Generalized Minimum Bias Models

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Abstract:
In this research, we propose a flexible and comprehensive approach for minimum bias models -- "Generalized Minimum Bias Models"(GMBM). Unlike the Generalized Linear Models (GLMs) that require the exponential family distribution assumption of response variables, the GMBM approach relaxes the distribution assumption. In addition, due to its model selection flexibility, we believe that GMBM will improve the accuracy and the goodness of fit of classification rates. All the multiplicative minimum bias models published to date and the commonly used multiplicative GLMs (such as Gamma, Poisson, normal, inverse Gaussian) can be proved as special cases of GMBM.

Keywords: GMBM, GLMs, Classification Ratemaking, Weighted Average.

1. INTRODUCTION

Minimum bias models have had a long history for property and casualty actuaries. Until recent interest in generalized linear models (GLM), minimum bias approach was the major technique used by actuaries in determining the rate relativities for a multiple rating variables class plan. Numerous studies have shown that these two related multivariate procedures can reduce the estimation errors from one-way analysis.

Bailey and Simon (1960) originally considered the biases in the classification ratemaking and introduced the minimum bias models. Bailey (1963) summarized the minimum bias theory and proposed two iterative methods (one multiplicative and one additive), which later became popular with the property and casualty actuaries. Because multiplicative models are more popular than additive ones, the following sections will focus on multiplicative models, and the discussion for the additive models will be given in the appendix.

Let $r_{i,j}$ and $w_{i,j}$ be the observed relativity and weight (earned exposure or number of claims) for the classification i and j, respectively; and x_i and y_j be the relativities for the classification i and classification j, respectively. The multiplicative formula proposed by Bailey (1963) is:

$$\text{Model 1:} \quad \hat{x}_i = \frac{\sum_j w_{i,j} r_{i,j}}{\sum_j w_{i,j} y_j} \qquad (1).$$

where i=1,2, to m; and j=1,2, to n. Similarly, $\hat{y}_j = \dfrac{\sum_i w_{i,j} r_{i,j}}{\sum_i w_{i,j} x_i}$.

Model 1 can be derived by the maximum likelihood (ML) method assuming Poisson distribution, and is also called "ML Poisson model".

Brown (1988) expanded the minimum bias method by using additional types of bias functions. He linked the minimum bias method to statistical theories by maximizing the likelihood functions to calculate the parameter relativity, and introduced four more minimum bias models (three multiplicative and one additive). In addition to the Poisson model in Equation (1), the ML exponential model is:

$$\text{Model 2:} \quad \hat{x}_i = \frac{1}{n}\sum_j \frac{r_{i,j}}{y_j} \qquad (2);$$

and the ML normal model is:

$$\text{Model 3:} \quad \hat{x}_i = \frac{\sum_j w_{i,j}^2 r_{i,j} y_j}{\sum_j w_{i,j}^2 y_j^2} \qquad (3);$$

and the least-square multiplicative model is:

$$\text{Model 4:} \quad \hat{x}_i = \frac{\sum_j w_{i,j} r_{i,j} y_j}{\sum_j w_{i,j} y_j^2} \qquad (4).$$

The formats of Models 1-4 are simple and straightforward. So, compared to GLM, one main advantage of the minimum bias approach is that it is easy to understand and easy to use.

Another minimum bias model by Bailey and Simon (1960) has a relatively complicated format:

$$\text{Model 5:} \quad \hat{x}_i = \left(\frac{\sum_j w_{i,j} r_{i,j}^2 y_j^{-1}}{\sum_j w_{i,j} y_j} \right)^{1/2} \qquad (5).$$

Feldblum and Brosius (2002) summarized these minimum bias models into four categories: "balance principle", "least squares", "χ-squared", and "maximum likelihood":

- Model 1 could be derived from the so-called "balance principle", that is, *"the sum of the indicated relativity = the sum of observed relativity"*. Such balance relationship is:
$$\sum_j w_{i,j} r_{i,j} = \sum_j w_{i,j} x_i y_j \ .$$

- Model 4 can be derived by minimizing the sum of square-error:
$$\underset{x,y}{Min} \sum_{i,j} w_{i,j} (r_{i,j} - x_i y_j)^2 \ .$$

- Models 1, 2, and 3 can be derived from the associated log likelihood functions of observed loss (or relativity).

- Model 5 can be derived by minimizing the " χ -squared" error, the square error divided by the indicated relativity:
$$\underset{x,y}{Min} \sum_{i,j} w_{i,j} \frac{(r_{i,j} - x_i y_j)^2}{x_i y_j} \ .$$

Mildenhall (1999) in his milestone paper further demonstrated that classification rates determined by various linear bias functions are essentially the same as those from GLM models. One main advantage of using statistical models such as GLM is that the characteristics of the models, such as parameters' confidence intervals and hypothesis testing, can be thoroughly studied and determined by statistical theories. Another advantage is that GLM models may be more efficient because they do not require actuaries to program the iterative process in determining the parameters[1]. However, this advantage can be discounted due to the powerful calculation capability associated with modern computers. Due to these advantages, GLMs are becoming more popular in recent years. Of course, actuaries need to acquire the necessary statistical knowledge in understanding and applying the GLM models.

One issue associated with most previous works on the minimum bias models and GLM is the model-selection limitation. GLMs assume the underlying distributions are from the

[1] GLMs may also involve iterative approach. The most commonly used numerical method to solve the GLM is the "iterative reweighted least square" algorithm. The discussion of calculation efficiency is given in the appendix.

exponential family, such as Poisson, Gamma, normal, negative binomial, and inverse Gaussian. On the other hand, only five types of multiplicative models and four types of additive models are available from previous minimum bias work[2]. These limitations, we believe, may reduce estimation accuracy in practice since insurance and actuarial data are rarely perfect and may not fit the exponential family of distributions or existing bias models well.

It is with this motivation that in this study, we propose a more flexible and comprehensive approach within the minimum bias framework, called "Generalized Minimum Bias Model". The key features of GMBM are:

- It does not assume a specific form of distributions, which increases the application appropriateness and model-selection flexibility.
- Due to its flexibility, it will improve the accuracy and the goodness of fit of classification rates. We will show the empirical evidence later.
- Similar to past minimum bias models, it is easy to understand and does not require advanced statistical knowledge.
- While GMBM still requires the iterative process in determining the parameters, we believe that the effort is not significant with today's powerful modern computers. We will prove in the appendix that the iterative process required to calculate the GMBM parameters can converge rapidly.

All five existing multiplicative models are proved to be the special cases of GMBM. In addition, we will show several more bias models that actuaries may consider for ratemaking based on insurance data.

The numerical analysis given later is based on severity data for private passenger auto collision given in Mildenhall (1999) and McCullagh and Nelder (1989). The results for selected generalized minimum bias models will be compared to those from the GLM models. Following Bailey and Simon (1960), the weighted absolute bias and the Pearson Chi-Squared statistic are used to measure the goodness of fit. We also calculate the weighted absolute percentage bias, which indicates how much the errors are relative to the predicted values. The empirical results indicate that actuaries can improve the accuracy of classification rates by using the appropriate generalized minimum bias models.

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[2] Feldblum and Brosius (2002) listed six multiplicative minimum bias models in their summary table. However, the balance principle model is the same as the maximum likelihood Poisson model.

The paper is organized as follows:

- Section 2 discusses the details of 2-parameter and 3-parameter GMBM models.
- Section 3 reviews numerical results for a severity case study.
- Section 4 outlines our conclusions.
- More details and insights for the statistical theories associated with GMBM will be given in Appendix 1.
    - Appendix 1.1 analyzes the bias function of GMBM, proves that GLM with log link are special cases of GMBM, and explores GMBM from the perspective of generalized balance principle.
    - Appendix 1.2 discusses the relativity link functions of GMBM, addresses the difference in the link functions between GLM and GMBM, and analyzes GMBM from the perspective of maximum likelihood method.
    - Appendix 1.3 investigates the possibility of further generalizing GMBM models, explores GMBM from the perspective of deviance functions.
    - Appendix 1.4 explores the additive models for GMBM.
    - Appendix 1.5 discusses the calculation efficiency of GMBM. It shows that GMBM could converge rapidly and is not necessarily inefficient in numerical calculations.
- Appendix 2 reports the numerical results for the severity example discussed in Section 3 with several selected GMBM models.
- Appendix 3 shows the numerical iterative results in Appendix 1.5 for selected GMBM and GLM models.

2. GENERALIZED MINIMUM BIAS MODELS - GMBM

2-Parameter GMBM

Following the notation used previously, in the multiplicative framework for two rating factors, the expected relativity for cell (i,j) should be equal to the product of x_i and y_j:

$$E(r_{i,j}) = \mu_{i,j} = x_i y_j \qquad (6).$$

By (6), there are a total of n alternative estimates for x_i and a total of m estimates for y_j:

$$\begin{aligned} \hat{x}_{i,j} &= r_{i,j} / y_j, \quad j = 1, \quad 2, \quad to \quad n \\ \hat{y}_{j,i} &= r_{i,j} / x_i, \quad i = 1, \quad 2, \quad to \quad m, \end{aligned} \qquad (7).$$

Following actuarial convention, the final estimates of x_i and y_j could be calculated by the weighted average of $\hat{x}_{i,j}$ and $\hat{y}_{j,i}$. If we use the straight average to estimate the relativity:

$$\hat{x}_i = \sum_j \frac{1}{n}\hat{x}_{i,j} = \frac{1}{n}\sum_j \frac{r_{i,j}}{y_j} \qquad (8).$$

Similarly, $\hat{y}_j = \sum_i \frac{1}{m}\hat{y}_{j,i} = \frac{1}{m}\sum_i \frac{r_{i,j}}{x_i}$. This is Model 2, the ML exponential model introduced by Brown (1988).

If the relativity-adjusted number of claims, $w_{i,j}x_i$ or $w_{i,j}y_j$, is used as the weight in determining the estimates:

$$\hat{x}_i = \sum_j \frac{w_{i,j}y_j}{\sum_j w_{i,j}y_j}\hat{x}_{i,j} = \sum_j \frac{w_{i,j}y_j}{\sum_j w_{i,j}y_j}\frac{r_{i,j}}{y_j} = \frac{\sum_j w_{i,j}r_{i,j}}{\sum_j w_{i,j}y_j} \qquad (9).$$

Similarly, $\hat{y}_j = \sum_i \frac{w_{i,j}x_i}{\sum_i w_{i,j}x_i}\hat{y}_{i,j} = \sum_i \frac{w_{i,j}x_i}{\sum_i w_{i,j}x_i}\frac{r_{i,j}}{x_i} = \frac{\sum_i w_{i,j}r_{i,j}}{\sum_i w_{i,j}x_i}$. The resulting model is the same as Model 1, the "balance principle" or ML Poisson model.

If the square of the relativity-adjusted number of claims, $w_{i,j}^2x_i^2$ or $w_{i,j}^2y_j^2$, is used as the weight:

$$\hat{x}_i = \sum_j \frac{w_{i,j}^2y_j^2}{\sum_j w_{i,j}^2y_j^2}\hat{x}_{i,j} = \frac{\sum_j w_{i,j}^2r_{i,j}y_j}{\sum_j w_{i,j}^2y_j^2} \qquad (10).$$

The resulting model is the same as Model 3, the ML normal model.

If the number of claims adjusted by the square of relativity, $w_{i,j}x_i^2$ or $w_{i,j}y_j^2$, is used as the weights:

$$\hat{x}_i = \sum_j \frac{w_{i,j} y_j^2}{\sum_j w_{i,j} y_j^2} \hat{x}_{i,j} = \frac{\sum_j w_{i,j} r_{i,j} y_j}{\sum_j w_{i,j} y_j^2} \qquad (11).$$

The resulting model is the same as Model 4, the least-square model.

From the above results, we propose the 2-parameter generalized minimum bias approach by using $w_{i,j}^p x_i^q$ and $w_{i,j}^p y_j^q$ as the weights for the bias function:

$$\text{2-Parameter GMBM: } \hat{x}_i = \sum_j \frac{w_{i,j}^p y_j^q}{\sum_j w_{i,j}^p y_j^q} \hat{x}_{i,j} = \frac{\sum_j w_{i,j}^p r_{i,j} y_j^{q-1}}{\sum_j w_{i,j}^p y_j^q} \quad (12).$$

When,

- p=q=0, it is the ML exponential model, Model 2;
- p=q=1, it is the ML Poisson model, Model 1;
- p=q=2, it is the ML normal model, Model 3
- p=1 and q=2, it is the least-square model, Model 4.

In addition, there are two more models that correspond to GLM with the exponential family of Gamma and inverse Gaussian distributions[3]. When the number of claims is used as the weights, that is, p=1 and q=0, the GMBM model is a GLM Gamma model and becomes:

$$\text{Model 6: } \hat{x}_i = \sum_j \frac{w_{i,j}}{\sum_j w_{i,j}} \hat{x}_{i,j} = \frac{\sum_j w_{i,j} r_{i,j} y_j^{-1}}{\sum_j w_{i,j}} \qquad (13).$$

When p=1 and q=-1, the GMBM model is a GLM Inverse Gaussian model and becomes:

$$\text{Model 7: } \hat{x}_i = \sum_j \frac{w_{i,j} y_j^{-1}}{\sum_j w_{i,j} y_j^{-1}} \hat{x}_{i,j} = \frac{\sum_j w_{i,j} r_{i,j} / y_j^2}{\sum_j w_{i,j} / y_j} \qquad (14).$$

[3] For detailed information, please refer to Section 7 of Mildenhall (1999).

Equation (12) indicates that in theory there is no limitation for the values of p and q to be used. It is with this feature that GMBM should greatly enhance the flexibility for actuaries when they apply the models to fit their data. Of course, in reality we do not expect that extreme values for p and q will be found useful. In ratemaking applications, earned premium could be used if exposure is not available. Normalized premium (premium divided by relativity) is a reasonable option for the weight. This suggests that q could be negative. In general, p should be positive: the more claims/exposure/premium, the more weight assigned. In the numerical analysis given later, we will test the model with the p and q values ranging from 0 to 2.

In actuarial exercises, we often exclude the extremely high and low values from the weighted average to yield more robust results. In case of several rating variables, there may be thousands of alternative estimates. Actuaries have the flexibility to use the weighted average within selected ranges (e.g. the average without the highest and the lowest 1% percentile). This is similar to concept of "trimmed" regression used with GLMs whereby observations with undue influence on fitted value are removed.

3-Parameter GMBM

So far, we have used the 2-parameter GMBM in Equation (12) to represent several commonly used minimum bias models, Models 1 to 4, but not Model 5, the "χ-squared' multiplicative model. In order to represent Model 5, we further expand the 2-parameter GMBM to a 3-parameter GMBM using the link function concept from GLM.

One generalization of GLMs compared to linear model is to introduce a link function to link the linear predictor to the response variable. Similarly, we introduce a relativity link function, f, which links the minimum bias estimate to the relativity. Of course, this relativity link function is different in several aspects from the link function in GLMs. In GLMs, the link function determines the type of model: log link implies a multiplicative model and identity link implies an additive model. This is not the case for GMBM. A multiplicative GMBM model, for example, could have a log, power, or exponential link function. The detailed discussion of GMBM link function and its difference from GLM link function will be given in Appendix 1.2.

