Casualty Actuarial Society E-Forum, Spring 2009

Including the 2009 CAS Reinsurance Call Papers

The 2009 CAS Reinsurance Call Papers
Presented at the 2009 Reinsurance Seminar
May 18-19, 2009
Hamilton, Bermuda

The Spring 2009 Edition of the CAS *E-Forum* is a cooperative effort between the Committee for the CAS *E-Forum* and the Committee on Reinsurance Research.

The CAS Committee on Reinsurance Research presents for discussion four papers prepared in response to their 2009 Call for Papers.

This *E-Forum* includes papers that will be discussed by the authors at the 2009 Reinsurance Seminar May 18-19, 2009, in Hamilton, Bermuda.

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Unstable Loss Development Factors

Gary Blumsohn, FCAS, Ph.D., and Michael Laufer, FCAS, MAAA

Abstract

Most actuaries learn loss development on the job and pick up whatever techniques are being used by those around them. The experienced actuary is exposed to many varieties of methods and techniques. In dealing with unstable triangles, actuaries will employ myriad assumptions, judgments and tools along the way to selecting loss development factors. The authors describe a recent survey demonstrating the variety of methods and variability of selections of loss development factors (prior to consideration of the tail) and the variability of the resulting reserve projections.

Keywords: Reinsurance Analysis, Trend and Loss Development, Reserving Methods, Reserve Variability, Uncertainty and Ranges.

1. INTRODUCTION

A major challenge in day-to-day reinsurance actuarial work is selecting loss development factors when triangles are unstable. This topic does not receive significant attention in exams and papers, and yet it's something reinsurance actuaries encounter regularly. Most actuaries learn how to select loss development factors on the job, picking up rules of thumb and helpful approaches along the way. Whether these ad hoc approaches are good or bad depend on the particulars of the underlying data.

In the early part of 2008, we asked a group of actuaries to select loss development factors for a 12-year triangle of umbrella business (disguised in various ways to avoid divulging proprietary information). The selection of the group was not random: it consisted of people signed up to attend the 2008 Casualty Actuarial Society Seminar on Reinsurance, as well as various acquaintances of the authors. There was nothing special about this triangle, other than that it was deemed to be sufficiently unstable for the purpose at hand.

The triangle was provided in an Excel spreadsheet, and the participants were asked to select age-to-age factors. To keep the topic focused on the triangle, participants were instructed to ignore the tail factor. They were also asked to describe how they selected their factors, including such items as what types of averages they relied on, how they dealt with outliers, and how they dealt with reversals in the pattern. The original request, including the triangle, is shown in Appendix A.

Originally, the project was intended to lead to work that would provide some guidance on the selection of factors. This paper does not provide such guidance, except indirectly. Rather, this paper reports the results of what we received and catalogs the high variation in people's responses.

While we provide some commentary along the way, we largely allow the results to speak for themselves.

We received 52 responses, although only 51 of them gave us selected factors. The other one, from senior actuary with many years of experience, gave us a list of questions that one would need to ask before even beginning to select factors. Though this actuary had a point, we continued with our project nevertheless.

Initial reactions from people varied from the positive

> "*Great and gutsy project!*"

to the horrified

> *"I believe the whole notion of 'picking factors' with no statistical guidance is something of a disgrace to the profession…."*

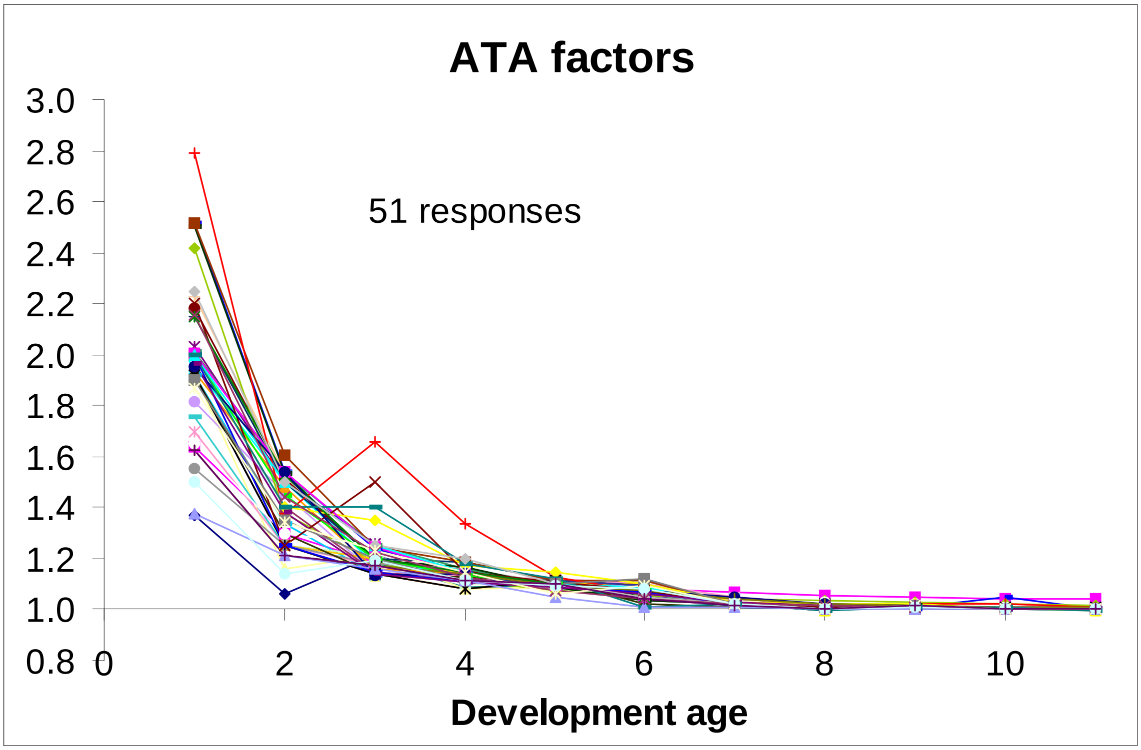
to the concerned

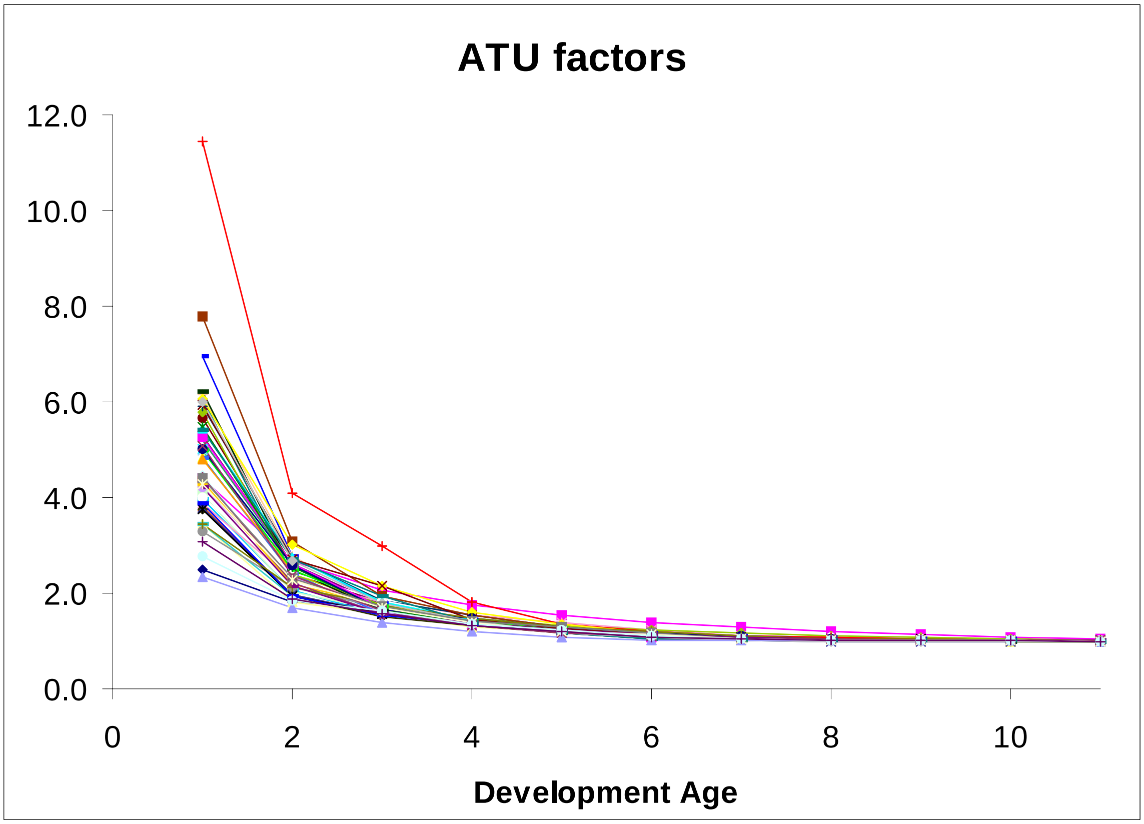
> *"While it may be helpful to share ideas on how to pick LDFs, it is vital that more information than just the triangle at hand be considered… I wouldn't make selections without other information such as individual claim information, changes in the underlying business, comparison to competitor or industry triangles if available, etc. Of course you can't always get the information you want…but I would hate to see people come to the seminar and learn some new selection techniques that don't look beyond the triangle."*

2. THE RESULTS

This section of the paper summarizes the results we received. For now, we content ourselves with describing the outputs, leaving for later people's explanations of their loss development factor picks.

The graphs below show the age-to-age (ATA) factors and the age-to-ultimate (ATU) factors selected by each of the 51 participants. While it isn't easy to draw detailed conclusions simply from looking at the graphs, one can quite easily see that the range of selections is wide.



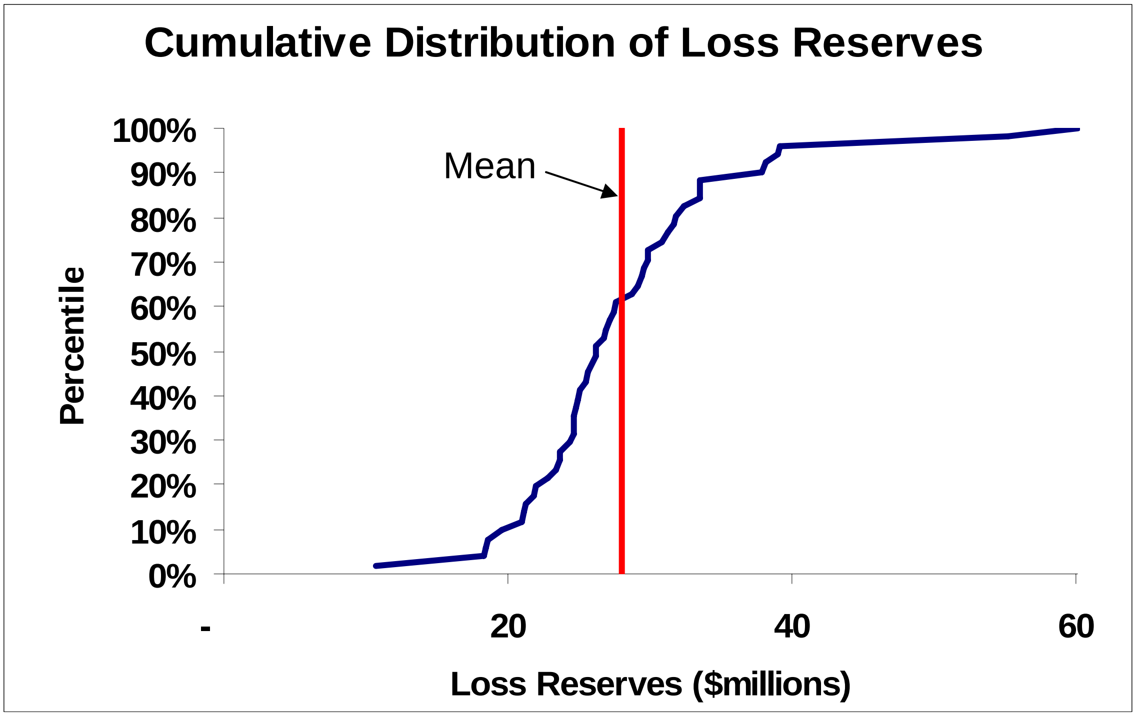


While one could measure the dispersion among selections by looking, for example, at the coefficient of variation of the various selections, we chose not to take that approach. In the spirit of a paper that is more psychology and rules of thumb than technical actuarial work, we are interested in the practical import of the dispersion among actuaries' selections. In that vein, our measure of dispersion will rely on looking at the expected reserve from a chain-ladder projection – in this case, the dispersion of the expected chain-ladder reserve that results from the different factor selections. We are, of course, aware that most actuaries would not use only one method to get the reserves, and would quite likely rely on a more stable method, like the Bornhuetter-Ferguson, to estimate the reserves for the more recent years. However, for our purposes we choose to ignore this because we are focusing on the development factors, rather than on the full range of reserving procedures.

The table below and the graph that follows show the implied reserves from the 51 respondents, ranging from a low of $10.7 million to a high of $60.2 million. The mean is $28.0 million, with a standard deviation of $8.3 million – a coefficient of variation of a whopping 30%.

Implied chain-ladder reserves, sorted from lowest to highest (in $millions)

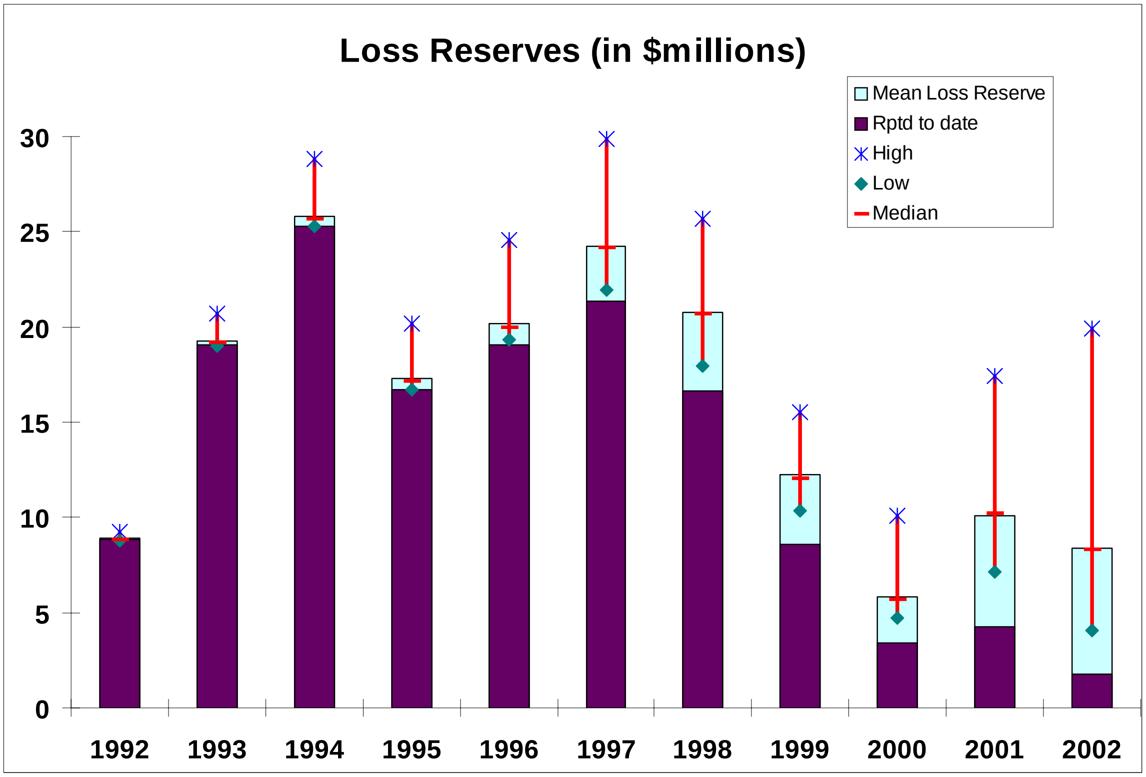
1	10.7		18	24.7		35	29.5
2	18.3		19	24.8		36	29.8
3	18.5		20	25.0		37	29.8
4	18.6		21	25.1		38	30.9
5	19.5		22	25.4		39	31.3
6	21.0		23	25.6		40	31.7
7	21.2		24	25.9		41	31.8
8	21.3		25	26.2		42	32.4
9	21.8		26	26.2		43	33.5
10	22.0		27	26.7		44	33.5
11	22.8		28	26.9		45	33.5
12	23.4		29	27.1		46	38.0
13	23.7		30	27.5		47	38.2
14	23.7		31	27.6		48	39.0
15	24.4		32	28.7		49	39.2
16	24.6		33	29.1		50	55.2
17	24.7		34	29.4		51	60.2
Mean				28.0			
Median				26.2			
Std. deviation				8.3			
Coefficient of variation				30%			



The range and the standard deviation of the reserves are greatly widened by three apparent outliers: one on the low end with reserves of $10.7 million, and two high-end selections of $60.2 million and $55.2 million.

Half the responses implied reserves in a fairly tight range between $23.7 million and $30.9 million. However, one cannot ignore that half the actuaries made picks that implied loss reserves outside of this range. In a world in which many employers, regulators, auditors, and investors seem to think actuaries can make loss picks that are within 10% of the "truth," this should be sobering. And it should not be forgotten that we deliberately ignored the tail factor in this exercise, which reduces the volatility.

The following graph shows the distribution of the reserves by accident year. It is not surprising that the greatest variability is in the most recent years.

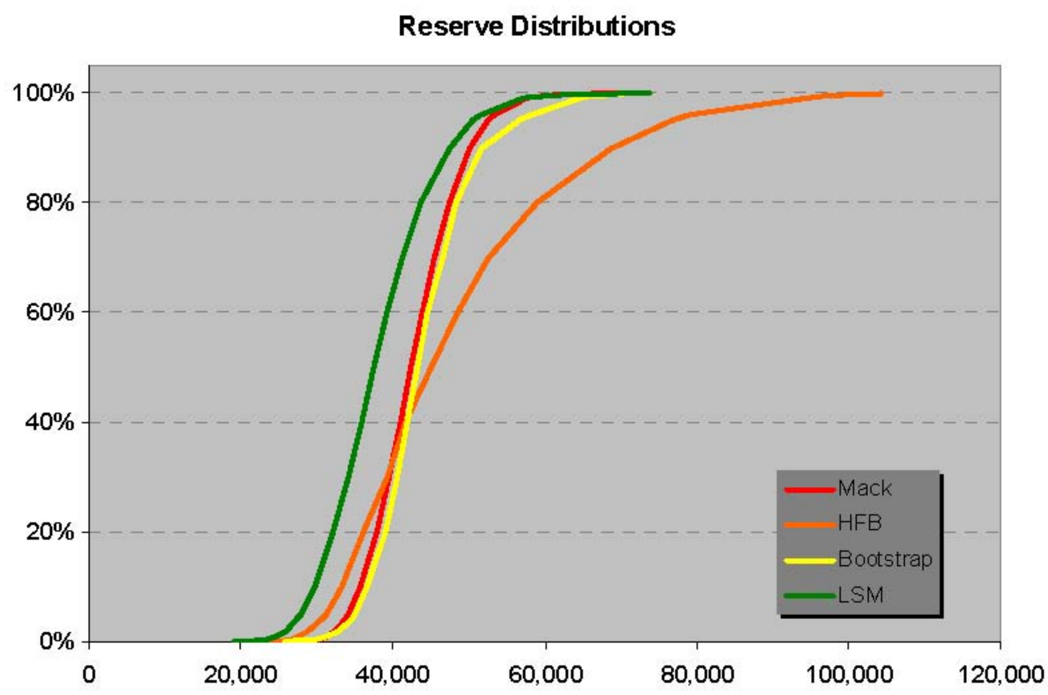


This graph has profound implications for pricing actuaries. It is not unusual for an actuary pricing a quota share to have 10 or fewer years of data. One would typically trend and develop losses and on-level the premiums, to bring everything to current level. One would then typically review the series of loss ratios in an attempt to divine the most likely loss ratio for the year being priced. Many pricing actuaries put more weight on the recent years than on the older years, on the assumption that more recent information is more valuable. However, the volatility in the estimates of the most recent years is extremely high, so that there is a high probability that the pricing estimate will differ significantly from the true mean.

In interpreting these results, it should not be forgotten that the distribution shown here is a distribution of *estimates of the mean* of the distribution of the unpaid losses. It is not a distribution of the unpaid losses. Typically, although not necessarily, the distribution of the unpaid losses will be considerably wider than distribution of estimates of the mean of the distribution of unpaid losses. For example, if we assume the estimate of the mean of the unpaid losses has a standard error of 30%, and the distribution of unpaid losses, given the expected reserve, has a coefficient of variation of 15%, then one would have to mix these two distributions to get an overall distribution of outcomes. The potential dispersion of loss reserve outcomes is extremely high.

2.1 A Side-Note on Reserve Variability and the Reputation of Casualty Actuaries

In March 2008, the CAS Task Force on Enhancing the Reputation of Casualty Actuaries produced a report ([2]) that discussed, among other things, the need for actuaries to communicate the uncertainty of their estimates to ensure that users understand it. They also called for an improvement in actuarial methodologies and terminology, all of which is to be applauded. What struck us, relative to our work in this paper, was a section titled "Comparison of Methods" on pp. 16 – 18 of that report, and, in particular, a graph on p. 17, reproduced below.



The graph shows the distribution of reserves that emerge from four approaches actuaries have used for getting a distribution of reserves, as applied to some unspecified general liability data.[1]

While we endorse the task force's suggestion that actuaries be concerned with the variability of reserves as demonstrated by a particular method, we want to stress the need to account for both the variability that results from a particular method of calculation as well as the variability among particular methods of calculation. At each step of the process, there is variability, and all the variability gets compounded. For example, if my analysis indicates that the mean loss reserve for a block of business is 100, and my approach to calculating the standard deviation of the reserve indicates that the standard deviation is 15, I could proceed to use those assumptions to estimate, say,

1 For Mack method, see [7]; for HFB method see [6]; for Bootstrap method see [3]; and for LSM method see [8].

the 95th percentile of the distribution of losses. However, I must recognize that if another actuary were to estimate the mean, that estimate could easily be, say, 92 or 112. Even if the other actuary's estimate of the standard deviation were the same as mine, the range of possible outcomes from these two actuaries will be considerably wider than the range of possible outcomes from my estimate alone. And if we recognize that the difficulties in estimating the standard deviation and the shape of the distribution are even greater than the difficulties in estimating the mean, we should recognize that our ability to accurately pin down the tail probabilities is very limited.

For example, in the GL example used by the CAS Task Force on Enhancing the Reputation of Casualty Actuaries, some of the methods gave a 95th percentile that is about 25% above the mean. If we assume, for convenience, that the reserves are lognormally distributed, this implies a coefficient of variation of about 15%. However, if we assume that the mean itself is lognormally distributed with a mean of 30%, then the mixture of the two distributions gives a 95th percentile that is more than 60% higher than the mean. To make things even harder, once one realizes that estimates of the coefficients of variation are themselves just estimates, and that the shape of the distribution is usually not much more than an educated guess, one realizes that one can place little faith in one's estimates of the tails of the distribution. If one is worried about the credibility of casualty actuaries, this is a key point that must be made.

2.2 How Did Participants Get Their Factors?

Most of the participants (42 of the 51) either wrote some explanation of how they had derived their factors or set up a spreadsheet so that it could be inferred how they thought. Broadly, these can be divided into those who picked factors based on reviewing the various averages of the factors (34 of the 42) and those who used some statistical approach to the problem (8 of the 42), though it must be noted that this classification is somewhat arbitrary, as there were a number of responses that were in a gray area, and where we used our judgment. A complete listing, edited for spelling and anonymity, together with the selected factors, is in Appendix B.

In comparing the factors from the various methods, we calculated the means and standard deviations of the implied loss reserves:

Method classification	**# of Respondents**	**Mean implied reserve (in $millions)**	**Standard deviation of the implied reserve (in $millions)**
Pick	34	27.1	5.9
Statistical	8	36.0	13.2
Not clear	9	24.6	5.0
Total	51	28.0	8.3

The statistical approaches gave a mean and standard deviation that were both much higher than from the pick approaches. However, given the small number of statistical approaches, and the possibly random fact that both of the high-end outliers are statistical approaches, this might be noise, rather than a signal that statistical methods generally lead to higher answers.

2.3 What Methods Did Participants Consider in Developing Their Factors?

We found broad categories in which participants described what they considered in making their factor selections, and they are summarized in the table below. Any single respondent might have given thought to more than one of these considerations, and some respondents did not tell us how they got their selections. Many respondents gave thought to why different methods were yielding different results, and they described why they believed a particular method to be appropriate to the case at hand.

The three responses well out of range mentioned above are removed from this table. The average estimate for the remaining 48 respondents is $27.2 million with a standard deviation of $5.3 million.

Table: Considerations in Selecting Factors for Unstable Triangles

	Average Reserve (in $millions)		Standard Deviation (in $millions)		# of Respondents	
Consideration	**Yes**	**No**	**Yes**	**No**	**Yes**	**No**
(1) Removed Outliers (including ex-high-low)	27.9	27.8	5.3	5.5	15	21
(2) Smoothed Links	28.7	24.0	8.1	3.3	23	13
(3) Would Use Only Factors > 1.00	27.5	25.0	5.7	1.0	12	5
(4-a) Used Volume Weighted Average	26.6	34.7	6.9	7.6	30	3
(4-b) Used Straight Average	25.2	27.6	11.3	5.1	8	24
(4-c) Used Ex-High-Low Average	27.1	28.0	5.6	5.0	12	17
(5) Used All (or Nearly All) Years	29.2	21.7	5.0	8.4	19	11
(6) Used Industry Data	32.4		4.8		3	
(7) Used Regression or Other Curve Fitting	28.0		4.9		9	
(8) No Risk Margin	24.7		7.3		2	
(9) Wanted More Information	26.2		8.3		17	

The decision whether or not to consider a certain method might have resulted in significantly different results; for example, whether to smooth the data, rely on all-year averages, use industry data, use curve-fitting techniques, or use volume-weighted averages.

We now address each category listed above in more detail.

2.3.1 Removing outliers (15 responses)

Some respondents began by checking the data for anomalies, and by excluding certain years or links; for example by excluding certain links when calculating link averages, by removing certain accident years, or by not using an average that contained the outlier. The questions here are (1) the extent to which outliers in the data can be identified, (2) the potential causes of the outliers, (3) how the outliers should be treated, and (4) the impact of removing or including the outliers on the result.

These questions are somewhat beyond the scope of this paper. Without resolving these issues, it seems we should consider the following:

- From a purely statistical standpoint, one might argue that removing outliers will tend to bias the results, especially seeing the distribution of development factors is probably positively skewed, so that outliers are more likely to be identified in the right-hand tail of the distribution. If these outliers are discarded, there will be a tendency for results to be biased downwards.[2]
- It must be recognized, though, that most triangles consist of a small number of points. A 12-year triangle, such as we are using as the basis for this paper, has only 66 loss-development points – hardly a large sample with which to apply fancy techniques. Outliers, particularly towards the tail of the triangle, can sharply change the results.
- If one is dealing with a large number of triangles, one might be most concerned with having an unbiased set of development factors. This might be the case if one is doing a reserve study, has divided the book of business into 20 segments, and is selecting development factors for each segment. By retaining the outliers, one acknowledges that the reserves on some segments will be high, and others will be low, but overall, one hopes to have an unbiased result. On the other hand, suppose one is a pricing actuary, separately pricing 20 reinsurance transactions. If one retains the outliers, the results on the 20 transactions may well be unbiased, but since clients are more likely to accept quotes that are based on low development factors than quotes based on high development factors (the well-known Winner's Curse – see [10], for example), the business that ends up on the books will more likely be where the loss estimates were biased downwards. In this situation, reducing the variance of the results by eliminating outliers may be more important than introducing a (hopefully) small amount of bias in the results. The uncomfortable upshot is that actuaries must recognize that there is no "best way to select development factors" isolated from the purpose for which those factors are being selected. Context matters.
- Until now, we've assumed the outlier question is a statistical one. Beyond the statistical question, there's the question of data errors. Reinsurance actuaries are familiar with the problems of data quality, and our well-honed intuition is alert for outliers that are not the result

[2] For a recent discussion of the impact of skewed distributions and small sample sizes in actuarial work, see [4].

of statistical fluctuation, but are the result of claim-department policy changes, changes in underwriting approach, changes in the mix of business, changes in the approach to settling claims or setting case reserves, coding errors, claim personnel errors, or simply errors without a cause known to the actuary. As a practical matter, it's virtually impossible for reinsurance actuaries to uncover the causes of many of the fluctuations in the data, or even to identify whether they are statistical fluctuations rather than data errors. We can speculate as to the causes, but without details of the underlying policies and claims, it's impossible to pin them down. Our assumption in eliminating outliers is that we are able to improve the quality of the data, and hence the quality of the answers, through "actuarial judgment," but one wonders whether it would be possible to somehow set up an experiment that would test this assumption.[3]

2.3.2 Link smoothing (23 responses)

Some respondents smoothed selected links across ages. A few believed it was appropriate to consider adjusting for reversals in the factors where links did not show a smooth pattern of decreasing with maturity; however others believed it unnecessary to smooth links at all.

Some respondents may have adjusted the data when they perceived takedowns followed later by upwards adjustments, or the reverse. In general, it probably isn't a bad a priori assumption that age-to-age factors decrease monotonically and smoothly until they reach 1, but it's only an assumption, and there are significant situations where the assumption has been wrong.

One of the authors worked on an instructive situation where this assumption proved wrong. It involved pricing working-layer excess workers compensation reinsurance. In reviewing the incurred-loss triangle, a surprising number of the 24-to-36 factors developed downwards, despite all the surrounding factors developing upwards. At first, we were tempted to smooth out this downward development. However, this was one of the relatively rare situations where we had complete access to the underlying data, and, once we examined the data, the downward development made perfect sense. The layer was high enough that most of the dollars of claims in

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[3] The recent book *Super Crunchers* by Ian Ayres [1] devotes a chapter to "Experts Versus Equations" (pp. 112 – 139), which presents evidence that mechanically applied equations seem to come up with better predictions than "experts" in a wide variety of situations. Ranging from models of legal decision-making that were better at predicting Supreme Court decisions than a group of experts, to medical studies where models were better at predicting how patients with schizophrenia would respond to electroshock therapy, to studies showing that models made better purchasing decisions than professional purchasing managers, Ayres contends that models work better than experts. He attributes the failures by experts to the cognitive biases and overconfidence that are by now well known.

In the actuarial case, it would be intriguing to have an experiment to test whether, over a large number of triangles, one would get better answers from purely mechanical application of some algorithm for selecting development factors. We like to think we add value in our analysis of loss triangles, and no doubt some data errors would not be revealed by an algorithm. But one wonders whether the data "errors" we find, and the outliers we throw out, are too often valuable bits of data that we choose to ignore more frequently than we should – at our peril and at the peril of our employers.

the layer, and especially those that were reported early, were from very severe injuries, such as brain and spinal-cord injuries. When the case reserves were first put up, they covered an "average" claim amount for that injury type. However, a significant fraction of severely injured people die within a year or two of the injury, and when that happened, the case reserve dropped to zero, or perhaps a small amount paid to the claimant's survivors. Meanwhile, the case incurred amount on those claimants who survived was kept the same as when it had first been set, with no increase. This explained the downward case development observed in the 24-to-36 factor. The subsequent upward case development was explained by less obviously severe claims, such as back or knee injuries, bleeding into the excess layer, as well as eventual recognition of increased costs due to medical inflation and increased life expectancies for those who survived the first couple of years. As actuaries, we may have suggestions for better ways to set the case reserves so that the loss development is smoother, but the bottom line is that we were not the ones setting the case reserves, and the downward development was real. The real world isn't always smooth.

2.3.3 Adjusting for factors less than 1.00

Akin to smoothing links, some respondents believed it was appropriate to adjust for reversals in the data where development factors were less than 1.00. Some expressed a tolerance for development factors that were marginally less than 1.00.

The workers compensation example given above applies to this situation as well. Sometimes downward development is real. Reinsurance actuaries sometimes learn about the case reserving habits of their clients – which clients tend to under-reserve and which tend to over-reserve. It is not altogether unusual to have downward development on incurred triangles (and occasionally on paid triangles). Most actuaries will concede this point, but when given a triangle where the case reserving practices are unknown, they are often resistant to allowing downward development, and perhaps for good reason. Downward case development is often a tipoff to bad data, or the existence of some unusual situation that is unlikely to be predictive of future downward development. While we cannot be sure that downward development is wrong, it may be correct to underweight it.

Another valid downward development situation occurs, of course, at mature ages in lines with significant salvage or subrogation, when data is given net of paid salvage and subrogation.

2.3.4 Type of average

Many respondents thought it was appropriate to calculate a variety of averages using varying weights and varying years. After doing so, respondents felt better equipped to discern trends, spot outliers, or raise other questions about the data.

Preferences for how to weight the averages ranged widely. (1) Some participants expressed a preference for volume-weighted averages on unstable triangles to avoid over-weighting erratic low-volume years. (2) Others were apprehensive about using volume-weighted averages without additional information such as large claim information, and preferred to use straight averages. (3) Some thought it was appropriate to use averages-excluding-high-low. (4) Some weighted their averages using time-sensitive weights.

Daniel Murphy's paper "Unbiased Loss Development Factors" [8] treats development factors as regression coefficients from the equation $y = bx$ + error, and shows the various assumptions that would make weighted or unweighted averages the best choice. Various authors have concluded that it generally appears that weighted averages are better than unweighted averages. See, for example, Struzzieri and Hussian's nice summary of these various results in their section "Best Link Ratio Averages" (pp. 384 – 388 of [9]). However, the interesting thing is that, with all of these papers by Murphy and Mack and others stressing that the key item is to check the assumptions, only one of the respondents to our survey appeared to have actually checked the assumptions (respondent number 32 – see Appendix B).

Though, as Struzzieri and Hussian note (p. 386), the literature generally supports weighted averages over unweighted averages, it is not perfectly settled in our minds. A counter to weighted averages is that if, say, high development factors correspond to years with high volume, the weighted average could bias the answer high. Another counter would be that in a time of high inflation, more recent years get more weight, even though they may be no different from the earlier years in real terms. While these counter-arguments are correct, our casual empiricism says that for the typically unstable triangles that we see in reinsurance, weighted averages are almost always superior.

2.3.5 Appropriate number of years to use in the average

Many participants preferred to use all (or nearly all) valuation years of data when reviewing unstable triangles. Others gave more weight to recent valuations, sometimes looking for patterns that they believed were different and discernible in the most recent valuations.

We are not aware of literature that answers the question of whether to put more weight onto recent information. Intuitively, it seems reasonable, since we believe the statistical process underlying the loss development changes over time, so more recent data is likely to be more representative of the future. However, the guidance on when to use each type of average seems to be fairly informal.

On the other hand, volatile data might be the result of claim situations that occur infrequently. For example, if a particular type of claim is encountered only once or twice a decade, and the development pattern for this type of claim is unique, it makes sense to include as many years as possible, to capture the data from this infrequent claim type. If this particular claim situation and its resulting development would not be captured in the most recent three or five-year average, using these short-term averages could yield inaccurate results; or if it is captured in the most recent years, it could be overweighted.

In determining the appropriate number of years to use, it is important for the actuary to assess the issues of (a) using later information, which presumably provides better insight into changes in the claims development process or changes in the claims environment, and (b) using more data points, which would contain development information on more (types of) claims.

2.3.6 Industry information (3 responses)

Some participants used industry link information culled from their own sources. These respondents generally weighted industry data with the data in the exercise for some or all of the links. These respondents may have considered adjusting their industry link set to be consistent with the exercise data.

While it's always useful to compare one's data to industry information, there was very little information provided about the triangle. Participants were told that it was "umbrella" data, but were not told whether it was personal umbrella or commercial umbrella, supported or unsupported, the limits being offered, and other important information. In the absence of this information, one must wonder how much weight one can reasonably give to industry information.

2.3.7 Regression or other curve fitting (9 responses)

Some respondents used limited regression techniques on the data to come up with selections or used regression techniques as a check on their selections. While each of these respondents used some regression technique, each one seemed to be using a different technique and applying it to different data, so that there wasn't a consistent method for us to describe.

Other respondents fit curves to the data, relying on a number of curves and fitting techniques. Curves mentioned include the inverse power/loglogistic, lognormal, exponential and Weibull. Again, there was no single, consistent method for us to describe.

2.3.8 Risk margin (2 responses)

Two respondents wondered whether to include a risk margin in their selections; they both ultimately decided not to include a risk margin.

We had not expected to encounter risk margins in this exercise, so it was interesting that more than one person raised the question. We aren't sure of the benefits of including a risk margin in development factors, rather than building risk margin in at the end of the procedure. For pricing, one would want a risk margin, though it's hard to see that the best place for it is in the development factors, rather than directly building it into the final price. A risk margin may be warranted in reserving, although, for example, US GAAP requires that companies carry the best estimate of the reserves – presumably precluding a risk margin.

2.3.9 Needs more information

We did not ask participants to tell us if they would have liked to see additional information; nevertheless, many participants described what additional information they felt would have been useful.

The type of information requested can be categorized into the following groups: (a) company loss data, (b) claims department information, (c) underwriting data, (d) industry data, and (e) prior selections.

A. Loss Data

   a. Paid triangle

   b. Claim count triangle

      i. Open claim count triangle

      ii. Paid closed claim count triangle

   c. Average claim size triangle

   d. Individual claim development

Participants thought the paid data could provide additional insight into the claims settlement and reserving process. Similarly, participants would have liked to have reviewed claim count data.

B. Qualitative claims information

   a. Changes in company case reserving philosophy

   b. Changes in company claim processing

   c. Claims audit report

C. Underwriting data

   a. Premium or other exposure proxy by year

   b. How long the company has been in the line

   c. Information on the underlying book

i. Retention and limit (and/or their changes over time)

ii. Type of umbrella

1. personal vs. commercial
2. supported vs. unsupported

d. Mix of business changes

Some respondents said they made assumptions with regard to exposures. Others said they would have liked to know more about the underlying book, with a view to guiding their selections.

D. Industry Information

a. Industry default development factors
b. Underwriting cycle information
c. Loss ratio benchmarks
d. Legal/legislative trends

Many participants said they would have liked to have had industry development factors, hoping to apply weights to company data vs. industry data. As mentioned above, some responders did assume an industry pattern and considered this industry pattern in their selection.

Some respondents said they would consider adjusting their factors by accident year based on where each year stood in terms of the underwriting cycle. This is an interesting notion that needs to be examined further. While some stable lines of business, like primary workers compensation, sometimes exhibit have cyclical loss development, we are not aware of anyone successfully applying cycles on top of unstable factors. It would be interesting to know whether the assumptions of cycles can be shown in fairly general cases to mute the instability of factors, or whether they might add to the noise.

Some respondents would have liked to have had company premium information and industry loss ratios in order to compare company loss ratio indications to industry loss ratio benchmarks. While loss ratios are always helpful, it isn't clear to us how to change development factors based on this information: if the company's loss ratios are much higher than industry loss ratios, does that mean higher or lower development factors?

Others mentioned they needed more information on recent industry trends. This request seems reasonable. An understanding of industry trends, and more generally, social and legal developments, might explain some apparently odd loss development, or might lead one to select development factors that are higher or lower than those suggested by the triangle.

E. Prior information

One respondent thought an appropriate method of validating selections would be to review prior selections that were based on this data. If, say, one were updating a reserve study, this would be a reasonable request, although one that obviously isn't always possible.

2.4 What lessons can we draw from the three respondents we removed from our review?

As mentioned above, three respondents were well out of range of the other participants. One of these participants was very low and two were very high. Each of these respondents discussed their methodology.

2.4.1 Second highest respondent

The respondent who produced the second highest loss reserve fit a single curve to all maturities.[4] Without knowing the details of the distribution that was fit, it's hard to comment definitively on this approach. However, the respondent produced low or moderate link ratios through the first six or so maturities and high factors beyond age six. In fitting the later maturities a reasonability check might have provided this respondent with additional insight into his fitting procedure. For example other respondents used a reasonability check on fitted results by comparing the fitted links to average links from the data over a multiple maturity period. In this case the respondent's fitted links from age 7:12 results in a total multi-age link of 1.29 while the data shows volume-weighted 7:12 link of about 1.03.

While fitting a distribution may seem more statistically sophisticated, there is the ticklish problem of selecting a distribution. Given the limited number of data points in a typical loss triangle, it's quite likely that many distributions will pass a goodness-of-fit test. It's not obvious that any of the "usual" distributions should be believed a priori to provide a good fit to loss development factors.

2.4.2 Lowest respondent

The respondent who produced very low results selected simple averages based on only the most recent valuations for the first four links.[5] For these earlier maturities, selecting only the most recent

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[4] This respondent described his process as follows: "I model a distribution of the 'time until reported' random variable. I then fit the incremental amounts to what the distribution would imply as a fraction of the cumulative amount for the accident period. I then inspect the errors for consistent bias and make adjustments if appropriate."

[5] This respondent also noted that he generally prefers weighted averages, but "since I just came back from vacation and will soon be on the road again, I did mostly a straight average analysis." We are not sure what lessons should be drawn from this comment. If the comment is meant to say he was pressed for time, so that he took some short-cuts, then the lesson would be that short-cuts are dangerous, and one should be especially careful of making rapid selections when

valuations, of course, ignores the development of the earlier accident years. Many other respondents believed that for unstable triangles it was important to consider development beyond the most recent accident years.

Also after the first four links, this respondent selected factors based on judgment, believing the data to have little credibility beyond this point. This judgment resulted in factors that were much lower than what the data indicated. For example using the reasonability check described above we see that for this respondent the multi-age link over the period 5:12 is a factor of 1.08, while the data shows a volume-weighted 5:12 link of 1.21.

2.4.3 Highest respondent

The respondent who produced the highest results described used a regression analysis on the first four links. In particular, noting a negative correlation between the age-to-age factor and the dollars of reported loss, he fit a regression line with the age-to-age factor (say the 12-24 factor) as the dependent variable and the dollars of reported loss (at age 12, in this example) as the independent variable. This regression was then used to fill out succeeding points within the lower half-triangle. After the first four years, he selected age-to-age factors judgmentally, "with the consideration of weighted averages, simple averages, smoothness, conservatism, and historical LDF range." In fact, it was in large part due to the judgmental factors after age 4 that led to the high result. If he had simply used the all-year weighted averages, the 5:12 factor would have been 1.21, and the reserve would have been $45.3 million, which would still have been high, but much lower than the selected 5:12 factor of 1.35 that gave reserves of $60.2 million.

The point regarding the negative correlation for the first four factors is correct, and is not addressed by most other participants. It is notable, however, that the dollars of loss could be low either because i) there is light reporting at, say, age 12, which is then offset by "more normal" reporting at age 24, and hence the negative correlation that was noted, or ii) the exposure is lower. Exposures were not given in our example, so it is not possible for the actuary to distinguish between them. This lack of exposure data was noted by a few of the respondents.

2.4.4 Lessons from the outliers?

It's hard to draw firm lessons from the three outliers. One apparent lesson is for the need to use reasonableness checks when heavily relying on judgment or models. If the selected factors or the fitted model differ significantly from the mean of the data, one must be sure there is a good reason

under pressure. On the other hand, if the comment was meant to say that he had just returned from vacation, and was thus in a good mood, then it is reminiscent of some of the literature on cognitive biases that has found that traders are more optimistic on sunny days, and that their trades reflect it. (See [5].) Perhaps the lesson is that actuaries should work in windowless offices.

for the discrepancy. There's an ever-present danger of getting so engrossed in the details of one's favorite approach that the big picture of fitting to data can be missed. A well-constructed graph will often visually draw one back to the underlying data, and provide some reasonableness to the work.

3. WHAT CONCLUSIONS CAN WE DRAW FROM THIS EXERCISE?

As we noted at the start of the paper, this exercise began with a hope that we could draw on the results that we received to provide some guidance to actuaries faced with selecting development factors from unstable loss triangles. This paper has turned out to be more of a description of the results we received, together with some commentary. The number of approaches varied widely – and so did the selected factors. It has proved difficult to find many general themes in the approaches; rather, it has provided much food for thought, and a guide to the many things that actuaries need to keep in mind when working with development factors.

We found that actuaries who used what sounded like similar methods when they were being described, sometimes ended up with results that proved to be significantly different. Conversely, much of the time, actuaries using what sounded like very different methods, came up with results that were very similar. Some actuaries saw patterns where others saw only noise, and there was a fair amount of disagreement as to what constituted reasonable or appropriate assumptions.

The guidance we hoped to provide is for another paper. Perhaps the best advice that we can provide for now is for actuaries to keep open minds, and to approach unstable triangles from a variety of perspectives. Having many tools in one's toolkit, and tempering all of these tools with good common sense and careful judgment, is more likely to give consistently reasonable answers over the variety of situations in which we find ourselves.

Acknowledgment

Many thanks to the 50+ participants for sharing their techniques and thoughts on this exercise. The authors wish to acknowledge David Homer and Joel Vaag for their insightful review of the paper and suggestions for improvements.

Disclaimer:

The opinions expressed in this paper are those of the authors, not of their employers.

Appendix A

Here is the e-mail that was sent to people attending the 2008 CAS Reinsurance Seminar:

From: Vincent Edwards [mailto:vedwards@casact.org]

Sent: Mon 5/12/2008 4:23 PM

To: cas@lists.casact.org

Subject: 2008 Reinsurance Seminar Research Opportunity! (File Attached)

Dear CAS Reinsurance Seminar Attendee,

The CAS Reinsurance Research Committee is conducting research to look at how practicing actuaries select loss development factors when dealing with unstable triangles. Some of the early results of this work will be presented at the "Loss Triangle Philosophy" session at the Reinsurance Seminar. To increase our sample size, we want to extend an invitation for attendees to participate in this study. In addition, session attendees may find it interesting to participate before they see what others have done.

Here's what we're trying to do:

One of the biggest challenges in day-to-day reinsurance actuarial work is selecting loss development factors when triangles are unstable. This topic does not receive significant attention in exams and papers, and yet it's something reinsurance actuaries do on a regular basis. Most actuaries learn how to select loss development factors on the job, pick up rules of thumb and helpful approaches along the way. Whether these approaches are good or bad probably depends on the particulars of the underlying data.

Taking a look at how different actuaries do things can be beneficial, even if it simply shows how many different approaches there are, and how widely the ultimate results vary. The attached Excel spreadsheet has an umbrella triangle (disguised in various ways to avoid divulging anything proprietary), but there is no special meaning to the choice, other than that it's unstable.

To participate, here are your instructions:

Select age-to-age factors from the triangle, and insert them in row 33 of the spreadsheet, where indicated. To keep the topic focused, we're ignoring the tail factor. Also, give a few sentences on how you selected your factors: all year averages? 5 year averages? Weighted or un-weighted? How many different averages did you look at before making your selection?

How did you deal with apparent outliers or with reversals in the nice smooth pattern you might have expected?

If you would like to participate and have your results included at the Seminar, please respond by emailing the spreadsheet to LDFs@comcast.net by May 17. Responses will be kept anonymous.

Thanks.

Gary Blumsohn

Chair

Committee on Reinsurance Research

The attached Excel file contained the following

Umbrella incurred loss triangle

Accident Year	1	2	3	4	5	6	7	8	9	10	11	12
1991	1,782	3,000	6,924	10,167	12,369	14,047	13,577	14,289	13,831	14,419	14,563	14,484
1992	430	2,814	3,557	5,745	9,033	7,884	8,715	8,982	9,048	8,934	8,856	
1993	2,234	3,902	10,841	14,262	17,666	19,154	19,411	19,021	18,854	19,085		
1994	3,335	12,937	23,694	20,477	19,715	23,689	23,955	25,066	25,269			
1995	2,006	5,406	9,802	8,949	10,611	10,623	16,633	16,699				
1996	7,640	8,485	12,085	13,515	15,418	18,894	19,029					
1997	6,643	13,184	18,530	17,782	20,867	21,358						
1998	2,474	9,684	10,636	16,266	16,649							
1999	4,229	6,135	5,972	8,613								
2000	2,065	2,982	3,384									
2001	3,448	4,240										
2002	1,736											

Age-to-age

1991	1.684	2.308	1.468	1.217	1.136	0.967	1.052	0.968	1.043	1.010	0.995
1992	6.544	1.264	1.615	1.572	0.873	1.105	1.031	1.007	0.987	0.991	
1993	1.747	2.778	1.316	1.239	1.084	1.013	0.980	0.991	1.012		
1994	3.879	1.831	0.864	0.963	1.202	1.011	1.046	1.008			
1995	2.695	1.813	0.913	1.186	1.001	1.566	1.004				
1996	1.111	1.424	1.118	1.141	1.225	1.007					
1997	1.985	1.405	0.960	1.173	1.024						
1998	3.914	1.098	1.529	1.024							
1999	1.451	0.973	1.442								
2000	1.444	1.135									
2001	1.230										

Insert your selected ATA factors below. Ignore the tail factor.

Selected ATA	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**

Appendix B

The table below shows the selected factors that the various participants provided, sorted in increasing order of the implied reserve from the factors.

		Selected age-to-age factors										
Reserve ranking	Implied reserve	1	2	3	4	5	6	7	8	9	10	11
1	10,707	1.375	1.213	1.163	1.113	1.050	1.010	1.007	1.005	1.003	1.001	1.000
2	18,323	1.921	1.300	1.142	1.124	1.112	1.021	1.013	1.008	1.005	1.003	1.002
3	18,512	1.626	1.213	1.173	1.113	1.103	1.043	1.013	1.002	1.017	1.003	1.000
4	18,561	1.921	1.157	1.214	1.081	1.085	1.050	1.021	1.002	1.017	1.000	1.000
5	19,512	1.499	1.138	1.194	1.113	1.094	1.082	1.020	1.002	1.010	1.002	1.000
6	21,028	2.000	1.250	1.150	1.110	1.090	1.070	1.021	1.010	1.005	1.002	1.001
7	21,163	1.921	1.250	1.142	1.113	1.112	1.093	1.021	1.000	1.000	1.000	1.000
8	21,304	1.758	1.250	1.142	1.113	1.094	1.075	1.021	1.013	1.007	1.003	1.000
9	21,807	1.371	1.063	1.214	1.113	1.085	1.120	1.013	1.008	1.017	1.003	1.000
10	21,958	2.000	1.250	1.210	1.100	1.100	1.075	1.020	1.005	1.005	1.003	1.000
11	22,827	1.921	1.340	1.142	1.112	1.093	1.060	1.021	1.014	1.008	1.005	1.003
12	23,398	2.000	1.400	1.150	1.120	1.100	1.080	1.020	1.005	1.003	1.002	1.001
13	23,703	2.005	1.383	1.142	1.134	1.112	1.093	1.021	1.000	1.000	1.000	1.000
14	23,729	2.014	1.535	1.249	1.163	1.089	1.034	1.023	0.994	1.014	1.001	0.995
15	24,433	1.550	1.250	1.182	1.163	1.100	1.049	1.030	1.018	1.011	1.007	1.004
16	24,636	1.962	1.380	1.138	1.111	1.090	1.065	1.030	1.015	1.010	1.005	1.002
17	24,706	2.005	1.538	1.135	1.142	1.094	1.075	1.021	0.995	1.017	1.003	0.995
18	24,706	2.005	1.538	1.135	1.142	1.094	1.075	1.021	0.995	1.017	1.003	0.995
19	24,835	1.866	1.343	1.232	1.146	1.062	1.093	1.021	1.002	1.014	1.003	1.000
20	25,002	1.921	1.250	1.142	1.081	1.112	1.093	1.021	1.000	1.017	1.010	1.005
21	25,056	1.921	1.250	1.142	1.081	1.112	1.050	1.025	1.020	1.015	1.010	1.010
22	25,439	2.153	1.443	1.227	1.127	1.074	1.043	1.025	1.014	1.008	1.004	1.005
23	25,619	1.816	1.457	1.140	1.112	1.104	1.094	1.020	1.014	1.008	1.002	1.000
24	25,856	2.500	1.535	1.195	1.165	1.090	1.035	1.023	0.995	1.014	1.000	1.000
25	26,235	2.000	1.400	1.400	1.180	1.120	1.010	1.020	0.995	1.012	1.010	0.995
26	26,247	2.005	1.538	1.247	1.142	1.094	1.050	1.021	1.010	1.005	1.003	1.000
27	26,730	2.513	1.538	1.188	1.090	1.094	1.075	1.011	1.005	1.007	1.005	1.002
28	26,880	2.000	1.250	1.200	1.140	1.100	1.100	1.025	1.010	1.010	1.005	1.000
29	27,137	2.250	1.500	1.250	1.200	1.100	1.050	1.025	1.000	1.000	1.000	1.000
30	27,468	1.900	1.350	1.200	1.115	1.100	1.120	1.020	1.005	1.015	1.001	1.000
31	27,599	2.150	1.525	1.200	1.150	1.100	1.050	1.025	1.015	1.005	1.003	1.002
32	28,712	2.005	1.538	1.135	1.142	1.094	1.075	1.021	1.018	1.017	1.003	1.002
33	29,107	1.626	1.213	1.173	1.142	1.094	1.075	1.044	1.021	1.016	1.012	1.009
34	29,444	2.000	1.450	1.200	1.135	1.100	1.075	1.025	1.015	1.010	1.008	1.005
35	29,528	2.179	1.515	1.197	1.156	1.094	1.075	1.021	1.017	1.010	1.003	1.000
36	29,805	1.934	1.482	1.158	1.141	1.095	1.073	1.026	1.021	1.017	1.006	1.004
37	29,837	2.033	1.500	1.256	1.138	1.078	1.047	1.029	1.019	1.013	1.009	1.006

		Selected age-to-age factors										
Reserve ranking	**Implied reserve**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
38	30,883	1.950	1.500	1.180	1.140	1.100	1.075	1.040	1.020	1.010	1.005	1.003
39	31,336	1.650	1.300	1.250	1.150	1.100	1.060	1.040	1.030	1.020	1.010	1.005
40	31,715	2.000	1.500	1.250	1.150	1.100	1.090	1.025	1.010	1.010	1.010	1.000
41	31,819	1.951	1.538	1.135	1.142	1.094	1.075	1.050	1.025	1.017	1.003	1.000
42	32,416	2.200	1.250	1.500	1.150	1.110	1.075	1.025	1.010	1.010	1.005	1.000
43	33,487	2.225	1.535	1.249	1.163	1.094	1.050	1.029	1.020	1.015	1.010	1.005
44	33,520	2.150	1.500	1.200	1.185	1.115	1.095	1.025	1.015	1.010	1.005	1.000
45	33,539	1.700	1.250	1.175	1.130	1.120	1.100	1.050	1.025	1.015	1.010	1.010
46	37,958	2.517	1.538	1.241	1.142	1.094	1.055	1.021	1.015	1.010	1.050	1.000
47	38,201	2.419	1.376	1.188	1.115	1.078	1.057	1.043	1.034	1.027	1.023	1.019
48	38,973	2.517	1.603	1.247	1.189	1.112	1.112	1.023	1.014	1.014	1.001	1.000
49	39,156	2.000	1.400	1.350	1.175	1.150	1.100	1.030	1.025	1.020	1.000	1.000
50	55,197	1.641	1.297	1.187	1.134	1.103	1.083	1.069	1.058	1.051	1.045	1.040
51	60,152	2.788	1.378	1.656	1.336	1.130	1.070	1.040	1.020	1.020	1.020	1.010

The next table shows an edited version of the narratives from the respondents, sorted in the same order as the table above. We have tried to edit lightly when the narrative wasn't clear. We have also edited spelling and egregious grammatical errors, as well as editing to retain the anonymity of the participants. While we have done our best to avoid changing the meaning of what was intended by the participants, it is quite possible that we have misunderstood something and have misrepresented some of what was intended. We apologize for any such errors. We also freely admit that we do not fully understand all of the explanations. In general, participants dashed off a quick few sentences – rather than a detailed documentation of what they had done. Where the explanation is blank, the respondent didn't provide one.

