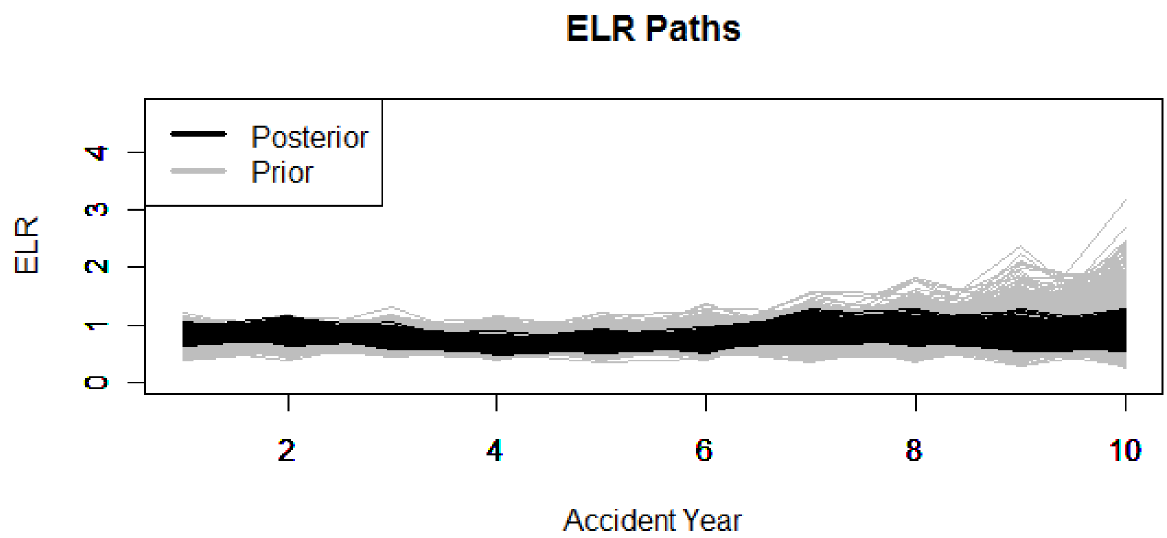


Exhibit 4b
ELR and Dev Paths for Insurer #2



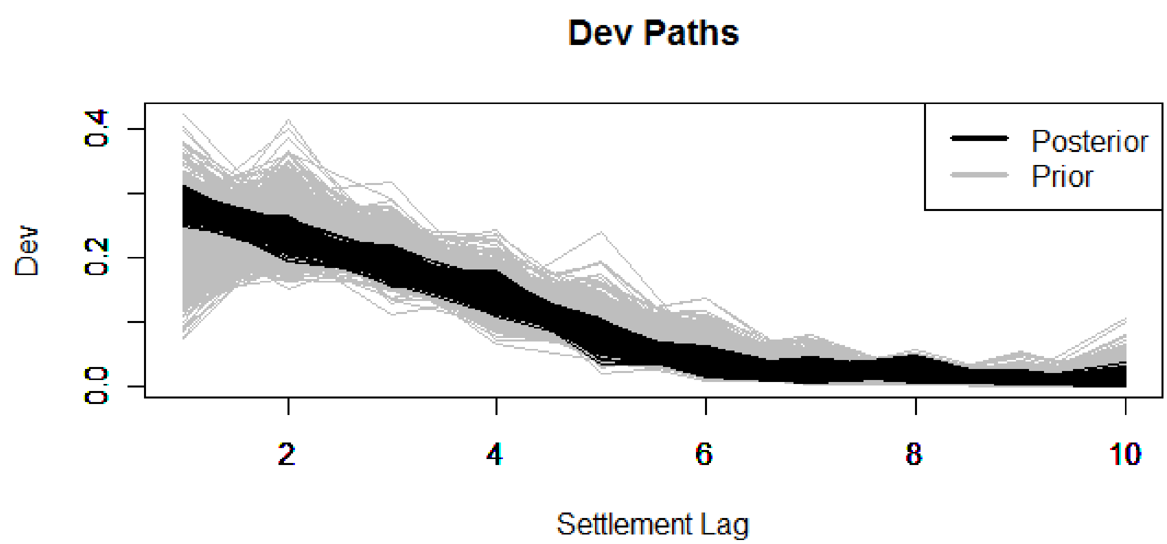
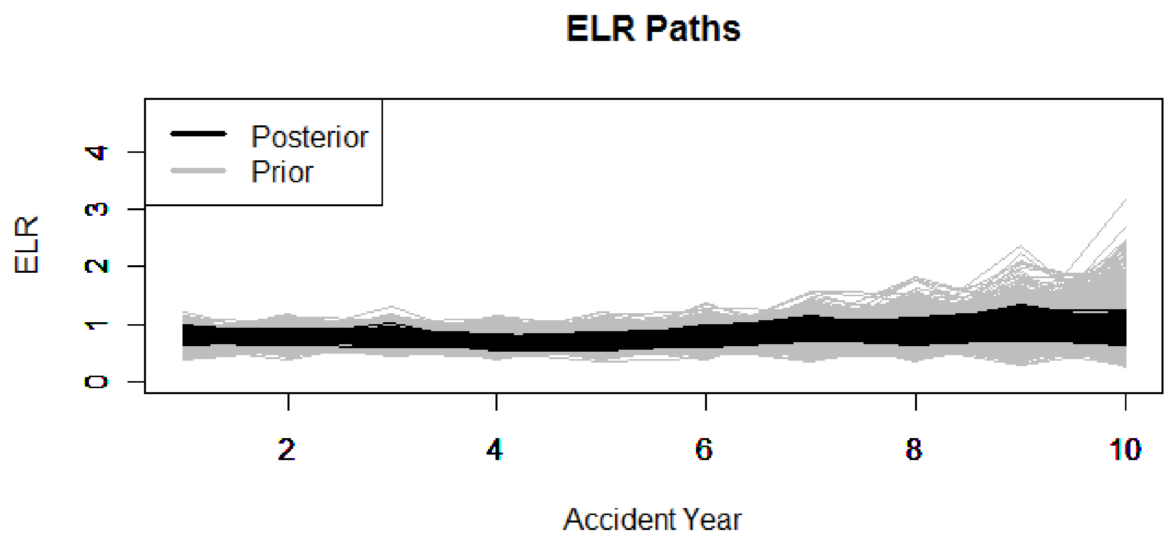


Exhibit 4c
ELR and Dev Paths for Insurer #3



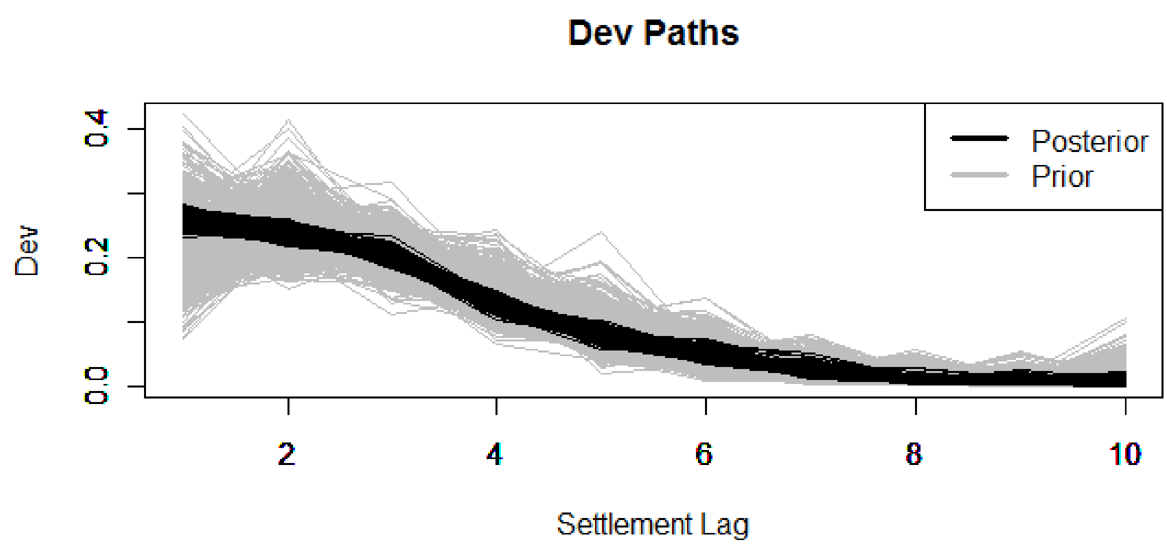
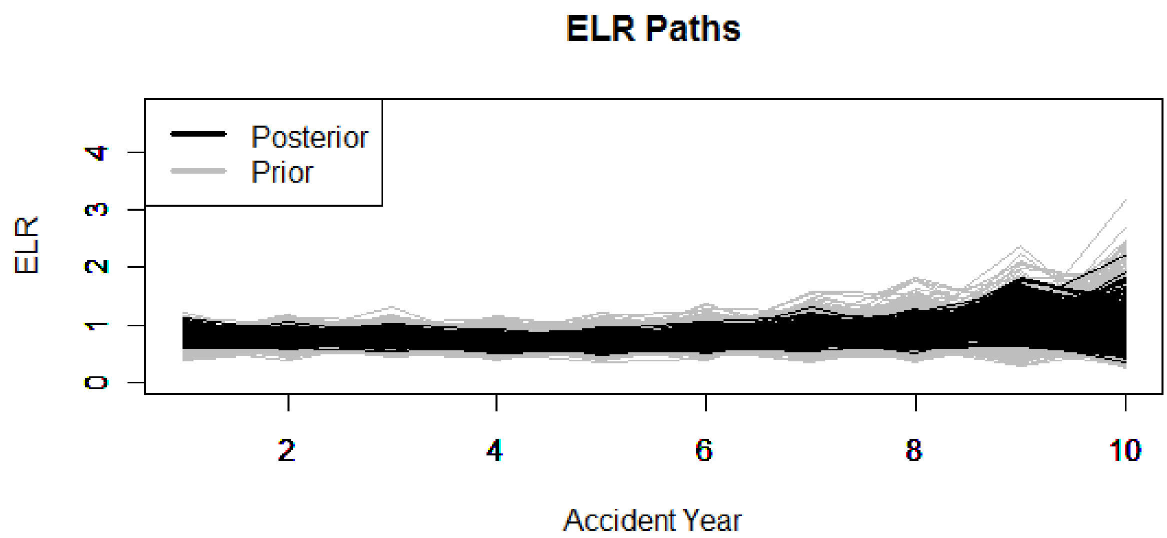
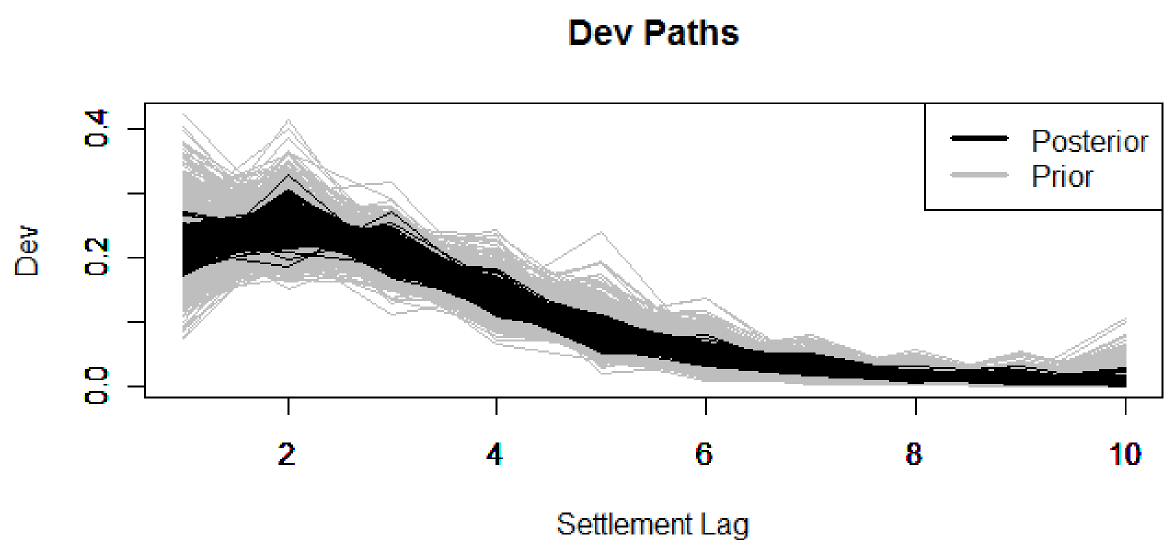


Exhibit 4d
ELR and Dev Paths for Insurer #3





5. MODEL DIAGNOSTICS

The model specified in Sections 2 and 3 predicts that the losses in each (*AY, Lag*) cell are a mixture of 500 Tweedie distributions. For a given value *x* in an (*AY*, *Lag*) cell, the cumulative probability is given by:

$$F_{AY,Lag}(x) = \frac{1}{500}\sum_{n=1}^{500} ptweedie\left(x \mid p, \mu_{n,AY,Lag}, \phi_{n,AY,Lag}\right), \tag{10}$$

and the mean loss for each (*AY*, *Lag*) cell is given by:

$$\mu_{AY,Lag} = \frac{1}{500}\sum_{n=1}^{500} \mu_{n,AY,Lag}, \tag{11}$$

where $\mu_{n,AY,Lag}$ and $\phi_{n,AY,Lag}$ are given by Equations 1 and 6 for each $Parm_n$, and *ptweedie* is the cumulative distribution function for the Tweedie distribution.

Denote the cumulative probabilities of each observed data point $x_{AY,Lag}$ by $p_{AY,Lag} = F_{AY,Lag}(x_{AY,Lag})$. Both the $\mu_{AY,Lag}$s and the $p_{AY,Lag}$s are given in Appendix B. Table 3 shows that the sum of the actual losses and the predicted losses are in excellent agreement.

Table 3

Insurer	Actual $\sum_{AY=1}^{10}\sum_{Lag=11-AY}^{10} x_{AY,Lag}$	Expected $\sum_{AY=1}^{10}\sum_{Lag=11-AY}^{10} \mu_{AY,Lag}$	Ratio $\frac{\text{Actual}}{\text{Expected}}$
1	269,804	269,916	0.9996
2	114,873	114,202	1.0059
3	394,629	394,854	0.9994
4	1,793,604	1,822,626	0.9841

For a well-fitting model one should expect that the collection of probabilities $\{p_{AY,Lag}\}$ will be uniformly distributed on the interval from zero to one. Following Meyers (2007b) this can be checked graphically with P-P plots. These plots compare the sorted probabilities, $\{p_{AY,Lag}\}$, with the expected probabilities. If the sorted probabilities are indeed uniform, the points in these plots will lie on a 45° line.

Exhibits 5a-5d provide P-P plots for each of the four insurers. One should expect random variation from the 45° line, and so the P-P plots also include confidence bands at the 99% and the 95% level based on the Kolmogorov-Smirnov test.

If the probabilities, $\{p_{AY,Lag}\}$, are truly random, one should also expect these probabilities to be independent of accident year, settlement lag, and calendar year (i.e., $AY+Lag-1$). Exhibits 5a-5d also contain plots of the probabilities against these variables. These plots are analogous to those described by Barnett and Zehnwirth (2000).

Here are some casual observations about the diagnostics.

- The P-P plots for all four insurers lie within the 99% confidence bands. The plots for Insurers 1, 2 and 3 all lie within the 95% confidence band, although the plot for Insurer 1 is just barely inside that band. The plot for Insurer 4 lies outside the 95% band.
- For Insurer 1, the set $\{p_{AY,Lag}\}$ for the first two accident years appears to be less spread out than expected.
- For Insurer 3, the small amount of overlap in the $p_{AY,Lag}$s in the later calendar years shows evidence of instability in the calendar year trend.
- For Insurer 4, the clearly nonrandom pattern in the calendar year plot leads to rather strange-looking patterns in the accident year and settlement lag plots.

In spite of the excellent agreement between the sum of the actual and the expected losses as identified in Table 3, the statistical diagnostics identify some potential problems with the model fits.

Exhibit 5a
Diagnostic Plots for Insurer #1

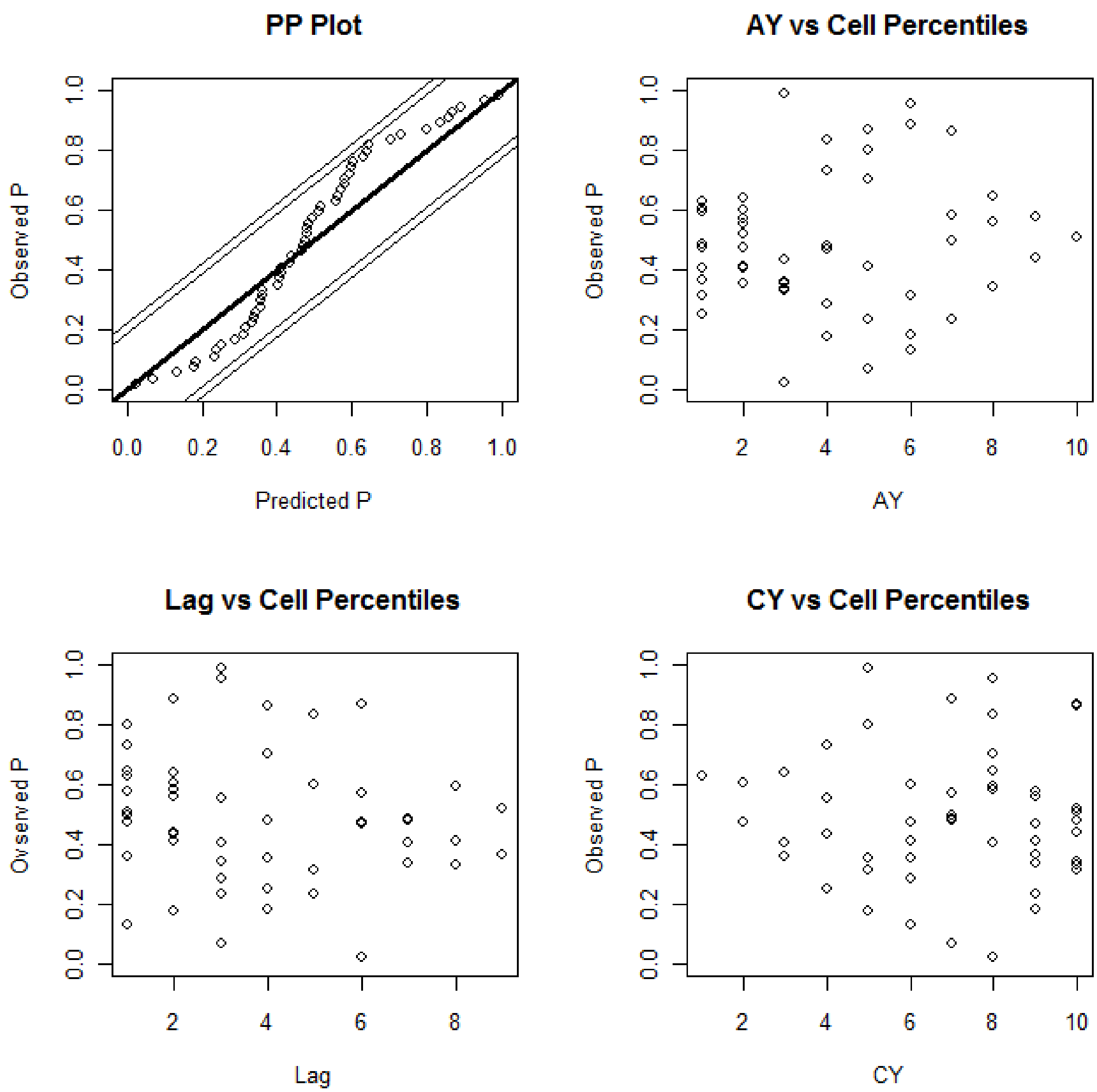


Exhibit 5b
Diagnostic Plots for Insurer #2

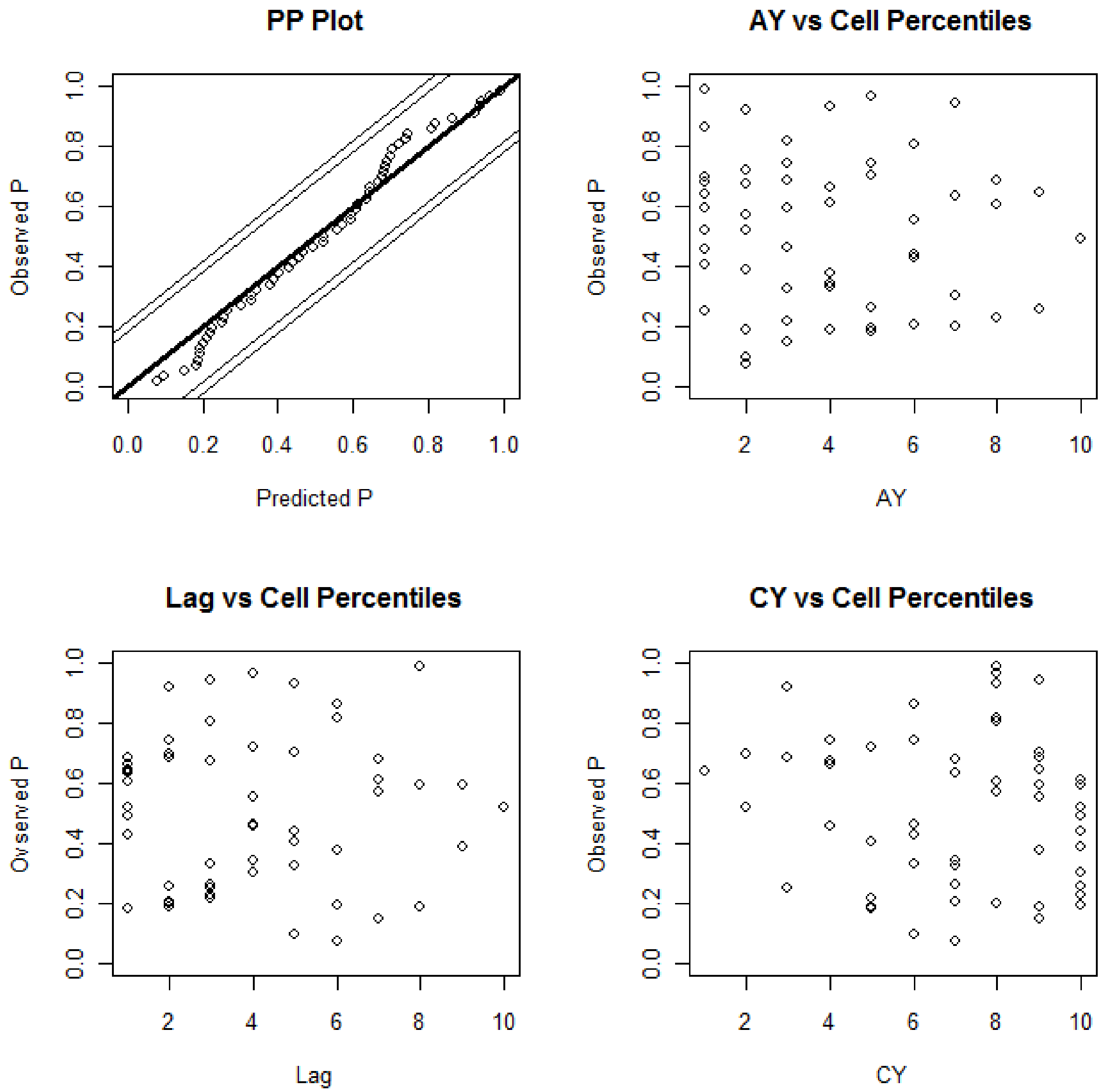


Exhibit 5c
Diagnostic Plots for Insurer #3

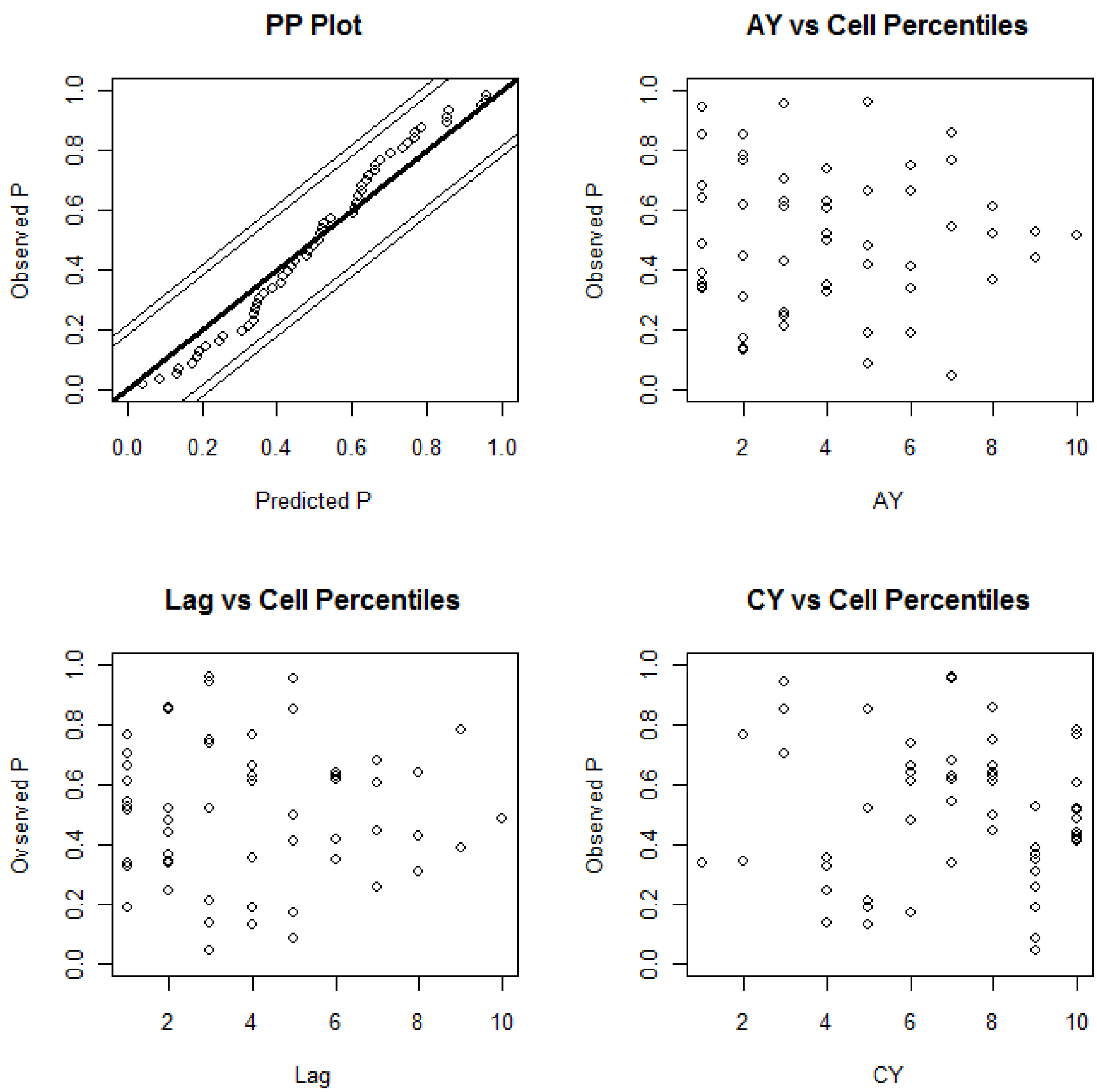
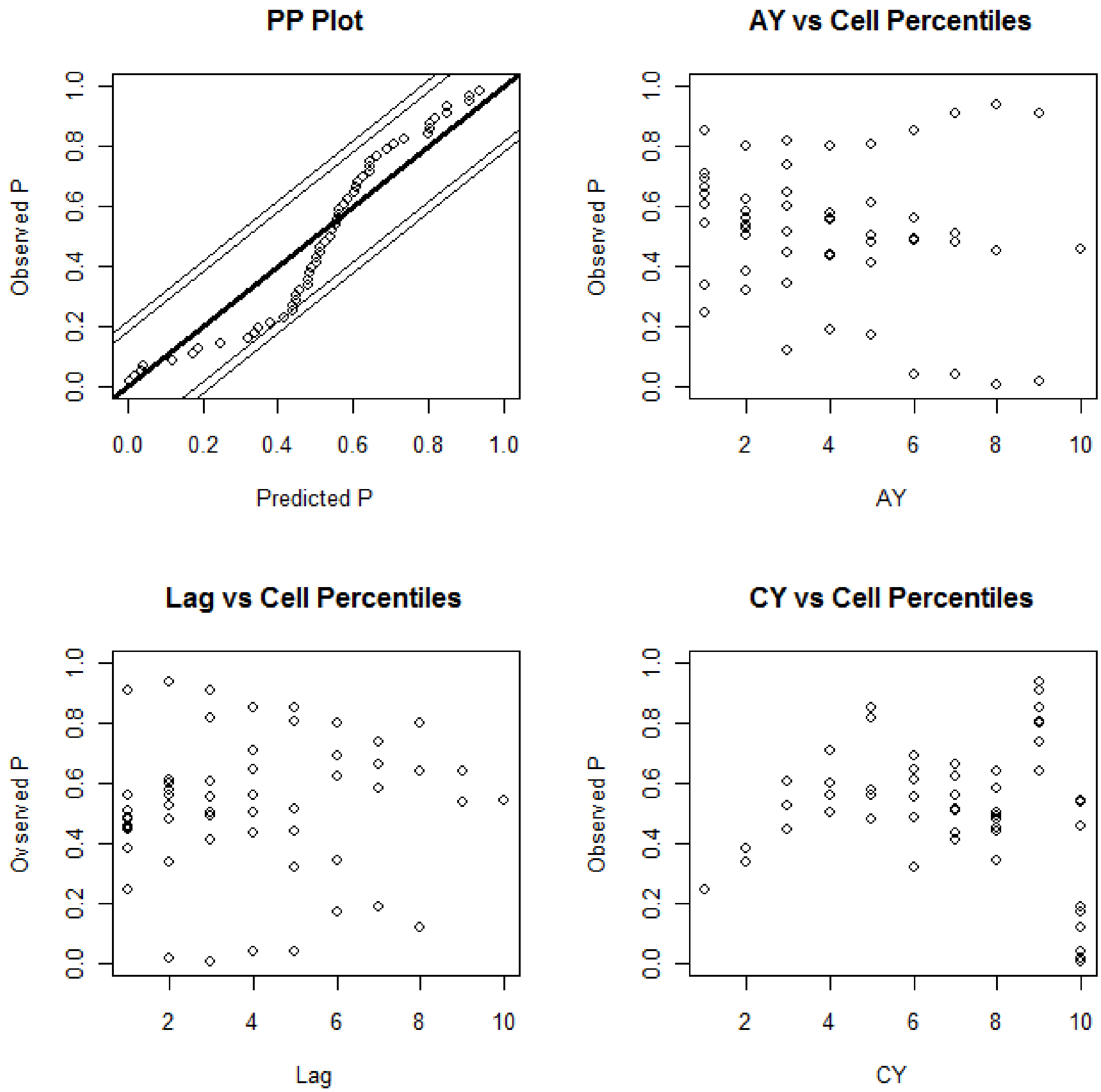


Exhibit 5d
Diagnostic Plots for Insurer #4



6. RETROSPECTIVE TESTS

As stated in the introduction, Solvency II requires insurers to have "processes and procedures in place to insure that best estimates, and the assumptions underlying the calculation of the best estimates, are regularly compared against experience." This section shows how to use the model to predict the distribution of paid loss outcomes for the next calendar year. Observing the next calendar year's total paid loss, $\sum_{AY=2}^{10} x_{AY,12-AY}$, one can check to see if the cumulative probability of that sum, as determined by its predictive distribution, lies within a normal range, say 0.05 to 0.95.

One way to determine this predictive distribution is to take a large sample, say 10,000 or so, of random *X*s from the following simulation algorithm.

Simulation Algorithm 1

1. Select a random parameter set from the list, $\mathsf{P}_n = \left\{ sev_n, t_n, c_n, \left\{ ELR_{n,AY} \right\}, \left\{ Dev_{n,Lag} \right\} \right\}$.
2. For each (*AY,Lag*) cell in next calendar year (*AY* = 2,…,10, *Lag* = 12 – *AY*):
    a. Calculate $\mu_{AY,Lag}$ from Equation 1.
    b. Calculate $\phi_{AY,Lag}$ from Equation 6.
    c. Select a random loss $X_{AY,Lag}$ from a Tweedie distribution with parameters p = 1.67, $\mu_{AY,Lag}$, and $\phi_{AY,Lag}$.
3. Set $\mathrm{X} = \sum_{AY=2}^{10} \mathrm{X}_{AY,12-AY}$

Following Meyers (2009), this paper uses the fast Fourier transform (FFT) to calculate the predictive distributions. It is faster and more numerically precise, but admittedly harder to implement. The R code for doing this is included in Appendix A.

When comparing the predictive distributions of this paper with predictive distributions derived from formulas in other papers, e.g., Mack (1993), one should be careful to distinguish between the predictive distribution of estimates, $\left\{ \sum_{AT=2}^{10} \mu_{AY,12-AY} \right\}$, and the predictive distribution of outcomes, $\left\{ \sum_{AY=2}^{10} X_{AY,12-AY} \right\}$. For retrospective tests we need the latter. Exhibits 6a-6d below provide the predictive distributions for both random variables.

After fitting the model to the 1997 paid loss triangle, I then obtained test data consisting of incremental paid loss data from the 1998 Schedule P and calculated the implied *p*-value for

$\left\{\sum_{AY=2}^{10} x_{AY,12-AY}\right\}$. That and other summary statistics are in Table 4. P-values for individual cell losses in the test data are given in Appendix B.

Table 4

Insurer	Actual $\sum_{AY=2}^{10} x_{AY,12-AY}$	Expected $\sum_{AY=2}^{10} \mu_{AY,12-AY}$	Ratio $\frac{\text{Actual}}{\text{Expected}}$	*p*-value
1	41,403	40,240	102.89%	0.6408
2	11,082	13,089	84.67%	0.1080
3	46,735	57,389	81.44%	0.0019
4	102,257	212,926	48.02%	0.0000

Here are some casual observations about the results.

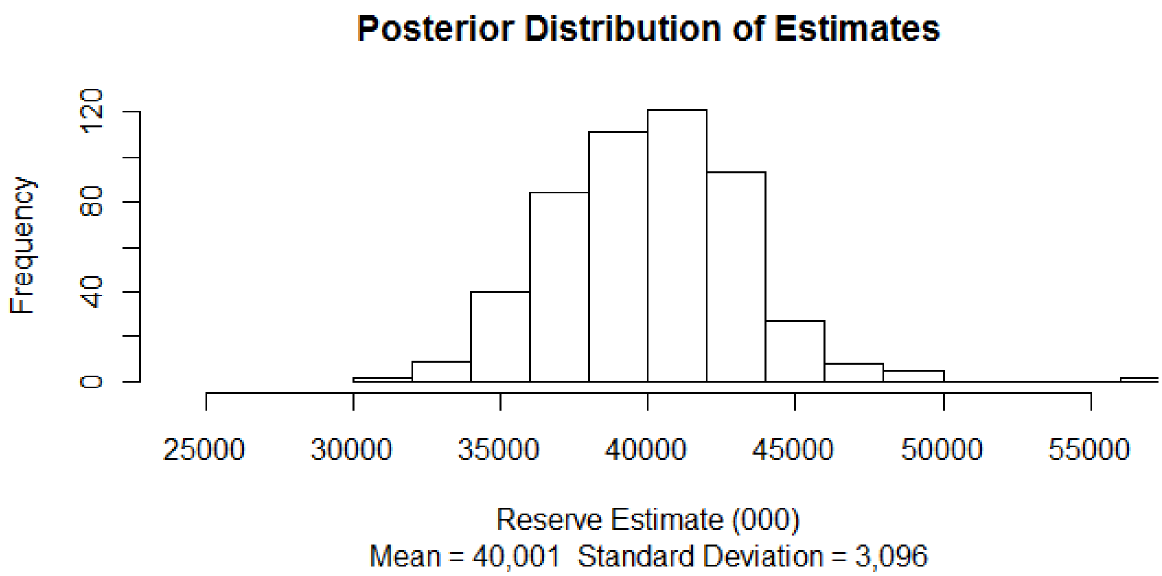
- The agreement between actual and expected results is not as good as obtained when fitting the data. Taken by itself, that is not necessarily a bad result. The test data contained only a single calendar year of data, while the data used for fitting contained 10 calendar years of data. The law of large numbers does not have a large enough number to work its magic.
- The *p*-values for Insurers 1 and 2 appear to be in the normal range. Thus, no change in assumptions seems necessary at this time.
- The *p*-value for Insurer 3 appears to be out of the normal range. If we examine the cell p-values for the test data in Appendix B, we see that all except the (*AY,Lag*) = (9,3) appear to be normal. The abnormality for the total calendar year loss appears to be caused by one bad cell. To test this, I calculated the predictive distribution for that same calendar year without the (9,3) cell. The results of this calculation are in Table 5 below. With that adjustment, the p-value moves into the normal range. An investigation into the (9,3) cell is called for. It may be a simple miscode, or some unusual event that caused the outlier.

Table 5

Insurer	Actual $\sum_{AY=2,\neq 9}^{10} x_{AY,12-AY}$	Expected $\sum_{AY=2,\neq 9}^{10} \mu_{AY,12-AY}$	Ratio $\frac{\text{Actual}}{\text{Expected}}$	p-value
3	35,861	39,063	91.80%	0.1646

- The extraordinarily low p-value for Insurer 4 cannot be explained by a single outlier. In looking at the cell p-values for the test data in Appendix B, one can see several cells with low *p*-values. This indicates there is something wrong with the structure of the model. This was apparent in the diagnostics, Exhibit 5d, of the previous section. The extraordinarily high *c* parameter and the very noticeable swings in the cell p-values by calendar provide an early indication of the problems with the model when applied to Insurer 4.

Exhibit 6a
Predictive Distributions for Insurer #1



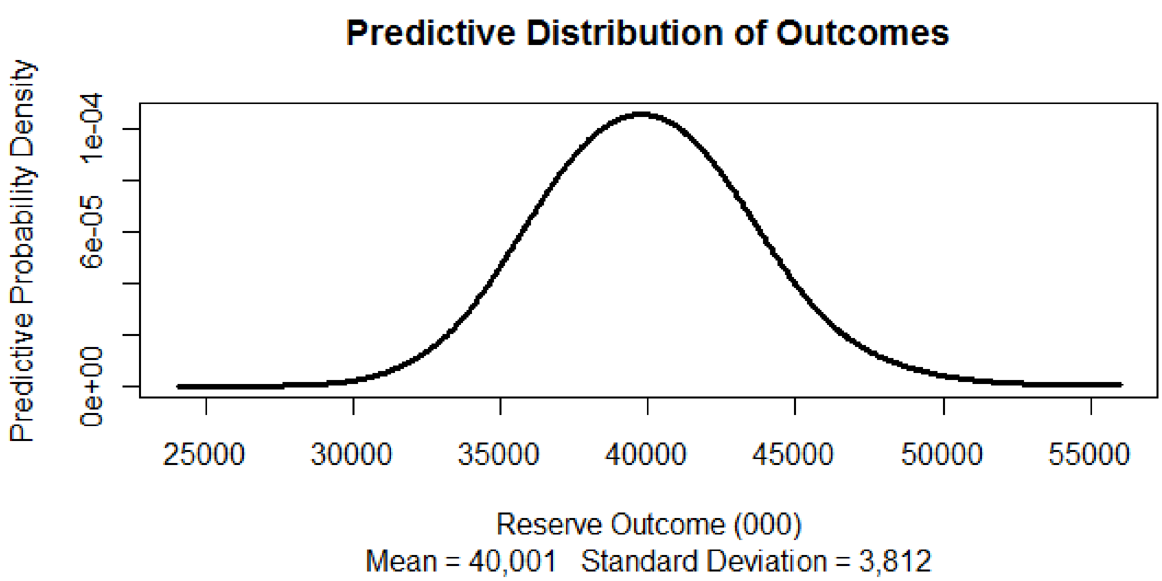
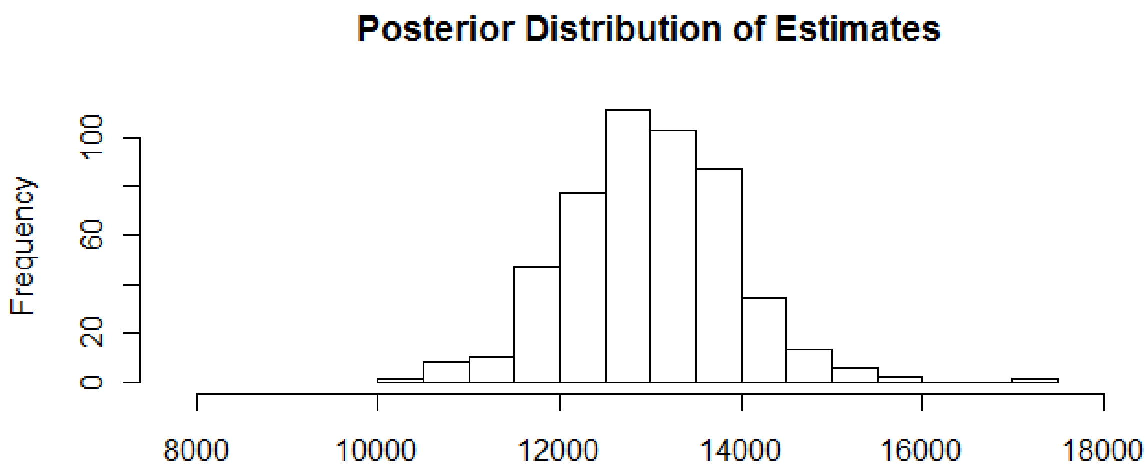
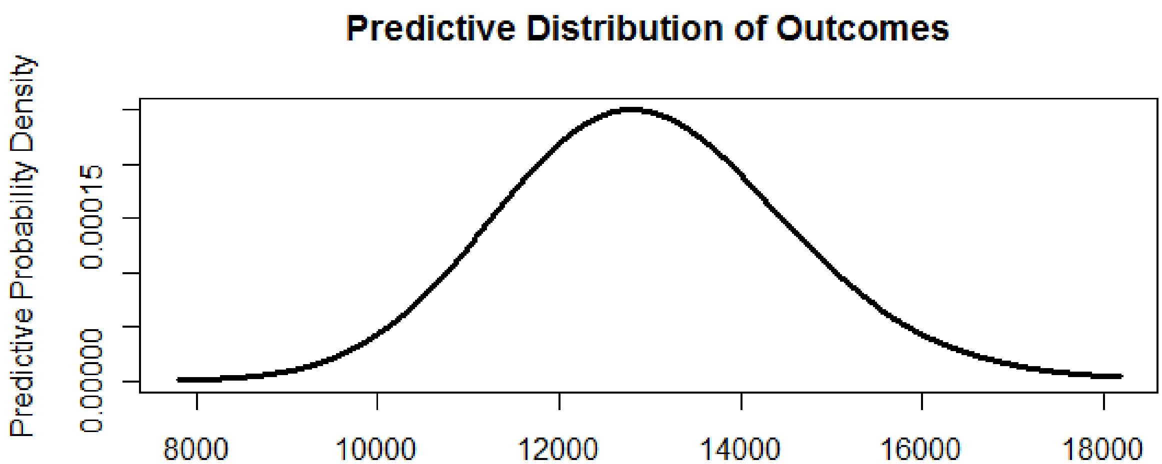


Exhibit 6b
Predictive Distributions for Insurer #2

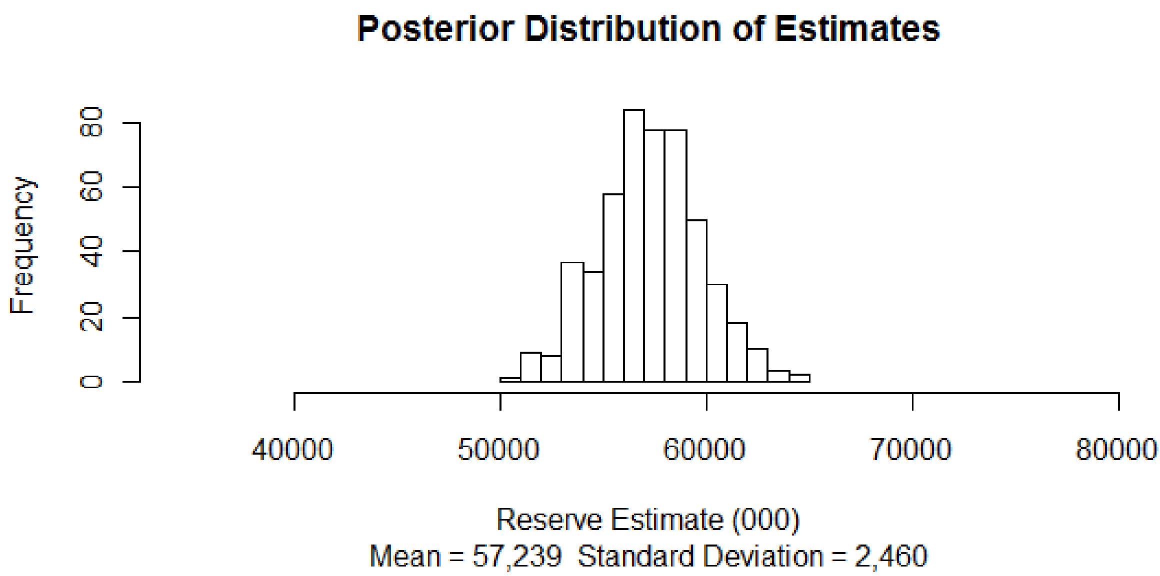


Reserve Estimate (000)
Mean = 12,996 Standard Deviation = 891



Reserve Outcome (000)
Mean = 12,996 Standard Deviation = 1,639

Exhibit 6c
Predictive Distributions for Insurer #3



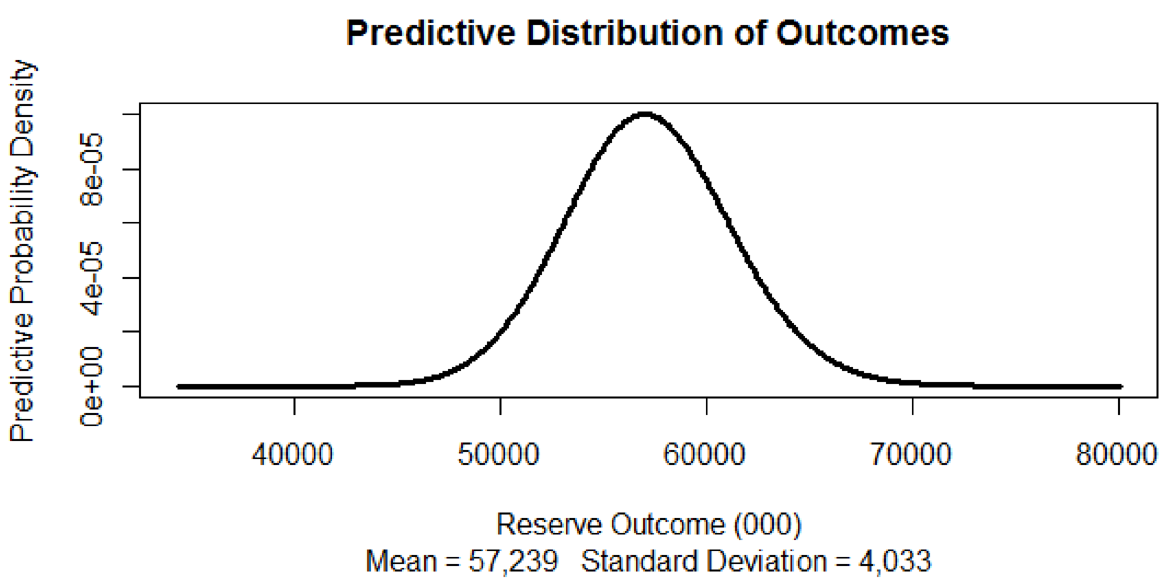
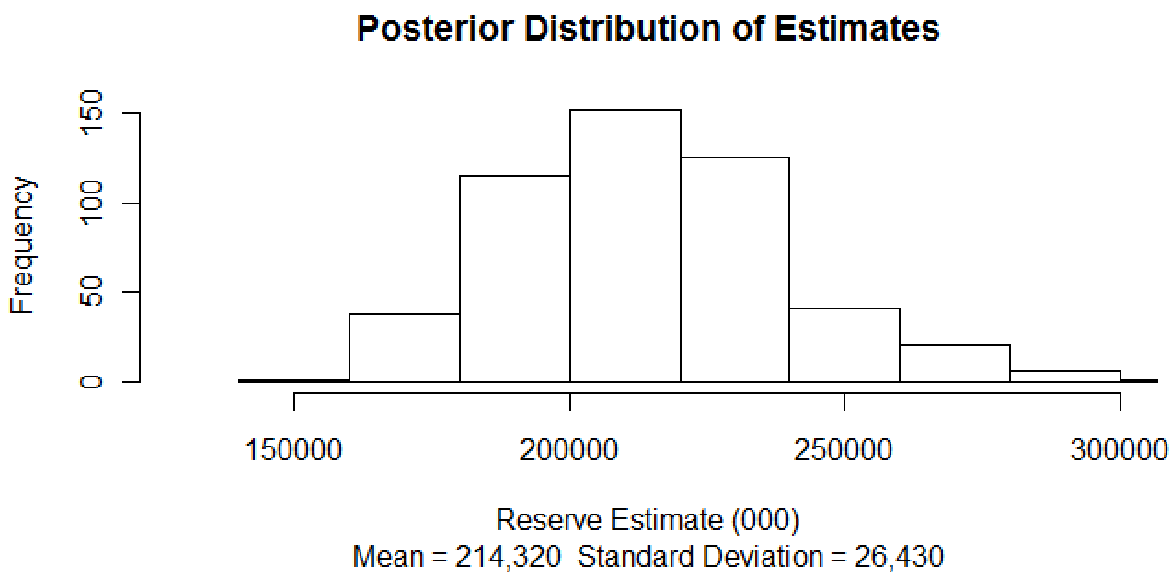
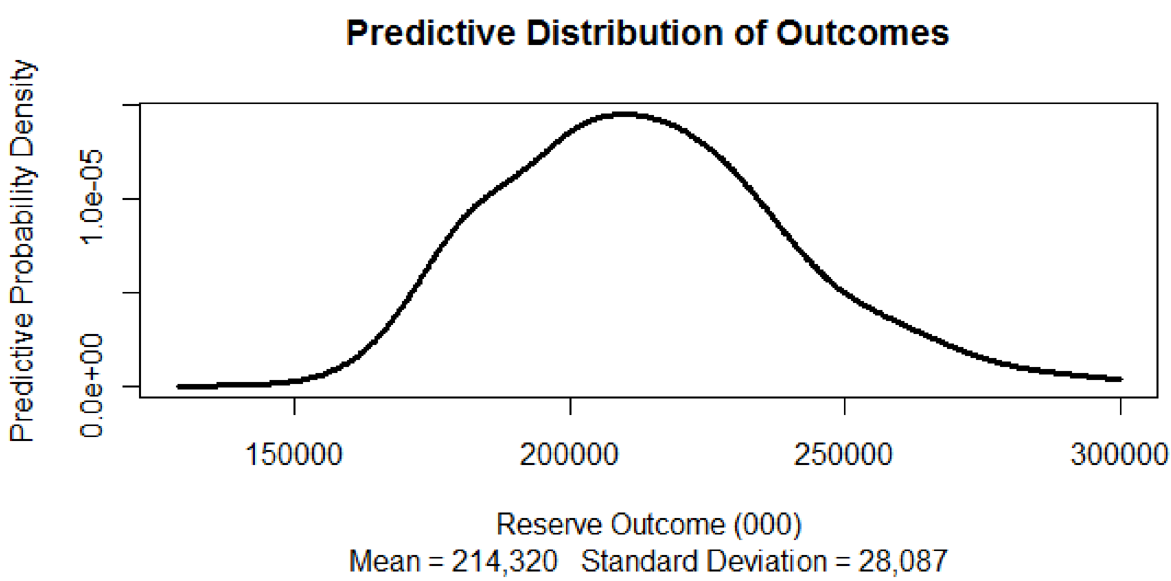


Exhibit 6d
Predictive Distributions for Insurer #4





7. BEST ESTIMATES AND RISK MARGINS

As stated in the introduction, according to the Solvency II Framework Directive:

"The value of the technical provisions shall be equal to the sum of a best estimate and a risk margin."