For a 3-parameter GMBM model, instead of using (7), we estimate the relativity link functions of $f(\hat{x}_i)$ and $f(\hat{y}_j)$ from $f(\hat{x}_{i,j})$ and $f(\hat{y}_{j,i})$ first; and then calculate $\hat{x}_i$ and

$\hat{y}_j$ by inverting the relativity function, $f^{-1}(f(\hat{x}_i))$ and $f^{-1}(f(\hat{y}_j))$. The functions $f(\hat{x}_{i,j})$ and $f(\hat{y}_{j,i})$ can be estimated by:

$$\begin{aligned} f(\hat{x}_{i,j}) &= f(r_{i,j}/y_j), \quad j=1, \quad 2, \quad to \quad n; \\ f(\hat{y}_{j,i}) &= f(r_{i,j}/x_i), \quad i=1, \quad 2, \quad to \quad m \end{aligned} \qquad (15).$$

Taking the weighted average using parameters p and q:

$$\begin{aligned} f(\hat{x}_i) &= \sum_j \frac{w_{i,j}^p y_j^q}{\sum_j w_{i,j}^p y_j^q} f(\hat{x}_{i,j}) = \frac{\sum_j w_{i,j}^p y_j^q f(\frac{r_{i,j}}{y_j})}{\sum_j w_{i,j}^p y_j^q} \\ f(\hat{y}_j) &= \sum_i \frac{w_{i,j}^p x_i^q}{\sum_i w_{i,j}^p x_i^q} f(\hat{y}_{j,i}) = \frac{\sum_i w_{i,j}^p x_i^q f(\frac{r_{i,j}}{x_i})}{\sum_i w_{i,j}^p x_i^q} \end{aligned} \qquad (16).$$

Thus,

$$\begin{aligned} \hat{x}_i &= f^{-1}\left(\frac{\sum_j w_{i,j}^p y_j^q f(\frac{r_{i,j}}{y_j})}{\sum_j w_{i,j}^p y_j^q} \right) \\ \hat{y}_j &= f^{-1}\left(\frac{\sum_i w_{i,j}^p x_i^q f(\frac{r_{i,j}}{x_i})}{\sum_i w_{i,j}^p x_i^q} \right) \end{aligned} \qquad (17).$$

One possible selection of the relativity link function is the power function, $f(\hat{x}_i) = \hat{x}_i^k$ and $f(\hat{y}_j) = \hat{y}_j^k$. In this case, equation (17) becomes a 3-parameter GMBM:

$$\text{3-Parameter GMBM: } \hat{x}_i = \left(\frac{\sum_j w_{i,j}^p r_{i,j}^k y_j^{q-k}}{\sum_j w_{i,j}^p y_j^q} \right)^{1/k} \qquad (18).$$

When k=2, p=1, and q=1, Equation (18) is equivalent to:

$$\hat{x}_i = \left(\frac{\sum_j w_{i,j} r_{i,j}^2 y_j^{-1}}{\sum_j w_{i,j} y_j} \right)^{1/2} \qquad (19),$$

and this is Model 5, the "χ-squared' multiplicative model.

Another example of a new iterative model is for k=1/2, p=1, and q=1:

$$\text{Model 8:} \quad \hat{x}_i = \left(\frac{\sum_j w_{i,j} r_{i,j}^{1/2} y_j^{1/2}}{\sum_j w_{i,j} y_j} \right)^2 \qquad (20).$$

In the numerical analysis given next, we will test a series of models with the value of k ranging from 0.5 to 3.

3. NUMERICAL ANALYSIS WITH A SEVERITY CASE STUDY

The numerical analysis is based on the severity data for private passenger auto collision given in Mildenhall (1999) and McCullagh and Nelder (1989). It includes thirty-two severity observations for two classification variables: eight age groups and four types of vehicle-use. The GMBMs are tested at hundreds of cases: k at 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0; and p at 0, 0.5, 1.0, 1.5, and 2.0; and q varies from –2.5 to 4.0. The results from multiplicative GLMs with Poisson, Gamma, and inverse Gaussian distributions are compared to those from GMBMs.

Four criteria are used to evaluate the performance of GMBMs: the absolute bias, the absolute percentage bias, the Pearson Chi-Squared statistic, and the combination of absolute bias and the Chi-Squared statistic:

- The weighted absolute bias (wab) criterion is proposed by Bailey and Simon (1960). It is the weighted average of absolute dollar difference between the observations and fitted values:

$$wab = \frac{\sum w_{i,j} \, | \, r_{i,j} - x_i y_j \, |}{\sum w_{i,j}}$$

- The second one, weighted absolute percentage bias ($wapb$), measures the absolute bias relative to the predicted values:

$$wapb = \frac{\sum w_{i,j} \frac{|r_{i,j} - x_i y_j|}{x_i y_j}}{\sum w_{i,j}}$$

- The weighted Pearson Chi-square ($wChi$) statistic is also proposed by Bailey and Simon (1960) and it is appropriate to test "differences between the raw data and the estimated relativities should be small enough to be caused by chance":

$$wChi = \frac{\sum w_{i,j} \frac{(r_{i,j} - x_i y_j)^2}{x_i y_j}}{\sum w_{i,j}}$$

- Lastly, we combine the absolute bias and Pearson Chi-square statistic, $\sqrt{wab * wChi}$, to be the fourth criterion for the model selection.

The numeric results of the study are given in Appendix 2. Tables 1-6 report the GMBM relativities for k=0.5, 1.0, 1.5, 2.0, 2.5, and 3.0, respectively. Table 7 lists the GLM relativities for Gamma, Poisson, and inverse Gaussian distributions. In addition, Table 8 specifically shows GMBM relativities for several selected cases when k=1 and p=1 in Table 2 because these cases are corresponding to the GLM results in Table 7. In all the cases, class "age 60+" and "pleasure" are used as the base. Tables 9-14 show the weighted absolute bias (wab), weighted absolute percentage bias ($wapb$), Chi-square statistic ($wChi$), and combined $\sqrt{wab * wChi}$ for the corresponding GMBM models in Tables 1-6.

The p and q values in Tables 1-6 and Tables 9-14 for each k value are selected by the following rules. For each k, five p values (0, 0.5, 1.0, 1.5, and 2.0) are used; and for each combination of k and p, at least five q values (0, 0.5, 1.0, 1.5, and 2.0) are calculated. If there exists a model with local minimized $\sqrt{wab * wChi}$ among the five values, no further q is used. If $\sqrt{wab * wChi}$ is strictly decreasing (or increasing) with q, we will try additional higher (or lower) q values until the local minimization with respect to $\sqrt{wab * wChi}$ is found. The local optimal model for each combination of k and p is reported in bold and the global optimal model for each k is reported in underlying bold.

From Tables 9-14, we can see that if wab or $wapb$ is used to measure the model performance, for all the k values tested, p=2, and q=0 is the best, and the global minimum error occurs when k=3. The result suggests that the best-fit model, in this example, does not occur with any of the 5 commonly used minimum models of which the underlying distributions are from the exponential family. It clearly demonstrates the fact the insurance data may not be perfect for predetermined distributions. Therefore, the GMBM approach will provide actuaries a more flexible and comprehensive approaching in analyzing their data.

On the other hand, if $wChi$ is used, "χ-squared' model (k=2, p=1, and q=1) provides the best solution. This is expected because "χ-squared' model is calculated by minimizing the Pearson Chi-square statistic.

If we use the criterion of $\sqrt{wab * wChi}$ to select models, k=2.5, p=1, q=-0.5 offers best overall result. The best solution for different k values is different: p=1.5 and q=1 is best when k=0.5 and 1; p=1 and q=-0.5 is best when k=1.5, 2, 2.5, and 3. Again, the 5 commonly used minimum bias models are not the best solution when absolute bias and Chi-square statistic are considered simultaneously.

As stated before, we find that GLMs with common exponential family distribution assumptions are special cases of GMBM (k=1 and p=1). Therefore, the results between Table 7 and Table 8 are the same for the corresponding models:

- when k=1, p=1, and q=2, the "least-square" GMBM has the same results as GLM with normal distribution[4];
- k=1, p=1, and q=1 is the same as Poisson GLM;
- k=1, p=1, and q=0 is the same as Gamma GLM.
- k=1, p=1 and q=-1 is the same as GLM with inverse Gaussian distribution.

We will prove in the appendix that GMBM with k=1 and p=1 ($\hat{x}_i = \frac{\sum_j w_{i,j} r_{i,j} y_j^{q-1}}{\sum_j w_{i,j} y_j^q}$) is

equivalent to the multiplicative GLMs with the variance function of $V(\mu) = \mu^{2-q}$ for assumed exponential family distribution.

It is well known that insurance and actuarial data is generally positively skewed. The skewness for the symmetric normal distribution is zero, and is increasingly positive from Poisson to Gamma to inverse Gaussian distribution. For the GMBM models, the skewness

[4] The underlying assumption of "least-square" regression is that the residuals follow normal distribution. So "least-square" method is same as GLM normal.

can be represented by the difference between p and q (i.e., p-q). The value of p-q is -1 for normal, 0 for Poisson, 1 for Gamma, and 2 for inverse Gaussian. Thus, larger difference between p and q should be selected in GMBM for more skewed data. We expect that GMBM with negative values of p-q will not have good performance in fitting actuarial data.

Based on the result in this research and our experience, we suggest for actuarial applications the following ranges of values for k, p, q:

- k is between 1 and 3.
- p>=q and 0.5<=p<=2.
- The higher the skewness of the data, the larger p-q to use.

4. CONCLUSIONS

In this research, we propose a generalized minimum bias approach by including different weighting functions and relativity link functions in the approach. As indicated by the severity example given previously, insurance and actuarial data are rarely perfect, so we expect that the best fitted results typically will not be based on a predetermined distribution, such as exponential family distributions. Therefore, GMBM can provide actuaries a great deal of flexibility in data fitting and model selection.

In theory, there is no limitation for different weighting functions or relativity link functions to try when GMBM is applied to a dataset. However, due to the fact that insurance and actuarial data is positively skewed in nature, we do not expect that a very wide range of weighting or relativity functions need to be used in practice.

For the severity example used in the study, we tried hundreds of different combinations and identified the best model with the minimum fitted error among the trial models. Two issues may exist for the example. The first issue is that since minimum bias models use an iterative process in determining the parameters, the fact that GMBM further requires multiple models in trial may make the approach even more time consuming and inefficient. However, we do not believe this issue is significant because of the powerful computation capability with modern computers.

Another issue is that the "true" best model with the minimum error may not be one of the models in the trial. This is very possible in practice. Resolving such global minimum error issue requires additional in-depth research and is beyond the scope of this paper.

With the fast development of information technology, people can analyze data in ways they could not imagine a decade ago. Currently there is a strong interest in data mining and predictive modeling in the insurance industry, and this calls for more powerful data analytical tools for actuaries. While some new tools, such as GLM, neural networks, decision Trees, and MARS, have emerged recently and have received a great deal of attention, we believe that the decades-old minimum bias models still have several advantages over other techniques, including easy to understand and easy to use. We hope that our work in improving the flexibility and comprehensiveness for the minimum bias approach is a timely effort and the approach will continue to be a useful tool for actuaries in the future.

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Appendix 1: Additional Discussions of GMBM Theories

1.1 Bias Functions of GMBM

Bailey and Simon (1960) lay out four criteria for classification relativities. The first criterion is that the rates are balanced for each class and in total. This results in an overall zero ordinary bias, which is the difference between the observed relativity and its fitted value $r_{i,j} - \mu_{i,j}$. Mildenhall (1999) showed that Bailey and Simon's balance criteria is equivalent to a minimum deviance criteria of GLMs of which the bias is measured by linear bias functions; and the bias functions of GLMs are ordinary biases weighted by exposure adjusted by the first order derivative of the GLM link function.

Let Z be the design matrix with rows z_i, h(x) be the inverse function of GLM link function, V(x) be the variance function of the GLM assumed distribution, W be the diagonal matrix of weights with the ith diagonal element of $w_i h'(z_i \beta)/V(\mu_i)$. Mildenhall showed that the bias function of GLM is:

$$Z'W(r-\mu)=0 \qquad \text{(A.1).}$$ [5]

Let a_i and b_j be the GLM coefficients. A.1 is equivalent to:

$$\sum_{j=1}^{n} \frac{w_{i,j} \dfrac{\partial h(a_i + b_j)}{\partial a_i}(r_{i,j} - \mu_{i,j})}{V(\mu_{i,j})} = 0 \quad for \quad j = 1,2,\ldots,n;$$
$$\sum_{i=1}^{m} \frac{w_{i,j} \dfrac{\partial h(a_i + b_j)}{\partial b_j}(r_{i,j} - \mu_{i,j})}{V(\mu_{i,j})} = 0 \quad for \quad i = 1,2,\ldots,m. \qquad \text{(A.2).}$$

For GLM multiplicative models, h(x) is the exponential function, $h(x) = e^x$, and the most commonly used variance functions are power functions $V(\mu) = \mu^c$. So, A.2 becomes:

$$\sum_{j=1}^{n} w_{i,j} \mu_{i,j}^{1-c} (r_{i,j} - \mu_{i,j}) = 0 \qquad \text{(A.3).}$$

[5] For the detailed explanation, please refer to Section 5 of Mildenhall (1999).

For normal distribution, c=0; for Poisson distribution, c=1, for Gamma distribution, c=2, and for inverse Gaussian distribution, c=3.

From A.3, Mildenhall (1999) further showed that, as an example, the GLM model with log link and Poisson distribution is equivalent to Model 1 $\hat{x}_i = \dfrac{\sum_j w_{i,j} r_{i,j}}{\sum_j w_{i,j} y_j}$. For more general cases of exponential family distributions with the variance function of $V(\mu) = \mu^c$, A.3 can be transformed into the following to estimate the relativity:

$$\sum_j w_{i,j} \mu_{i,j}^{1-c} (r_{i,j} - \mu_{i,j}) = 0 \Rightarrow x_i^{1-c} \sum_j w_{i,j} y_j^{1-c} (r_{i,j} - x_i y_j) = 0$$
$$\Rightarrow \hat{x}_i = \frac{\sum_j w_{i,j} r_{i,j} y_j^{1-c}}{\sum_j w_{i,j} y_j^{2-c}} \qquad \text{(A.4).}$$

Similar to GLM, the bias function of GMBM for a rating variable x can be represented as follows:

$$\sum_j w_{i,j}^P \mu_{i,j}^{q-k} (r_{i,j}^k - \mu_{i,j}^k) = 0 \qquad \text{(A.5).}$$

From A.5 we can derive the 3-parameter GMBM models as given in Equation (17):

$$\sum_j w_{i,j}^P \mu_{i,j}^{q-k} (r_{i,j}^k - \mu_{i,j}^k) = 0 \Rightarrow \sum_j w_{i,j}^P y_j^{q-k} (r_{i,j}^k - x_i^k y_j^k) = 0$$
$$\Rightarrow \sum_j w_{i,j}^P y_j^{q-k} r_{i,j}^k - x_i^k \sum_j w_{i,j}^p y_j^q = 0 \Rightarrow \hat{x}_i = \left(\frac{\sum_j w_{i,j}^p r_{i,j}^k y_j^{q-k}}{\sum_j w_{i,j}^p y_j^q} \right)^{1/k} \qquad \text{(A.6).}$$

Comparing A.3 and A.5, we can see that when k=1, the bias for GMBM is measured by the difference between observed and fitted values, and is an average of ordinary bias. Therefore, the GMBM models generalize GLMs on the weights assigned to each ordinary bias since the GLM models for c=0, 1, 2, and 3 can be represented by the GMBM models with the corresponding p and q values. On the other hand, when $k \neq 1$, the GMBM bias is measured by the powered difference of observed and fitted values, which is a further generalization. Compared to GLM, the GMBM models are more general in use in how to measure the bias and how to assign the weights.

