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Spring 2005
Including the Reinsurance Call Papers

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NOTICE

The Casualty Actuarial Society *Forum*
Spring 2005 Edition
Including the 2005 Reinsurance Call Papers

To CAS Members:

This is the Spring 2005 Edition of the Casualty Actuarial Society *Forum*. It contains 10 Reinsurance Call Papers and two additional papers.

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The CAS *Forum* is edited by the CAS Committee for the Casualty Actuarial Society *Forum*. Members of the committee invite all interested persons to submit papers on topics of interest to the actuarial community. Articles need not be written by a member of the CAS, but the paper's content must be relevant to the interests of the CAS membership. Members of the Committee for the Casualty Actuarial Society *Forum* request that the following procedures be followed when submitting an article for publication in the *Forum*:

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5. Authors should format graphs, tables, exhibits, and text in solid black and white. Avoid using color or gray-shaded graphs, tables, or exhibits as the printing process will produce muddy and moire patterns.
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The CAS *Forum* is printed periodically based on the number of call paper programs and articles submitted. The committee publishes two to four editions during each calendar year.

All comments or questions may be directed to the Committee for the Casualty Actuarial Society *Forum*.

Sincerely,

Glenn M. Walker, CAS *Forum* Chairperson

The 2005 CAS Reinsurance Call Papers
Presented at the
2005 CAS Reinsurance Seminar
June 6-7, 2005
Fairmont Hamilton Princess
Hamilton, Bermuda

The Spring 2005 Edition of the CAS *Forum* is a cooperative effort between the Committee for the CAS *Forum* and Committee on Reinsurance Research.

The CAS Committee on Reinsurance Research present for discussion 10 papers prepared in response to their 2005 call for papers, entitled "Pricing Low-Frequency, High-Risk Exposures."

This *Forum* includes papers that will be discussed by the authors at the 2005 CAS Reinsurance Seminar, June 6-7, 2005, in Hamilton, Bermuda.

2005 CAS Reinsurance Call Papers

Additional Papers

On Optimal Reinsurance Arrangement

Yisheng Bu, Ph.D.

Abstract:

The purpose of this paper is to develop a theoretical framework within which the optimal reinsurance arrangement for catastrophic risks is explored and derived. In the model, it is assumed that the insurer values the stability of its underwriting results in purchasing reinsurance protection. The optimal solutions to the model are obtained through numerical simulation and intend to provide justifications and explanations for the profile of reinsurance purchase that has been observed in practice.

Keywords: Catastrophic Risk, Excess-of-Loss Reinsurance, Optimality, Contingent Capital Calls

1. INTRODUCTION

Optimal reinsurance arrangement has been extensively studied in a series of papers from various perspectives. Borch (1961) examined risk sharing between insurers through quota-share reinsurance arrangements. Some of the recent studies have focused on the pricing and optimal design of excess-of-loss reinsurance contracts. Cummins et al. (1999) developed a pricing methodology for index-based catastrophe loss contracts. Gajek and Zagrodny (2004) derived optimal forms for stop-loss contracts when the insurer attempts to minimize the probability of ruin.

The purpose of this paper is to develop a theoretical framework within which the optimal reinsurance arrangement for catastrophic risks is explored and derived. In the model, it is assumed that the insurer values the stability of its underwriting results in purchasing reinsurance protection. The optimal solutions to the model are obtained through numerical simulation and intend to provide justifications and explanations for the profile of reinsurance purchase that has been observed in practice. From over 4,000 catastrophe reinsurance layers transacted during the period 1970-1998, Froot (2001) observed that: (i) reinsurance contracts had been more often used to cover lower catastrophic risk layers (which have higher probability to be penetrated) rather than more severe but lower-probability layers; and (ii) reinsurance contracts had been priced in such a way that higher reinsurance layers had higher ratios of premium to expected losses.

The rest of the paper is organized as follows: The next section sets up a model of reinsurance purchase from an insurer's standpoint and derives the optimality conditions. Section 3 numerically solves the model, specifies the simulation methodology and discusses the results. Section 4 assumes a discrete loss distribution and derives the analytical solutions for the optimal design of reinsurance contract. Section 5 suggests possible ways to modeling the reinsurer's behavior and endogenizing the rule of reinsurance pricing. Section 6 concludes.

2. THE MODEL

This section introduces a simple model in which the reinsurance-pricing rule is exogenously given in deriving the system of optimal solutions for the insurer, while Section 5 discusses modeling of the reinsurer's behavior and choices. Specifically, the model makes the following simplifying assumptions:

(i) The reinsurance market consists of one insurer and one reinsurer;

(ii) The reinsurer sets its own pricing rule which may be a function of its own cost of capital;

(iii) The insurer has perfect information about the reinsurance pricing rule, and chooses the reinsurance layer for full coverage; and

(iv) The insurer and reinsurer have access to the same information on the underlying loss distribution.

The Reinsurer. The reinsurer underwrites an excess-of-loss contract i for catastrophic risks (shortened as "cat" hereafter) and assumes a certain portion of cat losses arising from the underlying insurance contracts. The reinsurance layer is defined by $[a, b]$, where a denotes the insurer's retention and b the retention plus limit. Cat losses occur with a continuous distribution function $F(x)$, where $x \in [0, \infty)$. For the reinsurer, the expected value and variance of loss payment from underwriting contract i is given, respectively, as

$$E[L_R^i(x_i; a_i, b_i)] = \int_{a_i}^{b_i} (x_i - a_i) dF(x_i) + (b_i - a_i) \int_{b_i}^{\infty} dF(x_i) \tag{1}$$

and

$$Var[L_R^i(x_i;a_i,b_i)] = (E[L_R^i])^2 \int_0^{a_i} dF(x_i) + \int_{a_i}^{b_i} (x_i - a_i - E[L_R^i])^2 dF(x_i) \\ + (b_i - a_i - E[L_R^i])^2 \int_{b_i}^{\infty} dF(x_i) \quad . \tag{2}$$

It is further assumed that loss payments under a marginal reinsurance contract is stochastically independent of those under all other reinsurance contracts in the existing portfolio held by the reinsurer. Naturally, the assumption of stochastic independence among risks in the reinsurer's portfolio may not be realistic[1], as a cat event may impact on many of the risk exposures under different contracts covered by the reinsurer. This assumption, however, will simplify the following exposition and simulation but not change the nature of the results to be derived. Based on the Capital Asset Pricing Model, the reinsurance pricing formula can be formulated as:

$$Z^i(x_i;a_i,b_i) = E[L_R^i(x_i;a_i,b_i)] + \gamma_R \cdot Var[L_R^i(x_i;a_i,b_i)], \tag{3}$$

where γ_R $(\gamma_R > 0)$ is the price of risk determined by the reinsurer's existing portfolio, and mathematically, it can be expressed as $\gamma_R = (Z_R^{(i)} - E[L_R^{(i)}])/Var[L_R^{(i)}]$, where (i) refers to all risks excluding contract i. As stated in Borch (1982), one advantage of this formulation is that it ensures the additive property of reinsurance contacts so that the price of risk will not be altered by the addition of stochastically independent risks. There are several issues that are worthy of comments. First, the risk load as specified in (3) does not explicitly take into account parameter uncertainty associated with the underlying loss distribution, nor is it directly modeled as a function of the "down-side" variance that may seem to be the more reasonable and appropriate one than the total variance. Nevertheless, the formulation in (3) has been supported by many empirical findings on reinsurance pricing (for instance, Kreps and Major 2001, Lane 2004). Second, , Kreps (2004) suggested a probability-weighted average of the deviations of loss from its expected value multiplied by a "riskiness leverage ratio" as a more general form for risk load. The riskiness leverage ratio can be a function of

[1] The correlation among the risk exposures in a reinsurer's portfolio was analyzed via copula approach in Venter (2003).

higher moments of loss function. As pricing reinsurance contracts is not the main focus of this paper, further studies should explore optimal reinsurance arrangements using other forms of risk load specifications. For the simplicity of exposition, the subscript i will be dropped from all following mathematical expressions.

The Insurer. The insurer knows about the reinsurer's pricing rule, and purchases the optimal reinsurance layer, or makes the optimal choices about a and b. By choosing a and b, the insurer attempts to minimize the sum of the reinsurance premium the insurer pays for reinsurance coverage and the expected loss payment net of reinsurance recovery. Besides, the objective function of the insurer also includes a penalty term for the variation of net loss payment. The penalty for loss variations is assumed to be a function of the variance of net loss payment. Note that this paper abstracts from the consideration of probability ruin in deriving optimal reinsurance arrangements. To summarize, the insurer attempts to minimize the following objective function subject to the budget constraint (denoted by B) on reinsurance purchase,

$$\underset{a,b}{MIN}: Z + E[L_S(x;a,b)] + \gamma_S \cdot Var[L_S(x;a,b)] \tag{4}$$

$$s.t.\ Z \le B,$$

where

$$E[L_S(x;a,b)] = \int_0^a x dF(x) + a\int_a^b dF(x) + \int_b^\infty (x-b+a)dF(x),$$

$$\begin{aligned} Var[L_S(x;a,b)] = & \int_0^a (x - E[L_S])^2 dF(x) + (a - E[L_S])^2 \int_a^b dF(x) \\ & + \int_b^\infty (x-b+a-E[L_S])^2 dF(x) \end{aligned},$$

and γ_S ($\gamma_S > 0$) measures the extent to which the insurer values the stability of its underwriting results, or the degree of its risk aversion. Since $E[L_R] + E[L_S] = \int_0^\infty x dF(x)$, for a given loss distribution, the amount of gross insurance premium received under the underlying insurance contacts can be fixed if the insurance contracts are priced so that the expected loss ratio remains roughly constant over time. As such, the problem stated in (4) would be equivalent to a problem of maximizing the expected net income minus some function of its variance to account for the associated uncertainty.

Optimal Conditions. Maximizing (4) with respect to a and b yields the following first-order conditions

$$\begin{aligned}&[1-\int_a^\infty dF(x)]\int_b^\infty \{\gamma_R(b-a-E[L_R])-\gamma_S(x-b+a-E[L_S])\}dF(x)+\\&\quad[1-\int_a^\infty dF(x)]\int_a^b \{\gamma_R(x-a-E[L_R])-\gamma_S(a-E[L_S])\}dF(x)\qquad,\\&\quad-\int_a^\infty dF(x)\int_0^a \{\gamma_R(-E[L_R])-\gamma_S(x-E[L_S])\}dF(x)=\lambda\cdot(\partial Z/\partial a)\end{aligned}\tag{5}$$

$$\begin{aligned}&[1-\int_b^\infty dF(x)]\int_b^\infty \{\gamma_R(b-a-E[L_R])-\gamma_S(x-b+a-E[L_S])\}dF(x)\\&\quad-\int_b^\infty dF(x)\int_a^b \{\gamma_R(x-a-E[L_R])-\gamma_S(a-E[L_S])\}dF(x)\\&\quad-\int_b^\infty dF(x)\int_0^a \{(\gamma_R(-E[L_R])-\gamma_S(x-E[L_S])\}dF(x)=\lambda\cdot(\partial Z/\partial b)\end{aligned}\tag{6}$$

and

$$Z\le B,\lambda\ge 0\ \text{ c.s.,}\tag{7}$$

where

$$\frac{\partial Z}{\partial a}=-\int_a^\infty dF(x)-2\gamma_R(1-\int_a^\infty dF(x))\cdot E(L_R)$$

and

$$\frac{\partial Z}{\partial b}=\int_b^\infty dF(x)+2\gamma_R\cdot\int_b^\infty (b-a-E(L_R))dF(x)\,.$$

In equations (5) and (6), $\int_a^b dF(x)$ and $\int_b^\infty dF(x)$ are the exceedence and exhaustion probabilities, respectively. Complicated at first glance, equations (5) and (6) virtually state that by choosing the reinsurance coverage, the insurer attempts to achieve the optimal balance between the reduction in the cost of loss variation because of reinsurance coverage and the price for shifting such variation to the reinsurer. Many of the terms in (5) and (6) describe the deviations of loss payment from the expected values in each interval for the insurer or the reinsurer. For instance, $b - a - E[L_R]$ is the amount of loss payment by the reinsurer and $x - b + a - E[L_S]$ is the amount of loss payment by the insurer when the layer limit is exceeded, in excess of their respective expected loss payment. To the extent that the parameters γ_R and γ_S measure the cost of reinsurance and insurance capital, respectively, the terms multiplied by these two parameters should be interpreted as the cost of such deviations.

Note that $\partial Z / \partial a < 0$ and $\partial Z / \partial b > 0$, which are quite intuitive in that higher reinsurance coverage demands higher price. However, it is not obvious how the reinsurance premium responds to the retention while holding the layer limit constant. Substituting $a + l$ for b in equations (3), and differentiating the resulting equation with respect to a gives

$$\partial Z / \partial a = 2\gamma_R \int_a^{a+l} (-x + a + E(L_R))dF(x) - \int_a^{a+l} dF(x),$$

where l denotes the layer limit. The sign of $\partial Z / \partial a$ is uncertain, which depends on the layer boundaries and γ_R.

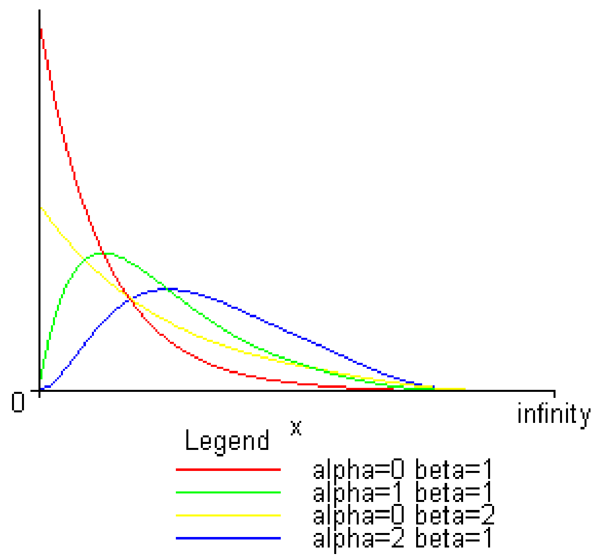
3. NUMERICAL SIMULATION

Methodology

While it is difficult to obtain the closed form solutions to the equations (5) and (6), the optimal values of a and b can be numerically solved for through simulation. In the simulation, it is assume that the insurer has no budget constraint on purchasing reinsurance. For illustrative purposes, the cat loss is assumed to be described by a Gamma distribution with the following probability density function:

$$f(x) = \frac{x^{\alpha} e^{(-x/\beta)}}{\alpha!\,\beta^{(\alpha+1)}}. \tag{8}$$

Figure 1. Gamma Distribution



In Figure 1, the probability density curves are plotted for several combinations of α and β. As the benchmark case in the simulation, $\alpha = 1$ and $\beta = 1$, and (8) then is simplified to $f(x) = x \cdot e^{-x}$ with mean and variance both equal to 2. For the benchmark case, the reinsurance premium and the value of the insurer's objective function, as functions of a and b, are plotted respectively in two three-dimensional figures (see Figures 2 and 3). In plotting the two figures, $\gamma_R = 2$ and $\gamma_S = 2$ are assumed.

To obtain the numerical solutions to equations (5) and (6), the following simulation procedures are used:

(i) Specify the values of γ_R and γ_S and choose the initial values of a and b (which are denoted by a_0 and b_0, respectively);

Figure 2. Reinsurance Premium as a Function of a and b (assuming $\gamma_R = 2$, $\gamma_S = 2$ $a \in (0, b), b \in (0,10)$)

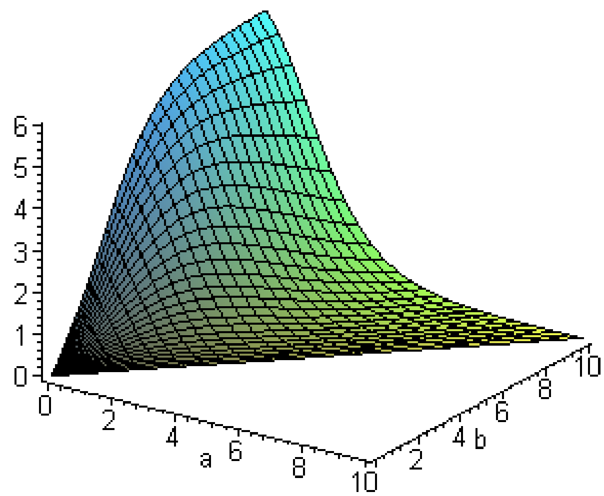
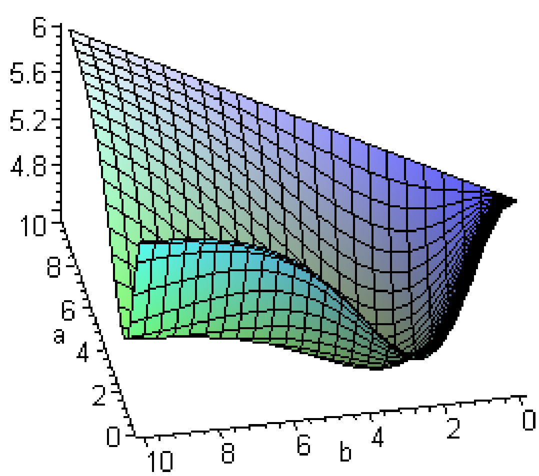


Figure 3. The Value of the Insurer's Objective Function (assuming $\gamma_R = 2$, $\gamma_S = 2$, $a \in (0, b), b \in (0,10)$)



(ii) Hold b_0 constant, and with a_0 as the starting value, apply the Newton's iteration method to find the "optimal" value of a (denoted by a_1) that satisfies Equation (5);

(iii) Holding a_1 constant, and with b_0 as the starting value, apply the Newton's iteration method to find the "optimal" value of b (denoted by b_1) that satisfies Equation (6); and

(iv) Repeat (ii) and (iii) a number of times (usually 50 times would be sufficient) until the differences between a_t and a_{t+1}, and between b_t and b_{t+1} are sufficiently small. Then a_{t+1} and b_{t+1} are the optimal solutions to (5) and (6) (denoted by a^* and b^*).

Results

Table 1. Numerical Simulation Results with $\gamma_R = 2$, $\gamma_S = 2$

Parameter Values, Expected Value and Variance of Underlying Loss Distribution

(1) alpha	(2) beta	(3) E(x)	(4) Var(x)
0	1	1	1
0	2	2	4
1	1	2	2
2	1	3	3

Simulation Results

(5) a	(6) b	(7) limit	(8) Z	(9) E(Lr)	(10) Obj	(11) ROL	(12) Z/E(Lr)
1.018	2.611	1.594	0.806	0.288	2.180	0.506	2.798
2.035	5.222	3.187	2.648	0.576	6.719	0.831	4.597
1.805	3.813	2.008	1.531	0.497	4.367	0.762	3.078
2.631	4.982	2.351	2.218	0.670	6.556	0.943	3.311

The numerical simulation results are summarized in Table 1. The optimal reinsurance layers are obtained through simulation for four sets of α and β values. In the table, the first two columns are the parameter values of α and β. Columns (3) and (4) are the expected value and variance of the cat loss distribution. For the cases where $\beta = 1$, the loss distribution is equi-dispersed, and over-dispersed when $\beta > 1$. The simulated values of the layer boundaries and policy limit are given in columns (5)-(7), the rate-on-line (ROL) and ratio of premium to expected loss in columns (11) and (12). Note that in the simulation, the budget constraint and the probability of ruin have been ignored, and the values of α and β are chosen arbitrarily so that the implied probability distribution of cat losses does not mirror the ones forecasted by engineering cat models in reality.

As the simulation results show, for all the four cases as presented in Table 1, the insurer who aims to stabilize its book of business should optimally use reinsurance protection against risks of moderate sizes, but leave the most severe loss scenarios uncovered or self-insured. This result justifies the aggregate profile of reinsurance purchases observed in Froot (2001)[2]. Also, as observed from the simulation results, the insurer's retention is set to be comparable with the expected value of ground-up cat losses under the underlying insurance contracts that are covered by the reinsurance treaty. As compared with the benchmark case ($\alpha = 1, \beta = 1$), the insurer, at the optimum, should purchase higher retention and higher limit for the case $\alpha = 0$, $\beta = 2$,the distribution which has the same expected loss but is more dispersed. The optimal layer in the latter case also has higher ROL and higher ratio of premium to expected loss.

It may be helpful to look at how the optimal choices of the insurer change while varying the parameters of the loss distribution. Table 1 reports the simulation results for the cases with different values of α ($\alpha = 0, 1, 2$), while holding β constant at 1. For the density function specified in (8), the value of α determines the shape of the distribution; the coefficient of variation decreases with the value of α, even though the loss distribution remains equi-dispersed ($Var(x)/E(x) = 1$). The optimal choices of the reinsurance layer can be very sensitive to the chosen values of the model parameters. With higher values of α, events of higher severity occur with larger probabilities(see Figure 1), and the insurer should have more protection (as shown by higher limit of reinsurance layer) against more severe events. On the other hand, the reinsurer would demand higher ROL and ratio of premium to expected loss for worse cat loss scenarios. Similar conclusions can be drawn by comparing the simulation results for different values of β; for instance, comparing the results between the cases with $\beta = 1$ and $\beta = 2$ (while $\alpha = 0$). To the extent that higher

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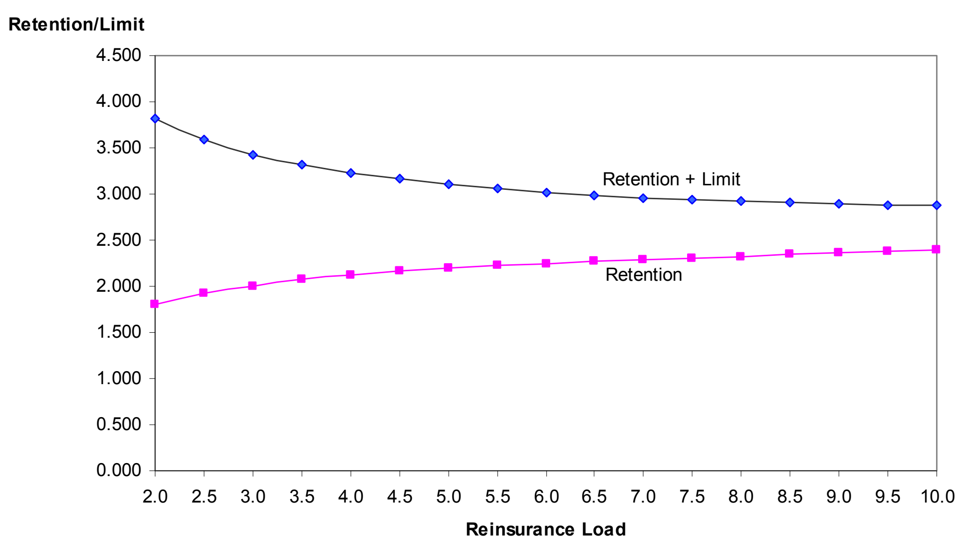
[2] As observed in Froot (2001, p. 536), "reinsurance coverage as a fraction of exposure is high at first (after some small initial retention) and then declines markedly with the size of the event, falling to a level of less than 30% for events of only about $8 billion (the author's note: $8 billion refers to the industry-wide loss).

reinsurance layers are more vulnerable to prediction errors from engineering models, parameter uncertainty may well explain high prices for low-probability layers as noted in Froot (2001).

Discussion

Varying the value of γ_R. Varying the value of γ_R between 2 and 10, Figure 4 plots the optimal values of a and b. The figure shows that when the price per unit of risk charged by the reinsurer increases relative to that by the insurer, or equivalently, as γ_R / γ_S increases, less reinsurance coverage will be purchased in terms of lower b and higher a and thus lower policy limit.

Figure 4. Retention and Limit As Reinsurance Load Changes (assuming $\gamma_S = 2$)



4. DISCRETE LOSS DISTRIBUTION: AN EXAMPLE

Table 2. Discrete Loss Distribution

	Scenario 1	Scenario 2	Scenario 3
Total Cat Loss	s1	s2	s3
Probability	f1	f2	f3

This section examines the optimal reinsurance arrangement when the loss distribution is discrete. Assume that there are a finite number of states for cat losses. Table 2 gives a simple discrete distribution of cat losses, where one of the following states of the world could occur: little or no occurrence (Scenario 1), moderate (Scenario 2), and most severe (Scenario 3). For the loss distribution given in Table 2, it is assumed that $0 \le s_1 < s_2 < s_3$, $f_1 \ge f_2 \ge f_3 \ge 0$ and $f_1 + f_2 + f_3 = 1$. For the simplicity of illustration, further set $s_1 = 0$ (no cat loss) and write $f_1 = 1 - f_2 - f_3$. There could be three choices regarding the sizes of retention and limit relative to loss severities: (i) $0 \le a \le s_2 \le b \le s_3$, (ii) $0 \le a \le b \le s_2$, and (iii) $s_2 \le a \le b \le s_3$. For this discrete distribution, it can be mathematically shown that

1. the optimal solutions always come from (i), or at the optimum,

   $0 < a^* < s_2 < b^* < s_3$. Specifically, the optimal reinsurance layer boundaries are

   given by $a^* = \dfrac{s_2 \gamma_R}{\gamma_R + \gamma_S}$ and $b^* = a^* + \dfrac{s_3 \gamma_S}{\gamma_R + \gamma_S} = \dfrac{s_2 \gamma_R + s_3 \gamma_S}{\gamma_R + \gamma_S}$.

2. the layer limit is independent of the probability with which each event occurs,

   and satisfies that $l^* = b^* - a^* = \dfrac{s_2 \gamma_R + s_3 \gamma_S}{\gamma_R + \gamma_S}$,

3. The minimum (optimal) value of the insurer's value function is equal to the rate on line of the reinsurance contract.

The proof is provided in the Appendix.

The optimal solutions satisfy $0 < a^* < s_2 < b^* < s_3$, provided that the ratio of γ_R to γ_S does not take extreme values. This implies that it is advisable for the insurer to purchase some reinsurance protection against both moderate and most severe cat loss scenarios rather than against any one particular scenario only, even though the coverage for any one of the scenarios in the latter case may be larger than that in the former case. Further, the insurer has more reinsurance protection against cat losses when $0 \le a \le s_2 \le b \le s_3$. As shown in the Appendix, the optimal reinsurance arrangement for (i) has the highest layer limit as compared with the other two choices. It is also observed that the comparative static results, $\partial b^* / \partial \gamma_R < 0$ and $\partial a^* / \partial \gamma_R > 0$, are consistent with the simulation results obtained for the continuous loss distribution (see Figure 4).

At the optimum, $Obj_{\min} = Z^* /(b^* - a^*)$, or in words, the objective function has its minimum value equal to the ratio of reinsurance premium to the layer limit, or the "rate on line". As compared, the simulation results for the continuous loss distribution (see Table 1) do not imply such a relationship between the two elements.

However, it is not intuitively clear why the optimal layer limit is independent of the occurrence probability of each cat loss scenario.

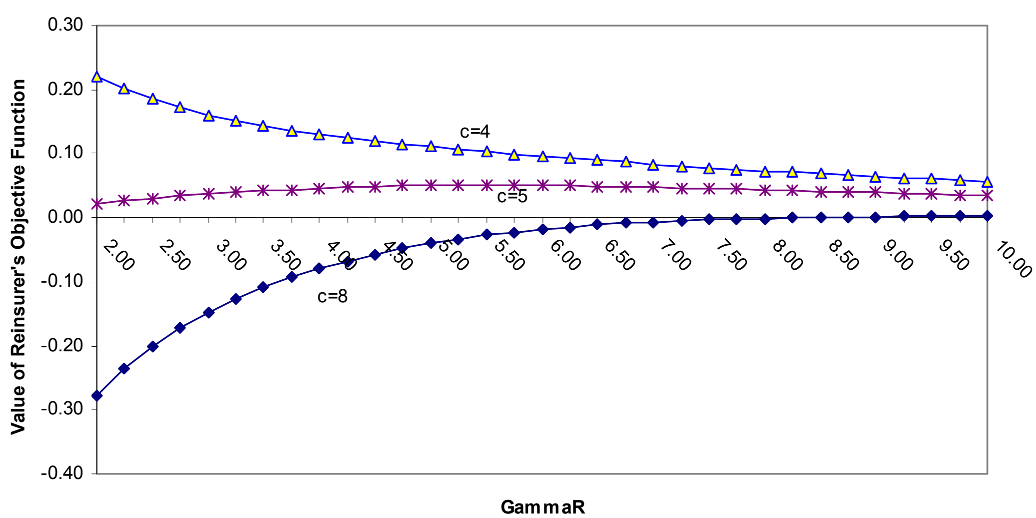
5. THE VALUE OF γ_R AND CONTINGENT CAPITAL CALLS

Mango (2004) introduced a capital consumption methodology for pricing reinsurance contracts, which in essence uses the value of potential capital usage as the risk load. Such potential access to surplus account is called contingent capital calls in that paper and other relevant studies. The discussion in the previous sections has been focused on the situation where the insurer makes its optimal decisions on reinsurance purchase subject to the pricing rule of the reinsurer who has been assumed not to consequently respond to the optimal choices of the insurer. In other words, the reinsurance pricing rule has been assumed exogenously given and fixed. Since reinsurers are also profit maximizing firms just like insurers, it is reasonable to assume that the reinsurer attempts to maximize the firm's expected net income after adjusting for the capital costs in the unprofitable states. For instance, using the methodology proposed in Mango (2004), the objective function of the reinsurer is formulated as:

$$\underset{\gamma_R}{MAX}: \gamma_R \cdot Var[L_R] - \int_{a+z}^{b} g(x-(a+Z))dF(x) - g(b-(a+Z)) \cdot \int_{b}^{\infty} dF(x), \tag{9}$$

where Z ($Z < b-a$) is a function of a and b as formulated in (3), and the function $g(\cdot)$ is the capital call charge function and convex so that $g'(\cdot) > 0$, $g''(\cdot) \geq 0$. The condition that $g''(\cdot) \geq 0$ requires nondecreasing marginal cost of capital calls. The reinsurer chooses the value of γ_R to maximize the problem in (9), or maximize the risk load minus the cost of contingent capital calls.

Figure 5. The Choice of γ_R and the Value of the Reinsurer's Objective Function



For a given reinsurance layer, a higher value of γ_R necessarily implies a higher ratio of reinsurance premium to expected loss. Figure 4 graphically shows that simply raising the value of γ_R would influence reinsurance purchase by increasing the insurer's retention and lowering the policy limit. As a results, it may well be the case that the reinsurer's objective function as specified in (9) is non-monotonic in γ_R. The optimal value of γ_R may be a function of the parameters of the underlying cat loss distribution and of the cost function of capital calls. For instance, the cost of access to surplus account is assumed to take the following functional form

$$g(\varepsilon) = \varepsilon + c \cdot \varepsilon^2 , \tag{10}$$

where ε ($\varepsilon \geq 0$) is the amount of capital calls and c ($c > 0$) is the rate at which the marginal cost of capital calls increases. With higher values of c, the reinsurer would find it more costly to underwrite more severe cat events. By assuming (10), Figure 5 graphs the trajectories of the value of (9) for $\gamma \in [2,10]$ for $c = 4$, 5 and 8, respectively. For the case of $c = 5$, the value of (9) is maximized when γ_R is around 6.25, while for $c = 4$ (or $c = 8$), smaller (or larger) values of γ_R always yield higher values of (9) when γ_R is within the stated range. Comparing the three curves in the figure would show: when the marginal cost of capital calls increases relatively faster for the reinsurer, the reinsurer sets higher γ_R and the insurer tends to purchase reinsurance protection for moderate losses only and leave higher layers uncovered.

6. CONCLUDING REMARKS

This paper has examined the optimal reinsurance arrangement for cat risks when the insurer values the stability of its underwriting results, subject to the reinsurance-pricing rule set by the reinsurer. In the model, the optimal solutions for reinsurance coverage purchase are obtained through numerical simulation, and the analytical solutions derived for the case when the loss distribution is discrete. Using the model results, the aggregate profile of reinsurance purchase observed for industry practice in previous studies is explained and justified.

As Froot and Posner (2000) stated, the risk pricing for cat reinsurance contracts is largely determined by the reinsurer. The first author in his 2001 paper further found some evidence implying that reinsurers possess certain market power in the reinsurance market. The general equilibrium model of reinsurance market was studied in Borch (1962), in which reinsurance capital market was assumed to be perfectly competitive and the pricing of quota share contracts were examined. Future research should develop a conceptual framework in which the reinsurer's behavior is systematically modeled and analytical solutions can be derived, and focus on the empirical measurement and determination of the cost of reinsurance capital in industry practice.

7. REFERENCES

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Appendix

This appendix provides proof for the three statements made in Section 4 for the discrete distribution. First, solve the maximization problem for each of the three cases (i) $0 \le a \le s_2 \le b \le s_3$, (ii) $0 \le a \le b \le s_2$, and (iii) $s_2 \le a \le b \le s_3$. For instance, for case (i), the maximization problem can be written as

$$\underset{a,b}{MIN} : Z + E[L_S(x;a,b)] + \gamma_S \cdot Var[L_S(x;a,b)]$$

$$s.t.\ \ 0 \le a \le s_2 \le b \le s_3,$$

where

$$\begin{aligned} Z = {} & f_2(s_2 - a) + f_3(b-a) + \gamma_R\{(1 - f_2 - f_3)(f_2(s_2 - a) + f_3(b-a))^2 \\ & + f_2(s_2 - a - f_2(s_2 - a) - f_3(b-a))^2 + f_3(b - a - f_2(s_2 - a) - f_3(b-a))^2\}, \end{aligned}$$

$$E[L_S(x;a,b)] = f_2 a + f_3(s_3 - b + a),$$

and

$$\begin{aligned} Var[L_S(x;a,b)] = {} & (1 - f_2 - f_3)(f_2 a + f_3(s_3 - b + a))^2 \\ & + f_2(a - f_2 a - f_3(s_3 - b + a))^2 + f_3(s_3 - b + a - f_2 a - f_3(s_3 - b + a))^2. \end{aligned}$$

The optimal solutions for each case are given below:

$$\text{Case 1: } a^* = \frac{s_2\gamma_R}{\gamma_R + \gamma_S},\ b^* = a^* + \frac{s_3\gamma_S}{\gamma_R + \gamma_S} = \frac{s_2\gamma_R + s_3\gamma_S}{\gamma_R + \gamma_S}, \tag{A.1}$$

$$Z^* = (b^* - a^*)\cdot((f_2 s_2 + f_3 s_3) + \frac{\gamma_R\gamma_S}{\gamma_R + \gamma_S}\cdot(f_2 s_2^2 + f_3 s_3^2 - (f_2 s_2 + f_3 s_3)^2)), \text{ and}$$

$$Obj_{\min} = Z^* /(b^* - a^*)$$
$$= (f_2 s_2 + f_3 s_3) + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} \cdot (f_2 s_2^2 + f_3 s_3^2 - (f_2 s_2 + f_3 s_3)^2)) ;$$

Case 2: $$b^* = a^* + \frac{\gamma_S}{\gamma_R + \gamma_S} \cdot \frac{f_2 s_2 + f_3 s_3}{f_2 + f_3}, \tag{A.2}$$

$$Z^* = (b^* - a^*) \cdot (f_2 + f_3 + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} \cdot (f_2 s_2 + f_3 s_3) \cdot (1 - f_2 - f_3)), \text{ and}$$

$$Obj_{\min} = f_2 s_2 + f_3 s_3 + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} (f_2 s_2^2 + f_3 s_3^2 - (f_2 s_2 + f_3 s_3)^2) + M ,$$

where $M = \frac{\gamma_S^2}{\gamma_R + \gamma_S} \cdot \frac{f_2 f_3}{f_2 + f_3} \cdot (s_3 - s_2)^2 > 0$;

Case 3: $$b^* = a^* + \frac{\gamma_S}{\gamma_R + \gamma_S} \cdot \frac{s_3 - f_2 s_2 - f_3 s_3}{1 - f_3}, \tag{A.3}$$

$$Z^* = (b^* - a^*) \cdot (f_3 + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} \cdot f_3 \cdot (s_3 - f_2 s_2 - f_3 s_3)), \text{ and}$$

$$Obj_{\min} = f_2 s_2 + f_3 s_3 + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} (f_2 s_2^2 + f_3 s_3^2 - (f_2 s_2 + f_3 s_3)^2) + N ,$$

where $N = \frac{\gamma_S^2}{\gamma_R + \gamma_S} \cdot \frac{f_2 s_2^2 \cdot (1 - f_2 - f_3)}{1 - f_3} > 0$.

Obviously, $Obj_{\min}$ obtained in case (i) has the lowest value among all three cases. Therefore, the insurer's objective function has its global minimum value when $0 < a^* < s_2 < b^* < s_3$. Note that for the cases $0 \le a \le b \le s_2$ and $s_2 \le a \le b \le s_3$, there exist multiple solutions for a and b, as only the layer limit ($b - a$), but not the limit boundaries (b, a) individually, matters for the insurer's value function. It is also easy to observe that the optimal layer limit for case (i) is larger than those for the other two cases.

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Exposure Rating Casualty Reinsurance Excess Layers With Closed Form Annuity Models

Jonathan Evans, FCAS, MAAA

Abstract

Casualty excess reinsurance terms are typically stated in fixed attachment and limit amounts. Unless a lump sum settlement or commutation is made ultimate recoveries are settled years later as total payments penetrate the excess layer. This paper demonstrates that annuity models incorporating claim life, late reporting, benefit inflation, and discounting can be formulated with simple functional form components that lead to closed form solutions, or at least efficient numerical integration solutions, for exposure rating quantities. Applications are shown to problems such as commutation of existing claims, prospective reinsurance rating, excess loss development factors, and volatility of excess layers.

Keywords. Annuity, reinsurance, exposure rating, casualty

1. INTRODUCTION

Consider a very motivating practical example along the lines of (Ferguson [6]) or (Bluhmson [1]). Suppose a permanent disability claim is reported to an insurer. The disabled beneficiary is 40 years old and expected to live 40 more years, annual indemnity and medical benefits are currently $100,000 but subject to future inflation throughout the life of the beneficiary. The claim is covered by one reinsurance treaty for $15 million excess of $10 million reinsurance treaty and by a second treaty $25 million excess of $25 million. What is the gross reserve for the claim, nominal and discounted, respectively? Traditional tools of life contingencies do a respectable job of answering this type of question (Bowers [2]).

What are the reserves, ceded to each reinsurance treaty and net, nominal and discounted, respectively? How much should the reinsurers pay to commute the claim? The reinsurers say the ceded amount will be about 0 since even if average benefit inflation is 4% the beneficiary would have to live to 82 years old to even begin to penetrate the first treaty layer. The insurer believes that average benefit inflation will be 11%, mostly driven by exploding medical expenses, and that both treaty layers will be totally exhausted if the beneficiary lives to 80. How are the answers affected by uncertainty in mortality and interest rates? For questions involving IBNR, prospective rating of treaties, and risk adjusted discounting the confusion and disagreements are even worse. Actuarial tools have not been well developed to answer these types of questions.

Large casualty claims, absent of a lump sum settlement, involve long streams of indemnity and medical payments that can last many decades. The ultimate value of these claims is highly uncertain due to many risk factors, including:

1. claim life or mortality of the disabled individual
2. inflation of future benefit amounts
3. interest rate (or appropriate discount rate) where present values are calculated

Excess reinsurance layers are generally stated in fixed nominal dollar amounts for attachment points and limits. Consequently, losses in these layers can be even more uncertain. This phenomenon has been dealt with through the use of discrete spreadsheet simulation models in (Bluhmsohn [1]).

However, closed form solutions for exposure rate estimates for excess reinsurance layers can be obtained if simple analytic functional forms are assumed for components such as claim life, reporting delay, inflationary trends, etc. of a life contingent annuity model.

1.1 Research Context

The net and ceded problem for annuity reserves was considered in (Ferguson[6]). The sensitivity of layers of a casualty claim to parameter assumptions has been considered in (Bluhmsohn [1]), and annuity models for paid tail factor development have been considered in (Corro [5]). This paper falls primarily into CAS research categories I.G.3, I.G.9, I.G.12, I.I.1.a, and I.I.1.b.

1.2 Objective

The techniques demonstrated in this paper connect the methods of life contingencies (Bowers [2]), as used for annuity calculations, with the methods of loss distributions (Klugman [8]), as used for per claim layered exposure rating calculations, to solve practical casualty reinsurance problems (Carter [4], Kiln [7], Strain [12]). The techniques are also somewhat applicable to low frequency per occurrence layers, but effectively inapplicable to high frequency catastrophic clash layers. Applications are demonstrated for commutation of claims, exposure rating of reinsurance treaties, loss development by layer, and volatility of excess layers.

1.3 Outline

This paper will explore annuity models with closed form solutions with the help of the symbolic manipulation and numerical calculation program MATHEMATICA. Attention will also be paid to parameter uncertainty – in addition to process variance.

Section 2 contains general setup for annuity models and several specific models with closed form solutions. Then a series of practical applications is presented in Section 3. Section 4 compares the advantages and disadvantages of closed form models versus simulation models. Section 5 discussions possibilities for further development of closed form models.

2. BACKGROUND AND METHODS

2.1 General Framework

The general notation and framework assumptions of this paper are presented in this section. Some readers, anxious to see the results in action, may benefit by skipping ahead to the examples in Section 3 and referring to Section 2.2 and Appendix C for model and formulae details.

The general framework and notation is:

1. A claim is reported at the time R and closed at time $R + T$, where $R \geq 0$ and $T \geq 0$, and the units of time are years.
2. Payments on the claim begin at time R and end at time $R + T$.
3. The survival function for R is $S_R(t) = 1 - F_R(t)$, where $F_R(t)$, is the probability distribution of R, and similarly the survival function of T is $S_T(t) = 1 - F_T(t)$.

4. The instantaneous rate of payment on a claim at time t is $P(t)$ dollars per year, where $P(t) \geq 0$.
5. The cumulative payment through time t, where $R+T \geq t \geq R$, is given by Formula 2.1.1.

$$C(t, R) = \int_R^t P(t)\, dt \tag{2.1.1}$$

6. The risk adjusted present value at time 0 of one dollar paid at time t is $D(t) \geq 0$.
7. The earliest time for the claim to close, or equivalently the minimum value for $R + T$, so that $C(t, R) \geq m$, where $m \geq 0$, is denoted $C^{-t}(l, R)$.
8. The ground up ultimate nominal claim payment is denoted Y and the layered payment is Y – A, limited to a minimum of 0 and a maximum of L.

This framework allows for process variance in the report time and claim life, which are stochastic, but leaves the benefit amount, benefit inflation, and discount rate deterministic. However, parameter uncertainty can be incorporated for all of these quantities. The examples in the next section will rely primarily on examination of a few parameter scenarios. Models including continuous distributions for parameters are also of interest, but are not demonstrated in this paper.

There are two key exposure rating formulae that form the focus of the rest of this paper. These formulae transform integration through the layer into an integral through an interval of time. This is possible since for a given report time the total value of payments, with or without discount, is a strictly increasing function of time as long as the claim is open and constant after the claim is closed. First, the expected nominal value of losses ceded to an excess layer with attachment $A \geq 0$ and limit $L \geq 0$ is defined by Formula 2.1.2.

$$NX(A, L) = \int_0^\infty d\, F_R(r) \int_{C^{-t}(A,R)}^{C^{-t}(A+L,R)} P(t)\, S_T(t-R)\, dt \tag{2.1.2}$$

Second, the expected present value of losses ceded to an excess layer with attachment A

≥ 0 and limit $L \geq 0$ is defined by Formula 2.1.3.

$$PX(A, L) = \int_0^\infty dF_R(r) \int_{C^{-t}(A,R)}^{C^{-t}(A+L,R)} D(t)\, P(t)\, S_T(t-R)\, dt \tag{2.1.3}$$

Formulae 2.1.2 and 2.1.3 have convenient closed form solutions for some basic but reasonable models for $S_R(t)$, $S_T(t)$, $P(t)$, and $D(t)$.

2.2 Some Basic Models with Closed Form Solutions

Formulae 2.1.2 and 2.1.3 have convenient closed form solutions for some basic models for $S_R(t)$, $S_T(t)$, $P(t)$, and $D(t)$, such as the following extremely simple model (MOD1):

- $R = 0$, claims reported immediately
- $S_T(t) = e^{-t/l}$, exponential distribution for time the claim is open with mean l
- $P(t) = B\ e^{at}$, constant payment rate B with constant force of inflation a
- $D(t) = e^{-dt}$, constant force of discount d

Formula 2.1.1 becomes:

$$C(t, R) = \frac{B(e^{aT}-1)}{a} \tag{2.2.1}$$

The closure time inversion formula becomes:

$$C^{-t}(m, R) = \frac{\log\left(\frac{a\,m}{B}+1\right)}{a} \tag{2.2.2}$$

Formula 2.1.2 becomes:

$$\mathrm{NX}(A, L) = \frac{l\left(B\left(\frac{B+a\,(A+L)}{B}\right)^{1-\frac{1}{a\,l}} - \left(\frac{a\,A}{B}+1\right)^{-\frac{1}{a\,l}}(a\,A+B)\right)}{al-1} \tag{2.2.3}$$

Formula 2.1.3 becomes:

$$\mathbf{PX}(A, L) = \frac{l\left(\left(1 + \frac{a A}{B}\right)^{-\frac{d+\frac{1}{l}}{a}} (-a A - B) + B\left(\frac{B + a\,(A+L)}{B}\right)^{1-\frac{d+\frac{1}{l}}{a}}\right)}{a l - d l - 1} \tag{2.2.4}$$

Notice that formulae 2.2.3 and 2.2.4 are pretty simple, involving only basic arithmetic operations and exponentiation. These formulae are very easy to program in a spreadsheet.

Next consider a slight generalization to a constant report lag (MOD2):

- $R = s$, claims reported with a constant lag

Now the formulae become:

$$C(t, R) = \frac{B e^{a s} (e^{a T} - 1)}{a} \tag{2.2.5}$$

$$C^{-t}(m, R) = \frac{\log\left(\frac{a m}{B} + e^{a s}\right)}{a} - s \tag{2.2.6}$$

$$\mathbf{NX}(A, L) = \frac{l\left(-(1 + a A \mathrm{B} e^{-\mathrm{as}})^{-\frac{1}{a l}} (a A + \mathrm{B} e^{\mathrm{as}}) + \mathrm{B} e^{\mathrm{as}} (1 + a \mathrm{B} e^{-\mathrm{as}} (A + L))^{1-\frac{1}{a l}}\right)}{a l - 1} \tag{2.2.7}$$

Formula 2.1.3 becomes:

$$\mathbf{PX}(A, L) = \frac{e^{-\mathrm{ds}}\, l\left((1 + a A \mathrm{B} e^{-\mathrm{as}})^{-\frac{d+\frac{1}{l}}{a}} (-a A - \mathrm{B} e^{\mathrm{as}}) + \mathrm{B} e^{\mathrm{as}} (1 + a \mathrm{B} e^{-\mathrm{as}} (A + L))^{1-\frac{d+\frac{1}{l}}{a}}\right)}{a l - d l - 1} \tag{2.2.8}$$

So far the solution expressions are pretty simple. For calculation these formulae can be programmed into spreadsheets or supporting macro programs easily, and use trivial computing capacity when running.

Another natural extension of the report time assumption leads to MOD3:

- $S_R(t) = e^{-t/s}$, exponential distribution for report time with mean s

MOD3 still produces a closed form solution, but the expressions are much longer and use hypergeometric functions (Appendix A). The reader is cautioned that although these formulae may superficially seem intimidating, they are still relatively easy to program for calculation and consume trivial computer resources, typically only a few hundred floating point calculations, and offer many other advantages over simulation that will be discussed in Section 4.

One concern about the exponential distribution as used for reporting time and claim life is that it has constant mean residual life (i.e. a claim that begins with expected time to closure of 20 years but happens to remain open after 60 years, as 5% of such claims do, still is expected to remain open 20 more years.). Generally beyond some time limit, for example 100 years, it is reasonable to expect that all claims have been reported and closed. The functional forms chosen for report time and claim life should have very low survival probability beyond this time limit. So the exponential distribution model would tend to be appropriate for situations where average claim life (or report lag) is relatively short, perhaps 5-10 years or less, but a significant fraction of claims do remain open for (or are reported at) periods several times longer, perhaps 30-60 years. A better model for claim life in most situations would be:

- $S_T(t) = e^{-\frac{\pi x^2}{4 l^2}}$, for time the claim is open with mean l.

However, the author was unable to find a corresponding continuous probability distribution for report time that led to a final closed form solution. The author would be very appreciative to any reader who can find such a model – or any other interesting model with a closed form solution – and forward it to him. Nevertheless, a closed form solution can still be obtained by using a less elegant finite discrete model for R. As a simple example:

- R = 0, s, or $2s$ each with probability $1/3$ respectively

The last two assumptions for T and R lead to MOD6, whose solution formulae are detailed in Appendix C, along with MOD4 and MOD5, which have the same report lag assumptions as MOD1 and MOD2, respectively. This illustrates an important point. In cases where continuous distributions form expressions so complicated that closed form solutions cannot be found, a finite distribution can be substituted. Unfortunately these finite distributions must be limited to a few allowed values to avoid long and unwieldy expressions with many terms to sum.

Another modeling concern is that report lag and time between report and closure are independent. In some real world situations claims reported at long lags after an exposure period would likely involve much older beneficiaries and would tend to remain open for shorter periods of time than claims reported earlier. A model that allowed for anti-correlation between T and R might be desirable but will not be considered in this paper.

3. EXAMPLE APPLICATIONS

This section presents applications to hypothetical situations that are representative of real world situations. The solution applications are somewhat simplified for expository purposes. In practice closed form annuity models should not be applied with overconfidence. Much attention should be paid to parameter and model uncertainties. Prominent reinsurance industry chief executives, with actuarial backgrounds, have pointed out historical pitfalls due overconfidence in actuarial models (Stanard and Wacek [11]). Another important consideration is that the models allow for the return on capital and income taxes through the discount rate (Butsic [3]), with a lower discount rate corresponding to a higher rate of return or more required capital.

3.1 Individual Claim Commutation Net vs Ceded Exposure Rating

MOD4 is applicable to the example in the introduction. Tables 1 and 2 show the expected values of the layered losses (See formula details in Appendix A), at benefit inflation assumptions that vary from 4% to 11%, on a undiscounted and a 3% discounted basis, respectively. The annuity model itself reduces the spread between expected losses in the reinsurance layers the 4% and 11% inflation assumptions. For a discounted reserve the spread is a bit wider since the higher inflation rate simultaneously increases the expected losses to the layers and speeds up the timing, decreasing the discount.

Table 1

Example Commutation Loss Nominal Expected Values, MOD 4

Benefit Inflation	10m xs 0	15m xs 10m	25m xs 25m
4%	7,098,059	4,204,456	2,526,378
5%	7,512,175	5,599,403	4,519,238
6%	7,839,130	6,798,060	6,573,467
7%	8,101,604	7,806,251	8,503,398
8%	8,315,646	8,649,693	10,236,941
9%	8,492,693	9,356,646	11,760,854
10%	8,641,010	9,952,287	13,087,796
11%	8,766,671	10,457,463	14,239,711

Table 2

Example Commutation Loss Present Values at 3% Discount, MOD4

Benefit Inflation	10m xs 0	15m xs 10m	25m xs 25m
4%	3,804,800	957,399	337,133
5%	4,161,881	1,506,256	786,403
6%	4,481,880	2,083,394	1,395,349
7%	4,768,343	2,656,426	2,106,887
8%	5,025,315	3,207,666	2,871,281
9%	5,256,633	3,728,593	3,652,392
10%	5,465,706	4,215,998	4,426,331
11%	5,655,473	4,669,605	5,178,355

Tables 1 and 2 tend to suggest compromise a nominal reserve value and commutation value of about \$8m and \$3m, respectively, for the 15m xs 10m treaty and about \$9.5m and \$2.5m, respectively, for the 25m xs 25m treaty.

3.2 Prospective Per Claim Excess Exposure Rating

Consider prospectively rating the \$15m xs \$10m layer from the example in the introduction. MOD6 is applicable to the example in the introduction. Since the layer is per claim frequency will not affect the exposure rates. Various parameter assumption scenarios are shown in Table 3 and corresponding exposure rates are shown in Table 4.

Table 3

Example Scenarios

Scenario	Average Claim Life	Benefit Inflation	Discount	Average Report Lag	Annual Benefits
1	30	5%	0%	1	75,000
2	30	5%	0%	1	150,000
3	30	5%	0%	3	75,000
4	30	5%	0%	3	150,000
5	30	5%	3%	1	75,000
6	30	5%	3%	1	150,000
7	30	5%	3%	3	75,000
8	30	5%	3%	3	150,000
9	30	9%	0%	1	75,000
10	30	9%	0%	1	150,000
11	30	9%	0%	3	75,000
12	30	9%	0%	3	150,000
13	30	9%	3%	1	75,000
14	30	9%	3%	1	150,000
15	30	9%	3%	3	75,000
16	30	9%	3%	3	150,000
17	50	5%	0%	1	75,000
18	50	5%	0%	1	150,000
19	50	5%	0%	3	75,000
20	50	5%	0%	3	150,000
21	50	5%	3%	1	75,000
22	50	5%	3%	1	150,000
23	50	5%	3%	3	75,000
24	50	5%	3%	3	150,000
25	50	9%	0%	1	75,000
26	50	9%	0%	1	150,000
27	50	9%	0%	3	75,000
28	50	9%	0%	3	150,000
29	50	9%	3%	1	75,000
30	50	9%	3%	1	150,000
31	50	9%	3%	3	75,000
32	50	9%	3%	3	150,000

Table 4

Scenario Estimates, MOD6

Scenario	Expected Loss 15m xs 10m	Expected Loss xs 0	Exposure Rate
1	1,909,437	8,561,971	22.3%
2	4,605,080	17,123,942	26.9%
3	2,210,921	9,462,441	23.4%
4	5,097,247	18,924,882	26.9%
5	451,287	3,521,110	12.8%
6	1,534,216	7,042,221	21.8%
7	517,975	3,664,810	14.1%
8	1,676,107	7,329,619	22.9%
9	5,809,299	49,344,415	11.8%
10	8,347,250	98,688,829	8.5%
11	6,447,687	59,075,990	10.9%
12	9,016,099	118,151,980	7.6%
13	2,124,825	12,390,469	17.1%
14	3,765,838	24,780,937	15.2%
15	2,347,693	13,970,214	16.8%
16	4,036,931	27,940,428	14.4%
17	7,010,277	56,319,066	12.4%
18	9,718,957	112,638,132	8.6%
19	7,402,612	62,242,194	11.9%
20	10,091,418	124,484,387	8.1%
21	1,597,912	8,284,288	19.3%
22	3,164,089	16,568,576	19.1%
23	1,675,854	8,622,376	19.4%
24	3,248,388	17,244,752	18.8%
25	10,630,733	5,168,591,731	0.2%
26	12,127,826	10,335,447,376	0.1%
27	11,041,391	6,187,721,601	0.2%
28	12,472,307	12,372,969,781	0.1%
29	3,856,218	138,561,666	2.8%
30	5,439,707	277,122,982	2.0%
31	3,989,718	156,227,807	2.6%
32	5,555,503	312,455,115	1.8%

Aside from the extreme Scenarios 25-32, which have both high benefit inflation and high average claim life, possibly raising doubts about the implicit assumption of primary rate adequacy, the exposure rates in Table 4 seem to cluster around the 15% to 25% range. In practice these exposure rates would need to be reduced by primary insurer underwriting expenses not passed to the reinsurer and the fraction of losses resulting from severe disability claims that are capable of penetrating the layer. For example if primary underwriting expense credit is 20% and only about 30% of losses in the underlying casualty exposures are severe disability then the rate, then range of final cession rates corresponds to about (100% - 20%) x 30% = 24% of the exposure rate, for a cession rate range of about 4% to 6%. Another possible refinement would be to use a lower discount rate for the excess layer to account for a higher cost of capital.

3.3 Prospective Non-Catastrophic Per Occurrence Exposure Rating

Although the technique in this paper does not readily lend itself to the per occurrence context, allowing for several dramatically different values in the annual benefit provides a crude adaptation to low frequency multiple claim occurrence situations. The simplifying assumption can be made that each claim clashing into the layer from the same occurrence is identical with the same benefit amount, lifetime, and discount. Then for a fixed number of claims the single claim model can be used with the total annual benefit for all of the claims substituted for the annual benefit. Although this does not let the claim characteristics vary for a given occurrence, it is easy to let them vary by the number of claims from a given occurrence. This is highly desirable as in many situations multiple claim occurrences tend to result in more severe and longer term disabilities. Tables 5 and 6 show this technique using MOD6 with the fixed assumptions that benefit inflation is 6% and the discount rate is 3%.

Table 5

Example Per Occurrence Undiscounted, MOD6

Number of Claims From Occurrence	Probability	Annual Benefit Per Claim	Average Claim Life
1	50.00000%	50,000	10
2	25.00000%	75,000	14
3	12.50000%	100,000	18
4	6.25000%	125,000	22
5	3.12500%	150,000	26
6	1.56250%	175,000	30
7	0.78125%	200,000	34
8	0.78125%	225,000	38

Table 6

Example Per Occurrence 3% Discounted, MOD6

Number of Claims	Expeced Loss 15m xs 10m	Expected Loss xs 0	Exposure Rate
1	1	650,485	0.0%
2	6,799	1,490,633	0.5%
3	170,650	2,799,830	6.1%
4	752,307	4,706,863	16.0%
5	1,653,234	7,378,498	22.4%
6	2,625,240	11,031,574	23.8%
7	3,522,071	15,949,257	22.1%
8	4,297,517	22,502,976	19.1%
Overall	223,824	2,045,413	10.9%

3.4 Excess Layer Paid Loss Development

Empirical triangles of loss development for excess layers are extremely sparse (Pinto [9]). Worse still they are more vulnerable than ground up triangles to changes over time in average claim severities. It has shown how a loss distribution can be interpreted to be an annuity density assuming a constant payment rate and this can then be used to produce tail factors for loss development (Corro [5]). The framework of this paper can be used to estimate loss development factors for excess layers. The expected amount paid in the layer at time T can be determined by limiting the claim life time limits of integration to T in Formula 2.1.2 as shown in Formula 3.4.1.

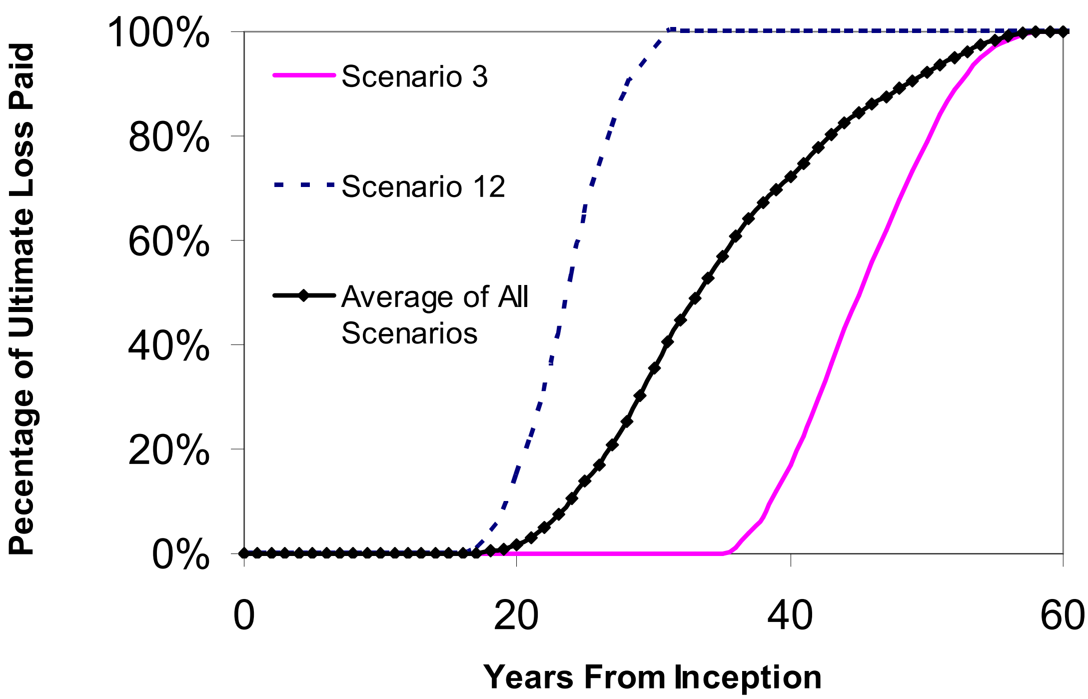
$$NXT(A, L, T) = \int_0^T d\, F_R(r) \int_{\mathrm{Min}\left(T, C^{-t}(A,R)\right)}^{\mathrm{Min}\left(T, C^{-t}(A+L,R)\right)} P(t)\, S_T(t-R)\, dt \tag{3.4.1}$$

NXT(A, L, T) / NXT(A, L, Infinity) is a reasonable proxy for the expected percentage of ultimate losses paid by time *T*.

Figure 1 shows the paid development patterns for MOD6 using Scenarios 3 and 12, and the overall average for all of the scenarios in Table 3. Scenario 3 does not begin to penetrate the layer until 36th year after inception, but totally exhausts the layer by the 58th year. Scenario 12 begins to penetrate the layer in the 17th year and exhausts the layer by the 31st year. Any of the Scenarios will produce 0 losses in the layer for many years and then exhaust the layer in a relatively quick period of time thereafter, even accounting for the stochastic pattern of reporting time and claim life. The timing of the layer payment is very sensitive to the scenario assumptions. For overall paid development percentages it would make sense to average over all the scenarios. These percentages can then be used in loss development factor, Bornhuetter-Ferguson, or other aggregate loss reserving methods. Generally, the Bornhuetter-Ferguson would need to be used as actual paid loss experience for the layer is very sparse and generally 0 for a long period of time after inception.

Figure 1

Example Paid Development Patterns for $15m xs $10m, MOD6



3.5 Excess Layer Case Incurred Loss Development

Whereas estimating paid loss development with an annuity model is straightforward in principal, the incurred loss development problem is very ambiguous. Case reserves should generally always reflect expect ultimate loss, and the only systematic development over time should be due to late reporting of claims and possibly the unraveling of tabular discount. However, it is often the case that case reserves at any point in time reflect an implicit discounting beyond that sometimes allowed for tabular discount. The expected case discounted amount paid in the layer up to time T can be determined by limiting the report time and payment time limits of integration to T in Formula 2.1.3 as shown in Formula 3.5.1.

$$PXT(A, L, T) = \int_0^T dF_R(r) \int_{\mathrm{Min}\left(T, C^{-t}(A,R)\right)}^{\mathrm{Min}\left(T, C^{-t}(A+L,R)\right)} D(t)\, P(t)\, S_T(t-R)\, dt \qquad (3.5.1)$$

The expected discounted case incurred amount in the layer based on claims reported through time T can be determined by limiting the report time limits of integration to T in Formula 2.1.3 as shown in Formula 3.5.2.

$$PXTR(A, L, T) = \int_0^T dF_R(r) \int_{C^{-t}(A,R)}^{C^{-t}(A+L,R)} D(t)\, P(t)\, S_T(t-R)\, dt \qquad (3.5.2)$$

Assuming case adjusters have a pretty good idea of ultimate nominal amount on cases reported, but effectively apply an implicit discount to the layered reserve $\left(NXT(A, L, T) + e^{dT}(PXTR(A, L,T) - PXT(A, L, T)) \right) / NX(A, L)$ is a reasonable proxy for the ratio of case incurred losses to ultimate losses incurred at time *T*. In this expression case reserves at time T correspond to the expected nominal total payments to date in the layer and the discounted expected future payments in the layer. This is not the same as combining the expected ground-up payments to date at time T with the discounted reserve and then layering. There is an argument for the latter but it requires a much more complicated expression and tends to greatly delay the recognition of any reserves in the layer.

Figure 2 shows all scenario averages for paid and case incurred development patterns. Only the scenarios with the 3% discount, Scenarios 5-8, 13-16, 21-24, 29-32, are used for the case incurred patterns. The incurred line shows a jagged pattern early on. This is due to the three discrete values allowed for report time. The report times are very early relative to the actual payments in the layers. Of course, MOD6 could be easily modified to include more report times and later report times.