"The best estimate shall correspond to the probability-weighted average of future cash flows, taking account of the time value of money using the relevant risk-free interest rate term structure."

"The risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance obligations over the lifetime thereof."

This section shows how to use the model developed above to calculate the current estimate and the risk margin.

Let's start with the best estimate. Given that the future cash flows generated by the Metropolis-Hastings algorithm are equally likely, the formula for the best estimate becomes.

$$\sum_{AY=2}^{10}\sum_{Lag=12-AY}^{10}\left(\frac{1}{500}\sum_{n=1}^{500}\mu_{n,AY,Lag}\right)\cdot\frac{1}{\left(1+i\right)^{AY+Lag-11.5}}, \tag{12}$$

where *i* is the "relevant risk-free interest rate." This formula assumes that the liabilities expire mid-year.

Articles 104 and 105 of the Framework Directive call for the Solvency Capital Requirement to have sufficient capital to cover losses over the next 12 months with a probability (Value-at-Risk or VaR) of 99.5%. Both the time horizon of one year and the VaR standard are controversial among actuaries.

Instead of the VaR requirement, many actuaries prefer the Conditional Tail Expectation (CTE), which is the average of all outcomes above a given percentile (say 99%) of the outcomes. Another common name for the CTE is the Tail Value at Risk (TVaR). My speculation on why the EU chose the VaR requirement is that many feel uncomfortable calculating tail probabilities at the high end of the distribution of outcomes. I believe that when one calculates the distribution of outcomes as described above, the VaR and TVaR calculations are equally reliable. So the examples in this paper will use the TVaR at 99% to calculate the Solvency Capital Requirement.

A rationale for the one-year time horizon is that it will provide regulators sufficient time to take corrective action if necessary. Not everybody agrees. As we shall see below, the choice of the time horizon can make a significant difference in the risk margin. This paper will calculate the risk margin assuming both a single year and a 100-year time horizon.

The first risk margin formula discussed here is called the Capital Cash Flow (CCF) risk margin. In words, this formula assumes that the insurer's investors need to put up capital to take on the loss reserve risk. As claims are settled, the insurer expects to release capital over time. The CCF risk margin is the profit that the insurer's investors would need to be persuaded to take on this risky venture.

We will now discuss the details. Let:

- i = Risk-free rate of return on investments.
- r = Total rate of return demanded by the reinsurer for taking additional insurance risk.
- t = Time the loss reserve liability is set.
- C_t = Amount of capital required to support an insurance portfolio at time t.

First look at the cash flow of the insurance transaction.

- At time t = 0, investors contribute a sum of C_0 to the insurer, which earns a risk-free rate of return, i, over the next year.
- At time t = 0, the investors collects a (market value) risk margin, MVM_{CCF}. Equivalently, one could say that the investor contributes $C_0 - MVM_{CCF}$ to the insurer.
- At time t = 1, the investors expect to keep C_1 invested in the insurer, and they expect to receive a cash flow $C_0(1+i) - C_1$ at the end of year 1. Since the loss the insurers are required to pay and C_1 is uncertain, the investors discount the value of the amount returned at the risky rate of return $r > i$.
- Continuing on to time t, the investors expect to keep C_t invested in the insurer, and they expect a cash flow of $C_{t-1}(1+i) - C_t$ at the end of year t.

Since the cash flows are uncertain, it is appropriate to discount the cash flow at the risky rate of return, r. This leads to the following expression,

$$C_0 = MVM_{CCF} + \sum_{t=1}^{\infty} \frac{C_{t-1}(1+i) - C_t}{(1+r)^t}. \tag{13}$$

This equation implies

$$\begin{aligned} MVM_{CCF} &= C_0 - \sum_{t=1}^{\infty} \frac{C_{t-1}(1+i) - C_t}{(1+r)^t} \\ &= \frac{C_0(1+r-1-i)}{1+r} + \frac{C_1(1+r-1-i)}{(1+r)^2} + \frac{C_2(1+r-1-i)}{(1+r)^3} + \ldots \\ &= (r-i)\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^{t+1}}. \end{aligned} \tag{14}$$

There are two other risk margin formulas that involve slightly similar calculations. Let's call the next formula MVMSST because of its similarity to that used in the Swiss Solvency Test

$$MVM_{SST} = (r-i) \cdot \sum_{t=1}^{\infty} \frac{C_t}{(1+i)^{t+1}}. \tag{15}$$

MVM_{SST} differs from MVM_{CCF} in two ways. First it discounts the C_ts at the risk-free rate i, rather than the risky rate r. Second, it starts at time $t = 1$ rather than at time $t = 0$.

Let's call the last risk margin formula MVM_{QIS4} because of its resemblance to that used by some in their response to the CEIOPS Quantitative Impact Survey #4,

$$MVM_{QIS4} = (r-i) \cdot \sum_{t=0}^{\infty} \frac{C_t}{(1+i)^{t+1}}. \tag{16}$$

MVM_{QIS4} differs from MVM_{SST} in that it starts at time $t = 0$.

I used the term "resemblance" in the description of MVM_{SST} and MVM_{QIS4} because we now use a different calculation of C_t.

For a one-year time horizon, C_0 depends upon the distribution of the sum of outcomes in calendar year 11, i.e., $\sum_{AY-2}^{10} X_{AY,12-AY}$. Simulation Algorithm 1 describes the distribution of these losses.

Other calendar years and other time horizon involve random sums over different sets of (*AY*,*Lag*) cells, and Simulation Algorithm 1 can be modified to accommodate any given set of cells. As in the previous section, this paper uses the FFT methodology to calculate the predictive distribution of outcomes and the TVaR statistics.

Tables 7 and 8 below describe the calculation of the C_rs for the one year and the 10 year time horizons for Insurer 1. The calculation accounts for the time value of money. Table 6 shows the result of the best estimate and risk margin calculations for Insurer 1 for two time horizons and the three risk margin formulas above.

Table 6

Insurer 1

r = 10%	i =4%		Best Estimate = 91,220			
Time Horizon	MVM_{CCF}	%	MVM_{SST}	%	MVM_{QIS4}	%
1	1,994	2.2%	1,854	2.0%	2,411	2.6%
10	5,082	5.6%	4,736	5.2%	6,129	6.7%

Table 7

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
t	L_t^{Nom}	ΔL_t^{Nom}	L_t^{Disc}	TVaR_t^{Nom}	$\Delta\text{TVaR}_t^{Nom}$	TVaR_t^{Disc}	C_t
0	40,375	13,882	37,526	52,875	15,933	48,415	10,889
1	26,493	12,004	24,870	36,942	15,641	34,103	9,233
2	14,490	6,867	13,624	21,301	8,603	19,516	5,893
3	7,622	3,661	7,165	12,698	4,741	11,524	4,358
4	3,962	1,919	3,719	7,957	2,606	7,150	3,432
5	2,042	766	1,910	5,352	834	4,779	2,869
6	1,276	484	1,205	4,517	230	4,119	2,914
7	792	341	760	4,287	190	4,050	3,290
8	451	451	442	4,097	4,097	4,017	3,575

(1) The time, *t*, after the liability is set.

(2) The expected value of payments in the next calendar year, $L_t^{Nom} = \sum_{AY=2+t}^{10} \mu_{AY,12+t-AY}$.

(3) $\Delta L_t^{Nom} = L_t^{Nom} - L_{t+1}^{Nom}$.

(4) The discounted liability, $L_t^{Disc} = \sum_{k=t}^{8} \frac{\Delta L_k^{Nom}}{\left(1+i\right)^{k-t+0.5}}$.

(5) The Tail-Value-at-Risk, i.e., the conditional expected value of the random loss, $\sum_{AY=2+t}^{10} X_{AY,12+t-AY}$, given that the loss exceeds the 99th percentile.

(6) $\Delta\text{TVaR}_t^{Nom} = \text{TVaR}_t^{Nom} - \text{TVaR}_{t+1}^{Nom}$.

(7) The discounted $\text{TVaR}_t^{Disc} = \sum_{k=t}^{8} \frac{\Delta TVaR_k^{Nom}}{\left(1+i\right)^{k-t+0.5}}$.

The needed capital at time *t* is expected to be $C_t = \text{TVaR}_t^{Disc} - L_t^{Disc}$.

Table 8

(1) t	(2) L_t^{Nom}	(3) ΔL_t^{Nom}	(4) L_t^{Disc}	(5) TVaR_t^{Nom}	(6) $\Delta\text{TVaR}_t^{Nom}$	(7) TVaR_t^{Disc}	(8) C_t
0	97,503	40,375	91,220	128,894	48,491	118,529	27,309
1	57,128	26,493	53,695	80,403	31,742	73,819	20,124
2	30,635	14,490	28,824	48,661	17,133	44,401	15,576
3	16,145	7,622	15,201	31,528	9,412	28,705	13,504
4	8,523	3,962	8,035	22,116	6,225	20,255	12,219
5	4,561	2,042	4,317	15,891	4,321	14,717	10,400
6	2,519	1,276	2,407	11,570	3,673	10,899	8,493
7	1,243	792	1,202	7,898	3,801	7,590	6,388
8	451	451	442	4,097	4,097	4,017	3,575

(1) The time, *t*, after the liability is set.

(2) The expected value of all future payments, $L_t^{Nom} = \sum_{AY=2+t}^{10} \sum_{Lag=12+t-AY}^{10} \mu_{AY,Lag}$.

(3) $\Delta L_t^{Nom} = L_t^{Nom} - L_{t+1}^{Nom}$.

(4) The discounted liability, $L_t^{Disc} = \sum_{k=t}^{8} \frac{\Delta L_k^{Nom}}{(1+i)^{k-t+0.5}}$.

(5) The Tail-Value-at-Risk, i.e., the conditional expected value of the random loss,

$\sum_{AY=2+t}^{10} \sum_{Lag=12+t-AY}^{10} X_{AY,Lag}$, given that the loss exceeds the 99th percentile.

(6) $\Delta\text{TVaR}_t^{Nom} = \text{TVaR}_t^{Nom} - \text{TVaR}_{t+1}^{Nom}$.

(7) The discounted $\text{TVaR}_t^{Disc} = \sum_{k=t}^{8} \frac{\Delta\text{TVaR}_k^{Nom}}{(1+i)^{k-t+0.5}}$.

(8) The needed capital at time *t* is expected to be $C_t = \text{TVaR}_t^{Disc} - L_t^{Disc}$.

8. NEXT STEPS

The goal of this paper was to demonstrate how publicly available data can be used to calculate the technical provisions in Solvency II. This is a purely hypothetical exercise, since the publicly available data is in America, and Solvency II applies to the European Union.

Even if the Americans were to adopt something like Solvency II, or the Europeans were to adopt reporting requirements similar to the American Schedule P, there is more work to be done. The 10 years of paid data reported in Schedule P are reasonably close to final for commercial auto. But losses in other lines of insurance can take longer than 10 years to settle. Schedule P does have incurred data that can be useful in getting estimates of outstanding losses beyond the 10-year maturity reporting limit of Schedule P. There are loss reserving methods now available that integrate both paid and incurred data. See, for example, Quarg and Mack (2008) or Posthuma, Cator, Veerkamp, and van Zwet (2008). One thing that could be done is to integrate Schedule P incurred losses into the empirical Bayesian framework developed in this paper.

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APPENDIX A – ANNOTATED R CODE

The methodology in this paper follows that of Meyers (2009). This appendix assumes that the reader is familiar with the methodology of that paper. I think the methodology needs further development before it can be considered to be mature. This paper makes a few evolutionary steps along that path.

This paper makes two improvements over the code in Meyers (2009).

First it adds the *sev*, *t*, and *c* parameters to the model. Note that Simulation Algorithm 4 or Meyers (2009) introduces the {*ELR*} and {*Dev*} parameters into the Metropolis-Hastings algorithm in two separate steps. This paper introduces the *sev* and *t* parameters into the algorithm as an additional step, and then introduces the *c* parameter as a second additional step.

Next it revises the "speedy Tweedie" approximation of Appendix B of Meyers (2009). The function "dtweedie" in R's Tweedie package is relatively slow compared to other density functions available in R. Appendix B makes use of the fact that the dtweeedie works nearly as fast on vectors as it does on single numbers. So it calculates the function dtweedie($\boldsymbol{y}$,p,$\boldsymbol{y}$,ϕ) over a vector $\boldsymbol{y}$ that spans the range needed. It then approximates the function by a single cubic polynomial. This paper attains a more accurate approximation with a piecewise cubic interpolation that is just as fast.

To run the program, you input the name of a comma separated value file containing the first four columns of the data in Appendix B. You then specify the names of the various output files (identified with various tables in the paper. Finally you have to provide a list of cells whose random sum you want to predict. It generally consists of cells that make up one or more calendar years. When testing against holdout data, you must take care to match the cells in the holdout data with the list of cells the go into the predictive distribution.

Hopefully the program comments make this clear.

```
#
# Input
#
insurer="Insurer 1 Data.csv"        # input file
adata=read.csv(insurer)
outname="Insurer 1 Summary"         # Table 3 and Table 6
#outname2="Insurer 1 Cells.csv"     # Appendix B  comment out if not testing
#outname3="Insurer 1 Test.csv"      # Table 4     comment out if not testing
tweedie.p=1.67
npost=500
#
# set up the (AY,Lag) pairs included in the predictive distribution
#
# in ayXX and lagXX below, the XX refers to the calendar year
#
ay11=2:10
lag11=12-ay11
ay12=3:10
lag12=13-ay12
ay13=4:10
lag13=14-ay13
ay14=5:10
lag14=15-ay14
ay15=6:10
lag15=16-ay15
ay16=7:10
lag16=17-ay16
ay17=8:10
lag17=18-ay17
ay18=9:10
lag18=19-ay18
ay19=10:10
lag19=20-ay19
#
# select which (AY,Lag) cells to include in predictive distribution
#
# examples
# use for the next calendar year
pred.ay=ay11
pred.lag=lag11
# use for all outstanding losses
#pred.ay=c(ay11,ay12,ay13,ay14,ay15,ay16,ay17,ay18,ay19)
#pred.lag=c(lag11,lag12,lag13,lag14,lag15,lag16,lag17,lag18,lag19)
# use for insurer 1 retro test (missing ay=3)
#ayins1=c(2,4,5,6,7,8,9,10)
#lagins1=12-ayins1
# use for insurer 3 retro test (missing ay=2)
#ayins3=3:10
#lagins3=12-ayins3
cys=unique(pred.lag+pred.ay-1)
#
# discretized gamma severity distribution
#
library(actuar)
```

```
discrete.gamma<-function(tau,p,h,fftn){
    alpha=(2-p)/(p-1)
    theta=tau/alpha
    m=2^fftn
    dpar<-rep(0,m)
    x<-h*0:(m-1)
    lev=levgamma(x,alpha,scale=theta)
    dpar[1]=1-lev[2]/h
    dpar[2:(m-1)]=(2*lev[2:(m-1)]-lev[1:(m-2)]-lev[3:(m)])/h
    dpar[m]=1-sum(dpar[1:(m-1)])
    return(dpar)
    } # end discrete.gamma function
#
# model with variable dev,elr,sev,con
#
fact.crm.llike1=function(dev,elr,sev,con){
  cyt=sev[2]^(rdata$ay+rdata$lag-1)
  eloss=rdata$premium*dev[rdata$lag]*elr[rdata$ay]*cyt
  phi=(eloss^(1-tweedie.p)*sev[1]*tau[rdata$lag])/(2-tweedie.p)+
            con*eloss^(2-tweedie.p)
  llike=ldtweedie.scaled(rdata$loss,eloss,phi)
  return(sum(llike))
  }
num=250
front=matrix(0,num,10)
log.y1=front
log.ybot=0
library(statmod)
library(tweedie)
ldtweedie.front=function(y,lyf,lf){
  ly=log(y)
  del=lyf[2]-lyf[1]
  low=pmax(floor((ly-lyf[1])/del),1)
  d01=(lf[low+1]-lf[low])/del
  d12=(lf[low+2]-lf[low+1])/del
  d23=(lf[low+3]-lf[low+2])/del
  d012=(d12-d01)/2/del
  d123=(d23-d12)/2/del
  d0123=(d123-d012)/3/del
  ld=lf[low]+(ly-lyf[low])*d01+(ly-lyf[low])*(ly-lyf[low+1])*d012+
           (ly-lyf[low])*(ly-lyf[low+1])*(ly-lyf[low+2])*d0123
  return(ld)
  }
#
ldtweedie.scaled=function(y,mu,phi){
  dev=y
  ll=y
  k=(1/phi)^(1/(2-tweedie.p))
  ky=k*y
  yp=ky>0
  dev[yp]=2*((k[yp]*y[yp])^(2-tweedie.p)/((1-tweedie.p)*
     (2-tweedie.p))-k[yp]*y[yp]*(k[yp]*mu[yp])^(1-tweedie.p)/
     (1-tweedie.p)+(k[yp]*mu[yp])^(2-tweedie.p)/(2-tweedie.p))
  ll[yp]=log(k[yp])+ldtweedie.front(ky[yp],log.y1,front)-dev[yp]/2
  ll[!yp]=-mu[!yp]^(2-tweedie.p)/phi[!yp]/(2-tweedie.p)
  return(ll)
  }
```

```
#
# log prior and proposal density functions
#
log.prior=function(dev,elr,sev,con){
  ld=dgamma(dev,alpha.dev,scale=theta.dev,log=T)
  le=dgamma(elr,alpha.elr,scale=theta.elr,log=T)
  ls=dgamma(sev,alpha.sev,scale=theta.sev,log=T)
  lc=dgamma(con,alpha.con,scale=theta.con,log=T)
  return(sum(ld,le,ls,lc))
  }

log.proposal.den=function(x,m,alpha){
  d=dgamma(x,alpha,scale=m/alpha,log=T)
  return(sum(d))
  }
#
# main program
#
# initialize variables for metropolis hastings
#
set.seed(12345)
nmh=11000                    # number of MH scenarios
#
# parameters for the prior distribution
#
alpha.sev=c(1.367644674,1290.230651)
theta.sev=c(136.2478465,0.00076972)
alpha.con=0.074005011
theta.con=0.139142639
alpha.elr=c(29.85060994,33.8347283, 35.33377535,24.49077508,28.66183085,
            25.63407528,16.80427236,14.36801632,9.305348568,6.366703316)
theta.elr=c(0.023695076,0.022680106,0.021353992,0.028504884,0.025371532,
            0.030388169,0.050089616,0.060203232,0.101715232,0.160927171)
alpha.dev=c(15.80995889,42.85381689,56.49438570,30.45284406,10.23093999,
            5.809417079,3.695390712,2.393367923,1.355938768,0.455240196)
theta.dev=c(0.013514659,0.005874493,0.003588986,0.004605868,0.008501860,
            0.008263645,0.006753167,0.005653256,0.006622295,0.020023956)
tau=1-(1-(1:10)/10)^3
alpha.prop.elr=500
alpha.prop.sev=500
alpha.prop.con=500
alpha.prop.dev=2000*alpha.dev*theta.dev
#
# get insurer data and set up tweedie model
#
rdata=subset(adata,adata$ay+adata$lag<12)  #separate  test  data  from  fitting
data
#
# set up the 'speedy tweedie' calculation
#
eloss.max=max(rdata$loss)
phi.min=eloss.max^(1-tweedie.p)*tau[1]/2/(2-tweedie.p)
k.max=(1/phi.min)^(1/(2-tweedie.p))
log.ytop=log(eloss.max*k.max)
log.ybot=0
```

```
del=(log.ytop-log.ybot)/num
log.y1=seq(from=log.ybot,to=log.ytop,length=num)
front=log(dtweedie(exp(log.y1),tweedie.p,exp(log.y1),1))
#
# initialize metropolic hastings arrays and select starting values
#
mh.dev=matrix(0,nmh,10)
mh.elr=mh.dev
mh.sev=matrix(0,nmh,2)
mh.con=mh.sev
mh.dev[1,]=alpha.dev*theta.dev  # use prior mean for mh starting values
mh.elr[1,]=alpha.elr*theta.elr
mh.sev[1,]=alpha.sev*theta.sev
mh.con[1]=alpha.con*theta.con
prev.log.post=fact.crm.llike1(mh.dev[1,],mh.elr[1,],mh.sev[1,],mh.con[1])+
              log.prior(mh.dev[1,],mh.elr[1,],mh.sev[1,],mh.con[1])
#
# generate samples using mh algorithm
#
  for (i in 2:nmh){
    devmh=rgamma(10,shape=alpha.prop.dev,scale=mh.dev[i-1,]/alpha.prop.dev)
    devmh=devmh/sum(devmh)
    u=log(runif(1))
    log.post=fact.crm.llike1(devmh,mh.elr[i-1,],mh.sev[i-1,],mh.con[i-1])+
               log.prior(devmh,mh.elr[i-1,],mh.sev[i-1,],mh.con[i-1])
    r=log.post-prev.log.post+
      log.proposal.den(mh.dev[i-1,],devmh,alpha.prop.dev)-
      log.proposal.den(devmh,mh.dev[i-1,],alpha.prop.dev)
    mh.dev[i,]=mh.dev[i-1,]
    if(u<r){
      mh.dev[i,]=devmh
      prev.log.post=log.post
      }
    #
    elrmh=rgamma(10,shape=alpha.prop.elr,scale=mh.elr[i-1,]/alpha.prop.elr)
    u=log(runif(1))
    log.post=fact.crm.llike1(mh.dev[i,],elrmh,mh.sev[i-1,],mh.con[i-1])+
               log.prior(mh.dev[i,],elrmh,mh.sev[i-1,],mh.con[i-1])
    r=log.post-prev.log.post+
      log.proposal.den(mh.elr[i-1,],elrmh,alpha.prop.elr)-
      log.proposal.den(elrmh,mh.elr[i-1,],alpha.prop.elr)
    mh.elr[i,]=mh.elr[i-1,]
    if(u<r){
      mh.elr[i,]=elrmh
      prev.log.post=log.post
      }
    #
    sevmh=rgamma(2,shape=alpha.prop.sev,scale=mh.sev[i-1,]/alpha.prop.sev)
    u=log(runif(1))
    log.post=fact.crm.llike1(mh.dev[i,],mh.elr[i,],sevmh,mh.con[i-1])+
               log.prior(mh.dev[i,],mh.elr[i,],sevmh,mh.con[i-1])
    r=log.post-prev.log.post+
      log.proposal.den(mh.sev[i-1,],sevmh,alpha.prop.sev)-
      log.proposal.den(sevmh,mh.sev[i-1,],alpha.prop.sev)
    mh.sev[i,]=mh.sev[i-1,]
    if(u<r){
      mh.sev[i,]=sevmh
```

```
      prev.log.post=log.post
      }
    conmh=rgamma(1,shape=alpha.prop.con,scale=mh.con[i-1]/alpha.prop.con)
    u=log(runif(1))
    log.post=fact.crm.llike1(mh.dev[i,],mh.elr[i,],mh.sev[i,],conmh)+
                   log.prior(mh.dev[i,],mh.elr[i,],mh.sev[i,],conmh)
    r=log.post-prev.log.post+
      log.proposal.den(mh.con[i-1],conmh,alpha.prop.con)-
      log.proposal.den(conmh,mh.con[i-1],alpha.prop.con)
    mh.con[i]=mh.con[i-1]
    if(u<r){
      mh.con[i]=conmh
      prev.log.post=log.post
      }
    }
#
# sample mh parameters
#
samp=sample(1001:nmh,size=npost)
#
# calculate predited percentiles of observed losses in training data
#
pctloss=rep(0,dim(rdata)[1])
meanloss=pctloss
tpct=rep(0,npost)
for (i in 1:dim(rdata)[1]){
  cyt=mh.sev[samp,2]^(rdata$ay[i]+rdata$lag[i]-1)
  mu=rdata$premium[i]*mh.elr[samp,rdata$ay[i]]*mh.dev[samp,rdata$lag[i]]*cyt
  meanloss[i]=mean(mu)
  phi=(mu^(1-tweedie.p)*mh.sev[samp]*tau[rdata$lag[i]])/(2-tweedie.p)+
       mh.con[samp]*mu^(2-tweedie.p)
  for (j in 1:npost){
    tpct[j]=ptweedie(rdata$loss[i],tweedie.p,mu[j],phi[j])
    }
  pctloss[i]=mean(tpct)
  if (rdata$loss[i]==0) pctloss[i]=pctloss[i]*runif(1)
  }
#
# plot results
#
windows(record=T)
#
# trace plot of estimates
#
nmh.pred=rep(0,nmh)
ay.prem=rep(0,10)
for (j in unique(pred.ay)){
  ay.prem[j]=mean(rdata$premium[rdata$ay==j])
  }
pred.mean=rep(0,nmh)
for (i in 1:nmh){
  for (j in unique(pred.ay)){
    ayp=(pred.ay==j)
    for (k in pred.lag[ayp]){
      cyt=mh.sev[i,2]^(j+k-1)
      nmh.pred[i]=nmh.pred[i]+ay.prem[j]*mh.elr[i,j]*mh.dev[i,k]*cyt
      }
```

```
    }
  }
plot(1:nmh,nmh.pred,type="l",main="Trace Plot for Mean Loss")
#
# plot of elr paths
#
set.seed(12345)
prior.elr=matrix(0,1000,10)
for (j in 1:1000){
  prior.elr[j,]=rgamma(10,shape=alpha.elr,scale=theta.elr)
  }
par(mfrow=c(2,1))
plot(1:10,prior.elr[1,],ylim=range(0,1.5*prior.elr),
       main="ELR Paths",
       xlab="Accident Year",ylab="ELR",type="n")
legend("topleft",legend=c("Posterior","Prior"),
       col=c("black","grey"),lwd=c(3,3))
for (j in 1:1000){
  par(new=T)
  plot(1:10,prior.elr[j,],ylim=range(0,1.5*prior.elr),main="",
         xlab="",ylab="",col="grey",type="l")
  }
for (j in samp){
  par(new=T)
  plot(1:10,mh.elr[j,],ylim=range(0,1.5*prior.elr),main="",
         xlab="",ylab="",col="black",type="l",lwd=1)
    }
#
# plot of dev paths
#
prior.dev=matrix(0,1000,10)
for (j in 1:1000){
  prior.dev[j,]=rgamma(10,shape=alpha.dev,scale=theta.dev)
  }

plot(1:10,prior.dev[1,],ylim=range(0,prior.dev),
       main="Dev Paths",
       xlab="Settlement Lag",ylab="Dev",type="n")
legend("topright",legend=c("Posterior","Prior"),
       col=c("black","grey"),lwd=c(3,3))
for (j in 1:1000){
  par(new=T)
  plot(1:10,prior.dev[j,],ylim=range(0,prior.dev),main="",
         xlab="",ylab="",col="grey",type="l")
  }
for (j in samp){
  par(new=T)
  plot(1:10,mh.dev[j,],ylim=range(0,prior.dev),main="",
         xlab="",ylab="",col="black",type="l",lwd=1)
    }
#
# plot of severity parameters
#
par(mfrow=c(2,1))
prior.sev1=rgamma(1000,alpha.sev[1],scale=theta.sev[1])
hist(prior.sev1,main="Prior Distribution of 'sev' Parameter",
  xlim=range(prior.sev1,mh.sev[,1]),xlab="sev")
```

```
hist(mh.sev[samp,1],xlim=range(prior.sev1,mh.sev[,1]),xlab="sev",
 main="Posterior Distribution of 'sev' Parameter")
#
# plot of calendar year trend parameters
#
par(mfrow=c(2,1))
prior.sev2=rgamma(1000,alpha.sev[2],scale=theta.sev[2])
hist(prior.sev2,main="Prior Distribution 't' Parameter",
  xlim=range(prior.sev2,mh.sev[,2]),xlab="t")
hist(mh.sev[samp,2],xlim=range(prior.sev2,mh.sev[,2]),xlab="t",
 main="Posterior Distribution of 't' Parameter")
#
# plot of contagion parameters
#
par(mfrow=c(2,1))
prior.con=rgamma(1000,alpha.con,scale=theta.con)
hist(prior.con,main="Prior Distribution of 'c' Parameter",
  xlim=range(prior.con,mh.con),xlab="c")
hist(mh.con[samp],xlim=range(prior.con,mh.con),xlab="c",
 main="Posterior Distribution of 'c' Parameter")
#
# pp plot of cell loss percentiles for training data
#
par(mfrow=c(2,2))
#
# pp plot of cell loss percentiles for training data
#
plot(sort(pctloss),1:length(pctloss)/
    (1+length(pctloss)),
    xlim=c(0,1),ylim=c(0,1),xlab="Predicted P",ylab="Observed P",
    main="PP Plot")
crit.val1=1.63/sqrt(length(pctloss)) # 1.36 for 5%, 1.63 for 1%
crit.val2=1.36/sqrt(length(pctloss))
abline(0,1,lwd=3)
abline(crit.val1,1)
abline(-crit.val1,1)
abline(crit.val2,1)
abline(-crit.val2,1)
#
#  plots of ay, lag and calendar year vs percentile for training data
#
plot(rdata$ay,pctloss,main="AY vs Cell Percentiles",ylim=c(0,1),
     xlab="AY",ylab="Observed P")
plot(rdata$lag,pctloss,main="Lag vs Cell Percentiles",ylim=c(0,1),
     xlab="Lag",ylab="Ovserved P")
plot(rdata$ay+rdata$lag-1,pctloss,main="CY vs Cell Percentiles",
     ylim=c(0,1),xlab="CY",ylab="Observed P")
#
# calculate predictive distributions of outcomes - takes some time
#

fftn=14
h=max(rdata$premium)*10/2^fftn
niceh=c(5,10,20,25,40,50,75,100,125,150,200,250,500,750,1000)
h=min(subset(niceh,niceh>h))
x=h*(0:(2^fftn-1))
phiz=matrix(0,2^fftn,9)
```

```
phix=complex(2^fftn,0,0)
postnum=0
eloss=matrix(0,length(samp),length(pred.ay))
for (k in 1:npost){
  i=samp[k]
  phixp=complex(2^fftn,1,0)
  for (j in 1:length(pred.ay)){
    ay=pred.ay[j]
    lag=pred.lag[j]
    premium=min(subset(rdata$premium,rdata$ay==ay))
    tau1=mh.sev[i,1]*tau[lag]*mh.sev[i,2]^(ay+lag-1)
    phiz=fft(discrete.gamma(tau1,tweedie.p,h,fftn))
    eloss[k,j]=premium*mh.elr[i,ay]*mh.dev[i,lag]*mh.sev[i,2]^(ay+lag-1)
    lam=eloss[k,j]/tau1
    phixp=phixp*exp(lam*(phiz-1))
    }
  phix=phix+phixp
  postnum=postnum+1
  print(postnum)
  }
pred=round(Re(fft(phix/npost,inverse=TRUE)),12)/2^fftn
mean.outcome=sum(x*pred)
sd.outcome=sqrt(sum(x*x*pred)-mean.outcome^2)
pred.range=(x>.6*mean.outcome)&(x<1.4*mean.outcome)
#
# plot distribution of estimates
#
par(mfrow=c(2,1))
pred.mean=rowSums(eloss)
hist(pred.mean,
   main="Posterior Distribution of Estimates",
   xlim=range(x[pred.range]),xlab="Reserve Estimate (000)",
   sub=paste("Mean =",format(round(mean(pred.mean)),big.mark=","),
   " Standard Deviation =",format(round(sd(pred.mean)),big.mark=",")))
#
# plot distribution of outcomes
#
xb=(x[cumsum(pred)>.99])
pb=pred[cumsum(pred)>.99]
tvar=sum(xb*pb)/sum(pb)
predb=pred[pred.range]
plot(x[pred.range],predb/h,type="l",col="black",lwd=3,
     ylim=c(0,max(predb/h)),
     xlim=range(x[pred.range]),
     main="Predictive Distribution of Outcomes",
     xlab="Reserve Outcome (000)",ylab="Predictive Probability Density",
     sub=paste("Mean =",format(round(mean.outcome),big.mark=","),
     "  Standard Deviation =",format(round(sd.outcome),big.mark=",")))
#
# write out summary statistics including tvar
#
outlab=c("input  data","train  sum  actual","train  sum  predicted","train  sum
ratio")
outlab=c(outlab,"pred mean","pred sd est","pred.sd out","pred tvar","cyid")
results=rep(0,9)
results[1]=insurer
results[2]=sum(rdata$loss)
```

```
results[3]=sum(meanloss)
results[4]=sum(rdata$loss)/sum(meanloss)
results[5]=mean.outcome
results[6]=sd(pred.mean)
results[7]=sd.outcome
results[8]=tvar
results[9]=sum(unique(pred.ay+pred.lag-1))
df.results=data.frame(outlab,results)
df.results
outname=paste(outname,results[9],".csv")
write.csv(df.results,file=outname,row.names=F)
#
# calculate predited percentiles of observed losses in test data
#
edata=subset(adata,adata$ay+adata$lag>11)  #separate  test  data  from  fitting
data
e.pctloss=rep(0,dim(edata)[1])
e.meanloss=e.pctloss
tpct=rep(0,npost)
for (i in 1:dim(edata)[1]){
  cyt=mh.sev[samp,2]^(edata$ay[i]+edata$lag[i]-1)
  mu=edata$premium[i]*mh.elr[samp,edata$ay[i]]*mh.dev[samp,edata$lag[i]]*cyt
  e.meanloss[i]=mean(mu)
  phi=(mu^(1-tweedie.p)*mh.sev[samp]*tau[edata$lag[i]])/(2-tweedie.p)+
       mh.con[samp]*mu^(2-tweedie.p)
  for (j in 1:npost){
    tpct[j]=ptweedie(edata$loss[i],tweedie.p,mu[j],phi[j])
    }
  e.pctloss[i]=mean(tpct)
  if (edata$loss[i]==0) e.pctloss[i]=e.pctloss[i]*runif(1)
  }
#
# calculate p-value for test data
#
actual=sum(edata$loss)
predicted=round(sum(e.meanloss))
ratio=round(100*actual/predicted,2)
b=(x<actual)
pvalue=round(max(cumsum(pred)[b]),4)
pvalue
testout=data.frame(actual,predicted,ratio,pvalue)
write.csv(testout,file=outname3,row.names=F)
#
# reproduce Insurer X Data
#
test=rep(0,dim(rdata)[1])
e.test=rep(1,dim(edata)[1])
ay=c(rdata$ay,edata$ay)
lag=c(rdata$lag,edata$lag)
premium=c(rdata$premium,edata$premium)
loss=c(rdata$loss,edata$loss)
pctloss=c(pctloss,e.pctloss)
meanloss=c(meanloss,e.meanloss)
test=c(test,e.test)
cell.results=data.frame(ay,lag,premium,loss,pctloss,meanloss,test)
cell.results
write.csv(cell.results,outname2,row.names=F)
```

APPENDIX B – INSURER DATA

This appendix contains the data for the four insurers analyzed in this paper, along with selected results particular to the accident year and settlement lag. The first four columns were used to fit the model. What follows is a description of each data element.

1. Accident Year (1987=1)
2. Settlement Lag
3. Net Premium
4. Incremental Paid Net Loss
5. P-value – $F_{AY,Lag}(x_{AY,Lag})$ (Equation 10)
6. Mean of the predictive distribution –$\mu_{AY,Lag}$ (Equation 11)
7. Test Indicator (= 0 if used for fitting, =1 in used for testing)

APPENDIX B – INSURER #1

AY	Lag	Premium	Loss	*p*-value	E[Loss]	Test
1	1	29,701	5,234	0.62800	4,979.25	0
1	2	29,701	5,172	0.59792	4,978.05	0
1	3	29,701	3,708	0.40137	4,025.16	0
1	4	29,701	1,783	0.25697	2,297.84	0
1	5	29,701	923	0.31750	1,224.53	0
1	6	29,701	537	0.47430	627.50	0
1	7	29,701	175	0.45819	266.75	0
1	8	29,701	145	0.58558	175.37	0
1	9	29,701	8	0.36715	110.23	0
2	1	27,526	5,234	0.46129	5,383.23	0
2	2	27,526	5,683	0.62545	5,385.46	0
2	3	27,526	4,392	0.54172	4,355.15	0
2	4	27,526	2,134	0.35274	2,485.43	0
2	5	27,526	1,377	0.58789	1,323.74	0
2	6	27,526	673	0.56343	678.62	0
2	7	27,526	155	0.38298	287.97	0
2	8	27,526	81	0.40826	189.45	0
2	9	27,526	47	0.50786	118.20	0
3	1	30,750	5,702	0.37457	6,117.66	0
3	2	30,750	5,865	0.44487	6,111.99	0
3	3	30,750	7,966	0.98836	4,945.90	0
3	4	30,750	2,472	0.36650	2,822.51	0
3	6	30,750	143	0.03003	770.53	0
3	7	30,750	152	0.32027	326.31	0
3	8	30,750	73	0.33895	215.34	0
4	1	35,814	6,349	0.73986	5,706.63	0
4	2	35,814	4,611	0.18397	5,704.71	0
4	3	35,814	3,959	0.29187	4,617.25	0
4	4	35,814	2,522	0.47976	2,635.60	0
4	5	35,814	1,924	0.82504	1,402.37	0
4	6	35,814	622	0.46919	719.95	0
4	7	35,814	206	0.45015	303.85	0
5	1	42,277	8,377	0.80655	7,291.36	0
5	2	42,277	6,890	0.41615	7,290.34	0
5	3	42,277	4,055	0.06994	5,897.94	0
5	4	42,277	3,795	0.70103	3,363.19	0
5	5	42,277	1,292	0.24508	1,791.99	0
5	6	42,277	1,422	0.85865	918.92	0
6	1	50,088	9,291	0.13432	11,322.40	0
6	2	50,088	13,836	0.88970	11,316.67	0
6	3	50,088	12,441	0.95467	9,154.43	0
6	4	50,088	4,086	0.19012	5,224.03	0
6	5	50,088	2,293	0.31767	2,780.94	0
7	1	56,921	12,029	0.51493	12,071.18	0
7	2	56,921	12,462	0.59261	12,068.66	0
7	3	56,921	8,369	0.24543	9,759.61	0
7	4	56,921	7,034	0.85965	5,567.06	0
8	1	61,406	13,119	0.65267	12,416.19	0
8	2	61,406	12,618	0.56142	12,410.83	0
8	3	61,406	9,117	0.33922	10,037.48	0
9	1	67,983	15,860	0.56056	15,630.32	0
9	2	67,983	14,893	0.41787	15,622.43	0
10	1	73,359	16,498	0.51160	16,687.70	0
2	10	27,526	0	0.36054	143.28	1
4	8	35,814	194	0.62159	201.01	1
5	7	42,277	324	0.51047	390.57	1
6	6	50,088	1,769	0.74071	1,427.91	1
7	5	56,921	4,783	0.95281	2,972.68	1
8	4	61,406	7,954	0.92811	5,735.63	1
9	3	67,983	12,655	0.52867	12,650.30	1
10	2	73,359	13,724	0.20078	16,718.34	1

APPENDIX B – INSURER #2

AY	Lag	Premium	Loss	*p*-value	E[Loss]	Test
1	1	15,274	3,718	0.63752	3,538.62	0
1	2	15,274	3,243	0.68616	2,962.24	0
1	3	15,274	1,889	0.25142	2,350.47	0
1	4	15,274	1,697	0.45221	1,826.75	0
1	5	15,274	731	0.39982	904.88	0
1	6	15,274	770	0.85561	434.59	0
1	7	15,274	287	0.67785	242.21	0
1	8	15,274	1,086	0.98831	245.97	0
1	9	15,274	49	0.58684	97.24	0
1	10	15,274	20	0.55321	89.22	0
2	1	15,722	3,844	0.53156	3,819.48	0
2	2	15,722	4,196	0.92276	3,197.64	0
2	3	15,722	2,806	0.67816	2,536.67	0
2	4	15,722	2,310	0.72096	1,973.39	0
2	5	15,722	414	0.09853	977.83	0
2	6	15,722	78	0.08000	467.88	0
2	7	15,722	232	0.56845	261.61	0
2	8	15,722	36	0.18582	265.54	0
2	9	15,722	5	0.38171	104.54	0
3	1	16,266	3,854	0.68477	3,594.26	0
3	2	16,266	3,378	0.72771	3,010.23	0
3	3	16,266	1,860	0.21581	2,387.61	0
3	4	16,266	1,736	0.45833	1,857.23	0
3	5	16,266	662	0.32273	920.45	0
3	6	16,266	697	0.81186	439.60	0
3	7	16,266	20	0.14912	245.46	0
3	8	16,266	228	0.58356	249.83	0
4	1	17,017	3,184	0.66632	2,975.79	0
4	2	17,017	1,948	0.18750	2,493.52	0
4	3	17,017	1,670	0.32708	1,979.80	0
4	4	17,017	1,257	0.33866	1,538.59	0
4	5	17,017	1,433	0.92792	761.65	0
4	6	17,017	217	0.37523	364.88	0
4	7	17,017	190	0.60559	203.25	0
5	1	18,016	2,837	0.19607	3,335.78	0
5	2	18,016	3,180	0.74147	2,793.63	0
5	3	18,016	1,794	0.26688	2,216.25	0
5	4	18,016	2,923	0.96200	1,723.16	0
5	5	18,016	1,035	0.70247	853.66	0
5	6	18,016	136	0.19534	408.70	0
6	1	18,395	3,380	0.44921	3,482.92	0
6	2	18,395	2,394	0.21652	2,916.78	0
6	3	18,395	2,859	0.80912	2,313.58	0
6	4	18,395	1,836	0.56140	1,798.69	0
6	5	18,395	763	0.44185	891.65	0
7	1	18,932	4,948	0.64629	4,715.55	0
7	2	18,932	3,288	0.20046	3,948.47	0
7	3	18,932	4,385	0.93990	3,130.72	0
7	4	18,932	2,024	0.30727	2,437.57	0
8	1	20,857	5,116	0.60750	4,936.96	0
8	2	20,857	4,466	0.67641	4,131.59	0
8	3	20,857	2,659	0.22562	3,276.81	0
9	1	24,348	5,702	0.63089	5,466.01	0
9	2	24,348	3,953	0.24678	4,577.60	0
10	1	24,030	5,450	0.48231	5,527.01	0
2	10	15,722	0	0.32000	95.88	1
3	9	16,266	124	0.73523	99.20	1
4	8	17,017	99	0.42612	207.58	1
5	7	18,016	42	0.23582	228.69	1
6	6	18,395	497	0.66174	427.07	1
7	5	18,932	1,118	0.48390	1,208.99	1
8	4	20,857	2,558	0.53458	2,553.63	1
9	3	24,348	2,531	0.09281	3,632.89	1
10	2	24,030	4,113	0.30838	4,635.13	1

APPENDIX B – INSURER #3

AY	Lag	Premium	Loss	*p*-value	E[Loss]	Test
1	1	39,383	7,701	0.32545	8,174.66	0
1	2	39,383	7,072	0.32563	7,615.81	0
1	3	39,383	8,473	0.94002	6,589.57	0
1	4	39,383	3,549	0.34291	3,969.54	0
1	5	39,383	3,327	0.85339	2,512.85	0
1	6	39,383	1,804	0.64478	1,630.75	0
1	7	39,383	817	0.69365	669.91	0
1	8	39,383	330	0.64372	298.40	0
1	9	39,383	105	0.33264	281.69	0
1	10	39,383	63	0.48387	160.76	0
2	1	44,770	9,609	0.78559	8,816.87	0
2	2	44,770	9,540	0.86835	8,213.95	0
2	3	44,770	5,755	0.12805	7,108.68	0
2	4	44,770	3,198	0.12712	4,282.46	0
2	5	44,770	1,898	0.15681	2,711.84	0
2	6	44,770	1,912	0.63151	1,756.90	0
2	7	44,770	602	0.45221	723.08	0
2	8	44,770	122	0.29190	320.84	0
2	9	44,770	462	0.76641	303.63	0
3	1	50,914	10,780	0.70914	10,187.56	0
3	2	50,914	8,570	0.24187	9,491.07	0
3	3	50,914	7,062	0.19366	8,214.06	0
3	4	50,914	5,220	0.62343	4,946.45	0
3	5	50,914	4,849	0.95961	3,134.63	0
3	6	50,914	2,220	0.63779	2,031.78	0
3	7	50,914	488	0.24846	834.47	0
3	8	50,914	239	0.41814	371.71	0
4	1	56,904	9,098	0.32524	9,619.73	0
4	2	56,904	8,974	0.51734	8,959.88	0
4	3	56,904	8,522	0.73617	7,753.12	0
4	4	56,904	4,985	0.63832	4,672.87	0
4	5	56,904	2,864	0.48789	2,955.96	0
4	6	56,904	1,576	0.33841	1,917.89	0
4	7	56,904	857	0.62105	787.71	0
5	1	62,551	9,446	0.19963	10,441.16	0
5	2	62,551	9,620	0.48453	9,727.04	0
5	3	62,551	10,928	0.96008	8,417.18	0
5	4	62,551	5,506	0.67673	5,069.52	0
5	5	62,551	1,973	0.07126	3,209.63	0
5	6	62,551	1,858	0.41349	2,082.31	0
6	1	67,205	13,791	0.67421	13,226.53	0
6	2	67,205	11,656	0.34160	12,322.62	0
6	3	67,205	11,664	0.75153	10,661.42	0
6	4	67,205	5,323	0.18643	6,421.71	0
6	5	67,205	3,731	0.39758	4,067.67	0
7	1	74,056	16,783	0.53513	16,707.39	0
7	2	74,056	17,370	0.85083	15,560.35	0
7	3	74,056	10,413	0.03875	13,467.69	0
7	4	74,056	9,144	0.77486	8,113.31	0
8	1	81,035	17,389	0.62141	16,946.46	0
8	2	81,035	15,132	0.36836	15,783.79	0
8	3	81,035	13,653	0.51292	13,658.91	0
9	1	90,568	22,871	0.54331	22,723.13	0
9	2	90,568	20,819	0.45069	21,167.46	0
10	1	99,940	22,916	0.47903	23,057.99	0
3	9	50,914	169	0.35782	352.10	1
4	8	56,904	372	0.62074	351.26	1
5	7	62,551	607	0.34127	856.37	1
6	6	67,205	2,079	0.26920	2,638.39	1
7	5	74,056	6,121	0.79950	5,138.10	1
8	4	81,035	8,253	0.52544	8,230.16	1
9	3	90,568	10,874	0.00027	18,325.98	1
10	2	99,940	18,260	0.10827	21,496.17	1

APPENDIX B – INSURER #4

AY	Lag	Premium	Loss	*p*-value	E[Loss]	Test
1	1	267,666	33,810	0.23507	45,787.91	0
1	2	267,666	45,318	0.34147	54,471.20	0
1	3	267,666	46,549	0.60459	44,198.73	0
1	4	267,666	35,206	0.71245	30,137.93	0
1	5	267,666	23,360	0.84899	16,758.58	0
1	6	267,666	12,502	0.68697	10,818.72	0
1	7	267,666	6,602	0.65542	5,883.45	0
1	8	267,666	3,373	0.64375	3,018.67	0
1	9	267,666	2,373	0.63989	2,176.39	0
1	10	267,666	778	0.50883	1,138.65	0
2	1	274,526	37,663	0.36921	44,076.07	0
2	2	274,526	51,771	0.53039	52,532.45	0
2	3	274,526	40,998	0.50420	42,610.85	0
2	4	274,526	29,496	0.56482	29,101.82	0
2	5	274,526	12,669	0.32337	16,191.71	0
2	6	274,526	11,204	0.62130	10,418.89	0
2	7	274,526	5,785	0.58105	5,664.48	0
2	8	274,526	4,220	0.80284	2,908.03	0
2	9	274,526	1,910	0.54331	2,088.72	0
3	1	268,161	40,630	0.44228	44,604.21	0
3	2	268,161	56,318	0.61502	53,036.35	0
3	3	268,161	56,182	0.82143	43,011.27	0
3	4	268,161	32,473	0.65752	29,363.66	0
3	5	268,161	15,828	0.52041	16,328.33	0
3	6	268,161	8,409	0.35092	10,522.64	0
3	7	268,161	7,120	0.73419	5,721.56	0
3	8	268,161	1,125	0.13471	2,948.32	0
4	1	276,821	40,559	0.54905	40,480.52	0
4	2	276,821	49,755	0.58209	48,176.91	0
4	3	276,821	39,323	0.55355	39,099.08	0
4	4	276,821	24,081	0.44189	26,694.41	0
4	5	276,821	13,209	0.43920	14,806.69	0
4	6	276,821	12,655	0.79588	9,561.64	0
4	7	276,821	2,921	0.19429	5,198.07	0
5	1	270,214	37,515	0.47765	39,993.47	0
5	2	270,214	51,068	0.62719	47,562.16	0
5	3	270,214	34,410	0.42354	38,619.62	0
5	4	270,214	25,529	0.51768	26,323.12	0
5	5	270,214	19,433	0.80611	14,676.52	0
5	6	270,214	5,728	0.17976	9,452.97	0
6	1	280,568	41,454	0.47870	44,192.46	0
6	2	280,568	53,552	0.57022	52,642.06	0
6	3	280,568	40,599	0.49452	42,665.27	0
6	4	280,568	40,026	0.85032	29,100.24	0
6	5	280,568	6,750	0.04135	16,206.07	0
7	1	344,915	57,783	0.50911	60,172.70	0
7	2	344,915	68,136	0.49687	71,598.64	0
7	3	344,915	86,915	0.90859	58,013.32	0
7	4	344,915	18,328	0.04564	39,731.67	0
8	1	371,139	62,011	0.43460	69,225.13	0
8	2	371,139	132,553	0.94135	82,082.69	0
8	3	371,139	21,083	0.00703	66,759.41	0
9	1	323,753	112,592	0.91665	72,997.37	0
9	2	323,753	33,783	0.02117	87,133.82	0
10	1	221,448	38,181	0.44006	43,939.91	0
2	10	274,526	887	0.56864	1,116.16	1
3	9	268,161	1,662	0.46527	2,132.64	1
4	8	276,821	1,043	0.15090	2,686.03	1
5	7	270,214	2,898	0.19996	5,153.72	1
6	6	280,568	5,513	0.11725	10,453.71	1
7	5	344,915	11,551	0.09181	22,134.60	1
8	4	371,139	17,129	0.01972	45,665.08	1
9	3	323,753	24,089	0.01220	70,644.07	1
10	2	221,448	37,485	0.28817	52,940.26	1

Anatomy of Actuarial Methods of Loss Reserving

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Abstract: This paper evaluates the foundation of loss reserving methods currently used by actuaries in property casualty insurance. The chain-ladder method, also known as the weighted loss development method in North America, is the most commonly used actuarial technique for loss reserving and setting liabilities for property/casualty insurers. Many actuaries believe that the basic assumption underlying this model is the future development of losses is dependent on losses to date for each accident year. We shall see that this is not the case and the method may be rooted in the complete independence of future loss development. The alternative assumptions are, in this author's opinion, a more natural way of analyzing the loss triangle. We shall also show that most of the methods used by actuaries are based on one common basic model, and the differences lie in how and which of the parameters are being estimated. The exposition provides some new insight to reserving methods. While it enriches our understanding of the loss reserving process and defines the common thread among various methods, it challenges some commonly held views in the actuarial profession. The exposition here points out a flaw in the Bornhuetter-Ferguson methodology as well as questions the basic framework of the loss development methodology. We shall show that we can obtain the same results as the loss development method under the assumption that the future losses are independent of what we know currently.