Next, we compare A.4 for GLM models and A.6 for GMBM models. With k=1, p=1, and q=2-c, we can see that:

- when c=0, the normal GLM model is the same as the GMBM Model 4 in Equation (4);
- when c=1, the Poisson GLM model is the same as the GMBM Model 1 in Equation (1);
- when c=2, the Gamma GLM model is the same as the GMBM Model 6 in Equation (13);
- when c=3, the Inverse Gaussian GLM model is the same a the GMBM Model 7 in Equation (14).

The GMBM bias function can also be related to a previous minimum bias work from the perspective of "balance principle". Let's rewrite A.4:

$$\sum_j w_{i,j}^P \mu_{i,j}^{q-k} (r_{i,j}^k - \mu_{i,j}^k) = 0 \Rightarrow \sum_j w_{i,j}^P \mu_{i,j}^{q-k} r_{i,j}^k = \sum_j w_{i,j}^P \mu_{i,j}^{q-k} \mu_{i,j}^k \qquad \text{(A.7).}$$

When p=1, q=1, and k=1, A.7 is

$$\sum_j w_{i,j} r_{i,j} = \sum_j w_{i,j} \mu_{i,j} \qquad \text{(A.8).}$$

This is the balance principle by Feldblum and Brosius (2002): "the sum of indicated pure premiums = the sum of the observed loss cost". GMBM further expands in A.7 the balance principle to the "generalized balance principle", that is, "the sum of weighted functions of indicated pure premiums = the sum of weighted functions of the observed loss costs".

1.2 Link Functions of GMBM

In GLMs, the purpose of the link function is to establish the relationship between the predicted value, which is the linear combination of GLM coefficients, and the response variable, such as severity, frequency or pure premium in our typical ratemaking applications. The link functions determine whether the model is multiplicative, additive, or other.

For example, if a GLM with a log link function has coefficients a_i and b_j, then the predicted value is $\hat{Y} = Intercept + a_i + b_j$. The link function then links the predicted value, $\hat{Y}$, to the indicated value, $\mu_{i,j}$, for the response variable:

$$\hat{Y} = \log(\mu_{i,j}) \Rightarrow \mu_{i,j} = e^{Intercept} e^{a_i + b_j} \Rightarrow e^{Intercept} x_i y_j \qquad \text{(A.9).}$$

Thus, GLM with a log link function results in a multiplicative model. On the other hand, if the link function is a power function, then

$$\hat{Y} = (\mu_{i,j})^k \Rightarrow \mu_{i,j} = (\text{int}\,ercept + a_i + b_i)^{1/k} \qquad \text{(A.10).}$$

For this case, the GLM model is neither an additive model nor a multiplicative model.

In GMBM, the link function links the iterative coefficients to the classification relativity. Let's assume the power link function to be $A_i = x_i^k$ and $B_j = y_j^k$ for a GMBM model. Then, the relativity is estimated through the following iterative weighted average procedure:

$x_i^k = \dfrac{\sum_j w_{i,j}^p r_{i,j}^k y_j^{q-k}}{\sum_j w_{i,j}^p y_j^q}$ and $y_j^k = \dfrac{\sum_i w_{i,j}^p r_{i,j}^k x_i^{q-k}}{\sum_i w_{i,j}^p x_i^q}$, and the GMBM coefficients are directly estimated by inverting the link function:

$$A_i = x_i^k \Rightarrow x_i = A_i^{1/k} = \left(\frac{\sum_j w_{i,j}^p r_{i,j}^k y_j^{q-k}}{\sum_j w_{i,j}^p y_j^q} \right)^{1/k} \qquad \text{(A.11).}$$

If we assume a log link function, the iterative process of GMBM is:

$$A_i = \log(x_i) = \frac{\sum_j w_{i,j}^p y_j^q \log(r_{i,j} / y_j)}{\sum_j w_{i,j}^p y_j^q} \qquad \text{(A.12).}$$

and the GMBM coefficients are estimated by

$$x_i = \exp(A_i) = \exp\left(\frac{\sum_j w_{i,j}^p y_j^q \log(r_{i,j} / y_j)}{\sum_j w_{i,j}^p y_j^q} \right) \qquad \text{(A.13).}$$

One difference between GLM and GMBM for the link function is that in GLM, the link function determines the type of model: log link implies a multiplicative model and identity link implies an additive model. This is not the case for GMBM. A multiplicative GMBM model, for example, could have a log, power, exponential, or any other formats, in theory, for the link function.

Finally, we will show how the GMBM link functions are related to the underlying distribution assumptions in the maximum likelihood procedure. For example, let $L_{i,j}$ be the loss of classification i and j, and B be the base rate. Assuming loss follows a normal distribution, $L_{i,j} = w_{i,j}Br_{i,j}$ and $L_{i,j} \sim N(w_{i,j}B\mu_{i,j}, \sigma^2)$. Following the method in Brown (1980), the density function is:

$$\begin{aligned} f(L_{i,j}) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(L_{i,j} - w_{i,j}B\mu_{i,j})^2 \right) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(w_{i,j}Br_{i,j} - w_{i,j}Bx_i y_j)^2 \right) \end{aligned} \qquad \text{(A.14).}$$

Minimizing the log likelihood function, we will get:

$$\begin{aligned} l &= \sum_i \sum_j \{-\ln(\sigma\sqrt{2\pi})\} - \frac{B^2}{2\sigma^2} \sum_i \sum_j w_{i,j}^2 (r_{i,j} - x_i y_j)^2 \\ \frac{\partial l}{\partial x_i} &= 0 \Rightarrow x_i = \frac{\sum_j w_{i,j}^2 r_{i,j} y_j}{\sum_j w_{i,j}^2 y_j^2} \end{aligned} \qquad \text{(A.15).}$$

A.15 is the GMBM model with p=2, q=2, and k=1.

Similarly, if we assume the loss square $L_{i,j}^2$ follows a normal distribution, then $L_{i,j}^2 \sim N((w_{i,j}BR_{i,j})^2, \sigma^2)$, where $R_{i,j}^2 = E(r_{i,j}^2)$ and $R_{i,j}^2 = X_i^2 * Y_j^2$
The density function is:

$$\begin{aligned} f(L_{i,j}^2) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(L_{i,j}^2 - w_{i,j}^2 B^2 R_{i,j}^2)^2 \right) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(w_{i,j}^2 B^2 r_{i,j}^2 - w_{i,j}^2 B^2 X_i^2 Y_j^2)^2 \right) \end{aligned} \qquad \text{(A.16).}$$

Minimizing the log likelihood function is:

$$l=\sum_i\sum_j\{-\ln(\sigma\sqrt{2\pi})\}-\frac{B^4}{2\sigma^2}\sum_i\sum_j w_{i,j}^4(r_{i,j}^2-X_i^2Y_j^2)^2$$

$$\frac{\partial l}{\partial X_i}=0\Rightarrow\frac{\partial l}{\partial X_i^2}\frac{dX_i^2}{dX_i}=0\Rightarrow\frac{\partial l}{\partial X_i^2}=0 \qquad \text{(A.17).[6]}$$

$$\Rightarrow X_i^2=\frac{\sum_j w_{i,j}^4 r_{i,j}^2 Y_j^2}{\sum_j w_{i,j}^4 Y_j^4}\Rightarrow X_i=\left(\frac{\sum_j w_{i,j}^4 r_{i,j}^2 Y_j^2}{\sum_j w_{i,j}^4 Y_j^4}\right)^{1/2}$$

A.17 is the GMBM model with p=4, q=4, and k=2.

Generalizing the maximum likelihood estimation of classification relativity as in A.16 and A.17, we assume that the link function of loss ($L_{i,j}^{k_0}$) follows the same distribution as $L_{i,j}$ in the 2-parameter model with $p=p_0$ and $q=q_0$, then the maximum likelihood estimation of relativities becomes 3-parameter GMBM:

$$\hat{X}_i=\left(\frac{\sum_j w_{i,j}^{p_0k_0}r_{i,j}^{k_0}Y_j^{(q_0-1)k_0}}{\sum_j w_{i,j}^{p_0k_0}Y_j^{q_0k_0}}\right)^{1/k_0} \qquad \text{(A.18).[7]}$$

This is the GMBM with $k=k_0$, $p=p_0k_0$, and $q=q_0k_0$.

1.3 GMBM based on Deviance Functions

Before Mildenhall (1999), the difference between a measure of bias and a measure of deviance was not discussed. Mildenhall pointed out that the bias could be positive or negative and should be proportional to the difference between the predicted value and the observed value. On the other hand, the deviance is always positive, and is used to measure the goodness of fit. Mathematically, the GLM classification relativities could be obtained by solving zero bias function or minimizing the deviance function. Mildenhall showed that the GLM deviance function with a linear bias function has the following format:

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[6] Because $X_iY_j=R_{i,j}\neq\mu_{i,j}=x_iy_j$, X_i and Y_j are biased estimates of x_i and y_j.

[7] Following the derivation for normal distribution, $w_{i,j}^{k_0}$, $r_{i,j}^{k_0}$, $X_i^{k_0}$, and $Y_j^{k_0}$ take the place of $w_{i,j}$, $r_{i,j}$, x_i, and y_j in the likelihood function and solution.

$$d(r,\mu) = 2w\int_{\mu}^{r} \frac{r-t}{V(t)}dt \qquad \text{(A.19).}$$

As given in A.4 and A.6, all of the GMBM models have a generalized linear bias function, $\sum_j w_{i,j}^P y_j^{q-k} r_{i,j}^k - x_i^k \sum_j w_{i,j}^p y_j^q$, and the parameters could be solved by the iterative procedure in minimizing the bias function. However, Mildenhall discussed that the deviance functions may not correspond to linear bias functions (i.e. $d(r,\mu) = w|r-\mu|$). One popular criterion to measure the goodness of fit is the Chi-square statistic, which could also be used as deviance function:

$$d(r,\mu) = \sum_i \sum_j w_{i,j} \frac{(r_{i,j} - \mu_{i,j})^2}{\mu_{i,j}} \qquad \text{(A.20).}$$

Feldblum and Brosius (2002) showed that Model 5 (the "χ-squared" model) could be derived by minimizing the deviance function in A.20.

Following the same generalization work previously for the balance principle, the deviance functions could be also generalized through the weights and the measurement of bias:

$$d(r,\mu) = \sum_i \sum_j w_{i,j}^p \mu_{i,j}^q \frac{(r_{i,j}^k - \mu_{i,j}^k)^2}{\mu_{i,j}^k} \qquad \text{(A.21).}$$

So, A.21 is our proposed generalized deviance function for GMBM, and the classification relativities can be solved through an iterative process by minimizing the generalized deviance function. The iterative procedure starts with setting the first order derivative of A.21 to zero:

$$\frac{\partial d(r,\mu)}{\partial x_i} = 0 \Rightarrow$$

$$x_i^k (q+k) \sum_j w_{i,j}^p y_j^{q+k} + x_i^{-k} (q-k) \sum_j w_{i,j}^p r_{i,j}^{2k} y_j^{q-k} = 2\text{q} \sum_j w_{i,j}^p r_{i,j}^k y_j^q \qquad \text{(A.22).}$$

Because A.22 (or its bias function) does not follow a linear format, the iterative process may not have a simple iterative weighted average formula. Only for certain special values for q, the relativities can be solved by the conventional minimum bias format:

- When q=0,

$$\hat{x}_i = \left(\frac{\sum_j w_{i,j}^p r_{i,j}^{2k} y_j^{-k}}{\sum_j w_{i,j}^p y_j^k} \right)^{\frac{1}{2k}} \qquad \text{(A.23).}$$

A special case of A.23 is the "χ-squared' model when k=1.

- When q=k,

$$\hat{x}_i = \left(\frac{\sum_j w_{i,j}^p r_{i,j}^k y_j^k}{\sum_j w_{i,j}^p y_j^{2k}} \right)^{\frac{1}{k}} \qquad \text{(A.24).}$$

- When q=-k,

$$\begin{aligned} \hat{x}_i &= \left(\frac{\sum_j w_{i,j}^p r_{i,j}^{2k} y_j^{-2k}}{\sum_j w_{i,j}^p r_{i,j}^k y_j^{-k}} \right)^{\frac{1}{k}} \\ &= \left(\sum_j \frac{w_{i,j}^p r_{i,j}^k y_j^{-k}}{\sum_j w_{i,j}^p r_{i,j}^k y_j^{-k}} (r_{i,j}^k / y_j^k) \right)^{\frac{1}{k}} \end{aligned} \qquad \text{(A.25).}$$

In A.25, $w_{i,j}^p r_{i,j}^k y_j^{-k}$ is used as the weight. Compared to $w_{i,j}^p y_j^q$ in Section 2, A.25 adds observed relativity into the weight.

1.4. Additive GMBM Models

Following the same notations given in the text, the expected cost for classification cell (i,j) with an additive model should be equal to the sum of x_i and y_j:

$$E(r_{i,j}) = \mu_{i,j} = x_i + y_j \qquad \text{(A.26).}$$

Thus,

$$\begin{aligned} \hat{x}_{i,j} &= r_{i,j} - y_j, \quad j = 1, \quad 2, \quad to \quad n \\ \hat{y}_{j,i} &= r_{i,j} - x_i, \quad i = 1, \quad 2, \quad to \quad m, \end{aligned} \qquad \text{(A.27).}$$

In GMBM multiplicative models, we use the relativity-adjusted claim number, $w_{i,j}^p x_i^q$ and $w_{i,j}^p y_j^q$, as the weighting function and introduce the power link function. However, the weighting functions and the link functions cannot be applied in an additive process.

For the additive GMBM models, we are limited to the following one-parameter model using $w_{i,j}^p$ as the weight:

$$\hat{x}_i = \sum_j \frac{w_{i,j}^p}{\sum_j w_{i,j}^p} \hat{x}_{i,j} = \frac{\sum_j w_{i,j}^p (r_{i,j} - y_j)}{\sum_j w_{i,j}^p} \qquad \text{(A.28).}$$

When p=1, it is the model introduced by Bailey (1963) or the "Balance Principle" model in Feldblum and Brosius (2002). Mildenhall (1999) also proved that it is equivalent to additive normal GLM model. When p=2, it is the ML additive normal model introduced by Brown (1988). When p=0, it is the least squares model by Feldblum and Brosius (2002). There is no further generalization for the additive models with additional parameters or link functions.

We can also extend the model through the deviance function. Because of its additive feature, we can only generalize the "χ-squared" deviance function through the weights:

$$d(r, \mu) = \sum_i \sum_j w_{i,j}^p \frac{(r_{i,j} - \mu_{i,j})^2}{\mu_{i,j}} \qquad \text{(A.29).}$$

Using the first order condition, we solve the numerical results for A.29 with Newton's method as follows:

$$\Delta x_i = \frac{\sum_j w_{i,j}^p \left(\frac{r_{i,j}}{x_i + y_j} \right)^2 - \sum_j w_{i,j}^p}{2 \sum_j w_{i,j}^p \left(\frac{r_{i,j}}{x_i + y_j} \right)^2 \left(\frac{1}{x_i + y_j} \right)} \qquad \text{(A.30).}$$

When p=1, it is the additive model introduced by Bailey and Simon (1960) or the "χ-squared" model by Feldblum and Brosius (2002).

Appendix 1.5: Calculation Efficiency of GMBM

Mildenhall (1999) discussed that one advantage of GLMs compared to minimum bias model is the calculation efficiency because GLMs do not require an iterative process in estimating the parameters. He showed that the additive minimum bias model by Bailey (1963), or GMBM with p=1, does not converge even after 50 iterations[8].