Figure 2

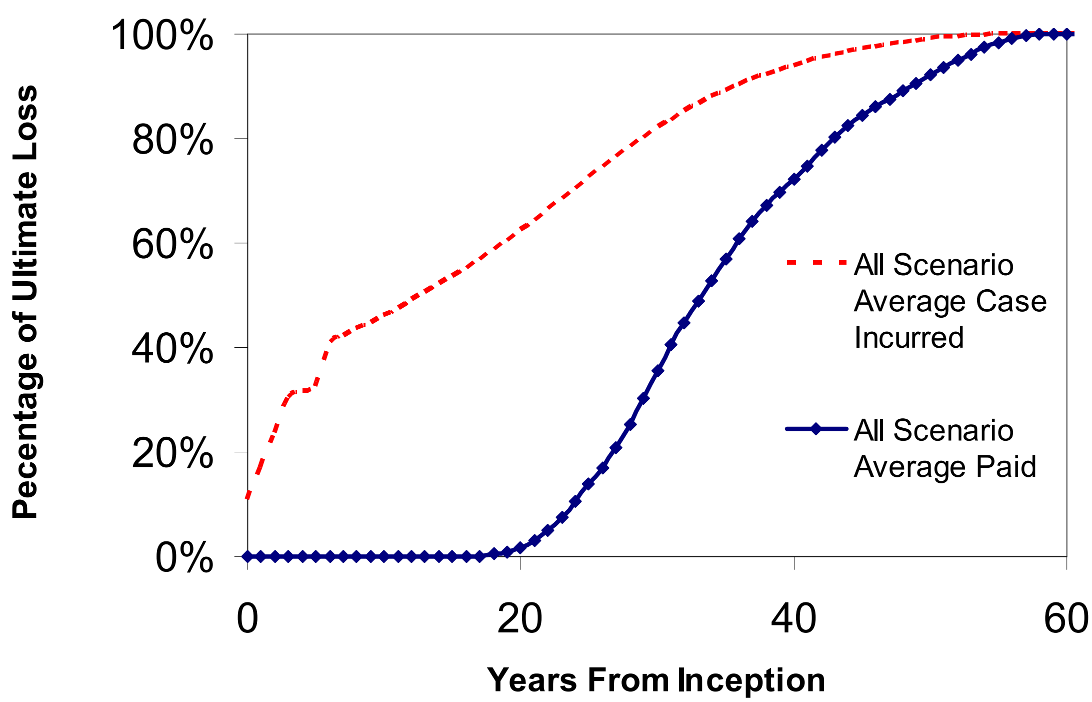


Table 7
Example Loss Development Patterns, MOD6

	% of Ultimate Paid			% of Ultimate Case Incurred
Years From Inception	Scenario 12	Scenario 3	All Scenario Average	All Discounted Scenario Average
1	0.0%	0.0%	0.0%	17.0%
2	0.0%	0.0%	0.0%	23.7%
3	0.0%	0.0%	0.0%	30.7%
4	0.0%	0.0%	0.0%	31.6%
5	0.0%	0.0%	0.0%	32.6%
6	0.0%	0.0%	0.0%	41.0%
7	0.0%	0.0%	0.0%	42.2%
8	0.0%	0.0%	0.0%	43.5%
9	0.0%	0.0%	0.0%	44.8%
10	0.0%	0.0%	0.0%	46.2%
11	0.0%	0.0%	0.0%	47.6%
12	0.0%	0.0%	0.0%	49.1%
13	0.0%	0.0%	0.0%	50.6%
14	0.0%	0.0%	0.0%	52.1%
15	0.0%	0.0%	0.0%	53.7%
16	0.0%	0.0%	0.0%	55.3%
17	0.0%	1.0%	0.1%	57.0%
18	0.0%	4.5%	0.5%	58.8%
19	0.0%	8.3%	1.0%	60.7%
20	0.0%	14.9%	1.8%	62.6%
21	0.0%	22.3%	3.1%	64.5%
22	0.0%	31.1%	5.0%	66.6%
23	0.0%	41.9%	7.5%	68.6%
24	0.0%	53.4%	10.5%	70.6%
25	0.0%	65.4%	13.8%	72.7%
26	0.0%	74.2%	17.1%	74.6%
27	0.0%	81.7%	20.7%	76.6%
28	0.0%	89.6%	25.2%	78.6%
29	0.0%	93.2%	30.2%	80.4%
30	0.0%	96.9%	35.6%	82.1%
31	0.0%	100.0%	40.5%	83.7%
32	0.0%	100.0%	44.8%	85.3%
33	0.0%	100.0%	48.9%	86.7%
34	0.0%	100.0%	52.8%	87.9%
35	0.0%	100.0%	56.8%	89.2%
36	1.1%	100.0%	60.7%	90.3%
37	4.0%	100.0%	64.3%	91.3%
38	6.9%	100.0%	67.4%	92.2%
39	11.8%	100.0%	69.7%	93.1%
40	17.0%	100.0%	72.1%	94.0%
41	22.6%	100.0%	74.7%	94.7%
42	29.6%	100.0%	77.7%	95.4%
43	36.4%	100.0%	80.3%	96.1%
44	43.0%	100.0%	82.6%	96.6%
45	49.5%	100.0%	84.5%	97.1%

46	55.8%	100.0%	86.1%	97.6%
47	61.9%	100.0%	87.6%	98.0%
48	67.8%	100.0%	89.1%	98.4%
49	73.5%	100.0%	90.6%	98.7%
50	79.0%	100.0%	92.1%	99.0%
51	84.3%	100.0%	93.6%	99.3%
52	89.0%	100.0%	95.0%	99.5%
53	92.0%	100.0%	96.2%	99.7%
54	94.9%	100.0%	97.4%	99.8%
55	97.1%	100.0%	98.5%	99.9%
56	98.3%	100.0%	99.3%	100.0%
57	99.5%	100.0%	99.8%	100.0%
58	100.0%	100.0%	100.0%	100.0%
59	100.0%	100.0%	100.0%	100.0%
60	100.0%	100.0%	100.0%	100.0%

3.6 Comparison of Different Reinsurance Treaties

A table like Table 4 can be useful in comparing reinsurance treaties with different underlying exposures or even different treaty terms. Even when actuarial models for reinsurance may be unreliable for absolute numerical estimates, these may be very valuable at determining relative differences in expected losses due to differences in underlying exposures or contract terms (Stanard [10]). Suppose the details known about Treaties A and B are:

1. Both treaties cover the layer $15m xs $10m on an effectively per claim basis, as clash effects are expected to be trivial.
2. Treaty A covers very much younger beneficiaries and claims will tend to remain open about 50 years versus 30 years for B.
3. Treaty A's younger beneficiaries tend to require a much lower annual medical benefit, bringing the total annual benefit closer to 75,000, versus 150,000 for Treaty B.
4. Treaty B benefits are much more dominated by higher inflation medical expenses leading to an overall benefit inflation of about 9% versus about 5% for Treaty A.
5. Both treaties involve $100m subject premium, 20% underwriting expense credits, and 30% of losses coming from serious disabilities, and a market cession rate of 2.8%.

Given a choice which treaty should the reinsurer participate in? As the details say nothing about reporting lag or discount the relevant scenarios are 17, 19, 21, 23 for Treaty A and 10, 12, 14, 16 for Treaty B. The exposure rate ranges for these groups of scenarios are 11.9% to 19.4% and 7.6% to 15.2%, respectively. The corresponding cession rate ranges are 2.9% to 4.7% and 1.8% to 3.6%, respectively. Since the market cession rate is just below the modeled range for Treaty A but above the modeled range midpoint for Treaty B, Treaty B appears to be the more attractive offer. The layer exposure to longer average claim life for Treaty A tends to dominate the higher benefit and higher benefit inflation of Treaty B.

3.7 Excess Layer Frequency, Severity, and Variance

Excess layer frequencies, severities, and variances can be calculated from the distribution of ultimate layer amount, and it is easy to determine this distribution from derivative of Formula 2.1.2 (Appendix B). The excess loss functions NX and PX only accounted for variability (MOD1-6) in the claim life and the report lag (MOD3, 6). Variability in the other parameters, such as benefit inflation, has been dealt with through scenario testing in the previous examples. The ultimate layer amount distribution underlying MOD6 is given by Equation 3.7.1.

$$F_{\text{layer loss}}(L) = \frac{1}{3}\left(-e^{-\frac{\pi\left(4a^2 s^2 + \log^2\left(\frac{aL}{B} + e^{2as}\right)\right)}{4a^2 l^2}} \left(\frac{aL}{B} + e^{2as}\right)^{\frac{\pi s}{a l^2}} - e^{-\frac{\pi\left(a^2 s^2 + \log^2\left(\frac{aL}{B} + e^{as}\right)\right)}{4a^2 l^2}} \left(\frac{aL}{B} + e^{as}\right)^{\frac{\pi s}{2a l^2}} - e^{-\frac{\pi \log^2\left(\frac{aL}{B} + 1\right)}{4a^2 l^2}} + 3 \right) \tag{3.7.1}$$

For Scenario 1 from Section 3.2, 25% of all claims penetrate the layer and 6.2% of claims exhaust the layer. The average severity of claims penetrating the layer is 1,909,437 / 25% = 7,637,748.

The cumulative ultimate amount density underlying MOD6 is given by Equation 3.7.2.

$$f_{\text{layer loss}}(L) = \frac{\pi}{6al^2}\left(\frac{e^{-\frac{\pi\left(4a^2s^2+\log^2\left(\frac{aL}{B}+e^{2as}\right)\right)}{4a^2l^2}}\left(\log\left(\frac{aL}{B}+e^{2as}\right)-2as\right)\left(\frac{aL}{B}+e^{2as}\right)^{\frac{\pi s}{al^2}}}{e^{2as}B+aL}\right.$$

$$+\frac{e^{-\frac{\pi\log^2\left(\frac{aL}{B}+1\right)}{4a^2l^2}}\log\left(\frac{aL}{B}+1\right)}{B+aL}+ \tag{3.7.2}$$

$$\left.\frac{\left(\frac{aL}{B}+e^{as}\right)^{\frac{\pi s}{2al^2}}e^{-\frac{\pi\left(a^2s^2+\log^2\left(\frac{aL}{B}+e^{as}\right)\right)}{4a^2l^2}}\left(\log\left(\frac{aL}{B}+e^{as}\right)-as\right)}{e^{as}B+aL}\right)$$

The conditional second moment of the layer loss is given by Equation 3.7.3.

$$\frac{(1-F(25000000))\,(25000000-10000000)^2+\int_{10000000}^{25000000}(L-10000000)^2\,f(L)\,dL}{1-F(10000000)} \tag{3.7.3}$$

Numerically this is 8.85 x 10^{13}, making the severity variance 3.02 x 10^{13} and the severity standard deviation 5,490,000. If the number of ground-up claims is Poisson with mean 10, then the number of claims penetrating the layer is Poisson with mean 2.5 and variance 2.5. The total layer variance is then easily determined by the standard formula:

E[N]Var[L]+Var[N]E[L]^2 = 2.5 (3.02 x 10^{13}) + 2.5 (7,637,748)^2 = 2.21 x 10^{14},

and total layer standard deviation is 1.48 x 10^7, for a coefficient of variation of 1.48 x 10^7 / 7,637,748 = 195%.

The analysis above omitted the very important considerations of parameter uncertainty in the benefit amount, claim life, report lag, and benefit inflation. These can be incorporated by running the calculation for all of Scenarios 1-32 in Table 3. Each scenario can be given equal weight and the total variance for the layer is then the average variance for each scenario plus the variance of the scenario layer means.

The analysis above can also be performed fairly easily on a discounted basis for MOD1, 2, 4, 5 where the report lag is fixed. However, when the report time is stochastic the discounted basis analysis becomes much harder because the discounted layered amount of the claim is no longer uniquely determined by the total discounted amount of the claim. Two claims with the same total discounted amount, but different report times, can have different layered amounts, nominal or discounted.

4. CLOSED FORM MODELS VERSUS SIMULATION

Closed form solutions, or even numerical integration solutions, are often much more computationally efficient than simulation. This higher efficiency makes testing many different parameter assumptions much easier. Such solutions also avoid concerns about structural patterns or biases in pseudorandom numbers. Simulation also may require tremendous amounts of memory or disk storage space if simulated data is not tabulated in bins during the simulation. The primary limitation of models with closed form, or numerical integration, solutions is that they do not allow for the kind of opened ended structural formulations that may include all sorts of special limitations or dependencies and can always be programmed into a simulation.

5. POSSIBLE FURTHER DEVELOPMENTS

5.1 Other Models and Standardized Software Modules

There are several obvious areas for improvement on the models presented in this paper.

- The most important area for improvement is probably incorporating good functional forms for stochastic uncertainty in the parameters.
- Also, of interest probability models for report lag R and claim life T. It

would be desirable to have simple functional forms that fit actual mortality or decrement experience tables well and possibly even involve a dependence relationship between R and T.

- Another concern is that in many situations long term disability claims have large medical expense early on. The models in this paper assume that the initial annual benefit simply increases with inflation. An extra benefit amount paid only at the time of report could be added.

The most burdensome part of using these models is actually finding models with closed form or efficient numerical integration solutions and programming software to calculate the solution values. Once found, the solutions can be programmed into standardized software modules, or macros in spreadsheets, and efficiently used. A major concern is debugging such modules - and even verifying the solution itself – as the complexity of the solution expressions can easily hide errors. A key tool that greatly facilitates debugging is testing the software calculation for several key properties that must hold (Appendix C).

5.2 Simplified Benchmarks for Non-actuaries

The mathematical sophistication required to define models, find solutions, and program them is on par with that of a credentialed actuary or a Ph.D. in a mathematical science. For practical use by managerial decision makers the model solutions must be turned into numerical exhibits and/or graphs, fixed in print or interactively generated through user friendly software. Since the initially there is a significant cost in high skilled labor to produce the software, care should be taken in determining what practical situations typically arise and what kind of condensed presentation will be most useful. Tables 1-6 and Figure 1 are very modest examples of such formats and much more can be done.

6. CONCLUSIONS

Annuity models offer a practical solution to problems of analyzing casualty excess reinsurance. Finding models with closed form solutions or easy numerical integration has been greatly facilitated by the advent of software, such as MATHEMATICA, capable of performing mathematical symbolic manipulation and efficient numerical analysis. These models are not subject to several problems of simulation analysis such as: biases of random number generators, large overly complex spreadsheets, extremely long times to run simulations and test parameters, and huge sizes of output. The primary challenge in implementation is to define a reasonable model that has closed form and then program a calculation module in a spreadsheet. Once the module is checked for reliability, spreadsheet analyses, including parameter sensitivity and graphical inspection, may be done in small simple spreadsheets that use very little computational capacity. Annuity models, like most actuarial models, will never be "right" and should not be blindly relied on for point estimates. However, if used properly these models do facilitate the understanding of qualitative effects, and the quantification of relative differences and magnitudes of uncertainty.

Appendix A – Other Model Solutions

MOD3

$$\begin{aligned}\mathbf{NX}(A, L) = -&\frac{l}{(al-1)(as-1)} \\ &-B\,{}_2F_1\left(\frac{1}{al}, \frac{1}{\mathrm{as}}-1; \frac{1}{as}; -\frac{aA}{B}\right) \\ &+B\,{}_2F_1\left(\frac{1}{al}, \frac{1}{\mathrm{as}}-1; \frac{1}{as}; -\frac{a(A+l)}{B}\right) \\ &+(as-1)\,a\left(A\,{}_2F_1\left(\frac{1}{al}, \frac{1}{as}; \frac{\mathrm{as}+1}{as}; -\frac{aA}{B}\right)\right. \\ &\qquad\left.-(A+l)\,{}_2F_1\left(\frac{1}{al}, \frac{1}{as}; \frac{\mathrm{as}+1}{as}; -\frac{a(A+l)}{B}\right)\right)\end{aligned} \tag{A.1}$$

$$\begin{aligned}\mathbf{PX}(A, L) = -&\frac{l}{(al-dl-1)(as-ds-1)(ds+1)} \\ &-(dsB+B)\,{}_2F_1\left(\frac{dl+1}{al}, \frac{\mathrm{sd}+1}{\mathrm{sa}}-1; \frac{ds+1}{as}; -\frac{aA}{B}\right) \\ &+(dsB+B)\,{}_2F_1\left(\frac{dl+1}{al}, \frac{\mathrm{sd}+1}{\mathrm{sa}}-1; \frac{ds+1}{as}; -\frac{a(A+l)}{B}\right) \\ &+(as-ds-1)\,a\left(A\,{}_2F_1\left(\frac{dl+1}{al}, \frac{ds+1}{as}; \frac{(a+d)s+1}{as}; -\frac{aA}{B}\right)\right. \\ &\qquad\left.-(A+l)\,{}_2F_1\left(\frac{dl+1}{al}, \frac{ds+1}{as}; \frac{(a+d)s+1}{as}; -\frac{a(A+l)}{B}\right)\right)\end{aligned} \tag{A.2}$$

$_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} (a)_k\,(b)_k / (c)_k\, z^k / k\,!$ is the hypergeometric function where $(a)_k = a\,(a + 1)\ldots(a + k - 1)$.

MOD4

$$\mathbf{NX}(A, L) = B\, e^{\frac{a^2 l^2}{\pi}}\, l \left(\operatorname{erf}\left(\frac{\pi \log\left(\frac{a\,(A+L)}{B} + 1 \right) - 2\, a^2 l^2}{2\, a \sqrt{\pi}\, l} \right) - \operatorname{erf}\left(\frac{\pi \log\left(\frac{a\, A}{B} + 1 \right) - 2\, a^2 l^2}{2\, a \sqrt{\pi}\, l} \right) \right) \tag{A.3}$$

$$\mathbf{PX}(A, L) = B\, e^{\frac{(a-d)^2 l^2}{\pi}}\, l \left(\operatorname{erf}\left(\frac{\pi \log\left(\frac{a\,(A+L)}{B} + 1 \right) - 2\, a\,(a - d)\, l^2}{2\, a \sqrt{\pi}\, l} \right) - \operatorname{erf}\left(\frac{\pi \log\left(\frac{a\, A}{B} + 1 \right) - 2\, a\,(a - d)\, l^2}{2\, a \sqrt{\pi}\, l} \right) \right) \tag{A.4}$$

$\operatorname{erf}(z) = 2 \big/ \sqrt{\pi} \int_0^z e^{-t^2}\, d\,t$ is the error function.

MOD5

$$\mathbf{NX}(A, L) = B\,e^{\frac{a\left(a\,l^2+\pi\,s\right)}{\pi}}\,l\left(\operatorname{erf}\left(\frac{-2a^2 l^2 - a\pi\,s + \pi\log\left(\frac{a\,(A+L)}{B} + e^{as}\right)}{2a\sqrt{\pi}\,l}\right) - \operatorname{erf}\left(\frac{-2a^2 l^2 - a\pi\,s + \pi\log\left(\frac{a\,A}{B} + e^{as}\right)}{2a\sqrt{\pi}\,l}\right)\right) \tag{A.5}$$

$$\mathbf{PX}(A, L) = B\,e^{\frac{(a-d)\left((a-d)\,l^2+\pi\,s\right)}{\pi}}\,l\left(\operatorname{erf}\left(\frac{-2a(a-d)\,l^2 - a\pi\mathrm{s} + \pi\log\left(\frac{a\,(A+L)}{B} + e^{a\,s}\right)}{2a\sqrt{\pi}\,l}\right) - \operatorname{erf}\left(\frac{-2a(a-d)\,l^2 - a\pi\,s + \pi\log\left(\frac{a\,A}{B} + e^{a\,s}\right)}{2a\sqrt{\pi}\,l}\right)\right) \tag{A.6}$$

MOD6

NX(A,L) and PX(A,L) are simply the formulas for MOD5 averages over the values 0, s, 2s substituted for s.

Appendix B – Converting Limited Pure Premiums into Loss Distributions

The nominal expected losses in a layer are equal to the integral of the survival function in the layer. Therefore the distribution can be derived from the derivative with respect to L of the expected losses in the layer [0, L].

$$NX(0, L) = \int_0^L (1 - F_{\text{total loss}}(l))\, dl \tag{B.1}$$

$$F_{\text{total loss}}(L) = 1 - \frac{d\, NX(0, L)}{dL} \tag{B.2}$$

Formula B.2 is particularly useful, as it allows easy conversion of the limited pure premium into the loss distribution. Assuming a constant report time s, the total discounted losses in a layer is an increasing function of the total losses in the layer, and hence uniquely determined by the total losses in the layer.

$$\mathrm{PV}(L) = \int_s^{C^{-t}(L,s)} D(t)\, P(t)\, dt \tag{B.3}$$

$$F_{\text{discounted total loss}}(L) = F_{\text{total loss}}\left(\mathrm{PV}^{-1}(L)\right) \tag{B.4}$$

When the report time is stochastic determining the loss distribution is more difficult.

Appendix C –Test Checklist for Solution Software

The following questions should all be answered affirmatively to check a solution calculation program. Affirmative answers do not guarantee correctness, but negative answers indicate an error.

1. Do PX(A, L) and NX(A, L) both increase as L increases and decrease as A increases?
2. Does PX(A, L) = NX(A, L) when D(t) = 1 for all t?
3. Is PX(A, L) < NX(A, L) when D(t) < 1 for all t?
4. Is PX(A, L) > NX(A, L) when D(t) > 1 for all t?
5. Does PX(A, L) increase when D(t) increases for all t?
6. Does NX(A, L) increase as P(t) increases for all t?
7. Does NX(A, L) increase as $S_T(t)$ increases for all t?

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Simple Practical Estimation of Sub-Portfolio Catastrophe Loss Exceedance Curves with Limited Information

Jonathan Evans, FCAS, MAAA

Abstract

A very simple method is shown for the estimation of the catastrophe loss exceedance curve of a sub-portfolio, when information available is limited to a total portfolio catastrophe loss exceedance curve, and just enough information about the sub-portfolio to make reasonable selections for two parameters: relative frequency and relative severity. Practical examples are shown in the contexts of exposure rating catastrophe excess reinsurance, catastrophe risk based capital requirements, and catastrophe deductible credits. The relationship of relative frequency and relative severity to the concentration diversification structure of sub-portfolios is described.

Keywords: reinsurance, catastrophe, modeling, solvency

1. INTRODUCTION AND BACKGROUND

1.1 Research Context

This paper falls primarily into the CAS research taxonomy categories I.G.9, II.Q.2, and III.I. The CAS literature contains a number of papers on the general theory of catastrophe modeling (Woo [15], Boisonnade et al[2], Friedman [6], Clark[5], Kozlowski[8]), and the estimation of risk loads (for example Meyers et al [10]), particularly by per occurrence layer for catastrophe reinsurance, using loss exceedance curves. Little or nothing has been written addressing the decomposition of catastrophe loss exceedance curves.

1.2 Objective

A very simple method is shown for the estimation of the catastrophe loss exceedance curve of a sub-portfolio, when information available is limited to a total portfolio catastrophe loss exceedance curve, and just enough information about the sub-portfolio to make reasonable selections for two parameters: relative frequency and relative severity. This method can greatly facilitate analysis of real world catastrophe sub-portfolios, particularly in reinsurance context (Carter [4], Kiln [7], Strain [14]). Even where the method may not provide meaningful absolute numbers, it may be valuable for determining relative differences between different reinsurance transactions (Stanard et al [12]).

1.3 Outline

Section 1.4 provides general background and caveats are listed in Section 1.5. Section 1.6 provides an introduction to loss exceedance curve representations of catastrophe model output. Section 2 presents several example applications and a description of the model underlying the method. Section 3 is a discussion of generalization to variable relative frequency and severity. Conclusions are presented in Section 4. Appendix A provides details of an interesting measure of loss correlation between sub-portfolios. Appendix B provides details of a key consistency constraint for choosing relative frequencies and severities. Readers only interested in a quickly understanding the method at a functional level need only read Section 1.4 and 1.6 and Section 2 through 2.6, and Appendix B.

1.4 Background

Modern catastrophe modeling software programs accept exposure profiles for a single risk or a portfolio of risks. For example, a portfolio might be all the homes in Florida insured by a large national insurer, a single risk might be just one of these homes in Miami, and a more interesting sub-portfolio might be all of these homes which are in the Florida Keys. Exposure profiles are sometimes very detailed, including information such as geocodes (latitude, longitude), construction type, insured value, etc. for every single risk included. If the modeling software, detailed exposure profiles, adequate computers, enough time, and staff trained to use the software are all simultaneously available, it is straightforward to produce loss exceedance curves, either for entire the portfolio combined or any sub-portfolio. Loss exceedance curves show the annual probability that one or more events of a given size will occur, and can be easily translated into frequencies or return periods for losses of a given size (Section 1.6).

However, in many real world situations an actuary may need to make some sort of catastrophe sub-portfolio estimates, having access to neither the modeling software nor the detailed exposure profiles, but only a total portfolio loss exceedance curve and a little bit of sub-portfolio descriptive information (i.e., percentage of total portfolio premium, percentage of total portfolio policy limits, general location, etc.). For example, the actuary might have the loss exceedance curve for the portfolio of every home in the state of Florida, the Florida homeowners market share for two different insurers, and a little bit of descriptive information about relative geographical spread of the two insurers. Detailed exposure information is often considered of high proprietary value by insurers and kept as confidential as possible. Sometimes detailed exposure information is available, but even with the dramatic increase in computer speed and capacity in recent years, catastrophe models can still take a very long time to run (Major[9]). This paper demonstrates a simple method for estimating the loss exceedance curve of the sub-portfolio in such situations.

1.5 Caveats

1. Examples in this paper are entirely hypothetical illustrations of methodology, using artificial numbers together with situations which are not based on any actual insurers, reinsurers, brokers, catastrophe modeling firms, or catastrophe modeling results.
2. The term "risk based capital" is used in this paper in the generic sense of how much capital an insurer must provide to cover the risk of higher than expected losses, not the specific context of the National Association of Insurance Commissioners' RBC requirements.
3. Most statements in this paper are the generalizations of a simple practical model which is not always guaranteed to be mathematically consistent. Counterexamples and inconsistencies can often be generated. (See Appendix B)

1.6 CATASTROPHE LOSS EXCEEDANCE CURVES

The output of catastrophe models is usually stated in terms of a finite number of points (L_i, $T(L_i)$), where $L_{i+1} > L_i$, called a loss exceedance curve. This curve shows the probability that one or more loss events will occur in a year that are at least as great as a given loss amount. For example, if T(50 million) = 1% there is a 99% chance that the largest event that occurs in a single year will be less than $50 million. Usually the exceedance curve is consistent with a collective risk model of event frequency and severity (Bowers et al [3]), with the number of events following a Poisson distribution. The sum of independent

Poisson distributed random variables is a Poisson distributed random variable. Similarly, Poisson distributed random variables can be decomposed into sums of independent Poisson distributed random variables. Each point on the exceedance curve can be thought of as representing a single event, independent of all other events. For a Poisson distribution with mean λ, the probability of at least one occurrence is $1 - e^{-\lambda}$. So the mean annual frequency of events at least as great as L_i, is $\Lambda(L_i) = -\mathrm{Ln}(1 - T(L_i))$ and the annual frequency of L_i specifically is $\lambda(L_i) = \Lambda(L_i) - \Lambda(L_{i+1})$. If the total frequency of all events is $\Lambda(0)$ then the size of loss distribution for individual events is $F(L_i) = 1 - \Lambda(L_i) / \Lambda(0)$. The return period of a loss amount is simply define as the inverse of the mean annual frequency of losses at least as great, $R(L_i) = 1/\Lambda(L_i)$. Sometimes the return period is defined as the inverse of the probability of exceedance, $R(L_i) = 1/T(L_i)$. This definition is numerically very close to the inverse of annual frequency for low frequency events but can become fairly meaningless for very high frequency events. Table 1.6.1 shows an example of the various components common in the representation of a loss exceedance curve.

Table 1.6.1

Example of Various Components of Common Representations of Loss Exceedance Curves

Event Loss	Probability of Exceedance	Frequency of Exceedance	Return Period
[1]	[2]	[3]	[4]
= Model Output	= Model Output	= -ln(1-[2])	= 1 / [3]
1,000,000,000,000	0.1998%	0.00200	500
100,000,000,000	0.9950%	0.01000	100
10,000,000,000	9.5163%	0.10000	10
1,000,000,000	18.1269%	0.20000	5
Total	18.1269%		

Event Loss	Incremental Frequency	Severity Distribution	Severity Density
[1] = Model Output	[5] = Difference ([4])	[6] = [3] / Total([5])	[7] = Difference ([6])
1,000,000,000,000	0.00200	100%	1.0%
100,000,000,000	0.00800	99%	4.0%
10,000,000,000	0.09000	95%	45.0%
1,000,000,000	0.10000	50%	50.0%
Total	0.20000		100.0%

Catastrophe excess reinsurance treaties often limit the number of separate events covered to a finite number of "reinstatements" m. Summing the product of loss amounts multiplied by incremental frequencies corresponds to unlimited reinstatements. However, Formula 1.6.1 shows an example adjustment factor, where reinstatements are limited by the number of occurrence and L_1 is the attachment point, that the unlimited reinstatement expected losses can be multiplied by to account for the finite limit on reinstatements. Further details on reinstatement adjustment factors can be found in Anderson[1] and Simon[11].

$$A(\Lambda(L_1), m) = 1 - \frac{e^{-\Lambda(L_1)}}{\Lambda(L_1)} \sum_{n=m+1}^{\infty} (n-m) \frac{(\Lambda(L_1))^n}{n!} \tag{1.6.1}$$

2. EXAMPLES AND THE UNDERLYING MODEL

The examples in this section use loss exceedance curves with only a handful of points for illustration. Real world situations typically involve curves with hundreds of points. The curves are presented in the very useful representation of loss amount, return period, and incremental frequency. Loss exceedance curves in other representations may be easily converted using the methods in Section 1.6. Calculations are made only for expected losses by layer, but loss exceedance curves contain much more probabilistic information that can be useful for calculating many other quantities such as variances. Expected loss is defined to be the sum of loss amounts multiplied by corresponding incremental frequencies without adjustment for reinstatement limitations, but a formula for an adjustment factor is also shown in Section 1.6.

2.1 A Motivational Example: Exposure Rating Property Catastrophe Treaties

REINSURER, a multi-line reinsurance company, assumes large accounts of reinsurance for two companies, MUTUAL and COMMERCIAL. Both accounts involve almost comprehensive programs of quota share, liability per occurrence, and property per risk cessions, but no catastrophe per occurrence excess. One day before renewal each cedant company expresses interest in adding $200 million xs $100 million catastrophe coverage on its Florida homeowners portfolio. There is no readily available detailed exposure profile ready for either company's Florida homeowners portfolio. Neither is there enough time to run models and review the output for a cession rate quote, even if the profile data was available.

However, the catastrophe modeling firm CATMOD has published an estimate that the total industry hurricane loss exceedance curve for homeowners insurance in Florida in a trade publication as shown in Table 2.1.1

Table 2.1.1

Hypothetical Loss Exceedance Curve for Florida Hurricanes

Event Loss	Return Period in Years	Incremental Frequencies
1,000,000,000,000	500	0.00200
100,000,000,000	100	0.00800
10,000,000,000	10	0.09000
1,000,000,000	5	0.10000

MUTUAL and COMMERCIAL each report $100 million of Florida homeowners premium, and total industry Florida homeowners premium underlying the CATMOD curve is estimated to be $5 billion. The ACTUARY at REINSURER reviews the article with the

CATMOD loss exceedance curve and begins to consider what relative frequency and severity assumptions should made for MUTUAL and COMMERCIAL, respectively.

MUTUAL's exposures tend to be spread throughout the state of Florida. Some of its business is urban, but most is suburban or rural. ACTUARY decides that 100% of Florida hurricane events will noticeably affect MUTUAL. All other things being equal this assumption, together with MUTUAL's 2% of premium, would imply about 2% of losses from every event go to MUTUAL, but ACTUARY thinks that the share of event losses should be reduced to 1%, since MUTUAL probably has a higher proportion of the market that is far inland and in the northern part of Florida. ACTUARY decides that the loss exceedance curve for MUTUAL should look something like Table 2.1.2.

Table 2.1.2

Estimated MUTUAL Florida Hurricane Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
10,000,000,000	500	0.00200
1,000,000,000	100	0.00800
100,000,000	10	0.09000
10,000,000	5	0.10000

COMMERCIAL's exposures tend to be heavily concentrated in Miami and Tampa. ACTUARY thinks that only about 20% of Florida hurricane events will noticeably affect COMMERCIAL. All other things being equal COMMERCIAL's 2% of premium would imply about 2% of overall expected losses and hence 10% of event losses from those events for which it is noticeably affected. ACTUARY thinks that the share of event losses should be increased to 20%, since COMMERCIAL has a higher proportion of the costal and near costal market. ACTUARY decides that the loss exceedance curve for MUTUAL should look something like Table 2.1.3.

Table 2.1.3

Estimated COMMERCIAL Florida Hurricane Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
200,000,000,000	2500	0.00040
20,000,000,000	500	0.00160
2,000,000,000	50	0.01800
200,000,000	25	0.02000

ACTUARY multiplies layered loss amounts by incremental frequencies and sums them up to estimate expected losses for the two treaties in Table 2.1.4.

Table 2.1.4

Exposure Rates For MUTUAL and COMMERCIAL

MUTUAL

Layered Loss	Incremental Frequencies	Expected Layered Loss
200,000,000	0.00200	400,000
200,000,000	0.00800	1,600,000
0	0.09000	0
0	0.10000	0
	Total Expected Layered Loss	2,000,000

COMMERCIAL

Layered Loss	Incremental Frequencies	Expected Layered Loss
200,000,000	0.00040	80,000
200,000,000	0.00160	320,000
200,000,000	0.01800	3,600,000
100,000,000	0.02000	2,000,000
	Total Expected Layered Loss	6,000,000

Overall REINSURER usually targets a loss ratio of about 60% corresponding to its target ROE of 15%. ACTUARY considers how the risk/return tradeoffs of the two new cat treaties, respectively, will marginally REINSURER's overall risk/return profile (Stanard et al [13]). ACTUARY believes that the lower frequency, higher severity COMMERCIAL treaty will place a higher than average burden on REINSURER's marginal risk based capital and retrocessional costs, and MUTUAL will cause a correspondingly lower than average burden. So, ACTUARY selects 55% and 65% as the target loss ratios for COMMERCIAL and MUTUAL, respectively, resulting in the quotes in Table 2.1.5.

Table 2.1.5

Quotes for MUTUAL and COMMERCIAL

Ceding Company	MUTUAL	COMMERCIAL
Expected Layered Loss	2,000,000	6,000,000
Target Loss Ratio	65%	55%
Ceded Premium Quote	3,076,923	10,909,091
Subject Premium	100,000,000	100,000,000
Cession Rate Quote	3.1%	10.9%

2.2 The Underlying Model

Suppose a sub-portfolio is described by three parameters:

r = relative frequency or the fraction of portfolio catastrophe loss events, at each size of loss level, in which the sub-portfolio experiences losses.

s = relative severity or the fraction of portfolio losses which are incurred by the sub-portfolio for each catastrophe loss event in which the sub-portfolio experience losses.

p = relative exposure or the ratio of sub-portfolio expected catastrophe losses to portfolio expected catastrophe losses.

These definitions immediately imply the following constraints in Formulae 2.2.1.

$$0 < r, s, p < 1 \qquad \qquad (2.2.1)$$
$$p = rs$$

Notice that two of the three quantities r, s, and p, uniquely determine the third.

Suppose that (L_i , λ (L_i)) one of a finite number of points on a discrete catastrophe loss exceedance curve for the entire portfolio, where $L_{i+1} > L_i$, and λ (L_i) is the incremental frequency of events of size L. Since F strictly non-decreasing as L increases, (r L_i, s λ (L_i)) is a point on the loss exceedance curve of the sub-portfolio.

The key implicit assumptions of this model are:

1. The severity distribution for the sub-portfolio is equal to the severity distribution of the total portfolio for losses, with losses rescaled by a positive number.
2. The sub-portfolio is only affected a certain fraction of the events affecting the total portfolio.

In reality the r and s would vary by Li, but for practical purposes in situations with limited information constant values often the best estimate that can be made. Variability of r and s by size of loss event will be dealt with in Section 3.

It is useful to conceptually relate relative frequency and severity to the diversification and concentration structure both for risks inside the sub-portfolio and for the sub-portfolio relative to the rest of the total portfolio. Sections 2.3 and 2.4 describe the correlation implications of r = 1 and s = 1, respectively and Appendix A presents a more detailed relationship between r, s, and correlation of sub-portfolios.

2.3 Extreme Situation: 100% Relative Frequency, All Events Affect All Sub-Portfolios, Highly Diversified Sub-Portfolios

A portfolio could be partitioned into non-overlapping sub-portfolios, each of which is assumed to share losses from any event which are in exactly the same proportion as its proportion of overall expected losses. In this case the losses of the sub-portfolio are 100% correlated, the relative frequencies are all 100% and the relative severities are equal to the relative exposures. Any sub-portfolio experiences the same frequency and a lower severity of losses. Generally speaking, this would be the case where risks in each sub-portfolio were randomly from the total portfolio.

2.4 Extreme Situation: Individual Events Affect Only One Sub-Portfolio, Highly Concentrated Sub-Portfolios

Alternatively, each sub-portfolio in a partition of the total portfolio could be assumed to experience all of the losses for only a certain fraction total portfolio events. In this case the losses of the sub-portfolio are 0% correlated, and the sum of relative frequencies of sub-portfolios is 100%. Generally speaking, this would be the case where risks in each sub-portfolio were selected to be all the risks with the same key risk characteristics, such as geographical location, which determine whether a particular risk is affected by a certain catastrophe event.

2.5 Characteristic Relationships Between Risk Correlations, Relative Frequency, and Relative Severity

For a fixed relative exposure p as relative frequency r goes up and relative severity s goes down, the sub-portfolio is more correlated with the entire portfolio. Correspondingly, one can expect that as the correlation between a sub-portfolio and the rest of the total portfolio goes up, the relative frequency tends to be higher (sub-portfolio affected by more events} and relative severities tends to be lower (affected less severely by events).

The higher correlation between a sub-portfolio and the rest of the total portfolio tends to accompany lower correlation of risks within the sub-portfolio. So diversification within a sub-portfolio often corresponds to a high correlation with the rest of the portfolio. Similarly, concentration inside the sub-portfolio often corresponds to low correlation with the rest of the total portfolio.

The general relationships described above are important considerations for making practical selections of relative frequencies, relative severities, and/or relative exposures to estimate sub-portfolio loss exceedance curves. Formula 2.5.1 shows two particularly useful heuristic formulae (derived from a model in Appendix A) for the correlation coefficient between the sub-portfolio and the rest on the total portfolio when constant relative frequency and severity are assumed.

$$\begin{aligned} \rho &= (r-p)/(1-p) \\ \rho &= r(1-s)/(1-rs) \end{aligned} \tag{2.5.1}$$

For example, if r =0.5 and s = 0.5 then the sub-portfolio losses will tend to have a 33% correlation with losses in the rest of the total portfolio.

2.6 Example – Risk Based Capital and Solvency

Since September 11, 2001 the insurance industry has recognized terrorism as a potential catastrophic peril. The insurance company COMMERCIAL is doing a general review of its capitalization and risk based capital adequacy with consideration to terrorism exposure. The primary catastrophe solvency criterion used by COMMERCIAL is the rule of thumb that its 100 year return period catastrophic loss, net of reinsurance and alternative risk transfers (ARTs), should be less than its surplus.

COMMERCIAL has modeled all of its property and casualty business for natural perils such as earthquakes and hurricanes, but not for terrorism, to produce the overall loss exceedance curve in Table 2.6.1.

Table 2.6.1

COMMERCIAL Natural Peril Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
220,000,000,000	350	0.00286
10,000,000,000	100	0.00714
5,000,000,000	50	0.01000
2,000,000,000	10	0.08000
900,000,000	5	0.10000

Since COMMERCIAL's surplus is $15 billion and its 100 year return period loss is $10 billion it easily satisfies its rule of thumb criterion, except possibly for terrorism exposure.

The cat modeler CATMOD has modeled terrorism for the entire United States property and casualty market to produce the loss exceedance curve in Table 2.6.2.

Table 2.6.2

Hypothetical Industry Terrorism Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
600,000,000,000	1000	0.00100
150,000,000,000	300	0.00233
60,000,000,000	50	0.01667
1,000,000,000	25	0.02000
500,000,000	10	0.06000

In the absence of model results specifically for COMMERCIAL which include terrorism, COMMERCIAL performs a simple estimate based on selections for relative frequency and relative severity applied to the CATMOD industrywide terrorism loss exceedance curve. The CATMOD model assumes that 50% of all future terrorism events will happen in just four of the largest U.S. cities: New York, Los Angeles, Chicago, and Washington, D.C. As COMMERCIAL's market share is around 35% in these cities, COMMERCIAL selects a terrorism relative frequency of 55% and relative severity of 30% for itself. These parameters imply that since the industrywide 50 year return period terrorism loss is $60 billion, COMMERCIAL's 91 year return period loss, just for terrorism, is $18 billion, as shown in Table 2.6.3.

Table 2.6.3

Estimated COMMERCIAL Terrorism Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
180,000,000,000	1818	0.00055
45,000,000,000	545	0.00128
18,000,000,000	91	0.00917
300,000,000	45	0.01100
150,000,000	18	0.03300

COMMERCIAL decides to perform more detailed terrorism modeling and analysis to determine whether its terrorism exposure now requires additional surplus, reinsurance, alternative risk transfers, and/or other risk management adjustments to bolster its solvency position.

2.7 Example – Individual Risk Catastrophe Deductible Credits

COMMERCIAL offers a special comprehensive commercial package policy, which excludes catastrophic perils, but offers the option of a large dollar deductible. A policyholder PHOLDER with 500 employees and $150 million of annual revenue is charged a no deductible premium of $2 million. Based on a frequency severity simulation model of non-catastrophic losses COMMERCIAL offers a deductible credit of 60% for an aggregate annual deductible of $2 million. So with the $2 million deductible the premium is $0.8 million for the non-catastrophe policy.

Because of increasing concern about natural catastrophe perils, PHOLDER is requesting extra coverage for natural catastrophes. COMMERCIAL does not use a catastrophe model for individual risks, but believes that extra premium for such extra coverage with no deductible should be about $100,000 for PHOLDER based on consideration of certain key risk characteristics. Since all 500 work in a single building in downtown Miami, which is the PHOLDER's only property, COMMERCIAL believes that catastrophe losses for the PHOLDER will be low frequency and high severity, and that the 60% deductible is far too high for the catastrophe coverage.

Since COMMERCIAL's total annual premium for natural catastrophe losses is $2 billion, the relative exposure of this policyholder is 0.005%. Due to PHOLDER's highly concentrated risk characteristics COMMERCIAL selects a relative frequency of 1%, making relative severity 0.5%. So, the estimated natural catastrophe loss exceedance curve for the PHOLDER derived from Table 2.6.1 is shown in Table 2.7.1.

Table 2.7.1

Estimated PHOLDER Natural Catastrophe Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
1,100,000,000	35,000	0.000029
50,000,000	10,000	0.000071
25,000,000	5,000	0.000100
10,000,000	1,000	0.000800
4,500,000	500	0.001000

This leads to an estimated deductible credit based on expected pure losses eliminated of 4,000/50,000 = 8% from the calculations in Table 2.7.2.

Table 2.7.1

Estimated PHOLDER Deductible Loss Elimination

Gross Loss	Deductible Loss	Incremental Frequency	Expected Gross Loss	Expected Deductible Loss
1,100,000,000	2,000,000	0.000029	31,429	57
50,000,000	2,000,000	0.000071	3,571	143
25,000,000	2,000,000	0.000100	2,500	200
10,000,000	2,000,000	0.000800	8,000	1,600
4,500,000	2,000,000	0.001000	4,500	2,000
			50,000	4,000

3. VARIABLE RELATIVE FREQUENCY AND SEVERITY

The assumption of constant relative frequency and severity across all loss sizes is grossly unrealistic in some circumstances. Consider a single $4 million property risk PROPERTY located in Brooklyn, New York with annual premium of $20,000. Suppose a terrorism loss exceedance curve for all property in New York City with total annual premium of $2 billion is given by Table 3.1.

Table 3.1

Hypothetical NYC Terrorism Property Loss Exceedance Curve

Event Loss	Return Period in Years	Incremental Frequencies
1,000,000,000,000	10000	0.00010
100,000,000,000	1000	0.00090
10,000,000,000	100	0.00900
1,000,000,000	10	0.09000
1,000,000	1	0.90000

For the 1 year return period event a reasonable selection might be r = 0.001% and s = 100%, as such a small event would probably only affect about a single property policy and if PROPERTY were the policy hit it would probably sustain about all of the $1 million in damages. However, the 10,000 year event would tend to saturate the entire New York City area, representing an event comparable to a hydrogen bomb detonation in Manhattan.

PROPERTY would stand much greater than a 1 in 100,000 chance of being affected and obviously could only experience a tiny fraction of the $1 trillion in damages. A better selection for this level of loss would be r =100% and s =0.0004%.

In this sort of a situation, where sufficient information about the nature of the sub-portfolio and the type of events underlying the total loss exceedance curve is available, there is no reason why the constant relative frequency and severity model cannot be generalized to functions $r(L_i)$ and $s(L_i)$ as shown in Table 3.2 and used to derive an estimated sub-portfolio loss exceedance curve such as shown in Table 3.3.

Table 3.2

Terrorism Variable Relative Frequency and Severity for PROPERTY

Return Period in Years	$r(L_i)$	$s(L_i)$.
10000	1.00000	0.0000040
1000	0.10000	0.0000313
100	0.01000	0.0002500
10	0.00100	0.0020000
1	0.00001	1.0000000

Table 3.3

Estimated Terrorism Loss Exceedance for PROPERTY

Event Loss	Return Period in Years	Incremental Frequencies
4,000,000	10000	0.00010
3,125,000	1000	0.00090
2,500,000	100	0.00900
2,000,000	10	0.09000
1,000,000	1	0.90000

4. CONCLUSIONS

Modern catastrophe modeling computer programs can be tremendously valuable. These software models quantify the exposure of insurance portfolios to catastrophic losses based on what physical scientists, engineers, and social scientists know about perils which may cause catastrophic losses, such as earthquakes, hurricanes, floods, tornados, and terrorism. In many practical situations the resources, time, and detailed portfolio exposure information needed to fully utilize such models for analysis of sub-portfolios – or even the model software itself – may not be available. In such situations it is possible to use total portfolio loss exceedance curves and other information about a sub-portfolio's risk concentration diversification characteristics to select a relative frequency and a relative severity. Then the calculation of a loss exceedance curve for the sub-portfolio and a variety of analyses based on it can easily be performed in very little time with a simple spreadsheet. The resulting loss estimated exceedance curve is by no means equivalent to that which would be produced by the actual catastrophe model. However, in some situations with limited information, time, or resources it may be a practical and reasonable substitute.

Appendix A - The Correlation Coefficient Interpretation

If the total portfolio loss random variable is Z with frequency given by the Poisson random variable N and severity given by the random variable X the variance of losses is given by the Formula A.1.

$$Var[Z]=E[N]\ E[X^2] \tag{A.1}$$

We can think of Z as a sum of loss random variables for disjoint sub-portfolios. If these sub-portfolios were 100% pair wise correlated then the total portfolio standard deviation would be equal to the sum of the standard deviations of the sub-portfolios. Alternatively if the sub-porfolios were 0% pair-wise correlated then the total portfolio standard deviation would be equal to the square root of the sum of the variances (standard deviations squared) of the sub-portfolios.

For a sub-portfolio with r and s relative frequency and severity, respectively, and loss random variable Y the variance of losses is given by the Formula A.2.

$$Var[Y] = E[rN]\ E[(sX)^2]=rs^2\ E[N]\ E[X^2] = rs^2\ Var[Z] \tag{A.2}$$

Now if Z is equal to the sum of k sub-portfolios identical to Y and the pair-wise correlation coefficient between different sub-portfolios is equal to the constant ρ the total portfolio variance of losses is given by the Formula A.3.

$$\begin{aligned} Var[Z] &= k\ Var[Y] + k\ (k\text{-}1)\ \rho\ (Var[Y])^{1/2}\ (Var[Y])^{1/2} \\ &= (k + k\ (k\text{-}1)\ \rho)\ Var[Y] \\ &= (k + k\ (k\text{-}1)\ \rho)\ rs^2\ Var[Z] \end{aligned} \tag{A.3}$$

Algebraically this implies Formula A.4.

$$(k + k\ (k\text{-}1)\ \rho)\ rs^2 = 1 \tag{A.4}$$

Since $1/k = p = rs$ Formula A.4 can be rewritten as Formula A.5.

$$(p + (1\text{-}p)\ \rho) / r = 1 \tag{A.5}$$

Formula A.5 can be restated into the particularly interesting forms in Formulae A.6.

$$\rho = (r-p)/(1-p)$$
$$\rho = r(1-s)/(1-rs) \tag{A.6}$$

So, if r = 1 then s = p and the correlation between the sub-portfolios is 100%. Similarly, if r = p then s = 1 and the correlation is 0%.

Appendix B - Consistency of Estimates of Relative Frequency and Relative Severity

Since expected losses for the total portfolio must equal the sum of the individual expected losses for a partition into disjoint sub-portfolios we have the constraint on constant relative frequencies and severities given by Formulae B.1.

$$\boldsymbol{E[N]\,E[X] = \sum_{j} E[r_j\,N]\,E[s_j\,X] = E[N]\,E[X]\sum_{j} r_j\,s_j}$$

$$\boldsymbol{\sum_{j} r_j\,s_j = 1} \tag{B.1}$$

However, this constraint is not sufficient to prevent inconsistencies. For example Table B.1 satisfies Formulae B.1 but still leads to an inconsistent interpretation.

Table B.1

Inconsistent Partition into Sub-portfolios

Sub-Portfolio	r_j	s_j
A	0.50	0.50
B	1.00	0.75

These selections satisfy the previous constraint but imply that 50% loss events affect both A and B which together experience 125% of the losses from such events. To avoid this type of inconsistency in selecting constant relative frequencies and severities for a partition of the total portfolio into a finite set of sub-portfolios, the selections of r and s must also meet following tiling condition:

- The set of rectangles with length r_j and width s_j, respectively, must completely cover the unit

The set of rectangles with length r_j and width s_j, respectively, must completely cover the unit square (length and width both equal to 1) without any overlap and without rotation. For example the relative frequencies and severities for the partition into 6 sub-portfolios given in Table B.2 is shown to be consistent by the tiling in Figure B.1.

Table B.2

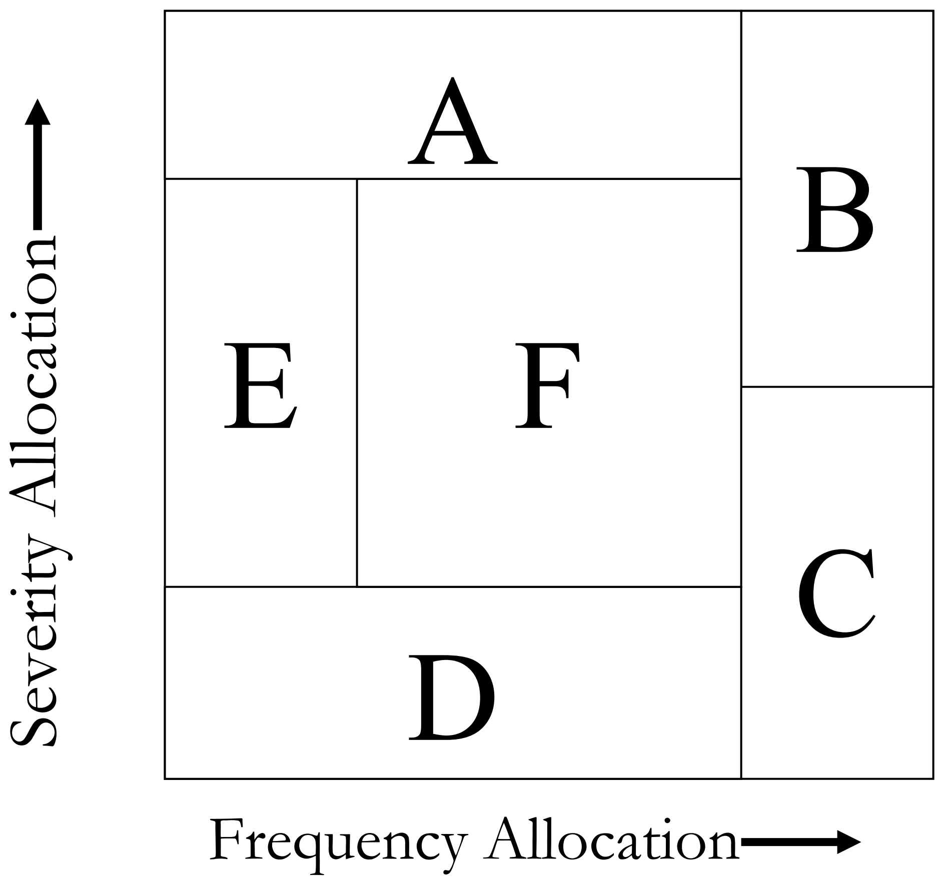
Consistent Partition Into Sub-portfolios

Sub-Portfolio	r_j	s_j
A	0.75	0.25
B	0.25	0.50
C	0.25	0.50
D	0.75	0.25
E	0.50	0.25
F	0.50	0.50

In the general case where r_j and s_j are allowed to vary as functions of the size of loss L_i, a tiling must exist for each L_i for consistency. As a practical matter, using judgment to divide up a unit square into rectangles such as in Figure X may be a good way to select r_j and s_j. The unit square offers a graphical tool where more horizontal coordinate overlap corresponds between rectangles tends to indicate higher correlation and the area of a rectangle is proportional to the share of overall expected losses.

Figure B.1

Tiling Demonstration of Relative Frequency and Severity Consistency



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Biography of Author

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Reinsuring for Catastrophes through Industry Loss Warranties – A Practical Approach

Ali Ishaq, FCAS, MAAA

Abstract:

Within the last couple of decades natural and man-made catastrophes have become a source of increasing concern for the insurance industry. Industry Loss Warranties (ILWs) are reinsurance products whose payout is triggered by catastrophic insured loss. There is a growing market for ILWs because they provide a viable alternative to traditional reinsurance and catastrophe bonds for mitigating losses from such events.

This growing market requires a consistent and sound way of pricing ILWs. The process is made simpler because pricing ILWs does not require knowledge of individual client's exposures but only the expected industry losses. Available catastrophe models provide a ready source of industry loss distributions. Conceptually it is simple to go from a given industry loss distribution to pricing an ILW, but ILWs can vary in their terms and conditions depending on the needs of a particular client. This paper shows how to account for some of these terms and conditions to price ILWs and provides an example of such calculations.

Keywords: Industry Loss Warranty (ILW); OLW; Catastrophe Reinsurance; Pricing Rare Events; Empirical Loss Distributions

"Everything should be made as simple as possible, but not simpler."

--Albert Einstein

1. REINSURANCE AND RARE EVENTS

Extremely rare events, by their very nature, are hard to insure. Insurance companies like to cover events that are rare for the insured, but in aggregate have a distribution that is stable and predictable. In one class of rare events, i.e., natural catastrophes, the loss is either very severe or the damage so wide-spread that there is a real need for primary insurers to spread the risk of loss from these events through some reinsurance mechanism. Until the large losses from natural catastrophes in the last few decades from events like Hurricane Andrew, primary insurers were content to mitigate the risks from natural catastrophes through traditional reinsurance instruments including treaty and facultative reinsurance. Before this time there were a few who had raised the alarm of losses from natural catastrophes large enough to shake the financial foundations of the P&C insurance industry, but not much attention was paid to their fears. Since then, the concern over catastrophic property exposure has been continually growing and newer market mechanisms, such as catastrophe bonds and industry loss warranties, are becoming more established.

2. WHAT IS AN INDUSTRY LOSS WARRANTY?

There are many kinds of instruments gathered under the rubric of industry loss warranties (ILW) also known as original loss warranties (OLW). Essentially they all cover losses from events where the industry-wide insured loss exceeds some pre-agreed threshold[1]. This structure, i.e., where the operative trigger is an industry loss rather than the company's own loss, implies some risk that there could be a loss to the reinsured portfolio without triggering the ILW if the corresponding industry loss is smaller than the industry trigger amount. This is the 'basis risk' for the reinsured. This risk is higher for companies whose exposure concentrations are farther away from the industry averages. Therefore ILW covers are typically bought by companies whose portfolios closely follow the market. This disconnect can be mitigated to some extent by choosing the right kind of trigger. The trigger amount can vary by geography, level, and the kinds of events that contribute to it.

There are many kinds of industry loss warranties available in the market. The variety comes from the kind and level of the industry loss chosen. The industry loss considered as a trigger can vary by amount or geographic scope. For example, an ILW may promise to pay when one of the following happens:

1. A hurricane with industry-wide insured loss in Florida in excess of $15 billion but less than $25 billion.
2. A winter freeze with industry-wide insured loss in North America in excess of $20 billion.
3. An earthquake with industry-wide insured property loss in excess of $35 billion anywhere in the world.
4. Second wind loss with industry-wide insured loss in excess of $10 billion anywhere in the US and territories.

[1] In order for these contracts to qualify as insurance there may be a second condition for loss payment that the insured company's loss exceed a pre-determined amount in addition to the industry being in excess of the given threshold. Usually the insured company's loss trigger is set to such a low level that for our analysis we can ignore it.

In the first case if the hurricane does a lot of damage to property that is not insured but the insured amount is small or if the damage is outside the State of Florida, the ILW may not be triggered. In the second example the total of $20 billion may arise from damage in both US and Canada. And in the third instance, the location of the earthquake does not matter but the loss would have to exceed $35 billion after casualty losses have been excluded before the ILW is triggered. There can be industry loss warranties that respond to a second-event of its type. The fourth example above is an instance of such a second event ILW.

All the above types together can be thought of as Occurrence ILWs, because they respond to (first or second) occurrence of single large events. ILWs can also be structured so that the coverage applies, not in the case of one large event, but when a series of catastrophes in a year add to exceed a pre-determined amount. For these ILWs, the industry losses contributing to the total are limited so that a single event would not trigger the coverage. Only losses above a certain amount are considered towards the total because there are few industry mechanisms to keep track of smaller losses. For example one could construct an aggregate ILW that pays when all losses in California that cause insured damage of least $100 million but not in excess of $5 billion sum to more than $3 billion in a twelve month period. In either case there may be a provision for reinstatement of the ILW limit upon payment of an agreed premium.

3. INDUSTRY LOSS WARRANTIES COMPARED TO OTHER CATASTROPHE REINSURANCE INSTRUMENTS

Catastrophe bonds, traditional reinsurance, and industry loss warranties each have strengths and weaknesses with respect to their ability to address the risk from catastrophic events. The following table summarizes some of these comparisons:

	Traditional Reinsurance	**Catastrophe Bonds**	**Industry Loss Warranties**
Availability	Wide	Limited	Increasing
Transaction Cost	Medium	High	Low
Risk Charge	High	High	Low
Basis Risk	Small	Small	Variable
Pricing Risk	Medium	Large	Small

Traditional reinsurance is widely available but the risk charge for layers that cover catastrophic risks may be large. Part of this is due to the difficulty in estimating the reinsured company's future losses in these very high layers. For the reinsured the basis risk is small, but the reinsurer has to estimate future losses from a historical portfolio that may be different from the future distribution of insured exposures: there is usually a lag between the information the reinsurer uses to price and the actual exposure that emerges over the insured period. This creates a kind of reverse basis risk and the reinsurer has to charge a larger risk load to cover this risk.

Successful catastrophe bond offers have allowed some large P&C insurers to access the large capital capacity of the world bond market, but these products are still considered non-traditional and complex by many bond traders. These bond offerings are more complex than traditional bond offerings and so this skepticism, on the part of the bond traders, may translate into limited marketability and a higher risk charge. Catastrophe bond offerings are generally not standardized and each offering has to be individually structured and underwritten. This translates into high transaction and fixed costs, which may put these instruments outside the reach off all but the largest insurers.

Industry loss warranties are unfamiliar to many primary companies but, properly structured, may provide an inexpensive solution to many of the catastrophic reinsurance needs that these companies may have. These products have low transaction costs because the pricing risk for the sellers is comparatively low; they do not have to evaluate the expected losses to the reinsured portfolio from a given trigger, but only the loss distribution of the industry portfolio. This lowers the uncertainty and thus the needed risk margin.

The pricing risk for ILWs is lower but not zero. There is still the inherent parameter risk from trying to estimate the loss distribution of events whose frequency is not known and may be changing. Some scientists have postulated that there is a long-term climate cycle which governs the changes in frequencies of large weather events. One study estimates that this climate cycle be as long as 100,000 years[2]. Since the data available to formulate the frequency distributions is rarely longer than a few decades, there can be large error in our estimates of future probabilities of these catastrophic events.

[2] See study by Mukul Sharma in the June 10th issue of Earth and Planetary Science Letters (Elsevier, volume 199, issues 3-4)

4. THE MARKET FOR INDUSTRY LOSS WARRANTIES

ILWs can be used to reduce a company's exposure to sharp losses from large events or a collection of events thus controlling the tail of the aggregate loss distribution at a reasonable price. For many companies the tradeoff between cost and stability gained can be very favorable, even when comparing to other reinsurance products. As hinted above, there needs to be care in selecting the appropriate trigger to reduce the basis risk for the ceding company.

ILWs have generally been bought by the larger national insurers but they may be even more useful to the regional insurers. The large insurers have generally been the first ones to utilize this market because they tend to have a more sophisticated view of risk and price within the reinsurance market. This product provides these large buyers another reinsurance option for spreading the risk from large events over a larger market capacity. For the regional companies, the ability to cede the risk from extreme events in a concentrated area may enable them to allow larger geographical accumulations, thus allowing them to concentrate on their areas of expertise while staying within their capital constraints.

The relatively low cost of ILWs is due to the lower information asymmetry as compared to most other reinsurance products. Normally as we move from primary insurance to facultative reinsurance to treaty reinsurance to retrocession the information asymmetry increases sharply; the reinsurer has to base its pricing decisions on less and less current information as compared to the information that the reinsured is using to make its buying decisions. This implies that the reinsurer has to build an increasingly larger margin for error (or risk) into their estimates of expected loss under the contract.

Since ILWs are priced based on the industry loss distribution this information asymmetry is suddenly reduced (or even reversed, since the reinsured has to estimate the basis risk it may be taking on[3]), and thus the margin for risk can be correspondingly smaller.

5. PRICING AN INDUSTRY LOSS WARRANTY

It is important that the pricing be consistent as well as accurate. The more consistently a product can be priced, the better the prospects of a successful market. If there is no

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[3] See the paper 'On the Basis Risk of Industry Loss Warranteis' by Lixin Zeng in the Summer 2000 issue of The Journal of Risk Finance for a discussion of estimating ILW basis risk.

consistent way of pricing a product the market prices may vary widely, which can lead to a fractured market or a race towards the lowest price provided. This can lead to a lack of confidence and consequently a lack of viability of a market for the product. In order to price ILWs consistently we need an acceptable way to calculate the loss distribution for a given industry loss warranty. For the illustration here an industry loss warranty is completely defined by its trigger. We will assume that every time a trigger is met, we need to pay the limit to the insured or that every loss is a full limit loss. This is not too much of a stretch because partial payments are comparatively rare. This assumption helps simplify the analysis and is not essential for the pricing methodologies described. Since we are using the actual industry loss distribution instead of the frequency alone, these methods can be easily adjusted to account for scenarios in which partial payments are allowed.

We start with the industry loss distribution and the definition of the industry loss warranty we want to price. The first step is to extract the conditional distribution of industry losses that meet the requirements of the trigger. Then we can combine the probabilities of triggering the ILW with the payout conditions in the contract to estimate the expected loss distribution for the contract.

6. CONCEPTUAL SIMPLICITY, PRACTICAL DIFFICULTIES

Conceptually the pricing is simple: calculate the distribution of losses under the contract. The expected value from this distribution is the expected loss under the contract and the shape of the distribution can be used to set a risk load. The sum of the expected loss, expenses, and the risk load is the theoretical premium needed.

Even with the conceptual simplicity it is clear that it would require some work to implement these steps in an actual pricing model. The first difficulty is in obtaining the industry loss distribution. This distribution or set of distributions would indicate the probability of industry-wide insured losses from various types of catastrophes that we want to insure. The most obvious possibility may be to use historical industry losses but unadjusted historical losses are not a good predictor of future industry losses. To be used in our calculations, these losses would have to be adjusted for exposure changes, inflation, and any other factors that would make the expected outcome of a historical event different in the prospective year from the historical year. We would also have to adjust for the fact that historical data is limited and incomplete. This adjustment is compounded if we consider that the historical data is a small slice from a possibly changing distribution of extreme events. If we want to price international ILWs we would also have to compensate for the fact that reliable historical loss data for most jurisdictions outside the US and Japan are not generally available.

A more practical source for industry loss distributions is catastrophe modeling data from commercial catastrophe models. This data is already adjusted for changes in exposure and has a good fit to historical record. In addition, the various probabilities coming out of these models are quickly converging to the industry consensus return time estimates for various kinds of events. There is still considerable variation between the various commercial models, but there is great pressure to estimate probabilities that are in line with the consensus estimates. As these models come closer, pricing derived from the inherent distributions would serve to make the pricing of the ILWs more consistent across the market, and this consistency will in turn make the product more marketable.

Generally the industry loss data from the commercial catastrophe models is available in the form of empirical distributions. Our first thought might be to fit a theoretical size of loss distribution to the output from a simulation model. But this may not be the best course because it raises new difficulties. The shape of the distributions for various perils (wind, earthquake, etc) may be very different and may not allow the use of a simple class of loss distributions. Secondly, since we have to censor or otherwise manipulate these distributions to extract the distributions under the various triggers, this may prove to be difficult or lack closed form solutions. So, we have a tradeoff between realism (fit to actual or prospective losses) and ease of computation. Theoretical distributions that are robust may be hard to use and the analysis may be hard to extend as we develop layers of analyses to sum from individual regions to multiple regions and triggers to the distributions of portfolios. Even after the availability of derivative distributions for the various types of triggers we are still left with the task of farther manipulating these distributions to achieve a consistent pricing. The advantage of symbolic distributions would be that a unique expected loss value and corresponding distribution around this mean would presumable exist for each trigger type and size. But the complexity of the process may make this scenario impractical.

7. EMPIRICAL DISTRIBUTIONS

"God does not care about our mathematical difficulties. He integrates empirically."
--Albert Einstein

As an alternative to fitting theoretical size of loss distributions to the empirical distributions derived from historical losses or from the output of a catastrophe model, we could use these empirical distributions directly in our pricing. If empirical distributions are used wisely, they can provide sound answers and allow us to get to the answers much more quickly the calculation of the final premium starting with one such empirical distribution is illustrated below.

After we obtain the expected future industry-wide insured loss distribution we can modify it to extract a loss distribution that corresponds to a particular trigger. For example, if we know the prospective industry loss distribution from hurricanes in Florida and we are pricing an ILW with a trigger of $20 billion industry loss, we can censor the distribution at $ 20 billion to determine the industry loss distribution above this point. The expected loss under an ILW can be calculated from the probability alone, but we can use the additional information in the distribution of losses that trigger the contract to estimate the appropriate risk load. To these we add the company's expenses to arrive at an estimate of the premium to be charged.

8. AN EXAMPLE

The first step is the output from either the historical loss analysis or a simulated distribution from a catastrophe model. Exhibit 1 shows what such a distribution might look like. Here the distribution is shown as a list of industry loss amounts along with a description of the geographic scope and peril. This distribution represents the expected losses from a 1000-year simulation period. So, if there are 100 losses above $20 billion in this list, we are assuming that the probability of a $20 billion or larger loss is 10% over the next year. The exact format of the loss distribution is not important and as long as enough information is available, the various formats are convertible from one into another.