We introduce a new method, termed the exposure development method, which has some advantages over traditional loss development methods in some situations. The proposed methodology allows us to construct several new estimators. One can estimate the ultimate losses by combining the information gleaned from paid losses and the incurred loss triangles. Most importantly, this methodology provides better analytical tools to examine the model, look for outliers, and provides an alternative method of estimating the variability of reserves.

INTRODUCTION

The results presented in this paper are quite basic and there is no need to review the current state of knowledge to proceed. For brevity, it will be appropriate to refer to them as needed in our exposition. Let $X_{i,j}$ denote the losses paid for the accident year i in the j^{th} year of development, where $i, j = 1, 2 \ldots n$. We assume that we have observed $X_{i,\,j}$ for $i + j < n + 2$ and are interested in estimating $X_{i,\,j}$ for $i + j = n + 2, n + 3 \ldots 2n$. Once we have estimated these, we could add them and compute the ultimate losses. In this paper, we restrict our attention to the development period n and assume that the losses are fully developed by that time. Any development beyond period n is outside the scope of the results presented here. Although we will mainly focus on the paid loss triangle, the methodology presented here can equally be applied to incurred or reported loss triangles. We also assume that we have some information available about the exposure for each accident year. For example, the earned premium for each accident year may be known. Although any measure of exposure will suffice for our purpose. If we have prior information about the ultimate losses, that may be used as an exposure base as well and might possibly be the best exposure base. The ultimate losses are exposure times a rate, and they are identical if the loss rate is constant. Sometimes we have used these interchangeably and the author assumes that does not cause any misunderstanding. As we shall see, the assumed knowledge of exposures is for exposition of the ideas presented here and is not necessary. Let us denote E_i be the exposure amount for the accident year i. We shall use the Buhlman (1967) method to estimate the average loss by development period.

We compute

$$r_j = \frac{\sum_{i=1}^{n-j+1} X_{i,j}}{\sum_{i=1}^{n-j+1} E_i}, j = 1, 2 \ldots n. \tag{1.1}$$

However, we do not need to compute r_1, so the number of parameters we need and use is only $n - 1$. If we use earned premium as a proxy for the exposure, the method is known as the partial loss ratio method. One should note that this method does not assume any relationship between development periods. We estimate

$$\hat{X}_{i,j} = E_i \times r_j \quad \text{for} \quad i + j > n + 1. \tag{1.2}$$

This method, although somewhat popular in Europe, is seldom used in North America. However, we shall see that this method can be used as the building block of the loss development method.

Now let us assume that the exposures E_is are not known and we want to estimate them from the data itself. It will suffice for our purpose if we have the estimates of relative exposure levels for each accident year, and that information is sufficient to compute r_j and hence the values of the unpaid losses, which is our primary goal. We assume that the exposure level for the first accident year is unity ($E_1 = 1$) and try to estimate the future accident years' exposure relative to the first accident year's exposure. We compute what we call exposure development factors (EDFs).

$$d_k = \frac{\sum_{i=1}^{k+1} \sum_{j=1}^{n-k} X_{i,j}}{\sum_{i=1}^{k} \sum_{j=1}^{n-k} X_{i,j}}. \tag{1.3}$$

It may be easy to relate these factors to weighted loss development factors. All we have done is changed the process of loss development from operating in columns to operating in rows.

Let us define

$$D_k = d_1 \times d_2 \times \ldots \times d_k. \tag{1.4}$$

D_k is the estimated total earned exposure by accident year k +1 relative to accident year 1.

These exposure development factors can then be used to estimate the relative individual accident year exposures. The exposure for accident year k +1 relative to the first accident year is $D_k - D_{k-1}$.

We could use these estimated relative exposures to compute r_k and then using equation (1.2) compute the unknown elements of the loss rectangle.

One should note that we have estimated 2(*n* - 1) parameters in the process, (*n* - 1) parameters for the exposure level and another (*n* - 1) parameters for the development period rates.

It is interesting to note that one need not compute the payment year rates. One can directly estimate the unobserved element by computing

$$\hat{X}_{i,j} = d_{i-1} \times \sum_{j=1}^{i-1} X_{i,j} - \sum_{j=1}^{i-1} X_{i,j}, \quad i + j > n + 1. \tag{1.5}$$

One can easily verify that the results so obtained are the same that one would obtain by the more elaborate procedure stated earlier. Similar to the loss development method, this requires computing only (*n* - 1) parameters. We will call this method the exposure development method. The exposure development method has its advantages over the loss development method and may be a better way of analyzing loss triangles, as we shall see further on. We have defined our computational scheme based on incremental loss data. For computational purpose, it may be better to use cumulative loss triangles as we do in the loss development method. The computational procedure for the exposure development method is similar to the weighted loss development method. The difference is that we first transpose the incremental loss triangle and use this triangle to compute the cumulative loss triangle and carry out the same computation as for the weighted loss development method.

A quite surprising observation is that the estimates so obtained are those that one would obtain if the weighted loss development method had been used. The proof is trivial and one can easily verify that the formula for estimating $X_{i,j}$ for the exposure development method is equivalent to the weighted loss development method, where the unobserved $X_{i,j}$ are estimated by the formula

$$\hat{X}_{i,j} = \sum_{k=1}^{j-1} X_{i,k} \times \frac{\sum_{l=1}^{i-1} \sum_{k=1}^{j} X_{l,k}}{\sum_{l=1}^{i-1} \sum_{k=1}^{j-1} X_{l,k}} - \sum_{k=1}^{j-1} X_{i,k}. \tag{1.6}$$

Where unobserved values of $X_{i,j}$ used in equation (1.6) are estimated first and then are treated as the observed values in the equation. The pictorial view shown in Figure 1 helps illustrate the approach better. The symbols A, B, C and D represent the sum of incremental losses of the area they cover. The right top formula in the figure 1, represents the estimate when weighted loss development method is used. The bottom left is the formula for exposure development, and the bottom right is the formula when we first estimated the exposure levels and then use Buhlman's method. We do not show the calculation of exposures (F in the formula in Figure 1) as it cancels out.

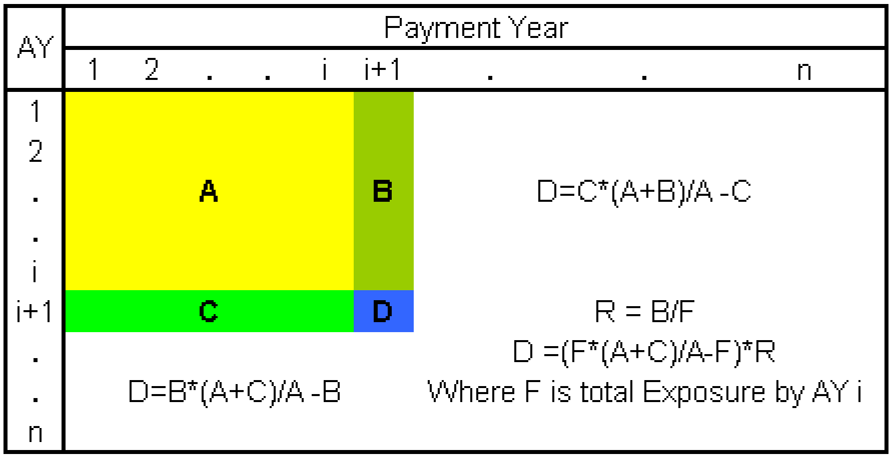

Figure1

The important point to note is that by using the alternate derivation (i.e., if we compute the relative exposures first and then use equation (1.2)) we have estimated $2(n - 1)$ parameters and arrive at the same answer as the weighted loss development method or the exposure development method, which appear to have $(n - 1)$ parameters. The contrast in the number of parameters is puzzling. The only explanation I have come up with is based on our misunderstanding of what we are trying to estimate. The general belief that our aim in loss reserving is to find a number for the value of ultimate losses that will be paid when all the claims arising from that accident year are finally settled does not follow statistical logic. In a statistical framework, the ultimate losses are a random variable. A random variable cannot be estimated. The statistical methods are not meant to estimate a random outcome or the results of a flip of a coin. All one can do is to estimate the parameters associated with the random process that are generating the random variable based on the observed data. To predict a random variable, first we compute (in most cases) the expected value of the random variable we want to predict. Then we try to estimate that expected value based on the available information or the estimated parameters of the random process. It should be clear that the estimator itself is a function of observed data and hence a random variable and its expected value need not match the expected value of the random variable we want to predict. If the two quantities are equal, the estimate is an unbiased estimator. The unbiasedness may be desirable criteria and in many cases, it may be preferred, but it is not always a best estimate and in many cases, it may not be possible to find an unbiased estimator. If we accept this notion of estimating the parameters of the loss process, the discrepancy we observe in the number of parameters can be explained. We are estimating both the relative exposures and the payout pattern and the true number of parameters is $2(n - 1)$. The individual year ultimate losses are themselves parameters of the random process and should be counted as such when we use the weighted loss development method or the exposure development method. I would like to add one other observation that is relevant to our discussion of number of the parameters. Technically, if we are interested in total ultimate losses for all accident years combined, we need to compute just one parameter. The estimated ultimate loss for all accident year by the weighted loss development method is same as the exposure development factor $D_{n\text{-}1}$ times the first accident year total paid losses by age n. The result can also be obtained by multiplying the sum of paid losses for all accident years in the first year with the age 1 to n ultimate weighted loss development factor. This will imply that we need only one parameter in estimating the all accident years combined ultimate loss.

I would like to point out that Lehigh (2007) has expressed similar views. He states that we use losses of prior development years as a proxy for exposure. However, the fact may be that we are estimating the exposure levels as well and not realizing it.

The exposure-based method does not assume any relationship between future losses and the paid losses to date. After the Mack (1993) paper, there is strong feeling among actuaries that the use of loss development methods has an implicit assumption that future development is dependent on current observation. It was one of the basic assumptions of Mack's method that future losses depend on losses paid to date by a constant factor. Chu, and Venter (1998) discusses methods to test this assumption.

It is well known that under the assumption that $X_{i,j}$ are independently distributed Poisson or multinomial variates, the same results as the weighted loss development method are obtained and the proof can be found in Renshaw and Verrall (1998). Therefore, the claim that $2(n - 1)$ parameters are being estimated, or the losses to be paid in future are independent of paid to date, is not new. One important difference in the method presented here is that our assumptions are slightly less restrictive. Renshaw and Verral require that both the column and row sum for the observed data be positive whereas we require only row sum to be positive.

The exposure development introduced here can also use simple averages of the exposure development factors, similar to what is done in the simple average loss development method. However, the two results from loss development and exposure development will not coincide. As we shall see, in the weighted loss development method, there is a balancing going on and that causes the exposure development and loss development results to coincide. Actuaries generally prefer weighted loss development factors over simple average loss development factors. Using simple averages of the exposure development factors will be confusing if the incremental loss is negative and is therefore not recommended. However, simple averages can be used for estimating rates. It may provide an alternative estimate of the ultimate losses and can be used in making a selection of the reserve requirements. We shall return to these issues later in the paper.

In the next section, we introduce yet another alternative computational procedure that reinforces the same idea and further strengthens the view that we are estimating both exposure and payout of the ultimate losses. That computational scheme has its own merit and utility besides strengthening the ideas presented here. The computational scheme is quite versatile, and helps us in assessing the validity or the appropriateness of the model. It identifies any outliers in our data and opens up a new area for further research, as well as provides a tool for estimating the variability of our reserve estimates.

In section 3, we define the basic model of loss reserving and discuss the common thread among most of the classical actuarial methods of loss reserving. The model presented is not new and one form or another has been presented by many authors, however the perspective here is different. The reader is encouraged to read Mack and Venter to get a better understanding of the issues and controversies.

Section 4 is quite brief and focused on the basic assumptions of loss development methods and some of the actuarial adjustments that are made in practice. We also discuss the validity of the method for policy year and report year losses.

Section 5 is devoted to an example where we carry out an analysis of a selected paid loss triangle and test its appropriateness.

In section 6, we discuss variability in the estimation of ultimate losses. We provide a simple simulation approach to attack the problem but most of the details are left to the reader to extend and modify the approach as needed for analyzing the data in hand.

In section 7, we focus on the exposure development method and see how it can be used to deal with another important issue, which is using both paid and incurred loss data. As we shall see the new methodology provides us a variety of different ways to achieve it. We define several new estimators and see how information available, from incurred loss data, can be used along with paid loss data to refine our results.

SECTION 2: INDEPENDENCE OF ACCIDENT YEAR

Most actuaries are familiar with categorical contingency tables and Chi Square test of independence. If we classify a population in two or more different categories and each of these classifications have two or more groups and we count the number of observations by category, we have a contingency table. For example, we may be interested in whether education level depends on gender. We may take a sample and count the number of people that have high school degree, a two-year college degree, a four-year college degree or a postgraduate degree separately for males and females and carry out a test to see whether education level differs for males and females. We shall not get into the computational details here, as that is not the purpose of the presentation. However, one can see the similarity and the differences with a loss triangle. The categories are accident years and development years and instead of counts we have paid loss amounts. The most important difference is that the loss dollars are not scalars and the lower half triangle of the loss rectangle is not known and our aim is to estimate them. However, it should not deter us from computing the expected value of each cell as we do in analyzing a contingency table.

Let us assume that we have all the observations in our loss rectangle. Let us define

$$R_i = \sum_{j=1}^{n} X_{i,j} \tag{2.1}$$

$$C_j = \sum_{i=1}^{n} X_{i,j}. \tag{2.2}$$

$$T = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{i,j}. \tag{2.3}$$

Define

$$\hat{X}_{i,j} = R_i \times \frac{C_j}{T}. \tag{2.4}$$

However, we do not know some of the $X_{i,j}$ and aim to estimate them from the observed data to date. We shall use an iterative procedure to achieve this. We assign the value 0 to all unknown $X_{i,j}$ and use equation (2.4) to compute them. This is our first iteration and will give us an estimate of unobserved $X_{i,j}$. We substitute these estimated values in place of the previously assigned values of zero for unobserved $X_{i,j}$. We update the values of R_i, C_j, and T and use equation 2.4 again to revise our estimate for unknown $X_{i,j}$. We repeat the process until it converges. The process will converge as long as each of the original R_is are positive (i.e., each accident year has positive exposure). The proof is messy and left to the reader. We only state that the estimates obtained by the weighted loss development method are a solution satisfying the stated criterion. The important point to note is that the process converges to the same values as the exposure development method and the weighted loss development method. Clearly we have estimated $2(n - 1)$ parameters.

This computing method is estimating the losses to be paid for accident years 2, 3 … n assuming that the loss payments are independent of accident year and that losses paid so far have nothing to do with future loss payments. A typical question one may ask is whether it is possible to test the assumption of independence. The answer is unfortunately no. One can compute statistics similar to Chi-Square as we do for contingency tables, but loss amounts are not scalar (i.e., if we restated the loss amounts in cents rather than dollars the value of the statistics so computed will be 100 times larger). We need a suitable scaling factor to test the assumption of independence. There is no satisfactory solution to the problem and we leave it as a challenge to the actuarial profession. One solution the author suggests is, if the claim count data is also available, the scaling factor can be approximated by the ratio of estimated total loss dollars for all accident years divided by the estimated total claim count for all accident years. One will divide the computed Chi-Square type statistics by this number and consider it distributed as Chi Square with $n^2 - 2n$ degrees of freedom. This technique has two problems. First, the estimated scaling factor is a random variable and second the scaling factors may be different for each cell due to inflation and varying average claim size by payment lag.

We cannot test the appropriateness of the assumption of independence of accident year and payment year lag. However, it does not prevent us from testing the suitability of the model. We have estimated both exposure and payment patterns and can obtain the estimates for each of the observed values and compute the residuals. These residuals can be tested for randomness, any pattern in accident year and payment year lag, as well as any outliers in the data. We can also compute the explained variation of the model and other statistics for goodness of fit of the model. We have analyzed a paid loss triangle data and shall discuss these results later in the paper.

One additional advantage of this iterative procedure is that we can use it when some data points are missing or when we believe the residuals are too large for some data elements and want to remove them from the analysis. These data points can be treated in the same manner as unobserved data points in the iterative estimation process. The only data elements one cannot remove are $X_{n,1}$ and $X_{1,n}$ for the obvious reasons. The removal of individual data elements and the ability to fit the original model allows us to compute model skill as introduced by Jing, Lebens, and Lowe (2009) in the actuarial field. There are additional advantages to removing a data element, as we shall see later.

SECTION 3: BASIC MODEL OF LOSS RESERVING METHODS

We shall define a model that is basic to almost all of the classical actuarial methods.

$$X_{i,j} = a_i \times b_j + e_{ij} . \tag{3.1}$$

Where

a_i is the accident year i total loss,

b_j is proportion of losses to be paid in payment lag j and is constant for all accident years, and

e_{ij} are error terms with mean zero and variance that may not be constant.

This model has $2n$ - 2 parameters, as there are 2 constraints

$$\sum_{j=1}^{n} b_j = 1 \text{ and}$$

a_1 is presumed known and equals R_1 defined earlier.

This model can be re-parameterized as

$$X_{i,j} = \mu \times a_i' \times b_j + e_{ij}, \qquad (3.2)$$

where $\mu = \sum_{i=1}^{n} a_i$, a_i' is a_i / μ and μ represents total expected loss amount for all accident years combined.

Now we shall explore the various actuarial methods and see how these are related to this basic model.

3.1 Weighted Loss Development Method: In this method the parameters of the model are estimated such that

$$\sum_{i=1}^{n+1-j} X_{i,j} = \sum_{i=1}^{n+1-j} a_i \times b_j \text{ , } j = 1, 2 \ldots n, \qquad (3.3)$$

$$\sum_{j=1}^{n+1-i} X_{i,j} = \sum_{j=1}^{n+1-i} a_i \times b_j \text{ , } i = 1, 2 \ldots n. \qquad (3.4)$$

The weighted loss development method or the exposure development method introduced here can be used to solve the above system of equations. The iterative procedure may be a systematic approach to find the same solution. We call it a systematic method merely to convey the idea that a mathematician given the problem and not exposed to actuarial methods will probably proceed that way.

3.2 Buhlman Method: We have already seen this method. In this method, *a*s are known and we estimate *b* parameters.

3.3 Bornhuetter-Ferguson Method: In this method we assume to have prior knowledge of ultimate losses. However, we do not use this information to compute the payment pattern. The payment pattern is derived as in the weighted loss development method, which presumes no knowledge of exposure or loss amounts. We then use this computed payment pattern and the prior known ultimate losses to estimate unknown loss values. The method is sometimes referred to as the combining of observed data and prior knowledge. However, this prior knowledge is not fully utilized to estimate the parameters to be used in the forecast. The method will be the same as the Buhlman method if the prior knowledge of ultimate loss is used in estimating the payment pattern.

3.4 Cape-Cod Method: This method is similar to Bornhuetter-Ferguson (B-F) method. We assume that we know the premium amount for each accident year but not the loss ratio. The loss ratio is derived from equating the actual paid to date losses for all accident years to the estimated percentage of earned premium. This method has the same basic flaw that the B-F method has. The knowledge of premium is not used in estimating the earned percentage or the payment pattern.

3.5 Least Squares Method: This method is also not that common in North America. We try to estimate a_i and b_js such that the residual sum of squares (RSS) is minimum, i.e.,

$$RSS = \sum_{i=1}^{n} \sum_{j=1}^{n+1-i} (X_{i,j} - a_i \times b_j)^2 \tag{3.5}$$

To solve for *a*s and *b*s, we differentiate equation 3.5 with respect to a_is and b_js and equate them to zero. The derived set of equations requires an iterative procedure for solution. We shall not pursue it here. A variation of this method is to weigh the individual error term by some predefined weighting factors.

3.6 Log Regression Model: This is a new trend in the last few decades but it is still not widely used in practice. The basic model is the same as equation (3.2) with one basic difference. The error terms are assumed to be multiplicative and have mean 1 rather than additive with mean 0. One takes the logarithm of the paid incremental losses, and the model becomes linear in parameter. These new parameters can be estimated much more easily. Interested readers are referred to Verral (1994). The modeling process breaks down if some of the paid values are negative and a variety of ad hoc adjustments are made to the data are made to fit the model and estimate the model parameters and the unpaid losses. The main drawback of this method is that it requires transforming the data by taking logarithms. Once we have estimated the parameters we have to convert the estimates to original units. There are many advantages as we can test the significance of the various parameters and can define the parameters in some functional form and reduce the number of the parameters to be estimated. The transformed equation (3.2) can be modified to include the calendar year parameters. There is vast literature on this methodology and we will not pursue it here. Alternative transformations other than logarithmic are also investigated by a few authors.

It may be worthwhile to add that the iterative procedure introduced in section 2 provides many of the advantages of this methodology. In section 5 we have a numerical example and discuss it in detail.

SECTION 4: INFLATION EFFECT

We have seen that for most of the actuarial methods, the basic underlying model is the same. In this section, we discuss the effect of inflation on the basic model as well as some of the simple approaches used by actuaries to deal with it.

The basic model presumes that each accident year has an exposure level (ultimate losses); losses are paid by a fixed pattern and that pattern remains constant over time. These are the implications of the assumption that the claims reporting and handling process is same for all accident years. Any changes we may observe are due to randomness and not due to systematic changes in the loss process or claims handling. We know that inflationary changes affect the loss payments. Under the assumption that inflation affects the loss payment by accident year only, the basic model is not affected. Inflation affects the losses paid uniformly for each delay and the payment pattern will remain the same for all accident years. The inflation impact will be in parameters a_is only and will be captured by the estimation process. However, the losses paid may be impacted by both the accident year as well as the year losses are paid. Bustic (1988) discusses these issues in detail. Under this scenario, the payment pattern is affected and the model (3.1) is distorted. The best way to handle such a situation is to restate the loss triangle by removing the inflationary effect, estimate the parameters, and adjust the estimated losses for the inflation. However, this may add more estimation error in our analysis. First, we have to estimate the inflation by accident year and how the loss payment is affected by payment delay and the accident year. There is no simple solution to these estimations, thus adjusting the loss triangle for inflation may add more distortion in the results rather than improving it. One common technique used by most actuaries is to compute the loss development factors based on more recent data (latest three years' average development factors). If we assume that either inflation changes for each year but changes are moderate or the effect of the payment lag is small or both, this adjustment works well. One of the advantages of the approach that we estimate both exposure level and the payment pattern is that the use of the latest years in estimating parameters can be modified. We could use it for exposures only or rates or both and as such providing us with alternative estimators. The concept is made clearer when we analyze a loss triangle later in the paper.

The assumption that we are estimating both the exposure level and the payment patterns raises another issue of great importance. Actuarial literature encourages the use of the loss development method for policy year loss triangles as well as report year loss triangles. Under the assumption that the exposure level is also being estimated, the loss development methodology is inappropriate for analyzing report year loss triangles. Each element of a report year loss triangle will have losses generated from a different number of accident years and the exposure level keeps changing for such a loss triangle. For policy year loss triangles, the inflationary changes will distort the data much more severely as they are affected by two years of inflationary impact. Unless inflation is fairly constant, the use of exposure development method on a policy year loss triangle may be questionable. However, it will lead to the same result as the weighted loss development method and indirectly raises questions about the suitability of using the loss development method for the policy year loss triangle. The inflationary distortion will be much more significant in a policy year loss triangle if the inflationary changes are large. Although, this author has no serious objection to the use of loss development method to the policy year loss triangle, however the additional analysis carried out in the next section, especially the testing the model validity and defining outliers, may not be appropriate for such data. We have also provided a method for computing variability in the loss reserve. Such an analysis for policy year loss triangles may be distorted.

SECTION 5: NUMERICAL EXAMPLE

We now focus on analyzing a real data set. This will help create a clearer understanding of the ideas presented in this paper.

We have selected a data set for use in this example; the main reason for selecting this data was that both the paid and incurred loss triangles are available. We can see how the information from both triangles is combined to estimate ultimate losses. In this section we focus on paid losses only.

We shall use model (3.2) for our discussion. We use a paid loss triangle from Quarg and Mack (2008) that has seven years of data. The incremental paid loss triangle, the development factors, and some additional computations are given below in table 1.

Table 1

	Payment Lag									
AY	1	2	3	4	5	6	7	R_i	**Ultimate**	a_i
1	576	1228	166	54	50	28	29	2131	2,131	0.0677
2	866	1082	214	70	52	64		2348	2,380	0.0757
3	1412	2346	494	164	78			4494	4,652	0.1479
4	2286	3006	432	126				5850	6,182	0.1965
5	1868	1910	870					4648	5,056	0.1607
6	1442	2568						4010	4,934	0.1568
7	2044							2044	6,128	0.1948
Total	10,494	12,140	2,176	414	180	92	29	**25,525**	**31,463**	
DF	2.437	1.131	1.029	1.021	1.021	1.014				
CDF	2.998	1.23	1.088	1.057	1.035	1.014	1			
b_j	0.334	0.479	0.106	0.027	0.02	0.02	0.014			
μ x bj	**10,494**	**15,077**	**3,356**	**849**	**618**	**642**	**428**	31,463		

For simplicity, we have computed ultimate losses using the loss development method. They could have easily been computed using an iterative procedure. The column $\mathbf{a}_i$ is accident year ultimate losses divided by the sum of estimated ultimate losses for all accident years, and represents the proportion of total losses for the accident year. We shall use the term exposure level to represent this quantity. The bottom two rows are the payment pattern and the total losses for the payment lag respectively. If we used the iteration procedure, the solution would converge at these values. In table 2 below, we give the residuals for each accident year and payment year. These are computed by subtracting the estimated values from observed data. The estimated values are the bottom row times the $\mathbf{a}_i$s for the corresponding row and columns.

Table 2

Residuals	Payment Year							
AY	1	2	3	4	5	6	7	Sum
1	(134.76)	206.86	(61.30)	(3.49)	8.14	(15.46)	-	0.00
2	72.06	(58.64)	(39.90)	5.78	5.24	15.46		0.00
3	(139.65)	116.76	(2.21)	38.49	(13.38)			0.00
4	224.23	43.89	(227.35)	(40.77)				0.00
5	181.79	(512.55)	330.76					0.00
6	(203.68)	203.68						0.00
7	0.00							0.00
Sum	0.00	0.00	0.00	0.00	0.00	0.00	0.00	**0.00**

Looking at these residuals, the second payment for accident year 5 seems to be an outlier. One can remove this observation and revise the estimate. We will be constructing this estimate later in the paper for estimating the variability of our reserve estimates. The residuals can be further analyzed as to whether there is a systematic variation from the model and some adjustments to the model can be made as needed. For the current data set the model seems quite good. The model statistics are given in the table below in table 3.

Table 3

	Sum of Squares	DF	MSS
Total	23,568,917	27	872,923
Error	704,033	12	58,669
Explained	22,864,884	15	1,429,055
		F	24.36
		R^2	0.97

The R^2 is unusually high for this data set and tells us that the estimated parameters fit the model very well. We have computed some basic model testing statistics. One may compute a host of other statistics for testing the appropriateness of the model. We shall not pursue these in detail, as that is not the theme of the paper. We shall focus on skill of the model statistics recently introduced by Yi Jing, Joseph R. Lebens, and Stephen P. Lowe (2009) to the actuarial field. However, they used it quite differently by computing it through the observed future with predicted future. The modeling procedure presented here allows us to compute it for a current data set and test how good the model will be for predicting the future. It may be a bit confusing that we need to look for additional statistics even if the explained ratio is quite high or other statistics indicate that the model is a good fit. One can think of the skill of the model as testing for model specification error. The assumption that we estimated both the exposure level as well as the payment pattern allows us to estimate the model skill. We have mentioned before that the iterative procedure can be used by removing individual observations. The skill of a model is defined as

$$Skill = 1 - \frac{SSA}{SSE} \tag{5.1}$$

where SSE is the average squared error of estimation by fitting all observed data points,

and SSA is the average squared error of estimation error of individual observations estimated by removing that observation and estimating it from the remaining observations. This following example will help clarify. We remove the first observed value from our data set and estimate the parameters. These parameters provide a new estimate for X_{11}. The original estimate of X_{11} was obtained by using all data points including observed X_{11}. We do this for each of the other observations. The square of the error of the second estimate from the observed value is averaged over all data points to compute SSA. In our case we can compute it for all but two observations. The following table displays the results of this computation along with some additional data that we will need for analysis in the next section.

Table 4

				Exposure Proportion						
AY	Lag	ESS	Error	AY 1	AY 2	AY 3	AY 4	AY 5	AY 6	AY 7
1	1	697,832	226.5	0.0749	0.0755	0.1474	0.1957	0.16	0.1559	0.1907
1	2	818,383	-438	0.0544	0.0767	0.1503	0.1999	0.1637	0.1602	0.1948
1	3	698,045	77	0.07	0.0754	0.1473	0.1957	0.16	0.1568	0.1948
1	4	703,998	4	0.0678	0.0756	0.1478	0.1964	0.1607	0.1568	0.1948
1	5	704,770	-11	0.0675	0.0757	0.148	0.1965	0.1607	0.1568	0.1948
1	6	702,858	30	0.0682	0.0751	0.1479	0.1965	0.1607	0.1568	0.1948
2	1	715,150	-121	0.0678	0.0718	0.1481	0.1969	0.161	0.1573	0.197
2	2	718,065	129	0.0674	0.0795	0.1472	0.1955	0.1598	0.1558	0.1948
2	3	704,950	51	0.0676	0.0771	0.1475	0.196	0.1602	0.1568	0.1948
2	4	703,358	-7	0.0678	0.0755	0.1479	0.1966	0.1607	0.1568	0.1948
2	5	703,700	-7	0.0678	0.0755	0.148	0.1965	0.1607	0.1568	0.1948
2	6	703,051	-33	0.0682	0.0751	0.1479	0.1965	0.1607	0.1568	0.1948
3	1	723,121	267	0.0675	0.0754	0.156	0.1956	0.1599	0.1557	0.1899
3	2	740,956	-286	0.0685	0.0765	0.1399	0.1987	0.1626	0.159	0.1948
3	3	703,886	3	0.0677	0.0756	0.1479	0.1964	0.1607	0.1568	0.1948
3	4	702,123	-57	0.068	0.0759	0.1466	0.1972	0.1607	0.1568	0.1948
3	5	704,873	28	0.0675	0.0754	0.1483	0.1965	0.1607	0.1568	0.1948
4	1	894,943	-453	0.0682	0.0762	0.1489	0.1825	0.162	0.1586	0.2036
4	2	725,306	-119	0.0681	0.076	0.1486	0.1934	0.1615	0.1577	0.1948
4	3	712,011	375	0.0665	0.0742	0.1451	0.2051	0.1575	0.1568	0.1948
4	4	702,149	71	0.0674	0.0753	0.1471	0.1979	0.1607	0.1568	0.1948
5	1	667,123	-352	0.0682	0.0761	0.1488	0.1977	0.1494	0.1582	0.2015
5	2	2,404,396	1436	0.0639	0.0714	0.1395	0.1854	0.1984	0.1466	0.1948
5	3	674,610	-485	0.0695	0.0777	0.1518	0.2017	0.1477	0.1568	0.1948
6	1	771,868	438	0.067	0.0748	0.1462	0.1942	0.1589	0.172	0.187
6	2	938,021	-598	0.0698	0.0779	0.1523	0.2024	0.1655	0.1374	0.1948

The first two columns represent the accident year and the payment year of the observation that was removed from the estimation process. The third column is the total error sum of squares for all observed values and column four is the estimation error of the observed value that was removed from the fitting. One can see that the error sum of squares are comparable to the error sum of squares of 704.03, which was computed based on fitting the model to all data points except for the error sum of squares for the second payment for accident year 5. Most of this variation is coming from the estimation error of this observation itself, as the corresponding residual is quite high (1,436 in the table). This observation is over-estimated a little more when it is removed from the fitting. This gives further credence to the previous statement that this observed value is probably an outlier in the data set. The data set overall appears to be well-behaved and the model appears to perform quite well as the total error sum of squares remains fairly constant when other individual data points are removed from the estimation process. We also captured the estimated accident year contribution to the all accident year estimated ultimate loss in each scenario, which we shall be using in estimating variance. These values are in columns 5 to 11.

The skill of the model is one minus the average of sum of squares of column 4 divided by the average error sum of squares with all data points included in the analysis. Its value is 0.79 for this data.

We will not pursue here the removal of the outliers and revising the estimates. We only broach this issue to point out that the modeling process presented allows us to identify such data elements and adjustments can be made as warranted. However, removal of the second payment for accident year 5 will result in accident year 5 ultimate losses of 6,617 instead of 5,056.

In table 5, we provide our analysis for the corresponding incurred loss triangle.

Table 5

	Payment Lag									
AY	1	2	3	4	5	6	7	**Sum**	**Ult**	a_i
1	978	1126	30	10	30	8	-8	2174	2,174	
2	1844	708	-86	14	28	-54		2454	2,445	0.0739
3	2904	1450	344	-98	44			4644	4,582	0.1385
4	3502	2456	112	72				6142	6,126	0.1852
5	2812	2070	-30					4852	4,839	0.1463
6	2642	1764						4406	4,476	0.1353
7	5022							5022	8,429	0.2549
Sum	19,704	9,574	370	-2	102	-46	-8	**29,694**	**33,071**	
DF	1.65	1.02	1	1.01	0.99	1				
CDF	1.68	1.02	1	1	0.99	1	1			
b_j	0.596	0.389	0.018	0.000	0.011	-0.010	-0.004			
$\mu \times bj$	**19,704**	**12,849**	**607**	**-4**	**367**	**-329**	**-122**	33,071		

The estimated ultimate losses from the incurred loss triangle are higher than the paid loss triangle. Accident year 7 is contributing for most of this difference. There is a significant increase in first year incurred loss for accident year 7 compared to earlier accident years. The paid loss triangle does not show such an increase. One will probably give less credence to the ultimate losses derived from incurred loss triangle for accident year 7 unless there is significant increase in the volume of business and is known from some alternative sources.

SECTION 6: VARIABILITY IN LOSS RESERVES

The estimation of variability in loss reserves is becoming an important issue. Although there are some methods available to achieve this, there is no consensus in the actuarial profession. Ad hoc methods are commonly used to derive a range of estimates. One uses a variety of methods or a different data set, paid and incurred loss triangles for example, to derive a range for ultimate losses. A range for ultimate losses is achieved but the assigning of a confidence level is not possible when these types of methods are used. We shall develop a simulation methodology to estimate the variability of the reserve estimates.

We shall again assume that the exposure levels are known and compute its variability. We shall use model (3.2) and further assume that

$$V(e_{ij}) = a_i \times \sigma_j^2 . \tag{6.1}$$

Under these assumptions

$$\hat{\mu} \times b_j = \frac{\sum_{i=1}^{n-j+1} X_{i,j}}{\sum_{i=1}^{n-j+1} a_i} , \tag{6.2}$$

$$\hat{\sigma}_j^2 = \left(\sum_{i=1}^{n-j+1} \frac{X_{i,j}^2}{a_i} - \frac{\sum_{i=1}^{n-j+1} X_{i,j}^2}{\sum_{i=1}^{n-j+1} a_i} \right) \times \frac{1}{n-j+1} \tag{6.3}$$

Since we have only one observation for payment year *n*, the variance cannot be estimated for that period. For our computational example, we have estimated the variance for b_n by the maximum of the variance estimates of b_{n-1} and the average of the variance estimates of b_{n-1} and b_{n-2}.

It must be noted that the variance assumption in equation (6.1) may not be valid. Exposure changes are caused by two factors: changes in volume cause the variance to increase linearly, which is consistent with equation (6.1), and changes in inflation cause variance to increase exponentially. Our formulation of the model is consistent with the way parameters are being estimated. Large changes in inflation may cause this variance to be underestimated slightly.

Under the assumption of independence of future payments,

$$\hat{X}_{i,j} = a_i \times \hat{\mu} \times b_j\,, \tag{6.4}$$

$$\hat{V}(X_{i,j}) = a_i \times \hat{\sigma}_j^2 \times \left(1 + a_i \times \left(\sum_{i=1}^{n-j+1} a_i\right)^{-1}\right). \tag{6.5}$$

However, a_is are not known and are estimated from the same data. Hence our estimate of the variance is understated. We will attack this problem by using bootstrap and simulation methods and use the following well-known equation. It is worth mentioning that equation (6.5) defines the variance for individual incremental payments. The all accident year variance estimates will be larger than the sum of individual accident years due to correlation introduced in accident year estimates by the estimation process.

$$V(X) = E\left(V\left(\frac{X}{A}\right)\right) + V\left(E\left(\frac{X}{A}\right)\right). \tag{6.6}$$

In the previous section we computed values a_i by reducing our observation set by one observation at a time. We can use the results for the exposure levels captured there for estimating the variance of the estimation through simulation. Steps of our simulation approach are as follows.

Step 1. Find minimum and maximum values for each accident year for columns 5 to 11 from table 4.

Step 2. Generate a uniform random variable in the range between minimum and maximum values for each accident year. These are preliminary relative exposures for each of the accident years.

Step 3. These exposure levels will not add to 1. Normalize them by dividing each preliminary exposure by the sum of the preliminary exposure levels.

Step 4. Use the normalized exposure levels in equation (6.2) to (6.5) to estimate the $X_{i,j}$ and its variance.

Step5. Repeat the process 1,000 times and use these to estimate the terms in equation (6.6); treat the result of each iteration as an observation of the corresponding variable.

One can increase the number of iterations if the data has larger variation. One thousand iterations for the current data set were sufficient.

The results for the paid loss triangle are summarized below for each accident year as well as totals for all accident years. One should note that the variance for all accident years is larger than the sum of individual accident years.

Table 6

Simulation Results

AY	Ultimate	Sim-d Ave Ultimate	Expected Variance	Variance Expected	Estimated Variance	ST Dev
1	2,131	2,131	0	0	-	-
2	2,380	2,382	628	11	639	25
3	4,652	4,659	3,677	125	3,802	62
4	6,182	6,188	6,310	373	6,683	82
5	5,056	5,097	6,704	1,672	8,376	92
6	4,934	4,928	45,332	4,091	49,423	222
7	6,128	6,131	295,912	20,055	315,967	562
Total	**31,463**	**31,515**	**450,987**	**26,873**	**477,860**	**691**

The all year total variation is larger than sum of individual accident years because of correlation.

SECTION 7: EXPOSURE DEVELOPMENT METHOD

The concept of the exposure development factor (EDF) method introduced in this paper is very useful. One important area where a lot of attention is being paid is combining the information from paid and incurred loss triangles to refine our estimates. In the 2009 CLRS meeting, there was a full session devoted to this topic. The EDF method provides an elegant way to achieve this. The important characteristic of the EDF method is that, unlike loss development factors, the EDFs for paid and incurred loss triangles are measuring the same quantity and provide two estimates of the relative exposure levels. This property can be exploited with significant improvement in our analysis of loss triangles. One extreme will be to use exposure levels derived from the paid loss triangle to the incurred loss triangle and vice versa. A better way would be to average the exposure levels determined by the paid and incurred loss triangles. The exposure levels from two triangles will be correlated, as the paid losses are included in the incurred losses. The average of the two factors will still be a better estimate. The averaging can be done in a variety of ways. One can average the year-to-year exposure development factors or the normalized exposure levels. One could use differential weights as well.

Once the selection of exposure level for each accident year is made, we use it to determine the payout pattern. In the examples presented earlier, we have used combined payout for all years. However, one can determine each accident year's payout rate separately and then make a selection.

In the loss development method, actuaries use a variety of averaging procedures and professional judgment to select a development factor. Similar analysis can be carried out in determining rates for the selected exposure level. One can take an average after removing high and low values for rates, for example.

In the following table we provide an example. The main purpose of this is to show how the data from the different triangles can be combined and used in a systematic way. In the table below we have adopted an arbitrary weighting scheme to select accident year exposure levels.

Table 7

AY	Paid Exposure Level	Incurred Exposure Level	Weight	Weighted Exposure Level	Selected Exposure Level
1	0.0677	0.0657	0.5	0.0667	0.0684
2	0.0757	0.0739	0.5	0.0748	0.0766
3	0.1479	0.1385	0.5	0.1432	0.1467
4	0.1965	0.1852	0.5	0.1909	0.1955
5	0.1607	0.1463	0.25	0.1499	0.1536
6	0.1568	0.1353	0.25	0.1407	0.1442
7	0.1948	0.2549	0.75	0.2098	0.215
Total	**1.0000**	**1.0000**		**0.9760**	**1.0000**

We have changed weights for accident year 5, 6, and 7. We saw before that the second payment for accident year 5 might be an outlier. It will affect EDFs 4 and 5 and exposure levels so less weight is assigned to the exposure level derived from the paid triangle for these years. The incurred loses for accident year 7 is quite high compared to accident year 6. We do not see that magnitude of increase in paid losses. More weight is therefore given to the exposure level derived from the paid loss triangle.

Now we use these selected exposure levels and the total observed payout by delay for each accident year and select a payout judgmentally. We are a bit conservative in our selection. This is obvious from the fact that the total estimated payout is less than the selected payout.

Table 8

AY	Delay1	Delay2	Delay3	Delay4	Delay5	Delay6	Delay7	Exposure Level
1	576	1,228	166	54	50	28	29	0.0684
2	866	1,082	214	70	52	64		0.0766
3	1,412	2,346	494	164	78			0.1467
4	2,286	3,006	432	126				0.1955
5	1,868	1,910	870					0.1536
6	1,442	2,568						0.1442
7	2,044							0.2150
Sum	10,494	12,140	2,176	414	180	92	29	1.0000

Accident Year Payout

Payout	Delay1	Delay2	Delay3	Delay4	Delay5	Delay6	Delay7	
1	8,424	17,960	2,428	790	731	410	424	
2	11,301	14,119	2,793	913	679	835		
3	9,624	15,990	3,367	1,118	532			
4	11,690	15,372	2,209	644				
5	12,162	12,435	5,664					
6	10,002	17,812						
7	9,509							
Average	10,387	15,615	3,292	866	647	622	424	**31,854**
Weighted	10,494	15,464	3,395	850	617	634	424	**31,879**
Selected	**10,450**	**15,600**	**3,350**	**860**	**640**	**630**	**424**	**31,954**

Estimated Payout all Accident Years

AY	Delay1	Delay2	Delay3	Delay4	Delay5	Delay6	Delay7	Ultimate
1	576	1,228	166	54	50	28	29	2,131
	866	1,082	214	70	52	64	32	2,380
	1,412	2,346	494	164	78	92	62	4,649
	2,286	3,006	432	126	125	123	83	6,181
	1,868	1,910	870	132	98	97	65	5,040
	1,442	2,568	483	124	92	91	61	4,861
	2,044	3,353	720	185	138	135	91	6,666
Sum Proj	10,494	15,493	3,379	855	633	631	424	**31,909**

The incurred loss triangle can be analyzed similarly using the selected exposure levels. We shall not do it here.

Actuaries often use recent accident year data for loss development factor calculations and projections of ultimate losses. Such results are responsive to changes that are too complex to model. The exposure development method is much more flexible and therefore can achieve this. Some care is needed, as the loss payment amount in later lags may be quite thin. It is advisable to use all payment lag data of an accident year for computing the exposure development factors. In the example below, we use the available latest three accident years to compute our exposure development factors. One can directly use these development factors to compute ultimate losses.

However, we have computed payout rates as there is flexibility here. One can use all years' data or the latest three years to determine rates. If we use the latest three years' data, the results will match with the latest three-year weighted loss development method.

One alternative approach that this author prefers is to use all accident year data for exposure development factors and use the latest years' observations for selecting payout rates. Of course, one would use exposure levels derived from incurred loss triangles if available, and compute payout rates based on the latest years or by excluding Hi-Low rates as is done in selecting development factors.

One other possible variation is indicated by examining the incurred loss triangle. The incremental incurred losses for some accident years are negative possibly due to some recoveries or subrogation. These just add additional variation in EDFs. One could compute the EDFs without these values. These data points could be included in computing rates.

SECTION 8: CONCLUSION AND FUTURE RESEARCH

In this paper we have a methodology that in some sense diverges from the common way actuaries look at loss triangles. Results are, however, consistent with loss development method and extend it in several ways. In practice, actuaries use a lot of professional judgment. Allowing judgment to be applied to both the exposure level and payment pattern, we have a two-dimensional selection processes rather than one. Knowledge of both the paid and incurred loss triangles extends that even further. The fact that the EDF method measures the same thing for paid and incurred losses has one other nice implication for excess and reinsurance writers. The paid loss experience is thin and not credible in the first few years. However, the exposure levels derived from incurred loss triangles for early years can be used on paid loss data. We had avoided the issue of tail losses. Perhaps one can use both the paid and incurred rates to derive a suitable decay function.

The author believes that the ideas presented will stimulate other researchers to modify and extend it further. There is ample opportunity to do so. We defined a range of exposure levels by removing one observation at a time and re-computing exposure levels. There may be different ways to achieve this result. One may define a range based on paid and incurred loss triangles or use information from both data sets or premium data. The simulation results in our example assumed uniform distribution in the range. One could use alternative distributions somehow derived from the data. Uniform distributions increase the variance estimates and, in that sense, are conservative estimates of the variance. Estimation of tail factors is another area where further research will be helpful.

The methodology presented in this paper is simple and is for practical use. How it fares in practice can only be determined by practicing actuaries.

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Estimation of Adjusting and Other Expense Reserves Utilizing Limited Historical Claim Report, Payment, and Closing Transaction Patterns

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Abstract: The estimation of adjusting and other expense (AOE) reserves can be constrained by the availability of historical AOE payment data, lack of uniformity of data, and lack of consensus of what AOE represents. Adjusting and other expenses are incurred when claims are first reported and opened, throughout the life of the claims when partial payments and revisions are made, and finally when claims are closed and final payments are issued by the insurer. Our paper will present a variation of the count-based methodology whereby we utilize the limited data presented to us by an insurer to estimate AOE reserves. This paper will attempt to describe how one can use historical reporting, payment (including partial payment), and closing patterns to estimate AOE reserves associated with IBNR counts and future payments on open claims. Assumptions are also made with respect to the average cost of each AOE per claim payment throughout the life of a claim using current AOE payment information and the number of claim payment and closing transactions. The cost of each transaction is developed using historical calendar year AOE amounts, and assumptions of the relative cost of each transaction type.

Keywords: Unallocated loss adjustment expense reserves, adjusting and other expense reserves

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1. INTRODUCTION

1.1 Research Context

In the annals of reserving literature, the review of adjusting and other expenses (AOE) seems to have become relegated to second-tier status. In the current CAS syllabus only one paper (R. Conger/A. Nolibos) on the reserving exam addresses the topic, and in the 2009 Casualty Loss Reserve Seminar there were no sessions that covered it. Furthermore, there has been a limited number of existing actuarial subject literature on this topic, particularly in recent years.

The limited existing actuarial literature (see Section 5 – References) on this topic either assumes that detailed, historical payment information is readily available or reviews and/or modifies the existing paid to paid methodologies on this topic.

There are a number of reasons why there is such little focus by actuaries on these expenses. Based on year-end 2008 Schedule P data for the U.S. P&C industry, adjusting and other unpaid losses represented only 5.2% of total net unpaid losses and expenses. Generally, there are no specific case reserves set up for AOE reserves as there are for the loss and defense and cost containment (DCC) reserves.

In addition, payment information used to develop estimates of AOE reserves is not generally available in a format that lends itself to a traditional actuarial analysis. This is due to a number of factors discussed below including:

- Inconsistencies in definition of AOE
- Lack of "triangular" data
- Availability of internal studies
- Operational variations among companies

1.2 Objective

The objective of this paper is to describe a methodology we developed in order to estimate AOE reserves for a single line of business based on a unique situation presented to us by an insurer where their historical expense data and detailed expense information (such as payment information) was limited or not available.

Whereas most actuarial literature on this topic assumes that information to estimate AOE reserves is readily available, there are situations where the insurer does not capture certain information, does not capture this information in the requisite detail, or undergoes changes in their claim handling function making it impractical to use historical information. The methodology described in this paper relies on certain assumptions and calculations related to the limited historical information available in this situation, based on discussions with insurance company personnel, particularly around the costs associated with various claim handling transactions, and actuarial judgment.

1.2.1 Lack of clarity in the definition of AOE

According to statement of Statutory Accounting Principles (SSAP) # 55, AOE are those expenses other than DCC that are assigned to the expense group "Loss Adjustment Expense." Whereas DCC are defined in SSAP #55 as defense, litigation, and medical cost containment expenses, AOE include, but are not limited to, the following items:

(a) fees and expenses of adjusters and settling agents;

(b) loss adjustment expenses for participation in voluntary and involuntary market pools if reported by calendar year;

(c) attorney fees incurred in the determination of coverage, including litigation between the reporting entity and the policyholder; and

(d) fees and salaries for appraisers, private investigators, hearing representatives, inspectors and fraud investigators, if working in the capacity of an adjuster.