<table>
<tr><td>1</td><td>I generally prefer weighted averages, 3-5 years for incurred and 5-7 for paid, but since I just came back from vacation and will soon be on the road again I did mostly a straight average analysis. With an unstable pattern, one technique I frequently use is a 5 or 7 year average excluding hi-lo.

For year 1, I did a 3 year average, seeing that the numbers were reasonably close and showing a downward trend, possibly suggesting that claims are being settled more quickly.

Year 2, 4: 5 year ex-hi/lo

Year 3: No clue what to do here, just made it halfway between 2 and 4.

As I typically do in later years where the observations are less credible, I just choose round numbers that slowly go down to 1. I've seen others use something like a Bondi curve for similar purposes.

At the end of the day, everything we do in reserving is "wrong" according to management; reserving actuaries take the hit when adverse development rears its ugly head. That's probably why it receives so little attention on the syllabus.</td></tr>
<tr><td>2</td><td></td></tr>
<tr><td>3</td><td>For 12-24 and 24-36 months, average ATA factors are lower in the more recent years (1996-2001). Considering this, I selected 5-point ex hi-lo averages for 12-24 through 72-84. For 84-96 I selected a 3-year weighted average; beyond 96 I selected all year weighted averages. I looked at simple and weighted averages for 1 to 7 years plus all years. I also looked at 12 to 36 months, 36 to 84 months, and 84 months to current as reasonability checks of my selections.</td></tr>
<tr><td>4</td><td>Apparently something going on in the latest diagonal. Hard to tell without paid triangle or claims audit report. Would like to use some sort of industry default as a test to see how the pattern here compares. No idea as to how exposures have increased/decrease => can't tell whether the 2002 year is uncharacteristically low, as expected, or higher than expected.

I went with the standard 3/5/All Year Averages. I'm not convinced that alternative methods for selecting LDFs do a better job than standard averages of capturing the full gamut of frequency/severity scenarios which could impact loss development. Perhaps this is just ignorance on my part.</td></tr>
<tr><td>5</td><td>For the most part I rely upon an average of 3-year loss-weighted and 5-year ex-hi-lo, with some regression for factors in the tail. I might be inclined to add part of a cumulative standard deviation for a volatile line such as umbrella.</td></tr>
<tr><td>6</td><td></td></tr>
<tr><td>7</td><td></td></tr>
<tr><td>8</td><td>My first pass was to select the median of the 3, 5, 7 year weighted averages.
I smoothed the tail, using 7-8 and 10-11 as anchors. Then I graphed the individual ATA factors and tweaked/smoothed the 6-7 selection. Then I graphed the whole range and tweaked 7-8 from median to higher 5-weighted average.</td></tr>
<tr><td>9</td><td></td></tr>
</table>

<table>
<tr><td>10</td><td></td></tr>
<tr><td>11</td><td>Generally used the 5 year weighted average. Exception 2-3 ATA factor is the average of the 5-year and 7-year. The 5-year point on its own was a bit low and uncomfortable so it was judgmentally increased.

The selections were smoothed to decrease with maturity. The 5-year 4-5 ATA was reduced from 1.081 to 1.060 --- the difference was used with the more mature points (e.g., 8-9 and 11-12) for additional smoothing.

In situations with unstable data, we like using dollar weighted averages versus straight averages so years with small volume/high development do not receive undo weight. And we're more likely to use more years rather than fewer years. With more stable data we're more likely to rely on shorter/more recent averages; and also use straight averages and not weighted averages.

Like to use more recent information rather then older diagonals. Legal and legislative trends or changes in company claim processing. Company maybe starting out and the earlier stuff they are "just cutting their teeth." In fact when there is a lot of data, we will show/use the latest 10 or 12 diagonals and not use the diagonals prior to that. I do realize this point contradicts the prior point -- I guess there is a balance that needs to be struck --- and in fact we like to have two selections one based on a longer term average and one based on shorter term averages. In this case we went with mid- to longer-term averages given the "instability" of the data.

We like to see decreasing factors --- so we smooth. Will borrow from one ATA to smooth another ATA.

Would like more information:
Have the retentions and limits been changing over time?
Can we get the individual claims at each valuation date? If so we would trend losses (perhaps even retentions and limits) to current date and then recast development triangle.
Are there any changes in payments or case reserving philosophy?

Important to look at the analysis every year (or more frequently) and adjust selections with latest information. Test prior assumptions and adjust when necessary. This point is the most important point here --- perhaps even more important than the ATA selections themselves (assuming that they were selected within reason).</td></tr>
</table>

12	I dislike using straight averages to make LDF selections for umbrella, or excess in general. I calculated an all-year and a 5-year weighted average, and made selections for smoothing purposes, trying to ensure my selections kept the cumulative factors somewhat consistent with the experience. If benchmark data had been made available, I would have looked at that as well. Also, I spent very little time with this. True it's just a sample, but I try not to overanalyze loss triangles. I will never know the "true" LDFs (assuming they even exist) regardless of the amount of time I spend analyzing the data, and I don't want to fool myself that my work is more predictive than it really is.
13	Early years: I like all-year weighted averages for highly variable books like umbrella. In this triangle, early years straight average > weighted average, implies larger years develop less. I would like to see premium normally, as well as previous selections and actual vs. expected to see how well previous selections are holding up. I also look at the year of the development - for example, I will put more or less weight on a year if it's clearly a soft market year whose development won't be repeated in the hard market. In this example, it really looks like a lot of losses got put up, then brought down, then developed upward. I tried to pick factors for 2:5 whose product was generally close to the weighted average development I saw on 94-98 for 2:5 - though I built in some conservatism. One other comment: This looks pretty short tailed for umbrella business. Is it personal umbrella? Based on what I see here, it's hard to justify development beyond 8 years. With more time, I'd poke into the whys and wherefores around that.
14	For the first evaluation, I excluded what I considered an outlier, the AY 1992 point and then took the average of the remaining ratios, excluding the remaining high and low values. For the next 5 evaluations, I took the all point average, excluding the high and low points. For the last 5 evaluations, the ratios were more stable and I used an all point average.
15	This is a formulaic approach giving some extra weight to more recent experience. Some judgmental smoothing is then applied.
16	Looked at various averages

17	I am basically selecting the "volume weighted all" factors. The logic behind the selection is based on the recent paper http://www.casact.org/pubs/forum/08fforum/1Bardis_Majidi_Murphy.pdf. The main idea is to build a statistical framework which would help test the underlying assumptions of the chain-ladder method. Given a set of selected development factors you can find a linear regression (with "good" assumptions about the variance of its error terms) that produces as best linear unbiased estimators (BLUE) the selected development factors. The significance of the previous statement is that a practitioner can use the robust statistical regression framework to check the reasonability of his/her selections. So residuals in the AY/DY/CY dimensions will provide the underlying trends and normality tests and the AIC/BIC fitting criteria will provide a statistical evidence of how well the chain ladder method "fits" the historical data. The important point is that all these statistics are within the confines of the chain ladder method that practitioners are comfortable with. I looked at the volume weighted, straight average (all vs. 5 years) and judgmental selections (with and without outliers) and the visual inspection suggested that the "volume weighted all" model performs as well as anybody. This should not come as a total surprise given the change in the volume of losses by accident year. Surprisingly also, it does not produce any outliers (within 1.5 interquartile distance) and the normality graph is pretty "tight" around the 45 degrees line. The data exhibits some decreasing trends on the accident year dimension which could suggest that an exposure adjustment is in order. Both the calendar year trend and standardized residuals vs. fitted values suggests that the selections understate the low historical losses (not surprisingly since some of the low volume years exhibit higher development than the average).
18	For the age to age factors I selected the all year weighted averages since the individual ATA factors are volatile. I did not make adjustments for reversals (ages 8 to 9 and 11 and 12) since they were minimal. If the AGA factor in that case was 0.800 instead of 0.995, I would make an adjustment.
19	I will often look at the triangle and if there appear to be some big takedowns following big increases, I will "smooth out" the triangle by removing the reversals. Then I'll tend to look at the last 3, 5 and all years both weighted and unweighted.

20	I generally went with 5 year weighted averages, until the factors close to the tail where I ignored the one age with negative development and selected a small positive factor. I looked at 3, 5, and all-year weighted and simple and hi-lo out averages. In real life I would look at the paid triangles to see what kind of ultimates that was producing as compared to incurred. Also might look at some diagnostics such as change in average claim size, etc if I was feeling really crazy. I would also get industry development factors. I also noticed that for the first 2 ages, the recent diagonals appear to be lower than the older ones - would have to investigate if this was random or a trend. Since I went with 5 year averages, I am giving it some, but not full credit, as my selections are still higher than the recent 3 diagonals for these ages. This isn't the case for the later evaluations - recent diagonals are not generally lower. For the 91 year it looks like the 0.967 and 0.968 get 'erased' by the factor after them, reserve takedown gets put back up the next year, so I judgmentally selected 1.0 for the 8-9 year even though the 5 year avg was <1, although I didn't completely throw out that year.
21	
22	My usual technique is to compare Weighted-All with Weighted for a certain number of years. I believe that straight Average is too subject to biases caused by years with very little loss reported. Also, I believe that removing the Min-Max is inherently biased since LDFs are capped from below by 0, but uncapped from above, so removing the Max will tend to have a larger effect than removing the min. I often use the Brosius Linear method as a "sanity check", but in this case it is prone to distortion for two reasons: It is most accurate when applied to trended loss ratios to on-leveled earned premium to best remove effects of loss trends and changes in exposures, and the linear statistical model assumes that variations from the "best fit" are due to random noise (independent, identically normally distributed for that matter) that is not always the case. For the purposes of this exercise, I also ran Markus Gesman's package in 'R' to calculate the bootstrap ultimates based on England and Verrall. However, this too is based on a generalized linear model that less accounts for changes in underlying factors as opposed to finding the best generalized linear fit to the data supplied. I will often choose a final "raw" selection and then fit a smoothed weibull and lognormal to that data. If the fitted curve fits he raw data "well," then the argument can be made that the fluctuations are the "noise" and the fit is the signal. Certain lines of business, however, which are known to over-reserve and then take down, are not suited to this kind of smoothing. In this case, the distribution of each age's ATA seems random, and the all-year weighted is a decent selection for each age. It also is a good candidate for smoothing, so the final selected ATA's would be the weibull-smoothed version of the weighted all-year ATU.

23	I calculated weighted 3, 5, and all-year averages; simple 3, 5, and all-year averages; and 5, 7 excluding hi-lo averages. I calculated the average of averages. I also calculate the median of these averages. Then I ran regression on losses at evaluation 1 through 6, assuming zero intercept. The final selection is basically a weighted average of these three methods for ATA from 1 to 5, and using average of averages for the rest, with adjustments and linear interpolation to smooth out the LDF curve.
24	Looked at volume weighted averages 3 years, 5 years and all years. Then simple averages 3 years, 5 years and all years. Finally simple average excluding the highest and lowest data points. In the end I selected RTRs according to the following: 1-2: simple average all years 2-3: simple average ex high/low 3-4: simple average 5 years 4-5: simple average ex high/low 5-6: simple average ex high/low 6-7: simple average ex high/low 7-8: simple average all 8-9: simple average all 9-10: simple average all 10-11: simple average all 11-12: simple average all I ended up not focusing on the volume weighted averages. Generally I would use these when information is available on premium volume by year, or number of claims by year. Large dollars of loss do not necessarily increase predictive power, as this might be the result of one or two unusually large claims, the development of which might not be representative of the average.

25	Assumption: I assumed the policy count and class/limit profile of the book remained relatively stable during over the history provided. Methods Employed: 1) Evaluated magnitude of losses for each year at each maturity and ranked as "H" high, "M" medium and "L" low. With low volume data I've found that the link ratio methods are very sensitive to the actual magnitude of losses at a point in time. Determined separate average link ratios for L, M, and H. 2) All years weighted average 3) All years straight average 4) All years straight average excluding minimum and maximum 5) All years weighted average excluding 1992. After review the 1992 year seemed very different from the others due to the low magnitude of losses and the up and down development pattern. Selection for older years looked primarily at the all year weighted average excluding 1992. Selection for more recent years looked at "L, M, H" averages based upon the magnitude of losses along the diagonal. Other data would have been helpful: Premium / Policy Counts for each calendar/accident year and perhaps a limit profile Paid closed claim count triangle Open claim count triangle Separate paid and outstanding loss triangles Any kind of class of business distribution (SIC, NAICS, GL Class Code, CMP Program)
26	

27	1st Pass 2.005 1.538 1.135 1.142 1.094 1.075 1.021 0.995 1.017 1.003 0.995 1.140. All-years weighted averages were selected because they behaved reasonably in a 48 month model of early development which attempts to spot unreasonable relationships between the early ATA factors. The 3 & 5 year weighteds did not pass this test. On volatile triangles like this, I also want to use as many years as I can. I never look at simple averages of ATA factors – only weighted. 2nd Pass 2.005 1.538 1.188 1.090 1.094 1.075 1.011 1.005 1.007 1.005 1.002 1.140. The modest reversals were eliminated using exponential smoothing. I generally do this as I find reversals to be abhorrent. When possible, I develop gross losses and salvage and subrogation separately in order to avoid analyzing downward development. The fact that I left 1 small reversal shows that I am not a fanatic! 3rd Pass 2.513 1.538 1.188 1.090 1.094 1.075 1.011 1.005 1.007 1.005 1.002 1.140. The last revision of the 12-24 ATA factor is based on the early development model. As a last comment, when I develop losses I want both Paid and Reported as I develop them together. This helps dramatically in estimating tail factors and the general quality of the data. I often hear, especially regarding volatile groups, that the paid data is just not useful. When developed on its own that may be true but when developed alongside the reported losses, I usually find it very helpful in making judgment calls at the very least.
28	I used simple and weighted averages for all years, most recent 3, and most recent 5 as summary references. While I see some volatility in the age-age factors, I didn't see many that were so freakish as to cause me to censor them out. I generally do not select to allow reversals unless the evidence is clear, and that they aren't re-reversed back later.
29	Generally I look at the average ex-hi/lo and the weighted average, and pick something close. I prefer to pick round numbers, as they won't fluctuate as much from year to year. In certain cases I will ignore the weighted average if one observation is influencing it too much (e.g. the 0.963 4-5 factor in 1994). I selected 1.00 for the last 4 factors, as I would normally cover these developments in a tail factor.
30	I considered all averages (weighted and unweighted) that end with the factor in the last diagonal. No ex-ante averages were considered. Most selections rely on the long term average (all-years or close to all-years) with tempered movements in the direction of the latest calendar years' link ratios. There is no smoothing between consecutive link ratios. Smoothing seems appropriate when there is an outlier. With erratic data, it's not too clear what points are outliers and to what degree they are outliers.
31	I selected mostly the volume weighted ex hi/lo with some judgment applied.
32	Weighted average all years, with a bias against factors < 1. No obvious calendar-year correlation effects so Mack gives the answer. I am not a reserving actuary and I don't get called upon to pick factors very often (I can recall no instances in the last three years, in fact) so you should underweight my choices.

33	As part of the selection process, I looked at the all-year weighted averages, the weighted averages of the last 5 years, of the last 4 years and the last 3 years, as well as the mean of the last 5 excluding high and low. For the 1-2 ATA factor, there appears to be a downward trend, but the 2001 factor seems to be very low. Rather than allowing that factor or the 1998 factor (which is very high) unduly influence the selection, I selected the mean of the last 5 excluding the lowest (2001) and highest (1998). I selected the next 3 ATA's in the same manner (i.e. last 5 ex high-low).For the 5-6 and 6-7 ATA's, the all-year weighted averages appeared to be reasonable. After that age, the limited number of data points were indicating even more erratic patterns. The 7-12 ATA is indicated as 1.031, whereas an indicated decay rate off the 6-7 ATA of 1.075 would generate 1.056. I decided to weight 50-50 the ATA's indicated by the data with the ATA's generated by a decay rate. In general, I look to select ATA's that are monotonically decreasing.
34	I also would include information that I have regarding the industry. It was not clear if the umbrella was supported or unsupported, but looking at the information I would guess supported. My selections are based on looking at all the averages and my general expectations regarding the line of business. I will typically select a smooth pattern, with the overall pattern based on the overall data.
35	I looked at 5 different averages: weighted and unweighted and for all years and 5 years, as well as weighted average ex-hi-lo. I did not calculate an average for less than 5 years, since the line is umbrella and due to the long tailed nature with volatility more experience is best. I prefer weighted averages as opposed to straight because it smoothes out the data. The weighted average of all and excluding high and low are very close. This gave me comfort in the selections, but I am still skeptical about the difference in the older years prior to 1996. I selected the weighted average ex hi-lo for 12-60 months and then went to all year weighted average for 72 and 84 months. If you kick out the high and low in the 72 and 84 months, you are only left with 3 and 4 factors, so I went with the all year weighted average for these selections. I smoothed out the 96 ATA factor and put the 108 factor in its place. For the 108 factor, I averaged the 108 and 120 factor. For 132, I assumed one based on the data, but given this is umbrella there will be a tail factor. I still am not sure about the data, since it does seem like it changed starting in 1996. The factors starting dropping off dramatically in 1996. There also seem to be reserve takedowns at 12/31/96 for the older years. It also seems like they are setting higher initial reserves at 12 months starting for 1996 and 1997. They strengthened reserves at 12/31/00 for '94-'97, but not so high as to match the cumulative products for the early 1990's. The danger of my selections would be if reserves are weaker now, since my selected factors more reflect the more recent years because I selected the weighted average and the hi-lo which often kicks out the older years. I would want to see more info such as: large loss listing, paid data, claim counts, premium volume, mix of business changes.

36	I'm not sure whether you were specifically looking for implicit or explicit inclusion of some risk margin in the selections (to anticipate for when things invariably go wrong, e.g., for reserving) or whether you were looking for "best estimate," without regard to a margin. I took the latter approach. In answer to your additional information request, I used the following 4 averages: Unweighted all year and weighted all year, 6 year and 3 year. In my selection, I assumed that weighted averages are more representative than unweighted. That is, I assumed more volume in one AY would indicate more credibility than an AY with less volume. This assumption would not be true if larger AY volumes were only due to larger shares in that AY, e.g. larger reinsurance shares but same underlying book. I gave a majority of the weight to the all year volume-weighted average. I did not include a tail factor, per your instructions. To help smooth selections and adjust for reversals, I selected an "industry" pattern based upon excess layer, LOB, and lagging assumptions for this umbrella book. To do this right, would naturally need more information on the underlying book, but nonetheless I took a stab at it. I first scaled the industry factors based on the volume weighted averages underlying my assumptions above, and comparing actual vs. expected development at each maturity. I then generated a scaled industry factor set, in this case using a factor of 0.9. For purposes of illustration for this survey, I then credibility weighted this scaled industry factor set using a claim count based formulaic approach and an estimate of excess claim counts underlying the umbrella triangle. Of course, would ideally have all this information including all individual claims and their histories in the data and not have to make these assumptions. The credibility formula uses actual claim counts (vs. e.g. expected claim counts using premiums, expected loss ratios, claim count reporting patterns, etc.) and a $z=n/(n+K)$ form. I adjusted the K from a default based upon the extra variability in the LDFs. These selected factors were slightly overridden to produce the ATA factors that I put into your spreadsheet. I did not include a specific risk margin in the selections, for the "things that invariably go wrong." To do this, I might not have scaled back the industry factors, nor included weights for the more recent (in general lower) averages. May also have selected factors higher than the averages and of course also included a tail factor.
37	(1) I assumed the data is on-level with a constant level of exposure. (2) I fit an inverse power function utilizing David Clark's maximum likelihood approach. http://www.casact.org/pubs/forum/03fforum/03ff041.pdf (3) I examined the total variance (parameter and process risk) for the two year weighted average, three year weighted average, etc… and selected the weighting with the lowest variance. This happened to be an all year weighted average. (4) I did not exclude any "outliers."

38	Attached are my picks. However, let me add that given paid losses and claim counts, I might have done something completely different. Due to the volatility, I tended to use the long term weighted averages as my guide but looked at all years, last five years and last three years weighted averages. I also did some smoothing of some of the points by moving dollars between generally contiguous evaluations (but not necessarily contiguous), recomputing LDFs and looking at the same statistics. After selecting, I compared the product of my selections through the first 7 points to the product of the all weighted average through the first 7 points and adjusted my selections slightly. The points after 7 were based on a judgmentally selected decay in the first 7 selections and the same set of six statistics noted above. The latter LDFs were selected more conservatively given it is umbrella (although did not have attachment point and limits) and given additional information on paid losses and open claims would probably have selected differently. I also compared my results to the fitted values from inverse power, exponential and Weibull curves. This has not been subjected to our quality control process and should be considered my thoughts.
39	I looked at 3 and 4 year averages, both weighted and unweighted, the 5 year excluding the high and low, the median, and an all year simple average. I tend to ignore outliers and reversals in the pattern. Loss data that comprises few losses, driven by severity rather than frequency, which this appears to be, cannot be relied upon for reasonable patterns – smoothing helps this out.
40	I mainly looked at weighted averages and also looked at how close these methods were. I may also give some credibility to an industry or default pattern.
41	In general, I used an all-year weighted average, but also looked at 3, 5, and 8 year averages. For unstable triangles, I tend to use as many years as I can. For the 1-2 ATA factor, I excluded the 6.544 because it was such an outlier, although it had little impact since the dollar amount was small. Beginning with the 7-8 ATA factor, I started smoothing the pattern out judgmentally - no real scientific reason as to why, just because there were so few points.
42	
43	The selections primarily use all year link ratios excluding high and low. Links are further smoothed.
44	I looked at straight, weighted, truncated, and geometric averages over a wide range of time frames (from 1 to 11 years) to see what type of pattern emerged. Without knowing the underlying business, it's tough to select factors. The use of the geometric average gave a quick look at the impact of outliers. For those older ages where it appeared to reverse, I judgmentally smoothed out the pattern.
45	

46	In the absence of an obvious pattern in any of the averages, will initially select overall weighted average, with the following exceptions: Age 1 - weighted average looks more reasonable relative to Age 2 Age 3 - weighted 75%/25% with Ages 4 & 2 to smooth the reversal (those are assumed more reasonable relative to industry/company default patterns, so Age 3 is replaced). Also concern over several negative points in the data. Age 6 - appears to be driven up by an outlier, however if I exclude the outlier it looks artificially low, so I gave some credit off the weighted average with an eye on the overall progression. Ages 5 & 7 appear reasonable relative to industry/company default XS pattern. (Age 5 highs are balanced by a low "outlier.") Ages 8-10: set to industry/company default (#'s changed slightly and rounded for proprietary reasons).
47	I looked at five weighted averages, but based my selections on an inverse power fit of the entire triangle of incremental reported losses, using the method described in Dave Clark's 2003 CAS Forum paper on "LDF Curve-Fitting and Stochastic Reserving: A Maximum Likelihood Approach." I also considered fitting only the last few diagonals, but ultimately decided that it made sense to fit all the data. I ignored loss trend in this example. Graphs of the factors (actual all years versus fitted all years, columns of actuals by development period compared to fitted, standardized residuals) increased my confidence that the maximum-likelihood approach was doing a good job fitting the entire triangle with all its volatility.
48	I used unweighted and no less than following age-to-age. For this case, it looks like better experience in recent years. If I find that there is a solid reason for the better experience, I will apply an adjustment factor for all ATA factors.
49	I didn't look at any averages formally. I eye-balled each column and struck repeated compromises between what the column suggested and my feeling that the age-to-age factors should decrease monotonically.
50	I model a distribution of the "time until reported" random variable. I then fit the incremental amounts to what the distribution would imply as a fraction of the cumulative amount for the accident period. I then inspect the errors for consistent bias and make adjustments if appropriate. I find this method to deal well with volatile development data. Because it is looking across the entire curve for the parameterization, the impact of an outlier is generally smoothed across development periods. In this example, my factors are larger for the later development periods than what history would indicate. However with a 6-7 factor on the 2nd most recent diagonal of 1.5, clearly the potential for late period development is there, and I feel this should impact the selection for other periods. This method is only appropriate when negative incremental amounts are observed sporadically, not consistently. I prefer to use a method that models case reserve balances and incremental paid losses within a single model framework, which therefore does not suffer this problem.

51	I looked at two kinds of averages, weighted averages and simple averages. If I had some extra knowledge on this book of business, I might have done some time weighted averages. I usually prefer the weighted average to simple average because the simple average has inherent bias. However, I didn't use either of them for LDFs of age 1-4. I found them to be disturbingly low and feel uncomfortable to go with any of them (5-year weighted average etc). I found the age 1-4 LDFs have a wider range then those of age 5 and on. I also noticed that there is a negative correlation between the LDF factors and reported amount at the same age. This is very typical for LOBs of high variance. For age 1-4, I used a linear regression to predict the LDF factors. For example: 2002 factor age 1 factor (3.14) was calculated using age one reported amounts and age 1-2 ATA factors. 2002 age 2 factor (1.65) was calculated using age two reported amounts and age 2-3 ATA factors. For age 5 and on, I judgmentally picked LDF with the consideration of weighted averages, simple averages, smoothness, conservatism, and historical LDF range. I did this for each year and each age. I did the age to ultimate factors by using ultimate divided by reported. ATA factors were derived from age to ultimate factors. There is a bump in the ATA curve, which is mostly caused by the low reported amount of year 2000.
52	*Authors' note: This participant didn't provide selected factors, and was thus not included in the earlier tables. However, we felt that the thought process was worth including.* Being the pain in the (you know where) actuary I am, I would first ask the underwriters/claim handlers a series of questions (either my own or via an underwriting/claim audit): What are the attachment and limit profiles of the underlying umbrella business and how have these changed over time? Is the data net or gross of reinsurance and how has the company's retention changed over time? What is the company's case reserving policy and how has this policy changed over time? What is the nature of the underlying umbrella business (i.e. commercial or personal)? Absent any answers to my questions, then I would probably take a series of averages (3 to 5 year simple average, 3 to 5 year weighted average, 5 year excluding high & low simple average). I would then probably take an average of the averages (yes I've done this) and then make a selection. In other words smooth the data to the point where a selection is easier and more formulaic and less arbitrary. While making selections, I would also note where in the underwriting cycle I am (helps to select a calibration period for the averages) - or perhaps use loss ratio benchmarks to see if ultimate loss projections take sense.

5. REFERENCES

[1] Ayres, Ian, *Super Crunchers: Why Thinking-by-Numbers Is the New Way To Be Smart*, 2007, Bantam.

[2] CAS Task Force for Enhancing the Reputation of Casualty Actuaries, Unpublished report, 2008, presented to the CAS Board in March 2008 and distributed to CAS Research Committees in April 2008.

[3] England, Peter and Richard Verrall, "Analytic and Bootstrap Estimates of Prediction Errors in Claim Reserving," *Insurance: Mathematics & Economics* 25, 1999, pp. 281.

[4] Fleming, Kirk G., "Yep, We're Skewed," *Variance* 2:2, 2008, pp. 179-183.

[5] Hirshleifer, David A. and Tyler Shumway, "Good Day Sunshine: Stock Returns and the Weather" *Dice Center Working Paper No. 2001-3*, 2001. Available at SSRN: http://ssrn.com/abstract=265674 or DOI: 10.2139/ssrn.265674.)

[6] Hodes, Douglas, Sholom Feldblum, and Gary Blumsohn, "Workers Compensation Reserve Uncertainty," *PCAS* LXXXVI, 1999, pp. 263-392.

[7] Mack, Thomas, "Measuring the Variability of Chain Ladder Reserve Estimates," *CAS Forum*, Spring 1994, Vol. 1, p. 247.

[8] Murphy, Daniel M., "Unbiased Loss Development Factors," *CAS Forum*, Spring 1994, Vol. 1, p. 183.

[9] Struzzieri, Paul J. and Paul R. Hussian, "Using Best Practices to Determine a Best Reserve Estimate," *CAS Forum*, Fall 1998, pp. 353-413

[10] Thaler, Richard H., *The Winner's Curse: Paradoxes and Anomalies of Economic Life*, 1994, Princeton University Press.

An Analysis of the Market Price of Cat Bonds

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Abstract

Motivation. Existing models of the market price of cat bonds are often overly exotic or too simplistic. We intend to offer a model that is grounded in theory yet also tractable. We also intend for our analysis of cat bond pricing to shed light on broader issues relating to the theory of risk pricing.

Method. We analyze several years of cat bond prices "when issued."

Results. We describe the market clearing issuance price of cat bonds as a linear function of expected loss, with parameters that vary by peril and zone.

Conclusions. The results provide a compact form of describing market prices of cat bonds and thus provide a framework for measuring differences in prices across various perils and zones; the results also allow us to measure changes in the price function over time. The results also suggest an overarching theory of risk pricing, in which price depends on two factors: the first factor is the required rate of return on downside risk capital in a portfolio context, and the second factor is the uncertainty of the estimate of the expected loss.

Keywords. Cat bonds; Insurance Linked Securities (ILS); market price of risk; reinsurance

1. INTRODUCTION

Describing the market price of property catastrophe (cat) bonds is important on two planes: the practical and theoretical. On the practical plane, firms desire to know how prices have behaved in the past, how prices vary by type of risk, and, potentially, how prices will behave in the future. Moreover, a model that accurately describes prices can also assist in benchmarking an observed price relative to a predicted price. On the theoretical plane, describing the market price of cat bonds illuminates the more general question of risk pricing, which relates to reinsurance contracts and other risk bearing transactions.

1.1 Research Context

Cat bond pricing has been investigated in Lane [6]. More recently, Gatumel [2] has reviewed Lane's model as well as other models of risk pricing.

1.2 Objective

Our objective is to propose a model that describes the market clearing price of cat bonds.[1] We propose a model that builds on theory, parsimoniously conforms to empirical data, and accentuates

[1] We do not intend for our model to address the full spectrum of complex issues that affect the prices of cat bonds. Rather, our proposed model, like all models, serves only as an approximation to reality.

practicability.

2. BACKGROUND

Insurance and reinsurance companies have used "cat bonds" to transfer, for a price, the risk of property catastrophe (cat) loss to investors.[2] Essentially, investors supply capital equal (usually) to the amount of the bond; the capital is then available to pay any covered losses from property catastrophe as defined in the bond. The insurance and reinsurance companies who sponsor the bonds thus hedge their exposure to cat risk, while investors earn return on capital via the coupon payments on the bonds.[3] If no cat event takes place, the investors receive all the coupon payments and return of principal, whereas if a cat loss does occur, the investors will typically lose out on some coupons and also sustain loss of principal.

The coupon rate received by the investors is usually split into two components. First, because the investors contribute money for one (or more) years, the investors receive interest payments for the time value of their money, which is usually based on the LIBOR rate.[4] In addition, the investors are subject to a potential cat loss, so they receive an additional coupon rate for taking on this risk; this additional coupon rate, quoted as a percentage of the amount of the bond, can be referred to as "risk premium," "risk spread," "spread over LIBOR," and "spread." In this paper, we will use the term "spread." Thus, we can say that:

$$\text{Total coupon rate \% to investors} = \text{LIBOR \%} + \text{spread \%} \tag{2.1}$$

The LIBOR rate is intended to compensate investors primarily for the holding of their money but not for cat risk; thus the spread is the component of the coupon rate that relates to the event

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[2] This section serves as basic introduction and background for the purpose of discussing the market price of cat risk. It is not intended as a comprehensive text on the cat bond market. Therefore, we caution the reader that some statements that are generally true may have caveats and exceptions; we typically choose not to highlight these caveats and exceptions, because our concerns are materiality and brevity.

[3] We caution the reader that this section serves as a streamlined background and does not address all the various technicalities of cat bonds. One example of a technicality is that an insurance company typically uses a Special Purpose Vehicle (SPV) to "issue" the bond; the company only "sponsors" the bond. In this paper, we use the terms "sponsor" and "issue" interchangeably.

[4] Technically, the time value of money should be based on a risk-free rate. The LIBOR rate, which is slightly higher than the risk-free rate, incorporates a modest amount of credit risk as well. Thus the reach for higher yield has served as a Trojan horse to insinuate credit risk into the cat bond market; these issues are beyond the scope of this paper.

risk of a cat loss. Therefore, we will generally use the spread to measure the "price" of risk transfer of cat risk:

$$\text{Price of risk transfer} = \text{spread } \% \tag{2.2}$$

While the spread represents the price of the bond, it does not measure the "net cost" to the sponsor of the bond. After all, the sponsor has a mathematical expectation of receiving some cat loss recoveries from the bond; the "annual average loss" (AAL) or "expected value" measures this quantity. In fact, as part of the bond issuance process, the sponsor will typically hire a third-party cat modeling firm to estimate the expected loss (which is then usually expressed as a percentage of the amount of the bond, a convention we follow in this paper). Usually, the spread should exceed the modeled expected loss, because the spread should be large enough to provide for the mathematically average loss and still provide some additional rate of return (above zero). Thus we can say:

$$\text{Spread } \% = \text{expected loss } \% + \text{additional rate of return } \% \tag{2.3}$$

$$\text{Spread } \% = \text{expected loss } \% + \text{margin } \% \tag{2.4a}$$

$$\text{Margin } \% = \text{spread } \% - \text{expected loss } \% \tag{2.4b}$$

Generally, these values are quoted as percentages of the amount of the bond; thus in this paper the terms "spread," "expected loss," and "margin" will typically be used in the context of "as a percentage of the bond amount." We also note that the bond amount is analogous to the occurrence limit and the aggregate limit of a property cat reinsurance contract; we will therefore use the term "limit" interchangeably with "bond amount."[5]

The question we investigate in this paper relates to the market pricing of cat bonds: how can we explain and predict the spreads of cat bonds? Do models of spread behavior conform to conceptual frameworks and also conform to empirical evidence? How does a theory of risk pricing inform our

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[5] Where "limit" is the "100% limit" reduced for "co-participation" or "coinsurance."

choice of model? Simultaneously, how does our inspection of empirical data affect our theory of risk pricing? These are the themes we explore in this paper.

3. MODELS OF CAT BOND PRICES

Each buyer and seller in the market uses his own risk preferences to evaluate price. Models of cat bond prices do not necessarily attempt to replicate the exact risk preferences and decisions of each market participant; rather, using a macro-level perspective, they describe the observed market clearing price, which is the outcome of all the risk preferences of all the individual buyers and sellers.

Before proceeding with our analysis, we discuss several pre-existing models of cat pricing and describe what motivates us to find an alternative model. Because the issue of pricing for cat risk arises in both the cat bond market as well as the traditional reinsurance market, we discuss models of cat risk pricing that derive from both sources.

3.1 Some Existing Models

One existing model of spreads is "multiple of expected loss." Practitioners in the cat bond market often measure, report, and benchmark cat bond spreads as a "multiple of expected loss." Implicitly, they espouse a model such that:

$$\text{Spread \%} = \text{expected loss \%} * \text{multiple} \tag{3.1}$$

In this model, the parameter "multiple" varies quite significantly: when expected loss is large, the multiple is small, and when expected loss is small, the multiple is large. As a result, the "multiple of expected loss" model is neither a complete nor an accurate description of spread. Thus one of our central motivations is to find an alternative model that better describes spread behavior, yet preserves both the parsimony and tractability of the "multiple of expected loss" model.

A different class of existing models focuses on some form of volatility metric or risk measure of the individual bond (or layer or "tranche") in order to model the spread. Thus:

$$\text{Spread \%} = \text{expected loss \%} + \text{margin \% based on standalone risk} \tag{3.2}$$

This family of models includes:

1. Margin % = function of standalone standard deviation.[6]
2. Margin % = function of conditional expected loss (i.e., conditional severity).[7]

One problem with using standard deviation is that for highly skewed distributions, which are prevalent in property cat reinsurance, standard deviation is not an accurate description of extreme downside risk; rather, the skewed downside risk must be measured using other metrics.[8] Thus we hypothesize that the following model, which focuses on the extreme downside of total amount of capital at risk, might be a suitable candidate:

$$\text{Spread \%} = \text{expected loss \%} + \frac{(\text{amount of capital at risk} * \text{required rate of return on capital \%})}{\text{amount of the bond}} \qquad (3.3)$$

For typical cat bonds, a severe downside loss can wipe out the entire principal; so the amount of "capital at risk" equals the full amount of the bond. Returning to equation (3.3), if we replace the term "amount of capital at risk" with "amount of the bond" and cancel the term in the numerator and the denominator, we derive:

$$\text{Spread \%} = \text{expected loss \%} + \text{required rate of return on capital \%} \qquad (3.4)$$

Another problem with the standalone standard deviation and conditional severity models is that they violate a key principle of risk pricing: that one ought to measure risk not on a standalone basis but rather in a portfolio context. Thus the standard deviation or conditional severity of a particular bond should be much less important in a portfolio context—what matters is the bond's contribution to the total risk of the portfolio, which may differ from its standalone volatility.[9]

What attribute of a cat bond can approximately indicate its contribution to the risk of the overall

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[6] See Kreps [5]; this model enjoys widespread popularity in the traditional reinsurance market.
[7] See Lane [6].
[8] See Kozik [4].
[9] Indeed, Kreps [5] states quite explicitly that a price based on standalone standard deviation should be viewed only as an upper bound of the price, whereas the actual market price should be lower.

portfolio?[10] In the context of property catastrophe risk, it seems that different perils ought to behave independently of one another; thus, we would expect virtually no connection or correlation between losses on a bond covering Southeast USA Hurricane and losses on a bond covering California Earthquake. At the same time, two bonds that both cover California Earthquake would likely tend to be correlated—if there's a loss on one bond, there will likely be a loss on the second bond as well. So a bond's covered "peril and geographical zone" (often "peril" for short), such as Southeast USA Wind, California Earthquake, etc., ought to be important for understanding a bond's contribution to the risk of the total portfolio.

3.2 Initial Hypothesis

As a result of the discussion above, our initial hypothesis is that cat bond pricing ought to conform to the following model:

$$\text{Spread \%} = \text{expected loss \%} + \text{peril specific required rate of return on capital \%} \qquad (3.5)$$

$$\text{Spread \%} = \text{expected loss \%} + \text{peril specific margin \%} \qquad (3.6)$$

Such a model states that a bond's spread over LIBOR must be large enough to cover the expected loss and also provide an "additional rate of return on capital" to compensate for the bond's contribution to total portfolio risk, which varies based upon the covered peril. One advantage of this type of model, in which the compensation for risk is expressed as an additional rate of return, is its similarity to other bond market models.[11] In addition, such a model would satisfy the intuition of practitioners that:

1. If a bond's expected loss is small, then the spread's "multiple of expected loss" is relatively high.

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[10] This "overall portfolio" could theoretically be as diverse as the portfolio of all investments opportunities, but we note that taking cat risk may require critical mass of time, money, and expertise, which implies an "overall portfolio" that is concentrated in cat bonds. At the same time, cat bond pricing can be influenced by pricing in the traditional reinsurance market. So the "overall portfolio" in which we evaluate the risk of a cat bond may range from the portfolio of all investment opportunities to the portfolio of all cat bonds to the portfolio of all reinsured exposures; or it may reflect a mixture of these various perspectives. This issue requires further research.

[11] Hull [3].

2. If a bond's expected loss is large, then the spread's "multiple of expected loss" is relatively low.

3.3 Revised Hypothesis

A cursory glance at cat bond market prices, however, shows that this model does not fully describe the data. Rather, as the expected loss increases, not only does the spread increase, but the margin itself (which equals spread minus expected loss) tends to increase as well. This surprising phenomenon occurs in the corporate bond market as well, where it is sometimes referred to as the "credit spread puzzle."[12]

We therefore revise our hypothesis and propose the following model:

1. Spread % = expected loss % + peril specific margin %
2. Peril specific margin % = increasing function of expected loss %

To describe margin as an increasing function of expected loss, we begin with a basic model, a linear relationship:

Peril specific margin % = peril specific flat margin % + peril specific factor * expected loss % (3.7)

Combining all the pieces of equations (3.6) and (3.7), we obtain:

Spread % = expected loss % + peril specific flat margin % + peril specific factor * expected loss % (3.8)

Or, more concisely, we have a straightforward linear function to describe spread:

Spread % = peril specific flat margin % + expected loss % * (1+ peril specific factor) (3.9)

[12] Hull [3].

Or, for each peril, we can say:

$$\text{Spread \%} = \text{constant \%} + \text{loss multiplier} * \text{expected loss \%} \qquad (3.10)$$

Because the variables of "spread," "constant," and "expected loss" are defined as "% of bond amount," we also wish to show the equation in dollar terms. Multiplying both sides of the equation by "bond amount," we obtain:

$$\text{Spread \$} = \text{bond amount \$} * \text{constant \%} + \text{loss multiplier} * \text{expected loss \$} \qquad (3.11)$$

This form of the model clarifies that the total dollar price of risk transfer in the cat bond market is a linear function of expected loss and bond amount.

Similarly, we note that we can rewrite equation (3.11) in the terminology of traditional reinsurance[13]:

$$\text{Premium \$} = \text{aggregate limit \$} * \text{constant \%} + \text{loss multiplier} * \text{expected loss \$} \qquad (3.12)$$

This form of the model clarifies that the total dollar price of risk transfer in the reinsurance market is a linear function of expected loss and aggregate limit.

One favorable aspect of this type of model, as described in equations (3.10), (3.11), and (3.12), is that it satisfies the "no arbitrage principle of pricing" as described by Venter [9]. In contradistinction, a pricing model based on standalone standard deviation violates the principle of "no arbitrage."

3.4 Conjecture on Revised Hypothesis

Why does the original hypothesis, "spread % = constant % + expected loss %," fail? Why do we

[13] Here the "spread" is the "price" (or "premium") and the "bond amount" is the aggregate limit. This analogy holds for reinsurance contracts that have no reinstatement premium and no reinstatement of limit. We require further research to determine how to adapt the cat bond pricing formula for a reinsurance contract with reinstatable limit and premium.

need the revised model, "spread % = constant % + loss multiplier * expected loss %"? Why does the expected loss need to be multiplied by a factor to obtain a viable model for spread? What does the "loss multiplier" represent? On one hand, we do not strictly need to answer these questions; so long as one model does a superior job of approximating, describing, and predicting reality, we should generally choose the better model (all else equal). On the other hand, formulating a reasonable conjecture about why a promising initial model fails, and why a modified model works better, can provide insight and potentially assist us in our later analysis.

Our conjecture is that the "loss multiplier" parameter relates to uncertainty in the expected loss estimate. Recall that we do not actually know the true underlying value of expected loss; rather, cat modeling firms, using computer software, provide values that are merely estimates of the true expected loss.[14] Perhaps if we knew the precise value of expected loss, then we could say that a reasonable model is "spread % = constant % + expected loss %." But given the uncertainty in the estimated expected loss, we must amend the model to say that "spread % = constant % + loss multiplier * expected loss %."[15]

4. ANALYSIS OF EMPIRICAL DATA

In this section, we discuss the underlying data that we use in our analysis and investigate the results of fitting parameters of our proposed model to the empirical data.

4.1 Data and Limitations

When investigating the price of risk in the cat bond market, we can analyze the spreads of bonds "when issued" (when they are first bought by investors in the "primary market") and also later on when the bonds are resold and traded in the "secondary market." Although trading in the secondary market has become more active, many initial investors prefer to buy and hold their bonds to maturity. Thus, we view the pricing of "when issued" bonds to be more informative and robust, whereas the secondary market, still in its formative stages, may not be sufficiently reliable (yet) for analyzing the price of risk. As a result, we use only the data points for cat bonds in the primary market, when they are originally issued.

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[14] A telling manifestation of this loss estimation uncertainty is the fact that various major modeling firms calculate values for expected loss that are sometimes quite different from each other.

[15] The uncertainty of the estimated expected loss may explain the "credit spread puzzle" as well. Several competing explanations for the "credit spread puzzle," however, would not explain our observations in the cat bond market.

The data for this study comprises various tranches of bonds, their expected loss, spreads, and the perils they cover, for the years 1998–2008. Before proceeding to analyze the data, we applied a number of filters to the data. First, because we describe cat bond spreads with peril-specific parameters, we could use only single peril bonds; we excluded from this analysis any bonds that covered more than one peril.[16] This initial filtering left us with approximately 150 useable data points. Next, some bonds can be issued for a longer duration than the time that they are exposed to property cat risk; thus, the published spread, which corresponds to the entire lifespan of the bond, does not correspond to the time that the bond is "on risk." To avoid this problem, we excluded any bonds whose issuance date preceded the inception date by more than 30 days. A similar complication arises when the bond covers a seasonal risk such as Wind: because the risk of cat loss is not uniform throughout the year, there can be a difference between the lifespan of the bond and the amount of time it is "on risk." To deal this problem, we excluded any bonds whose covered peril was Wind and whose duration exceeded a whole number of years by more than 30 days. After applying the various data filters, we began the analysis with 115 data points. We also mapped the data to "issuance year" based on a 12-month period ending June 30; thus the "2008 issuance year" comprises bonds issued between July 1, 2007 and June 30, 2008.[17]

4.2 Results

In this section we use the empirical data to fit the parameters of our proposed model:

$$\text{Spread } \% = \text{constant } \% + \text{loss multiplier} * \text{expected loss } \% \tag{4.1}$$

4.2.1 Wind: USA

We begin by inspecting results for USA Wind.[18] Exhibit 1 shows the fitted parameters:

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[16] With sufficiently detailed information, one could include multi-peril bonds in the analysis; however, because we could not obtain reliable data quantifying how much the various perils contributed to the total expected loss of multi-peril bonds, we could not include these bonds in our analysis.

[17] This mapping is used in AON Capital Markets [1].

[18] The bonds in this category generally cover some combination of Florida, Southeast USA, and/or Northeast USA. We could not split this large category into more granular subcategories.

Exhibit 1

Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Wind	USA	All years	Full cycle	Constant %	3.33%	0.45%	2.38%	4.27%
Wind	USA	All years	Full cycle	Loss Multiplier	2.40	0.17	2.05	2.76

The parameters in Exhibit 1 show that we can approximate the spread (when issued) of any cat bond that covers USA Wind as follows:

Spread % = 3.33% + 2.40 * expected loss %

The model provides an approximation for describing spread; one can use expert judgment to refine the modeled spread by incorporating the many additional factors that influence the actual issuance spread (market conditions, trigger type, etc.).

As noted in Exhibit 1, the regression applies to USA Wind, using all years of data (1998-2008); the time horizon of the historical data covers market conditions ranging from the high prices of a "hard market" to the low prices of a "soft market." We note that the model's intercept (Constant %) and slope (Loss Multiplier) are significant variables.

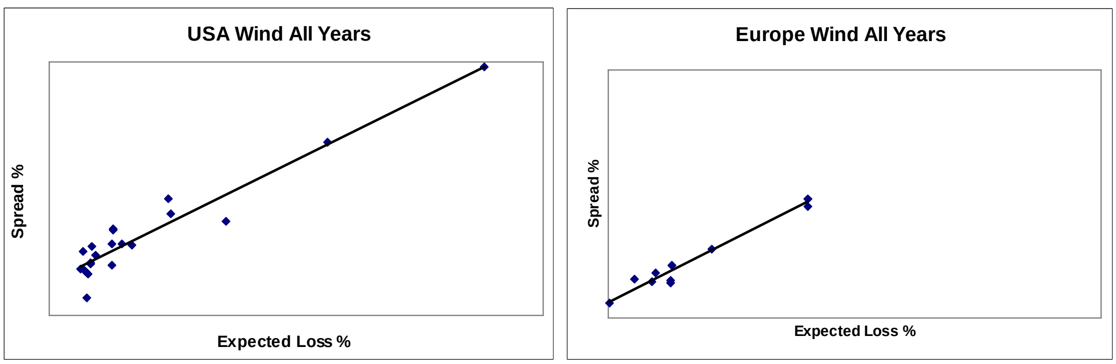
One benefit of having a mathematical model of cat bond pricing is that it allows us to take the wide array of cat bond prices and summarize them in compact form (two variables). Such a model also enables us to compare and contrast price behavior for various different perils, zones, time periods, and market conditions.

4.2.2 USA Wind vs. Europe Wind

We now inspect the results for the peril Wind in two different geographical zones: USA and Europe.

Exhibit 2

Peril	Zone	Years	Market Condition	# of Observations	R Square
Wind	USA	All years	Full cycle	21	91.6%
Wind	Europe	All years	Full cycle	12	96.8%



Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Wind	USA	All years	Full cycle	Constant %	3.33%	0.45%	2.38%	4.27%
Wind	USA	All years	Full cycle	Loss Multiplier	2.40	0.17	2.05	2.76
Wind	Europe	All years	Full cycle	Constant %	1.61%	0.33%	0.88%	2.33%
Wind	Europe	All years	Full cycle	Loss Multiplier	2.49	0.14	2.17	2.81

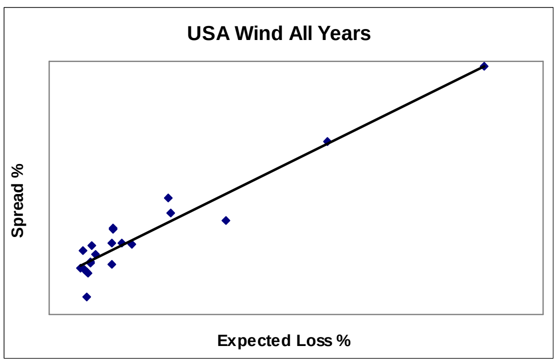
We note that the parameter "Constant %" for Wind is significantly higher for USA than for Europe. Given the very large accumulation of exposure in USA, it is reasonable that USA Wind contributes much more than Europe Wind to the total risk of an overall portfolio; thus the higher value of "Constant %" for USA is consistent with our hypothesis that this parameter relates to the "peril-specific required rate of return on capital." We also note that the second parameter, "Loss Multiplier," is significantly different than unity. Interestingly, the "Loss Multiplier" for Wind does not vary much between USA and Europe. This may suggest a similar magnitude of uncertainty for expected loss estimates for USA Wind and Europe Wind.

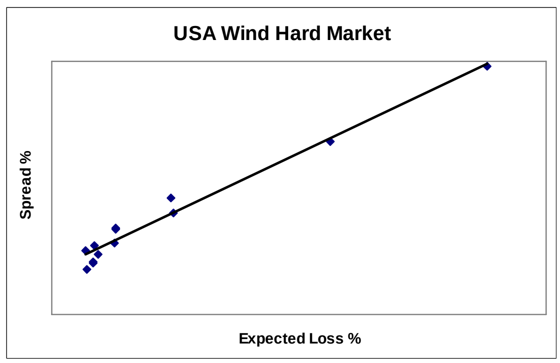
4.2.3 USA Wind All Years vs. USA Wind Hard Market

Having inspected two different zones, we now turn to analyzing two different time periods.

Exhibit 3

Peril	Zone	Years	Market Condition	# of Observations	R Square
Wind	USA	All years	Full cycle	21	91.6%
Wind	USA	2006 - 2007	Hard Market	13	97.3%





Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Wind	USA	All years	Full cycle	Constant %	3.33%	0.45%	2.38%	4.27%
Wind	USA	All years	Full cycle	Loss Multiplier	2.40	0.17	2.05	2.76
Wind	USA	2006 - 2007	Hard Market	Constant %	4.28%	0.37%	3.47%	5.09%
Wind	USA	2006 - 2007	Hard Market	Loss Multiplier	2.33	0.12	2.07	2.58

Exhibit 3 shows results for the peril Wind and the zone USA on two bases:

1. Using all years of data across a full cycle of market conditions.
2. Using the 2006 and 2007 years of data, which correspond to a "hard market," a time

period of increased risk aversion and higher prices.