The covered event for an ILW may be limited as to geographic area as well as peril. For next step, therefore, we construct a new loss distribution selecting only those events from exhibit 1 that meet the definition of our trigger. In this example we would exclude all events that have a Florida loss of less than \$20 Billion. Table 2 illustrates this censored distribution.

The third step is to summarize this distribution by simulation year. In this example this step is used to calculate an annual cost for the ILW. For each year of simulation we calculate the payout as well as the reinstatement premium if any. This calculation can be simplified by our assumption that each time the ILW is triggered there would be a full limit payment and a full reinstatement if there is a reinstatement still available for that year.

Thus for each of the simulated years the model calculates whether a pay-out would happen:

$$PayoutTrigger_{event,year} = \begin{Bmatrix} 1 & \begin{Bmatrix} \text{if } Loss_{event,industry} > Trigger_{\min} \\ and \\ \text{if } Loss_{event,industry} < Trigger_{\max} \end{Bmatrix} \\ 0 & \text{otherwise} \end{Bmatrix}$$

Here:

$$Loss_{event,industry} = Industy\ loss\ from\ a\ given\ event$$
$$Trigger_{\min} = Minimum\ Industry\ loss\ that\ will\ trigger\ a\ payout$$
$$Trigger_{\max} = Industry\ loss\ above\ wich\ there\ will\ be\ no\ payout$$

From the above the total number of losses payments for each year of the simulation is calculated as:

$$PayoutTrigger_{year} = \sum_{event} PayoutTrigger_{event,year}$$

The loss payments for each year are limited by the number of reinstatements allowed for the contract and then multiplied by the limit sold by the reinsurer to derive the loss pay-out for each year of the simulation. This 1000-point loss distribution for the expected pay-out can then be directly used to calculate the expected loss and the variance around the expected value for the treaty being priced. These calculations are shown in Exhibit 3.

Other types of ILWs including second-event covers, corridor covers, collar covers, and a consideration for a 'no-claim bonus' can be priced in a similar manner.

9. CALCULATING THE LOAD FOR RISK

Once the expected loss has been estimated the next step is to determine how much risk or profit margin to charge for a given exposure. In general, we should charge higher premium for higher risk. This can be done using various measures of risk. One way would be to use the coefficient of variation (CV) of loss derived from the pervious analysis to target a profit margin. Once we have decided the relationship between profit load and the CV based on the market and company appetite for risk and the required return on capital the result could be summarized as a relationship between ROE and the CV. If the result is a linear relationship between risk and ROE and if we want to limit the minimum and maximum return to realistically achievable levels this relationship may look like the following:

$$ROE_{\text{Target}} = \begin{cases} ROE_{\min} & \text{if } CV \le CV_{\min} \\ ROE_{\max} & \text{if } CV \ge CV_{\max} \\ \dfrac{CV - CV_{\min}}{CV_{\max} - CV_{\min}} (ROE_{\max} - ROE_{\min}) & \text{if } CV_{\min} < CV < CV_{\max} \end{cases}$$

Where:

ROE_{Target} = Target Rate of Return

CV = *Coefficient of Variation* of loss under the contract

Given the expense and brokerage information, the ROE target can be converted into a loss ratio or a combined ratio target.

Another possible risk load could be based on the risk of loss on the contract once the reinstatement premium is taken into account. One way of calculating this amount is to calculate the profit or loss by year by summarizing the profit or loss from each event in each of years of simulation. If we calculate the expected value and CV of this distribution, this gives us another measure of return that we could target. We could also eliminate all years in which there is a profit which would leave years in which the contract is in a deficit. If our risk appetite is more in line with limiting the loss from a contract in any one year we could use the expected value of this distribution as a measure of expected downside and use this as another constraint on our final price or the desirability of a given contract and the market price.

10. OTHER ISSUES

10.1 International ILWs

Reinsurance markets outside the US are smaller and less developed. Therefore, even though exposure is much higher in US the risk margins in ILWs are generally lower. This is partly due to the higher parameter uncertainty in Asia and Europe. The catastrophe data and modeling are have more inherent uncertainty outside the US and therefore the reinsurer should charge a premium for taking up the risk arising from this uncertainty. As a result, the product may get too expensive compared to the traditional reinsurance and government guarantees. To the extent that ILWs compete on the basis of price with traditional reinsurance, they may be at a disadvantage in the international market, until either the traditional reinsurance prices rise or the catastrophe potential and the corresponding models improve. Some markets are providing worldwide coverage without charging much of a premium for the high parameter risk inherent in the worldwide models. For large buyers this may be an excellent opportunity in the short term, but one wonders if this aggressive stance is sustainable in the long run.

10.2 Aggregate ILWs

An example of pricing Aggregate ILWs has not been provided in this paper, but a methodology very similar to the one used here can be employed to calculate the loss distribution of an Aggregate ILW. Typically these products pay when the sum of industry losses between a minimum and maximum in a year exceed the trigger amount. For example, if an aggregate ILW gets triggered when the sum of industry losses that exceed $100 million but limited to $500 million exceeds $4 billion in a year, one would add losses exceeding $100 million limited to $500 million for each year of simulation and test if this sum exceeds the $4 billion trigger amount. This process would result in a loss and corresponding premium and profit distributions very similar to what result from an occurrence ILW calculation shown herein.

10.3 Lack of Protection from Unforeseen Events

Generally ILW sellers look backwards when deciding what kinds of triggers to offer. Before Hurricane Andrew there was probably not a big market for Florida Hurricane ILWs with triggers in excess of $15 billion. And more recently some buyers look for second and third event coverages, but these generally have triggers above $2 billion. Aggregate ILWs generally exclude losses with industry loss amounts above $500 million. Therefore none of these ILWs would have protected companies in 2004 when a series of hurricanes hit the Southern US Coast, with industry losses in the range of $1 billion each so that none of the ILWs mentioned above would have triggered. This again illustrates how ILWs have less parameter risk than some other reinsurance products where unforeseen losses may account for a large amount of loss in bad years.

10.4 Model Creep

On a related note, a one may notice that the commercial catastrophe models generally get recalibrated after almost every extreme event. This will probably mean that if before 2004 the models indicated four large hurricanes hitting Southern US as extremely improbable, these distributions would be revised.

10.5 Managing a Book of Industry Loss Warranty Business

Once we have appropriately priced a book of ILWs the next question is how to manage this book and the aggregate risk it contributes to the reinsurer's portfolio. If we have used a consistent set of loss distributions to price the ILWs, the aggregate loss distribution for the portfolio can be calculated as the sum of distributions of individual ILWs. This loss distribution can then be combined with loss distributions from other product portfolios to generate a companywide loss distribution. This combined loss distribution could act as a robust input into a DFA or ERM model to gauge the overall risk–return profile of the company.

11. CONCLUSION

Industry Loss Warranties provide an alternative to traditional reinsurance and catastrophe bonds when insurers are trying to smooth their results from the impact of catastrophic events. We have looked at one way to price for these instruments that takes advantage of the empirical industry loss distributions available as an output from many commercial catastrophe reinsurance models to simplify this task.

Exhibit 1: Simulated 1000-year Industry Loss Distribution

Year Number	Industry Loss (Millions)	Catastrophe Description
4	4,679	FL Hurricane
4	2,586	FL Hurricane
5	2,948	Winter Storm
7	19,000	FL Hurricane
8	3,438	FL Hurricane
10	9,242	CA EQ
10	3,304	FL Hurricane
12	3,293	FL Hurricane
14	5,234	Winter Freeze
15	4,636	FL Hurricane
16	2,949	FL Hurricane
17	26,424	CA EQ
17	7,532	FL Hurricane
17	5,419	NY Hurricane
17	4,426	FL Hurricane
19	24,939	CA EQ
20	2,739	FL Hurricane
20	2,603	FL Hurricane
20	2,165	Winter Freeze
23	3,912	FL Hurricane
24	2,441	FL Hurricane
26	20,638	FL Hurricane
26	5,507	FL Hurricane
27	2,573	CA Landslide
28	4,946	FL Hurricane
29	9,626	CA EQ
.	.	.
.	.	.
.	.	.

Exhibit 2: Losses Meeting ILW Trigger

Year Number	Industry Loss (Millions)	Catastrophe Description
26	20638	FL Hurricane
42	24801	FL Hurricane
63	24323	FL Hurricane
153	20977	FL Hurricane
179	30669	FL Hurricane
205	22307	FL Hurricane
232	23976	FL Hurricane
288	27315	FL Hurricane
343	34381	FL Hurricane
431	33108	FL Hurricane
438	20223	FL Hurricane
467	28063	FL Hurricane
467	26904	FL Hurricane
518	70029	FL Hurricane
614	28195	FL Hurricane
640	22597	FL Hurricane
725	29006	FL Hurricane
730	22173	FL Hurricane
779	22259	FL Hurricane
793	20996	FL Hurricane
811	47370	FL Hurricane
866	22261	FL Hurricane
893	56128	FL Hurricane
897	37107	FL Hurricane
908	21207	FL Hurricane
966	20701	FL Hurricane

Terms:

ILW Limit = 100 million

1 reinstatement at 150% initial premium

Epense = 20% of Premium

Initial ROL 5%

Exhibit 3: Simulated Results for the ILW

Year Number	ILW loss	Premium	Profit/ Loss
1	0	5	4
2	0	5	4
3	0	5	4
4	0	5	4
5	0	5	4
6	0	5	4
7	0	5	4
8	0	5	4
9	0	5	4
10	0	5	4
.	.	.	.
.	.	.	.
.	.	.	.
26	100	12.5	-90
.	.	.	.
.	.	.	.
.	.	.	.
467	200	12.5	-190
.	.	.	.
.	.	.	.
.	.	.	.
1000	0	5	4
μ	2.6	5.19	1.55
δ	16.83		
μ/δ	6.47		

Biography of the Author

Ali Ishaq went to school at the University of Texas at Austin where he studied Mathematics. He started his career at the Texas Department of Insurance where he built models to predict insurance company insolvencies. Later, at Aetna Property & Casualty he constructed underwriting models based on Neural Networks. At CNA Reinsurance Company he helped manage the property catastrophe aggregates and built pricing models for various reinsurance products. Currently, Mr. Ishaq is a Regional Actuary at Zurich American Insurance where he works on a diverse book of construction related insurance business.

On the Optimality of Proportional Reinsurance

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Abstract:

Proportional reinsurance is often thought to be a very simple method of covering the portfolio of an insurer. Theoreticians have not been particularly interested in analysing the optimality properties of these types of reinsurance covers. In this paper, we will use a real-life insurance portfolio in order to compare four proportional structures: quota share reinsurance, variable quota share reinsurance, surplus reinsurance and surplus reinsurance with a table of lines. We adopt the point of view of the ceding company and propose ways to optimize the proportional covers of the primary insurer.

Keywords: Proportional Reinsurance, Quota Share Reinsurance, Variable Quota Share, Surplus Reinsurance, Table of Lines, Optimality, RORAC, de Finetti, Individual Risk Model.

1. INTRODUCTION

It is well-known in literature that non-proportional reinsurance is more efficient compared to proportional reinsurance. See e.g. Vermandele and Denuit (1998) where it is proved that the retention of an insurer covered by an excess of loss treaty is smaller in the stop-loss order than the retention covered by any other reinsurance of the individual type (i.e. compensation on a claim by claim basis) under the hypothesis that the expected retained loss is the same in both situations as well as the loading of the reinsurer. Vermandele and Denuit (1998) also show that the retention of an insurer covered by a stop-loss treaty is smaller in the stop-loss order than the retention covered by any other reinsurance treaty, under the hypothesis that the expected retained loss is the same in both situations as well as the loading of the reinsurer.

At first sight, it therefore seems that proportional reinsurance is less efficient than excess of loss and stop-loss covers, which are of the non proportional type.

In practice this is not the case for multiple reasons such as:

1. stop-loss covers are difficult to obtain due to the possible moral hazard behaviour that the ceding company may adopt after buying such a cover
2. stop-loss covers are extremely difficult to price by reinsurers
3. the loading for a stop-loss cover will clearly differ from a proportional cover (e.g. due to the first two points)

4. excess of loss covers are sometimes difficult to price
5. the loading for an excess of loss cover will also differ from a proportional cover.

Proportional covers can be quite desirable and it is worth analysing their optimality properties.

The main objective of this paper is to illustrate by means of a numerical example that the traditional belief that surplus treaties with a table of lines are better (more efficient) than standard surplus treaties is wrong. We will take this opportunity to compare all the proportional types of reinsurance.

The rest of the paper is organized as follows. Section 2 describes the data we will use for the numerical application. Section 3 explains how the individual risk model will be used as well as approximations of the aggregate claims distribution within the individual risk model. Section 4 describes the four types of proportional reinsurance to be compared in section 5 where we will look for optimal reinsurance structures. Section 6 concludes.

2. DATA

For the calculations a real-life data set will be used. It is obtained from one of the leading Belgian insurance companies and contains $27\,551$ fire policies, covering industrial risks.

The $27\,551$ policies are divided into four classes ($j = 1,2,3,4$), depending on their frequency (q_{ij}) as well as their relative claims severity (X_{ij}), $i = 1,\ldots,n_j$ where n_j is the number of policies in class j. Knowing the sum insured SI_{ij}, we can obtain the loss amount: $L_{ij} = SI_{ij} \times X_{ij}$. We will assume the X_{ij} to be identically distributed within a given risk class ($j = 1,2,3,4$) : $X_{ij} \approx X_j, i = 1,\ldots,n_j, j = 1,2,3,4$. We also assume that the probability of having a loss is identical within a class : $q_{ij} = q_j, i = 1,\ldots,n_j, j = 1,2,3,4$.

For the density of X_j we will use the MBBEFD distribution class introduced by Bernegger (1997). Using the following notations

$$b(c) = e^{3.1-0.15c(1+c)}$$
$$g(c) = e^{c(0.78+0.12c)}$$

we assume the density function of X_j to be

$$f(x) = \frac{(b-1)(g-1)\ln(b)b^{1-x}}{\left((g-1)b^{1-x}+(1-gb)\right)^2}, 0 \le x < 1$$
$$f(1) = \frac{1}{g}.$$

We then have a family of distributions indexed by the parameter c. According to Bernegger (1997), $c = 2,3,4,5$ corresponds to the Swiss Re exposure curves 2, 3, 4 and the Lloyd's industrial exposure curve respectively. We will assume that we have the following characteristics for our portfolio:

Class	q	c
1	0.75%	2
2	1.00%	3
3	1.25%	4
4	1.50%	5

Table 2.1 Claims characteristics of the portfolio

Regarding the sum insured, we have the following information:

$Class(j)$	n	$\mu_j(SI)$	$\sigma_j(SI)$	$\gamma_j(SI)$
1	3 933	13 457 022	10 752 926	8.51
2	17 472	12 034 729	7 960 092	2.23
3	3121	11826 858	9119 825	4.62
4	3 025	10 879 648	7 826 747	11.98

Table 2.2 Characteristics of the sums insured

where

$$\mu_j(SI) = \frac{\sum_{i=1}^{n_j} SI_{ij}}{n_j}$$

= expected insured sum

$$\sigma_j(SI) = \sqrt{\frac{\sum_{i=1}^{n_j} (SI_{ij} - \mu_j(SI))^2}{n_j}}$$

=standard deviation of insured sum

$$\gamma_j(SI) = \frac{\sum_{i=1}^{n_j} (SI_{ij} - \mu_j(SI))^3}{\sigma_j^3(SI)}$$

= skewness insured sum.

3. INDIVIDUAL RISK MODEL AND APPROXIMATIONS

Clearly our portfolio fits into the definition of the individual risk model (see e.g. Klugman et al. (1998)). We have $n = \sum_{j=1}^{4} n_j$ policies with a different sum insured, which are divided into four classes according to their claims behaviour (frequency and severity).

Therefore the aggregate claims amount is given by

$$S^{ind} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} D_{ij} L_{ij}$$

where

1. D_{ij} is the indicator function taking value 1 when there is a claim and 0 when there is no claim. We have $P[D_{ij} = 1] = P[D_j = 1] = q_j$.
2. $L_{ij} = SI_{ij} X_{ij}$ is the conditional loss value.
3. $S_{ij} = D_{ij} L_{ij}$ is the loss associated to policy ij.

Obtaining the exact distribution of S^{ind} is possible by using recursive formulae (see e.g. Dhaene and Vandebroek (1995)) but the computing time will be very long due to the size of the portfolio. Moreover a discretization of distribution of L_{ij} is required.

An approximation of the individual risk model is provided by the collective risk model (see e.g. Klugman et al. (1998)) leading to the use of the Panjer recursive formula (see Panjer (1981)). Once again the computing time will be long and discretization will be required.

In this paper, as the porfolio is large, and its skewness less than 2 (see further for the calculations) we will concentrate on a parametric approximation, namely the shifted gamma distribution, that will reproduce the first three moments of the original distribution. We therefore need to obtain the first three moments of S^{ind}.

The shifted gamma distribution (S) (see e.g. Dufresne and Niederhauser (1997)) has the form

$$S \approx Z + x_0$$

where $Z \approx Gam(\alpha, \beta)$, i.e.

$$f_Z(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad , \quad x > 0$$

$$F_Z(x) = \int_0^x f_Z(s)ds$$

where $\Gamma(x)$ is the gamma function. By abuse of notation, we will also write $F(\alpha, \beta, x)$ the cumulative density function of Z.

Central moments are given by

$$\mu = \sum_{j=1}^{4} \left[q_j EX_j \right] \sum_{i=1}^{n_j} SI_{ij}$$

$$\mu_2 = \sum_{j=1}^{4} \left[q_j VarX_j + q_j(1-q_j)(EX_j)^2 \right] \sum_{i=1}^{n_j} SI_{ij}^2$$

$$\mu_3 = \sum_{j=1}^{4} \left[q_j EX_j^3 - 3q_j^2 EX_j EX_j^2 + 2q_j^3 (EX_j)^3 \right] \sum_{i=1}^{n_j} SI_{ij}^3$$

Using numerical integration, it is possible to obtain the first three moments of X_j, as a function of the parameter c:

Class	EX_j	EX_j^2	EX_j^3
1	0.2260909	0.1623865	0.1474579
2	0.0871796	0.0479373	0.0407141
3	0.031852	0.0123161	0.0094975
4	0.0121457	0.0030479	0.0020178

Table 3.1 Moments of X_j

An analytical formula exists for EX : $EX = \frac{\ln(gb)(1-b)}{\ln(b)(1-gb)}$ but not for higher moments.

The $\sum_{i=1}^{n_j} SI_{ij}^x, x = 1,2,3$ terms are easily obtained from table 2.2.

From this we can obtain the mean (μ), the standard deviation (σ), the coefficient of variation ($CV = \frac{\sigma}{\mu}$) and the skewness ($\gamma = \frac{E(S-\mu)^3}{\sigma^3}$) of S^{ind} :

$$\mu = 293\,751\,934$$
$$\sigma = 57\,364\,022$$
$$CV = 0.20$$
$$\gamma = 0.6$$

The corresponding shifted gamma approximation has the following parameters:

$$\alpha = \frac{4}{\gamma^2} = 10.44$$
$$\beta = \frac{2}{\gamma\sigma} = 5.6310^{-8}$$
$$x_0 = \mu - \frac{2\sigma}{\gamma} = 108\,404\,392$$

4. PROPORTIONAL REINSURANCE

Proportional reinsurance is the easiest way of covering an insurance portfolio. In proportional reinsurance, the ceding company and the reinsurer agree on a cession percentage, say τ_i, for each policy in portfolio. The premium corresponding to the policy i, say P_i, is then shared proportionally between the insurer and the reinsurer. The reinsurer receives $\tau_i P_i$ whereas the insurer keeps the premium $(1-\tau_i)P_i$. If S_i is a claim hitting policy i, the reinsurer is liable for $\tau_i S_i$ whereas the insurer retains $(1-\tau_i)S_i$.

Clearly the way a proportional reinsurance works is extremely simple. Moving to the way of fixing the cession percentage τ_i, we can distinguish between four subtypes of proportional reinsurance: quota share reinsurance, variable quota share reinsurance, surplus reinsurance and surplus reinsurance with a table of lines.

Note that proportional reinsurance is sometimes called pro-rata reinsurance.

We will use the following notations:

- S_{ij} is the loss associated with policy ij.
- $S = \sum_{j=1}^{4} \sum_{i=1}^{n_j} S_{ij}$ is the aggregate loss of the insurer.
- $S^{Re} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} \tau_{ij} S_{ij}$ is the aggregate liability of the reinsurer.
- $S^{R} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1-\tau_{ij}) S_{ij}$ is the aggregate loss in retention when a reinsurance cover is bought.
- P_{ij} is the premium associated with policy ij.
- $P = \sum_{j=1}^{4} \sum_{i=1}^{n_j} P_{ij}$ is the total premium of the insurer.
- $P^{Re} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} \tau_{ij} P_{ij}$ is the total ceded premium.
- $P^{R} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1-\tau_{ij}) P_{ij}$ is the total retained premium.

It is clear that only the risk premium has to be considered. In practice the insurer cedes on the basis of the commercial premium and the reinsurer pays a reinsurance commission representing the management expenses and acquisition costs of the ceding company. To keep things simple, we will always refer to the risk premium in the following and not to the reinsurance commission.

4.1 Quota Share Reinsurance

In quota share reinsurance τ_i is the same for the whole insurance portfolio. Quota share reinsurance is therefore extremely simple as the cession percentage does not vary among policies: we note it as τ. As a consequence the administration of a quota share treaty is straightforward: it suffices to obtain the total premium and the total claims in order to share the premium and the claims with the reinsurer. Quota share reinsurance is of the individual type (i.e. the reinsurance compensation applies claim by claim) and of the global type (i.e. the reinsurance compensation applies on the yearly aggregate loss) at the same type:

$$S^{Re} = \sum_{j=1}^{4}\sum_{i=1}^{n_j} \tau S_{ij} = \tau \sum_{j=1}^{4}\sum_{i=1}^{n_j} S_{ij} = \tau S.$$

Quota share reinsurance has a nice property if we compare its use to the use of the allocated capital (u).

Let ε be the ruin probability without quota share reinsurance:

$$\varepsilon = P[S > u + P].$$

Let ε_R be the ruin probability after quota share reinsurance:

$$\begin{aligned} \varepsilon_R &= P[(1-\tau)S > u + (1-\tau)P] \\ &= P[S > \frac{u}{1-\tau} + P] \\ &< \varepsilon. \end{aligned}$$

We observe that buying a quota share treaty has the same effect as increasing the economic capital in the same proportion as the cession percentage.

Now let us analyse the retained risk of a portfolio covered by a quota share treaty

$$\begin{aligned} ES^R &= (1-\tau)ES \\ VarS^R &= (1-\tau)^2 VarS \\ \sigma(S^R) &= (1-\tau)\sigma(S) \\ CV(S^R) &= CV(S) \\ E(S^R - ES^R)^3 &= (1-\tau)^3 E(S - ES)^3 \\ \gamma(S^R) &= \gamma(S) \end{aligned}$$

Here we can observe that the variability and the skewness of the retained risk is the same as if there were no quota share reinsurance. Obviously quota share reinsurance does not provide a reduction in the relative homogeneity of the portfolio.

It is nevertheless very much used for multiple reasons such as

- Financing management and acquisition costs by means of the reinsurance commission (in case of a new product or a start-up insurance company).

- Reinsurance against underpricing (new classes of business). It limits the danger of new (unknown) risks.
- Reduction of the required solvency margin.
- Compensation for less balanced treaties of the cedant.

Note that quota share reinsurance is sometimes referred to as participating reinsurance.

4.2 Variable Quota-Share Reinsurance

Sometimes, the cession percentage may vary within the portfolio. This is called variable quota share reinsurance. In our example, we will assume that the percentage may vary in function of the class of risk. This is equivalent to analysing four different quota share treaties.

We then have the following relations:

$$ES^R = \sum_{j=1}^{4} (1-\tau_j) ES_j$$

$$VarS^R = \sum_{j=1}^{4} (1-\tau_j)^2 VarS_j$$

$$\sigma(S^R) = \sqrt{\sum_{j=1}^{4} (1-\tau_j)^2 VarS_j}$$

$$CV(S^R) \neq CV(S)$$

$$E(S^R - ES^R)^3 = \sum_{j=1}^{4} (1-\tau_j)^3 E(S_j - ES_j)^3$$

$$\gamma(S^R) \neq \gamma(S)$$

It becomes impossible to compare the coefficient of variation and the skewness analytically. This will be done numerically.

4.3 Surplus Reinsurance

In surplus reinsurance the cession percentage is a function of both the sum insured and the line, or retention, chosen by the ceding company.

The line (R) is the maximal amount that the insurer wants to pay in case of a loss. If one wants to make use of proportional reinsurance and of the property that the maximal loss will never be larger than the line, the cession percentage must be defined as

$$\tau_{ij} = \max\left(0, 1 - \frac{R}{SI_{ij}}\right).$$

The retention percentage is

$$(1 - \tau_{ij}) = \min\left(1, \frac{R}{SI_{ij}}\right).$$

In case of a total loss, the retained loss is

$$\min\left(1, \frac{R}{SI_{ij}}\right) \times SI_{ij} = SI_{ij} if SI_{ij} < R$$
$$\min\left(1, \frac{R}{SI_{ij}}\right) \times SI_{ij} = R if SI_{ij} > R$$

It is clear that surplus reinsurance is appealing from an optimality point of view. In surplus reinsurance, the loss amount may not exceed the line. Furthermore, the smallest risks are not reinsured. Therefore, one feels that the retained risk will be more homogeneous than it is in case of a quota share reinsurance.

The retained risk has the following central moments:

$$S^R = \sum_{j=1}^{4} \sum_{i=1}^{n_j} (1 - \tau_{ij}) D_{ij} L_{ij}$$

$$\mu = \sum_{j=1}^{4} \left[q_j EX_j \right] \sum_{i=1}^{n_j} (1 - \tau_i) SI_{ij}$$

$$\mu_2 = \sum_{j=1}^{4} \left[q_j VarX_j + q_j (1 - q_j)(EX_j)^2 \right] \sum_{i=1}^{n_j} (1 - \tau_i)^2 SI_{ij}^2$$

$$\mu_3 = \sum_{i=1}^{4} \left[q_j EX_j^3 - 3q_j^2 EX_j EX_j^2 + 2q_j^3 (EX_j)^3 \right] \sum_{i=1}^{n_j} (1 - \tau_i)^3 SI_{ij}^3$$

It is not possible to make analytical comparisons with these formulae. We will therefore concentrate on the numerical application in order to make further comments.

One should note that surplus reinsurance is far more expensive from an administrative point of view since each policy must be closely examined in order to compute the ceded premium and the possible recovery from the reinsurer, based on its own cession percentage, which is a function of the insured sum.

Note that surplus reinsurance is sometimes referred to as surplus share reinsurance.

4.4 Surplus Reinsurance with a Table of Lines

We now move on to surplus reinsurance with a table of lines. In the above definition of surplus reinsurance, the same retention R is used for the whole portfolio. In practice however, it may happen that a surplus programme is presented with a table of lines. This means that a retention is fixed per group of similar risks. In this way the portfolio that the ceding company retains is qualitatively more homogeneous. It is especially the fire risks in an insurer's portfolio that may differ in quality. Determining factors are the location of the risk, the building's construction, its use, the loss prevention and protection measures, ... The quality of the risk is translated into a frequency and severity distribution: the better the risk, the smaller the frequency and the less dangerous the claims severity. So we have four classes of risks with different characteristics. If we choose the same retention for the entire portfolio as we described above, the expected loss per risk would not be homogeneous. With the same retention the yearly expected loss of the ceding company would depend upon the kind of risk that has been affected. We therefore choose a different retention per class, in order to make the expected loss per risk independent of the kind of risk. As a consequence the insurer is able to retain more of the good risks and less of the bad risks. For the reinsurer however there is always the risk that only the dangerous risks are transferred. When the cedant's rate is wrong, this implies a danger to the reinsurer. This phenomenon is called antiselection.

Thus, in surplus reinsurance with a table of lines, the cession percentage is

$$\tau_{ij} = \max\left(0, 1 - \frac{R_j}{SI_{ij}}\right)$$

where the line R_i may vary among the policies.

In order to fix the lines, certain practitioners use one of the following methods with no real justification.

A first method to construct a table of lines is to determine a retention for each class of business by aiming at an equal maximum loss throughout the entire portfolio. This means that the lines will be such that

$$R_1 \times q_1 = R_2 \times q_2 = R_3 \times q_3 = R_4 \times q_4 .$$

This is the method of the inverse claim frequency.

A second method takes into account not only the frequency but also the claims severity. This table of lines is constructed in order to reach the same average loss for all policies, contrary to the same maximum loss of the first method. This means that the lines will be such that

$$R_1 \times rate_1 = R_2 \times rate_2 = R_3 \times rate_3 = R_4 \times rate_4 .$$

where $rate_j = q_j EX_j$. This is the method of the inverse rate.

5. OPTIMAL REINSURANCE

In this section we will compare the original portfolio with the retained portfolio after a proportional cession of the four types described in the previous sections. We will use two criteria:

1. a de Finetti criterion, i.e. we will minimize the variance of the retained loss under the constraint that the expected gain is fixed.
2. a RORAC criterion, i.e. we will maximize the return on risk adjusted capital of the retained risk.

We will assume that the insurer is using a loading ξ. That loading contains only the capital charge. All administrative expenses must be charged on top of that loading. We will also assume that the reinsurer is using a loading ξ^{Re}. That loading includes the capital charge of the reinsurer as well as the administrative expenses. It is clear that the insurer pays for the administrative expenses of the reinsurer in the reinsurance premium. For the numerical application, we will use $\xi = 5\%$ and $\xi^{Re} = 7\%$.

5.1 de Finetti's Type Results

Following de Finetti (1940), we will minimize the variance of the gain of the retained portfolio by assuming that the four subportfolios are covered by a quota-share with a possible different cession rate. The gain of the retained portfolio is

$$Z(\tau)=\sum_{j=1}^{4}\sum_{i=1}^{n_j}((1+\xi_{ij})ES_{ij}-(1+\xi_{ij}^{Re})\tau_{ij}ES_{ij}-(1-\tau_{ij})S_i).$$

where τ is the vector of cession percentages $\{\tau_{11},...,\tau_{n_1 1},\tau_{12}...,\tau_{n_2 2},\tau_{13},...,\tau_{n_3 3},\tau_{14},...,\tau_{n_4 4}\}$. The de Finetti problem is the following:

$$\min_{\tau} VarZ(\tau)$$

under the constraint that

$$EZ(\tau)=k.$$

de Finetti (1940) showed that the solution is given by

$$\tau_{ij}=\max\left(0,1-\frac{b\xi_{ij}^{Re}ES_{ij}}{VarS_{ij}}\right)\quad,\quad j=1,\cdots,4\quad,\quad i=1,...,n_j,$$

where b is a constant given by the condition $EZ(\tau)=k$.

Assuming that we want to keep an expected gain equal to $5\,000\,000$, the solution provided by de Finetti is the following:

Case	τ_1	τ_2	τ_3	τ_4	E	σ
1	64.98%	41.75%	24.05%	0.00%	5000000	29173126

Table 5.1. Optimal variable quota share treaty with expected gain = $5\,000\,000$

If we cover the whole portfolio by a uniform quota share, i.e. with the same cession percentage for all risks in all four classes, we obtain

Case	τ_1	τ_2	τ_3	τ_4	E	σ
2	47.11%	47.11%	47.11%	47.11%	5000000	30338327

Table 5.2 Quota share treaty with expected gain = $5\,000\,000$

The volatility is larger than it is in case of the variable quota share treaty, which shows the optimality of de Finetti's result.

Now let us move on to surplus reinsurance. We will analyse the following cases:

3. surplus with one line
4. surplus with table of lines corresponding to the quota share treaty
5. surplus with table of lines corresponding to the variable quota share (the lines are chosen such that the global cession for the subportfolio is the same for both covers)
6. surplus with table of lines obtained by the inverse rate method
7. surplus with table of lines obtained by the inverse frequency method

Case	R_1	R_2	R_3	R_4	E	σ
3	7304175	7304175	7304175	7304175	5000000	24858743
4	7989249	7065148	6963402	6167660	5000000	25111701
5	4886924	8036122	12533770	333398280	5000000	24700617
6	4246111	8258874	18083771	39520454	5000000	24913398
7	9084700	6813525	5450820	4542350	5000000	25693734

Table 5.3 Comparison of surplus treaties with expected gain = $5\,000\,000$

We can make the following comments:

1. the surplus treaty (case 3) is optimal compared to the surplus treaty with table of lines obtained by the practitioners method (cases 6 and 7). This is clearly against the traditional belief.
2. the surplus treaty corresponding to the cessions of the variable quota share treaty (case 5) is the best treaty. This is a sign that building the table of lines according to the shares of de Finetti's solution is probably more sensible than using the practitioners formula which has no theoretical justification.

By minimizing the objective function numerically, we were able to obtain two situations that are more efficient than the previous ones:

Case	R_1	R_2	R_3	R_4	E	σ
8	6001860	7940027	7514596	7483510	5000000	24689131
9	5819865	7592990	10774593	333398280	5000000	24597666

Table 5.4. Trying to obtain the optimal table of lines with expected gain = $5\,000\,000$ numerically

Note that the objective function seems to be very flat. It is therefore difficult to make sure that the global minimum has been achieved. We observe that the two proposed solutions are very different.

In fact, it is not difficult to write the de Finetti's formulae for a surplus treaty or a surplus treaty with a table of lines.

Indeed, we have the following results:

a. Surplus treaty

$$Z(\mathbf{R}) = \sum_{j=1}^{4} \sum_{i=1}^{n_j} ((1+\xi_{ij})ES_{ij} - (1+\xi_{ij}^{Re})\tau_{ij} ES_{ij} - (1-\tau_{ij})S_{ij})$$

with

$$\tau_{ij} = \max\left(0, 1 - \frac{R_{ij}}{SI_{ij}}\right).$$

Glineur and Walhin (2004) have used convex optimization to prove that the optimal lines are

$$R_{ij} = b\xi_{ij}^{Re} \frac{ED_{ij}X_{ij}}{VarD_{ij}X_{ij}}$$

where b is a constant that is determined by the constraint on the expected gain.

Clearly this result is not useful as it will not be possible from an administrative point of view to apply a different line to each policy in the portfolio. We then move on to the more interesting case of the table of lines.

b. Surplus treaty with a table of lines

$$Z(\mathbf{R}) = \sum_{j=1}^{4} \sum_{i=1}^{n_j} ((1+\xi_{ij})ES_{ij} - (1+\xi_{ij}^{Re})\tau_{ij} ES_{ij} - (1-\tau_{ij})S_{i})$$

with

$$\tau_{ij} = \max\left(0, 1 - \frac{R_j}{SI_{ij}}\right).$$

Glineur and Walhin (2004) have used convex optimization to prove that the optimal lines are

$$R_j = b\frac{\sum_{i=1}^{n_j}\xi_{ij}^{Re}E[D_{ij}L_{ij}]SI_{ij}}{\sum_{i=1}^{n_j}Var[D_{ij}L_{ij}]} \quad , \quad j=1,2,3,4$$

where b is a constant that is determined by the constraint.

On the reasonable assumption that the X_{ij} and D_{ij} are identically distributed within the class j and that the reinsurance loading is the same for each risk within the class j, the formula is reduced to

$$R_j = b\xi_j^{Re}\frac{ED_jX_j}{VarD_jX_j}$$

where b is a constant that is determined by the constraint on the expected gain.

We obtain

Case	R_1	R_2	R_3	R_4	E	σ
10	5844858	7628597	10842395	16701348	5000000	24511489

Table 5.5 Optimal surplus treaty with a table of line with expected gain = 5000000

5.2 RORAC's Type Results

Now let us compute the RORAC (Return On Risk Adjusted Capital) for different reinsurance structures.

Let us assume that the required solvency level, RSL, is given by the Tail Value at Risk at the level $\varepsilon = 99\%$.

Using our shifted gamma approximation, we have

$$\begin{aligned} RSL &= E[S|S > VaR_S(\varepsilon)] \\ &= E[Z|Z > VaR_Z(\varepsilon)] + x_0 \\ &= \frac{\alpha}{\beta}\frac{1}{1-\varepsilon}\left(1 - F(\alpha+1,\beta,VaR_Z(\varepsilon))\right) + x_0 \end{aligned}$$

where $VaR_Z(\varepsilon) = F^{-1}(\alpha,\beta,\varepsilon)$.

The retained premium is equal to

$$P^R = (1+\xi)ES - (1+\xi^{Re})ES^{Re}.$$

The risk adjusted capital is obtained by deducting the retained premium from RSL. In other words, the risk adjusted capital is the required solvency level minus the premium that is charged to the policyholders plus the premium that is charged by the reinsurers:

$$RAC = RSL - P^R$$

and RORAC is defined as

$$RORAC = \frac{P^R - ES^R}{RAC}$$

For the direct (i.e. before any reinsurance) portfolio, we obtain the following:

$$ES = ES^R = 293\,751\,934$$
$$CV = 0.20$$
$$\gamma = 0.62$$
$$VaR = 452\,547\,891$$
$$TVaR = 483\,141\,978$$
$$P = P^R = 308\,439\,531$$
$$RAC = 174\,702\,447$$
$$RORAC = 8.41\%.$$

Now let us compare the RORAC for the reinsurance structures that have been analysed in the previous section:

Case	*CV*	γ	*TVaR*	*RORAC*
1	0.20	0.62	255521124	5.25%
2	0.19	0.51	248418187	5.68%
3	0.16	0.24	227868224	7.41%
4	0.16	0.25	228686935	7.32%
5	0.16	0.30	228769300	7.31%
6	0.16	0.28	228915033	7.29%
7	0.17	0.26	230640846	7.11%
8	0.16	0.25	227430799	7.45%
9	0.16	0.29	228247460	7.36%
10	0.16	0.25	226965212	7.51%

Table 5.6. RORAC for our 10 alternatives

We can make the following observations:

1. as for the de Finetti's criterion, the differences are not that large
2. the ranking is not exactly the same as the de Finetti's one. In particular the second best alternative under de Finetti's criterion (case 9) is now outperformed by the classical surplus (case 3). This is due to the fact that the de Finetti's criterion does not account for the skewness. Case 9 is penalized in the RORAC criterion due to its larger skewness.
3. we also observe that the RORAC in this reinsurance structure is less than in the case of no reinsurance. Obviously buying reinsurance at that level destroys value. This is due to the fact that the reduction in risk is not counter-balanced by the cost of reinsurance ($\xi^{Re} = 7\% > 5\% = \xi$).

Now let us analyse other situations:

Case	*Line*	*CV*	γ	*RORAC*	$\frac{ES^{Re}}{ES}$
1	5 000 000	0.16	0.24	4.16%	61.31%
2	7 500 000	0.16	0.24	7.58%	46.03%
3	10 000 000	0.16	0.25	9.05%	33.91%
4	12 500 000	0.17	0.26	9.71%	24.82%
5	15 000 000	0.17	0.28	9.98%	18.12%
6	17 500 000	0.17	0.29	10.06%	13.20%
7	20 000 000	0.18	0.30	10.06%	9.59%
8	22 500 000	0.18	0.31	10.00%	6.91%

Table 5.7 RORAC as a function of the line of a surplus treaty

We observe that the optimal line is about $20\,000\,000$ providing a $RORAC = 10.06\%$, instead of 8.41% without reinsurance.

Now we consider the RORAC for surplus treaties with a table of lines. We choose the method of the inverse rate and we choose the lines so as to get the same global cession as in the previous table.

Case	R_1	R_2	R_3	R_4	*CV*	γ	*RORAC*	$\frac{ES^{Re}}{ES}$
1	2 792 144	5 430 844	11 891 468	25 987 731	0.16	0.29	4.06%	61.31%
2	4 373 473	8 506 598	18 626 192	40 705 865	0.16	0.28	7.47%	46.03%
3	6 066 679	11 799 959	25 837 392	56 465 292	0.16	0.28	8.93%	33.91%
4	7 857 669	15 283 513	33 465 040	73 134 831	0.17	0.29	9.58%	24.82%
5	9 739 358	18 943 481	41 478 968	90 648 548	0.17	0.30	9.86%	18.12%
6	11 697 749	22 752 639	49 819 564	108 876 170	0.17	0.31	9.96%	13.20%
7	13 736 088	26 717 298	58 500 649	127 847 900	0.18	0.32	9.96%	9.59%
8	15 858 279	30 845 054	67 538 854	147 600 082	0.18	0.33	9.92%	6.91%

Table 5.8. RORAC in function of the table of lines (inverse rate method)

We observe that the RORAC is smaller in case of a table of lines than it is in case of a classical surplus with one fixed line.

Now we consider the RORAC for surplus treaties with a table of lines that is built using the de Finetti optimal table of lines:

Case	R_1	R_2	R_3	R_4	*CV*	γ	*RORAC*	$\frac{ES^{Re}}{ES}$
1	3949974	5155430	7327325	11286824	0.15	0.24	4.22%	61.31%
2	6007752	7841203	11144567	17166807	0.16	0.25	7.68%	46.03%
3	8113889	10590093	15051518	23184974	0.16	0.26	9.15%	33.91%
4	10247187	13374433	19008852	29280752	0.17	0.27	9.79%	24.82%
5	12397936	16181549	22998558	35426392	0.17	0.28	10.05%	18.12%
6	14573268	19020751	27033868	41642281	0.17	0.29	10.13%	13.20%
7	16743363	21853117	31059460	47843201	0.18	0.30	10.11%	9.59%
8	18964227	24751746	35179233	54189193	0.18	0.31	10.05%	6.91%

Table 5.9. RORAC in function of the table of lines (de Finetti's optimal table)

Previous results are confirmed: this method of building up a table of lines is more efficient than the two methods of practitioners. Note that in our numerical example, it becomes more efficient than the surplus with a single line.

6. CONCLUSION

We have analysed the optimality properties of an insurance portfolio covered by a proportional reinsurance. The numerical application has confirmed that quota share reinsurance is suboptimal when compared to all other types of proportional reinsurance. In fact, quota share reinsurance will only be of interest to the ceding company when the loading of the reinsurer is smaller than the loading of the insurer. This is possible if one refers to the diversification possibilities that are offered to the reinsurer. So one may argue that less capital needs to be remunerated from the reinsurer's point of view. On the other hand, one may argue that the reinsurer's shareholders may require a higher cost of capital due to the agency costs (see Hancock et al. (2001) for details) that apply when underwriting a business that is less well understood by the reinsurer than the primary insurer. This means that ceding companies should provide as much information as possible to reinsurers in order to reduce these agency costs.

We have also observed that surplus reinsurance with a table of lines based on the inverse frequency method, or inverse rate method, is not, in our numerical example, optimal when compared to surplus reinsurance with one single line. This goes against the traditional belief of practitioners. Obviously we have not proved that it is always true but we have simply shown that a table of lines is not always optimal.

On the other hand we have derived the optimal table of lines using the de Finetti's criterion. This table of lines is more efficient, in our numerical example, than the other proportional reinsurance programmes.

Eventually, one should note that the reinsurer's loading would most probably not remain constant in case of surplus treaties with increasing retentions. Indeed, when increasing the retentions, the reinsured business becomes less well balanced, implying a larger volatility for the reinsurer. Clearly the reinsurer will apply higher capital charges in these cases. Moreover the fixed management expenses of the reinsurer will be more important in those treaties where the cession is small. One therefore has to be cautious with the previous conclusions and always ask quotes from the reinsurer when analysing the optimality of a reinsurance programme.

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Transition Matrix Theory And Individual Claim Loss Development

John B. Mahon

Abstract

Motivation. Individual claim development is important for creating the average severity distributions that underlie most increased limits, and reinsurance pricing analyses, but most current methods do not adequately represent the true process.

Method. Transition Matrix Theory is applied to a large database of reinsurance data. The data is processed to isolate GL data, and the Transition Matrix process is described in detail.

Results. Individual claim size development is characterized as a distributional process. The effect of this distributional process on pricing parameters is contrasted with traditional methods.

Conclusions. Individual Claim Size development is a distributional process, and can be measured and introduced into procedures for calculating average severity distributions. A simple five parameter formula can model this process. The Transition Matrix process may overstate the distribution of the ultimate distribution, but this can be measured and corrected. Pricing parameters are affected by this process and its effect should be factored in when possible.

Keywords. Transition Matrix, Average Severity, Individual Claim Loss Development, Distributional Loss Development

1. INTRODUCTION

Loss development has long been considered to be an aggregate phenomenon, and not applicable to individual claims. Rating procedures require accurate estimates of individual claim size development in order to estimate the average severity distribution curves that underlie increased limits ratemaking, and reinsurance excess layer pricing. Current methods have limitations based on sparse data at high layers, or are based on assumptions that may introduce errors. This study applies the Transition Matrix Theory approach to a large collection of reinsurance individual large losses, and, characterizes individual claim size development as a distributional procedure. It was found that a simple five parameter distribution will model the process.

1.1 Research Context

Several approaches have been used to apply loss development to individual claims to adjust them for increased limits calculations and for pricing reinsurance excess layers. Transition matrix theory as applied to losses was introduced at the International Congress of Actuaries in 1980 by Charles Hachemeister [1]. A more recent presentation of this method can be found in Ole Hesselager's [2] 1994 paper where he presents a time continuous method for computing transition matrices. The present research uses a time discrete method of computing Markov transition matrices to represent the age to age loss development of a large body of reinsurance general liability claims.

A weakness of the transition matrix approach is that it generates a large number of parameters which make it unwieldy, and prone to parameter error.

1.2 Objective

This study uses a straightforward interpretation of the transition matrix theory and applies it to a large body of reinsurance individual large losses. This yields vectors which can be used to develop individual claims from an arbitrary size and evaluation to ultimate. The behavior of the loss development forecasts suggests a five parameter model that can be used to characterize the development of an open claim as a future distribution. This model is modified to reflect the fact that observed variation appears to be smaller than that provided by the Transition Matrix process.

1.3 Outline

The remainder of the paper proceeds as follows**.** Section 2.1 will provide a background for individual claim development. Section 2.2 describes the details of the Transition Matrix method as applied here. Section 2.3 describes the application of the Transition Matrix method to aollection of reinsurance data. Section 2.4 describes a comparison between transition matrix results and initial to final transitions. It discusses an adjustment to the Transition Matrix results to reduce excess variation introduced by the Transition Matrix method, and, proposes a model for distributional loss development. Section 2.5 describes an effect of the distributional loss development method on pricing parameters. Section 3 discusses the applicabililty and limitations of the method used here, and Section 4 collects the conclusions.

2. BACKGROUND AND METHODS

In this section, the background, method, data and application are described.

2.1 Background

Loss development has long been important to both reserving and pricing activities. Loss development for reserving has concentrated on the aggregate behavior of the losses.

Triangles of sums of losses or claim counts are subjected to procedures which measure the behavior of aggregate losses. The behavior of individual claims, for the most part, is not important. It only becomes important where large individual claims are near or at limits, and further development may distort the result. Treatment in this situation usually involves isolating these claims from the aggregate data, and handling them on an individual and ad hoc basis.

The rating discipline needs to address individual claim loss development at a more detailed level. Increased limits pricing for primary business, and excess layer pricing for reinsurance business require the correct estimate of large size losses. The issue of individual claim loss development becomes a critical factor in determining the correct probability of large losses used to determine pricing in these two business applications.

A variety of solutions been developed to deal with this problem, some, better than others. Elimination of the problem by using closed claim data has been successful to the extent that the data is available, and, not too stale. Fitting immature loss size data to severity distributions and measuring loss development by counts within empirical intervals of size, or changes in the parameters has been successful for creating increased limits factors for subline pricing for many years. It is limited by the fact that it requires large amounts of data and many man-hours to complete. This eliminates it from use in reinsurance pricing exercises.

This most common experience rating method used in reinsurance involves combining features of aggregate loss development that can be applied to individual losses. The losses are trended, then layered into the excess layer of rating interest, and then, the appropriate excess layer loss development factor is applied. This method suffers from two problems. One is that the excess loss development may be very different from the factors that are used, and the other is that there may be no losses in the higher layers after trending. Both of these can lead to significant errors.

Another method commonly used is to apply trend and average severity development factors to individual claims, then use the adjusted claims to fit a theoretical severity distribution. This severity distribution is then used to evaluate excess layers using exposure rating techniques. The first thing to say here is that it is incorrect to apply average severity loss development factors to individual losses, and call the result the ultimate value of that claim. This has to do with the nature of loss development of an individual claim. A claim can have a wide range of outcomes as it matures to ultimate. To simply say that when it matures to ultimate, it will have a value some "X" percent larger than current, misses the variability of the loss development process.

Exhibit 1 shows a typical adjustment for an individual loss to prepare it for fitting

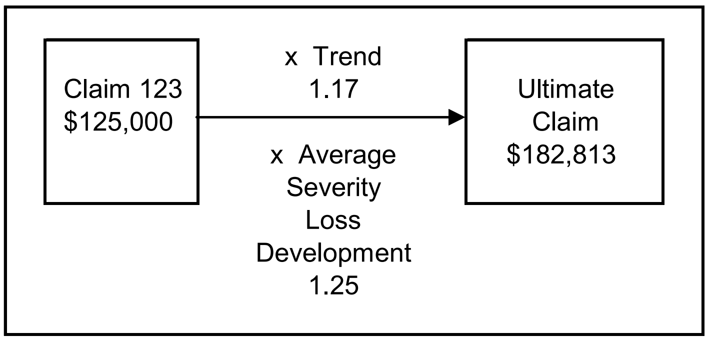

Exhibit 1 showing a typical trend and development adjustment to an indivdual claim

a severity curve. The reality of the situation is that an open claim has four possible outcomes at ultimate, it may stay the same size, it will grow in

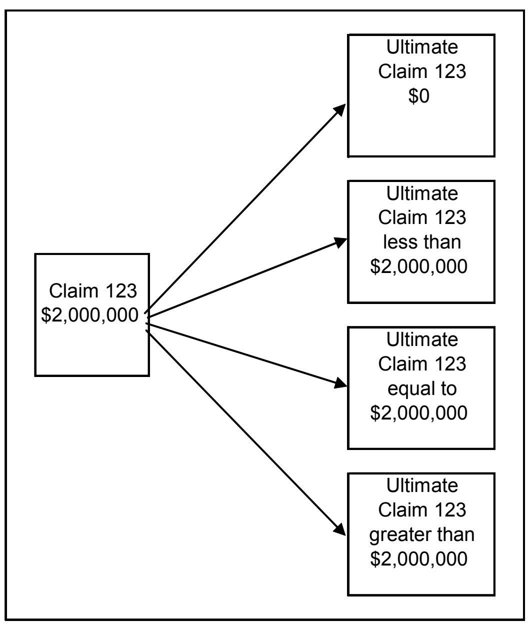

Exhibit 2. The four possible states for the ultimate settlement of a claim.

size, it will settle for a lesser amount, or it will close with no payment as shown in exhibit 2. Transition matrix theory accommodates the variation of possible outcomes.

2.2 The Transition Matrix Approach

A Markov Transition Matrix is a square matrix that contains the probabilities of moving from one state to another state [3]. For our purposes, the states will be the combination of open or closed, and size of loss. Exhibit 3 shows the complete list of states for our example. Note that the endpoints of the size intervals are determined exponentially. They increase by a constant factor, two, in this case. The interval end points can be arbitrary, but selecting exponential ones will provide additional insight into the results of this study.

Class	Open/ Closed	Interval bottom	Interval top	Count
0	Open	0	0	0
1	Open	0	200,000	0
2	Open	200,000	400,000	0
3	Open	400,000	800,000	0
4	Open	800,000	1,600,000	0
5	Open	1,600,000	3,200,000	0
6	Open	3,200,000	6,400,000	1
7	Open	6,400,000	12,800,000	0
8	Open	12,800,000	25,600,000	0
9	Open	25,600,000	51,200,000	0
10	Closed	0	0	0
11	Closed	0	200,000	0
12	Closed	200,000	400,000	0
13	Closed	400,000	800,000	0
14	Closed	800,000	1,600,000	0
15	Closed	1,600,000	3,200,000	0
16	Closed	3,200,000	6,400,000	0
17	Closed	6,400,000	12,800,000	0
18	Closed	12,800,000	25,600,000	0
19	Closed	25,600,000	51,200,000	0
As of 36 months				

Exhibit 4. This shows the state of the same claim shown in exhibit 3, but it is now $3,500,000 with a maturity of 36 months. It is now in class 6.

Class	Open/ Closed	Interval bottom	Interval top	Count
0	Open	0	0	0
1	Open	0	200,000	0
2	Open	200,000	400,000	0
3	Open	400,000	800,000	0
4	Open	800,000	1,600,000	0
5	Open	1,600,000	3,200,000	1
6	Open	3,200,000	6,400,000	0
7	Open	6,400,000	12,800,000	0
8	Open	12,800,000	25,600,000	0
9	Open	25,600,000	51,200,000	0
10	Closed	0	0	0
11	Closed	0	200,000	0
12	Closed	200,000	400,000	0
13	Closed	400,000	800,000	0
14	Closed	800,000	1,600,000	0
15	Closed	1,600,000	3,200,000	0
16	Closed	3,200,000	6,400,000	0
17	Closed	6,400,000	12,800,000	0
18	Closed	12,800,000	25,600,000	0
19	Closed	25,600,000	51,200,000	0
As of 24 months				

Exhibit 3. This shows our claim of $2,000,000 with a maturity of 24 months. All of the possible states of size and open or closed are shown. These states are labeled with a Class number which will be used to track them.

Also shown is an open claim of $2,000,000 as a count of one in class 5.

We now consider this claim as it matures to the 36 month evaluation, and it changes in value to $3,500,000. Exhibit 4 shows its state as a class 6.

Transition from 24 to 36 months																				
Final	Initial Class																			
Class	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0																				
1																				
2																				
3																				
4																				
5																				
6						1														
7																				
8																				
9																				
10																				
11																				
12																				
13																				
14																				
15																				
16																				
17																				
18																				
19																				

Exhibit 5. The transition matrix for the sample claim that is a class 5 at 24 months and a class 6 at 36 months

We can now construct a transition matrix for the transition from 24 to 36 months for this loss as shown in exhibit 5. Consider if we have 755 class 5 losses at 24 months and they are entered into the transition matrix. This would result in the matrix shown in exhibit 6.

Transition from 24 to 36 months																				
Final	Initial Class																			
Class	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0						13														
1						22														
2						35														
3						57														
4						91														
5						146														
6						91														
7						57														
8						35														
9						22														
10						4														
11						7														
12						11														
13						19														
14						30														
15						48														
16						30														
17						19														
18						11														
19						7														
Total						755														

Exhibit 6. The matrix showing all 24 month class 5 claims populated into their final class at 36 months.

Now we consider a complete collection of fictitious claims, in our example there are 4,259, of all sizes and open or closed status. These claims are mapped into this transition matrix and this results in the matrix shown in exhibit 7.

Transition from 24 to 36 months																					
Final Class	Initial Class																				
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	Total
0	0	2	3	5	8	13	22	5	1	0	0	0	0	0	0	0	0	0	0	0	59
1	1	3	5	8	13	22	35	8	2	0	0	0	0	0	0	0	0	0	0	0	97
2	2	2	8	13	22	35	57	13	3	0	0	0	0	0	0	0	0	0	0	0	155
3	3	1	5	22	35	57	91	22	5	1	0	0	0	0	0	0	0	0	0	0	242
4	5	0	3	13	57	91	146	35	8	2	0	0	0	0	0	0	0	0	0	0	360
5	9	0	2	8	35	146	234	57	13	3	0	0	0	0	0	0	1	0	0	0	508
6	15	0	1	5	22	91	375	91	22	5	0	0	0	0	0	0	2	0	0	0	629
7	9	0	0	3	13	57	234	146	35	8	0	0	0	0	0	0	1	0	0	0	506
8	5	0	0	2	8	35	146	91	57	13	0	0	0	0	0	0	0	0	0	0	357
9	3	0	0	1	5	22	91	57	35	22	0	0	0	0	0	0	0	0	0	0	236
10	0	0	1	1	2	4	7	1	0	0	0	0	0	0	0	0	0	0	0	0	16
11	0	1	1	2	4	7	11	2	0	0	0	0	0	0	0	0	0	0	0	0	28
12	0	0	2	4	7	11	19	4	1	0	0	0	1	0	0	0	0	0	0	0	49
13	1	0	1	7	11	19	30	7	1	0	0	0	0	2	0	0	0	0	0	0	79
14	1	0	1	4	19	30	48	11	2	0	0	0	0	0	6	0	0	0	0	0	122
15	3	0	0	2	11	48	78	19	4	1	0	0	0	0	0	17	0	0	0	0	183
16	5	0	0	1	7	30	125	30	7	1	0	0	0	0	0	0	45	0	0	0	251
17	3	0	0	1	4	19	78	48	11	2	0	0	0	0	0	0	0	17	0	0	183
18	1	0	0	0	2	11	48	30	19	4	0	0	0	0	0	0	0	0	6	0	121
19	1	0	0	0	1	7	30	19	11	7	0	0	0	0	0	0	0	0	0	2	78
Total	67	9	33	102	286	755	1905	696	237	69	0	0	1	2	6	17	49	17	6	2	4259

Exhibit 7. This is a transition matrix populated with a complete inventory of losses starting at a 24 month maturity, and ending in a 36 month maturity. The value shown is claim count.

This matrix can be converted to a Markov transition matrix by dividing each column by the total at the bottom of the column. This normalizes each column so that it sums to one, and each value represents the probability that a selected initial class will make the transition to the selected final class. The complete Markov transition matrix is shown in exhibit 8.

Transition from 24 to 36 months																				
Final Class	Initial Class																			
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	0.22	0.09	0.05	0.03	0.02	0.01	0.01	0	0	0	0	0	0	0	0	0	0	0	0
1	0.01	0.33	0.15	0.08	0.05	0.03	0.02	0.01	0.01	0	0	0	0	0	0	0	0	0	0	0
2	0.03	0.22	0.24	0.13	0.08	0.05	0.03	0.02	0.01	0	0	0	0	0	0	0	0	0	0	0
3	0.04	0.11	0.15	0.22	0.12	0.08	0.05	0.03	0.02	0.01	0	0	0	0	0	0	0	0	0	0
4	0.07	0	0.09	0.13	0.2	0.12	0.08	0.05	0.03	0.03	0	0	0	0	0	0	0	0	0	0
5	0.13	0	0.06	0.08	0.12	0.19	0.12	0.08	0.05	0.04	0	0	0	0	0	0	0.02	0	0	0
6	0.22	0	0.03	0.05	0.08	0.12	0.2	0.13	0.09	0.07	0	0	0	0	0	0	0.04	0	0	0
7	0.13	0	0	0.03	0.05	0.08	0.12	0.21	0.15	0.12	0	0	0	0	0	0	0.02	0	0	0
8	0.07	0	0	0.02	0.03	0.05	0.08	0.13	0.24	0.19	0	0	0	0	0	0	0	0	0	0
9	0.04	0	0	0.01	0.02	0.03	0.05	0.08	0.15	0.32	0	0	0	0	0	0	0	0	0	0
10	0	0	0.03	0.01	0.01	0.01	0	0	0	0	1	0	0	0	0	0	0	0	0	0
11	0	0.11	0.03	0.02	0.01	0.01	0.01	0	0	0	0	1	0	0	0	0	0	0	0	0
12	0	0	0.06	0.04	0.02	0.01	0.01	0.01	0	0	0	0	1	0	0	0	0	0	0	0
13	0.01	0	0.03	0.07	0.04	0.03	0.02	0.01	0	0	0	0	0	1	0	0	0	0	0	0
14	0.01	0	0.03	0.04	0.07	0.04	0.03	0.02	0.01	0	0	0	0	0	1	0	0	0	0	0
15	0.04	0	0	0.02	0.04	0.06	0.04	0.03	0.02	0.01	0	0	0	0	0	1	0	0	0	0
16	0.07	0	0	0.01	0.02	0.04	0.07	0.04	0.03	0.01	0	0	0	0	0	0	0.92	0	0	0
17	0.04	0	0	0.01	0.01	0.03	0.04	0.07	0.05	0.03	0	0	0	0	0	0	0	1	0	0
18	0.01	0	0	0	0.01	0.01	0.03	0.04	0.08	0.06	0	0	0	0	0	0	0	0	1	0
19	0.01	0	0	0	0	0.01	0.02	0.03	0.05	0.1	0	0	0	0	0	0	0	0	0	1
Total	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Exhibit 8. A Markov transition matrix for the transition from 24 to 36 months. Note that each column sums to one.

With this matrix populated, it is possible to observe structural details of loss development. To do this, we consider four different types of transitions, open to open, open to closed, closed to closed, and closed to open. This corresponds to the 4 quadrants of the transition matrix. The section of the matrix that shows open to open is shown in exhibit 9. Here we see that transitions with no size change (same initial and final class) has the highest probability. This forms a diagonal ridge across the matrix. Note that the columns do not sum to one because some of the probability is carried in the part of the matrix representing the open to closed transitions which is shown in Exhibit 10. This shows a similar diagonal ridge which represents claims that close in the same size range that they were open at the beginning of the transition.

Transition from 24 to 36 months										
Final Class	Initial Class									
	0	1	2	3	4	5	6	7	8	9
0	0	0.22	0.09	0.05	0.03	0.02	0.01	0.01	0	0
1	0.01	0.33	0.15	0.08	0.05	0.03	0.02	0.01	0.01	0
2	0.03	0.22	0.24	0.13	0.08	0.05	0.03	0.02	0.01	0
3	0.04	0.11	0.15	0.22	0.12	0.08	0.05	0.03	0.02	0.01
4	0.07	0	0.09	0.13	0.2	0.12	0.08	0.05	0.03	0.03
5	0.13	0	0.06	0.08	0.12	0.19	0.12	0.08	0.05	0.04
6	0.22	0	0.03	0.05	0.08	0.12	0.2	0.13	0.09	0.07
7	0.13	0	0	0.03	0.05	0.08	0.12	0.21	0.15	0.12
8	0.07	0	0	0.02	0.03	0.05	0.08	0.13	0.24	0.19
9	0.04	0	0	0.01	0.02	0.03	0.05	0.08	0.15	0.32

Exhibit 9 Transition Matrix for open to open losses.

Transition from 24 to 36 months										
Initial Class	Final Class									
	10	11	12	13	14	15	16	17	18	19
0	0	0	0	0.01	0.01	0.04	0.07	0.04	0.01	0.01
1	0	0.11	0	0	0	0	0	0	0	0
2	0.03	0.03	0.06	0.03	0.03	0	0	0	0	0
3	0.01	0.02	0.04	0.07	0.04	0.02	0.01	0.01	0	0
4	0.01	0.01	0.02	0.04	0.07	0.04	0.02	0.01	0.01	0
5	0.01	0.01	0.01	0.03	0.04	0.06	0.04	0.03	0.01	0.01
6	0	0.01	0.01	0.02	0.03	0.04	0.07	0.04	0.03	0.02
7	0	0	0.01	0.01	0.02	0.03	0.04	0.07	0.04	0.03
8	0	0	0	0	0.01	0.02	0.03	0.05	0.08	0.05
9	0	0	0	0	0	0.01	0.01	0.03	0.06	0.1

Exhibit 10 Transition Matrix for open to closed losses.

A third type of transition to be considered is the closed to closed transition as shown in exhibit 11. This looks as expected, where, the transitions with the same initial and final size form a 100 percent ridge forming a diagonal across the page. There is one exception in

Transition from 24 to 36 months										
Final Class	Initial Class									
	10	11	12	13	14	15	16	17	18	19
10	1	0	0	0	0	0	0	0	0	0
11	0	1	0	0	0	0	0	0	0	0
12	0	0	1	0	0	0	0	0	0	0
13	0	0	0	1	0	0	0	0	0	0
14	0	0	0	0	1	0	0	0	0	0
15	0	0	0	0	0	1	0	0	0	0
16	0	0	0	0	0	0	0.92	0	0	0
17	0	0	0	0	0	0	0	1	0	0
18	0	0	0	0	0	0	0	0	1	0
19	0	0	0	0	0	0	0	0	0	1

Exhibit 11 Transition Matrix for close to closed losses

our example, in the initial class of 16 where the probability is less then 100 percent. The rest of the probability is carried in the fourth type of transition the closed to open transitions shown in exhibit 12. Although this type of transition is rather rare, they are shown to illustrate that this portion of the matrix can contain real data and should not be ignored. This quadrant of the matrix must exist in order to accommodate the few claims that may fall into it. Otherwise, when coding to process data, errors can appear.

Transition from 24 to 36 months										
Final Class	Initial Class									
	10	11	12	13	14	15	16	17	18	19
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0.02	0	0	0
6	0	0	0	0	0	0	0.04	0	0	0
7	0	0	0	0	0	0	0.02	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0

Exhibit 12 Transition Matrix for close to open losses

A second quantity that needs to be developed is a vector representing the probability of a claim being in a class state at an evaluation. This is performed as follows: One selects the evaluation of interest and assigns class values to all claims based on size and open status at the evaluation. Then, the count for each class is divided by the total number of claims in the evaluation. This will produce a vector of probabilities, an example of which, is shown in exhibit 13.

If we take the square Markov transition matrix for the transition from 24 to 36 months shown in exhibit 8 and multiply it by the 24 month initial vector shown in exhibit 13, the result will be a one dimensional vector that contains the final probabilities at 36 months. The Markov transition matrix chain is then

Initial Values 24 month evaluation		
Class	Claim Count	Prob.
0	67	0.016
1	9	0.002
2	33	0.008
3	102	0.024
4	286	0.067
5	755	0.177
6	1905	0.447
7	696	0.163
8	237	0.056
9	69	0.016
10	0	0.000
11	0	0.000
12	1	0.000
13	2	0.000
14	6	0.001
15	17	0.004
16	49	0.012
17	17	0.004
18	6	0.001
19	2	0.000
Total	4259	1.000

Initial Vector for Matrix multiplicatio
0.016
0.002
0.008
0.024
0.067
0.177
0.447
0.163
0.056
0.016
0.000
0.000
0.000
0.000
0.001
0.004
0.012
0.004
0.001
0.000

Exhibit 13. A vector of initial probabilities for 24 month evaluation.

established by using this final at 36 months vector, and using it as the initial vector at 36 months and multiplying the 36 to 48 month transition matrix by it resulting in the final at 48 month probability vector. This process is continued in maturity order until the oldest transition matrix is used. The last transition matrix may have to be judgmentally adjusted so that all claims are closed after it is used. This is accomplished by using a matrix where the first and last quadrants, exhibits 9 and 12 contain all zeros, and 100% of the probability is in the other two quadrants, exhibits 10 and 11.