In this study, we propose a simple but improved iterative algorithm. With the improved algorithm we find that GMBM is not necessarily inefficient in numerical calculations. Applying the improved algorithm to the same severity data studied previously in Section 3 and by Mildenhall, most of the GMBM models reported in Appendix 2 converge within 10 iterations. The algorithm could be implemented in major statistical languages, such as SAS, Splus, and Matlab. If the data is not large, it can also be conducted in EXCEL with straightforward VBA codes.

To illustrate the calculation efficiency of GMBM, we first use multiplicative Gamma model because of its popularity in fitting severity data. As for additive model, we will show the iterative process for GMBM with p=1. We will show that the minimum bias model with the proposed algorithm converges in 5 iterations compared to the 50 iterations in Mildenhall (1999). The corresponding GLM models are also solved numerically for comparison using the iterative reweighted least square method[9].

Before showing the detailed results, we would like to discuss the iterative procedure. For the GMBM Gamma model: $x_i = \frac{\sum_j w_{i,j} r_{i,j} y_j^{-1}}{\sum_j w_{i,j}}$. In the iterative process, it is better to include as much updated information as possible. Let t be the t-th iteration step,

[8] For detailed information, please refer to Exhibit 5 of Mildenhall (1999).

[9] Iterative reweighted least square method is the most commonly used algorithm for GLMs. The major statistical languages, such SAS and Splus, apply this method to solve GLMs numerically. The discussion of the algorithm is beyond the scope of this paper.

$$x_{i,t} = \frac{\sum_j w_{i,j} r_{i,j} y_{j,t-1}^{-1}}{\sum_j w_{i,j}}$$
$$y_{j,t} = \frac{\sum_j w_{i,j} r_{i,j} x_{i,t}^{-1}}{\sum_j w_{i,j}} \qquad \text{(A.31)}$$

In A.31, $x_{i,t}$ is calculated based on the latest iterative result for $y_{j,t-1}$. When calculating $y_{j,t}$, the latest result of $x_{i,t}$ is used so that $y_{j,t} = \frac{\sum_j w_{i,j} r_{i,j} x_{i,t}^{-1}}{\sum_j w_{i,j}}$. Similarly, for a model that has 3 rating factors, the iterative process is:

$$x_{i,t} = \frac{\sum_j w_{i,j,k} r_{i,j,k} y_{j,t-1}^{-1} z_{k,t-1}^{-1}}{\sum_j w_{i,j,k}}$$
$$y_{j,t} = \frac{\sum_j w_{i,j,k} r_{i,j,k} x_{i,t}^{-1} z_{k,t-1}^{-1}}{\sum_j w_{i,j,k}} \qquad \text{(A.32)}$$
$$z_{k,t} = \frac{\sum_j w_{i,j,k} r_{i,j,k} x_{i,t}^{-1} y_{j,t}^{-1}}{\sum_j w_{i,j,k}}$$

In GLMs, a specific classification is usually selected as the base (i.e. age 60+ and pleasure). Mildenhall (1999) also follows this method. We suggest using the average severity as the base in GMBM numerical analysis. Using a specific classification as the base, the numerical value of base varies in each iteration, and the factor of base class has to be forced as one. These may cause additional iterations to solve the functions.

Another well-known issue in numerical analysis is how to set the starting point for t=0. Using average severity as the base, the average factor is one for multiplicative models and average discount is zero for the additive models. Therefore, in this study, we chose the starting values of $x_{i,0}$ and $y_{j,0}$ to be one for the multiplicative models and zero for the additive models.

For both GLMs and GMBM models, the iteration will stop if the errors of factors (or GLM coefficients) are less than 10^{-7} and the errors of dollar values are less than 10^{-4}.

The numerical iterative results are report as follows:

- Table 15 shows the multiplicative factors for the Gamma GMBM using average severity as the base.
- Table 16 transfers those factors using the classification age 60+ and pleasure as the base.
- Table 17 reports the iterative process for the coefficients of GLM with the Gamma distribution and log link.
- Table 18 transfers those coefficients to the multiplicative factors of Gamma GLM.
- Table 19 lists the additive factors for the GMBM with p=1.
- Table 20 shows the additive dollar values for the GMBM with p=1 and uses the classification age 60+ and pleasure as the base.
- Table 21 reports the coefficients of additive normal GLM.

From Tables 15-18, the multiplicative Gamma GMBM converges in 4 iterations. This is as fast as the corresponding GLM model. As expected, the numerical solutions between the two models are identical, and the solutions are also identical to the previous results given in Table 2 for k=1, p=1, and q=0.

Tables 19 and 20 report the iterative process for the GMBM additive model with p=1. Mildenhall (1999) used this model as an example to show that minimum bias model is not efficient. He showed that the minimum bias model still did not converge to the GLM results and the dollar values at the 50^{th} iteration are about 2 cents different from those by GLM. However, using our algorithm, the GMBM model converges completely in 5 iterations with solutions identical to GLM results.

Appendix 2: Numerical Results of GMBM

Table 1: The Age and Vehicle-use Relativities for Selected GMBMs with k=0.5

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
0	-2.5	1.336	1.231	1.228	1.159	0.885	1.026	1.035	1.000	1.744	1.288	1.093	1.000
0	**-2**	**1.348**	**1.227**	**1.219**	**1.156**	**0.884**	**1.023**	**1.032**	**1.000**	**1.749**	**1.282**	**1.090**	**1.000**
0	-1.5	1.365	1.223	1.210	1.152	0.882	1.019	1.029	1.000	1.755	1.275	1.089	1.000
0	-1	1.386	1.216	1.199	1.148	0.879	1.017	1.026	1.000	1.763	1.269	1.088	1.000
0	-0.5	1.414	1.209	1.188	1.144	0.874	1.014	1.023	1.000	1.773	1.263	1.087	1.000
0	0	1.448	1.201	1.175	1.139	0.869	1.012	1.020	1.000	1.788	1.256	1.086	1.000
0	0.5	1.490	1.191	1.162	1.134	0.862	1.011	1.017	1.000	1.809	1.248	1.086	1.000
0	1	1.543	1.180	1.147	1.129	0.854	1.010	1.014	1.000	1.839	1.238	1.086	1.000
0	1.5	1.611	1.167	1.130	1.122	0.844	1.010	1.011	1.000	1.886	1.224	1.086	1.000
0	2	1.700	1.151	1.110	1.115	0.831	1.011	1.008	1.000	1.959	1.206	1.086	1.000
0.5	-1.5	1.313	1.270	1.217	1.156	0.912	1.014	1.028	1.000	1.665	1.280	1.063	1.000
0.5	**-1**	**1.320**	**1.263**	**1.209**	**1.154**	**0.909**	**1.012**	**1.026**	**1.000**	**1.665**	**1.277**	**1.062**	**1.000**
0.5	-0.5	1.328	1.256	1.200	1.151	0.905	1.010	1.024	1.000	1.665	1.274	1.062	1.000
0.5	0	1.340	1.247	1.191	1.148	0.900	1.009	1.021	1.000	1.667	1.272	1.062	1.000
0.5	0.5	1.356	1.237	1.181	1.144	0.894	1.007	1.019	1.000	1.669	1.269	1.062	1.000
0.5	1	1.376	1.226	1.170	1.140	0.887	1.006	1.017	1.000	1.673	1.266	1.062	1.000
0.5	1.5	1.401	1.215	1.158	1.136	0.879	1.005	1.014	1.000	1.680	1.263	1.062	1.000
0.5	2	1.433	1.202	1.145	1.131	0.870	1.004	1.012	1.000	1.691	1.260	1.063	1.000
1	-1	1.291	1.313	1.220	1.160	0.938	1.010	1.026	1.000	1.645	1.267	1.043	1.000
1	**-0.5**	**1.291**	**1.305**	**1.213**	**1.158**	**0.934**	**1.008**	**1.024**	**1.000**	**1.643**	**1.266**	**1.043**	**1.000**
1	0	1.291	1.297	1.206	1.156	0.929	1.007	1.022	1.000	1.642	1.265	1.043	1.000
1	0.5	1.294	1.287	1.198	1.154	0.924	1.005	1.021	1.000	1.640	1.264	1.043	1.000
1	1	1.298	1.276	1.190	1.152	0.918	1.004	1.019	1.000	1.639	1.263	1.043	1.000
1	1.5	1.305	1.265	1.181	1.149	0.911	1.004	1.017	1.000	1.638	1.262	1.042	1.000
1	2	1.314	1.253	1.171	1.146	0.904	1.003	1.015	1.000	1.638	1.261	1.042	1.000

Table 1: The Age and Vehicle-use Relativities for Selected GMBMs with k=0.5, continued

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1.5	0	1.281	1.343	1.218	1.162	0.953	1.006	1.024	1.000	1.643	1.253	1.030	1.000
1.5	0.5	1.277	1.334	1.211	1.161	0.948	1.005	1.022	1.000	1.641	1.252	1.030	1.000
1.5	**1**	**1.274**	**1.324**	**1.205**	**1.159**	**0.943**	**1.003**	**1.021**	**1.000**	**1.639**	**1.252**	**1.030**	**1.000**
1.5	1.5	1.272	1.313	1.197	1.157	0.937	1.002	1.019	1.000	1.638	1.252	1.030	1.000
1.5	2	1.271	1.301	1.190	1.155	0.930	1.002	1.018	1.000	1.636	1.252	1.029	1.000
2	0	1.288	1.385	1.228	1.167	0.972	1.007	1.025	1.000	1.651	1.239	1.022	1.000
2	0.5	1.282	1.376	1.223	1.166	0.968	1.006	1.024	1.000	1.650	1.240	1.022	1.000
2	1	1.277	1.367	1.217	1.165	0.963	1.004	1.023	1.000	1.648	1.240	1.022	1.000
2	1.5	1.272	1.357	1.211	1.163	0.958	1.003	1.022	1.000	1.646	1.240	1.022	1.000
2	2	1.266	1.346	1.205	1.162	0.952	1.002	1.020	1.000	1.644	1.241	1.021	1.000
2	2.5	1.262	1.335	1.198	1.160	0.945	1.001	1.019	1.000	1.643	1.241	1.021	1.000
2	**3**	**1.258**	**1.322**	**1.190**	**1.157**	**0.938**	**1.000**	**1.018**	**1.000**	**1.641**	**1.241**	**1.020**	**1.000**
2	3.5	1.255	1.308	1.182	1.155	0.930	0.999	1.016	1.000	1.640	1.241	1.020	1.000

Table 2: The Age and Vehicle-use Relativities for Selected GMBMs with k=1

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
0	-2.5	1.353	1.235	1.229	1.159	0.887	1.027	1.036	1.000	1.757	1.290	1.093	1.000
0	**-2**	**1.369**	**1.231**	**1.221**	**1.156**	**0.886**	**1.023**	**1.032**	**1.000**	**1.762**	**1.284**	**1.090**	**1.000**
0	-1.5	1.389	1.226	1.212	1.153	0.884	1.020	1.029	1.000	1.768	1.278	1.089	1.000
0	-1	1.414	1.220	1.201	1.149	0.881	1.017	1.026	1.000	1.776	1.272	1.088	1.000
0	-0.5	1.445	1.213	1.190	1.144	0.877	1.014	1.023	1.000	1.787	1.267	1.087	1.000
0	0	1.483	1.204	1.178	1.140	0.872	1.012	1.020	1.000	1.801	1.260	1.087	1.000
0	0.5	1.528	1.195	1.165	1.135	0.865	1.010	1.017	1.000	1.823	1.253	1.087	1.000
0	1	1.582	1.184	1.150	1.129	0.858	1.009	1.014	1.000	1.854	1.244	1.087	1.000
0	1.5	1.649	1.171	1.134	1.123	0.848	1.009	1.011	1.000	1.899	1.233	1.087	1.000
0	2	1.731	1.156	1.116	1.116	0.836	1.010	1.008	1.000	1.967	1.217	1.087	1.000
0.5	-1.5	1.330	1.275	1.218	1.156	0.914	1.014	1.028	1.000	1.670	1.280	1.062	1.000
0.5	**-1**	**1.340**	**1.268**	**1.210**	**1.153**	**0.911**	**1.012**	**1.026**	**1.000**	**1.670**	**1.277**	**1.062**	**1.000**
0.5	-0.5	1.352	1.260	1.201	1.150	0.906	1.010	1.023	1.000	1.671	1.274	1.061	1.000
0.5	0	1.368	1.250	1.191	1.147	0.901	1.009	1.021	1.000	1.673	1.272	1.061	1.000
0.5	0.5	1.388	1.240	1.181	1.144	0.895	1.007	1.019	1.000	1.676	1.270	1.062	1.000
0.5	1	1.412	1.230	1.170	1.140	0.889	1.006	1.016	1.000	1.681	1.267	1.062	1.000
0.5	1.5	1.442	1.218	1.158	1.135	0.881	1.005	1.014	1.000	1.689	1.265	1.063	1.000
0.5	2	1.479	1.205	1.146	1.130	0.872	1.004	1.011	1.000	1.702	1.262	1.064	1.000
1	-1	1.303	1.318	1.220	1.159	0.939	1.010	1.026	1.000	1.647	1.266	1.042	1.000
1	**-0.5**	**1.304**	**1.310**	**1.213**	**1.158**	**0.935**	**1.008**	**1.024**	**1.000**	**1.646**	**1.265**	**1.042**	**1.000**
1	0	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
1	0.5	1.312	1.291	1.198	1.154	0.925	1.006	1.020	1.000	1.643	1.263	1.042	1.000
1	1	1.319	1.280	1.190	1.151	0.919	1.005	1.019	1.000	1.642	1.262	1.042	1.000
1	1.5	1.329	1.269	1.181	1.148	0.912	1.004	1.017	1.000	1.641	1.261	1.042	1.000
1	2	1.343	1.256	1.171	1.145	0.905	1.003	1.015	1.000	1.641	1.260	1.042	1.000
1.5	0	1.290	1.348	1.218	1.162	0.955	1.006	1.023	1.000	1.645	1.251	1.029	1.000
1.5	0.5	1.287	1.339	1.212	1.160	0.950	1.005	1.022	1.000	1.643	1.251	1.029	1.000
1.5	**1**	**1.286**	**1.328**	**1.205**	**1.159**	**0.944**	**1.004**	**1.021**	**1.000**	**1.641**	**1.251**	**1.029**	**1.000**

Table 2: The Age and Vehicle-use Relativities for Selected GMBMs with k=1, continued

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1.5	1.5	1.286	1.317	1.198	1.157	0.938	1.003	1.019	1.000	1.639	1.251	1.029	1.000
1.5	2	1.287	1.305	1.190	1.154	0.931	1.002	1.018	1.000	1.638	1.251	1.029	1.000
2	0	1.294	1.389	1.228	1.167	0.973	1.007	1.025	1.000	1.653	1.238	1.021	1.000
2	0.5	1.289	1.380	1.223	1.166	0.969	1.006	1.024	1.000	1.652	1.238	1.021	1.000
2	1	1.284	1.371	1.217	1.165	0.964	1.004	1.023	1.000	1.650	1.239	1.021	1.000
2	1.5	1.280	1.361	1.211	1.163	0.959	1.003	1.021	1.000	1.648	1.239	1.020	1.000
2	2	1.276	1.351	1.205	1.161	0.953	1.002	1.020	1.000	1.646	1.239	1.020	1.000
2	2.5	1.272	1.339	1.198	1.159	0.946	1.001	1.019	1.000	1.644	1.240	1.020	1.000
2	3	1.270	1.326	1.190	1.157	0.939	1.000	1.017	1.000	1.643	1.240	1.020	1.000
2	**3.5**	**1.268**	**1.312**	**1.182**	**1.154**	**0.931**	**0.999**	**1.016**	**1.000**	**1.641**	**1.240**	**1.019**	**1.000**
2	4	1.269	1.298	1.173	1.151	0.922	0.999	1.014	1.000	1.640	1.240	1.018	1.000

