Manually Adjustable Link Ratio Model for Reserving

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Abstract: The chain ladder method is very popular in General/Property-Casualty Insurance actuarial circles. Mack [1] expanded the deterministic algorithm to include calculations for the variance of the chain ladder projections. The assumptions underlying the chain ladder method are important in regards to the appropriateness of the deterministic projections; they are even more important in regards to the appropriateness of the stochastic results. The purpose of this paper is to introduce more statistical rigor to this popular method and help close the link between practice and statistical theory. We will discuss residual analysis and other statistical measures as they apply to the chain ladder method so that the appropriateness of its deterministic and stochastic results can be objectively measured based on statistically rigorous principles. We will also show how the regression approach of Murphy [2] can be expanded so that link ratios "selected judgmentally" can be seen as conforming to an underlying statistical model.

Keywords: chain ladder; selection; residuals; Mack; Murphy

1. INTRODUCTION

A big part of the actuarial research in the last two decades is dedicated to reserving. While many statistical methods have been dedicated to this problem, none of them is broadly accepted by the practitioners. The aim of this paper is to reduce, or even to close, the gap between practice and theory by embedding this practice into a theoretical flexible framework. The most popular method to solve the central problem of reserving, namely to estimate an "expected value for the outstanding payments=Best Estimate," is the chain ladder method. This is the reason of the popularity of the analysis of Mack [1], where the standard chain-ladder approach is discussed. Murphy [2] considers the more general question of "loss development method," where the chain ladder method is treated as a special case of a more general linear regression approach. Zenwirth [3] calls this family the "extended link ratio family," he criticizes its prediction power and suggests the "probability trend family" instead. However Zenwirth's approach is not consistent with the traditional chain ladder method and the user input associated with this method. The incorporation of user judgement is a typical Bayesian problem, and the approach suggested from Verall [4] is a theoretical rigorous way to tackle the inflexibility of the previous methods. The necessity of the MCMC algorithm (Markov Chain Monte Carlo) in this method makes Verall's approach hard to describe and the basic assumptions of prior distributions for the link ratios are not easily verified.

The purpose of this paper is to present an appropriate model, which

1. Is compatible with the way practitioners implement the chain ladder method and

2. Provides a statistical framework that will help test the underlying assumptions of the chain ladder method (for example for approval of an internal model[1] in Solvency II-context, or the use of benchmarks for the reserving exercise).

In the first section we will propose a model that is built around the regression interpretation of the chain ladder method similar to Murphy [2]. It turns out that a flexible formulation of the chain ladder method along the lines of a regression model satisfies the above-mentioned requirements. Furthermore we will demonstrate how this embedded statistical process can be used to test the appropriateness of the "actuarial selected link ratios" both visually and statistically. Finally we will suggest how to proceed if the approach taken is not appropriate and demonstrate with an example.

2. THE LINK RATIO APPROACH

We start with the usual notation, where the observed cumulative paid losses are denoted by the set $\boldsymbol{D} = \{\boldsymbol{C}_{ij} \mid 1 \le \boldsymbol{i} \le \boldsymbol{I}, 1 \le \boldsymbol{j} \le \boldsymbol{I} + 1 - \boldsymbol{i}\}$. A regression model equivalent to the chain ladder method is

$$\boldsymbol{C}_{ik+1} = \boldsymbol{f}_k \boldsymbol{C}_{ik} + \sigma_k \varepsilon_{i,k} \boldsymbol{C}_{i,k}^{\alpha_k/2} \qquad (1)$$

$$\varepsilon_{i,k} \sim \aleph(0,1), 1 \le \boldsymbol{i} \le \boldsymbol{I}, 1 \le \boldsymbol{k} \le \boldsymbol{I} + 1 - \boldsymbol{i} \qquad (2)$$

Thereby the set $\{\varepsilon_{ik} \mid 1 \le \boldsymbol{i} \le \boldsymbol{I}, 1 \le \boldsymbol{k} \le \boldsymbol{I} + 1 - \boldsymbol{i}\}$ is assumed to be "noise" or independent identical distributed (i.i.d.) normal[2] random variables with mean 0 and standard deviation 1. Making explicit the implicit assumption of the error term is crucial for assessing the appropriateness of a model because it provides a data set of residuals for model testing. Under these assumptions the least square estimate of the link ratio, given the set of observations $\boldsymbol{D}$, can be calculated through weighted averages of the observed link ratios:

The optimal solution of model (1), (2) is specified by the parameters $(\hat{f}, \hat{\sigma}, \hat{\alpha})$ (the "model specification") where the solution for the values of the αs is discussed below.