We now have all the tools necessary to evaluate loss development by transition matrix theory. Let us consider the ultimate loss development of an individual claim. For an example, let us select a $2,000,000 claim that is open at 24 months. Then, we raise the question, what does the ultimate development for this claim look like? To arrive at the answer we simply place this claim in an initial vector, and we multiply this vector by all the transition matrices in maturity sequence forming a Markov transition chain.

Initial Values 24 month evaluation		
Class	Claim Count	Prob.
0	0	0
1	0	0
2	0	0
3	0	0
4	0	0
5	1	1
6	0	0
7	0	0
8	0	0
9	0	0
10	0	0
11	0	0
12	0	0
13	0	0
14	0	0
15	0	0
16	0	0
17	0	0
18	0	0
19	0	0
Total	1	1

Exhibit 14. Initial vector for a claim of $2,000,000 size and a maturity of 24 months.

The initial vector for this is a special case where all the probability is concentrated in one class, and our example is shown in exhibit 14. When this vector is multiplied by the transition matrix for the 24 to 36 month transition, the result is a final value vector which contains the contents of the initial class 5 column of the transition matrix, as shown in exhibit 15.

Final Value 36 Month Maturity	
Class	Prob.
0	0.017
1	0.029
2	0.046
3	0.075
4	0.121
5	0.193
6	0.121
7	0.075
8	0.046
9	0.029
10	0.005
11	0.009
12	0.015
13	0.025
14	0.040
15	0.064
16	0.040
17	0.025
18	0.015
19	0.009
Total	0.000

Exhibit 15. Final vector for a claim of $2,000,000 size and a maturity of 24 months at 36 months.

This final at 36 month vector serves as the initial vector for 36 months to multiply with the transition matrix for 36 to 48 months forming the next step in the Markov chain. This process is repeated until all of the transition matrices are used. The final value vector that results is the ultimate loss development for the claim in our example. In order to illustrate this example, artificial data in transition matrices is used. The results of these various transitions are shown in a graph in Exhibit 16. One obvious feature of this graph is the two distinct peaks. The first one is in the open claims range, (Classes 0 to 9) and the second one in the closed range, (Classes 10 to 19) This shows that as claims mature, the peak decreases on the left, and increases on the right, corresponding with a decrease in open claims and an increase in closed claims. The "Final" line, indicated by triangles, shows a peak, centered around class 15 (the same size as our starting size class 5) and classes 0 through 9 have zero probability signifying there are no open claims. It is interesting to note that there is about 3% probability in class 10 which is the closed, zero size class. This allows us to make a statement about the potential loss development of an open claim. We can say that its ultimate value will be distributed with the probabilities contained in the "Final" line shown on this graph. It has some probability of closing with no payment, and the rest of the probability is distributed with the indicated distribution of size.

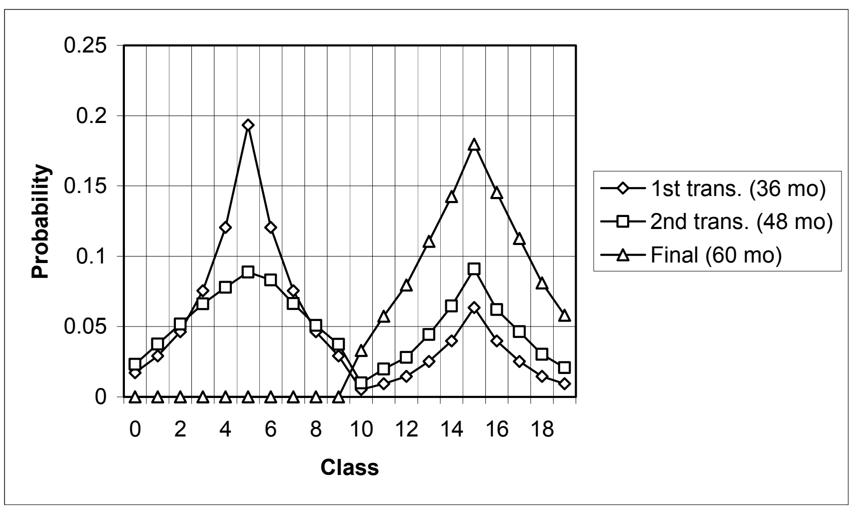

Exhibit 16. Graph showing intermediate and final results for a initial class 5 claim at 24 months maturity.

It is possible to select the points corresponding to the closed with non-zero size on the "Final" line and renormalize the probabilities and fit a distribution. This will be discussed further when the process is applied to real data.

2.3 Transition Matrix Applied to Real Data

A very large database was available for testing out this procedure. This data consists of all the claims that are submitted to a very large reinsurance intermediary for claims processing. The details of processing this data is contained in appendix A.

			Interval	Loss Average	Interval	Cumulative		% Exceeding
Class	Lower Limit	Upper Limit	Average	in Interval	Count	Count	Percent	Upper Limit
001	-	5,423	2,712	1,461	2039	2,039	7.3%	92.7%
002	5,423	9,714	7,569	7,422	528	2,567	9.1%	90.9%
003	9,714	17,400	13,557	13,171	735	3,302	11.8%	88.2%
004	17,400	31,168	24,284	23,963	1009	4,311	15.4%	84.6%
005	31,168	55,828	43,498	42,707	1212	5,523	19.7%	80.3%
006	55,828	100,000	77,914	76,343	1850	7,373	26.3%	73.7%
007	100,000	179,121	139,561	137,018	2366	9,739	34.7%	65.3%
008	179,121	320,845	249,983	246,534	2916	12,655	45.1%	54.9%
009	320,845	574,702	447,774	438,115	3266	15,921	56.7%	43.3%
010	574,702	1,029,416	802,059	772,980	3667	19,588	69.8%	30.2%
011	1,029,416	1,843,905	1,436,661	1,364,345	3273	22,861	81.4%	18.6%
012	1,843,905	3,302,830	2,573,368	2,434,316	2071	24,932	88.8%	11.2%
013	3,302,830	5,916,079	4,609,455	4,382,069	1367	26,299	93.7%	6.3%
014	5,916,079	10,596,969	8,256,524	7,850,304	875	27,174	96.8%	3.2%
015	10,596,969	18,981,451	14,789,210	13,711,652	467	27,641	98.5%	1.5%
016	18,981,451	33,999,861	26,490,656	24,389,644	258	27,899	99.4%	0.6%
017	33,999,861	60,901,062	47,450,462	43,658,304	106	28,005	99.8%	0.2%
018	60,901,062	109,086,897	84,993,980	83,690,124	38	28,043	99.9%	0.1%
019	109,086,897	195,398,091	152,242,494	138,646,237	17	28,060	100.0%	0.0%
020	195,398,091	350,000,000	272,699,046	262,891,307	9	28,069	100.0%	0.0%
021	350,000,000	626,923,500	488,461,750	457,301,309	2	28,071	100.0%	0.0%

Exhibit 17. This is the size of loss profile of the claims used in this study after trend and at latest evaluation. Also shown are the interval end-points for defining the size classes.

2.3.1 Data Attributes

It is interesting to explore the size of loss distribution of the claims used in this study. Exhibit 17 contains the loss size distribution of the 28,000 claims in the study as of 2003. The size boundaries in this LSD, at first, appear to be unusual. They were selected to provide 14 intervals between \$100,000 and \$350,000,000. They were also selected to increase exponentially, and each is 1.79121 times the last one. Here we see about 3/4 of the claims are over \$100,000, about 43% are over \$574,702, 1/10 are over \$3,302,830, and about 1% are over \$18,981,451. It would appear that there is enough population in all the size bands to allow a valid study.

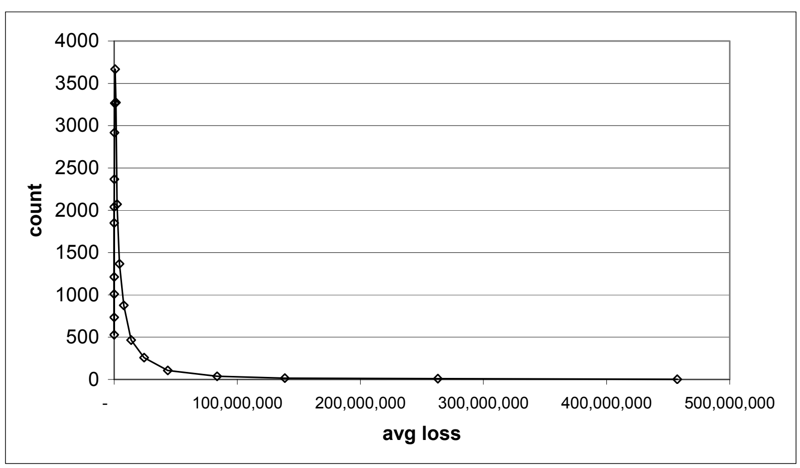

Exhibit 18. Graph of the distribution of the claim sizes used in the study.

2.3.2 Lognormal Behavior

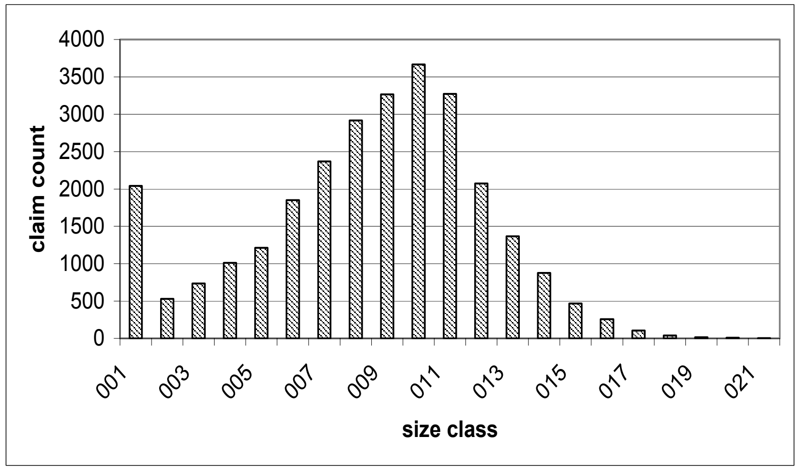

Exhibit 19. Histogram plot of claim counts by size class for losses in study.

Exhibit 18 shows a graph of the size of loss profile. This shows the claim count in asize interval verses the average size of the interval (the average of the upper and lower bound) This view shows a typical heavy tailed distribution with a significant skew to the right, but, one can get little other insight from it. Exhibit 19 shows the claim count by interval, plotted as a histogram of the intervals. This reveals much more about the distribution of the data. We see a bell shaped curve with little skewing left or right except for the elevated first interval. This occurs because the boundaries of the intervals increase exponentially. This behavior suggests that the losses are log normally distributed. Exhibit 20 shows the claim count plotted verses the interval average on a log scale. Again, we see the bell shaped curve with little skewing. The elevated first interval is probably due to the fact that it does not follow the exponential pattern of the other intervals. It contains all losses between 0 and $5,000 which would have been distributed over several intervals had they been defined with narrower (and exponential) boundaries. We find further support for the hypothesis that these losses are distributed log normally when we check the moments of this distribution. If we use the grouped data and take the log of the interval average as the distribution, we get a mean of 12.15, a standard deviation of 2.04, a skewness of -.4 and a kurtosis of 2.9. The last two are of particular interest as a distribution with a skewness between -0.5 and 0.5 is considered to be symmetrical and a normal distribution has a kurtosis of 3. This suggests that the lognormal distribution is consistent with this data.

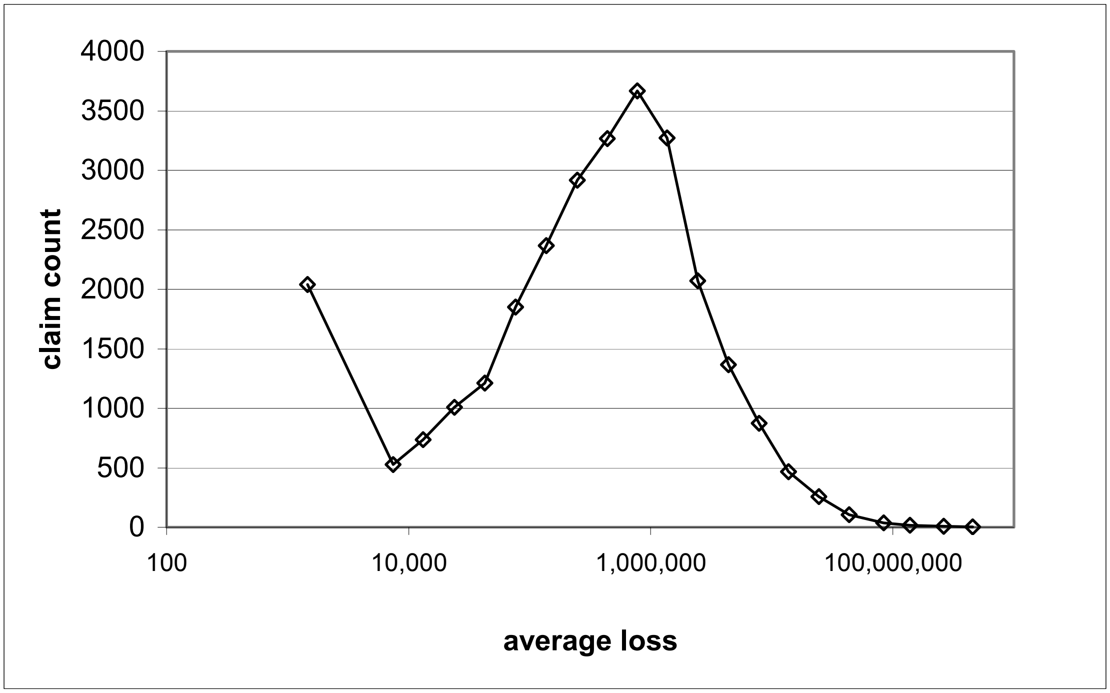

Exhibit 20. Log plot of claim counts of losses used in study.

Final	Initial size category ("O" for open, "C" for closed)																																										
Class	O000	O001	O002	O003	O004	O005	O006	O007	O008	O009	O010	O011	O012	O013	O014	O015	O016	O017	O018	O019	O020	O021	C000	C001	C002	C003	C004	C005	C006	C007	C008	C009	C010	C011	C012	C013	C014	C015	C016	C017	C018	C019	C020	C021
O000	0.00																																											
O001		0.35		0.01	0.01		0.00																																					
O002		0.01	0.34				0.01		0.00																																			
O003		0.00		0.32		0.01																																						
O004		0.00	0.03	0.04	0.40	0.01	0.01						0.01																															
O005		0.01		0.01	0.02	0.39	0.00																																					
O006		0.00		0.01	0.02	0.03	0.42	0.01				0.00																																
O007		0.00	0.02	0.03	0.02	0.05	0.03	0.43	0.01	0.01		0.00																																
O008		0.02		0.01	0.02	0.02	0.03	0.07	0.46	0.02	0.00	0.01																																
O009		0.01		0.02		0.02	0.01	0.03	0.06	0.48	0.04	0.01		0.03																														
O010		0.01		0.03		0.01	0.02	0.03	0.03	0.07	0.51	0.05																																
O011		0.03					0.00	0.02	0.03	0.03	0.05	0.48	0.08	0.03																														
O012		0.01					0.00	0.01	0.01	0.00	0.04	0.08	0.54	0.04	0.02																													
O013		0.00				0.01		0.00		0.00	0.01	0.02	0.05	0.61	0.10																													
O015		0.00						0.00			0.00	0.00	0.01		0.45	0.15																												
O016					0.01			0.00				0.00		0.01	0.02	0.41	0.18																											
O017										0.00				0.01		0.07	0.47																											
O018																	0.06	0.75																										
O019																			0.33																									
O020																				0.67	1.00																							
O021																						1.00																						
C000		0.41	0.39	0.29	0.31	0.16	0.14	0.12	0.12	0.11	0.09	0.11	0.03	0.07	0.17	0.04							1.00																					
C001		0.05	0.08	0.09	0.04	0.03	0.02	0.01	0.03	0.01	0.00	0.01	0.01											1.00																				
C002		0.01	0.02	0.01		0.01	0.01	0.02		0.00	0.00	0.00													1.00																			
C003		0.00	0.05	0.03	0.02	0.01	0.02	0.01		0.00																1.00																		
C004		0.00	0.02	0.01	0.02	0.02	0.00		0.01			0.00															1.00																	
C005		0.00		0.01	0.03	0.09	0.03	0.01	0.01	0.00	0.00																	1.00																
C006		0.00	0.02	0.02	0.02	0.07	0.10	0.03	0.02	0.01	0.00																		1.00															
C007		0.01	0.02	0.03	0.02	0.01	0.03	0.13	0.02	0.02	0.00	0.00																		1.00														
C008		0.01		0.03	0.02	0.01	0.04	0.03	0.12	0.05	0.02	0.00		0.01																	1.00													
C009		0.01		0.01	0.01	0.01	0.01	0.03	0.04	0.11	0.04	0.02	0.01																			1.00												
C010		0.01	0.02	0.01	0.01		0.02	0.01	0.02	0.04	0.14	0.02	0.02																				1.00											
C011		0.00			0.01	0.01	0.00		0.01	0.02	0.03	0.15	0.05	0.01	0.02																			1.00										
C012			0.02						0.00	0.00	0.01	0.02	0.15	0.03	0.02																				1.00									
C013		0.00								0.00	0.00	0.01	0.04	0.11	0.02	0.04	0.06																			1.00								
C014							0.00					0.00	0.01	0.03	0.17	0.15																					1.00							
C015														0.01		0.07																						1.00						
C016		0.00					0.00									0.07	0.18		0.17																				1.00					
C017																	0.06	0.25																						1.00				
C018																			0.50																						1.00			
C019																																										1.00		
C020																				0.33																							1.00	
C021																																												1.00

Exhibit 21 An example of a transition matrix for GL data for the transition of 24 to 36 months

Final Size Class	Initial Size Class																					
Class	000	001	002	003	004	005	006	007	008	009	010	011	012	013	014	015	016	017	018	019	020	021
000	0.00	0.80	0.54	0.45	0.42	0.42	0.41	0.36	0.36	0.35	0.33	0.29	0.29	0.27	0.30	0.19	0.10	0.14	0.17	0.06	0.23	
001		0.04	0.10	0.14	0.13	0.09	0.06	0.03	0.02	0.02	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
002		0.00	0.09	0.02	0.03	0.02	0.02	0.02	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
003		0.01	0.03	0.05	0.05	0.03	0.03	0.02	0.01	0.01	0.00	0.01	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
004		0.01	0.05	0.07	0.10	0.05	0.04	0.02	0.02	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
005		0.01	0.05	0.08	0.08	0.14	0.05	0.04	0.02	0.02	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
006		0.01	0.03	0.07	0.06	0.08	0.14	0.06	0.04	0.02	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
007		0.01	0.04	0.02	0.04	0.06	0.08	0.19	0.08	0.03	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
008		0.01	0.01	0.03	0.04	0.04	0.06	0.10	0.23	0.08	0.03	0.02	0.01	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	
009		0.02	0.01	0.01	0.01	0.03	0.04	0.06	0.10	0.24	0.08	0.04	0.02	0.02	0.00	0.01	0.00	0.00	0.00	0.00	0.00	
010		0.02	0.01	0.02	0.02	0.01	0.03	0.05	0.07	0.12	0.28	0.11	0.04	0.02	0.01	0.00	0.00	0.00	0.00	0.00	0.00	
011		0.03	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.05	0.13	0.32	0.12	0.05	0.02	0.03	0.00	0.00	0.00	0.00	0.01	
012		0.01	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.02	0.03	0.09	0.29	0.08	0.04	0.01	0.01	0.01	0.00	0.00	0.01	
013		0.01	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.04	0.11	0.34	0.11	0.05	0.01	0.02	0.00	0.01	0.05	
014		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.05	0.13	0.38	0.10	0.04	0.04	0.00	0.02	0.01	
015		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.05	0.09	0.45	0.15	0.06	0.01	0.08	0.01	
016		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.03	0.11	0.51	0.16	0.02	0.07	0.01	
017		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.05	0.15	0.55	0.21	0.72	0.04	
018		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.39	0.03	0.01	
019		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.19	0.00	0.04	
020																					0.57	0.67
021		0.00			0.00	0.00	0.00	0.00		0.00	0.00	0.00	0.00	0.00	0.00	0.00					0.00	0.33

Exhibit 22. This is an ultimate matrix for the transition from 24 months maturity to ultimate. A claim open at 24 months will have an ultimate size that has a probability distribution described by the probabilities under its initial class size.

2.3.3 Transition Matrix Results

This data was processed through a Markov transition matrix analysis, that produced transition matrices which were multiplied together to yield ultimate matrices. Exhibit 21 shows an example of a transition matrix from this study. This one is for the transition from 24 to 36 months. Note that the classes indicating size and open status have been modified from the earlier example. Open claims are indicated with a class starting with an "O" and closed with a "C". Size ranges from 0 to 20 where 0 is a loss size of $0.00.

2.3.4 Distributional Development

An Ultimate Matrix is shown in exhibit 22. Note that the open and closed status has been collapsed into the closed status. This was accomplished by assuming that the final status of the last set of transitions was always closed. Since, at 23 years more than 95% of the transitions were closed to closed, this is not thought to be an unreasonable assumption. This Ultimate Matrix can be thought of a series of one dimensional vectors stacked next to each other. Each vector provides the prediction of ultimate size based on the initial size. This provides a critical insight. This suggests that it is possible to describe the loss development of an individual open claim. This vector of final possible outcomes provides some probability of closing with no payment, or an array of probabilities of closing at various sizes. We can study the conditional probability given that a claim closes with some payment by removing the probability of closing with no payment into a separate category. After the zero claims are removed, what is left is a probability distribution of final size given an initial size and initial maturity. Exhibit 23 shows the final distribution of a loss that had an initial size category 7 at a maturity of 24 months.

Size Category	Probability
000	0.362
001	0.029
002	0.019
003	0.020
004	0.024
005	0.036
006	0.059
007	0.189
008	0.101
009	0.064
010	0.047
011	0.030
012	0.009
013	0.007
014	0.002
015	0.001
016	0.000
017	0.000
018	0.000
019	0.000
020	0.000
021	0.000

Exhibit 23 - Ultimate size vector for size category 7 at 24 months.

Size Category	Lower Limit	Upper Limit	Average Loss size	Log of avg Loss Size	Probability (class) 007	Normalized Probability	x*prob	x^2*prob	x^3*prob	x^4*prob
000	0	0		0.00	0.3617					
001	0	5,423	2,712	7.91	0.0294	0.0460	0.36	2.88	22.74	179.73
002	5,423	9,714	7,569	8.93	0.0192	0.0301	0.27	2.40	21.48	191.86
003	9,714	17,400	13,557	9.51	0.0204	0.0320	0.30	2.90	27.59	262.51
004	17,400	31,168	24,284	10.10	0.0244	0.0383	0.39	3.90	39.39	397.78
005	31,168	55,828	43,498	10.68	0.0363	0.0568	0.61	6.48	69.21	739.18
006	55,828	100,000	77,914	11.26	0.0591	0.0927	1.04	11.75	132.39	1,491.19
007	100,000	179,121	139,561	11.85	0.1888	0.2958	3.50	41.51	491.77	5,825.66
008	179,121	320,845	249,983	12.43	0.1005	0.1575	1.96	24.33	302.40	3,758.64
009	320,845	574,702	447,774	13.01	0.0642	0.1005	1.31	17.02	221.52	2,882.39
010	574,702	1,029,416	802,059	13.59	0.0469	0.0735	1.00	13.59	184.77	2,512.00
011	1,029,416	1,843,905	1,436,661	14.18	0.0304	0.0476	0.67	9.57	135.68	1,923.60
012	1,843,905	3,302,830	2,573,368	14.76	0.0085	0.0134	0.20	2.91	42.94	633.78
013	3,302,830	5,916,079	4,609,455	15.34	0.0066	0.0103	0.16	2.42	37.08	568.94
014	5,916,079	10,596,969	8,256,524	15.93	0.0021	0.0033	0.05	0.83	13.28	211.51
015	10,596,969	18,981,451	14,789,210	16.51	0.0010	0.0015	0.02	0.41	6.70	110.69
016	18,981,451	33,999,861	26,490,656	17.09	0.0003	0.0005	0.01	0.14	2.42	41.30
017	33,999,861	60,901,062	47,450,462	17.68	0.0001	0.0001	0.00	0.04	0.72	12.74
018	60,901,062	109,086,897	84,993,980	18.26	0.0000	0.0000	0.00	0.01	0.22	4.02
019	109,086,897	195,398,091	152,242,494	18.84	0.0000	0.0000	0.00	0.00	0.03	0.58
020	195,398,091	350,000,000	272,699,046	19.42	0.0000	0.0000	0.00	0.00	0.00	0.00
021	350,000,000	626,923,500	488,461,750	20.01	0.0000	0.0000	0.00	0.01	0.23	4.59
Total					1.000		mean (mu)	E(x^2)	E(x^3)	E(x^4)
							11.86	143.12	1,752.57	21,752.68
								variance	3rd moment	4th moment
								2.39	-1.95	21.95
								std dev(sigma)	skewness	kurtosis
								1.55	-0.53	3.84

initial class	007
average size	139,561
log of avg size	11.85
mu	11.86
sigma	1.55
ratio mu/log(avg)	1.00
ratio sigma/mu	0.13
skewness	-0.53
kurtosis	3.84

Exhibit 24 - Moments of ultimate size vector for Category 7 size at 24 months.

We can apply the same log normal analysis, used previously, to this distribution, and we get the results shown on Exhibit 24. Here we see a mu of 11.86, a sigma of 1.55, a skewness of -0.53, and a kurtosis of 3.84. The skewness close to zero suggests that the ultimate values are lognormally distributed.

It is instructive to divide the final mu and sigma by the log of the initial value. Here we find that the final mean is very nearly equal to the initial value of the loss. Also, the standard deviation is a small fraction of the initial mean. This result suggests the remarkable conclusion. It seems that the loss development potential of a claim is that its ultimate value will be log normally distributed with a mu equal to the natural log of the initial value, and a sigma that can be predicted. Although this seems to be counter intuitive, we must remember that the formula for the mean of a lognormal has the following formula:

mean = e^(mu+((sigma^2)/2))

which leads to an increase in the mean as sigma grows. So, the resulting average loss at ultimate will be greater then the current evaluation.

2.3.5 Variation Over Initial Size

The next question to investigate is how mu and sigma of the ultimate distribution vary with initial size and maturity. Shown in exhibit 25 are the ultimate distributions for three initial claims sizes, Classes 6, 7, and 8, initially at 24 months maturity. This shows a peak of 25 to 30 percent in the Class 0 (closed with no payment) category. To the right, each curve has a distinctive bell shaped curve that peaks in the final size category that is the same as the initial size category. Each curve looks symmetrical, and, remarkably like each other.

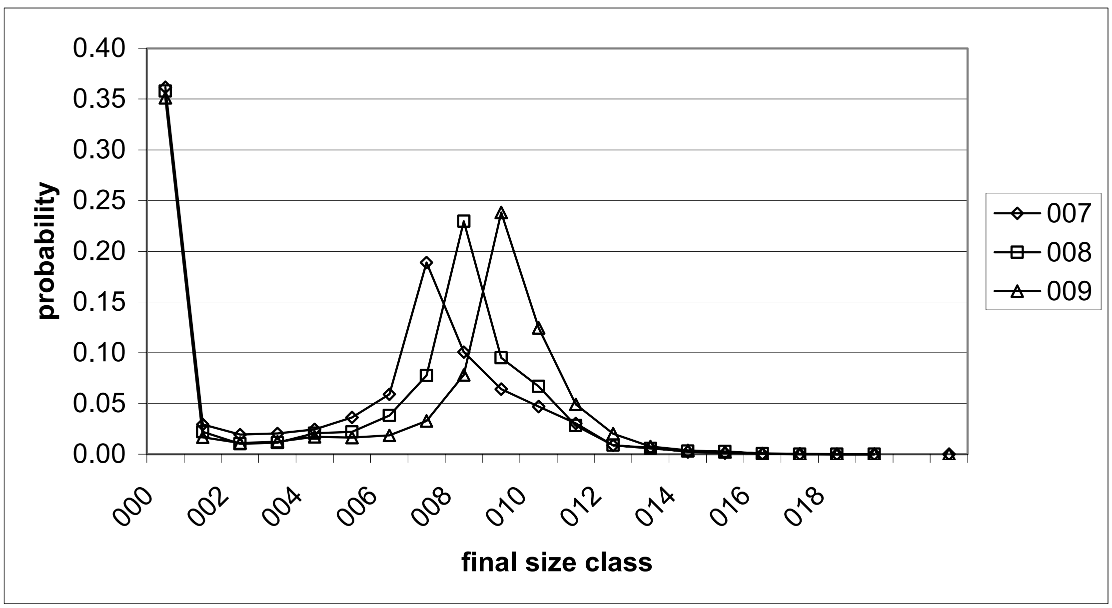

Exhibit 25 Graph of final distribution of 24 month initial open claims with intital sizes of Class 6, Class 7, and Class 8

The graphical appearance of this data alone suggests a possible behavior where the final size is related to the initial size, and the spread of the distribution does not vary with initial size.

Exhibit 26 is similar to the previous graph with a wider range of initial sizes. This covers size classes 7 to 14. The next exhibit is a graph of all the ultimate distributions for the 24month to ultimate transition. Each curve is for a different initial size class. They all demonstrate the bell shaped appearance seen previously, suggesting a lognormal distribution.

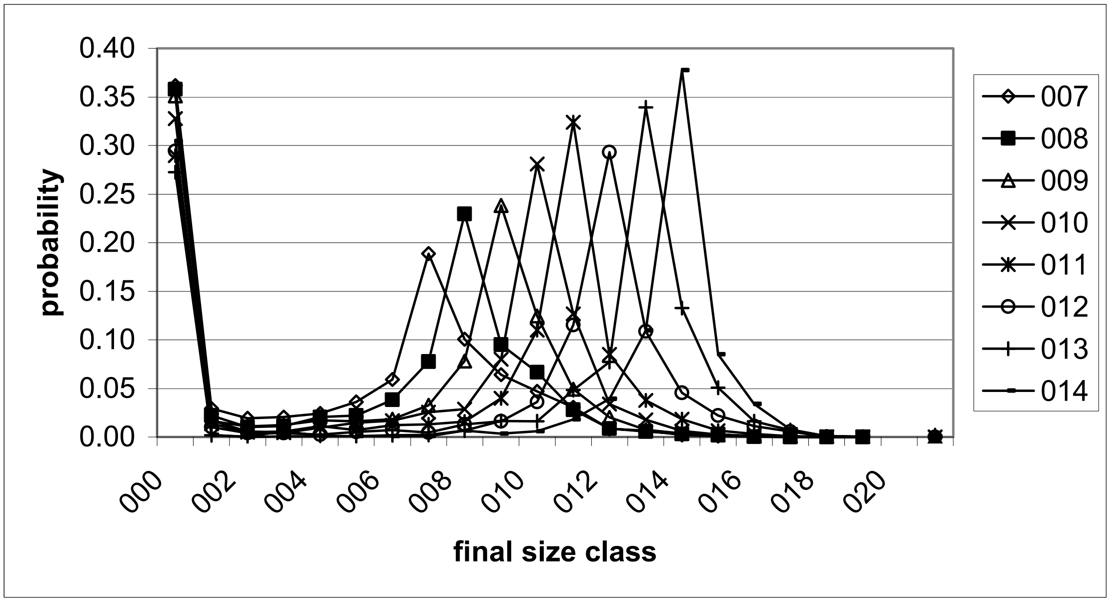

Exhibit 26 Graph of final distribution of 24 month initial open claims with intital sizes of Class 4 to Class 11.

The striking aspect of this graph is how much the bell shaped curves resemble each other in the range of size class 5 to 10. At first glance, these appear as identical curves which are offset from each other by a constant amount. Since the boundaries of the classes were defined with a multiplicative factor, the constant spacing occurs because of the logarithmic nature of the scale, and the fact that the mu of each distribution has a relation to the initial loss size. This relationship is that, the mu is nearly the same as the logarithm of the original loss size. The fact that the shape of the curves do not change as they progress from left to right suggest that the spread parameters are very similar for all the curves.

	average	log of	Parms of ult dists.			ratio	ratio
initial class	class size	avg size	mu	sigma	Skewness	mu/log(avg)	sigma/mu
001	2,712	7.91	11.93	2.57	-0.34	1.51	0.22
002	7,569	8.93	10.16	1.95	0.78	1.14	0.19
003	13,557	9.51	10.37	1.97	0.45	1.09	0.19
004	24,284	10.10	10.49	2.00	0.62	1.04	0.19
005	43,498	10.68	10.84	1.84	0.21	1.01	0.17
006	77,914	11.26	11.24	1.77	-0.04	1.00	0.16
007	139,561	11.85	11.86	1.55	-0.53	1.00	0.13
008	249,983	12.43	12.27	1.43	-0.90	0.99	0.12
009	447,774	13.01	12.75	1.45	-1.31	0.98	0.11
010	802,059	13.59	13.28	1.43	-1.76	0.98	0.11
011	1,436,661	14.18	13.81	1.46	-1.94	0.97	0.11
012	2,573,368	14.76	14.54	1.44	-2.16	0.98	0.10
013	4,609,455	15.34	15.25	1.14	-2.07	0.99	0.08
014	8,256,524	15.93	15.75	0.95	-3.14	0.99	0.06

Exhibit 27 - Parameters for ultimate distributions by size class for 24 month initial losses

Exhibit 27 shows the mu, sigma, skewness, and kurtosis of all the 24 month to ultimate distributions by initial size classification. Here we see that the mu's are all very close to the log of the initial, and the sigma values are all very similar. In exhibit 28 we graph the mu's and the ratio of mu to the natural log of the average loss in the interval. This shows a strong relationship between these two values.

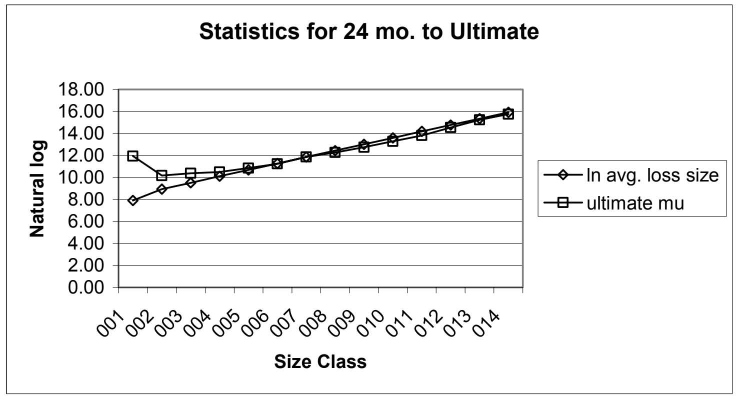

Exhibit 28 - This graph shows the natural log of the average loss size, and the mu of the lognormal distribution of the ultimate loss size distribution for claims with a current maturity of 24 months.

The sigma values shown in exhibit 27, and graphed in exhibit 29, show a gradual decrease as the loss size increases. It may be possible to find a relationship between the loss size and sigma. It appears that a linear relationship between sigma and the natural log of the initial loss may describe the behavior of sigma.

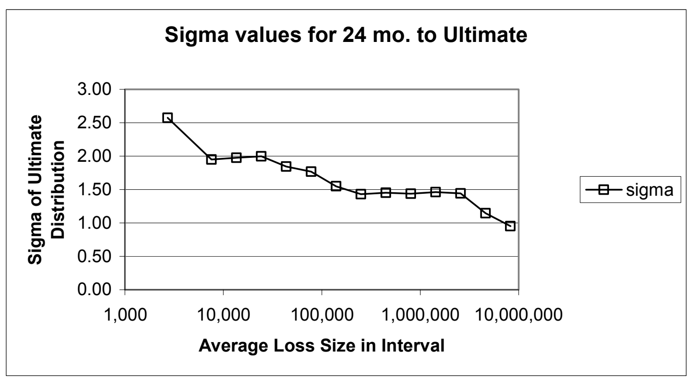

Exhibit 29 - Graph showing the sigma values for the lognormal distributions for the ultimate values for claims at 24 months.

The sknewness values shown in exhibit 27, and, graphed in exhibit 30, are positive for the small losses and negative for the large losses, and close to zero for the mid size losses. This is a relatively complex behavior but it can be understood based on the nature of the reinsurance claims that constitute the data.

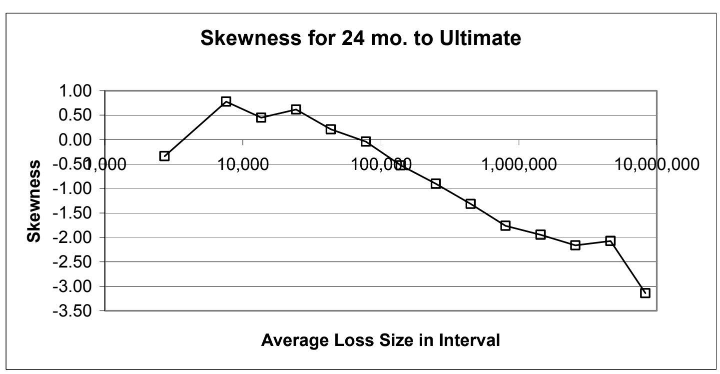

Exhibit 30 - Graph showing the skewness values for the lognormal distributions for the ultimate values for claims at 24 months.

Remember that these are only claims that are submitted for reinsurance recoveries. The full inventory of claims are not represented here. The positive skewing of the smaller claims can be understood as being caused by the submission of small claims that are expected to settle are larger amounts and enjoy a reinsurance recovery.

The larger claims may skew negatively because they are large enough to feel the effect of policy limits. As large claims settle, they are always free to settle at smaller then currently reserved amounts, but, any tendency to settle at higher values may be limited when the policy limit is reached.

Thus, we can understand the appearance of the positive skewing of the small claims and the negative skewing of the large claims as being due to data reporting and policy limit effects and is not an essential element of the loss development. The use of a lognormal distribution to describe the ultimate development of an individual claim continues to be consistent with the observations.

At this point, there is enough evidence to postulate a model for ultimate loss development for open claims at 24 months maturity. A lognormal distribution with a mu equal to the natural log of the open claim size, and a sigma which is described by a linear relationship between sigma and the natural log of the open claim size is consistent with the current observations.

One must remember that this is a conditional distribution, based on the condition that the claim does not close with no payment. We must remember that the transition matrix process contains an ultimate size category 000 which contains a significant number of claims that close with no payment. One needs only to refer back to exhibit 23, and pick the value from the first line under the correct initial loss size, to get the probability that the claim closes with no payment.

Another consideration in the use of this model is that the primary policy limit distribution must be applied after the loss development is applied.

Size Category	Average Size in Interval	Ratio of Mu/natural log of average size							
		Maturity in Months							
		24	36	48	60	72	84	96	108
001	2,712	1.51	1.52	1.48	1.46	1.44	1.46	1.43	1.44
002	7,569	1.14	1.15	1.14	1.18	1.11	1.10	1.17	0.99
003	13,557	1.09	1.11	1.08	1.06	1.07	1.01	1.05	0.99
004	24,284	1.04	1.02	1.04	1.05	1.06	0.99	0.98	0.96
005	43,498	1.01	1.00	1.00	0.98	0.98	0.95	0.96	0.99
006	77,914	1.00	0.99	0.98	0.99	0.99	0.99	1.00	0.99
007	139,561	1.00	0.98	0.96	0.98	0.98	0.97	0.97	0.97
008	249,983	0.99	0.98	0.98	0.98	0.97	0.97	0.98	0.98
009	447,774	0.98	0.98	0.98	0.97	0.98	0.98	0.98	0.99
010	802,059	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98
011	1,436,661	0.97	0.98	0.98	0.98	0.98	0.98	0.98	0.98
012	2,573,368	0.98	0.99	0.99	0.98	0.99	0.99	0.99	0.99

Exhibit 31 - This shows the ratio of the mu of the ultimate distribution divided by the natural log of average size in the initial size category.

2.3.6 Exploration over Maturities

Thus far, we have explored loss development behavior at one selected initial maturity. In order to create a system that can accommodate a complete selection of real data, we need to be able to describe this process for losses over the entire range of maturities. In order to do this, we first look at the comparison of the modeled mu to the natural log of the average loss size in each initial size category. We are looking at the ability of the initial loss size to forecast the mu of the ultimate distribution. Shown in exhibit 31 is the ratio of the mu of the ultimate distribution divided by the natural log of the average loss within the size category. If the forecast of mu is perfect, then this ratio should be 1.00, according to the postulated model. What we find is a relative flat surface except for the turned edge, as shown in exhibit 32.

The turned edge may be caused by the origin of the data. Since it is a collection of claims

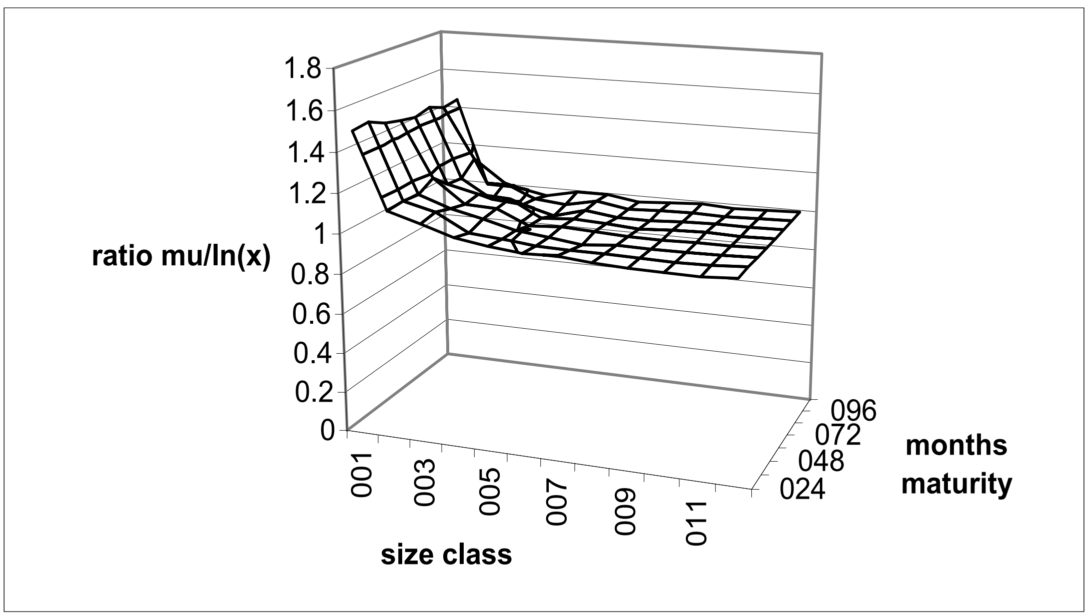

Exhibit 32 - Surface of ultimate mu / ln(avg loss size) over the range of size categories and months maturity of initial size observation.

that anticipate a reinsurance collection, it may be biased to develop larger. A more complete collection of claims may not have this bias. This suggests that the natural log of the current value of an open claim is a good predictor of the mu for the lognormal distribution describing the ultimate loss size.

If we conclude that this surface is a plane, and we ignore the first size category (001), all the remaining points have an average of 1.005 and a standard deviation of 0.05 Our model for estimating mu then becomes mu= 1.005 * ln(x), where x is the current open claim size. If one is not satisfied with the accuracy of applying one number over all size, maturity combinations, one could interpolate over the surface given in exhibit 31 and apply interpolated numbers to individual claims.

2.3.6 The Sigma Surface

We can also review the fitted sigma values as they vary by initial loss size and by initial maturity to see if a pattern emerges. These values are shown in exhibit 33, and graphed as a

Initial Size	Sigma at Ultimate Initial Maturity in Months									
Class	012	024	036	048	060	072	084	096	108	120
001	2.668	2.572	2.653	2.809	2.803	2.682	2.604	2.512	2.530	2.708
002	2.075	1.950	1.847	1.917	2.414	2.313	2.392	2.714	1.727	1.892
003	2.071	1.973	2.159	2.257	2.423	2.344	2.115	2.367	1.936	1.712
004	2.011	1.997	1.995	2.000	2.223	2.382	2.008	2.048	1.948	2.217
005	1.734	1.844	1.800	1.765	1.867	1.919	1.793	1.849	1.733	1.785
006	1.795	1.766	1.766	1.811	1.866	1.746	1.655	1.632	1.838	1.519
007	1.532	1.547	1.578	1.575	1.510	1.599	1.580	1.338	1.441	1.211
008	1.544	1.430	1.397	1.369	1.353	1.413	1.389	1.353	1.306	1.172
009	1.426	1.452	1.370	1.374	1.317	1.261	1.247	1.161	1.134	1.117
010	1.363	1.435	1.352	1.260	1.230	1.198	1.138	1.195	1.111	1.050
011	1.516	1.460	1.309	1.271	1.207	1.143	1.035	1.019	0.961	0.943
012	1.444	1.444	1.183	1.184	1.105	1.011	1.029	0.984	0.867	0.861
013	1.144	1.144	1.072	1.029	1.027	0.999	1.033	0.896	0.849	0.894
014	0.951	0.951	1.013	1.136	1.222	1.179	1.181	1.125	0.925	0.859
015	0.982	0.982	0.956	1.051	0.968	0.954	0.998	1.056	0.993	1.127

Exhibit 33 - These are the sigma's of the ultimate distributions
by initial size class and by initial maturity

surface in exhibit 34. This data appears to have some structure associated with it. There is a pronounced decrease in sigma as the size of the loss increases, and there is a modest decrease as the maturity of the claim increases. To further understand how sigma varies, it is instructive to graph it as one of the other variables change. First we look as the size changes.

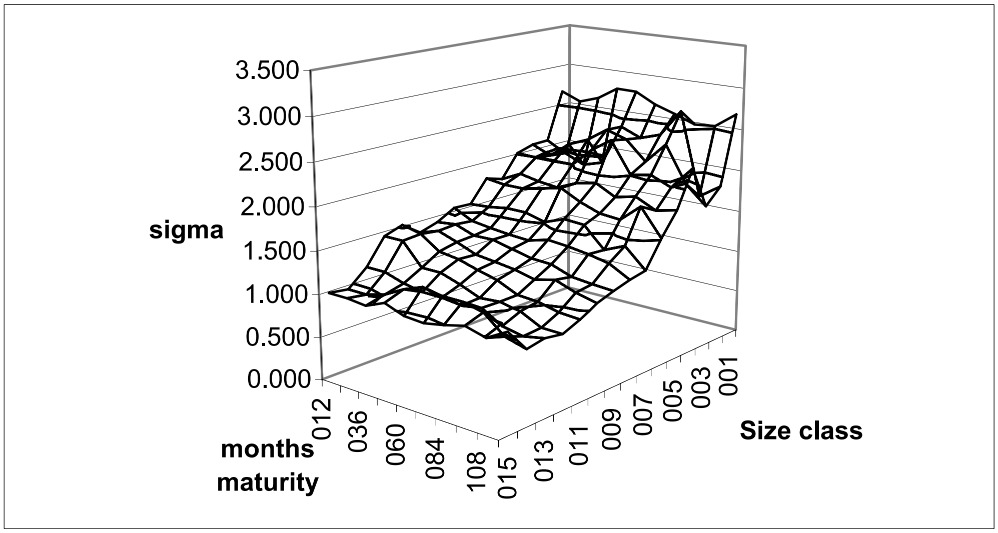

Exhibit 34 - 3D surface map of the sigma's of the ultimate distributions plotted verses maturity and size of current open claim.

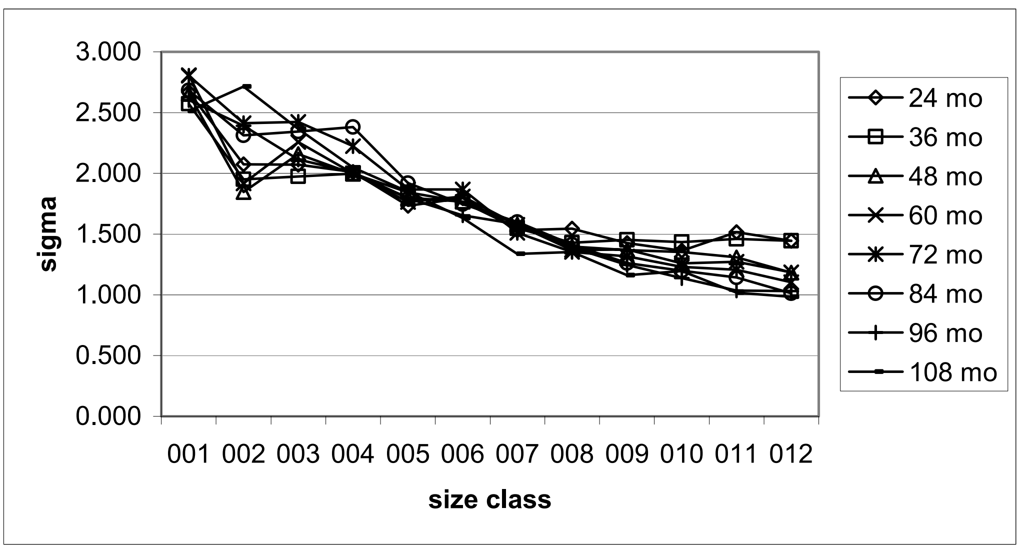

Exhibit 35 - Graph showing the variation of sigma with the size of the claim.

In exhibit 35, sigma is plotted as the size changes, and, each line represents a different maturity. This shows the decrease over time, which becomes more gradual as time progresses. If we assume that there is no structure in the maturity direction and all the variation is noise, we can average each size evaluation and plot the result. This is shown in exhibit 36. This shows more clearly the slowing of the decrease over time.

Exhibit 36 - Sigma's averaged over maturities and plotted verses size class.

It was found that the inverse of sigma behaves in a more orderly fashion. This variable appears to be linear when plotted against size class. The plot of the inverse of the average sigma is shown in exhibit 37. Here we see it is increasing at a constant rate. This variable has an additional benefit in that it behaves well at its extremes. At very large class sizes sigma becomes small, which is a believable result, and at very small class sizes it becomes very large, and then undefined. This is acceptable because there is no interest in modeling very, very small claims.

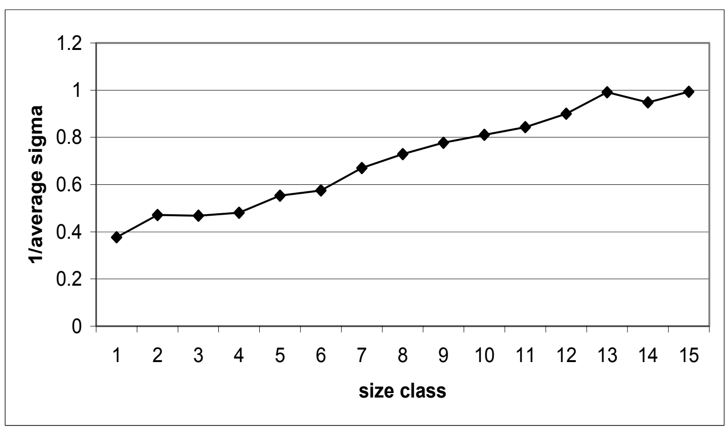

Exhibit 37 - Plot of 1/(avg. sigma) verses loss size.

The favorable behavior of this variable encourages us to explore the behavior of this transformed variable verses maturity. Exhibit 38 shows the plot of the inverse of sigma verses maturity, where, each line represents a size class. The overall impression is there is a bit of increase as maturity progresses. We average over the sizes and produce one value for each maturity and check to see if this varies as maturity. These values are plotted in exhibit 40. This shows a slight increase with maturity. There is nothing to suggest that this increase is more complicated than a first order linear effect.

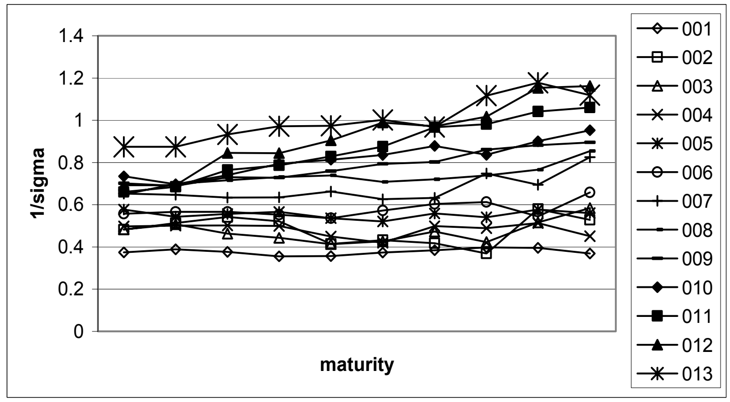

Exhibit 38 - Plot of 1/sigma verses maturity. Each line represents a different size class.

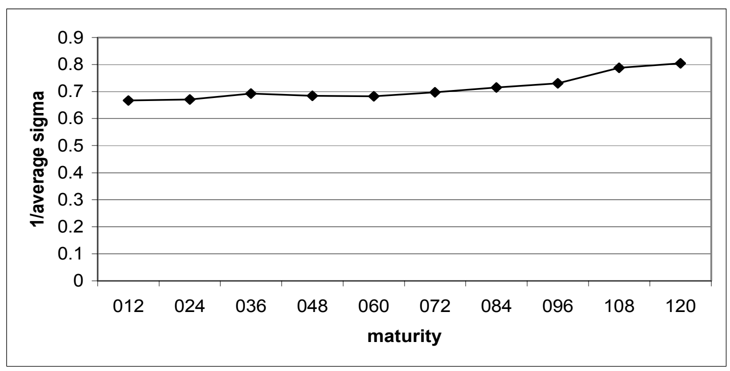

Exhibit 39 Plot of average of 1/sigma verses maturity.

2.3.7 Fitting the Sigma Surface

These two observations of linear behavior of 1/sigma verses maturity and verses loss size suggests that a linear fit to the surface will allow use

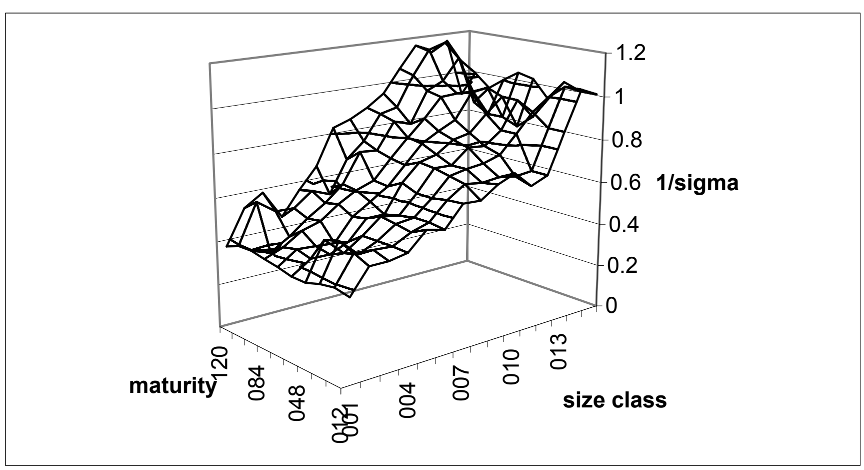

Exhibit 40 - Plot of 1/sigma verses maturity and size class.

to model sigma as a function of size and maturity. The sigma values in exhibit 33 are inverted and are graphed in exhibit 40. These values are analyzed by general linear regression against the dependent variables of, maturity in months, and, natural log of the average interval claims size. This resulted in a fitted regression of:

$$\text{Sigma} = 1/(\text{maturity} *0.001205+\ln(\text{loss size})*0.078874-0.34447)$$

A table of fitted sigma values are shown in exhibit 41, and these values are plotted in exhibit 42.

initial size class	initial maturity									
	012	024	036	048	060	072	084	096	108	120
001	3.41	3.25	3.10	2.97	2.85	2.73	2.63	2.53	2.44	2.36
002	2.67	2.57	2.48	2.39	2.31	2.24	2.17	2.10	2.04	1.98
003	2.38	2.30	2.23	2.16	2.09	2.03	1.97	1.92	1.87	1.82
004	2.14	2.08	2.02	1.96	1.91	1.86	1.81	1.76	1.72	1.68
005	1.95	1.90	1.85	1.80	1.75	1.71	1.67	1.63	1.59	1.56
006	1.79	1.75	1.70	1.66	1.62	1.59	1.55	1.52	1.48	1.45
007	1.65	1.62	1.58	1.54	1.51	1.48	1.45	1.42	1.39	1.36
008	1.54	1.50	1.47	1.44	1.41	1.38	1.36	1.33	1.31	1.28
009	1.44	1.41	1.38	1.35	1.33	1.30	1.28	1.25	1.23	1.21
010	1.35	1.32	1.30	1.27	1.25	1.23	1.21	1.19	1.17	1.15
011	1.27	1.25	1.22	1.20	1.18	1.16	1.14	1.12	1.11	1.09
012	1.18	1.18	1.16	1.14	1.12	1.10	1.09	1.07	1.05	1.04
013	1.12	1.12	1.10	1.08	1.07	1.05	1.03	1.02	1.00	0.99
014	1.06	1.06	1.05	1.03	1.02	1.00	0.99	0.97	0.96	0.95
015	1.01	1.01	1.00	0.98	0.97	0.96	0.94	0.93	0.92	0.91

Exhibit 41 - Table of fitted sigma's.

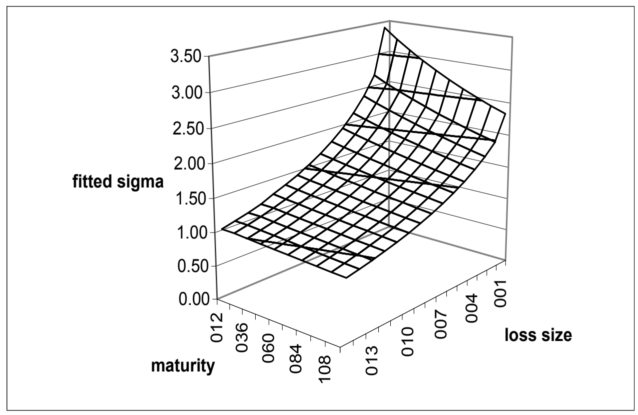

Exhibit 42 - Plot of fitted sigma's. Compare this with exhibit 34.

The error values for this regression are shown in exhibit 43. These values are the fitted sigma values minus the actual sigma values observed at each maturity and size class. The average of all these values is 0.00. These values are graphed in exhibit 44.

The only structure revealed in this graph is a sharp rise at early maturities and small size classes. This model tends to overestimate sigma in this region, but, since there is little interest in this region, this is an acceptable error. For those who need high accuracy in this area, it would be best to interpolate values directly from exhibit 33.

A review of the skewness as size and maturity is varied shows the same tendencies as noted earlier, positive skew for small losses and negative skew for large losses. The skewness values are shown in exhibit 45 and the surface is shown in exhibit 46. It may well be that the negative skewness and the decrease in sigma for large claims is caused by policy limit censoring. It is a long held view that small claims tend to develop larger, and large claims tend to develop smaller. This evidence certainly supports that view. One might be concerned that the proposed model will overdevelop large claims. One should reexamine exhibit 26 and observe that the graphed distributions for the large initial size classes

size class	Error values of sigma regression initial maturity 012	024	036	048	060	072	084	096	108	120
001	0.74	0.67	0.45	0.16	0.04	0.05	0.03	0.02	-0.09	-0.35
002	0.60	0.62	0.63	0.48	-0.10	-0.07	-0.22	-0.61	0.31	0.09
003	0.31	0.33	0.07	-0.10	-0.33	-0.31	-0.14	-0.45	-0.07	0.10
004	0.13	0.08	0.02	-0.04	-0.32	-0.53	-0.20	-0.29	-0.23	-0.54
005	0.22	0.05	0.05	0.03	-0.11	-0.21	-0.12	-0.22	-0.14	-0.23
006	0.00	-0.02	-0.06	-0.15	-0.24	-0.16	-0.11	-0.12	-0.35	-0.07
007	0.12	0.07	0.00	-0.03	0.00	-0.12	-0.13	0.08	-0.05	0.15
008	-0.01	0.07	0.08	0.07	0.06	-0.03	-0.03	-0.02	0.00	0.11
009	0.01	-0.05	0.01	-0.02	0.01	0.04	0.03	0.09	0.10	0.09
010	-0.02	-0.11	-0.06	0.01	0.02	0.03	0.07	-0.01	0.05	0.10
011	-0.25	-0.21	-0.08	-0.07	-0.02	0.02	0.11	0.11	0.15	0.15
012	-0.27	-0.27	-0.02	-0.04	0.02	0.09	0.06	0.09	0.19	0.18
013	-0.03	-0.03	0.03	0.05	0.04	0.05	0.00	0.12	0.15	0.10
014	0.11	0.11	0.03	-0.10	-0.21	-0.18	-0.19	-0.15	0.03	0.09
015	0.03	0.03	0.04	-0.07	0.00	0.00	-0.05	-0.12	-0.07	-0.22

Exhibit 43 - This table contains the error values of the regression for sigma. The values shown are (fitted sigma - actual sigma).

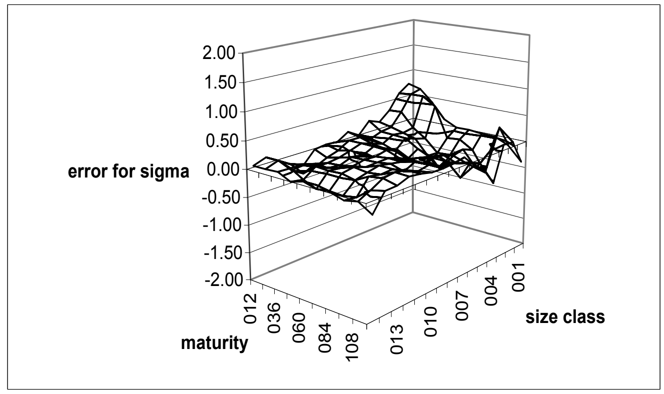

Exhibit 44 Plot of error values of regression for sigma. Vertical scale is : (modeled sigma - actual sigma)

Initial Size class	average size in interval	Skewness Initial Maturity in Months 024	036	048	060	072	084	096	108	120
001	2,712	-0.34	-0.34	-0.15	-0.09	-0.17	-0.30	-0.21	-0.15	-0.07
002	7,569	0.78	0.86	1.05	0.54	1.12	1.29	0.95	2.67	2.46
003	13,557	0.45	0.56	0.61	0.85	0.81	1.27	0.94	1.32	1.70
004	24,284	0.62	0.57	0.65	0.84	0.90	1.23	1.32	1.21	1.13
005	43,498	0.21	0.14	0.23	0.41	0.53	0.65	0.81	0.82	0.85
006	77,914	-0.04	0.03	0.15	0.19	0.27	0.44	0.59	0.76	0.21
007	139,561	-0.53	-0.59	-0.40	-0.32	-0.14	0.08	0.49	0.63	0.09
008	249,983	-0.90	-0.81	-0.74	-0.52	-0.36	-0.54	-0.67	-0.64	0.30
009	447,774	-1.31	-1.34	-1.21	-0.96	-0.99	-0.86	-0.87	-0.67	-0.93
010	802,059	-1.76	-1.68	-1.78	-1.88	-1.88	-1.93	-2.10	-2.17	-2.10
011	1,436,661	-1.94	-2.02	-1.97	-2.05	-2.14	-2.11	-2.26	-2.23	-2.36
012	2,573,368	-2.16	-2.34	-2.27	-2.24	-2.41	-2.60	-2.54	-2.66	-3.18
013	4,609,455	-2.07	-1.51	-2.49	-2.71	-2.92	-3.15	-2.62	-3.42	-3.73
014	8,256,524	-3.14	-3.20	-3.25	-3.57	-3.68	-3.15	-3.61	-2.79	-2.78
015	14,789,210	-2.81	-3.30	-3.02	-3.34	-3.63	-3.76	-4.21	-5.15	-5.11

Exhibit 45 Table of skewness values of ultimate distributions verses maturity and size.

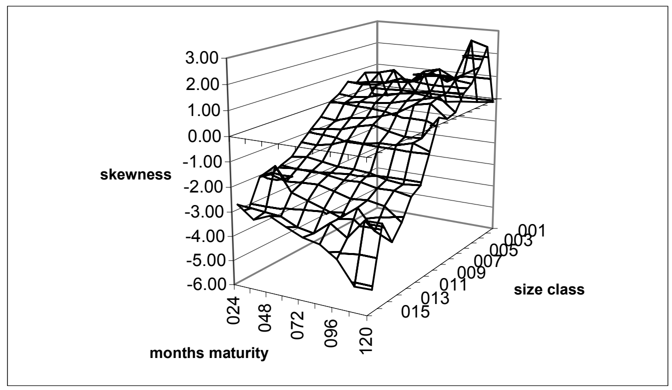

Exhibit 46 Graph of skewness values of ultimate distributions verses maturity and size.

demonstrate a high level of symmetry. It seems that the skewness is resulting from an extended negative tail of small values. The use of a model that does not pick up the negative skewness of large initial claims may only be missing a small probability of these small ultimates.

The concern about underdeveloping smaller claims may be unnecessary. It may be observed here because it is a characteristic of the reinsurance nature of the data. More often, this process is applied to " primary" data which will contain the complete inventory of small losses, not just the ones anticipating a reinsurance recovery. These claims should develop in a less skewed manner.

The Transition Matrix analysis of this data provides us with a method to model the future ultimate distribution of an individual open claim of a given size x, and maturity m. An open claim can be represented at ultimate as a lognormal distribution with:

$$\text{mu} = 1.005 * \ln(\text{loss size})$$

and,

$$\text{Sigma} = 1/(\text{maturity} * 0.001205 + \ln(\text{loss size}) * 0.078874 - 0.34447)$$

Where maturity is in months and loss size is in US dollars

2.3.8 Effect of Policy Limits

This study assumed that policy limits affected large losses and sought to avoid its effect. Due to this, the resulting distribution of ultimate losses, have no policy limit censoring. It is necessary to introduce it to arrive at the correct final ultimate value. The final distribution will be the lognormal distribution, given earlier, which has been censored by the policy limit for the claim. If this is not available, then a reasonable assumed policy limit must be used.