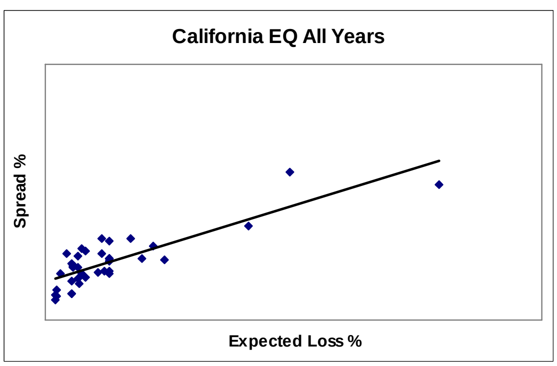
We note that the "Constant %" tends to be significantly higher during the hard market than the all years average, which conforms to our expectations that the required rate of return on capital increases during a hard market. In contrast, the fitted value for "Loss Multiplier" does not vary much for USA Wind between the hard market and the all years average.

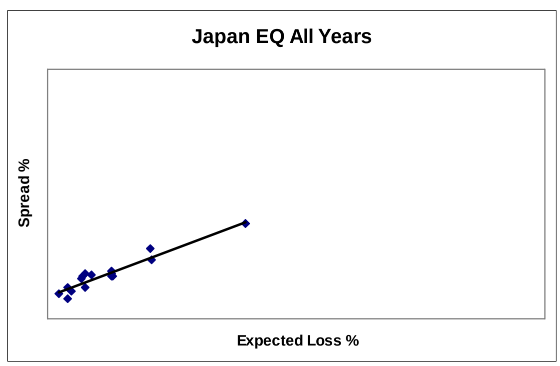
4.2.4 Earthquake: California vs. Japan

We now turn to the other major catastrophic peril, Earthquake (EQ). The exhibit below shows a comparison between California EQ and Japan EQ:

Exhibit 4

Peril	Zone	Years	Market Condition	# of Observations	R Square
Earthquake	California	All years	Full cycle	42	69.4%
Earthquake	Japan	All years	Full cycle	18	93.9%





Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Earthquake	California	All years	Full cycle	Constant %	3.78%	0.29%	3.19%	4.36%
Earthquake	California	All years	Full cycle	Loss Multiplier	1.48	0.16	1.16	1.79
Earthquake	Japan	All years	Full cycle	Constant %	2.28%	0.20%	1.85%	2.70%
Earthquake	Japan	All years	Full cycle	Loss Multiplier	1.85	0.12	1.60	2.10

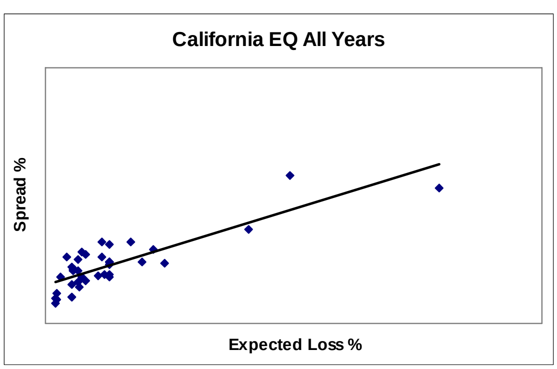
California, with its peak level of exposure accumulation, has a significantly higher "Constant %" than Japan. The value of the parameter "Loss Multiplier" does vary between USA and Japan, although the difference is not as significant as the difference in the "Constant %" parameter.

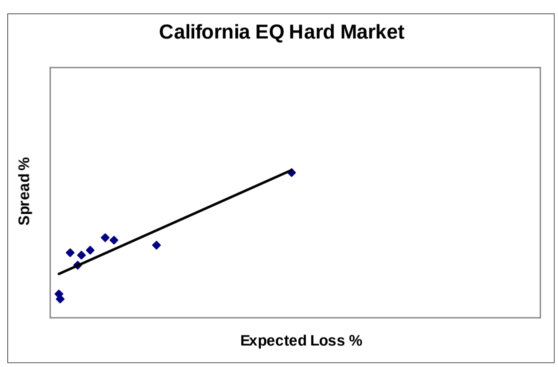
4.2.5 California EQ All Years vs. California EQ Hard Market

We now analyze EQ pricing during different time periods.

Exhibit 5

Peril	Zone	Years	Market Condition	# of Observations	R Square
Earthquake	California	All years	Full cycle	42	69.4%
Earthquake	California	2006 - 2007	Hard Market	10	85.1%





Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Earthquake	California	All years	Full cycle	Constant %	3.78%	0.29%	3.19%	4.36%
Earthquake	California	All years	Full cycle	Loss Multiplier	1.48	0.16	1.16	1.79
Earthquake	California	2006 - 2007	Hard Market	Constant %	4.40%	0.55%	3.12%	5.67%
Earthquake	California	2006 - 2007	Hard Market	Loss Multiplier	2.04	0.30	1.34	2.73

Exhibit 5 shows that for California Earthquake, the "Constant %" and the "Loss Multiplier" both increased significantly during the hard market of 2006–2007 relative to the all years average.

4.2.6 Wind and EQ, USA and Europe, California and Japan

We now examine our results for Wind and EQ in one combined context:

Exhibit 6a

Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Wind	USA	All Years	Full Cycle	Constant %	3.33%	0.45%	2.38%	4.27%
Wind	USA	All Years	Full Cycle	Loss Multiplier	2.40	0.17	2.05	2.76
Wind	Europe	All Years	Full Cycle	Constant %	1.61%	0.33%	0.88%	2.33%
Wind	Europe	All Years	Full Cycle	Loss Multiplier	2.49	0.14	2.17	2.81
Earthquake	California	All Years	Full Cycle	Constant %	3.78%	0.29%	3.19%	4.36%
Earthquake	California	All Years	Full Cycle	Loss Multiplier	1.48	0.16	1.16	1.79
Earthquake	Japan	All Years	Full Cycle	Constant %	2.28%	0.20%	1.85%	2.70%
Earthquake	Japan	All Years	Full Cycle	Loss Multiplier	1.85	0.12	1.60	2.10

Exhibit 6a displays the critical parameters that summarize the behavior of four major peril/zone combinations: USA Wind, Europe Wind, California EQ, and Japan EQ. Each unique combination of peril and zone contributes in a different way to the risk of the total portfolio; thus each peril/zone requires its own linear model with different parameters. Despite the differences in the models, however, there appear to be some similarities. We begin by focusing on the "Constant %"

parameter of the linear models:

Exhibit 6b

Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Wind	USA	All Years	Full Cycle	Constant %	3.33%	0.45%	2.38%	4.27%
Earthquake	California	All Years	Full Cycle	Constant %	3.78%	0.29%	3.19%	4.36%
Wind	Europe	All Years	Full Cycle	Constant %	1.61%	0.33%	0.88%	2.33%
Earthquake	Japan	All Years	Full Cycle	Constant %	2.28%	0.20%	1.85%	2.70%

Exhibit 6b shows that for the parameter "Constant %," which is the intercept of the linear models, the values for the "peak" perils/zones of USA Wind and California EQ are quite similar. Additionally, the values for the significant yet "non-peak" perils of Europe Wind and Japan EQ are simultaneously similar to each other and also dissimilar to the values for the two peak perils.[19] This phenomenon is consistent with our hypothesis that the "Constant %" relates to the "required rate of return on capital": peak zones with the largest accumulation of exposure tend to contribute the most to the total portfolio risk and thus ought to have the highest "required rate of return on capital"; non-peak zones, which have less acute accumulation of exposure, tend to correlate less directly with the overall portfolio, will receive some credit for their diversification effect, and will have lower "required rate of return on capital."

We now turn to the second parameter of the model, "Loss Multiplier."

[19] See MMC Securities [8], which categorizes perils/zones into three major buckets:
1. "Peak" (USA Wind and USA EQ)
2. "Non-Peak" (Europe Wind and Japan EQ)
3. "Pure Diversifying Perils" (other perils such as Australia EQ, Mexico EQ, and Japan Wind)

Exhibit 6c

Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Wind	USA	All Years	Full Cycle	Loss Multiplier	2.40	0.17	2.05	2.76
Wind	Europe	All Years	Full Cycle	Loss Multiplier	2.49	0.14	2.17	2.81
Earthquake	California	All Years	Full Cycle	Loss Multiplier	1.48	0.16	1.16	1.79
Earthquake	Japan	All Years	Full Cycle	Loss Multiplier	1.85	0.12	1.60	2.10

In exhibit 6c, parameter values are similar not based on "peak" and "non-peak" but rather they are similar based on geophysical peril. The value for Loss Multiplier for the peril Wind hardly varies, whether in the USA zone or in the Europe zone. In addition, the value for Loss Multiplier for the peril Earthquake for the California zone is somewhat similar to its value for the Japan zone; moreover, these values for Earthquake are dissimilar to the values for Wind. Returning once again to our conjecture: if the Loss Multiplier is greater than 1.0 because of the uncertainty in the cat model's estimated expected loss, then this uncertainty would likely be similar within a common peril (Wind) and likely dissimilar across different perils (Wind vs. EQ).

Until now we have advocated the use of an individual linear model for each unique combination of major peril and zone, as described in Exhibit 6a; using two parameters to describe each major peril/zone combination, we have a total of eight parameters to describe cat bond pricing for these major perils. However, our discussion of the partial similarities of the linear models (the intercept is similar by zone, the slope is similar by peril) suggests the possibility of combining the various peril/zone combinations into one single linear model. We've seen that the Loss Multiplier varies by peril (Wind versus EQ) and that the Constant % varies by zone (peak USA Wind and California EQ, versus non-peak Europe Wind and Japan EQ). So a single linear model combining all the individual linear models ought to be:

$$\begin{aligned}\text{Spread \%} = {} & \text{Constant}_{\text{All}}\ \% \\ & + \text{Additional Constant}_{\text{Peak}}\ \% * \text{Peak Peril Indicator} \\ & + \text{Loss Multiplier}_{\text{EQ}} * \text{Expected Loss}_{\text{EQ}}\ \% \\ & + \text{Loss Multiplier}_{\text{Wind}} * \text{Expected Loss}_{\text{Wind}}\ \%\end{aligned} \tag{4.2}$$

For this model, we assign each data point's expected loss to either EQ or Wind. We also use an indicator variable to classify the data point as peak or non-peak (1 or 0). Now we can include all data points from single peril bonds covering USA Wind, California EQ, Europe Wind, and Japan EQ in one model and fit the parameters:

Exhibit 7a

Peril	Zone	Years	Market Condition	# of Observations	R Square	Adjusted R Square
Multiple	Multiple	All years	Full cycle	93	87.3%	86.9%

Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Multiple	Multiple	All Years	Full Cycle	$\text{Constant}_{\text{All}}$ %	2.31%	0.26%	1.79%	2.83%
Multiple	Multiple	All Years	Full Cycle	Additional $\text{Constant}_{\text{Peak}}$ %	1.24%	0.28%	0.70%	1.79%
Multiple	Multiple	All Years	Full Cycle	Loss $\text{Multiplier}_{\text{EQ}}$	1.63	0.11	1.41	1.85
Multiple	Multiple	All Years	Full Cycle	Loss $\text{Multiplier}_{\text{Wind}}$	2.32	0.10	2.12	2.52

Exhibit 7a shows how the spread varies based upon expected loss, peril (Wind vs. EQ), and zone (Peak vs. Non-Peak). The intercept of the line for a non-peak zone is $\text{Constant}_{\text{All}}$ %, whereas the intercept for a peak zone is the sum of $\text{Constant}_{\text{All}}$ % and Additional $\text{Constant}_{\text{Peak}}$ %; thus peak zones have a larger intercept value. The slope of the line depends upon the Loss Multiplier, which varies by peril; thus the slope of the line is steeper for Wind than for Earthquake.

We can also examine such a model for a restricted time period, when market conditions are more homogeneous:

Exhibit 7b

Peril	Zone	Years	Market Condition	# of Observations	R Square	Adjusted R Square
Multiple	Multiple	2006 - 2007	Hard Market	32	95.7%	95.3%

Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
Multiple	Multiple	2006 - 2007	Hard Market	Constant_{All} %	2.07%	0.41%	1.23%	2.91%
Multiple	Multiple	2006 - 2007	Hard Market	Additional Constant_{Peak} %	2.30%	0.38%	1.51%	3.09%
Multiple	Multiple	2006 - 2007	Hard Market	Loss Multiplier_{EQ}	1.94	0.14	1.65	2.24
Multiple	Multiple	2006 - 2007	Hard Market	Loss Multiplier_{Wind}	2.34	0.09	2.15	2.53

Exhibit 7b, together with Exhibit 7a, shows that when using the data of the hard market years of 2006–2007, the parameters of the linear model change in various ways. The parameter "Additional Constant_{Peak} %," which reflects the incremental additional price for peak zones, roughly doubles, from a 1.17% all years average to a hard market value of 2.30%; meanwhile, the parameter "Constant_{All} %", which serves as the intercept for "non-peak" zones, hardly changes. The parameter "Loss Multiplier_{Wind}," which already has a high value for the all years data, does not change when fitted to the hard market data; "Loss Multiplier_{EQ}," which has a lower prevailing value for the all years data, increases significantly when fitted to hard market data.

4.2.7 All Perils

When we describe bond spreads using individual linear models, there are certain peril/zone combinations that will not have sufficient data to support reliable parameters. For example, Australia EQ, Mexico EQ, Mediterranean EQ, and Japan Wind are some of the perils for which we do not have enough data points to support standalone linear price functions. However, one of the advantages of a "combined" model such as equation (4.2) is the ability to include many of these ancillary perils in one overall linear function. In order to do so, we first note that these ancillary perils are likely less correlated with the overall portfolio than "peak" and "non-peak" perils; as a result, they ought to have a lower required rate of return on capital and thus a materially different value for "Constant %." We thus expand our categories of perils to three buckets[20]:

1. Peak (USA Wind, California EQ)
2. Non-peak (Europe Wind, Japan EQ)
3. Diversifying (Japan Wind, Australia EQ, Mexico EQ, Mediterranean EQ, Central USA EQ, and Pacific Northwest USA EQ)

We now can augment equation (4.2) to apply to all peril/zone combinations, as follows:

$$\begin{aligned}\text{Spread \%} = {} & \text{Constant}_{\text{All}}\ \% \\ & + \text{Additional Constant}_{\text{Peak}}\ \% * \text{Peak Peril Indicator} \\ & + \text{Additional Constant}_{\text{Diversifying}}\ \% * \text{Diversifying Peril Indicator} \\ & + \text{Loss Multiplier}_{\text{EQ}} * \text{Expected Loss}_{\text{EQ}}\ \% \\ & + \text{Loss Multiplier}_{\text{Wind}} * \text{Expected Loss}_{\text{Wind}}\ \%\end{aligned} \qquad (4.3)$$

In equation (4.3), the slope of the linear price function depends on whether the covered peril is EQ or Wind; the intercept depends upon the peril/zone being peak, non-peak, or diversifying.

For a diversifying peril, the intercept is the sum of "$\text{Constant}_{\text{All}}$ %" and "Additional $\text{Constant}_{\text{Diversifying}}$ %." We expect that the parameter value for "Additional $\text{Constant}_{\text{Diversifying}}$ %" should be negative, because a diversifying peril should have a lower required rate of return on capital and thus a lower intercept than other perils.

Exhibit 8a below shows the results of fitting parameters to the model in equation (4.3), using data from bonds covering all perils/zones:

[20] See MMC Securities [8].

Exhibit 8a

Peril	Zone	Years	Market Condition	# of Observations	R Square	Adjusted R Square
All	All	All years	Full cycle	115	87.4%	87.0%

Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
All	All	All Years	Full Cycle	$\text{Constant}_{\text{All}}$ %	2.35%	0.25%	1.85%	2.85%
All	All	All Years	Full Cycle	Additional $\text{Constant}_{\text{Peak}}$ %	1.28%	0.27%	0.76%	1.81%
All	All	All Years	Full Cycle	Additional $\text{Constant}_{\text{Diversifying}}$ %	-1.09%	0.35%	-1.79%	-0.39%
All	All	All Years	Full Cycle	Loss $\text{Multiplier}_{\text{EQ}}$	1.60	0.10	1.40	1.81
All	All	All Years	Full Cycle	Loss $\text{Multiplier}_{\text{Wind}}$	2.29	0.10	2.10	2.48

The parameters displayed in Exhibit 8a describe the spreads of property cat bonds covering all perils and zones. They tell us that one can approximate the spread of any "single peril" cat bond by taking the product of the expected loss and a "Loss Multiplier" (which depends on whether the peril is Wind or EQ) and then adding a "Constant %" (which depends upon the whether the covered peril/zone is "peak," "non-peak," or "diversifying").[21] Exhibit 8a also confirms our expectations that the linear function for a diversifying peril has a significant additional negative parameter (Additional $\text{Constant}_{\text{Diversifying}}$ %) and thus a lower intercept than other perils.

We now inspect the results of fitting parameters to the same model but using the more homogenous market conditions prevalent during the hard market years of 2006–2007:

[21] The equation should be easily extendable to apply to multi-peril bonds as well.

Exhibit 8b

Peril	Zone	Years	Market Condition	# of Observations	R Square	Adjusted R Square
All	All	2006 - 2007	Hard Market	43	95.5%	95.1%

Peril	Zone	Years	Market Condition	Parameter Name	Parameter Value	Standard Error	Confidence Interval (95%) Lower Bound	Confidence Interval (95%) Upper Bound
All	All	2006 - 2007	Hard Market	$\text{Constant}_{\text{All}}$ %	2.20%	0.40%	1.38%	3.02%
All	All	2006 - 2007	Hard Market	Additional $\text{Constant}_{\text{Peak}}$ %	2.31%	0.38%	1.54%	3.08%
All	All	2006 - 2007	Hard Market	Additional $\text{Constant}_{\text{Diversifying}}$ %	-1.66%	0.45%	-2.56%	-0.76%
All	All	2006 - 2007	Hard Market	Loss $\text{Multiplier}_{\text{EQ}}$	1.87	0.13	1.60	2.14
All	All	2006 - 2007	Hard Market	Loss $\text{Multiplier}_{\text{Wind}}$	2.31	0.09	2.12	2.50

Exhibit 8b, together with Exhibit 8a, shows that when using the data of the hard market years of 2006–2007, the parameters of the model change in various ways. The parameter "Additional $\text{Constant}_{\text{Peak}}$ %," which reflects the incremental additional price for peak zones, increases sharply. The parameter "$\text{Constant}_{\text{All}}$ %," which serves as the intercept for non-peak zones, hardly changes. Finally, the parameter "Additional $\text{Constant}_{\text{Diversifying}}$ %" becomes even more negative when using hard market data, implying that the price of a "diversifying peril" is lower when the additional cost of peak perils is higher; stated differently, the "benefit" of a diversifying peril is larger when the incremental cost of peak perils is larger. However, the large standard error for this negative parameter indicates that this change may not be significant, so this issue requires further investigation. Finally, we note that the parameter "Loss $\text{Multiplier}_{\text{EQ}}$" increases when fitted to hard market data, while the parameter "Loss $\text{Multiplier}_{\text{Wind}}$" does not change. While the future seldom duplicates the past, these results may provide some hints about how key pricing parameters may behave during future hard markets.

5. AREAS FOR FURTHER RESEARCH

Some areas for further research are as follows:

1. Our analysis uses simple regression, which weights all squared errors equally. Future

research may consider a linear model that allows for varying weights on the squared error terms when fitting parameters.

2. Because of data limitations, we included only single peril bonds in our analysis. For multi-peril bonds, one requires information about the amount of expected loss that various perils and zones contribute to the total expected loss. With such data, one can include price information from multi-peril bonds when selecting models and fitting parameters. One could also then quantify to what extent (if any) a multi-peril bond suffers a price penalty relative to what the price "should have been" based on its expected loss and covered perils. Such a model could help quantify the tradeoff of sponsoring several bonds that each cover a single peril (e.g., better price but higher transactional costs) versus the advantages of sponsoring one bond covering multiple perils (e.g., worse price but lower transactional costs).

3. The parameters of the proposed linear model tend to vary based on market conditions, which are constantly changing. With sufficient data, one may be able to fit parameters to many incremental time periods and produce a time series of fitted parameters; such a data set would allow one to analyze how the parameters drift over time. If one could identify the catalysts that drive the changes in the parameters over time, one could develop a forward-looking model that predicts the likely values of the key parameters of the price function for the next time period.

4. Our focus thus far has been on the price of transferring cat risk via the cat bond market. What about the price of transferring cat risk in the traditional reinsurance market? We note that reinsurance contracts, which typically have reinstatable limit and premium, have different contractual features than cat bonds. Still, would some form of linear model adequately capture the market price of reinsurance contracts? Would such a model for the price of reinsurance contracts be similar or dissimilar to the model for cat bond prices? What would the similarity or dissimilarity of these models tell us? What would these models tell us about which types of cat risk are best handled via balance sheet equity capital, reinsurance capital, and cat bond capital? For example, our analysis suggests that two forces affect the price of cat risk in the cat bond market: the first factor is required rate of return on capital, and the second factor is uncertainty in the estimate of expected loss. Now, the broad asset portfolios that hold cat bonds may provide excellent diversification, which may lower the required rate of return on capital and reduce the

price of risk transfer for cat bonds. But the uncertainty in the estimated expected loss raises the price of risk transfer for cat bonds. So cat bonds may be relatively attractive in situations in which price is dominated by the "required rate of return" factor, but not in situations in which price is dominated by the "uncertainty in the expected loss" factor. This suggests that cat bonds will likely continue to be relevant mainly for cat layers that have low expected loss and/or cover peak perils, whereas other forms of capital may be preferable in other situations. The implication is that insurers may be able to enhance their capital structure by mixing together equity capital, reinsurance capital, and cat bond capital in an optimal combination.

6. CONCLUSIONS

In this paper, we describe the market clearing price of cat bonds by modeling cat bond spreads as a linear function of the bonds' expected loss. This relationship between spread and expected loss, however, differs by cat peril and geographic zone; each unique combination of peril and zone sports its own "price line" with a different intercept and slope. We also present an approach that combines these individual models into one unified model. Whether using individual models or a combined model, the parameters change over time as market conditions change. We hypothesize that the key parameters in the linear models relate to two main drivers of price: required rate of return on capital and uncertainty of the expected loss. These two factors provide a roadmap for indentifying situations that are most suitable for reinsurance versus cat bonds and vice versa. We also note that the factor relating to uncertainty of the expected loss may help explain the broader issue of the "credit spread puzzle," which appears in the corporate bond market.

Using the proposed linear models, we can compare the market clearing price functions for cat bonds for various perils and zones, how they compare and contrast to each other, and how they change over time. Such models help us understand the drivers of the price of cat risk and help us describe how prices have behaved in the past and, potentially, how they may behave in the future.

Acknowledgment

The authors thank all those who commented on earlier drafts of this paper.

7. REFERENCES

[1] AON Capital Markets, "Insurance Linked Securities 2008,"

http://www.aon.com/attachments/capitalmarketsreport.pdf.
[2] Gatumel, M., and D. Guégan, "Towards an Understanding Approach of the Insurance Linked Securities Market," Presented at the 2008 ASTIN Colloquium, July 13-18, 2008, Manchester, United Kingdom.
[3] Hull, J., M. Predescu, and A. White, "Bond Prices, Default Probabilities and Risk Premiums," *Journal of Credit Risk* 1:2, 2005, pp. 53-60.
[4] Kozik, T., and A. Larson, "The *N*-Moment Insurance CAPM," *Proceedings of the Casualty Actuarial Society* LXXXVIII, 2001, pp. 39-63.
[5] Kreps, R., "Investment-Equivalent Reinsurance Pricing," Chapter 6 in *Actuarial Considerations Regarding Risk and Return In Property-Casualty Insurance Pricing,* (Arlington, Va.: Casualty Actuarial Society, 1999) pp. 77-104.
[6] Lane, M., "Pricing Risk Transfer Transactions," *ASTIN Bulletin* 30:2, 2000, pp. 259-293.
[7] Lane, M., and O. Mahul, "Catastrophe Risk Pricing: An Empirical Analysis," (November 1, 2008) World Bank Policy Research Working Paper Series, http://ssrn.com/abstract=1297804.
[8] MMC Securities, "The Catastrophe Bond Market at Year-End 2006: Ripples into Waves," March 2007.
[9] Venter, G., "Premium Calculation Implications of Reinsurance Without Arbitrage," *ASTIN Bulletin* 21:2, 1991, pp. 223-230.

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Common Pitfalls and Practical Considerations in Risk Transfer Analysis

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The current papers available on risk transfer have provided background and a general description of the tools available for analysis. Risk transfer analysis has many nuances that can trip up an actuary testing a contract. This paper discusses several of these pitfalls and provides direction on how to address them based on previously published materials from the accounting boards, the American Academy of Actuaries (AAA), and the Casualty Actuarial Society (CAS). This paper also addresses several outstanding risk transfer concerns that have no easy answers. While these issues do not have obvious solutions, the intent of the paper is to shed some light on these topics and open the door for further discussion.

To facilitate the discussion of these common pitfalls and practical considerations two example contracts are reviewed with an Expected Reinsurer Deficit (ERD) calculated for both.

Keywords: Risk transfer, Expected Reinsurer Deficit (ERD), FAS 113, Reinsurance Attestation Supplement (RAS), SSAP 62.

1. INTRODUCTION

Current papers available on risk transfer have provided background and a general description of the tools available for analysis. However, risk transfer analysis has many seemingly minor nuances that can trip up an actuary testing a contract. In this paper, we will discuss several of these pitfalls and provide direction on how to address them based on previously published materials from the accounting boards, the American Academy of Actuaries (AAA), and the Casualty Actuarial Society (CAS). We will also highlight a number of practical considerations that have not received as much attention in the available literature. While these practical considerations do not have obvious solutions, we hope to shed some light on the available options and open the door for further discussion on the topic.

1.1 Risk Transfer in Current Literature

This discussion is derived from a review of existing risk transfer literature, most notably "Reinsurance Attestation Supplement 20-1: Risk Transfer Testing Practice Note" from the AAA Committee on Property and Liability Financial Reporting and "Risk Transfer Testing of Reinsurance Contracts: Analysis and Recommendations" from the CAS Research Working Party on Risk Transfer Testing [1][2]. We also relied heavily on the accounting standards, Financial Accounting Standard No. 113, "Considerations in Risk Transfer Testing" (FAS 113) and SSAP 62, "Property and Casualty Reinsurance." While some discussion of the CAS Working Party paper and the AAA Practice Note is necessary, this paper is an attempt to go beyond the framework provided in the current literature and review the more routine issues faced by actuaries in reviewing reinsurance transactions for risk transfer.

1.2 Objective

In this paper, we will discuss several pitfalls and practical considerations with risk transfer analyses. We will provide direction on how to address the pitfalls based on previously published materials and we hope to shed some light on the available options concerning the practical considerations and open the door for further discussion on the topics.

1.3 Outline

In Section 2 of this paper we will present a brief history and background of risk transfer, including a discussion of the terms "substantially all" and "self-evident," as well as discussion on measuring risk transfer and risk transfer thresholds.

Section 3 will contain a discussion on the pitfalls and practical considerations. We will start by showing two sample contracts that will be used as a basis for much of the discussion, and how to analyze risk transfer. Next we will cover various pitfalls, including discussion on the following topics:

- Profit Commissions
- Reinsurer Expenses
- Interest Rates and Discount Factors
- Premiums
- Evaluation Date
- Commutation and Timing of Payments

In the last part of Section 3, we will highlight some of the practical considerations in risk transfer testing, including discussion on:

- Parameter Selection
- Interest Rate
- Payment Pattern
- Loss Distribution

- Parameter Risk
- Use of Pricing Assumptions
- Commutation Clauses

The fourth and final section of the paper will contain a short wrap up, conclusions and a reminder that risk transfer testing is a principle-based exercise and not just a "plug and chug" methodological exercise.

2. BRIEF HISTORY OF RISK TRANSFER

Since the reinsurance goals of ceding companies are as different as the risks reinsured, reinsurance contracts contain a variety of terms and conditions that can impact the economic structure of the reinsurance transaction. When a contract qualifies as reinsurance there are certain accounting benefits that a ceding company can realize.

The demonstration of risk transfer for reinsurance is required by FAS 113 in order for the contract to receive reinsurance accounting treatment under Generally Accepted Accounting Principles (GAAP). Statutory Accounting Principles (SAP) defined in SSAP 62 are similar in guidance to FAS 113. Generally, both standards for risk transfer require that:

1. The reinsurer assumes significant insurance risk under the reinsured portion of the underlying insurance agreement; and
2. It is reasonably possible that the reinsurer may realize a significant loss from the transaction.

Because the terms "significant insurance risk," "reasonably possible," and "significant loss" are not defined in either accounting standard, the challenge is to appropriately interpret and apply the accounting standards to each reinsurance transaction.

The abuses of the past several years in the use of finite reinsurance contracts have highlighted the need to document and quantify risk transfer. An increase in scrutiny of reinsurance contracts led to the introduction of the "Reinsurance Attestation Supplement," in the 2005 NAIC Annual Statement.

The supplement requires the chief executive officer (CEO) and chief financial officer (CFO) to confirm that:

1. There are no separate written or oral agreements between the reporting entity and assuming

reinsurer.

2. There is documentation for every reinsurance contract for which risk transfer is not reasonably self-evident that details the transaction's economic intent and that documentation evidencing risk transfer is available for review.

3. The reporting entity complies with all requirements set forth in the Statement of Statutory Accounting Principles No. 62, "Property and Casualty Reinsurance" (SSAP 62).

4. The appropriate controls are in place to monitor the use of reinsurance.

CEOs and CFOs have the responsibility to attest to risk transfer in reinsurance transactions. However, since actuaries are uniquely qualified to quantify and evaluate risk transfer, they are increasingly being called upon to quantify risk transfer and provide the necessary documentation.

As mentioned above, GAAP and SAP accounting standards contain similar wording about what is required for risk transfer to be present. Most notably, both require the presence of insurance risk. Insurance risk has two components, underwriting risk and timing risk. If both of these types of risk are not present, then insurance risk has not been transferred. While risk transfer is independently defined in each standard, we are unaware of any examples of a contract that would meet the requirements of one standard, but not the other. Contracts that qualify according to one standard are generally considered to meet the requirements of the other standard as well.

2.1 One Exemption from Risk Transfer Requirements – "Substantially All"

Both GAAP and SAP accounting standards specifically require that it be reasonably possible that the reinsurer may realize a significant loss from the transaction, except in cases where the reinsurer meets the "substantially all" requirement. This is meant to exempt a very narrow definition of contracts where the reinsurer assumes "substantially all of the insurance risk relating to the reinsured portions of the underlying insurance contracts." The most common examples are straight quota share or individual risk contracts with no loss ratio caps or other risk limiting features. The reason for this exemption is that it allows companies to acquire qualifying reinsurance on inherently profitable books of business where it may not be reasonably possible that the reinsurer will realize a significant loss.

2.2 Required Risk Transfer Documentation and Reasonably Self-Evident

When the NAIC introduced the "Reinsurance Attestation Supplement" (RAS) in 2005 they also introduced a new term to the risk transfer lexicon, "reasonably self-evident." The RAS requires documentation "for every reinsurance contract for which risk transfer is not reasonably self-evident." This classification of contracts is meant to reduce the need to rigorously test every reinsurance contract for risk transfer. Unfortunately, very little guidance was offered on what "reasonably-self evident" encompasses. The AAA Practice Note followed the introduction of the RAS and laid out some general guidelines for establishing when the presence of risk transfer is reasonably self-evident. The guidelines were general in nature and provided characteristics to look for in contracts to determine when risk transfer is reasonably self-evident and when it is not.

The CAS Working Party paper took these guidelines one step further and provided a list of specific contract categories where risk transfer is reasonably self-evident based on meeting a 1% Expected Reinsurer Deficit (ERD) threshold. They point out that this list is preliminary and expect it could be considerably expanded. They also point out that there are exceptions to the list, such as when a contract looks contrived. We feel that it can be dangerous to attempt to codify this terminology with explicit definitions. Every contract is different and must have its terms thoroughly reviewed.

Specifically, the CAS Working Party paper lists a couple of categories that we do not agree are always reasonably self-evident such as individual risk contracts and certain long tail excess of loss treaties. Individual risk treaties with no significant risk limiting features would likely be exempt from the accounting standards since the reinsurer assumes "substantially all" of the underlying risk. For individual risk contracts that do not qualify for this exemption, it is not hard to imagine special features that would restrict risk transfer.

For long tail excess of loss treaties, the CAS Working Party paper provides a few numerical qualifications to meet the reasonably self-evident standard. For excess of loss contracts that are not on short tail exposures, the CAS Working Party paper finds that any contract with aggregate limits no less than one per occurrence limit or twice the premium, meets the reasonably self-evident criteria if there are no ceding commissions and the rate on line is below 500%. It is not difficult to construct a contract around these parameters that clearly does not transfer risk. An extreme example would be a single doctor paying $1M for a $1M x $5M medical malpractice treaty with a $2M aggregate limit. This contract passes the established criteria for the risk transfer to be reasonably self-evident, but I think most would agree that not enough risk is transferred in this contract for it to qualify as reinsurance. This is obviously an unrealistic example, but it shows how applying specific parameters on the terminology can lead to unintended results.

The RAS requires documentation "for every reinsurance contract for which risk transfer is not reasonably self-evident." It seems obvious that any contract requiring a more rigorous review would also require documentation for the model results. However, it is our recommendation that documentation be kept on all reinsurance contracts reviewed for risk transfer. We think it is valuable to have documentation for those contracts found to be exempt for any reason, although the most notable are those that meet the "substantially all" clause. We find it to be just as important to document any contract where the risk transfer is found to be reasonably self-evident. While the term reasonably self-evident might lead one to believe the conclusion is obvious and anyone who picks up the contract will reach the same conclusion, not all contracts that meet this standard are clear cut. This is of particular importance if you are using any reference, such as the previously discussed list from the CAS Working Party Paper, to make your determination. The AAA Practice Note also recommends keeping documentation for reasonably self-evident contracts. The practice note also includes several example checklists in the appendix from companies who have made this type of documentation standard.

2.3 Selected Risk Measuring Method – Expected Reinsurer Deficit (ERD)

Neither SSAP 62 nor FAS 113 provide a clear numeric trigger of when risk transfer fails. The "10-10" rule was developed as a benchmark to give meaning to the criteria in the two accounting standards. The "10-10" rule says that a reinsurance contract exhibits risk transfer if there is at least a 10% chance of a 10% or greater loss for the reinsurer.

Another method that has gained acceptance and overcomes some shortcomings of the "10-10" rule is the Expected Reinsurer Deficit (ERD). ERD can be viewed as the probability of a net present value (NPV) underwriting loss for the reinsurer multiplied by the NPV of the average severity of the underwriting loss. A treaty is typically considered to exhibit risk transfer if ERD is greater than 1%, which is consistent with the "10-10" rule (10% loss multiplied by 10% chance is a 1% ERD). Therefore, contracts that qualify for risk transfer under the "10-10" rule generally qualify under a 1% ERD. We will discuss thresholds more in the next section.

ERD has not been explicitly endorsed by any professional body. However, while the CAS Working Party paper stopped short of endorsing ERD, they did prefer its use as a de facto standard over the "10-10" rule. There are a handful of other methods, but none of them are as widely used as the two previously mentioned. Some methods, such as Value at Risk (VaR) and Tail Value at Risk (TVaR) are generalizations of methodologies we have already discussed. Others, such as the

Right Tail Deviation (RTD) method by Wang outlined in the CAS practice note, have not caught on due to the complexity of the model [4][5]. There are also methods, such as the Risk Coverage Ratio (RCR) by Ruhm, which have not caught on due to the exclusion of key variables [3]. RCR does an adequate job of evaluating risk in the losses that are transferred, but it does not make any comparison to premium.

In this paper we will test for risk transfer using a simple cash flow simulation and calculating the Expected Reinsurer Deficit (ERD). While some of these other measures could be used in our example analysis we will use only ERD in the interest of consistency.

2.4 Risk Transfer Thresholds

The CAS Working Party paper began some brief discussion about what the appropriate guideline threshold percentage should be and suggested that further research be done. Currently, because it is consistent with the "10-10" rule, the most commonly recognized threshold for ERD is 1%. Some have suggested that a 2% threshold would be more appropriate. Our recommendation is to continue using the 1% threshold until a more thorough analysis suggests otherwise. Using 2% would be a more stringent guideline, but the 2% threshold does not appear to be any less arbitrary than the current 1% threshold. While the 1% threshold is based on the somewhat arbitrary "10-10" rule, there is some reasoning behind it. The "10-10" rule was loosely derived from the accounting standard language that required that the reinsurer face a "reasonable chance of a significant loss." For the purposes of risk transfer, it has been commonly accepted that a 10% chance is a "reasonable chance" and that a 10% loss is a "significant loss." From these two accepted values, the ERD of 1% has been derived and this threshold continues to gain acceptance.

The CAS Working Party paper also mentions the possibility of including other requirements, such as a required maximum loss, in order to show risk transfer. We recommend not complicating the methodology with extra arbitrary requirements. While adding a maximum loss requirement may feel intuitive, it begins to complicate the process and makes explaining results to the decision-makers more difficult. Adding requirements can also lead to more engineering of contrived contracts. If a maximum loss is required, any contract can be rewritten to incorporate a rare maximum loss.

3. COMMON PITFALLS AND PRACTICAL CONSIDERATIONS DISCUSSION

In order to illustrate the common pitfalls that can affect a risk transfer analysis it is first important to demonstrate how a basic risk transfer analysis is completed, highlighting many of the issues that can surface along the way. Many of the pitfalls referenced in this section are further emphasized later in the paper.

To demonstrate risk transfer analysis two reinsurance contracts are used. Contract #1 is a quota share contract while Contract #2 is an excess of loss contract.

The terms for Contract #1 are summarized in Table 1:

Table 1 - Summary of Terms - Contract #1

Inception Date	1/1/2008
Estimated Subject Premium	10,000,000
Reinsurance Premium	8,000,000
Cession	80.0%
Ceding Commission	25.0%
Profit Commission	
Loss Ratio	66.0%
Profit Swing	5.0%
Loss Ratio Cap	100.0%
Reinsurers Expenses as % of Prem.	
Brokerage	*2.0%*
Underwriting Exp.	*2.0%*
Federal Excise Taxes	*1.0%*

The underlying exposure for Contract #1 is multi-state workers compensation. The company has written workers compensation for a number of years. The cession is a straightforward quota share with a loss ratio cap of 100%. This loss ratio cap has the potential to significantly affect risk transfer. The presence of the loss ratio cap does not always indicate a lack of risk transfer. Contracts, with loss ratio caps at 200% to 300% can clearly result in a significant loss ot the reinsurer. Secondly, there is a profit commission provision whereby the ceding company will receive a profit commission if the underlying loss ratio is 66% or less with maximum profit provision of 5.0%. The profit provision swings on a one-to-one basis with the loss ratio. The impact of profit provisions on risk transfer is discussed later in the paper.

The terms of the second contract are summarized in Table 2:

Table 2 - Summary of Terms - Contract #2

Inception Date	1/1/2008
Estimated Subject Premium	10,000,000
Provisional Reinsurance Rate	8.50%
Provisional Premium	800,000
Maintenance Fee	50,000
Retention	250,000
Limit	250,000
Swing Rate	
Swing Loss Ratio	75.0%
Minimum Rate	6.00%
Maximum Rate	11.00%
Reinsurers Expenses as % of Prem.	
Brokerage	*10.0%*
Underwriting Exp.	*7.0%*
Federal Excise Taxes	*1.0%*

This is an excess of loss contract covering workers compensation exposure that has a number of potential risk limiting features. The contract is swing rated with a provisional rate of 8.5% which can swing up or down by 2.5%. The swing is based on a ceded loss ratio of 75.0%. Secondly, there is a feature that states that the contract is automatically commuted after five years unless the ceding company pays an additional maintenance fee of $50,000.

For the two example contracts it is not reasonably "self-evident" that risk transfer exists due to the presence of such features as low loss ratio caps and swing-rated premiums.

3.1 Analyzing Risk Transfer

The first step in any risk transfer review is to understand the reinsurance contract's terms and conditions, focusing especially on the terms that can affect the amount of risk being transferred. Care must be taken to understand not only the terms of the treaty but also when those terms will be triggered. In Contract #2 there is a commutation clause that requires a maintenance fee to avoid early commutation that is triggered after five years.

Next the reporting dates and premium due dates need to be determined. In both example contracts the reinsurance premium is payable in quarterly installments due one month after quarter end , i.e., on April 30, July 31, October 31, and January 31 of the following year.

In both contracts there is not a pre-defined loss payment schedule and therefore losses are reimbursed as they occur. To determine the net present value of the losses, a loss payment pattern reflecting the underlying exposure being reinsured is applied. It is further assumed that losses in any given calendar year are paid at the midpoint of the year.

For Contract #2, it is assumed that the first swing rate adjustment is applied two years after the contract's effective date. Most contracts will define the timing of the experience adjustments to the premium. It is also assumed in the model that the impact of the adjustment is correctly identified for the first adjustment with no further changes to the ceding commission necessary. This assumption implies that the ultimate loss ratio is known at the first adjustment.

The second assumption is that the commutation fee will be paid by the ceding company after five years. This is a reasonable assumption since the ceding company may not want to commute the contract and reassume the risk of changes in the unpaid claims estimates.

The risk transfer analysis was completed using Monte Carlo simulation, modeling first the direct loss payments and then projecting the treaty cessions from the direct loss payments. The ceded losses are then discounted to the effective date of the treaty. Next, the final premium amounts are determined based upon the nominal treaty results, not on the discounted premiums or losses. Any premium adjustments are determined from the modeled results. Care must be taken so that the premium payment dates are appropriately modeled. Like the losses, premium payments are discounted to the treaty effective date. The reinsurer profit/loss is then calculated for each iteration of the simulation as the net present value (NPV) of all payments made from the ceding company to the reinsurer minus the NPV of all the payments made from the reinsurer to the ceding company.

All cash flows between the ceding company and reinsurer need to be represented in the model whether they are called premiums, fees, or experience adjustments. Reinsurer expenses are not included in the model since this is not a cash flow between the ceding company and the reinsurer. For instance in Contract #2 the maintenance fee is included in the analysis and the reinsurer expenses are not. The reinsurer expenses are not part of the risk assumed by the reinsurer from the ceding company.

Finally, the Expected Reinsurer Deficit (ERD) is calculated. ERD can be viewed as the probability of a net present value (NPV) underwriting loss for the reinsurer multiplied by the NPV of the average severity of the reinsurer underwriting losses. The resulting ERD values are 2.85% for Contract #1 and 2.09% for Contract #2. Details of the simulation and ERD calculation can be found in Appendices A and B. These results indicate that both of these contracts appear to exhibit risk transfer. This conclusion is based on the calculated ERD values and the commonly accepted threshold of 1.0%. As with any risk transfer decision, the ultimate determination must be made by the company CEO or CFO or both.

3.2 Common Pitfalls

This section will highlight easy-to-make mistakes or common pitfalls. Most of these come from our own experience in reviewing contracts for risk transfer and reviewing risk transfer analyses of other actuaries. It is our intent to provide concrete solutions citing previously published materials.

3.2.1 Profit Commissions

Profit commissions generally should not be considered in risk transfer analysis. When determining if risk transfer is present, the analysis focuses only on the scenarios resulting in a loss for the reinsurer. While profit commissions can affect the economic results of a treaty, they usually are not triggered during a reinsurer loss.

This exclusion of profit commissions and focus on reinsurer loss scenarios is not necessarily intuitive. However, the accounting standards clearly state that the presence of risk transfer requires a "reasonable chance of a significant loss" to the reinsurer. Therefore, the results of the ceding company should not be considered in a risk transfer analysis.

It is important to remember that contract features like profit commissions can still have an indirect impact on risk transfer. This impact on risk transfer stems from how these features may affect other aspects of the contract, most notably the premium. Reinsurance contracts are priced while considering any and all expected payments paid and received by the reinsurer. Any addition of a profit commission clearly increases the amount of future expected payments by the reinsurer to the ceding company and may result in a higher premium for the contract.

In the example analysis for Contract #1, the profit commissions were included in the simulation to demonstrate that they did not affect the reinsurer in any loss scenarios. However, if the contract failed to meet risk transfer requirements, the ceding company and the reinsurer may consider potential changes that would allow the contract to be accounted for as reinsurance. One potential change would be to eliminate or reduce the profit commissions with a corresponding decrease in premium. This change in premium may result in the contract meeting risk transfer requirements.

Another way profit commissions can affect risk transfer is through carryforwards. Carryforwards may be used in multi-year contracts where the profits or losses from prior years may affect the results of the future years. A contract for periods of more than one year usually requires further testing for risk transfer and any carryforwards that may impact a loss position for the reinsurer would need to be incorporated into the model. Carryforwards can also be used in one-year contracts where the primary company and reinsurer agree to terms each year and at that time choose whether or not results will be carried forward. In this case each contract renewal may require a specific analysis. If there is a carryforward from a previous year that would affect results when there is a loss for the reinsurer, then it must be incorporated into the cash flow model. However, when considering one-year contracts with no impact from prior carryforwards there is no need to incorporate potential future carryforwards since they have no impact on the contract being reviewed.

3.2.2 Reinsurer Expenses

Only cash flows between the ceding company and the reinsurer should be considered in a risk transfer analysis. According to SSAP 62, "The evaluation is based on the present value of all cash flows between the ceding and assuming enterprises under reasonably possible outcomes." This means that broker expenses, operating expenses, fees related to letters of credit, and taxes should bear no impact on the analysis. As can be seen in the Appendices, the analyses of the example contracts did not incorporate any of these expenses that did not result in a cash flow between the reinsurer and the ceding company.

3.2.3 Interest Rates and Discount Factors

SSAP 62 requires a constant interest rate to be used for discounting across all simulated scenarios. The interest rate should not vary by scenario because risk transfer analysis should only consider insurance risk. Non-insurance risks such as investment risk, currency risk, and credit risk should not be included. The AAA Practice Note interprets this to also mean that the same interest rate should be applied to all cash flows, including premiums and losses.

SSAP 62 only requires the selection of the interest rate to be reasonable and appropriate. The AAA Practice Note recommends the risk free rate as a reasonable choice. This is not necessarily a conservative selection. Because the risk free rate is commonly below a reinsurer's expected investment returns, it will actually result in higher projected present valued losses. However, the investment abilities of the reinsurer should not affect the presence of risk transfer, so the risk-free rate is a consistent and reasonable selection for the analysis. The selection of other interest rates is considered later in the paper.

SSAP 62 states that a reasonable and appropriate interest rate "generally would reflect the expected timing of payments to the reinsurer and the duration over which those cash flows are expected to be invested by the reinsurer." Therefore the duration used to select an interest rate should be based on the net cash flows to the reinsurer.

There has been a lot of guidance on interest rate selection and there is very little room for deviation from the use of a constant interest rate in all risk transfer analyses. However, in the selection of the interest rate the accounting standards do not prescribe a set framework and note that judgment is involved. While using a risk-free rate with duration equal to that of the reinsurers net cash flows is recommended, a selected rate could still be considered a "reasonable and appropriate rate".

Page 4 of Appendix A provides an example of calculating a duration using loss and premium payments and then selecting a risk-free rate based on that duration. To get the duration of the net cash flows we performed two duration calculations. First we determined the duration of the premium payments. This was straight forward since the premium payment schedule is laid out in the contract. Next the loss duration is calculated using an industry payment pattern. The duration of the net cash flows is then the difference between the two. This calculation may not be exact, but it is a good approximation of the "duration over which those cash flows are expected to be invested by the reinsurer," as the standard requires. The calculated duration of net cash flows was then used to select an interest rate based on the years of maturity and yield curve rates from the U.S. Treasury in Columns (7) and (8). This interest rate was used in the analysis for Contract #1.

For Contract #2 an interest rate was selected with consideration given to the current risk-free rates and longer expected payment pattern for an excess of loss contract.

3.2.4 Premiums

The premium paid by the ceding company is one of the most significant inputs when determining if risk transfer is present. When using the "10-10" rule or ERD all potential loss situations are going to be compared against the premium to calculate a percent of loss. While its importance is clear, what the premium should include is not nearly as straightforward.

First, the premiums used in risk transfer analysis should be gross premiums. This is specifically pointed out in SSAP 62. Gross premiums entail all premium paid to the reinsurer before the consideration of any payments back such as a ceding commission.

When making comparisons against premium to determine a reinsurer's profit or loss, it is required that the present value of the premium be used. Reinsurance contracts often lay out specific payment plans for premium. The same interest rate used to discount losses should be applied to calculate the present value of the premium. While the risk transfer analysis is a present value calculation, it is important to model the actual functioning of the contract. This means that the application of the loss ratio caps and experience adjustments are based upon the nominal premium and loss amounts. As shown in Appendix A, the loss ratio cap in Contract #1 is applied to nominal losses and premiums in the simulation. The discounting of premium and losses happens after the contract losses and premiums are determined and any caps or experience based features are applied.

When the premium of a reinsurance contract is dependent upon future events, using the proper premium in a cash flow simulation is slightly more complicated.

There are a number of premiums that could be considered for this purpose. The initial deposit premium is an intuitive and simple choice, but it does not account for future payments from the ceding company to the reinsurer and could therefore be easily manipulated. The other options are to use an expected premium or the actual premium in each scenario.

The use of expected premiums may also seem intuitive, but can be troublesome as well. The most significant concern with using expected premiums is the potential over detection of risk transfer. When premium is dependent upon loss experience, the highest premium levels often occur when the loss experience is the poorest and the reinsurer's losses are at their highest. If the reinsurer's percent of loss is calculated using an average expected premium, it is likely that the resulting reinsurer loss percentage will be a larger negative value than what is actually possible. Because of this it is imperative that actual premiums are developed along with the losses for each scenario and that each scenario has a corresponding percent of reinsurer loss developed. From these simulated results, percentiles and values such as ERD can be calculated.

It is not uncommon for a reinsurance contract to include fees other than premium. When there are fees that depend upon future events, the impact of these events should be included in the model. If it is not possible to include certain events in the model, a general assumption about their impact on any future cash flows may be necessary. The conservative decision would be to include all fees that the ceding company may be required to pay to the reinsurer. There is an example of this in Contract #2, which requires a fee to delay mandatory commutation of the contract after five years. In the example it is assumed that the primary company will not want to commute the contract and reassume the risk after five years and therefore will be required to pay a fee of $50,000. When this type of fee is expected to occur, it should be considered as premium in any calculation of reinsurer loss. While the fee may be entirely administrative and related to the reinsurer's claim handling costs, any cash flows from the ceding company to the reinsurer should be considered as premium. If this were not the case, the determination of risk transfer could be manipulated based upon the labeling of certain cash flows as premiums or fees.

3.2.5 Evaluation Date

The date used in risk transfer analysis will likely only be used in the selection of an interest rate or in determination of how much was known about potential losses when the contract was entered into. SSAP 62 states that "risk transfer assessment is made at the inception date based on facts and circumstances known at the time." Therefore any parameters that may be affected by the date at which they were determined should be considered from the time of the contract's inception. The contract inception date is the date the contract comes into force, or the original effective date. According to SSAP 62 it is not necessary to retest for risk transfer at every renewal unless there are any significant amendments made to the treaty. If a contract is tested at inception, the results of that test are unlikely to change. In the case of an amendment that makes a material change to the amount of risk being transferred, the amendment date should be treated as the inception date of the contract and the contract should be reviewed again for risk transfer.

3.2.6 Commutations and Timing of Payments

According to SSAP 62, any reinsurance contracts that have prescribed payment patterns do not meet the risk transfer requirements. In order to have risk transfer in a reinsurance contract, there must be timing risk as well as underwriting risk. Prescribed payment plans remove the timing risk necessary for risk transfer. In order for the contract to contain timing risk the reinsurer must make "timely reimbursement payments."

Contracts with commutation clauses may still meet risk transfer requirements, but to the extent they affect the cash flows between the ceding company and reinsurer, they must be modeled. If a fee is required to avoid an early forced commutation, this fee should be considered as part of the expected premium paid. If the commutation decision is unilateral, it may be necessary to incorporate the commutation decision into the model based on economically rational decision making. To the extent the commutation clause impacts the payment pattern, this too should be considered in the cash flow model.

3.3 Practical Considerations

This section is meant to highlight a number of practical considerations that commonly appear in risk transfer analyses and have not been thoroughly addressed in the current literature. While not all of these practical considerations have obvious solutions, we hope to shed some light on the available options and open the door for further discussion on the topics.

3.3.1 Parameter Selection

One of the first and most important steps in performing a cash flow simulation for risk transfer analysis is choosing the parameters. Any parameters that are not given by the contract must be selected after some contemplation. This includes the interest rate, payment pattern, and any loss distributions used for projecting cash flows.

3.3.2 Interest Rate

Making the appropriate interest rate selection was previously addressed in the Common Pitfalls section. Using a risk-free rate based upon a duration calculation and the expected premium and loss payments is recommended by the AAA Practice Note. It is also required by the accounting standards that the same rate be used throughout the analysis.