2.1 Chain Ladder

The model introduced in Mack [1] is a special case of the model (1), (2) with $\alpha_k=1$, $k=1,\ldots,I$. Mack noted in this model the minimum variance estimator

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[1] "Proposal for a DIRECTIVE OF THE EUROPEAN PARLIAMENT AND OF THE COUNCIL on the taking-up and pursuit of the business of Insurance and Reinsurance" http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=CELEX:52007PC0361:EN:NOT

[2] The normality assumption is made to assure that the chain ladder link ratios correspond to ML estimators. Other distributions can be assumed as well, but that might lead to an ML solution other than the least squares solution.

$$\hat{f}_k = LR_k(1) = \sum_{i=1}^{n-k} \frac{C_{i,k}}{\sum_{j=1}^{n-k} C_{j,k}} \cdot \frac{C_{i,k+1}}{C_{i,k}} = \frac{\sum_{i=1}^{n-k} C_{i,k+1}}{\sum_{i=1}^{n-k} C_{i,k}},$$

and derived estimators for uncertainty, popularly known as "The Mack Formula." In other words, if we specify a "variance assumption" by selecting the alpha parameter, then the link ratios in this model as well as the uncertainty of the estimators are also selected. This *model* embeds, by making these extensions, the traditional chain ladder *method* in a statistical framework.

Hereby it is important to distinguish between a model and a method. A model is a mathematical description of an observation, phenomenon, etc. and produces "best-fitted" parameters based on the underlying data characteristics. A method, on the other hand, is an algorithm that makes certain assumptions and produces estimates based on a number of predetermined steps. Thus a method can always be used to calculate some estimates, whereas a model is based on assumptions that need to be tested, before the model is used. The traditional chain ladder *method* is "consistent" with many stochastic models that have been created around it, such as the Mack/Murphy Model or the over-dispersed Poisson model. By "consistent" we mean that, given the model that is appropriate for the data on hand, the chain ladder method is a reasonable algorithm to produce reserve estimates that are similar to the estimates of these models. However, actuaries are used to *selecting* link ratio judgmentally because estimated link ratios by averaging methods can be inappropriate in cases when the stochastic component of the loss generation process is made complex by the influence of many unknown and unobserved parameters. An experienced actuary recognizes, for example, trends in the triangles and adjusts the link ratios manually, or uses benchmark pattern instead. There is no doubt that such a manual extension of the model makes sense, but no matter how experienced an actuary is, the appropriateness of his or her selection is always open to question. The model framework of this paper can be used to answer that question with more objective statistical tests.[3]

2.2 Residuals and Model Selection

In the traditional world, actuaries' methods and selections are defended by their expertise and experience. However, mathematical and graphical tools can provide more objective ways to defend their selections and to communicate their answers. One of the most important diagnosis and validation tools are residuals, which are in general the difference between a "data set" and its "formulaic representation." In the chain ladder case, the formulaic representation of the data is given by the specifications of the model parameters.

[3] Furthermore, we mention here that the residuals are often used to simulate the distribution of the stochastic reserving process through the bootstrapping approach. The core of the bootstrapping method is the "independent identical" assumption in (2). The bootstrapping results will be wrong if this assumption is violated.

If we reformulate (1) with respect to $\varepsilon_{i,k}$, we obtain

$$\varepsilon_{i,k} = (C_{ik+1} - f_k C_{ik}) / (\sigma_k C_{i,k}^{\alpha_k/2}).$$

This residual assumption can be validated with the data set.

We define the "corresponding" residuals of a model specification $(\hat{f}, \hat{\sigma}, \hat{\alpha})$ by

$$r_{i,k} := r_{i,k}(\hat{f}, \hat{\sigma}, \hat{\alpha}) := (C_{ik+1} - \hat{f}_k C_{ik}) / (\hat{\sigma}_k C_{i,k}^{\hat{\alpha}_k/2}) \quad (4)$$

We start by selecting the parameters in this model and proposing a certain estimate, which corresponds to a hypothesis for the future liabilities that leads to an estimate for the reserves. The question is now, how confident are we in that estimate? Taking (2) and (3) together our chosen estimates need to fulfill the hypothesis

"The data set $\{r_{i,k}\}$ looks like noise."

Although we have a subjective feeling for a data set looking like noise, we could hardly test it without further clarifications. However the hypothesis "i.i.d. normal distributed" can be tested through visual tests (e.g., QQ-Plots) as well as statistical tests (e.g., Shapiro-Francia-test for normality [5]).

Now one can raise the question: What should we do if the test fails? We change the link ratios manually. Of course this is not new. Actuaries have always selected link ratios manually by employing experience, judgment, benchmarks, etc.

Assuming we manually change the link ratios, the next question is: Is the new set of link ratios more appropriate than those selected initially?

In the next sections we describe an approach to answer these questions and show how to use the approach to fine-tune the selected link ratios in a controlled work flow way.