2.3.9 Effect of Zero Dollar Claims

The transition matrix process produces an estimate for claims that close with no payment at every maturity. The reader will remember in exhibit 24 all the statistical values are calculated using a conditional probability, after the probability of closing with no payment is removed. Any estimate of future development must reflect this. When taking an open non-zero claim to ultimate, the exhaustive range of outcomes must include zero value of probability P(x=0) and the proposed lognormal with a mu and sigma as previously discussed that has a probability of (1-(P(x=0)). A table of probabilities of closed with no payments is not included in this study because of the biased nature of the data. Since the claims used here are only those submitted for a possible reinsurance recovery, it is expected that the numbers of CWNP claims will differ from a general population of claims.

2.4 Comparison with Direct Transitions

2.4.1 Lack of Memory

The transition matrix process suffers the possibility of a troublesome error. This is due to the implied independence from transition to transition. It is as if each transition has no memory of earlier transitions. A claim, at some value x, does not care how it came to have this value. Its future transitions only depend on its current value. This is clearly not the real world situation. Each claim has a complete history from occurrence to settlement, and each succeeding value has some dependence on earlier values.

It is easy to imagine a scenario in the transition matrix approach that might be counter intuitive. Say, a $1,000,000 claim that goes through ten transitions where each transition happens to reduce the value by half. This results in the value of the claim reducing to $976. Remember that this transition is simply the movement from one class to the adjacent lower one. The transition matrix approach allows this possibility (abet with low probability), but, intuition tells us that this doesn't occur. One might guess that the dispersion caused by future development is exaggerated by the transition matrix approach.

Final	Initial Class																						Grand
Class	000	001	002	003	004	005	006	007	008	009	010	011	012	013	014	015	016	017	018	019	020	021	Total
000		2162	146	315	396	432	547	702	821	1038	919	722	443	263	134	44	18	8	3	2			9115
001		1478	33	68	84	95	83	59	43	35	31	16	8	3	2	1							2039
002		18	287	9	25	27	46	49	22	21	11	11	1	1									528
003		18	14	448	27	34	44	51	30	38	14	9	3	2	2	1							735
004		22	10	27	590	66	79	70	57	46	19	17	3	2	1								1009
005		18	10	23	41	784	87	84	59	49	33	17	4		2	1							1212
006		18	8	24	45	84	1302	117	99	73	50	19	8	1	2								1850
007		29	13	15	24	49	117	1701	194	117	65	24	13	2	1	1		1					2366
008		35	1	13	20	47	89	199	2058	275	112	41	12	9	3	2							2916
009		43	4	9	7	35	55	130	233	2352	265	91	26	12	2	2							3266
010		57	5	10	11	9	35	71	135	379	2588	293	55	14	3	1	1						3667
011		53	6	10	15	19	20	46	71	160	324	2302	192	37	13	4	1						3273
012		35	2	5	4	4	11	13	23	62	100	227	1434	123	24	1	2	1					2071
013		20		5	2	7	7	10	15	19	44	75	173	878	88	18	5	1					1367
014		10		1	3	4	6	3	7	14	5	27	47	98	588	52	8	1			1		875
015		5	1		4	4	5	3	3	5	9	22	19	28	58	282	16	2	1				467
016		2	1		1		2		2	5	2	6	9	8	19	35	157	5	3	1			258
017		1										1	4	4	7	16	12	55	4	2			106
018					2							1	1	1			2		30	1			38
019													1						3	13			17
020																				1	6	2	9
021														1								1	2
Grand Total		4024	541	982	1301	1700	2535	3308	3872	4688	4591	3921	2456	1487	949	461	222	74	44	20	7	3	37186

Exhibit 47 - Counts of claims for initial report to final report (current) classified by
by initial and final size. All initial maturities are shown here.

2.4.2 An Alternative Method

In order to assess how much distortion might be caused by this, a study involving direct observation was conducted. The previous transition matrix approach involved observing each transition from year to year, and multiplying each transition until ultimate matrices were created. This study utilized direct observation of the transition from first report to current value which is taken as a proxy for ultimate. These observations were categorized as to initial size class, initial maturity, and final (current) size class at the latest evaluation, which allows these observations to be directly compared with the ultimate loss development matrices.

This study used the same data as previously and was adjusted by the same trend. To provide an overview of the outcome of this study, exhibit 47 shows the number of claims for all initial maturities when classified by initial and final (current) size.

When we view any of the initial size class columns, the bell shaped distribution becomes obvious. We can take the counts of initial claims in any column and divide it by the total counts in that column, and it represents the probability of a final size outcome given the initial size. This is graphed in exhibit 48. For clarity only the odd numbered size classes

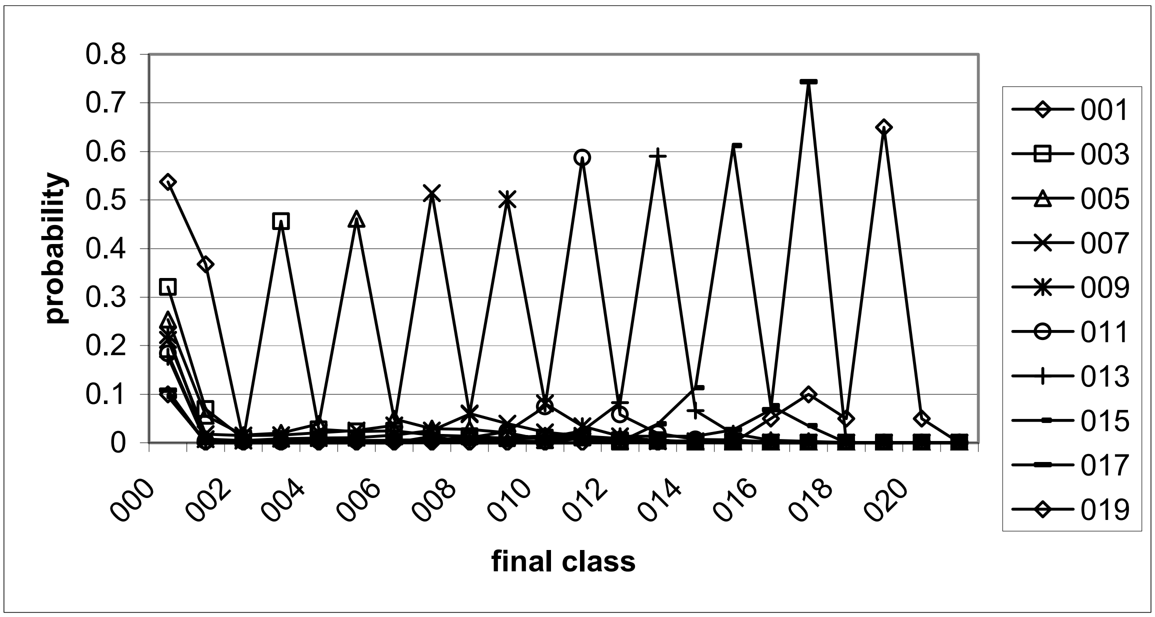

Exhibit 48 - Plot of probability for final size class for a selection of initial size classes for the initial to final transition.

are shown. The graph continues to show the strong "normal distribution like" behavior that we have seen previously. We will look at the distribution statistics for each "ultimate" distribution to see if this empirical ultimate transition is similar to the multiplied results from the transaction matrix method. The counts are further classified by initial maturity, and the statistics, mean, standard deviation, skewness, and kurtosis are calculated allowing us to compare the results directly with those of the transition matrix study.

2.4.3 Similar Results

The mu for these distributions are computed and then divided by the natural log of the average size within the interval similarly to what was done earlier for the transition matrix results. Exhibit 49 is a table of the mu to ln(x) ratio for the initial to "ultimate" transition data.

Size Category	Average Size in Interval	Ratio of Mu/natural log of average size Maturity in Months											
		012	024	036	048	060	072	084	096	108	120	132	144
001	2,712	1.60	1.55	1.57	1.53	1.60	1.45	1.59	1.49	1.42	1.40	1.69	1.43
002	7,569	1.08	1.06	1.10	1.04	1.06	1.10	1.02	1.00	1.07	1.10	1.08	1.16
003	13,557	1.07	1.05	1.04	1.06	1.06	1.09	1.00	1.08	1.04	1.06	1.08	1.03
004	24,284	1.04	1.05	1.05	1.02	1.03	1.03	1.03	1.04	1.00	1.01	1.03	1.06
005	43,498	1.03	1.03	1.03	1.02	1.01	1.00	1.01	1.00	0.99	1.02	1.01	1.02
006	77,914	1.01	1.02	1.01	1.01	1.00	1.00	1.01	1.00	0.99	0.98	0.98	1.00
007	139,561	1.00	1.01	1.00	1.00	1.01	0.99	0.99	0.99	1.00	1.00	1.00	1.00
008	249,983	1.00	1.00	0.99	1.00	0.99	1.00	0.99	1.00	0.98	0.99	0.99	1.00
009	447,774	0.99	1.00	0.99	0.99	0.99	1.00	1.00	1.00	0.99	0.99	1.00	1.00
010	802,059	1.00	1.00	1.00	1.00	1.00	1.00	0.99	1.00	0.99	1.00	0.99	1.00
011	1,436,661	0.99	0.99	0.99	0.99	0.99	0.99	1.00	0.99	0.99	1.00	0.99	1.00
012	2,573,368	0.98	1.00	0.99	1.00	1.00	0.99	0.98	0.99	0.99	0.99	0.98	0.99
013	4,609,455	0.99	1.00	1.00	1.00	0.99	0.98	0.99	1.00	0.97	1.00	0.99	1.00
014	8,256,524	0.99	1.00	0.99	0.99	0.99	1.00	1.00	0.99	1.00	1.00	1.00	0.96
015	14,789,210	0.99	1.00	0.99	0.97	0.98	1.01	0.99	1.00	1.00	1.00	1.00	1.00

Exhibit 49 - Table of ratio of mu / ln(x) for distributions of initial to final transitions.

This appears very similar to its transition matrix counterpart shown in exhibit 31. The edge towards the small size classes is turned up as it is in the transition matrix study, though it doesn't seem to be any pattern to the differences. If one accepts this elevated mu's for small initial claims to be caused by the data collection process, then this data supports the assertion that the mu of the distribution can be estimated by the natural log of the loss size. This surface is graphed in

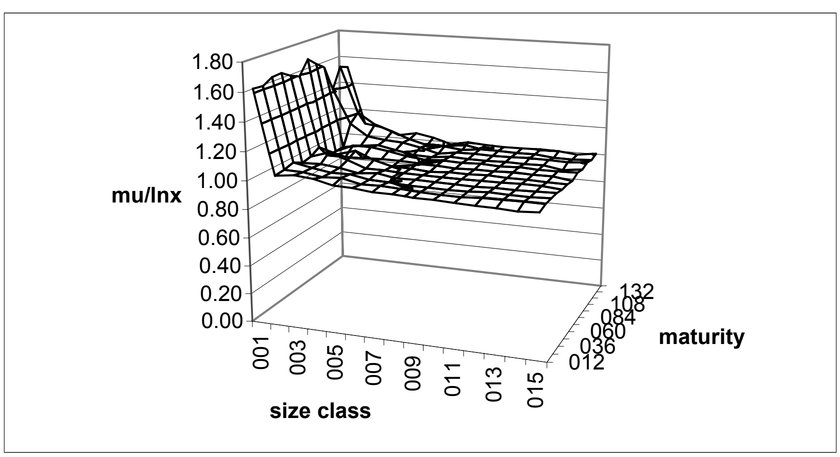

Exhibit 50 - Graph of ratio of mu / ln(x) for distributions of initial to final transitions.

Exhibit 50.

The other important statistical parameter to check is the standard deviation of the log of the "ultimate" loss size. Exhibit 51 shows the sigma's of the ultimate distributions of the

Initial Size Class	Sigma at Ultimate Initial Maturity in Months											
	012	024	036	048	060	072	084	096	108	120	132	144
001	2.40	2.20	2.29	2.15	1.88	2.09	1.99	1.74	1.54	1.09	1.83	2.56
002	1.15	1.25	1.51	0.93	1.18	2.01	0.48			2.16	2.86	1.71
003	1.43	1.23	1.32	1.34	1.98	1.01	0.36	1.55	0.86	1.48	2.12	1.40
004	1.42	1.22	1.18	1.38	1.45	2.24	0.45	1.80	1.57	1.50	1.88	1.59
005	1.18	1.45	1.28	1.14	1.14	1.30	1.08	1.43	1.00	1.25	0.90	1.47
006	1.22	1.24	1.17	1.27	1.02	0.89	1.07	0.89	0.81	1.29	1.46	1.45
007	1.01	1.18	1.17	0.99	1.16	1.01	0.93	1.00	0.86	0.70	0.82	0.85
008	1.10	1.08	1.08	0.92	0.94	0.86	0.89	0.99	0.58	0.72	0.69	1.20
009	1.13	1.07	1.07	1.03	0.96	0.88	1.04	0.96	0.89	0.68	0.59	0.85
010	1.03	1.06	1.01	0.97	0.94	0.92	0.80	0.66	0.79	0.90	0.42	0.48
011	1.16	1.22	1.00	1.01	0.75	0.76	0.95	0.68	0.34	0.52	0.69	0.17
012	1.23	1.21	0.86	0.89	0.77	0.77	0.55	0.48	0.77	0.72	0.40	0.32
013	1.11	1.63	0.94	0.99	0.55	0.47	0.82	0.50	0.37	0.21	0.53	0.34
014	1.08	0.57	1.26	0.63	0.96	0.79	0.47	0.53	1.12	0.62	1.18	0.48
015	1.25	0.54	0.75	0.71	1.30	0.96	1.19	0.30	1.06	0.17	0.42	0.17

Exhibit 51 - These are the sigma's of the ultimate distributions for the emperical initial to "ultimate" observations by initial size class and by initial maturity.

empirically observed initial to "ultimate" transitions, and, these are graphed in exhibit 52. Note that two values, initial size class 002, maturity 96 and 108 months, are missing due to sparse data.

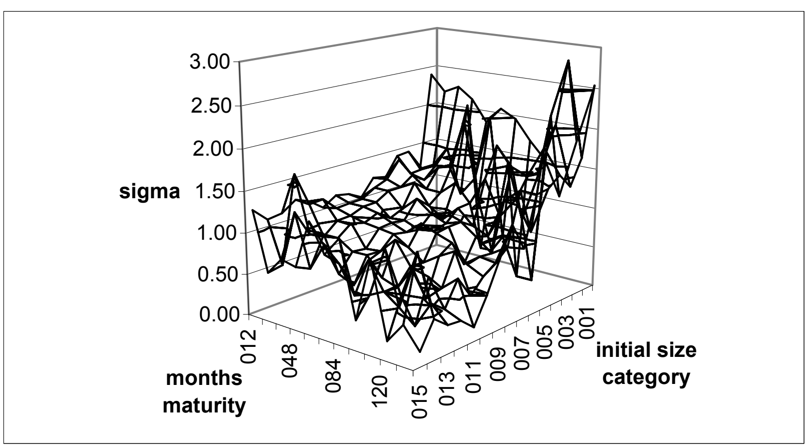

Exhibit 52 - Graph of the sigma's of the ultimate distributions for the emperical initial to "ultimate" observations by initial size class and by initial maturity.

When one compares them to the transition matrix values in exhibit 33 we see that the general shape of the surface is similar, but the values are somewhat higher. On average the transition matrix values are 1.4 times higher then the empirical values. When we look at the surface of the empirical sigma's we see a similar structure to that observed earlier for the sigma's of the transition matrix study. Exhibit 52 shows the plot of the values as a surface, and is comparable to the plot in exhibit 34. This shows higher values and a higher volatility in the small size classes.

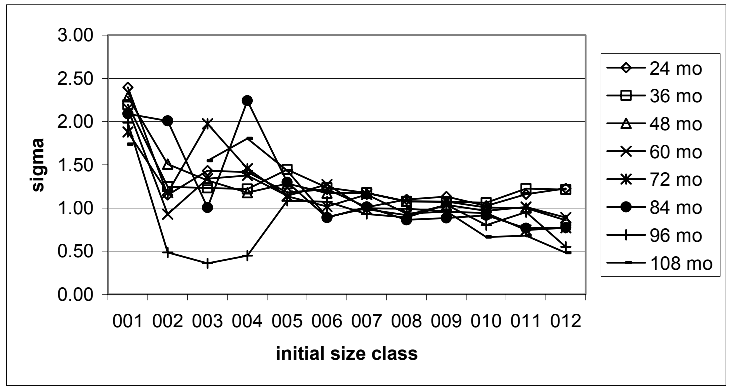

Exhibit 53 - Plot of emperical sigma's by initial size class. Each line represents a different initial maturity.

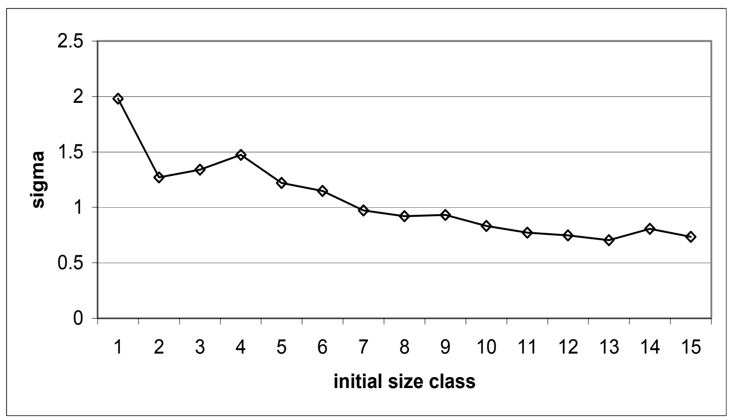

Exhibit 54 - Plot of emperical sigma's averaged over maturity and plotted by initial size class

The "sideways" view of this plot shown in exhibit 53 shows these higher values at small size classes, and then a leveling off as initial size class increases

If we take the average across the maturities and plot these averages verses the initial size class, we get a better sense of how sigma changes with initial size. A plot of this is shown in exhibit 54 and we see a gradual decrease with increasing size. If we take the inverse of this average sigma we see an increasing linear relation as shown in exhibit 55. Again, this is consistent with our earlier model for sigma.

Looking at the other dimension, change in maturity, we find that the sigma values show a gradual decrease with increasing maturity. These values are plotted in exhibit 56, which is comparable to what was seen in the transition matrix values. This is confirmed with a review of exhibit 39, which shows a gradual increase in 1/sigma.

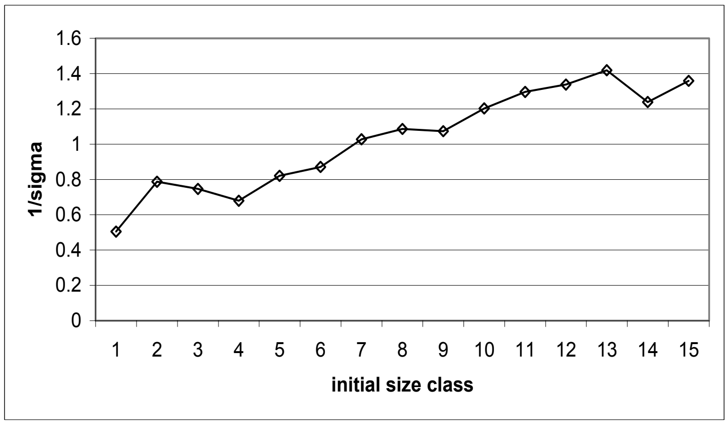

Exhibit 55 - Plot of inverse of average sigma's verses initial class size.

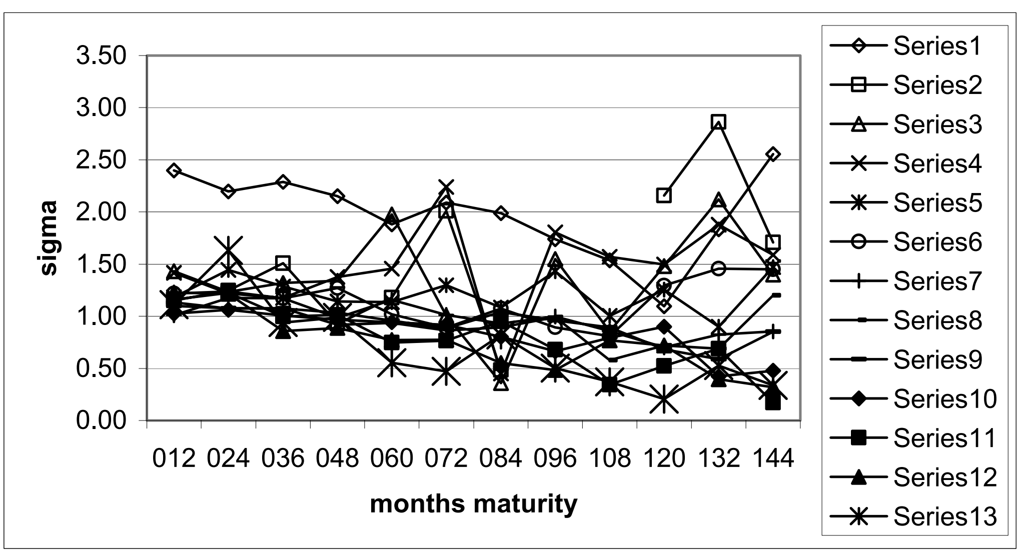

Exhibit 56 - Plot of emperical sigma's by months maturity. Each line represents a different initial size class.

The empirical sigma data is averaged over all sizes and plotted by maturity to show the decreasing trend with maturity as shown in exhibit 57. The inverse of the average

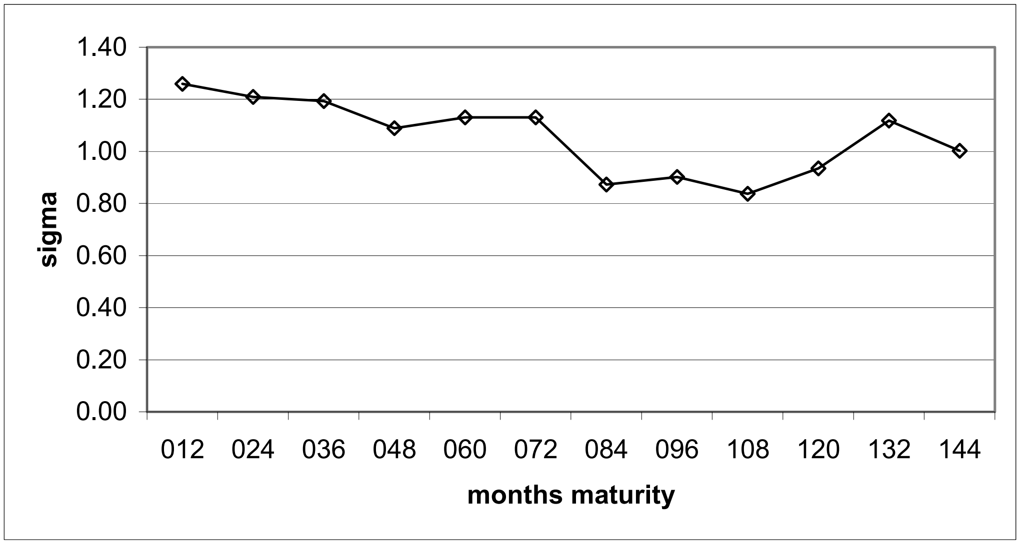

Exhibit 57 - Plot of emperical sigma's averaged over initial size class and ploted by maturity

sigma when plotted verses maturity shows the same increasing trend observed in the transition matrix data. This is plotted in exhibit 58 and can be compared with exhibit 39.

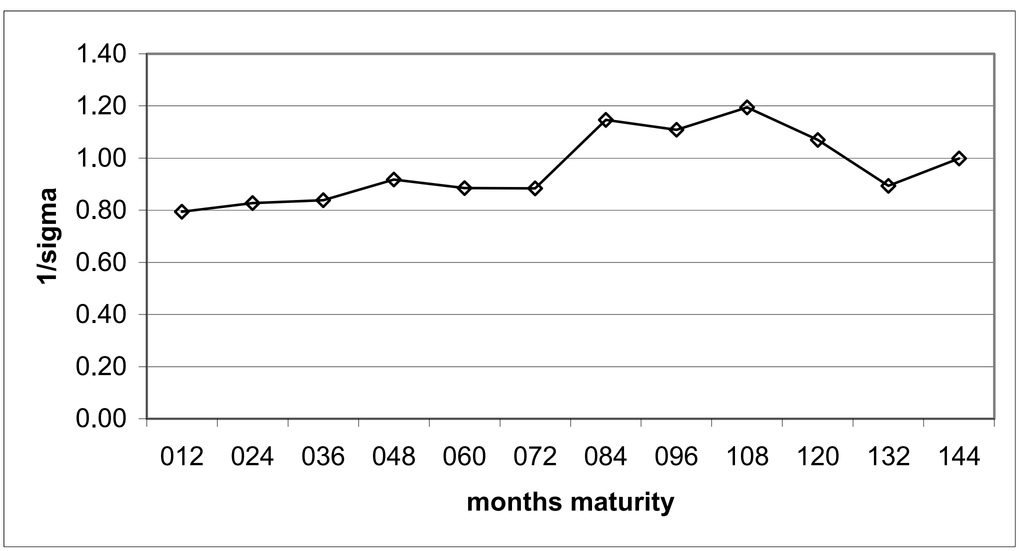

Exhibit 58 - Plot of inverse of average sigma's verses maturity.

A review of the skewness statistics from the empirical study shows similar behavior to the transition matrix study. The skewness is positive for small initial sizes and negative for larger sizes. It has no trend as maturity varies. The average for all observations is -0.33.

2.4.4 Sigma Differs

The overall impression provided by the empirical study is that the lognormal model of developed from the transition matrix study does a good job of describing the loss development, but it needs adjustment. The estimates of mu's from both are very similar, and the skewness follows the same pattern. The estimates of sigma follow a similar pattern but the values of the estimates differ. The transition matrix values are about 1.4 higher than the empirical estimates. If we accept the earlier argument that the transition matrix process may generate more variability then is present in reality, then it is necessary to find a way to reduce the variability. We can accept the observed sigma values in the empirical study, but this has limited application. Since the data has only one observation per claim it is limited and contains more noise. Since sigma behaved similarly in both studies, and differed only by scale, it is better to accept the aggregate level of the empirical sigmas and to try to adjust the fitted sigma surface from the transition matrix study. To do this we need to take a detailed look at the difference between the transition matrix and the empirical sigmas.

We can measure the difference between these two by dividing the empirical sigmas from exhibit 51, by the transition matrix sigmas from exhibit 41, which results in the table and

Initial Size Class	Sigma Ratio Surface Emperical Sigma's divided by TransitionMatrix Sigma's Initial Maturity in Months										
	012	024	036	048	060	072	084	096	108	120	avg.
001	0.90	0.85	0.86	0.77	0.67	0.78	0.76	0.69	0.61	0.40	0.73
002	0.56	0.64	0.82	0.48	0.49	0.87	0.20			1.14	0.65
003	0.69	0.62	0.61	0.59	0.82	0.43	0.17	0.65	0.44	0.86	0.59
004	0.70	0.61	0.59	0.69	0.65	0.94	0.22	0.88	0.81	0.68	0.68
005	0.68	0.78	0.71	0.65	0.61	0.68	0.60	0.77	0.58	0.70	0.68
006	0.68	0.70	0.66	0.70	0.55	0.51	0.65	0.55	0.44	0.85	0.63
007	0.66	0.76	0.74	0.63	0.77	0.63	0.59	0.74	0.60	0.58	0.67
008	0.71	0.75	0.77	0.67	0.69	0.61	0.64	0.73	0.44	0.61	0.66
009	0.79	0.74	0.78	0.75	0.73	0.70	0.83	0.83	0.79	0.61	0.76
010	0.75	0.74	0.74	0.77	0.77	0.77	0.70	0.56	0.71	0.86	0.74
011	0.76	0.84	0.76	0.80	0.62	0.67	0.92	0.67	0.36	0.55	0.69
012	0.85	0.84	0.72	0.75	0.70	0.77	0.53	0.49	0.88	0.84	0.74
013	0.97	1.42	0.88	0.96	0.54	0.47	0.79	0.56	0.44	0.23	0.73
014	1.14	0.60	1.25	0.55	0.79	0.67	0.39	0.47	1.21	0.72	0.78
015	1.27	0.55	0.79	0.67	1.34	1.00	1.19	0.28	1.06	0.16	0.83
avg.	0.81	0.76	0.78	0.70	0.71	0.70	0.61	0.63	0.67	0.65	

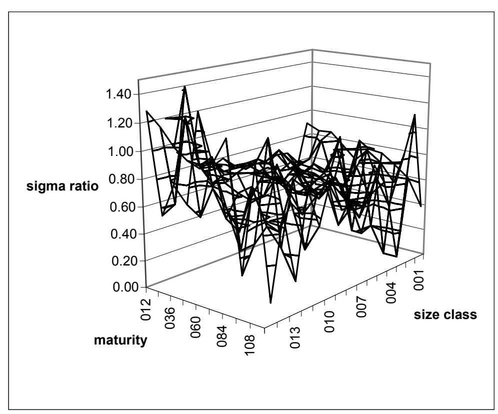

Exhibit 59 - This is the ratio of the emperical sigmas in exhibit 52 divided by the transition matrix sigmas in exhibit 34. A plot of this surface is shown at right. The average of this surface is 0.704

graph shown in exhibit 59. This is the surface of the ratio that is the correction factor to take the fitted transition matrix sigma value to the actual empirical sigma value. In a perfect world, every single value would be equal to each other. But, since there is noise in the data, variation is observed across the surface.

2.4.5 Correction Factor

We want to look for structure by taking the average across maturity and the average across size, which are displayed in the last column and the bottom row respectively. First, we consider changes with size, and we plot the individual maturity data, and then the averages as shown in exhibit 60. This reveals a slight upward trend with increasing initial loss size, however, the fluctuations in this line is well within the noise of the individual data points. Looking at the behavior as maturity varies in exhibit 61, we see a similar result, a slight decreasing trend as maturity increases which is much smaller than the noise of the original data. In interest of parsimony we select this surface to be a level plane with a value of its average, 0.704.

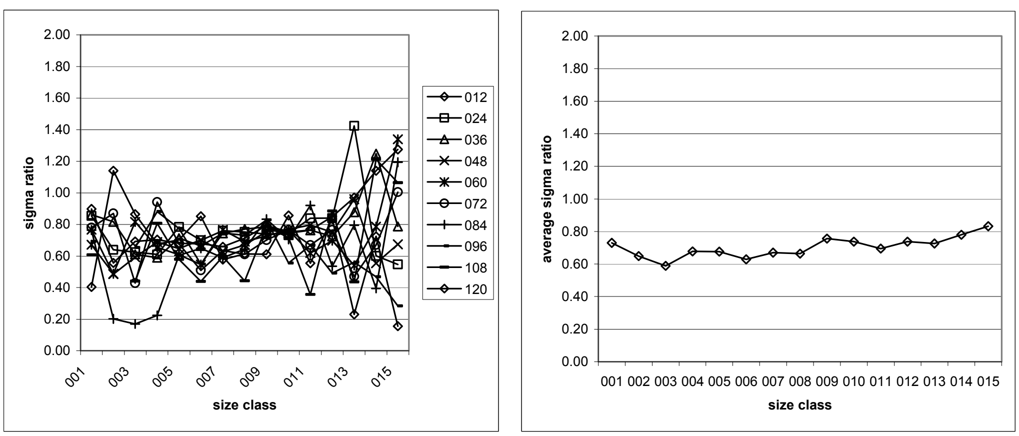

Exhibit 60 - Plot of sigma ratio surface showing how it changes with size. Individual maturities are shown at left, and values averaged over maturity is shown at right

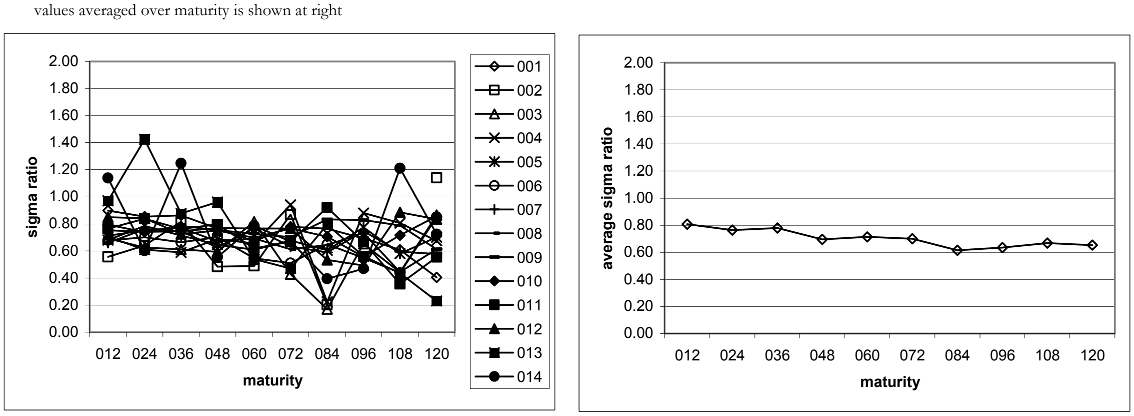

Exhibit 61 - Plot of sigma ratios as maturity varies. The left shows each size class as an individual line. The right shows the values averaged over maturity.

initial size class	fitted sigma values after factor adjustment initial maturity									
	012	024	036	048	060	072	084	096	108	120
001	2.40	2.29	2.18	2.09	2.00	1.92	1.85	1.78	1.72	1.66
002	1.88	1.81	1.74	1.68	1.63	1.58	1.53	1.48	1.44	1.39
003	1.67	1.62	1.57	1.52	1.47	1.43	1.39	1.35	1.31	1.28
004	1.51	1.46	1.42	1.38	1.34	1.31	1.27	1.24	1.21	1.18
005	1.37	1.34	1.30	1.27	1.23	1.20	1.17	1.15	1.12	1.10
006	1.26	1.23	1.20	1.17	1.14	1.12	1.09	1.07	1.04	1.02
007	1.16	1.14	1.11	1.09	1.06	1.04	1.02	1.00	0.98	0.96
008	1.08	1.06	1.04	1.01	0.99	0.97	0.95	0.94	0.92	0.90
009	1.01	0.99	0.97	0.95	0.93	0.92	0.90	0.88	0.87	0.85
010	0.95	0.93	0.91	0.90	0.88	0.86	0.85	0.83	0.82	0.81
011	0.89	0.88	0.86	0.85	0.83	0.82	0.80	0.79	0.78	0.77
012	0.83	0.83	0.82	0.80	0.79	0.78	0.76	0.75	0.74	0.73
013	0.79	0.79	0.77	0.76	0.75	0.74	0.73	0.72	0.71	0.70
014	0.75	0.75	0.74	0.73	0.72	0.70	0.69	0.69	0.68	0.67
015	0.71	0.71	0.70	0.69	0.68	0.67	0.66	0.66	0.65	0.64

Exhibit 62 - Adjusted fitted sigma surface (exhibit 42) after the application of the adjustment factor.

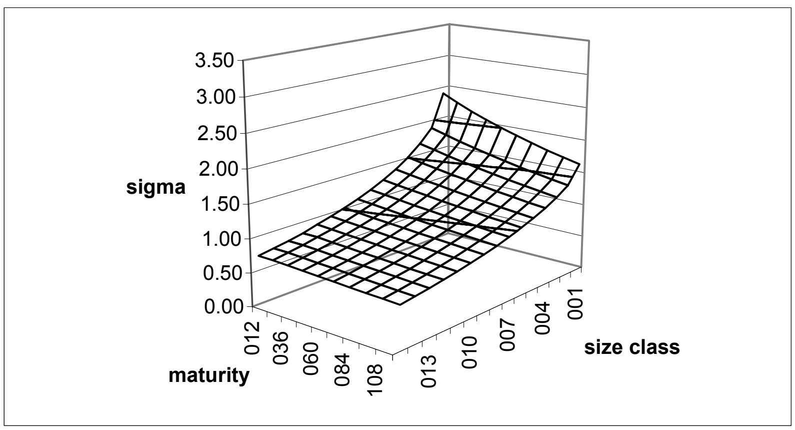

Exhibit 63 - Plot of fitted sigma values after adjusting to the level of the emperical sigmas. This is the data in exhibit 63 and can be compared to plot in exhibit 43.

With this surface approximated as a single value, we adjust the fitted sigma values in exhibit 41, and compare them to the empirical values. Exhibit 62 contains the revised fitted sigmas and this surface is plotted in exhibit 63.

In order to test how well we have approximated the empirical sigmas we can subtract the two surfaces, the adjusted fitted sigmas in exhibit 62 and the empirical sigmas in exhibit 51.

The resultant table of differences is shown in exhibit 64 and plotted in exhibit 65. Here we see the overall average of 0.01 of this surface is very close to zero indicating that the adjusted fitted sigmas are a good approximation to the average level of the empirical sigmas.

initial size class	Error Surface - Difference between adjusted, fitted TM sigmas and Emperical sigmas (TM - E) initial maturity										
	012	024	036	048	060	072	084	096	108	120	avg.
001	0.00	0.09	-0.10	-0.06	0.12	-0.17	-0.14	0.04	0.18	0.57	0.05
002	0.73	0.56	0.24	0.76	0.45	-0.43	1.04			-0.76	0.32
003	0.24	0.39	0.25	0.18	-0.50	0.42	1.03	-0.20	0.46	-0.20	0.21
004	0.09	0.25	0.24	0.00	-0.11	-0.93	0.82	-0.56	-0.36	-0.32	-0.09
005	0.19	-0.11	0.02	0.12	0.10	-0.09	0.09	-0.28	0.12	-0.16	0.00
006	0.04	-0.01	0.03	-0.10	0.12	0.23	0.02	0.17	0.24	-0.27	0.05
007	0.16	-0.04	-0.06	0.10	-0.09	0.03	0.09	0.00	0.12	0.26	0.06
008	-0.02	-0.02	-0.04	0.10	0.06	0.11	0.06	-0.05	0.34	0.18	0.07
009	-0.12	-0.08	-0.10	-0.08	-0.02	0.03	-0.14	-0.08	-0.03	0.17	-0.05
010	-0.08	-0.13	-0.09	-0.07	-0.06	-0.05	0.05	0.17	0.03	-0.09	-0.03
011	-0.27	-0.35	-0.14	-0.17	0.09	0.05	-0.15	0.11	0.44	0.24	-0.01
012	-0.40	-0.38	-0.04	-0.09	0.02	0.00	0.21	0.27	-0.03	0.01	-0.04
013	-0.32	-0.84	-0.17	-0.23	0.20	0.27	-0.09	0.21	0.34	0.49	-0.01
014	-0.34	0.18	-0.53	0.10	-0.24	-0.08	0.23	0.16	-0.44	0.04	-0.09
015	-0.54	0.18	-0.05	-0.02	-0.61	-0.28	-0.53	0.36	-0.41	0.46	-0.14
avg.	-0.04	-0.02	-0.04	0.04	-0.03	-0.06	0.17	0.02	0.07	0.04	
									overall average		0.01

Exhibit 64 - Error Surface of difference between adjusted fitted Transition Matrix sigma's and emperical sigmas.

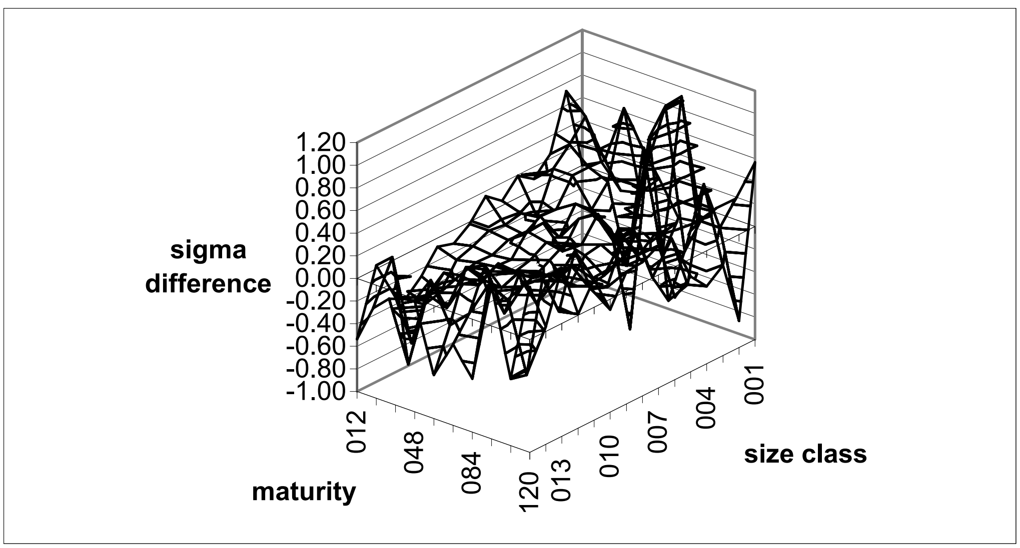

Exhibit 65 - Plot of Error surface of difference between adjusted, fitted Transition Matrix sigma's and emperical sigma's.

Observing the average column and row in exhibit 64 there may be some residual behavior, the average difference seems to go from positive to negative with increasing size, and from negative to positive with increasing maturity. These trends are small, and may be noise. We also must consider that the empirical study used the transition from initial to current report for a proxy for ultimate development. By adjusting the variation in transition matrix model to the level of the empirical data we have the shape of the transition matrix model, but the value levels of the empirically observed data.

2.4.6 The Distributional Loss Development Model

With the sigmas estimated, it is now possible to propose a model that describes the loss development of an open claim of a given size at a given maturity.

An open claim of a given loss size x and a maturity m its ultimate size can be expressed as a log normal distribution with:

Mu = ln(loss size x) * 1.005

and

Sigma = 0.701*(1/(maturity *0.001205+ln(loss size)*0.078874-0.34447))

Where maturity is in months and loss size is in US dollars

2.5 Ratemaking Considerations

2.5.1 Synthetic Data

It is instructive to explore the effect of distributional loss development on estimation of limited expected values and increased limits. In order to do this, a collection of claim values were simulated using a lognormal distribution with a mu of 13 and a sigma of 1.0. Ten thousand values were simulated and 50 were selected using stratified sampling. This was done by sorting them in order and selecting the first percentile, and then every second percentile thereafter, ending with the 99th percentile value. These values are shown in exhibit 66. When these values are graphed in order on a log scale the lognormal

42,151	201,660	357,243	595,631	1,059,857
68,964	215,841	374,978	628,044	1,143,719
86,054	231,946	393,343	657,576	1,236,435
104,213	245,987	414,947	689,340	1,353,120
119,440	260,120	435,752	730,062	1,503,952
134,623	273,892	457,429	772,238	1,670,692
148,901	289,921	480,840	818,901	1,916,141
161,543	305,308	505,404	869,830	2,341,089
175,838	321,387	535,753	925,229	2,897,278
189,464	339,692	567,292	993,142	4,564,144

Exhibit 66 - Simulated loss values, lognormal distribution mu = 13, sigma = 1.0

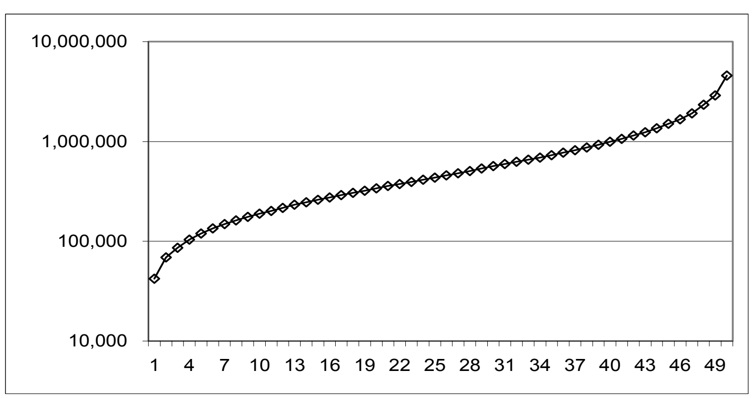

Exhibit 67 - Simulated values graphed in order on a log scale.

distribution becomes obvious as shown in exhibit 67.

2.5.2 Application of Loss Development

We select to apply distributional loss development to these claims using the lognormal model. This process is illustrated in exhibit 68 where eight of the fifty claims are shown for illustration. The claim value is shown in column 1 and its log is shown in column 2.

posted value (1)	dev mu (2) ln(1)	dev sigma (3) see below	ln(dev mean) (4) (2)+((3)^2)/2	dev mean (5) exp(4)	dev factor (6) (5)/(1)
42,151	10.65	1.37	11.59	108,440	2.57
161,543	11.99	1.14	12.64	308,745	1.91
260,120	12.47	1.07	13.04	462,449	1.78
374,978	12.83	1.03	13.36	635,643	1.70
535,753	13.19	0.99	13.68	871,713	1.63
772,238	13.56	0.95	14.01	1,210,565	1.57
1,236,435	14.03	0.90	14.44	1,858,590	1.50
4,564,144	15.33	0.80	15.65	6,270,935	1.37

formula for (3) =0.701*(1/(12*0.001205+LN(1)*0.078874-0.34447))

Exhibit 68 - Application of distributional loss development to eight of the 50 claim values. Note that in the formula for column (3), the log value is of the posted value in column (1).

assume that mu for the loss development model is 1.00 time the log of the loss size. Columns four and five are used to calculate the average loss size of the developed loss. The ratio of column 5 divided by column 1 is the implied loss development factor for the traditional loss development method. These will be used to create developed losses to compare with the distributional developed losses. In this way, we will compare losses whose averages are the same, and differ only in the change in the shape of the distribution. This process was applied to all fifty selected claims. When done, we have three lists of losses, the original shown in column 1, the traditionally developed losses in column 5 and the distributional developed losses represented by mu in column 2 and sigma in column 3.

2.5.3 Comparing Cumulative Density Functions

A good technique to compare the different distributions is to look at the cumulative density functions. Since we have a collection of losses, we can calculate an empirical cumulative density function as shown in exhibit 69. This method involves counting the number of

Original Claims					
Probabliity that the Limit exceeds the Loss					
	Limit				
Loss	25,000	100,000	500,000	1,000,000	5,000,000
42,151	0	1	1	1	1
161,543	0	0	1	1	1
260,120	0	0	1	1	1
374,978	0	0	1	1	1
535,753	0	0	0	1	1
772,238	0	0	0	1	1
1,236,435	0	0	0	0	1
4,564,144	0	0	0	0	1
Count	0	1	4	6	8
Probability	0.000	0.125	0.500	0.750	1.000

Exhibit 69 - Computation of cumulative probability for original claims.

claims that exceed a collection of arbitrary selected limits. The limits run across the top of the table, and the claims are in the first column. A count of one is placed in the field of the table for each intersection representing a claim exceeding a limit. The counts are totaled at the bottom and divided by the number of claim to yield the probability. Again, for display purposes we show eight claims, where fifty claims were used in the study. The final cumulative probability is the ordered pair of the limits running across the top of the table, and the probability running across the bottom of the table.

Claims with Traditional Loss Developemnt					
Probabliity that the Limit exceeds the Loss					
	Limit				
Loss	25,000	100,000	500,000	1,000,000	5,000,000
108,440	0	0	1	1	1
308,745	0	0	1	1	1
462,449	0	0	1	1	1
635,643	0	0	0	1	1
871,713	0	0	0	1	1
1,210,565	0	0	0	0	1
1,858,590	0	0	0	0	1
6,270,935	0	0	0	0	0
Count	0	0	3	5	7
Probability	0.000	0.000	0.375	0.625	0.875

Exhibit 70 - Computation of cumulative probability for claims with traditional loss development.

Exhibit 70 shows a similar treatment for the losses with traditional loss development. Even with the small sample of eight we see a shift in the distribution.

2.5.4 CDF for Distributional Development

The computation of the cumulative probability distribution for the losses with the distributional loss development applied is a bit more complicated. In this case, each developed loss is a distribution. But, this allows us to estimate a probability that a claim is less than a limit. Exhibit 71 shows the detail of this calculation. Across the top is the limit,

Probabliity that the loss is less than the Limit.							
			Limit, ln of limit				
			25,000	100,000	500,000	1,000,000	5,000,000
Loss	mu	sigma	10.13	11.51	13.12	13.82	15.42
42,151	10.65	1.37	0.35	0.74	0.96	0.99	1.00
161,543	11.99	1.14	0.05	0.34	0.84	0.95	1.00
260,120	12.47	1.07	0.01	0.19	0.73	0.90	1.00
374,978	12.83	1.03	0.00	0.10	0.61	0.83	0.99
535,753	13.19	0.99	0.00	0.04	0.47	0.74	0.99
772,238	13.56	0.95	0.00	0.02	0.32	0.61	0.98
1,236,435	14.03	0.90	0.00	0.00	0.16	0.41	0.94
4,564,144	15.33	0.80	0.00	0.00	0.00	0.03	0.55
Sum			0.42	1.42	4.10	5.44	7.44
Probability			0.05	0.18	0.51	0.68	0.93

Exhibit 71 - Illustration of method to calculate cumulative probability of claims with log normal distributional development applied.

and also shown is the log of the limit. We capitalize on the characteristic of the lognormal distribution that the log of the value is normally distributed. We can take the mu and sigma of the lognormal as the mean and standard deviation of a normal, respectively, and calculate the probability interval represented by the

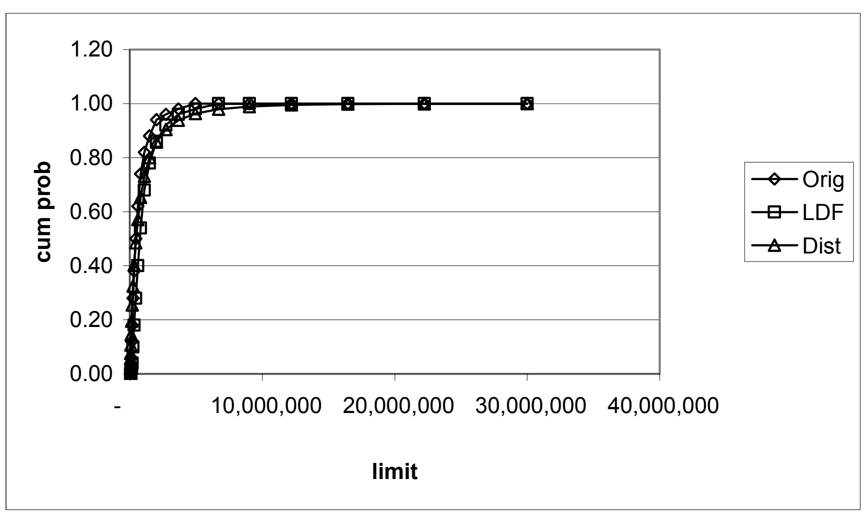

Exhibit 72 - Plot of cumulative probability curves for original losses, traditionally developed losses, and distributionally developed losses.

log of the limit using a normal distribution table. By choosing the correct tail of the normal, we will get the probability that the developed loss is less than the indicated limit. These probabilities are summed down the column and the total is divided by the total number of claims yielding the cumulative probability verses the limit.

Note that our example assumes that all the claims are open, and are subject to development. In a real life situation the collection of claims would be a mixture of open and closed claims. The open claims would be treated in the manner shown in exhibit 71 while the closed claims would be treated as in exhibit 70 where a zero or one is assigned to the probability in the table. The interesting fact is that, by simply adding the count and the "sum" values of exhibits 70 and 71 and then dividing by the total number of claims, one has the probability for the open and closed claims. This provides a method of creating the cumulative probability distribution of the mixed claims. With a cumulative probability available for the three types of losses they can be compared by plotting as shown in exhibit 72. We get a clearer picture of the different behaviors of the various loss development methods when we rearranging the horizontal scale to a log scale, as shown in exhibit 73.

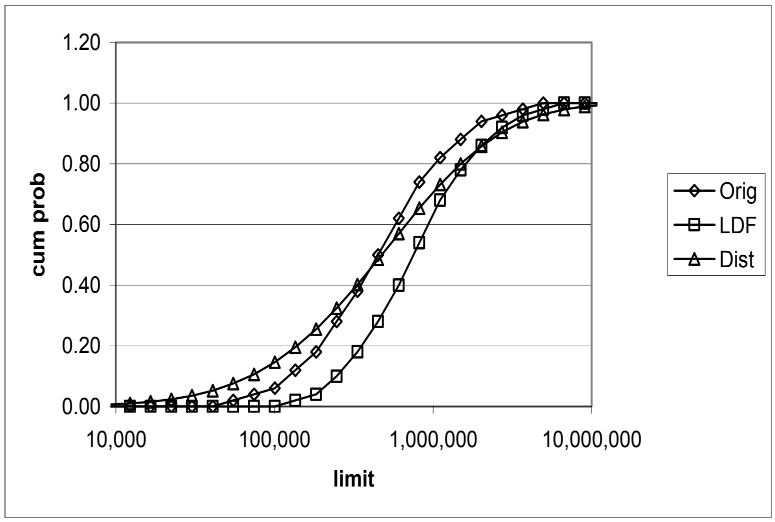

Exhibit 73 - Plot of cumulative probability curves for original losses, traditionally developed losses, and distributionally developed losses with a log scale for the limit.

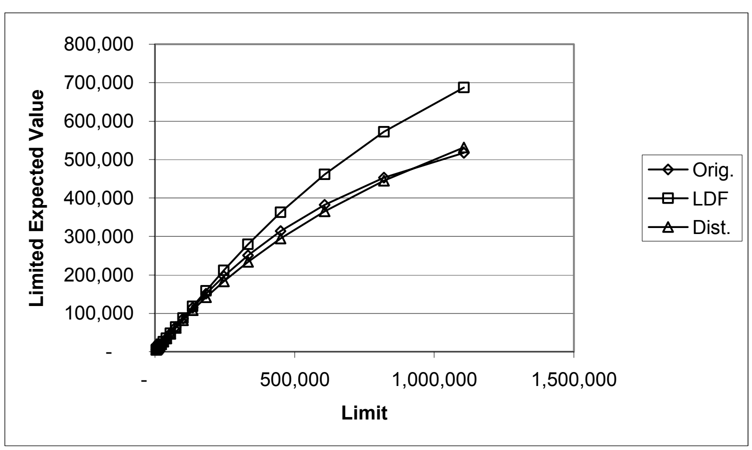

Exhibit 74 - Limited expected value in the range 0 to 1,000,000.

There are two comparisons in this graph. The first comparison is between the original claim data (diamonds) and the loss development factor data (boxes). Here, it would appear that the original line is shifted to the right, but maintains the same shape. The distributional development line (triangles) crosses the original line at the 50% range, but it shows more dispersion at small and large loss sizes.

2.5.5 Comparing Limited Expected Values

Limited expected values were calculated from the empirical cumulative density functions to see how they would behave relative to each other. Exhibit 74 shows the LEV for the range zero to one million in limit. Here we see similarity between the original and the distributional developed losses, while the LDF developed rises quickly. As we look at the range to $5,000,000 in limit we see the distributional rising up to meet the LDF adjusted data as shown in exhibit 75. Looking over the entire range up to $30,000,000 in limit, as shown in exhibit 76, we see that the distributional curve rises up and meets the LDF adjusted limited expected value. One can conclude that this is an expected conclusion since the values were formulated to have an equal mean. Remember, we selected the loss development factors so

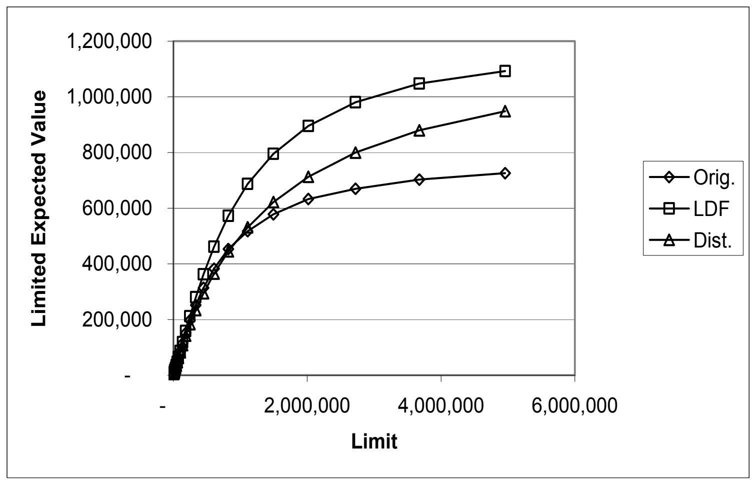

Exhibit 75 - Limited expected value in the range 0 to 5,000,000.

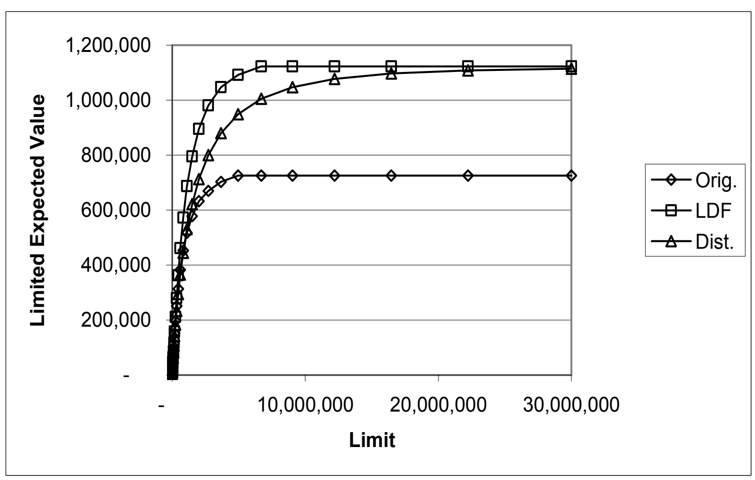

Exhibit 76 - Limited expected value in the range 0 to 30,000,000.

that the average for each individual loss would be the same for the LDF data and the distributional data. We only find this average converging at very high limits where they have little impact on the distribution.

2.5.6 Comparing Increased Limits Factors

With limited expected value curves available we can calculate pure loss increased limits factors. First we select a basic limit of \$100,000 and compute the increased limits. In exhibit 77 we show the increased limits for the range of \$100,000 to \$1,000,000. This shows that the factor method produces ILFs higher by 20 to 30% as compared to the distributional method. The undeveloped and distributional adjusted ILF's are very similar. If we look over a wider range, up to \$5,000,000, as shown in exhibit 78, we see a change in behavior. The distributional curve rises from the original curve and begins to approach the factor curve. This is exactly what is seen in the limited expected value curves, and, it is no surprise since increased limits are simply ratios of LEV's with the same denominator. The last range to explore is increased limits factors up to \$30,000,000

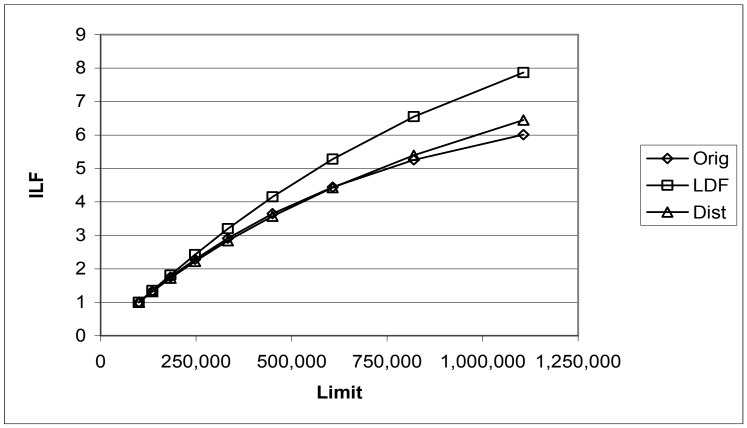

Exhibit 77 - Increased Limits Factors up to \$1,000,000 where basic limit is \$100,000.

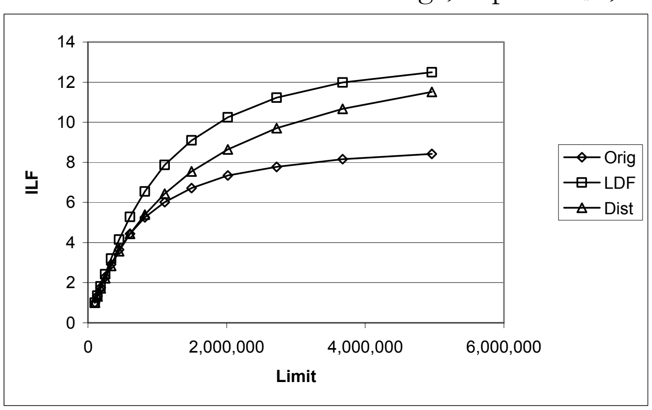

Exhibit 78 - Increased Limits Factors up to \$5,000,000 where basic limit is \$100,000.

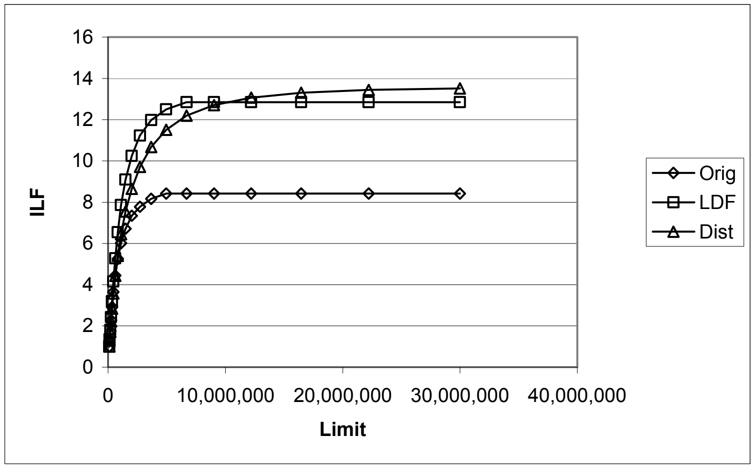

Exhibit 79 - Increased Limits Factors up to $30,000,000 where basic limit is $100,000.

as shown in exhibit 79. Here we see that the distributional line has risen up and exceeded the factor line. This is because the denominator for the distributional line is less at the $100,000 basic limit.

Varying the basic limit will change the behavior of the increased limits factors. Selecting $1,000,000 as basic limit and recalculating the increased limits factors produces the results

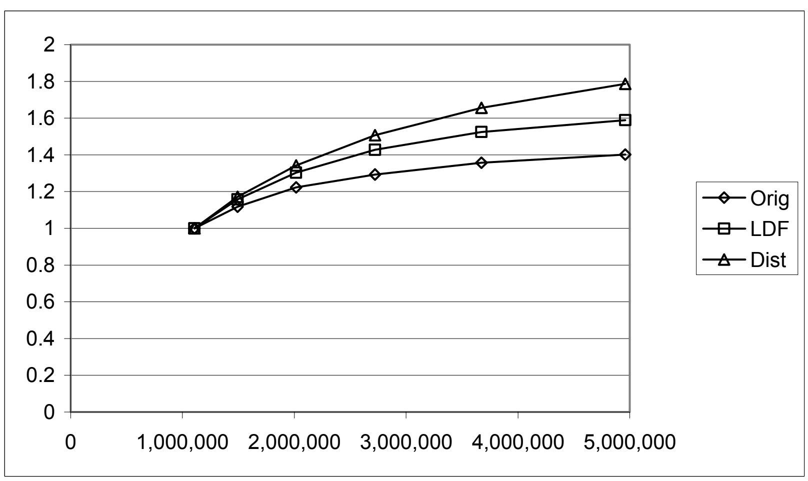

Exhibit 80 - Increased Limits Factors up to $5,000,000 where basic limit is $1,000,000.

shown in exhibit 80. Here we see that the distributional result is higher than the factor result, and this is consistent over this range and over the larger range, up to $30,000,000 as

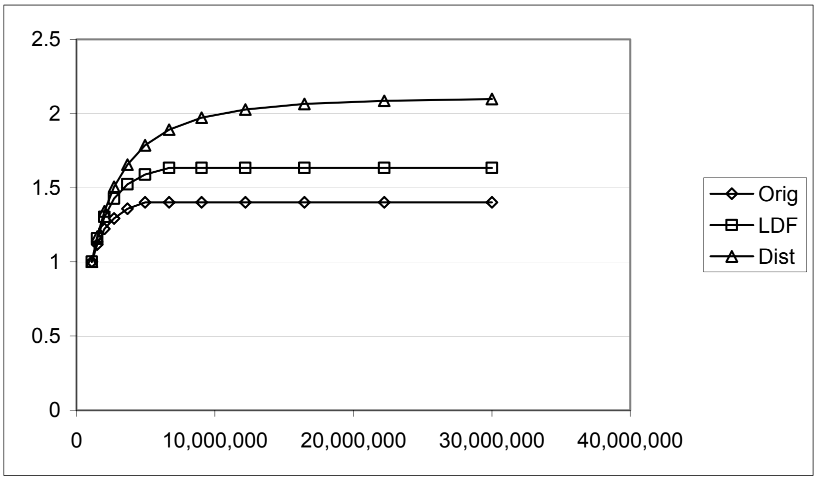

Exhibit 81 - Increased Limits Factors up to $30,000,000 where basic limit is $1,000,000.

shown in exhibit 81. Here we see a difference that is again in 30 percent range.

3. RESULTS AND DISCUSSION

It is well known that individual open claims will develop to an ultimate value that may be more, less, or the same as the current value. The exact nature of this distribution has never been clear. This study shows, with two different approaches, that it is a skewed distribution that can be modeled with well known severity distributions.

The lognormal distribution is particularly suited to modeling the severity distribution of the ultimate of an open claim. One difficulty of exploring loss development of individual claims is the excessive parameterization that occurs. The empirical transition matrix approach results in a very large number of parameters that vary by initial maturity, and claim size. Using them in a practical system to apply loss development to individual claims with the intention of arriving at an ultimate severity distribution would be cumbersome at the least. Applying the lognormal distribution to the transition matrix approach greatly reduces the parameters needed by characterizing the ultimate distribution of an individual loss as a lognormal with a mu which is a function of the initial size, and a sigma which a function of initial size and maturity. The math for combining a collection of individual lognormal distributions into an aggregate severity distribution is well known. This results in a practical method to apply loss development to individual claims that results in the ultimate severity distribution where the loss development recognizes the potential for claims to increase or decrease from the current size.