There are at least two issues associated with this definition that may arise. The first relates to the fact that AOE includes all expenses other than DCC assigned to the expense group "Loss Adjustment Expense." Although four examples are given, it is less clear how specific overhead costs should be included and how they should be reserved for if a company goes into runoff. The second arises when claims handling services are outsourced to a third party and the overall, inseparable fee may include costs for both AOE and DCC services.

1.2.2 Lack of "triangular" data

Since typical triangular data by accident year is not available for AOE, standard actuarial methodologies employing loss triangles (or loss expense triangles in this case) cannot be used. The actuary is forced to use calendar-year data, and apply various assumptions specific to what that data represents, to both determine the AOE reserve amounts required, and allocate it to accident year.

1.2.3 Availability of internal studies

There is a wide variation among companies related to how they determine the portion of expenses attributable to AOE. Based on some of the detailed methodologies, it would appear that some of the more sophisticated procedures involve internal studies that relate salaries and other costs to time spent on specific activities. This might vary significantly among companies in their frequency (if conducted at all) and methodology. Often it may be a byproduct of other operational efficiency studies conducted by the company when they reorganize to reduce expenses.

1.2.4 Operational variations among companies

Although all companies go through some type of process to open claims, set initial reserves, revise those reserves as needed, make payments and close claims, how they go about doing it may vary considerable. Some have their own claim departments, while others may use third-party administrators (TPAs), or have the insured settle their own claims below a certain dollar amount. Even within a company's claim department, relative costs could vary significantly based on lines of business written, the degree of automation, the use of predictive modeling, and other factors.

Recognizing these limitations, the actuary is often forced to come up with procedures to take advantage of the data provided, and make assumptions as to exactly what AOE is meant to represent, and how it relates to various measurable claim activities. It is clear that, even if the best data were available, the methodologies that would be appropriate to determine the AOE reserves for one company might not be the best for another company with a different book of business, maturity of the book, or corporate structure. Thus it is best for the actuary to develop and consider a menu of various techniques to establish reserves.

The determination of AOE reserves is also different from other reserve reviews in that it often involves both allocation and reserve determination calculations. Initially a portion of the calendar year expense payments needs to be allocated to AOE. It may also need to be allocated to individual lines of business as well. This is then used to develop the overall reserve requirement, and that reserve then needs to be allocated to accident year, line of business (if not done previously), etc. For the methodology described in this paper, we assume there is only one line of business, but it could be expanded to include multiple lines.

For the actuary, the calculation of the AOE reserves often comes down to the information available, knowledge of the claim department structure and how it handles claims, and selected assumptions based on information from management. As a result there may be a wide variation in methodologies used. This paper presents a methodology we deployed faced with one such unique set of constraints.

1.3 Outline

The remainder of the paper proceeds as follows. Section 2 will discuss the background of our methodology, and then go into further details of the calculations and formulas used. A sample calculation is presented to explain the methodology and clarify the discussion. Our methodology will also discuss two different approaches an actuary can take – one without trended AOE costs, and another assuming trended AOE costs. Section 3 includes the results of this methodology using the sample data shown throughout this paper, along with a discussion of additional enhancements that could be made to the methodology. Section 4 presents our conclusions and main findings of this paper. An appendix is included, providing exhibits relating to the sample calculation referred to throughout this paper. Finally, references are provided in Section 5.

2. BACKGROUND AND METHODS

2.1 Information Available

The methodology and approach described below was developed to estimate AOE reserves for a single line of business with a moderate tail based on a unique situation presented to us by an insurer where their historical expense data was limited.

For this situation transaction counts (closings, partial payments, etc.) and AOE payments were only useful for the most recent 1-3 years because of recent changes in the insurer's claim handling operations. Thus the limited information that we did use in our methodology included:

1. Historical accident year claim reporting patterns
2. Open claim counts by accident year for the last three calendar year ends
3. Number of claims closed in the last two calendar years
4. Number of claim payment transactions in the last calendar year
5. Estimated calendar year AOE payments for the last year

In our methodology, we had to make certain assumptions around the costs associated with various claim handling transactions. These assumptions are described throughout the methodology discussion below.

2.2 Methodology Overview

In our methodology, based on our approach and assumptions made, estimates of AOE reserves are determined for expenses associated with:

1. Initial reporting of IBNR claims,
2. Future payments for IBNR claims and claims already reported and currently considered open, and
3. Future closings for IBNR claims and claims already reported and currently considered open.

It should be noted that the closings and interim payments include those on both known and IBNR claims combined, as this process does not allow them to be identified separately.

Our methodology assumes that the insurer does not incur any expenses when claims are reopened. Furthermore, we assume that no expenses are incurred for claim maintenance on outstanding claims for which no payments are issued to the insured during the year.

Utilizing the available information described above, and incorporating assumptions regarding the relative average AOE costs associated with various actions taken on a claim and the impact of inflation on future such costs, various estimates of AOE reserves were made. Based on our understanding of claim handling expenses for this particular insurer, they incur considerably more expenses when a claim is first reported to them as opposed to when payments are made (including interim payments) or when a claim is closed. This is inconsistent with the 50-50 rule actuaries typically use to establish ULAE/AOE reserves, which assumes that 50% of ULAE/AOE is paid when the claim is first opened and the other 50% when it is closed.

In addition to the assumption we made regarding the relative average AOE cost for a given transaction, our methodology calculates the following to develop the indicated AOE reserve:

1. the ultimate claim reporting pattern
2. an estimate of the ratio of number of claims closed to the number of payment and closing transactions

3. percentage of total reported claims that remain open at the end of each calendar year based on the maturity of the accident year

As described in more detail below, the average transaction cost assumptions and the selected AOE trend factor are applied to the various estimated transaction counts. To calculate these counts, the items listed above were used to project:

1. estimated IBNR claim counts,
2. number of payment and closing transactions, for both known and IBNR claims combined, and
3. the timing of the reporting, payment and closing transactions.

2.2.1 Estimate of IBNR Claim Counts

The estimate of IBNR claim counts is determined by using a reported claim count loss development triangle. Age-to-age claim development factors are selected at 12-month intervals, and age-to-ultimate claim development factors are computed, as shown in Appendix Exhibit A.1. Using the selected age-to-age factors, reported claim counts are developed to ultimate, as shown in Appendix Exhibit A.2. Taking the differences in the implied cumulative reported claim counts at future year ends from Appendix Exhibit A.2, the incremental claims to be reported in each future calendar year can be obtained (Appendix Exhibit A.3). The number of reported claim counts, ultimate claim counts, and IBNR counts by accident year are shown in Table 1 below:

Table 1 – Summary of Reported, IBNR, and Ultimate Claim Counts

Accident Year	(1) Reported Claims at 12/31/08	(2) Ultimate Claim Count	(3) IBNR Claims (2)-(1)
1993	995	995	0
1994	979	979	0
1995	816	816	0
1996	1,182	1,182	0
1997	1,376	1,377	1
1998	1,442	1,444	2
1999	1,418	1,421	3
2000	1,534	1,542	8
2001	1,572	1,582	10
2002	1,758	1,772	14
2003	1,999	2,019	20
2004	2,610	2,659	49
2005	2,888	2,970	82
2006	2,684	2,886	202
2007	2,226	2,526	300
2008	1,744	2,670	926
Total	27,223	28,840	1,617

IBNR claim counts are computed by subtracting reported claim counts from the ultimate claim counts estimated. These counts are used to estimate the AOE associated with the first reporting of claims and are combined with current open claim counts to estimate the AOE associated with future claim payments (for both partial payments and payments to close claims).

2.3 Methodology – No Trending

2.3.1 AOE Reserves Associated with the Initial Reporting of Future Claims

A key assumption we used to estimate the AOE reserves is the relative relationship between the average AOE cost when a claim is first reported to an insurer versus when a payment is made or a claim is closed. Using calendar year AOE payments, the number of claims reported in a calendar year, the number of payments and closings made in a calendar year, and the relationship of average costs by type of transaction, we estimated the average AOE per reported claim and per claim payment/closing transaction.

The average AOE incurred based on the type of transaction (when a claim is first reported or a payment or closing is made) is not readily available to most insurers. The average costs by transaction type can be approximated based on an understanding of the claim handling function of an insurance company and interviews with those involved in the process. This would allow one to better discern the AOE costs incurred when a claim is first reported to an insurer relative to the costs incurred when a claim payment or closing is made. As a result of these considerations we utilized a ratio of five to one (5:1). In other words, for every $5 spent on AOE when a claim is first reported, $1 is spent on AOE every time a claim payment is made or a claim is closed.

Using this relationship, along with the amount of paid AOE and the number of claim payments and closings in a calendar year, we can estimate the average AOE to open a claim, as shown in Table 2. When utilizing calendar year data, information from the most recent calendar year or an average from a number of calendar years can be used based on the availability of data.

Table 2 – Average AOE Cost to Open a Claim

	5:1 Ratio
(1) CY AOE Payments	$6,105,000
(2) Number of Claims Reported	2,594
(3) Number of Payments & Closings Made	3,339
(4) Average AOE to Open a Claim	$1,871.67

Row (4), the average AOE cost to open a claim, is calculated as follows: 5 x Row (1) ÷ [Row (2) x 5 + Row (3)] ***or*** (5 x $6,105,000 / [2,594 x 5 + 3,339] = $1,871.67.

In order to estimate the AOE reserves associated with opening of the IBNR claims, we multiply the average AOE per claim reported in Row (4) of Table 2 by the IBNR counts in Column (3) of Table 1 ($1,871.67 x 1,617 = $3,026,484). This produces an estimate of the AOE reserves associated with opening of IBNR claims based on the five to one cost relativity and assuming no increase in future costs to set a claim in the future (i.e. no trend).

2.3.2 Projection of Claim Settlement and Payment Pattern

The claim settlement pattern can be projected in a similar manner to the claim reporting pattern as discussed above by utilizing closed claim development triangles. Unfortunately, for our situation, we were limited to two years of calendar year closed claim count data. Thus, using this limited data, we devised a method to estimate the claim settlement pattern using only recent claim reporting and open claim count information as described in the following paragraphs.

Table 3 displays the open claim count and reported claim count information for the last three years with evaluation dates as of December 31.

Table 3 – Open Claim and Reported Claim Count

	(1)	(2)	(3)	(4)	(5)	(6)
Accident		Open Claim Counts			Reported Claim Counts	
Year	As of YE08	As of YE07	As of YE06	As of YE08	As of YE07	As of YE06
1993	3	4	4	995	995	995
1994	4	8	12	979	979	979
1995	7	13	14	816	816	816
1996	5	9	11	1,182	1,181	1,172
1997	17	22	28	1,376	1,376	1,376
1998	13	20	25	1,442	1,442	1,442
1999	19	28	38	1,418	1,418	1,418
2000	23	34	49	1,534	1,534	1,532
2001	47	64	99	1,572	1,569	1,568
2002	74	129	255	1,758	1,757	1,752
2003	197	285	440	1,999	1,981	1,962
2004	405	601	876	2,610	2,579	2,456
2005	875	1,134	1,459	2,888	2,678	2,477
2006	987	1,222	1,553	2,684	2,622	1,982
2007	1,455	1,612		2,226	1,702	
2008	1,350			1,744		
Total	**5,481**	**5,185**	**4,863**	**27,223**	**24,629**	**21,927**

Using the claim count information summarized in Table 3, we calculate the percentage of reported claims that were still open as of the three most recent evaluation dates (December 31, 2006, 2007, and 2008) by dividing Columns (1) through (3) by Columns (4) through (6) respectively. The results are displayed in Table 4, Columns (1) through (3) below. Next, based on this three-year history, we select the percentage of reported claims that we would expect to be open at different evaluation periods (12 months, 24 months, etc.) as shown in Column (4).

Table 4 – Claim Settlement Pattern

Maturity (months)	(1) in 2008	(2) Percentage of Reported Claims That are Open in 2007	(3) in 2006	(4) % of Claims Open Selection
192	0.3%			**0.5%**
180	0.4%	0.4%		**0.7%**
168	0.9%	0.8%	0.4%	**1.0%**
156	0.4%	1.6%	1.2%	**1.5%**
144	1.2%	0.8%	1.7%	**2.0%**
132	0.9%	1.6%	0.9%	**2.2%**
120	1.3%	1.4%	2.0%	**2.5%**
108	1.5%	2.0%	1.7%	**3.0%**
96	3.0%	2.2%	2.7%	**3.5%**
84	4.2%	4.1%	3.2%	**4.2%**
72	9.9%	7.3%	6.3%	**8.5%**
60	15.5%	14.4%	14.6%	**15.0%**
48	30.3%	23.3%	22.4%	**25.0%**
36	36.8%	42.3%	35.7%	**35.0%**
24	65.4%	46.6%	58.9%	**55.0%**
12	77.4%	94.7%	78.4%	**85.0%**

Using the selected percentages of reported claims that are open in Column (4), adjusting it for the actual percentage at the latest point in time (Column (1)) and utilizing the actual number of claims reported and open as of year-end 2008 (from Table 3) and estimated to be reported in calendar year 2009 (from Appendix Exhibit A.3), we are able to project the number of claims that will close in calendar year 2009. Thus, for example, the number of claim closings in calendar year 2009 for accident year 2008 is calculated as follows:

Reported Claims in CY 2009 – [Estimated Reported Claims at YE 2009 x Column (1) {12 Months} x Column (4) {24 Months] / Column (4) {12 Months}] + Open Claims at YE 2008, ***or***

610 – [2,354 x 0.774 x 0.55 / 0.85] + 1,350 = 781, where:

- 610 is the number of AY 2008 claims expected to be reported in calendar year 2009
- 2,354 is the number of AY 2008 total reported claims expected through 12/31/09
- 1,350 is the number of open claims as of 12/31/2008
- 77.4% is the actual percentage of AY 2008 reported claims that are open as of 12/31/08
- 55% is the selected percentage of reported claims that are open for a typical AY at 24 months based on historical data

- 85% is the selected percentage of reported claims that are open for a typical AY at 12 months based on historical data

The calculations used to project the number of claims to be closed in calendar year 2010 and subsequent are made in conjunction with the estimation of the number of open claims at the beginning of each calendar year along with the number of claims that are expected to be reported during the calendar year. The number of new claims reported in future calendar years (from Appendix Exhibit A.3) is reduced by these closings (as shown in Table 5 below), and is then added to the open claim counts from the beginning of the year to determine the open claim counts for the beginning of the following year (as shown in Table 6 below). Thus, for accident year 2008, in our example we estimate that there will be 1,179 open claims as of 12/31/09 (as shown in Table 6), which is equal to:

$$1{,}350 + 610 - 781 = 1{,}179, \text{ where:}$$

- 1,350 claims that are open as of 12/31/08 (from Table 3),
- 610 claims that are projected to be reported in calendar year 2009 (as shown in Appendix Exhibit A.3),
- 781 claims that are estimated to be closed in calendar year 2009 (as calculated previously and shown in Table 5 below).

Using this estimate of open claims for AY 2008 as of 12/31/09 and a new ratio of open to reported claims (50.1%) is calculated for AY 2008 at 24 months. This is shown in Appendix Exhibit A.4. The same calculation of calendar year closings and the resulting number of open claims at the end of the year (and the resulting new ratio of open to reported claims) is repeated for each subsequent calendar year until all claims reported have been closed.

For AY 2008, in calendar year 2010, we expect 517 claims to close based on the following calculation:

$$129 – [2{,}483 \times 0.501 \times 0.35 / 0.55] + 1{,}179 = 517, \text{ where:}$$

- 129 is the number of AY 2008 claims expected to be reported in calendar year 2010
- 2,483 is the number of AY 2008 total reported claims expected through 12/31/10
- 1,179 is the previously calculated number of open claims as of 12/31/2009
- 50.1% is the calculated percentage of AY 2008 reported claims that are open as of 12/31/09

- 35% is the selected percentage of reported claims that are open for a typical AY at 36 months based on historical data
- 55% is the selected percentage of reported claims that are open for a typical AY at 24 months based on historical data

The same calculation is also made for each of the accident years prior to 2008.

Thus, Tables 5 and 6, which display the projected closed and open claim patterns by accident year and calendar year, respectively, can be built up as a result of this calculation using the reported claim pattern in Appendix Exhibit A.2, and the selected percentages of claims open from Table 4, Column (4).

Table 5 – Projected Future Closed Claim Counts

AY	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024
1993	3															
1994	1	3														
1995	2	1	4													
1996	2	1	1	2												
1997	5	4	3	2	4											
1998	2	4	3	2	1	3										
1999	3	3	5	4	2	2	4									
2000	6	4	4	6	4	2	2	4								
2001	9	9	6	5	9	7	4	3	7							
2002	16	11	11	7	6	11	9	5	4	9						
2003	105	20	14	14	9	7	14	12	7	5	12					
2004	195	125	24	17	17	11	8	17	14	8	6	14				
2005	374	250	162	31	21	21	14	10	21	18	11	7	18			
2006	371	318	214	138	27	18	18	12	9	18	15	9	6	15		
2007	600	354	312	209	134	26	18	18	12	9	18	15	9	6	15	
2008	781	517	313	259	175	111	22	15	15	10	8	15	12	7	5	12

Table 6 – Projected Open Claim Counts

	Number of Claims at Open Year-End														
AY	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023
1993	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1994	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1995	5	4	-	-	-	-	-	-	-	-	-	-	-	-	-
1996	3	2	2	-	-	-	-	-	-	-	-	-	-	-	-
1997	13	9	6	4	-	-	-	-	-	-	-	-	-	-	-
1998	12	9	6	4	3	-	-	-	-	-	-	-	-	-	-
1999	17	15	11	8	5	4	-	-	-	-	-	-	-	-	-
2000	19	17	15	12	8	5	4	-	-	-	-	-	-	-	-
2001	40	34	30	27	20	14	9	7	-	-	-	-	-	-	-
2002	62	53	44	39	35	27	18	12	9	-	-	-	-	-	-
2003	98	82	70	58	51	47	35	23	16	12	-	-	-	-	-
2004	231	115	96	82	69	60	55	41	28	19	14	-	-	-	-
2005	530	303	150	125	108	90	79	72	54	36	25	18	-	-	-
2006	737	446	255	126	106	91	76	67	61	45	30	21	15	-	-
2007	977	729	442	252	125	104	90	75	66	60	45	30	21	15	-
2008	1,179	791	591	358	205	101	85	73	61	53	49	36	24	17	12

The next step is to determine the total number of claim payment and closing transactions by accident year and calendar year using the calculated closed counts. Again, we had limited information to work with, since only the 2008 interim claim payment counts were available. Using this information, we examined the ratio of closings to total claim payment and closing transactions. This is shown on Table 7 below.

Table 7 – Claim Payments and Closed Claim Patterns

Accident Year	(1) Claims Closed in 2008	(2) 2008 Claim Payment & Closing Counts	(3) Percentage of Closings to Total Payments & Closings (1)/(2)	(4) Selected Percentage of Closings to Total Payments & Closings
1993	1	1	100%	**100%**
1994	4	4	100%	**100%**
1995	6	6	100%	**100%**
1996	5	6	83%	**100%**
1997	5	7	71%	**98%**
1998	7	7	100%	**95%**
1999	9	10	90%	**93%**
2000	11	13	85%	**92%**
2001	20	27	74%	**90%**
2002	56	66	85%	**90%**
2003	106	122	87%	**87%**
2004	227	259	88%	**85%**
2005	469	586	80%	**75%**
2006	297	540	55%	**60%**
2007	681	1,012	67%	**50%**
2008	394	673	59%	**40%**
Total	**2,298**	**3,339**		

As shown above in Table 7, the percentage of closings to total payments and closings during the year increases as the accident year matures. Using the selected percentages in Column (4), we then project the pattern of claim payments and closings to be made by accident year and calendar year as shown in Table 8 below.

Table 8 – Projected Number of Claim Payments and Closings to be Made

AY	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	Total
1993	3																3
1994	1	3															4
1995	2	1	4														7
1996	2	1	1	2													5
1997	5	4	3	2	4												18
1998	2	4	3	2	1	3											15
1999	3	3	5	4	2	2	4										22
2000	6	5	4	6	4	2	2	4									32
2001	9	9	6	5	9	7	4	3	7								59
2002	18	12	12	8	6	11	9	5	4	9							92
2003	117	22	15	15	9	7	14	12	7	5	12						234
2004	224	139	27	18	18	12	9	17	14	8	6	14					503
2005	440	288	180	34	23	22	14	10	21	18	11	7	18				1,086
2006	495	374	246	153	30	20	19	13	9	18	15	9	6	15			1,423
2007	1,001	471	367	241	149	28	19	19	13	9	18	15	9	6	15		2,381
2008	1,562	861	417	304	201	124	24	16	16	11	8	15	12	7	5	12	3,595
Total (G)	3,890	2,197	1,288	792	456	237	118	99	90	78	69	60	45	28	20	12	9,479

The projected number of combined claim payments and closings to be made in Table 8 is determined by dividing the projected number of claims closed by accident year and calendar year in Table 5 by the selected ratio of payments and closings made to closed claims from Table 7, Column (4). Thus, for accident year 2008, we project 1,562 claim payments and closings to be made in 2009, by dividing the 781 claims we expect to close in the year by 50%, which is the percentage of claims that we expect to close from 12 to 24 months after policy inception.

Thus, the projected total number of payment and closing transactions is the sum of all the future calendar year claim payment and closing counts in Table 8, or 9,479. Using the average AOE to open a claim as estimated in Table 2, and the five to one relationship, the implied cost of each payment and closing transaction is $374.33 ($1,871.67 ÷ 5). We can then estimate the AOE reserves associated with the future claim payment and closing transactions in Table 8 (9,479 x $374.33, or $3,548,248). It should be noted that the closings and interim payments include those on both known and IBNR claims combined, as this process does not allow them to be identified separately.

By combining the estimates of AOE reserves associated with the cost of opening future IBNR claims from Section 2.3.1 ($3,026,484) with the estimates of AOE reserves in Table 8 associated with the projected claim payments and closings ($3,548,248), we obtain an estimate of AOE reserves ($6,574,732) that does not reflect the impact of inflation.

2.4 Methodology – Reflecting Trend

Realistically, AOE costs increase over time as a result of inflationary pressures. Using the timing of the claim reporting and closing patterns as outlined in Appendix Exhibit A.2 and Table 8, respectively, along with the average AOE per transaction type, we are able to apply a trending procedure to develop additional estimates for AOE reserves. The following calculations and tables assume that the average AOE cost to open a claim is five times higher than the AOE cost associated with each subsequent payment or claim closing. First, we apply the average AOE cost related to opening a claim of $1,871.67 (from Table 2, Row (4)) trended by 4.0% annually, to the incremental claim counts to be reported in each future calendar-year diagonal shown in Appendix Exhibit A.3. This provides an estimate of the AOE reserves associated with the opening of the IBNR claims assuming trended AOE costs, which are shown in Appendix Exhibit A.5.

For example, for accident year 2008, the 610 incremental claim counts expected to be reported in 2009 are multiplied by the expected cost to open a claim in 2009 ($1,946.53 or $1,871.67 x 1.04), to produce $1,187,385 in required AOE reserves. Summing all the values in Appendix Exhibit A.5 gives the total AOE reserve required for the opening of future IBNR claims of $3,287,931. (Note: differences may exist due to rounding)

Next, the estimate of AOE reserves associated with the future claim payments and closures is made using the projected pattern of claim payment and closing transactions from Table 8. The same procedure used for the newly reported claims is used here, except that the underlying starting cost that is used is $374.33 rather than $1,871.67 (assuming an AOE cost relativity of five times). The resulting estimate of the AOE reserves for all future payments (partial and closing) is shown in Appendix Exhibit A.6.

For example, for accident year 2008, the 1,562 claim payments to be made in 2009 are multiplied by the expected cost to pay a claim in 2009 ($389.30 or $374.33 x 1.04), to produce $608,047 in required AOE reserves. Summing all the values in Appendix Exhibit A.6 gives the total AOE reserve required for the future claim payments of $3,956,876.

By combining the total estimates of AOE reserves in Appendix Exhibits A.5 and A.6 we get the total AOE reserves estimate of $7,244,807, which assumes an annual trend of 4.0%.

3. RESULTS AND DISCUSSION

The methodology described above provides us with an estimate of AOE reserves using a specific set of assumptions made by the actuary. To test the sensitivity of the results, we ran the model changing one of our assumptions (the relativity of the cost of paying/settling a claim and opening it) from a 5:1 ratio to a 3:1 ratio. The results from each of these are summarized in Table 9 below.

Table 9 – Summary of AOE Reserves

		AOE Relativity 5X	AOE Relativity 3X
No Trend	Reporting of IBNR Claims	3,026,484	2,663,012
	Future Payments and Closings on Open & IBNR Claims	3,548,248	5,203,522
	Total	**6,574,732**	**7,866,534**
Trend	Reporting of IBNR Claims	3,287,931	2,893,060
	Future Payments and Closings on Open & IBNR Claims	3,956,876	5,802,778
	Total	**7,244,807**	**8,695,838**

The results from Table 9 demonstrate the sensitivity of estimates of AOE reserves to the AOE relativity and application of a trend factor. As the ratio of AOE incurred when a claim is first reported versus when a payment is made or claim is settled decreases (from 5:1 to 3:1) the estimate of the AOE reserves will increase because the average AOE cost associated with claim payment and closing transactions is higher, and the number of claim payment and closing transactions estimated are greater than the number of IBNR claims. Likewise, the methodology assuming an annual trend of 4.0% produces estimates of AOE reserves that are approximately 10% higher than the estimates without annual trend.

One consideration for modifying this methodology is to also reflect the average AOE cost related to maintaining a claim that remains open during a calendar year regardless of whether there were payments or revisions made to that claim. This could be reflected in the model by assuming a fixed cost to maintain any claim that is open during the year. A relativity could be established between this fixed cost per open claim and the cost to open or make payments on a claim. Similarly, the AOE cost of revising a claim estimate could be included provided the appropriate transaction information is available. Consideration can also be given to different costs associated with the reopening of claims previously closed.

This methodology could also be modified to reflect different assumptions associated with the relativity of AOE costs for claim payment and closing transactions, and also to reflect partial accident years (i.e., for a quarterly evaluation). In order to reflect partial accident years for the most recent 12-month period, we would have to apply several changes to the calculations in our methodology, particularly around those calculations working off of the ultimate claim counts for the most recent accident year, 2008.

4. CONCLUSIONS

For the actuary, the calculation of the AOE reserves often comes down to the information available, knowledge of the claim department structure and how it handles claims, and selected assumptions based on information from management. As a result there may be a wide variation in methodologies and assumptions used by the actuary. An actuary needs to recognize the limitations to him or her and work with the information that is available to come up with a reasonable methodology, which was the primary reason for us developing the methodology described in this paper.

As noted throughout the paper, considerable uncertainty may be due to assumptions made because certain information is not available to the actuary. Resulting estimates of AOE reserves may vary based on differing assumptions of:

- Annual trend percentage
- Relationship between AOE costs by transaction type
- Claim reporting age to age development factors
- Selected claim payment pattern
- Selected percentage of claims that will close during a calendar year

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Appendix Exhibit A.1

Reported Claim Development Triangle

Accident Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192
1993	406	767	901	945	965	975	975	980	980	980	990	995	995	995	995	995
1994	390	755	881	940	945	964	964	964	969	969	979	979	979	979	979	
1995	408	618	738	774	798	804	804	804	804	810	810	816	816	816		
1996	581	882	945	980	1,043	1,099	1,132	1,144	1,157	1,170	1,172	1,181	1,182			
1997	737	1,231	1,300	1,360	1,368	1,368	1,368	1,376	1,376	1,376	1,376	1,376				
1998	865	1,264	1,341	1,379	1,386	1,411	1,442	1,442	1,442	1,442	1,442					
1999	847	1,282	1,322	1,370	1,378	1,418	1,418	1,418	1,418	1,418						
2000	936	1,388	1,482	1,496	1,526	1,532	1,532	1,534	1,534							
2001	1,005	1,485	1,518	1,560	1,566	1,568	1,569	1,572								
2002	1,201	1,553	1,722	1,748	1,752	1,757	1,758									
2003	1,498	1,769	1,866	1,962	1,981	1,999										
2004	1,749	2,367	2,456	2,579	2,610											
2005	1,847	2,477	2,678	2,888												
2006	1,982	2,622	2,684													
2007	1,702	2,226														
2008	1,744															

Link Ratios	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192
1993	1.889	1.175	1.049	1.021	1.010	1.000	1.005	1.000	1.000	1.010	1.005	1.000	1.000	1.000	1.000	
1994	1.936	1.167	1.067	1.005	1.020	1.000	1.000	1.005	1.000	1.010	1.000	1.000	1.000	1.000		
1995	1.515	1.194	1.049	1.031	1.008	1.000	1.000	1.000	1.007	1.000	1.007	1.000	1.000			
1996	1.518	1.071	1.037	1.064	1.054	1.030	1.011	1.011	1.011	1.002	1.008	1.001				
1997	1.670	1.056	1.046	1.006	1.000	1.000	1.006	1.000	1.000	1.000	1.000					
1998	1.461	1.061	1.028	1.005	1.018	1.022	1.000	1.000	1.000	1.000						
1999	1.514	1.031	1.036	1.006	1.029	1.000	1.000	1.000	1.000							
2000	1.483	1.068	1.009	1.020	1.004	1.000	1.001	1.000								
2001	1.478	1.022	1.028	1.004	1.001	1.001	1.002									
2002	1.293	1.109	1.015	1.002	1.003	1.001										
2003	1.181	1.055	1.051	1.010	1.009											
2004	1.353	1.038	1.050	1.012												
2005	1.341	1.081	1.078													
2006	1.323	1.024														
2007	1.308															
2008																
All Per Avg	1.484	1.082	1.042	1.016	1.014	1.005	1.003	1.002	1.003	1.004	1.004	1.000	1.000	1.000	1.000	
All Per Wtd Avg	1.404	1.067	1.043	1.013	1.013	1.005	1.003	1.002	1.002	1.003	1.004	1.000	1.000	1.000	1.000	
5 Yr Avg	1.301	1.061	1.045	1.010	1.009	1.005	1.002	1.002	1.004	1.002	1.004					
5 Yr Wtd Avg	1.306	1.057	1.049	1.010	1.009	1.004	1.002	1.002	1.003	1.002	1.004					
3 Yr Avg	1.324	1.047	1.060	1.008	1.004	1.000	1.001	1.000	1.000	1.001	1.005	1.000	1.000			
3 Yr Wtd Avg	1.324	1.047	1.061	1.009	1.005	1.000	1.001	1.000	1.000	1.001	1.004	1.000	1.000			
Selected A-A	**1.350**	**1.055**	**1.045**	**1.010**	**1.008**	**1.003**	**1.002**	**1.001**	**1.001**	**1.001**	**1.001**	**1.001**	**1.000**	**1.000**	**1.000**	
Selected A-U	1.530	1.134	1.075	1.028	1.018	1.010	1.007	1.005	1.004	1.003	1.002	1.001	1.000	1.000	1.000	**1.000**

Appendix Exhibit A.2

Developed Reported Claim Counts to Ultimate

AY	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	Ult
1993	406	767	901	945	965	975	975	980	980	980	990	995	995	995	995	995	**995**
1994	390	755	881	940	945	964	964	964	969	969	979	979	979	979	979	*979*	**979**
1995	408	618	738	774	798	804	804	804	804	810	810	816	816	816	*816*	*816*	**816**
1996	581	882	945	980	1,043	1,099	1,132	1,144	1,157	1,170	1,172	1,181	1,182	*1,182*	*1,182*	*1,182*	**1,182**
1997	737	1,231	1,300	1,360	1,368	1,368	1,368	1,376	1,376	1,376	1,376	1,376	*1,377*	*1,377*	*1,377*	*1,377*	**1,377**
1998	865	1,264	1,341	1,379	1,386	1,411	1,442	1,442	1,442	1,442	1,442	*1,443*	*1,444*	*1,444*	*1,444*	*1,444*	**1,444**
1999	847	1,282	1,322	1,370	1,378	1,418	1,418	1,418	1,418	1,418	*1,419*	*1,420*	*1,421*	*1,421*	*1,421*	*1,421*	**1,421**
2000	936	1,388	1,482	1,496	1,526	1,532	1,532	1,534	1,534	*1,536*	*1,538*	*1,540*	*1,542*	*1,542*	*1,542*	*1,542*	**1,542**
2001	1,005	1,485	1,518	1,560	1,566	1,568	1,569	1,572	*1,574*	*1,576*	*1,578*	*1,580*	*1,582*	*1,582*	*1,582*	*1,582*	**1,582**
2002	1,201	1,553	1,722	1,748	1,752	1,757	1,758	*1,762*	*1,764*	*1,766*	*1,768*	*1,770*	*1,772*	*1,772*	*1,772*	*1,772*	**1,772**
2003	1,498	1,769	1,866	1,962	1,981	1,999	*2,005*	*2,009*	*2,011*	*2,013*	*2,015*	*2,017*	*2,019*	*2,019*	*2,019*	*2,019*	**2,019**
2004	1,749	2,367	2,456	2,579	2,610	*2,631*	*2,639*	*2,644*	*2,647*	*2,650*	*2,653*	*2,656*	*2,659*	*2,659*	*2,659*	*2,659*	**2,659**
2005	1,847	2,477	2,678	2,888	*2,917*	*2,940*	*2,949*	*2,955*	*2,958*	*2,961*	*2,964*	*2,967*	*2,970*	*2,970*	*2,970*	*2,970*	**2,970**
2006	1,982	2,622	2,684	*2,805*	*2,833*	*2,856*	*2,865*	*2,871*	*2,874*	*2,877*	*2,880*	*2,883*	*2,886*	*2,886*	*2,886*	*2,886*	**2,886**
2007	1,702	2,226	*2,348*	*2,454*	*2,479*	*2,499*	*2,506*	*2,511*	*2,514*	*2,517*	*2,520*	*2,523*	*2,526*	*2,526*	*2,526*	*2,526*	**2,526**
2008	1,744	*2,354*	*2,483*	*2,595*	*2,621*	*2,642*	*2,650*	*2,655*	*2,658*	*2,661*	*2,664*	*2,667*	*2,670*	*2,670*	*2,670*	*2,670*	**2,670**

* Reported Claim Counts in italics are projected.

Appendix Exhibit A.3

Incremental Future Reported Claim Counts by Accident Year and Calendar Year

AY	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192
1993	406	361	134	44	20	10	0	5	0	0	10	5	0	0	0	0
1994	390	365	126	59	5	19	0	0	5	0	10	0	0	0	0	*0*
1995	408	210	120	36	24	6	0	0	0	6	0	6	0	0	*0*	*0*
1996	581	301	63	35	63	56	33	12	13	13	2	9	1	*0*	*0*	*0*
1997	737	494	69	60	8	0	0	8	0	0	0	0	*1*	*0*	*0*	*0*
1998	865	399	77	38	7	25	31	0	0	0	0	*1*	*1*	*0*	*0*	*0*
1999	847	435	40	48	8	40	0	0	0	0	*1*	*1*	*1*	*0*	*0*	*0*
2000	936	452	94	14	30	6	0	2	0	*2*	*2*	*2*	*2*	*0*	*0*	*0*
2001	1,005	480	33	42	6	2	1	3	*2*	*2*	*2*	*2*	*2*	*0*	*0*	*0*
2002	1,201	352	169	26	4	5	1	*4*	*2*	*2*	*2*	*2*	*2*	*0*	*0*	*0*
2003	1,498	271	97	96	19	18	*6*	*4*	*2*	*2*	*2*	*2*	*2*	*0*	*0*	*0*
2004	1,749	618	89	123	31	*21*	*8*	*5*	*3*	*3*	*3*	*3*	*3*	*0*	*0*	*0*
2005	1,847	630	201	210	*29*	*23*	*9*	*6*	*3*	*3*	*3*	*3*	*3*	*0*	*0*	*0*
2006	1,982	640	62	*121*	*28*	*23*	*9*	*6*	*3*	*3*	*3*	*3*	*3*	*0*	*0*	*0*
2007	1,702	524	*122*	*106*	*25*	*20*	*7*	*5*	*3*	*3*	*3*	*3*	*3*	*0*	*0*	*0*
2008	1,744	*610*	*129*	*112*	*26*	*21*	*8*	*5*	*3*	*3*	*3*	*3*	*3*	*0*	*0*	*0*

* Incremental reported claim counts in italics are projected.

Appendix Exhibit A.4

Projected Percentages of Reported Claims that will be Open at Subsequent Calendar Year Ends

AY	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024
1993	*0.3%*	0.0%															
1994	*0.4%*	0.3%	0.0%														
1995	*0.9%*	0.6%	0.4%	0.0%													
1996	*0.4%*	0.3%	0.2%	0.1%	0.0%												
1997	*1.2%*	0.9%	0.6%	0.4%	0.3%	0.0%											
1998	*0.9%*	0.8%	0.6%	0.4%	0.3%	0.2%	0.0%										
1999	*1.3%*	1.2%	1.1%	0.8%	0.5%	0.4%	0.3%	0.0%									
2000	*1.5%*	1.2%	1.1%	1.0%	0.7%	0.5%	0.3%	0.2%	0.0%								
2001	*3.0%*	2.6%	2.1%	1.9%	1.7%	1.3%	0.9%	0.6%	0.4%	0.0%							
2002	*4.2%*	3.5%	3.0%	2.5%	2.2%	2.0%	1.5%	1.0%	0.7%	0.5%	0.0%						
2003	*9.9%*	4.9%	4.1%	3.5%	2.9%	2.6%	2.3%	1.7%	1.2%	0.8%	0.6%	0.0%					
2004	*15.5%*	8.8%	4.3%	3.6%	3.1%	2.6%	2.3%	2.1%	1.6%	1.0%	0.7%	0.5%	0.0%				
2005	*30.3%*	18.2%	10.3%	5.1%	4.2%	3.6%	3.0%	2.7%	2.4%	1.8%	1.2%	0.8%	0.6%	0.0%			
2006	*36.8%*	26.3%	15.8%	8.9%	4.4%	3.7%	3.2%	2.6%	2.3%	2.1%	1.6%	1.1%	0.7%	0.5%	0.0%		
2007	*65.4%*	41.6%	29.7%	17.8%	10.1%	5.0%	4.2%	3.6%	3.0%	2.6%	2.4%	1.8%	1.2%	0.8%	0.6%	0.0%	
2008	*77.4%*	50.1%	31.9%	22.8%	13.7%	7.7%	3.8%	3.2%	2.7%	2.3%	2.0%	1.8%	1.4%	0.9%	0.6%	0.5%	0.0%

* Calendar Year-End 2008 percentages in italics are actual

Appendix Exhibit A.5

Estimates of AOE Reserves Associated with the Opening of IBNR Claims Assuming Trended AOE

AY	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	Ult
1993																	**0**
1994																0	**0**
1995															0	0	**0**
1996														0	0	0	**0**
1997													1,947	0	0	0	**1,947**
1998												1,947	2,024	0	0	0	**3,971**
1999											1,947	2,024	2,105	0	0	0	**6,076**
2000										3,893	4,049	4,211	4,379	0	0	0	**16,532**
2001									3,893	4,049	4,211	4,379	4,554	0	0	0	**21,086**
2002								7,786	4,049	4,211	4,379	4,554	4,737	0	0	0	**29,716**
2003							11,679	8,098	4,211	4,379	4,554	4,737	4,926	0	0	0	**42,583**
2004						40,877	16,195	10,527	6,569	6,832	7,105	7,389	7,685	0	0	0	**103,178**
2005					56,449	46,561	18,948	13,138	6,832	7,105	7,389	7,685	7,992	0	0	0	**172,098**
2006				235,530	56,683	48,424	19,706	13,663	7,105	7,389	7,685	7,992	8,312	0	0	0	**412,488**
2007			237,477	214,586	52,634	43,792	15,940	11,841	7,389	7,685	7,992	8,312	8,644	0	0	0	**616,291**
2008		1,187,385	261,147	235,801	56,929	47,821	18,946	12,315	7,685	7,992	8,312	8,644	8,990	0	0	0	**1,861,966**
																Total	**3,287,931**

Appendix Exhibit A.6

Estimates of AOE Reserves Associated with Claim Closings and Payments, Assuming AOE Trend

	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	Total
1993	1,168																**1,168**
1994	445	1,157															**1,602**
1995	818	567	1,474														**2,858**
1996	649	405	281	730													**2,064**
1997	2,040	1,722	1,075	745	1,937												**7,519**
1998	863	1,599	1,246	777	539	1,401											**6,426**
1999	1,339	1,037	2,020	1,668	1,041	721	1,876										**9,702**
2000	2,431	1,825	1,511	2,555	1,755	1,095	759	1,974									**13,905**
2001	3,666	3,779	2,660	2,083	3,973	3,200	1,997	1,385	3,600								**26,343**
2002	7,004	4,739	4,885	3,349	2,558	5,134	4,374	2,729	1,892	4,920							**41,586**
2003	45,578	9,047	6,214	6,404	4,291	3,202	6,728	5,996	3,742	2,594	6,745						**100,540**
2004	87,103	56,092	11,195	7,893	8,136	5,562	4,236	8,551	7,328	4,573	3,170	8,243					**212,082**
2005	171,170	116,538	75,677	14,968	10,295	10,610	7,096	5,286	11,148	9,972	6,223	4,314	11,217				**454,516**
2006	192,690	151,615	103,779	66,969	13,587	9,261	9,544	6,471	4,887	10,028	8,737	5,452	3,780	9,828			**596,628**
2007	389,530	190,860	154,651	105,441	67,989	13,494	9,538	9,833	6,671	5,043	10,337	8,996	5,613	3,892	10,119		**992,007**
2008	608,047	348,623	175,517	133,318	91,361	58,498	11,896	8,357	8,613	5,955	4,585	9,052	7,578	4,729	3,278	8,524	**1,487,929**
																Total	**3,956,876**

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Abbreviations and Notations

AOE, Adjusting and Other Expenses
DCC, Defense and Cost Containment
IBNR, Incurred but Not Reported
SSAP, Statement of Statutory Accounting Principles
ULAE, Unallocated Loss Adjustment Expenses

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Bootstrap Modeling: Beyond the Basics

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Abstract

Motivation. Bootstrapping is a very versatile model for estimating a distribution of possible outcomes for the unpaid claims, is relatively easy to use and explain to others, and can be readily "generalized" to be more flexible and combined with other related models that can be used to assess risk for a wide variety of enterprise risk management issues. While the CAS literature includes several papers that describe the bootstrap model, all of these papers are limited to the basic calculations of the model or focus on a particular aspect of the model. In contrast, this paper outlines the modifications to the basic algorithm that are required in order to put the bootstrap model into practical everyday use.

Method. This paper will start by pulling all of the issues from different papers into the complete basic bootstrap modeling framework using a standard notation. Then it will describe some of the enhancements required for practical usage and it will show how the output of the model can be easily "extended" to address other risk management issues. It will then expand the basic model and generalize the approach, as well as address many common modeling issues that arise during the diagnostic testing of the model parameters and assumptions. Finally, it will summarize testing of the model using simulated data and suggest possible areas for further research.

Results. The paper will illustrate the practical implementation of the bootstrap modeling framework as a powerful tool for estimating a distribution of unpaid claims.

Conclusions. The paper outlines the full versatility of the bootstrap model for the practicing actuary.

Availability. A set of companion Excel files are available at http://www.casact.org/pubs/forum/10fforum/, which contains the calculations illustrated in this paper as well as serving as a learning tool for the student or practicing actuary.

Keywords. Bootstrap, Over-Dispersed Poisson, Reserve Variability. Reserve Range, Distribution of Possible Outcomes.

1. INTRODUCTION

The term "bootstrap" has a colorful history that dates back to German folk tales of the 18th-century. It is aptly conveyed in the familiar cliché admonishing laggards to "pull oneself up by their own bootstraps." A physical paradox and virtual impossibility, the idea has nonetheless caught the imagination of scientists in a broad array of fields, including physics, biology and medical research, computer science, and statistics.

Bradley Efron, Chairman of the Department of Statistics at Stanford University, is most often associated as the source of expanding bootstrapping into the realm of statistics, with his notion of taking one available sample and using it to arrive at many others through resampling. His essential strategy involves duplicating the original sample and then treating the expanded sample that results from the process as a virtual population. Samples are then drawn with replacement from this population to verify the estimators.

In actuarial science, bootstrapping has become increasingly common in the process of loss reserving. The most commonly cited examples point to England and Verrall [9, 10], Pinheiro, et al. [25], and Kirschner, et al. [15], who suggest using a basic chain ladder technique to square a triangle of paid losses, repeating that randomly and stochastically over a large number of trials. The model generates a distribution of possible outcomes, rather than the chain ladder's typical point estimate, thus providing more information about the potential results. For example, without an estimated distribution it is impossible to directly estimate the amount of capital required[1] or how likely it is that the ultimate value of the claims will exceed a certain amount.

Another advantage of a bootstrap model is that it can be specifically tailored to the statistical features found in the data under analysis. This is particularly important as the results of any simulation model are only as good as the model used in the simulation process. If the model does not fit the data then the results of the simulation may not be a very good estimate of the distribution of possible outcomes. Like all models and methods, the quality of a bootstrap model depends on the quality of the assumptions. Thus, we will elaborate on the model diagnostics in Section 4.

A third advantage of a bootstrap model is that it can reflect the fact that insurance loss distributions are generally "skewed to the right." Rather than use the commonly recognized normal distribution (which is sometimes used as a simplifying assumption in other models), the bootstrap sampling process does not require a distributional assumption. Instead, the level of skewness in the underlying data is automatically reflected back into the resampled or pseudo data.

Another aspect of bootstrap models that could be considered a disadvantage is that they are more complex than other methods and thus more time consuming to create. However, once a flexible model has been developed they can be used as efficiently as most standard methods.

There are several disadvantages of bootstrap models that we will discuss in due course as we describe how this framework can be modified for a variety of practical uses.[2]

1.1 Objectives

The world of enterprise risk management is changing the horizon for actuaries. Understanding the central estimate for insurance claims is no longer adequate when managing risk. Actuaries must now measure and understand the distribution of the insurance claims in order to better understand and explain risk to management. On the pricing and dynamic risk modeling fronts, the actuarial

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[1] Without an estimated distribution, required capital could be "estimated" using industry benchmark ratios or other rules of thumb, but these do not directly account for the specific risk profile under review.

[2] This section is based in large part on [22].

models have already embraced this new reality.

Unfortunately, in the reserving area the vast majority of actuaries are focused on deterministic point estimates for reserving. This is not surprising, as our primary standard of practice for reserving, ASOP 36, seems to be focused exclusively on deterministic point estimates and the regulators, via the actuarial opinion, are also focused on deterministic estimates. However, actuaries are free to estimate distributions instead of point estimates.[3] But nothing seems to be forcing the profession towards unpaid claim distributions.

This is changing due to a number of factors:

- the SEC is looking for more reserving risk information in the 10-K reports filed by publicly traded companies;
- all of the major rating agencies have built or are building dynamic risk models to help with their insurance rating process and welcome the input of company actuaries regarding unpaid claim distributions; and
- companies that use dynamic risk models to help their internal risk management processes need unpaid claim distributions.

One objective of this paper is to show how the bootstrap modeling framework can be used in practice, to help the wider adoption of unpaid claim distributions.

Another potential roadblock seems to be the notion that actuaries are still searching for the perfect model to describe "the" distribution of unpaid claims, as if imperfections in a model remove it from all consideration since it can't be "the one." This notion can also manifest itself when an actuary settles for a model that seems to work the best or is the easiest to use, or with the idea each model must be used in its entirety or not at all. Interestingly, this notion was dispelled long ago with respect to practice for deterministic point estimates as actuaries commonly use many different methods, which range from easy to complex, and judgmentally weight the results by accident year (i.e., use only parts of a method) to arrive at their best estimate. Thus, another objective of this paper is to show how stochastic reserving needs to be similar to deterministic reserving when it comes to analyzing and using the best parts of multiple models.

Finally, most of the papers describing stochastic models, including bootstrap models, tend to focus primarily on the theoretical aspects of the model while ignoring the data issues that commonly

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[3] Indeed, ASOP 43 opened the door a bit further by defining "actuarial central estimate" in such a way that it could include either deterministic point estimates or a first moment estimate from a distribution.

arise in practice. As a result, most models described in papers can be quite elegantly implemented yet can suffer from practical limitations such as only being useful for complete triangles or only for positive incremental values. This could also act as a deterrent by limiting the usability of a model to specific situations and by giving the impression that using the model is not worth the effort. Thus, while keeping as close to the theoretical foundation as possible, another objective of the paper is to illustrate how a variety of practical adjustments can be made to accommodate common data issues.

1.2 Outline

This paper will start by reviewing the notation from the CAS Working Party on Quantifying Variability in Reserve Estimates Summary Report [6] which we will use in this paper. Then we will illustrate and expand the foundation developed in other papers for the basic calculations of the bootstrap model, including showing how the GLM framework of the model can be "generalized" to include diagonal parameters. In order to be consistent with the theoretical foundation yet recognize practical needs, we will describe data issues that require enhancements to the basic algorithm. With a complete modeling framework established, we can then review the diagnostic tests to ensure that the model assumptions are consistent with the statistical features in the data. Should the assumptions appear inconsistent, we will suggest adjustments to the model that can be made.

Even though bootstrapping is a very versatile framework, it is still important to draw from the strengths of different models and weight distributions, similar to weighting point estimates, in order to get a best estimate of the distribution. Thus, we will briefly explore ways to combine the results of different models, including non-bootstrap models with bootstrap models. Since the analysis of enterprise risk involves all sources of risk, we will also explore correlation issues for the bootstrap model and then describe extensions to the model output and how they can be used for assessing risks in addition to reserve risk. In order to use the results with confidence, we will briefly discuss some findings related to testing of the model compared to another commonly used model (Mack). Finally, we will close with some possible areas for future research.

2. NOTATION

The papers that describe the basic bootstrap model use different notation, despite sharing common steps. Rather than pick the notation in one of the papers, we will use the notation from the CAS Working Party on Quantifying Variability in Reserve Estimates Summary Report [6] since it is intended to serve as a basis for further research and the bootstrap model is also described in that paper.

Many models visualize loss statistics as a two-dimensional array. The row dimension is the annual period by which the loss information is subtotaled, most commonly an accident year or policy year. For each accident period, w, the (w,d) element of the array is the total of the loss information as of development age d.[4] Here the development age is the accounting year[5] of the loss information expressed as the number of time periods after the accident or policy year. For example, the loss statistic for accident year 2 as of the end of calendar year 4 has development age 3 years.