Table 3: The Age and Vehicle-use Relativities for Selected GMBMs with k=1.5

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
0	-2	1.391	1.235	1.223	1.156	0.888	1.024	1.033	1.000	1.774	1.286	1.090	1.000
0	**-1.5**	**1.415**	**1.230**	**1.214**	**1.153**	**0.886**	**1.020**	**1.030**	**1.000**	**1.780**	**1.281**	**1.089**	**1.000**
0	-1	1.444	1.224	1.203	1.149	0.884	1.017	1.026	1.000	1.788	1.275	1.088	1.000
0	-0.5	1.478	1.217	1.192	1.145	0.880	1.014	1.023	1.000	1.798	1.270	1.087	1.000
0	0	1.519	1.208	1.181	1.140	0.875	1.012	1.020	1.000	1.813	1.264	1.087	1.000
0	0.5	1.566	1.199	1.168	1.135	0.869	1.010	1.017	1.000	1.834	1.258	1.087	1.000
0	1	1.621	1.188	1.154	1.130	0.861	1.009	1.014	1.000	1.865	1.250	1.088	1.000
0	1.5	1.685	1.176	1.138	1.124	0.852	1.009	1.011	1.000	1.908	1.241	1.088	1.000
0	2	1.762	1.162	1.121	1.118	0.842	1.009	1.008	1.000	1.972	1.228	1.089	1.000
0.5	-1.5	1.350	1.279	1.218	1.156	0.916	1.015	1.028	1.000	1.675	1.279	1.061	1.000
0.5	**-1**	**1.362**	**1.272**	**1.210**	**1.153**	**0.912**	**1.012**	**1.025**	**1.000**	**1.675**	**1.277**	**1.061**	**1.000**
0.5	-0.5	1.378	1.264	1.202	1.150	0.908	1.010	1.023	1.000	1.677	1.274	1.061	1.000
0.5	0	1.398	1.254	1.192	1.147	0.903	1.009	1.021	1.000	1.679	1.272	1.061	1.000
0.5	0.5	1.422	1.244	1.182	1.143	0.897	1.007	1.018	1.000	1.682	1.270	1.061	1.000
0.5	1	1.451	1.233	1.171	1.139	0.890	1.006	1.016	1.000	1.688	1.268	1.062	1.000
0.5	1.5	1.485	1.221	1.159	1.134	0.883	1.005	1.013	1.000	1.698	1.266	1.063	1.000
0.5	2	1.526	1.208	1.146	1.129	0.874	1.004	1.011	1.000	1.712	1.264	1.065	1.000
1	-1	1.315	1.323	1.220	1.159	0.941	1.010	1.025	1.000	1.650	1.264	1.041	1.000
1	**-0.5**	**1.319**	**1.315**	**1.214**	**1.157**	**0.937**	**1.008**	**1.024**	**1.000**	**1.648**	**1.264**	**1.041**	**1.000**
1	0	1.325	1.306	1.206	1.155	0.932	1.007	1.022	1.000	1.647	1.263	1.041	1.000
1	0.5	1.333	1.295	1.198	1.153	0.927	1.006	1.020	1.000	1.645	1.262	1.041	1.000
1	1	1.344	1.284	1.190	1.150	0.921	1.005	1.018	1.000	1.644	1.261	1.041	1.000
1	1.5	1.357	1.273	1.181	1.148	0.914	1.004	1.016	1.000	1.644	1.261	1.041	1.000
1	2	1.375	1.260	1.171	1.144	0.906	1.003	1.015	1.000	1.644	1.260	1.042	1.000
1.5	0	1.299	1.353	1.218	1.162	0.956	1.006	1.023	1.000	1.647	1.250	1.028	1.000
1.5	0.5	1.299	1.343	1.212	1.160	0.951	1.005	1.022	1.000	1.645	1.250	1.028	1.000
1.5	**1**	**1.299**	**1.333**	**1.205**	**1.158**	**0.945**	**1.004**	**1.020**	**1.000**	**1.643**	**1.250**	**1.028**	**1.000**

Table 3: The Age and Vehicle-use Relativities for Selected GMBMs with k=1.5, continued

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1.5	1.5	1.301	1.322	1.198	1.156	0.939	1.003	1.019	1.000	1.641	1.250	1.028	1.000
1.5	2	1.305	1.310	1.190	1.154	0.933	1.002	1.017	1.000	1.639	1.250	1.028	1.000
2	0	1.300	1.393	1.228	1.167	0.974	1.007	1.025	1.000	1.655	1.236	1.019	1.000
2	0.5	1.295	1.384	1.223	1.166	0.970	1.005	1.024	1.000	1.654	1.237	1.019	1.000
2	1	1.292	1.375	1.217	1.164	0.965	1.004	1.022	1.000	1.652	1.237	1.019	1.000
2	1.5	1.288	1.366	1.211	1.163	0.960	1.003	1.021	1.000	1.650	1.238	1.019	1.000
2	2	1.286	1.355	1.205	1.161	0.954	1.002	1.020	1.000	1.648	1.238	1.019	1.000
2	2.5	1.284	1.343	1.198	1.159	0.947	1.001	1.018	1.000	1.646	1.239	1.019	1.000
2	3	1.283	1.330	1.190	1.156	0.940	1.000	1.017	1.000	1.644	1.239	1.019	1.000
2	**3.5**	**1.284**	**1.316**	**1.182**	**1.153**	**0.932**	**0.999**	**1.016**	**1.000**	**1.642**	**1.239**	**1.018**	**1.000**
2	4	1.286	1.302	1.174	1.150	0.924	0.999	1.014	1.000	1.641	1.239	1.018	1.000

Table 4: The Age and Vehicle-use Relativities for Selected GMBMs with k=2

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DriveLong	DriveShort	Pleasure
0	-2	1.414	1.240	1.225	1.157	0.891	1.024	1.033	1.000	1.786	1.289	1.090	1.000
0	**-1.5**	**1.442**	**1.235**	**1.216**	**1.153**	**0.889**	**1.021**	**1.030**	**1.000**	**1.791**	**1.283**	**1.089**	**1.000**
0	-1	1.475	1.229	1.206	1.149	0.886	1.017	1.027	1.000	1.798	1.278	1.088	1.000
0	-0.5	1.513	1.221	1.195	1.145	0.882	1.015	1.023	1.000	1.808	1.274	1.087	1.000
0	0	1.556	1.213	1.183	1.141	0.878	1.012	1.020	1.000	1.822	1.268	1.087	1.000
0	0.5	1.605	1.203	1.171	1.136	0.872	1.010	1.017	1.000	1.843	1.263	1.088	1.000
0	1	1.659	1.193	1.157	1.131	0.865	1.009	1.014	1.000	1.873	1.256	1.088	1.000
0	1.5	1.721	1.181	1.143	1.125	0.857	1.008	1.011	1.000	1.914	1.248	1.089	1.000
0	2	1.791	1.168	1.127	1.119	0.847	1.008	1.008	1.000	1.973	1.238	1.090	1.000
0.5	-1.5	1.371	1.284	1.219	1.155	0.918	1.015	1.028	1.000	1.680	1.279	1.060	1.000
0.5	**-1**	**1.387**	**1.277**	**1.211**	**1.153**	**0.914**	**1.013**	**1.025**	**1.000**	**1.681**	**1.276**	**1.060**	**1.000**
0.5	-0.5	1.407	1.268	1.202	1.150	0.910	1.010	1.023	1.000	1.682	1.274	1.060	1.000
0.5	0	1.431	1.258	1.193	1.146	0.905	1.009	1.021	1.000	1.684	1.273	1.060	1.000
0.5	0.5	1.459	1.248	1.183	1.143	0.899	1.007	1.018	1.000	1.689	1.271	1.061	1.000
0.5	1	1.492	1.237	1.172	1.139	0.892	1.006	1.016	1.000	1.695	1.269	1.062	1.000
0.5	1.5	1.530	1.224	1.160	1.134	0.884	1.004	1.013	1.000	1.705	1.268	1.064	1.000
0.5	2	1.573	1.211	1.147	1.129	0.876	1.003	1.011	1.000	1.721	1.266	1.065	1.000
1	-1	1.330	1.328	1.220	1.159	0.943	1.010	1.025	1.000	1.652	1.263	1.039	1.000
1	**-0.5**	**1.336**	**1.320**	**1.214**	**1.157**	**0.939**	**1.008**	**1.024**	**1.000**	**1.651**	**1.262**	**1.039**	**1.000**
1	0	1.345	1.310	1.206	1.155	0.934	1.007	1.022	1.000	1.649	1.262	1.040	1.000
1	0.5	1.356	1.300	1.199	1.153	0.928	1.006	1.020	1.000	1.648	1.261	1.040	1.000
1	1	1.371	1.289	1.190	1.150	0.922	1.005	1.018	1.000	1.647	1.261	1.040	1.000
1	1.5	1.389	1.277	1.181	1.147	0.915	1.004	1.016	1.000	1.647	1.260	1.041	1.000
1	2	1.410	1.264	1.172	1.144	0.908	1.003	1.014	1.000	1.648	1.260	1.041	1.000
1.5	0	1.310	1.357	1.218	1.161	0.957	1.006	1.023	1.000	1.649	1.248	1.026	1.000
1.5	0.5	1.311	1.348	1.212	1.160	0.952	1.005	1.022	1.000	1.647	1.248	1.027	1.000
1.5	**1**	**1.314**	**1.337**	**1.205**	**1.158**	**0.947**	**1.004**	**1.020**	**1.000**	**1.645**	**1.249**	**1.027**	**1.000**
1.5	1.5	1.319	1.326	1.198	1.156	0.941	1.003	1.019	1.000	1.643	1.249	1.027	1.000

Table 4: The Age and Vehicle-use Relativities for Selected GMBMs with k=2, continued

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DriveLong	DriveShort	Pleasure
1.5	2	1.326	1.314	1.190	1.153	0.934	1.002	1.017	1.000	1.641	1.249	1.027	1.000
2	0	1.306	1.396	1.228	1.167	0.975	1.007	1.024	1.000	1.657	1.235	1.018	1.000
2	0.5	1.303	1.388	1.223	1.165	0.971	1.005	1.023	1.000	1.655	1.236	1.018	1.000
2	1	1.300	1.379	1.217	1.164	0.966	1.004	1.022	1.000	1.654	1.236	1.018	1.000
2	1.5	1.298	1.369	1.212	1.162	0.961	1.003	1.021	1.000	1.652	1.237	1.018	1.000
2	2	1.297	1.359	1.205	1.160	0.955	1.001	1.020	1.000	1.650	1.237	1.018	1.000
2	2.5	1.297	1.347	1.198	1.158	0.948	1.001	1.018	1.000	1.648	1.238	1.018	1.000
2	3	1.298	1.334	1.191	1.156	0.941	1.000	1.017	1.000	1.646	1.238	1.018	1.000
2	**3.5**	**1.301**	**1.321**	**1.183**	**1.153**	**0.934**	**0.999**	**1.015**	**1.000**	**1.644**	**1.238**	**1.018**	**1.000**
2	4	1.307	1.306	1.174	1.150	0.925	0.999	1.014	1.000	1.642	1.238	1.017	1.000

Table 5: The Age and Vehicle-use Relativities for Selected GMBMs with k=2.5

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DriveLong	DriveShort	Pleasure
0	-2	1.439	1.244	1.227	1.157	0.893	1.025	1.033	1.000	1.797	1.290	1.090	1.000
0	**-1.5**	**1.471**	**1.239**	**1.218**	**1.154**	**0.891**	**1.021**	**1.030**	**1.000**	**1.801**	**1.285**	**1.088**	**1.000**
0	-1	1.507	1.233	1.208	1.150	0.888	1.018	1.027	1.000	1.807	1.281	1.087	1.000
0	-0.5	1.548	1.226	1.197	1.146	0.885	1.015	1.024	1.000	1.816	1.276	1.087	1.000
0	0	1.594	1.217	1.186	1.141	0.880	1.012	1.020	1.000	1.829	1.272	1.087	1.000
0	0.5	1.643	1.208	1.174	1.137	0.875	1.010	1.017	1.000	1.849	1.267	1.088	1.000
0	1	1.697	1.198	1.161	1.132	0.869	1.009	1.014	1.000	1.877	1.261	1.089	1.000
0	1.5	1.755	1.186	1.147	1.126	0.861	1.008	1.011	1.000	1.917	1.255	1.089	1.000
0	2	1.819	1.174	1.132	1.121	0.852	1.008	1.008	1.000	1.972	1.246	1.090	1.000
0.5	-2	1.379	1.296	1.227	1.157	0.923	1.017	1.030	1.000	1.684	1.280	1.059	1.000
0.5	**-1.5**	**1.395**	**1.289**	**1.219**	**1.155**	**0.920**	**1.015**	**1.028**	**1.000**	**1.685**	**1.278**	**1.059**	**1.000**
0.5	-1	1.415	1.281	1.212	1.152	0.916	1.013	1.025	1.000	1.686	1.276	1.059	1.000
0.5	-0.5	1.439	1.273	1.203	1.149	0.912	1.011	1.023	1.000	1.687	1.274	1.059	1.000
0.5	0	1.467	1.263	1.194	1.146	0.907	1.009	1.020	1.000	1.690	1.273	1.060	1.000
0.5	0.5	1.499	1.252	1.184	1.142	0.901	1.007	1.018	1.000	1.694	1.271	1.061	1.000
0.5	1	1.535	1.241	1.173	1.138	0.894	1.006	1.015	1.000	1.701	1.270	1.062	1.000
0.5	1.5	1.576	1.228	1.161	1.133	0.886	1.004	1.013	1.000	1.712	1.269	1.064	1.000
0.5	2	1.621	1.215	1.149	1.128	0.878	1.003	1.010	1.000	1.728	1.268	1.066	1.000
1	-1	1.346	1.333	1.221	1.159	0.944	1.010	1.025	1.000	1.655	1.262	1.038	1.000
1	**-0.5**	**1.355**	**1.324**	**1.214**	**1.157**	**0.940**	**1.008**	**1.023**	**1.000**	**1.653**	**1.261**	**1.038**	**1.000**
1	0	1.367	1.315	1.207	1.155	0.935	1.007	1.022	1.000	1.652	1.261	1.038	1.000
1	0.5	1.382	1.304	1.199	1.152	0.930	1.006	1.020	1.000	1.651	1.260	1.039	1.000
1	1	1.401	1.293	1.191	1.149	0.924	1.005	1.018	1.000	1.650	1.260	1.040	1.000
1	1.5	1.423	1.281	1.181	1.146	0.917	1.004	1.016	1.000	1.650	1.260	1.040	1.000
1	2	1.449	1.268	1.172	1.143	0.909	1.003	1.014	1.000	1.652	1.259	1.041	1.000
1.5	0	1.322	1.361	1.218	1.161	0.958	1.006	1.023	1.000	1.651	1.247	1.025	1.000
1.5	0.5	1.325	1.352	1.212	1.159	0.954	1.005	1.021	1.000	1.649	1.247	1.025	1.000
1.5	1	1.331	1.342	1.205	1.157	0.948	1.003	1.020	1.000	1.647	1.247	1.026	1.000

Table 5: The Age and Vehicle-use Relativities for Selected GMBMs with k=2.5, continued