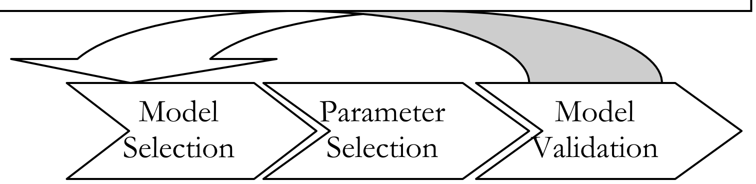
3. SELECTED LINK RATIO MODEL

Consider the regression approach (1) to the chain ladder method. The problem with the common actuarial practice is that when the selected link ratios are not the volume weighted average, then they are not consistent with the best linear unbiased estimators calculated by the statistical models employed in stochastic reserving exercises. In particular non-volume-weighted-average selected link ratios are not proper estimators for f_k according to Mack's model, and his associated uncertainty estimators employing such selections will be incorrect.[4] A matter of a greater concern though is that the residual definition is not valid for the new model and thus the selected model cannot be tested.

[4] In Mack's 1999 paper he expanded his formulas to incorporate simple averages in addition to weighted averages.

In the remainder of the paper we close this gap in a sense that for each "reasonably" selected link ratio set we provide a statistical model which has this set of selected link ratios as its best linear unbiased regression estimators. Using this tool, we are now able to incorporate a statistical work flow cycle into the reserving process:

Figure 1: Actuarial validation loop



This diagram shows of course only the work flow assuming that the data is appropriate. However one major part of the reserving exercise is reviewing the underlying data. We will see that the residuals can help the actuary identify outliers and trends.

As actuaries select, evaluate, and re-select link ratios, they are implicitly reformulating the model (1) by "selecting" a different α parameter each time. This correspondence is established by the following two theorems that prove the existence of the α parameters that solve model (1) for selected link ratios that are reasonable. By reasonable selected link ratios we mean selected link ratios within the range produced from the various average link ratios based on the empirical data.

3.1 Theorem (Link Ratio Function)

We consider for a given triangle the corresponding link ratio function as in (3) and denote the set of all *reasonable* link ratios with $\mathrm{LR}_{\boldsymbol{k}}(\Re) := \{\mathrm{LR}_{\boldsymbol{k}}(\alpha) \mid \alpha \in \Re\}$ and $i_{\min,k}, i_{\max,k}$ be the index of $\min\{C_{j,k}, j < n-k\}, \max\{C_{j,k}, j < n-k\}$ respectively. Then

1. If $c, d \in LR_k(\Re)$, then the whole interval $[c, d] \subseteq LR_k(\Re)$
2. $\mathrm{LR}_{\boldsymbol{k}}(\alpha) \to \boldsymbol{F}_{\boldsymbol{i}_{\min}, \boldsymbol{k}}, (\alpha \to \infty)$
3. $\mathrm{LR}_{\boldsymbol{k}}(\alpha) \to \boldsymbol{F}_{\boldsymbol{i}_{\max}, \boldsymbol{k}}, (\alpha \to -\infty)$
4. In particular, every value between the straight average link ratio, the weighted average link ratio and the link ratios corresponding to the minimum and maximum weight $\min\{C_{j,k}, j < n-k\}, \max\{C_{j,k}, j < n-k\}$ respectively, is reasonable.

3.2 Theorem (Existence)

Let $\{\boldsymbol{h_k}; \boldsymbol{k} \leq \boldsymbol{n}-2\}$ be a set of reasonable link ratios (as defined in 3.1) with $\boldsymbol{h_k} \in \mathrm{LR}_{\boldsymbol{k}}(\Re), \boldsymbol{k} \leq \boldsymbol{n}-2$. Then for each *k* there is at least one α such that $\boldsymbol{h_k}$ is the ML-estimator of (1). We define

$$\hat{\alpha}_k := \max(\min\{\alpha > 0 \mid h_k = \mathrm{LR}(\alpha)\}, \max\{\alpha \leq 0 \mid h_k = \mathrm{LR}(\alpha)\}).$$

Then $\hat{\alpha}_k$ is well defined and can be calculated using a solver.[5] In other words among all possible α we take the one with smallest absolute value, and in cases, where two possible α have exactly the same absolute value, we choose the positive.

The proofs of both theorems are relegated to the appendix.

The condition $k \leq n$-2 is necessary because for the last development period ($k=n$-1) a regression type of approach is not useful as there is only one observation.

Remark 1:

In the original chain ladder method modeled in Mack (1993) the standard deviations of payments of all development periods is assumed to be proportional to the square root of payments of the previous development year. But why is it the square root, and why should this hold for all development years? Theorem 3.2 Theorem (Existence) relaxes this requirement. It shows that even with judgmentally selected link ratios an underlying statistical model exists such that the selected link ratios are the optimal parameters.[6]

Although assumptions cannot be tested, residuals can, which enables us to find the appropriate chain ladder model that is consistent with the actuary's link ratio selections. This underlines the thought that models offer "proposals" to understand the data structure. To cite George Box: "Essentially, all models are wrong, but some are useful."