For this discussion, we assume that the loss information available is an "upper triangular" subset of the two-dimensional array for rows $w = 1,2,\ldots,n$. For each row, w, the information is available for development ages 1 through $n-w+1$. If we think of year n as the most recent accounting year for which loss information is available, the triangle represents the loss information as of accounting dates 1 through n. The diagonal $k = w + d$ represents the loss information for each accident period w as of accounting year k.[6]

The paper uses the following notation for certain important loss statistics:

$c(w,d)$:	cumulative loss from accident[7] year w as of age d. (Think w = "when" and d = "delay")
$c(w,n) = U(w)$:	total loss from accident year w when claims are at ultimate values.
$R(w,d)$:	future development after age d for accident year w, i.e., = $U(w) - c(w,d)$.
$q(w,d)$:	incremental loss for accident year w from d - 1 to d.
$f(d)$:	factor applied to $c(w,d)$ to estimate $q(w,d+1)$ or can be used more generally to indicate any factor relating to age d.
$F(d)$:	factor applied to $c(w,d)$ to estimate $c(w,n)$ or can be used more generally to indicate any cumulative factor relating to age d.
$G(w)$:	factor relating to accident year w – capitalized to designate ultimate loss level.
$h(w+d)$:	factor relating to the diagonal k along which $w+d$ is constant.

4 Depending on the context, the (w,d) cell can represent the cumulative loss statistic as of development age d or the incremental amount occurring during the d th development period.

5 The development ages are assumed to be in yearly intervals for ease of discussion. However, they can be in different time units such as half-years, quarters, or months.

6 For a more complete explanation of this two-dimensional view of the loss information, see the *Foundations of Casualty Actuarial Science* [12], Chapter 5, particularly pages 210-226.

7 The use of accident year is also used for ease of discussion. All of the discussion could also apply to underwriting year, policy year, report year, etc.

$e(w,d)$: a mean zero random fluctuation which occurs at the w, d cell.

$E(x)$: the expectation of the random variable x.

$Var(x)$: the variance of the random variable x.

What are called factors here could also be summands, but if factors and summands are both used, some other notation for the additive terms would be needed. The notation does not distinguish paid vs. incurred, but if this is necessary, capitalized subscripts P and I could be used.

3. THE BOOTSTRAP MODEL

Even though many variations of the bootstrap model framework are possible, we will focus primarily on the most common example that essentially reproduces the basic chain ladder method. It will also be helpful to briefly review the assumptions that underpin the basic chain ladder method.

The foundation for any model is the data being modeled. Like many commonly used models, then, we will start with a triangle array of cumulative data:

		d					
		1	**2**	**3**	**…**	**n-1**	**n**
w	**1**	c(1,1)	c(1,2)	c(1,3)	…	c(1,n-1)	c(1,n)
	2	c(2,1)	c(2,2)	c(2,3)	…	c(2,n-1)	
	3	c(3,1)	c(3,2)	c(3,3)	…		
	…	…	…				
	n-1	c(n-1,1)	c(n-1,2)				
	n	c(n,1)					

A typical deterministic analysis of this data will start with an array of age-to-age ratios or development factors:

$$F(w,d)=\frac{c(w,d)}{c(w,d-1)}. \tag{3.1}$$

Then two key assumptions are made in order to make a projection of the known elements to their respective ultimate values. First, it is assumed that each accident year has the same development factor. Equivalently, for each $w=1,2,\ldots,n$:

$$F(w,d)=F(d).$$

Under this first assumption, one of the more popular estimators for the development factor is the weighted average:

$$\hat{F}(d)=\frac{\sum_{w=1}^{n-d+1}c(w,d)}{\sum_{w=1}^{n-d+1}c(w,d-1)}. \tag{3.2}$$

Certainly there are other popular estimators in use, but they are beyond our scope at this stage yet most are still consistent with our first assumption that each accident year has the same factor. Projections of the ultimate values, or $\hat{c}(w,n)$ for *w* = 1, 2, 3, … , *n*, are then computed using:

$$\hat{c}(w,n)=c(w,d)\prod_{i=d+1}^{n}\hat{F}(i). \tag{3.3}$$

This part of the claim projection algorithm relies explicitly on the second assumption, namely that each accident year has a parameter representing its relative level. These level parameters are the current cumulative values for each accident year, or $c(w,n-w+1)$. Of course variations on this second assumption are also common, but the point is that every model has explicit assumptions that are an integral part of understanding the quality of that model.

One variation on the second assumption is to assume that the accident years are completely homogeneous. In this case we would estimate the level parameter of the accident years using:

$$\frac{\sum_{w=1}^{n-d+1} c(w,d)}{n-d+1}. \tag{3.4}$$

Complete homogeneity implies that the observations $c(1,d)$, $c(2,d)$, …, $c(n-d+1,d)$ are generated by the same mechanism. Interestingly, the basic chain ladder algorithm explicitly assumes that the mechanisms generating the observations are NOT homogeneous and effectively that "pooling" of the data does not provide any increased efficiency.[8] In contrast, it could be argued that the Bornhuetter-Ferguson and Cape Cod methods are a "blend" of these two extremes as the homogeneity of the future expected result depends on the consistency of the a priori loss ratios and decay rate, respectively.

3.1 Origins of Bootstrapping

Possibly the earliest development of a stochastic model for the actuarial array of cumulative development data is attributed to Kremer [16]. The basic model described by Kremer can be defined by the multiplicative representation:

$$P(w,d) = G'(w) \times F'(d) \times e'(w,d). \tag{3.5}$$

Where: $G'(w)$ is a parameter representing the effect of accident year *w*,
$F'(d)$ is a parameter representing the effect of development period *d*, and
$e'(w,d)$ is a random error term.

Taking logarithms of both sides of equation (3.5), the model can be formulated as a two-way analysis of variance:

$$Y(w,d) = \log[P(w,d)] = \mu + G(w) + F(d) + e(w,d). \tag{3.6}$$

Where: μ is the overall mean effect on a log scale,
$G(w)$ is the residual effect due to accident year *w*,
$F(d)$ is the residual effect due to development period *d*,
$e(w,d)$ represent zero mean uncorrelated errors with $Var[e(w,d)] = \sigma^2$, and

$$\sum G(w) = \sum F(d) = 0. \tag{3.7}$$

This model is further described by England and Verrall [9] and Zehnwirth [39], so we will not elaborate further here. It should be noted, however, that the model in (3.6) can be extended by considering alternatives. This log-normal model, and generalizations thereof, has also been discussed in Zehnwirth [1, 40], Renshaw [30], Christofides [7], and Verrall [37, 38], among others.

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8 For a more complete discussion of these assumptions of the basic chain ladder model see Zehnwirth [39].

3.2 The Over-Dispersed Poisson Model

The genesis of this model into a bootstrap framework originated with Renshaw and Verrall [31] when they proposed modeling the incremental claims $q(w,d)$ directly as the response, with the same linear predictor as Kremer [16], but using a generalized linear model (GLM) with a log-link function and an over-dispersed Poisson (ODP) error distribution. Then, England and Verrall [9] discuss how this model can be used to estimate parameters and use bootstrapping (sampling the residuals with replacement) to estimate the complete distribution. More formally, using the following:

$$E[q(w,d)] = m_{w,d} \text{ and } Var[q(w,d)] = \phi E[q(w,d)] = \phi m_{w,d}^{z} \tag{3.8}$$

$$\ln[m_{w,d}] = \eta_{w,d} \tag{3.9}$$

$$\eta_{w,d} = c + \alpha_w + \beta_d \text{, where: } w = 1, 2, \ldots, n;\ d = 1, 2, \ldots, n; \text{ and } \alpha_1 = \beta_1 = 0. \tag{3.10}$$

In this case the α parameters function as adjustments to the constant, *c*, level parameter and the β parameters adjust for the development trends after the first development period. The power, z, is used to specify the error distribution with $z=0$ for normal, $z=1$ for Poisson, $z=2$ for Gamma and $z=3$ for inverse Gaussian. Alternatively, we can remove the constant which will cause the α parameters to function as individual level parameters while the β parameters continue to adjust for the development trends after the first development period:

$$\eta_{w,d} = \alpha_w + \beta_d \text{, where: } w = 1, 2, \ldots, n; \text{ and } d = 2, \ldots, n. \tag{3.11}$$

Standard statistical software can be used to estimate parameters and goodness of fit measures. The parameter ϕ is a scale parameter that is estimated as part of the fitting procedure while setting the variance proportional to the mean (thus "over-dispersed" Poisson for $z=1$). For educational purposes, we have included the calculations to solve these equations for a 10 x 10 triangle in the "Bootstrap Models.xls" file, but we will illustrate the calculations here for a 3 x 3 triangle for ease of exposition and in the "Simple GLM.xls" file. Consider the following incremental data triangle:

	1	**2**	**3**
1	$q(1,1)$	$q(1,2)$	$q(1,3)$
2	$q(2,1)$	$q(2,2)$	
3	$q(3,1)$		

In order to set up the GLM model to fit parameters to the data we need to do a log-link or transform which results in:

	1	**2**	**3**
1	$\ln[q(1,1)]$	$\ln[q(1,2)]$	$\ln[q(1,3)]$
2	$\ln[q(2,1)]$	$\ln[q(2,2)]$	
3	$\ln[q(3,1)]$		

The model is then specified using a system of equations with vectors of α_w and β_d parameters as follows:

$$\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\beta_2 + 0\beta_3 \\
\ln[q(2,1)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\beta_2 + 0\beta_3 \\
\ln[q(3,1)] &= 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 0\beta_2 + 0\beta_3 \\
\ln[q(1,2)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 + 0\beta_3 \\
\ln[q(2,2)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 1\beta_2 + 0\beta_3 \\
\ln[q(1,3)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 + 1\beta_3 .
\end{aligned} \tag{3.12}$$

Converting this to matrix notation we have:

$$\text{Y} = \text{X x A} \tag{3.13}$$

Where:

$$\text{Y} = \begin{bmatrix} \ln[q(1,1)] & 0 & 0 & 0 & 0 & 0 \\ 0 & \ln[q(2,1)] & 0 & 0 & 0 & 0 \\ 0 & 0 & \ln[q(3,1)] & 0 & 0 & 0 \\ 0 & 0 & 0 & \ln[q(1,2)] & 0 & 0 \\ 0 & 0 & 0 & 0 & \ln[q(2,2)] & 0 \\ 0 & 0 & 0 & 0 & 0 & \ln[q(1,3)] \end{bmatrix}, \tag{3.14}$$

$$\text{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \text{ and} \tag{3.15}$$

$$\text{A} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_2 \\ \beta_3 \end{bmatrix}. \tag{3.16}$$

In this form we can use the Newton-Raphson method[9] to solve for the parameters in the A vector that minimize the difference between the Y matrix and the W matrix:

[9] Other methods, such as orthogonal decomposition, can also be used to solve for the parameters.

$$W = \begin{bmatrix} \ln[m_{1,1}] & 0 & 0 & 0 & 0 & 0 \\ 0 & \ln[m_{2,1}] & 0 & 0 & 0 & 0 \\ 0 & 0 & \ln[m_{3,1}] & 0 & 0 & 0 \\ 0 & 0 & 0 & \ln[m_{1,2}] & 0 & 0 \\ 0 & 0 & 0 & 0 & \ln[m_{2,2}] & 0 \\ 0 & 0 & 0 & 0 & 0 & \ln[m_{1,3}] \end{bmatrix}. \tag{3.17}$$

Typically, X is known as the design matrix and W is known as the weight matrix. After solving the system of equations we will have:

$$\begin{aligned} \ln[m_{1,1}] = \eta_{1,1} &= \alpha_1 \\ \ln[m_{2,1}] = \eta_{2,1} &= \alpha_2 \\ \ln[m_{3,1}] = \eta_{3,1} &= \alpha_3 \\ \ln[m_{1,2}] = \eta_{1,2} &= \alpha_1 + \beta_2 \\ \ln[m_{2,2}] = \eta_{2,2} &= \alpha_2 + \beta_2 \\ \ln[m_{1,3}] = \eta_{1,3} &= \alpha_1 + \beta_2 + \beta_3 . \end{aligned} \tag{3.18}$$

This solution can then be shown as a triangle:

	1	**2**	**3**
1	$\ln[m_{1,1}]$	$\ln[m_{1,2}]$	$\ln[m_{1,3}]$
2	$\ln[m_{2,1}]$	$\ln[m_{2,2}]$	
3	$\ln[m_{3,1}]$		

These results can then be exponentiated to the fitted, or expected, incremental results of the GLM model:

	1	**2**	**3**
1	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$
2	$m_{2,1}$	$m_{2,2}$	
3	$m_{3,1}$		

We will refer to this as the "GLM framework" and have illustrated this model for a simple 3 x 3 triangle in the "Simple GLM.xls" file. While the GLM framework is used to solve these equations for the fitted results, the usefulness of this framework is that the fitted results (with the Poisson error distribution assumption) will exactly equal the results that can be derived from volume-weighted average age-to-age ratios. That is, it can be reproduced by using the last cumulative diagonal, dividing backwards successively by each age-to-age factor and subtracting to get the fitted incremental results. We will refer to this method as the "simplified GLM". This has three very useful consequences.

First, GLM portion of the algorithm can be replaced with a simpler link ratio algorithm while still being based on the underlying GLM framework. Second, the use of the age-to-age ratios serves as a "bridge" to the deterministic framework and allows the model to be more easily explainable to others. And, third, for the GLM algorithm the log-link process means that negative incremental values can often cause the algorithm to not have a solution, whereas using the link ratios will generally allow for a solution.[10]

With a model fitted to the data, the bootstrap process involves sampling with replacement from the residuals. England and Verrall [9] note that the deviance, Pearson, and Anscombe residuals could all be considered for this process, but the Pearson residuals are the most desirable since they are calculated consistently with the scale parameter. The unscaled Pearson residuals and scale parameter are calculated as follows:

$$r_{w,d} = \frac{q(w,d) - m_{w,d}}{\sqrt{m_{w,d}^{z}}}. \tag{3.19}$$

$$\phi = \frac{\sum r_{w,d}}{n-p}. \tag{3.20}$$

Where n = the number of data cells in the triangle and p = the number of parameters, which is typically equal to $2*n - 1$.[11] Sampling with replacement from the residuals can then be used to create new sample triangles of incremental values using formula 3.16. Sampling with replacement assumes that the residuals are independent and identically distributed, but it does not require the residuals to be normally distributed. Indeed, this is often cited as an advantage of the ODP bootstrap model since whatever distributional form the residuals have will flow through the simulation process. Some authors have referred to this a "semi-parametric" bootstrap model since we are not parameterizing the residuals.

$$q'(w,d) = r^{*} \times \sqrt{m_{w,d}^{z}} + m_{w,d}. \tag{3.21}$$

The sample triangle of incremental values can then be cumulated, new average age-to-age factors and loss development factors can be calculated for the sample and applied to calculate a point estimate for this data. This process could be described as getting a distribution of point estimates, which includes incorporating process variance and parameter variance in the simulation of the

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[10] More specifically, individual negative cell values may not be a problem. If the total of all incremental cell values in a development column is negative, then the GLM algorithm will fail. This situation will not cause a problem fitting the model as a link ratio less than one will be perfectly useful. However, this may still cause other problems, which we will address in section 4.

[11] The number of parameters could be less than $2*n - 1$. For example, if the incremental values are zeros for the last three columns in a triangle then there will be three fewer β parameters since none are needed to fit to these zero values as the development process is completed already.

historical data. In England and Verrall [9] this is the end of the process, but at the end of the appendix they note that you should also multiply the resulting distribution by the degrees of freedom adjustment factor (3.22), to effectively allow for over-dispersion of the residuals in the sampling process.

$$f = \sqrt{\frac{n}{n-p}}\,. \tag{3.22}$$

Later, in England and Verrall [10], the authors note that the Pearson residuals (3.19) could be multiplied by the degrees of freedom adjustment factor (3.22) in order to correct for a bias in the residuals. They also expand the simulation process by adding process variance to the future incremental values from the point estimates. To add this process variance, they assume that each future incremental value $m_{w,d}$ is the mean and the mean times the scale parameter, $\phi m_{w,d}$, is the variance of a gamma distribution.[12] This revised model could now be described as estimating a distribution of possible outcomes, which incorporates process variance and parameter variance in the simulation of the historical and future data.

However, Pinheiro et al. [25, 26] noted that the bias correction for the residuals using the degrees of freedom adjustment factor (3.22) does not create standardized residuals, which is an important step for making sure that the residuals all have the same variance. In order to have standardized Pearson residuals, the GLM framework requires the use of a hat matrix adjustment factor.

$$H = X\left(X^T W X\right)^{-1} X^T W\,. \tag{3.23}$$

$$f_{w,d}^{H} = \sqrt{\frac{1}{1-H_{i,i}}}\,. \tag{3.24}$$

The hat matrix (3.23) is calculated using matrix multiplication of the design matrix (3.15) and the weight matrix (3.17). The hat matrix adjustment factor (3.24) uses the diagonal of the hat matrix. In Pinheiro, et al. [26] the authors note two important points about the bootstrap process as described by England and Verrall [9, 10]. First, the sampling of the residuals should not include any zero-value residuals, which are typically in the corners of the triangle.[13] The exclusion of the zero-value residuals is accounted for in the hat matrix adjustment factor (3.24), but another common explanation is that the zero-value cells will have some variance but we just don't know what it is yet so we should sample from the remaining residuals but not the zeros. Second, the hat matrix

[12] The Poisson distribution could be used, but it is considerably slower to simulate, so gamma is a close substitute that performs much faster in simulation.

[13] Technically, the two "corner" residuals are zero because they each have a parameter that is unique to that incremental value which causes the fitted incremental value to exactly equal the actual incremental value.

adjustment factor (3.24) is a replacement for the degrees of freedom factor (3.22), which improves the calculation of the residuals.[14]

Thus, the unscaled Pearson residuals (3.19) should be replaced by the standardized Pearson residuals:

$$r_{w,d}^{H} = \frac{q(w,d) - m_{w,d}}{\sqrt{m_{w,d}^{z}}} \times f_{w,d}^{H} . \tag{3.25}$$

However, the scale parameter (3.20) is still calculated as before, although the standardized Pearson residuals could be used to approximate the scale parameter as follows:

$$\phi^{H} = \frac{\sum r_{w,d}^{H}}{n} . \tag{3.26}$$

At this point we have a complete basic "ODP bootstrap" model, as it is often referred to, although various stages of this complete model have been in popular use and formally tested. It is also important to note that the two key assumptions mentioned earlier, each accident year has the same development factor and each accident year has a parameter representing its relative level, are equally applicable to this model.

In order for the reader to test out the different "combinations" of this modeling process the "Bootstrap Models.xls" file includes options to allow these historical algorithms to be simulated. Our purpose in describing this evolution of the bootstrap model framework is threefold: first, to allow the interested reader to better understand the details of the algorithm and how these papers have contributed to the model framework; second, to illustrate the value of collaborative research via different published papers and the contributions of different authors; and, third, to provide a solid basis for us to continue the evolutionary process.

3.3 Variations on the ODP Model

When estimating insurance risk it is generally considered desirable to focus on the claim payment stream in order to measure the variability of the actual cash flows that directly affect the bottom line. Clearly, changes in case reserves and IBNR reserves will also impact the bottom line, but to a considerable extent the changes in IBNR are intended to counter the impact of the changes in case reserves. To some degree, then, the total reserve movements can act to mask the underlying changes due to cash flows. On the other hand, the case reserves represent potential future payments so we

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[14] This second point was not addressed clearly in Pinheiro et al. [25], but as the authors updated and clarified the paper in [26] this issue was more clearly addressed.

should not just ignore them and focus exclusively on paid data.

3.3.1 Bootstrapping the incurred loss triangle

The ODP model, as described, can be used to model both paid and incurred data. However, the resulting distribution from using incurred data will be possible outcomes of the IBNR so they will not be directly comparable to the distribution of possible outcomes of the total unpaid (i.e., from using paid data). A convenient way of converting the results of an incurred data model to a payment stream is to use payment patterns applied to the ultimate value of the incurred claims. This is consistent with how a deterministic incurred ultimate can be converted using a paid development pattern. If a paid data model is run in parallel with the incurred data model the possible outcomes from the paid data model can be used to convert incurred ultimate values to a payment pattern for each iteration (and for each accident year individually).

The "Bootstrap Models.xls" file illustrates this concept. It is worth noting, however, that this process allows the "added value" of using the case reserves to help predict the ultimate results to work its way into the calculations, thus perhaps improving the estimates, while still focusing on the payment stream for measuring risk. In effect, it allows a distribution of IBNR to become a distribution of IBNR and case reserves. This process could be made more sophisticated by correlating the residual sampling and/or process variance portions of the parallel models. Correlations must be considered if, for example, you wanted the iterations showing long payment streams to be compared with the iterations with high incurred results. It is also possible to use other modeling algorithms such as the Munich chain ladder (see [27]), although that is beyond the scope of this paper.

3.3.2 Bootstrapping the Bornhuetter-Ferguson and Cape Cod models

Another common issue with using the ODP bootstrap process is that iterations for the latest few accident years can produce results with more variance than you would expect given what you simulated for the earlier accident years. This is usually due to the fact that age-to-age factors are used to extrapolate the sampled values prior to adding process variance, which is completely analogous to one of the weaknesses of the deterministic paid chain ladder method.

As for the deterministic chain ladder method, the ODP bootstrap process can be modified by changing the extrapolation of future incremental values by using the Bornhuetter-Ferguson or generalized Cape Cod algorithms, among others. These deterministic methods can be converted into stochastic models while still using the underlying ODP assumptions and process, and that the deterministic assumptions of these methods can also be converted to stochastic assumptions. For example, instead of simply using a vector of deterministic a priori loss ratios for the Bornhuetter-Ferguson model, we could add a vector of standard deviations to go with these means, assume a distribution and simulate a different a priori loss ratio for every iteration of the model. Finally, it is worth noting that these "new" models can be set up separately for paid and incurred data and that the paid and incurred assumptions should be internally consistent with each other and with other models, as they should be for deterministic methods.

The "Bootstrap Models.xls" file also illustrates the Bornhuetter-Ferguson and Cape Cod models.

3.4 Generalizing the ODP Model

Using deterministic algorithms to enhance the flexibility of the basic ODP bootstrap process is a straightforward way to create additional models and to overcome many of the limitations of using bootstrapping. However, some limitations are more difficult to overcome just by using these algorithms. For example, calendar-year effects can be adjusted using a Berquist-Sherman algorithm but it is hard to make the assumptions more stochastic.

Rather than add essentially deterministic algorithms to a stochastic model, another approach is to go back to the original GLM framework and generalize the basic model. Returning to formulas (3.8) to (3.11), the GLM framework does not require a certain number of parameters so we are actually free to specify only as many parameters as we need to get a robust model. Indeed, it is ONLY when we specify a parameter for EVERY accident year and EVERY development year and specify a Poisson error distribution that we end up exactly replicating the volume weighted average age-to-age factors that allow us to substitute the deterministic algorithm instead of solving the GLM fit.

Thus, using the original GLM framework we can specify a model with only a few parameters, but there are two drawbacks to doing so. First, the GLM must be solved for each iteration of the bootstrap model (which may slow down the simulation process) and, second, the model is no longer directly explainable to others using age-to-age factors.[15] While the impact of these drawbacks should be considered, the potential benefits of using the GLM framework can be much greater.

First, having fewer parameters will help avoid the potential of over-parameterizing the model.[16] For example, if we use only one accident year parameter then the model specified using a system of equations is as follows (which is analogous to formula 3.12):

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[15] However, age-to-age factors could be calculated for the fitted data to compare to the actual age-to-age factors and used as an aid in explaining the model to others.

[16] Over-parameterization is a common criticism of the ODP bootstrap model. This will be addressed more completely in Section 5.

$$\begin{aligned}\ln[q(1,1)] &= 1\alpha_1 + 0\beta_2 + 0\beta_3 \\ \ln[q(2,1)] &= 1\alpha_1 + 0\beta_2 + 0\beta_3 \\ \ln[q(3,1)] &= 1\alpha_1 + 0\beta_2 + 0\beta_3 \\ \ln[q(1,2)] &= 1\alpha_1 + 1\beta_2 + 0\beta_3 \\ \ln[q(2,2)] &= 1\alpha_1 + 1\beta_2 + 0\beta_3 \\ \ln[q(1,3)] &= 1\alpha_1 + 1\beta_2 + 1\beta_3\end{aligned} \tag{3.27}$$

In this case we will only have one level parameter and *n-1* development trend parameters, but it will only be coincidence that we would end up with the equivalent of average age-to-age factors. Interestingly, this model parameterization moves us away from one of the common basic assumptions (i.e., each accident year has its own level) and substitutes the assumption that all accident years are homogeneous.

Another example of using fewer parameters would be to only use one development year parameter (while continuing to use an accident-year parameter for each year), which would equate to the following system of equations:

$$\begin{aligned}\ln[q(1,1)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\beta_2 \\ \ln[q(2,1)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\beta_2 \\ \ln[q(3,1)] &= 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 0\beta_2 \\ \ln[q(1,2)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 \\ \ln[q(2,2)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 1\beta_2 \\ \ln[q(1,3)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 2\beta_2\end{aligned} \tag{3.28}$$

In this example the model parameterization would continue to follow the two common assumptions (i.e., each accident year has its own level and uses the same development factor), although again it would be pure coincidence to end up with the equivalent of average age-to-age factors.[17] It is also interesting to note that for both of these two examples there will be one additional non-zero residual that can be used in the simulations because in each case one of the incremental values no longer has a unique parameter – i.e., for (3.27) $q(3,1)$ is no longer uniquely defined by α_3, and for (3.28) $q(1,3)$ is no longer uniquely defined by β_3.

This flexibility allows the modeler to use enough parameters to capture the statistically relevant level and trend changes in the data without forcing a specific number of parameters.[18]

The second benefit, and depending on the data perhaps the most significant, is that this

[17] If we were to generalize the development factor assumption to focus on the number of parameters instead, then we would have only one parameter instead of a different parameter for each development period.

[18] How to determine which parameters are statistically relevant will be discussed in Section 5.

framework allows us the ability to add parameters for calendar-year trends. Adding diagonal parameters to (3.11) we now have:

$$\eta_{w,d} = \alpha_w + \beta_d + \gamma_k \text{, where: } w = 1, 2, \ldots, n; \; d = 2, \ldots, n; \text{ and } k = 2, \ldots, n. \tag{3.29}$$

A complete system of equations for the (3.29) framework would look like the following:

$$\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\beta_2 + 0\beta_3 + 0\gamma_2 + 0\gamma_3 \\
\ln[q(2,1)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\beta_2 + 0\beta_3 + 1\gamma_2 + 0\gamma_3 \\
\ln[q(3,1)] &= 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 0\beta_2 + 0\beta_3 + 1\gamma_2 + 1\gamma_3 \\
\ln[q(1,2)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 + 0\beta_3 + 1\gamma_2 + 0\gamma_3 \\
\ln[q(2,2)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 1\beta_2 + 0\beta_3 + 1\gamma_2 + 1\gamma_3 \\
\ln[q(1,3)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 + 1\beta_3 + 1\gamma_2 + 1\gamma_3
\end{aligned} \tag{3.30}$$

However, there is no unique solution for a system with seven parameters and six equations, so some of these parameters will need to be removed. A logical starting point would be to start with a model with one accident year (level) parameter, one development trend parameter and one calendar trend parameter and then add or remove parameters as needed. The system of equations for this basic model is as follows:

$$\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\beta_2 + 0\gamma_2 \\
\ln[q(2,1)] &= 1\alpha_1 + 0\beta_2 + 1\gamma_2 \\
\ln[q(3,1)] &= 1\alpha_1 + 0\beta_2 + 2\gamma_2 \\
\ln[q(1,2)] &= 1\alpha_1 + 1\beta_2 + 1\gamma_2 \\
\ln[q(2,2)] &= 1\alpha_1 + 1\beta_2 + 2\gamma_2 \\
\ln[q(1,3)] &= 1\alpha_1 + 2\beta_2 + 2\gamma_2
\end{aligned} \tag{3.31}$$

A fourth benefit of the GLM framework is that it can be used to model data shapes other than triangles. For example, missing incremental data for the first few diagonals would mean that the cumulative values could not be calculated and the remaining values in those first few rows would not be useful for the simplified GLM. However, since the GLM framework uses the incremental values the entire trapezoid can be used to fit the model parameters.[19]

It should also be noted that the GLM framework allows the future expected values to be directly estimated from the parameters of model for each sample triangle in the bootstrap simulation process. However, we must solve the GLM within each iteration for the same parameters as we originally set up for the model rather than using age-to-age factors to project future expected values.

The additional modeling power that the flexible GLM framework adds to the actuary's toolkit

[19] We will examine this issue in more detail in Section 4.

cannot be overemphasized. Not only does it allow one to move away from the two basic assumptions of a deterministic chain ladder method, it allows for the ability to match the model parameters to the statistical features you find in the data and to extrapolate those features. For example, modeling with fewer development trend parameters means that the last parameter can be assumed to continue past the end of the triangle which will give the modeler a "tail" of the incremental values beyond the end of the triangle without the need for a specific tail factor.

While we have continued to illustrate the GLM framework in the body of the paper with a 3 x 3 triangle, also included in the companion Excel files are a set of "Simple GLM 6___.xls" files that illustrate the calculations for these different models using a 6 x 6 triangle. Also, the "Bootstrap Models.xls" file contains a "flexible" model for a 10 x 10 triangle that can be used to specify any combination of accident year, development year, and calendar year parameters, including setting parameters to zero. The flexible GLM model is akin to the incremental log model described in Barnett and Zehnwirth [1], so we will leave it to the reader to explore this flexibility by using the Excel file.

4. PRACTICAL ISSUES

Now that we have expanded the basic ODP bootstrap model in a variety of ways, we also want to address some of the key assumptions of the ODP model and some common data issues.

4.1 Negative Incremental Values

As noted in Section 3.2, because of the log-link used in the GLM framework the incremental values must be greater than zero in order to parameterize a model. However, a slight modification to the log-link function will help this common problem become a little less restrictive. If we use (4.1) as the log-link function, then individual negative values are only an issue if the total of all incremental values in a development column is negative, as the GLM algorithm will not be able to find a solution in that case.

$$\begin{gathered} \ln[q(w,d)] \text{ for } q(w,d) > 0, \\ 0 \text{ for } q(w,d) = 0, \\ -\ln[abs\{q(w,d)\}] \text{ for } q(w,d) < 0. \end{gathered} \tag{4.1}$$

Using (4.1) in the GLM framework will help in many situations, but it is quite common for entire development columns of incremental values to be negative, especially for incurred data. To give the GLM framework the ability to solve for a solution in this case we need to make another modification to the basic model to include a constant.

$$\ln[m_{w,d}] + \psi = \eta_{w,d} \tag{4.2}$$

Whenever a column or columns of incremental values sum to a negative value, we can find the largest negative[20] in the triangle, add the absolute value of the largest negative to every incremental value in the triangle, set ψ equal to the largest negative, and solve the GLM using formulas (3.10), (3.11), or (3.29). Then when we use (4.2) to calculate the fitted incremental values, the constant ψ is used to reduce each fitted incremental value by the largest negative.

The combination of formulas (4.1) and (4.2) allow the GLM framework to handle all negative incremental values, which overcomes a common criticism of the ODP bootstrap. Incidentally, these formulas can also be used to allow the incremental log model described by Barnett and Zehnwirth [1] to handle negative incremental values.

When using the age-to-age factors to simplify the ODP bootstrap simulation process, the solution to negative incremental values needs to focus on the residuals and sampled incremental values since an age-to-age factor less than 1.00 will create negative incremental values in the fitted values. More specifically, we need to modify formulas (3.19) and (3.21) as follows:

$$r_{w,d} = \frac{q(w,d) - m_{w,d}}{\sqrt{abs\{m_{w,d}\}}}. \tag{4.3}$$

$$q'(w,d) = r^* \times \sqrt{abs\{m_{w,d}\}} + m_{w,d}. \tag{4.4}$$

While the fitted incremental values and residuals using the age-to-age simplification will generally not match the GLM framework solution using (4.1) and (4.2) they should be reasonably close. While the "purists" may object to these practical solutions, we must keep in mind that every model is an approximation of reality so our goal is to find reasonably close models rather than only restrict ourselves to "pure" models. After all, the assumptions of the "pure" models are themselves approximations.

4.1.1 Negative values during simulation

Even though we have solved problems with negative values when parameterizing a model, negative values can still affect the process variance in the simulation process. When each future incremental value (using $m_{w,d}$ as the mean and the mean times the scale parameter, $\phi m_{w,d}$, as the variance) is sampled from a gamma distribution to add process variance, the parameters of a gamma distribution must be positive. In this case we have two options for using the gamma distribution to

20 The largest negative value can either be the largest negative among the sums of development columns (in which case there may still be individual negative values in the adjusted triangle) or the largest negative incremental value in the triangle.

simulate from a negative incremental value, $m_{w,d}$.

$$-Gamma\left[abs\{m_{w,d}\},\phi abs\{m_{w,d}\}\right] \tag{4.5}$$

$$Gamma\left[abs\{m_{w,d}\},\phi abs\{m_{w,d}\}\right]+2m_{w,d} \tag{4.6}$$

Using formula (4.5) is more intuitive as we are using absolute values to simulate from a gamma distribution and then changing the sign of the result. However, since the gamma distribution is skewed to the right, the resulting distribution using (4.5) will be skewed to the left. Using formula (4.6) is a little less intuitive, but seems more logical since subtracting twice the mean, $m_{w,d}$, will result in a distribution with a mean of $m_{w,d}$ while keeping it skewed to the right (since $m_{w,d}$ is negative).

Negative incremental values can also cause extreme outcomes. This is most prevalent when resampled triangles are created with negative incremental losses in the first few development periods, causing one column of cumulative values to sum close to zero and then next column sum to a much larger number and, consequentially, age-to-age factors that are extremely large. This can result in one or two extreme iterations in a simulation (for example, outcomes that are multiples of 1,000s of the central estimate). These extreme outcomes cannot be ignored, even if the high percentiles are not of interest, because they are likely to significantly affect the mean of the distribution.

In these instances, you have several options. You can 1) remove these iterations from your simulation and replace them with new iterations, 2) recalibrate your model, 3) limit incremental values to zero, or 4) use more than one model.

The first option is to identify the extreme iterations and remove them from your results. Care must be taken that only truly unreasonable extreme iterations are removed, so that the resulting distribution does not understate the probability of extreme outcomes.

The second option is to recalibrate the model to fix this issue. First you must identify the source of the negative incremental losses. For example, it may be from the first row in your triangle, which was the first year the product was written, and therefore exhibit sparse data with negative incremental amounts. One option is to remove this row from the triangle if it is causing extreme results and does not improve the parameterization of the model.

The third option is to limit incremental losses to zero, where any negative incremental is replaced with a zero incremental. This can be done in many ways. Negative incremental values can be replaced with zeros in the original data triangles. Negative incremental values can be kept in the original data triangles, but replaced with zeros if they appear in the sampled triangles. Negative incremental losses can be kept in the historical sampled triangle but replaced with zeros in the projected future incremental losses. Finally, negative incremental values can be replaced with zeros based on which development column they are in (this option is used in the "Bootstrap Models.xls" file). Judgment is required when deciding amongst these options.

The most theoretically sound method to deal with negative incremental values is to consider the source of these losses. If they are caused by reinsurance or salvage and subrogation, then you can model the losses gross of salvage and subrogation, model the salvage and subrogation separately, and combine the iterations assuming 100% correlation.

4.2 Non-Zero Sum of Residuals

The residuals that are calculated in the bootstrap model are essentially error terms, and should be identically distributed with a mean of zero. Generally, however the average of all the residuals is non-zero. The residuals are random observations of the true residual distribution, so this observation is not necessarily incompatible with the true residual distribution having a mean of zero. The real issue is whether these residuals should be adjusted so that their average is zero. For example, if the average of the residuals is positive, then re-sampling from the residual pool will not only add variability to the resampled incremental losses, but may increase the resampled incremental losses such that the average of the resampled loss will be greater than the fitted loss.

The reason why residuals may not sum to zero is due to differing magnitudes of losses in each accident year. If the magnitude of losses is higher for a particular accident year that shows higher development than the weighted average, then the average of all the residuals will be negative. If the magnitude of losses is lower for a particular accident year that shows higher development than the weighted average, then the average of all the residuals will be positive.

It can be argued that the non-zero average of residuals is a characteristic of the data set, and therefore should not be removed. However, if a zero residual average is desired, then one option is the addition of a single constant to all residuals, such that the sum of the shifted residuals is zero.

4.3 Using an *N*-Year Weighted Average

The basic ODP bootstrap model can be simplified by using volume-weighted average age-to-age factors for all years in the triangle. It is quite common, however, for actuaries to use weighted averages that are less than for all years. Thus, it is also important to be able to adjust the ODP bootstrap model to use N-year average age-to-age factors.

For the GLM framework, we can use N years of data by excluding the first few diagonals in the triangle so that we only use N+1 diagonals (since an N-year average uses N+1 diagonals) to parameterize the model. The shape of the data to be modeled essentially becomes a trapezoid instead of a triangle, the excluded diagonals are given zero weight in the model and we have fewer calendar year trend parameters if we are using formula (3.29). When running the bootstrap simulations we will only need to sample residuals for the trapezoid that we used to parameterize the model as that is all that will be needed to estimate parameters for each iteration.

Using the simplified GLM we can also calculate N-year average factors instead of all-year factors and exclude the first few diagonals when calculating residuals. However, when running the bootstrap simulations we would still need to sample residuals for the entire triangle so that we can calculate cumulative values. To be consistent with the assumptions of the simplified GLM in this case, we would still want to use N-year average factors for projecting the future expected values.

The calculations for the GLM framework are illustrated in the companion "Simple GLM 6 with 3yr avg.xls" file. Note that because the GLM framework estimates parameters for the incremental data, the fitted values will no longer match the fitted values from the simplified GLM using volume-weighted average age-to-age factors. However, the fitted values are generally close so the simplified GLM will still be a reasonable approximation to the GLM framework.

4.4 Missing Values

Sometimes the loss triangle will have missing values. For example, values may be missing from the middle of the triangle. Another example is a triangle that is missing the oldest diagonals, if loss data was somehow lost or not kept in the early years of writing the book of business.

If values are missing, then the following calculations will be affected:

- Loss development factors
- Fitted triangle – if the missing value lies on the last diagonal
- Residuals
- Degrees of freedom

There are several solutions. The missing value may be estimated using the surrounding values. Or, the loss development factors can be modified to exclude the missing value, and there will not be a corresponding residual for this missing value. Subsequently, when triangles are resampled, the simulated incremental corresponding to the missing value should not be resampled to reproduce the uncertainty in the original dataset.

If the missing value lies on the last diagonal, the fitted triangle cannot be calculated in the usual way. A solution is to estimate the value, or use the value in the second to last diagonal to construct the fitted triangle. These are not strictly mathematically correct solutions, and judgment will be needed as to their affect on the resulting distribution.

4.5 Outliers

There may be extreme or incorrect values in the original triangle dataset that would be considered outliers. These may not be representative of the variability of the dataset in the future and, if so, the modeler may want to remove their impact from the model.

There are several solutions. If these values formed the first row of the data triangle, which is common, then this whole first row could be deleted, and the model run on a smaller triangle. Alternatively, these values could be removed, and dealt with in the same manner as missing values. Another alternative is to identify outliers and exclude them from the average age-to-age factors (either the numerator, denominator, or both) and residual calculations, as when dealing with missing values, but re-sample the corresponding incremental when simulating triangles.

The calculations for the GLM framework are illustrated in the companion "Simple GLM 6 with Outlier.xls" file. Again the GLM framework fitted values will no longer exactly match the fitted values from the simplified GLM using volume weighted average age-to-age factors.

4.6 Heteroscedasticity

As noted earlier, the ODP model is based on the assumption that the Pearson residuals are independent and identically distributed. It is this assumption that allows the model to take a residual from one development period/accident period and apply it to the fitted loss in any other development period/accident period, to produce the sampled values. In statistical terms this is referred to as homoscedasticity and it is important that this assumption is validated.

A problem is commonly observed when some development periods have residuals that appear to be more variable than others – i.e., they appear to have different distributions or variances. If this observation is correct, then we have multiple distributions within the residuals (statistically referred to as heteroscedasticity) and it is no longer possible to take a residual from one development/accident period and deem it suitable to be applied to any other development/accident period. In making this assessment, you must account for the credibility of the observed difference, and also to note that there are fewer residuals as the development years become older, so comparing development years is difficult, particularly near the tail-end of the triangle.[21]

To adjust for heteroscedasticity in your data there are at least two options, 1) stratified sampling, or 2) calculating variance parameters. Stratified sampling is accomplished by organizing the development periods by group with homogeneous variances within each group and then sampling with replacement only from the residuals in each group. While this process is straightforward and easy to accomplish, quite often some groups may only have a few residuals in them, which limits the amount of variability in the possible outcomes.

The second option is to sort the development periods into groups with homogeneous variances and calculate the standard deviation of the residuals in each of the "hetero" groups. Then calculate h_i, which is the hetero-adjustment factor, for each group, i:

$$h_i = \frac{Max[stdev(r^i_{w,d})]}{stdev(r^i_{w,d})}. \tag{4.7}$$

All residuals in group i are multiplied by h_i.

$$r^{iH}_{w,d} = \frac{q(w,d) - m_{w,d}}{\sqrt{m_{w,d}}} \times f^H_{w,d} \times h^i. \tag{4.8}$$

Now all groups have the same standard deviation and we can sample with replacement from among all $r^{iH}_{w,d}$. The original distribution of residuals has been altered, but this can be remedied. When the residuals are resampled, the residual is divided by the hetero-adjustment factor that applies to the development year of the incremental value, as shown in (4.9).

$$q^i{}'(w,d) = \frac{r^*}{h^i} \times \sqrt{m_{w,d}} + m_{w,d}. \tag{4.9}$$

By doing this, the heteroscedastic variances we observed in the data are replicated when the sample triangles are created, but we are able to freely resample with replacement from the entire pool of residuals. Also note that we have added more parameters so this will affect the degrees of freedom, which impacts the scale parameter (3.20) and the degrees of freedom adjustment factor (3.22). Finally, the hetero group parameters should also be used to adjust the variance when simulating the future process variance.

It is possible to modify the GLM framework to also include "hetero group" parameters, but that is beyond the scope of this paper.

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[21] We will illustrate how to use residual graphs and other statistical tests to evaluate heteroscedasticity in Section 5.

4.7 Heteroecthesious Data

The basic ODP bootstrap model requires both a symmetrical shape (e.g., annual by annual, quarterly by quarterly, etc. triangles) and homoecthesious data (i.e., similar exposures).[22] As discussed above, using an *N*-year weighted average in the simplified GLM model or adjusting to a trapezoid shape allow us to "relax" the requirement of a symmetrical shape. Other non-symmetrical shapes (e.g., annual x quarterly data) can also be modeled with either the simplified GLM or GLM framework, but they will not be discussed in detail in this paper.

Most often, the actuary will encounter heteroecthesious data (i.e., incomplete or uneven exposures) at interim evaluation dates, with the two most common data triangles being either a partial first development period or a partial last calendar period. For example, with annual data evaluated as of June 30, partial first development period data would have development periods ending at 6, 18, 30, etc. months, while partial last calendar period data would have development periods as of 12, 24, 36, etc. months for all of the data in the triangle except the last diagonal, which would have development periods as of 6, 18, 30, etc. months. In either case, not all of the data in the triangle has full annual exposures – i.e., it is heteroecthesious data.

4.7.1 Partial first development period data

For partial first development period data, the first development column has a different exposure period than the rest of the columns (e.g., in the earlier example the first column has six months of development exposure while the rest have 12). In a deterministic analysis this is not a problem as the age-to-age factors will reflect the change in exposure. For parameterizing an ODP bootstrap model, it also turns to be a moot issue. In addition, since the Pearson residuals use the square root of the fitted value to make them all "exposure independent" that part of an ODP bootstrap model is likewise unaffected.

The only adjustment for this type of heteroecthesious data is the projection of future incremental values. In a deterministic analysis, the most recent accident year needs to be adjusted to remove exposures beyond the evaluation date. For example, continuing the previous example the development periods at 18 months and later are all for an entire year of exposure whereas the six month column is only for six months of exposure. Thus, the 6-18 month age-to-age factor will effectively extrapolate the first six months of exposure in the latest accident year to a full accident year's exposure. Accordingly, it is common practice to reduce the projected future payments by half

[22] To our knowledge, the terms *homoecthesious* and *heteroecthesious* are new. They are a combination of the Greek *homos* (or ὁμός) meaning the same or *hetero* (or ἑτερο) meaning different and the Greek *ekthesē* (or ἐκθεση) meaning exposure.

to remove the exposure from June 30 to December 31.

The simulation process for the ODP bootstrap model can be adjusted similarly to the way a deterministic analysis would be adjusted. After the age-to-age factors from each sample triangle are used to project the future incremental values the last accident year's values can be reduced (in the previous example by 50%) to remove the future exposure and then process variance can be simulated as before. Alternatively, the future incremental values can be reduced after the process variance step.

4.7.2 Partial last calendar period data

For partial last calendar period data, most of the data in the triangle has annual exposures and annual development periods, except for the last diagonal which, continuing our example, only has a six-month development period (and a six-month exposure period for the bottom cell). For a deterministic analysis, it is quite common in this situation to exclude the last diagonal when calculating average age-to-age factors, interpolate those factors for the exposures in the last diagonal and use the interpolated factors to project the future values. In addition, the last accident year will also need to have the future incremental values reduced to remove exposures beyond the evaluation date.

Similarly to the adjustments for partial first development period data, we could adjust the calculations and steps in the simplified GLM model, but adjustments to the GLM framework are more problematic. Instead of ignoring the last diagonal during the parameterization of the model, an alternative is to adjust or annualize the exposures in the last diagonal to make them consistent with the rest of the triangle.

During the bootstrap simulation process, age-to-age factors can be calculated from the fully annualized sample triangles and interpolated. Then, the last diagonal from the sample triangle can be adjusted to de-annualize the incremental values in the last diagonal – i.e., reversing the annualization of the original last diagonal. The new cumulative values can be multiplied by the interpolated age-to-age factors to project future values. Again, the future incremental values for the last accident year must be reduced (in the previous example by 50%) to remove the future exposure.[23]

4.8 Exposure Adjustment

Another common issue in real data is exposures that have changed dramatically over the years.

[23] These heteroecthesious data issues are not illustrated in the "Bootstrap Models.xls" file.

For example, in a line of business that has experienced rapid growth or is being run off. If the earned exposures exist for this data, then a useful option for the ODP bootstrap model is to divide all of the claim data by the exposures for each accident year – i.e., effectively using pure premium development instead of total loss development. Quite often this will improve the fit of the model to the data.

During the bootstrap simulation process, all of the calculations would be done using the exposure-adjusted data and only after the process variance step has been completed would you multiply the results by the exposures by year to restate them in terms of total values again.

4.9 Parametric Bootstrapping

Because the number of data points used to parameterize the ODP bootstrap model are limited (in the case of a 10x10 triangle to 53 residuals), it is hard to determine whether the most extreme observation is a one-in-100 or a one-in-1,000 event (or simply, in this example, a one-in-53 event). Of course, the nature of the extreme observations in the data will also affect the level of extreme simulations in the results. Judgment is involved here, but the modeler will either need to be satisfied with the level of extreme simulations in the results or modify the bootstrap algorithm.

One way to overcome a lack of extreme residuals for the ODP bootstrap model would be to parameterize a distribution for the residuals and resample using the distribution (e.g., use a normal distribution if the residuals are normally distributed). This option for "sampling residuals" is beyond the scope of the companion Excel files, but this is commonly referred to as parametric bootstrapping.

5. DIAGNOSTICS

The quality of a bootstrap model depends on the quality of the underlying assumptions. When any model fails to "fit" the data, it cannot produce a good estimate of the distribution of possible outcomes.[24]

One of the advantages of the ODP bootstrap model is how readily it can be tailored to some of the statistical features of the data using the GLM framework and considerations described in the previous two sections. The CAS Working Party, in the third section of their report on quantifying variability in reserve estimates [6], identified 20 criteria or diagnostic tools for gauging the quality of

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[24] While the examples are different, significant portions of sections 5 and 6 are based on [22] and [14].

a stochastic model. The Working Party also noted that, in trying to determine the optimal "fit" of a model, or indeed an optimal model, no single diagnostic tool or group of tools can be considered definitive. Depending on the statistical features found in the data, a variety of diagnostic tools are necessary to best judge the quality of the model assumptions and to change or adjust the parameters of the model. In this sense, the diagnostic tools are used to help find the models that ultimately provide the best fit to the data. We will discuss some of these tools in detail in this paper.

The key diagnostic tests are designed for three purposes: to test various assumptions in the model, to gauge the quality of the model fit, or to help guide the adjustment of model parameters. Some tests may be considered relative in nature, enabling results from one set of model parameters to be compared to those of another, for a specific model. In turn, by analyzing these results a modeler may then be able to improve the fit of the model. For the most part, however, the tests generally can't be used to compare different models. The objective, consistent with the goals of a deterministic analysis, is not to find the one best model, but rather a set of reasonable models.

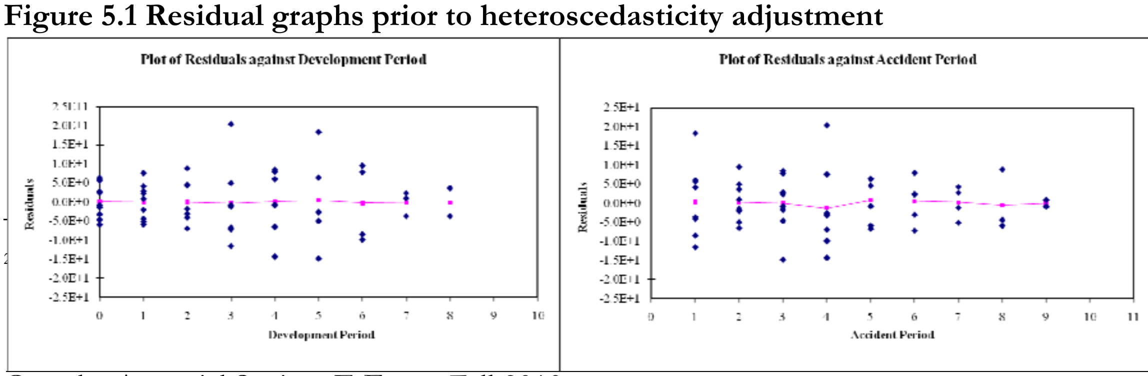
Some diagnostic measures include statistical tests, providing a pass/fail determination for some aspects of the model assumptions. This can be useful even though a "fail" does not necessarily invalidate an entire model; it only points to areas where improvements can be made to the model or its parameterization. The goal is to find the sets of models and parameters that will yield the most realistic, most consistent simulations, based on statistical features found in the data.