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DriveLong	DriveShort	Pleasure
1.5	**1.5**	**1.338**	**1.330**	**1.198**	**1.155**	**0.942**	**1.003**	**1.018**	**1.000**	**1.645**	**1.248**	**1.026**	**1.000**
1.5	2	1.348	1.318	1.190	1.153	0.935	1.002	1.017	1.000	1.643	1.248	1.026	1.000
2	0	1.312	1.400	1.228	1.166	0.976	1.007	1.024	1.000	1.659	1.234	1.017	1.000
2	0.5	1.310	1.392	1.223	1.165	0.972	1.005	1.023	1.000	1.657	1.235	1.017	1.000
2	1	1.309	1.383	1.217	1.164	0.967	1.004	1.022	1.000	1.655	1.235	1.017	1.000
2	1.5	1.308	1.373	1.212	1.162	0.962	1.002	1.021	1.000	1.653	1.236	1.017	1.000
2	2	1.309	1.363	1.205	1.160	0.956	1.001	1.019	1.000	1.651	1.236	1.017	1.000
2	2.5	1.311	1.351	1.198	1.158	0.950	1.000	1.018	1.000	1.649	1.237	1.017	1.000
2	3	1.315	1.338	1.191	1.155	0.942	1.000	1.016	1.000	1.647	1.237	1.017	1.000
2	**3.5**	**1.321**	**1.325**	**1.183**	**1.152**	**0.935**	**0.999**	**1.015**	**1.000**	**1.645**	**1.237**	**1.017**	**1.000**
2	4	1.329	1.310	1.174	1.149	0.927	0.999	1.013	1.000	1.644	1.238	1.017	1.000

Table 6: The Age and Vehicle-use Relativities for Selected GMBMs with k=3

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DriveLong	DriveShort	Pleasure
0	-2	1.466	1.249	1.228	1.158	0.895	1.025	1.034	1.000	1.806	1.292	1.089	1.000
0	**-1.5**	**1.501**	**1.244**	**1.220**	**1.154**	**0.893**	**1.022**	**1.030**	**1.000**	**1.808**	**1.287**	**1.088**	**1.000**
0	-1	1.541	1.238	1.210	1.150	0.891	1.018	1.027	1.000	1.813	1.283	1.087	1.000
0	-0.5	1.585	1.230	1.200	1.146	0.887	1.015	1.024	1.000	1.820	1.279	1.087	1.000
0	0	1.631	1.222	1.189	1.142	0.883	1.013	1.020	1.000	1.833	1.275	1.087	1.000
0	0.5	1.681	1.212	1.177	1.137	0.878	1.010	1.017	1.000	1.852	1.271	1.088	1.000
0	1	1.733	1.202	1.165	1.132	0.872	1.009	1.014	1.000	1.879	1.266	1.089	1.000
0	1.5	1.787	1.191	1.152	1.127	0.865	1.007	1.011	1.000	1.917	1.260	1.090	1.000
0	2	1.845	1.179	1.137	1.122	0.857	1.007	1.008	1.000	1.968	1.253	1.091	1.000
0.5	-2	1.401	1.301	1.227	1.157	0.925	1.017	1.030	1.000	1.689	1.280	1.058	1.000
0.5	**-1.5**	**1.421**	**1.294**	**1.220**	**1.155**	**0.922**	**1.015**	**1.027**	**1.000**	**1.689**	**1.278**	**1.057**	**1.000**
0.5	-1	1.445	1.286	1.212	1.152	0.918	1.013	1.025	1.000	1.690	1.276	1.058	1.000
0.5	-0.5	1.472	1.277	1.204	1.149	0.914	1.011	1.023	1.000	1.691	1.274	1.058	1.000
0.5	0	1.504	1.267	1.194	1.146	0.909	1.009	1.020	1.000	1.694	1.273	1.059	1.000
0.5	0.5	1.540	1.256	1.184	1.142	0.903	1.007	1.018	1.000	1.699	1.272	1.060	1.000
0.5	1	1.578	1.245	1.174	1.138	0.896	1.006	1.015	1.000	1.706	1.271	1.062	1.000
0.5	1.5	1.621	1.232	1.162	1.133	0.889	1.004	1.013	1.000	1.717	1.270	1.064	1.000
0.5	2	1.667	1.219	1.150	1.128	0.880	1.003	1.010	1.000	1.734	1.270	1.066	1.000
1	-1	1.364	1.338	1.221	1.158	0.946	1.010	1.025	1.000	1.657	1.260	1.036	1.000
1	-0.5	1.377	1.329	1.214	1.156	0.942	1.008	1.023	1.000	1.656	1.260	1.037	1.000
1	0	1.392	1.319	1.207	1.154	0.937	1.007	1.021	1.000	1.655	1.260	1.037	1.000
1	0.5	1.411	1.309	1.199	1.152	0.931	1.006	1.019	1.000	1.654	1.259	1.038	1.000
1	1	1.434	1.297	1.191	1.149	0.925	1.005	1.017	1.000	1.653	1.259	1.039	1.000
1	1.5	1.460	1.285	1.182	1.146	0.918	1.004	1.015	1.000	1.654	1.259	1.040	1.000
1	2	1.490	1.272	1.172	1.142	0.911	1.003	1.013	1.000	1.655	1.259	1.041	1.000
1.5	0	1.335	1.366	1.218	1.161	0.960	1.006	1.023	1.000	1.652	1.246	1.024	1.000
1.5	0.5	1.341	1.356	1.212	1.159	0.955	1.005	1.021	1.000	1.650	1.246	1.024	1.000
1.5	1	1.349	1.346	1.205	1.157	0.949	1.003	1.020	1.000	1.648	1.246	1.024	1.000

Table 6: The Age and Vehicle-use Relativities for Selected GMBMs with k=3, continued

p	q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DriveLong	DriveShort	Pleasure
1.5	**1.5**	**1.360**	**1.334**	**1.198**	**1.155**	**0.943**	**1.002**	**1.018**	**1.000**	**1.647**	**1.247**	**1.025**	**1.000**
1.5	2	1.374	1.322	1.190	1.152	0.936	1.002	1.016	1.000	1.645	1.247	1.025	1.000
2	0	1.319	1.404	1.228	1.166	0.977	1.007	1.024	1.000	1.661	1.233	1.016	1.000
2	0.5	1.318	1.396	1.223	1.165	0.973	1.005	1.023	1.000	1.659	1.233	1.016	1.000
2	1	1.318	1.387	1.217	1.163	0.968	1.004	1.022	1.000	1.657	1.234	1.016	1.000
2	1.5	1.320	1.377	1.212	1.162	0.963	1.002	1.020	1.000	1.655	1.235	1.016	1.000
2	2	1.322	1.366	1.205	1.160	0.957	1.001	1.019	1.000	1.653	1.235	1.016	1.000
2	2.5	1.327	1.355	1.198	1.157	0.951	1.000	1.018	1.000	1.651	1.236	1.016	1.000
2	3	1.334	1.342	1.191	1.155	0.944	1.000	1.016	1.000	1.649	1.236	1.016	1.000
2	**3.5**	**1.343**	**1.329**	**1.183**	**1.152**	**0.936**	**0.999**	**1.014**	**1.000**	**1.647**	**1.237**	**1.016**	**1.000**
2	4	1.354	1.314	1.175	1.149	0.928	0.999	1.013	1.000	1.645	1.237	1.016	1.000

Table 7: The Age and Vehicle-use Relativities for Selected GLMs

Distribution	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
Inverse Gaussian	1.303	1.318	1.220	1.159	0.939	1.010	1.026	1.000	1.647	1.266	1.042	1.000
Gamma	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
Poisson	1.319	1.280	1.190	1.151	0.919	1.005	1.019	1.000	1.642	1.262	1.042	1.000
Normal	1.343	1.256	1.171	1.145	0.905	1.003	1.015	1.000	1.641	1.260	1.042	1.000

Table 8: The Age and Vehicle-use Relativities for Selected GMBMs with k=1 and p=1

q	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
-1	1.303	1.318	1.220	1.159	0.939	1.010	1.026	1.000	1.647	1.266	1.042	1.000
-0.5	1.304	1.310	1.213	1.158	0.935	1.008	1.024	1.000	1.646	1.265	1.042	1.000
0	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
0.5	1.312	1.291	1.198	1.154	0.925	1.006	1.020	1.000	1.643	1.263	1.042	1.000
1	1.319	1.280	1.190	1.151	0.919	1.005	1.019	1.000	1.642	1.262	1.042	1.000
1.5	1.329	1.269	1.181	1.148	0.912	1.004	1.017	1.000	1.641	1.261	1.042	1.000
2	1.343	1.256	1.171	1.145	0.905	1.003	1.015	1.000	1.641	1.260	1.042	1.000

Table 9: The Model Performance of GMBM with K=0.5

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2.5	13.747	5.63%	1.262	4.1651
0	**-2**	**13.545**	**5.56%**	**1.255**	**4.1225**
0	-1.5	13.588	5.58%	1.260	4.1384
0	-1	13.770	5.65%	1.283	4.2024
0	-0.5	14.050	5.77%	1.328	4.3198
0	0	14.618	6.01%	1.411	4.5418
0	0.5	15.337	6.32%	1.559	4.8894
0	1	16.467	6.80%	1.832	5.4920
0	1.5	18.211	7.58%	2.382	6.5860
0	2	21.788	9.33%	3.642	8.9078
0.5	-1.5	11.593	4.67%	1.056	3.4992
0.5	**-1**	**11.584**	**4.69%**	**1.052**	**3.4912**
0.5	-0.5	11.735	4.76%	1.052	3.5131
0.5	0	11.972	4.88%	1.056	3.5559
0.5	0.5	12.261	5.01%	1.068	3.6180
0.5	1	12.594	5.17%	1.089	3.7036
0.5	1.5	12.974	5.35%	1.126	3.8221
0.5	2	13.410	5.57%	1.187	3.9900
1	-1	10.696	4.18%	1.046	3.3453
1	**-0.5**	**10.739**	**4.22%**	**1.040**	**3.3420**
1	0	10.851	4.28%	1.035	3.3506
1	0.5	11.018	4.37%	1.031	3.3699
1	1	11.208	4.47%	1.029	3.3967
1	1.5	11.422	4.59%	1.032	3.4330
1	2	11.658	4.72%	1.040	3.4818
1.5	0	10.376	3.96%	1.077	3.3435
1.5	0.5	10.452	4.01%	1.067	3.3391
1.5	**1**	**10.531**	**4.06%**	**1.057**	**3.3359**

1.5	1.5	10.657	4.14%	1.048	3.3419
1.5	2	10.820	4.23%	1.042	3.3572
2	0	10.310	3.84%	1.155	3.4515
2	0.5	10.374	3.88%	1.140	3.4389
2	1	10.443	3.93%	1.124	3.4262
2	1.5	10.518	3.98%	1.108	3.4143
2	2	10.600	4.04%	1.093	3.4040
2	2.5	10.688	4.10%	1.080	3.3969
2	**3**	**10.781**	**4.16%**	**1.069**	**3.3943**
2	3.5	10.877	4.23%	1.062	3.3982

Table 10: The Model Performance of GMBM with K=1

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2.5	13.873	5.65%	1.276	4.2068
0	**-2**	**13.631**	**5.56%**	**1.268**	**4.1575**
0	-1.5	13.614	5.56%	1.274	4.1642
0	-1	13.753	5.61%	1.296	4.2221
0	-0.5	14.018	5.72%	1.343	4.3381
0	0	14.588	5.96%	1.426	4.5612
0	0.5	15.344	6.27%	1.573	4.9128
0	1	16.480	6.74%	1.838	5.5038
0	1.5	18.011	7.43%	2.353	6.5095
0	2	21.263	9.00%	3.460	8.5778
0.5	-1.5	11.563	4.64%	1.049	3.4825
0.5	**-1**	**11.554**	**4.66%**	**1.045**	**3.4746**
0.5	-0.5	11.722	4.74%	1.045	3.4998
0.5	0	11.977	4.86%	1.050	3.5470
0.5	0.5	12.274	5.00%	1.064	3.6134
0.5	1	12.613	5.16%	1.089	3.7054
0.5	1.5	12.998	5.34%	1.130	3.8331
0.5	2	13.434	5.56%	1.199	4.0139

1	-1	10.669	4.15%	1.043	3.3358
1	**-0.5**	**10.716**	**4.19%**	**1.036**	**3.3313**
1	0	10.826	4.26%	1.029	3.3376
1	0.5	10.996	4.35%	1.024	3.3556
1	1	11.190	4.45%	1.022	3.3815
1	1.5	11.408	4.57%	1.024	3.4178
1	2	11.664	4.70%	1.032	3.4696
1.5	0	10.350	3.94%	1.079	3.3423
1.5	0.5	10.424	3.99%	1.067	3.3353
1.5	**1**	**10.503**	**4.04%**	**1.055**	**3.3295**
1.5	1.5	10.637	4.12%	1.045	3.3340
1.5	2	10.804	4.21%	1.037	3.3471
2	0	10.290	3.82%	1.164	3.4614
2	0.5	10.353	3.86%	1.148	3.4471
2	1	10.422	3.91%	1.130	3.4323
2	1.5	10.495	3.96%	1.113	3.4179
2	2	10.577	4.01%	1.096	3.4051
2	2.5	10.663	4.07%	1.081	3.3947
2	3	10.753	4.14%	1.068	3.3883
2	**3.5**	**10.846**	**4.21%**	**1.058**	**3.3881**
2	4	10.983	4.30%	1.054	3.4028

Table 11: The Model Performance of GMBM with K=1.5

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2	13.717	5.56%	1.288	4.2028
0	**-1.5**	**13.638**	**5.53%**	**1.293**	**4.1988**
0	-1	13.760	5.58%	1.315	4.2535
0	-0.5	14.017	5.68%	1.361	4.3678
0	0	14.549	5.90%	1.444	4.5833
0	0.5	15.379	6.23%	1.587	4.9399
0	1	16.443	6.67%	1.839	5.4984
0	1.5	17.737	7.25%	2.310	6.4013
0	2	20.678	8.65%	3.273	8.2267
0.5	-1.5	11.539	4.61%	1.044	3.4712
0.5	**-1**	**11.559**	**4.64%**	**1.041**	**3.4683**
0.5	-0.5	11.718	4.72%	1.042	3.4938
0.5	0	11.980	4.84%	1.049	3.5449
0.5	0.5	12.283	4.98%	1.065	3.6168
0.5	1	12.627	5.14%	1.094	3.7161
0.5	1.5	13.014	5.33%	1.141	3.8531
0.5	2	13.445	5.54%	1.217	4.0455
1	-1	10.642	4.12%	1.041	3.3290
1	**-0.5**	**10.692**	**4.17%**	**1.033**	**3.3231**
1	0	10.798	4.23%	1.025	3.3272
1	0.5	10.971	4.33%	1.019	3.3443
1	1	11.169	4.43%	1.017	3.3700
1	1.5	11.423	4.56%	1.019	3.4119
1	2	11.710	4.71%	1.028	3.4699
1.5	0	10.326	3.91%	1.083	3.3435
1.5	0.5	10.400	3.96%	1.069	3.3341
1.5	**1**	**10.478**	**4.01%**	**1.056**	**3.3257**
1.5	1.5	10.617	4.10%	1.043	3.3283

1.5	2	10.787	4.19%	1.034	3.3393
2	0	10.271	3.80%	1.174	3.4725
2	0.5	10.333	3.84%	1.156	3.4566
2	1	10.404	3.89%	1.138	3.4404
2	1.5	10.480	3.94%	1.119	3.4241
2	2	10.559	3.99%	1.100	3.4085
2	2.5	10.643	4.05%	1.083	3.3950
2	3	10.730	4.12%	1.068	3.3851
2	**3.5**	**10.821**	**4.18%**	**1.056**	**3.3810**
2	4	10.970	4.28%	1.050	3.3938