The lognormal individual claim loss development model appears to describe behavior in the absence of policy limits. It provides ample upward development of even very large claims. Policy limits must be applied after the loss development in order to correctly represent the potential loss of the developed severity distribution.

The observed dispersion in development appears to decrease and skewness becomes less positive as claim size increases. This is caused by an extended negative tail of small claims while the main peak remains symmetrical. The lognormal model does not capture this negative tail. Some practitioners may be uncomfortable in relying on policy limits to explain this and may want to adopt a more complex model that reflects this detail. A bi-modal lognormal treatment of the ultimate distributions may more accurately reflect this.

A comparison between Transition Matrix ultimate development and empirically observed first to last report transitions indicate that the Transition Matrix approach may result in more variation than actually present. This could be due to the independent nature of the Transition Matrix approach. It is reasonable to expect a certain amount of dependency as real claims progress to ultimate. It is interesting to note that the two models have the same shape and differ from each other only in the scale of one parameter, sigma. It is possible to adjust the lognormal development model resulting from the Transition Matrix approach so that it "balances" to the average values of the empirical loss development model. Another factor to consider is that the empirical loss development may very well underestimate sigma since it is missing data, but, does provides a minimum boundary. Further study is indicated to measure how much sigma is underestimated by the initial to final transitions, and, one may find that the truth may lay somewhere in between the empirical and the transition matrix approaches.

Both the Transition Matrix model and the empirically derived model exhibit a positive skewness for small claims and a negative skewness for larger claims. This conveniently fits with long held opinions in the casualty actuary community that small claims have a tendency to develop larger, and large claims have a tendency to develop smaller. The tendency for small claims to develop larger, in this case, may be a characteristic of the data because it is a subset that has been submitted for reinsurance recovery. It is not hard to imagine that the process of selecting claims for submission will exclude small simple claims that are not expected to develop. This results in a collection of claims that is biased towards larger development for the smaller claims.

The larger initial claims exhibit a symmetrical distribution (in the log transform space) about the most populated final size interval, with a small percentage of claims filling in the lower size intervals. Typically you see 2 to 4 percent of the claims in lower intervals

which are more that three size classes lower than the mode. Though not planned, the width of the size classes are about one standard deviation. So, the mode size class and the four adjacent classes account for about 95% of the claims. So, this 2% to 4% spread evenly in the lower size classes causes the negative skew values. If one ignores the extreme outliers, then the lognormal loss development model is a very good fit. In the future, it may do well to revisit the fit of these distributions with a bi-modal lognormal distribution. One mode will pick up the sharp peak around the initial loss size, and the other will be low probability with wide dispersion to pick up the claims that develop to much smaller values.

It is important to note that the study to measure ratemaking impacts was designed to stress the difference in the result between the two methods. It uses a claim set of all open claims. In most realistic situations, 60 to 75 percent of the claims would be closed and not experiencing any additional development. This would cut down this observed difference from the 30 percent range to less than 10 percent.

When considering the effect on pricing measures, it is important to point out that this study only compares the distributional loss development with single factor development, which is a method used in an ad-hoc manner to adjust small datasets for reinsurance rating. It does not imply a comparison with published industry standard increased limits factors, which are prepared with sophisticated methods that correctly reflect the distributional nature of individual loss development.

And last, the reader is guarded against a direct comparison of these increased limits factors with published increased limits factors since industry factors are prepared with a ballasting of a large collection of small losses, and that is clearly not the case in the losses used here.

4. CONCLUSIONS

Applying Transition Matrix theory to individual claims allows one to build up a picture of individual loss development, which has been seldom seen before. Using this method, it is possible characterize individual claim development as, distributional process where, the claim, at a known current and open amount, will, at ultimate, be a value which is forecast by a claim distribution. For general liability claims in the United States, the ultimate loss development of an individual claim can be represented as a heavy tailed skewed distribution, which closely resembles a log-normal distribution. It is possible to develop a simple functional relationship between the size and maturity of the open claim, and its ultimate lognormal distribution using four parameters.

The Transition Matrix approach may introduce excessive dispersion into the forecast of ultimate loss due to its independence assumption, but, it is possible to measure and adjust for it. This results in a model that allows one to take individual open claims, and adjust for development to ultimate, before fitting these claims to a severity curve. It can be shown that the distributional loss development process will change the shape of the ultimate size of loss distribution in a way that will affect loss cost estimates in a range of a few percent to 10 to 20 percent. It is important to reflect the distributional nature of loss development when evaluating individual loss data in order to avoid these errors.

Acknowledgement

The author wishes to acknowledge the assistance of, Jack Barnett who provided me his procedure for implementing the transition matrix method which is describe in this paper, Gary Venter who conceived the idea of applying the transition matrix method to our internal data, and whose comments have guided this paper, Spencer Gluck, and Jose Couret, for their review and comments which have influenced this paper.

Appendix A

The following describes the data used in this study,and, its treatment. This data was submitted as first dollar, 100% ground up insured loss. These claims were entered into the initial computer system as incremental transactions. The critical timing elements captured were date of loss, date the claim notice was submitted to the intermediary, and the date the claim was entered into the system. The various money types were maintained separately, such as indemnity, expense, subrogation, etc. A line of business code was entered for each claim, which was used for segregation into broad categories. A description of cause of loss field contained a detailed text description of the loss. This was used to isolate claims into sublines. It was found that simple terse descriptions were used repeatedly, and these were useful for identifying sublines. All the unique descriptions were isolated, and each was assigned to sublines. The goal was to create an Other Liability collection of data by identifying and removing Workers Comp, Medical Professional, Lawyers Professional, Pollution, and Auto. Also removed were claims arising from special events or circumstances such as the World Trade Center Disaster, toxic waste, environmental clean up, tobacco, cancer, etc. The remainder was deemed to be the Other Liability.

It was possible to isolate claims labeled as "other liability" using the Line of Business field. Inspection of this subset of the data revealed that it contained more than ordinary liability losses. A list of string fragments was assembled to eliminate claims based on the likelihood that the claim was another subline. For these claims, the description of cause of loss gave a good indication that the loss might be workers comp, legal liability, medical professional liability, auto liability, products etc. It also allows removal of special incidents, such as the World Trade Center disaster, tobacco losses, hazardous waste, environmental cleanup etc. The entire list of string fragments is shown in exhibit 17.

Strings for eliminating Claims				
FIRE	BREAST	IMPLANT	LEGAL	SURGERY
MOLD	CLASH	INS RAN	LIQUOR	SURGICAL
LAWYER	CLASS	INS REAR	M.V.	TABACCO
SURETY	D & O	INS VEH	M/	TOBBACL
ASBES	D & 0	INS. STRUCK	MED MAL	TRACTOR
AGG	D&O	INS.BACKED	MED. MAL	TRAILER
WASTE	D &O	INSD BACKED	MED PROF	TRUCK
ENV	D*O	COLLIDED	MED MAP	TYPHOON
CLEAN	D+O	CROSSED	MED NEG	VEH
SEX	D. & O.	INSD DRV	MED.MAL	VESSEL
CONTAM	D. AND O.	DRIVER	MED/MAL	W C
POL	D/O	INSD DV	MOLEST	W.C.
SITE	DIR &	INSD FAIL	MOTOR VEH	W/C
REMED	DIRECTOR	INSD HIT	MOTOR ACC	WC
WTC	E & O	INSD LOST	MOTORCYCLE	WORK
TOBAC	E&0	INSD R/E	MOTORIST	WORLD TRADE CENTER
CANCER	E&O	INSD RAN	MOTORVEHICLE	CAR CO
MEDICAL	CAR AC	INSD RE	MOTORYCLE	CAR CR
ACCOUNT	TRUCK	INSD REAR	MV	CAR FL
ACCT	VEH.	INSD ROLLED	NURSE	CAR HIT
AIDS	VAN	INSD RENTED	PEDEST	CAR IN
ATTNY	FEN PHEN	INSD SKID	PROD	CAR R
ATTORNEY	FIDELITY	INSD STRUCK	PROF	CAR S
ATTY	H.I.V.	INSD STUCK	REAR END	CAR T
AUDIT	HIV	INSD TRUCK	REAR-END	CAR/
AUTO	HOSPITAL	INSD TURN	REAREND	CAR\
Agg	HURRICANE	INTERSECTION	DRIVING	

Exhibit 17 This is a list of string fragments that were used to eliminate claims from the study.

Claims were eliminated if the string fragment was contained in the "Description of Cause of Loss" field.

The sum of the indemnity and ALAE was used as the loss in this study. The incremental transactions at irregular times were accumulated into year end evaluations for each claim. A claim was deemed to be closed if it's paid and incurred amounts were the same. The closure event was deemed to have occurred when the claim first arrived at this amount, and a flag was entered into the data, for each claim, to mark this.

The losses were trended using the Masterson trend factors published in Best's. The General Liability Bodily Injury trend indications were used. These were available from 1984 to present. 1980 to 1983 were adjusted by an additional annual trend of 8%.

Claims from accident years 1979 and earlier were excluded from this study in order to reduce the amount of computation. This left about 37,000 claims, of which, about 28,000 were non zero in 2003.

5. REFERENCES

Footnotes

[1] Hachemeister, Charles A., "A Stochastic Model for Loss Reserving," Trans. 21st Int. Congress of Actuaries, 1980, Vol. 1, pp. 185-194

[2] Hesselager, Ole, "A Markov Model for Loss Reserving,", ASTIN Bulletin, 1994, Vol 24-2, pp. 183-193

[3] http://economics.about.com/library/glossary/bldef-markov-transition-matrix.htm

Biography of the Author

John Mahon is a Vice President with Guy Carpenter Instrat, within Guy Carpenter & Co., Inc., where he has been providing actuarial services for 12 years. Prior to this, he was with American Re-Insurance Co., Inc. where he provided actuarial services for 7 years. Prior to this he was with ISO for 3 years in the increased limits and research areas. He was trained as a scientist, and holds a B.S. and a M.S. from Stevens Institute of Technology

An Improved Method for Experience Rating Reinsurance Treaties using Exposure Rating Techniques

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Mark A. Verheyen, FCAS, MAAA

Abstract

This paper deals with two of the most common disadvantages of standard excess of loss experience rating methods: lack of complete individual claim history and significant changes in the underlying book of business due to shifts in limit profile during the experience period. We develop a methodology to estimate an trend factor by layer of loss based on the unlimited trend factor, the severity distribution and the limit profile. This excess trend factor can then be applied to the nominal losses in the layer, overcoming the problem of having incomplete individual claim detail to exposure rate lower more credible layers. The excess trend is split into its frequency and severity components. We also present a methodology to estimate exposure adjustment factors necessaries to bring the experience of excess layer to the projected limit profile distribution. The impact of shifts in limit profile is also analyzed in terms of its frequency and severity components.
Keywords: Treaty Reinsurance, Experience Rating, Exposure Rating, Excess Trend, Exposure Adjustment.

1 INTRODUCTION

Exposure rating and experience rating are the two most prevalent and widely documented approaches to pricing excess of loss reinsurance contracts. Each method has its own strengths and weaknesses in any given situation, and frequently these methods are used in tandem to price a contract, with the final loss estimate usually being a credibility-weighted average of the two methods. Excess of loss exposure rating relies on a current snapshot of the policies subject to the reinsurance contract. This snapshot will often include some measure of the percentage of policies or premium exposing the reinsurance contract (the "limit profile"), an estimate of the gross losses (i.e. before reinsurance) for the policies exposing the reinsurance contract, and a severity distribution used to allocate the portion of the gross losses to the various layers ceding to the reinsurance contract. Excess of loss experience rating relies on the historical losses of the cedant, adjusted for trend to the current claim cost level and adjusted to the current exposure level. Trending of losses is typically accomplished on a per claim basis, with adjustments made for policy deductibles and limits when this information is available. Exposure adjustments are typically accomplished through on-leveling of premium or through the use of historical exposure information (policy counts, revenue, etc.). There are several drawbacks of relying entirely on the experience method. Two of the most common problems found in practice are incomplete data to experience rate lower layers and significant shifts in the mix of business in the underlying portfolio during the experience period. In this paper we present an improved method based on the mathematics of exposure rating that helps us overcome these two common practical problems. The problem of incomplete claim information is dealt with by calculating trend factors by layer of loss based on the selected unlimited trend factor, the severity distribution and the ceding company's limit profile. The methodology to estimate excess trends had been previously introduced in Miccolis [9] and Lange [5]; however those two articles did not take into account the issue of different policy limits, which is the case with excess of loss treaties. Although most relevant reinsurance pricing articles mentioned that the impact of changes in limit profile should be taken into account when experience rating excess of loss treaties very few has been written about how to quantify these changes. The impact of shifts in limit profile by layer of loss is estimated with the method presented in Section 3.3. The paper is outlined as follows: Section 2 presents a brief summary of the mathematics of exposure and experience rating. Section 3.1 outlines the assumptions and notation used throughout the paper. Section 3.2 presents the method developed to estimate the trend factor by layer of loss given the unlimited trend factor. Section 3.3 presents the methodology to estimate the exposure adjustment by layer of loss due to changes in Limits Profile. Detailed worked examples are presented in Sections 3.2.3 and 3.3.2.

2 BACKGROUND

In this section we present a brief overview of the basics principles of first dollar ratemaking as well as excess of loss experience and exposure rating. Sections 2.1 and 2.2 below present the standard methods for loss trending and premium adjustment used in practice and some of the disadvantages of these methods. Section 3 then presents the proposed approach to overcome the drawbacks of standard loss and premium trending methods.

2.1 Loss Trending

Adjusting losses for underlying trends is in essence making adjustments to the loss experience to reflect changes in the expected cost per claim (severity) and the expected frequency between the experience period and the prospective or future period. In the remainder of this paper we assume trend is related to ground-up severity trend only. Adjustments for ground-up frequency trend can be incorporated in a straight-forward fashion, but are not addressed here. In first dollar ratemaking the prospective period usually refers to the period when rates are going to be in effect whereas in reinsurance pricing it refers to the policy or treaty period. In this paper the prospective period is assumed to be the treaty period. McClenahan [8] presents in details the fundamental principles of adjusting losses for severity and frequency trend for ratemaking purposes. In brief the methodology involves estimating from the data (or from other sources) an unlimited trend or a basic limit trend, calculating the time difference between the average loss date of the prospective period and the average loss date of the experience period and then applying the trend factor to the aggregate losses (unlimited or basic limit losses depending on the situation). In his paper McClenahan [8] also discusses the effect of limits on severity trend and makes the following remark:

"Where severity trend has been analyzed based upon unlimited loss data or loss data including limits higher than the basic level, the resulting indicated severity trend

must be adjusted before it is applied to basic limit losses. Because such adjustment will require knowledge of the underlying size-of-loss distribution, it is generally preferable to use basic limit data in the severity trend".[1]

However there are situations in practice where one needs to analyze total limit losses when each loss is subject to a different limit of liability (quota share treaty, for example). In this case neither a basic limit trend nor an unlimited trend is appropriate. Hence one needs to estimate a total limits trend which is consistent with the unlimited trend and the basic limit trend. As stated by McClenahan [8] the development of such trend requires knowledge of the size-of-loss distribution. In Section 3.2.1 we present a methodology to estimate trend factors for different layers of losses based on an unlimited trend factor and the severity distribution. When experience rating excess of loss reinsurance one also needs to adjust losses for frequency and severity trends in order to project the expected loss cost in the layer. To do so one requires individual losses with policy limit and deductible details. The trend factor is applied to each ground up loss (gross of the deductible), each trended loss is capped at the policy limit and the deductible is netted out. The resulting loss is then applied to the reinsurance layer. However, underlying policy information is frequently not available for each individual claim and therefore the standard per claim trending methodology can lead to misestimation of the loss cost in the layer. In Section 3.2.1 we develop a methodology to estimate an excess of loss trend based on the limit profile of the ceding company. This methodology helps us to overcome the problem of lack of policy detail by claim.

2.2 Premium Trending

The second fundamental aspect of ratemaking is adjusting historical premium for rate changes as well as other factors that affect the average premium per exposure over

[1] McClenahan [8] p.110.

time. The first step to adjust for changes in average premium is to adjust for rate changes. The methodology for bringing premium to current rate levels is presented in detail in McClenahan [8] and therefore will not be repeated here. If rates can be accurately measured the difference between historical on-level premium would be due to other factors such as growth, changes in rating plans or simply increases in exposure. Jones [3] presents in detail the methodology to adjust historical premium for trends other than rate changes that affect the average premium per exposure. In his paper he identifies four changes that affect premium levels:

1. Past rate changes.
2. Past rating plan changes.
3. The existence of rating plans which change the premium level over time.
4. Past and expected future shifts in the mix of business.

The primary focus of the methodology presented in Jones [3] is first dollar ratemaking and it is based on estimating a trend factor net of rate changes and then applying it in a similar fashion as loss trend is applied to aggregate losses. In this paper we focus our attention on the fourth item identified in Jones [3] : Past and expected future shifts in the mix of business. However we do so from the reinsurer's view point. When insurance companies change their mix of business they can do so in many ways. Two of the most important changes in mix of business affecting reinsurers are: changes in policy limit and deductible distribution and changes in line of business. In a soft market insurance companies tend to offer higher limits of liability to remain competitive and to maintain certain target premium levels. In a hard market capacity is reduced and rates increase, hence insureds tend to buy lower limits because either higher limits are no longer available or they are prohibitively expensive relative to the additional cover they provide. This change in limit distribution has a significant impact on both premium and losses, and the impact is usually magnified in excess layers often ceded to reinsurers. Similarly when companies change their core business from low severity high frequency lines to high severity low frequency lines premium levels tend to change and this is also reflected in the loss experience. This change in line of business also has a significant impact when pricing excess of loss reinsurance. If one does not appropriately adjust the experience to reflect the difference between expected mix of business and the historical mix of business the loss cost in the excess layers can be materially misestimated. In Section 3.3 we present a methodology to adjust excess of loss experience for changes in limit profile based on the mathematics of excess of loss exposure rating. The methodology can be extended to quantify changes in mix of business.

2.3 Excess of Loss Pricing: an Overview

Exposure and experience rating are the two most prevalent and widely documented approaches to pricing excess of loss reinsurance contracts. Each method has its own strengths and weaknesses in any given situation, and frequently these methods are used in tandem to price a contract, with the final loss estimate being a credibility-weighted average of the two methods. In this section we present a brief overview of the two methods and we introduce the notation that will be used in the remainder of the paper.

2.3.1 Experience Rating

Experience rating relies on the historical losses of the cedant, adjusted to the prospective claim cost and exposure level. Trending of losses is typically accomplished on a per claim basis, with adjustments made for policy deductibles and limits when this information is available. Exposure adjustments are typically accomplished through on-leveling premium or through the use of historical exposure information (policy count, revenue, etc.). Clark [1] presents a detailed overview of the basics of reinsurance pricing. Below are the basic steps and data requirements to experience rate excess of loss treaties.

1. Individual loss information with policy deductible and limit: each ground up loss (gross of deductible) is trended to the future average date of loss, the deductible is netted out and the loss is capped at the policy limit. The resulting loss is the applied to the reinsurance layer.

2. Losses in the layer are then aggregated by year (treaty year or accident year, depending on the basis of the analysis).

3. Aggregate losses in the layer are then developed to ultimate using excess development factors. In this paper we do not discuss excess layers loss development factors. Pinto and Gogol [11] and Siewert [12] discuss methods for estimating loss development factors for excess layers.

4. Historical subject premium (earned or written) is adjusted for rate changes and for exposure changes to the prospective premium level.

5. The loss cost to the layer by year is calculated by dividing the ultimate trended losses in the layer by the corresponding adjusted premium.

6. An average of the loss cost is taken between appropriate years of experience.

7. The reinsurance rate is developed by loading the loss cost for reinsurer's expenses and profit.

There are several disadvantages to relying entirely on the results of experience rating as outlined above. Some of the problems are related to the availability of the data required to perform the experience rating and some problems are related to the methodology.

Disadvantages of experience rating methods

1. Often reinsurers receive individual loss detail for large losses only, i.e. incurred losses that are greater than a certain value below the attachment point. This

generates the problem of incomplete information for lower layers in particular for older years. For example if we use 8% annual trend and we receive losses greater than \$200,000 since 1995, the smallest trended loss in 2005 terms is $\$200,000 * (1.05)^{10} = \$325,779$. Hence, we could not use this data to experience rate layers with attachments lower than \$325,779. In practice this problem is dealt with by selecting a data limit or threshold and layers that attach below that threshold are not experience rated. This methodology has two disadvantages: all data below the threshold are eliminated from the analysis and one cannot perform experience rating of lower layers (which is more credible as there is more data and results are usually more stable than higher layers). In our example, we would be unable to experience rate a \$200k *xs* \$200k layer using standard loss trending due to the incomplete data. We would also be unaware if losses in that \$200k *xs* \$200k layer were deteriorating in recent years. Figure 1 shows the impact of severity trend on the data limit and the amount of data lost due to having incomplete claim history.

2. Another problem encountered in practice is the lack of policy information for each claim. If this information is not available, applying a trend and not capping at policy limit can significantly overstate the expected loss cost. Similarly if deductible information is not available, applying a trend factor to a loss net of the deductible can significantly understate the expected loss cost.

3. If there have been significant changes in the book of business during the experience period such as expanding or contracting limits, then each experience period is on a different mix basis and therefore it is not appropriate to simply average between years. Furthermore, the projected loss cost would not be a true projection of the expected loss cost in the future period. When experience rating excess of loss layers for a book of business that has experienced significant changes

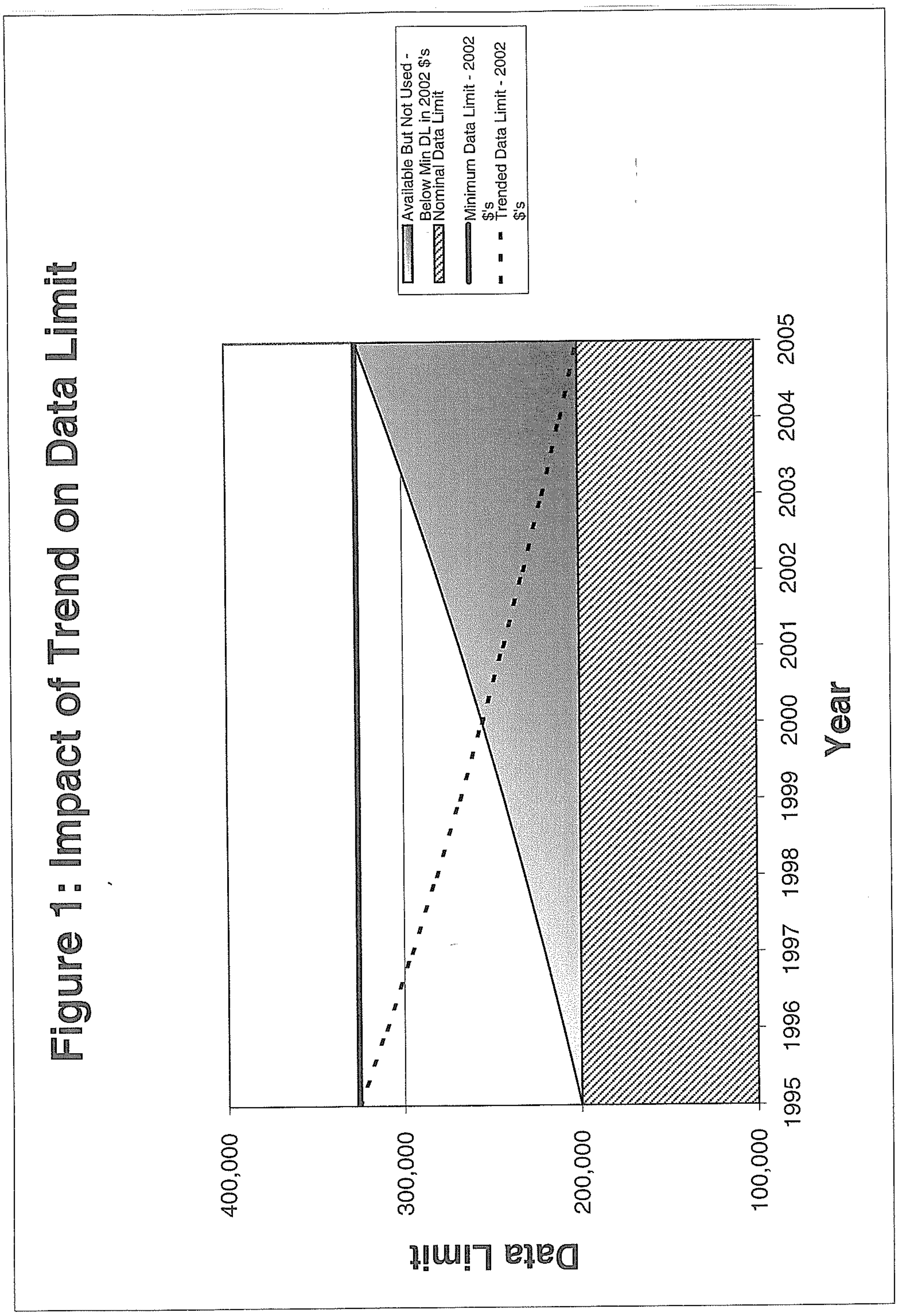
Figure 1 : Impact of Trend on Data Limit
Available But Not Used - Below Min DL in 2002 $'s
Nominal Data Limit
Minimum Data Limit - 2002 $'s
Trended Data Limit - 2002 $'s
400,000
300,000
200,000
100,000
Data Limit
1995
1996
1997
1998
1999
2000
2001
2002
2003
2004
2005
Year

in the mix of business actuaries often load or give credit for these changes in exposure in their final experience rating results. However very little has been written in the literature about quantifying these changes in a more systematic and rigorous way.

In Section 3.2 we present a method that will help us overcome the practical problems in items 1 and 2 above and in Section 3.3 we present a method to help us quantify the impact of changes in limits therefore overcome the problem in item 3 above.

2.3.2 Exposure Rating

The exposure rating method relies on a current snapshot of the policies subject to the reinsurance contract. This snapshot will include some measure of the percentage of policies or premium exposing the reinsurance contract, usually called the "limit profile", and an estimate of the gross losses (i.e. before reinsurance) for such policies. To perform an exposure rating one requires a size of loss distribution (severity distribution), an Increased Limit Factor table or an exposure curve. In the US, many actuaries rely on ISO severity or exposure curves as benchmarks for those lines of business ISO covers. For non-ISO lines of business reinsurers often use ceding company data to fit severity distributions. Outside the US, the use of exposure rating is more cumbersome given the lack of industry benchmark severity distributions by line of business. The objective of the exposure rating method is to estimate the proportion of the loss for the underlying policy that is expected in the excess layer.

Assumptions and Notation:

1. Let X be the random variable representing the ground up cost per claim.
2. Let $f_X(x)$ and $F_X(x)$ denote the probability density function and cumulative density function of X.
3. Assume an underlying policy with limit PL and attachment or deductible D.[2]

[2] In practice deductibles and attachments are not the same and there are many variations on

This is the policy written by the ceding company and we assume the ceding company takes 100% of this policy. In other words there is no coinsurance either with the insured or with other insurers.

4. Assume the expected gross loss ratio for the underlying book is ELR (before reinsurance).

5. The ceding company's projected subject premium for the prospective period is SP. The subject premium is the premium written or earned to which the reinsurance rate applies, e.g. premium net of facultative insurance or net of commissions. It is usually defined in the treaty wording.

6. We denote by α the proportion of the total subject premium that corresponds to policies with limits PL and deductible D. This is often presented in the limit profile.

7. The reinsurance layer is $L\ xs\ A$, i.e. losses for the underlying policy in excess of A subject to a limit L.

The following definition and notation will be widely used in the remainder of the paper.

Definition 2.1 *Let X be a random variable with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. The Limited Expected Value of X up to a limit a, i.e.* $\min(X, a)$, *is given by*

$$E[X \wedge a] = \int_0^a x f_X(x)dx + a(1 - F_X(a)) = \int_0^a (1 - F_X(x))dx. \tag{2.1}$$

See Klugman, Panjer and Willmot [4]. Note that the notation $X \wedge a$ stands for $\min(X, a)$. The exposure rating method can be expressed in terms of Limited Expected Values, Increased Limit Factors or Exposure Curves, see for example Clark [1]

how deductibles and attachments apply. The notation deductible refers to primary business whereas attachment refers to excess business. For simplicity and consistency we will assume in this paper that a policy with limit PL and deductible D covers losses excess of D up to a limit PL.

and Ludwig [7]. The expected loss cost in the layer for a policy with limit PL and deductible D can be expressed as follows:

$$\text{Loss Cost} = (SP)(ELR)\alpha \frac{E_X[X \wedge T] - E_X[X \wedge B]}{E_X[X \wedge PL + D] - E_X[X \wedge D]}, \tag{2.2}$$

where $T = \min(PL + D, L + A + D)$ (i.e. top of the layer, allowing for the fact that some policies may only partially expose the layer) and $B = \min(PL + D, A + D)$ (i.e. the bottom of the layer, allowing for the fact that some policies may not expose the layer in which case $T = B$ and the Loss Cost is 0). The ratio of expected severity in the layer to expected severity for the underlying policy in equation (2.2) is usually referred to as "the % of loss in the layer". To estimate the total loss cost in the layer for the underlying book of business one adds across all combinations of policy limits and deductibles presented in the limit profile of the ceding company.

3 A METHOD FOR IMPROVING EXPERIENCE RATING TECHNIQUES

3.1 Assumptions and Notation

1. Let X denote the ground up cost per claim for the experience period.

2. Let Y denote the ground up cost per claim for the projected or future period. Hence, $Y = rX$, where r is the trend factor necessary to bring experience losses to future claim cost level.

3. We assume that $F_Y(y)$ the distribution function of Y is given. $E_Y[.]$ will denote the expectation calculated using the distribution function of Y.

4. PL and D are the policy limit and deductible for policies written by the ceding company. We assume there are $p = 1, \ldots, P$ combinations of policy limits and deductibles.

5. Let α_p denote the proportion of premium expected to be written for the pth combination of limit and deductible in the experience period. Let β_p be the proportion of premium written for the pth combination of policy and deductible in the prospective period.

6. Let OLP denote the on-level subject premium for the experience period and SP the projected subject premium for the prospective period.

7. Let ELR denote the expected gross loss ratio (i.e. before reinsurance) for the prospective period.

8. Let N_p and M_p be the number of claims for the ceding company for the pth combination of policy limit and deductible with projected subject premium and on-level subject premium respectively.

9. Let $S_{X,p}$ and $S_{Y,p}$ denote the cost per claim for the pth policy type for the experience and prospective period respectively.

10. The reinsurance layer is $L\ xs\ A$.

11. Let $S_{X,p}(A,L)$ and $S_{Y,p}(A,L)$ denote the loss cost per claim in the layer for experience and future period respectively.

12. Let LC denote the projected expected gross loss cost for the ceding company before reinsurance in the future period, and $LC(A,L)$ the projected expected loss cost in the reinsurance layer.

Using the notation and assumptions above it can be seen that

$$\begin{aligned} LC = &\ (SP)(ELR) = \textstyle\sum_{p=1}^{P} E[N_p]E[S_{Y,p}] \\ LC(A,L) = &\ \textstyle\sum_{p=1}^{P} E[N_p]E[S_{Y,p}(A,L)] \end{aligned} \tag{3.1}$$

where the last equality is the standard result of the Collective Risk Model. It then follows that:

(a) The expected cost per claim for the pth policy type is given by

$$E_Y[S_{Y,p}] = E_Y[Y \wedge PL + D] - E_Y[Y \wedge D]. \tag{3.2}$$

(b) The expected number of claims for the pth policy type is given by

$$E[N_p] = \frac{(SP)(ELR)\beta_p}{E_Y[S_{Y,p}]}, \tag{3.3}$$

i.e. the total loss cost for the pth policy type divided by the expected cost per claim.

(c) The expected severity in the layer for the pth policy type is given by

$$E_Y[S_{Y,p}(A, L)] = E_Y[Y \wedge T] - E_Y[Y \wedge B]. \tag{3.4}$$

where T and B are as defined in equation (2.2). Note that this is not the conditional expected severity in the layer (i.e. the severity in the layer given that the loss has exceeded the layer attachment). To calculate the conditional expectation, this quantity needs to be divided by $(1 - F_Y(D + A))$.

Thus we can re-write the exposure rating equation (2.2) as follows:

$$\text{Loss Cost} = E[N_p]E_Y[S_{Y,p}(A, L)], \tag{3.5}$$

and therefore

$$LC(A, L) = \sum_{p=1}^{P} E[N_p]E_Y[S_{Y,p}(A, L)]. \tag{3.6}$$

Expressing the exposure rating method in terms of expected frequency and severity will help us present all the methods below in terms of frequency and severity.

3.2 Aggregate Trend

Miccolis [9] and Lange [5] present detailed discussion on the mathematics of Increased Limits Factors and Excess of Loss pricing. Miccolis [9] presents the mathematical foundations of pricing increased limits based on the basic limit loss cost and then ties the concept of increased limits factors to pricing excess of loss coverage. Lange [5] presents the methodology to estimate increased limits factors given loss experience by policy limit. Although in both articles Miccolis and Lange discuss the effect of trend and inflation in excess layers and how the excess trend can be calculated from the unlimited trend factor and the severity distribution, no numerical examples are presented in their discussions and they do not discuss how the calculation of excess trends can help improve excess of loss experience methods. In this paper we present the methodology for calculating excess trends by layer of loss, both frequency and severity excess trends, and how the resulting excess trend could be used for experience rating excess layers. The following results are stated in Miccolis [9] and will be used in the worked examples below.

Result 1 *Let X be a random variable with cumulative distribution function $F_X(x)$. Define $Y = aX$, where $a \geq 0$. Then the following results hold:*

$$F_Y(y) = F_X(y/a) \qquad \text{and} \tag{3.7}$$

$$E_Y[Y \wedge y] = aE_X[X \wedge y/a] \tag{3.8}$$

In other words given the distribution of X one can easily deduce the distribution of Y by either re-scaling the distribution of X dividing by a or by trending the parameters of the distribution of X. Most of the commonly used loss distribution functions can be expressed in terms of trended parameters. For example if X follows a lognormal distribution with parameters μ and σ then aX also follows a lognormal distribution with parameters $\mu' = \mu + \ln(a)$ and $\sigma' = \sigma$.

3.2.1 Layer Excess Trend

Actuaries are already familiar with the fact that although a ground up or unlimited trend factor may apply to all sizes of loss the leverage effect of inflation varies greatly by layer of loss and is highly dependent on the attachment or retention. In this section we address the problem of consistency between unlimited trend, basic limits trend and excess trend. The method presented below is not new to the CAS, Miccolis [9] describes the leverage effect of inflation for excess of loss layers, whereas Siewert [12] and Pinto and Gogol [11] have used similar techniques to estimate excess of loss development factors based on ground up loss development factors and a severity distribution. The fundamental idea of adjusting losses for trends is to reflect the change of the average loss cost between the experience period and the future period. Therefore, if one can calculate the expected loss cost for the future period and the average loss cost for the experience period, the loss trend will be given by the ratio of these two figures. The methodology presented below is based on this principle. In the methodology presented below we assume that limit profile and mix of business have remained constant throughout the experience period and will remain constant during the prospective period. Thus we are only adjusting for changes in average loss cost.

Steps to calculate excess trend for the layer $L\ xs\ A$ Using the notation outlined in Section 3.1 the following are the steps to estimate the trend factor applicable to the excess layer.

1. Assume the following items for the prospective period are given or can be estimated: ELR, Limits Profile with $p = 1, \ldots, P$ limit and deductible combinations, expected subject premium SP, the severity distribution for the claim cost Y is given by $F_Y(y)$ and the unlimited trend factor between the experience period and the prospective period is r.

2. Using the standard exposure rating method as described in Section 2.3.2 calculate the expected loss cost in the layer for each policy type. We denote this expected loss cost by LC_p, for $p = 1, \ldots, P$.

3. For each policy type calculate the expected cost per claim in the layer at the

future claim cost level and at the experience period claim cost level. Therefore the change in average cost in the layer per policy type is given by:

$$\text{(Excess Trend)}_p = \frac{E_Y[Y \wedge T] - E_Y[Y \wedge B]}{E_X[X \wedge T] - E_X[X \wedge B]} = \frac{E[S_{Y,p}(A,L)]}{E[S_{X,p}(A,L)]}, \tag{3.9}$$

where T and B are the top and bottom of the layer as defined above. Note that $X = Y/r$ and since the distribution of Y is given the expected values on X can be calculated using the distribution of Y re-scaled as stated in Result 1.

4. Then the total layer trend can be calculated as the weighted average of the excess trend per policy type, where the weights are given by the contribution to the total loss cost of each policy type. In other words the total layer trend is calculated as:

$$\text{Layer Trend} = \frac{\sum_{p=1}^{P}(LC_p)\text{(Excess Trend)}_p}{\sum_{p=1}^{P}(LC_p)} \tag{3.10}$$

As discussed in Miccolis [9] , the leverage effect of inflation in the excess layer is controlled by the attachment. The reason for this is that losses in excess of the attachment will increase for two reasons: smaller losses that on an incurred basis did not reach the attachment can potentially exceed the attachment once the trend factor is applied and losses that exceeded the attachment have most of the inflation effect within the layer. Hence, more losses will trend into the layer and losses in the layer become larger. When pricing excess of loss reinsurance one is usually interested not only in estimating the total loss cost but also in estimating how much of the loss cost is due to frequency and how much is severity. Is frequency in the layer increasing or are claims in the layer getting larger? What is the impact of trend in excess frequency and severity? In the following sections we estimate how much of the excess trend calculated in (3.10) is due to frequency and how much is due to severity.

3.2.2 Excess Frequency and Severity Trend

In order to calculate the excess frequency trend in the layer we need to estimate the change in expected claim count excess of the attachment between the prospective period and the experience period. Note that when we discuss excess frequency trend in this context, we are referring to the increased frequency in an excess layer resulting from ground-up severity trend. Ground-up frequency trends are not specifically addressed in this paper, although they can be incorporated into this framework. The following result will be used to determine the excess frequency trend in the layer.

Result 2 *Let N denote the number of claims in a given portfolio of policies and let $F_Y(y)$ denote the claim size distribution. The expected number of losses that exceed a certain threshold A, $N(A)$, is given by:*

$$E[N(A)] = E[N](1 - F_Y(A)). \tag{3.11}$$

Using this result it can be easily seen that the frequency trend is given by the following equation:

$$\begin{aligned} \text{Excess Frequency Trend} &= \frac{\sum_{p=1}^{P} \frac{E[LC_p]}{E[S_{Y,p}]} \Pr(S_{Y,p} > A)}{\sum_{p=1}^{P} \frac{E[LC_p]}{E[S_{Y,p}]} \Pr(S_{X,p} > A)} \\ &= \frac{\sum_{p=1}^{P} E[N_p](1 - F_Y(A+D))\mathbf{1}_{\{PL>A\}}}{\sum_{p=1}^{P} E[N_p](1 - F_X(A+D))\mathbf{1}_{\{PL>A\}}}, \end{aligned} \tag{3.12}$$

where $\mathbf{1}_{\{PL>A\}}$ is the indicator function that takes the value of 1 if a policy exposes the layer. The formula in (3.12) can be interpreted as follows: the expected number of claims in the layer given the loss distribution for the prospective period relative to the expected number of claims in the layer given the loss distribution of the experience period. Note that in equation (3.12) we are assuming the same number of claims for the underlying policy, but more claims will penetrate the layer as severity increases. Note that in the formulas above we have assumed constant ground up frequency trend. However, if ground up frequency is also increasing the assumed frequency trend would naturally flow through the calculations in a multiplicative fashion. Equations (3.10) and (3.12) give us the total excess trend factor and the excess frequency trend factor. To estimate the severity trend factor in the layer we simply divide the total layer trend factor by the excess frequency trend factor. Once we have a trend factor by layer we can apply this factor to the **nominal** losses in the layer. In other words, we take the actual incurred losses in the layer, develop them to ultimate with appropriate development factors and then apply the layer trend factor as developed in this section. There are various advantages of this method:

1. When large losses do not include policy and deductible information one does not need to make assumptions about the underlying policy. The limits profile provides the assumptions about the distribution of underlying policies and applies an average trend for the layer in light of this. One simply takes the nominal aggregate losses in the layer and then applies the layer trend factor.

2. The approach provides consistency between unlimited trend and trend at various layers of loss. This is not only useful for experience rating excess layers, but it also helps quantify the differences in ground-up trend for two books of comparable business with differing limits.

3. Since we are not trending individual losses it is no longer necessary to select a data limit or threshold, hence all data available can be used in the analysis. Since no data are eliminated we could perform an experience rating for lower layers and then use the severity distribution to extrapolate losses to higher layers. This method has the advantage that experience is usually more stable in lower layers than higher layers.

The principal disadvantage of the method is that it is more difficult to explain to a non-technical audience. The typical underwriter may not understand why a loss that has settled for policy limits continues to receive a trend adjustment, which is one of the results of using this approach. This is necessary under this approach, as it counteracts the under-trending of a loss that has barely penetrated the layer attachment.

The following section provides a worked example to illustrate step by step the trending methodology presented above.

3.2.3 Worked Example

All Tables related to this example are presented in the appendix. Assume that a ceding company writes limits between \$250,000 and \$5,000,000 with projected written premium distribution as in Table 1. We assume in this example that all policies are primary and there are no deductibles. We assume that a reinsurer is interested in pricing an excess of loss treaty for this ceding company for underwriting year 2005. The loss distribution for this line of business for underwriting year 2005 is assumed to follow a lognormal distribution with parameters μ=9.31 and σ=2.29. The annual unlimited trend factor for this line of business is assumed to be 8%. Therefore, the loss distribution for losses in 2000 values is also a lognormal distribution with parameters $\mu' = \mu - \ln(1.08^5) = 8.93$ and $\sigma' = \sigma = 2.29$. In this example we work out the trend factor for various layers to be applied to experience losses for underwriting year 2000. The unlimited trend factor between 2000 and 2005 is given by $1.08^5 = 1.47$. Table 1 presents the limited expected value or expected severity for each policy in 2005 and 2000 loss cost values. As per our notation in Section 3.2.1 we have $E[S_{Y,p}] = E_Y[Y \wedge PL]$ represents the expected severity in 2005 values and $E[S_{X,p}] = E_X[X \wedge PL]$ the severity in 2000 values, since there are no deductibles.

Table 2 shows the results of applying the standard exposure rating method using the limit profile as in Table 1 and the distribution with 2005 parameters and an expected gross loss ratio of 60%. (We have assumed a loss ratio for completeness, though it can be seen that the loss ratio cancels in all the equations in Section 3.2.1 and 3.2.2). The results shown in Table 3.2 were calculated using equation (2.2) and adding across policy limits.

Table 3 shows the excess trend per policy across layers using equation (3.9). For example, the trend factor for the layer \$250,000 *xs* \$250,000 is zero for policies with limit of \$250,000 as they do not expose the layer, and the trend factor is the same for all other policies as the remaining policies fully exposed the layer. The trend factor in this case is calculated as follows:

$$\frac{E_{2005}[Y \wedge 500,000] - E_{2005}[Y \wedge 250,000]}{E_{2000}[X \wedge 500,000] - E_{2000}[X \wedge 250,000]} = \frac{64,416 - 48,539}{50,191 - 38,900} = 1.248.$$

We also note from Table 3 the trend factor is different for policies that partially expose the layer. We note this in the case of the \$750,000 policy limit in the layer \$500,000 *xs* \$500,000 and all policies in the \$5MM *xs* \$0.

The total trend in Table 3 is calculated as the weighted average of the trend factor per policy with the corresponding exposure rating loss cost as given in Table 2. From Table 3 we observe the effect of capping at policy limits. For example, if we had not taken into account the effect of policy limits the trend factor in the first \$5MM would have been 1.389 instead of 1.328. We also observe the leverage effect of loss trend in the various layers, for example the annual trend factor for the layer \$4MM *xs* \$1MM is 9.59% compared to an unlimited ground up trend factor of 8%. If we were analyzing aggregate gross losses for the ceding company using a trend factor of 8% would be inadequate since we would not be taking into account the fact that the majority of the business is written at lower limits. In this case it would be more adequate to use a trend factor of 5.83% which takes into account the limit profile of the ceding company's portfolio. Table 4 shows the frequency trend in the various layers. The second column of Table 4 shows the expected number of claims by policy limit. These figures were calculated by taking the projected premium times the expected loss ratio and divided by the limited expected value at the corresponding policy limit. The expected number of claims in the layer were calculated using equation (3.11) with the curves in 2005 and 2000 values respectively. The frequency trend is then the ratio of the expected number of claims in the layer using 2005 severity distribution and the expected number of claims in the layer using 2000 severity distribution. It can be seen from Table 4 that although in this example we assumed no ground up frequency trend (hence the frequency trend for primary layers is zero) the majority of the increase in loss cost in excess layers is expected from additional claims trending into the layer. Table 5 shows a summary of the trend factors by layer as well as the frequency and severity trend factors. As discussed above most of the expected increase in loss cost in excess layers is due to an increase in claims in the layer rather than an increase in severity in the layer. Having the excess trend split into frequency and severity has several advantages when pricing excess of loss reinsurance. In practice frequency tends to be more stable than severity, thus often one models frequency from the ceding company's experience and uses severity from the loss distribution and then multiplies the two to estimate the total loss cost. There are several variations of this "mix" method used in practice.

3.3 Adjustment for Changes in Limit Profile

While primary pricing is concerned with total exposure growth, excess of loss reinsurance pricing is concerned with exposure growth by layer. Traditional pricing assumes exposure growth by layer is consistent with total exposure growth: an assumption that frequently fails in a stable operating environment, let alone an environment where limits usage expands or contracts greatly due to significant shifts in underwriting appetite and pricing adequacy. One of the fundamental components of a typical reinsurance submission is the Limits Profile. The Limits Profile will typically show the amount of business the ceding company has written during the latest calendar period segmented by the limits of the primary policies. It may be based on written or earned premium, or it may be based on policy or exposure counts. If there is significant variability in deductibles, or if the book of business is written on an excess basis with varying attachment points, the Limits Profile may also be segmented by deductible or attachment. While the form of the Limits Profile may differ depending on the characteristics of the business, its purpose is the same: to provide the reinsurer with an estimate of the ceding company's limits usage. By comparing Limits Profiles across time periods, the reinsurer can estimate not only current period limits usage, but also the change in limits usage over time. The experience of the ceding company will greatly vary depending on the composition of the book. For example if the maximum limit capacity of a ceding company in year 2000 was \$3MM the maximum loss for any one claim will be \$3MM. If the ceding company has expanded its capacity to a maximum limit of \$5MM for year 2005 and the reinsurer is interested in pricing the layer \$2.5MM xs \$2.5MM then using the experience for 2000 to project the loss cost in the layer for year 2005 will result in an understatement of the the loss cost for the reinsurers as the maximum exposure for the layer in 2000 was \$500,000 whereas for 2005 the layer is fully exposed. In this section we develop a method based on the mathematics of exposure rating that will help us estimate the differences in exposure across layers. A very similar version of the method presented below was presented by Robert Giambo at the CARe seminar in 2004. In his presentation he used the results of the exposure rating method to adjust for changes in limits profile. There are two main differences between the approach presented by Giambo and the approach presented in this paper:

1. Giambo's approach is based on the experience loss cost expressed as a percentage of the historical on-level premium. Hence his adjustment factor did not include an adjustment for the total limits exposure change. In this paper the estimated exposure adjustment factor includes overall exposure change (through the relative change in on-level premium) as well as exposure in the layer due to shifts in limit profile.

2. We take the methodology one step further and we split the exposure adjustment into its frequency and severity component. This split is helpful when one is interested in applying a mixed method where frequency is estimated from the experience and severity is estimated from the exposure method.

We follow the notation outline in Section 3.1.

Steps to estimate exposure adjustment by layer

1. Given rate changes for the ceding company during the experience period calculate on-level factors with standard on-level methodologies. See, for example, McClenahan [8] .

2. Calculate on-level historic limit profile. In absence of additional information we assume that the same rate changes (hence on-level factors) apply to all policy limits.

3. Using the severity distribution with parameters for the prospective period calculate the expected severity by policy limit $E_Y[S_{Y,p}]$.

4. With the results from item 3 above calculate the expected severity in the layer for each policy limit $E_Y[S_{Y,p}(A,L)]$.

5. Using the projected limit profile with premium distribution by policy given by β_p, for $p = 1, \ldots, P$, calculate the exposure rate as follows:

$$(\text{Loss Cost})_{projected} = (SP)(ELR) \sum_{p=1}^{P} \beta_p \frac{E_Y[S_{Y,p}(A,L)]}{E_Y[S_{Y,p}]}. \tag{3.13}$$

6. Using the historic on-level limit profile with premium distribution given by α_p for $p = 1, \ldots, P$ calculate the exposure rate as follows:

$$(\text{Loss Cost})_{historic} = (OLP)(ELR) \sum_{p=1}^{P} \alpha_p \frac{E_Y[S_{Y,p}(A,L)]}{E_Y[S_{Y,p}]}. \tag{3.14}$$

7. The exposure adjustment in the layer L xs A is given by the ratio of equations (3.13) and (3.14):

$$\begin{aligned}\text{Layer Exposure Adjustment} &= \frac{(\text{Loss Cost})_{projected}}{(\text{Loss Cost})_{historic}} \\ &= \frac{(SP)(ELR)}{(OLP)(ELR)} \frac{\sum_{p=1}^{P} \beta_p \frac{E_X[S_{X,p}(A,L)]}{E_X[S_{X,p}]}}{\sum_{p=1}^{P} \alpha_p \frac{E_X[S_{X,p}(A,L)]}{E_X[S_{X,p}]}}.\end{aligned} \tag{3.15}$$

Note from equation (3.15) the term $(SP)/(OLP)$ represents the total limits exposure adjustment. This term is then multiplied by the change in exposure in the layer given by changes in limit profile distribution across policies. If the ceding company's premium or exposure distribution by policy type has remained constant the right hand side term of equation (3.15) would equate $(SP)/(OLP)$, i.e. the total limit exposure adjustment. The method presented in this section only takes into account changes in limit profile, it does not reflect changes in average loss cost as we have assumed the severity distribution is in future value terms. The exposure adjustment as presented in equation (3.15) is applied to ultimate trended losses in the layer, where trended losses can either be calculated using the standard per claim trending method or the aggregate excess trend method as presented in Section 3.2.1. The exposure adjustment and the trend factor are multiplicative.

3.3.1 Frequency and Severity Exposure Adjustment

In this section we extend the method of adjusting for changes in limit profile to frequency and severity. In other words we estimate the increase in exposure due to increase in frequency and severity separately. The following steps provide the methodology to estimate the frequency exposure adjustment for a generic layer $L\ xs\ A$.

1. Calculate the expected number of claims by policy type for the prospective period as follows:

$$E[N_p] = \frac{(SP)(ELR)\beta_p}{E_Y[S_{Y,p}]}, \tag{3.16}$$

i.e. dividing the projected total expected loss cost by policy type by the expected severity for that policy type.

2. Calculate the "as if" expected number of claims by policy type for the experience period as follows:
$$E[M_p] = \frac{(OLP)(ELR)\alpha_p}{E_Y[S_{Y,p}]}, \tag{3.17}$$
i.e. the expected loss cost for the experience period in current rate level by the expected severity by policy type using the severity distribution for the prospective period.

3. Calculate the expected claim count in the layer by multiplying the expected claim count by policy type times the probability of claims penetrating the layer as follows:
$$\begin{aligned} E[N_p(A,L)] = &\quad E[N_p](1 - F_Y(A + D)) \\ E[M_p(A,L)] = &\quad E[M_p](1 - F_Y(A + D)) \end{aligned} \tag{3.18}$$

4. Add the expected claim count in the layer across policies for both the prospective and the experience period:
$$\begin{aligned} (\text{Claim Count})_{projected} = &\quad \textstyle\sum_{p=1}^{P} E[N_p(A,L)] \\ (\text{Claim Count})_{historic} = &\quad \textstyle\sum_{p=1}^{P} E[M_p(A,L)] \end{aligned} \tag{3.19}$$

5. Calculate the frequency exposure adjustment as:
$$\text{Frequency Exposure Adjustment} = \frac{(\text{Claim Count})_{projected}}{(\text{Claim Count})_{historic}}$$

The severity exposure adjustment is then calculated as the ratio between the layer exposure adjustment and the frequency adjustment. The following section presents a detailed worked example showing the exposure adjustment for various layers of loss.

3.3.2 Worked Example

We continue to use the same assumptions as in the example shown in Section 3.2.3. Table 6 shows the ceding company's Limits Profile for business written during underwriting year 2000 and the projected Limits Profile for underwriting year 2005.

The Limits Profile and rate increase information are typically provided by the ceding company as a part of the reinsurance submission. Assume we know that rates for business written in underwriting year 2000 will have increased a cumulative 50% by underwriting year 2005.

In this example, we have assumed that the 50% cumulative rate increase has equally impacted policies written at each limit. On-level written premium for underwriting year 2000 is 50% higher in Table 7 (Historic On-Level Limits Profile) than its counterpart in Table 3.6 (Historic Limits Profile). If rate changes have varied by limits due to a change in Increased Limits Factors, an appropriate differential increase should be applied separately to each limit.

We then calculate the expected losses in the layer using the standard exposure rating method. Table 8 shows the % of loss by layer by policy limit using the severity distribution with 2005 parameter. As in the example in Section 3.2.3 we are using a lognormal with $\mu = 9.31$ and $\sigma = 2.29$. The % of loss in the layer is calculated as the ratio of the expected severity in the layer $E_Y[S_{Y,p}(A, L)]$ and the expected severity for the underlying policy $E_Y[S_{Y,p}]$. Using this percent of loss in layer as a proxy for the exposure in each layer, we then multiply Table 8 by our 2000 limit profile (on-level) and the assumed 60% expected loss ratio to estimate historic exposure by layer. This yields Table 9.

Then we calculate the projected exposure by multiplying the results from Table 8 by the ELR of 60% and by the 2005 Limit Profile from Table 7. The results are shown in Table 10.

Taking the relativity between Table 9 and Table 10 yields the exposure adjustment to be applied to the trended ultimate losses in the layer for experience year 2000. These results are shown in Table 11. Note that the exposure in 2005 for the \$5MM *xs* \$0 layer is 1.09 times the exposure in 2000 for the \$5MM *xs* \$0 layer. As this layer represents total limits (there are no policies in this example greater than \$5 million), this is our total limits change in exposure. It is simply the ratio of on-level total limits premium in 2005 relative to that in 2000, or \$25,875,000 divided by \$23,737,500. This is the total limits change in exposure yielded by standard on-leveling procedures.

It is interesting to note how this 9% increase in total limits exposure differs by layer. Exposure in the \$250,000 *xs* \$0 layer is actually down slightly, while exposure in the \$4MM *xs* \$1MM layer has doubled. Table 12 shows the expected frequency by policy for underwriting years 2000 and 2005 respectively. These results were calculated by taking the total expected loss cost by policy type and dividing it by the expected severity for such policy. Table 13 then shows the expected frequency in the layer. These results are calculated by multiplying the total expected frequency by the probability of a loss penetrating the layer given that the underlying policy exposes the layer.

The severity adjustment is calculated as the ratio of the layer adjustment and the frequency adjustment. Table 14 shows a summary of the exposure adjustment by layer of loss. As we can see from Table 14 there is a significant difference in exposure changes across layers. Standard experience rating method would assume that the total limit exposure change apply consistently across layers. In our example above the total limit exposure changes is 1.09, however we can see that although the total exposure has increased by 9% the exposure in the layer \$4MM *xs* \$1MM has doubled. Hence, if we had adjusted the experience in the \$4MM *xs* \$1MM layer by a factor of 1.09 we would have significantly understated the experience rate. It is also interesting to note from Table 14 the contribution to the exposure change of frequency and severity separately. As we can see most of the exposure change for total limits is due to an increase in severity since frequency has reduced slightly.

4 CONCLUSIONS

We have presented in this paper an improved method for experience rating excess of loss treaties based on the mathematics of exposure rating. The methodology presented in Section 3.2 to estimate a trend factor by layer based on the unlimited trend factor presents several advantages over the standard per claim trending procedures widely used in practice. First, the aggregate trend method is useful when individual claim information does not contain policy details such as limit and deductible. In this case we simply calculate the excess trend factor and apply it to nominal losses in the layer. Second, when reinsurers receive individual losses greater than a certain amount the aggregate trend method is useful as it helps us overcome the problem of having to eliminate all data that falls below a selected threshold. Hence, by using an aggregate trend instead of the per claim trending method we can experience rate lower layers that would have been otherwise impossible to do. Third, the aggregate excess trend methodology brings consistency between unlimited trend factors and trend factors by layer of loss. Finally, the aggregate trend method helps us to quantify the impact of frequency and severity trend in excess layers which is useful for measuring the impact of increased frequency in excess layers even when ground up frequency is stable. The methodology presented in Section 3.3 helps us to quantify the exposure adjustment by layer of loss due to shifts in limit profile for the ceding company. Traditional experience rating methods quantify the exposure change for the underlying book of business through changes in on-level premium or other measure of exposure. However, this method assumes that the total limits exposure change is consistent across layers. Through the example presented in Section 3.3.2 we have seen that using the total limit exposure change across layers can result in a significant misestimation of the expected loss cost. The exposure adjustment method is based on the mathematics of exposure rating, which are widely used in reinsurance pricing, and data that is typically available in a standard reinsurance submission, making the method relatively easy to implement in practice.

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Biographies of the Authors[3]

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Appendix: Numerical Results

Table 1: Limits Profile and Limited Expected Values

	Underwriting Year	2005	2000
Policy Limit (PL)	Premium (SP)	$E_Y[Y \wedge PL]$	$E_X[X \wedge PL]$
250,000	2,250,000	48,539	38,900
500,000	5,400,000	64,416	50,191
750,000	2,925,000	74,252	56,947
1,000,000	6,300,000	81,301	61,681
5,000,000	9,000,000	117,221	84,401
Total	25,875,000		

[3]The methods set forth in this paper are the opinion of the authors and they do not necessarily represent the views of the ACE Group of Insurance and Reinsurance companies or Carvill and any of its subsidiaries. The worked examples provided in this paper were derived from purely hypothetical assumptions and any similarity between these examples and any existing insurance company are coincidental only.

Table 2: Exposure Rating using 2005 severity distribution

	Layer of Loss				
Policy Limit	\$250k *xs* 0	\$250k *xs* \$250k	\$500k *xs* \$500k	\$4MM *xs* \$1MM	\$5MM *xs* 0
250,000	1,350,000	0	0	0	1,350,000
500,000	2,441,429	798,571	0	0	3,240,000
750,000	1,147,250	375,256	232,495	0	1,755,000
1,000,000	2,256,760	738,167	785,072	0	3,780,000
5,000,000	2,236,034	731,388	777,862	1,654,717	5,400,000
Total	9,431,473	2,643,382	1,795,428	1,654,717	15,525,000

Table 3: Trend Factor per Policy and Total Layer Trend

	Layer of Loss				
Policy Limit	\$250k *xs* 0	\$250k *xs* \$250k	\$500k *xs* \$500k	\$4MM *xs* \$1MM	\$5MM *xs* 0
250,000	1.248				1.248
500,000	1.248	1.406			1.283
750,000	1.248	1.406	1.456		1.304
1,000,000	1.248	1.406	1.470		1.318
5,000,000	1.248	1.406	1.470	1.581	1.389
Total	1.248	1.406	1.468	1.581	1.328
Annualized	4.53%	7.06%	7.98%	9.59%	5.83%

Table 4: Expected number of claims by policy and frequency trend by layer

		\$250k xs 0		\$250k xs \$250k		\$500k xs \$500k		\$4MM xs \$1MM		\$5MM xs 0	
Limit ('000)	$E[N_p]$	2000	2005	2000	2005	2000	2005	2000	2005	2000	2005
250	27.81	27.81	27.81							27.81	27.81
500	50.30	50.30	50.30	3.17	4.35					50.30	50.30
750	23.64	23.64	23.64	1.49	2.05	0.79	1.13			23.64	23.64
1,000	46.49	46.49	46.49	2.93	4.03	1.55	2.23			46.49	46.49
5,000	46.07	46.07	46.07	2.90	3.99	1.54	2.21	0.75	1.13	46.07	46.07
Total	194.31	194.31	194.31	10.49	14.42	3.88	5.57	0.75	1.13	194.31	194.31
Excess Frequency Trend		1.00		1.37		1.44		1.50		1.00	
Annualized Trend		0%		6.57%		7.50%		8.47%		0%	

Table 5: Summary of trend factors by layer of loss

	\$250k *xs* 0	\$250k *xs* \$250k	\$500k *xs* \$500k	\$4MM *xs* \$1MM	\$5MM *xs* 0
Layer Trend Factor	1.248	1.406	1.468	1.581	1.328
Annual Trend Factor	4.53%	7.06%	7.98%	9.59%	5.83%
Frequency Trend Factor	1.00	1.37	1.44	1.50	1.00
Annual Frequency Trend	0.00%	6.57%	7.50%	8.47%	0.00%
Severity Trend Factor	1.248	1.023	1.022	1.053	1.328
Annual Severity Trend	4.53%	0.46%	0.44%	1.04%	5.83%

Table 6: Historic Limits Profile

	Underwriting Year	
Limit	2000	2005
250,000	2,250,000	2,250,000
500,000	4,500,000	5,400,000
750,000	2,925,000	2,925,000
1,000,000	3,150,000	6,300,000
5,000,000	3,000,000	9,000,000
Total	15,825,000	25,875,000
OL Factor	1.50	1.00
OL Written Premium	23,737,500	25,875,000

Table 7: Historic On-Level Limits Profile

	Underwriting Year	
Limit	2000	2005
250,000	3,375,000	2,250,000
500,000	6,750,000	5,400,000
750,000	4,387,500	2,925,000
1,000,000	4,725,000	6,300,000
5,000,000	4,500,000	9,000,000
OL Written Premium	23,737,500	25,875,000

Table 8: Percentage of Loss in Layer using 2005 Severity Distribution

	Layer of Loss				
Limit	\$250k *xs* 0	\$250k *xs* \$250k	\$500k *xs* \$500k	\$4MM *xs* \$1MM	\$5MM *xs* 0
250,000	100.0%	0.0%	0.0%	0.0%	100.0%
500,000	75.4%	24.6%	0.0%	0.0%	100.0%
750,000	65.4%	21.4%	13.2%	0.0%	100.0%
1,000,000	59.7%	19.5%	20.8%	0.0%	100.0%
5,000,000	41.4%	13.5%	14.4%	30.6%	100.0%

Table 9: Expected Loss Cost by layer: 2005 Curve 2000 Limit Profile

	Layer of Loss				
Limit	\$250k *xs* 0	\$250k *xs* \$250k	\$500k *xs* \$500k	\$4MM *xs* \$1MM	\$5MM *xs* 0
250,000	2,025,000	0	0	0	2,025,000
500,000	3,051,788	998,212	0	0	4,050,000
750,000	1,720,876	562,883	348,741	0	2,632,500
1,000,000	1,692,572	553,625	588,803	0	2,835,000
5,000,000	1,118,015	365,693	388,929	827,363	2,700,000
Total	9,608,250	2,480,413	1,326,474	827,363	14,242,500

Table 10: Expected Loss Cost by layer: 2005 Curve 2005 Limit Profile

	Layer of Loss				
Limit	\$250k *xs* 0	\$250k *xs* \$250k	\$500k *xs* \$500k	\$4MM *xs* \$1MM	\$5MM *xs* 0
250,000	1,350,000	0	0	0	1,350,000
500,000	2,441,430	798,570	0	0	3,240,000
750,000	1,147,250	375,255	232,494	0	1,755,000
1,000,000	2,256,763	738,167	785,071	0	3,780,000
5,000,000	2,236,030	731,385	777,858	1,654,726	5,400,000
Total	9,431,473	2,643,377	1,795,423	1,654,726	15,525,000

Table 11: Exposure Adjustment by Layer of Loss

	Layer of Loss				
Limit	\$250k *xs* 0	\$250k *xs* \$250k	\$500k *xs* \$500k	\$4MM *xs* \$1MM	\$5MM *xs* 0
250,000	0.667				0.667
500,000	0.800	0.800			0.800
750,000	0.667	0.667	0.667		0.667
1,000,000	1.333	1.333	1.333		1.333
5,000,000	2.000	2.000	2.000	2.000	2.000
Total	0.982	1.066	1.354	2.000	1.090

Table 12: Expected Frequency by Policy Limit

Limit	$E[M_p]$	$E[N_p]$
250,000	27.81	41.72
500,000	50.30	62.87
750,000	23.64	35.45
1,000,000	46.49	34.87
5,000,000	46.07	23.03
Total	194.31	197.95

Table 13: Frequency Exposure Adjustment by Layer of Loss

	\$250k xs 0		\$250k xs \$250k		\$500k xs \$500k		\$4MM xs \$1MM		\$5MM xs 0	
$(1 - F_Y(A))$	1.000		0.087		0.048		0.025		1.000	
Limit ('000)	2000	2005	2000	2005	2000	2005	2000	2005	2000	2005
250	41.72	27.81							41.72	27.81
500	62.87	50.30	5.44	4.35					62.87	50.30
750	35.45	23.64	3.07	2.05	1.70	1.13			35.45	23.64
1,000	34.87	46.49	3.02	4.03	1.67	2.23			34.87	46.49
5,000	23.03	46.07	1.99	3.99	1.11	2.21	0.57	1.13	23.03	46.07
Total	197.95	194.31	13.53	14.42	4.48	5.57	0.57	1.13	197.95	194.31
Freq. Adj.	0.98		1.07		1.24		2.00		0.98	

Table 14: Summary of Exposure Adjustment by Layer of Loss

	\$250k *xs* 0	\$250k *xs* \$250k	\$500k *xs* \$500k	\$4MM *xs* \$1MM	\$5MM *xs* 0
Layer	0.982	1.066	1.354	2.000	1.090
Frequency	0.982	1.066	1.245	2.000	0.982
Severity	1.000	1.000	1.087	1.000	1.110

On Predictive Modeling for Claim Severity

Glenn Meyers, FCAS, MAAA, Ph.D.