To illustrate some of the diagnostic tests for the ODP bootstrap model we will consider data from England and Verrall [9].[25]

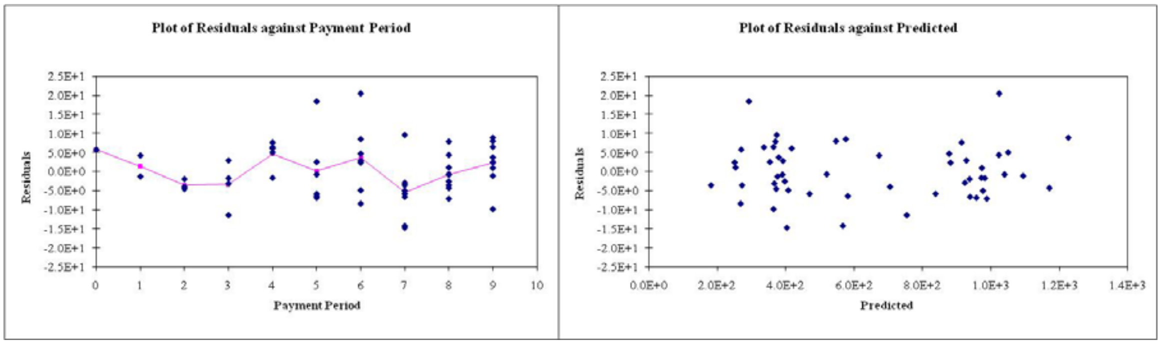
5.1 Residual graphs

The ODP bootstrap model does not require a specific type of distribution for the residuals, but they are assumed to be independent and identically distributed. Because residuals will be sampled with replacement during the simulations, this requirement becomes important and thus it is necessary to test this assumption. A look at graphs of residuals is a good way to do this.

Figure 5.1 Residual graphs prior to heteroscedasticity adjustment



is data is



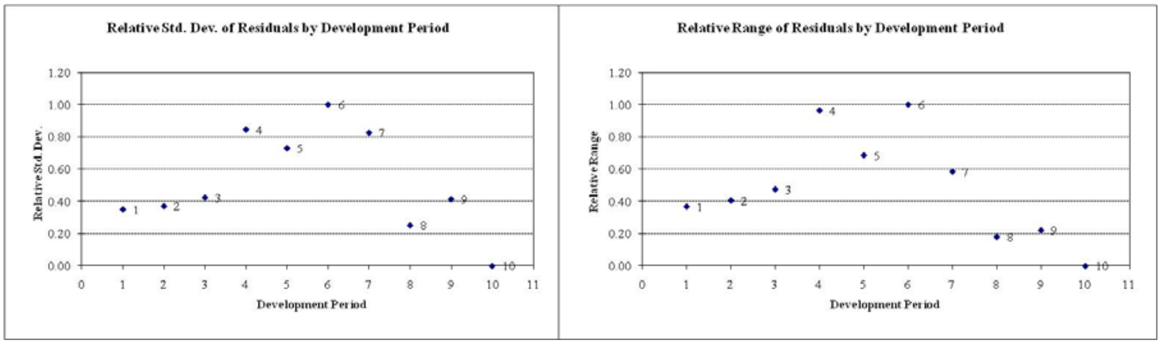
Going clock-wise, and starting from the top-left-hand corner, the graphs in Figure 5.1 show the residuals (blue dots) by development period, accident period, and calendar period and against the fitted incremental loss (in the lower-right-hand corner). In addition, the graphs include a trend line (in pink) that highlights the averages for each period.

At first glance, the residuals in the graphs appear reasonably random, indicating the model is likely a good fit of the data. But a closer look may also reveal potential features in the data that, with the benefit of further analysis, may indicate ways to improve the model fit.

The graphs in Figure 5.1 do not appear to indicate issues with trends, even if the trends for the development and accident periods are both essentially straight. That's because the simplified GLM specifies a parameter for every row and column of the triangle. The development-period graph does, however, reveal a potential heteroscedasticity issue associated with the data. Heteroscedasticity is when random variables have different variances. Note how the upper left graph appears to show a variance of the residuals in the first three periods that differs from those of the middle four or last two periods.

Adjustments for heteroscedasticity can be made with the “Bootstrap Models.xls” file, which enables us to recognize groups of development periods and then adjust the residuals to a common standard deviation value. As an aid to visualizing how to group the development periods into “hetero” groups, graphs of the standard deviation and range relativities can then be developed. Figure 5.2 represents pre-adjusted relativities for the residuals shown in Figure 5.1.

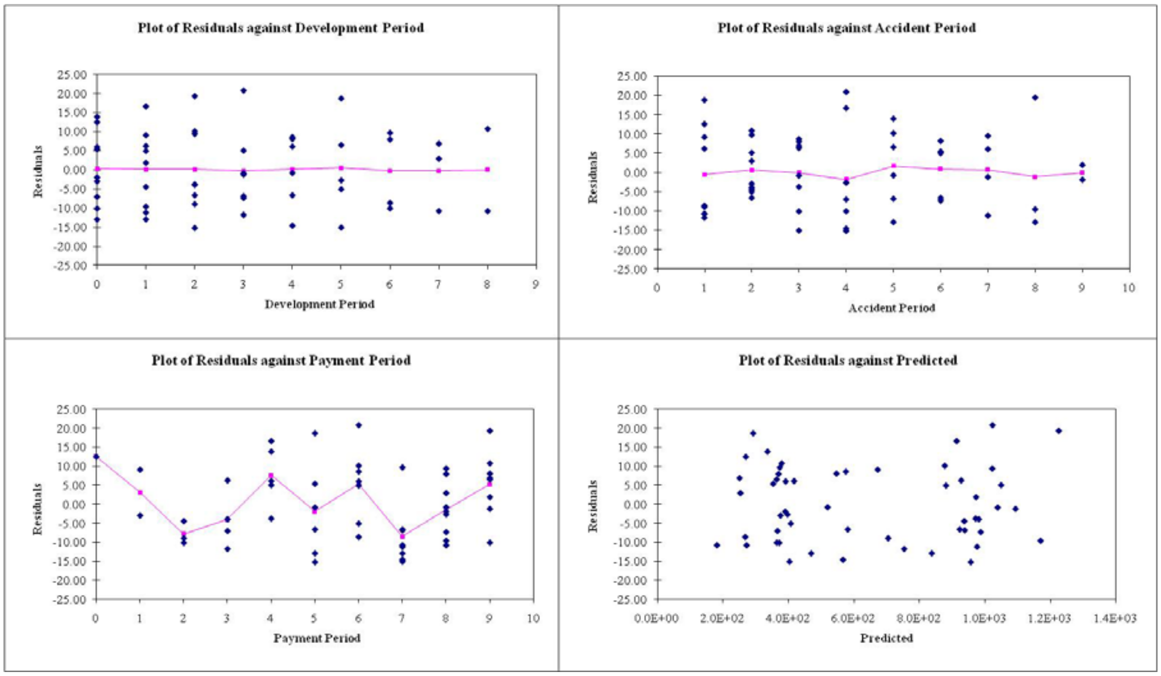
Figure 5.2 Residual relativities prior to heteroscedasticity adjustment



The relativities illustrated in Figure 5.2 help to clarify the veracity of this test, indicating that the residuals in the first three periods are different from those in the middle four or the last two. However, further testing will be required to assess the optimal groups, which can be performed using the other diagnostic tests noted below.

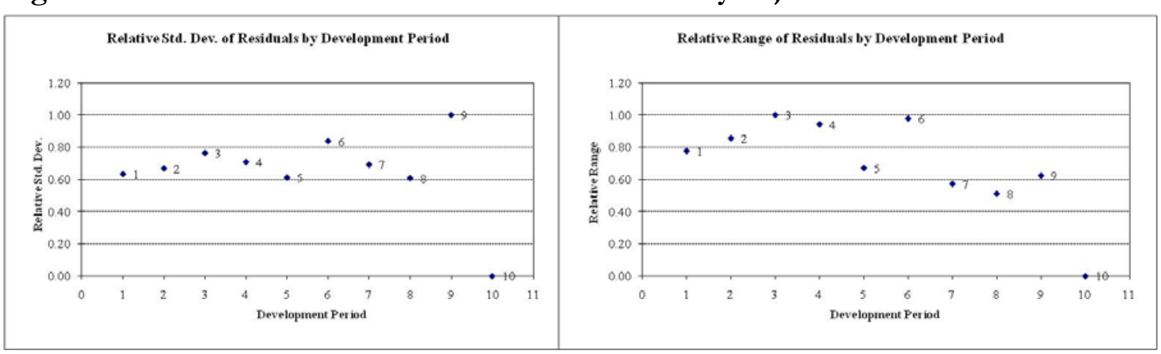
The residual plots in Figure 5.3 originate from the same data model after setting up "hetero" groups for the same array: the first three, middle four, and last two development periods, respectively. Determining whether this "hetero" grouping has improved the model fit will require review of other diagnostic tests.

Figure 5.3 Residual graphs after heteroscedasticity adjustment



Comparing the residual plots in Figures 5.1 and 5.3 does show that the general "shape" of the residuals has not changed and the "randomness" is still consistent. But the residuals now appear to exhibit the same standard deviation, or homoscedasticity. More consistent relativities may also be seen in a comparison of the residual relativities in Figures 5.2 and 5.4.

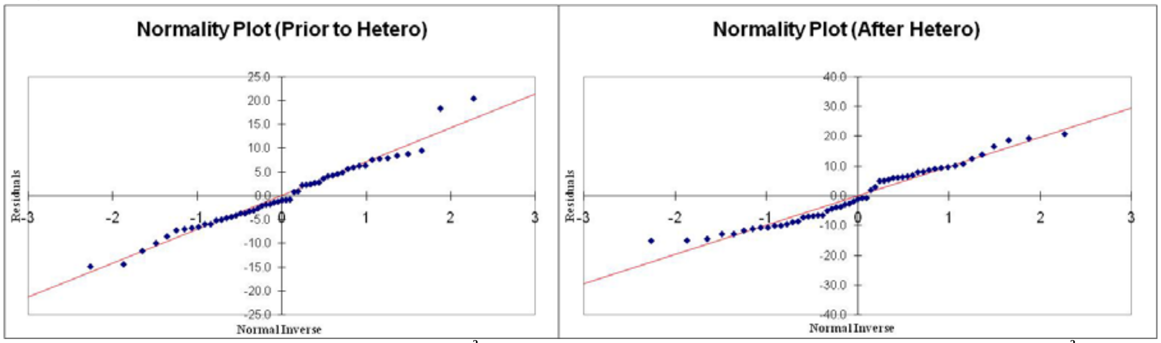
Figure 5.4 Residual relativities after heteroscedasticity adjustment



5.2 Normality test

The ODP bootstrap model does not depend on the residuals being normally distributed, but even so, comparing residuals against a normal distribution remains a useful test, enabling comparison of parameter sets and gauging skewness of the residuals. This test uses both graphs and calculated test values. Figure 5.5 is based on the same heteroscedasticity groups used earlier.

Figure 5.5 Normality plots prior to and after heteroscedasticity adjustment



N = 53 P-Value = 20.5% R^2 = 97.2%
Normal: MU = 0.04, Sigma = 7.09 AIC = 206.1, BIC = 93.1

N = 53 P-Value = 9.9% R^2 = 96.4%
Normal: MU = 0.07, Sigma = 9.85 AIC = 257.8, BIC = 148.8

Even before the heteroscedasticity adjustment, the residual plots appear close to normally distributed, with the data points tightly distributed around the diagonal line. The p-value, a statistical pass-fail test for normality, came in at 20.5%, which far exceeds the value generally considered a "passing" score of the normality test, which is greater than 5.0%.[26] The graphs in Figure 5.5 also show N (the number of data points) and the R^2 test. After the hetero adjustment, the p-value and R^2 don't appear to improve, which indicates that the tested "hetero" groups have not made the residual distribution more normally distributed.

While the p-value and R^2 tests are straightforward and easy to apply, neither adjusts for additional parameters used in the model, a critical limitation. Two other tests, the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC), address this limitation, using the difference between each residual and its normal counterpart from the normality plot to calculate the Residual Sum Squared (RSS) and include a penalty for additional parameters, as shown in (5.1) and (5.2), respectively.[27]

$$AIC = 2 \times p + n \times \left[\ln\left(\frac{2 \times \pi \times RSS}{n}\right) + 1 \right] \tag{5.1}$$

$$BIC = n \times \ln\left(\frac{RSS}{n}\right) + p \times \ln(n) \tag{5.2}$$

A smaller value for the AIC and BIC tests indicate residuals that fit a normal distribution more

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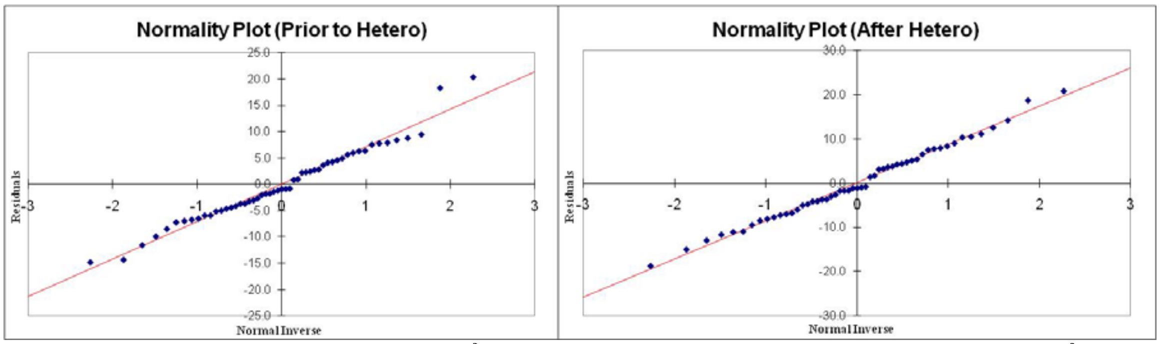
26 Remember that this doesn't indicate whether the bootstrap model itself passes or fails – the bootstrap model doesn't require the residuals to be normally distributed. While not included in the "Bootstrap Models.xls" file, as discussed in section 4.9, it could be used to determine whether to switch to a parametric bootstrap process using a normal distribution.

27 There are different versions of the AIC and BIC formula from various authors and sources, but the general idea of each version is consistent.

closely, and this improvement in fit overcomes the penalty of adding a parameter. With some trial and error, a better "hetero" grouping was found with the normality results shown in Figure 5.6.[28] For the new "hetero" groups, all of the statistical tests improved dramatically.

Figure 5.6 Normality plots prior to and after heteroscedasticity adjustment



N = 53 P-Value = 20.5% R^2 = 97.2%
Normal: MU = 0.04, Sigma = 7.09 AIC = 206.1, BIC = 93.1

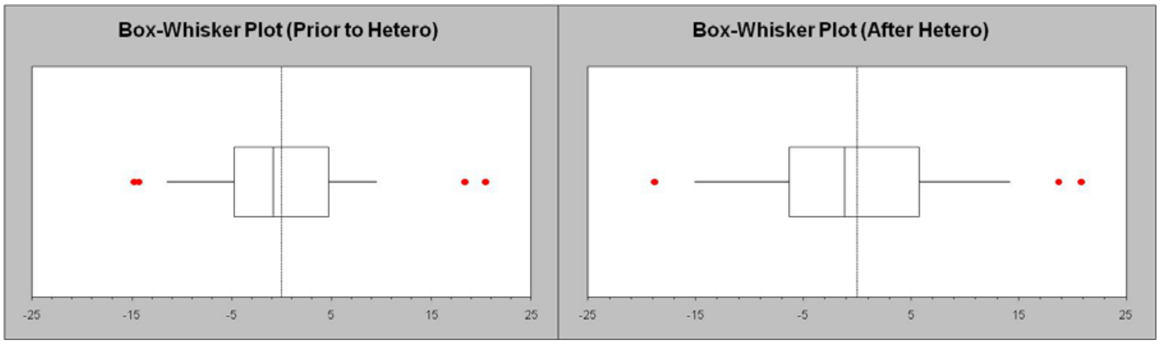
N = 53 P-Value = 95.4% R^2 = 99.2%
Normal: MU = 0.05, Sigma = 8.51 AIC = 160.8, BIC = 51.7

5.3 Outliers

Identifying outliers in the data provides another useful test in determining model fit. Outliers can be represented graphically in a box-whisker plot, which shows the inter-quartile range (the 25th to 75th percentiles) and the median (50th percentile) of the residuals—the so-called box. The whiskers then extend to the largest values within three times this inter-quartile range. Values beyond the whiskers may generally be considered outliers and are identified individually with a point.

[28] In the "Bootstrap Models.xls" file the England and Verrall data was entered as both paid and incurred. The first set of "hetero" groups are illustrated for the "incurred" data and the second set of "hetero" groups are illustrated for the "paid" data.

Figure 5.7 Box-Whisker Plots Prior to and After Heteroscedasticity Adjustment



Interquartile Range = [-4.73, 4.74] Median = -0.87
Outliers = 4

Interquartile Range = [-6.33, 5.75] Median = -1.14
Outliers = 3

Figure 5.7 shows an example of the residuals for the second set of "hetero" groups (Figure 5.6). A pre-hetero adjustment plot returns four outliers (red dots) in the data model, corresponding to the two highest and two lowest values in the previous graphs in Figures 5.1, 5.3, 5.5, and 5.6.

Even after the hetero adjustment, the residuals still appear to contain three outliers. Now comes a very delicate and often tricky matter of actuarial judgment. If the data in those cells genuinely represent events that cannot be expected to happen again, the outliers may be removed from the model (by giving them zero weight). But extreme caution should be taken even when the removal of outliers seems warranted. The possibility always remains that apparent outliers may actually represent realistic extreme values, which, of course, are critically important to include as part of any sound analysis.

Additionally, when residuals are not normally distributed a significant number of "outliers" tend to result, which may be only an artifact of the function of the distributional shape of the residuals. Again, it is preferable to let these stand in order to enable the simulation process to replicate this shape.

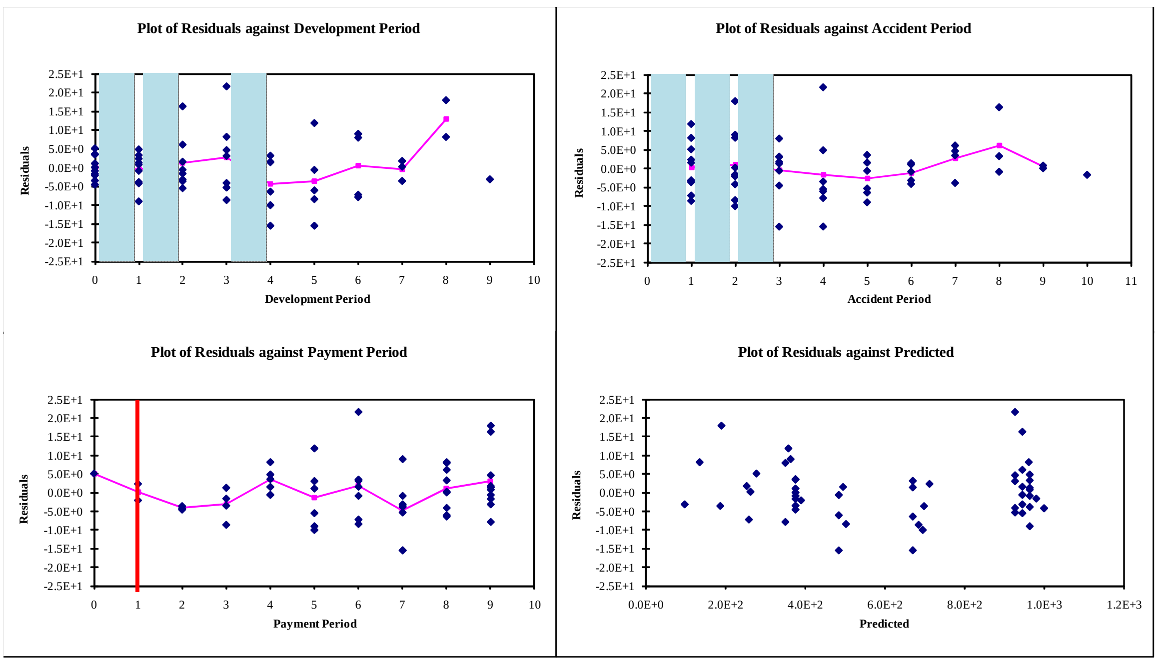
While the three diagnostic tests shown above demonstrate techniques commonly used with most types of models, they are not the only tests available. Next, we'll take a look at the flexibility of the GLM framework and some of the diagnostic elements of the simulation results. For a more extensive list of other tests available, see the report, CAS Working Party on Quantifying Variability in Reserve Estimates [6].

5.4 Parameter adjustment

As noted in section 5.1 the relatively straight average lines in the development and accident period graphs are a reflection of having a parameter for every accident and development period. In some instances, this is also an indication that the model may be over parameterized. Using the "flexible" model in the "Bootstrap Models.xls" file we can illustrate the power of removing some of the parameters.

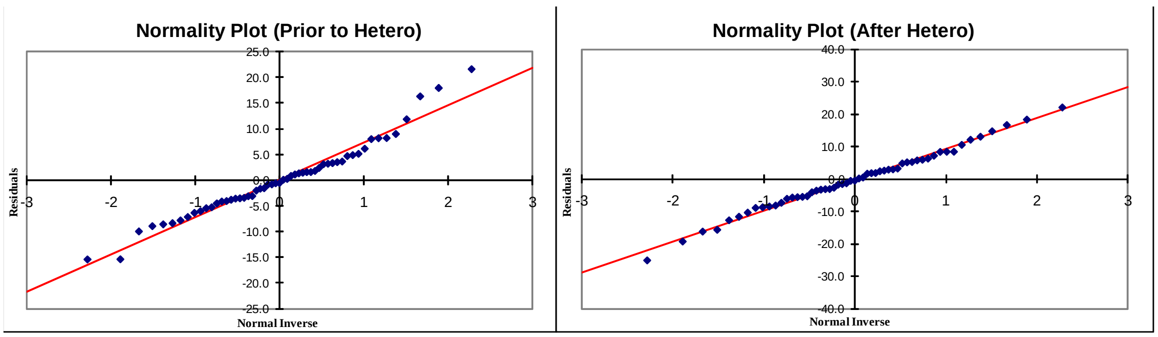
Starting with the "basic" model which includes only one parameter for accident, development and calendar periods (i.e., only one α , β and γ parameter), with a little trial and error we can find a reasonably good fit to the data using only three accident, three development and no calendar parameters. Adding blue bars to signify a parameter and red bars to signify no parameter (i.e., parameter of zero), the residual graphs for the "flexible" model are shown in figure 5.8.

Figure 5.8 Residual graphs for "flexible" model



Using the second set of "hetero" groups we can also check the normality graphs and statistics in figure 5.9 and outliers in figure 5.10. Comparing the statistics to the simplified GLM values shown in figures 5.6 and 5.7, some values improved while others did not. However, the values are not significantly different, yet the "flexible" model is far more parsimonious.

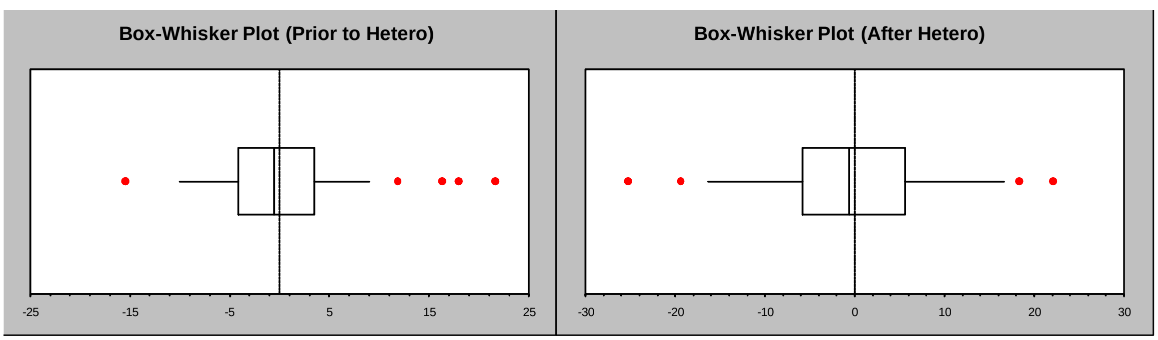
Figure 5.9 Normality plots for "flexible" model



N = 55 P-Value = 7.6% R^2 = 96.2%
Normal: MU = 0.07, Sigma = 7.26 AIC = 204.9, BIC = 60.9

N = 55 P-Value = 94.8% R^2 = 99.2%
Normal: MU = -0.31, Sigma = 9.40 AIC = 150.3, BIC = 10.2

Figure 5.10 Box-Whisker plots for "flexible" model



Interquartile Range = [-4.16, 3.46] Median = -0.58
Outliers = 6

Interquartile Range = [-5.78, 5.61] Median = -0.59
Outliers = 4

5.5 Model results

Once diagnostics have been reviewed, simulations should be run for each model. These simulation results may often provide an additional diagnostic tool to aid in evaluation of the model. As one example, we will review the results for the England and Verrall data using the simplified GLM model. The estimated-unpaid results shown in Figure 5.11 were simulated using 1,000 iterations with the hetero adjustments from Figure 5.6.

Figure 5.11 Estimated-Unpaid Model Results

England & Verrall Data
Accident Year Unpaid
Paid Chain Ladder Model

Accident Year	To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	3,901	-	-		-	-	-	-	-	-
2000	5,339	93	125	134.0%	(377)	900	62	156	306	502
2001	4,909	479	246	51.3%	(115)	1,694	447	615	940	1,186
2002	4,588	723	276	38.2%	(51)	1,892	691	899	1,220	1,515
2003	3,873	984	293	29.7%	267	2,160	976	1,176	1,453	1,802
2004	3,692	1,430	366	25.6%	434	2,888	1,400	1,670	2,072	2,405
2005	3,483	2,183	484	22.2%	896	3,812	2,140	2,497	3,038	3,483
2006	2,864	3,909	749	19.2%	1,793	6,482	3,875	4,402	5,175	5,935
2007	1,363	4,261	830	19.5%	1,757	7,865	4,221	4,789	5,700	6,321
2008	344	4,672	1,839	39.4%	617	11,009	4,523	5,853	7,878	9,509
Totals	34,358	18,737	2,769	14.8%	11,019	29,190	18,647	20,533	23,611	25,486

5.5.1 Estimated-Unpaid Results

It's recommended to start diagnostic review of the estimated-unpaid table with the standard error (standard deviation) and coefficient of variation (standard error divided by the mean), shown in Figure 5.11. Keep in mind that the standard error should increase when moving from the oldest years to the most recent years, as the standard errors (value scale) should follow the magnitude of the mean of unpaid estimates. In Figure 5.11, the standard errors conform to this pattern. At the same time, the standard error for the total of all years should be larger than any individual year.

Also, the coefficients of variation should generally decrease when moving from the oldest years to the more recent years and the coefficient of variation for all years combined should be less than for any individual year. With the exception of the 2008 accident year, the coefficients of variation in Figure 5.11 seem to also conform, although some random fluctuations may be seen.

The main reason for the decrease in the coefficient of variation has to do with the independence in the incremental claim-payment stream. Because the oldest accident year typically has only a few incremental payments remaining, or even just one, the variability is nearly all reflected in the coefficient. For more current accident years, random variations in the future incremental payment stream may tend to offset one another, thereby reducing the variability of the total unpaid loss.

While the coefficients of variation should go down, they could also start to rise again in the most recent years, which can been seen in Figure 5.11 for 2008. Such a reversal could result from a couple of issues:

- With an increasing number of parameters used in the model, the parameter uncertainty tends to increase when moving from the oldest years to the more recent years. In the most recent years, parameter uncertainty can grow to "overpower" process uncertainty, which may cause the coefficient of variation to start rising again. At a minimum, increasing parameter uncertainty will slow the rate of decrease in the coefficient of variation.
- The model may be overestimating the uncertainty in recent accident years if the increase is significant. In that case, the Bornhuetter-Ferguson or Cape Cod model may need to be used instead of a chain-ladder model.

Keep in mind also that the standard error or coefficient of variation for the total of all accident years will be less than the sum of the standard error or coefficient of variation for the individual years. This is because the model assumes that accident years are independent.

Minimum and maximum results are the next diagnostic element in our analysis of the estimated-unpaid claims in Figure 5.11, representing the smallest and largest values from all iterations of the simulation. These values will need to be reviewed in order to determine their veracity. If any of them seem implausible, the model assumptions would need to be reviewed. Their effects could materially alter the mean indication.

5.5.2 Mean and Standard Deviation of Incremental Values

The mean and standard deviation of every incremental value from the simulation process also provide useful diagnostic results, enabling us to dig deeper into potential coefficient of variation issues that may be found in the estimated-unpaid results. Consider, for example, the mean and standard deviation results shown in Figures 5.12 and 5.13, respectively.

Figure 5.12 Mean of incremental values

England & Verrall Data
Accident Year Incremental Values by Development Period
Paid Chain Ladder Model

Accident	Mean Values									
Year	12	24	36	48	60	72	84	96	108	120
1999	266	675	694	767	421	294	267	180	274	67
2000	375	945	973	1,030	588	400	376	251	383	93
2001	372	926	987	1,040	572	406	373	249	385	94
2002	369	916	967	1,037	576	395	366	253	380	90
2003	333	837	893	936	508	362	334	222	342	86
2004	351	876	943	983	546	384	354	237	362	93
2005	395	973	1,028	1,093	606	425	389	266	400	97
2006	463	1,165	1,218	1,297	721	511	472	315	476	116
2007	393	964	1,020	1,075	601	422	388	262	396	97
2008	340	861	913	974	543	359	345	233	361	84

The mean values in Figure 5.12 appear consistent throughout and support the increases in estimated unpaid by accident year that are shown in Figure 5.11. In fact, the future mean values, which lay beyond the stepped diagonal line in Figure 5.12, sum to the results in Figure 5.11. The standard deviation values in Figure 5.13, however, only appear consistent up to 2007; 2008 has larger standard deviations, which again are consistent with the standard deviations seen in Figure 5.11. But contrariwise the standard deviations can't be added because the standard deviations in Figure 5.11 represent those for aggregated incremental values by accident year, which are less than perfectly correlated.

Figure 5.13 Standard deviation of incremental values

England & Verrall Data
Accident Year Incremental Values by Development Period
Paid Chain Ladder Model

Accident	Standard Error Values									
Year	12	24	36	48	60	72	84	96	108	120
1999	106	120	233	232	138	143	103	88	137	69
2000	126	142	264	272	153	173	128	106	162	125
2001	131	134	260	281	156	174	127	102	205	129
2002	126	139	256	274	150	171	124	115	195	128
2003	122	131	246	249	148	158	125	107	178	118
2004	128	134	252	259	151	167	132	108	187	127
2005	133	144	281	283	180	190	142	123	207	134
2006	141	161	302	356	202	207	168	140	232	153
2007	124	142	297	326	185	180	138	115	208	124
2008	120	340	410	458	256	210	190	137	230	121

6. USING MULTIPLE MODELS

So far we have focused only on one model. In practice, multiple stochastic models should be used in the same way that multiple methods should be used in a deterministic analysis. First the results for each model must be reviewed and finalized, after an iterative process of diagnostic testing and reviewing model output. Then these results can be combined by assigning a weight to the results of each model.

Two primary methods exist for combining the results for multiple models:

- **Run models with the same random variables.** For this algorithm, every model uses the exact same random variables. In the "Bootstrap Models.xls" file, the random values are simulated before they are used to simulate results, which means that this algorithm may be accomplished by reusing the same set of random variables for each model. At the end, the incremental values for each model, for each iteration by accident year (that have a partial weight), can be weighted together.
- **Run models with independent random variables.** For this algorithm, every model is run with its own random variables. In the "Bootstrap Models.xls" file, the random values are simulated before they are used to simulate results, which means that this algorithm may be accomplished by simulating a new set of random variables for each model. At the end, the weights are used to randomly select a model for each iteration by accident year so that the result is a weighted "mixture" of models.

Both algorithms are similar to the process of weighting the results of different deterministic methods to arrive at an actuarial best estimate. The process of weighting the results of different stochastic models produces an actuarial best estimate of a distribution.

The second method of combining multiple models can be illustrated using combined Schedule P data for five top 50 companies.[29] Data for all Schedule P lines with 10 years of history may be found in the "Industry Data.xls" file, but we will confine our examination to Parts A, B, and C. For each line of business we ran simplified GLM models for paid and incurred data (labeled Chain Ladder), as well as paid and incurred data for the Bornhuetter-Ferguson and Cape Cod models described in section 3.3. For this section, we will only focus on the results for Part A (Homeowners/Farm owners).

By comparing the results for all six models (or fewer, depending on how many are used)[30] a qualitative assessment of the relative merits of each model may be determined. Bayesian methods

[29] The five companies represent large, medium and smaller companies that have been combined to maintain anonymity. For each Part, a unique set of five companies were used.
[30] Other models in addition to a bootstrap model could also be included in the weighting process.

can be used to determine weighting based on the quality of each model's forecasts. The weights can be determined separately for each year. The table in Figure 6.1 shows an example of weights for the Part A data.[31] The weighted results are displayed in the "Best Estimate" column of Figure 6.2. As a parallel to a deterministic analysis, the means from the six models could be considered a reasonable range (i.e., from $4,059 to $5,242).

Figure 6.1 Model weights by accident year

Accident	Model Weights by Accident Year						
	Chain Ladder		Bornhuetter-Ferguson		Cape Cod		
Year	Paid	Incurred	Paid	Incurred	Paid	Incurred	TOTAL
1999	50.0%	50.0%					100.0%
2000	50.0%	50.0%					100.0%
2001	50.0%	50.0%					100.0%
2002	50.0%	50.0%					100.0%
2003	50.0%	50.0%					100.0%
2004	50.0%	50.0%					100.0%
2005	50.0%	50.0%					100.0%
2006	12.5%	12.5%	18.8%	18.8%	18.8%	18.8%	100.0%
2007	12.5%	12.5%	18.8%	18.8%	18.8%	18.8%	100.0%
2008	12.5%	12.5%	18.8%	18.8%	18.8%	18.8%	100.0%

Figure 6.2 Summary of results by model

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Summary of Results by Model

Accident	Mean Estimated Unpaid						
	Chain Ladder		Bornhuetter Ferguson		Cape Cod		Best Est.
Year	Paid	Incurred	Paid	Incurred	Paid	Incurred	(Weighted)
1999	-	-	-	-	-	-	-
2000	2	1	1	2	2	2	1
2001	38	36	25	25	25	32	37
2002	42	40	36	36	36	42	41
2003	57	60	56	57	57	66	59
2004	98	98	94	92	92	106	99
2005	212	219	164	166	166	189	218
2006	290	292	327	318	318	371	339
2007	677	665	715	701	701	823	739
2008	3,826	3,826	2,642	2,840	2,840	3,324	3,192
Totals	5,242	5,239	4,059	4,236	4,236	4,953	4,726

With our focus on the entire distribution, the weights by year were used to randomly sample the specified percentage of iterations from each model. A more complete set of the results for the "weighted" iterations can be created similar to the tables shown in section 5. The companion "Best Estimate.xls" file can be used to weight six different models together in order to calculate a weighted

[31] For simplicity, the weights are judgmental and not derived using Bayesian methods.

best estimate. An example for Part A is shown in the table in Figure 6.3.

Figure 6.3 Estimated-unpaid model results (best estimate)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Best Estimate (Weighted)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	5,234	-	-		-	-	-	-	-	-
2000	6,470	1	11	745.3%	(52)	84	0	2	19	51
2001	7,848	37	39	104.5%	(68)	263	28	56	112	164
2002	7,020	41	36	86.7%	(48)	230	33	58	108	155
2003	7,291	59	41	69.1%	(41)	276	49	78	136	191
2004	8,134	99	49	49.1%	(14)	377	90	121	188	259
2005	10,800	218	78	36.0%	28	666	209	259	359	457
2006	7,522	339	129	38.1%	37	1,227	321	402	570	739
2007	7,968	739	259	35.1%	112	1,981	722	875	1,196	1,557
2008	9,309	3,192	920	28.8%	1,090	11,122	3,128	3,629	4,792	5,722
Totals	77,596	4,726	999	21.1%	2,528	13,422	4,632	5,209	6,554	7,442
Normal Dist.		4,726	999	21.1%			4,726	5,400	6,369	7,050
logNormal Dist.		4,725	968	20.5%			4,628	5,307	6,461	7,419
Gamma Dist.		4,726	999	21.1%			4,656	5,356	6,480	7,354
TVaR							5,454	6,003	7,311	8,915
Normal TVaR							5,523	5,996	6,786	7,388
logNormal TVaR							5,484	6,021	7,054	7,964
Gamma TVaR							5,518	6,049	7,018	7,824

6.1 Additional Useful Output

Three rows of percentile numbers for the normal, lognormal, and gamma distributions, which have been fitted to the total unpaid-claim distribution, may be seen at the bottom of the table in Figure 6.3. These fitted mean, standard deviation, and selected percentiles are in their respective columns; the smoothed results can be used to assess the quality of fit, parameterize a DFA model, or used to estimate extreme values,[32] among other applications.

Four rows of numbers indicating the Tail Value at Risk (TVaR), defined as the average of all of the simulated values equal to or greater than the percentile value, may also be seen at the bottom of Figure 6.3. For example, in this table, the 99th percentile value for the total unpaid claims for all accident years combined is 7,442, while the average of all simulated values that are greater than or equal to 7,442 is 8,915. The Normal TVaR, Lognormal TVaR, and Gamma TVaR rows are calculated similarly, except that they use the respective fitted distributions in the calculations rather than actual simulated values from the model.

An analysis of the TVaR values is likely to help clarify a critical issue: if the actual outcome exceeds the X percentile value, how much will it exceed that value on average? This type of assessment can have important implications related to risk-based capital calculations and other technical aspects of enterprise risk management. But it is worth noting that the purpose of the

[32] Of course the use of the extreme values assumes that the models are reliable.

normal, lognormal, and gamma TVaR numbers is to provide "smoothed" values—that is, that some of the random statistical noise is essentially prevented from distorting the calculations.

6.2 Estimated Cash Flow Results

An ODP bootstrap model's output may also be reviewed by calendar year (or by future diagonal), as shown in the table in Figure 6.4. A comparison of the values in Figures 6.3 and 6.4 indicates that the total rows are identical, because summing the future payments horizontally or diagonally will produce the same total. Similar diagnostic issues (as discussed in Section 5) may be reviewed in the table in Figure 6.4, with the exception of the relative values of the standard errors and coefficients of variation moving in opposite directions for calendar years compared to accident years. This phenomenon makes sense on an intuitive level when one considers that "final" payments, projected to the furthest point in the future, should actually be the smallest, yet relatively most uncertain.

Figure 6.4 Estimated Cash Flow (best estimate)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Calendar Year Unpaid
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2009	3,093	726	23.5%	1,445	9,809	3,024	3,428	4,355	5,101
2010	799	186	23.3%	312	2,057	786	900	1,125	1,329
2011	362	97	26.7%	124	856	356	422	528	601
2012	191	63	32.9%	52	507	183	224	312	386
2013	118	52	44.0%	(14)	430	110	144	212	285
2014	64	34	52.8%	(61)	205	60	80	127	175
2015	50	36	71.1%	(14)	332	42	67	116	191
2016	41	39	95.9%	(93)	296	31	56	112	177
2017	7	17	257.3%	(60)	175	0	9	40	64
2018	-	-		-	-	-	-	-	-
Totals	4,726	999	21.1%	2,528	13,422	4,632	5,209	6,554	7,442

6.3 Estimated Ultimate Loss Ratio Results

Another output table, Figure 6.5, shows the estimated ultimate-loss ratios by accident year. Unlike the estimated-unpaid and estimated-cash-flow tables, the values in this table are calculated using all simulated values, not just the values beyond the end of the historical triangle. Because the simulated sample triangles represent additional possibilities of what could have happened in the past, even as the "squaring of the triangle" and process variance represent what could happen as those same past values are played out into the future, we are in possession of sufficient information to enable us to estimate the complete variability in the loss ratio from day one until all claims are completely paid and settled for each accident year.[33]

Figure 6.5 Estimated-loss-ratio (best estimate)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Ultimate Loss Ratios
Best Estimate (Weighted)

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	66.3%	23.9%	36.0%	-1.4%	155.5%	65.5%	71.1%	118.7%	146.5%
2000	78.4%	24.1%	30.8%	-0.7%	189.8%	77.6%	83.9%	123.8%	157.3%
2001	87.9%	25.5%	29.0%	12.9%	260.1%	88.5%	94.1%	136.1%	175.3%
2002	72.2%	21.9%	30.3%	-31.1%	170.8%	71.6%	76.3%	117.4%	143.5%
2003	64.7%	19.2%	29.7%	15.1%	227.3%	63.4%	68.3%	104.6%	125.7%
2004	64.1%	17.3%	27.1%	-5.8%	130.7%	62.9%	67.1%	102.1%	118.6%
2005	80.3%	18.8%	23.4%	16.4%	165.7%	79.1%	84.5%	119.6%	139.1%
2006	55.1%	16.3%	29.5%	7.9%	205.9%	53.8%	57.6%	89.7%	106.1%
2007	56.7%	16.2%	28.6%	10.1%	123.8%	56.8%	60.7%	89.0%	106.4%
2008	83.6%	20.6%	24.6%	33.1%	307.1%	81.8%	87.9%	123.5%	150.9%
Totals	70.1%	6.7%	9.5%	50.0%	114.7%	69.9%	74.1%	81.0%	87.7%

The use of all simulated values indicates that the standard errors in Figure 6.5 should be proportionate to the means, while the coefficients of variation should be relatively constant by accident year. In terms of diagnostics, any increases in standard error and coefficient of variation for the most recent years would be consistent with the reasons previously cited in Section 5.4 for the estimated-unpaid tables. Risk management-wise, the loss ratio distributions have important implications for projecting pricing risk.

6.4 Distribution Graphs

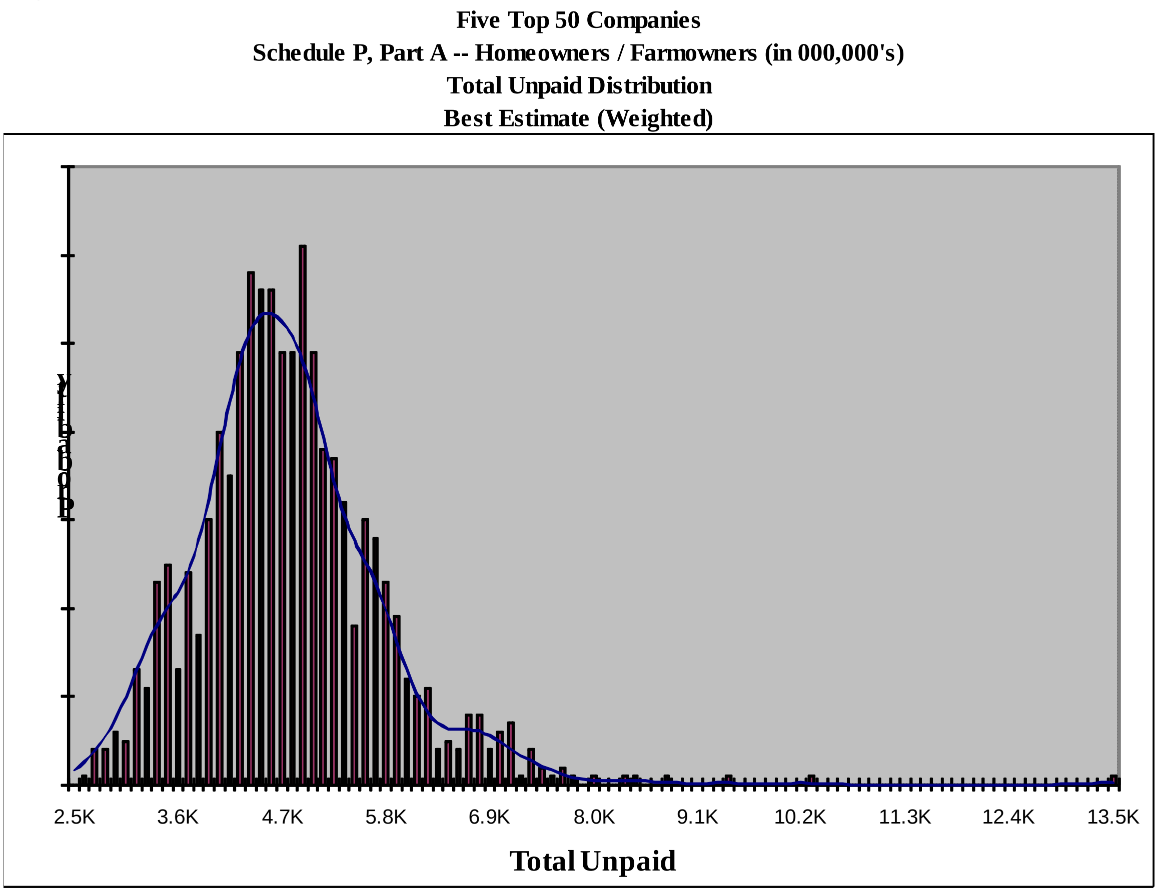
The final model output to consider is a histogram of the estimated-unpaid amounts for the total of all accident years combined, as shown in the graph in Figure 6.6. This total-unpaid-distribution histogram was created by dividing the range of all values generated from the simulation into 100

[33] If we are only interested in the "remaining" volatility in the loss ratio, then the values in the estimated-unpaid table (Figure 6.3) can be added to the cumulative paid values by year and divided by the premiums.

buckets of equal size and then counting the number of simulations that fall within each bucket. Dividing the number of simulations in each bucket by the total number of simulations (1,000 in this case) enables us to arrive at the frequency or probability for each bucket or bar in the graph.

Because the simulation results typically appear jagged, as they do in Figure 6.6, a Kernel density function (the blue line) is also used to calculate a smoothed distribution fit to the histogram values.[34] A Kernel density function may be conceptualized as a weighted average of values close to each point in the jagged distribution, with systematically less weight being given to values furthest from the points evaluated.[35]

Figure 6.6 Total Unpaid Claims Distribution



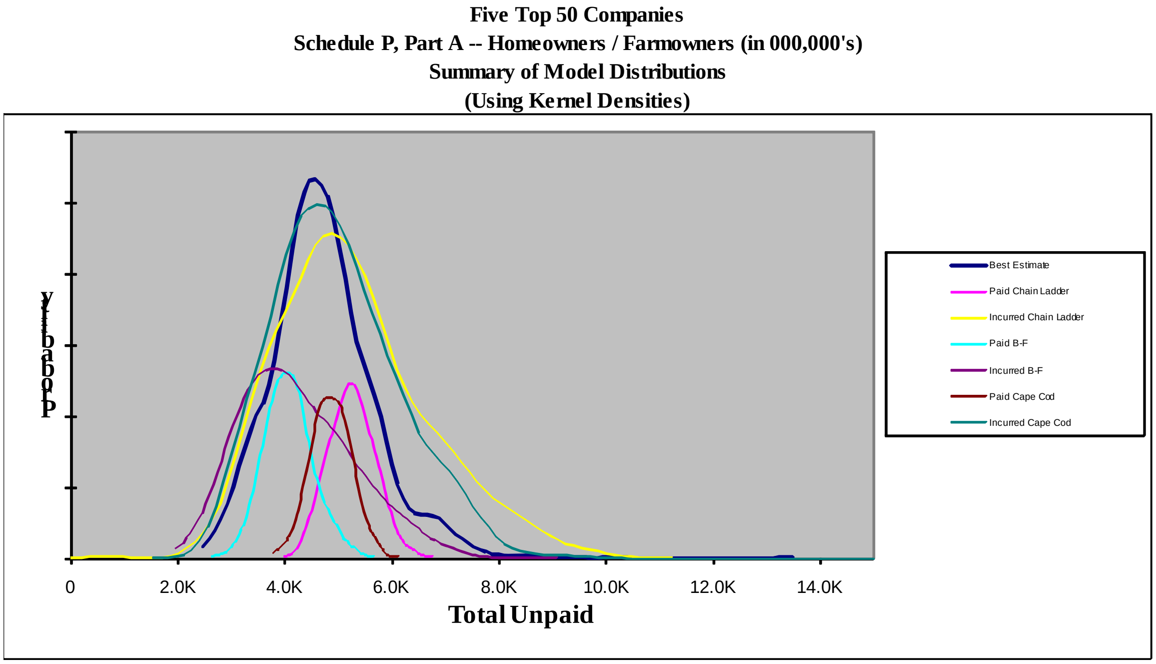
Another useful strategy for graphing the total unpaid distribution may be accomplished by creating a summary of the six model distributions used to determine the weighted "best estimate"

[34] Essentially, a Kernel density function will estimate each point in the distribution by weighting all of the values near that point, with less weight given the further the other points are from each respective point.

[35] For a more detailed discussion of Kernel density functions, see Wand & Jones, *Kernel Smoothing*, Chapman & Hall, 1995.

and distribution. An example of this graph using the kernel density functions is shown in Figure 6.7.

Figure 6.7 Summary of model distributions



The corresponding tables and graphs for the Part B and Part C results are shown in Appendices A and B, respectively.

6.5 Correlation

Results for an entire business unit can be estimated, after each business segment has been analyzed and weighted into best estimates, using aggregation. This represents another area where caution is warranted. The procedure is not a simple matter of "adding up" the distributions for each segment. In order to estimate the distribution of possible outcomes for the company as a whole a process that incorporates the correlation of results among segments must be used.[36]

Simulating correlated variables is commonly accomplished with a multivariate distribution whose parameters and correlations have been previously specified. This type of simulation is most easily applied when distributions are uniformly identical and known in advance (for example, all derived from a multivariate normal distribution). Unfortunately, these conditions do not exist for the ODP bootstrap model, a process that does not allow us to know the characteristics of distributions in

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[36] This section assumed the reader is familiar with correlation.

advance. If their shapes turn out, indeed, to be different, then another approach will be needed.

Two useful correlation processes for the bootstrap model are location mapping and re-sorting.[37]

With location mapping, each iteration will include sampling residuals for the first segment and then going back to note the location in the original residual triangle of each sampled residual.[38] Each of the other segments is sampled using the residuals at the same locations for their respective residual triangles. Thus, the correlation of the original residuals is preserved in the sampling process.

The location-mapping process is easily implemented in Excel and does not require the need to estimate a correlation matrix. There are, however, two drawbacks to this process. First, it requires all of the business segments to come with data triangles that are precisely the same size with no missing values or outliers when comparing each location of the residuals.[39] Second, the correlation of the original residuals is used in the model, and no other correlation assumptions can be used for stress testing the aggregate results.