While the risk-free rate is recommended, there are other possibilities to consider. It is difficult to envision a scenario were it would be reasonable to use an interest rate that is lower than the risk-free rate. This may seem conservative, but using a lower interest rate would lead to higher losses at present value and could result in over-detecting risk transfer. It is also difficult to construct an argument for why a company would not have the risk-free rate available to them. Therefore, it seems reasonable to treat the risk-free rate as the lowest possible choice, or floor, when selecting an interest rate.

A better argument could be made for selecting an interest rate above the risk-free rate. The most logical argument is that the reinsurer in the contract has a higher expected return on investments and this expected return should be used when determining if they face a "reasonable chance of a significant loss." While this argument is intuitive, it does have its flaws. First, this is not likely an available parameter if the risk transfer analysis is being done on behalf of the ceding company. Next, if a reinsurer's expected investment returns are used in the risk transfer analysis, it will create the situation where a contract may be found to exhibit risk transfer for a reinsurer with poor investment strategy, but be found not to transfer risk for a reinsurer with superior investment strategies. This type of counter-intuitive result is also why cash flows that are not between the ceding company and the reinsurer are not considered.

Based on these considerations it is difficult to construct an argument for using anything that is not at least loosely based upon the risk-free rate. For consistency and to provide support for the interest rate selected, it may be worthwhile to base the selection on the treasury yields available at the inception date of the contract and the expected duration of the cash flows, as was done in the example for Contract #1. This approach is consistent with the recommendation from the AAA Practice Note. However, depending on the situation and in an effort to keep an analysis simple, it may also be just as reasonable to select an appropriate approximation of the current risk-free rate, as was done in the example for Contract #2.

An alternative to selecting a duration-matched interest rate, which has been used by some practitioners, is the selection of a constant yield curve. Use of a yield curve is common in company planning and in making economic decisions on contracts. However, the use of yield curves in risk transfer analysis does not appear to be consistent with the accounting standards. The AAA Practice Note finds that SSAP 62 requires, "that a single interest rate be used to present-value the cash flows."

A constant yield curve would generally result in a more stringent risk transfer analysis since interest rates tend to be higher at longer durations. The typical yield curve would lead to more discount being applied to losses in comparison to the premiums, which are often paid much quicker. While the use of a yield curve may seem like an improvement to the analysis, the language in the accounting standards clearly leads to a similar conclusion to the AAA Practice Note. Both standards refer to the use of "a constant interest rate," through all cash flow scenarios. The intent of the standards appears to be that interest rate risk should not be incorporated in the model. Thus, an interest rate that varies by scenario is not allowed. Capturing interest rate risk is not the intent of incorporating a yield curve into the analysis. A constant yield curve across all scenarios would only result in a different interest rate when the timing of the cash flows differed, which reflects risk due to the timing of losses and premiums, not the interest rate. However, the use of a yield curve to discount cash flows would result in a different effective interest rate when no losses are paid compared to a situation where significant losses are paid. This appears to violate the requirement in SSAP 62 that the "same interest rate shall be used to compute the present value of cash flows for each reasonable possible outcome tested."

3.3.3 Payment Pattern

Payment patterns are often based on previous experience for the ceding company or industry benchmarks or both. While this can be a simple parameter to select, it is important to remember that there is uncertainty involved in the payment pattern. While this risk is more difficult to measure than the risk involved in a loss distribution, the timing of payments can play a significant role in the amount of risk transferred. For example, when a constant payment pattern is applied to a loss distribution, the results will not recognize the potential impact of quicker than expected payments. This will have the most significant impact on the tails of the distribution, which is often the portion we are the most interested in for determining risk transfer. While introducing variability into a payment pattern may be too complicated for the benefit it provides, it is important to at least consider this risk as you complete your analysis.

3.3.4 Loss Distribution

Loss distributions are often based on previous company experience, industry benchmarks, pricing information, or judgment, or all of these factors. For transactions covering large books of business with several years of historical experience available, selecting a loss distribution can be as easy as fitting a distribution to the available data. For books of business with low premium volume or immature loss experience, selecting the appropriate distribution can be much more difficult. Even for mid-size books of business it can be difficult to select a loss distribution because risk transfer testing focuses on the right tail of the distribution. This concern is compounded when working with high-level excess of loss contracts. However the loss distribution is determined, it is important to test the reasonableness of the tail results. Having an adequate comfort level with the tail results produced by the selected distribution is crucial.

When a company does not have enough historical loss experience to base a distribution upon, it is typical to turn to industry benchmarks or the information used to price the reinsurance contract. The use of pricing assumptions in risk transfer analyses is discussed later in the paper. Industry data can provide a starting point for overall expected loss ratios or frequencies and severities. However, it is difficult to select a distribution and develop a variance using only industry results. Individual companies can experience significantly higher variance in their loss than the industry as a whole. In these instances it may be necessary to rely on some generally accepted distributions. Likewise a selected variance will be required. This selection will depend on a number of considerations, such as the size of the book of business, the type of coverage, the type of business being underwritten, and a variety of other factors.

3.3.5 Parameter Risk

A key consideration for any simulation model is parameter risk. Cash flow simulations for risk transfer are no different. As we previously discussed, selecting parameters to simulate future loss payments is a difficult process and it is important to account for the risk that the selected parameters or model are incorrect. Accounting for this increased variability in your simulation will increase the likelihood that your analysis will determine risk transfer is present. This is a reasonable result when you consider that the reinsurer is clearly accepting this same parameter risk when entering into the contract.

Parameter risk can be accounted for explicitly or implicitly. Implicitly it can be reflected in a slightly higher expected loss selection or in an increase to the expected volatility of losses. In the case of explicit recognition it is common to see a probability distribution assigned to key parameters and then to have them simulated also. This provides some variability to the selected parameters to help account for parameter risk. While this is a more concrete method than including it implicitly, it also depends on judgment and the selection of more distributions and parameters. There is not much information available about incorporating parameter risk into cash flow simulation models. Currently, there are no widely accepted methods and the costs of more complicated techniques may tend to outweigh the benefits.

Parameter risk is going to have the greatest impact on the losses simulated, but it can affect other facets of the analysis as well. When premium projections must be estimated based on the treaty terms, there is some additional parameter risk, but it will rarely affect the result of the analysis. There is also parameter risk in the discounting function used in the analysis. However, not all of that risk should be accounted for in a risk transfer analysis.

The majority of the parameter risk in discounting comes from two key inputs, the payment pattern and the interest rate. As we previously discussed, there is real risk in not incorporating an accurate payment pattern. This risk relates to timing risk, which is a part of insurance risk and should be considered in a risk transfer analysis. The second piece of the discount, the interest rate, however, should not contribute any risk, parameter or process, to the analysis. SSAP 62 clearly states that "the possibility of investment income varying from expectations is not an element of insurance risk."

Because there are no widely accepted methods and because the methods available either require some arbitrary selections or may add more cost than benefit to the analysis, we do not feel that parameter risk must be explicitly shown in a risk transfer analysis. We would strongly encourage practitioners to at least include it implicitly if not explicitly. Regardless, we recommend documenting the existence of parameter risk and, whether or not it is included in the analysis, documenting how it could affect the results. This documentation can be beneficial if another actuary needs to review the analysis. More importantly, parameter risk is too important to entirely exclude from both the analysis and the report when the analysis may be directly used to make the decision on risk transfer.

3.3.6 Use of Pricing Assumptions

One potential resource, if available, for selecting parameters for small or immature books of business is the reinsurance pricing assumptions. This concept is very attractive since a properly priced reinsurance agreement is likely to be based on an appropriate expected loss assumption with an appropriate risk load and payment pattern. While we are often more interested in a loss distribution than just the expected losses for testing risk transfer, these assumptions can help provide some of the necessary parameters for our simulation.

Pricing assumptions can also be helpful in parameter selection since they reflect how risky the market views a particular piece of business. The reinsurance market may provide a better indication of the amount of risk involved in a small new primary company searching for reinsurance than what you could find based on industry benchmarks. Of course, this market-driven view of a reinsurance contract is also one of the biggest drawbacks to using pricing assumptions. Simulation testing for risk transfer should be based on expected loss experience and should not be market-driven. Pricing assumptions should only be used in selecting parameters when reasonable. A hard insurance market with higher premiums does not mean that companies do not need to meet the same risk transfer standards. Because of this, when available, the underlying data that the pricing assumption was based upon can be even more beneficial than the parameters actually used in the pricing of the reinsurance.

To correctly apply the expected loss assumptions from a pricing model to a risk transfer analysis, it is important to properly account for the risk load in the pricing. In many reinsurance contracts, risk load is a significant piece of the puzzle. It may be implicitly added into the expected loss ratio or explicitly stated in the development of the rate. If it is implicit in the expected losses, it is important not to blindly carry forward the expected losses without recognizing the extra loaded amount. If it is explicitly stated, intuitively there should be a relationship between this risk load amount and the level of risk inherent in the underlying coverage. While this risk load reflects the amount of variability the reinsurer anticipates in the contract, it is not easy to translate this load into a variance for your loss distribution. However, it is worthwhile to at least consider the size of this risk load when selecting the loss distribution and variance.

Another caveat to remember when using pricing information to select parameters for risk transfer testing is that while both practices are generally aimed at determining expected future losses, they both are doing so for very different reasons. The differences in intent can lead to different approaches and selections. Notably, when pricing a reinsurance contract, it might be considered prudent to make conservative selections. This might lead to slightly higher expected losses and risk load. These selections would not be considered conservative in a risk transfer analysis. Selecting higher expected losses and increasing the expected variability would lead to over-detecting risk transfer. For risk transfer testing the more conservative approach would be to use lower expected losses and variability. These differences in approach are important to remember anytime you are relying on assumptions from an analysis developed for a different purpose.

While pricing assumptions can clearly provide valuable input to any risk transfer analysis, it should also be clear that there are variety of reasons one may deviate from them. This is true even for reinsurance analysts who may be testing the same contracts they priced. These two exercises might require different assumptions about the modeled losses. Loss models used for pricing are often optimized based on their projections of all the potential results. Risk transfer, on the other hand, requires a model that is optimized on the right tail of the distribution. Due to this distinct difference in focus, the resulting selections for loss distribution and/or parameters may not be the same for pricing and risk transfer analysis.

3.3.7 Commutation Clauses

As previously discussed, any mandatory fees to delay a required commutation should be included when determining if risk transfer is present. Commutation clauses should be read carefully to determine their entire impact on risk transfer. While commutation clauses do not often prohibit a contract from exhibiting risk transfer, it is important to recognize that any commutation requirement does restrict the amount of risk transferred. It is not uncommon for these clauses to set a predetermined date for commutation based on an actuarial determination of the unpaid claim estimates at that time. While this is a fair method for completing a commutation, it does require the ceding company to reassume the risk of any changes in the unpaid claims after the predetermined commutation date. This clearly returns some risk back to the ceding company, limiting the amount of risk transferred in the original transaction.

If a commutation clause states that the future commutation will be based on a mutually agreed upon value or on an actuarial determination, the payment pattern used to discount losses in the risk transfer analysis may not need to be adjusted. While the commutation may result in an earlier payment than anticipated by the reinsurer for any outstanding claims, the payment should reflect the present value of expected payments at that time and the impact on the original payment pattern assumption should be minimal. If there are explicit rules for the calculation of the value of outstanding claims at commutation, these rules may need to be included in the original analysis and may affect the selected payment pattern.

4. CONCLUSIONS

It is important to remember that none of the methods to test risk transfer provide a "bright line" indicator for its existence. While actuaries have the necessary skill set to evaluate the existence of risk transfer in any reinsurance contract, the final decision belongs to the CEO or CFO of the company. Risk transfer analysis, and more specifically ERD, is a tool to aid them in that decision. If a risk transfer analysis produces a borderline result, such as an ERD of 0.95% or 1.05%, it will likely require further consideration and documentation to show that risk transfer does or does not exist in the contract being reviewed. Risk transfer testing is a principle-based exercise and the existence of risk transfer is entirely based upon there being a "reasonable chance of a significant loss" to the reinsurer. ERD and other methodologies are just tools to help determine if a contract meets this standard.

Acknowledgment

The authors wish to thank Robert Harnatkiewicz both for his suggestions and his help throughout the process. The authors also wish to thank Rob Walling, Laura Maxwell, and Greg Fears for their reviews of the paper. Any remaining errors are those of the authors.

5. REFERENCES

[1] AAA Committee on Property and Liability Financial Reporting, "Reinsurance Attestation Supplement 20-1: Risk Transfer Testing Note," January 2007.

[2] CAS Research Working Party on Risk Transfer Testing, "Risk Transfer Testing of Reinsurance Contracts: Analysis and Recommendations," Casualty Actuarial Society Forum, Winter 2006.

[3] Ruhm, David L., "Risk Coverage Ratio: A Leverage-Independent Method of Pricing based on Distribution of Return," Presentation at the ASTIN Colloquium, July 2001, http://www.actuaries.org/ASTIN/Colloquia/Washington/Ruhm.pdf

[4] Wang, Shaun S., "Premium Calculation by Transforming the Layer Premium Density," *ASTIN Bulletin* 26:1 (1996), pp. 71-92, http://www.casact.org/library/astin/vol26no1/71.pdf

[5] Wang, Shaun S., "A Universal Framework for Pricing Financial and Insurance Risks," *ASTIN Bulletin* 32:2 (2002), pp. 213-234, http://www.casact.org/library/astin/vol32no2/213.pdf

Abbreviations and notations

AAA, American Academy of Actuaries
ERD, Expected Reinsurer Deficit
RAS, Reinsurance Attestation Supplement
CAS, Casualty Actuarial Society
FAS 113, Financial Accounting Standard No. 113
SSAP, Statement of Statutory Accounting Principles

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An Update to D'Arcy's "A Strategy for Property-Liability Insurers in Inflationary Times"*

Richard Krivo, FCAS

In 1980, D'Arcy wrote a paper to provide insurers with a strategy to immunize against inflation. Over the past year (2008), it appeared that inflation was going to be a significant obstacle for the insurance industry on the basis of a sharp increase in the cost of commodities and increasing severity trends for property coverage as a result. These inflationary concerns were eclipsed by a massive credit crises spurred on by years of questionable lending practices and commodity prices dropped precipitously in response. At this moment, the U.S. government is planning on spending initiative to stave off a long recession. To finance this initiative, the government could both print and borrow money, which could possibly lead to the kinds of inflationary figures last seen in the late 70s. Given the threat inflation poses to insurance firms, the following is an update to this seminal paper.

Insurance company profits are the sum of underwriting profit (premium less incurred loss) and investment income on the premium prior to paying incurred loss amounts as well as the capital necessary to support the operation. Each of these components will be discussed separately.

DATA CAVEATS

A majority of the analysis relies on calendar year data taken from Best's *Aggregates & Averages*. Accident and policy year data would be preferable since these are closely tied to the years in which the business was assumed or written. An example of bias in calendar year data this is that during the soft market years of the late 90s, the largest underwriting losses were realized years later when the insurance companies took large reserve hits.

The Consumer Price Index (CPI) is the measure used as a proxy for inflation of insurers. As measured by industry sources the insurance industry has absorbed higher cost increases than the CPI would suggest but it's directionally consistent.

UNDERWRITING PROFIT MARGIN[1]

Insurance companies are unique in that they sell their product without truly knowing the underlying costs for the coverage afforded. When prices of goods rise at unanticipated rates,

[1] Underwriting Profit Margin is defined as 100% less the Combined Ratio before dividends. The Combined Ratio is the sum of the Net Loss Ratio and the Underwriting Expense Ratio. The Net Loss Ratio is defined as Loss plus Loss Adjustment Expense divided by Net Earned Premium. The Underwriting Expense Ratio is defined as Underwriting Expenses over Net Written Premium.

*The views expressed in this article are those of the writer and do not necessarily reflect the views of PartnerRe Ltd. or its affiliated companies.

insurance companies find they have undercharged for their coverage. In addition, regulators may put more scrutiny on insurers charging higher rates or from restricting coverage in a tough economic climate.

Below is a regression equation D'Arcy fitted to explore the relationship between underwriting profit margin (UPM) and the inflation rate.

$$\text{UPM} = -0.617*\text{Inflation} + [2.955] \rightarrow R^2 = 27.7\% \{ 1951 - 1976 \} \quad (1)$$

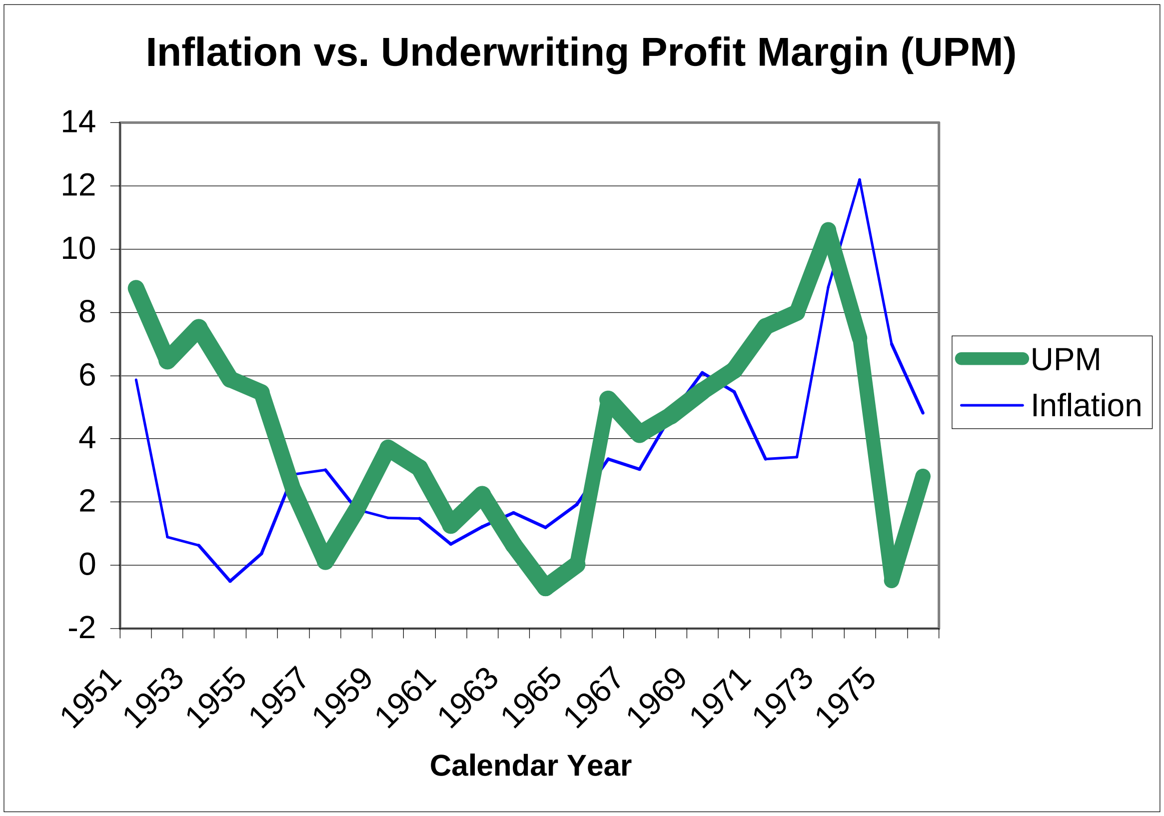

Exhibit 1 – Graph of Equation #1, which shows a negative correlation between inflation and underwriting profit margin.

This equation displays that underwriting profits and inflation are negatively correlated since the coefficient [-0.617] applied to the inflation variable is negative. Performing the same regression analysis on data from 1977-2006 resulted in the following.

$$\text{UPM} = 0.593*\text{Inflation} + [-9.586] \rightarrow R^2 = 8.7\% \{ 1977 - 2006 \} \quad (2)$$

The correlation coefficient, from here out referred to as R^2, is so low that any observed relationship is essentially meaningless. In addition, the equation suggests that the UPM and inflation are positively correlated!

In response to the poor fit, an attempt was made to model the UPM with the "Year-Over-Year Change in Inflation" rather than the absolute value. Unfortunately no meaningful relationship was observed after this transformation. The fitted equations are in Appendix B.

In an attempt to replace the calendar year statistics, Equation (3) is based on Industry Schedule P Accident Year data. [2] This equation displayed no measurable relationship between UPM and inflation over the years listed. Accident Year data is a better measure of insurer profitability but the fit over this period is not meaningful.

$$\text{UPM} = 1.201 * \text{Inflation} + [-1.735] \rightarrow R^2 = 0.4\% \; \{ \; 1995 - 2007 \; \} \quad (3)$$

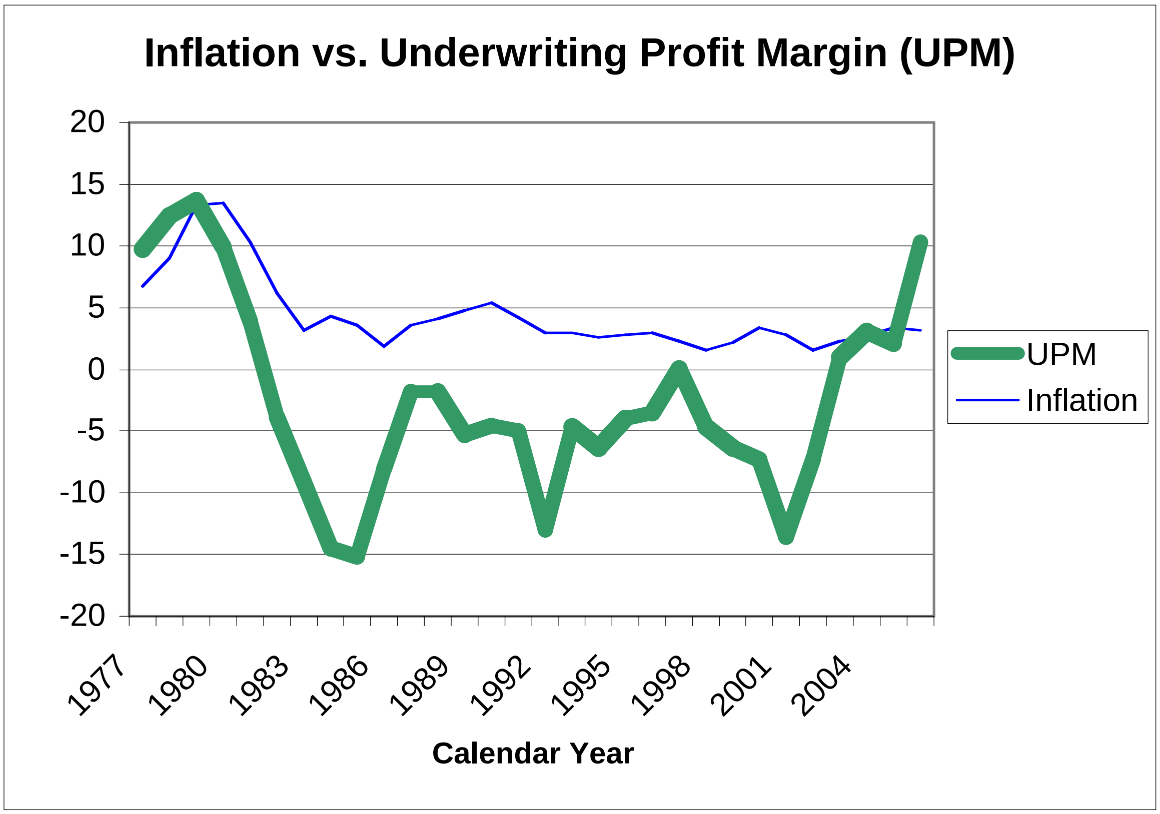

Exhibit 2 – Graph of Equation #2, which shows a no meaningful relationship between inflation and underwriting profit margin.

Is There Any Possible Explanation For The Lack Of A Meaningful Fit?

Inflation has remained in a narrow band over the subsequent 30 years since D'Arcy scripted his paper. Since 1983, the CPI has stayed between 1.6% (1998) and 5.4% (1990) after exceeding 13% in 1979 and 1980. On the other hand, profit margins have varied greatly. Notably there have been two

[2] The Underwriting Profit Margin was derived replacing the Calendar Year with Accident Year Schedule P Loss and LAE Ratios. The same calendar year expense ratios from Best's were used.

soft markets (1981-1985: 1997-2001) followed by two hard markets. These cycles appear uncorrelated to inflation.

The underwriting cycle is the key driver behind the dramatic changes in profit margins. Trying to appropriately model the underwriting cycle and its varied causes are outside the scope of this paper and doing so would relegate the important questions regarding inflation and its impact on insurance profitability to obscurity. The prevailing wisdom behind the cause of the underwriting cycle is the amount of excess capital in the industry. [3].

Autoregression Model

In an attempt to factor out the underwriting cycle's impact on the underwriting profit margin (UPM) and directly observe the inflationary effect, a two-factor autoregressive model was fitted to the insurance data compiled.

$$UPM_{(t)} = a*UPM_{(t\text{-}1)} + b*Inflation_{(t)} + c \quad (4a)$$

The premise of this model is that the current year's underwriting profit margin (UPM) is based on last year's. When actuaries price business, the assumption is that the best predictor for the current year is the prior year's data appropriately adjusted. This model should be somewhat familiar except for its over-reliance on a single prior year rather than multiple years. An obvious shortcoming of this model is that it completely misses the inflection points where the market hardens (see 1987). The last two soft market corrections have been rather abrupt as can be observed on Exhibit 3 below.

$$UPM_{(t)} = 0.84*UPM_{(t\text{-}1)} - 0.364 * Inf_{(t)} + 0.911 \rightarrow R^2 = 70\% \{ 1951 - 2006 \} \quad (4b)$$

$\underline{t = 11.130}$ $\underline{t = -2.285}$

[3] One belief is that the first soft market was, in part, caused by cash-flow underwriting (a term that means writing business to a loss knowing that extremely high returns on bonds/assets would compensate for underwriting losses) . These high returns were caused in response to the high inflation experienced in the late 70s/early 80s.

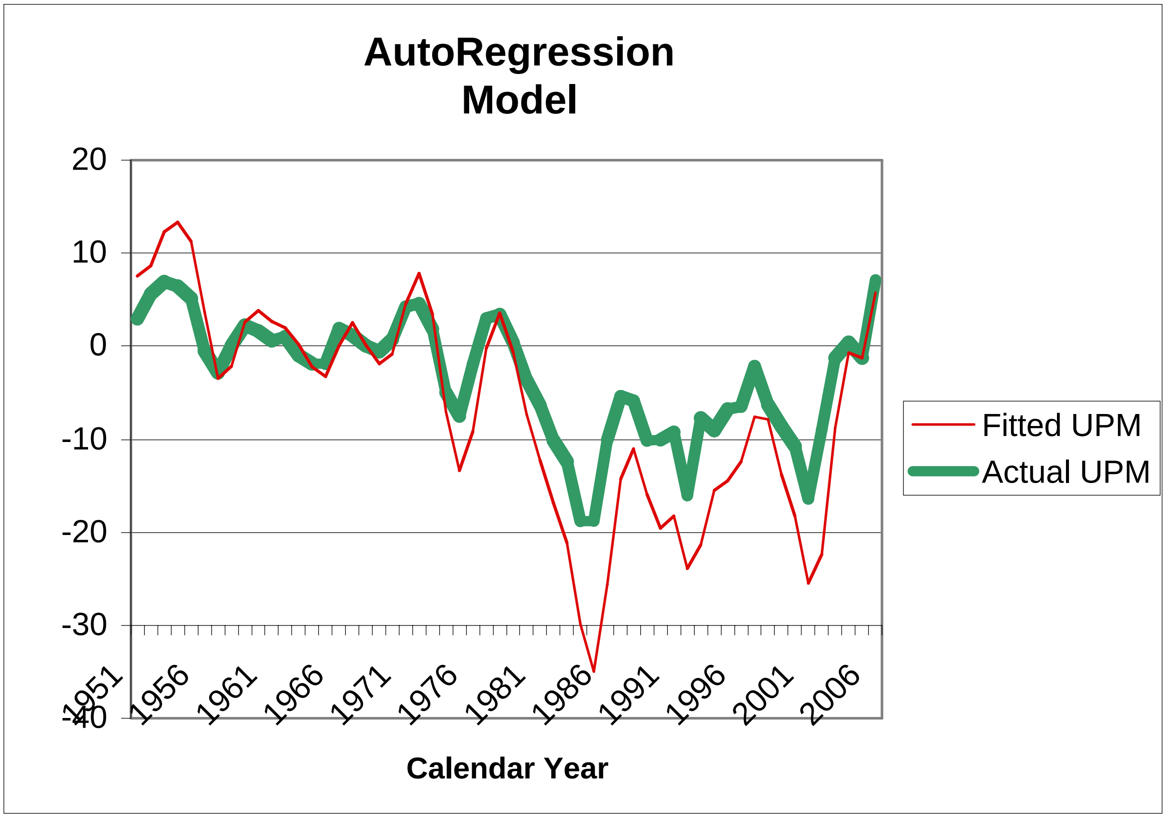

Exhibit 3 – Graph of Equation #4b, which displays the actual and fitted underwriting profit margins.

The R^2 of the equation is reasonably good and the coefficient applied to the prior years' UPM (0.84) shows a strong positive correlation. The *t*-values on each independent variable above are significant above the 95% level. The ability to model a proxy for the underwriting cycle has given us another indication that inflation and underwriting profits are negatively correlated (the negative coefficient in front of the inflation variable implies a negative correlation).

Inflation is likely a second order type variable in terms of its impact on the UPM. The impact inflation has on Underwriting Profit Margins is likely to be obscured unless there's a large spike as there was in the late 70s. A big increase could catch insurance companies flat-footed; especially because that would necessitate obtaining substantial rate increases in lines of insurance that may be heavily regulated. For longer tailed lines, delays in recognizing the inflationary trend could lead to significant underwriting losses and an inability to properly price their products. The next section speaks to inflations' impact on specific lines of insurance.

Personal Auto

Personal auto is the largest line of insurance and drives the overall industry result. It is a completely data-driven line and has been both profitable and predictable. At this point, industry trend factors lag only six months behind the closing date of claims so any inflationary spikes should be recognizable by insurers. Decisions these firms make not to file for rate increases (to ignore the loss cost trends) would be of a competitive nature. I believe the regulatory hurdles personal auto carriers face are much lower than in the early 80s since they have the data and wherewithal to file and support increases in rates. Massachusetts and New Jersey (whom historically had a hard time attracting carriers) have opened up due to some easing on regulatory restrictions. Given the time it takes to settle claims, there is some reserve risk facing personal auto insurers. In an inflationary environment, an outstanding reserve for $100,000 could grow to $120,000 in the 18 months it takes to settle. The threat inflation poses to this line is moderate in light of the reserve risk but lower than in the past due to carriers having the ability to read and react and lower regulatory hurdles.

Property Coverage

Property coverage already experienced an inflationary spike in the past year due to substantial increases in commodity costs and the high cost of gasoline and coal needed to transport building materials. The hurricane catastrophe models already build in "demand surge" since after big events scarcity drives up the cost of key materials. By considering such inflationary forces in the policies they sell, insurers should be better braced to respond to a significant inflationary spike. Many homeowners policies have a built-in inflation guard that automatically increases the amount of coverage purchased. One concern is that in certain jurisdictions, Homeowners is still a highly regulated line with rates that, even without inflationary pressures, are inadequate. The threat inflation poses to this line is moderate but the short time to settlement gives insurers the ability to react.

Workers Compensation

Workers compensation is particularly troubling from an inflationary perspective due to the absence of a limit on medical coverage. Even without additional inflationary pressures, new expensive procedures and medications have increased costs at a pace much higher than the medical CPI. The additional danger for workers compensation is the effect of compounding inflationary increases due to the amount of time it takes to settle claims. There is also the indemnity (wage replacement) component of the workers compensation policy that has recently served to temper the inflationary impacts caused by the aforementioned medical increases. Even if wages were to increase at a higher pace, most states have caps based on the average weekly wage in the state. Workers compensation is highly regulated line and prices can be constrained by states. Generous changes in coverage can be applied retroactively without allowing companies financial recourse. The largest market, California, had some coverage expansion in the mid-90s that resulted in high severity trends that took years to be properly implemented into the bureau's rates. The threat inflation poses to workers compensation is high for the aforementioned reasons.

Other Casualty

Casualty lines of insurance such as professional and general liability along with umbrella are long-tailed. The danger in assuming risk in these lines is that any trend shortfall in the pricing won't be discovered or factored into policies until multiple renewals have ensued. Losses on policies written today may incur five years of higher than anticipated inflation. Underwriting losses for "other casualty" are typically realized at a later date when the reserves are adjusted to factor in these higher loss ratios. The reserve risk companies assume in writing long-tailed lines is due to the most common reserving methods' reliance on initial expected loss ratios to determine IBNR. Following the last two soft markets, IBNR reserves were increased dramatically due to inadequate pricing. The risk posed by inflation is moderate to high depending on the specific circumstances of the line of business covered.

Reinsurance

Reinsurance companies can be exposed to significant inflation risk due to "trend leveraging" where more claims exceed a fixed retention and the increases above the retention are greater than the underlying ground-up inflation rate. One of the dangers in writing reinsurance is that these companies aren't always privy to what is happening in the primary market so they can assume business without have the ability to properly gauge the underlying trends or primary rate changes. Reinsurance companies have many ways to limit the realized inflationary impact in the coverage they offer. Loss ratio caps, aggregate limits and limited reinstatements are all common treaty features that can insulate reinsurers from the impacts of inflation. On the plus side, reinsurers are not subject to the same regulatory constraints primary companies are. In particular, the ability to cancel a treaty indiscriminately or offer the primary carrier more restrictive coverage (than the underlying policies provide) can protect reinsurers from assuming inflationary risky business. In Europe, a response to dealing with inflation is a common treaty feature referred to as an index clause. This is an agreement to index the retention amount with an inflationary adjustment.[4] This serves to keep the cost of reinsurance and the number of expected ceded claims in a layer flat over a number of years. The threat inflation poses to this line is high since any coverage response requires an awareness of changes at the primary level.

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[4] Stephen D'Arcy mentioned that in response to the last inflation spike, General Re tried to negotiate index clauses in some U.S. contracts but these never gained traction.

INVESTMENT INCOME RETURNS (IIR)[5]

Insurance companies can target low or even negative underwriting profit margins due to the income earned on their invested assets. However when inflation rates rise, their portfolios of bond assets lose value. Insurance companies have an incentive to own bonds since statutory accounting rules allow them to keep bonds at par value even if they have a significant unrealized loss position. If companies sell these bonds in an inflationary environment, they would realize a loss. Similar to the approach D'Arcy used with underwriting profit margins, he fitted insurance companies investment income returns (IIR) with inflation rates.

$$\text{IIR} = -0.818 * \text{Inflation} + [7.815] \rightarrow R^2 = 22.6\% \ \{ 1951 - 1976 \} \quad (5)$$

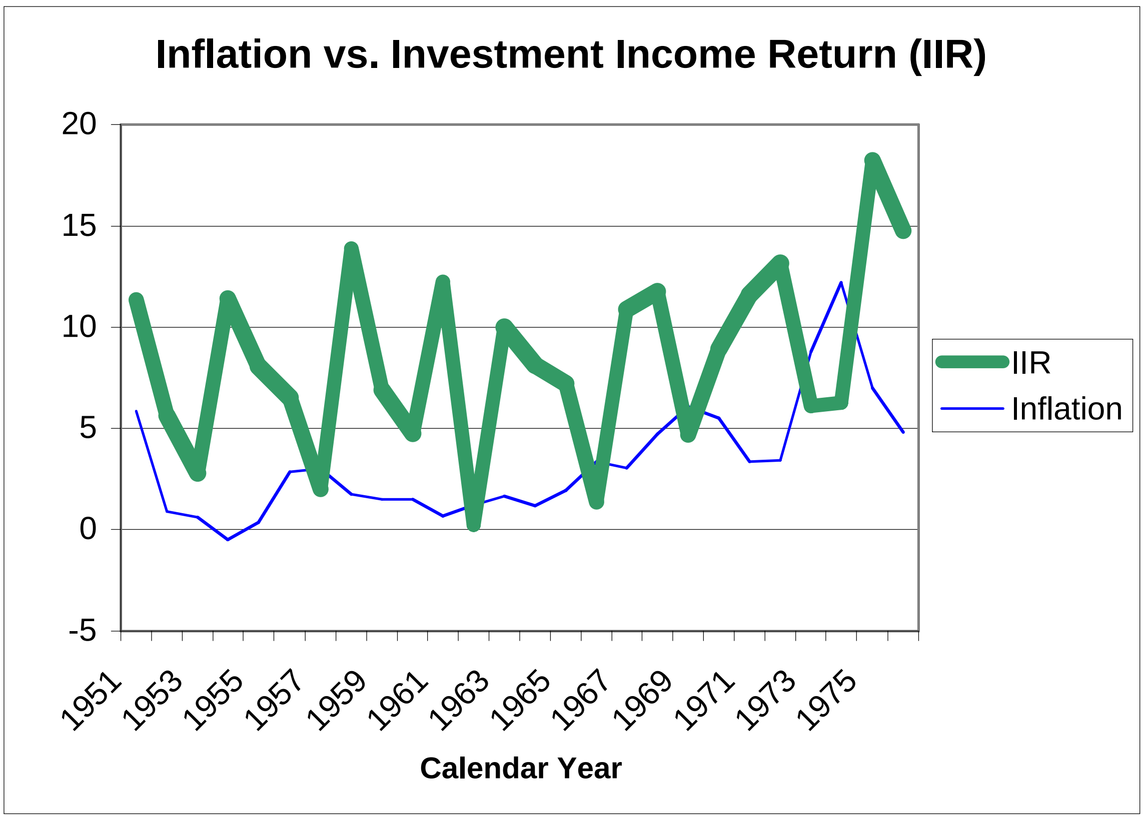

Exhibit 4 – Graph of Equation #5, which shows a negative correlation between inflation and investment income returns.

This equation displays that investment income return and inflation are negatively correlated. Performing the same regression analysis on data from 1977-2006 resulted in the following. Note that the negative correlation still holds.

[5] Investment Income Return (IIR) is the total of income earned through dividends, interest, realized capital gains or losses for bonds and real estate, and both realized and unrealized gains and losses for stock. Unrealized capital gains or losses on bonds are not a factor in statutory investment profit or loss for insurers.

$$IIR = -0.297*Inflation + [14.333] \rightarrow R^2 = 14.5\% \; \{ 1977 - 2006 \} \quad (6)$$

The limitation to utilizing statutory investment income for this study is that unrealized losses on bond portfolios are not captured in the investment returns. If the investment returns were to capture these unrealized losses, the likely result would be a stronger negative correlation. One observation about the 1977-2006 investment income returns is that they don't fluctuate to the same extent as in the earlier period. Also, the R^2 of Eq. (6) (R^2 = 14.5% for 1977-2006) is significantly lower than Eq. (5) (R^2 = 22.6% for 1951-1976) and could possibly be the result of insurance companies becoming more sophisticated in managing their portfolios of assets with respect to inflation and other sources of volatility. The amount of knowledge and use of financial instruments at these companies disposal is much greater today than in the past. One thing worth noting, as the insurance industry heads towards fair value accounting, unrealized losses on bond holdings will find their way into quarterly earnings and seemingly increase the impact inflation has on insurers' returns. Assuming most insurers hold their bonds to maturity, there will be more volatility in earnings. The next section updates correlations performed on different classes of assets with inflation.

Treasury Bills (T-Bills)[6]

As we have entered this credit crisis, Treasury (T-Bill) yields have gone down dramatically due to the desire for safety over return. When yields fall, there are realized gains for those who already own them. T-Bills rise in value at times of economic uncertainty due to their short duration. An investor fearful of locking up capital in longer duration or riskier assets could invest in T-Bills until the uncertainty passes. In doing so, the investor typically sacrifices yield for security. The following equations display the positive correlations between government securities and the inflation rate. This relationship is the key to D'Arcy's immunization strategy.

[6] Treasury Bills (or T-Bills) are government securities that mature in one year or less. Like zero-coupon bonds, they do not pay interest prior to maturity; instead they are sold at a discount of the par value to create a positive yield to maturity. Many regard T-Bills as the least risky investment available to U.S. investors. The data used for this study relied on average yields for 3-month T-Bill returns for data from 1977-2007. For the prior data (1926-1976), holding period returns on the shortest term bills not less than one month to maturity held for one month were used.

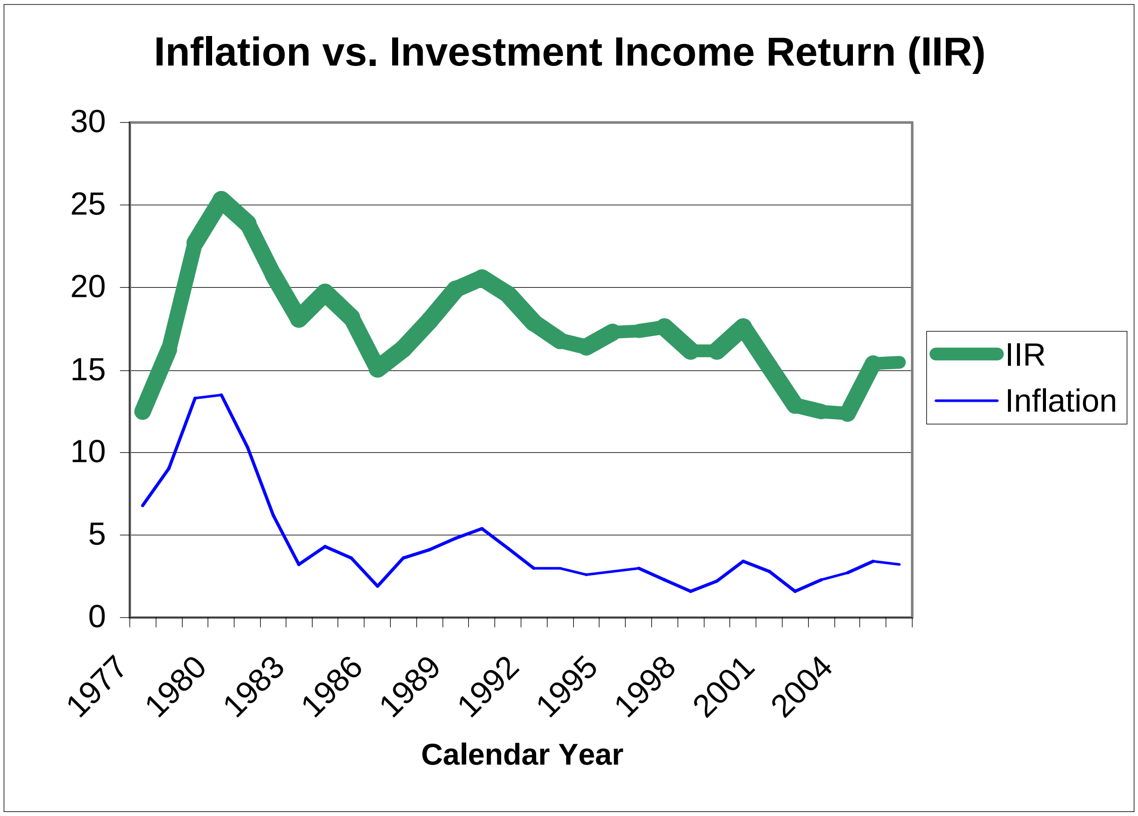

Exhibit 5 – Graph of Equation #6, which shows a weak but negative correlation between inflation and investment income returns.

T-Bill = 0.556*Inflation + [1.873] → R^2 = 70.6% { 1951 - 1976 } (7)

T-Bill = 0.770*Inflation + [3.177] → R^2 = 47.0% { 1977 - 2006 } (8)

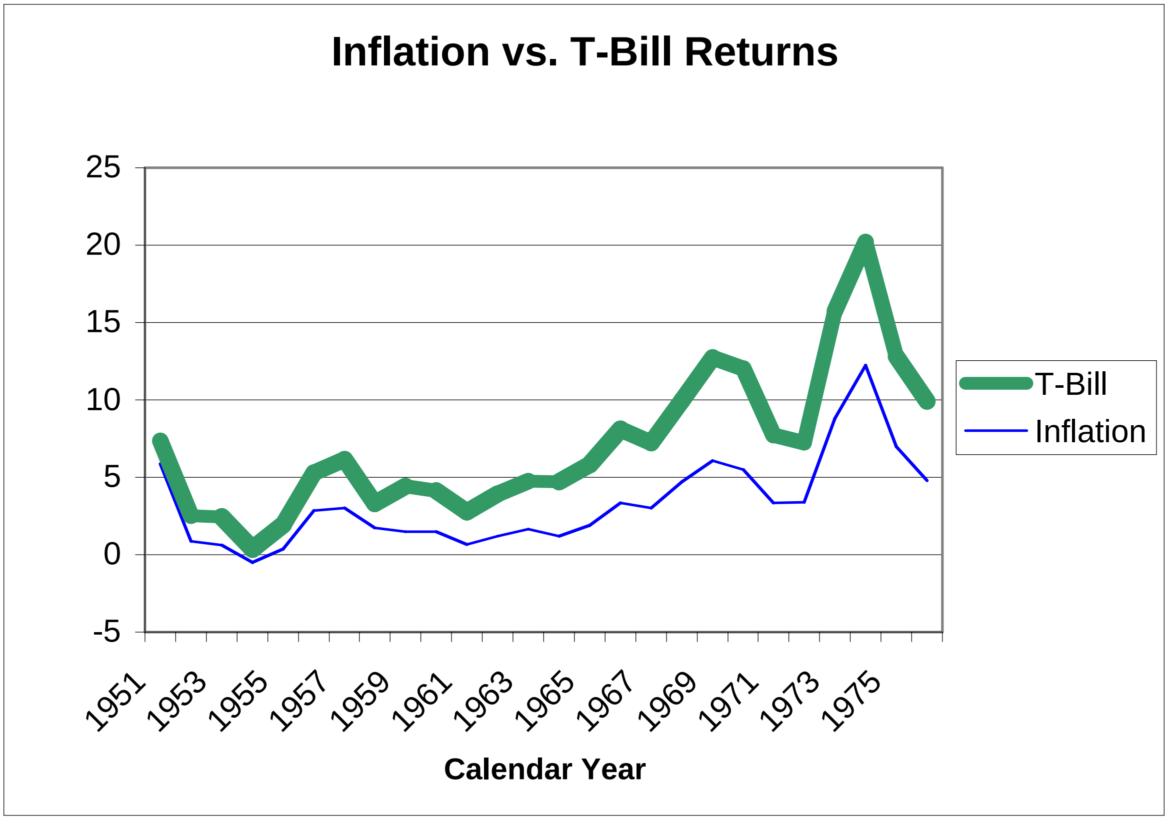

Exhibit 6 – Graph of Equation #7, which displays a positive correlation between inflation and T-Bill returns.

Long-Term Corporate Bonds

Long-Term Corporate Bonds (LTCorp) show little correlation with inflation with R^2 values close to 0%. It would be interesting to see how bonds from the less-cyclical sectors or companies with stronger ratings perform on relative basis against inflation since these may provide an alternative immunization source offering potentially higher yields for a portfolio.

$$\text{LTCorp} = -0.084*\text{Inflation} + [3.929] \rightarrow R^2 = 0.1\% \ \{ 1951 - 1976 \} \qquad (9)$$

$$\text{LTCorp} = 1.321*\text{Inflation} + [4.582] \rightarrow R^2 = 5.4\% \ \{ 1977 – 2006 \} \qquad (10)$$

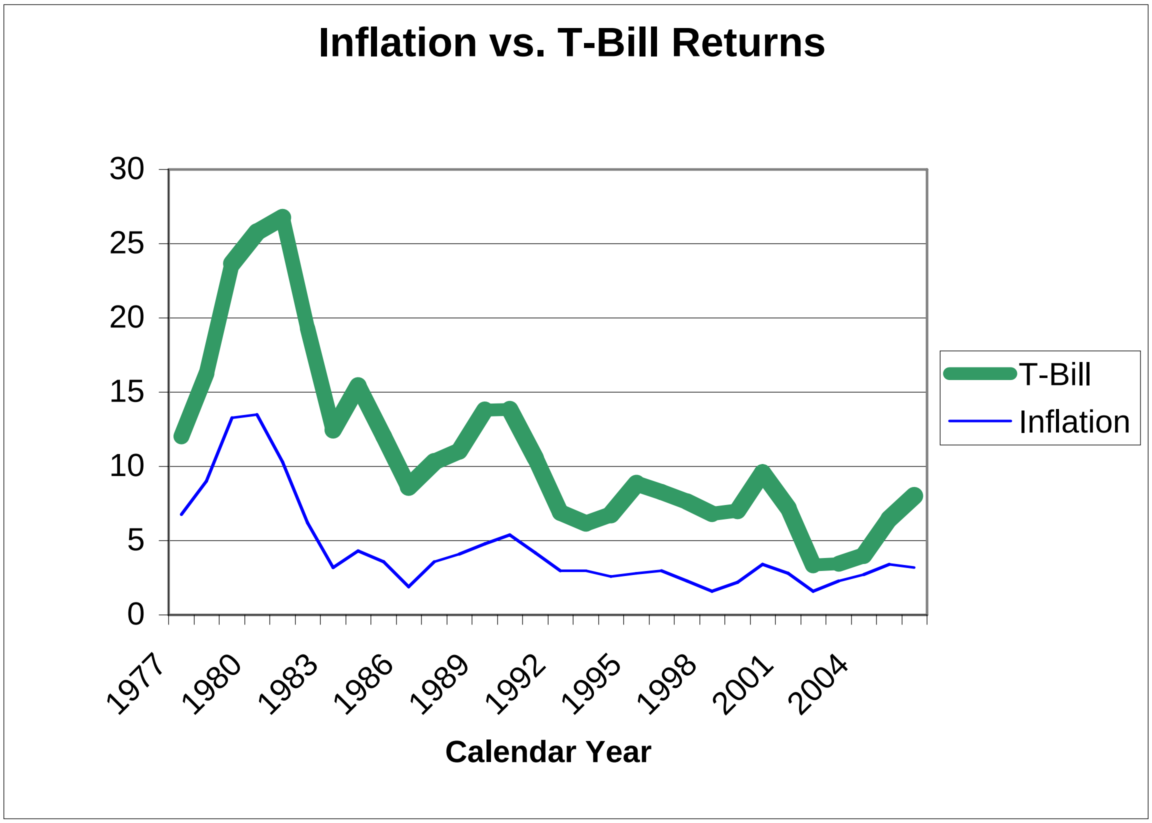

Exhibit 7 – Graph of Equation #8, which displays a positive correlation between inflation and T-Bill returns.

Stocks

Stock returns in the latest period show no correlation with inflation. In the past, significant negative correlations existed. There are sectors that are less sensitive to inflation shocks. Utility and health care stocks are considered by experts to be defensive. These may have a place in a portfolio looking for an effective hedge against inflation.

$$\text{Stock} = -3.114*\text{Inflation} + [22.773] \rightarrow R^2 = 23.0\% \ \{ 1951 - 1976 \} \quad (11)$$

$$\text{Stock} = -0.087*\text{Inflation} + [13.183] \rightarrow R^2 = 0.0\% \ \{ 1977 - 2006 \} \quad (12)$$

THE STRATEGY

D'Arcy recommended using T-Bills to immunize PC insurers against deteriorating profit margins as well as portfolio losses on bonds due to inflationary pressures. Based on the data since 1977, T-Bills are still the most effective hedge. There are certainly questions as to how significant inflationary effects are on insurer profit margins. In response to those questions, there are too many factors for profit margins not to be affected. The data over the past 20 years didn't have the inflationary hurdles that would test the insurers' success in immunizing their risk and asset portfolios.

Inflation needs to be an element of concern for insurance companies. Primary companies need to be proactive in analyzing and selecting trend factors in order to keep prices and reserves adequate. Reinsurance companies need to consider the frequency of claims over fixed retentions and/or utilize treaty features to insulate their risk portfolios against the inflationary risks they face.

As noted in the paper, insurers' asset portfolio returns have been much more consistent than in the past and have significantly exceeded risk-free. Purchasing T-Bills to immunize an asset portfolio is like a form of insurance. It provides a hedge against inflation and economic uncertainty but a price is paid in terms of sacrificing yield (i.e., expected income). D'Arcy has regressed three different asset types but there are a slew of different hedges against inflation. Commodities, derivatives, treasury inflation protected securities (TIPS), real estate, and ETFs can all assist in diversifying the risk inflation poses.

This paper was written in response to increased inflation risk observed in the summer of 2008. Between stocks, bonds, and derivatives, a large amount of capital has exited the industry over the past six months. The danger of inflation has been replaced with that of an economic depression. In his seminal paper, D'Arcy excludes the depression years from his analysis despite there being huge reductions in the cost of goods (negative inflation). Despite insurers' abilities to save on losses in this environment, sharp premium declines resulted in high expense ratios and profit margins suffered.[7] One asset class that has held up, especially on a relative basis, have been government securities. Currently, the yields on these securities are paltry thus making the cost of immunization higher than ever. In 2009, insurance carriers need to consider the impact of the U.S. government's stimulus package, which involves huge deficit spending. In this highly volatile climate, finding assets that can immunize insurers against inflation is as critical a challenge for the industry as it was when D'Arcy wrote this paper some 30 years ago.

Acknowledgment

I would like to thank Gary Bluhmson, Robert Giambo and Jeff Englander who all contributed time and intellectual capacity to this paper. All errors are my own.