Table 12: The Model Performance of GMBM with K=2

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2	13.813	5.56%	1.313	4.2583
0	**-1.5**	**13.667**	**5.51%**	**1.316**	**4.2415**
0	-1	13.803	5.56%	1.337	4.2964
0	-0.5	14.015	5.65%	1.382	4.4014
0	0	14.564	5.86%	1.463	4.6157
0	0.5	15.373	6.18%	1.600	4.9588
0	1	16.366	6.58%	1.835	5.4794
0	1.5	17.553	7.11%	2.260	6.2987
0	2	20.067	8.30%	3.091	7.8758
0.5	-1.5	11.538	4.59%	1.043	3.4683
0.5	**-1**	**11.564**	**4.62%**	**1.040**	**3.4681**
0.5	-0.5	11.715	4.70%	1.043	3.4948
0.5	0	11.983	4.82%	1.052	3.5508
0.5	0.5	12.290	4.96%	1.072	3.6292
0.5	1	12.637	5.12%	1.105	3.7364
0.5	1.5	13.023	5.30%	1.157	3.8821
0.5	2	13.448	5.51%	1.240	4.0832
1	-1	10.614	4.10%	1.042	3.3248
<u>1</u>	**<u>-0.5</u>**	**<u>10.666</u>**	**<u>4.14%</u>**	**<u>1.032</u>**	**<u>3.3178</u>**
1	0	10.767	4.20%	1.024	3.3199
1	0.5	10.957	4.30%	1.017	3.3390
1	1	11.192	4.42%	1.015	3.3705
1	1.5	11.458	4.56%	1.018	3.4156
1	2	11.756	4.71%	1.029	3.4783
1.5	0	10.304	3.89%	1.087	3.3470
1.5	0.5	10.376	3.94%	1.072	3.3350
1.5	**1**	**10.454**	**3.99%**	**1.057**	**3.3241**
1.5	1.5	10.596	4.07%	1.043	3.3250

1.5	2	10.768	4.17%	1.032	3.3342
2	0	10.257	3.78%	1.184	3.4855
2	0.5	10.320	3.82%	1.166	3.4683
2	1	10.390	3.87%	1.146	3.4503
2	1.5	10.464	3.92%	1.125	3.4317
2	2	10.542	3.97%	1.105	3.4135
2	2.5	10.624	4.03%	1.086	3.3970
2	3	10.709	4.09%	1.069	3.3838
2	**3.5**	**10.797**	**4.16%**	**1.056**	**3.3762**
2	4	10.955	4.26%	1.047	3.3873

Table 13: The Model Performance of GMBM with K=2.5

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2	13.931	5.57%	1.342	4.3236
0	**-1.5**	**13.770**	**5.52%**	**1.343**	**4.3003**
0	-1	13.840	5.54%	1.362	4.3415
0	-0.5	14.040	5.62%	1.405	4.4413
0	0	14.575	5.82%	1.482	4.6474
0	0.5	15.333	6.12%	1.611	4.9699
0	1	16.259	6.49%	1.827	5.4508
0	1.5	17.354	6.97%	2.208	6.1896
0	2	19.455	7.95%	2.922	7.5395
0.5	-2	11.570	4.57%	1.050	3.4852
0.5	**-1.5**	**11.536**	**4.57%**	**1.045**	**3.4715**
0.5	-1	11.571	4.60%	1.044	3.4749
0.5	-0.5	11.712	4.68%	1.048	3.5037
0.5	0	11.986	4.80%	1.061	3.5655
0.5	0.5	12.298	4.94%	1.084	3.6510
0.5	1	12.647	5.10%	1.121	3.7658
0.5	1.5	13.030	5.28%	1.179	3.9188
0.5	2	13.477	5.49%	1.266	4.1300
1	-1	10.584	4.07%	1.044	3.3236
1	**-0.5**	**10.639**	**4.11%**	**1.034**	**3.3159**
1	0	10.761	4.18%	1.025	3.3207
1	0.5	10.974	4.29%	1.019	3.3435
1	1	11.218	4.41%	1.017	3.3779
1	1.5	11.495	4.55%	1.022	3.4273
1	2	11.804	4.70%	1.035	3.4961
1.5	0	10.288	3.87%	1.093	3.3531
1.5	0.5	10.353	3.91%	1.076	3.3381
1.5	1	10.430	3.97%	1.060	3.3252

1.5	**1.5**	**10.573**	**4.05%**	**1.045**	**3.3245**
1.5	2	10.748	4.15%	1.033	3.3326
2	0	10.252	3.76%	1.196	3.5009
2	0.5	10.308	3.80%	1.176	3.4811
2	1	10.377	3.85%	1.155	3.4613
2	1.5	10.449	3.90%	1.133	3.4406
2	2	10.526	3.96%	1.111	3.4200
2	2.5	10.605	4.01%	1.091	3.4008
2	3	10.688	4.07%	1.072	3.3847
2	**3.5**	**10.773**	**4.13%**	**1.057**	**3.3741**
2	4	10.940	4.24%	1.047	3.3841

Table 14: The Model Performance of GMBM with K=3

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2	14.081	5.59%	1.373	4.3971
0	**-1.5**	**13.870**	**5.52%**	**1.371**	**4.3603**
0	-1	13.884	5.53%	1.387	4.3882
0	-0.5	14.110	5.61%	1.427	4.4879
0	0	14.553	5.78%	1.500	4.6729
0	0.5	15.266	6.05%	1.621	4.9744
0	1	16.133	6.40%	1.818	5.4164
0	1.5	17.149	6.84%	2.156	6.0804
0	2	18.860	7.63%	2.769	7.2268
0.5	-2	11.563	4.55%	1.054	3.4916
0.5	**-1.5**	**11.537**	**4.55%**	**1.050**	**3.4813**
0.5	-1	11.581	4.58%	1.051	3.4892
0.5	-0.5	11.711	4.65%	1.058	3.5208
0.5	0	11.992	4.78%	1.074	3.5892
0.5	0.5	12.308	4.92%	1.101	3.6818
0.5	1	12.659	5.08%	1.143	3.8036
0.5	1.5	13.038	5.25%	1.204	3.9619
0.5	2	13.498	5.46%	1.293	4.1783
1	-1	10.560	4.04%	1.048	3.3269
1	**-0.5**	**10.645**	**4.10%**	**1.038**	**3.3236**
1	0	10.771	4.17%	1.029	3.3291
1	0.5	10.992	4.28%	1.024	3.3546
1	1	11.246	4.41%	1.024	3.3930
1	1.5	11.533	4.55%	1.031	3.4480
1	2	11.851	4.70%	1.048	3.5238
1.5	0	10.272	3.85%	1.100	3.3613
1.5	0.5	10.330	3.89%	1.082	3.3435
1.5	1	10.407	3.94%	1.065	3.3291

1.5	**1.5**	**10.557**	**4.03%**	**1.049**	**3.3285**
1.5	2	10.765	4.14%	1.037	3.3413
2	0	10.247	3.75%	1.207	3.5170
2	0.5	10.296	3.79%	1.186	3.4947
2	1	10.363	3.83%	1.164	3.4733
2	1.5	10.435	3.88%	1.141	3.4508
2	2	10.509	3.94%	1.118	3.4282
2	2.5	10.587	3.99%	1.096	3.4066
2	3	10.667	4.05%	1.076	3.3882
2	**3.5**	**10.750**	**4.11%**	**1.060**	**3.3753**
2	4	10.930	4.22%	1.049	3.3859

Appendix 3: Numerical Iterative Process of GMBM

Table 15: Numerical Iterations for Multiplicative Gamma GMBM Factors Using Average Severity as Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	241.46	1.203551	1.207636	1.154385	1.123670	0.890524	0.970979	0.953411	0.921830	1.393434	1.073693	0.887450	0.854173
2	241.46	1.239832	1.234406	1.144728	1.097253	0.883394	0.955745	0.969955	0.948637	1.398901	1.075486	0.886537	0.850980
3	241.46	1.240574	1.234754	1.144648	1.096890	0.883231	0.955539	0.970167	0.949079	1.398980	1.075513	0.886525	0.850929
4	241.46	1.240585	1.234759	1.144647	1.096885	0.883229	0.955536	0.970170	0.949086	1.398981	1.075513	0.886525	0.850928

Table 16: Numerical Iterations for Multiplicative Gamma GMBM Factors Using 60+ and Pleasure as the Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	190.126	1.306	1.310	1.252	1.219	0.966	1.053	1.034	1.000	1.631	1.257	1.039	1.000
2	194.924	1.307	1.301	1.207	1.157	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
3	195.003	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
4	195.004	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000

Table 17: Numerical Iterations for Multiplicative Gamma GLM Coefficients Using 60+ and Pleasure as the Base

Iteration	Intercept	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	5.2710091	0.2447301	0.2563505	0.1870363	0.1454066	-0.0752765	0.0065328	0.0224012	0	0.4944179	0.2360242	0.0430371	0
2	5.2729277	0.2683345	0.2629985	0.1872796	0.1447046	-0.0720638	0.0068159	0.0219726	0	0.497298	0.2342915	0.0411756	0
3	5.2730182	0.2678326	0.263122	0.1873523	0.1447304	-0.0719202	0.0067721	0.0219717	0	0.4971672	0.234231	0.0409872	0
4	5.2730202	0.2678398	0.2631314	0.1873525	0.14473	-0.0719157	0.0067735	0.0219717	0	0.4971718	0.2342254	0.040982	0

Table 18: Numerical Iterations for Multiplicative Gamma GLM Factors Using 60+ and Pleasure as the Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	194.612	1.277	1.292	1.206	1.157	0.927	1.007	1.023	1.000	1.640	1.266	1.044	1.000
2	194.986	1.308	1.301	1.206	1.156	0.930	1.007	1.022	1.000	1.644	1.264	1.042	1.000
3	195.004	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
4	195.004	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000

Table 19: Numerical Iterations for Additive Factors of Gamma GMBM Using Average Severity as the Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	241.46	1.203551	1.207636	1.154385	1.123670	0.890524	0.970979	0.953411	0.921830	0.393760	0.072094	-0.111752	-0.145133
2	241.46	1.247269	1.219244	1.138179	1.101277	0.875751	0.958654	0.972661	0.955564	0.398400	0.074096	-0.113076	-0.149284
3	241.46	1.248066	1.219512	1.137964	1.100911	0.875520	0.958409	0.972929	0.956185	0.398478	0.074130	-0.113097	-0.149360
4	241.46	1.248079	1.219517	1.137960	1.100904	0.875516	0.958405	0.972934	0.956196	0.398479	0.074131	-0.113097	-0.149361
5	241.46	1.248080	1.219517	1.137960	1.100904	0.875516	0.958405	0.972934	0.956196	0.398479	0.074131	-0.113097	-0.149361

Table 20: Numerical Iterations for Additive Dollar Values of Gamma GMBM Using Age 60+ and Pleasure as Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	187.5412	68.0244	69.0107	56.1527	48.7365	-7.5591	11.8675	7.6257	0.0000	130.1212	52.4515	8.0601	0.0000
2	194.6844	70.4351	63.6680	44.0942	35.1839	-19.2717	0.7462	4.1282	0.0000	132.2437	53.9373	8.7428	0.0000
3	194.8161	70.4774	63.5829	43.8922	34.9454	-19.4776	0.5369	4.0430	0.0000	132.2809	53.9639	8.7561	0.0000
4	194.8184	70.4781	63.5815	43.8887	34.9412	-19.4811	0.5333	4.0414	0.0000	132.2815	53.9644	8.7563	0.0000
5	194.8185	70.4781	63.5814	43.8887	34.9412	-19.4812	0.5332	4.0414	0.0000	132.2815	53.9644	8.7563	0.0000

Table 21: Additive Normal GLM Coefficients Using 60+ and Pleasure as the Base

GLM	Intercept	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
Coefficient	194.8185	70.4781	63.5814	43.8887	34.9412	-19.4812	0.5332	4.0414	0.0000	132.2815	53.9644	8.7563	0.0000

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Table 9: The Model Performance of GMBM with K=0.5

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2.5	13.747	5.63%	1.262	4.1651
0	**-2**	**13.545**	**5.56%**	**1.255**	**4.1225**
0	-1.5	13.588	5.58%	1.260	4.1384
0	-1	13.770	5.65%	1.283	4.2024
0	-0.5	14.050	5.77%	1.328	4.3198
0	0	14.618	6.01%	1.411	4.5418
0	0.5	15.337	6.32%	1.559	4.8894
0	1	16.467	6.80%	1.832	5.4920
0	1.5	18.211	7.58%	2.382	6.5860
0	2	21.788	9.33%	3.642	8.9078
0.5	-1.5	11.593	4.67%	1.056	3.4992
0.5	**-1**	**11.584**	**4.69%**	**1.052**	**3.4912**
0.5	-0.5	11.735	4.76%	1.052	3.5131
0.5	0	11.972	4.88%	1.056	3.5559
0.5	0.5	12.261	5.01%	1.068	3.6180
0.5	1	12.594	5.17%	1.089	3.7036
0.5	1.5	12.974	5.35%	1.126	3.8221
0.5	2	13.410	5.57%	1.187	3.9900
1	-1	10.696	4.18%	1.046	3.3453
1	**-0.5**	**10.739**	**4.22%**	**1.040**	**3.3420**
1	0	10.851	4.28%	1.035	3.3506
1	0.5	11.018	4.37%	1.031	3.3699
1	1	11.208	4.47%	1.029	3.3967
1	1.5	11.422	4.59%	1.032	3.4330
1	2	11.658	4.72%	1.040	3.4818
1.5	0	10.376	3.96%	1.077	3.3435
1.5	0.5	10.452	4.01%	1.067	3.3391
<u>1.5</u>	**<u>1</u>**	**<u>10.531</u>**	**<u>4.06%</u>**	**<u>1.057</u>**	**<u>3.3359</u>**
1.5	1.5	10.657	4.14%	1.048	3.3419
1.5	2	10.820	4.23%	1.042	3.3572
2	0	10.310	3.84%	1.155	3.4515
2	0.5	10.374	3.88%	1.140	3.4389
2	1	10.443	3.93%	1.124	3.4262
2	1.5	10.518	3.98%	1.108	3.4143
2	2	10.600	4.04%	1.093	3.4040
2	2.5	10.688	4.10%	1.080	3.3969
2	**3**	**10.781**	**4.16%**	**1.069**	**3.3943**
2	3.5	10.877	4.23%	1.062	3.3982

Table 10: The Model Performance of GMBM with K=1

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2.5	13.873	5.65%	1.276	4.2068
0	**-2**	**13.631**	**5.56%**	**1.268**	**4.1575**
0	-1.5	13.614	5.56%	1.274	4.1642
0	-1	13.753	5.61%	1.296	4.2221
0	-0.5	14.018	5.72%	1.343	4.3381
0	0	14.588	5.96%	1.426	4.5612
0	0.5	15.344	6.27%	1.573	4.9128
0	1	16.480	6.74%	1.838	5.5038
0	1.5	18.011	7.43%	2.353	6.5095
0	2	21.263	9.00%	3.460	8.5778
0.5	-1.5	11.563	4.64%	1.049	3.4825
0.5	**-1**	**11.554**	**4.66%**	**1.045**	**3.4746**
0.5	-0.5	11.722	4.74%	1.045	3.4998
0.5	0	11.977	4.86%	1.050	3.5470
0.5	0.5	12.274	5.00%	1.064	3.6134
0.5	1	12.613	5.16%	1.089	3.7054
0.5	1.5	12.998	5.34%	1.130	3.8331
0.5	2	13.434	5.56%	1.199	4.0139
1	-1	10.669	4.15%	1.043	3.3358
1	**-0.5**	**10.716**	**4.19%**	**1.036**	**3.3313**
1	0	10.826	4.26%	1.029	3.3376
1	0.5	10.996	4.35%	1.024	3.3556
1	1	11.190	4.45%	1.022	3.3815
1	1.5	11.408	4.57%	1.024	3.4178
1	2	11.664	4.70%	1.032	3.4696
1.5	0	10.350	3.94%	1.079	3.3423
1.5	0.5	10.424	3.99%	1.067	3.3353
1.5	**1**	**10.503**	**4.04%**	**1.055**	**3.3295**
1.5	1.5	10.637	4.12%	1.045	3.3340
1.5	2	10.804	4.21%	1.037	3.3471
2	0	10.290	3.82%	1.164	3.4614
2	0.5	10.353	3.86%	1.148	3.4471
2	1	10.422	3.91%	1.130	3.4323
2	1.5	10.495	3.96%	1.113	3.4179
2	2	10.577	4.01%	1.096	3.4051
2	2.5	10.663	4.07%	1.081	3.3947
2	3	10.753	4.14%	1.068	3.3883
2	**3.5**	**10.846**	**4.21%**	**1.058**	**3.3881**
2	4	10.983	4.30%	1.054	3.4028