4. EXAMPLES

Example 1:

We first consider the following triangle, which is discussed in Mack (1993) and in Zehnwirth (2004). The weighted averages link ratios are shown below:

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[5] For example the Newton-Algorithm with starting point 0.

[6] In fact in some cases there can be more than one $\hat{\alpha}_k$ for the same link ratio. In other words, it is possible to have more than one standard deviation assumption associated with the same link ratio.

Table 1

5,012	8,269	10,907	11,805	13,539	16,181	18,009	18,608	18,662	18,834
106	4,285	5,396	10,666	13,782	15,599	15,496	16,169	16,704	
3,410	8,992	13,873	16,141	18,735	22,214	22,863	23,466		
5,655	11,555	15,766	21,266	23,425	26,083	27,067			
1,092	9,565	15,836	22,169	25,955	26,180				
1,513	6,445	11,702	12,935	15,852					
557	4,020	10,946	12,314						
1,351	6,947	13,112							
3,133	5,395								
2,063									

Simple Average	8.206	1.696	1.315	1.183	1.127	1.043	1.034	1.018	1.009
Weighted Average	2.999	1.624	1.271	1.172	1.113	1.042	1.033	1.017	1.009

Although this triangle is quite well understood, we try to "analyze" it again.

First we declare our *goal*, which is to find a model, which describes our data with a certain confidence.

- Model Selection: We start with the link ratio model, which means that we believe

$$\boldsymbol{C}_{ik+1} = \boldsymbol{f}_k \boldsymbol{C}_{ik} + \sigma_k \varepsilon_{i,k} \boldsymbol{C}_{i,k}^{\alpha_k/2}.$$

- Parameter Selection: This means in our case, that we choose a set of link ratios and *calculate* the corresponding variance assumption. We start here with the simple averages.

- Model Validation: Now we need to test the corresponding residuals.

Table 2

-0.5313	-0.7949	-0.7322	-0.5395	0.9132	1.3861	-0.1275	-0.7071
2.6108	-0.9210	2.0882	1.6351	0.0653	-0.9937	1.0576	0.7071
-0.4513	-0.3229	-0.4763	-0.3326	0.7867	-0.2809	-0.9301	
-0.4994	-0.6992	0.1083	-1.2187	-0.1807	-0.1115		
0.0448	-0.0850	0.2693	-0.1818	-1.5844			
-0.3198	0.2526	-0.6596	0.6376				
-0.0801	2.1662	-0.5977					
-0.2483	0.4040						
-0.5254							

The following plot graphs the residuals along the accident-year dimension and helps the practitioner to identify the existence, or absence, of any trends. The graph below suggests that the residuals are, for the most part, random.

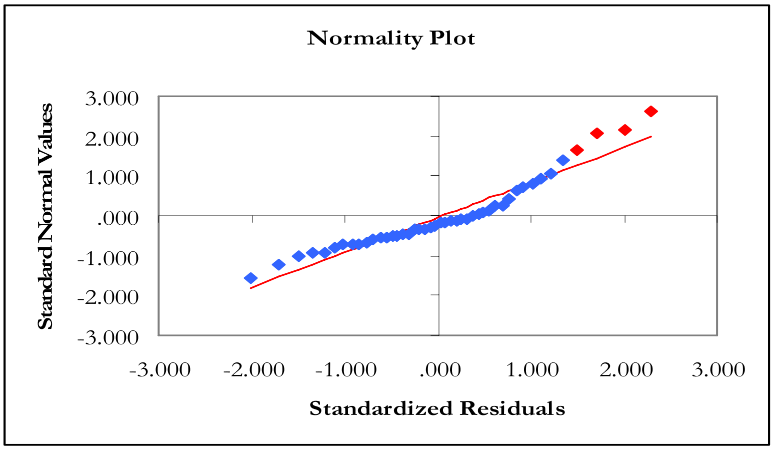


Figure 2 The residuals for the first example with the selected simple average link ratios against the quantiles of the normal distribution (red line)

Additionally we could test the data in several different other ways to make sure we are confident about the "noise hypothesis." In particular the Shapiro-Francia *P*-Value is 2.6%, which suggests that the assumption of normality of the residuals is rejected at the 5.0% confidence level. This means we would need to go back to one of the previous steps.