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7 D'Arcy writes on page 118: "The pre-1933 period does not conform with the negative relations outline above. Underwriting profitability actually declined in 1930, 1931, and 1932 as price levels dropped substantially. One possible explanation for this atypical correspondence is the pervasive effect of the Depression. Despite price level reductions, economic conditions were so poor that insurance premium receipts declined, causing expense ratios to climb…The usefulness of this model will thus be restricted to inflationary conditions and would not necessarily apply to deflationary situations."

REFERENCES

[1.] D'Arcy, Stephen P., "A Strategy for Property-Liability Insurers in Inflationary Times," CAS *Discussion Paper Program* 1981, pp. 110-147.

[2.] *Best's Aggregates and Average: Property/Casualty*, (Oldwick, New Jersey: A.M Best and Company) 2007.

APPENDIX A

An Update to D'Arcy's "A Strategy for Property-Liability Insurers in Inflationary Times"

Inf — Inflation (CPI)
UPM — Underwriting Profit Margin
IIR — Investment Income Return
IIR — Investment Income ReturnIns_Ret — Insurance
Return [= IIR + UPM]
T-Bill — Treasury Bills
LTCorp – Longer Term Corporate Bonds

Variable	**1951-2006**	**1951-1976**	**1977-2006**
UPM	UPM = -0.163*Inf + [-2.658]	UPM = -0.617*Inf + [2.955]	UPM = 0.593*Inf + [-9.586]
R^2	0.6%	27.7%	8.7%
Ins_Ret	Ins_Ret = -0.422*Inf + [7.693]	Ins_Ret = -1.434*Inf + [10.77]	Ins_Ret = 0.296*Inf + [4.748]
R^2	5.2%	42.4%	3.4%
IIR	IIR = -0.259*Inf + [10.351]	IIR = -0.818*Inf + [7.815]	IIR = -0.297*Inf + [14.333]
R^2	2.1%	22.6%	14.5%
LTCorp	LTCorp = -0.014*Inf + [6.067]	LTCorp = -0.084*Inf + [3.929]	LTCorp = 1.321*Inf + [4.582]
R^2	0.0%	0.1%	5.4%
Stock	Stock = -1.192*Inf + [17.701]	Stock = -3.114*Inf + [22.733]	Stock = 0.087*Inf + [13.183]
R^2	4.6%	23.0%	0.0%
TBill	T-Bill = 0.743*Inf + [2.344]	T-Bill = 0.556*Inf + [1.873]	T-Bill = 0.77*Inf + [3.177]
R^2	50.0%	70.6%	47.0%
Adapt UPM	UPM = 0.84*UPM(-1)-0.364*Inf + [0.911] for the 1951 to 2006 Period		
R^2	69.1%		

APPENDIX B

An Update to D'Arcy's "A Strategy for Property-Liability Insurers in Inflationary Times"

DInf - Change in Inflation (CPI)
UPM - Underwriting Profit Margin
Ins_Ret - Insurance Return [= IIR + UPM]

Ins_Ret - Insurance Return [= IIR + UPM]

Variable	1951-2006	1951-1976	1977-2006
UPM	UPM = 0.59*DInf + [-3.273]		UPM = 1.528*DInf + [-6.855]
R^2	3.0%		16.2%
Ins_Ret	Ins_Ret = -0.551*DInf + [6.003]	Ins_Ret = -1.441*DInf + [5.927]	Ins_Ret = 0.783*DInf + [6.111]
R^2	3.5%	24.4%	6.8%

Modeling Paid and Incurred Losses Together

Leigh J. Halliwell, FCAS, MAAA

Abstract

The modeling skills of actuaries and academicians have developed to the point of their seeking joint models for paid and incurred losses, i.e., models in which paid and incurred losses will inform each other so that their confidence intervals will narrow and the two sets of ultimate losses will be equal. The key to such models is covariance; heteroskedastic models cannot serve the purpose. Properly accounting for covariance in the linear statistical model will provide an exact, sound, and elegant solution to the problem. Moreover, covariance is what distinguishes the same information from like information, and prevents the creation of information out of nothing.

Key concepts: linear statistical model, paid and incurred losses, seemingly unrelated regression (SUR), covariance, variance structure

The author thanks David R. Clark, FCAS, MAAA, for his helpful editing and critique of this paper.

1. INTRODUCTION

For setting the loss reserves of most casualty lines of insurance, actuaries must turn loss triangles into rectangles. Until the mid-1990s this was largely a deterministic exercise, which involved selecting development factors – perhaps with certain adjustments and sensitivity testing. Since then, business needs have demanded, and advances in theory and computing have allowed, probabilistic modeling of loss triangles. Deterministic methods are waning as actuaries are increasingly asked to estimate statistical properties of loss reserves, especially their probability distributions. However, attempts to answer this need are hampered by the duality (sometimes the multiplicity) of triangles for the same line of business. Triangles usually come in pairs, one of paid losses and another of (case-) incurred losses, which ultimately must reach equality. But when paid and incurred losses are modeled separately, any equality is accidental, and even then devoid of jointly statistical properties. A crasis of models, however artful, is not science, despite appeals to actuarial judgment.[1]

According to Gary Venter [2008, 348], "Formal modeling of paid and incurred simultaneously appears to have begun with Halliwell." But the idea received scant attention until a paper by Quarg and Mack [2004 and 2008], which spurred the Venter paper, as well as a yet unpublished paper by Zhang and Clark [2009]. Unlike the Quarg and Mack approach, which "reduces the gap" between the

[1] Actuarial judgment functions *within* actuarial science, not to the transcendence or out-guessing of science. "Actuarial judgment is no antidote …, as if actuaries possessed some expertise or intuition to herd or prod methods into correctness. … Actuaries must not presume to judge what they cannot scientifically model." [Halliwell, 2007, footnote 5]

projections, the Halliwell [1997] approach offered an exact solution. Moreover, Quarg and Mack sought the solution in the design matrix of their model, whereas the variance structure was the key for Halliwell. We believe that Halliwell was on the right track, but with some flaws. Our solution to the problem of modeling paid and incurred losses together improves on his variance approach, correcting its flaws. Furthermore, it is easier to understand, simpler to program, as well as theoretically streamlined and elegant.

In the next section we will introduce and comment upon Halliwell's version of the linear statistical model. In Section 3, while discussing basic properties of the model, we will introduce the distinction between statistical sameness and statistical similarity, which arises from differing variance structures and which ensures that information cannot be created *ex nihilo*. Section 4 will apply this theory to a simple example of a joint model of paid and incurred loss triangles, and Section 5 will show a more elaborate application to industry Workers' Compensation losses. Appendix A will deepen insights into the ideas of Section 3 by treating simultaneous equations as a subset of the linear statistical model. Finally, Appendix B will elaborate on permissible variance structures, viz., how with two random variables of known correlation a third may be correlated.

2. GENERAL FORMULATION OF THE LINEAR STATISTICAL MODEL

Many writers present versions of the linear statistical model; however, the version found in Halliwell [1997], despite its initial complexity, is most versatile and general. Moreover, most presentations stop at the estimation of the parameter β. But in his version this is just an intermediate step; the focus is on predictions based on β and their prediction-error variances. The basic form of the linear model is $\mathbf{y}_{t\times 1} = X_{t\times k}\beta_{k\times 1} + \mathbf{e}_{t\times 1}$. It is the error term $\mathbf{e}$ that makes the model statistical (otherwise called probabilistic and stochastic); it is a random vector whose moments are: $E[\mathbf{e}] = 0_{t\times 1}, Var[\mathbf{e}] = \Sigma_{t\times t} = \sigma^2\Phi_{t\times t}$. For those unfamiliar with multivariate means and variances, especially with the quadratic form $Var[A\mathbf{e}] = AVar[\mathbf{e}]A'$ and with non-negative definite and positive definite matrices, the reader is advised to read Halliwell [1997; Appendix A], Healy [1986; Chapter 7], and Judge [1988, Appendix A].

The matrix X is known as the design, or regressor, matrix; its columns are called regressor variables and independent variables. The vector $\mathbf{y}$ is called the response variable and the dependent variable, and β is the parameter of the model. But in this formulation, which emphases predictions rather than parameters, the t rows of the model are partitioned into t_1 observations and t_2 predictions. In matrix-partitioned form it is expressed:

$$\begin{bmatrix} \mathbf{y}_{1(t_1\times 1)} \\ \hline \mathbf{y}_{2(t_2\times 1)} \end{bmatrix} = \begin{bmatrix} X_{1(t_1\times k)} \\ \hline X_{2(t_2\times k)} \end{bmatrix} \beta_{k\times 1} + \begin{bmatrix} \mathbf{e}_{1(t_1\times 1)} \\ \hline \mathbf{e}_{2(t_2\times 1)} \end{bmatrix},$$

where:

$$E\begin{bmatrix} \mathbf{e}_{1(t_1\times 1)} \\ \hline \mathbf{e}_{2(t_2\times 1)} \end{bmatrix} = \begin{bmatrix} 0_{t_1\times 1} \\ \hline 0_{t_2\times 1} \end{bmatrix}, \quad Var\begin{bmatrix} \mathbf{e}_{1(t_1\times 1)} \\ \hline \mathbf{e}_{2(t_2\times 1)} \end{bmatrix} = \left[\begin{array}{c|c} \Sigma_{11(t_1\times t_1)} & \Sigma_{12(t_1\times t_2)} \\ \hline \Sigma_{21(t_2\times t_1)} & \Sigma_{22(t_2\times t_2)} \end{array}\right] = \sigma^2 \left[\begin{array}{c|c} \Phi_{11(t_1\times t_1)} & \Phi_{12(t_1\times t_2)} \\ \hline \Phi_{21(t_2\times t_1)} & \Phi_{22(t_2\times t_2)} \end{array}\right].$$

The noun 'observations' is imprecise. What distinguishes observations from predictions is that predictions are missing, or blank, elements of $\mathbf{y}$ (which signals that predictions are desired), whereas "observations" are non-missing, real-valued elements. Usually, they happen to come from observation, such as with loss amounts. However, the key to a joint model of paid and incurred losses is what we will call "tautologous observations," i.e., elements of $\mathbf{y}$ that contain zeroes as the differences by exposure period of incurred ultimate losses from paid ultimate losses. We do not actually need to observe their ultimate equality to know that it will obtain; this is information that we know *a priori* and of which we should make use. The observed and predicted elements can appear in any order; the clearest presentation may not place all the observations in rows above those of the predictions; however, our software will reorder them. Of course, since the variance structure is symmetric, the columns of $Var[\mathbf{e}]$ must likewise be reordered.

The known, or specified, elements of the linear model are the entire[2] design matrix X, the entire variance structure, whether in absolute form Σ or in relative form Φ, and the observed elements of $\mathbf{y}$. The modeler desires an estimate of $\mathbf{y}_2$, viz., $\hat{\mathbf{y}}_2$. However, that estimate will turn out to be in error by the vector $\mathbf{y}_2 - \hat{\mathbf{y}}_2$, which we will call the prediction error. The formulæ for an estimator of $\mathbf{y}_2$ and the variance of its prediction error, viz., $\hat{\mathbf{y}}_2$ and $Var[\mathbf{y}_2 - \hat{\mathbf{y}}_2]$, are:

[2] This conflicts with many linear models of loss triangles, in particular with the Quarg/Mack [2004 and 2008] model, whose predictions two or more periods into the future depend on predictions of the previous periods. The feedback loop 'predictions → regressors → predictions', which Judge [1988; Chapter 13] calls "Stochastic Regressors," is undesirable for theoretical, numerical-analytic, and æsthetic reasons, some of which Halliwell [2007; Section 5] discusses.

$$\hat{\mathbf{y}}_2 = X_2\hat{\boldsymbol{\beta}} + \Phi_{21}\Phi_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)$$

$$Var\left[\mathbf{y}_2 - \hat{\mathbf{y}}_2\right] = \left(X_2 - \Phi_{21}\Phi_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(X_2 - \Phi_{21}\Phi_{11}^{-1}X_1\right)' + \sigma^2\left(\Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12}\right),$$

where $\hat{\boldsymbol{\beta}} = \left(X_1'\Phi_{11}^{-1}X_1\right)^{-1}X_1'\Phi_{11}^{-1}\mathbf{y}_1$ and $Var\left[\hat{\boldsymbol{\beta}}\right] = \sigma^2\left(X_1'\Phi_{11}^{-1}X_1\right)^{-1}$. Or one may use the absolute-variance form, in which 'σ^2' is omitted and 'Σ' replaces 'Φ'. This estimator algebraically reduces to a linear function of $\mathbf{y}_1$, and Halliwell [1997; Appendix C] gives a version of the Gauss-Markov theorem in proof that $\hat{\mathbf{y}}_2$ is the best linear unbiased estimator (BLUE) of $\mathbf{y}_2$.[3]

A few minor conditions need to be made explicit. First, the variance structure (Σ or Φ) must be non-negative definite. Otherwise some random variable consisting of a linear combination of the elements of $\mathbf{e}$ would have a negative variance. At the very least this implies that that none of the diagonal elements of the variance structure is negative. Second, Σ_{11} or Φ_{11} must be positive definite. Being a block-diagonal part of the variance structure, it must be non-negative definite. However, a positive definite Σ_{11} or Φ_{11} has no variance degeneracy, which guarantees the existence of its inverse. Third, X_1 must be of full column rank,

[3] This estimator is unbiased in that $E\left[\hat{\mathbf{y}}_2\right] = \mathbf{y}_2$. There are infinitely many linear-in-$\mathbf{y}_1$, unbiased estimators (LUE) of $\mathbf{y}_2$. But according to the Gauss-Markov theorem, none of them is as good as $\hat{\mathbf{y}}_2$; $\hat{\mathbf{y}}_2$ is the best (B). This means that its prediction-error variance is less than theirs, in the sense that the difference of its prediction-error variance from theirs is positive definite. To the philosophically inclined it is amazing for an estimator to exist that positive-definitely dominates in the linear unbiased universe. It is no less amazing that this BLUE estimator is identical to the maximum-likelihood estimator under the assumption that the error terms are multivariate-normally distributed – an assumption not necessary to the linear statistical model. Such feelings of amazement incline most mathematicians toward a Platonic belief that mathematics is discovered, rather than toward a formalist belief that it is invented. Then again, in this abstruse realm what might be the difference between discovery and invention?

i.e., rank(X_1) = k. The second and third conditions together guarantee that $\left(X_1'\Phi_{11}^{-1}X_1\right)^{-1}$ exists.

Usually the variance structure is known only to within a scale factor; most models posit relative, not absolute, variances. In that case one must estimate σ^2 as $\hat{\sigma}^2 = \left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)'\Phi_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)\Big/\left(t_1 - k\right)$, a matrix-weighted "sum of squared residuals" divided by the degrees of freedom. Our software will display a 3×3 matrix titled "SSCP," which stands for "(matrix-weighted) sums of squares and cross products." Define the t_1×3 partitioned matrix $V = \left[\mathbf{y}_1 \mid X_1\hat{\boldsymbol{\beta}} \mid \mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right]$. Then $\text{SSCP} = V'\Phi_{11}^{-1}V$, and $\hat{\sigma}^2 = \text{SSCP}_{33}/\left(t_1 - k\right)$. Theorems of the linear model state that $\text{SSCP}_{11} = \text{SSCP}_{22} + \text{SSCP}_{33}$ and $\text{SSCP}_{32} = \text{SSCP}_{23} = 0$. $\text{SSCP}_{22}/\text{SSCP}_{11}$ is the portion of the observations that the model "explains."[4]

3. BASIC PROPERTIES OF THE LINEAR STATISTICAL MODEL

Before introducing the joint model of paid and incurred losses we must discuss some basic properties of the model and its predictions. First consider the model:

$$\begin{bmatrix} \mathbf{y}_1 \\ \hline \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ \hline I_{k\times k} \end{bmatrix}\beta + \begin{bmatrix} \mathbf{e}_1 \\ \hline 0 \end{bmatrix}, \quad Var\begin{bmatrix} \mathbf{e}_1 \\ \hline 0 \end{bmatrix} = \left[\begin{array}{c|c} \Sigma_{11} & 0 \\ \hline 0 & 0 \end{array}\right]$$

One here wants just to predict the parameter ($\mathbf{y}_2 = \beta$), so this is a check of the parameter of the model. Applying the formulæ, we confirm:

[4] Though we will call this a rho-square statistic, it differs from the commonly defined statistic that was devised for regression models containing an intercept and that strips away the explanatory power of the intercept. Our definition is appropriate to our intercept-free models.

$$\begin{aligned}
\hat{\mathbf{y}}_2 &= X_2\hat{\boldsymbol{\beta}} + \Sigma_{21}\Sigma_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right) \\
&= I\,\hat{\boldsymbol{\beta}} + 0\Sigma_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right) \\
&= \hat{\boldsymbol{\beta}} \\
Var\left[\mathbf{y}_2 - \hat{\mathbf{y}}_2\right] &= \left(X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1\right)' + \left(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right) \\
&= \left(I - 0\Sigma_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(I - 0\Sigma_{11}^{-1}X_1\right)' + \left(0 - 0\Sigma_{11}^{-1}0\right) \\
&= Var\left[\hat{\boldsymbol{\beta}}\right]
\end{aligned}$$

In fact, Halliwell [1997; 331] began with a "wordier" version of the solution and derived in this manner the streamlined version of the solution that employs $\hat{\boldsymbol{\beta}}$.

The second model is:

$$\begin{bmatrix} A\mathbf{y}_1 \\ \hline \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} AX_1 \\ \hline X_2 \end{bmatrix}\beta + \begin{bmatrix} A\mathbf{e}_1 \\ \hline \mathbf{e}_2 \end{bmatrix}, \quad Var\begin{bmatrix} A\mathbf{e}_1 \\ \hline \mathbf{e}_2 \end{bmatrix} = \left[\begin{array}{c|c} A\Sigma_{11}A' & A\Sigma_{12} \\ \hline \Sigma_{21}A' & \Sigma_{22} \end{array}\right]$$

The observed part of this model has been transformed by matrix A. The transformation affects even the error term, $\mathbf{e}_1 \rightarrow A\mathbf{e}_1$, and one purpose of this exercise is to sensitize the reader to the variance structure. For in general, $Cov\left[A\mathbf{e}_1, B\mathbf{e}_2\right] = ACov\left[\mathbf{e}_1, \mathbf{e}_2\right]B'$. If A is a nonsingular, or invertible, matrix:

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= \left(\left(AX_1\right)'\left(A\Sigma_{11}A'\right)^{-1}\left(AX_1\right)\right)^{-1}\left(AX_1\right)'\left(A\Sigma_{11}A'\right)^{-1}\left(A\mathbf{y}_1\right) \\
&= \left(X_1'A'A'^{-1}\Sigma_{11}^{-1}A^{-1}AX_1\right)^{-1}X_1'A'A'^{-1}\Sigma_{11}^{-1}A^{-1}A\mathbf{y}_1 \\
&= \left(X_1'\Sigma_{11}^{-1}X_1\right)^{-1}X_1'\Sigma_{11}^{-1}\mathbf{y}_1 \\
&= Var\left[\hat{\boldsymbol{\beta}}\right]X_1'\Sigma_{11}^{-1}\mathbf{y}_1
\end{aligned}$$

At this point we've demonstrated that the parameter estimate is unaffected. A similar cancellation of A with its inverse continues into the rest of the prediction, as the reader can verify. Hence, a one-to-one transformation of the observations has no effect on the predictions. A corollary to this is that incremental or cumulative models of loss triangles will yield the same predictions, as long as the variance structure is correctly handled.

Now consider a transformation of the predictions:

$$\begin{bmatrix} \mathbf{y}_1 \\ B\mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ BX_2 \end{bmatrix}\beta + \begin{bmatrix} \mathbf{e}_1 \\ B\mathbf{e}_2 \end{bmatrix}, \quad Var\begin{bmatrix} \mathbf{e}_1 \\ B\mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12}B' \\ B\Sigma_{21} & B\Sigma_{22}B' \end{bmatrix}$$

In this case:

$$\begin{aligned} \widehat{B\mathbf{y}}_2 &= BX_2\hat{\boldsymbol{\beta}} + B\Sigma_{21}\Sigma_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right) \\ &= B\left\{X_2\hat{\boldsymbol{\beta}} + \Sigma_{21}\Sigma_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)\right\} \\ &= B\hat{\mathbf{y}}_2 \end{aligned}$$

$$\begin{aligned} Var\left[B\mathbf{y}_2 - \widehat{B\mathbf{y}}_2\right] &= \left(BX_2 - B\Sigma_{21}\Sigma_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(BX_2 - B\Sigma_{21}\Sigma_{11}^{-1}X_1\right)' + \left(B\Sigma_{22}B' - B\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}B'\right) \\ &= B\left\{\left(X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(X_2 - \Sigma_{21}\Sigma_{11}^{-1}X_1\right)' + \left(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)\right\}B' \\ &= BVar\left[\mathbf{y}_2 - \hat{\mathbf{y}}_2\right]B' \end{aligned}$$

So the BLUE of a linear combination of predictions is the linear combination of the BLUE of the predictions. Note that here B, unlike the previous A, does not have to be invertible. The most common linear combinations of loss-triangle cells are exposure-period subtotals, i.e., unpaid, IBNR, or even ultimate losses. When we care only for these subtotals, as in the Workers' Compensation model of Section 5, this theorem allows us to bypass the extra time and space of cell-by-cell prediction and to predict them directly.

A special pair of models that will illustrate the effect of covariance is the following:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_1 \end{bmatrix}\beta + \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix}, \quad Var\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{11} \end{bmatrix}$$

Here the prediction is *like* the observation, the solution being:

$$\begin{aligned}\hat{\mathbf{y}}_2 &= X_1\hat{\boldsymbol{\beta}} + \Sigma_{21}\Sigma_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)\\ &= X_1\hat{\boldsymbol{\beta}} + 0\Sigma_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)\\ &= X_1\hat{\boldsymbol{\beta}}\end{aligned}$$

$$\begin{aligned}Var[\mathbf{y}_2 - \hat{\mathbf{y}}_2] &= \left(X_1 - \Sigma_{21}\Sigma_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(X_1 - \Sigma_{21}\Sigma_{11}^{-1}X_1\right)' + \left(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)\\ &= \left(X_1 - 0\Sigma_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(X_1 - 0\Sigma_{11}^{-1}X_1\right)' + \left(\Sigma_{11} - 0\Sigma_{11}^{-1}0\right)\\ &= X_1 Var\left[\hat{\boldsymbol{\beta}}\right]X_1' + \Sigma_{11}\end{aligned}$$

The prediction is *like*, because the variance of $\mathbf{e}_2$ is like that of $\mathbf{e}_1$, 'like' in the sense of identically distributed, but nonetheless uncovaried. But changing the variance structure to $Var\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{bmatrix} = \left[\begin{array}{c|c}\Sigma_{11} & \Sigma_{11}\\ \hline \Sigma_{11} & \Sigma_{11}\end{array}\right]$ alters the solution drastically:

$$\begin{aligned}\hat{\mathbf{y}}_2 &= X_1\hat{\boldsymbol{\beta}} + \Sigma_{21}\Sigma_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)\\ &= X_1\hat{\boldsymbol{\beta}} + \Sigma_{11}\Sigma_{11}^{-1}\left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)\\ &= X_1\hat{\boldsymbol{\beta}} + \left(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}}\right)\\ &= \mathbf{y}_1\end{aligned}$$

$$\begin{aligned}Var[\mathbf{y}_2 - \hat{\mathbf{y}}_2] &= \left(X_1 - \Sigma_{11}\Sigma_{11}^{-1}X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(X_1 - \Sigma_{11}\Sigma_{11}^{-1}X_1\right)' + \left(\Sigma_{11} - \Sigma_{11}\Sigma_{11}^{-1}\Sigma_{11}\right)\\ &= \left(X_1 - X_1\right)Var\left[\hat{\boldsymbol{\beta}}\right]\left(X_1 - X_1\right)' + \left(\Sigma_{11} - \Sigma_{11}\right)\\ &= 0\end{aligned}$$

This version predicts not something *like* $\mathbf{y}_1$, but rather something *identical* to $\mathbf{y}_1$. Covariance is the key to distinguishing between likeness and sameness.[5]

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[5] Appendix A shows covariance to be the solution to the "paradox" of why writing data twice does not increase information. It would increase, if it were *like*, or similar, information; just as repeated sampling of independent, identically-distributed random variables increases information. But repetitions of the *same* information are perfectly correlated with the original, and provide nothing new, not even when the repetition is disguised by a linear transformation. In matrix algebra, $Var\begin{bmatrix}\mathbf{e}\\ A\mathbf{e}\end{bmatrix} = \begin{bmatrix}\Sigma & \Sigma A'\\ A\Sigma & A\Sigma A'\end{bmatrix} = \begin{bmatrix}I\\ A\end{bmatrix}\Sigma\begin{bmatrix}I & A'\end{bmatrix}$, and the rank of this larger matrix is still equal to the rank of Σ. If the off-block-diagonal elements were zero the rank would increase, depending on *A*, to as much as twice the rank of Σ.

And finally, we consider pooling two models. For this purpose and for here only, the subscripts '1' and '2' identify the models (e.g., paid and incurred), rather than observations and predictions (here dismissed with a dot '•'). We might be tempted to solve the "super" model:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \hline \bullet \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \\ \hline \bullet & \bullet \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \hline \bullet \end{bmatrix}, \quad Var\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \hline \bullet \end{bmatrix} = \left[\begin{array}{cc|c} \Sigma_{11} & 0 & \bullet \\ 0 & \Sigma_{22} & \bullet \\ \hline \bullet & \bullet & \bullet \end{array}\right]$$

However, the solution for the parameter is:

$$\begin{aligned} Var\begin{bmatrix} \hat{\boldsymbol{\beta}}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} &= \left(\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}' \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \right)^{-1} \\ &= \left(\begin{bmatrix} X_1' & 0 \\ 0 & X_2' \end{bmatrix} \begin{bmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \right)^{-1} \\ &= \left(\begin{bmatrix} X_1'\Sigma_{11}^{-1}X_1 & 0 \\ 0 & X_2'\Sigma_{22}^{-1}X_2 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \left(X_1'\Sigma_{11}^{-1}X_1\right)^{-1} & 0 \\ 0 & \left(X_2'\Sigma_{22}^{-1}X_2\right)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} Var\left[\hat{\boldsymbol{\beta}}_1\right] & 0 \\ 0 & Var\left[\hat{\boldsymbol{\beta}}_2\right] \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \hat{\boldsymbol{\beta}}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} &= Var\begin{bmatrix} \hat{\boldsymbol{\beta}}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} \left(\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}' \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} Var\left[\hat{\boldsymbol{\beta}}_1\right] & 0 \\ 0 & Var\left[\hat{\boldsymbol{\beta}}_2\right] \end{bmatrix} \left(\begin{bmatrix} X_1'\Sigma_{11}^{-1}\mathbf{y}_1 \\ X_2'\Sigma_{22}^{-1}\mathbf{y}_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} Var\left[\hat{\boldsymbol{\beta}}_1\right] X_1'\Sigma_{11}^{-1}\mathbf{y}_1 \\ Var\left[\hat{\boldsymbol{\beta}}_2\right] X_2'\Sigma_{22}^{-1}\mathbf{y}_2 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\boldsymbol{\beta}}_1 \\ \hat{\boldsymbol{\beta}}_2 \end{bmatrix} \end{aligned}$$

Hence, simply juxtaposing two (or more) models provides no additional information.[6] There is no covariance between the two; they are "frictionless" and "like ships passing in the night." But if the off-block-diagonal variance were not zero, the combined model would not reduce to the separate submodels. Judge [1988; Section 11.2] calls covariance-linked models "seemingly unrelated regression" (SUR) models, for they seem to be unrelated if one considers only the model design and ignores the variance structure. The Zhang/Clark [2009] model is an SUR model, tying paid and incurred losses together with covariance; but it does not guarantee the equality of ultimate paid and incurred. The Halliwell model [1997] also qualifies as SUR. The model that we will present in the next section is not an SUR one; rather, tautologous equations will be the glue between paid and incurred, and the only tricky part will be the effect of these additional equations on the variance structure.

4. TAUTOLOGY AND A SIMPLE JOINT PAID-INCURRED MODEL

The following example comes from Halliwell [1997; Exhibit 1]. It consists of incremental paid and incurred losses for three accident years at three years of development, development being complete at the third year:

	Paid			Incurred		
	@1	@2	@3	@1	@2	@3
AY1	50	30	20	75	15	10
AY2	60	25		75	25	
AY3	45			50		

[6] However, one caveat: The combined model in relative-variance format would invite the modeler to estimate an overall variance scale $\hat{\sigma}^2$, which would be an average of the scale estimates of the submodels weighted according to their respective degrees of freedom.

The AY exposures are equal, and the cells are homoskedastic. The first accident year is mature, and both paid and incurred losses accumulate to 100. The paid and incurred models have the same design matrix (viz., additive, cf. Halliwell [2007; 228]), and the parameter elements are pure premiums by loss type and by age or development period. All cells have the same variance relativity and zero covariance. In symbols, the model of the *hij*th cell is $\boldsymbol{y}_{hij} = 1 \cdot \beta_{hj} + \boldsymbol{e}_{hij}$, $Var[\boldsymbol{e}_{hij}] = \sigma^2 \cdot 1$, where $h \in$ {1 = Paid, 2 = Incurred}, $i \in$ {1 = AY1, 2 = AY2, 3 = AY3}, and $j \in$ {1 = Age1, 2 = Age2, 3 = Age3}. If the accident years were not of equal exposure, the model would be $\boldsymbol{y}_{hij} = \xi_i \beta_{hj} + \boldsymbol{e}_{hij}$, $Var[\boldsymbol{e}_{hij}] = \sigma^2 \xi_i$, where ξ_i is the exposure of the *i*th accident year. In this simple, juxtaposed model one would estimate paid development at ages 2 and 3 as 27.5 and 20, and incurred development likewise as 20 and 10. The paid ultimate losses of AY2 and AY3 would be 95 and 92.5, as compared with incurred ultimate losses of 110 and 80.

Exhibit 1 contains the joint model in the standard form:

$$\begin{bmatrix} \mathbf{y}_{1\,(14\times1)} \\ \hdashline \mathbf{y}_{2\,(6\times1)} \end{bmatrix} = \begin{bmatrix} X_{1\,(14\times6)} \\ \hdashline X_{2\,(6\times6)} \end{bmatrix} \beta_{6\times1} + \begin{bmatrix} \mathbf{e}_{1\,(14\times1)} \\ \hdashline \mathbf{e}_{2\,(6\times1)} \end{bmatrix}, \quad Var\begin{bmatrix} \mathbf{e}_1 \\ \hdashline \mathbf{e}_2 \end{bmatrix} = \sigma^2 \left[\begin{array}{c:c} \Phi_{11\,(14\times14)} & \Phi_{12\,(14\times6)} \\ \hdashline \Phi_{21\,(6\times14)} & \Phi_{22\,(6\times6)} \end{array}\right]$$

The reader will recognize the $\mathbf{y}$, X, and Φ matrices within the exhibit, and the dotted lines partition the matrices into observations and predictions. Zeroes are present, but not shown. Except for rows 13 and 14, marked 'Diff', and corresponding columns 13 and 14 of Φ, the model would consist of two, unrelated paid and incurred homoskedastic submodels. The '1's in the design matrix represent the unitary exposure slotted into the *hj*th column of X so as to interact with the *hj*th element of the model parameter β (implicit in the exhibit, but not shown).

But it is the rows marked 'Diff' that join, or laminate, the paid and the incurred losses. Although we have not observed six of the eighteen cells, we do know that total paid must equal total incurred by AY. For AY2 the difference is:

$$\begin{aligned}0 &= \boldsymbol{y}_{121} + \boldsymbol{y}_{122} + \boldsymbol{y}_{123} - \boldsymbol{y}_{221} - \boldsymbol{y}_{222} - \boldsymbol{y}_{223} \\ &= \beta_{11} + \beta_{12} + \beta_{13} - \beta_{21} - \beta_{22} - \beta_{23} + \{\boldsymbol{e}_{121} + \boldsymbol{e}_{122} + \boldsymbol{e}_{123} - \boldsymbol{e}_{221} - \boldsymbol{e}_{222} - \boldsymbol{e}_{223}\} \\ &= [1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1]\beta + \{\boldsymbol{e}_{\text{Diff2}}\}\end{aligned}$$

So this tautology, or difference equation, is equivalent to a zero observation, the design $[1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1]$, and error term that involves six other error terms. Because of heteroskedasticity, the variance of this error term is 6. However, because these are *same* error terms, not *like* ones, $Cov[\boldsymbol{e}_{\text{Diff2}}, \pm\boldsymbol{e}_{hij}] = \pm Var[\boldsymbol{e}_{hij}]$. So the reader should now understand row 13 of the model, and similarly, row 14, the tautology for AY3. We could have added a tautology for AY1; however, its variance structure involves no predictions. It would add no new observation, and the resulting 15×15 matrix Φ_{11} would still be of rank 14 (see footnote 5) and thus non-invertible.

In Exhibit 2 we solve for the estimates of β and σ, and derive the SSCP matrix, as explained in Section 2. In the spreadsheet we simplified the notation by dropping subscripts (which are all '1', pertaining to observations) and carets. Both paid and incurred total pure premiums equal $97.91\overline{6}$, which is the average of the stand-alone total pure premiums of paid $99.1\overline{6}$ and incurred $96.\overline{6}$.

The predictions are estimated in Exhibit 3 according to the formulæ:

$$\hat{\mathbf{y}}_2 = X_2\hat{\boldsymbol{\beta}} + \Phi_{21}\Phi_{11}^{-1}(\mathbf{y}_1 - X_1\hat{\boldsymbol{\beta}})$$

$$Var[\mathbf{y}_2 - \hat{\mathbf{y}}_2] = (X_2 - \Phi_{21}\Phi_{11}^{-1}X_1)Var[\hat{\boldsymbol{\beta}}](X_2 - \Phi_{21}\Phi_{11}^{-1}X_1)' + \sigma^2(\Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12})$$

To help the reader who wishes to reproduce the results, in the bottom half of the exhibit are intermediate calculations. The prediction and the variance of the prediction error are:

Type	AY	Age	$\hat{\mathbf{y}}_2$	$Var[\mathbf{y}_2 - \hat{\mathbf{y}}_2]$					
Paid	2	3	22.50	79.95	0	39.97	79.95	0	39.97
Paid	3	2	23.75	0	89.94	-29.98	0	29.98	29.98
Paid	3	3	17.50	39.97	-29.98	109.93	39.97	29.98	49.97
Incd	2	3	7.50	79.95	0	39.97	79.95	0	39.97
Incd	3	2	23.75	0	29.98	29.98	0	89.94	-29.98
Incd	3	3	12.50	39.97	29.98	49.97	39.97	-29.98	109.93

In the topmost figure of the exhibit AY subtotals are formed and combined with the paid and incurred to date. For example, the calculation of the prediction-error standard deviation of the AY3 IBNR is $11.83 = \sqrt{89.94 - 29.98 - 29.98 + 109.93}$. Although it may seem a wonder that the means and variances of the ultimate losses are identical whether one builds them from the paid to date or from the incurred, the model was constructed for this purpose. The identity serves only to confirm that the model was solved without mistake.

This technique is more general than the tautology of $0 = Paid - Incd$. In addition to the observations $\mathbf{y}_1$ of the model $\mathbf{y} = X\beta + \mathbf{e}$, one may know by means other than observation that $\mathbf{z}_{t_3\times 1} = Q_{t_3\times t}\mathbf{y}_{t\times 1}$. Hence:

$$\mathbf{z} = Q(X\beta + \mathbf{e}) = QX\beta + Q\mathbf{e}$$

$$Var[\mathbf{z}] = Var[Q\mathbf{e}] = Q\Sigma Q'$$

$$Cov[\mathbf{z}, \mathbf{e}_1] = Cov[Q\mathbf{e}, \mathbf{e}_1] = QCov[\mathbf{e}, \mathbf{e}_1] = QCov\left[\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{bmatrix}, \mathbf{e}_1\right] = Q\begin{bmatrix}Cov[\mathbf{e}_1, \mathbf{e}_1]\\ Cov[\mathbf{e}_2, \mathbf{e}_1]\end{bmatrix} = Q\begin{bmatrix}\Sigma_{11}\\ \Sigma_{21}\end{bmatrix}$$

$$Cov[\mathbf{z}, \mathbf{e}_2] = Cov[Q\mathbf{e}, \mathbf{e}_2] = \ldots = Q\begin{bmatrix}\Sigma_{12}\\ \Sigma_{22}\end{bmatrix}$$

Augmenting the model for the t_3 new observations, we arrive at the form:

$$\begin{bmatrix}\mathbf{y}_1\\ \mathbf{z}\\ \hline \mathbf{y}_2\end{bmatrix} = \begin{bmatrix}X_1\\ QX\\ \hline X_2\end{bmatrix}\beta + \begin{bmatrix}\mathbf{e}_1\\ Q\mathbf{e}\\ \hline \mathbf{e}_2\end{bmatrix}, \quad Var\begin{bmatrix}\mathbf{e}_1\\ Q\mathbf{e}\\ \hline \mathbf{e}_2\end{bmatrix} = \left[\begin{array}{cc|c}\Sigma_{11} & [\Sigma_{11} \;\; \Sigma_{12}]Q' & \Sigma_{12}\\ Q\begin{bmatrix}\Sigma_{11}\\ \Sigma_{21}\end{bmatrix} & Q\Sigma Q' & Q\begin{bmatrix}\Sigma_{12}\\ \Sigma_{22}\end{bmatrix}\\ \hline \Sigma_{21} & [\Sigma_{21} \;\; \Sigma_{22}]Q' & \Sigma_{22}\end{array}\right]$$

From the juxtaposed model the simple joint model will arise according to this form, if $\mathbf{z}$ is zero and:

$$Q = \begin{bmatrix}0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & -1\end{bmatrix}$$

It is a powerful extension of the linear statistical model; nevertheless, the same two questions inform every statistical model: "What is the equation for each row?" and "How does each row covary with itself and the other rows?" [7] Next we will apply the joint model to ten accident years of industry Workers' Compensation losses.

5. TAUTOLOGY AND A JOINT WORKERS' COMPENSATION MODEL

Exhibit 4 contains net paid and case-incurred (incurred less bulk and IBNR) triangles for U.S. Worker's Compensation, along with net earned premium and ultimate loss. Ignoring the prior-to-1998 line, we have ten accident years at ten evaluations. But we will project beyond the tenth evaluation (@120 months) to ultimate, our models assuming that at 120 months paid losses are 85.5% of ultimate and incurred are 95.0%. So our model will work with two 10×11 rectangles, each with fifty-five observations. However, to save space we will not predict each future cell, but only the unpaid and IBNR totals by AY. Thus, each of the paid and incurred submodels will have fifty-five observations and ten predictions.

It is not necessary to do so, but we will use the same design matrix for both submodels, an additive design with pure premiums by age (cf. Halliwell [2007;

[7] Halliwell's solution [1997; Exhibit 14] differs from ours only in the prediction error variance, and there only because of a disagreement over the estimate of σ^2: his 106.597 versus our 79.948. And this is due to a difference of degrees of freedom, his six versus our eight (106.597/79.948 = 8/6). Halliwell [1997; 247-249] both constrained the variance structure and imposed constraints on β. This seems to have double-counted some observations and reduced the degrees of freedom. Our approach is more easily understood, stays closer to the empirical data, and does not require eigen-decompostion of variance matrices. Moreover, it will not disturb the variance relativities of non-homoskedastic models.

Section 7 and Exhibits 7A and 7B]). Because we have no exposure data, net earned premium will have to suffice. However, due to the underwriting cycle, we must adjust it to a constant loss ratio. Without rate-change information, we needed to develop the triangles to ultimate with standard deterministic methods. It's not desirable, perhaps it even smacks of cheating; but frequently it's a necessary evil, and its circularity does not seem to be vicious. The adjusted, or on-level, premium summary appears in Exhibit 5. The "Selected" losses are the simple average of the booked ultimates and four development methods. The overall loss ratio is 71%, and "Adj Prem" is simply the selected losses divided by this loss ratio, which conserves total premium. Since premium is the exposure base, our pure-premium betas are actually loss ratios.

Recognizing that the volatility, or unit-variance, of the incremental losses varies by age, we must also derive variances, or at least variance relativities, for a heteroskedastic model. We use the additive method on adjusted premium in Exhibits 6. First we derive pure premiums by age in Exhibit 6.1. The pure premium from 120 months to ultimate is calculated as:

$$\beta_{ult} = \left(\frac{1}{m} - 1\right)\left(\beta_{12} + \ldots + \beta_{120}\right),$$

where m, the maturity at 120 months, is 0.855 for paid and 0.950 for incurred. We assume that variance is proportional to exposure, and the "Selected" rows of Exhibit 6.2 derive the paid and incurred unit variances with some judgmental smoothing and extrapolation. The selected unit variances are then multiplied in Exhibit 6.3 by the adjusted premiums. We will treat these as absolute variances, and the "Unpaid" and "IBNR" columns contain the sums of the variances of AY predictions.

Exhibits 7.1 and 7.2 present the separate paid and incurred models. As mentioned, they share the same design matrix. The column marked 'Σ' contains the homoskedastic variances; the column header 'Σ', rather than 'Φ', signals our modeling software to take these as absolute variances. Each model has sixty-six rows, fifty-five observations, ten predictions of AY totals, and one row marked 'Constraint'. Since there is no observation beyond 120 months, the constraint (note its zero variance) allows for the estimation of β_{ult}. In keeping with the two maturities, each constraint is $0 = (1 - m)(\beta_{12} + \ldots + \beta_{120}) - m\beta_{ult}$.

Part of the joint paid-incurred model appears in Exhibit 8.1. A new column 'Type' identifies the paid and incurred submodels, which one can recognize in the block diagonal form. To these one hundred thirty-two rows were added the tautologous observations, 'Ult =' for each accident year. The negative exposure in the incurred half of these ten rows indicates that the zero '**y**' values are the difference of incurred from paid (paid minus incurred). The variance 'Σ' of each new row is the sum of all the paid and incurred variances of its AY. However, the variance structure of the model at this point is the 'Σ' column distributed down the main diagonal of a 142×142 matrix. Variance matrices, though large (sometimes exceeding the limit of 256 columns of a Excel spreadsheet) are often sparse. Our software allows us to specify only the columns of non-zero covariance, and to treat the rest of the covariance as zero. In Exhibit 8.2 we have specified how the last ten rows, the tautologous observations covary with all 142 rows. Each column (e.g., the 1998 'Covariance' column) records the 'Σ' values of its AY, except that 'Type' = 'Incd' rows must be negated, since incurred rows are subtracted in the tautology. Let C be the 142×10 covariance matrix of this exhibit. The software knows to insert C into the last ten columns and to insert C' into the last ten rows of the 142×142 variance structure. This completes the joining of the two types of losses. The two modeling questions are answered: "What is the expected value of each row?" and "How does each row vary with itself and covary with the others?"

Exhibit 9.1 presents some diagnostics. The model had $122 = 2(55+1)+10$ observations; however, two of these were constraints. After transformation (see footnote 9), a model with 122 observations in 22 parameters becomes one with 120 observations and 20 parameters. Readers wishing to solve the model in Excel can do so by reformulating '@Ult' predictions in terms of paid and incurred $\beta_{12}, \ldots, \beta_{120}$ (if Excel will invert a 120×120 matrix). Two paid observations, AY 1998@12 and AY 2001@60, are more than two standard units away from expected. This is apparent even from the paid residuals of Exhibit 6.2. However, the diagnostics are not unusual, and we will focus on the estimates of β and $\mathbf{y}_2$, as found in Exhibit 9.2. As for 'Betahat', the first eleven elements are paid, and the second eleven incurred. They both sum to 0.696. This pure-premium equality would not obtain, if the exposures were adjusted for inflation and changes in claim processing; nonetheless, the ultimate AY equality would still be preserved. The reader may verify that $\hat{\beta}_{12} + \ldots \hat{\beta}_{120}$ equals 0.855×0.696 for paid and 0.950×0.696 for incurred, as required by the constraints.

The unpaid and IBNR predictions are transferred from Exhibit 9.2 to summary Exhibit 10. The 'Joint Paid-Incurred' box shows agreement of ultimate loss and the standard deviation of its prediction error throughout the ten accident years and in total. The total 'Std Dev' of ±1,196,054 equals the square root of the sum of the diagonal 11×11 blocks of the 'VarPrdErr' matrix of Exhibit 9.2. We also ran separate paid and incurred models from Exhibits 7, and summarized them in the right side of Exhibit 10. Not surprisingly, the joint model mediates between the separate models: $218{,}374{,}758 \in (217{,}725{,}562,\ 219{,}198{,}836)$. This holds true by accident year except for AY 2000. But the joint modeling produces second-moment estimates that dominate those of the submodels, both in total and by accident year.

6. CONCLUSION

In a footnote of the introduction we quoted, "Actuaries must not presume to judge what they cannot scientifically model." For a time science may endure competing theories, but eventually one will prevail. Likewise, the *ad hoc* blending of models, especially those arising from paid and incurred data sources, is a stopgap. For at some point it puts knowledge at the mercy of intuition at best, and of whim at worst. Hence actuaries have begun to search for joint models. Here we have shown that the linear statistical model is versatile enough to satisfy the search. The key, as always, is to ask first what all the equations are and second how they covary with each other. We may dub these the "first and second moment" questions. Actuaries have made rapid progress on the first-moment question; we hope that this paper will spur progress on the second.[8]

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[8] Since 2000 the topic of generalized linear models (GLM) has received much attention from actuaries and academicians. One is easily lulled into thinking that generalizing is only in one direction. Consider a linear statistical model (LSM) $\mathbf{y} = Xb + \mathbf{e}, Var[\mathbf{e}] = diag(\sigma)$, i.e., a heteroskedastic model. GLM generalizes it with a link function and with a distributional form of **e**, for example as: $\mathbf{y} = g^{-1}(Xb) + \mathbf{e} - E[\mathbf{e}], \mathbf{e} \sim distribution(\Theta)$ [Anderson, 2004; 13-14]. The LSM is a GLM whose link is the identity function and whose distribution is multivariate normal. Now the link function can be accommodated with a *non-linear* statistical model (Halliwell [1997; 325-326] and Judge [1988; Chapter 12]). Hence, many consider the advantage of GLM over LSM to reside in non-normal error terms. However, the density of **e** is invariably assumed to be the product of the densities of the elements of **e**, which implies zero covariance. GLM does not generalize the variance structure beyond heteroskedasticity. Our linear model generalizes the LSM in a different direction from the one in which GLM generalizes it. We do not wish to gainsay GLM; certainly, no one has a panacea. However, to him whose only tool is a hammer everything looks like a nail. Enthusiasm over GLM may distract actuaries from asking the second-moment question. If covariance is key to the joint paid-incurred model, GLM will not provide an acceptable solution.

REFERENCES

Anderson, Duncan, Feldblum, Sholom, *et al*., "A Practitioner's Guide to Generalized Linear Models," *2004 Discussion Paper Program: Applying and Evaluating Generalized Linear Models*, Casualty Actuarial Society, 2004, 1-116, www.casact.org/pubs/dpp/dpp04/04dpp1.pdf.

Halliwell, Leigh J., "Conjoint Prediction of Paid and Incurred Losses," *1997 Loss Reserving Discussion Papers*, Casualty Actuarial Society, 1997, 241-379, www.casact.org/pubs/forum/97sforum/97sf1241.pdf.

"Chain-Ladder Bias: Its Reason and Meaning," *Variance*, 1:2, 2007, 214-247, www.variancejournal.org/issues/01-02/214.pdf.

Healy, M. J. R., *Matrices for Statistics*, Oxford, Clarendon Press, 1986.

Judge, George G., Hill, R. C., *et al*., *Introduction to the Theory and Practice of Econometrics* (Second Edition), New York, John Wiley & Sons, 1988.

Quarg, Gerhard, and Mack, Thomas, "Munich Chain Ladder: A Reserving Method that Reduces the Gap between IBNR Projections Based on Paid Losses and IBNR Projections Based on Incurred Losses," *Variance*, 2:2, 2008, 266-299, www.variancejournal.org/issues/02-02/266.pdf. Reprinted from Blätter der Deutschen Gesellschaft für Versicherungs- und Finanzmathematik, 26:4, 2004, 597-630.

Venter, Gary G., "Distribution and Value of Reserves Using Paid and Incurred Triangles," Casualty Actuarial Society *E-Forum*, Fall 2008, 348-375, www.casact.org/pubs/forum/08fforum/15Venter.pdf.

Zhang, Yanwei, and Clark, David R., "A Bivariate Approach to Setting Reserves," unpublished as of March 2009.