The second correlation process, re-sorting, can be accomplished with algorithms such as Iman-Conover or Copulas, among others. The primary advantages of re-sorting include:

- The triangles for each segment may have different shapes and sizes
- Different correlation assumptions may be employed
- Different correlation algorithms may also have other beneficial impacts on the aggregate distribution

For example, using a *t*-distribution Copula with low degrees of freedom rather than a normal-distribution Copula, will effectively "strengthen" the focus of the correlation in the tail of the distribution. This type of consideration is important for risk-based capital and other risk modeling issues.

To induce correlation among different segments in the bootstrap model, a calculation of the correlation matrix using Spearman's Rank Order and use of re-sorting based on the ranks of the total unpaid claims for all accident years combined may be done. The calculated correlations for Parts A, B, and C based on the paid residuals after hetero adjustments may be seen in the table in

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37 For a useful reference see Kirschner, et al. [15].

38 For example, in the "Bootstrap Models.xls" file the locations of the sampled residuals are shown in Step 15, which could be replicated iteration by iteration for each business segment.

39 It is possible to fill in "missing" residuals in another segment using a randomly selected residual from elsewhere in the triangle, but in order to maintain the same amount of correlation the selection of the other residual would need to account for the correlation between the residuals, which complicates the process.

Figure 6.8.

Figure 6.8 Estimated Correlation and P-values

Rank Correlation of Residuals after Hetero Adjustment - Paid

LOB	1	2	3
1	1.00	0.52	0.23
2	0.52	1.00	0.25
3	0.23	0.25	1.00

P-Values of Rank Correlation of Residuals after Hetero Adjustment - Paid

LOB	1	2	3
1	0.00	0.00	0.09
2	0.00	0.00	0.08
3	0.09	0.08	0.00

Using these correlation coefficients, the "Aggregate Estimate.xls" file, and the simulation data for Parts A, B, and C, we can then calculate the aggregate results for the three lines of business that are summarized in the table in Figure 6.9. A more complete set of tables for the aggregate results is shown in Appendix C.

Figure 6.9 Aggregate estimated unpaid

Five Top 50 Companies
Aggregate All Lines of Business
Accident Year Unpaid

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	18,613	-	-		-	-	-	-	-	-
2000	20,618	31	12	37.5%	(21)	117	30	34	50	77
2001	22,866	115	40	34.8%	12	354	108	137	189	234
2002	22,842	211	43	20.3%	107	419	205	234	293	333
2003	22,351	387	52	13.4%	221	660	381	415	478	547
2004	22,422	741	86	11.5%	439	1,097	735	791	894	981
2005	24,350	1,514	150	9.9%	874	2,062	1,507	1,600	1,788	1,911
2006	19,973	2,958	264	8.9%	1,944	4,153	2,945	3,087	3,427	3,753
2007	18,919	5,533	475	8.6%	3,623	7,612	5,506	5,778	6,356	6,845
2008	15,961	12,565	1,195	9.5%	8,649	20,314	12,526	13,230	14,542	15,970
Totals	208,915	24,056	1,324	5.5%	19,572	32,759	24,008	24,852	26,193	27,644

Note that using residuals to correlate the lines of business, as in the location mapping method, and measuring the correlation between residuals, as in the re-sorting method, are both liable to create correlations that are close to zero. For reserve risk, the correlation that is desired is between the total unpaid amounts for two segments. The correlation that is being measured is the correlation between each incremental future loss amount, given the underlying model describing the overall trends in the data. This may or may not be a reasonable approximation.

Correlation is often thought of as being much stronger than "close to zero." For pricing risk, the correlation that is desired is between the loss ratio movements by accident year between two segments. This correlation is not as likely to be close to zero, so correlation of loss ratios (e.g., for the data in Figure 6.5) is often done with a different correlation assumption compared to reserving risk.

7. MODEL TESTING

Work on testing stochastic unpaid claim estimation models is still in its infancy. Most papers on stochastic models display results, and some even compare a few different models, but they tend to be void of any statistical evidence regarding how well the model in question predicts the underlying distribution. This is quite understandable since we don't know what the underlying distribution is, so with real data the best we can hope for is to retrospectively test a very old data set to see how well a model predicted the actual outcome.[40]

Testing a few old data sets is better than not, but ideally we would need many similar data sets to perform meaningful tests. One recent paper authored by the General Insurance Reserving Oversight Committee (GI ROC) in their papers for the General Insurance Research Organizing (GIRO) conference in 2007 titled "Best Estimates and Reserving Uncertainty" [28] and their updated in 2008 titled "Reserving Uncertainty" [29] took a first step in performing more meaningful statistical testing of a variety of models.

A large number of models were reviewed and tested in these studies, but one of the most interesting portions of the studies were done by comparing the unpaid liability distributions created by the Mack and ODP bootstrap model against the "true" artificially generated unpaid loss percentiles. To accomplish these tests, artificial datasets were constructed so that all of the Mack and ODP bootstrap assumptions, respectively, are satisfied. While the artificial datasets were recognized as not necessarily realistic, the "true" results are known so the Working Parties were able to test to see how well each model performed against datasets that could be considered "perfect".

7.1 Mack model results

To test the Mack model, incremental losses were simulated for a 10 x 10 square of data based on the assumptions of the Mack model. For the 30,000 datasets simulated, the upper triangles were used and the Mack model was applied to estimate the expected results and various percentiles. The actual results (lower triangle) for each iteration were then compared to the Mack estimates to see

[40] For example, data for accident years 1990 to 2000 could be completely settled and all results known as of 2010. Thus, we could use the triangle as it existed at year end 2000 to test how well a model predicts the final results.

how often they exceeded each tested percentile. If the model is working well, then the actual results should exceed the estimated percentiles one minus the percentile percent of the time – e.g., for the 90^{th} percentile, the actual results should exceed the estimated 10% of the time.

In the test, the proportion of simulated scenarios in which the "true" outcome exceeded the 99^{th} percentile of the Mack method's results was around 8-13%. If the Mack method's distribution was accurate, this should be 1%. However, it appears that the distribution created by the Mack method underestimates tail events.

7.2 Bootstrap model results

To test the ODP bootstrap model, incremental losses were simulated for a 10 x 10 square of data based on the assumptions of the ODP bootstrap model. For the 30,000 datasets simulated, the upper triangles were used and the OPD bootstrap model from England and Verrall [9 and 10] were used to estimate the expected results and various percentiles. Similarly, the proportion of simulated scenarios in which the "true" outcome exceeded the 99^{th} percentile of the Bootstrap method's results was around 2.6-3.1%.

Thus, the bootstrap model performed better than the Mack model for "perfect" data, even though the results for both models were somewhat deficient in the sense that they both seem to underpredict the extremes of the "true" distribution. In fairness, it should be noted however, that the ODP bootstrap model that was tested did not include many of the "advancements" described in section 3.2.

7.3 Future testing

The testing done for GIRO was a significant improvement over simply looking at results for different models, without knowing anything about the "true" underlying distribution. The next step in the testing process will be to test models against "true" results for realistic data instead of "perfect" data. The CAS Loss Simulation Model Working Party is testing a model that will create datasets from the claim transaction level up. The goal is to create thousands of datasets based on characteristics of real data that can be used for testing various models.

8. FUTURE RESEARCH

With testing of stochastic models in its infancy, much work in the area of future research is needed. We only offer a few such areas.

- Expand testing of the ODP bootstrap model with realistic data using the CAS loss simulation model.
- Expand the ODP bootstrap model in other ways, for example use of the Munich chain ladder with an incurred/paid set of triangles, or the use of claim counts and average severities.
- Research other risk analysis measures and how the ODP bootstrap model can be used for enterprise risk management.
- Research how the ODP bootstrap model can be used for Solvency II requirements in Europe and the International Accounting Standards.
- Research into the most difficult parameter to estimate: the correlation matrix.

9. CONCLUSIONS

With this paper we endeavored to show how the ODP bootstrap model can be used in a variety of practical ways, and to illustrate the diagnostic tools the actuary needs to assess whether the model is working well. By doing so, we believe that this toolset can become an integral part of the actuaries regular estimation of unpaid claim liabilities, rather than just a "black box" to be used only if necessary.

Acknowledgment

The authors gratefully acknowledge the many authors listed in the References (and others not listed) that contributed to the foundation of the ODP bootstrap model, without which our research would not have been possible.

Supplementary Material

There are several companion files designed to give the reader a deeper understanding of the concepts discussed in the paper. The files are all in the "Beyond the Basics.zip" file. The files are:

Model Instructions.doc – this file contains a written description of how to use the primary bootstrap modeling files.

Primary bootstrap modeling files:

Industry Data.xls – this file contains Schedule P data by line of business for the entire U.S. industry and five of the top 50 companies, for each LOB that has 10 years of data.

Bootstrap Model.xls – this file contains the detailed model steps described in this paper as well as various modeling options and diagnostic tests. Data can be entered and simulations run and saved for use in calculating a weighted best estimate.

Best Estimate.xls – this file can be used to weight the results from six different models to get a "best estimate" of the distribution of possible outcomes.

Aggregate Estimate.xls – this file can be used correlate the best estimate results from 3 LOBs/segments.

Correlation Ranks.xls – this file contains the ranks used to correlate results by LOB/segment.

Simple example calculation files:

Simple GLM.xls – this file illustrates the calculation of the GLM framework for a simple 3 x 3 triangle.

Simple GLM 6.xls – this file illustrates the calculation of the GLM framework for a simple 6 x 6 triangle.

Simple GLM 6 with Outlier.xls – this file illustrates how the calculation of the GLM framework for a simple 6 x 6 triangle is adjusted for an outlier.

Simple GLM 6 with 3yr avg.xls – this file illustrates how the calculation of the GLM framework for a simple 6 x 6 triangle is adjusted to only use the equivalent of a three-year average (i.e., the last four diagonals).

Simple GLM 6 with 1 Acc Yr Parameter.xls – this file illustrates the calculation of the GLM framework using only one accident year (level) parameter, a development year trend parameter for every year and no calendar year trend parameter for a simple 6 x 6 triangle.

Simple GLM 6 with 1 Dev Yr Parameter.xls – this file illustrates the calculation of the GLM framework using only one development year trend parameter, an accident year (level) parameter for every year and no calendar year trend parameter for a simple 6 x 6 triangle.

Simple GLM 6 with 1 Acc Yr & 1 Dev Yr Parameter.xls – this file illustrates the calculation of the GLM framework using only one accident year (level) parameter, one development year trend parameter and no calendar year trend parameter for a simple 6 x 6 triangle.

Simple GLM 6 with 1 Acc Yr 1 Dev Yr & 1 Cal Yr Parameter.xls – this file illustrates the calculation of the GLM framework using only one accident year (level) parameter, one development year trend parameter and one calendar year trend parameter for a simple 6 x 6 triangle.

Appendix A – Schedule P, Part B Results

In this appendix the results for Schedule P, Part B (Private Passenger Auto Liability) are shown.

Figure A.1 Estimated-unpaid model results (best estimate)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Best Estimate (Weighted)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	11,816	-	-		-	-	-	-	-	-
2000	12,679	27	4	14.8%	14	41	27	29	33	37
2001	13,631	66	8	12.4%	35	96	66	70	81	88
2002	14,472	142	21	14.9%	73	225	141	153	178	201
2003	13,717	270	32	11.7%	146	390	269	286	324	361
2004	13,090	525	68	12.9%	277	767	526	559	641	709
2005	12,490	1,048	127	12.2%	553	1,503	1,048	1,100	1,278	1,387
2006	11,598	2,148	222	10.4%	1,124	3,066	2,150	2,249	2,511	2,865
2007	10,306	3,960	383	9.7%	2,115	5,421	3,962	4,103	4,611	5,158
2008	6,357	8,195	778	9.5%	4,554	11,486	8,174	8,549	9,434	10,682
Totals	120,157	16,380	898	5.5%	12,811	19,377	16,341	16,836	17,863	18,955
Normal Dist.		16,380	898	5.5%			16,380	16,986	17,857	18,469
logNormal Dist.		16,380	904	5.5%			16,355	16,975	17,909	18,595
Gamma Dist.		16,380	898	5.5%			16,364	16,976	17,884	18,541

Figure A.2 Estimated cash flow (best estimate)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Calendar Year Unpaid
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2009	8,090	459	5.7%	6,153	9,603	8,081	8,335	8,843	9,338
2010	3,944	225	5.7%	3,127	4,787	3,935	4,079	4,311	4,560
2011	2,162	132	6.1%	1,586	2,672	2,165	2,239	2,376	2,514
2012	1,125	77	6.9%	864	1,450	1,124	1,171	1,252	1,326
2013	546	42	7.7%	404	697	545	571	617	672
2014	275	20	7.4%	205	371	274	288	310	327
2015	137	15	11.2%	97	192	137	146	162	179
2016	71	6	8.2%	50	93	71	74	80	86
2017	30	3	11.0%	15	41	31	32	35	38
2018	-	-		-	-	-	-	-	-
Totals	16,380	898	5.5%	12,811	19,377	16,341	16,836	17,863	18,955

Figure A.3 Estimated-loss-ratio (best estimate)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Ultimate Loss Ratios
Best Estimate (Weighted)

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	75.6%	9.2%	12.1%	38.2%	103.8%	75.6%	77.5%	92.8%	100.2%
2000	81.8%	9.7%	11.9%	43.8%	112.6%	81.9%	83.9%	99.7%	107.4%
2001	83.5%	9.6%	11.5%	47.6%	116.1%	83.4%	85.6%	101.0%	110.3%
2002	79.4%	9.0%	11.4%	45.3%	108.7%	79.4%	81.3%	97.0%	103.1%
2003	68.9%	7.2%	10.5%	39.3%	94.7%	68.7%	70.7%	82.7%	89.0%
2004	65.7%	7.4%	11.3%	36.3%	89.1%	65.5%	67.3%	80.1%	85.6%
2005	66.5%	7.6%	11.4%	36.2%	91.6%	66.3%	68.1%	80.9%	87.3%
2006	66.4%	5.8%	8.7%	33.6%	92.3%	66.4%	67.3%	76.6%	86.4%
2007	70.1%	6.2%	8.8%	40.4%	95.8%	69.9%	71.0%	81.1%	90.6%
2008	71.1%	6.4%	8.9%	41.5%	98.6%	71.2%	73.0%	81.3%	91.2%
Totals	72.3%	2.4%	3.3%	63.7%	80.6%	72.2%	73.8%	76.3%	78.0%

Figure A.4 Mean of incremental values

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Mean Values									
Year	12	24	36	48	60	72	84	96	108	120
1999	5,257	3,374	1,464	850	460	225	113	59	32	25
2000	5,625	3,613	1,566	908	490	240	121	61	34	27
2001	6,086	3,906	1,690	981	531	261	131	67	37	29
2002	6,489	4,168	1,805	1,044	567	279	140	71	39	31
2003	6,233	3,996	1,730	1,003	544	268	134	68	38	30
2004	6,073	3,894	1,689	978	528	261	131	67	37	29
2005	6,035	3,869	1,679	973	527	259	130	66	37	29
2006	6,050	3,882	1,685	1,032	571	271	138	66	40	30
2007	6,301	4,042	1,798	1,037	577	272	139	66	41	30
2008	6,361	4,202	1,811	1,048	581	276	140	66	41	30

Figure A.5 Standard deviation of incremental values

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Standard Error Values									
Year	12	24	36	48	60	72	84	96	108	120
1999	643	417	188	109	62	35	14	14	4	3
2000	677	437	195	115	66	36	15	14	5	4
2001	708	456	201	119	70	38	16	14	5	4
2002	745	481	215	127	72	40	16	16	5	4
2003	663	423	188	115	65	38	15	15	5	4
2004	691	441	201	122	67	39	15	17	5	4
2005	692	448	200	119	69	39	16	15	5	4
2006	530	348	156	112	70	36	14	14	5	4
2007	556	363	178	109	67	37	14	13	5	3
2008	571	406	179	109	66	37	14	14	5	3

Figure A.6 Total unpaid claims distribution

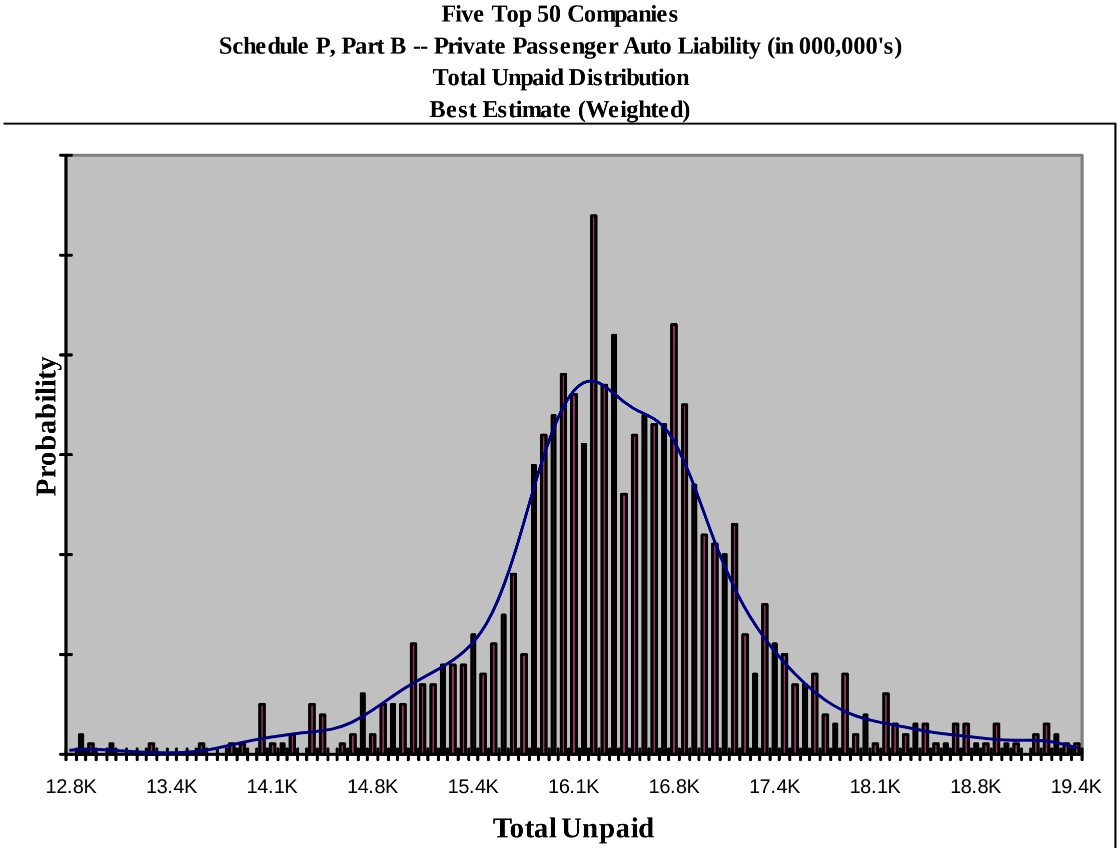
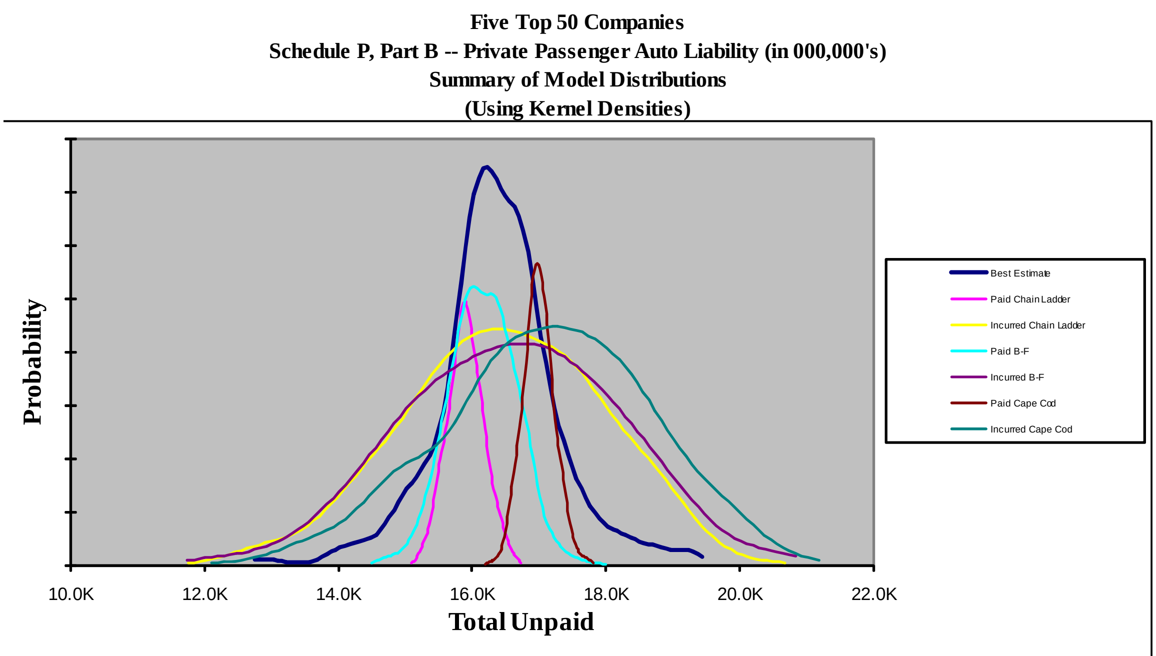


Figure A.7 Summary of model distributions



Appendix B – Schedule P, Part C Results

In this appendix the results for Schedule P, Part C (Commercial Auto Liability) are shown.

Figure B.1 Estimated-unpaid model results (best estimate)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Best Estimate (Weighted)

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	1,563	-	-		-	-	-	-	-	-
2000	1,469	3	2	70.8%	(1)	12	2	4	6	8
2001	1,387	12	4	31.4%	3	31	12	14	19	22
2002	1,350	28	5	19.5%	12	53	28	31	37	41
2003	1,342	58	8	13.9%	33	84	59	64	71	79
2004	1,198	116	17	14.9%	61	191	115	127	146	158
2005	1,061	249	34	13.8%	151	334	250	272	304	322
2006	853	472	56	11.9%	323	628	479	516	553	577
2007	645	834	73	8.8%	605	1,015	844	891	937	965
2008	294	1,178	106	9.0%	904	1,484	1,181	1,262	1,337	1,366
Totals	11,162	2,950	149	5.0%	2,434	3,363	2,949	3,055	3,186	3,276
Normal Dist.		2,950	149	5.0%			2,950	3,050	3,194	3,295
logNormal Dist.		2,950	150	5.1%			2,946	3,048	3,202	3,314
Gamma Dist.		2,950	149	5.0%			2,947	3,048	3,198	3,306

Figure B.2 Estimated cash flow (best estimate)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Calendar Year Unpaid
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2009	1,171	65	5.5%	974	1,374	1,172	1,214	1,280	1,321
2010	806	46	5.7%	657	960	806	838	882	911
2011	488	35	7.1%	364	595	490	512	544	571
2012	256	27	10.7%	174	343	255	274	303	324
2013	125	15	12.2%	73	177	124	136	150	160
2014	58	8	13.7%	35	90	57	63	71	76
2015	30	5	15.8%	17	47	30	33	39	42
2016	14	3	24.6%	4	25	13	16	19	22
2017	3	2	55.4%	(0)	12	3	4	6	7
2018	-	-		-	-	-	-	-	-
Totals	2,950	149	5.0%	2,434	3,363	2,949	3,055	3,186	3,276

Figure B.3 Estimated-loss-ratio (best estimate)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Accident Year Ultimate Loss Ratios
Best Estimate (Weighted)

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	89.5%	3.2%	3.5%	80.2%	99.5%	89.5%	91.8%	94.5%	96.3%
2000	81.3%	2.8%	3.5%	72.9%	90.1%	81.2%	83.3%	86.0%	87.4%
2001	73.1%	2.6%	3.5%	63.8%	81.8%	73.2%	74.8%	77.4%	79.2%
2002	60.6%	2.1%	3.5%	53.7%	66.2%	60.6%	62.1%	64.1%	65.5%
2003	55.5%	1.9%	3.5%	48.9%	61.4%	55.5%	56.9%	58.7%	59.8%
2004	53.8%	2.1%	3.9%	47.5%	60.2%	53.8%	55.3%	57.3%	58.5%
2005	51.5%	2.2%	4.3%	43.1%	57.9%	51.6%	53.0%	55.1%	56.5%
2006	53.7%	2.9%	5.3%	43.5%	62.1%	53.9%	55.7%	58.1%	59.8%
2007	59.6%	3.6%	6.1%	46.9%	68.6%	59.9%	62.4%	65.0%	66.6%
2008	61.8%	4.6%	7.5%	49.4%	75.3%	62.1%	65.3%	68.7%	70.2%
Totals	62.5%	0.9%	1.5%	59.6%	65.3%	62.5%	63.1%	64.0%	64.6%

Figure B.4 Mean of incremental values

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Mean Values									
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**
1999	332	384	345	244	135	64	29	17	12	3
2000	312	360	326	229	127	61	27	15	11	3
2001	297	343	311	218	121	57	26	15	9	3
2002	292	339	305	214	118	56	25	16	10	3
2003	296	343	309	218	119	58	28	17	11	3
2004	276	322	288	203	111	62	26	15	10	3
2005	270	312	280	199	128	64	27	16	10	3
2006	265	308	278	224	128	65	27	16	10	3
2007	299	348	331	239	136	68	29	17	11	3
2008	294	370	320	231	132	67	27	16	11	3

Figure B.5 Standard deviation of incremental values

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Standard Error Values									
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**
1999	18	35	18	15	21	14	5	4	3	2
2000	17	34	18	15	19	13	5	4	3	2
2001	17	34	17	15	20	13	5	4	3	2
2002	16	32	17	14	19	14	5	4	3	1
2003	17	32	17	14	20	13	6	4	3	2
2004	16	32	16	14	19	14	6	3	3	1
2005	16	32	16	14	22	15	6	4	3	2
2006	16	31	16	25	22	15	6	4	3	1
2007	17	35	29	22	22	14	7	4	3	2
2008	17	45	27	21	21	13	6	4	3	2

Figure B.6 Total unpaid claims distribution

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Best Estimate (Weighted)

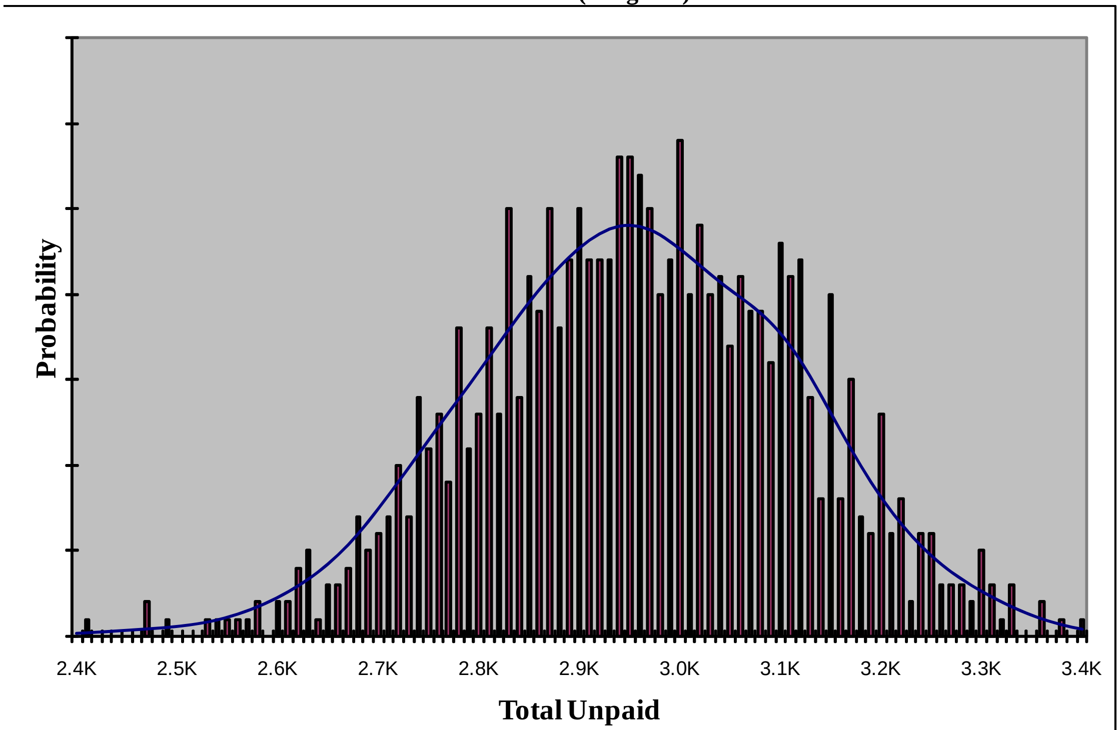
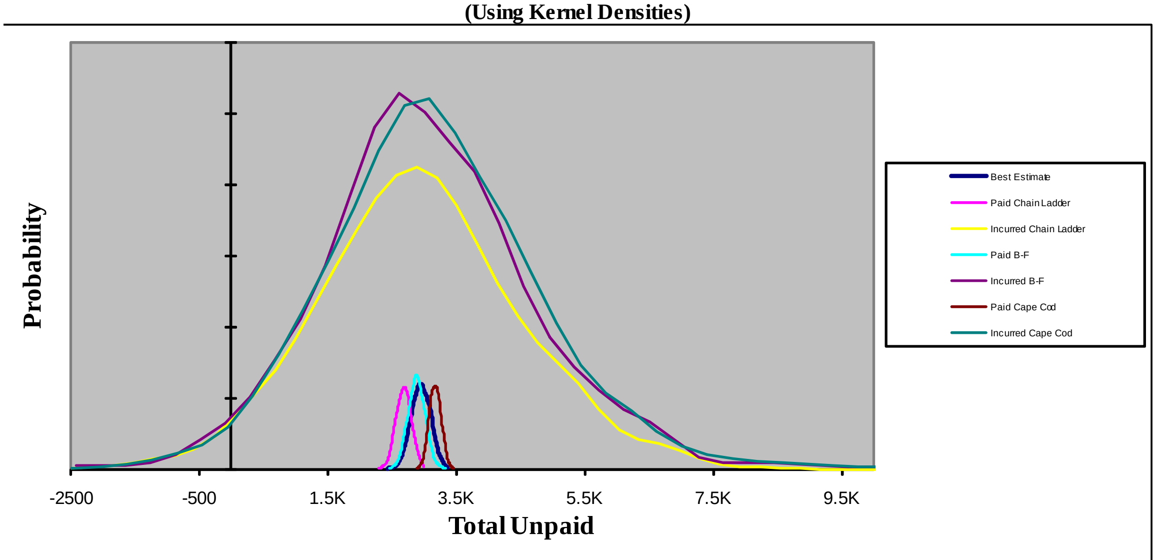


Figure B.7 Summary of model distributions

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Summary of Model Distributions
(Using Kernel Densities)



Appendix C – Aggregate Results

In this appendix the results for the correlated aggregate of the three Schedule P lines of business (Parts A, B, and C) are shown, using the correlation calculated from the paid data after adjustment for heteroscedasticity.

Figure A.1 Estimated-unpaid model results (best estimate)

Five Top 50 Companies
Aggregate All Lines of Business
Accident Year Unpaid

Accident Year	Paid To Date	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	18,613	-	-		-	-	-	-	-	-
2000	20,618	31	12	38.2%	(26)	111	30	35	49	78
2001	22,866	115	40	34.7%	5	348	106	136	190	240
2002	22,842	211	43	20.5%	109	397	205	234	295	348
2003	22,351	387	51	13.2%	230	624	381	416	482	532
2004	22,422	741	86	11.6%	432	1,080	732	788	883	1,003
2005	24,350	1,514	156	10.3%	876	2,079	1,506	1,601	1,779	1,908
2006	19,973	2,958	267	9.0%	1,771	3,970	2,942	3,092	3,428	3,704
2007	18,919	5,533	487	8.8%	3,472	7,657	5,525	5,770	6,402	6,981
2008	15,961	12,565	1,410	11.2%	7,894	21,492	12,527	13,260	14,919	16,794
Totals	208,915	24,056	1,644	6.8%	18,197	34,272	23,963	25,008	26,726	28,724
Normal Dist.		24,056	1,644	6.8%			24,056	25,164	26,760	27,880
logNormal Dist.		24,055	1,635	6.8%			24,000	25,124	26,835	28,105
Gamma Dist.		24,056	1,644	6.8%			24,018	25,143	26,822	28,044

Figure A.2 Estimated cash flow (best estimate)

Five Top 50 Companies
Aggregate All Lines of Business
Calendar Year Unpaid

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2009	12,354	1,018	8.2%	9,070	19,805	12,305	12,931	13,951	15,231
2010	5,549	348	6.3%	4,293	7,187	5,535	5,768	6,123	6,499
2011	3,012	188	6.2%	2,349	3,740	3,012	3,129	3,329	3,491
2012	1,572	114	7.3%	1,262	2,009	1,563	1,641	1,769	1,865
2013	789	73	9.3%	583	1,117	785	830	913	1,019
2014	397	42	10.5%	260	552	395	420	470	512
2015	217	39	18.1%	133	505	211	234	289	351
2016	125	40	32.3%	(13)	396	116	142	200	266
2017	40	18	44.6%	(25)	208	36	42	71	98
2018	-	-		-	-	-	-	-	-
Totals	24,056	1,644	6.8%	18,197	34,272	23,963	25,008	26,726	28,724

Figure A.3 Estimated loss ratio (best estimate)

Five Top 50 Companies
Aggregate All Lines of Business
Accident Year Ultimate Loss Ratios

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
1999	73.7%	9.5%	12.9%	39.8%	110.8%	73.4%	77.6%	91.2%	99.7%
2000	80.7%	9.6%	11.9%	50.0%	122.2%	80.5%	84.5%	97.4%	107.9%
2001	84.2%	10.1%	12.0%	53.4%	139.7%	84.2%	88.4%	101.0%	112.9%
2002	75.7%	9.0%	11.9%	42.2%	119.1%	75.6%	79.2%	92.6%	100.5%
2003	66.5%	7.8%	11.7%	40.7%	116.9%	66.2%	70.0%	80.3%	89.2%
2004	64.3%	7.4%	11.6%	37.4%	97.3%	64.0%	67.8%	78.4%	86.0%
2005	70.7%	8.4%	11.9%	41.1%	104.5%	70.2%	74.0%	86.5%	94.6%
2006	61.2%	6.9%	11.3%	38.4%	119.4%	60.8%	63.1%	74.3%	82.4%
2007	64.0%	7.5%	11.7%	40.4%	93.5%	63.9%	66.4%	78.7%	86.4%
2008	75.5%	9.8%	13.0%	46.7%	168.1%	74.8%	78.4%	92.6%	107.2%
Totals	70.8%	2.9%	4.0%	62.0%	88.1%	70.7%	72.5%	75.7%	78.0%

Figure A.4 Mean of incremental values

Five Top 50 Companies
Aggregate All Lines of Business
Accident Year Incremental Values by Development Period

Accident	Mean Values									
Year	12	24	36	48	60	72	84	96	108	120
1999	9,292	4,873	2,023	1,183	635	310	153	81	67	30
2000	10,524	5,354	2,154	1,254	666	326	162	84	75	31
2001	11,868	5,902	2,324	1,333	713	348	174	90	82	34
2002	11,809	6,018	2,398	1,382	740	363	181	94	82	35
2003	11,790	5,934	2,343	1,355	722	354	178	93	82	34
2004	12,243	5,985	2,313	1,327	703	356	175	92	85	33
2005	14,191	6,560	2,419	1,369	741	367	181	94	98	32
2006	11,951	5,878	2,290	1,418	775	366	186	92	81	39
2007	12,654	6,215	2,525	1,440	793	371	188	93	83	39
2008	16,129	6,928	2,585	1,463	801	378	191	94	86	40

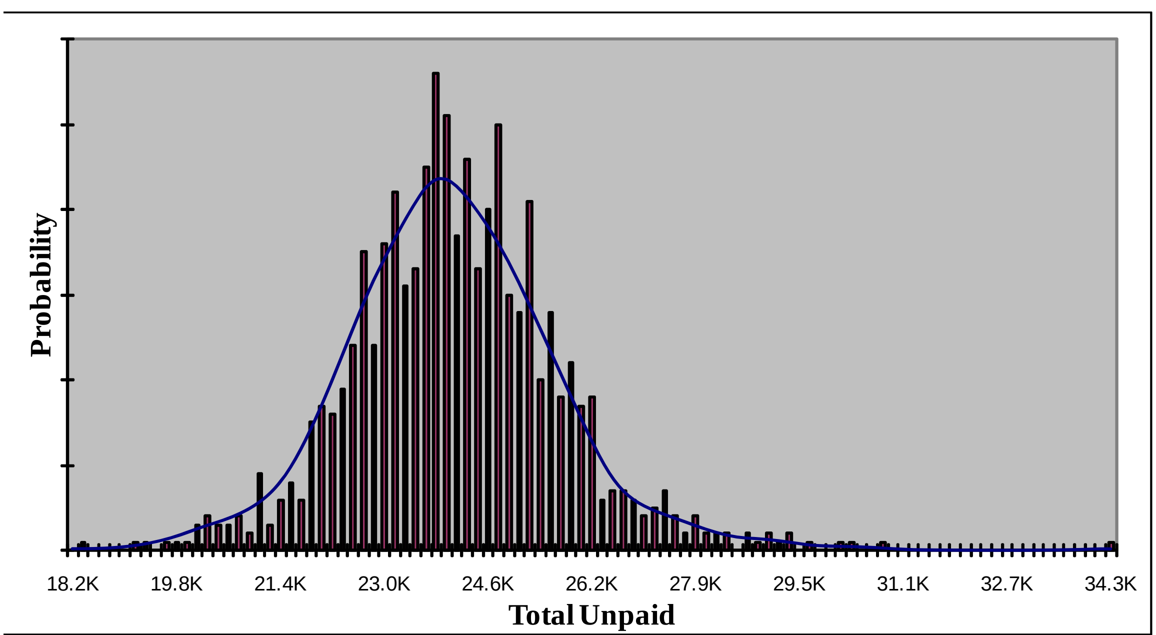
Figure A.5 Standard deviation of incremental values

Five Top 50 Companies
Aggregate All Lines of Business
Accident Year Incremental Values by Development Period

Accident	Standard Deviation Values									
Year	12	24	36	48	60	72	84	96	108	120
1999	1,503	607	216	123	72	39	16	15	27	7
2000	1,549	615	224	132	76	40	17	14	28	12
2001	1,755	681	242	137	81	43	18	15	37	14
2002	1,704	702	252	145	81	44	18	17	34	12
2003	1,714	657	230	135	76	42	17	16	35	15
2004	1,743	662	231	138	79	43	18	17	38	13
2005	1,998	762	253	148	83	44	19	16	47	16
2006	1,745	624	201	141	86	40	19	15	29	16
2007	1,850	678	246	138	85	42	18	14	32	17
2008	2,534	918	270	147	88	45	20	15	37	18

Figure A.6 Total unpaid claims distribution

Five Top 50 Companies
Aggregate All Lines of Business
Total Unpaid Distribution



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Abbreviations and notations

Collect here in alphabetical order all abbreviations and notations used in the paper

AIC: Akaike Information Criteria
APD: Automobile Physical Damage
BIC: Bayesian Information Criteria
BF: Bornhuetter-Ferguson
CC: Cape Cod
CL: Chain Ladder
CoV: Coefficient of Variation
DFA, Dynamic Financial Analysis
ELR: Expected Loss Ratio
GLM: Generalized Linear Models
ERM, Enterprise Risk Management
MLE: Maximum Likelihood Estimate
ODP: Over-Dispersed Poisson
OLS: Ordinary Least Squares
RSS: Residual Sum Squared
SSE: Sum of Squared Errors

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Claims Development by Layer: The Relationship between Claims Development Patterns, Trend and Claim Size Models

Rajesh Sahasrabuddhe, FCAS, MAAA

The purpose of Charles Cook's 1970 paper *Trend and Loss Development Factors* was to address the "overlap fallacy." That is, the focus of that paper was to demonstrate that trend and claims development were mutually exclusive adjustments. While this is certainly true, it should also be understood that there is a relationship between limited claims development patterns and trend factors. The "connector" between claims development patterns and trend is the claim size model. This relationship is critical to analyzing "real word" data which is rarely available on a ground-up, unlimited basis and where the implicit assumption of trend in a single direction may not be appropriate.

This paper presents a demonstration of that relationship and also provides an approach to adjust development patterns for a particular claim size layer in order to calculate a development pattern for any other layer. As importantly, the approach discussed is designed to produce models that are internally consistent with respect to development patterns, trend factors and size of loss models (increased / decreased limit factors).

Keywords. development patterns, excess layer

1. INTRODUCTION

The purpose of this paper is to demonstrate the relationship between claims development, trend and claim size factors. Those relationships are then explored in order to provide a practical approach for adjusting a development pattern appropriate for any claim layer to produce a development pattern for any other layer. The approach also allows for adjustments related to cost level assumptions implicit in development patterns and ensures that assumptions related to claim size models, claims development and trend are internally consistent.

The procedure may be applied to either paid claims or reported claims. Additionally, although we use "claims" in the discussion, the procedure may also be applied to claims and allocated claim adjustment expenses (or only allocated claim adjustment expenses) assuming that all parameters and assumptions are defined consistently.

1.1 Research Context

The current approach for estimating excess layer development is based on Emanuel Pinto and Daniel Gogol's paper, "*An Analysis of Excess Loss Development*." The focus of that paper is the fitting of observed development factors as a function of retentions. The observed factors were developed using an analysis of a large industry database. Pinto/Gogol then present an approach for calculating excess layer development in Section 5 and this approach is explored further in George M. Levine's review. However, this approach requires that the actuary first calculate excess layer development using their fitting approach.

Many actuaries would not have access to such industry data and as such the Pinto/Gogol approach would not be practical. In addition to this issue, the methodology does not use the inherent relationship of claims size models, trend and claims development patterns.

1.2 Scope and Objective

This paper includes comments related to assumptions implicit in the determination of development patterns, trend and claim size distributions in practice. However, the development of these actuarial models and their parameters is beyond the scope of this paper. The objective of this paper is to provide a methodology to calculate development factors by layer once the actuary has already determined his/her assumptions with respect to a "base" development pattern, trend and claim size models.

1.3 Outline

The paper presents a discussion of a robust approach and then provides an example that incorporates simplifying assumptions that are common in actuarial practice. The remainder of the paper proceeds as follows. Section 2 will provide notation and define important algebraic definitions of model factors. Section 3 provides the discussion of the inter-relationship between claims development, trend and claim size models. Section 4 will provide implementation examples to the oft-studied *Mack* triangle and a simpler approach that may be sufficient for many analyses.

2. BACKGROUND

We begin by examining the implicit and explicit assumptions of claims development, trend and claim size models.

The discussion will assume that we are analyzing an $n \times n$ claims triangle. We generalize our discussion to allow for data that is truncated from below at d and censored from above at p. This is typical of data subject to deductibles and policy limits. Of course, if $d = 0$ and

$p = \infty$, then the claims data is provided on a ground-up, unlimited (GUU) basis. The notation used in this paper is as follows:

$C_{i,j}^{\boldsymbol{L}}$ = Cumulative claims in the layer *L*, for exposure period *i* as of the end of development interval *j*

$C_{i,\infty}^{\boldsymbol{L}}$ = Ultimate claims in the layer *L*, for exposure period *i* ($j = \infty$)

$L(d,p)$ = Claims layer truncated from below at *d* and censored from above at *p* where $0 \leq d < p \leq \infty$

Though it will be obvious that this is not a necessary assumption, in order to simplify notation, we will assume claims layer $\boldsymbol{L}$ is consistent throughout the data triangle. Claims data is typically organized as presented in Table 1.

TABLE 1
CUMULATIVE CLAIMS DATA

Exposure Period (*i*)	Development Interval (*j*)	1	2	3	…	*n*
	1	$C_{1,1}^{\boldsymbol{L}}$	$C_{1,2}^{\boldsymbol{L}}$	$C_{1,3}^{\boldsymbol{L}}$	…	$C_{1,n}^{\boldsymbol{L}}$
	2	$C_{2,1}^{\boldsymbol{L}}$	$C_{2,2}^{\boldsymbol{L}}$	$C_{2,3}^{\boldsymbol{L}}$	…	
	3	$C_{3,1}^{\boldsymbol{L}}$	$C_{3,2}^{\boldsymbol{L}}$	$C_{3,3}^{\boldsymbol{L}}$	…	
	…	…				
	n	$C_{n,1}^{\boldsymbol{L}}$				

Below we first discuss trend, claims size models and development patterns separately and then discuss their relationships.

2.1 Trend Factors

Trend rates typically refer to the annual change in cost level for a particular claims layer. In practice, trend rates often do not vary between accident periods. In addition, trend that acts in the development period or calendar period direction is often not considered. Finally, the consideration of the varying effects of trend applicable to different claims layer is often nonexistent.

Rather than using annual rates of change, we will use cost level indices, $\boldsymbol{T}$. Cost level indices are determined so as to apply to cumulative claims for accident year *i* as of development maturity *j*. The indices are an accumulation of the incremental changes relative to a "base cost level." Any accident year and maturity combination can be considered the "base." In practice, the base cost level will typically be defined as the cost level associated with ultimate claims for the oldest exposure period.

Our trend is explicitly defined to apply to the ground-up, unlimited claims layer. This is consistent with approaches in practice where the trend assumption is based on external cost information such as the Consumer Price Index. If trend is estimated from claims data that is subject to policy limits or deductibles then we will first need to adjust the data to a ground-up, unlimited basis using the claim size model.

Our model allows for trend that acts in multiple directions. We use the following notation for cost level indices.

$T_{i,j}$ = Trend indices for cumulative GUU claims for exposure period *i* at the end of development interval *j*

TABLE 2
COST LEVEL INDICES

Exposure Period (*i*)	Development Interval (*j*): 1	2	3	…	*n*
1	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	…	$T_{1,n}$
2	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	…	
3	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	…	
…	…				
n	$T_{n,1}$				

2.2 Claim Size Model

The claim size model describes the distribution of claim sizes. Though we do not restrict claim size models with respect to complexity, for practicality we require the following:

- that claims size model parameters can be adjusted for the impact of inflation (includes most common claim size models such as the lognormal and exponential)

- that limited expected values and unlimited means (first moments) can be calculated with reasonable effort.

2.2.1 Limit Adjustment Factors

The limit adjustment factors, $S(a,b)$, represents the ratio of expectations of claims between layer L_a and L_b.

$$S_{i,\infty}(\boldsymbol{L}_a, \boldsymbol{L}_b) = \{LEV(p_a; \boldsymbol{\Phi}_{i,\infty}) - LEV(d_a; \boldsymbol{\Phi}_{i,\infty})\} \ / \ \{LEV(p_b; \boldsymbol{\Phi}_{i,\infty}) - LEV(d_b; \boldsymbol{\Phi}_{i,\infty})\} \quad (2.1)$$

$$S_{i,j}(\boldsymbol{L}_a, \boldsymbol{L}_b) = \{LEV(p_a; \boldsymbol{\Phi}_{i,j}) - LEV(d_a; \boldsymbol{\Phi}_{i,j})\} \ / \ \{LEV(p_b; \boldsymbol{\Phi}_{i,j}) - LEV(d_b; \boldsymbol{\Phi}_{i,j})\} \quad (2.2)$$

$$S_{i,j}(L_a, L_b) = \mathrm{E}[C_{i,j}^{\boldsymbol{L}_a}] / \ [C_{i,j}^{\boldsymbol{L}_b}] \quad (2.3)$$

where LEV is the characteristic limited expected value function for the claim size model and $\boldsymbol{\Phi}$ represents the "name" (e.g. lognormal, Pareto, exponential) and parameters of the claim size model. We also acknowledge that the parameters of the claims size model, $\boldsymbol{\Phi}$, will vary by exposure period i and development interval j as a result of differences in cost level.

In later sections, we will use the notation $LEV(\boldsymbol{L}; \boldsymbol{\Phi})$ to refer to the limited expected value for the layer $L(d, p)$. This is calculated as follows:

$$LEV(\boldsymbol{L}; \boldsymbol{\Phi}) = LEV(p; \boldsymbol{\Phi}) - LEV(d; \boldsymbol{\Phi}) \quad (2.4)$$

2.2.2 Gross-up Factors

In the special case where p_a=∞ and d_a=0, $S(a,b)$ simplifies to a factor to gross-up claims to a GUU basis. We can then use the characteristic first moment (mean) function, M, in the numerator rather than the limited expected value function.

$$G_{i,.}(b) = M(\boldsymbol{\Phi}_{i,\infty}) \ / \ \{LEV(p_b; \boldsymbol{\Phi}_{i,\infty}) - LEV(d_b; \boldsymbol{\Phi}_{i,\infty})\} \quad (2.4)$$

$$G_{i,j}(b) = M(\boldsymbol{\Phi}_{i,j}) \ / \ \{LEV(p_b; \boldsymbol{\Phi}_{i,j}) - LEV(d_b; \boldsymbol{\Phi}_{i,j})\} \quad (2.5)$$

2.3 Claims Development

Claims development factors, F, represent the expected ratios of ultimate claims to claims at maturities prior to ultimate. That is:

$$F_{i,j}^{\boldsymbol{L}} = \mathrm{E}[\ C_{i,\infty}^{\boldsymbol{L}} \ / \ C_{i,j}^{\boldsymbol{L}}\ ] \quad (2.6)$$

3. RESULTS AND DISCUSSION

We can now explore the relationships between claims development, trend, and claim size models. The discussion assumes that we have been provided with unlimited claims trend factors and that we have developed the cost level indices as presented in Table 2.

3.1 Claim Size and Trend

As per the requirements of Section 2.2, for our selected claim size model, we can calculate model parameters for prior or future exposure periods using the trend indices.

$$\boldsymbol{\Phi}_{i,j} \sim f\left(\boldsymbol{\Phi}_{n,j}, T_{i,j}, T_{n,j}\right) \tag{3.1}$$

3.2 Claim Development Patterns, Claim Size and Trend

In practice, claims development patterns are estimated from unadjusted data and are applied to claims for all exposure periods. We should acknowledge that this is not appropriate unless (i) claims data are provided on a GUU basis and (ii) trend acts only in the accident year direction. Since this is oftentimes not the case, we address these issues by adjusting the triangle of claims data prior to analysis. Specifically, we adjust observed claim amounts for differences in cost level and limit using the limited expected value function.