- Model (Re)Selection: With an exception of a few outliers, the model was acceptable, so we might still stay with the same model.
- Parameter (Re)Selection: Obviously the first few link ratio "produces" outliers, so we might change the first three selected link ratios to be the volume-weighted ones. That means we would select:

Selection	2.999	1.624	1.271	1.183	1.127	1.043	1.034	1.018	1.009
alpha	1.000	1.000	1.000	2.000	2.000	2.000	2.000	2.000	

- Model Validation: The Shapiro Francia test delivers a *P*-Value of 12.0%, so dependent on our level of statistical confidence we could accept this model, the selected parameters (and the corresponding "best estimate" reserves, the standard deviation, etc.). By comparing Figure 2 and Figure 3, we see that the selected link ratio set is a much better approximation of the normal distribution than the simple average link ratios.

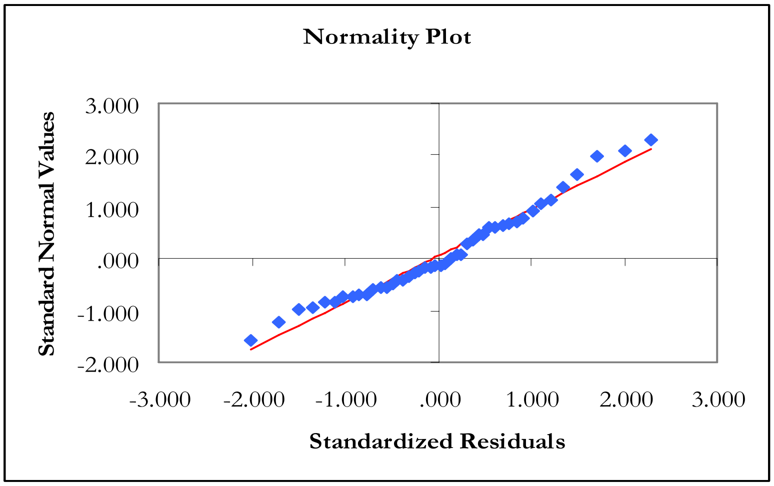


Figure 3 The residuals for the first example with the selected link ratios against the quantiles of the normal distribution (red line)

The Chain ladder link ratios, based on the volume weighted averages, deliver a *P*-Value of 23.4% for the Shapiro Francia test.

2.999	1.624	1.271	1.172	1.113	1.042	1.033	1.017	1.009

In other words the volume weighted link ratios are easily acceptable with our 5% level of confidence, but this demonstrates again that many models are "similarly wrong," but good enough for this task. We have chosen this well known-example to demonstrate the different steps in Figure 1.

Example 2:

We consider now the following triangle:

Table 3

29	97	216	388	580	764	930	1,119	1,322	1,526	1,657	1,720	1,739	1,748	1,752
30	102	227	403	631	849	1,046	1,270	1,518	1,703	1,820	1,877	1,894	1,901	
35	107	234	451	723	984	1,221	1,496	1,714	1,880	1,987	2,037	2,056		
34	112	268	526	850	1,162	1,447	1,689	1,888	2,037	2,134	2,178			
34	123	308	622	1,014	1,393	1,648	1,869	2,048	2,181	2,265				
42	152	373	745	1,216	1,555	1,786	1,984	2,145	2,265					
49	185	449	898	1,322	1,630	1,839	2,019	2,163						
58	217	537	939	1,319	1,594	1,779	1,938							
70	262	550	917	1,262	1,510	1,679								
88	261	518	846	1,154	1,379									
76	235	466	755	1,033										
68	207	411	673											
58	185	372												
53	167													

- Model Selection: We consider again the link ratio model.
- Parameter Selection: Before selecting the parameters, we might want to look at the link ratios and probably try the “latest year averages” because of the possible trend in the most recent calendar years.

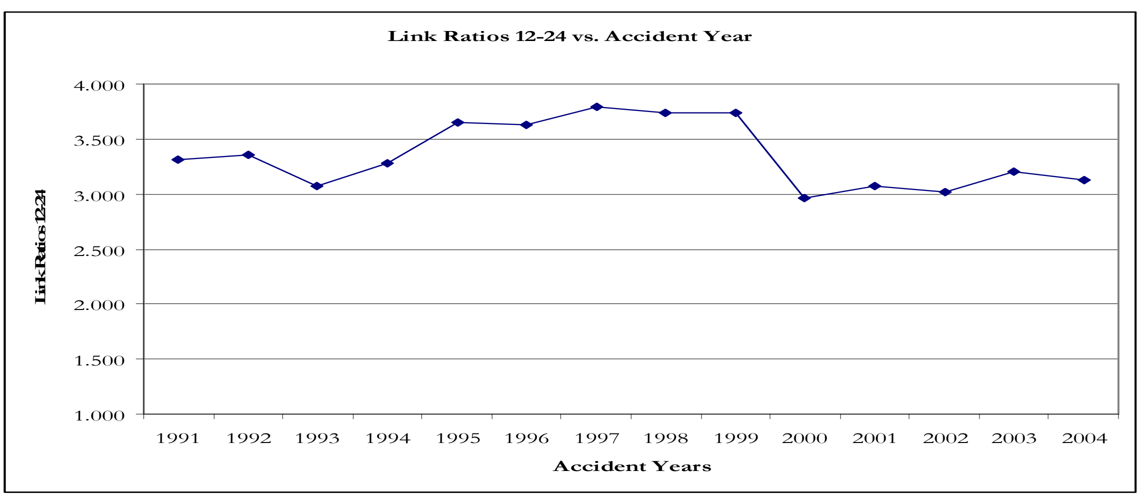


Figure 4 The link ratios for the first development period

➢ Model Validation: The following tables show the selected link ratios and the corresponding weights:

Chain Ladder
VW All Years

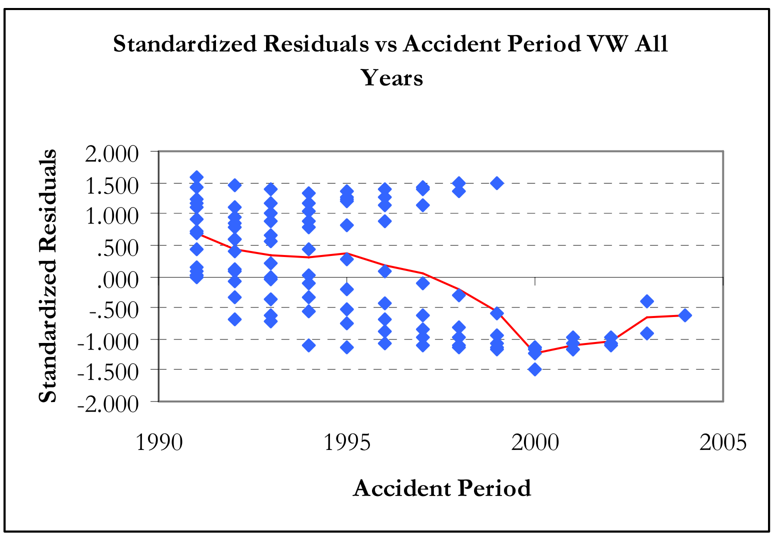
Link ratio	3.325	2.196	1.791	1.483	1.273	1.169	1.144	1.118	1.090	1.057	1.028	1.010	1.004	1.002
alpha	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
***P* Value**	0.0054%													

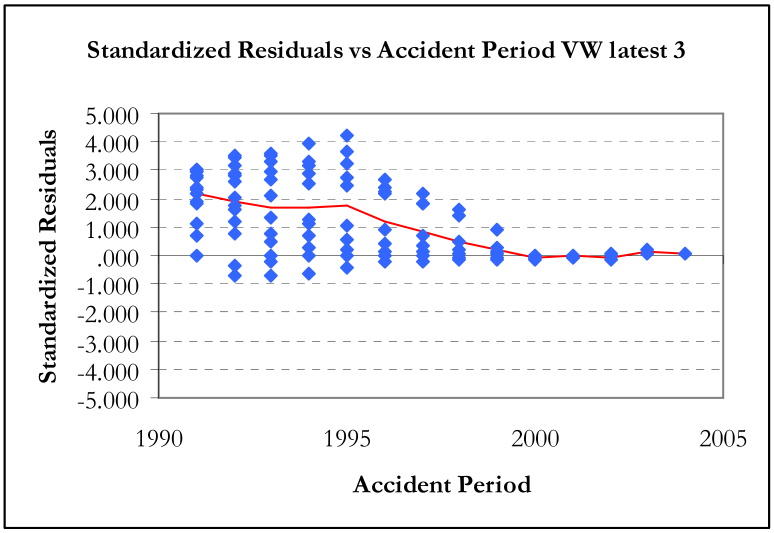
VW Latest 5

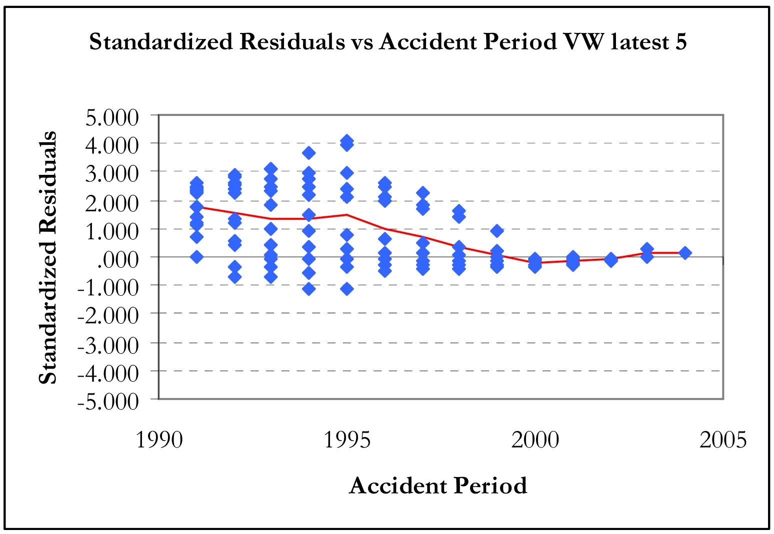
Link ratio	3.063	2.015	1.664	1.398	1.222	1.137	1.118	1.099	1.081	1.057	1.028	1.010	1.004	1.002
alpha	6.430	5.076	4.719	4.300	4.065	3.948	3.892	3.854	3.727	1.000	1.000	1.000	1.000	1.000
***P* Value**	0.0058%													