Exhibit 1

Joint Paid-Incurred Model

				Paid			Incd																						
				1	2	3	1	2	3																				
Type	AY	Age	**y**	*X*						Φ																			
Paid	1	1	50	1						1																			
Paid	1	2	30		1						1																		
Paid	1	3	20			1						1																	
Paid	2	1	60	1									1									1							
Paid	2	2	25		1									1								1							
Paid	3	1	45	1											1								1						
Incd	1	1	75				1									1													
Incd	1	2	15					1									1												
Incd	1	3	10						1									1											
Incd	2	1	75				1												1			-1							
Incd	2	2	25					1												1		-1							
Incd	3	1	50				1														1		-1						
Diff	2	Ult	0	1	1	1	-1	-1	-1				1	1					-1	-1		6		1			-1		
Diff	3	Ult	0	1	1	1	-1	-1	-1						1						-1		6		1	1		-1	-1
Paid	2	3				1																1		1					
Paid	3	2			1																		1		1				
Paid	3	3				1																	1			1			
Incd	2	3							1													-1					1		
Incd	3	2						1															-1					1	
Incd	3	3							1														-1						1

Exhibit 2

Solution of the Joint Paid-Incurred Model

$X'\Phi^{-1}\mathbf{y}$	$X'\Phi^{-1}X$					
155	3	0	0	0	0	0
56.25	0	2.25	0.25	0	-0.25	-0.25
28.75	0	0.25	1.75	0	-0.25	-0.75
200	0	0	0	3	0	0
38.75	0	-0.25	-0.25	0	2.25	0.25
1.25	0	-0.25	-0.75	0	0.25	1.75

	β	$(X'\Phi^{-1}X)^{-1}$					
Paid1	51.666667	0.3333333	0	0	0	0	0
Paid2	26.25	0	0.4583333	-0.041667	0	0.0416667	0.0416667
Paid3	20	0	-0.041667	0.7083333	0	0.0416667	0.2916667
Incd1	66.666667	0	0	0	0.3333333	0	0
Incd2	21.25	0	0.0416667	0.0416667	0	0.4583333	-0.041667
Incd3	10	0	0.0416667	0.2916667	0	-0.041667	0.7083333

Var[β]					
26.649306	0	0	0	0	0
0	36.642795	-3.331163	0	3.3311632	3.3311632
0	-3.331163	56.629774	0	3.3311632	23.318142
0	0	0	26.649306	0	0
0	3.3311632	3.3311632	0	36.642795	-3.331163
0	3.3311632	23.318142	0	-3.331163	56.629774

Type	AY	Age	**y**	Xβ	**e**
Paid	1	1	50	51.666667	-1.666667
Paid	1	2	30	26.25	3.75
Paid	1	3	20	20	0
Paid	2	1	60	51.666667	8.3333333
Paid	2	2	25	26.25	-1.25
Paid	3	1	45	51.666667	-6.666667
Incd	1	1	75	66.666667	8.3333333
Incd	1	2	15	21.25	-6.25
Incd	1	3	10	10	0
Incd	2	1	75	66.666667	8.3333333
Incd	2	2	25	21.25	3.75
Incd	3	1	50	66.666667	-16.66667
Diff	2	Ult	0	0	0
Diff	3	Ult	0	0	0

	SSCP	
24868.75	24229.167	639.58333
24229.167	24229.167	0
639.58333	0	639.58333
100.0%	97.4%	2.6%

t	14
k	6
df	8
σ^2	79.947917

Exhibit 3

Prediction of the Joint Paid-Incurred Model

AY	Paid	Incd	Unpaid		IBNR		Ultimate	
1	100	100	0	± 0	0	± 0	100	± 0
2	85	100	22.50	± 8.94	7.50	± 8.94	107.50	± 8.94
3	45	50	41.25	± 11.83	36.25	± 11.83	86.25	± 11.83
Total	230	250	63.75	± 17.31	43.75	± 17.31	293.75	± 17.31

Type	AY	Age	$\hat{\mathbf{y}}_2$	*Std* [PE]	*Var* [Prediction Error]					
Paid	2	3	22.50	± 8.94	79.95	0	39.97	79.95	0	39.97
Paid	3	2	23.75	± 9.48	0	89.94	-29.98	0	29.98	29.98
Paid	3	3	17.50	± 10.48	39.97	-29.98	109.93	39.97	29.98	49.97
Incd	2	3	7.50	± 8.94	79.95	0	39.97	79.95	0	39.97
Incd	3	2	23.75	± 9.48	0	29.98	29.98	0	89.94	-29.98
Incd	3	3	12.50	± 10.48	39.97	29.98	49.97	39.97	-29.98	109.93

$X_2 - \Phi_{21}\Phi_{11}^{-1}X_1$

0	0	0.5	0	0	0.5
0	0.75	-0.25	0	0.25	0.25
0	-0.25	0.75	0	0.25	0.25
0	0	0.5	0	0	0.5
0	0.25	0.25	0	0.75	-0.25
0	0.25	0.25	0	-0.25	0.75

$\Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12}$

0.5	0	0	0.5	0	0
0	0.75	-0.25	0	0.25	0.25
0	-0.25	0.75	0	0.25	0.25
0.5	0	0	0.5	0	0
0	0.25	0.25	0	0.75	-0.25
0	0.25	0.25	0	-0.25	0.75

$\Phi_{21}\Phi_{11}^{-1}$

0	0	0	-0.5	-0.5	0	0	0	0	0.5	0.5	0	0.5	0
0	0	0	0	0	-0.25	0	0	0	0	0	0.25	0	0.25
0	0	0	0	0	-0.25	0	0	0	0	0	0.25	0	0.25
0	0	0	0.5	0.5	0	0	0	0	-0.5	-0.5	0	-0.5	0
0	0	0	0	0	0.25	0	0	0	0	0	-0.25	0	-0.25
0	0	0	0	0	0.25	0	0	0	0	0	-0.25	0	-0.25

Exhibit 4

Industry Workers' Compensation Net Losses (000)

Cumulative Paid

AY	EarnPrem	@12	@24	@36	@48	@60	@72	@84	@96	@108	@120
1998	23,278,084	4,651,588	9,585,142	12,606,256	14,094,760	15,268,042	16,074,584	16,668,642	17,106,835	17,422,941	17,738,999
1999	21,555,421	4,211,880	9,632,480	12,750,495	14,618,989	15,637,068	16,221,974	16,753,957	17,200,779	17,557,257	
2000	23,495,444	4,553,584	10,366,172	13,709,157	15,579,342	16,724,292	17,365,134	17,961,104	18,432,885		
2001	25,864,065	4,556,995	10,343,323	13,761,573	15,619,782	16,358,074	16,800,979	17,289,118			
2002	29,134,414	4,262,115	9,525,796	12,527,871	14,177,862	15,284,598	15,899,281				
2003	32,391,860	4,274,440	9,451,725	12,390,213	14,138,206	15,283,538					
2004	36,533,278	4,624,395	9,798,635	12,473,626	14,134,508						
2005	39,208,849	4,865,363	9,946,876	12,789,801							
2006	42,065,555	5,130,174	10,724,002								
2007	40,220,014	5,211,936									

Cumulative Case-Incurred

AY	Ultimate	@12	@24	@36	@48	@60	@72	@84	@96	@108	@120
1998	20,815,720	10,440,449	14,526,669	16,215,164	17,259,403	18,111,150	18,727,822	19,147,843	19,469,090	19,540,774	19,765,070
1999	21,107,246	10,104,076	14,366,317	16,374,957	17,641,331	18,407,246	18,857,403	19,336,715	19,548,265	19,812,936	
2000	22,339,113	10,614,330	15,701,665	17,701,687	18,844,583	19,668,677	20,094,561	20,402,031	20,739,446		
2001	21,958,321	11,104,926	15,846,924	17,963,819	18,932,871	19,179,055	19,484,216	19,811,447			
2002	21,039,160	10,379,583	15,108,660	16,994,756	17,685,953	18,233,994	18,589,679				
2003	21,658,869	10,932,703	15,324,420	16,898,562	17,695,675	18,314,350					
2004	22,204,956	11,239,343	15,320,398	16,843,250	17,632,172						
2005	23,445,324	11,978,411	15,632,319	17,221,257							
2006	26,885,991	12,468,437	16,822,179								
2007	27,906,944	12,931,177									

Exhibit 5

Comparison of Ultimates and On-Level Premium

AY	EarnPrem	Ultimate	CL Paid	CL Incd	Add Paid	Add Incd	Selected
1998	23,278,084	20,815,720	20,747,367	20,805,337	20,226,368	20,881,833	20,695,325
1999	21,555,421	21,107,246	20,907,312	21,095,112	20,153,220	21,074,491	20,867,476
2000	23,495,444	22,339,113	22,380,335	22,271,937	21,614,963	22,286,578	22,178,585
2001	25,864,065	21,958,321	21,545,915	21,589,768	21,305,566	21,762,406	21,632,395
2002	29,134,414	21,039,160	20,472,774	20,661,071	21,107,195	21,042,136	20,864,467
2003	32,391,860	21,658,869	20,446,996	20,823,394	21,885,290	21,389,016	21,240,713
2004	36,533,278	22,204,956	20,265,744	20,762,892	23,064,618	21,607,790	21,581,200
2005	39,208,849	23,445,324	20,806,146	21,420,815	24,850,676	22,488,817	22,602,355
2006	42,065,555	26,885,991	22,848,638	23,395,295	28,090,567	24,939,425	25,231,983
2007	40,220,014	27,906,944	24,129,598	25,119,943	28,910,427	26,647,699	26,542,922
Total	313,746,984	229,361,644	214,550,827	217,945,563	231,208,890	224,120,191	223,437,423

AY	Adj Prem	Ultimate	CL Paid	CL Incd	Add Paid	Add Incd	Selected
1998	29,060,019	89%	89%	89%	87%	90%	89%
1999	29,301,751	98%	97%	98%	93%	98%	97%
2000	31,142,788	95%	95%	95%	92%	95%	94%
2001	30,375,837	85%	83%	83%	82%	84%	84%
2002	29,297,526	72%	70%	71%	72%	72%	72%
2003	29,825,844	67%	63%	64%	68%	66%	66%
2004	30,303,950	61%	55%	57%	63%	59%	59%
2005	31,737,838	60%	53%	55%	63%	57%	58%
2006	35,430,317	64%	54%	56%	67%	59%	60%
2007	37,271,114	69%	60%	62%	72%	66%	66%
Total	313,746,984	73%	68%	69%	74%	71%	71%

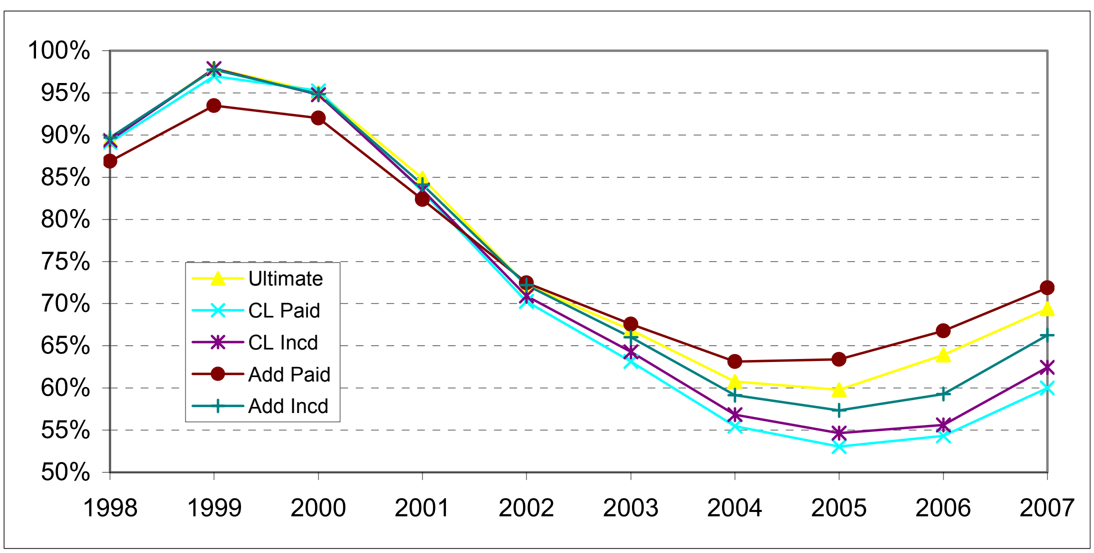

Exhibit 6.1

Additive Projections with On-Level Premium

Incremental Paid

AY	Adj Prem	@12	@24	@36	@48	@60	@72	@84	@96	@108	@120	@Ult	Ultimate	LR
1998	29,060,019	4,651,588	4,933,554	3,021,114	1,488,504	1,173,282	806,542	594,058	438,193	316,106	316,058	2,924,110	20,663,109	71%
1999	29,301,751	4,211,880	5,420,600	3,118,015	1,868,494	1,018,079	584,906	531,983	446,822	356,478	318,687	2,948,434	20,824,378	71%
2000	31,142,788	4,553,584	5,812,588	3,342,985	1,870,185	1,144,950	640,842	595,970	471,781	358,902	338,710	3,133,685	22,264,182	71%
2001	30,375,837	4,556,995	5,786,328	3,418,250	1,858,209	738,292	442,905	488,139	460,466	350,063	330,369	3,056,512	21,486,528	71%
2002	29,297,526	4,262,115	5,263,681	3,002,075	1,649,991	1,106,736	614,683	540,138	444,120	337,636	318,641	2,948,009	20,487,825	70%
2003	29,825,844	4,274,440	5,177,285	2,938,488	1,747,993	1,145,332	617,774	549,878	452,129	343,725	324,387	3,001,170	20,572,601	69%
2004	30,303,950	4,624,395	5,174,240	2,674,991	1,660,882	1,071,056	627,677	558,692	459,376	349,235	329,587	3,049,279	20,579,410	68%
2005	31,737,838	4,865,363	5,081,513	2,842,925	1,841,463	1,121,735	657,376	585,128	481,113	365,759	345,182	3,193,561	21,381,119	67%
2006	35,430,317	5,130,174	5,593,828	3,580,408	2,055,705	1,252,242	733,858	653,204	537,087	408,313	385,342	3,565,110	23,895,269	67%
2007	37,271,114	5,211,936	6,503,618	3,766,430	2,162,510	1,317,302	771,986	687,141	564,991	429,527	405,362	3,750,337	25,571,140	69%
	Pure Prem	0.148	0.174	0.101	0.058	0.035	0.021	0.018	0.015	0.012	0.011	0.101	217,725,562	69%

Incremental Case-Incurred

AY	Adj Prem	@12	@24	@36	@48	@60	@72	@84	@96	@108	@120	@Ult	Ultimate	LR
1998	29,060,019	10,440,449	4,086,220	1,688,495	1,044,239	851,747	616,672	420,021	321,247	71,684	224,296	1,015,137	20,805,337	72%
1999	29,301,751	10,104,076	4,262,241	2,008,640	1,266,374	765,915	450,157	479,312	211,550	264,671	226,162	1,023,581	21,093,787	72%
2000	31,142,788	10,614,330	5,087,335	2,000,022	1,142,896	824,094	425,884	307,470	337,415	179,484	240,372	1,087,893	22,272,950	72%
2001	30,375,837	11,104,926	4,741,998	2,116,895	969,052	246,184	305,161	327,231	295,330	175,064	234,452	1,061,102	21,596,099	71%
2002	29,297,526	10,379,583	4,729,077	1,886,096	691,197	548,041	355,685	374,902	284,846	168,850	226,129	1,023,434	20,678,323	71%
2003	29,825,844	10,932,703	4,391,717	1,574,142	797,113	618,675	430,571	381,663	289,983	171,895	230,207	1,041,889	20,861,756	70%
2004	30,303,950	11,239,343	4,081,055	1,522,852	788,922	652,563	437,473	387,781	294,631	174,650	233,897	1,058,590	20,855,966	69%
2005	31,737,838	11,978,411	3,653,908	1,588,938	1,015,906	683,441	458,173	406,129	308,573	182,914	244,964	1,108,680	21,601,428	68%
2006	35,430,317	12,468,437	4,353,742	2,114,552	1,134,100	762,954	511,478	453,379	344,473	204,195	273,464	1,237,667	23,811,342	67%
2007	37,271,114	12,931,177	5,309,716	2,224,415	1,193,022	802,594	538,052	476,935	362,370	214,804	287,672	1,301,970	25,621,849	69%
	Pure Prem	0.358	0.142	0.060	0.032	0.022	0.014	0.013	0.010	0.006	0.008	0.035	219,198,836	70%

Exhibit 6.2

Variances from Additive Method

Paid Residuals

AY	1/AdjPrem	@12	@24	@36	@48	@60	@72	@84	@96	@108	@120	@Ult
1998	3.44E-08	359,234	-137,270	84,456	-197,589	146,191	204,630	58,299	-2,327	-18,793	0	
1999	3.41E-08	-116,179	307,595	156,928	168,375	-17,556	-22,012	-8,233	2,638	18,793		
2000	3.21E-08	-46,408	378,332	195,853	63,247	44,246	-4,209	21,812	-311			
2001	3.29E-08	70,286	485,901	348,622	95,770	-335,305	-186,261	-71,879				
2002	3.41E-08	-65,320	151,413	41,415	-49,883	71,250	7,852					
2003	3.35E-08	-131,031	-27,171	-75,561	17,466	91,174						
2004	3.30E-08	148,305	-113,643	-387,373	-97,386							
2005	3.15E-08	177,478	-456,577	-364,340								
2006	2.82E-08	-103,114	-588,580									
2007	2.68E-08	-293,250										
	Zero check	0	0	0	0	0	0	0	0	0	0	
	WSSR	10,180	33,825	15,703	3,150	4,962	2,602	305	0	24	0	
	df	9	8	7	6	5	4	3	2	1	0	
	Unit Var	1,131	4,228	2,243	525	992	651	102	0	24		
	Selected	1,131	4,228	2,243	992	992	651	102	102	55	27	1,928

Incurred Residuals

AY	1/AdjPrem	@12	@24	@36	@48	@60	@72	@84	@96	@108	@120	@Ult
1998	3.44E-08	48,816	-53,727	-45,865	114,048	225,970	197,156	48,158	38,710	-95,797	0	
1999	3.41E-08	-373,998	87,856	259,853	328,446	134,933	27,152	104,356	-73,338	95,797		
2000	3.21E-08	-522,084	650,673	141,358	146,037	153,467	-23,699	-91,045	34,628			
2001	3.29E-08	242,767	414,597	304,004	-3,257	-407,927	-133,350	-61,469				
2002	3.41E-08	-96,980	555,294	137,561	-246,596	-82,850	-67,259					
2003	3.35E-08	267,218	142,669	-205,924	-157,591	-23,593						
2004	3.30E-08	402,891	-236,105	-285,748	-181,086							
2005	3.15E-08	629,211	-867,527	-305,240								
2006	2.82E-08	-201,163	-693,730									
2007	2.68E-08	-396,678										
	Zero check	0	0	0	0	0	0	0	0	0	0	
	WSSR	41,458	69,959	13,759	8,805	8,866	2,121	842	274	629	0	
	df	9	8	7	6	5	4	3	2	1	0	
	Unit Var	4,606	8,745	1,966	1,467	1,773	530	281	137	629		
	Selected	4,606	8,745	1,966	1,773	1,773	530	291	291	291	200	1,072

Exhibit 6.3

Absolute Variances from Additive Method

Paid Variance

AY	Adj Prem	12	24	36	48	60	72	84	96	108	120	@Ult	Unpaid
1998	29,060,019	3.29E+10	1.23E+11	6.52E+10	2.88E+10	2.88E+10	1.89E+10	2.95E+09	2.95E+09	1.59E+09	7.97E+08	5.60E+10	5.60E+10
1999	29,301,751	3.31E+10	1.24E+11	6.57E+10	2.91E+10	2.91E+10	1.91E+10	2.98E+09	2.98E+09	1.61E+09	8.04E+08	5.65E+10	5.73E+10
2000	31,142,788	3.52E+10	1.32E+11	6.99E+10	3.09E+10	3.09E+10	2.03E+10	3.16E+09	3.16E+09	1.71E+09	8.55E+08	6.01E+10	6.26E+10
2001	30,375,837	3.44E+10	1.28E+11	6.81E+10	3.01E+10	3.01E+10	1.98E+10	3.08E+09	3.08E+09	1.67E+09	8.33E+08	5.86E+10	6.42E+10
2002	29,297,526	3.31E+10	1.24E+11	6.57E+10	2.91E+10	2.91E+10	1.91E+10	2.98E+09	2.98E+09	1.61E+09	8.04E+08	5.65E+10	6.49E+10
2003	29,825,844	3.37E+10	1.26E+11	6.69E+10	2.96E+10	2.96E+10	1.94E+10	3.03E+09	3.03E+09	1.64E+09	8.18E+08	5.75E+10	8.54E+10
2004	30,303,950	3.43E+10	1.28E+11	6.80E+10	3.01E+10	3.01E+10	1.97E+10	3.08E+09	3.08E+09	1.66E+09	8.32E+08	5.84E+10	1.17E+11
2005	31,737,838	3.59E+10	1.34E+11	7.12E+10	3.15E+10	3.15E+10	2.06E+10	3.22E+09	3.22E+09	1.74E+09	8.71E+08	6.12E+10	1.54E+11
2006	35,430,317	4.01E+10	1.50E+11	7.95E+10	3.52E+10	3.52E+10	2.30E+10	3.60E+09	3.60E+09	1.94E+09	9.72E+08	6.83E+10	2.51E+11
2007	37,271,114	4.22E+10	1.58E+11	8.36E+10	3.70E+10	3.70E+10	2.42E+10	3.78E+09	3.78E+09	2.05E+09	1.02E+09	7.19E+10	4.22E+11

Incurred Variance

AY	Adj Prem	12	24	36	48	60	72	84	96	108	120	@Ult	IBNR
1998	29,060,019	1.34E+11	2.54E+11	5.71E+10	5.15E+10	5.15E+10	1.54E+10	8.45E+09	8.45E+09	8.45E+09	5.81E+09	3.12E+10	3.12E+10
1999	29,301,751	1.35E+11	2.56E+11	5.76E+10	5.20E+10	5.20E+10	1.55E+10	8.52E+09	8.52E+09	8.52E+09	5.86E+09	3.14E+10	3.73E+10
2000	31,142,788	1.43E+11	2.72E+11	6.12E+10	5.52E+10	5.52E+10	1.65E+10	9.06E+09	9.06E+09	9.06E+09	6.23E+09	3.34E+10	4.87E+10
2001	30,375,837	1.40E+11	2.66E+11	5.97E+10	5.39E+10	5.39E+10	1.61E+10	8.83E+09	8.83E+09	8.83E+09	6.08E+09	3.26E+10	5.63E+10
2002	29,297,526	1.35E+11	2.56E+11	5.76E+10	5.19E+10	5.19E+10	1.55E+10	8.52E+09	8.52E+09	8.52E+09	5.86E+09	3.14E+10	6.28E+10
2003	29,825,844	1.37E+11	2.61E+11	5.86E+10	5.29E+10	5.29E+10	1.58E+10	8.67E+09	8.67E+09	8.67E+09	5.97E+09	3.20E+10	7.98E+10
2004	30,303,950	1.40E+11	2.65E+11	5.96E+10	5.37E+10	5.37E+10	1.61E+10	8.81E+09	8.81E+09	8.81E+09	6.06E+09	3.25E+10	1.35E+11
2005	31,737,838	1.46E+11	2.78E+11	6.24E+10	5.63E+10	5.63E+10	1.68E+10	9.23E+09	9.23E+09	9.23E+09	6.35E+09	3.40E+10	1.97E+11
2006	35,430,317	1.63E+11	3.10E+11	6.96E+10	6.28E+10	6.28E+10	1.88E+10	1.03E+10	1.03E+10	1.03E+10	7.09E+09	3.80E+10	2.90E+11
2007	37,271,114	1.72E+11	3.26E+11	7.33E+10	6.61E+10	6.61E+10	1.98E+10	1.08E+10	1.08E+10	1.08E+10	7.45E+09	4.00E+10	6.31E+11

Exhibit 7.1

Paid Linear Model

				@12	@24	@36	@48	@60	@72	@84	@96	@108	@120	@Ult	
AY	Age	AdjPrem	y	X											Σ
1998	12	29,060,019	4,651,588	29,060,019											3.29E+10
1998	24	29,060,019	4,933,554		29,060,019										1.23E+11
1998	36	29,060,019	3,021,114			29,060,019									6.52E+10
1998	48	29,060,019	1,488,504				29,060,019								2.88E+10
1998	60	29,060,019	1,173,282					29,060,019							2.88E+10
1998	72	29,060,019	806,542						29,060,019						1.89E+10
1998	84	29,060,019	594,058							29,060,019					2.95E+09
1998	96	29,060,019	438,193								29,060,019				2.95E+09
1998	108	29,060,019	316,106									29,060,019			1.59E+09
1998	120	29,060,019	316,058										29,060,019		7.97E+08
1999	12	29,301,751	4,211,880	29,301,751											3.31E+10
1999	24	29,301,751	5,420,600		29,301,751										1.24E+11
1999	36	29,301,751	3,118,015			29,301,751									6.57E+10
1999	48	29,301,751	1,868,494				29,301,751								2.91E+10
1999	60	29,301,751	1,018,079					29,301,751							2.91E+10
1999	72	29,301,751	584,906						29,301,751						1.91E+10
1999	84	29,301,751	531,983							29,301,751					2.98E+09
1999	96	29,301,751	446,822								29,301,751				2.98E+09
1999	108	29,301,751	356,478									29,301,751			1.61E+09
2000	12	31,142,788	4,553,584	31,142,788											3.52E+10
2000	24	31,142,788	5,812,588		31,142,788										1.32E+11
2000	36	31,142,788	3,342,985			31,142,788									6.99E+10
2000	48	31,142,788	1,870,185				31,142,788								3.09E+10
2000	60	31,142,788	1,144,950					31,142,788							3.09E+10
2000	72	31,142,788	640,842						31,142,788						2.03E+10
2000	84	31,142,788	595,970							31,142,788					3.16E+09
2000	96	31,142,788	471,781								31,142,788				3.16E+09
2001	12	30,375,837	4,556,995	30,375,837											3.44E+10
2001	24	30,375,837	5,786,328		30,375,837										1.28E+11
2001	36	30,375,837	3,418,250			30,375,837									6.81E+10
2001	48	30,375,837	1,858,209				30,375,837								3.01E+10
2001	60	30,375,837	738,292					30,375,837							3.01E+10
2001	72	30,375,837	442,905						30,375,837						1.98E+10
2001	84	30,375,837	488,139							30,375,837					3.08E+09
2002	12	29,297,526	4,262,115	29,297,526											3.31E+10
2002	24	29,297,526	5,263,681		29,297,526										1.24E+11
2002	36	29,297,526	3,002,075			29,297,526									6.57E+10
2002	48	29,297,526	1,649,991				29,297,526								2.91E+10
2002	60	29,297,526	1,106,736					29,297,526							2.91E+10
2002	72	29,297,526	614,683						29,297,526						1.91E+10
2003	12	29,825,844	4,274,440	29,825,844											3.37E+10
2003	24	29,825,844	5,177,285		29,825,844										1.26E+11
2003	36	29,825,844	2,938,488			29,825,844									6.69E+10
2003	48	29,825,844	1,747,993				29,825,844								2.96E+10
2003	60	29,825,844	1,145,332					29,825,844							2.96E+10
2004	12	30,303,950	4,624,395	30,303,950											3.43E+10
2004	24	30,303,950	5,174,240		30,303,950										1.28E+11
2004	36	30,303,950	2,674,991			30,303,950									6.80E+10
2004	48	30,303,950	1,660,882				30,303,950								3.01E+10
2005	12	31,737,838	4,865,363	31,737,838											3.59E+10
2005	24	31,737,838	5,081,513		31,737,838										1.34E+11
2005	36	31,737,838	2,842,925			31,737,838									7.12E+10
2006	12	35,430,317	5,130,174	35,430,317											4.01E+10
2006	24	35,430,317	5,593,828		35,430,317										1.50E+11
2007	12	37,271,114	5,211,936	37,271,114											4.22E+10
		Constraint	0	0.145	0.145	0.145	0.145	0.145	0.145	0.145	0.145	0.145	0.145	-0.855	
1998	Unpd	29,060,019												29,060,019	5.60E+10
1999	Unpd	29,301,751											29,301,751	29,301,751	5.73E+10
2000	Unpd	31,142,788										31,142,788	31,142,788	31,142,788	6.26E+10
2001	Unpd	30,375,837									30,375,837	30,375,837	30,375,837	30,375,837	6.42E+10
2002	Unpd	29,297,526								29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	6.49E+10
2003	Unpd	29,825,844							29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	8.54E+10
2004	Unpd	30,303,950						30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	1.17E+11
2005	Unpd	31,737,838					31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	1.54E+11
2006	Unpd	35,430,317				35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	2.51E+11
2007	Unpd	37,271,114			37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	4.22E+11

Exhibit 7.2

Case_Incurred Linear Model

				@12	@24	@36	@48	@60	@72	@84	@96	@108	@120	@Ult	
AY	Age	AdjPrem	**y**	X											Σ
1998	12	29,060,019	10,440,449	29,060,019											1.34E+11
1998	24	29,060,019	4,086,220		29,060,019										2.54E+11
1998	36	29,060,019	1,688,495			29,060,019									5.71E+10
1998	48	29,060,019	1,044,239				29,060,019								5.15E+10
1998	60	29,060,019	851,747					29,060,019							5.15E+10
1998	72	29,060,019	616,672						29,060,019						1.54E+10
1998	84	29,060,019	420,021							29,060,019					8.45E+09
1998	96	29,060,019	321,247								29,060,019				8.45E+09
1998	108	29,060,019	71,684									29,060,019			8.45E+09
1998	120	29,060,019	224,296										29,060,019		5.81E+09
1999	12	29,301,751	10,104,076	29,301,751											1.35E+11
1999	24	29,301,751	4,262,241		29,301,751										2.56E+11
1999	36	29,301,751	2,008,640			29,301,751									5.76E+10
1999	48	29,301,751	1,266,374				29,301,751								5.20E+10
1999	60	29,301,751	765,915					29,301,751							5.20E+10
1999	72	29,301,751	450,157						29,301,751						1.55E+10
1999	84	29,301,751	479,312							29,301,751					8.52E+09
1999	96	29,301,751	211,550								29,301,751				8.52E+09
1999	108	29,301,751	264,671									29,301,751			8.52E+09
2000	12	31,142,788	10,614,330	31,142,788											1.43E+11
2000	24	31,142,788	5,087,335		31,142,788										2.72E+11
2000	36	31,142,788	2,000,022			31,142,788									6.12E+10
2000	48	31,142,788	1,142,896				31,142,788								5.52E+10
2000	60	31,142,788	824,094					31,142,788							5.52E+10
2000	72	31,142,788	425,884						31,142,788						1.65E+10
2000	84	31,142,788	307,470							31,142,788					9.06E+09
2000	96	31,142,788	337,415								31,142,788				9.06E+09
2001	12	30,375,837	11,104,926	30,375,837											1.40E+11
2001	24	30,375,837	4,741,998		30,375,837										2.66E+11
2001	36	30,375,837	2,116,895			30,375,837									5.97E+10
2001	48	30,375,837	969,052				30,375,837								5.39E+10
2001	60	30,375,837	246,184					30,375,837							5.39E+10
2001	72	30,375,837	305,161						30,375,837						1.61E+10
2001	84	30,375,837	327,231							30,375,837					8.83E+09
2002	12	29,297,526	10,379,583	29,297,526											1.35E+11
2002	24	29,297,526	4,729,077		29,297,526										2.56E+11
2002	36	29,297,526	1,886,096			29,297,526									5.76E+10
2002	48	29,297,526	691,197				29,297,526								5.19E+10
2002	60	29,297,526	548,041					29,297,526							5.19E+10
2002	72	29,297,526	355,685						29,297,526						1.55E+10
2003	12	29,825,844	10,932,703	29,825,844											1.37E+11
2003	24	29,825,844	4,391,717		29,825,844										2.61E+11
2003	36	29,825,844	1,574,142			29,825,844									5.86E+10
2003	48	29,825,844	797,113				29,825,844								5.29E+10
2003	60	29,825,844	618,675					29,825,844							5.29E+10
2004	12	30,303,950	11,239,343	30,303,950											1.40E+11
2004	24	30,303,950	4,081,055		30,303,950										2.65E+11
2004	36	30,303,950	1,522,852			30,303,950									5.96E+10
2004	48	30,303,950	788,922				30,303,950								5.37E+10
2005	12	31,737,838	11,978,411	31,737,838											1.46E+11
2005	24	31,737,838	3,653,908		31,737,838										2.78E+11
2005	36	31,737,838	1,588,938			31,737,838									6.24E+10
2006	12	35,430,317	12,468,437	35,430,317											1.63E+11
2006	24	35,430,317	4,353,742		35,430,317										3.10E+11
2007	12	37,271,114	12,931,177	37,271,114											1.72E+11
		Constraint	0	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	-0.95	
1998	IBNR	29,060,019												29,060,019	3.12E+10
1999	IBNR	29,301,751											29,301,751	29,301,751	3.73E+10
2000	IBNR	31,142,788										31,142,788	31,142,788	31,142,788	4.87E+10
2001	IBNR	30,375,837									30,375,837	30,375,837	30,375,837	30,375,837	5.63E+10
2002	IBNR	29,297,526								29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	6.28E+10
2003	IBNR	29,825,844							29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	7.98E+10
2004	IBNR	30,303,950						30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	1.35E+11
2005	IBNR	31,737,838					31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	1.97E+11
2006	IBNR	35,430,317				35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	2.90E+11
2007	IBNR	37,271,114			37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	6.31E+11

Exhibit 8.1

Joint Paid and Incurred Linear Model

Type	AY	Age	y	@12 X	@24	@36	@48	@60	@72	@84	@96	@108	@120	@Ult	@12	@24	@36	@48	@60	@72	@84	@96	@108	@120	@Ult	Σ
Paid	1998	12	4,651,588	29,060,019																						3.29E+10
Paid	1998	24	4,933,554		29,060,019																					1.23E+11
Paid	1998	36	3,021,114			29,060,019																				6.52E+10
Paid	1998	48	1,488,504				29,060,019																			2.88E+10
Paid	1998	60	1,173,282					29,060,019																		2.88E+10
Paid	1998	72	806,542						29,060,019																	1.89E+10
Paid	1998	84	594,058							29,060,019																2.95E+09
Paid	1998	96	438,193								29,060,019															2.95E+09
Paid	1998	108	316,106									29,060,019														1.59E+09
Paid	1998	120	316,058										29,060,019													7.97E+08
Paid	1999	12	4,211,880	29,301,751																						3.31E+10
Paid	1999	24	5,420,600		29,301,751																					1.24E+11
Paid	1999	36	3,118,015			29,301,751																				6.57E+10
Paid	1999	48	1,868,494				29,301,751																			2.91E+10
Paid	1999	60	1,018,079					29,301,751																		2.91E+10
Paid	1999	72	584,906						29,301,751																	1.91E+10
Paid	1999	84	531,983							29,301,751																2.98E+09
Paid	1999	96	446,822								29,301,751															2.98E+09
Paid	1999	108	356,478									29,301,751														1.61E+09
Paid	2000	12	4,553,584	31,142,788																						3.52E+10
Paid	2000	24	5,812,588		31,142,788																					1.32E+11
Paid	2000	36	3,342,985			31,142,788																				6.99E+10
Paid	2000	48	1,870,185				31,142,788																			3.09E+10
Paid	2000	60	1,144,950					31,142,788																		3.09E+10
Paid	2000	72	640,842						31,142,788																	2.03E+10
Paid	2000	84	595,970							31,142,788																3.16E+09
Paid	2000	96	471,781								31,142,788															3.16E+09
Paid	2001	12	4,556,995	30,375,837																						3.44E+10
Paid	2001	24	5,786,328		30,375,837																					1.28E+11
Paid	2001	36	3,418,250			30,375,837																				6.81E+10
Paid	2001	48	1,858,209				30,375,837																			3.01E+10
Paid	2001	60	738,292					30,375,837																		3.01E+10
Paid	2001	72	442,905						30,375,837																	1.98E+10
Paid	2001	84	488,139							30,375,837																3.08E+09
Paid	2002	12	4,262,115	29,297,526																						3.31E+10
Paid	2002	24	5,263,681		29,297,526																					1.24E+11
Paid	2002	36	3,002,075			29,297,526																				6.57E+10
Paid	2002	48	1,649,991				29,297,526																			2.91E+10
Paid	2002	60	1,106,736					29,297,526																		2.91E+10
Paid	2002	72	614,683						29,297,526																	1.91E+10
Paid	2003	12	4,274,440	29,825,844																						3.37E+10
Paid	2003	24	5,177,285		29,825,844																					1.26E+11
Paid	2003	36	2,938,488			29,825,844																				6.69E+10
Paid	2003	48	1,747,993				29,825,844																			2.96E+10
Paid	2003	60	1,145,332					29,825,844																		2.96E+10
Paid	2004	12	4,624,395	30,303,950																						3.43E+10
Paid	2004	24	5,174,240		30,303,950																					1.28E+11
Paid	2004	36	2,674,991			30,303,950																				6.80E+10
Paid	2004	48	1,660,882				30,303,950																			3.01E+10
Paid	2005	12	4,865,363	31,737,838																						3.59E+10
Paid	2005	24	5,081,513		31,737,838																					1.34E+11
Paid	2005	36	2,842,925			31,737,838																				7.12E+10
Paid	2006	12	5,130,174	35,430,317																						4.01E+10
Paid	2006	24	5,593,828		35,430,317																					1.50E+11
Paid	2007	12	5,211,936	37,271,114																						4.22E+10
Paid	Constraint		0	0.145	0.145	0.145	0.145	0.145	0.145	0.145	0.145	0.145	0.145	-0.855												
Paid	1998	Unpd												29,060,019												5.60E+10
Paid	1999	Unpd											29,301,751	29,301,751												5.73E+10
Paid	2000	Unpd										31,142,788	31,142,788	31,142,788												6.26E+10
Paid	2001	Unpd									30,375,837	30,375,837	30,375,837	30,375,837												6.42E+10
Paid	2002	Unpd								29,297,526	29,297,526	29,297,526	29,297,526	29,297,526												6.49E+10
Paid	2003	Unpd							29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844												8.54E+10
Paid	2004	Unpd						30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950												1.17E+11
Paid	2005	Unpd					31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838												1.54E+11
Paid	2006	Unpd				35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317												2.51E+11
Paid	2007	Unpd			37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114												4.22E+11
Incd	1998	12	10,440,449												29,060,019											1.34E+11
Incd	1998	24	4,086,220													29,060,019										2.54E+11
Incd	1998	36	1,688,495														29,060,019									5.71E+10
Incd	1998	48	1,044,239															29,060,019								5.15E+10
Incd	1998	60	851,747																29,060,019							5.15E+10
Incd	1998	72	616,672																	29,060,019						1.54E+10
Incd	1998	84	420,021																		29,060,019					8.45E+09
Incd	1998	96	321,247																			29,060,019				8.45E+09
Incd	1998	108	71,684																				29,060,019			8.45E+09
Incd	1998	120	224,296																					29,060,019		5.81E+09
Incd	1999	12	10,104,076												29,301,751											1.35E+11
Incd	1999	24	4,262,241													29,301,751										2.56E+11
Incd	1999	36	2,008,640														29,301,751									5.76E+10
Incd	1999	48	1,266,374															29,301,751								5.20E+10
Incd	1999	60	765,915																29,301,751							5.20E+10
Incd	1999	72	450,157																	29,301,751						1.55E+10
Incd	1999	84	479,312																		29,301,751					8.52E+09
Incd	1999	96	211,550																			29,301,751				8.52E+09
Incd	1999	108	264,671																				29,301,751			8.52E+09
Incd	2000	12	10,614,330												31,142,788											1.43E+11
Incd	2000	24	5,087,335													31,142,788										2.72E+11
Incd	2000	36	2,000,022														31,142,788									6.12E+10
Incd	2000	48	1,142,896															31,142,788								5.52E+10
Incd	2000	60	824,094																31,142,788							5.52E+10
Incd	2000	72	425,884																	31,142,788						1.65E+10
Incd	2000	84	307,470																		31,142,788					9.06E+09
Incd	2000	96	337,415																			31,142,788				9.06E+09
Incd	2001	12	11,104,926												30,375,837											1.40E+11
Incd	2001	24	4,741,998													30,375,837										2.66E+11
Incd	2001	36	2,116,895														30,375,837									5.97E+10
Incd	2001	48	969,052															30,375,837								5.39E+10
Incd	2001	60	246,184																30,375,837							5.39E+10
Incd	2001	72	305,161																	30,375,837						1.61E+10
Incd	2001	84	327,231																		30,375,837					8.83E+09
Incd	2002	12	10,379,583												29,297,526											1.35E+11
Incd	2002	24	4,729,077													29,297,526										2.56E+11
Incd	2002	36	1,886,096														29,297,526									5.76E+10
Incd	2002	48	691,197															29,297,526								5.19E+10
Incd	2002	60	548,041																29,297,526							5.19E+10
Incd	2002	72	355,685																	29,297,526						1.55E+10
Incd	2003	12	10,932,703												29,825,844											1.37E+11
Incd	2003	24	4,391,717													29,825,844										2.61E+11
Incd	2003	36	1,574,142														29,825,844									5.86E+10
Incd	2003	48	797,113															29,825,844								5.29E+10
Incd	2003	60	618,675																29,825,844							5.29E+10
Incd	2004	12	11,239,343												30,303,950											1.40E+11
Incd	2004	24	4,081,055													30,303,950										2.65E+11
Incd	2004	36	1,522,852														30,303,950									5.96E+10
Incd	2004	48	788,922															30,303,950								5.37E+10
Incd	2005	12	11,978,411												31,737,838											1.46E+11
Incd	2005	24	3,653,908													31,737,838										2.78E+11
Incd	2005	36	1,588,938														31,737,838									6.24E+10
Incd	2006	12	12,468,437												35,430,317											1.63E+11
Incd	2006	24	4,353,742													35,430,317										3.10E+11
Incd	2007	12	12,931,177												37,271,114											1.72E+11
Incd	Constraint		0												0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	-0.95	
Incd	1998	IBNR																							29,060,019	3.12E+10
Incd	1999	IBNR																						29,301,751	29,301,751	3.73E+10
Incd	2000	IBNR																					31,142,788	31,142,788	31,142,788	4.87E+10
Incd	2001	IBNR																				30,375,837	30,375,837	30,375,837	30,375,837	5.63E+10
Incd	2002	IBNR																			29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	6.28E+10
Incd	2003	IBNR																		29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	7.98E+10
Incd	2004	IBNR																	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	1.35E+11
Incd	2005	IBNR																31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	1.97E+11
Incd	2006	IBNR															35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	2.90E+11
Incd	2007	IBNR														37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	6.31E+11
Ult =	1998		0	29,060,019	29,060,019	29,060,019	29,060,019	29,060,019	29,060,019	29,060,019	29,060,019	29,060,019	29,060,019	29,060,019	-29,060,019	-29,060,019	-29,060,019	-29,060,019	-29,060,019	-29,060,019	-29,060,019	-29,060,019	-29,060,019	-29,060,019	-29,060,019	9.88E+11
Ult =	1999		0	29,301,751	29,301,751	29,301,751	29,301,751	29,301,751	29,301,751	29,301,751	29,301,751	29,301,751	29,301,751	29,301,751	-29,301,751	-29,301,751	-29,301,751	-29,301,751	-29,301,751	-29,301,751	-29,301,751	-29,301,751	-29,301,751	-29,301,751	-29,301,751	9.96E+11
Ult =	2000		0	31,142,788	31,142,788	31,142,788	31,142,788	31,142,788	31,142,788	31,142,788	31,142,788	31,142,788	31,142,788	31,142,788	-31,142,788	-31,142,788	-31,142,788	-31,142,788	-31,142,788	-31,142,788	-31,142,788	-31,142,788	-31,142,788	-31,142,788	-31,142,788	1.06E+12
Ult =	2001		0	30,375,837	30,375,837	30,375,837	30,375,837	30,375,837	30,375,837	30,375,837	30,375,837	30,375,837	30,375,837	30,375,837	-30,375,837	-30,375,837	-30,375,837	-30,375,837	-30,375,837	-30,375,837	-30,375,837	-30,375,837	-30,375,837	-30,375,837	-30,375,837	1.03E+12
Ult =	2002		0	29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	29,297,526	-29,297,526	-29,297,526	-29,297,526	-29,297,526	-29,297,526	-29,297,526	-29,297,526	-29,297,526	-29,297,526	-29,297,526	-29,297,526	9.96E+11
Ult =	2003		0	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	29,825,844	-29,825,844	-29,825,844	-29,825,844	-29,825,844	-29,825,844	-29,825,844	-29,825,844	-29,825,844	-29,825,844	-29,825,844	-29,825,844	1.01E+12
Ult =	2004		0	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	30,303,950	-30,303,950	-30,303,950	-30,303,950	-30,303,950	-30,303,950	-30,303,950	-30,303,950	-30,303,950	-30,303,950	-30,303,950	-30,303,950	1.03E+12
Ult =	2005		0	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	31,737,838	-31,737,838	-31,737,838	-31,737,838	-31,737,838	-31,737,838	-31,737,838	-31,737,838	-31,737,838	-31,737,838	-31,737,838	-31,737,838	1.08E+12
Ult =	2006		0	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	35,430,317	-35,430,317	-35,430,317	-35,430,317	-35,430,317	-35,430,317	-35,430,317	-35,430,317	-35,430,317	-35,430,317	-35,430,317	-35,430,317	1.20E+12
Ult =	2007		0	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	37,271,114	-37,271,114	-37,271,114	-37,271,114	-37,271,114	-37,271,114	-37,271,114	-37,271,114	-37,271,114	-37,271,114	-37,271,114	-37,271,114	1.27E+12

Exhibit 8.2

Covariance of Joint Paid and Incurred Linear Model

Type	AY	Age	Σ	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
				Covariance									
Paid	1998	12	3.29E+10	3.29E+10									
Paid	1998	24	1.23E+11	1.23E+11									
Paid	1998	36	6.52E+10	6.52E+10									
Paid	1998	48	2.88E+10	2.88E+10									
Paid	1998	60	2.88E+10	2.88E+10									
Paid	1998	72	1.89E+10	1.89E+10									
Paid	1998	84	2.95E+09	2.95E+09									
Paid	1998	96	2.95E+09	2.95E+09									
Paid	1998	108	1.59E+09	1.59E+09									
Paid	1998	120	7.97E+08	7.97E+08									
Paid	1999	12	3.31E+10		3.31E+10								
Paid	1999	24	1.24E+11		1.24E+11								
Paid	1999	36	6.57E+10		6.57E+10								
Paid	1999	48	2.91E+10		2.91E+10								
Paid	1999	60	2.91E+10		2.91E+10								
Paid	1999	72	1.91E+10		1.91E+10								
Paid	1999	84	2.98E+09		2.98E+09								
Paid	1999	96	2.98E+09		2.98E+09								
Paid	1999	108	1.61E+09		1.61E+09								
Paid	2000	12	3.52E+10			3.52E+10							
Paid	2000	24	1.32E+11			1.32E+11							
Paid	2000	36	6.99E+10			6.99E+10							
Paid	2000	48	3.09E+10			3.09E+10							
Paid	2000	60	3.09E+10			3.09E+10							
Paid	2000	72	2.03E+10			2.03E+10							
Paid	2000	84	3.16E+09			3.16E+09							
Paid	2000	96	3.16E+09			3.16E+09							
Paid	2001	12	3.44E+10				3.44E+10						
Paid	2001	24	1.28E+11				1.28E+11						
Paid	2001	36	6.81E+10				6.81E+10						
Paid	2001	48	3.01E+10				3.01E+10						
Paid	2001	60	3.01E+10				3.01E+10						
Paid	2001	72	1.98E+10				1.98E+10						
Paid	2001	84	3.08E+09				3.08E+09						
Paid	2002	12	3.31E+10					3.31E+10					
Paid	2002	24	1.24E+11					1.24E+11					
Paid	2002	36	6.57E+10					6.57E+10					
Paid	2002	48	2.91E+10					2.91E+10					
Paid	2002	60	2.91E+10					2.91E+10					
Paid	2002	72	1.91E+10					1.91E+10					
Paid	2003	12	3.37E+10						3.37E+10				
Paid	2003	24	1.26E+11						1.26E+11				
Paid	2003	36	6.69E+10						6.69E+10				
Paid	2003	48	2.96E+10						2.96E+10				
Paid	2003	60	2.96E+10						2.96E+10				
Paid	2004	12	3.43E+10							3.43E+10			
Paid	2004	24	1.28E+11							1.28E+11			
Paid	2004	36	6.80E+10							6.80E+10			
Paid	2004	48	3.01E+10							3.01E+10			
Paid	2005	12	3.59E+10								3.59E+10		
Paid	2005	24	1.34E+11								1.34E+11		
Paid	2005	36	7.12E+10								7.12E+10		
Paid	2006	12	4.01E+10									4.01E+10	
Paid	2006	24	1.50E+11									1.50E+11	
Paid	2007	12	4.22E+10										4.22E+10
Paid	Constraint												
Paid	1998	Unpd	5.60E+10	5.60E+10									
Paid	1999	Unpd	5.73E+10		5.73E+10								
Paid	2000	Unpd	6.26E+10			6.26E+10							
Paid	2001	Unpd	6.42E+10				6.42E+10						
Paid	2002	Unpd	6.49E+10					6.49E+10					
Paid	2003	Unpd	8.54E+10						8.54E+10				
Paid	2004	Unpd	1.17E+11							1.17E+11			
Paid	2005	Unpd	1.54E+11								1.54E+11		
Paid	2006	Unpd	2.51E+11									2.51E+11	
Paid	2007	Unpd	4.22E+11										4.22E+11
Incd	1998	12	1.34E+11	-1.34E+11									
Incd	1998	24	2.54E+11	-2.54E+11									
Incd	1998	36	5.71E+10	-5.71E+10									
Incd	1998	48	5.15E+10	-5.15E+10									
Incd	1998	60	5.15E+10	-5.15E+10									
Incd	1998	72	1.54E+10	-1.54E+10									
Incd	1998	84	8.45E+09	-8.45E+09									
Incd	1998	96	8.45E+09	-8.45E+09									
Incd	1998	108	8.45E+09	-8.45E+09									
Incd	1998	120	5.81E+09	-5.81E+09									
Incd	1999	12	1.35E+11		-1.35E+11								
Incd	1999	24	2.56E+11		-2.56E+11								
Incd	1999	36	5.76E+10		-5.76E+10								
Incd	1999	48	5.20E+10		-5.20E+10								
Incd	1999	60	5.20E+10		-5.20E+10								
Incd	1999	72	1.55E+10		-1.55E+10								
Incd	1999	84	8.52E+09		-8.52E+09								
Incd	1999	96	8.52E+09		-8.52E+09								
Incd	1999	108	8.52E+09		-8.52E+09								
Incd	2000	12	1.43E+11			-1.43E+11							
Incd	2000	24	2.72E+11			-2.72E+11							
Incd	2000	36	6.12E+10			-6.12E+10							
Incd	2000	48	5.52E+10			-5.52E+10							
Incd	2000	60	5.52E+10			-5.52E+10							
Incd	2000	72	1.65E+10			-1.65E+10							
Incd	2000	84	9.06E+09			-9.06E+09							
Incd	2000	96	9.06E+09			-9.06E+09							
Incd	2001	12	1.40E+11				-1.40E+11						
Incd	2001	24	2.66E+11				-2.66E+11						
Incd	2001	36	5.97E+10				-5.97E+10						
Incd	2001	48	5.39E+10				-5.39E+10						
Incd	2001	60	5.39E+10				-5.39E+10						
Incd	2001	72	1.61E+10				-1.61E+10						
Incd	2001	84	8.83E+09				-8.83E+09						
Incd	2002	12	1.35E+11					-1.35E+11					
Incd	2002	24	2.56E+11					-2.56E+11					
Incd	2002	36	5.76E+10					-5.76E+10					
Incd	2002	48	5.19E+10					-5.19E+10					
Incd	2002	60	5.19E+10					-5.19E+10					
Incd	2002	72	1.55E+10					-1.55E+10					
Incd	2003	12	1.37E+11						-1.37E+11				
Incd	2003	24	2.61E+11						-2.61E+11				
Incd	2003	36	5.86E+10						-5.86E+10				
Incd	2003	48	5.29E+10						-5.29E+10				
Incd	2003	60	5.29E+10						-5.29E+10				
Incd	2004	12	1.40E+11							-1.40E+11			
Incd	2004	24	2.65E+11							-2.65E+11			
Incd	2004	36	5.96E+10							-5.96E+10			
Incd	2004	48	5.37E+10							-5.37E+10			
Incd	2005	12	1.46E+11								-1.46E+11		
Incd	2005	24	2.78E+11								-2.78E+11		
Incd	2005	36	6.24E+10								-6.24E+10		
Incd	2006	12	1.63E+11									-1.63E+11	
Incd	2006	24	3.10E+11									-3.10E+11	
Incd	2007	12	1.72E+11										-1.72E+11
Incd	Constraint												
Incd	1998	IBNR	3.12E+10	-3.12E+10									
Incd	1999	IBNR	3.73E+10		-3.73E+10								
Incd	2000	IBNR	4.87E+10			-4.87E+10							
Incd	2001	IBNR	5.63E+10				-5.63E+10						
Incd	2002	IBNR	6.28E+10					-6.28E+10					
Incd	2003	IBNR	7.98E+10						-7.98E+10				
Incd	2004	IBNR	1.35E+11							-1.35E+11			
Incd	2005	IBNR	1.97E+11								-1.97E+11		
Incd	2006	IBNR	2.90E+11									-2.90E+11	
Incd	2007	IBNR	6.31E+11										-6.31E+11
Ult =	1998		9.88E+11	9.88E+11									
Ult =	1999		9.96E+11		9.96E+11								
Ult =	2000		1.06E+12			1.06E+12							
Ult =	2001		1.03E+12				1.03E+12						
Ult =	2002		9.96E+11					9.96E+11					
Ult =	2003		1.01E+12						1.01E+12				
Ult =	2004		1.03E+12							1.03E+12			
Ult =	2005		1.08E+12								1.08E+12		
Ult =	2006		1.20E+12									1.20E+12	
Ult =	2007		1.27E+12										1.27E+12