Table 11: The Model Performance of GMBM with K=1.5

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2	13.717	5.56%	1.288	4.2028
0	**-1.5**	**13.638**	**5.53%**	**1.293**	**4.1988**
0	-1	13.760	5.58%	1.315	4.2535
0	-0.5	14.017	5.68%	1.361	4.3678
0	0	14.549	5.90%	1.444	4.5833
0	0.5	15.379	6.23%	1.587	4.9399
0	1	16.443	6.67%	1.839	5.4984
0	1.5	17.737	7.25%	2.310	6.4013
0	2	20.678	8.65%	3.273	8.2267
0.5	-1.5	11.539	4.61%	1.044	3.4712
0.5	**-1**	**11.559**	**4.64%**	**1.041**	**3.4683**
0.5	-0.5	11.718	4.72%	1.042	3.4938
0.5	0	11.980	4.84%	1.049	3.5449
0.5	0.5	12.283	4.98%	1.065	3.6168
0.5	1	12.627	5.14%	1.094	3.7161
0.5	1.5	13.014	5.33%	1.141	3.8531
0.5	2	13.445	5.54%	1.217	4.0455
1	-1	10.642	4.12%	1.041	3.3290
1	**-0.5**	**10.692**	**4.17%**	**1.033**	**3.3231**
1	0	10.798	4.23%	1.025	3.3272
1	0.5	10.971	4.33%	1.019	3.3443
1	1	11.169	4.43%	1.017	3.3700
1	1.5	11.423	4.56%	1.019	3.4119
1	2	11.710	4.71%	1.028	3.4699
1.5	0	10.326	3.91%	1.083	3.3435
1.5	0.5	10.400	3.96%	1.069	3.3341
1.5	**1**	**10.478**	**4.01%**	**1.056**	**3.3257**
1.5	1.5	10.617	4.10%	1.043	3.3283
1.5	2	10.787	4.19%	1.034	3.3393
2	0	10.271	3.80%	1.174	3.4725
2	0.5	10.333	3.84%	1.156	3.4566
2	1	10.404	3.89%	1.138	3.4404
2	1.5	10.480	3.94%	1.119	3.4241
2	2	10.559	3.99%	1.100	3.4085
2	2.5	10.643	4.05%	1.083	3.3950
2	3	10.730	4.12%	1.068	3.3851
2	**3.5**	**10.821**	**4.18%**	**1.056**	**3.3810**
2	4	10.970	4.28%	1.050	3.3938

Table 12: The Model Performance of GMBM with K=2

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2	13.813	5.56%	1.313	4.2583
0	**-1.5**	**13.667**	**5.51%**	**1.316**	**4.2415**
0	-1	13.803	5.56%	1.337	4.2964
0	-0.5	14.015	5.65%	1.382	4.4014
0	0	14.564	5.86%	1.463	4.6157
0	0.5	15.373	6.18%	1.600	4.9588
0	1	16.366	6.58%	1.835	5.4794
0	1.5	17.553	7.11%	2.260	6.2987
0	2	20.067	8.30%	3.091	7.8758
0.5	-1.5	11.538	4.59%	1.043	3.4683
0.5	**-1**	**11.564**	**4.62%**	**1.040**	**3.4681**
0.5	-0.5	11.715	4.70%	1.043	3.4948
0.5	0	11.983	4.82%	1.052	3.5508
0.5	0.5	12.290	4.96%	1.072	3.6292
0.5	1	12.637	5.12%	1.105	3.7364
0.5	1.5	13.023	5.30%	1.157	3.8821
0.5	2	13.448	5.51%	1.240	4.0832
1	-1	10.614	4.10%	1.042	3.3248
1	**-0.5**	**10.666**	**4.14%**	**1.032**	**3.3178**
1	0	10.767	4.20%	1.024	3.3199
1	0.5	10.957	4.30%	1.017	3.3390
1	1	11.192	4.42%	1.015	3.3705
1	1.5	11.458	4.56%	1.018	3.4156
1	2	11.756	4.71%	1.029	3.4783
1.5	0	10.304	3.89%	1.087	3.3470
1.5	0.5	10.376	3.94%	1.072	3.3350
1.5	**1**	**10.454**	**3.99%**	**1.057**	**3.3241**
1.5	1.5	10.596	4.07%	1.043	3.3250
1.5	2	10.768	4.17%	1.032	3.3342
2	0	10.257	3.78%	1.184	3.4855
2	0.5	10.320	3.82%	1.166	3.4683
2	1	10.390	3.87%	1.146	3.4503
2	1.5	10.464	3.92%	1.125	3.4317
2	2	10.542	3.97%	1.105	3.4135
2	2.5	10.624	4.03%	1.086	3.3970
2	3	10.709	4.09%	1.069	3.3838
2	**3.5**	**10.797**	**4.16%**	**1.056**	**3.3762**
2	4	10.955	4.26%	1.047	3.3873

Table 13: The Model Performance of GMBM with K=2.5

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2	13.931	5.57%	1.342	4.3236
0	**-1.5**	**13.770**	**5.52%**	**1.343**	**4.3003**
0	-1	13.840	5.54%	1.362	4.3415
0	-0.5	14.040	5.62%	1.405	4.4413
0	0	14.575	5.82%	1.482	4.6474
0	0.5	15.333	6.12%	1.611	4.9699
0	1	16.259	6.49%	1.827	5.4508
0	1.5	17.354	6.97%	2.208	6.1896
0	2	19.455	7.95%	2.922	7.5395
0.5	-2	11.570	4.57%	1.050	3.4852
0.5	**-1.5**	**11.536**	**4.57%**	**1.045**	**3.4715**
0.5	-1	11.571	4.60%	1.044	3.4749
0.5	-0.5	11.712	4.68%	1.048	3.5037
0.5	0	11.986	4.80%	1.061	3.5655
0.5	0.5	12.298	4.94%	1.084	3.6510
0.5	1	12.647	5.10%	1.121	3.7658
0.5	1.5	13.030	5.28%	1.179	3.9188
0.5	2	13.477	5.49%	1.266	4.1300
1	-1	10.584	4.07%	1.044	3.3236
1	**-0.5**	**10.639**	**4.11%**	**1.034**	**3.3159**
1	0	10.761	4.18%	1.025	3.3207
1	0.5	10.974	4.29%	1.019	3.3435
1	1	11.218	4.41%	1.017	3.3779
1	1.5	11.495	4.55%	1.022	3.4273
1	2	11.804	4.70%	1.035	3.4961
1.5	0	10.288	3.87%	1.093	3.3531
1.5	0.5	10.353	3.91%	1.076	3.3381
1.5	1	10.430	3.97%	1.060	3.3252
1.5	**1.5**	**10.573**	**4.05%**	**1.045**	**3.3245**
1.5	2	10.748	4.15%	1.033	3.3326
2	0	10.252	3.76%	1.196	3.5009
2	0.5	10.308	3.80%	1.176	3.4811
2	1	10.377	3.85%	1.155	3.4613
2	1.5	10.449	3.90%	1.133	3.4406
2	2	10.526	3.96%	1.111	3.4200
2	2.5	10.605	4.01%	1.091	3.4008
2	3	10.688	4.07%	1.072	3.3847
2	**3.5**	**10.773**	**4.13%**	**1.057**	**3.3741**
2	4	10.940	4.24%	1.047	3.3841

Table 14: The Model Performance of GMBM with K=3

p	q	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
0	-2	14.081	5.59%	1.373	4.3971
0	**-1.5**	**13.870**	**5.52%**	**1.371**	**4.3603**
0	-1	13.884	5.53%	1.387	4.3882
0	-0.5	14.110	5.61%	1.427	4.4879
0	0	14.553	5.78%	1.500	4.6729
0	0.5	15.266	6.05%	1.621	4.9744
0	1	16.133	6.40%	1.818	5.4164
0	1.5	17.149	6.84%	2.156	6.0804
0	2	18.860	7.63%	2.769	7.2268
0.5	-2	11.563	4.55%	1.054	3.4916
0.5	**-1.5**	**11.537**	**4.55%**	**1.050**	**3.4813**
0.5	-1	11.581	4.58%	1.051	3.4892
0.5	-0.5	11.711	4.65%	1.058	3.5208
0.5	0	11.992	4.78%	1.074	3.5892
0.5	0.5	12.308	4.92%	1.101	3.6818
0.5	1	12.659	5.08%	1.143	3.8036
0.5	1.5	13.038	5.25%	1.204	3.9619
0.5	2	13.498	5.46%	1.293	4.1783
1	-1	10.560	4.04%	1.048	3.3269
1	**-0.5**	**10.645**	**4.10%**	**1.038**	**3.3236**
1	0	10.771	4.17%	1.029	3.3291
1	0.5	10.992	4.28%	1.024	3.3546
1	1	11.246	4.41%	1.024	3.3930
1	1.5	11.533	4.55%	1.031	3.4480
1	2	11.851	4.70%	1.048	3.5238
1.5	0	10.272	3.85%	1.100	3.3613
1.5	0.5	10.330	3.89%	1.082	3.3435
1.5	1	10.407	3.94%	1.065	3.3291
1.5	**1.5**	**10.557**	**4.03%**	**1.049**	**3.3285**
1.5	2	10.765	4.14%	1.037	3.3413
2	0	10.247	3.75%	1.207	3.5170
2	0.5	10.296	3.79%	1.186	3.4947
2	1	10.363	3.83%	1.164	3.4733
2	1.5	10.435	3.88%	1.141	3.4508
2	2	10.509	3.94%	1.118	3.4282
2	2.5	10.587	3.99%	1.096	3.4066
2	3	10.667	4.05%	1.076	3.3882
2	**3.5**	**10.750**	**4.11%**	**1.060**	**3.3753**
2	4	10.930	4.22%	1.049	3.3859

Appendix 3: Numerical Iterative Process of GMBM

Table 15: Numerical Iterations for Multiplicative Gamma GMBM Factors Using Average Severity as Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	241.46	1.203551	1.207636	1.154385	1.123670	0.890524	0.970979	0.953411	0.921830	1.393434	1.073693	0.887450	0.854173
2	241.46	1.239832	1.234406	1.144728	1.097253	0.883394	0.955745	0.969955	0.948637	1.398901	1.075486	0.886537	0.850980
3	241.46	1.240574	1.234754	1.144648	1.096890	0.883231	0.955539	0.970167	0.949079	1.398980	1.075513	0.886525	0.850929
4	241.46	1.240585	1.234759	1.144647	1.096885	0.883229	0.955536	0.970170	0.949086	1.398981	1.075513	0.886525	0.850928

Table 16: Numerical Iterations for Multiplicative Gamma GMBM Factors Using 60+ and Pleasure as the Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	190.126	1.306	1.310	1.252	1.219	0.966	1.053	1.034	1.000	1.631	1.257	1.039	1.000
2	194.924	1.307	1.301	1.207	1.157	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
3	195.003	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
4	195.004	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000

Table 17: Numerical Iterations for Multiplicative Gamma GLM Coefficients Using 60+ and Pleasure as the Base

Iteration	Intercept	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	5.2710091	0.2447301	0.2563505	0.1870363	0.1454066	-0.0752765	0.0065328	0.0224012	0	0.4944179	0.2360242	0.0430371	0
2	5.2729277	0.2683345	0.2629985	0.1872796	0.1447046	-0.0720638	0.0068159	0.0219726	0	0.497298	0.2342915	0.0411756	0
3	5.2730182	0.2678326	0.263122	0.1873523	0.1447304	-0.0719202	0.0067721	0.0219717	0	0.4971672	0.234231	0.0409872	0
4	5.2730202	0.2678398	0.2631314	0.1873525	0.14473	-0.0719157	0.0067735	0.0219717	0	0.4971718	0.2342254	0.040982	0

Table 18: Numerical Iterations for Multiplicative Gamma GLM Factors Using 60+ and Pleasure as the Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	194.612	1.277	1.292	1.206	1.157	0.927	1.007	1.023	1.000	1.640	1.266	1.044	1.000
2	194.986	1.308	1.301	1.206	1.156	0.930	1.007	1.022	1.000	1.644	1.264	1.042	1.000
3	195.004	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
4	195.004	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000

Table 19: Numerical Iterations for Additive Factors of Gamma GMBM Using Average Severity as the Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	241.46	1.203551	1.207636	1.154385	1.123670	0.890524	0.970979	0.953411	0.921830	0.393760	0.072094	-0.111752	-0.145133
2	241.46	1.247269	1.219244	1.138179	1.101277	0.875751	0.958654	0.972661	0.955564	0.398400	0.074096	-0.113076	-0.149284
3	241.46	1.248066	1.219512	1.137964	1.100911	0.875520	0.958409	0.972929	0.956185	0.398478	0.074130	-0.113097	-0.149360
4	241.46	1.248079	1.219517	1.137960	1.100904	0.875516	0.958405	0.972934	0.956196	0.398479	0.074131	-0.113097	-0.149361
5	241.46	1.248080	1.219517	1.137960	1.100904	0.875516	0.958405	0.972934	0.956196	0.398479	0.074131	-0.113097	-0.149361

Table 20: Numerical Iterations for Additive Dollar Values of Gamma GMBM Using Age 60+ and Pleasure as Base

Iteration	Base	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
1	187.5412	68.0244	69.0107	56.1527	48.7365	-7.5591	11.8675	7.6257	0.0000	130.1212	52.4515	8.0601	0.0000
2	194.6844	70.4351	63.6680	44.0942	35.1839	-19.2717	0.7462	4.1282	0.0000	132.2437	53.9373	8.7428	0.0000
3	194.8161	70.4774	63.5829	43.8922	34.9454	-19.4776	0.5369	4.0430	0.0000	132.2809	53.9639	8.7561	0.0000
4	194.8184	70.4781	63.5815	43.8887	34.9412	-19.4811	0.5333	4.0414	0.0000	132.2815	53.9644	8.7563	0.0000
5	194.8185	70.4781	63.5814	43.8887	34.9412	-19.4812	0.5332	4.0414	0.0000	132.2815	53.9644	8.7563	0.0000

Table 21: Additive Normal GLM Coefficients Using 60+ and Pleasure as the Base

GLM	Intercept	17-20	21-24	25-29	30-34	35-39	40-49	50-59	60+	Business	DTW Long	DTW Short	Pleasure
Coefficient	194.8185	70.4781	63.5814	43.8887	34.9412	-19.4812	0.5332	4.0414	0.0000	132.2815	53.9644	8.7563	0.0000

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