VW Latest 3

Link ratio	3.111	1.994	1.630	1.370	1.200	1.119	1.099	1.082	1.066	1.047	1.025	1.010	1.004	1.002
alpha	6.170	5.274	4.697	4.433	4.178	4.044	3.967	3.903	3.713	3.538	3.404	1.000	1.000	1.000
***P* Value**	0.0016%													







The test of normality rejected the assumption for all three types of selected link ratios. After these three loops of trying different levels of diagnosis, we might reconsider the model.

- <u>Model Selection:</u> We might now consider a more complex model, for example: $C_{ik+1} = g_k + f_k C_{ik} + \sigma_k \varepsilon_{i,k} C_{i,k}^{\alpha_k/2}$. For this model we refer the reader to Murphy [2].

The data might be even too complex for this model, but we demonstrate here the controlled way of actuarial work, which, of course, needs actuarial judgment, but also statistical tools to quantify the level of confidence for objective communication and assurance of quality (for example, for approval of an internal model in Solvency II-context).

5. CONCLUSION AND FURTHER RESEARCH

As already mentioned before, an alternative approach to ours would be the Bayesian approach, which means one could define a priori for the α_k and derive the a posteriori distribution for the variance assumption.

We have shown how to use the more flexible regression model (1) to reproduce the results of the traditional chain ladder methodology, which offers both consistency with the actuarial reserving work flow and statistical diagnostic tools. It is now quite obvious that the recursive formula of Mack/Murphy for the overall reserve uncertainty can be adapted to the selected link ratio model. In addition to that, a similar approach for the uncertainty of the BF method or Cape Cod method seems to be straight forward. We mention here also that any kind of bootstrapping can be done using the tested residuals. As we mentioned before, for bootstrapping purposes the residuals should be tested to assure proper results.

Even though the approach we introduced here is much more flexible than just employing average link ratios, there are many cases, where the model is not capable of modeling the structure in an appropriate way (such that the residuals looks like noise). In these cases, taking a more complicated method with more prediction power is necessary. The most natural way of making another step towards flexibility is to use the regression model of Murphy [2] with an intercept.

6. APPENDIX

Proof of Theorem 3.1 (Link Ratio Function)