Exhibit 9.1

Joint Paid and Incurred Linear Model: Diagnostics

TYPE	AY	AGE	y1	Fitted	Resid	Student	Cnstrnd
Paid	1998	12	4,651,588	4,295,141	356,447	2.06	0
Paid	1998	24	4,933,554	5,081,187	-147,633	-0.44	0
Paid	1998	36	3,021,114	2,939,870	81,244	0.34	0
Paid	1998	48	1,488,504	1,689,312	-200,808	-1.27	0
Paid	1998	60	1,173,282	1,034,360	138,922	0.89	0
Paid	1998	72	806,542	612,055	194,487	1.57	0
Paid	1998	84	594,058	538,316	55,742	1.18	0
Paid	1998	96	438,193	444,117	-5,924	-0.13	0
Paid	1998	108	316,106	337,302	-21,196	-0.74	0
Paid	1998	120	316,058	320,133	-4,075	0/0	0
Paid	1999	12	4,211,880	4,330,869	-118,989	-0.69	0
Paid	1999	24	5,420,600	5,123,455	297,145	0.89	0
Paid	1999	36	3,118,015	2,964,325	153,690	0.64	0
Paid	1999	48	1,868,494	1,703,364	165,130	1.04	0
Paid	1999	60	1,018,079	1,042,964	-24,885	-0.16	0
Paid	1999	72	584,906	617,146	-32,240	-0.26	0
Paid	1999	84	531,983	542,793	-10,810	-0.23	0
Paid	1999	96	446,822	447,812	-990	-0.02	0
Paid	1999	108	356,478	340,107	16,371	0.57	0
Paid	2000	12	4,553,584	4,602,979	-49,395	-0.28	0
Paid	2000	24	5,812,588	5,445,363	367,225	1.07	0
Paid	2000	36	3,342,985	3,150,575	192,410	0.78	0
Paid	2000	48	1,870,185	1,810,387	59,798	0.37	0
Paid	2000	60	1,144,950	1,108,494	36,456	0.23	0
Paid	2000	72	640,842	655,922	-15,080	-0.12	0
Paid	2000	84	595,970	576,897	19,073	0.39	0
Paid	2000	96	471,781	475,948	-4,167	-0.09	0
Paid	2001	12	4,556,995	4,489,622	67,373	0.38	0
Paid	2001	24	5,786,328	5,311,260	475,068	1.40	0
Paid	2001	36	3,418,250	3,072,986	345,264	1.41	0
Paid	2001	48	1,858,209	1,765,803	92,406	0.57	0
Paid	2001	60	738,292	1,081,195	-342,903	-2.16	0
Paid	2001	72	442,905	639,769	-196,864	-1.56	0
Paid	2001	84	488,139	562,690	-74,551	-1.55	0
Paid	2002	12	4,262,115	4,330,245	-68,130	-0.39	0
Paid	2002	24	5,263,681	5,122,716	140,965	0.42	0
Paid	2002	36	3,002,075	2,963,898	38,177	0.16	0
Paid	2002	48	1,649,991	1,703,119	-53,128	-0.34	0
Paid	2002	60	1,106,736	1,042,814	63,922	0.41	0
Paid	2002	72	614,683	617,057	-2,374	-0.02	0
Paid	2003	12	4,274,440	4,408,332	-133,892	-0.77	0
Paid	2003	24	5,177,285	5,215,093	-37,808	-0.11	0
Paid	2003	36	2,938,488	3,017,345	-78,857	-0.32	0
Paid	2003	48	1,747,993	1,733,831	14,162	0.09	0
Paid	2003	60	1,145,332	1,061,619	83,713	0.53	0
Paid	2004	12	4,624,395	4,478,997	145,398	0.83	0
Paid	2004	24	5,174,240	5,298,691	-124,451	-0.37	0
Paid	2004	36	2,674,991	3,065,713	-390,722	-1.60	0
Paid	2004	48	1,660,882	1,761,624	-100,742	-0.63	0
Paid	2005	12	4,865,363	4,690,929	174,434	0.97	0
Paid	2005	24	5,081,513	5,549,408	-467,895	-1.35	0
Paid	2005	36	2,842,925	3,210,773	-367,848	-1.48	0
Paid	2006	12	5,130,174	5,236,686	-106,512	-0.56	0
Paid	2006	24	5,593,828	6,195,043	-601,215	-1.66	0
Paid	2007	12	5,211,936	5,508,761	-296,825	-1.54	0
Paid	Constraint		0	0	0	0.00	1
Incd	1998	12	10,440,449	10,388,110	52,339	0.15	0
Incd	1998	24	4,086,220	4,135,376	-49,156	-0.10	0
Incd	1998	36	1,688,495	1,735,893	-47,398	-0.21	0
Incd	1998	48	1,044,239	928,957	115,282	0.55	0
Incd	1998	60	851,747	618,070	233,677	1.12	0
Incd	1998	72	616,672	413,144	203,528	1.82	0
Incd	1998	84	420,021	365,835	54,186	0.67	0
Incd	1998	96	321,247	273,968	47,279	0.62	0
Incd	1998	108	71,684	157,407	-85,723	-1.25	0
Incd	1998	120	224,296	198,262	26,034	0.79	0
Incd	1999	12	10,104,076	10,474,522	-370,446	-1.06	0
Incd	1999	24	4,262,241	4,169,776	92,465	0.19	0
Incd	1999	36	2,008,640	1,750,333	258,307	1.15	0
Incd	1999	48	1,266,374	936,684	329,690	1.55	0
Incd	1999	60	765,915	623,211	142,704	0.68	0
Incd	1999	72	450,157	416,581	33,576	0.30	0
Incd	1999	84	479,312	368,878	110,434	1.37	0
Incd	1999	96	211,550	276,247	-64,697	-0.84	0
Incd	1999	108	264,671	158,716	105,955	1.54	0
Incd	2000	12	10,614,330	11,132,639	-518,309	-1.44	0
Incd	2000	24	5,087,335	4,431,764	655,571	1.33	0
Incd	2000	36	2,000,022	1,860,307	139,715	0.60	0
Incd	2000	48	1,142,896	995,536	147,360	0.68	0
Incd	2000	60	824,094	662,368	161,726	0.75	0
Incd	2000	72	425,884	442,755	-16,871	-0.15	0
Incd	2000	84	307,470	392,055	-84,585	-1.03	0
Incd	2000	96	337,415	293,604	43,811	0.56	0
Incd	2001	12	11,104,926	10,858,477	246,449	0.69	0
Incd	2001	24	4,741,998	4,322,623	419,375	0.86	0
Incd	2001	36	2,116,895	1,814,493	302,402	1.32	0
Incd	2001	48	969,052	971,019	-1,967	-0.01	0
Incd	2001	60	246,184	646,056	-399,872	-1.88	0
Incd	2001	72	305,161	431,851	-126,690	-1.11	0
Incd	2001	84	327,231	382,400	-55,169	-0.68	0
Incd	2002	12	10,379,583	10,473,012	-93,429	-0.27	0
Incd	2002	24	4,729,077	4,169,174	559,903	1.17	0
Incd	2002	36	1,886,096	1,750,080	136,016	0.60	0
Incd	2002	48	691,197	936,549	-245,352	-1.16	0
Incd	2002	60	548,041	623,121	-75,080	-0.36	0
Incd	2002	72	355,685	416,521	-60,836	-0.54	0
Incd	2003	12	10,932,703	10,661,870	270,833	0.77	0
Incd	2003	24	4,391,717	4,244,357	147,360	0.30	0
Incd	2003	36	1,574,142	1,781,639	-207,497	-0.91	0
Incd	2003	48	797,113	953,438	-156,325	-0.73	0
Incd	2003	60	618,675	634,358	-15,683	-0.07	0
Incd	2004	12	11,239,343	10,832,779	406,564	1.14	0
Incd	2004	24	4,081,055	4,312,393	-231,338	-0.48	0
Incd	2004	36	1,522,852	1,810,199	-287,347	-1.26	0
Incd	2004	48	788,922	968,721	-179,799	-0.84	0
Incd	2005	12	11,978,411	11,345,353	633,058	1.75	0
Incd	2005	24	3,653,908	4,516,442	-862,534	-1.74	0
Incd	2005	36	1,588,938	1,895,852	-306,914	-1.32	0
Incd	2006	12	12,468,437	12,665,306	-196,869	-0.52	0
Incd	2006	24	4,353,742	5,041,899	-688,157	-1.32	0
Incd	2007	12	12,931,177	13,323,337	-392,160	-1.01	0
Incd	Constraint		0	0	0	0.00	1
Ult =	1998		0	-2,019	2,019	0.00	0
Ult =	1999		0	-2,035	2,035	0.00	0
Ult =	2000		0	-2,163	2,163	0.00	0
Ult =	2001		0	-2,110	2,110	0.00	0
Ult =	2002		0	-2,035	2,035	0.00	0
Ult =	2003		0	-2,072	2,072	0.00	0
Ult =	2004		0	-2,105	2,105	0.00	0
Ult =	2005		0	-2,205	2,205	0.00	0
Ult =	2006		0	-2,461	2,461	0.00	0
Ult =	2007		0	-2,589	2,589	0.00	0

SSCP	TTy1	Fitted	Resid
TTy1	21797.28	21712.24	85.03434
Fitted	21712.24	21712.24	0
Resid	85.03434	0	85.03434
rhosq	100.0%	99.6%	0.4%
df	120	20	100
s2hat	181.644	1085.612	0.850343
s2sel			**1**

Exhibit 9.2

Joint Paid and Incurred Linear Model: Predictions

TYPE	AY	AGE	y2hat	StdPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr	VarPrdErr
Paid	1998	Unpd	2,999,915	± 142,339	2.03E+10	3.9E+08	5.28E+08	5.95E+08	6.3E+08	8.06E+08	1.12E+09	1.43E+09	2.04E+09	3.23E+09	2.03E+10	3.9E+08	5.28E+08	5.95E+08	6.3E+08	8.06E+08	1.12E+09	1.43E+09	2.04E+09	3.23E+09
Paid	1999	Unpd	3,398,227	± 158,471	3.9E+08	2.51E+10	2.53E+09	2.43E+09	2.34E+09	2.60E+09	2.88E+09	3.27E+09	4.21E+09	5.59E+09	3.9E+08	2.51E+10	2.53E+09	2.43E+09	2.34E+09	2.60E+09	2.88E+09	3.27E+09	4.21E+09	5.59E+09
Paid	2000	Unpd	3,804,647	± 177,631	5.28E+08	2.53E+09	3.16E+10	3.96E+09	3.77E+09	4.09E+09	4.37E+09	4.85E+09	6.08E+09	7.67E+09	5.28E+08	2.53E+09	3.16E+10	3.96E+09	3.77E+09	4.09E+09	4.37E+09	4.85E+09	6.08E+09	7.67E+09
Paid	2001	Unpd	4,227,840	± 186,625	5.95E+08	2.43E+09	3.96E+09	3.48E+10	4.60E+09	4.95E+09	5.22E+09	5.73E+09	7.08E+09	8.71E+09	5.95E+08	2.43E+09	3.96E+09	3.48E+10	4.60E+09	4.95E+09	5.22E+09	5.73E+09	7.08E+09	8.71E+09
Paid	2002	Unpd	4,662,500	± 192,377	6.3E+08	2.34E+09	3.77E+09	4.60E+09	3.70E+10	5.45E+09	5.71E+09	6.24E+09	7.64E+09	9.26E+09	6.3E+08	2.34E+09	3.77E+09	4.60E+09	3.70E+10	5.45E+09	5.71E+09	6.24E+09	7.64E+09	9.26E+09
Paid	2003	Unpd	5,421,105	± 220,945	8.06E+08	2.60E+09	4.09E+09	4.95E+09	5.45E+09	4.88E+10	7.87E+09	8.50E+09	1.02E+10	1.19E+10	8.06E+08	2.60E+09	4.09E+09	4.95E+09	5.45E+09	4.88E+10	7.87E+09	8.50E+09	1.02E+10	1.19E+10
Paid	2004	Unpd	6,568,927	± 272,535	1.12E+09	2.88E+09	4.37E+09	5.22E+09	5.71E+09	7.87E+09	7.43E+10	1.25E+10	1.46E+10	1.67E+10	1.12E+09	2.88E+09	4.37E+09	5.22E+09	5.71E+09	7.87E+09	7.43E+10	1.25E+10	1.46E+10	1.67E+10
Paid	2005	Unpd	8,692,524	± 320,964	1.43E+09	3.27E+09	4.85E+09	5.73E+09	6.24E+09	8.50E+09	1.25E+10	1.03E+11	1.92E+10	2.15E+10	1.43E+09	3.27E+09	4.85E+09	5.73E+09	6.24E+09	8.50E+09	1.25E+10	1.03E+11	1.92E+10	2.15E+10
Paid	2006	Unpd	13,144,841	± 403,064	2.04E+09	4.21E+09	6.08E+09	7.08E+09	7.64E+09	1.02E+10	1.46E+10	1.92E+10	1.62E+11	3.07E+10	2.04E+09	4.21E+09	6.08E+09	7.08E+09	7.64E+09	1.02E+10	1.46E+10	1.92E+10	1.62E+11	3.07E+10
Paid	2007	Unpd	20,392,908	± 549,291	3.23E+09	5.59E+09	7.67E+09	8.71E+09	9.26E+09	1.19E+10	1.67E+10	2.15E+10	3.07E+10	3.02E+11	3.23E+09	5.59E+09	7.67E+09	8.71E+09	9.26E+09	1.19E+10	1.67E+10	2.15E+10	3.07E+10	3.02E+11
Incd	1998	IBNR	973,844	± 142,339	2.03E+10	3.9E+08	5.28E+08	5.95E+08	6.3E+08	8.06E+08	1.12E+09	1.43E+09	2.04E+09	3.23E+09	2.03E+10	3.9E+08	5.28E+08	5.95E+08	6.3E+08	8.06E+08	1.12E+09	1.43E+09	2.04E+09	3.23E+09
Incd	1999	IBNR	1,142,547	± 158,471	3.9E+08	2.51E+10	2.53E+09	2.43E+09	2.34E+09	2.60E+09	2.88E+09	3.27E+09	4.21E+09	5.59E+09	3.9E+08	2.51E+10	2.53E+09	2.43E+09	2.34E+09	2.60E+09	2.88E+09	3.27E+09	4.21E+09	5.59E+09
Incd	2000	IBNR	1,498,084	± 177,630	5.28E+08	2.53E+09	3.16E+10	3.96E+09	3.77E+09	4.09E+09	4.37E+09	4.85E+09	6.08E+09	7.67E+09	5.28E+08	2.53E+09	3.16E+10	3.96E+09	3.77E+09	4.09E+09	4.37E+09	4.85E+09	6.08E+09	7.67E+09
Incd	2001	IBNR	1,705,511	± 186,625	5.95E+08	2.43E+09	3.96E+09	3.48E+10	4.60E+09	4.95E+09	5.22E+09	5.73E+09	7.08E+09	8.71E+09	5.95E+08	2.43E+09	3.96E+09	3.48E+10	4.60E+09	4.95E+09	5.22E+09	5.73E+09	7.08E+09	8.71E+09
Incd	2002	IBNR	1,972,102	± 192,377	6.3E+08	2.34E+09	3.77E+09	4.60E+09	3.70E+10	5.45E+09	5.71E+09	6.24E+09	7.64E+09	9.26E+09	6.3E+08	2.34E+09	3.77E+09	4.60E+09	3.70E+10	5.45E+09	5.71E+09	6.24E+09	7.64E+09	9.26E+09
Incd	2003	IBNR	2,390,298	± 220,945	8.06E+08	2.60E+09	4.09E+09	4.95E+09	5.45E+09	4.88E+10	7.87E+09	8.50E+09	1.02E+10	1.19E+10	8.06E+08	2.60E+09	4.09E+09	4.95E+09	5.45E+09	4.88E+10	7.87E+09	8.50E+09	1.02E+10	1.19E+10
Incd	2004	IBNR	3,071,266	± 272,536	1.12E+09	2.88E+09	4.37E+09	5.22E+09	5.71E+09	7.87E+09	7.43E+10	1.25E+10	1.46E+10	1.67E+10	1.12E+09	2.88E+09	4.37E+09	5.22E+09	5.71E+09	7.87E+09	7.43E+10	1.25E+10	1.46E+10	1.67E+10
Incd	2005	IBNR	4,261,067	± 320,964	1.43E+09	3.27E+09	4.85E+09	5.73E+09	6.24E+09	8.50E+09	1.25E+10	1.03E+11	1.92E+10	2.15E+10	1.43E+09	3.27E+09	4.85E+09	5.73E+09	6.24E+09	8.50E+09	1.25E+10	1.03E+11	1.92E+10	2.15E+10
Incd	2006	IBNR	7,046,664	± 403,064	2.04E+09	4.21E+09	6.08E+09	7.08E+09	7.64E+09	1.02E+10	1.46E+10	1.92E+10	1.62E+11	3.07E+10	2.04E+09	4.21E+09	6.08E+09	7.08E+09	7.64E+09	1.02E+10	1.46E+10	1.92E+10	1.62E+11	3.07E+10
Incd	2007	IBNR	12,673,667	± 549,291	3.23E+09	5.59E+09	7.67E+09	8.71E+09	9.26E+09	1.19E+10	1.67E+10	2.15E+10	3.07E+10	3.02E+11	3.23E+09	5.59E+09	7.67E+09	8.71E+09	9.26E+09	1.19E+10	1.67E+10	2.15E+10	3.07E+10	3.02E+11

Betahat	StdBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta	VarBeta
0.148	± 0.002	3.59E-06	-5.2E-08	-3.5E-08	-2E-08	-2.7E-08	-2.8E-08	-7E-09	-1.2E-08	-1.3E-08	-1.7E-08	5.74E-07	1.5E-08	3.5E-08	0	1.6E-08	2.6E-08	1.4E-08	1.5E-08	2.8E-08	5.8E-08	1.07E-07	1.7E-08
0.175	± 0.004	-5.2E-08	1.48E-05	-2.93E-07	-1.49E-07	-1.82E-07	-1.61E-07	-3.7E-08	-5.9E-08	-6E-08	-7.4E-08	2.33E-06	6.6E-08	7.11E-07	1.75E-07	1.82E-07	2.25E-07	9.5E-08	8.2E-08	1.37E-07	2.68E-07	4.74E-07	1.27E-07
0.101	± 0.003	-3.5E-08	-2.93E-07	8.96E-06	-1.68E-07	-1.89E-07	-1.51E-07	-3.1E-08	-4.6E-08	-4.3E-08	-5.1E-08	1.35E-06	4.4E-08	3.97E-07	2.52E-07	2.44E-07	2.72E-07	1E-07	7.4E-08	1.11E-07	1.97E-07	3.26E-07	1.06E-07
0.058	± 0.002	-2E-08	-1.49E-07	-1.68E-07	4.6E-06	-1.5E-07	-1.14E-07	-2.2E-08	-3.1E-08	-2.7E-08	-3.1E-08	6.6E-07	2.5E-08	1.89E-07	1.16E-07	2.14E-07	2.3E-07	7.9E-08	5.5E-08	7.6E-08	1.26E-07	1.96E-07	6.9E-08
0.036	± 0.002	-2.7E-08	-1.82E-07	-1.89E-07	-1.5E-07	5.28E-06	-1.94E-07	-3.6E-08	-4.8E-08	-4.1E-08	-4.4E-08	7.41E-07	3.5E-08	2.1E-07	1.23E-07	2.24E-07	4.18E-07	1.39E-07	9.2E-08	1.21E-07	1.89E-07	2.82E-07	9.6E-08
0.021	± 0.002	-2.8E-08	-1.61E-07	-1.51E-07	-1.14E-07	-1.94E-07	4.13E-06	-4.3E-08	-5.5E-08	-4.4E-08	-4.6E-08	5.58E-07	3.5E-08	1.65E-07	8.9E-08	1.59E-07	2.94E-07	1.72E-07	1.1E-07	1.39E-07	2.08E-07	2.96E-07	8.8E-08
0.019	± 0.001	-7E-09	-3.7E-08	-3.1E-08	-2.2E-08	-3.6E-08	-4.3E-08	8.35E-07	-1.5E-08	-1.2E-08	-1.2E-08	1.05E-07	0	3.4E-08	1.7E-08	2.8E-08	5.2E-08	3E-08	3.2E-08	3.9E-08	5.7E-08	7.8E-08	2E-08
0.015	± 0.001	-1.2E-08	-5.9E-08	-4.6E-08	-3.1E-08	-4.8E-08	-5.5E-08	-1.5E-08	1.11E-06	-2.2E-08	-2.1E-08	1.35E-07	1.6E-08	4.8E-08	2.1E-08	3.5E-08	6.3E-08	3.6E-08	3.8E-08	7.2E-08	1.03E-07	1.38E-07	3E-08
0.012	± 0.001	-1.3E-08	-6E-08	-4.3E-08	-2.7E-08	-4.1E-08	-4.4E-08	-1.2E-08	-2.2E-08	9.17E-07	-2.3E-08	1.07E-07	1.7E-08	4.5E-08	1.8E-08	2.7E-08	4.8E-08	2.7E-08	2.8E-08	5.4E-08	1.13E-07	1.5E-07	2.8E-08
0.011	± 0.001	-1.7E-08	-7.4E-08	-5.1E-08	-3.1E-08	-4.4E-08	-4.6E-08	-1.2E-08	-2.1E-08	-2.3E-08	9.14E-07	1.01E-07	2.1E-08	5.2E-08	1.8E-08	2.7E-08	4.6E-08	2.6E-08	2.7E-08	5E-08	1.06E-07	1.95E-07	3E-08
0.101	± 0.001	5.74E-07	2.328E-06	1.35E-06	6.6E-07	7.41E-07	5.58E-07	1.05E-07	1.35E-07	1.07E-07	1.01E-07	1.13E-06	4.8E-08	3.2E-07	1.43E-07	1.96E-07	2.84E-07	1.22E-07	9.3E-08	1.4E-07	2.42E-07	3.8E-07	1.04E-07
0.357	± 0.004	1.5E-08	6.6E-08	4.4E-08	2.5E-08	3.5E-08	3.5E-08	0	1.6E-08	1.7E-08	2.1E-08	4.8E-08	1.47E-05	-4.4E-08	0	-2E-08	-3.3E-08	-1.8E-08	-1.8E-08	-3.5E-08	-7.3E-08	-1.35E-07	7.51E-07
0.142	± 0.006	3.5E-08	7.11E-07	3.97E-07	1.89E-07	2.1E-07	1.65E-07	3.4E-08	4.8E-08	4.5E-08	5.2E-08	3.2E-07	-4.4E-08	3.04E-05	-2.93E-07	-2.81E-07	-3.1E-07	-1.11E-07	-8E-08	-1.17E-07	-2.04E-07	-3.31E-07	1.51E-06
0.060	± 0.003	0	1.75E-07	2.52E-07	1.16E-07	1.23E-07	8.9E-08	1.7E-08	2.1E-08	1.8E-08	1.8E-08	1.43E-07	0	-2.93E-07	7.95E-06	-1.89E-07	-1.98E-07	-6.5E-08	-4.2E-08	-5.4E-08	-8.2E-08	-1.19E-07	3.63E-07
0.032	± 0.003	1.6E-08	1.82E-07	2.44E-07	2.14E-07	2.24E-07	1.59E-07	2.8E-08	3.5E-08	2.7E-08	2.7E-08	1.96E-07	-2E-08	-2.81E-07	-1.89E-07	8.12E-06	-3.7E-07	-1.19E-07	-7.4E-08	-9.1E-08	-1.3E-07	-1.77E-07	3.51E-07
0.021	± 0.003	2.6E-08	2.25E-07	2.72E-07	2.3E-07	4.18E-07	2.94E-07	5.2E-08	6.3E-08	4.8E-08	4.6E-08	2.84E-07	-3.3E-08	-3.1E-07	-1.98E-07	-3.7E-07	9.21E-06	-2.22E-07	-1.36E-07	-1.64E-07	-2.29E-07	-3E-07	3.81E-07
0.014	± 0.002	1.4E-08	9.5E-08	1E-07	7.9E-08	1.39E-07	1.72E-07	3E-08	3.6E-08	2.7E-08	2.6E-08	1.22E-07	-1.8E-08	-1.11E-07	-6.5E-08	-1.19E-07	-2.22E-07	3.42E-06	-8E-08	-9.5E-08	-1.3E-07	-1.68E-07	1.27E-07
0.013	± 0.002	1.5E-08	8.2E-08	7.4E-08	5.5E-08	9.2E-08	1.1E-07	3.2E-08	3.8E-08	2.8E-08	2.7E-08	9.3E-08	-1.8E-08	-8E-08	-4.2E-08	-7.4E-08	-1.36E-07	-8E-08	2.34E-06	-9.9E-08	-1.36E-07	-1.74E-07	7.9E-08
0.009	± 0.002	2.8E-08	1.37E-07	1.11E-07	7.6E-08	1.21E-07	1.39E-07	3.9E-08	7.2E-08	5.4E-08	5E-08	1.4E-07	-3.5E-08	-1.17E-07	-5.4E-08	-9.1E-08	-1.64E-07	-9.5E-08	-9.9E-08	3.06E-06	-2.58E-07	-3.29E-07	9.6E-08
0.005	± 0.002	5.8E-08	2.68E-07	1.97E-07	1.26E-07	1.89E-07	2.08E-07	5.7E-08	1.03E-07	1.13E-07	1.06E-07	2.42E-07	-7.3E-08	-2.04E-07	-8.2E-08	-1.3E-07	-2.29E-07	-1.3E-07	-1.36E-07	-2.58E-07	4.44E-06	-6.94E-07	1.32E-07
0.007	± 0.002	1.07E-07	4.74E-07	3.26E-07	1.96E-07	2.82E-07	2.96E-07	7.8E-08	1.38E-07	1.5E-07	1.95E-07	3.8E-07	-1.35E-07	-3.31E-07	-1.19E-07	-1.77E-07	-3E-07	-1.68E-07	-1.74E-07	-3.29E-07	-6.94E-07	5.6E-06	1.67E-07
0.035	± 0.000	1.7E-08	1.27E-07	1.06E-07	6.9E-08	9.6E-08	8.8E-08	2E-08	3E-08	2.8E-08	3E-08	1.04E-07	7.51E-07	1.51E-06	3.63E-07	3.51E-07	3.81E-07	1.27E-07	7.9E-08	9.6E-08	1.32E-07	1.67E-07	2.08E-07

Exhibit 10

Summary of Linear Models

				Joint Paid-Incurred						Paid		Case-Incurred	
AY	EarnPrem	Paid	CaseIncd	Unpaid	Std Dev	IBNR	Std Dev	Ultimate	=	Ultimate	Std Dev	Ultimate	Std Dev
1998	23,278,084	17,738,999	19,765,070	2,999,915	± 142,339	973,844	± 142,339	20,738,914	TRUE	20,663,109	± 239,110	20,780,207	± 177,147
1999	21,555,421	17,557,257	19,812,936	3,398,227	± 158,471	1,142,547	± 158,471	20,955,484	TRUE	20,824,378	± 244,026	21,062,679	± 209,846
2000	23,495,444	18,432,885	20,739,446	3,804,647	± 177,631	1,498,084	± 177,630	22,237,532	TRUE	22,264,182	± 257,624	22,247,195	± 248,293
2001	25,864,065	17,289,118	19,811,447	4,227,840	± 186,625	1,705,511	± 186,625	21,516,958	TRUE	21,486,528	± 262,929	21,577,395	± 268,263
2002	29,134,414	15,899,281	18,589,679	4,662,500	± 192,377	1,972,102	± 192,377	20,561,781	TRUE	20,487,825	± 265,436	20,667,840	± 282,302
2003	32,391,860	15,283,538	18,314,350	5,421,105	± 220,945	2,390,298	± 220,945	20,704,643	TRUE	20,572,601	± 310,529	20,860,557	± 317,408
2004	36,533,278	14,134,508	17,632,172	6,568,927	± 272,535	3,071,266	± 272,536	20,703,435	TRUE	20,579,410	± 367,486	20,871,758	± 408,036
2005	39,208,849	12,789,801	17,221,257	8,692,524	± 320,964	4,261,067	± 320,964	21,482,325	TRUE	21,381,119	± 424,534	21,630,036	± 491,580
2006	42,065,555	10,724,002	16,822,179	13,144,841	± 403,064	7,046,664	± 403,064	23,868,843	TRUE	23,895,269	± 547,487	23,858,441	± 597,042
2007	40,220,014	5,211,936	12,931,177	20,392,908	± 549,291	12,673,667	± 549,291	25,604,844	TRUE	25,571,140	± 709,931	25,642,727	± 867,816
Total	313,746,984	145,061,325	181,639,713	73,313,433	± 1,196,054	36,735,050	± 1,196,055	218,374,758	TRUE	217,725,562	± 1,572,984	219,198,836	± 1,866,883

APPENDIX A

The Effect of Covariance in a Simple Model

This appendix will present variations of an elementary model so as to illustrate the effect of covariance. The top part of Exhibit A.1 shows the basic form (Model 1). Eight quantities are observed (Obs 1–8), and two predictions are desired (Pred 1–2). The model for each $\boldsymbol{y}$, whether observed or predicted, is $\boldsymbol{y}_i = x_i\beta + \boldsymbol{e}_i$. Except for Pred 2, all x and ϕ values are one; Pred 2 is like a doubling of Pred 1, which makes its variance relativity four. Zeroes in the variance structure are not shown. This is a heteroskedastic model (homoskedastic in the observed part). The formulæ of the linear statistical model were explained in Section 2.

In the middle of the exhibit is shown the estimate of the parameter: $\hat{\beta} = 97.875 \pm 2.074$. Those unfamiliar with this formulation of the linear model should at least recognize that Model 1 is equivalent to the simple average. At the bottom of the exhibit are various diagnostics (left side) and weighted sums of squares and crossproducts ("SSCP", right side). The diagonal elements of the 3×3 SSCP matrix must satisfy the equation $d_1 = d_2 + d_3$, and the fit accounts for 99.7% of d_1. Eight (unrelated) observations and one parameter make for seven degrees of freedom, and the estimate of the variance scale is $\hat{\sigma}^2 = 240.875/7 = 34.411$.

The estimator of β linearly depends on the error terms of the observations. Because the error terms of the predictions do not covary with those of the observations, nether do they covary with $\hat{\beta}$. Hence:

$$
\begin{aligned}
Var[\text{Pred}] &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} Var[\hat{\beta}][1 \quad 2] + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \hat{\sigma}^2 \\
&= \begin{bmatrix} 4.301 & 8.603 \\ 8.603 & 17.205 \end{bmatrix} + \begin{bmatrix} 34.411 & 0 \\ 0 & 137.643 \end{bmatrix} \\
&= \begin{bmatrix} 38.712 & 8.603 \\ 8.603 & 154.848 \end{bmatrix}
\end{aligned}
$$

(We explain the model in this manner for the benefit of those unfamiliar with matrix algebra and multivariate statistics; nevertheless, we encourage them to study these fields until they become natural. See references.) We conclude Model 1 by saying that Pred 2 is *like* Pred 1, but on twice the scale. The covariance between the two predictions is solely due to their reliance on the estimate of β; it has to do with parameter variance or uncertainty.

Model 2 of Exhibit A.2 is identical to Model 1 except that the variance relativity of Obs 4 is zero, rather than one. The column 'Constrnd' signals a variance degeneracy with a '1' for this observation. Whether this be unrealistic, it is at least instructive as a limiting case. The model says that in one observation we were able to see β without the obfuscation of an error term. So $\hat{\beta} = 96 \pm 0$, plain and simple. The error terms readjust, and according to the formulation of our software[9] there are seven (stochastic) observations and zero estimated

---

[9] It would take us too far afield to detail how our software solves the linear statistical model when the variance of the observations Σ_{11} is not positive definite. But briefly, it eigen-decomposes the

parameters, which again makes for seven degrees of freedom. In this model it is easier to see that Pred 2 is like Pred 1, but on twice the scale. If, in addition, the variance of any other observation were set to zero, the resulting two equations would be inconsistent.

Things become very interesting with Model 3 (Exhibit A.3), in which Obs 2 and Obs 6 covary, as well as Obs 7 and Pred 2. Both covariances imply perfect positive correlations. In the latter case, we show non-negative definiteness by:

$$Var\left[\begin{bmatrix}2 & -1\end{bmatrix}\begin{bmatrix}\text{Obs 7}\\ \text{Pred 2}\end{bmatrix}\right]=\begin{bmatrix}2 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}=0$$

So the error term of Pred 2 equals (with probability one) the error term of Obs 7. It is not *like* twice the error term of Obs 7; rather, it is the *same* as twice the error term. And because its dependent variable is twice that of Obs 7, Pred 2 is not *like* twice Obs 7; it *is* twice Obs 7. Without even estimating β we know Pred 2 to be $2\cdot 90=180\pm 0$. Covariance makes the difference between like and same.

But within the observations themselves is a variance degeneracy. Though all the observation error terms are *alike*, the error term of Obs 7 *is* that of Obs 3. The 'Cnstrnd' column indicates with '0.5' the dependency of the two observations. In effect, it says that the same observation is written twice, and should be counted

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observations, and treats the once-transformed rows whose eigenvalues are zero as constraints on β. Then it transforms constrained β-space into a lower-dimensional, unconstrained γ-space. The twice-transformed model (cf. "TTy1" in SSCP, where "TT" stands for "twice-transformed") is solved, and transformed back. It is a theorem that the solution of a linear model is invariant to any invertible, or one-to-one, transformation of the observations, i.e., $A\mathbf{y}_1 = AX_1\beta + A\mathbf{e}_1$, for any nonsingular A. But one must not forget to transform the covariances: $\Sigma_{12} \to A\Sigma_{12}$ and $\Sigma_{21} \to \Sigma_{21}A'$.

once. Thus instead of eight observations, there are really seven, which with one parameter estimated makes for six degrees of freedom. When covariance is properly considered, one cannot create information *ex nihilo*, i.e., by repeating the same observation. One cannot fool the model even by repeating a linear combination of observations, for $Var\begin{bmatrix} \mathbf{y}_1 \\ A\mathbf{y}_1 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{11}A' \\ A\Sigma_{11} & A\Sigma_{11}A' \end{bmatrix}$ contains no more information than Σ_{11} contains. Note that although Obs 2 and Obs 6 are redundant, they are consistent. If they were not both 93, they would be inconsistent equations.

Lastly, Model 4 in Exhibit A.4 is a mixture of Models 2 and 3. The reason for changing Obs 6 from 93 to 94 will soon appear. Similarly to Model 3:

$$Var\left[\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} \text{Obs 2} \\ \text{Obs 6} \end{bmatrix}\right] = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0 \text{ ,}$$

i.e., $\text{Prob}[2\boldsymbol{e}_2 - 1\boldsymbol{e}_6 = 0] = 1$, or simply, $\boldsymbol{e}_6 = 2\boldsymbol{e}_2$. But here Obs 2 and Obs 6 are not redundant. Nevertheless, they are equivalent to three equations in three variables:

$$\begin{aligned} 93 &= \beta + \boldsymbol{e}_2 \\ 94 &= \beta + \boldsymbol{e}_6 \text{ ,} \\ \boldsymbol{e}_6 &= 2\boldsymbol{e}_2 \end{aligned}$$

whose solution is $\beta = 92, \boldsymbol{e}_2 = 1, \boldsymbol{e}_6 = 2$. In Model 2 β was like a gem lying on the surface; here we had to pan a little for it. So $\hat{\beta} = 92 \pm 0$. If we had not changed Obs 6, the equations still would have been consistent, but the error terms would have both been zero – a much less interesting result. The 'Cnstrnd' column indicates the variance degeneracy between the two observations; but their counting as one observation is apportioned inversely according to their 1:4 variance relativities.

Granted, the behavior of Models 2–4 depends on perfect correlation, and is akin to imagining relativistic effects at the speed of light. As long as Σ_{11} has no variance degeneracy, i.e., is positive definite, the observations consist of t_1 consistent equations in $t_1 + k$ variables. They comprise a system of equations that can be solved only probabilistically. However, these limiting cases confirm the conservation of information. Just as there is no magic, just illusion, so too only by trickery can someone produce information out of nothing.

Consequently, to include in a joint paid-incurred model tautologous observations for completely observed exposure periods, correctly accounting for covariance, will furnish no additional information. For it's simply a linear combination of old information. Moreover, for numerical-analytic reasons it's dangerous, since the software must decide when small eigenvalues should be treated as zeroes. Without these redundant equations the joint model will be of full rank.

Exhibit A.1

Model 1

ID	y	X	Φ									
Obs 1	106	1	1									
Obs 2	93	1		1								
Obs 3	99	1			1							
Obs 4	96	1				1						
Obs 5	105	1					1					
Obs 6	93	1						1				
Obs 7	90	1							1			
Obs 8	101	1								1		
Pred 1		1									1	
Pred 2		2										4

ID	y2hat	StdPrdErr	VarPrdErr	VarPrdErr
Pred 1	97.875	6.222	38.712	8.603
Pred 2	195.750	12.444	8.603	154.848

Betahat	StdBeta	VarBeta
97.875	2.074	4.301

ID	y1	Fitted	Resid	Student	Cnstrnd
Obs 1	106	97.875	8.125	1.48	0
Obs 2	93	97.875	-4.875	-0.89	0
Obs 3	99	97.875	1.125	0.21	0
Obs 4	96	97.875	-1.875	-0.34	0
Obs 5	105	97.875	7.125	1.30	0
Obs 6	93	97.875	-4.875	-0.89	0
Obs 7	90	97.875	-7.875	-1.44	0
Obs 8	101	97.875	3.125	0.57	0

SSCP	TTy1	Fitted	Resid
TTy1	76877	76636.13	240.875
Fitted	76636.13	76636.13	0
Resid	240.875	0	240.875
rhosq	100.0%	99.7%	0.3%
df	8	1	7
s2hat	9609.625	76636.13	34.41071
s2sel			**34.41071**

Exhibit A.2

Model 2

ID	y	X	Φ									
Obs 1	106	1	1									
Obs 2	93	1		1								
Obs 3	99	1			1							
Obs 4	96	1				0						
Obs 5	105	1					1					
Obs 6	93	1						1				
Obs 7	90	1							1			
Obs 8	101	1								1		
Pred 1		1									1	
Pred 2		2										4

ID	y2hat	StdPrdErr	VarPrdErr	VarPrdErr
Pred 1	96	6.199	38.429	0
Pred 2	192	12.398	0	153.714

Betahat	StdBeta	VarBeta
96	0	0

ID	y1	Fitted	Resid	Student	Cnstrnd
Obs 1	106	96	10	1.61	0
Obs 2	93	96	-3	-0.48	0
Obs 3	99	96	3	0.48	0
Obs 4	96	96	0	0.00	1
Obs 5	105	96	9	1.45	0
Obs 6	93	96	-3	-0.48	0
Obs 7	90	96	-6	-0.97	0
Obs 8	101	96	5	0.81	0

SSCP	TTy1	Fitted	Resid
TTy1	269	0	269
Fitted	0	0	0
Resid	269	0	269
rhosq	100.0%	0.0%	100.0%
df	7	0	7
s2hat	38.42857	0	38.42857
s2sel			**38.42857**

Exhibit A.3

Model 3

ID	y	X	Φ									
Obs 1	106	1	1									
Obs 2	93	1		1				1				
Obs 3	99	1			1							
Obs 4	96	1				1						
Obs 5	105	1					1					
Obs 6	93	1		1				1				
Obs 7	90	1							1			2
Obs 8	101	1								1		
Pred 1		1									1	
Pred 2		2							2			4

ID	y2hat	StdPrdErr	VarPrdErr	VarPrdErr
Pred 1	98.571	6.380	40.707	0
Pred 2	180	0	0	0

Betahat	StdBeta	VarBeta
98.571	2.256	5.088

ID	y1	Fitted	Resid	Student	Cnstrnd
Obs 1	106	98.571	7.429	1.34	0
Obs 2	93	98.571	-5.571	-1.01	0.5
Obs 3	99	98.571	0.429	0.08	0
Obs 4	96	98.571	-2.571	-0.47	0
Obs 5	105	98.571	6.429	1.16	0
Obs 6	93	98.571	-5.571	-1.01	0.5
Obs 7	90	98.571	-8.571	-1.55	0
Obs 8	101	98.571	2.429	0.44	0

SSCP	TTy1	Fitted	Resid
TTy1	68228	68014.29	213.7143
Fitted	68014.29	68014.29	0
Resid	213.7143	0	213.7143
rhosq	100.0%	99.7%	0.3%
df	7	1	6
s2hat	9746.857	68014.29	35.61905
s2sel			**35.61905**

Exhibit A.4

Model 4

ID	y	X	Φ									
Obs 1	106	1	1									
Obs 2	93	1		1				2				
Obs 3	99	1			1							
Obs 4	96	1				1						
Obs 5	105	1					1					
Obs 6	94	1		2				4				
Obs 7	90	1							1			
Obs 8	101	1								1		
Pred 1		1									1	
Pred 2		2										4

ID	y2hat	StdPrdErr	VarPrdErr	VarPrdErr
Pred 1	92	8.586	73.714	0
Pred 2	184	17.171	0	294.857

Betahat	StdBeta	VarBeta
92	0	0

ID	y1	Fitted	Resid	Student	Cnstrnd
Obs 1	106	92	14	1.63	0
Obs 2	93	92	1	0.12	0.8
Obs 3	99	92	7	0.82	0
Obs 4	96	92	4	0.47	0
Obs 5	105	92	13	1.51	0
Obs 6	94	92	2	0.12	0.2
Obs 7	90	92	-2	-0.23	0
Obs 8	101	92	9	1.05	0

SSCP	TTy1	Fitted	Resid
TTy1	516	0	516
Fitted	0	0	0
Resid	516	0	516
rhosq	100.0%	0.0%	100.0%
df	7	0	7
s2hat	73.71429	0	73.71429
s2sel			**73.71429**

APPENDIX B

Correlation Constraints among Three Random Variables

Our solution to the joint model involved the addition of tautologous observations which covary with certain of the paid and incurred observations. Often the loss observations are not inter-correlated. According to statistical and econometric terminology (e.g., Judge [1988], Chapter 9), the variance structure of such observations is homo- or heteroskedastic, as opposed to autocorrelated. In our simple example they were homoskedastic; in the Workers' Compensation example they were heteroskedastic. If z be a tautologous observation that involves loss observation x (so that $Cov[z,x] \neq 0$; in our models, $Cov[z,x] = \pm Var[x]$), and x does not covary with any other loss observation, then we may assume that z does not *secondarily* covary with any other loss observation. But in general, for z to covary with x and for x to covary with y places a transitive tendency for z to covary with y. Ignoring secondary covariance may lead one to create models whose variance structure is not non-negative definite. Such models would be defective, because a variance structure is legitimate if and only if it is non-negative definite.

So our task here is to solve an interesting problem: If the correlation between two random variables is ρ, what are legitimate values of x and y, the correlations of a third random variable with the first two? Mathematically expressed, for what values of x and y is $\begin{bmatrix} 1 & \rho & x \\ \rho & 1 & y \\ x & y & 1 \end{bmatrix}$ non-negative definite? To express the problem as correlation is simpler and no less general than to express it as covariance.

First, because the correlation coefficient is bounded, $-1 \le \rho, x, y \le 1$. Hence, regardless of ρ, allowable pairs (x, y) must be on or within the square whose four corners are (±1, ±1). And second, it is a theorem of matrix algebra that a matrix Σ is non-negative definite if and only if it has a "square root," i.e., a real-valued matrix W such that $\Sigma = WW'$. The Cholesky decomposition yields a suitable square W that is lower-triangular (i.e., zero above the main diagonal).[10]

The Cholesky decomposition of the correlation matrix is:

$$\begin{bmatrix} 1 & \rho & x \\ \rho & 1 & y \\ x & y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ x & a & b \end{bmatrix} \begin{bmatrix} 1 & \rho & x \\ 0 & \sqrt{1-\rho^2} & a \\ 0 & 0 & b \end{bmatrix},$$

where:

$$\begin{aligned} y &= \rho x + a\sqrt{1-\rho^2} \\ 1 &= x^2 + a^2 + b^2 \end{aligned}$$

Hence, the correlation matrix is non-negative definite if and only if real values of a and b exist that solve these two last equations (in which ρ, x, and y are given).

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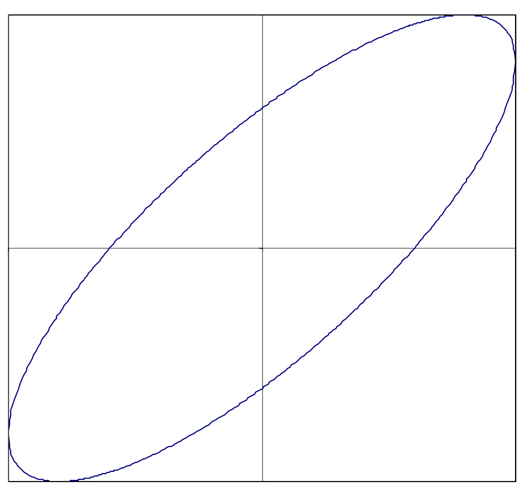
[10] Cf. Halliwell [1997; Appendix A], Healy [1986; 54f], and Judge [1988; 961].

Of course, $\sqrt{1-\rho^2} \geq 0$. If $\sqrt{1-\rho^2} = 0$, y must equal ρx, and a may be any real number. Setting a to zero in this case gives the most leeway for b to be real. Hence, a Cholesky decomposition exists if and only if $b^2 \geq 0$. With this information we derive the inequality:

$$\begin{aligned} x^2 - 2\rho xy + y^2 &= x^2 - \rho^2 x^2 + y^2 - 2\rho xy + \rho^2 x^2 \\ &= x^2\left(1-\rho^2\right) + \left(y - \rho x\right)^2 \\ &= x^2\left(1-\rho^2\right) + \left(a\sqrt{1-\rho^2}\right)^2 \\ &= \left(x^2 + a^2\right)\left(1-\rho^2\right) \\ &= \left(x^2 + a^2 + 0\right)\left(1-\rho^2\right) \\ &\leq \left(x^2 + a^2 + b^2\right)\left(1-\rho^2\right) \\ &= 1 \cdot \left(1-\rho^2\right) \end{aligned}$$

Therefore, the correlation matrix is non-negative definite if and only if $x^2 - 2\rho xy + y^2 \leq \left(1-\rho^2\right)$.

Legitimate (x, y) points satisfy the equation $x^2 - 2\rho xy + y^2 \leq \left(1-\rho^2\right)$. The region of these legitimate points is symmetric about the two lines $y = \pm x$. In fact, it is an ellipse whose axes are on those lines, its half lengths along the lines being $\sqrt{1 \pm \rho}$ respectively. Here is a graph is of the ellipse when $\rho = 0.8$:



Interior points of the ellipse, i.e., $x^2 - 2\rho xy + y^2 < (1-\rho^2)$, produce *positive* definite matrices; boundary points indicate a linear dependence among the three random variables. If $\rho = 0$, the ellipse becomes the unit circle. In the case that $\rho = \pm 1$, the ellipse degenerates into the respective lines $y = \pm x$, as expected. The origin is always a legitimate candidate for (x, y), since for any two random variables there exists a third uncorrelated with either one of them.

The area of the ellipse is $\pi\sqrt{1-\rho^2}$. That this is maximized for $\rho = 0$ means that one may accommodate new random variables most freely into a universe of uncorrelated random variables. We can integrate the area of ellipse(ρ) over ρ: $\int_{-1}^{1} \pi\sqrt{1-\rho^2}\, d\rho = \pi\frac{\pi}{2} = \frac{\pi^2}{2}$. From this we conclude that the probability of constituting a legitimate correlation structure by randomly sampling ρ, x, and y from a Uniform$[-1, 1]$ distribution is $\frac{\pi^2}{2} \Big/ 2^3 = \left(\frac{\pi}{4}\right)^2 \approx 61.7\%$.[11]

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[11] The legitimacy equation has the three-way symmetric form $x^2 + y^2 + z^2 - 2xyz \le 1$, as well as the determinant form $\begin{vmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{vmatrix} \ge 0$. However, non-negative (positive) definiteness means more than a non-negative (positive) determinant. A symmetric $n \times n$ matrix is non-negative (positive) definite if and only if all its subdeterminants, of which there are $2^n - 1$, are ≥ 0 (> 0). However, as a test of definiteness this is much less efficient than the Cholesky decomposition.

The Cost of Risk: A COTOR-VALCON[1] Discussion

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This note is a summary of a COTOR-VALCON discussion on the relationship between an insurer's risk and cost of capital. The focus is two fold: on the applicability of the capital asset pricing model (CAPM), and on the effects of financial frictions.

Keywords. CAPM; cost of capital; financial frictions; shareholder value; financial economics.

1. INTRODUCTION

A central concern of insurance company management is knowing whether, and how, they are creating value for shareholders. This leads to finance questions like, "What is our cost of capital?" and "What rate of profit distinguishes a division or line of business that creates value from one that does not?" The actuarial profession has spent over a century quantifying profit and loss in this inherently tricky business, what with its randomness, uncertainty, and delayed revelation of ultimate reality. But grappling with the relationship between the balance sheet and the stock ticker is a relatively recent challenge for actuaries.

In 2006, members of the COTOR-VALCON discussion group held a four month long on-line exchange of ideas concerning these issues. This note is a summary of some of that discussion. Section 2 introduces the basic elements: cost of capital, the capital asset pricing model (CAPM), and financial frictions. Section 3 explores CAPM more deeply, examining whether non-market risk might be priced in the capital markets. Section 4 addresses the sources and role of financial frictions. Section 5 summarizes the "emerging view" of the nature of shareholder value in the insurance firm.

2. IT ALL STARTED WITH THAT SWISS RE PAPER

"The Economics of Insurance: How Insurers Create Value for Shareholders" (Hancock, et al. 2001) was received warmly by most of the group. "Good meat and potatoes stuff" and "I'm 100% on-board" were some of the initial reactions. The essence of that article is the statement:

> ...an insurer's opportunity cost of capital is the return that shareholders could otherwise achieve by investing their risk capital directly themselves plus additional compensation

[1] Committee On the Theory Of Risk – VALuation of CONtingent liabilities

for various frictional costs that are specific to insurers.

But what does this mean?

Insurance companies are often compared to closed-end mutual funds (investment trusts) because their assets are primarily financial assets. The performance of a mutual fund is transparent: it is the sum of the performance of its investments, less a small amount of overhead expense. More specifically, insurers are compared to *levered* mutual funds. The issuance of policies creates a cash flow into the investments for a period of time before claims take cash back out, and this is analogous to the issuance of bonds. In the case of a levered fund, the analysis of performance is quite straightforward because all of its assets and liabilities have market values – and returns – that can be observed every trading day. That mix of assets and liabilities leads to a specific hurdle rate of return on equity that the firm is expected to make. That hurdle rate depends on, among other things, the systematic risk of returns, i.e. their correlation with the returns of the capital markets as a whole. Measuring systematic risk by a stock's "beta" is the basis of the Capital Asset Pricing Model (CAPM). More complex versions of CAPM, using higher moments (asymmetry and "fat tails"), or more than one priced factor, have also been developed.

If an investment trust's returns persistently fell short of the hurdle rate, say through poor stock-picking or mismanagement of expenses, then the market capitalization of the firm (the sum of its share values on the market) could be lower than its book value (the value of assets minus value of liabilities) and shareholders would rightly feel that they could have done better investing directly in the assets themselves. On the other hand, if returns are persistently higher, then market cap could be higher than book value – a situation where the difference, called *franchise value*, is positive. Possessing franchise value justifies the existence of the firm; the shareholders do *not* think that they could have done better by themselves.

Getting back to the quote from Hancock et al., that hurdle rate of return on equity, the "return that shareholders could otherwise achieve," is known as the *base cost of capital.*

But insurers are not levered investment trusts. Their liabilities are not (all) bonds. Their business is issuing insurance policies. This brings a few complications to the analysis.

First, insurance policies are not traded on an exchange, so do not have a market value. The problem of valuation of insurance liabilities is and has been a major concern of actuaries for a long, long time. In more recent decades, however, the increasing sophistication of financial markets has created pressure for *market consistent*, or so-called *fair* valuation of liabilities.

The second complication is what Hancock et al. refer to as *frictions*. This is a term used in finance theory (which of course borrowed it from physics). The idea can be traced back to the famous Modigliani-Miller (1963) Irrelevance Theorems that state that under the following conditions, neither capital structure, dividend policy, nor profit volatility (and by implication risk management) matter to the value of a firm (Ng and Varnell 2003):

- Financial markets are arbitrage-free
- Taxes are neutral (i.e. the same tax rate applies to all profits and losses)
- There are no transaction costs
- There are no distress costs (costs incurred when the firm goes bankrupt or is threatened with bankruptcy)
- There are no agency costs (costs incurred due to the fact that management and ownership of the firm are separated)
- Changes in financial structure or dividend policy do not convey new information about future profits.

Because capital structure, dividend policy, volatility, and risk management manifestly *do* affect the value of the firm, it must be because one or more of these requirements is not met. The failure of these conditions are known as *financial frictions*. Since failure brings negative contributions to firm value, those contributions are known as *frictional costs*.

Repeating the Hancock et al. quote in full, with added emphasis, "…an insurer's opportunity cost of capital is the return that shareholders could otherwise achieve by investing their risk capital directly themselves *plus additional compensation for various frictional costs that are specific to insurers*." In pseudomathematical shorthand, we might say: COST = CAPM + FRICTION.

Thus we see two areas where deeper understanding is needed in order to determine the cost of capital of an insurance firm. One is the valuation of liabilities consistent with financial economics; the other is the valuation of the impact of frictions, or, loosely, the estimation of frictional costs.

3. CAPM: IS NONSYSTEMATIC RISK PRICED IN THE MARKET?

Since its introduction in the 1960s, the Capital Asset Pricing Model (CAPM) has been a staple of market valuation. The insurance industry has been reluctant to use it, however. One problem is that its distributional assumptions are inconsistent with the "heavy tails" encountered in some lines of insurance. Fama and French (1992) report another problem with CAPM: other factors besides covariance are priced. In particular, larger companies, and companies with high market-to-book ratios, tend to have lower returns. Chung, Johnson and Schill (2006), following the work of Hung, Shackleton, and Xu (2004), find that the Fama-French phenomenon can be explained away if enough higher co-moments (up to 10) are included in a CAPM-style formula. Levy and Roll (2008) suggest that estimation methods may be to blame for the apparent inadequacies of CAPM. Despite these complexities, the unaugmented CAPM explains enough to be the bedrock stock pricing formula even today.

If insurance liabilities behaved like stocks and bonds, if they were traded in fluid markets or could be hedged by securities that were so traded, then their "market consistent" valuation would not be controversial. But they don't and they aren't and they can't, so it is. The discussion group analyzed several "thought experiments" and traded several papers that hinted at the breadth of this controversy. A few are recounted here.

3.1 Hole In One Insurance

The Hole-In-One Insurance example has been around for a while. (And there actually are companies that provide such insurance!) But its value as a thought experiment lies in its simplicity. The discussant who introduced the topic described it this way:

> Let's say that I have set up my own "Lloyds Syndicate" to provide hole-in-one insurance. The local golf course is my only policyholder, and pays me $50 of premium for the following coverage: If anyone hits a hole-in-one during the year, I pay that person $1,000 (but that's the aggregate policy limit, and thus the maximum loss). If nobody hits one, I pay nothing. The coverage period is from 1/1 to 12/31 (we golf year-round in Texas!). I estimate that the probability of a hole-in-one during the year is 5% (this is a tough course). I also contribute $1,000 of capital "funds" to my Lloyds Syndicate. The premium + capital (total of $1,050) is invested in a one-year bank CD at 3%. There are no underwriting or loss adjusment expenses. There is no IBNR, so I can easily determine my annual ROE on the following 1/1. There is a 95% chance that my return on the $1,000 investment will be 8.15%, and a 5% chance that it will be -91.85%. Thus, my expected return is 3.15%. Is this a good deal for me?

According to the canon of financial economics, it is a good deal. There are no sources of friction and the random outcome is independent of the financial markets.2 But would you accept this sort of deal? Most people would not. The risk is substantial, and even though it is not "systematic," i.e., correlated with the financial markets, most people would want a higher return than the additional 0.15% over the risk-free rate for bearing this risk. According to financial economics, non-systematic (also known as "idiosyncratic") risk, because it is diversifiable, should have a price of zero in the financial markets. Does this mean financial economics is wrong? Or is there something else going on?

3.2 Corporate Bond Spreads

A good place to look for real-world departures from theory is the phenomenon of corporate bond spreads. Here you have the relatively "pure" world of interest rate risk, copiously represented by risk-free, that is to say, credit-risk-free, government bonds, plus the risk of default on the part of the issuing corporation. Spreads between corporate bond yields and the corresponding government yields measure the market's pricing of the default risk; typically it is several times the actuarial value. Does correlation with the market (i.e., CAPM beta and all that) explain these spreads?

Amato & Remolona (2003) address this puzzle and conclude that the problem lies in the inability to properly diversify credit risk. That is, the CAPM principle does not apply because its assumptions cannot be met.

Elton et al. (2001) also address the credit spread issue, with extensive data analysis. They conclude that correlation between default and overall stock market returns is an adequate explanation for credit spreads, so the CAPM principle does apply.

So which is it?

3.3 Who Is The Investor?

Look at the finance argument for hole-in-one insurance again. If you have a well-diversified portfolio, and you consider whether to add this "investment" to it, you will find that it will improve the risk/return characteristics of your portfolio. So why not do it?