Abstract:

Reinsurers typically face two problems when they want to use insurer claim severity experience to experience rate their liability excess of loss treaties. First, the claim severity data has insufficient volume to make credible projections of excess layer costs. Second, the data they do receive is not fully developed. Most claims that pierce the excess layers can take at least a few years to settle. This paper sets forth a methodology for dealing with these problems. The paper starts with some introductory examples that illustrate how to quantify the inherent uncertainty in fitting claim severity distributions. Then the paper illustrates a Bayesian methodology to estimate the expected cost of excess layers, and shows how to quantify the uncertainty in these estimates. The Bayesian "prior models" are not derived from purely subjective considerations. Instead they are derived after examining the claim severity data of several insurers. Each "prior model" contains claim severity distributions of immature data that are used to calculate the posterior probabilities with comparable immature data submitted by an insurer. Each "prior model" also contains a fully developed claim severity distribution. The estimate of the cost of an excess layer is the average of the fully developed excess layer costs weighted by the posterior probabilities calculated with the immature data submitted by the insurer.

Keywords: Loss Distributions, Bayesian Estimation, Excess of Loss Reinsurance

1. INTRODUCTION

One of the many jobs an actuary is asked to do is to predict future claim costs in high layers. It is often the case that there are few claims from past experience. When this is the case, an actuary must resort to either one or both of the following.

- Try to discern a pattern in the claims that lie below the layer, and use this pattern to project claim costs in the layer. This is usually done by fitting a parametric probability distribution to these other claims.
- Examine claims from other insurers, or from an industry source, in the hope that these claims are similar to the claims you are trying to project. Often an actuary will make use of a probability distribution that has been fit to these claims.

There are difficulties with each of these approaches. The first approach can have credibility problems if there are not enough claims to get a reliable estimate of the parameters of the parametric probability distribution. And identifying the best distribution can also be a problem. The second approach can have relevance problems if the population that underlies the "industry" is different than the population that the actuary is addressing.

Liability insurance presents yet another problem in fitting claim severity distributions. It can take a considerable amount of time to settle the claims. A changing legal environment will force the actuary to compromise between completeness and relevance of the claim information.

This paper will address each of these problems. Here is a summary of what is to follow.

- I will begin with a description of how to construct a classical (non-Bayesian) confidence interval of parameters of a claim severity distribution using the likelihood ratio test.
- Next I will show how to use Bayes' Theorem to calculate posterior probabilities for a series of selected claim severity distributions. The "selected claim severity distributions" can come from different parameterizations of a selected model, such as a Pareto or a lognormal model. Or the "selected claim severity distributions" can come from different models. This allows us to incorporate what we informally call "model uncertainty" as well as "parameter uncertainty" into our estimation procedure.
- It is generally the case that particular claim severity models, or particular parameterizations of these models, are not of direct interest. What are of interest are functions of the models and their possible parameterizations. An example of such a function would be the expected cost of a particular layer of loss. I will show how to quantify uncertainty in the expected cost of a layer of loss in terms of the posterior probabilities of each of the models and their multiple parameterizations.
- It is possible to associate claim severity distributions developed to their ultimate value, with the immature claim severity distributions representing the data that is currently available. By fitting (i.e., determining the posterior probabilities) the immature data to the immature distributions and then applying the posterior weights to the associated fully developed distributions, it is possible to get estimates of the expected losses for a layer of loss. Furthermore, one can quantify the uncertainty in this estimate.

A major theme of this paper will be the importance of the likelihood function. Loosely stated, the likelihood function is the probability of observing a given set of data as a function of a parametric probability distribution. The likelihood function will play a key role in both the classical and Bayesian methodologies described below.

2. CONFIDENCE REGIONS FOR PARAMETERS

I will begin the discussion of confidence regions with a description of hypothesis testing using the likelihood ratio test.

Let:

- $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ be a parameter vector for a given parametric probability distribution;
- $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a set of observed losses;
- $\mathbf{p}^{\mathrm{ML}}$ be the maximum likelihood estimate of the parameter vector given the data $\mathbf{x}$;
- $\mathbf{p}^{\mathrm{T}}$ be the "true" parameter vector underlying the population of interest; and
- $\mathbf{p}^*$ be a parameter vector for a proposed model for a claim severity distribution.

Denote likelihood of a parameter vector, $\mathbf{p}$, given the data, x, by $L(\mathbf{x};\mathbf{p})$.

We want to test the null hypothesis:

$$\mathrm{H}_0\text{: } \mathbf{p}^* = \mathbf{p}^{\mathrm{T}};$$

against the alternative hypothesis:

$$\mathrm{H}_1\text{: } \mathbf{p}^* \neq \mathbf{p}^{\mathrm{T}}.$$

Theorem

If H_0 is true, then the statistic:

$$\ln(LR) \equiv 2 \cdot \left(L\left(\mathbf{x};\mathbf{p}^{\mathrm{ML}}\right) - L\left(\mathbf{x};\mathbf{p}^*\right)\right)$$

has a χ^2 distribution with k degrees of freedom.

This theorem is given in Section 13.4.4 of Klugman, Panjer and Willmot (KPW) [2004][1].

Informally, this theorem says that one should accept (or fail to reject) the hypothesis that $\mathbf{p}^*$ is the parameter vector for the population if the likelihood of $\mathbf{p}^*$ is sufficiently "close" to

[1] There are a number of technical conditions placed on the probability distribution for this result to hold. Most of the common distributions used by actuaries (such as Pareto, lognormal and gamma distributions) satisfy these conditions.

the maximum likelihood estimate, $\mathbf{p}^{ML}$, of the sample. More formally, "close" is defined by the above statistic and the critical values of the χ^2 distribution with k degrees of freedom.

To illustrate the likelihood ratio test I took a random sample of 1,000 claims from a Pareto distribution of the form

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha},$$

with $\alpha = 2$ and $\theta = 10{,}000$.

While it is not convenient to list all 1,000 claims in this random sample, here is a grouped summary of these claims.

Table 1

Range	Count
$x_i \leq 5{,}000$	562
$5{,}000 < x_i \leq 10{,}000$	181
$10{,}000 < x_i \leq 20{,}000$	134
$20{,}000 < x_i$	123

Here is the log-likelihood function for the grouped data.

$$l_G(\theta,\alpha) = 562 \cdot \ln\left(1 - \left(\frac{\theta}{\theta+5000}\right)^{\alpha}\right) + 181 \cdot \ln\left(\left(\frac{\theta}{\theta+5000}\right)^{\alpha} - \left(\frac{\theta}{\theta+10000}\right)^{\alpha}\right)$$
$$+134 \cdot \ln\left(\left(\frac{\theta}{\theta+10000}\right)^{\alpha} - \left(\frac{\theta}{\theta+20000}\right)^{\alpha}\right) + 123 \cdot \ln\left(\left(\frac{\theta}{\theta+20000}\right)^{\alpha}\right)$$

Using a general purpose maximizing tool, Excel Solver, I found the maximum likelihood estimate of the Pareto parameters for the grouped data to be equal to $\left(\theta_G^{ML}, \alpha_G^{ML}\right) = (7447.8, 1.6041)$.

Here is the log-likelihood function for the detailed data.

$$l_D(\theta,\alpha) = 1000 \cdot \left(\ln(\alpha) + \alpha \cdot \ln(\theta)\right) - (\alpha+1)\sum_{i=1}^{1000} \ln(x_i + \theta).$$

Using Excel Solver, I found the maximum likelihood estimate of the Pareto parameters for the detailed data to be equal to $\left(\theta_D^{ML}, \alpha_D^{ML}\right) = (9626.8, 1.8079)$.

Note that the parameter estimates in each case are different from the true parameters that I used to generate the simulated data. If I were to generate another simulation, I would get a different parameter estimate. Repeated simulations will yield samples from a bivariate distribution of parameter estimates. There is a formula that describes the bivariate distribution of the maximum likelihood estimates in terms of the parameters that are used to generate the original distribution.[2]

Our problem is different. In practice, we don't know the parameters of the underlying distribution.[3] A question to ask is the following: What are acceptable parameters for the distribution given the data we have? To answer this question one can invoke the likelihood ratio test by defining a *p*% confidence region of the parameters as the set of all parameters that pass the likelihood ratio test at the *p*% level.

Figures 1 and 2 are plots of the parameters that pass the likelihood ratio test at the 5% level for the grouped and detailed data, respectively. These plots were generated by calculating the likelihood ratio statistic for a grid of (θ,α) points and plotting them if the statistic was less than the 5% critical value, 5.99, of the χ^2 distribution with two degrees of freedom.

It is worth noting that the confidence region is wider for the grouped likelihood data than for the detailed likelihood data. This illustrates the additional information provided by the detailed data.

[2] The distribution of the maximum likelihood estimates has an asymptotic normal distribution with parameters given by the Fisher Information Matrix. See Section 12.3 of Klugman, Panjer and Willmot [2004].

[3] In practice, we don't even know the underlying distribution itself. I will get to that below.

Figure 1
Confidence Region
Grouped Likelihood

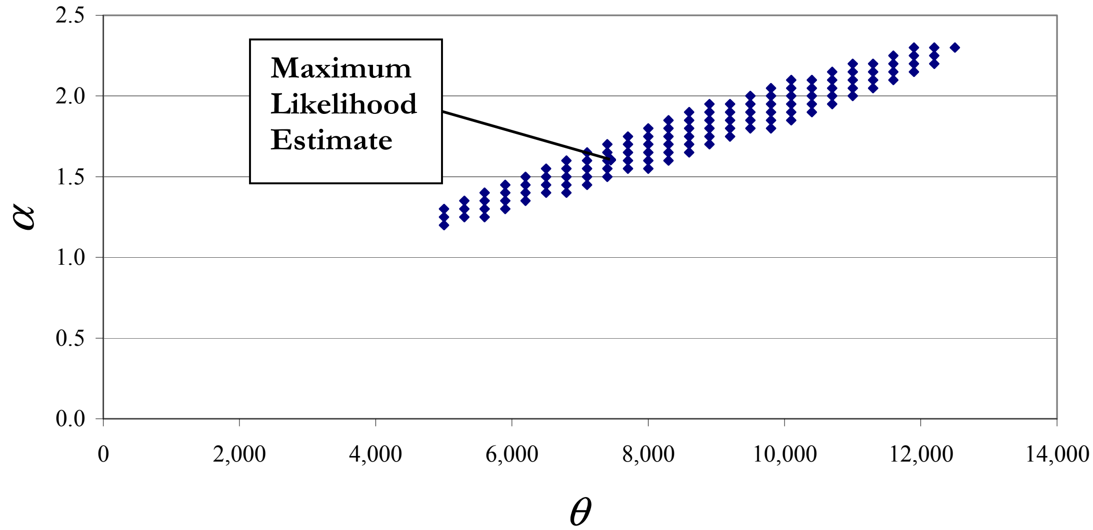
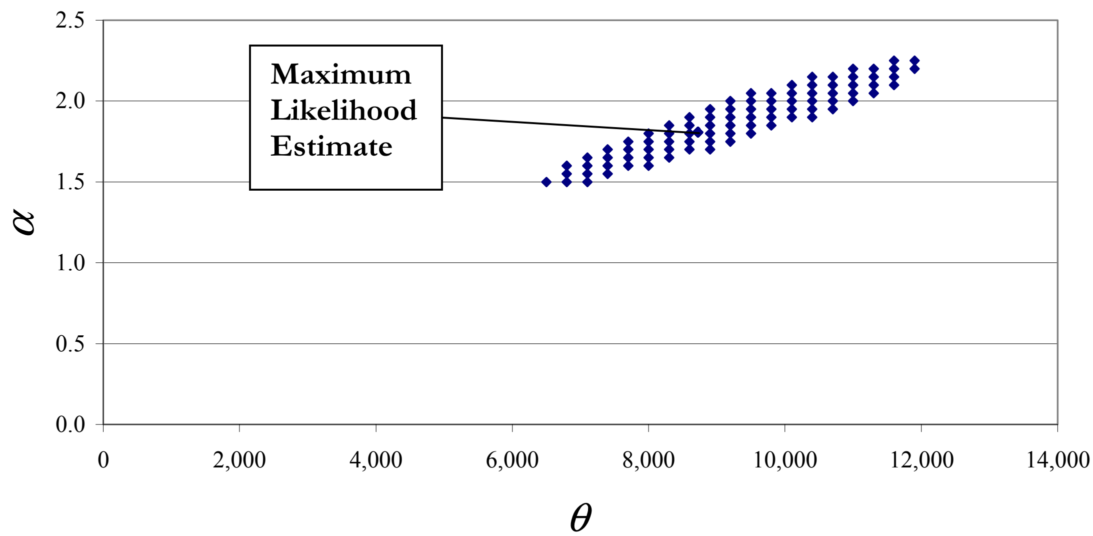


Figure 2
Confidence Region
Detailed Likelihood



This definition of a confidence region is somewhat unusual. The standard technique (described in KPW, Section 12.3) is to use the Fisher information matrix, substituting the maximum likelihood estimate of the parameters for the "true" parameters. However, the definition of confidence regions of the parameters used here has precedent. One of these is in the first edition of KPW[4].

[4] See Example 2.69 on page 131 of Klugman, Panjer and Willmot [1998]. I asked Professor Klugman why this example was not in the 2nd edition. He replied that it was an oversight, and made a note to put it back in the third edition.

3. THE COTOR CHALLENGE

Last year, the CAS Committee on the Theory of Risk (COTOR) issued a challenge. The committee published a list of 250 claims, and asked contestants to estimate the pure premium of a $5 million x $5 million layer. An additional requirement of the challenge was to put a 95% confidence interval around this estimate. A full description of the COTOR Challenge can be found on the CAS web site:

http://www.casact.org/cotor/round2.htm.

The "claims" were generated by a simulation from a transformed inverse gamma distribution, a fact that was not revealed until after all solutions were submitted.

The COTOR challenge has some of the elements that reinsurance actuaries face in their work. Most importantly:

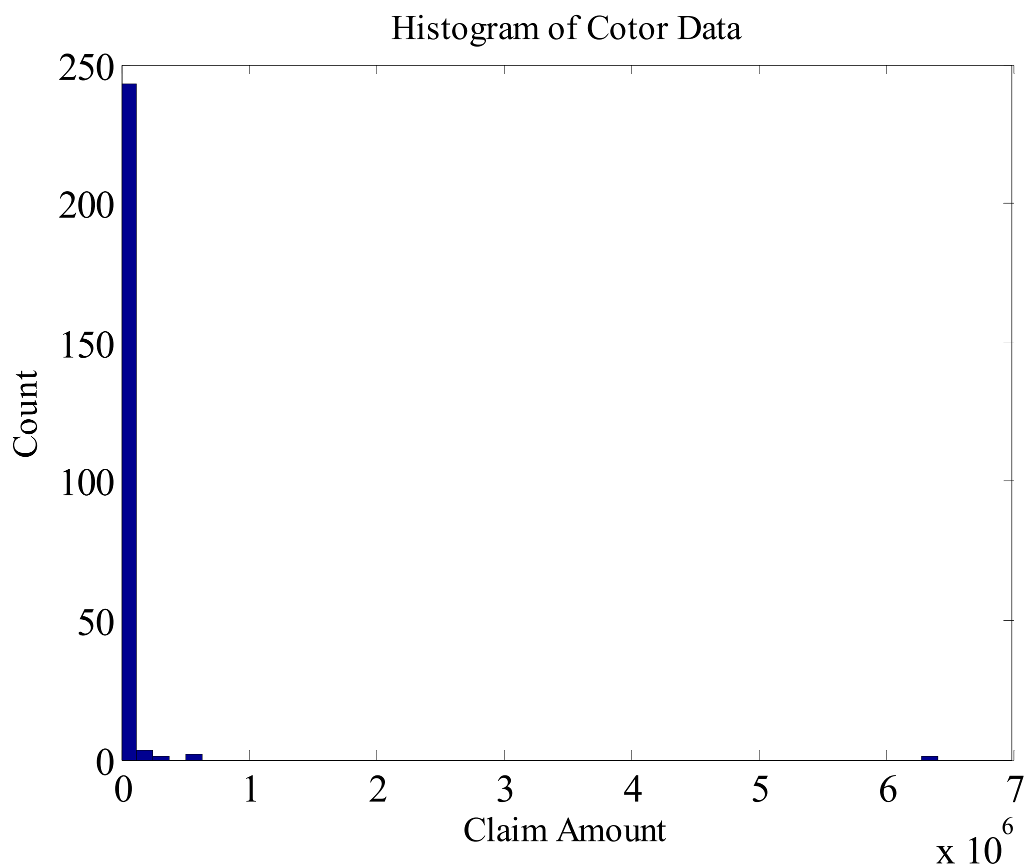
- The underlying loss distribution is unknown, and is very likely not one of the standard models that are in the typical actuarial toolbox.
- There are very few claims in the layer of interest. Actuaries typically try to project the frequency of claims in a high layer by looking at claims in a lower layer.

My solution, which is posted on the COTOR web site, provides an example that I believe to be of educational value as we move toward the ultimate goals stated in the introduction. I will describe it in some detail here.

The solution makes use on a software package called MATLAB. The software has tools for plotting histograms, calculating maximum likelihood estimates, and supporting statistics.

The first step one should almost always take when fitting a distribution to data is to plot a histogram.

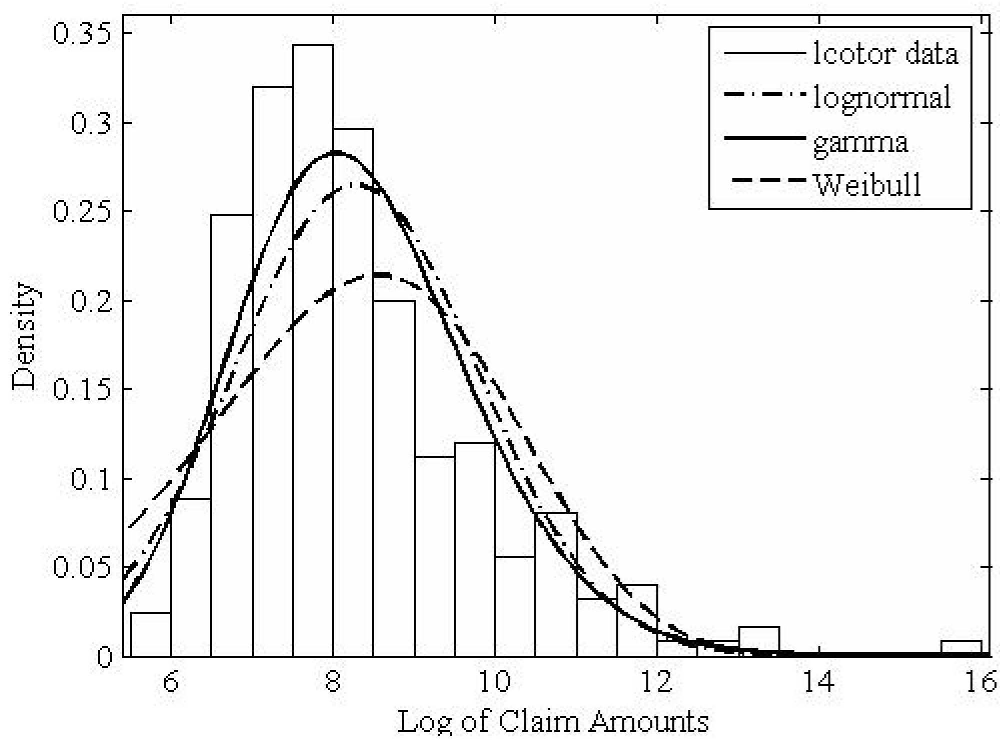
Figure 3



Note that there is only one claim in the $5 million x $5 million layer that contestants were asked to predict. The next highest claim was about $600,000. I did not even attempt to fit a distribution to this data.

The next step I took was to take the log of that data. The histogram looked promising so I tried to fit a couple of different distributions by maximum likelihood.

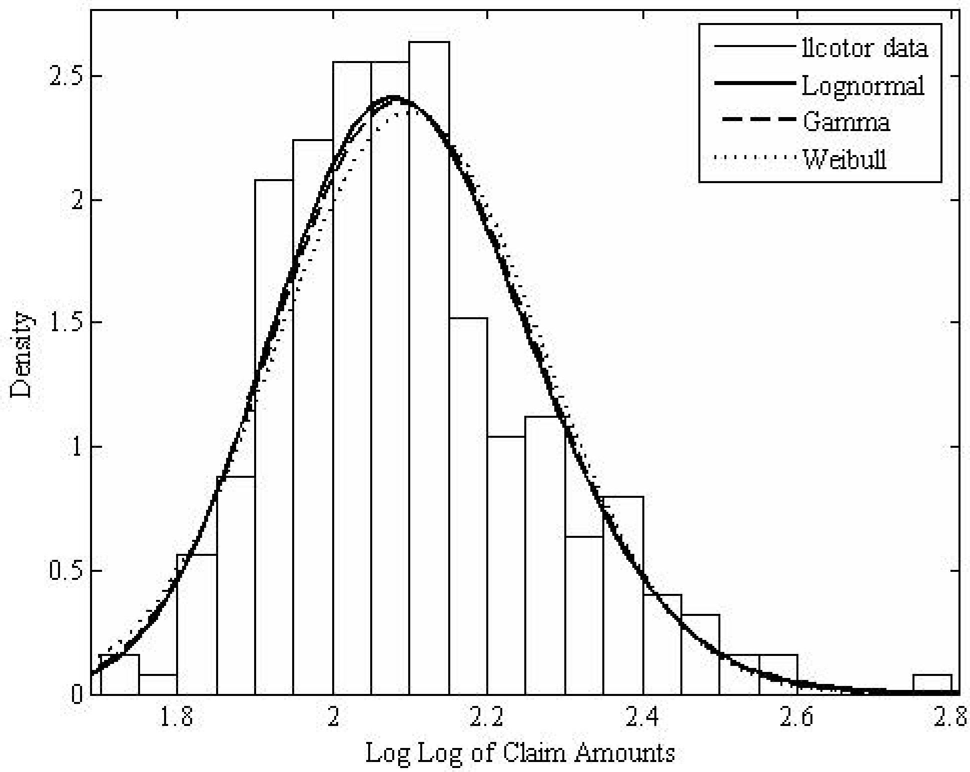
Figure 4



None of the selected distributions looked particularly good, with the Weibull providing the worst fit. The distribution was still skewed to the right.

In an attempt to reduce the skewness, I plotted a histogram of the double log of the data and fit some distributions to the double logged data by maximum likelihood.

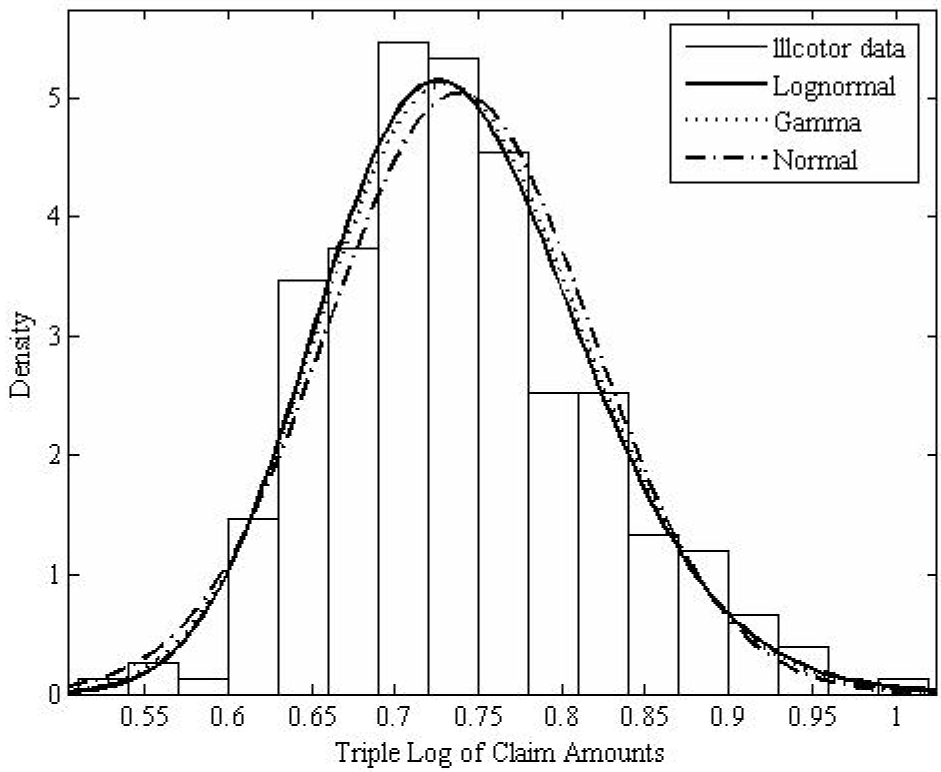
Figure 5



Here the fit of the three distributions is closer, but the double log of the data still looks more skewed than the distributions I tried.

Continuing the above logic, I took the triple log of the data and tried some more maximum likelihood fits.

Figure 6



At this stage, the maximum likelihood fits began to look reasonable. I examined these fits in more detail.

Here are some fitting statistics for the three distributions. Some observations:

- The lognormal distribution has the highest loglikelihood and the hence the best fit of the three.
- The loglikelihood of the gamma distribution is reasonably close to that of the lognormal distribution. The loglikelihood of the normal distribution is a bit lower, but not totally out of the running.
- Looking at Figure 7 below, we see that the maximum likelihood fit of all three distributions are nicely within the confidence bounds for lognormal fit.
- It is possible to go to a quadruple log transform since the triple logs of the claim amounts are still positive. But that is as far as we can go, since some triple logs are less than one. I stopped at the triple log transform since the lognormal is equivalent to a normal with the quadruple log transform.

Table 2

Distribution:	Lognormal	
Log likelihood:	283.496	
Mean:	0.73835	
Variance:	0.00619	
Parameter	Estimate	Std. Err.
μ	-0.30898	0.00672
σ	0.10625	0.00477
Estimated covariance of parameter estimates:		
	μ	σ
μ	4.52E-05	1.31E-19
σ	1.31E-19	2.27E-05

Distribution:	Gamma	
Log likelihood:	282.621	
Mean:	0.73836	
Variance:	0.00615	
Parameter	Estimate	Std. Err.
a	88.6454	7.91382
b	0.00833	0.00075
Estimated covariance of parameter estimates:		
	a	*b*
a	62.6286	-0.00588
b	-0.00588	5.56E-07

Distribution:	Normal	
Log likelihood:	279.461	
Mean:	0.738355	
Variance:	0.006285	
Parameter	Estimate	Std. Err.
μ	0.738355	0.005014
σ	0.079279	0.003556
Estimated covariance of parameter estimates:		
	μ	σ
μ	2.51E-05	-1.14E-19
σ	-1.14E-19	1.26E-05

Figure 7

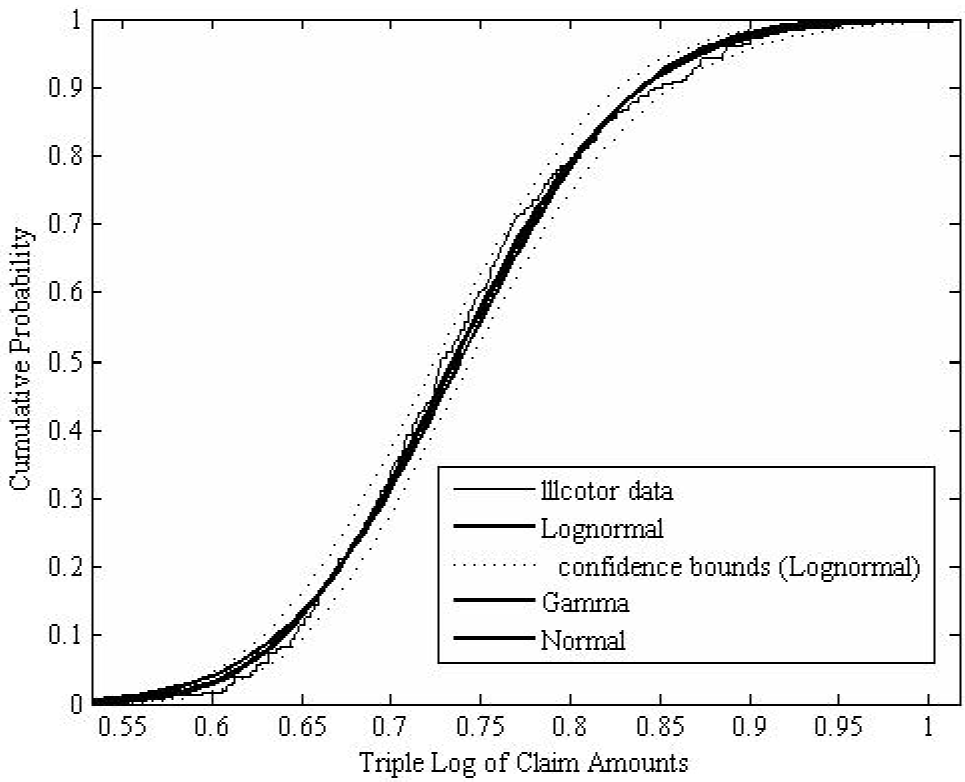


Figure 7 is a plot of the cumulative distribution functions of each of the fits and the data. The dotted lines give upper and lower confidence bounds for the best-fitting lognormal distribution. These confidence bounds contain the normal and gamma distributions and so we should consider all three of these distributions as potential models for the triple logs of the data.

Up to now, this analysis has been fairly classical. A typical classical analysis of the data would take the cumulative distribution function, $F(x)$, of the best-fitting model, (in this case the quadruple lognormal with the parameters in Table 2) and integrate the following formula to estimate the pure premium of the $5 million x $5 million layer.

$$\text{Layer Pure Premium}^{5} = \int_{5,000,000}^{10,000,000} \left(1 - F(x)\right) dx . \tag{1}$$

Such an estimate of the layer pure premium reflects the uncertainty of the loss given the model and the parameters of the model. It does not reflect the uncertainty of the model and the uncertainty in the parameters given the model.

If we are to reflect these additional uncertainties, we need to get the probability of the potential models and parameters. The only information we have to get these probabilities is the data. Now we have the ability to calculate the likelihood function (i.e., the probability of the data) for any given model and parameter set. To carry out this program, I made use of Bayes' Theorem to calculate the probability of each model and parameter set given the data.

Here is an outline of the methodology underlying my solution.

- I began by hypothesizing a series of "models" for the data. I interpret the term "model" broadly to include choices of the parametric form of the '*model*' (in the narrow sense; e.g., lognormal or gamma) as well as a choice of parameters for each '*model*.' I am intentionally blurring the distinction between parameter uncertainty and '*model*' uncertainty.
- For each model, I calculated the likelihood (or probability) of the data given each model. Using Bayes' theorem, the posterior probability of each model, given the data, is calculated by the following formula.

$$\text{Posterior}\left(\text{model}|\text{data}\right) \propto \text{Likelihood}\left(\text{data}|\text{model}\right) \cdot \text{Prior}\left(\text{model}\right).$$

- For each model, I calculated the cost of the $5 million x $5 million layer using Equation 1 above. I then calculated various statistics of the posterior distribution of the layer costs using posterior probabilities. For example:

[5] One should distinguish between the layer pure premium and the layer average severity. The layer average severity is the average severity given that the claim has pierced the layer. The layer pure premium is equal to the layer average severity times the probability of piercing the layer.

- The posterior expected cost of the layer was the posterior probability-weighted average of the layer cost for each model. Calculations such as this led to the mean and standard deviation in my solution.
- The posterior percentile of a selected layer cost is the sum of the posterior probabilities of all the models for which the layer cost is less than the selected layer cost. Calculations such as this led to the median and confidence interval for my solution.

Now let's look at the details.

The above analysis identified three potential '*models*' for the data with the triple log transform — the lognormal, the gamma and the normal. The fitting statistics give an indication of the range of possible parameters for each '*model*.'

Here are the steps in my calculations.

1. For each '*model*,' I calculated the confidence interval at the 0.1% level for both parameters.
2. I divided the confidence interval into 50 intervals and created a 51 by 51 grid of possible parameters for each '*model*.' The three '*models*' along with the 2,601 parameters yielded 7,803 "models" from which to do the Bayesian calculations.
3. I calculated the loglikelihood of each model. I assumed that the prior probabilities for each model were equal. I then exponentiated the posterior loglikelihoods and normalized them so that the posterior probabilities sum to one.
4. I then calculated the pure premiums for each model using Equation 1 and MATLAB's numerical quadrature function.
5. Finally I transferred the MATLAB arrays into Excel, sorted the models in increasing order of the pure premiums, and calculated the statistics reported in the results below.

The MATLAB code for executing the first four steps along with the spreadsheet for Step 5 can be downloaded from the COTOR web site.

I should point out that the '*model*' uncertainty did not have a significant effect on the final answer. The lognormal models got 95.33% of the posterior probability, the gamma models got 2.98% of the posterior probability and the normal models got the remaining 1.69%.

Here are the results.

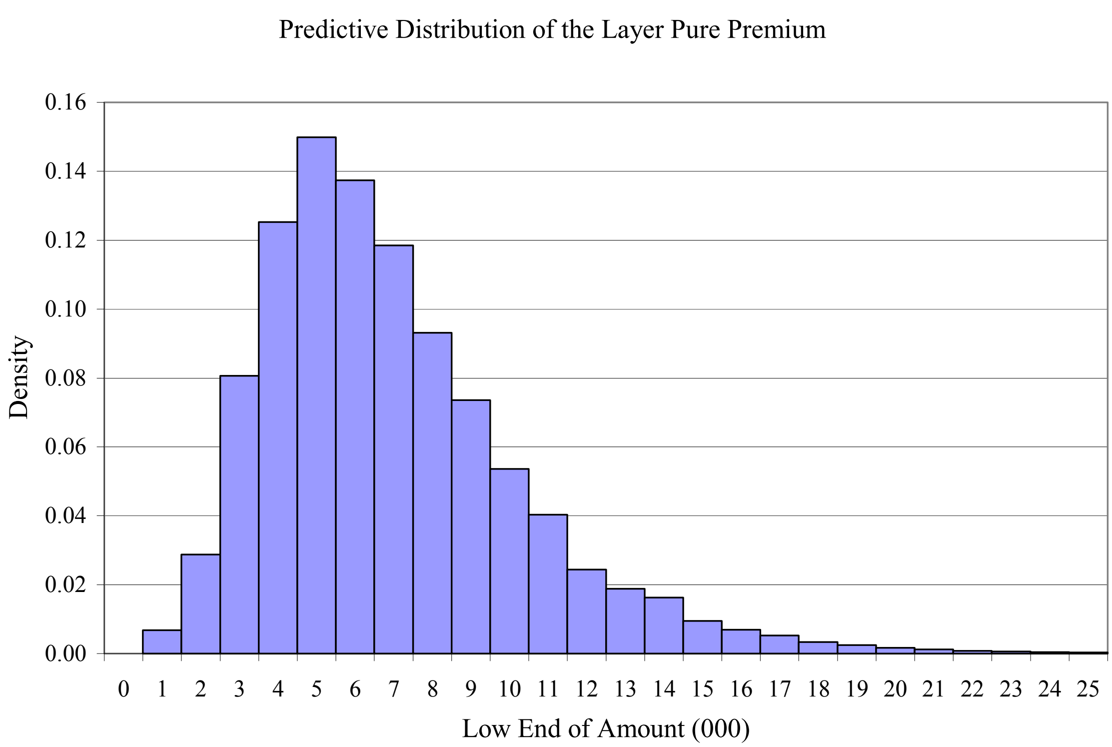
Table 3

Predictive Statistics for the Layer Pure Premium

Mean	6,430
Standard Deviation	3,370
Median	5,780
Range	
Low at 2.5%	1,760
High at 97.5%	14,710

Here is a histogram for the predictive distribution.

Figure 8



Using Bayes' Theorem in the solution above is similar to the likelihood ratio approach described in Section 2 in that both approaches use the likelihood function to identify potential models. The likelihood ratio test only provides you with a "yes/no" decision. And this "yes/no" decision is only correct if the underlying '*model*' is correct. But if you are comfortable with assigning prior probabilities to models, Bayes' Theorem allows you to use the likelihood function of the data associated with each model to calculate posterior probabilities for each model. And with probabilities assigned to each model, you can calculate any desired statistic of a function (e.g., layer pure premium) of the potential models. And the Bayesian approach can deal with '*model*' uncertainty.

Using Bayes' theorem to fit claim severity distributions is not new to the CAS literature. Meyers [1994], Klugman [1994], and Kreps [1997] have papers on this subject.

4. AN EXAMPLE BASED ON INSURANCE DATA

I believe the Bayesian methodology underlying the COTOR challenge is potentially useful for predicting pure premiums for high layers of insurance, but the methodology is far from complete. In this section, I will give an example that uses this Bayesian methodology that also addresses two of the more serious shortcomings.

1. The models that make up the prior information need careful consideration. In my solution to the COTOR Challenge, I developed the prior information using preliminary fits to the data and the standard errors of the parameter estimates. A true Bayesian would think hard and develop models that they believe are plausible in the absence of any data.
2. Liability claims can take a long time to settle. We are often given the task of predicting the ultimate claim severity distribution given an incomplete sample of claims. We do not know the ultimate values for many of the claims.

The fact that reinsurers go through great effort to examine the excess claims experience of their prospective contracts indicates that they believe that there are significant differences in the excess loss potential between insurers. Otherwise all reinsurance contracts would be priced using claim severity distributions based on industry aggregate experience such as those available from my employer, Insurance Services Office, Inc. (ISO).

To test this belief, I asked our (ISO's) increased limits ratemaking division to extract the empirical claim severity distributions for a liability coverage by individual insurer[6]. We the then fit mixed exponential distributions separately to 20 large insurers[7]. Each model had 10 parameters. Thus I think it is more appropriate to think of the "fitting" as "smoothing," and I do not expect each insurer's result to be necessarily predictive of future results.

The mixed exponential models were fit separately by settlement lag.

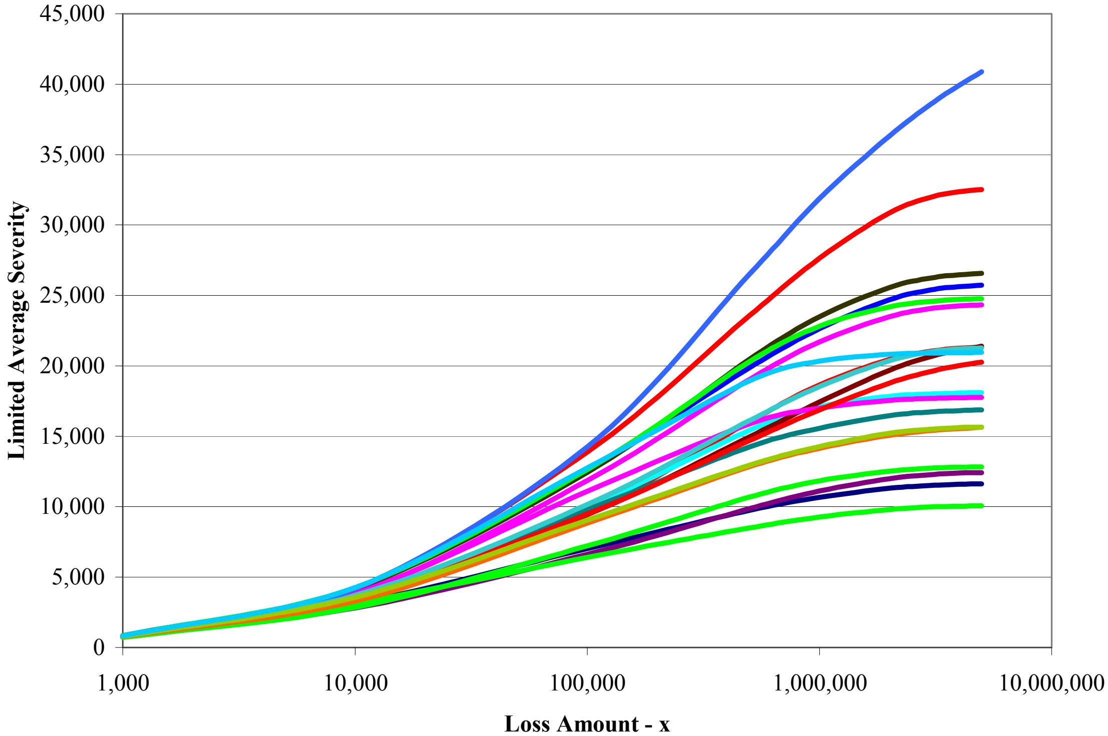
The limited average severity, $E[X^\wedge x]$, is the average severity of claims subject to a limit of x. Mathematically:

$$E\left[X \wedge x\right] = \int_0^x \left(1 - F(u)\right) du \; .$$

Figure 9 gives the ultimate limited average severity curves, based on the fitted mixed exponential distributions for each of the 20 insurers.

[6] ISO's standard increased limits ratemaking procedure also includes data from excess and umbrella claims that are reported separately to ISO. These claims were not included in this study.
[7] See Keatinge [1999] for information and details of fitting the mixed exponential distribution.

Figure 9 – Initial Insurer Models



If you are looking at Figure 9 in color, it should be apparent that the relationship between limited average severities at low loss amounts and high loss amounts is by no means perfect, but there does seem to be a general trend. The lack of correlation can be due to a lack of a fundamental relationship between losses at low levels and high levels, or it could be due to a lack of credibility of the data (as realized through the smoothing procedure.)

If there is a fundamental relationship between low-level losses and high-level losses, it makes the job of estimating high layer losses more reliable since low-level claims are more numerous. Ultimately this is a judgment call, and it is one that reinsurance actuaries routinely make.

The examples below will consist of estimates of the pure premium for the $500,000 x $500,000 layer, and the $1 million x $1 million layer. Figures 10 and 11 below respectively show how probabilities of exceeding $5,000 and $100,000 track with the pure premium for the $500,000 x $500,000 layer. The correspondence appears to be stronger in the latter case.

At this point in the analysis, I decided to use only claims that are in excess of $100,000 to estimate the cost of these layers.

Following the methodology of the previous section, the next step is to hypothesize a series of models for the data. Each model should represent the probability distribution of claims over $100,000. In developing this series of models, a good place to start is with models that were fit to individual insurer data. After all, the object is to project future losses to individual insurers.

I first attempted to use the fits directly. But in spite of the general pattern of higher layer losses increasing with lower layer losses observed in Figure 11, the Bayesian methodology would assign posterior probabilities to models where this was not the case. Given the general trend observed in Figure 11 and my prior actuarial experience (otherwise known as preconceptions) I decided to smooth out the initial set of company models. The process was informal. Loosely speaking, I dropped company models that did not behave "correctly" and replaced them with mixtures of company models that did behave "correctly." I was not able to reduce the noise entirely. Before putting this plan into practice, the choice of priors needs to be addressed more fully. I welcome debate on my notion of "correct" models. One of the advantages of the Bayesian methodology is that if forces one to make the assumptions explicit for all to see and open them to debate.

Figures 12 and 13 give the limited average severity curves and the layer pure premiums for the final set of models. These should be compared with Figures 9 and 11, respectively.

Figure 10 – Initial Insurer Models

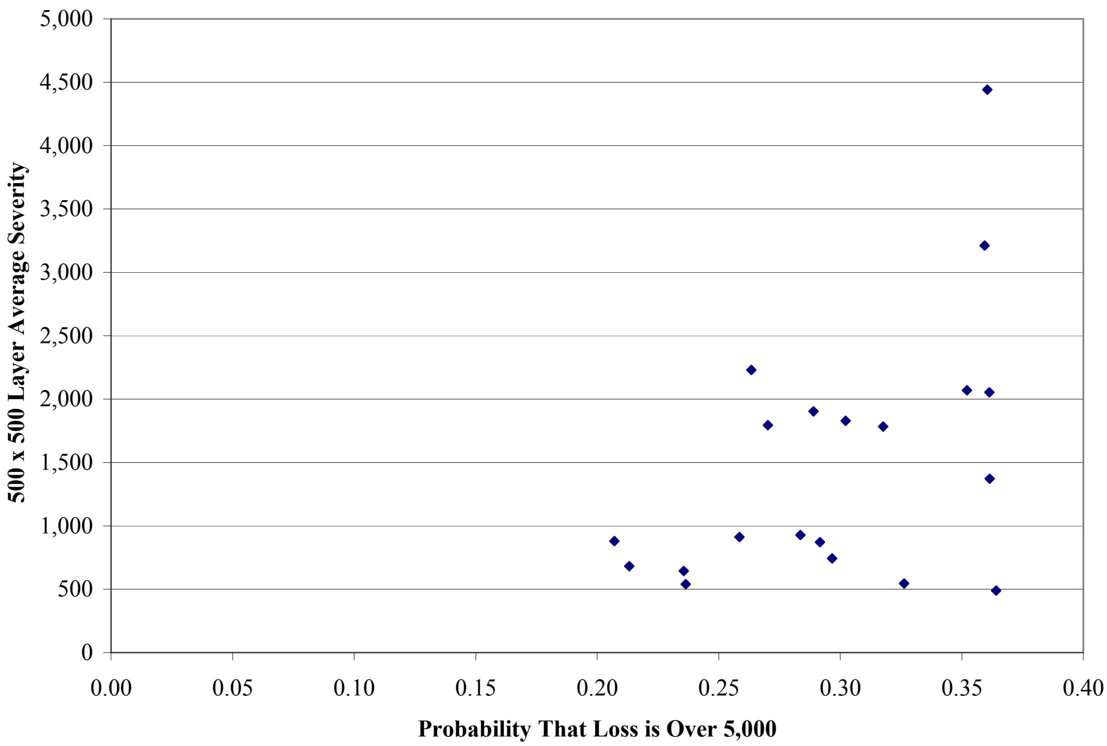


Figure 11 – Initial Insurer Models

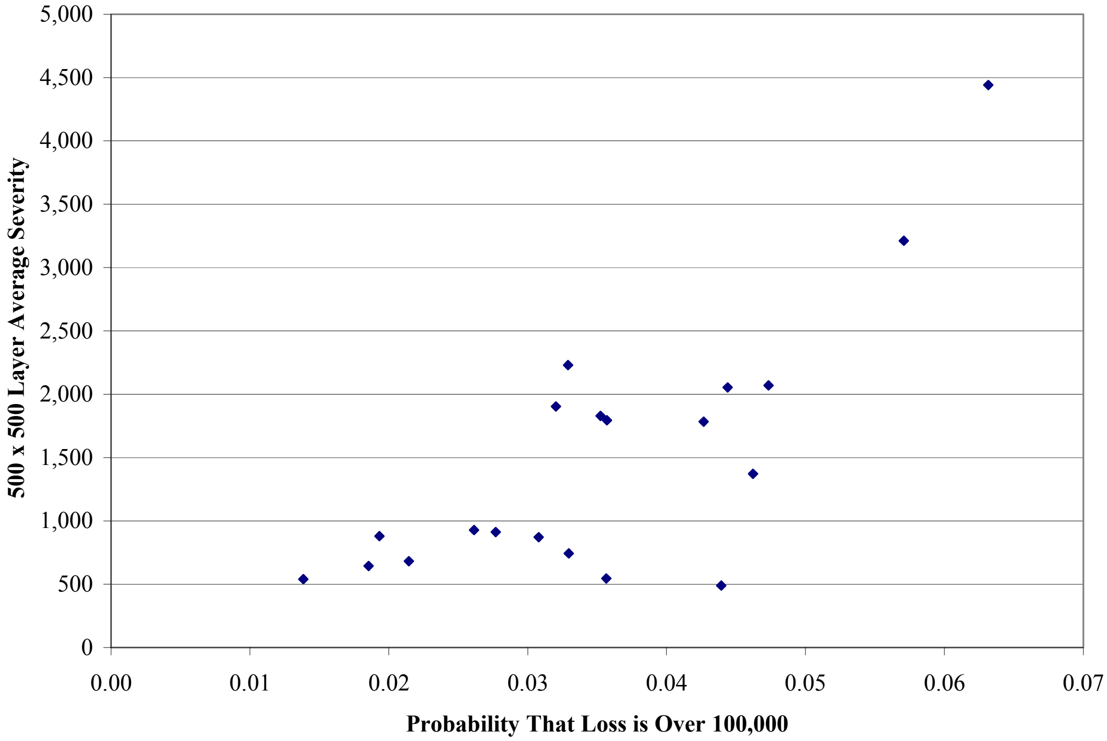


Figure 12 – Selected Insurer Models

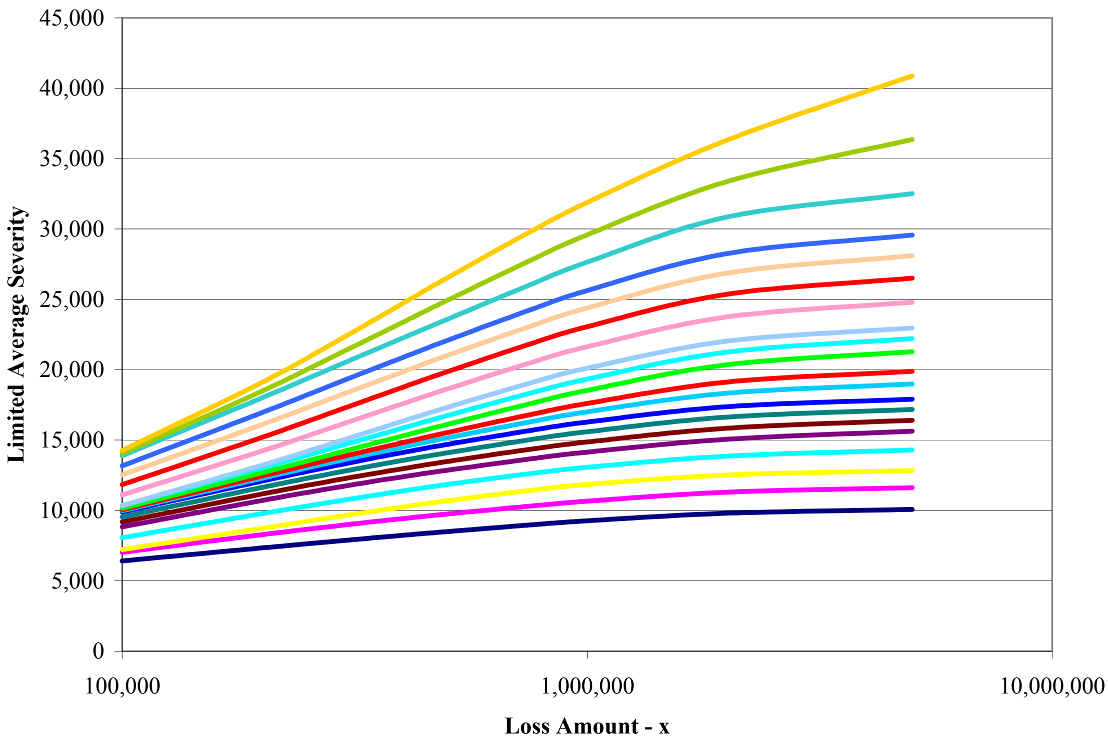
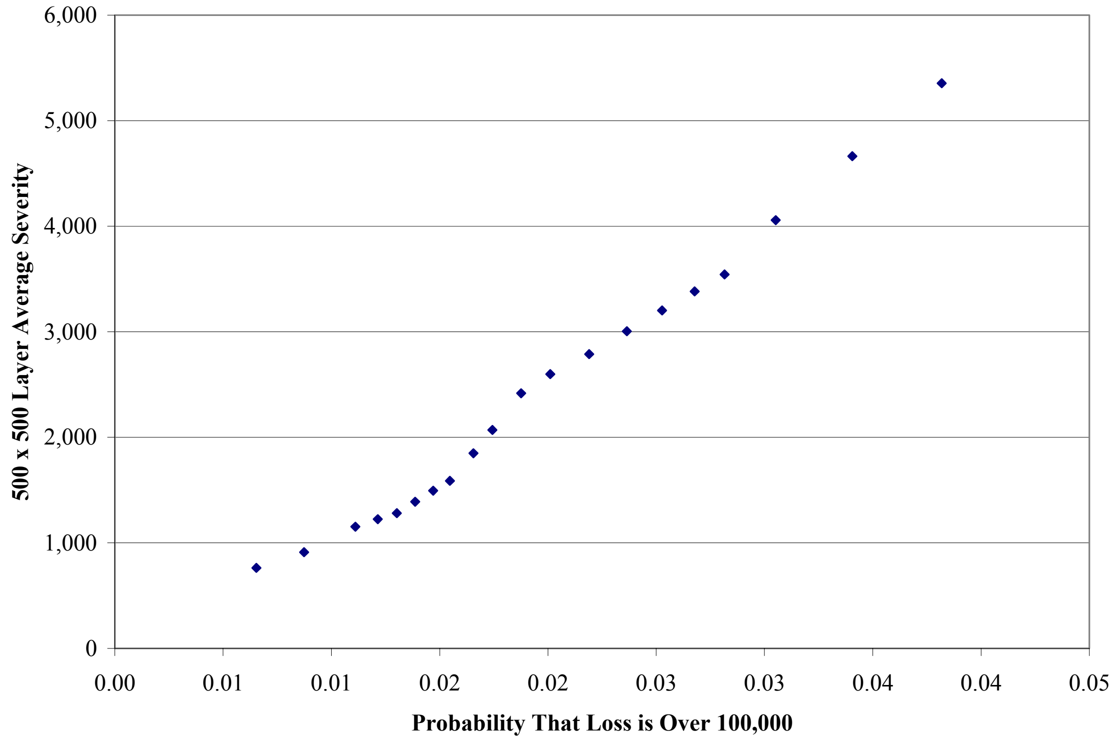


Figure 13 – Selected Insurer Models



The insurer models that underlie Figures 12 and 13 have been developed to ultimate. The insurer data that is presented for evaluation for an excess of loss treaty usually come from years that are too recent to contain all claims at their ultimate value. To make use of the data that reinsurers typically get, we need to have distributions of the data that are available for each insurer model.

Let's look at some examples. These examples will consist of three years of settled claim data. This data will be used to calculate the likelihood of each of 20 models. The prior probability for each model will be equal to 1/20. Then using the Bayesian methodology described in the previous section, I will calculate posterior layer pure premiums for the $500,000 x $500,000 layer, and for the $1 million x $1 million layer.

The 20 models used in this section's example will consist of the following distributions:

- The claim severity distribution for all claims settled within 1 year – Table 4.
- The claim severity distribution for all claims settled within 2 years – Table 5.

- The claim severity distribution for all claims settled within 3 years – Table 6.
- The ultimate claim severity distribution for all claims – Table 7.
- The ultimate limited average severity curve – Table 8.

As mentioned above, the models are a bit noisy, but I think they are good enough to illustrate the principles involved in this Bayesian methodology.

Table 4
Cumulative Probability for Lag 1

Claim Amount	Prior Model Number 1	2	3	4	5	6	7
100,000	0.998424	0.997549	0.999228	0.999234	0.999241	0.999097	0.998949
200,000	0.999522	0.999158	0.999779	0.999784	0.999789	0.999729	0.999668
300,000	0.999741	0.999563	0.999921	0.999923	0.999925	0.999888	0.999849
400,000	0.999837	0.999737	0.999971	0.999972	0.999973	0.999947	0.999921
500,000	0.999891	0.999827	0.999989	0.999990	0.999990	0.999972	0.999953
750,000	0.999954	0.999924	0.999999	0.999999	0.999999	0.999991	0.999982
1,000,000	0.999979	0.999962	1.000000	1.000000	1.000000	0.999996	0.999992
1,500,000	0.999996	0.999988	1.000000	1.000000	1.000000	0.999999	0.999998
2,000,000	0.999999	0.999995	1.000000	1.000000	1.000000	1.000000	0.999999

Claim Amount	Prior Model Number 8	9	10	11	12	13	14
100,000	0.998806	0.998725	0.998659	0.998562	0.998122	0.997094	0.996603
200,000	0.999609	0.999610	0.999611	0.999612	0.999422	0.998979	0.998818
300,000	0.999812	0.999817	0.999822	0.999828	0.999727	0.999492	0.999413
400,000	0.999895	0.999899	0.999903	0.999908	0.999848	0.999710	0.999663
500,000	0.999935	0.999938	0.999940	0.999943	0.999905	0.999817	0.999786
750,000	0.999974	0.999974	0.999974	0.999974	0.999960	0.999928	0.999911
1,000,000	0.999987	0.999987	0.999986	0.999985	0.999980	0.999968	0.999957
1,500,000	0.999996	0.999995	0.999994	0.999993	0.999993	0.999994	0.999987
2,000,000	0.999999	0.999998	0.999997	0.999996	0.999997	0.999999	0.999995

Claim Amount	Prior Model Number 15	16	17	18	19	20
100,000	0.996112	0.995621	0.995130	0.994197	0.995956	0.997715
200,000	0.998658	0.998498	0.998337	0.997573	0.998259	0.998944
300,000	0.999335	0.999256	0.999177	0.998684	0.999032	0.999381
400,000	0.999616	0.999570	0.999523	0.999183	0.999392	0.999601
500,000	0.999754	0.999722	0.999690	0.999443	0.999585	0.999728
750,000	0.999893	0.999875	0.999858	0.999730	0.999807	0.999884
1,000,000	0.999945	0.999933	0.999921	0.999848	0.999898	0.999948
1,500,000	0.999981	0.999975	0.999969	0.999939	0.999964	0.999989
2,000,000	0.999992	0.999988	0.999984	0.999969	0.999983	0.999998

Table 5
Cumulative Probability for Lags 1-2

Claim Amount	Prior Model Number 1	2	3	4	5	6	7
100,000	0.996249	0.994479	0.996598	0.995650	0.994980	0.994376	0.993730
200,000	0.998776	0.997967	0.998770	0.998387	0.998117	0.997858	0.997582
300,000	0.999329	0.998904	0.999393	0.999171	0.999012	0.998860	0.998698
400,000	0.999579	0.999315	0.999648	0.999499	0.999391	0.999291	0.999184
500,000	0.999717	0.999529	0.999768	0.999659	0.999579	0.999508	0.999432
750,000	0.999881	0.999770	0.999886	0.999824	0.999779	0.999744	0.999706
1,000,000	0.999947	0.999869	0.999931	0.999891	0.999862	0.999842	0.999821
1,500,000	0.999989	0.999947	0.999968	0.999948	0.999933	0.999926	0.999918
2,000,000	0.999998	0.999973	0.999983	0.999971	0.999962	0.999959	0.999955

Claim Amount	Prior Model Number 8	9	10	11	12	13	14
100,000	0.993080	0.993542	0.993901	0.994418	0.993722	0.992133	0.990703
200,000	0.997303	0.997541	0.997726	0.997995	0.997530	0.996472	0.995866
300,000	0.998534	0.998696	0.998821	0.999003	0.998640	0.997820	0.997472
400,000	0.999075	0.999190	0.999279	0.999407	0.999126	0.998490	0.998252
500,000	0.999355	0.999437	0.999501	0.999593	0.999375	0.998882	0.998699
750,000	0.999667	0.999704	0.999733	0.999775	0.999656	0.999388	0.999268
1,000,000	0.999800	0.999817	0.999830	0.999849	0.999780	0.999625	0.999538
1,500,000	0.999910	0.999913	0.999916	0.999920	0.999891	0.999827	0.999778
2,000,000	0.999952	0.999952	0.999953	0.999954	0.999939	0.999907	0.999877

Claim Amount	Prior Model Number 15	16	17	18	19	20
100,000	0.989248	0.987769	0.986267	0.983454	0.987292	0.991076
200,000	0.995249	0.994620	0.993980	0.992118	0.993669	0.995200
300,000	0.997118	0.996756	0.996388	0.995124	0.995968	0.996801
400,000	0.998009	0.997762	0.997510	0.996571	0.997128	0.997676
500,000	0.998513	0.998323	0.998129	0.997388	0.997808	0.998222
750,000	0.999146	0.999022	0.998895	0.998429	0.998698	0.998964
1,000,000	0.999449	0.999359	0.999266	0.998945	0.999134	0.999320
1,500,000	0.999727	0.999675	0.999622	0.999451	0.999539	0.999626
2,000,000	0.999847	0.999816	0.999784	0.999685	0.999717	0.999749

Table 6
Cumulative Probability for Lags 1-3

Claim Amount	Prior Model Number 1	2	3	4	5	6	7
100,000	0.993117	0.991183	0.991960	0.990009	0.988313	0.987309	0.986241
200,000	0.997248	0.996461	0.996618	0.995966	0.995416	0.994950	0.994452
300,000	0.998325	0.997996	0.998083	0.997766	0.997507	0.997228	0.996931
400,000	0.998843	0.998687	0.998752	0.998567	0.998419	0.998233	0.998034
500,000	0.999143	0.999056	0.999110	0.998984	0.998886	0.998753	0.998611
750,000	0.999529	0.999487	0.999527	0.999458	0.999404	0.999337	0.999265
1,000,000	0.999710	0.999680	0.999711	0.999664	0.999627	0.999589	0.999549
1,500,000	0.999865	0.999848	0.999867	0.999841	0.999819	0.999805	0.999790
2,000,000	0.999927	0.999917	0.999928	0.999911	0.999898	0.999891	0.999885

Claim Amount	Prior Model Number 8	9	10	11	12	13	14
100,000	0.985174	0.986141	0.986900	0.988014	0.987115	0.985052	0.982547
200,000	0.993953	0.994405	0.994760	0.995282	0.994564	0.992851	0.991672
300,000	0.996631	0.996928	0.997161	0.997502	0.996895	0.995426	0.994721
400,000	0.997834	0.998038	0.998197	0.998430	0.997942	0.996754	0.996259
500,000	0.998468	0.998608	0.998718	0.998879	0.998494	0.997557	0.997171
750,000	0.999193	0.999247	0.999289	0.999351	0.999139	0.998625	0.998370
1,000,000	0.999509	0.999526	0.999540	0.999559	0.999437	0.999140	0.998956
1,500,000	0.999774	0.999771	0.999769	0.999765	0.999715	0.999594	0.999488
2,000,000	0.999878	0.999873	0.999870	0.999864	0.999839	0.999777	0.999715

Claim Amount	Prior Model Number 15	16	17	18	19	20
100,000	0.980077	0.977642	0.975241	0.970705	0.974003	0.977352
200,000	0.990513	0.989375	0.988255	0.985651	0.986512	0.987325
300,000	0.994028	0.993349	0.992682	0.990980	0.991195	0.991346
400,000	0.995774	0.995298	0.994831	0.993577	0.993620	0.993606
500,000	0.996793	0.996421	0.996058	0.995061	0.995077	0.995047
750,000	0.998121	0.997876	0.997635	0.996980	0.997026	0.997049
1,000,000	0.998774	0.998596	0.998421	0.997952	0.997998	0.998030
1,500,000	0.999385	0.999283	0.999182	0.998920	0.998912	0.998892
2,000,000	0.999653	0.999592	0.999533	0.999378	0.999321	0.999251

Table 7
Ultimate Cumulative Probability

Claim Amount	Prior Model Number 1	2	3	4	5	6	7
100,000	0.986144	0.981451	0.978563	0.975297	0.972292	0.970836	0.969375
200,000	0.993462	0.991264	0.988893	0.987858	0.986981	0.986135	0.985300
300,000	0.995749	0.994592	0.992756	0.992349	0.992056	0.991480	0.990917
400,000	0.996907	0.996197	0.994830	0.994621	0.994495	0.994066	0.993649
500,000	0.997603	0.997108	0.996110	0.995958	0.995865	0.995527	0.995200
750,000	0.998547	0.998270	0.997838	0.997697	0.997579	0.997370	0.997172
1,000,000	0.999030	0.998845	0.998670	0.998530	0.998397	0.998261	0.998132
1,500,000	0.999499	0.999403	0.999387	0.999273	0.999155	0.999095	0.999039
2,000,000	0.999714	0.999659	0.999668	0.999582	0.999490	0.999467	0.999445

Claim Amount	Prior Model Number 8	9	10	11	12	13	14
100,000	0.967995	0.966848	0.965978	0.964755	0.962971	0.961104	0.957388
200,000	0.984525	0.983441	0.982566	0.981238	0.979893	0.978100	0.976354
300,000	0.990400	0.989463	0.988683	0.987462	0.986407	0.984925	0.983884
400,000	0.993269	0.992457	0.991774	0.990694	0.989877	0.988749	0.988008
500,000	0.994904	0.994197	0.993601	0.992655	0.992024	0.991202	0.990608
750,000	0.996993	0.996480	0.996049	0.995368	0.995031	0.994695	0.994273
1,000,000	0.998018	0.997636	0.997316	0.996813	0.996621	0.996506	0.996183
1,500,000	0.998990	0.998768	0.998583	0.998294	0.998218	0.998230	0.998036
2,000,000	0.999427	0.999294	0.999184	0.999012	0.998974	0.998998	0.998881

Claim Amount	Prior Model Number 15	16	17	18	19	20
100,000	0.953900	0.950641	0.947611	0.942936	0.940188	0.936834
200,000	0.974729	0.973226	0.971844	0.969483	0.965947	0.961818
300,000	0.982924	0.982042	0.981240	0.979667	0.976492	0.972815
400,000	0.987327	0.986706	0.986145	0.984848	0.982275	0.979308
500,000	0.990063	0.989568	0.989122	0.987937	0.985918	0.983600
750,000	0.993887	0.993537	0.993223	0.992192	0.991065	0.989781
1,000,000	0.995888	0.995620	0.995380	0.994510	0.993768	0.992925
1,500,000	0.997859	0.997698	0.997554	0.996986	0.996440	0.995814
2,000,000	0.998774	0.998677	0.998589	0.998237	0.997707	0.997093

Table 8
Ultimate Limited Average Severity

Claim	Prior Model Number						
Amount	1	2	3	4	5	6	7
100,000	6,412	7,021	7,217	8,067	8,835	9,181	9,540
200,000	7,340	8,276	8,735	9,774	10,712	11,168	11,637
300,000	7,864	8,961	9,629	10,735	11,725	12,251	12,788
400,000	8,226	9,413	10,241	11,375	12,385	12,960	13,546
500,000	8,498	9,744	10,689	11,841	12,861	13,474	14,097
750,000	8,964	10,302	11,417	12,604	13,650	14,329	15,016
1,000,000	9,261	10,655	11,842	13,065	14,142	14,864	15,591
1,500,000	9,612	11,073	12,297	13,583	14,725	15,493	16,264
2,000,000	9,802	11,300	12,524	13,860	15,054	15,842	16,631

Claim	Prior Model Number						
Amount	8	9	10	11	12	13	14
100,000	9,891	9,977	10,045	10,148	10,337	10,344	11,107
200,000	12,091	12,290	12,448	12,682	13,025	13,217	14,235
300,000	13,306	13,606	13,847	14,209	14,671	15,028	16,180
400,000	14,107	14,495	14,809	15,286	15,841	16,328	17,567
500,000	14,692	15,155	15,533	16,111	16,739	17,323	18,628
750,000	15,668	16,283	16,787	17,567	18,311	19,032	20,461
1,000,000	16,279	17,004	17,602	18,528	19,336	20,110	21,632
1,500,000	16,990	17,863	18,583	19,702	20,571	21,361	23,009
2,000,000	17,374	18,333	19,125	20,356	21,253	22,032	23,756

Claim	Prior Model Number					
Amount	15	16	17	18	19	20
100,000	11,832	12,518	13,166	13,884	14,036	14,257
200,000	15,197	16,105	16,958	18,006	18,489	19,102
300,000	17,267	18,289	19,246	20,485	21,307	22,315
400,000	18,735	19,830	20,854	22,234	23,344	24,684
500,000	19,856	21,006	22,080	23,583	24,922	26,526
750,000	21,803	23,057	24,223	26,002	27,721	29,758
1,000,000	23,058	24,388	25,623	27,641	29,585	31,880
1,500,000	24,550	25,985	27,314	29,688	31,939	34,581
2,000,000	25,367	26,865	28,251	30,851	33,369	36,320

We are now ready to work through our examples in detail. Exhibits 1-3 below give the results of the Bayesian methodology for a small, a medium, and a large insurer. The exhibits take claim severity data from each insurer and provide estimates of the layer pure premium for a \$500,000 x \$500,000 layer and for a \$1 million x \$1 million layer. For the record, I note that the "data" for each insurer was produced from a simulation taken from a single claim severity distribution. The "true" expected pure premiums for the layers are \$1,382 and \$1,015, respectively.