3.2.1 Development of Basic Limit Claims Development Pattern, Exposure Year *n* Cost Level

We first select a Basic Limit, $\boldsymbol{B}$, which is the threshold at which we believe the data is sufficiently credible for the purpose of estimating claims development patterns. Recall from Table 1 that $\boldsymbol{L}$ represents the layer for which data is available. We then adjust each observation of cumulative claims as follows[1]:

$$E\left[\acute{C}_{i,j}^{\boldsymbol{B}} | C_{i,j}^{\boldsymbol{L}}\right] = C_{i,j}^{\boldsymbol{L}} \times LEV(\boldsymbol{B}; \boldsymbol{\Phi}_{n,j}) / LEV(\boldsymbol{L}; \boldsymbol{\Phi}_{i,j}) \tag{3.2}$$

We note that there is no restriction that $\boldsymbol{B} \neq \boldsymbol{L}$. We should recognize that if $\boldsymbol{B} = \boldsymbol{L}$, then we are simply adjusting the data for differences due to the impact of trend in the layer. (Note the difference between the first subscript of $\boldsymbol{\Phi}$ in the numerator and denominator of Equation 3.2).

We then analyze this adjusted data, $\acute{C}_{i,j}^{\boldsymbol{B}}$, in order to estimate development patterns at a common (basic) limit and an exposure period i=n cost level. This pattern is denoted $F_{n,j}^{\boldsymbol{B}}$ and we have the following relationship:

$$F_{n,j}^{\boldsymbol{B}} = E\left[\acute{C}_{n,\infty}^{\boldsymbol{B}} / \acute{C}_{n,j}^{\boldsymbol{B}}\right] \tag{3.3}$$

---

[1] We presume that a triangle at the basic limit is not readily available.

As you review the following sections, keep in mind that this basic limit development pattern at exposure year *n* cost level will now be used to calculate basic limit development for any other layer and exposure period (cost level).

3.2.2 Calculation of Claims Development Pattern for Any Layer and Cost Level

Equation 3.2 also provides an important general relationship applicable to any layer $\boldsymbol{X}$ if we have data for layer $\boldsymbol{L}$.

$$E\left[C_{i,j}^{\boldsymbol{X}} | C_{i,j}^{\boldsymbol{L}}\right] = C_{i,j}^{\boldsymbol{L}} \times LEV(\boldsymbol{X}; \boldsymbol{\Phi}_{i,j}) / LEV(\boldsymbol{L}; \boldsymbol{\Phi}_{i,j}) \tag{3.4}$$

$$C_{i,j}^{\boldsymbol{L}} \times S_{i,j}(\boldsymbol{X}, \boldsymbol{L}) \tag{3.5}$$

Using this general relationship, we can calculate basic limit development factors for any exposure period for any layer $\boldsymbol{X}$ from the development factor for $\boldsymbol{B}$ at exposure year *n* cost levels:

$$F_{i,j}^{\boldsymbol{X}} = E\left[\frac{C_{i,\infty}^{\boldsymbol{B}}}{C_{i,j}^{\boldsymbol{B}}}\right] = E\left[\frac{C_{n,\infty}^{B}}{C_{n,j}^{B}} \times \frac{LEV(\boldsymbol{X}; \boldsymbol{\Phi}_{i,\infty}) / LEV(\boldsymbol{B}; \boldsymbol{\Phi}_{n,\infty})}{LEV(\boldsymbol{X}; \boldsymbol{\Phi}_{i,j}) / LEV(\boldsymbol{B}; \boldsymbol{\Phi}_{n,j})}\right] \tag{3.6}$$

$$F_{i,j}^{\boldsymbol{X}} = F_{n,j}^{\boldsymbol{B}} \times \frac{LEV(\boldsymbol{X}; \boldsymbol{\Phi}_{i,\infty}) / LEV(\boldsymbol{B}; \boldsymbol{\Phi}_{n,\infty})}{LEV(\boldsymbol{X}; \boldsymbol{\Phi}_{i,j}) / LEV(\boldsymbol{B}; \boldsymbol{\Phi}_{n,j})} \tag{3.7}$$

$$F_{i,j}^{\boldsymbol{X}} = F_{n,j}^{\boldsymbol{B}} \times \frac{S_{i,\infty}(\boldsymbol{X}, \boldsymbol{B})}{S_{i,j}(\boldsymbol{X}, \boldsymbol{B})} \tag{3.8}$$

However, as we demonstrated in Equation 3.1, $\boldsymbol{\Phi}_{i,j}$ is a function of trend indices and $\boldsymbol{\Phi}_{n,j}$. So, substituting Equation 3.1 into Equation 3.7, we have:

$$\boxed{F_{i,j}^{\boldsymbol{X}} = F_{n,j}^{\boldsymbol{B}} \times \frac{LEV(\boldsymbol{X}; T_{i,\infty}, T_{n,\infty}, \boldsymbol{\Phi}_{n,\infty}) / LEV(\boldsymbol{B}; \boldsymbol{\Phi}_{n,\infty})}{LEV(\boldsymbol{X}; T_{i,j}, T_{n,j}, \boldsymbol{\Phi}_{n,j}) / LEV(\boldsymbol{B}; \boldsymbol{\Phi}_{n,j})}} \tag{3.9}$$

Equations 3.8 and 3.9 are the primary findings of this research: **Development factors at different cost levels and different layers are related to each other based on claim size models and trend.**

3.3 Other Practical Uses

Oftentimes, we are simply provided with a development pattern. Although we are typically aware of the limits associated with the triangle and/or pattern, it is not stated at any particular cost level.

In Equation 3.9, we demonstrated that, for limited claims data, development patterns will vary with cost level. However, this relationship is often ignored usually because it is presumed immaterial. For convenience, we will simply assert that the cost level is that of the latest exposure period.

We also typically have a claim size model at ultimate (e.g. increased limit factors), but size models by age are usually not available. Let us also assume that we are only concerned with estimating development factors applicable to claims at the latest valuation date.

We can use a variation of Equation 3.6 to develop claims development patterns:

$$F_{i,j}^{\boldsymbol{X}} = F_{n,j}^{B} \times \frac{LEV(\boldsymbol{X}; \boldsymbol{\Phi}_{i,\infty})/LEV(\boldsymbol{B}; \boldsymbol{\Phi}_{n,\infty})}{R_j\ (X, B)} \tag{3.10}$$

The primary difference between Equations 3.8 and Equation 3.6 is that rather than using claim size models by age in the denominator, we use a quantity, $R_j\ (X, B)$, that is simpler to estimate approximately.

$R_j\ (X, B)$ is the ratio between limited expected values for layer $\boldsymbol{X}$ and $\boldsymbol{B}$ at the end of development interval *j*. $R_j\ (X, B)$ is only evaluated along a single diagonal since we typically have at least one diagonal (usually the current diagonal) where we can observe ratios of claims at various limits. It should be noted that *R* carries only one subscript, that for maturity. In using this latter approach, we assume that differences in cost level are immaterial to the calculation of **ratios** of claims by layer[2].

For the moment, we will ignore the possibility of negative development and assume that $R_j\ (X, B)$<1. The latter assumption indicates that we are trying to develop an estimate for a pattern at a lower layer given a pattern at a higher layer. We should recognize that $\boldsymbol{R}$ will have the following properties:

i. $\boldsymbol{R}_a > \boldsymbol{R}_b$ for a<b - At early maturities, there will be less development in the excess layer than at later maturities.
ii. $\boldsymbol{R}_a \geq U$, where $\boldsymbol{U} = \lim_{a\to\infty} \boldsymbol{R}_a$ - We should recognize that *U* can be calculated as the product of $\boldsymbol{R}$ and the ratio of ultimate claim development factors layer $\boldsymbol{X}$ and $\boldsymbol{B}$. Until we reach ultimate, the reported ratio will always be greater than ultimate ratio. This is because the there is more development associated with the denominator of $\boldsymbol{R}$ (claims in layer $\boldsymbol{B}$, the higher limit) than the numerator of $\boldsymbol{R}$ (claims in layer $\boldsymbol{X}$, the lower limit) and at ultimate $\boldsymbol{R} = \boldsymbol{U}$.

[2] Note that we are not asserting that they are immaterial with respect to absolute limited expected values.

iii. If our base development pattern is provided on an unlimited basis (i.e. ***B***=GUU), then the maximum value for ***R*** may be calculated as *U*Claims Development Factor*. The derivation of this maximum is presented in Appendix A.

It should be recognized that these conditions will be violated if there is negative development or if we assume that an excess layer might develop more quickly than a working layer. These conditions are not necessary for application of this approach. However, it is useful to review the results under the typical considerations described above to provide a more intuitive understanding of the dynamics of the calculation.

In the third example presented in Section 4, we use a simpler approach to calculating ***R***[3] which is then used to calculate development factors for a layer other than the layer associated with the development pattern provided.

3.4 Issues

Relative to common development method projections, the procedure described above requires additional assumptions and calculations. The use of certain assumptions and calculations would not appear to be overly onerous:

1. The procedure requires that the actuary select a basic limit. However, actuaries either explicitly or implicitly select a basic limit in applying the development method. That is, whenever a development triangle is analyzed there is an implicit assumption that the limit associated with that triangle is sufficiently credible to produce development factors.

2. The procedure requires the use of a(n ultimate) claim size model in order to implement a development method analysis. This may or may not result in an additional burden on the actuary. Oftentimes, claim size information (such as increased limit factors) or a claim size model is already available to the actuary. If not, we would submit that knowledge of the distribution of claim sizes is important in understanding the dynamics of claims development.

   We should also recognize that we use the claim size model only to calculate relative limited expected values near the deductible, basic limit, policy limit and

[3] Simpler than calculating claim size models by age.

limit underlying the development data. Deductibles generally would not be an issue for the types of exposures for which the actuary would be willing to invest the effort required of this approach. As such, what is important is that our claim size model produces reasonable ratios of limited expectations to unlimited means at higher values. It is less important that the absolute limited expected values are accurate and therefore a simpler size of loss model may be sufficient though we need to recognize its shortcomings and not use that model out of context.

3. The procedure requires that the data triangle be adjusted to a basic limit and common cost level. As demonstrated in Examples 1 & 2 of Section 4, given claim size and trend information, the calculation and application of adjustment factors would not seem to create a significant additional burden.

There are however two sets of assumptions that could be perceived as resulting in a significant additional burden.

1. Claim size models at maturities prior to ultimate are generally not available. In addition, these models would have limited application outside of this context. However, understanding changes in claims size models over time would be a significant benefit for actuaries to understand excess layer development.

   With an insurance company database or even a self-insured risk of sufficient size, we believe that an algorithm could be reasonably programmed to calculate these claim size models.

   Although a robust claim size model is required for full implementation of this approach (Examples 1 & 2), it should be recognized that only the ratio of expected values is required to adjust development patterns from one layer to another. This is a significantly reduced burden as will be demonstrated in Example 3 in the next section.

2. The procedure requires the calculation of a triangle of trend indices in order to implement a development method analysis. We would expect that a trend assumption exists in the analysis. The trend indices specify the cost level associated with cumulative claim observations. This becomes somewhat difficult to conceptualize in two respects:

a. Trend typically acts on incremental activity.

b. The impact of trend on reported incurred claims and, more specifically, the timing of the effect of trend on case basis reserves, is difficult to ascertain.

These difficulties are not an issue if we assume that development only acts in the exposure period direction. Even if we have trend also acting across calendar periods, we would submit that this will require the actuary to confront the assumption with respect to the direction(s) in which trend acts or (more importantly) does not act. In addition documenting this assumption produces greater transparency and better informs the consumer of actuarial information.

4. EXAMPLES

We now present three examples that implement the concepts described in Section 3. The first two examples are based on the oft-studied claims triangle included in the *Distribution-Free Calculation of the Standard Error of Chain Ladder Reserve Estimates* by Thomas Mack. Example 1 and Example 2 are identical except that in Example 1, the Basic Limit is well above the working claims layer; in Example 2, the Basic Limit is within the working layer. The third example presents the approach discussed in Section 3.3 where we adjust a development pattern provided to us to determine patterns for other layers.

4.1 Example 1 & 2

For Examples 1 & 2, we provide the following additional (contrived) information about the Mack triangle. This information is intended to be typical of that which might apply to actual data:

- We have selected a basic limit of $500 thousand
- The policy limit is $2 million
- The data in the triangle is for the ground-up layer to $1 million
- Trend acts at a rate of 2% each exposure period; but there was a one-time increase to 5% between exposure period 6 and 7.
- Trend acts at a rate of 1% each calendar period; but there was a one-time decrease of 5% between calendar period 2 and 3.

The calculations in the examples are presented as follows:

- In Section A, we present the claims data and relevant information. Both exposure periods and development intervals are annual. However, since this is not a strict requirement of our approach, we have retained the more generic labels: "Exposure Period" and "Development Interval."
- In Section B, we present the calculation of trend indices.
- In Section C, we present the claim size model. Section C1 provides the claim size model at Exposure Period 10 cost level. We use an exponential model for simplicity of presentation; however any model that meets the requirements of Section 2.2 could be used.

  In Section C2, we present the calculation of adjusted exponential parameters based on the Exposure Year 10 parameters and trend indices.

  In Sections C4 through C6, we present the calculation of limited expected values using the characteristic function of the exponential model.
- In Section D1, we present the adjusted cumulative claims triangle. This triangle adjusts all historical observations to the basic limit at Exposure Period 10 cost levels. The adjustments are based on ratios of limited expected values. In Sections D2 and D3, we calculate the incremental and cumulative development patterns.
- In Section E, we apply Equation 3.7 to calculate development factors for various layers at appropriate exposure year cost levels. In Section E7, we present the differences between factors calculated through examination of the (unadjusted) triangle in Section A1 and the factors resulting from our approach.

  Factors for certain excess layers are presented as "very large." This occurs since the expectation of claim in the layer at early maturities is very small.

  We note that the differences presented in Section of E7 of Example 1 are quite small. The differences will grow with the expectation of claims in the layer between the basic limit and layer under review. This is demonstrated in Example 2, where the resulting differences are quite a bit greater. We should also recognize that layers that are excess layers for an insurer (or self-insured) become working layers for reinsurers (excess insurers).

It will also grow in situations where trend and/or development act over longer periods or at higher rates.

4.2 Example 3

The third example presents the approach described in Section 3.3. This approach is intended to provide a simpler application of the theory in Section 3. As presented in Example 1, if the basic limit is sufficiently high and trend is contained, the impact of data adjustments is minimal.

The calculations in Example 3 are reasonably self-explanatory. However, readers should note the following:

- At ultimate, all claims development factors equal unity and the ratio at age (col. 9) equals the ratio at ultimate (col. 8).
- The *x* axis is labeled "maturity," not exposure period. The observed pattern should be viewed as one observation of a random process at a particular maturity and not viewed as the ratio applicable to an exposure period.
- We use an algorithm to select ratios by age. At the earliest maturity, we know that the ratio should be "high." That is because claims emergence in excess layers is still "low."

  Our selected ratios are calculated as follows:

  Selected Ratio = Ultimate Ratio + (1-Ultimate Ratio) * Decay Factor

  This approach recognizes that we want to "keep" a portion of the distance between the ultimate ratio and the maximum ratio (unity). This portion is determined through the use of a decay model where we keep most of the difference at the earliest maturity and none at ultimate.

  In practice, assuming we are analyzing development patterns at limits at or above the working layer, the ratios will be close to unity and the amount of error that could possibly be created by this approach is minimal.

5. CONCLUSION

In this paper we have demonstrated that there is a relationship between claim development patterns by layer and that that relationship is a function of trend and claim size models. This relationship can be used to calculate development patterns for a claims layer from a development pattern for any other claims layer.

These relationships also demonstrate that limited development factors are a function of not only maturity but also cost level. Therefore, the same pattern of limited factors should not always be applied to all exposure periods under review.

With short development patterns, low trend rates and limits above the working layer, the adjustment is small and often immaterial. Not all exposures exhibit these characteristics and for these exposures, the adjustments may be meaningful. For exposures where the adjustment may not be meaningful, we provided an alternative simpler approach to adjust development patterns.

Acknowledgment

The author acknowledges Katy Siu and Jason Shook, for their reviews of this paper. Any remaining errors in the paper are solely the responsibility of the author.

Appendix A: Calculation of Maximum Ratios of Basic Limit to Unlimited Claims

The maximum ratio is represented by the limiting case where all development in the unlimited layer occurs above the basic limit. The maximum ratio is calculated as follows:

Notation:

R = Ultimate ratio of basic limit to unlimited claims

A = Ratio of basic limit to unlimited claims prior to ultimate

D = Unlimited claim development factor

<u>Claims</u>

	Prior to Ultimate	At Ultimate
Limited to Basic Limit	B_a	B_r
Excess of Basic Limit	X_a	X_r
Unlimited	C_a	C_r

Identities:

I1: $B_a = B_r$ (All development in excess layer; basic limit layer at ultimate)

I2: $R = B_r / C_r$

I3: $C_r = C_a * D$

Then under maximum conditions:

$A_{\max} = B_a / C_a$

$$A_{\max} = B_a / (C_r/D) \quad \text{« per I3 »}$$
$$A_{\max} = D * B_a / C_r$$
$$A_{\max} = D * B_r / C_r \quad \text{« per I1 »}$$
$$A_{\max} = D * R \quad \text{« per I2 »}$$

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Biography of the Author

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Claims Development by Layer

Example 1

A. Data and Information

1 Cumulative Development Triangle (C_{ij})

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	357,848	1,124,788	1,735,330	2,218,270	2,745,596	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463
2	352,118	1,236,139	2,170,033	3,353,322	3,799,067	4,120,063	4,647,867	4,914,039	5,339,085	
3	290,507	1,292,306	2,218,525	3,235,179	3,985,995	4,132,918	4,628,910	4,909,315		
4	310,608	1,418,858	2,195,047	3,757,447	4,029,929	4,381,982	4,588,268			
5	443,160	1,136,350	2,128,333	2,897,821	3,402,672	3,873,311				
6	396,132	1,333,217	2,180,715	2,985,752	3,691,712					
7	440,832	1,288,463	2,419,861	3,483,130						
8	359,840	1,421,128	2,864,498							
9	376,686	1,363,294								
10	344,014									

2 Limit of Data in Triangle 1,000,000

3 Selected Basic Limit 500,000

4 Policy Limit 2,000,000

B. Trend Indices

1 Exposure Period Trend Index [2% EP Trend; 5% between EP 6 and 7]

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.020	1.020	1.020	1.020	1.020	1.020	1.020	1.020	1.020	1.020
3	1.040	1.040	1.040	1.040	1.040	1.040	1.040	1.040	1.040	1.040
4	1.061	1.061	1.061	1.061	1.061	1.061	1.061	1.061	1.061	1.061
5	1.082	1.082	1.082	1.082	1.082	1.082	1.082	1.082	1.082	1.082
6	1.104	1.104	1.104	1.104	1.104	1.104	1.104	1.104	1.104	1.104
7	1.159	1.159	1.159	1.159	1.159	1.159	1.159	1.159	1.159	1.159
8	1.182	1.182	1.182	1.182	1.182	1.182	1.182	1.182	1.182	1.182
9	1.206	1.206	1.206	1.206	1.206	1.206	1.206	1.206	1.206	1.206
10	1.230	1.230	1.230	1.230	1.230	1.230	1.230	1.230	1.230	1.230

2 Calendar Period Trend Index [1% Calendar Period Trend; -5% between CP 2 and 3]

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	1.000	1.010	0.960	0.969	0.979	0.989	0.998	1.008	1.019	1.029
2	1.010	0.960	0.969	0.979	0.989	0.998	1.008	1.019	1.029	1.039
3	0.960	0.969	0.979	0.989	0.998	1.008	1.019	1.029	1.039	1.049
4	0.969	0.979	0.989	0.998	1.008	1.019	1.029	1.039	1.049	1.060
5	0.979	0.989	0.998	1.008	1.019	1.029	1.039	1.049	1.060	1.070
6	0.989	0.998	1.008	1.019	1.029	1.039	1.049	1.060	1.070	1.081
7	0.998	1.008	1.019	1.029	1.039	1.049	1.060	1.070	1.081	1.092
8	1.008	1.019	1.029	1.039	1.049	1.060	1.070	1.081	1.092	1.103
9	1.019	1.029	1.039	1.049	1.060	1.070	1.081	1.092	1.103	1.114
10	1.029	1.039	1.049	1.060	1.070	1.081	1.092	1.103	1.114	1.125

3 Combined Trend Index [B1 * B2]

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	1.000	1.010	0.960	0.969	0.979	0.989	0.998	1.008	1.019	1.029
2	1.030	0.979	0.988	0.998	1.008	1.018	1.029	1.039	1.049	1.060
3	0.998	1.008	1.018	1.029	1.039	1.049	1.060	1.070	1.081	1.092
4	1.028	1.039	1.049	1.060	1.070	1.081	1.092	1.103	1.114	1.125
5	1.059	1.070	1.081	1.092	1.102	1.114	1.125	1.136	1.147	1.159
6	1.091	1.102	1.113	1.125	1.136	1.147	1.159	1.170	1.182	1.194
7	1.157	1.169	1.181	1.193	1.204	1.217	1.229	1.241	1.253	1.266
8	1.192	1.204	1.216	1.229	1.241	1.253	1.266	1.278	1.291	1.304
9	1.228	1.241	1.253	1.266	1.278	1.291	1.304	1.317	1.330	1.344
10	1.266	1.278	1.291	1.304	1.317	1.330	1.343	1.357	1.370	1.384

Claims Development by Layer

Example 1

C. Claim Size Model (Apply to Cumulative Claims)

1 Claims Size Model Parameters at Exposure Year 10 Cost Level [via claim size modeling]

	Development Interval (j)									
Exponential (θ)	1	2	3	4	5	6	7	8	9	10
i=10	28,138	84,242	133,998	182,460	204,649	228,245	252,830	265,063	275,707	280,000

2 Claims Size Model Parameters [C1 * $B3_{i,j}$ / $B3_{10,j}$]

Exposure Period (i)	Development Interval (j)									
Exponential (θ)	1	2	3	4	5	6	7	8	9	10
1	22,233	66,564	99,590	135,608	152,099	169,636	187,908	197,000	204,911	208,101
2	22,905	64,501	102,598	139,703	156,693	174,759	193,583	202,949	211,099	214,386
3	22,195	66,449	105,696	143,922	161,425	180,037	199,429	209,078	217,475	220,861
4	22,865	68,455	108,888	148,268	166,300	185,474	205,452	215,392	224,042	227,531
5	23,555	70,523	112,177	152,746	171,322	191,075	211,657	221,897	230,808	234,402
6	24,267	72,653	115,564	157,359	176,496	196,846	218,049	228,598	237,779	241,481
7	25,735	77,048	122,556	166,879	187,174	208,755	231,241	242,429	252,164	256,090
8	26,512	79,375	126,257	171,919	192,827	215,059	238,224	249,750	259,780	263,824
9	27,313	81,772	130,070	177,111	198,650	221,554	245,418	257,292	267,625	271,792
10	28,138	84,242	133,998	182,460	204,649	228,245	252,830	265,063	275,707	280,000

3 Unlimited Means

Exposure Period (i)	Development Interval (j)									
	1	2	3	4	5	6	7	8	9	10
1	22,233	66,564	99,590	135,608	152,099	169,636	187,908	197,000	204,911	208,101
2	22,905	64,501	102,598	139,703	156,693	174,759	193,583	202,949	211,099	214,386
3	22,195	66,449	105,696	143,922	161,425	180,037	199,429	209,078	217,475	220,861
4	22,865	68,455	108,888	148,268	166,300	185,474	205,452	215,392	224,042	227,531
5	23,555	70,523	112,177	152,746	171,322	191,075	211,657	221,897	230,808	234,402
6	24,267	72,653	115,564	157,359	176,496	196,846	218,049	228,598	237,779	241,481
7	25,735	77,048	122,556	166,879	187,174	208,755	231,241	242,429	252,164	256,090
8	26,512	79,375	126,257	171,919	192,827	215,059	238,224	249,750	259,780	263,824
9	27,313	81,772	130,070	177,111	198,650	221,554	245,418	257,292	267,625	271,792
10	28,138	84,242	133,998	182,460	204,649	228,245	252,830	265,063	275,707	280,000

4 Limited Expected Values at Policy Limits

Exposure Period (i)	Development Interval (j)									
	1	2	3	4	5	6	7	8	9	10
1	22,233	66,564	99,590	135,607	152,099	169,635	187,904	196,992	204,899	208,087
2	22,905	64,501	102,598	139,703	156,692	174,757	193,577	202,938	211,083	214,367
3	22,195	66,449	105,696	143,922	161,424	180,034	199,420	209,063	217,453	220,835
4	22,865	68,455	108,888	148,268	166,299	185,470	205,440	215,372	224,013	227,496
5	23,555	70,523	112,177	152,746	171,321	191,070	211,640	221,870	230,769	234,356
6	24,267	72,653	115,564	157,358	176,494	196,838	218,026	228,562	237,726	241,420
7	25,735	77,048	122,556	166,878	187,170	208,740	231,200	242,365	252,074	255,987
8	26,512	79,375	126,257	171,917	192,821	215,039	238,170	249,667	259,662	263,690
9	27,313	81,772	130,070	177,109	198,642	221,527	245,348	257,184	267,473	271,619
10	28,138	84,242	133,998	182,456	204,638	228,209	252,737	264,922	275,512	279,779

5 Limited Expected Values at Limits of Data Triangle

Exposure Period (i)	Development Interval (j)									
	1	2	3	4	5	6	7	8	9	10
1	22,233	66,564	99,586	135,522	151,887	169,169	186,990	195,770	203,355	206,398
2	22,905	64,501	102,592	139,594	156,428	174,187	192,478	201,479	209,249	212,366
3	22,195	66,449	105,688	143,784	161,096	179,340	198,105	207,328	215,285	218,474
4	22,865	68,455	108,877	148,094	165,893	184,629	203,871	213,318	221,461	224,723
5	23,555	70,523	112,161	152,527	170,822	190,056	209,778	219,448	227,777	231,112
6	24,267	72,652	115,544	157,085	175,885	195,621	215,826	225,719	234,233	237,640
7	25,735	77,048	122,521	166,462	186,279	207,020	228,179	238,510	247,385	250,932
8	26,512	79,375	126,211	171,407	191,748	213,003	234,644	245,194	254,248	257,866
9	27,313	81,772	130,010	176,486	197,356	219,126	241,247	252,014	261,246	264,932
10	28,138	84,241	133,921	181,699	203,105	225,390	247,987	258,969	268,375	272,128

6 Limited Expected Values at Basic Limit

Exposure Period (i)	Development Interval (j)									
	1	2	3	4	5	6	7	8	9	10
1	22,233	66,528	98,933	132,211	146,418	160,735	174,776	181,433	187,052	189,273
2	22,905	64,473	101,813	135,805	150,248	164,762	178,957	185,674	191,337	193,574
3	22,195	66,413	104,764	139,462	154,134	168,836	183,176	189,948	195,652	197,903
4	22,865	68,409	107,785	143,181	158,075	172,956	187,431	194,253	199,993	202,256
5	23,555	70,464	110,876	146,960	162,068	177,119	191,718	198,586	204,358	206,632
6	24,267	72,578	114,037	150,798	166,111	181,322	196,035	202,944	208,743	211,026
7	25,735	76,931	120,483	158,539	174,229	189,725	204,633	211,606	217,447	219,744
8	26,512	79,229	123,851	162,538	178,404	194,029	209,019	216,018	221,873	224,175
9	27,313	81,591	127,286	166,587	182,619	198,360	213,422	220,441	226,307	228,611
10	28,138	84,019	130,788	170,682	186,869	202,716	217,839	224,873	230,745	233,050

Claims Development by Layer

Example 1

D. Calculation of Development Factors at Basic Limit

1 Cumulative Triangle Exposure Year 10 Cost Levels and Basic Limit ($C_{i,j}$) [$A1_{i,j}$ * $C5_{10,j}$ / $C4_{i,j}$]

Exposure Period (*i*)	Development Interval (*j*) 1	2	3	4	5	6	7	8	9	10
1	452,881	1,419,731	2,279,040	2,793,772	3,377,947	3,978,376	4,038,185	4,142,394	4,349,865	4,405,265
2	432,566	1,610,197	2,766,439	4,100,116	4,538,378	4,794,873	5,260,266	5,484,617	5,887,561	
3	368,296	1,634,013	2,745,402	3,840,401	4,623,708	4,671,636	5,090,015	5,324,759		
4	382,236	1,741,436	2,636,785	4,330,559	4,539,482	4,811,270	4,902,613			
5	529,368	1,353,815	2,481,777	3,242,747	3,722,312	4,131,335				
6	459,320	1,541,795	2,468,413	3,244,186	3,922,258					
7	481,990	1,405,037	2,583,135	3,571,425						
8	381,903	1,504,277	2,968,365							
9	388,062	1,400,758								
10	344,014									

2 Exposure Year 10 Incremental Basic Limit Development Factors [per D1; Volume Weighted Averages]

	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10
i=10	3.511	1.714	1.399	1.147	1.076	1.057	1.039	1.063	1.013

3 Exposure Year 10 Cumulative Development Factors [per D2]

	1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
i=10	12.291	3.501	2.042	1.460	1.273	1.183	1.119	1.077	1.013	1.000

E. Calculation of Development Factors by Layer

1 Basic Limit [D_{3j} * ($C6_{i,10}/C6_{10,10}$) / ($C6_{i,j}/C6_{10,j}$)]

Exposure Period (*i*)	Development Interval (*j*) 1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
1	12.633	3.590	2.193	1.531	1.319	1.211	1.133	1.084	1.015	1.000
2	12.541	3.789	2.179	1.524	1.315	1.209	1.132	1.083	1.014	
3	13.232	3.761	2.165	1.517	1.310	1.206	1.130	1.083		
4	13.126	3.731	2.151	1.510	1.306	1.203	1.129			
5	13.017	3.701	2.136	1.503	1.301	1.200				
6	12.904	3.669	2.121	1.496	1.296					
7	12.671	3.605	2.090	1.482						
8	12.547	3.571	2.074							
9	12.421	3.536								
10	12.291									

2 Basic Limit to Policy Limit [D_{3j} * (($C4_{i,10}$-$C6_{i,10}$) / $C6_{10,10}$) / (($C4_{i,j}$-$C6_{i,j}$/$C6_{10,j}$)]

Exposure Period (*i*)	Development Interval (*j*) 1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
1	very large	652.420	32.802	5.924	3.380	2.175	1.499	1.257	1.057	1.000
2	very large	946.242	30.374	5.704	3.293	2.140	1.488	1.252	1.056	
3	very large	807.075	28.187	5.498	3.210	2.107	1.477	1.247		
4	very large	691.561	26.215	5.305	3.132	2.075	1.466			
5	very large	595.239	24.431	5.124	3.058	2.044				
6	very large	514.560	22.814	4.954	2.987					
7	very large	390.710	20.042	4.647						
8	very large	341.869	18.820							
9	very large	300.278								
10	very large									

3 Policy Limit to Unlimited [D_{3j} * (($C3_{i,10}$-$C4_{i,10}$) / $C6_{10,10}$) / (($C3_{i,j}$-$C4_{i,j}$) / $C6_{10,j}$)]

Exposure Period (*i*)	Development Interval (*j*) 1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
1	very large	very large	84,538.278	279.503	48.056	11.155	3.254	1.887	1.183	1.000
2	very large	very large	62,192.336	240.423	43.321	10.464	3.157	1.857	1.178	
3	very large	very large	46,166.664	207.723	39.172	9.835	3.066	1.829		
4	very large	very large	34,571.138	180.241	35.524	9.261	2.979			
5	very large	very large	26,108.458	157.047	32.309	8.735				
6	very large	very large	19,880.311	137.391	29.466					
7	very large	very large	11,880.695	106.724						
8	very large	very large	9,257.797							
9	very large	very large								
10	very large									

Claims Development by Layer

Example 1

4 Limit of Data in Triangle [D_{3j} * ($C5_{i,10}$ / $C6_{10,10}$) / ($C5_{i,j}$/$C6_{10,j}$)]

Development Interval (j)

Exposure Period (i)	1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
1	13.776	3.913	2.375	1.629	1.387	1.255	1.155	1.096	1.018	1.000
2	13.759	4.155	2.372	1.627	1.385	1.254	1.154	1.095	1.018	
3	14.607	4.149	2.369	1.625	1.384	1.253	1.154	1.095		
4	14.584	4.143	2.366	1.623	1.382	1.252	1.153			
5	14.559	4.136	2.362	1.620	1.381	1.251				
6	14.532	4.128	2.357	1.618	1.379					
7	14.469	4.110	2.347	1.612						
8	14.433	4.100	2.342							
9	14.394	4.089								
10	14.352									

5 Unadjusted Incremental Development Factors at Limits of Data Triangle[per A1; Volume Weighted Averages]

1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10
3.490	1.747	1.457	1.174	1.104	1.086	1.054	1.077	1.018

6 Unadjusted Cumulative Development Factors [per E5]

1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
14.445	4.139	2.369	1.625	1.384	1.254	1.155	1.096	1.018	1.000

Differences [E6 / E4, last diagonal -1]

7	+0.7%	+1.2%	+1.2%	+0.8%	+0.4%	+0.3%	+0.1%	+0.1%	+0.0%	+0.0%

Claims Development by Layer

Example 1

A. Data and Information

1 Cumulative Development Triangle (C_{ij})

Exposure Period (i)	Development Interval (j) 1	2	3	4	5	6	7	8	9	10
1	357,848	1,124,788	1,735,330	2,218,270	2,745,596	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463
2	352,118	1,236,139	2,170,033	3,353,322	3,799,067	4,120,063	4,647,867	4,914,039	5,339,085	
3	290,507	1,292,306	2,218,525	3,235,179	3,985,995	4,132,918	4,628,910	4,909,315		
4	310,608	1,418,858	2,195,047	3,757,447	4,029,929	4,381,982	4,588,268			
5	443,160	1,136,350	2,128,333	2,897,821	3,402,672	3,873,311				
6	396,132	1,333,217	2,180,715	2,985,752	3,691,712					
7	440,832	1,288,463	2,419,861	3,483,130						
8	359,840	1,421,128	2,864,498							
9	376,686	1,363,294								
10	344,014									

2 Limit of Data in Triangle 1,000,000

3 Selected Basic Limit 500,000

4 Policy Limit 2,000,000

B. Trend Indices

1 Exposure Period Trend Index [2% EP Trend; 5% between EP 6 and 7]

Exposure Period (i)	Development Interval (j) 1	2	3	4	5	6	7	8	9	10
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.020	1.020	1.020	1.020	1.020	1.020	1.020	1.020	1.020	1.020
3	1.040	1.040	1.040	1.040	1.040	1.040	1.040	1.040	1.040	1.040
4	1.061	1.061	1.061	1.061	1.061	1.061	1.061	1.061	1.061	1.061
5	1.082	1.082	1.082	1.082	1.082	1.082	1.082	1.082	1.082	1.082
6	1.104	1.104	1.104	1.104	1.104	1.104	1.104	1.104	1.104	1.104
7	1.159	1.159	1.159	1.159	1.159	1.159	1.159	1.159	1.159	1.159
8	1.182	1.182	1.182	1.182	1.182	1.182	1.182	1.182	1.182	1.182
9	1.206	1.206	1.206	1.206	1.206	1.206	1.206	1.206	1.206	1.206
10	1.230	1.230	1.230	1.230	1.230	1.230	1.230	1.230	1.230	1.230

2 Calendar Period Trend Index [1% Calendar Period Trend; -5% between CP 2 and 3]

Exposure Period (i)	Development Interval (j) 1	2	3	4	5	6	7	8	9	10
1	1.000	1.010	0.960	0.969	0.979	0.989	0.998	1.008	1.019	1.029
2	1.010	0.960	0.969	0.979	0.989	0.998	1.008	1.019	1.029	1.039
3	0.960	0.969	0.979	0.989	0.998	1.008	1.019	1.029	1.039	1.049
4	0.969	0.979	0.989	0.998	1.008	1.019	1.029	1.039	1.049	1.060
5	0.979	0.989	0.998	1.008	1.019	1.029	1.039	1.049	1.060	1.070
6	0.989	0.998	1.008	1.019	1.029	1.039	1.049	1.060	1.070	1.081
7	0.998	1.008	1.019	1.029	1.039	1.049	1.060	1.070	1.081	1.092
8	1.008	1.019	1.029	1.039	1.049	1.060	1.070	1.081	1.092	1.103
9	1.019	1.029	1.039	1.049	1.060	1.070	1.081	1.092	1.103	1.114
10	1.029	1.039	1.049	1.060	1.070	1.081	1.092	1.103	1.114	1.125

3 Combined Trend Index [B1 * B2]

Exposure Period (i)	Development Interval (j) 1	2	3	4	5	6	7	8	9	10
1	1.000	1.010	0.960	0.969	0.979	0.989	0.998	1.008	1.019	1.029
2	1.030	0.979	0.988	0.998	1.008	1.018	1.029	1.039	1.049	1.060
3	0.998	1.008	1.018	1.029	1.039	1.049	1.060	1.070	1.081	1.092
4	1.028	1.039	1.049	1.060	1.070	1.081	1.092	1.103	1.114	1.125
5	1.059	1.070	1.081	1.092	1.102	1.114	1.125	1.136	1.147	1.159
6	1.091	1.102	1.113	1.125	1.136	1.147	1.159	1.170	1.182	1.194
7	1.157	1.169	1.181	1.193	1.204	1.217	1.229	1.241	1.253	1.266
8	1.192	1.204	1.216	1.229	1.241	1.253	1.266	1.278	1.291	1.304
9	1.228	1.241	1.253	1.266	1.278	1.291	1.304	1.317	1.330	1.344
10	1.266	1.278	1.291	1.304	1.317	1.330	1.343	1.357	1.370	1.384

Claims Development by Layer

Example 1

C. Claim Size Model (Apply to Cumulative Claims)

1 Claims Size Model Parameters at Exposure Year 10 Cost Level [via claim size modeling]

Development Interval (j)

Exponential (θ)	1	2	3	4	5	6	7	8	9	10
i=10	60,295	168,483	267,996	348,332	409,299	456,490	505,660	530,125	551,415	565,000

2 Claims Size Model Parameters [C1 * $B3_{i,j}$ / $B3_{10,j}$]

Development Interval (j)

Exposure Period (i) / Exponential (θ)	1	2	3	4	5	6	7	8	9	10
1	47,642	133,128	199,180	258,887	304,199	339,272	375,816	393,999	409,822	419,919
2	49,081	129,001	205,195	266,705	313,386	349,518	387,166	405,898	422,199	432,600
3	47,560	132,897	211,392	274,760	322,850	360,073	398,858	418,156	434,949	445,665
4	48,996	136,911	217,776	283,058	332,600	370,948	410,904	430,784	448,085	459,124
5	50,476	141,046	224,353	291,606	342,644	382,150	423,313	443,794	461,617	472,990
6	52,000	145,305	231,129	300,413	352,992	393,691	436,097	457,197	475,558	487,274
7	55,146	154,096	245,112	318,588	374,348	417,509	462,481	484,857	504,329	516,754
8	56,812	158,750	252,514	328,209	385,654	430,118	476,448	499,500	519,560	532,360
9	58,527	163,544	260,140	338,121	397,300	443,108	490,837	514,585	535,250	548,437
10	60,295	168,483	267,996	348,332	409,299	456,490	505,660	530,125	551,415	565,000

3 Unlimited Means

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	47,642	133,128	199,180	258,887	304,199	339,272	375,816	393,999	409,822	419,919
2	49,081	129,001	205,195	266,705	313,386	349,518	387,166	405,898	422,199	432,600
3	47,560	132,897	211,392	274,760	322,850	360,073	398,858	418,156	434,949	445,665
4	48,996	136,911	217,776	283,058	332,600	370,948	410,904	430,784	448,085	459,124
5	50,476	141,046	224,353	291,606	342,644	382,150	423,313	443,794	461,617	472,990
6	52,000	145,305	231,129	300,413	352,992	393,691	436,097	457,197	475,558	487,274
7	55,146	154,096	245,112	318,588	374,348	417,509	462,481	484,857	504,329	516,754
8	56,812	158,750	252,514	328,209	385,654	430,118	476,448	499,500	519,560	532,360
9	58,527	163,544	260,140	338,121	397,300	443,108	490,837	514,585	535,250	548,437
10	60,295	168,483	267,996	348,332	409,299	456,490	505,660	530,125	551,415	565,000

4 Limited Expected Values at Policy Limits

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	47,642	133,128	199,171	258,773	303,774	338,338	373,981	391,539	406,709	416,332
2	49,081	129,001	205,183	266,558	312,855	348,374	384,956	402,957	418,499	428,352
3	47,560	132,897	211,376	274,570	322,191	358,680	396,209	414,655	430,569	440,653
4	48,996	136,911	217,754	282,816	331,786	369,258	407,742	426,635	442,921	453,234
5	50,476	141,045	224,323	291,300	341,645	380,112	419,557	438,896	455,554	466,096
6	52,000	145,305	231,088	300,027	351,770	391,243	431,653	451,439	468,466	479,234
7	55,146	154,096	245,042	317,989	372,558	414,040	456,358	477,020	494,769	505,979
8	56,812	158,749	252,423	327,468	383,496	426,005	469,287	490,388	508,497	519,926
9	58,527	163,543	260,021	337,208	394,713	438,252	482,494	504,028	522,492	534,136
10	60,295	168,482	267,843	347,214	406,209	450,779	495,975	517,938	536,750	548,605

5 Limited Expected Values at Limits of Data Triangle

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	47,642	133,056	197,865	253,447	292,836	321,470	349,552	362,866	374,105	381,110
2	49,081	128,946	203,626	260,430	300,495	329,523	357,913	371,347	382,674	389,729
3	47,560	132,826	209,527	267,544	308,268	337,673	366,352	379,896	391,303	398,402
4	48,996	136,819	215,569	274,786	316,150	345,913	374,861	388,507	399,986	407,123
5	50,476	140,928	221,752	282,155	324,136	354,239	383,436	397,173	408,715	415,886
6	52,000	145,156	228,075	289,646	332,222	362,644	392,070	405,887	417,485	424,685
7	55,146	153,862	240,967	304,782	348,458	379,450	409,265	423,212	434,894	442,134
8	56,812	158,458	247,701	312,616	356,809	388,057	418,038	432,035	443,746	450,999
9	58,527	163,183	254,572	320,556	365,237	396,720	426,844	440,882	452,614	459,875
10	60,295	168,038	261,575	328,598	373,738	405,433	435,677	449,745	461,490	468,753

6 Limited Expected Values at Basic Limit

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	47,641	130,016	182,998	221,361	245,406	261,556	276,465	283,245	288,835	292,261
2	49,079	126,327	187,251	225,794	249,827	265,920	280,743	287,475	293,020	296,416
3	47,558	129,810	191,537	230,232	254,237	270,263	284,992	291,670	297,168	300,532
4	48,994	133,360	195,853	234,671	258,632	274,581	289,207	295,830	301,277	304,609
5	50,473	136,974	200,196	239,109	263,009	278,872	293,389	299,953	305,347	308,645
6	51,997	140,651	204,561	243,542	267,367	283,134	297,533	304,035	309,374	312,637
7	55,140	148,090	213,237	252,269	275,900	291,453	305,601	311,973	317,197	320,386
8	56,803	151,944	217,653	256,671	280,182	295,615	309,626	315,928	321,092	324,242
9	58,516	155,855	222,080	261,056	284,435	299,739	313,608	319,837	324,938	328,049
10	60,280	159,819	226,514	265,423	288,655	303,824	317,544	323,700	328,736	331,806

Claims Development by Layer

Example 1

D. Calculation of Development Factors at Basic Limit

1 Cumulative Triangle Exposure Year 10 Cost Levels and Basic Limit ($C_{i,j}$) [$A1_{i,j}$ * $C5_{10,j}$ / $C4_{i,j}$]

Development Interval (j)

Exposure Period (i)	1	2	3	4	5	6	7	8	9	10
1	452,767	1,351,032	1,986,583	2,323,084	2,706,395	3,137,755	3,148,934	3,217,037	3,368,618	3,396,735
2	432,457	1,532,103	2,413,944	3,417,616	3,649,372	3,798,743	4,123,637	4,283,518	4,586,542	
3	368,203	1,554,934	2,398,380	3,209,537	3,732,390	3,718,630	4,012,222	4,183,099		
4	382,140	1,657,379	2,306,488	3,629,412	3,679,452	3,848,804	3,886,711			
5	529,235	1,288,675	2,174,035	2,725,977	3,030,200	3,322,067				
6	459,205	1,467,893	2,165,788	2,736,050	3,207,585					
7	481,869	1,338,349	2,274,720	3,033,321						
8	381,807	1,433,334	2,619,477							
9	387,965	1,335,194								
10	343,928									

2 Exposure Year 10 Incremental Basic Limit Development Factors [per D1; Volume Weighted Averages]

	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10
i=10	3.344	1.578	1.341	1.109	1.061	1.046	1.035	1.061	1.008

3 Exposure Year 10 Cumulative Development Factors [per D2]

	1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
i=10	9.639	2.883	1.827	1.363	1.229	1.158	1.107	1.069	1.008	1.000

E. Calculation of Development Factors by Layer

1 Basic Limit [D_{3j} * ($C6_{i,10}/C6_{10,10}$) / ($C6_{i,j}/C6_{10,j}$)]

Development Interval (j)

Exposure Period (i)	1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
1	10.743	3.121	1.992	1.439	1.273	1.185	1.120	1.077	1.011	1.000
2	10.576	3.258	1.975	1.431	1.269	1.182	1.119	1.076	1.011	
3	11.066	3.215	1.957	1.423	1.264	1.179	1.117	1.075		
4	10.888	3.172	1.940	1.415	1.259	1.177	1.116			
5	10.709	3.129	1.923	1.407	1.255	1.174				
6	10.529	3.086	1.906	1.400	1.250					
7	10.175	3.004	1.874	1.385						
8	9.996	2.963	1.858							
9	9.817	2.923								
10	9.639									

2 Basic Limit to Policy Limit [D_{3j} * (($C4_{i,10}$-$C6_{i,10}$) / $C6_{10,10}$) / (($C4_{i,j}$-$C6_{i,j}$/$C6_{10,j}$)]

Development Interval (j)

Exposure Period (i)	1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
1	very large	55.346	9.569	3.616	2.273	1.714	1.348	1.195	1.052	1.000
2	very large	68.491	9.178	3.529	2.238	1.697	1.342	1.192	1.050	
3	very large	63.024	8.810	3.445	2.205	1.681	1.335	1.189		
4	very large	58.114	8.465	3.366	2.172	1.665	1.329			
5	very large	53.693	8.140	3.289	2.141	1.649				
6	very large	49.704	7.834	3.216	2.111					
7	very large	42.907	7.279	3.079						
8	very large	39.930	7.020							
9	very large	37.221								
10	very large									

3 Policy Limit to Unlimited [D_{3j} * (($C3_{i,10}$-$C4_{i,10}$) / $C6_{10,10}$) / (($C3_{i,j}$-$C4_{i,j}$) / $C6_{10,j}$)]

Development Interval (j)

Exposure Period (i)	1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
1	very large	very large	515.543	34.213	9.036	4.072	2.071	1.521	1.151	1.000
2	very large	very large	441.637	31.367	8.568	3.939	2.037	1.507	1.147	
3	very large	very large	380.045	28.831	8.138	3.814	2.005	1.494		
4	very large	very large	328.487	26.566	7.741	3.697	1.974			
5	very large	very large	285.139	24.538	7.373	3.586				
6	very large	very large	248.540	22.717	7.034					
7	very large	very large	191.734	19.638						
8	very large	very large	169.080							
9	very large	very large								
10	very large									

Claims Development by Layer

Example 1

4 Limit of Data in Triangle [D_{3j} * ($C5_{i,10}$ / $C6_{10,10}$) / ($C5_{i,j}$/$C6_{10,j}$)]

Development Interval (*j*)

Exposure Period (*i*)	1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
1	14.008	3.977	2.403	1.639	1.392	1.257	1.155	1.096	1.018	1.000
2	13.905	4.197	2.387	1.632	1.387	1.254	1.154	1.095	1.017	
3	14.669	4.165	2.372	1.623	1.382	1.251	1.152	1.094		
4	14.551	4.132	2.356	1.615	1.377	1.248	1.151			
5	14.429	4.098	2.339	1.607	1.372	1.245				
6	14.302	4.063	2.323	1.599	1.367					
7	14.040	3.990	2.289	1.582						
8	13.902	3.952	2.271							
9	13.760	3.913								
10	13.614									

5 Unadjusted Incremental Development Factors at Limits of Data Triangle[per A1; Volume Weighted Averages]

1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10
3.490	1.747	1.457	1.174	1.104	1.086	1.054	1.077	1.018

6 Unadjusted Cumulative Development Factors [per E5]

1 to ult	2 to ult	3 to ult	4 to ult	5 to ult	6 to ult	7 to ult	8 to ult	9 to ult	10 to ult
14.445	4.139	2.369	1.625	1.384	1.254	1.155	1.096	1.018	1.000

Differences [E6 / E4, last diagonal -1]

7	+6.1%	+5.8%	+4.3%	+2.8%	+1.3%	+0.7%	+0.3%	+0.1%	+0.0%	+0.0%

Claims Development by Layer

Example 2

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
							$500K to $1m				
Exposure Period (*i*)	Maturity	Claims, Limited to $1m, as of End of EP 10	Claims, Limited to Basic Limit ($500K), as of End of EP 10	Observed Ratio	Exponential Claim Size Model Parameter (θ)	Limited Expected Value at Basic Limit at Ultimate	Ratio at Ultimate	Selected Ratio at Age	Ultimate Claims Development Factor at $1m	Ultimate Claims Development Factor at $500K	Ultimate Claims Development Factor $500K to $1m
1	10	3,901,463	3,846,592	0.986	208,000	189,203	0.910	0.910	1.000	1.000	1.000
2	9	5,339,085	4,692,053	0.879	214,985	193,978	0.902	0.909	1.018	1.010	1.094
3	8	4,909,315	4,695,780	0.957	222,204	198,788	0.895	0.905	1.096	1.083	1.213
4	7	4,588,268	3,795,644	0.827	229,665	203,628	0.887	0.902	1.155	1.135	1.335
5	6	3,873,311	3,873,311	1.000	237,377	208,493	0.878	0.901	1.254	1.222	1.546
6	5	3,691,712	3,670,631	0.994	245,348	213,379	0.870	0.904	1.384	1.332	1.878
7	4	3,483,130	2,750,008	0.790	253,587	218,283	0.861	0.912	1.625	1.534	2.563
8	3	2,864,498	1,771,896	0.619	262,102	223,199	0.852	0.927	2.369	2.175	4.833
9	2	1,363,294	1,363,294	1.000	270,903	228,123	0.842	0.954	4.139	3.652	14.296
10	1	344,014	344,014	1.000	280,000	233,050	0.832	0.998	14.445	12.043	1,444.501

(5) = (4) / (3)
(6) Via claim size model
(7) LEV [exponential(θ);x]= q * (1 - exp (x/θ))
(8) = (7) / (6)
(9) See Section 4.2
(10) Provided
(11) = (10) * (8) / (9)
(12) = (10) * (1- (8)) / (1- (9))