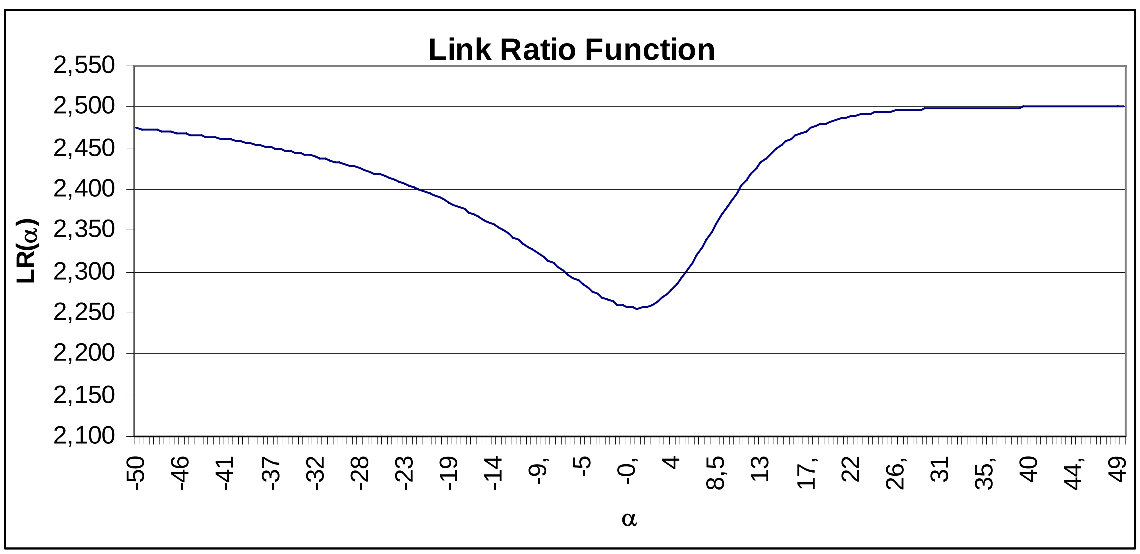
1. If $\mathrm{LR}_k : \Re \to \Re$ is a differentiable function and in particular continuous, its range is an interval in the set of real numbers.
2. We first note for arbitrary α that $\sum_{j=1}^{n-k} w_{j,k}^{\alpha} = 1$. Without loss of generality we assume $\boldsymbol{C}_{\boldsymbol{i}_{\min},\boldsymbol{k}} < \boldsymbol{C}_{\boldsymbol{j},\boldsymbol{k}}, (\boldsymbol{j} \le \boldsymbol{n} - \boldsymbol{k})$. It is now sufficient to prove $\boldsymbol{w}_{\boldsymbol{i}_{\min},\boldsymbol{k}}^{\alpha} \to 1$ as $\alpha \to \infty$. This can be seen by rewriting the weight
$$\boldsymbol{w}_{\boldsymbol{i}_{\min},\boldsymbol{k}}^{\alpha} = \boldsymbol{C}_{\boldsymbol{i}_{\min},\boldsymbol{k}}^{2-\alpha} / \sum_{\boldsymbol{j}=1}^{\boldsymbol{n}-\boldsymbol{k}} \boldsymbol{C}_{\boldsymbol{j},\boldsymbol{k}}^{2-\alpha} = \boldsymbol{C}_{\boldsymbol{i}_{\min},\boldsymbol{k}}^{2} / \sum_{\boldsymbol{j}=1}^{\boldsymbol{n}-\boldsymbol{k}} \boldsymbol{C}_{\boldsymbol{j},\boldsymbol{k}}^{2} \cdot (\boldsymbol{C}_{\boldsymbol{i}_{\min},\boldsymbol{k}} / \boldsymbol{C}_{\boldsymbol{j},\boldsymbol{k}})^{\alpha} .$$
Obviously all $\cdot(\boldsymbol{C}_{\boldsymbol{i}_{\min},\boldsymbol{k}} / \boldsymbol{C}_{\boldsymbol{j},\boldsymbol{k}}) < 1, \boldsymbol{j} \ne \boldsymbol{i}_{\min}$, thus all terms converge to 0 except for $\boldsymbol{j} = \boldsymbol{i}_{\min}$, so that we see $\sum_{\boldsymbol{j}=1}^{\boldsymbol{n}-\boldsymbol{k}} \boldsymbol{C}_{\boldsymbol{j},\boldsymbol{k}}^{2} \cdot (\boldsymbol{C}_{\boldsymbol{i}_{\min},\boldsymbol{k}} / \boldsymbol{C}_{\boldsymbol{j},\boldsymbol{k}})^{\alpha} \to \boldsymbol{C}_{\boldsymbol{i}_{\min},\boldsymbol{k}}^{2}$ as $\alpha \to \infty$.
3. Similar to 2, we can deduce $\boldsymbol{w}_{\boldsymbol{i}_{\max},\boldsymbol{k}}^{\alpha} \to 1$ as $\alpha \to -\infty$.
4. The weighted average and the simple average correspond to $\mathrm{LR}_{\boldsymbol{k}}(2), \mathrm{LR}_{\boldsymbol{k}}(1)$, respectively. This, with 1 above, proves the theorem.

The following example illustrates the function $\mathrm{LR}_k(\alpha)$ with an example, where $\boldsymbol{F}_{i_{\min},k} = \boldsymbol{F}_{i_{\max},k} = 2.5$. This is a case, where for all link ratios, except for the minimum for $\alpha = 0$, there are two different variance assumptions, which lead to the same link ratio. Also the infinitesimal behavior of the function is stated in the following graph.

Table 4: Link Ratio Example

152	380	**2.5000**
185	449	2.4270
217	537	2.4747
250	550	2.2000
262	655	**2.5000**
235	466	1.9830
207	411	1.9855
185	372	2.0108

Simple Average:	α=2	**2.2601**
VW Average:	α=1	**2.2563**
quadratic	α=0	**2.2559**



Proof of Theorem 3.2

Using Theorem 3.1 we observe that the set $\{\alpha \in \Re \mid h_k = \mathrm{LR}(\alpha)\}$ is not empty. Furthermore we note that $h_k = \mathrm{LR}(\alpha) \Leftrightarrow (h_k \cdot \sum_{j=1}^{n-k} C_{j,k}^{2-\alpha} - \sum_{i=1}^{n-k} C_{i,k+1} C_{i,k}^{1-\alpha}) = 0$, which can be solved with an appropriate numerical solver algorithm.

Consider again the example in Table 4. Then we get two solutions for the link ratio 2.400: -21.4 and 10.7, thus we set the variance estimation to max(-21.4, 10.7)=10.7.

7. REFERENCES

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