There is a stream of research known as "behavioral finance" that explores the psychological side of risk-taking and investment. You can watch an ongoing experiment in behavioral finance weekly,

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[2] And let's not quibble about the estimated 5% probability of a hole-in-one. It could be high, it could be low. All we require is that it be an unbiased estimate.

on television (Post et al. 2006). Examples of more scientific studies are (Shiv et al. 2005) and (Wakker et al. 1997). This research finds that most investors, being human, are not nearly the rational actors that economic theory builds its models upon. Perhaps more importantly, (and this is emphasized in Modigliani & Miller's 1963 paper), even though investors may be rational, they do not necessarily assume that other investors are, too – and this itself can lead to what appears to be "irrational" behavior (like bubbles).

How many rational investors does it take to make the finance argument fly? If the hole-in-one premium were higher and the expected return were, say, 6%, there would likely be many people vying for the opportunity to write it – so the premium would be bid down. Would it be bid down all the way to 3.15%?

And how long would it take? Much of financial economic theory is about equilibrium – that mythical state of affairs when everything has settled down after all those nasty random disturbances finally stop. Ask a bunch of day-traders about the relevance of CAPM, beta, and equilibrium economics and they just might tear themselves from their screens for a few seconds to laugh at you. For them, it's all about the short term, hedging and arbitrage, taking a position, and finding an edge. The eventual trends in pricing will matter to them, but not much. On the other hand, a buy-and-hold investor whose planning horizon goes out for decades will not be overly concerned with current market turbulence, but rather with the relationship between current and expected future prices.

So which perspective rules? That is hard to say. Price formation is not a function of all investors, it is a function of the so-called marginal investors – the investors at the margin, the ones who are ready to trade next. If an investor who is rational and holds a diversified portfolio knows about the hole-in-one opportunity and is in a position to act upon it, then a 3.15% expected return will be attractive and this will be the marginal investor. But what if no such investors are available? This appears to be the case for some types of investments. For example, Gabaix et al. (2007) discuss mortgage-backed securities where it seems the available marginal investors are specialists who hold overweighted (not well-diversified) portfolios. As a consequence, returns are higher than they would be according to CAPM-type arguments. This could well be happening to catastrophe insurers, too.

4. FRICTIONS AND FRICTIONAL COSTS

As one discussant put it, "There's more to financial theory than systematic covariance with a market portfolio of all risky assets." And a lot of that "more" seems to be covered under the heading of "frictions."

Recall that the Modigliani-Miller Irrelevance Theorems apply when certain conditions are met; when they are not, we refer to the reasons they are not as "frictions." Most relevant to the discussion group were the financial frictions experienced by insurance companies.

By negating the premises of M&M, we can construct a list of potential frictions:

- Barriers in the financial markets, such as institutional restrictions on trading
- Tax asymmetry and nonlinearity
- Transaction costs
- Distress costs and counterparty credit sensitivity
- Agency effects
- Information asymmetry
- Signaling

Some of these are fairly straightforward to quantify, such as direct tax and transaction costs. Some are a bit trickier, like tax nonlinearities. Some are wholly mysterious, such as agency effects – which are fundamentally psychological phenomena.

Generally, the impact of frictions can be seen as taking place on a continuum of the amount of risk capital held. Too much risk capital, and some frictions, such as the tax cost of holding capital, dominate. Too little risk capital and another set of frictions, such as distress costs and credit sensitivity, dominate instead. The art of managing frictions involves, among other things, finding the optimal level of capital to hold.

4.1 Too Much Capital

Feldblum (2006) argues that the main frictional cost that bears on the question of fair value accounting is the tax on investment income that risk capital earns. According to financial theory (basically, going back to M&M again), whatever risk capital is invested in, its rate of return would be sufficient for investors – as long as they got it all in their returns from the insurer. But they don't (usually) because the IRS takes about 35% of it.[3] Investors need to be compensated for that tax loss, so premiums need to be higher by an amount sufficient to raise those funds. To make a level playing field in the capital markets, *policyholders* need to pay that tax on risk capital – not just once,

[3] That is, in the USA. Bermuda-based firms, on the other hand, face approximately zero tax.

but twice, because the higher profits resulting from the higher premiums are themselves taxed, before they can flow to the investors!

While Feldblum acknowledges "principal agent problems" as another cost of holding capital, he writes, "[T]hey are rarely large, and they are not easily measured. We ignore them in this paper." (Neither does he address the costs of holding too little capital.)

Jensen & Meckling (1976) started the whole discussion of agency costs of holding capital. They were coming from the perspective of the typical industrial firm, where capital was about funding new, positive net-present-value projects – that is, projects that will earn profits higher than the required capital market hurdle rate. A firm that holds a bunch of cash evidently doesn't know what to do with it. While the firm is holding it, in government bonds, say, it is indeed earning an appropriate rate (taxes aside). But what happens when management decides to spend it on an ill-advised acquisition or other negative NPV project? What if management plays it safe and doesn't embark on certain risky but profitable ventures that investors would approve of? What if management decides to spend the money on an upgrade to the corporate jet fleet or other perks? That capital is at risk, and investors would rather get it back than trust management to eventually invest it in good, new projects. As a result, certain costs, which mostly go under the heading of "monitoring and bonding" are incurred to avoid these outcomes. Jensen & Meckling recognize the difficulty of quantifying this sort of risk. They state (p. 346),

Before proceeding further, we point out that the issue regarding the exact shapes of the functions drawn in fig. 5 and several others discussed below [agency costs vs. financial leverage] is essentially an open question at this time. In the end the shape of these functions is a question of fact and can only be settled by empirical evidence.

4.2 Too Little Capital

Why would an insurance firm hold too much capital? To avoid the adverse effects of holding too little capital. For the industrial firm, too little capital means flirting with the possibility of bankruptcy and the costs that financial distress brings (lawyers, management distractions, raising capital) and missed opportunities for investing in positive NPV projects. For the financial firm, it also means disapproval from ratings agencies and potential action by regulatory authorities.

At heart, this reflects more than the distress and bankruptcy costs facing an industrial firm; it reflects a significant amount of counterparty credit risk, or, more specifically, customer (policyholder) risk aversion. Policyholders buy insurance to protect themselves from certain adverse contingencies – like property damage, bodily injury, medical expense, and death – they cannot easily "diversify away." Being offered coverage with a material probability of not paying off (call it "probabilistic insurance") is not going to satisfy customers' desire for certainty in removing the risk. Wakker et al. cite survey data suggesting that people will pay much less in premiums than the actuarial cost of bearing the residual credit risk, perhaps twenty times less.

Looked at from the opposite perspective, customers are willing to pay more for the service of having certain risks transferred away from them than the actuarial value of those risks. That's what supports underwriting and claim expenses and even, sometimes, underwriting profits. Holding too little capital threatens the profitability of the insurance firm – not just probabilistically, but immediately.

5. THE EMERGING VIEW

The emerging view of the insurer's financial predicament (Harrington & Danzon 1994, Staking & Babbel 1995, Doherty 2000, Froot et al. 2004, Babbel & Merrill 2005, Epermanis & Harrington 2006, Panning 2006, Mango & Major 2007, Froot 2007, Major 2008, Yu et al. 2008) centers on its need to protect and enhance its franchise value. Recall, franchise value is defined as the difference between market capitalization and book value. If the firm earns only enough to cover its cost of capital, then (in theory) there will be no franchise value. As Babbel & Merrill (2005) explain it,

The franchise value stems from what economists call "economic quasi-rents." It is the present value of the "quasi-rents" that an insurer is expected to garner because it has scarce resources, scarce capital, charter value, licenses, a distribution network, personnel, reputation, and so forth. It includes renewal business. Franchise value is dependent on firm insolvency risk. The less insolvency risk there is, the more likely the firm is to remain solvent long enough to capture all the available economic rents arising from its renewal business, its distribution network, its reputation, and so forth.

As we saw earlier, threats to surplus can translate into threats to profitability. This means, therefore, threats to franchise value as well. Mathematically, MARKET CAP = BOOK VALUE + FRANCHISE VALUE. If franchise value is positive, but erodes when book value is too low (distress and credit frictions) or too high (tax and agency frictions), then the market value vs book value curve (which Froot et al. call the "M-Curve") will be concave, not linear. Thus, increased volatility in book value will translate into decreased expectations for future market value.

The source of that volatility does not matter; it can be systematic or idiosyncratic. Either way, it will have an impact on the value of the firm. So, a risk management program (e.g., investment hedges, reinsurance, cat bond, strategic changes to the business mix, etc.) that protects a million dollars of surplus, but appears to do so at greater than actuarial cost, may be worth engaging if it also protects a substantial amount of franchise value as well.

Acknowledgment

The author most gratefully acknowledges the participants in this discussion thread: Don Mango, Todd Bault, Trent Vaughn, Dan Heyer, David Ruhm, Leigh Halliwell, Lee Van Slyke, Keith Rogers, Phil Heckman, John Aquino, Jim Garven, Richard Goldfarb, Gary Venter, Steve Mildenhall, Richard Derrig, and possibly others to whom apologies are due for not naming.

5. REFERENCES

[1] Amato, Jeffery D. and Eli M. Remolona (2003), "The Credit Spread Puzzle," *BIS Quarterly Review*, December.

[2] Babbel, David F. and Craig Merrill (2005), "Real and Illusory Value Creation by Insurance Companies," *The Journal of Risk and Insurance*, Vol. 72 No. 1, March.

[3] Chung, Y. Peter, Herb Johnson and Michael J. Schill. 2006. "Asset Pricing when Returns Are Non-normal," *Journal of Business* 79(March):923-940.

[4] Doherty, Neil (2000), Integrated Risk Management, New York: McGraw-Hill.

[5] Elton, Edwin J., Martin J. Gruber, Deepak Agrawal, and Christopher Mann (2001), "Explaining the Rate Spread on Corporate Bonds," *The Journal Of Finance*, Vol. LVI, No. 1, February.

[6] Epermanis, Karen and Scott E. Harrington (2006), "Market Discipline in Property/Casualty Insurance: Evidence from Premium Growth Surrounding Changes in Financial Strength Ratings," *Journal of Money, Credit and Banking*, Vol 38, No. 6.

[7] Fama, Eugene and Kenneth French. 1992. "The Cross-Section of Expected Stock Returns," *Journal of Finance*, 47(June):427-465.

[8] Feldblum, Sholom (2006), "Fair Value Accounting For Property-Casualty Insurance Liabilities," *Casualty Actuarial Society Discussion Paper Program*.

[9] Froot, Kenneth, Gary Venter, and John Major (2004), "Capital and Value of Risk Transfer," 14th Annual International AFIR Colloquium.

[10] Froot, Kenneth (2007), "Risk Management, Capital Budgeting, and Capital Structure Policy for Insurers and Reinsurers," *The Journal of Risk and Insurance*, Vol. 74, No. 2, June.

[11] Gabaix, Xavier, Arvind Krishnamurthy, and Olivier Vigneron (2007) "Limits of Arbitrage: Theory and Evidence from the Mortgage-Backed Securities Market," *The Journal Of Finance*, Vol. LXII, No. 2, April.

[12] Hancock, John, Paul Huber, and Pablo Koch (2001), "The Economics of Insurance: How Insurers Create Value for Shareholders," Zurich: Swiss Re Technical Publishing.

[13] Harrington, Scott, and Patricia Danzon (1994), "Price Cutting in Liability Insurance Markets," *Journal of Business*, Vol. 67, No. 4.

[14] Hung, D. C., M. Shackleton, and X. Xu. 2004. "CAPM, higher comment and factor models of UK stock returns." *Journal of Business, Finance, and Accounting* 31:87–112.

[15] Jensen, Michael C., and William H. Meckling (1976), "Theory of the Firm: Managerial Behaviour, Agency Costs and Ownership Structure," *Journal of Financial Economics* Vol. 3.

[16] Levy, Moshe and Richard Roll (2008), "The Market Portfolio May Be Mean-Variance Efficient After All," working paper, Jerusalem School of Business Administration at the Hebrew University and Anderson School of Management at UCLA.

[17] Major, John A. (2008), "On a Connection between Froot-Stein and the de Finetti Optimal Dividends Models,"

National Bureau of Economic Research, Insurance Project Workshop, May 8.
[18] Mango, Don, and John A. Major (2007), "Measuring the Market Value of Risk Management," *Risk Management*, September.
[19] Modigliani, Franco, and Merton Miller (1958), "The Costs of Capital, Corporation Finance, and the Theory Of Investment," *American Economic Review* Vol. 48.
[20] Ng, Hui M, and E. M. Varnell (2003), "Frictional Costs," presented to the Staple Inn Actuarial Society, London, April 15.
[21] Panning, William H. (2006), "Managing the Invisible: ALM, Solvency, and Franchise Value," Willis Re.
[22] Post, Thierry , Guido Baltussen, and Martijn Van den Assem (2006), "Deal or No Deal? Decision-making under Risk in a Large Payoff Game Show," Tinbergen Institute Discussion Paper TI 2006-009/2.
[23] Shiv, Baba, George Loewenstein, Antoine Bechara, Hanna Damasio, and Antonio R. Damasio (2005), "Investment Behavior and the Negative Side of Emotion," *Psychological Science*, Vol. 16, No. 6, June.
[24] Staking, Kim B., and David F. Babbel, (1995) "The Relation Between Capital Structure, Interest Rate Sensitivity, and Market Value in the Property-Liability Insurance Industry," *The Journal of Risk and Insurance*, Vol. 62, No. 4.
[25] Wakker, Peter P., Richard H. Thaler, and Amos Tversky (1997), "Probabilistic Insurance," *Journal of Risk and Uncertainty*, vol. 15.
[26] Yu, Tong, Bingxuan Lin, Henry R. Oppenheimer, and Xuanjuan Chen (2008), "Intangible Assets and Firm Asset Risk Taking: An Analysis of Property and Liability Insurance Firms," *Risk Management & Insurance Review*, Vol. 11, No. 1, Spring.

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Quantifying Uncertainty In Reserve Estimates

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Abstract

Property/casualty reserves are estimates of losses and loss development and as such will not match the ultimate results. Sources of error include model error (the methodology used does not accurately reflect the development process), parameter error (model parameters are calibrated from the data), and process error (future development is random). This paper provides a comprehensive and practical methodology for quantifying risk that includes all three sources. The key feature is that variability is captured by examining historical changes in ultimate values rather than examining the underlying claim distribution. We present both the conceptual framework as well as practical examples.

1. THE VARIABILITY PROBLEM

The Challenge of Reserving

The Property/Casualty business model relies on the accurate measurement of risk. Of relevance to this paper is the measurement of reserving risk. Accurate actuarial loss reserving is one of the regulatory requirements in measuring solvency. Consequently, one of the most important tasks for an actuary is to estimate the proper amount of reserves to be set aside to meet future liabilities of current in-force business.

Because the stated reserve is an estimate and not the true number, there is error and it is important to measure this error. Quantifying the potential error allows for setting ranges around a best estimate, allows for measures of risk, and can assist in the setting of risk based capital requirements. Recent papers such as Hayne [8] and Shapland [11] make the importance of this issue clear.

In this paper we propose a method for measuring the total risk involved in reserve estimates. It is simple to apply and uses data that is almost always available from the reserve setting process. A key feature is that our measure of reserve variability does not depend on the method used to determine the reserves.

Literature Review

There are many papers regarding different loss reserving techniques, some deterministic, some stochastic. For a comprehensive review of existing deterministic methods the reader is referred to Wiser [17] and Brown and Gottlieb [2]. For an excellent overview of a wide range of stochastic reserving methods in general insurance, the reader is referred to England and Verrall [5]. Other references of interest include Bornhuetter and Ferguson [1], Finger [6], and Taylor

[12].

While the traditional chain ladder technique provides only a point estimate of the total reserve, it has become evident recently that actuaries also need a measure of variability in loss reserving estimation. Sound methodologies that quantify risks related to the balance sheet are important for attracting and retaining capital in the firm. Standard and Poor's ratings as well as investors are very interested in the value at risk (VaR) measurements. The NAIC and state regulators are interested in monitoring the reserve variability in the form of reserve ranges. Variability and sometimes the entire distribution of the loss reserving estimation are important for risk management purposes. Questions such as what is the 95th percentile of losses or the cost of loss portfolio transfer are important in managing and assessing risk. These questions can be properly addressed with a thorough analysis of the variability of the reserve estimate.

Over the last thirty years many researchers have made significant contributions to the study of the variability of reserving methods. The 2005 CAS working party paper [3] presents a comprehensive review that brings all of the important historical research together. A representative, but not exhaustive, list would include the following: Taylor and Ashe [13] introduced the second moment of estimates of outstanding claims. Hayne [7] provided an estimate of statistical variation in development factor methods when a lognormal distribution is assumed for these factors. Verrall [15] derived unbiased estimates of total outstanding claims as well as the standard errors of these estimates. Mack [10] used a distribution-free formula to calculate the standard errors of chain ladder reserve estimates. England and Verrall [5] presented analytic and bootstrap estimates of prediction errors in claims reserving. De Alba [4] gave a Bayesian approach to obtain predictive distribution of the total reserves. Taylor and McGuire [14] applied general linear model techniques to obtain an alternative method in cases where the chain ladder method performs poorly. Verrall [16] implemented Bayesian models within the framework of generalized linear models that led to posterior predictive distributions of quantities of interest.

However, the CAS 2005 working party [3] ultimately concluded, "there is no clear preferred method within the actuarial community." Actuaries need to select one or several methods that are considered appropriate for the specific situation. When it comes to the final decision, judgment still overrules.

The Risk Measurement Problem

The 2005 CAS working party report [3] notes that the sources of uncertainty in the reserve estimate come from three types of risk: process risk, parameter risk and model risk. Model risk is the uncertainty in the choice of model. Parameter risk is the uncertainty in the estimates. Process risk is the uncertainty in the observations given the model and its parameters.

Shapland [11, p.124] highlights the importance of all the risks:

> Returning to the earlier definition of loss liabilities … all three types or risk … should be included as part of the calculated expected value. Alternatively, some or all of these types of risk could be included in a 'risk margin' as defined under ASOP No. 36.

These three risks are intertwined and thus hard to separate. In particular, process risk is usually calculated as if the model and parameters were correct and parameter risk is calculated as if the model were correct.

In this paper the three risks will not be separately measured nor directly treated[1]. Often, when model risk is discussed it is in the context of the chosen model versus the true model or in the context of the chosen model versus alternative models. Rather, we capture the total risk from all three sources underlying the reserve estimate. The reserving model (or ultimate loss selection if no specific method is used) will be considered fixed and any errors measured will be a consequence of that choice.

From a risk management perspective this is appropriate. Now that the reserves have been established, what is the potential error that may result, given the current reserve review?

A Summary of Our Approach

Most approaches to risk measurement rely on the statistical properties of the data as reflected by the model selected for calculating the reserve. Those approaches attempt to capture the underlying distribution of losses. This will capture process error, but not model or estimation error. That is because any calculations will be done assuming the model and its estimated parameters are correct. In this setting, process error can be estimated using statistical measures such as Fisher information. We choose to look at the reserves (as reflected in the estimated ultimate losses) themselves as they evolve over time. This provides a way to reflect all the sources of error. Each reserve set in the past is an estimate of its distribution and thus its errors can be estimated from the historical errors made in the estimations. Because the ultimates will converge to the true value, the errors made along the way reflect all sources of error.

Our methodology will be introduced within a stable context. In particular, we assume that the underlying loss development processes have not changed over time and that the same reserving methodology has also not changed. Further, we assume we are working with indicated reserves that are the result of a specific methodology. Our method does not depend on the particular

[1] The model presented will capture the parameter and model risk in the actuary's estimate but will not measure the parameter uncertainty due to its own estimate.

methodology selected, just that it be consistently applied. After working through this situation we will discuss modifications that can allow our methods to be applied in more general settings.

Note that we model indicated reserves instead of held reserves. This is because indicated reserves do not have management or other ad hoc adjustments. Therefore, indicated reserves are more stable and thus easier to model. What is interesting about held reserves is where they fall within the probability distribution of the unknown true reserve. This can provide an indication of the degree to which held reserves are conservative or aggressive.

Because the method presented here is free of the choice of the reserving method used, it is not necessary to even have a specific method. The only requirement is a history of ultimate loss selections. Thus we rely on the actual error history of the reserving department.

We make a theoretical and practical case for the lognormal distribution for the errors in aggregate reserves, line or total. The focus on the aggregate distribution also removes the need to choose individual size of loss distributions.

Each of the following sections will take one step through the development of the risk measure. An example will be followed throughout to illustrate the formulas.

2. RESERVING PROCESS AND DATA

The reserve review process generates reserves based on raw data analysis. The indicated ultimate losses are selected by line using perhaps several methods as well as judgment. These line ultimates are then added to yield total indicated reserve.

Management adjustments called margins may be applied to the total reserve. These margins may then allocated by line and by accident year to the indicated ultimate losses and reported in Schedule P Part2. As noted earlier, our method does not work with these reserves.

There are three issues of interest relating to measuring reserve variability:

- The distribution of the true (but unknown) ultimate losses by line and in aggregate. This shows the volatility and the bias in the actuarial selections.
- The held reserves that are reported in Schedule P Part2 as a percentile on the distribution of reserves.
- A procedure to allocate the margin by line & accident year such that ultimate losses for all years are at a constant percentile.

It is instructive to understand several aspects of indicated reserves:

Indicated reserves are the actuary's best estimates based on the data and exclude management adjustments. Therefore errors are due only to actuarial selections, methods, or randomness in the data.

Data triangles used for reserve reviews can be quarterly or annual. Generally most companies like to have consistency in the indicated reviews as it is easier to update spreadsheets for each review if they are the same size. Also most companies like to track development to ultimate and prefer complete triangles if data is available.

Reserve reviews are done on a net and or direct basis and the underlying triangles are based on the relevant data. They are definitely conducted annually to report Annual Statement reserves but many companies do them quarterly.

Reserving actuaries refer to lines of business as "segments" as they can be custom defined by the actuary for reserve reviews. These can be different than the usual lines of business such as those defined in Schedule P. For consistency of notation in this paper, we will call reserve segments as lines of business. The methodology will be the same in both cases.

Generally the DCC (Defense & Cost Containment) reserve review (indicated reserves) is done using data where they are either a part of the loss triangle (loss + DCC) or are treated separately (DCC only). In the first case we treat DCC as part of the losses and in the latter case as another data segment (if broken out separately). In this paper loss shall mean whatever appears in the analysis being evaluated.

The variability for ULAE reserves is outside the scope of the paper and will not be discussed. For most companies ULAE reserves are a relatively small part of the total reserves.

Scope of Model

The approach presented in this paper is generic and applies to any type of triangle such as

Paid or incurred

Count or Severity

Accident year, policy year, or report year

Quarterly or annual.

Each of these triangles eventually leads to the appropriate ultimate losses. Depending on the choices made above and the selected reserve methodology, the distribution of errors will differ. For example if only the paid loss development method is used and we track its history of ultimate losses over time then the resulting reserve distribution will pertain to the paid loss development method. If several methods are used and the actuary finally selects reserves based on several indications (typically this is the case) then the triangle history of final selected ultimate losses will provide the distribution of the selected reserves.

An interesting point is that the method can even be applied to the raw data. This is equivalent to treating the data as the selected ultimate loss. In this case the resulting distribution will pertain to the data itself (paid or incurred losses etc). This essentially creates a new reserving method. However, the purpose of this paper is not to promote a new way of calculating reserves, rather to develop a method for determining the distribution of the indicated reserves resulting from actuarial selections.

Reasonable estimates

A standard assumption that is in line with Actuarial Standards of practice (ASOP) on reserves is that the reserve ranges are set around reasonable estimates. This is partly achieved by using indicated ultimate reserves instead of held reserves as these do not have management adjustments.

Cases where the indicated ultimate losses are unreasonable are outside the scope of this paper. This does not necessarily mean that the model will not apply but rather that the authors have not given consideration for such cases in this paper.

3. A MODEL FOR ERRORS

Measuring the error from the data

The notation will be illustrated with an example that will be carried through the paper. Suppose for a particular line we believe that losses are fully developed after 10 years. There have been twelve reserve reviews completed and in each year an ultimate loss has been estimated. For notation, let

U_i^k - the estimated ultimate loss as of calendar year k for accident year i.

The results for an example block of business are in Table 1. Note that the available data has i and k range from 1 through 12 but, for example the AY 1 losses were fully developed by CY 10.

AY(i) /CY(k)	1	2	3	4	5	6	7	8	9	10	11	12
1	148,741	103,058	100,010	98,001	96,280	95,579	95,176	95,161	95,150	95,113		
2		186,087	120,444	113,083	109,097	108,443	107,934	107,836	107,814	107,907	107,860	
3			139,092	94,318	89,032	86,552	85,584	85,532	85,557	85,655	85,626	85,650
4				58,441	52,585	52,136	51,375	51,501	51,799	51,870	51,914	51,933
5					22,738	30,670	32,948	33,986	34,363	34,329	34,467	34,642
6						24,134	37,035	42,981	42,688	42,894	43,052	43,533
7							26,695	51,849	57,143	57,817	58,200	58,647
8								57,397	67,688	74,995	75,793	76,736
9									94,537	94,281	98,453	98,055
10										93,784	104,539	100,257
11											116,443	124,781
12												172,224

Table 1 – Indicated ultimate losses from 12 calendar year valuations.

For example, the value 42,894 is the indicated ultimate loss estimated at the end of calendar year 10 for losses incurred in accident year 6. We are interested in the errors made in the estimates of the ultimate losses. Some of those errors can be determined from Table 1. We will use the logarithm of the ratio for the errors, for reasons to be explained later.

For example, the ultimate value for accident year 3 was estimated at the end of calendar year 11 (development year 8) to be 85,626. A year later, the actual value of 85,650 was known. The error is ln(85,650/85,626) = 0.00028. We cannot calculate the error for later accident years because they have not yet been fully developed. The errors we care about are $e_i^k = \ln(U_i^{i+9} / U_i^k)$ where $i + 9 > k$. The numerator is the fully developed ultimate loss and the denominator is the estimate as of CY k. For our example we are currently at $k = 12$ and so are concerned with the errors made in estimates from AY 4 onward.

In order to gather more information about errors from this information, begin with an error that is not immediately useful. Consider $e_i^{j*} = \ln(U_i^{j+1} / U_i^j)$ where available. This represents the error realized as the estimated ultimate value is updated one calendar year later. Here is one reason for using logarithms – the errors are additive. In fact,

$$
\begin{aligned}
e_i^k &= \ln(U_i^{i+9} / U_i^k) \\
&= \ln\left(\frac{U_i^{k+1}}{U_i^k} \frac{U_i^{k+2}}{U_i^{k+1}} \cdots \frac{U_i^{i+9}}{U_i^{i+8}} \right) \\
&= \sum_{g=0}^{i+8-k} \ln(U_i^{k+g+1} / U_i^{k+g}) \\
&= \sum_{j=k}^{i+8} e_i^{j*}.
\end{aligned}
$$

In this notation, the development year of the denominator value is $d = j - i + 1$. The advantage of this approach is that the available data in our example provides many estimated values as presented in Table 2.

AY(i)\DY(d)	1	2	3	4	5	6	7	8	9
1	-0.36691	-0.03003	-0.02029	-0.01772	-0.00731	-0.00423	-0.00016	-0.00012	-0.00039
2	-0.43503	-0.06306	-0.03589	-0.00601	-0.00471	-0.00090	-0.00021	0.00086	-0.00044
3	-0.38846	-0.05768	-0.02825	-0.01125	-0.00061	0.00029	0.00114	-0.00034	0.00028
4	-0.10559	-0.00858	-0.01470	0.00245	0.00577	0.00137	0.00085	0.00037	
5	0.29925	0.07165	0.03102	0.01103	-0.00099	0.00401	0.00506		
6	0.42824	0.14889	-0.00684	0.00481	0.00368	0.01111			
7	0.66386	0.09722	0.01173	0.00660	0.00765				
8	0.16492	0.10251	0.01058	0.01237					
9	-0.00271	0.04330	-0.00405						
10	0.10857	-0.04182							
11	0.06916								

Table 2 – year-to-year error values, e_i^{j*}

There is another interpretation of these errors. Regardless of the reserving method, a factor representing the ultimate development can be inferred. Suppose we are looking at accident year *i* and calendar year *j*. The factor is $u_i^j = U_i^j / L_i^j$ where L_i^j is the paid loss for that accident year at the end of year *j*. One year later the factor is $u_i^{j+1} = U_i^{j+1} / L_i^{j+1}$. These represent age to ultimate for two different development years. The ratio, u_i^j / u_i^{j+1} represents how losses were expected to develop while L_i^{j+1} / L_i^j is how they actually developed. The ratio of actual to expected is U_i^{j+1} / U_i^j which is the error measurement we have been using.

Now that the key data values have been calculated it is time to construct a model.

The Model

We propose that the error random variables have the normal distribution, and in particular,

$$e_i^{j*} \sim N\left(\mu_{j-i+1}^*, \sigma_{j-i+1}^{*2}\right).$$

Note that the mean and variance are constant for a given development year. The rest of this section is devoted to justifying these assumptions. It should be noted that justifying the assumptions through data analysis is not sufficient. If this method is to be useful regardless of the loss reserving method used, the justification must be based on our beliefs about the loss development and reserving processes and not any particular models or empirical evidence.

Normal distribution

After the ultimate loss is estimated at a particular time, what factors will cause it to change when it is re-estimated one year later? Day by day during that year a variety of events may take place. Among them are[2]:

Economic forces such as inflation and changes in the legal environment will alter the amounts paid on open claims or those newly reported.

The rate at which claims develop may change.

Purely random events may affect individual open claims.

Actuary's opinion on IBNR may change depending on the adequacy of case reserves.

These factors will tend to act proportionally on the current estimate of the ultimate loss. Because there are many such factors happening many times in the course of a year, it is reasonable to assume we are looking at the product of a variety of random variables, most with values near 1. The Central Limit Theorem indicates that the result is a lognormal distribution and

---

[2] At this point in the paper we focus on lognormality as a consequence of unchanging development processes. We relax this assumption later in the paper.

thus measuring the error in the logarithm will produce values with a normal distribution.

Normal approximation

The above arguments relating to the central limit theorem apply to a fictitious accident year with infinite claims. This accident year need not be evaluated infinitely many times but the changes at each valuation should be driven by infinite reasons underlying infinite claims.

In practice a finite subset of this hypothetical accident year is available. The reasons causing the change in the ultimate loss are finite and not infinite and thus the distribution is approximately normal[3]. The approximation can be improved by increasing the number of reasons driving the change in ultimate losses. This can be done in the following ways:

Increase the valuation time (example once a year rather than four times a year)

Increase the claim count (large volume line)

The claim count required for a good normal approximation in turn depends on the skewness of the underlying size of loss distribution. As a practical matter the model will work well even for a modest claim count.

Reserve valuations conducted less frequently (example once a year rather than four times a year) will allow greater "reasons" for changes and thus help with the normal approximation. The tradeoff is the loss of accuracy in ultimate loss estimation and consequential bias. Also note that both the parameters of the distribution and its shape (to which extent it is lognormal) will change depending on the frequency of reserve review.

A subtle point concerns the later development intervals. In these intervals the changes are usually driven by a handful of claims (few reasons) and thus violate the normality assumption. However at those valuations the aggregate errors are often close to zero with small volatility (constants) and thus departure from the normality has little impact on the total distribution for the entire accident year.

Mean

At first it may seem that the mean should be zero. There are two reasons that is not so. First, consider the expected revised ultimate loss given the current estimate:

$$E(U_i^{j+1} / U_i^j) = e^{\mu^*_{j-i+1} + \sigma^{*2}_{j-i+1}/2}.$$

For the reserve estimate to be unbiased, it is necessary that $\mu^*_{j-i+1} = -0.5\sigma^{*2}_{j-i+1}$ and not zero.

---

[3] Technically, the distribution remains approximately normal even for the infinite claim accident year but the distribution is closer to normal than in the finite case.

In addition, it is possible that the reserving method is biased. This may be a property of the method selected and may even be a deliberate attempt by the reserving actuary to adjust the reserves based on knowledge that is outside the data.

Having the mean be constant from one accident year to the next is a consequence of the assumptions that were made at the beginning. That is, there is no change in the underlying development process or reserving method.

Having the mean depend on the development year seems reasonable. As the development year increases any systematic bias is likely to be reduced in the expectation that the ultimate value will not be much different from the current value. In addition, we expect the variance of the errors to decrease and if the ultimate loss estimates are unbiased, the means will then also decrease (in absolute value).

Complete Triangle of Errors

For at least one accident year we need the losses to be fully developed. Even better would be to have a few fully developed years so there would be more data available for computing covariances and variances. There is a tradeoff in that older fully developed accident years that are not part of the in-force book may be less predictive and may not reflect the current prevailing business environment.

The opposite concern is when there are not enough accident years available to obtain fully developed losses. This can happen for relatively new companies or lines of business. Note that tail errors generally result from estimation errors of pending court cases or simply absence of data due to a new line. Since we are dealing with a year in aggregate, these errors are usually close to zero and therefore contribute less to the variance than earlier development intervals.

4. PARAMETER ESTIMATION

Return to the continuing example and recall Table 2. The numbers in this table are percentage changes (errors) of the estimated ultimate losses and have been assumed to have been drawn from six normal distributions. The upper half of the triangle is fixed and known and our goal is to obtain the normal means and variances for the yet-to-be observed errors in the lower right part of the table.

Estimation of the mean

We choose to calculate an initial estimate of the mean by calculating the sample mean,

$$\hat{\mu}_d^* = \frac{\sum_{i=1}^{12-d} e_i^{(i+d-1)*}}{12-d}$$

where d is the development year, the sum is taken over all accident years for which errors were available.[4]

The results for our example are in Table 3.

DY(d)	1	2	3	4	5	6	7	8	9
mu-hat	0.0396	0.0262	0.0063	0.0003	0.0005	0.0019	0.0013	0.0002	0.0002
mu-hat (Selected)	0.0396	0.0262	0.0063	0.0003	0.0005	0.0019	0.0013	0.0002	0.0002

Table 3 – Estimated means by development year

However, there are reasons why the sample mean may not be the appropriate choice. The results may be biased due to model risk. The estimate of the mean affects the estimate of the expected ultimate loss.

For example if $\mu < -\frac{\sigma^2}{2}$ then the estimated ultimate loss will be less than the indicated ultimate showing redundancy in resulting reserves. The opposite is true if the inequality is reversed. Thus any value of mean that is different than $-\frac{\sigma^2}{2}$ should be justified.

The estimation of the mean can effectively result in taking a position that the actuarial estimate is biased. This is not trivial and there should be careful analysis before making that assertion. We discuss the mean selection in greater detail under distribution reviews later in the paper.

Estimation of the Variance

The variance parameter measures the risk historically faced by the book in force. For very long tailed lines this would mean that it will capture the changes in reserving methods and other changes that have taken place.

The variance estimate is the usual unbiased estimate. The equation is,

[4] An alternative is to use a weighted average where the weights are the indicated ultimate values for that accident year. This allows more weight to be placed on those accident years in which there is more data.

$$\hat{\sigma}_d^{*2} = \frac{\sum_{i=1}^{12-d} \left[e_i^{(i+d-1)*} - \hat{\mu}_d \right]^2}{11-d}.$$

Note that the estimated mean must be the sample mean.

For our data the estimated standard deviations by development year are given in Table 4. As expected, for the most part, the standard deviations decrease by lag.

DY(d)	1	2	3	4	5	6	7	8	9
sd-hat	0.3502	0.0762	0.0213	0.0108	0.0055	0.0052	0.0022	0.0005	0.0004

Table 4 – Standard deviations by development year

Correlations

For a given accident year, it is likely that the errors for one development year are correlated with those from other development years. These must be estimated. A formula for the covariance is

$$\hat{\sigma}_{d,d'}^{*} = \frac{\sum_{i}^{12-d} \left[e_i^{(i+d-1)*} - \hat{\mu}_d \right] \left[e_i^{(i+d'-1)*} - \hat{\mu}_{d'} \right]}{11-d}$$

Where $d > d'$ represent two different development years and each sample mean is based only on the first $10 - d$ observations.

The matrix of covariances is given in Table 5.

d	1	2	3	4	5	6	7	8	9
1	0.12261	0.02379	0.00664	0.00356	0.00178	0.00175	0.00062	0.00000	0.00000
2	0.02379	0.00581	0.00122	0.00070	0.00027	0.00040	0.00011	0.00000	0.00000
3	0.00664	0.00122	0.00045	0.00020	0.00005	0.00006	0.00005	0.00000	0.00000
4	0.00356	0.00070	0.00020	0.00012	0.00004	0.00004	0.00002	0.00000	0.00000
5	0.00178	0.00027	0.00005	0.00004	0.00003	0.00002	0.00000	0.00000	0.00000
6	0.00175	0.00040	0.00006	0.00004	0.00002	0.00003	0.00001	0.00000	0.00000
7	0.00062	0.00011	0.00005	0.00002	0.00000	0.00001	0.00000	0.00000	0.00000
8	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
9	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5 – Covariances

The total variance for any given accident year can now be calculated from Table 5. For example the variance for AY 12 will be the sum of the values in the table (shown as 0.46129 in the table 6 below).

5. THE ERROR DISTRIBUTION FOR A GIVEN ACCIDENT YEAR

Now that we have a model for the errors from one calendar year to the next, we need to return to the error we care about. Recall that

$$e_i^k = \sum_{j=k}^{i+8} e_i^{j*} .$$

From the assumptions made previously, this error has a normal distribution and its mean depends only on the latest development year, $k - i + 1$.

$$e_i^k \sim N\left(\mu_{k-i+1}, \sigma_{k-i+1}^2\right)$$

$$\mu_{k-i+1} = \sum_{j=k}^{i+8} \mu_{j-i+1}^*$$

$$\sigma_{k-i+1}^2 = \sum_{j=k}^{i+8} \sigma_{j-i+1}^{*2} + 2 \sum_{j=k+1}^{i+8} \sum_{j'=k}^{j} \sigma_{j-i+1,\, j'-i+1}^*$$

These quantities can be estimated by adding the respective mean, variance, and covariance estimates.

For our example, we now have estimates of the distribution of the error in the ultimate loss as estimated from the data available at the end of calendar year 12. They are given in Table 6.

AY	mean	sd
4	-0.000181	0.000401
5	0.000013	0.000303
6	0.001352	0.002243
7	0.003294	0.006727
8	0.003791	0.010914
9	0.004077	0.021106
10	-0.002222	0.040233
11	0.024019	0.113210
12	0.063590	0.460129

Table 6 – Means and standard deviations by accident year

6. THE ERROR DISTRIBUTION FOR ALL ACCIDENT YEARS COMBINED

While we have been using logarithms to measure the error, when all is done we are interested in the ultimate losses themselves. To make the formulas easier to follow, rather than allow arbitrary values, we will follow the example and assume we are at the end of CY 12 and losses are fully developed after 10 years. In particular, we care about

$$U = U_4^{13} + \cdots + U_{12}^{21}.$$

Recall from the notation that the terms on the right hand side represent the fully developed losses at a time in the future when the ultimate results are known. Rewrite this expression as

$$
\begin{aligned}
U &= U_4^{12}\exp(e_4^{12}) + \cdots + U_{12}^{12}\exp(e_{12}^{12}) \\
&= V\sum_{i=4}^{12} r_i \exp(e_i^{12}) \\
r_i &= \frac{U_i^{12}}{U_4^{12} + \cdots + U_{12}^{12}} \\
V &= U_4^{12} + \cdots + U_{12}^{12}.
\end{aligned}
$$

Here V is the estimated ultimate loss as of CY 12 for all open years and the weights are the relative proportion in each accident year. This indicates that the ultimate loss is a weighted average of lognormal random variables. This can be painful to work with (though not hard to simulate). Because the error random variables are usually close to zero and will vary about their mean, consider the following Taylor series approximations.

$$
\begin{aligned}
X &= \ln\frac{U}{V} = \ln\left[\sum_{i=4}^{12} r_i \exp(e_i^{12})\right] \\
&\approx \ln\left[\sum_{i=4}^{12} r_i (1 + e_i^{12})\right] \\
&= \ln\left(1 + \sum_{i=4}^{12} r_i e_i^{12}\right) \\
&\approx \sum_{i=4}^{12} r_i e_i^{12}.
\end{aligned}
$$

The approximate log-ratio has a normal distribution and thus U has an approximate lognormal distribution. The moments of the normal distribution are:

$$
\begin{aligned}
E\left(\ln\frac{U}{V}\right) &= E\left(\sum_{i=4}^{12} r_i e_i^{12}\right) \\
&= \sum_{i=4}^{12} r_i \mu_{13-i} \\
&= \mu \\
Var\left(\ln\frac{U}{V}\right) &\approx Var\left(\sum_{i=4}^{12} r_i e_i^{12}\right) \\
&= \sum_{i=4}^{12} r_i^2 \sigma_{13-i}^2 \\
&= \sigma^2.
\end{aligned}
$$

For the example, V = 760,808, μ = 0.01927, σ^2 = 0.01123, and then the expected ultimate loss is 779,978 and the standard deviation is 82,892.

7. EXTENSION TO MULTIPLE LINES OF BUSINESS

Suppose there are two lines of business. Each can be analyzed separately using the method previously outlined. However, it is likely that the results for the lines are not independent. One approach would be to model the correlation structure between error values from the two lines. The problem with that approach is that there may not be corresponding cells to match and also the number of parameters may become prohibitive[5].

An alternative is to combine the data from the two lines into a single triangle and analyze it using the methods of this paper. When finished, there will be a distribution for each line separately and for the combined lines. Usually, only the total reserve is of importance and thus only the combined results are needed. The individual line results will be useful for internal analyses and also if it is desired to allocate the reserves back to the lines.

To illustrate this idea, we add a second line of business. The same analysis gives: V = 244,537, μ = -030759, σ^2 = 0.008933, and then the expected ultimate loss is 180,593 and the standard deviation is 17,107.

Combining the two lines creates a single table with the totals from each. An analysis of these tables produces V = 1,005,376, μ = -0.02674, σ^2 = 0.009582, and then the expected ultimate loss is 983,520 and the standard deviation is 96,506.

8. ALLOCATION OF ULTIMATE LOSSES

From the previous examples there is an interesting situation. If the two lines were independent, the standard deviation of the total would be

$$\sqrt{82,892^2 + 17,107^2} = 84,638$$

Which is less than the standard deviation from the combined lines, which was 96,506. There is thus a positive correlation between the lines and when setting reserves and then allocating them to the lines, something needs to be done.

[5] Combining distributions also requires the multivariate normality assumption. This is hard to test but is often made in practice.

Allocation of reserves

Suppose we set reserves with a margin for conservatism based on the results of the previous section. For example, suppose we set ultimate losses to be at the 95th percentile. That is,

$$1,005,376\exp(-0.02674+1.645\sqrt{0.009582})=1,149,833.$$

where the mean and standard deviation are for the corresponding normal distribution.

We also want to set ultimate losses for each line of business such that they add to 1,149,833. However, this cannot be achieved by setting each line at its 95th percentile. A reasonable approach is set each line at the same percentile, using the percentile that makes the sum work out. It turns out that if each line is set at the 96.28th percentile, the ultimate estimates will be 937,025 and 212,808.

9. MODEL EXTENTION TO PRACTICAL SETTINGS

Distribution reviews

Once the reserving actuary has completed the reserve review the distribution review should follow. The actuary will remember the considerations for selecting the ultimate losses. These include data considerations, coverage changes, mix changes etc. All of these can now be factored in the bias (mean parameter) estimation of resulting reserve distribution.

The distribution review allows the actuary to consider the possibility of estimation bias in the current reserve estimate. For example if the historical errors of the selected ultimate losses for a line are positive and the actuary has not changed the selection approach then the current estimate will likely have a positive error (too low an estimate).

The reserve diagnostics are particularly important in evaluating such errors as the past is not always indicative of the future. For example a sign of case reserve weakening should lead to a higher IBNR all else being equal.

These mechanisms of monitoring errors allow early warning signs for future potential reserve deficiencies.

Finally note that changes in conditions do not invalidate the lognormal property. However, changes in conditions will change the lognormal parameters, thereby increasing the degree of difficulty for projection accuracy.

Volatility measurement under changing conditions

While the distribution reviews and the resulting mean parameter will account for changes in reserve adequacy, the volatility parameter reflects historical data and is therefore slow to respond to changes in the errors. Thus if the reserving methods change abruptly or the reserving actuary is replaced, the volatility will change slowly as the data emerges. Such cases are handled in the following ways:

In many cases it is wise to let data lead the way as it may be pre-emptive to conclude about the volatility of the new process. This is especially relevant in light of the fact that processes themselves usually change slowly over time (such as experience of reserving actuary etc).

In rare cases, the change in reserve methodology is driven by an abrupt decision such as outsourcing the reserve review to a consultant. In that case, the historical error data will have to be restated with the new process.

10. APPLICATIONS

We now present a few applications of the previous results.

Loss estimation method

The chain ladder estimate is biased for long tailed lines partly due to the fact that the covariance by development intervals is not incorporated in its estimate. Using the parameter estimation technique presented in this paper (including the variance covariance matrix of a triangle) the expected ultimate loss for an accident year

$$E(U) = U_o \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

U = Ultimate loss, U_o = Incurred or paid loss, σ^2 = sum of variance covariance matrix (the number of terms depend on the age of the accident year).

Claim Commutations

Claim commutations involve transfer of reserves from a ceding carrier to an acquiring carrier. The risk to the acquiring company is that the indicated reserves may not be sufficient to pay all claims. One complication is that the agreed to reserve value may not be the same as that used in the determination of the reserve distribution. Let H be this arbitrary reserve value. A reasonable fair transaction price is this reserve plus the value of expected excess reserve cost above H. Let C be this cost. The total price is then given by $H + C$. The formula for C is:

$$C = \int_H^\infty (x - H)\, f_R(x)dx$$

where, R is the random true reserve. Let P be the amount paid and as before U is the random, true, ultimate loss. Then, $R = U - P$ and so

$$\begin{aligned} C &= \int_H^{\infty} (x-H)\, f_R(x)dx \\ &= \int_H^{\infty} (x-H)\, f_U(x+P)dx \\ &= \int_{P+H}^{\infty} (y-P-H)\, f_U(y)dy \\ &= E(U) - E(U \wedge P+H) \\ &= \exp[\mu + \ln(V) + \sigma^2/2]\left[1-\Phi\left(\frac{\ln(H+P)-\mu-\ln(V)-\sigma^2}{\sigma}\right)\right] \\ &\quad -(H+P)\left[1-\Phi\left(\frac{\ln(H+P)-\mu-\ln(V)}{\sigma}\right)\right] \end{aligned}$$

The second line is a change of variable using $R = U - P$.

The same concept applies in situations where one company acquires or merges with another company or a re-insurer acquires the reserve of the ceding carrier (Loss Portfolio Transfer).

Insurance market segmentation

The expected excess cost formula gives an insight into insurance economics. The assuming carrier will pool the assumed reserves into existing homogenous reserve segments. If the pooling decreases the total risk, captured in the estimated σ^2 for the line, the assuming carrier's expected excess cost will be lower than ceding carrier's. This assumes that the reserve adequacy of the reserve segment is estimated identically by both parties.

The above argument explains insurance market segmentation. Companies grow their business in a given line and continue to acquire business from smaller companies in the same line because they face different total risks. Unlike other businesses, the volatility of losses underlying estimates drives decisions to acquire and grow an existing business segment. This is also the reason why low risk (short tailed, fast developing) lines are seldom ceded to other companies.

Net and Direct Reserve Distributions

We can quantify the reserve distribution net of reinsurance and/or recoveries using the method explained in the paper. This is possible because we are following the net reserve reviews and simply measuring the uncertainty in the estimates.

One caveat when dealing with net distributions is that the true mean can be harder to estimate especially if treaties have changed recently. As stated earlier quarterly error triangles are more helpful in such cases as they are more responsive to changes. In other cases, the current actuary's estimate can be taken to be unbiased until further history is developed under the new treaty.

In many where reinsurance has a relatively small impact on total reserve a change in treaty provisions will not change the resulting net error distribution significantly. For examples quota share reinsurance on a loss occurring basis on a line with large claim count per accident year will not necessarily lead to a lower variability of the net aggregate *error* distribution (emphasis added). The reserving actuary will see proportionally lower losses for new accident years but this may not impact net error distribution.

Another example is casualty excess of loss coverage that attaches at a high layer. If the ground up claim count is large enough the change of reinsurance treaty many have very little impact on the net error distribution as few losses out of the total will be ceded in that layer.

In some cases especially for low frequency & high severity lines such as personal umbrella excess of loss coverage, the impact of a change in treaty can be significant and the error history will not be relevant to the current treaty. This will require the actuary to conduct net historical reserve reviews based on the new treaty. This can be time consuming should be done if the line forms a significant part of the total reserve.

Net distributions result in net reserve ranges and net reserving capital. These are relevant from a solvency, regulatory and company rating standpoint.

Regulatory & Rating Agency Applications

Regulators and rating agencies are interested in quantifying reserve ranges, percentiles, and reserving capital in order to monitor the solvency of the company. The regulator, in particular is interested in measuring the performance of the *held* reserve, shown in Schedule P.

Once the distribution of the reserves is known we can state the percentile of the held reserves. Note that the management bias will now make a difference as it will position the held reserves to the right or left of the true mean.

We can also calculate the reserving risk capital as the difference between a selected cutoff value (say 95th percentile) of the indicated reserves and the held reserves. The resulting reserving risk capital can be used to modify the RBC reserving risk charge. Note that the NAIC tests of reserve development to surplus ratios suggest comparing dollar ultimate loss errors to company surplus. This is very similar to our approach of measuring reserve uncertainty. Thus, the reserving risk charge measured using the method outlined in this paper would be consistent with the current annual statement and NAIC practices.

11. CONCLUSION

The paper presents a shift in paradigm from loss distributions of the underlying losses to distributions of the company's estimates. This represents a new way of measuring reserving risk. By being able to quantify risk both by line and for the company, effective management of capital, reinsurance, and other company functions becomes feasible.

We provide a framework for assessing reserve review accuracy as well as measuring the distribution of the current reserve review. This is done using a "distribution review" immediately after a reserve review using the same segments and models currently used by the reserving department.

Given the current complex reserving environment where reserving is both an art and a science there is no statistical formula to set reserves or its distribution. Rather we present a rigorous framework that involves the same considerations and process as the underlying reserve review itself.

Acknowledgement

We are indebted to Zhongzian Han (Statistics & Actuarial Science department of the University of Central Florida) for his furnishing the literature review. The authors would also like to thank William J Gerhardt (Nationwide Insurance) for being a great sounding board and providing invaluable feedback.

The authors are also grateful to an anonymous reviewer for extensive data testing and feedback.

REFERENCES

[1] Bornhuetter, R.L. and R.E. Ferguson, 1972. "The Actuary and IBNR," *Proceedings of the Casualty Actuarial Society*, LIX, pp. 181-195.

[2] Brown, R.L., Gottlieb, L.R., 2001. *Introduction to ratemaking and loss reserving for property and casualty insurance, second ed.* Winsted, CT: ACTEX.

[3] CAS Working Party on Quantifying Variability in Reserve Estimates, 2005. "The Analysis and Estimation of Loss & ALAE Variability: A Summary Report," *Casualty Actuarial Society Forum*, Fall 2005, pp. 29-146.

[4] De Alba, E., 2002. "Bayesian Estimation of Outstanding Claims Reserve,". *North American Actuarial Journal*, 6, pp. 1-20.

[5] England, P. D. and Verrall, R.J., 2002. "Stochastic Claims Reserving in General Insurance," *British Actuarial Journal* 8, pp.443-544.

[6] Finger, R, J.,1976. "Modeling Loss Reserve Developments.," *Proceedings of the Casualty Actuarial Society*, LXIII, pp. 90-105.

[7] Hayne, R.M., 1985. "An Estimate of Statistical Variation in Development Factor Methods," *Proceedings of the Casualty Actuarial Society*, LXXII, pp. 25-43.

[8] Hayne, R.M., 2004. "Estimating and Incorporating Correlation in Reserve Variability," *Casualty Actuarial Society Forum*, Fall 2004, pp. 53-72.

[9] Kreps, R.E., 2005. "Riskiness Leverage Models," *Proceedings of the Casualty Actuarial Society,* XCII, pp. 31-60.

[10] Mack, T., 1993. "Distribution-Free Calculation of the Standard Error of Chain Ladder Reserve Estimates," *ASTIN Bulletin* 23, pp. 213-25.

[11] Shapland, Mark, 2007. "Loss Reserve Estimates: A Statistical Approach for Determining Reasonableness," *Variance*, 1, pp. 120-148.

[12] Taylor, G.C., 2000. *Loss Reserving: An Actuarial Perspective.* Dordrecht, Netherlands: Kluwer Academic Publishers.

[13] Taylor, G.C., and Ashe, F.R., 1983. "Second moments of estimates of outstanding claims," *Journal of Econometric,s* 23 (1), pp.37-61.

[14] Taylor, G. C., and McGuire, G., 2004. "Loss Reserving with GLMs: A Case Study,". *Casualty Actuarial Society Discussion Paper Program, Applying and Evaluating Generalized Linear Model*, pp. 327-92.

[15] Verrall, R.J., 1991. "On the Estimation of Reserves from Loglinear Models," *Insurance: Mathematics and Economics*, 10 (1), pp. 75-80.

[16] Verrall, R. J., 2004. "A Bayesian Generalized Linear Model for the Bornhuetter-Ferguson Method of Claims Reserving," *North American Actuarial Journal* 8, no. 3:67-89.

[17] Wiser, R.F., 2001. *Foundations of Casualty Actuarial Science*, Alexandria, VA: Casualty Actuarial Society (Chapter 5), pp. 197-285.

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