Here is a step by step description of the calculations in those exhibits.

Lags – As mentioned above, the claim severity distributions underlying the models were fit by settlement lag. Claims from the most recent accident year consist of claims that were settled in Lag 1. Claims from the second most recent year consist of claims that were settled in Lags 1 and 2. The designation Lag 1 corresponds to accident year AY=1, Lags 1-2 corresponds to AY=2 and Lags 1-3 corresponds to AY=3.

Interval Lower Bound and Claim Count – We summarized the claim amounts in intervals, with the lower bound of the interval being specified to the left of the claim count. Let $n_{i,AY}$ be the observed claim count in the i^{th} interval for accident year AY. For example, in Exhibit 1, there were 15 claims in the interval (100,000, 200,000] and there were no claims more than 2,000,000 in Lag 1. The underlying exposure was the same for each accident year. Note that there are more high severity claims in the earlier accident years.

Prior Model # – These are the models described in Tables 4-8 above. Each table gives a different part of each model as described above.

Posterior Probability – This is calculated for each prior model. Let:

- $n_{i,AY}$ be the number of claims in the i^{th} interval of the AY^{th} accident year.
- $x_{i,\text{AY}}$ be the lower bound of the ith interval, $x_{i+1,\text{AY}}$ be the upper bound of the ith interval. Note $x_{10,AY}$ = □.
- Let $P_{i,\text{AY},m}$ be the probability that a claim is observed in i^{th} cell given that it is in the AY^{th} accident year for model m. Let x_i be the lower bound of the i^{th} interval. Let $F_{AY,m}(x_i)$ be the probability that a claim is $\leq x_i$ in accident year AY for model m. These probabilities are given in Tables 4-6 above. Then:

$$P_{i,AY,m} = \frac{F_{AY,m}\left(x_{i+1}\right) - F_{AY,m}\left(x_i\right)}{1 - F_{AY,m}\left(x_1\right)}.$$

- The likelihood of the data $\{n_{i,AY}\}$ for model m, l_m, is given by:

$$l_m = \prod_{i=1}^{9} \prod_{AY=1}^{3} \left(P_{i,AY,m}\right)^{n_{i,AY}} .$$

- Let Prior(m) be the prior probability associated with model m. (In this example, Prior(m) = 1/20 for all m.) Then according to Bayes' theorem:

$$\text{Posterior}(m) \propto l_m \cdot \text{Prior}(m) .$$

As was done in the COTOR Challenge example, you first calculate the product l_m·Prior(m) and then normalize.

Layer Pure Premium – The layer pure premium for each model is calculated from the limited average severity curves in Table 8. For example, the layer pure premium for the $500 thousand x $500 thousand layer is calculated as the difference between the limited average severity for $1 million and the limited average severity at $500 thousand[8].

Posterior Mean and Standard Deviation – These quantities are calculated by the following formulas.

$$\text{Posterior Mean} = \sum_{m=1}^{20} \text{Layer Pure Premium}(m) \cdot \text{Posterior}(m).$$

$$\text{Posterior Standard Deviation} = \sqrt{\sum_{m=1}^{20} \text{Layer Pure Premium}(m)^2 \cdot \text{Posterior}(m) - \text{Posterior Mean}^2} .$$

Note that as we increase the exposure, and hence the number of observations, the posterior probability tends to be concentrated on fewer models. As the posterior standard deviations indicate, increasing exposure leads to less uncertainty in the final estimate.

[8] It is often the case that the reinsurer will have an independent estimate of the probability that a claim is more than $100,000. To make use of this information, the reinsurer should multiply the layer pure premium times the ratio of this probability to each model's probability that a claim is more than $100,000. I did not do this in these examples.

5. CONCLUDING REMARKS

In this paper, I gave some examples showing how to use the likelihood function and Bayes' theorem to estimate the costs of high layers of reinsurance. Many of the assumptions need to be debated, and regardless of how the debate is resolved, much work is needed to complete the job. I hope this paper provides strong evidence that such an approach can succeed and provide a sound methodology for reinsures to use in pricing coverage of high layers of reinsurance.

Exhibit 1 – Small Insurer

Lags	Interval Lower Bound	Claim Count	Prior Model #	Posterior Probability	Layer Pure Premium $500K x $500K	Layer Pure Premium $1M x $1M
1	100,000	15	1	0.016406	763	541
1	200,000	2	2	0.041658	911	645
1	300,000	1	3	0.089063	1,153	682
1	400,000	2	4	0.130281	1,224	796
1	500,000	0	5	0.157593	1,281	912
1	750,000	0	6	0.110614	1,390	978
1	1,000,000	0	7	0.075702	1,494	1,040
1	1,500,000	0	8	0.053226	1,587	1,095
1	2,000,000	0	9	0.080525	1,849	1,328
			10	0.104056	2,069	1,523
			11	0.129925	2,417	1,828
1-2	100,000	40	12	0.010896	2,598	1,916
1-2	200,000	10	13	0.000007	2,788	1,922
1-2	300,000	1	14	0.000009	3,004	2,124
1-2	400,000	0	15	0.000011	3,202	2,309
1-2	500,000	2	16	0.000013	3,382	2,477
1-2	750,000	0	17	0.000014	3,543	2,628
1-2	1,000,000	2	18	0.000000	4,058	3,211
1-2	1,500,000	0	19	0.000000	4,663	3,784
1-2	2,000,000	0	20	0.000000	5,354	4,440
			Posterior Mean		**1,572**	**1,113**
1-3	100,000	76	**Posterior Std. Dev.**		**463**	**385**
1-3	200,000	26				
1-3	300,000	11				
1-3	400,000	3				
1-3	500,000	8				
1-3	750,000	0				
1-3	1,000,000	0				
1-3	1,500,000	0				
1-3	2,000,000	0				

Exhibit 2 – Medium Insurer

Lags	Interval Lower Bound	Claim Count	Prior Model #	Posterior Probability	Layer Pure Premium $500K x $500K	Layer Pure Premium $1M x $1M
1	100,000	31	1	0.000973	763	541
1	200,000	12	2	0.021135	911	645
1	300,000	4	3	0.221357	1,153	682
1	400,000	2	4	0.235280	1,224	796
1	500,000	1	5	0.209597	1,281	912
1	750,000	0	6	0.123874	1,390	978
1	1,000,000	0	7	0.059523	1,494	1,040
1	1,500,000	0	8	0.028986	1,587	1,095
1	2,000,000	0	9	0.037532	1,849	1,328
			10	0.037637	2,069	1,523
			11	0.023539	2,417	1,828
1-2	100,000	107	12	0.000567	2,598	1,916
1-2	200,000	33	13	0.000000	2,788	1,922
1-2	300,000	14	14	0.000000	3,004	2,124
1-2	400,000	3	15	0.000000	3,202	2,309
1-2	500,000	7	16	0.000000	3,382	2,477
1-2	750,000	2	17	0.000000	3,543	2,628
1-2	1,000,000	0	18	0.000000	4,058	3,211
1-2	1,500,000	0	19	0.000000	4,663	3,784
1-2	2,000,000	0	20	0.000000	5,354	4,440
			Posterior Mean		**1,344**	**909**
1-3	100,000	191	**Posterior Std. Dev.**		**278**	**245**
1-3	200,000	47				
1-3	300,000	31				
1-3	400,000	22				
1-3	500,000	6				
1-3	750,000	5				
1-3	1,000,000	1				
1-3	1,500,000	2				
1-3	2,000,000	1				

Exhibit 3 - Large Insurer

Lags	Interval Lower Bound	Claim Count
1	100,000	77
1	200,000	20
1	300,000	7
1	400,000	1
1	500,000	1
1	750,000	1
1	1,000,000	0
1	1,500,000	0
1	2,000,000	0
1-2	100,000	193
1-2	200,000	60
1-2	300,000	22
1-2	400,000	14
1-2	500,000	10
1-2	750,000	7
1-2	1,000,000	2
1-2	1,500,000	1
1-2	2,000,000	1
1-3	100,000	431
1-3	200,000	117
1-3	300,000	40
1-3	400,000	25
1-3	500,000	24
1-3	750,000	4
1-3	1,000,000	5
1-3	1,500,000	0
1-3	2,000,000	1

Prior Model #	Posterior Probability	Layer Pure Premium $500K x $500K	Layer Pure Premium $1M x $1M
1	0.000000	763	541
2	0.000193	911	645
3	0.000481	1,153	682
4	0.050204	1,224	796
5	0.689060	1,281	912
6	0.179377	1,390	978
7	0.015896	1,494	1,040
8	0.001625	1,587	1,095
9	0.009032	1,849	1,328
10	0.022443	2,069	1,523
11	0.031690	2,417	1,828
12	0.000000	2,598	1,916
13	0.000000	2,788	1,922
14	0.000000	3,004	2,124
15	0.000000	3,202	2,309
16	0.000000	3,382	2,477
17	0.000000	3,543	2,628
18	0.000000	4,058	3,211
19	0.000000	4,663	3,784
20	0.000000	5,354	4,440
Posterior Mean		**1,360**	**966**
Posterior Std. Dev.		**234**	**188**

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Biography of the Author

Glenn Meyers is the Chief Actuary for ISO Innovative Analytics. He holds a bachelor's degree in mathematics and physics from Alma College in Alma, Mich., a master's degree in mathematics from Oakland University, and a Ph.D. in mathematics from the State University of New York at Albany. Glenn is a Fellow of the Casualty Actuarial Society and a member of the American Academy of Actuaries. Before joining ISO in 1988, Glenn worked at CNA Insurance Companies and the University of Iowa.

Glenn has assumed positions of increasing responsibility at ISO, including leadership of increased limits and catastrophe ratemaking. He created ISO's Multi-Distributional Increased Limits Developer (MILD), ISO's increased limits software, and Property Size-of-Loss Database (PSOLD), ISO's model for commercial property size of loss distributions.

Glenn was the principal author of several books in the *ISO Issues Series*. His current projects include developing ISO's Dynamic Financial Analysis and insurance scoring products solutions.

Glenn's work has been published in *Proceedings of the Casualty Actuarial Society (CAS)*. He is a three-time winner of the Woodward-Fondiller Prize, a two-time winner of the Dorweiller Prize and a winner of the Dynamic Financial Analysis Prize. He is a frequent speaker at CAS meetings and seminars.

His service to the CAS includes long-term membership of the Examination Committee and the Committee on the Theory of Risk. He serves on the International Actuarial Association Solvency Committee and co-chairs the CAS Working Party on Correlation. He also serves on the CAS Dynamic Risk Modeling Committee and the Modeling Workshop Task Force.

Coherent Capital for Treaty ROE Calculations

Ira Robbin, Ph.D. and Jesse DeCouto

Abstract:

This paper explores how a coherent risk measure could be used to determine risk-sensitive capital requirements for reinsurance treaties. The need for a risk-sensitive capital calculation arises in the context of estimating the return on equity (ROE) for several treaties or different options on one treaty. Looking at the loss random variable alone is insufficient for a complete risk analysis since this would fail to incorporate the impact of adjustable premium and ceding commission provisions on the final net risk. The paper presents a framework for systematically reflecting treaty features by viewing capital as a function of the distribution of the final net underwriting loss. To avoid negative values for indicated capital, the concept of a risk quantity variable is introduced as a non-negative monotonically increasing function of the net underwriting loss variable. Two risk quantities are discussed: one obtained by capping the net underwriting loss from below at zero, and the other by taking the excess of the net underwriting loss above its expectation. A coherent risk measure is then applied to a risk quantity to obtain indicated capital. The approach is demonstrated in simple discrete distribution examples by applying a coherent measure, the Tail Value at Risk, to the two risk quantities. Sensitivity testing on the examples is presented showing how the different measures respond to changes in premium adequacy, swing rating, sliding scale commission plans, and layering. In summary, this paper is one attempt to bridge the gap between the theoretical results of coherence theory and the practical need for methods to set risk-sensitive capital in treaty ROE analysis.

Keywords: Coherence, Reinsurance, Capital Requirements, TVaR

1. INTRODUCTION

Current financial theory says the theoretically best way to measure risk is with a coherent risk measure. The theory views risk as uncertainty regarding the future net worth of an investment portfolio or company at a specified point in time ([1], [2]). The theory allows the net worth to possibly take on negative values. It measures risk as the additional amount of money needed to ensure the future net worth will fall within a predefined set of acceptable outcomes, called the acceptance set [2]. The measure could take on a negative value, indicating that risk-free assets, such as cash, could be withdrawn while still leaving the portfolio in the acceptable range. The measure and the acceptance set are directly related: the acceptance set is the set of all the net worth random variables on which the measure is less than or equal to zero. A coherent risk measure is one that satisfies several desirable properties. Some common measures of risk, such as variance, standard deviation, and the Value at Risk (VaR) fail to be coherent. On the other hand, other measures, including the Tail Value at Risk (TVaR) and the Proportional Hazard Transform (PHT), have been proved to be coherent. Several authors, including Artzner [2], Meyers [8], and Wirch and Hardy [14], have recommended using coherent risk measures to determine appropriate capital requirements. Our purpose in this paper is to explore how to apply coherent risk theory in order to obtain a coherent capital calculation for reinsurance treaty analysis.

While risk measures in insurance are often viewed as applying to loss random variables, that is insufficient for our purposes. The problem is that focusing solely on the loss random variables fails to capture the essential complexity of various adjustable features of reinsurance treaties. These features may alter the premium or change the ceding commission, so that these become functions of the loss outcome. Some of the features seem to reduce risk; others intuitively have no effect. Our goal is to arrive at a capital calculation that reflects the impact treaty features may have on the final net risk. In order to do this in a general and consistent fashion, we believe it is best to start with the final net underwriting loss random variable. Here the underwriting (technical) loss is defined as loss plus commission less premium.

But this raises a problem. The net underwriting loss random variable should often take on negative values, corresponding to underwriting gain scenarios. Yet, if we allow measures, even coherent measures, to operate on random variables that can become negative, we may well end up with a negative result. For an indicated capital algorithm to produce a negative answer is, in our view, flatly unacceptable. It is possible that our insistence on non-negative capital requirements is at odds with the basic conceptual structure of coherence theory in which it is well possible for the measure to be negative.

In any event, to handle the negative values problem in some generality, we introduce the concept of a risk quantity variable. We define a risk quantity variable as any non-negative random variable that is a monotonically increasing function of the net underwriting loss. Because the risk quantity is non-negative, we can never end up with a negative capital indication. To summarize, our general notion is to compute capital by applying a coherent measure such as TVaR or PHT to a risk quantity variable. We will call this a "coherent capital calculation" though our approach may differ in places with some of the basic structure of coherent risk measures.

We have found two plausible risk quantity variables. Both are based on the net underwriting loss random variable after application of all treaty provisions. The first is obtained by capping the underwriting loss from below at zero. Since underwriting gains correspond to negative underwriting losses, this capping collapses all underwriting gain scenarios to zero. The second approach uses a risk quantity variable obtained by taking the amount of underwriting loss in excess of its expected value.

To illustrate these methods, we will apply a coherent risk measure, TVaR, the Tail Value at Risk, to our risk quantity variables and thus obtain two different coherent capital formulas. We will show these coherent capital formulas have different behavior when viewed as functions of the loss, expense, and premium.

To show how these formulas work we will apply them to a hypothetical treaty with losses that follow a discrete loss distribution. We will then conduct sensitivity testing to see how indicated capital responds to changes in treaty pricing, treaty features, reinsurer share, and layering. For comparison purposes, we will also compute indicated capital based on a fixed leverage ratio against provisional premium and a fixed leverage ratio against initial expected layer loss. We will also run comparisons against the standard deviation of underwriting loss and the variance of underwriting loss. As previously stated, these are non-coherent risk measures.

In the end, we believe we will have shown with concrete examples that fixed leverage ratio methods are deficient and that net underwriting loss should be the basis for risk-sensitive capital calculations. Our work also casts doubt on the variance of underwriting loss and, to a lesser extent, on the standard deviation of underwriting loss. We will have demonstrated two different ways of implementing a coherent capital methodology, without concluding which one is best, but shown that they have quite different behavior.

In this paper our focus is on process risk and how to reflect changes in process risk induced by changes in treaty features. Because parameter risk, correlation, and portfolio effects have not been considered, our treatment is incomplete. Further, our approach to implementing coherent capital concepts may not be the only one. But nonetheless our larger conclusion is that at this point the introduction of coherent risk measures has not definitively settled the question of how to set capital. Though progress has been made, implementation of coherence concepts remains a topic open to further research in the future.

2. CAPITAL FOR TREATY ROE CALCULATIONS

Our interest in determining capital arises when computing the prospective ROE (return on equity) for a treaty. Typically we are asked to determine the ROE for a treaty at several premium rates. In our calculation, we must reflect any contingent commission, reinstatement, aggregate loss cap, swing rating, or other such provisions of the treaty. The amount of capital is a critical determinant of our results and so questions about how to set the capital become important. Because various treaty features that impact premiums and expenses can change the overall risk of the deal, a risk measure based on loss only is inadequate for our purpose. Our approach to capital requirements is, in this regard, similar to Feldblum's view that risk loads should not be based solely on the loss distribution [6].

When actually pricing a treaty, we would first model possible loss scenarios and use actuarial techniques to estimate the probability of each scenario. Depending on the terms and provisions of the treaty, each scenario leads to its own ultimate combined ratio, cash flows, and ROE. For each of these scenarios, we would hold the same amount of capital in our pricing model, because, a priori, we have no way of knowing which scenario will actually occur. We end up with a distribution of ROE values, not just a single point estimate. Also, the capital held in our models would not be a simple fixed block amount posted for one year, but would also include amounts varying over time to cover uncertainty in the reserves. In this paper, however, we will only consider the distribution of ultimate outcomes and will leave for others the question of how capital should be held over time to cover potential reserve inadequacy. Also we will assume in this paper that all values are at present value. This simplification will allow us to ignore the time value of money. In any real application, one should of course reflect the time value of money, payout pattern uncertainty, asset risk, and other related concerns.

We should also realize at the outset that use of any theoretically based measure to set capital may lead to an implicit leverage ratio on a treaty or block of treaties that is either higher or lower than industry rating agency or regulatory norms. While particular blocks of business may be more or less risky than presumed in deriving industry standards, there is a great deal of uncertainty and some subjectivity in selecting parameters for any model. Given that uncertainty, we are not suggesting that our estimates of required capital ought to lead to any revision of accepted industry capital benchmarks. Also, we are setting a theoretically appropriate level of capital by treaty that when aggregated over all treaties may differ from the actual amount of capital held by a company. In separating actually held capital from the capital used in pricing models, we are recognizing that no pricing penalties or subsidies should ensue from pre-existing under-capitalization or over-capitalization. Though in principle we should compute a benchmark amount of capital for our whole portfolio and then allocate this coherently [5] to individual treaties, our work here is focused on the simpler problem of computing benchmark capital for each treaty on a stand-alone basis. Our goal is see if coherent capital approaches can be used to appropriately model the impact of treaty features on the capital requirement. Thus we leave as a topic for future research the consideration of portfolio effects, parameter risk, and correlation. To summarize, our purpose is to study procedures that should guarantee a logical ordering of the capital requirements for alternative treaty structures, and not to resolve questions about overall calibration or allocation.

3. COHERENT RISK MEASURES

The theory of risk measures took a major step forward with the introduction of the concept of coherence by Artzner, Delbaen, Eber and Heath in 1999 [1] and their presentation of results on the representation of coherent risk measures. Their work successfully implemented a general program of listing desirable properties for a risk measure and then characterizing the types of measures that satisfy those properties. Before and since, others such as Wang [11] and Venter [10], have made critical contributions to the theory and understanding of arbitrage-free pricing, power transforms, distortion measures, stochastic dominance properties, and other related concepts. Wirch and Hardy [14] explained the relation between concave distortion measures and coherent risk measures. Meyers [8], [9] did a great service to the actuarial community by writing an intuitive and accessible introduction to the concept.

In applying the concept to insurance, what is sometimes unclear in the literature is whether the risk measure is being viewed as a premium calculation, risk load calculation, or required capital formula. We will defer consideration of this issue till later after we have defined coherence in a general setting.

To begin the mathematical development of risk measures for insurance capital, we define a **risk measure**, ρ, as a function that maps a non-negative random variable, B, to a non-negative number, $\rho(B)$. The reason we insist on having non-negative variables is to avoid negative values for the risk measure and the resulting capital requirement. For an example of how this could occur, consider an underwriting loss distribution that takes on the value, –70 with 90% probability, -50 with 9% probability, and +400 with 1% probability. As we will later learn when we consider TVaR in more detail, the TVaR measure associated with the 90^{th} percentile would be –5. But we would certainly not want that as a capital requirement. Assuming our restriction to non-negative random variables, we define coherence of the risk measure as follows:

Coherence Properties for Risk Measures (3.1)

A risk measure, ρ, is said to be coherent if it satisfies:

1. Zero has No Risk: If $B \equiv 0$, then $\rho(B)=0$
2. Monotonicity: If $B_1 \leq B_2$, then $\rho(B_1) \leq \rho(B_2)$
3. Scaling: If $\lambda > 0$, then $\rho(\lambda B) = \lambda\, \rho(B)$
4. Subadditivity: $\rho(B_1 + B_2) \leq \rho(B_1) + \rho(B_2)$
5. Translation Additivity: If $\alpha > 0$, $\rho(B + \alpha) = \rho(B) + \alpha$
6. Bounded from Below: $E[B] \leq \rho(B)$
7. Bounded from Above: If $\max(B) < \square$, then $\rho(B) \leq \max(B)$

This list was drawn from the lists of coherence properties that are contained in the papers by Meyers [8] and Wirch and Hardy [14]. We believe the translation additivity property and the bounds describe a coherent premium calculation operating on the loss distribution.

4. COHERENT CAPITAL

Our overall goal is to set capital, C, as a function of the loss, expense and premium. To apply the coherence properties in setting capital, we first define underwriting loss, U, as the sum of loss, L, plus expense, X, less the premium, P.

For our first method, we follow the suggestion of Wirch and Hardy [14] and define our risk quantity variable as the bounded underwriting loss, B, obtained by capping the underwriting loss from below at zero. Thus, in our notation, we have:

$$B = \max(0, U) = \max(0, L+X-P) \qquad (4.1)$$

We may sometimes write B(U), B(U(L,X,P)), or B(L,X,P): whatever is most convenient. Note, B is a non-negative random variable and that all underwriting gain scenarios collapse to the "zero" mass point of B. We will define capital as Level Sensitive Coherent if it can be expressed by applying a coherent risk measure to the bounded underwriting loss.

Level Sensitive Capital Coherence (LSCC) Definition (4.2)

A capital function C is called *level sensitive coherent* if there exists a coherent risk measure, ρ, such that $C(L,X,P) = \rho(B(L,X,P))$ where $B=\max(0, L+X-P)$.

In basing our definition on bounded underwriting loss, we are implicitly saying that all contracts with the same distribution of bounded underwriting losses will get the same capital, even if different premiums, expenses, and losses are involved. This is a key strength of the approach. Underwriters and brokers can sometimes fashion two alternatives, say one with a swing premium and the other with a larger provisional premium and a profit commission that yield the same underwriting loss for any given loss scenario. Neglecting some cash flow and security issues, it is hard to argue why theoretically one alternative should have a different capital requirement and a different ROE than the other.

We will now derive LSCC properties with respect to loss, expense, and premium. These will be based on the properties of the coherent risk measure and on the behavior of the bounded underwriting loss function.

Our first coherence property for a risk measure was that the measure is zero on the random variable identically equal to zero. For Level Sensitive Coherent Capital, this implies no capital is needed if there are no possible underwriting losses. This is potentially controversial, because it disconnects our risk measure from whatever volatility may exist in underwriting gain scenarios.

Using $\max(0,L+X-P) \le \max(0,L+\alpha+X-P) = \max(0,L+X-(P+\alpha)) \le \max(0,L+X-P) + \alpha$ and the monotonicity and translation additivity properties for a coherent risk measure, we can show:

$$\rho(B(L+\alpha, X, P)) \leq \rho(B(L, X, P)) +\alpha. \tag{4.3}$$

and

$$\rho(B(L, X, P)) - \alpha \leq \rho(B(L, X, P+\alpha)) \tag{4.4}$$

Note that despite translation additivity of our risk measure, the LSCC amount might go up by less than \$1 after all losses are increased by \$1. As well for premium, increasing the premium by \$1 will decrease the required capital, but by an amount that could be, but does not have to be, less than \$1. This sensitivity of required capital to fixed increments in premium or loss is why we call LSCC, "level sensitive". Note that LSCC still depends on the volatility of the underwriting losses as long as there is some possibility of an actual net underwriting loss.

Scaling carries over in the obvious way: if all losses, expenses, and premiums are scaled by a common factor, then the LSCC coherent capital scales up the same way.

With subadditivity of the max operator we can show:

$$B(L_1+ L_2 , X_1+ X_2, P_1 + P_2) =\max(0, L_1+X_1 -P_1 +L_2+X_2 -P_2) \leq$$

$$\max(0, L_1+X_1 -P_1) + \max(0, L_2+X_2 -P_2)$$

This, along with monotonicity and subadditivity of a coherent risk measure, implies

$$\rho(B(L_1+ L_2 , X_1+ X_2, P_1 + P_2)) \leq \rho(B(L_1,X_1,P_1)) + \rho(B(L_2,X_2,P_2)) \tag{4.5}$$

In other words, the LSCC needed for two treaties combined is less than or equal to the sum of the LSCC required for each treaty. Note the inequality is not strict. According to Wang [13], the case for strict inequality is only compelling when the separate underwriting losses are not comonotonic. Comonotonic means each of the random variables can be expressed as an increasing function of a third random variable. Under Wang's power transforms, the risk loads for separate layers sum up to the risk load of the combined layers. This suggests required capital ought to be similarly decomposable by layer. This is a question we will study later in our examples.

The following summarizes the properties of Level Sensitive Coherent Capital:

LSCC-Coherent Capital Properties with Respect to L, X, and P (4.6)

Let B = max(0, L+X-P) and assume ρ is a Coherent Risk Measure. Let C=ρ(B). Then:

1. No Capital Needed if No Risk of Underwriting Loss: If P-L-X >0, then C(L,X,P)=0.
2. Monotonicity: If $L_1+X_1-P_1 \leq L_2+X_2-P_2$, then $C(L_1, X_1, P_1) \leq C(L_2, X_2, P_2)$
3. Scaling: If $\lambda>0$, then $C(\lambda L, \lambda X, \lambda P)) = \lambda\, C(L,X,P)$
4. Subadditivity:

   $C(L_1+L_2, X_1+X_2, P_1+P_2) \leq C(L_1,X_1,P_1) + C(L_2,X_2,P_2)$
5. Translation Additivity Inequalities:

   i) $C(L+\alpha, X, P) \leq C(L, X, P) + \alpha$

   ii) $C(L, X+\alpha, P) \leq C(L, X, P) + \alpha$

   iii) $C(L, X, P) - \alpha \leq C(L, X, P+\alpha)$

Next we define our second notion of coherent capital, Deviation Sensitive Coherent Capital (DSCC). First we define the underwriting loss in excess of expectation, B*, via

$$B^* = \max(0, U - E[U]) \text{ where } U = L+X-P \tag{4.7}$$

Note that B* is unaffected by adding a fixed amount to the loss or to the premium. Also observe that B* can be strictly positive for scenarios where there are underwriting gains if those underwriting gains fall short of expectation. In defining B*, we are following a logic similar to that suggested by Bault [3] in which he discussed generalizing ruin theory for risk load calculations so that any adverse deviation from a target might be counted as contributing to the probability of ruin.

We will define capital as Deviation Sensitive Coherent if it can be expressed by applying a coherent risk measure to the bounded underwriting loss in excess of expectation.

Deviation Sensitive Capital Coherence (DSCC) Definition (4.8)

A capital function C is called deviation sensitive coherent if there exists a coherent risk measure, ρ, such that $C(L,X,P) = \rho(B^*(L,X,P))$ where $B^*(L,X,P) = \max(0, L+X-P-E[L+X-P])$.

Using modified versions of the arguments employed in analyzing LSCC properties, we obtain the following properties for DSCC with respect to loss, expense, and premium.

DSCC-Coherent Capital Properties with Respect to L, X, and P (4.8)

Let $B^*=\max(0, U-E[U])$ where $U = L+X-P$ and assume ρ is a Coherent Risk Measure. Let $C=\rho B^*$. Then:

1. No Risk if No Variability in Underwriting Loss: If $P-L-X \equiv \alpha$, then $C(L,X,P)=0$.
2. Monotonicity: If $L_1+X_1-P_1 \leq L_2+X_2-P_2$,

   then $E[U_1] + C(L_1, X_1, P_1) \leq E[U_2] + C(L_2, X_2, P_2)$
3. Scaling: If $\lambda>0$, then $C(\lambda L, \lambda X, \lambda P)) = \lambda C(L,X,P)$
4. Subadditivity: $C(L_1+L_2, X_1+X_2, P_1+P_2) \leq C(L_1,X_1,P_1)) + C(L_2,X_2,P_2)$
5. Translation Invariance:

   i) $C(L+\alpha, X, P) = C(L, X, P)$

   ii) $C(L, X+\alpha, P) = C(L, X, P)$

   iii) $C(L, X, P) = C(L, X, P+\alpha)$

The first major point to be made in comparing the DSCC and LSCC concepts of coherence is that they do actually differ: they are not merely different ways of saying the same thing. The difference shows up perhaps most strongly with respect to translation properties. As we saw previously, for LSCC adding $1 of premium decreases capital by an amount less than or equal to $1; but for DSCC this does change capital at all.

Another point of interest is that DSCC and LSCC are equal when the expected underwriting loss is zero. It follows that LSCC will be less than DSCC when there is a negative expected underwriting loss; in other words when there is an expected underwriting gain. Since reinsurers write a treaty expecting to make money, this will typically be true.

Now that we have defined two concepts of coherent capital and derived their properties, we will next demonstrate the concepts using the coherent risk measure, TVaR.

5. VAR AND TVAR

A common approach to setting capital is to set it at the 90th, 95th, 99th or other chosen percentile. Borrowing from financial terminology, the percentile is usually called the Value at Risk (VaR). Also, in managing catastrophe books, a similar idea is to control writings so as to keep the 100, 250, or 500-year event within acceptable bounds.

Given ε, we define VaR_ε as follows:

$$\text{VaR}_\varepsilon = \inf \{x \mid F(x) \geq \varepsilon\} \tag{5.1}$$

Here "inf" stands for infimum and the definition means that VaR is the lower bound of the set of all x such that the cumulative distribution at x is greater than or equal to ε.

While VaR has a great deal of appeal as a measure of risk, it is unfortunately not a coherent metric. This is shown by example in Exhibit 1: VaR vs. TVaR. This exhibit shows ten different loss scenarios for two different portfolios. The example is composed in such a way that the two different portfolios have the same loss distribution even though they suffer different amounts of loss for any particular event. In our example, VaR at the 80th percentile level for each portfolio is 50, but VaR for the combined portfolio is 110. So, VaR at the 80th percentile would indicate it is riskier to combine the two portfolios than it would be to double the losses for either portfolio. This fails to make intuitive sense and is in violation of the subadditivity property of coherence, 3.1.

The Tail Value at Risk is defined as the conditional expected value for points strictly above the Value at Risk.

$$\text{TVaR}_\varepsilon = E[X \mid X > \text{VaR}_\varepsilon] \tag{5.2}$$

TVaR is known to be coherent [9]. Thus, there is no example we can construct that will result in the sum of the TVaR for the individual portfolios being less than the TVaR for the combined portfolio. Continuing on with our specific numerical example, we see from Exhibit 1 that the sum of the TVaR for the individual portfolios exceeds TVaR for the combined portfolio (90+90>135). So, according to the TVaR measure, there is a risk benefit in combining the portfolios. If, on the other hand, the two portfolios in our example were 100% correlated, the sum of the TVaR would equal 180 or the sum of the two individual portfolios. This still satisfies the subadditivity property of coherence because the inequality in the definition is not strict.

Now that we have an understanding of TVaR, we will use it for demonstration purposes as the coherent measure in the definition of our two coherent capital formulas. We do not wish to suggest that TVaR is the only coherent measure appropriate for treaty pricing applications. One of the Proportional Hazards Transforms defined by Wang [12] would also be an excellent choice.

6. CAPITAL SENSITIVITY COMPARISONS

We will now study how our coherent capital formulas compare against each other and against other methods. The full list of methods we will examine is:

Fixed Leverage Ratio Against Provisional Premium

Fixed Leverage Ration Against Expected Loss

Standard Deviation of Underwriting Loss

Variance of Underwriting Loss

TVaR of Bound Underwriting Loss (LSCC)

TVaR of Underwriting Loss Excess of Expectation (DSCC)

First we will consider how the methods respond to a change in premium adequacy. This has practical importance for example in evaluating how much of a rate change is needed in order to achieve a target ROE. Using a fixed leverage ratio against premium effectively assigns more capital in response to an increase in rate. Why more capital is needed is unclear from a risk perspective. The effect is to make the ROE less sensitive to a rate change than it would otherwise be. In contrast, the amount of capital does not change with the rate when using either a fixed leverage ratio against expected loss, the standard deviation of underwriting loss, the variance of underwriting loss, or the DSCC method. While the amount of capital does not change, the resulting premium-to-capital leverage ratio will rise with a rate increase. With the LSCC method, the amount of indicated capital declines due to a rate increase. The ROE with LSCC is thus more sensitive to a rate change than the other methods as both the numerator and the denominator are affected. Table 1 summarizes the premium adequacy results shown in Exhibit 2.

Table 1	**Sensitivity of Capital to Premium Adequacy**			
Method		**Premium - 10%**	**Base Case**	**Premium +10%**
Fixed Premium Leverage	Capital	61	68	74
	Premium Leverage	1.48	1.48	1.48
Fixed Loss Leverage	Capital	68	68	68
	Premium Leverage	1.33	1.48	1.63
Standard Deviation	Capital	68	68	68
	Premium Leverage	1.33	1.48	1.63
Variance	Capital	68	68	68
	Premium Leverage	1.33	1.48	1.63
LSCC	Capital	70	63	55
	Premium Leverage	1.29	1.60	2.00
DSCC	Capital	68	68	68
	Premium Leverage	1.33	1.48	1.63

Note that we have set our base case so that all the methods yield the same answer except for the Level Sensitive Coherent Capital calculation. This was done because our base case has an expected net underwriting profit. As previously observed, in such a situation we will always have LSCC less than DSCC. Thus we cannot set all the methods equal. We are free to pick constants for the Standard Deviation and Variance methods, but once selected these constants are fixed and not allowed to change from scenario to scenario.

Next we look at scenarios involving a treaty that is priced by first agreeing on a net rate and then arriving at the final rate by grossing up for ceding commission. This "net rating" is not uncommon on excess of loss treaties. We consider how the methods respond if the ceding commission rate changes from a base case of 25% to either 20% or 30%. Table 2 summarizes the results from Exhibit 3.

Table 2		**Sensitivity to Changes in Cede on Net Rated Treaty**		
Method		**Cede = 20%**	**Cede = 25%**	**Cede = 30%**
Fixed Premium Leverage	Capital	63	68	72
	Premium Leverage	1.48	1.48	1.48
Fixed Loss Leverage	Capital	68	68	68
	Premium Leverage	1.39	1.48	1.59
Standard Deviation	Capital	68	68	68
	Premium Leverage	1.39	1.48	1.59
Variance	Capital	68	68	68
	Premium Leverage	1.39	1.48	1.59
LSCC	Capital	63	63	63
	Premium Leverage	1.50	1.60	1.71
DSCC	Capital	68	68	68
	Premium Leverage	1.39	1.48	1.59

The results are the same as for a change in premium adequacy except for the LSCC method. Because changing the ceding commission percentage on a net rated deal does not change the net underwriting loss, the LSCC method now agrees with the DSCC and the other underwriting loss based methods in indicating capital should not change between the scenarios.

Next we examine how the methods respond to a sliding scale commission plan. Results are shown in Table 3 for several different slides. The "Balanced" slide leads to no change in expected commission, the "Avg Inc" slide generates a net increase in expected commission, and the "Avg Dec" slide produces an average net decrease. The fixed premium and fixed loss leverage methods are of course totally unresponsive to changes in risk induced by any adjustable commission plan. LSCC and DSCC are responsive to the introduction of the slide, but then seem oblivious to the different slide options. The reason is that the minimum commission and corresponding loss ratio were picked to be the same for all the options. So all the slides yield the same net underwriting loss and risk quantity in the adverse scenarios that determine LSCC and DSCC. This underscores a positive feature of both our coherent capital methods: changing the distribution of favorable outcomes does not change the required capital.

Table 3	Sensitivity of Capital to Sliding Scale Commission				
Method		**No Slide**	**Balanced**	**Avg Inc Cede**	**Avg Dec Cede**
Fixed Premium Leverage	Capital	68	68	68	68
	Premium Leverage	1.48	1.48	1.48	1.48
Fixed Loss Leverage	Capital	68	68	68	68
	Premium Leverage	1.48	1.48	1.48	1.48
Standard Deviation	Capital	68	64	63	63
	Premium Leverage	1.48	1.56	1.59	1.59
Variance	Capital	68	61	59	58
	Premium Leverage	1.48	1.64	1.70	1.71
LSCC	Capital	63	58	58	58
	Premium Leverage	1.60	1.74	1.74	1.74
DSCC	Capital	68	63	63	63
	Premium Leverage	1.48	1.60	1.60	1.60

Now, we examine sensitivity under a Swing Rated Premium plan. In one scenario, the Max and Min are set so the average premium in the plan is balanced back to the premium in the Base Case without Swing Rating. In another, we reduce the Max and, in the other, we raise the Max. The swings we have in our example are more modest than those typically found in practice. Note we set capital under the Fixed Premium Leverage method relative to the Provisional Premium and not the expected Swing Premium. The swing in all scenarios shortens the tail of the underwriting loss distribution relative to the fixed premium Base Case. It does not change the shape of the tail as much as pull it towards the mean. However one would characterize it, the swing reduces volatility and as a result, all the methods, both coherent and non-coherent, that are based on the underwriting loss distribution indicate that less capital is needed. This is true even in the Balanced Case where there is no change in the expected underwriting loss. Table 4 summarizes the results found in Exhibit 5.

Table 4	Sensitivity of Capital to Swing Rated Premium				
Method		**No Swing**	**Balanced**	**Lower Max**	**Raise Max**
Fixed Premium Leverage	Capital	68	68	68	68
	Premium Leverage	1.48	1.48	1.48	1.48
Fixed Loss Leverage	Capital	68	68	68	68
	Premium Leverage	1.48	1.48	1.48	1.48
Standard Deviation	Capital	68	61	64	59
	Premium Leverage	1.48	1.65	1.56	1.70
Variance	Capital	68	55	61	51
	Premium Leverage	1.48	1.83	1.65	1.95
LSCC	Capital	63	52	59	48
	Premium Leverage	1.60	1.92	1.70	2.07
DSCC	Capital	68	57	64	53
	Premium Leverage	1.48	1.75	1.57	1.88

Next we examine how the methods respond to changing the share a reinsurer has in a deal. We know that both DSCC and LSCC have the scaling property and so their capital requirements scale up and down with the share and their indicated leverage ratios do not change. The Standard Deviation of Underwriting Loss scales as well, yet the Variance of Underwriting Loss does not. Results are shown in Table 5.

Table 5	Sensitivity of Capital to Changes in Share			
Method		**Base Share**	**2X Share**	**(1/2) X Share**
Fixed Premium Leverage	Capital	68	135	34
	Premium Leverage	1.48	1.48	1.48
Fixed Loss Leverage	Capital	68	135	34
	Premium Leverage	1.48	1.48	1.48
Standard Deviation	Capital	68	135	34
	Premium Leverage	1.48	1.48	1.48
Variance	Capital	68	270	17
	Premium Leverage	1.48	0.74	2.96
DSCC	Capital	63	125	31
	Premium Leverage	1.60	1.60	1.60
LSCC	Capital	68	135	34
	Premium Leverage	1.48	1.48	1.48

Finally, we consider different layering scenarios. In one, the reinsurer can take a lower per occurrence layer, in another a layer just above it, and in the last scenario it can take both layers together. We examine how the capital on the combined scenario compares with sum of the capital on the separate layer scenarios. Results are shown in Table 6.

Table 6	**Capital by Layer**				
Method		**Layer 1**	**Layer 2**	**Sum of Capital**	**Combined**
Fixed Premium Leverage	Capital	47	21	68	68
	Premium Leverage	1.48	1.48	1.48	1.48
Fixed Loss Leverage	Capital	47	21	68	68
	Premium Leverage	1.48	1.48	1.48	1.48
Standard Deviation	Capital	19	50	70	68
	Premium Leverage	3.58	0.62	1.44	1.48
Variance	Capital	5	38	43	68
	Premium Leverage	12.54	0.83	2.32	1.48
DSCC	Capital	16	47	63	63
	Premium Leverage	4.36	0.67	1.60	1.60
LSCC	Capital	19	48	68	68
	Premium Leverage	1.60	1.60	1.60	1.48

The sum of the variance-based capital requirements for the layers is less than the variance based capital for the combined layer, whereas the opposite is true for the standard deviation based capital requirement. Coherence would say combining the layers should not increase the capital. For our coherent capital measures, it so happened in our example that the sum of capital for the layers equaled the capital for the combined layer. Whether this is true in general, or is an artifact of the way our example was constructed is an issue that awaits further study. In reinsurance circles, many believe "ventilation" is an effective risk reduction strategy. Using a ventilation approach, the reinsurer takes shares of disconnected layers. It would be useful to see what coherence can tell us about such a strategy.

7. CONCLUSION

We have seen that basing a capital calculation on the net underwriting loss handles many of the problems that arise in setting capital for reinsurance treaty pricing applications. We have also found that implementation of coherent capital concepts does handle some of the problems that remain. Yet we have no theoretical reason to prefer one of our two coherent capital approaches above the other. So we present the results we have as interim steps taken to advance understanding of how to apply coherence concepts to tackle practical problems. We believe these initial results on how to implement coherent capital are useful in their own right and will provide a solid foundation and some direction for future research on the topic.

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Biographies of the Authors[1]

Ira Robbin

After receiving a BS in Mathematics from Michigan State University and a PhD in Mathematics from Rutgers University, the author took an actuarial research position with the Insurance Company of North America. Over his career, the author has directed pricing for casualty cash flow business, provided actuarial support for large risk property business, developed ROE and capital calculation models for pricing, and done other work where the latest theory often runs into difficulty when faced with the problems of practical application. The author has written papers on risk load, credibility estimation of IBNR, pricing ROE models, excess of aggregate insurance charges, loss development models, and other diverse topics. He has made presentations at CAS Meetings and Seminars, taught CAS examination courses, wrote a study note on profit provisions and served on several industry committees, including the ISO Increased Limits Committee and the NCCI Individual Risk Rating Plan Subcommittee. Currently, the author is a Senior Pricing Actuary with PartnerRe in Greenwich, where he has been pricing treaties and helping to develop improved pricing models.

Jesse De Couto

The author developed a strong interest in research and problem-solving while obtaining his BS in Biomedical Engineering at the University of Miami. There he designed and validated an active Laplacian bipolar concentric ring ECG sensor. The author then continued on to graduate from the College of Insurance with an MBA in Financial Risk Management. He took an actuarial position with PartnerRe in Greenwich and worked on pricing Regional and Specialty Lines treaties and on developing treaty ROE software. The author is now an Assistant Underwriter and Actuary for the Catastrophe Business Unit of PartnerRe in Bermuda.

[1] The opinions expressed in this paper are those of the authors. No representation of the corporate position of PartnerRe should be inferred.

Exhibit 1

TVaR versus VaR

Scenario	Probability	Portfolio 1 Losses	Portfolio 2 Losses	Combined Portfolio Losses
1	10%	100	10	110
2	10%	80	20	100
3	10%	50	80	130
4	10%	40	100	140
5	10%	30	40	70
6	10%	20	30	50
7	10%	10	50	60
8	10%	20	30	50
9	10%	30	5	35
10	10%	5	20	25
VaR at 80th Percentile		**50**	**50**	**110**
TVaR at 80th Percentile Level		**90**	**90**	**135**

Sensitivity of Capital to Changes in Premium Adequacy

Exhibit 2
Sheet 1

Scenario 1

Base Case	
Prov Premium	100.0
Prov Ceding %	25.0%

Results by Method	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	67.5	67.5	62.5	67.5
Prem Leverage	1.481	1.481	1.481	1.481	1.600	1.481
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	0.0	0.0	100	25.0	-50.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	0.0	100	25.0	-30.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	0.0	100	25.0	-20.0	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	0.0	100	25.0	-10.0	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	0.0	100	25.0	0.0	0.0	5.0
5%	85%	90.0	100.0	25.0	0.0	0.0	100	25.0	15.0	15.0	20.0
5%	90%	110.0	100.0	25.0	0.0	0.0	100	25.0	35.0	35.0	40.0
5%	95%	150.0	100.0	25.0	0.0	0.0	100	25.0	75.0	75.0	80.0
5%	100%	200.0	100.0	25.0	0.0	0.0	100	25.0	125.0	125.0	130.0
	Expected Values:	**70.0**	**100.0**	**25.0**	**0.0**	**0.0**	**100.0**	**25.0**	**-5.0**	**12.5**	**14.0**

Sensitivity of Capital to Changes in Premium Adequacy

Exhibit 2
Sheet 2

Scenario 2

Rate Change	10.0%
Prov Premium	110.0
Prov Ceding %	25.0%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	74.3	67.5	67.5	67.5	55.0	67.5
Prem Leverage	1.481	1.630	1.630	1.630	2.000	1.630
Loss Leverage	0.943	1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	25.0	110.0	27.5	0.0	0.0	110	27.5	-57.5	0.0	0.0
20%	30%	45.0	110.0	27.5	0.0	0.0	110	27.5	-37.5	0.0	0.0
25%	55%	55.0	110.0	27.5	0.0	0.0	110	27.5	-27.5	0.0	0.0
15%	70%	65.0	110.0	27.5	0.0	0.0	110	27.5	-17.5	0.0	0.0
10%	80%	75.0	110.0	27.5	0.0	0.0	110	27.5	-7.5	0.0	5.0
5%	85%	90.0	110.0	27.5	0.0	0.0	110	27.5	7.5	7.5	20.0
5%	90%	110.0	110.0	27.5	0.0	0.0	110	27.5	27.5	27.5	40.0
5%	95%	150.0	110.0	27.5	0.0	0.0	110	27.5	67.5	67.5	80.0
5%	100%	200.0	110.0	27.5	0.0	0.0	110	27.5	117.5	117.5	130.0
	Expected Values:	70.0	110.0	27.5	0.0	0.0	110.0	27.5	-12.5	11.0	14.0

Sensitivity of Capital to Changes in Premium Adequacy

Exhibit 2
Sheet 3

Scenario 3

Rate Change	-10.0%
Prov Premium	90.0
Prov Ceding %	25.0%

Results by Method	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	60.8	67.5	67.5	67.5	70.0	67.5
Prem Leverage	1.481	1.333	1.333	1.333	1.286	1.333
Loss Leverage	1.152	1.037				
Scale Constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	25.0	90.0	22.5	0.0	0.0	90	22.5	-42.5	0.0	0.0
20%	30%	45.0	90.0	22.5	0.0	0.0	90	22.5	-22.5	0.0	0.0
25%	55%	55.0	90.0	22.5	0.0	0.0	90	22.5	-12.5	0.0	0.0
15%	70%	65.0	90.0	22.5	0.0	0.0	90	22.5	-2.5	0.0	0.0
10%	80%	75.0	90.0	22.5	0.0	0.0	90	22.5	7.5	7.5	5.0
5%	85%	90.0	90.0	22.5	0.0	0.0	90	22.5	22.5	22.5	20.0
5%	90%	110.0	90.0	22.5	0.0	0.0	90	22.5	42.5	42.5	40.0
5%	95%	150.0	90.0	22.5	0.0	0.0	90	22.5	82.5	82.5	80.0
5%	100%	200.0	90.0	22.5	0.0	0.0	90	22.5	132.5	132.5	130.0
	Expected Values:	70.0	90.0	22.5	0.0	0.0	90.0	22.5	2.5	14.8	14.0

Sensitivity of Capital to Changes in Ceding Comm Rate on Net Rated Treaty

Exhibit 3
Sheet 1

Scenario 1

Base Case	
Prov Premium	100.0
Prov Ceding %	25.0%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	67.5	67.5	62.5	67.5
Prem Leverage	1.481	1.481	1.481	1.481	1.600	1.481
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	0.0	0.0	100	25.0	-50.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	0.0	100	25.0	-30.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	0.0	100	25.0	-20.0	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	0.0	100	25.0	-10.0	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	0.0	100	25.0	0.0	0.0	5.0
5%	85%	90.0	100.0	25.0	0.0	0.0	100	25.0	15.0	15.0	20.0
5%	90%	110.0	100.0	25.0	0.0	0.0	100	25.0	35.0	35.0	40.0
5%	95%	150.0	100.0	25.0	0.0	0.0	100	25.0	75.0	75.0	80.0
5%	100%	200.0	100.0	25.0	0.0	0.0	100	25.0	125.0	125.0	130.0
	Expected Values:	70.0	100.0	25.0	0.0	0.0	100.0	25.0	-5.0	12.5	14.0

Sensitivity of Capital to Changes in Ceding Comm Rate on Net Rated Treaty

Exhibit 3
Sheet 2

Scenario 2

Comm Change	5.0%
Prov Premium	107.1
Prov Ceding %	30.0%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	72.3	67.5	67.5	67.5	62.5	67.5
Prem Leverage	1.481	1.587	1.587	1.587	1.714	1.587
Loss Leverage	0.968	1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	25.0	107.1	32.1	0.0	0.0	107.1	32.1	-50.0	0.0	0.0
20%	30%	45.0	107.1	32.1	0.0	0.0	107.1	32.1	-30.0	0.0	0.0
25%	55%	55.0	107.1	32.1	0.0	0.0	107.1	32.1	-20.0	0.0	0.0
15%	70%	65.0	107.1	32.1	0.0	0.0	107.1	32.1	-10.0	0.0	0.0
10%	80%	75.0	107.1	32.1	0.0	0.0	107.1	32.1	0.0	0.0	5.0
5%	85%	90.0	107.1	32.1	0.0	0.0	107.1	32.1	15.0	15.0	20.0
5%	90%	110.0	107.1	32.1	0.0	0.0	107.1	32.1	35.0	35.0	40.0
5%	95%	150.0	107.1	32.1	0.0	0.0	107.1	32.1	75.0	75.0	80.0
5%	100%	200.0	107.1	32.1	0.0	0.0	107.1	32.1	125.0	125.0	130.0
	Expected Values:	70.0	107.1	32.1	0.0	0.0	107.1	32.1	-5.0	12.5	14.0

Sensitivity of Capital to Changes in Ceding Comm Rate on Net Rated Treaty

Exhibit 3
Sheet 3

Scenario 3

Comm Change	-5.0%
Prov Premium	93.8
Prov Ceding %	20.0%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	63.3	67.5	67.5	67.5	62.5	67.5
Prem Leverage	1.481	1.389	1.389	1.389	1.500	1.389
Loss Leverage	1.106	1.037				
Scale Constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	25.0	93.8	18.8	0.0	0.0	94	18.8	-50.0	0.0	0.0
20%	30%	45.0	93.8	18.8	0.0	0.0	94	18.8	-30.0	0.0	0.0
25%	55%	55.0	93.8	18.8	0.0	0.0	94	18.8	-20.0	0.0	0.0
15%	70%	65.0	93.8	18.8	0.0	0.0	94	18.8	-10.0	0.0	0.0
10%	80%	75.0	93.8	18.8	0.0	0.0	94	18.8	0.0	0.0	5.0
5%	85%	90.0	93.8	18.8	0.0	0.0	94	18.8	15.0	15.0	20.0
5%	90%	110.0	93.8	18.8	0.0	0.0	94	18.8	35.0	35.0	40.0
5%	95%	150.0	93.8	18.8	0.0	0.0	94	18.8	75.0	75.0	80.0
5%	100%	200.0	93.8	18.8	0.0	0.0	94	18.8	125.0	125.0	130.0
	Expected Values:	**70.0**	**93.8**	**18.8**	**0.0**	**0.0**	**93.8**	**18.8**	**-5.0**	**12.5**	**14.0**

Sensitivity of Capital to Sliding Scale Commission Plan

Exhibit 4
Sheet 1

Scenario 1

Slide	No Slide
Prov Premium	100.0
Prov Ceding %	25.0%

No Slide	Commission	LR
Min	25%	45%
Provision	25%	60%
Max	25%	80%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	67.5	67.5	62.5	67.5
Prem Leverage	1.481	1.481	1.481	1.481	1.600	1.481
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	0.0	0.0	100	25.0	-50.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	0.0	100	25.0	-30.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	0.0	100	25.0	-20.0	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	0.0	100	25.0	-10.0	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	0.0	100	25.0	0.0	0.0	5.0
5%	85%	90.0	100.0	25.0	0.0	0.0	100	25.0	15.0	15.0	20.0
5%	90%	110.0	100.0	25.0	0.0	0.0	100	25.0	35.0	35.0	40.0
5%	95%	150.0	100.0	25.0	0.0	0.0	100	25.0	75.0	75.0	80.0
5%	100%	200.0	100.0	25.0	0.0	0.0	100	25.0	125.0	125.0	130.0
	Expected Values:	70.0	100.0	25.0	0.0	0.0	100.0	25.0	-5.0	12.5	14.0

Sensitivity of Capital to Sliding Scale Commission Plan

Exhibit 4
Sheet 2

Scenario 2

Slide	Balanced
Prov Premium	100.0
Prov Ceding %	25.0%

Slide	Commission	LR
Min	20%	85%
Provision	25%	70%
Max	27%	50%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	64.2	61.1	57.5	62.6
Prem Leverage	1.481	1.481	1.557	1.637	1.739	1.597
Loss Leverage	1.037	1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	25.0	100.0	25.0	0.0	2.0	100.0	27.0	-48.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	2.0	100.0	27.0	-28.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	1.5	100.0	26.5	-18.5	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	0.5	100.0	25.5	-9.5	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	-1.7	100.0	23.3	-1.7	0.0	3.5
5%	85%	90.0	100.0	25.0	0.0	-5.0	100.0	20.0	10.0	10.0	15.1
5%	90%	110.0	100.0	25.0	0.0	-5.0	100.0	20.0	30.0	30.0	35.1
5%	95%	150.0	100.0	25.0	0.0	-5.0	100.0	20.0	70.0	70.0	75.1
5%	100%	200.0	100.0	25.0	0.0	-5.0	100.0	20.0	120.0	120.0	125.1
	Expected Values:	**70.0**	**100.0**	**25.0**	**0.0**	**-0.1**	**100.0**	**24.9**	**-5.1**	**11.5**	**12.9**

Sensitivity of Capital to Sliding Scale Commission Plan

Exhibit 4
Sheet 3

Scenario 3

Slide	Avg Inc Cede
Prov Premium	100.0
Prov Ceding %	25.0%

Slide	Commission	LR
Min	20%	85%
Provision	25%	70%
Max	30%	50%

Results by Method						
	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	63.1	58.9	57.5	62.6
Prem Leverage	1.481	1.481	1.586	1.697	1.739	1.597
Loss Leverage	1.037	1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	25.0	100.0	25.0	0.0	5.0	100	30.0	-45.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	5.0	100	30.0	-25.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	3.8	100	28.8	-16.3	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	1.3	100	26.3	-8.8	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	-1.7	100	23.3	-1.7	0.0	3.5
5%	85%	90.0	100.0	25.0	0.0	-5.0	100	20.0	10.0	10.0	15.1
5%	90%	110.0	100.0	25.0	0.0	-5.0	100	20.0	30.0	30.0	35.1
5%	95%	150.0	100.0	25.0	0.0	-5.0	100	20.0	70.0	70.0	75.1
5%	100%	200.0	100.0	25.0	0.0	-5.0	100	20.0	120.0	120.0	125.1
	Expected Values:	70.0	100.0	25.0	0.0	1.5	100.0	26.5	-3.5	11.5	12.9

Sensitivity of Capital to Sliding Scale Commission Plan

Exhibit 4
Sheet 4

Scenario 4

Slide	Avg Dec Cede
Prov Premium	100.0
Prov Ceding %	25.0%

Slide	Commission	LR
Min	20%	85%
Provision	25%	85%
Max	30%	50%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	62.8	58.4	57.5	62.6
Prem Leverage	1.481	1.481	1.593	1.713	1.739	1.597
Loss Leverage	1.037	1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	25.0	100.0	25.0	0.0	5.0	100	30.0	-45.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	5.0	100	30.0	-25.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	4.3	100	29.3	-15.7	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	2.9	100	27.9	-7.1	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	1.4	100	26.4	1.4	1.4	6.5
5%	85%	90.0	100.0	25.0	0.0	-5.0	100	20.0	10.0	10.0	15.1
5%	90%	110.0	100.0	25.0	0.0	-5.0	100	20.0	30.0	30.0	35.1
5%	95%	150.0	100.0	25.0	0.0	-5.0	100	20.0	70.0	70.0	75.1
5%	100%	200.0	100.0	25.0	0.0	-5.0	100	20.0	120.0	120.0	125.1
	Expected Values:	70.0	100.0	25.0	0.0	2.1	100.0	27.1	-2.9	11.6	13.2

Sensitivity of Capital to Swing Rated Premium Plan

Exhibit 5
Sheet 1

Scenario 1

Swing	No Swing
Prov Premium	100.0
Prov Ceding %	25.0%

No Swing	
Min	100%
Prov	100%
Max	100%
Margin	5%
Load Factor	133%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	67.5	67.5	62.5	67.5
Prem Leverage	1.481	1.481	1.481	1.481	1.600	1.481
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	0.0	0.0	100	25.0	-50.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	0.0	100	25.0	-30.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	0.0	100	25.0	-20.0	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	0.0	100	25.0	-10.0	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	0.0	100	25.0	0.0	0.0	5.0
5%	85%	90.0	100.0	25.0	0.0	0.0	100	25.0	15.0	15.0	20.0
5%	90%	110.0	100.0	25.0	0.0	0.0	100	25.0	35.0	35.0	40.0
5%	95%	150.0	100.0	25.0	0.0	0.0	100	25.0	75.0	75.0	80.0
5%	100%	200.0	100.0	25.0	0.0	0.0	100	25.0	125.0	125.0	130.0
	Expected Values:	70.0	100.0	25.0	0.0	0.0	100.0	25.0	-5.0	12.5	14.0

Sensitivity of Capital to Swing Rated Premium Plan

Exhibit 5
Sheet 2

Scenario 2

Swing	Balanced
Prov Premium	100.0
Prov Ceding %	25.0%

Balanced	
Min	95%
Prov	100%
Max	114%
Margin	7.0%
Load Factor	133%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	60.7	54.6	52.0	57.0
Prem Leverage	1.481	1.481	1.647	1.831	1.923	1.754
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	-5.0	-1.3	95	23.8	-46.3	0.0	0.0
20%	30%	45.0	100.0	25.0	-5.0	-1.3	95	23.8	-26.3	0.0	0.0
25%	55%	55.0	100.0	25.0	-5.0	-1.3	95	23.8	-16.3	0.0	0.0
15%	70%	65.0	100.0	25.0	-5.0	-1.3	95	23.8	-6.3	0.0	0.0
10%	80%	75.0	100.0	25.0	7.0	1.8	107	26.8	-5.3	0.0	0.0
5%	85%	90.0	100.0	25.0	14.0	3.5	114	28.5	4.5	4.5	9.5
5%	90%	110.0	100.0	25.0	14.0	3.5	114	28.5	24.5	24.5	29.5
5%	95%	150.0	100.0	25.0	14.0	3.5	114	28.5	64.5	64.5	69.5
5%	100%	200.0	100.0	25.0	14.0	3.5	114	28.5	114.5	114.5	119.5
	Expected Values:	70.0	100.0	25.0	0.0	0.0	100.0	25.0	-5.0	10.4	11.4

Sensitivity of Capital to Swing Rated Premium Plan

Exhibit 5
Sheet 3

Scenario 3

	Swing	Avg Dec Prem
Prov Premium		100.0
Prov Ceding %		25.0%

Balanced	
Min	95%
Prov	100%
Max	105%
Margin	7.0%
Load Factor	133%

Results by Method						
	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	64.0	60.7	58.8	63.8
Prem Leverage	1.481	1.481	1.562	1.647	1.702	1.569
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	-5.0	-1.3	95	23.8	-46.3	0.0	0.0
20%	30%	45.0	100.0	25.0	-5.0	-1.3	95	23.8	-26.3	0.0	0.0
25%	55%	55.0	100.0	25.0	-5.0	-1.3	95	23.8	-16.3	0.0	0.0
15%	70%	65.0	100.0	25.0	-5.0	-1.3	95	23.8	-6.3	0.0	0.0
10%	80%	75.0	100.0	25.0	5.0	1.3	105	26.3	-3.8	0.0	1.3
5%	85%	90.0	100.0	25.0	5.0	1.3	105	26.3	11.3	11.3	16.3
5%	90%	110.0	100.0	25.0	5.0	1.3	105	26.3	31.3	31.3	36.3
5%	95%	150.0	100.0	25.0	5.0	1.3	105	26.3	71.3	71.3	76.3
5%	100%	200.0	100.0	25.0	5.0	1.3	105	26.3	121.3	121.3	126.3
	Expected Values:	**70.0**	**100.0**	**25.0**	**-2.0**	**-0.5**	**98.0**	**24.5**	**-3.5**	**11.8**	**12.9**

Sensitivity of Capital to Swing Rated Premium Plan

Exhibit 5
Sheet 4

Scenario 4

Swing	Avg Inc Prem
Prov Premium	100.0
Prov Ceding %	25.0%

Balanced	
Min	95%
Prov	100%
Max	119%
Margin	7.0%
Load Factor	133%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	58.9	51.4	48.3	53.3
Prem Leverage	1.481	1.481	1.698	1.946	2.073	1.878
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	-5.0	-1.3	95	23.8	-46.3	0.0	0.0
20%	30%	45.0	100.0	25.0	-5.0	-1.3	95	23.8	-26.3	0.0	0.0
25%	55%	55.0	100.0	25.0	-5.0	-1.3	95	23.8	-16.3	0.0	0.0
15%	70%	65.0	100.0	25.0	-5.0	-1.3	95	23.8	-6.3	0.0	0.0
10%	80%	75.0	100.0	25.0	7.0	1.8	107	26.8	-5.3	0.0	0.0
5%	85%	90.0	100.0	25.0	19.0	4.8	119	29.8	0.8	0.8	5.8
5%	90%	110.0	100.0	25.0	19.0	4.8	119	29.8	20.8	20.8	25.8
5%	95%	150.0	100.0	25.0	19.0	4.8	119	29.8	60.8	60.8	65.8
5%	100%	200.0	100.0	25.0	19.0	4.8	119	29.8	110.8	110.8	115.8
	Expected Values:	70.0	100.0	25.0	1.0	0.3	101.0	25.3	-5.8	9.7	10.7

Sensitivity of Capital to Changes in Share

Exhibit 6
Sheet 1

Scenario 1

Base Case	
Prov Premium	100.0
Prov Ceding %	25.0%

Results by Method	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	67.5	67.5	62.5	67.5
Prem Leverage	1.481	1.481	1.481	1.481	1.600	1.481
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	0.0	0.0	100	25.0	-50.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	0.0	100	25.0	-30.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	0.0	100	25.0	-20.0	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	0.0	100	25.0	-10.0	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	0.0	100	25.0	0.0	0.0	5.0
5%	85%	90.0	100.0	25.0	0.0	0.0	100	25.0	15.0	15.0	20.0
5%	90%	110.0	100.0	25.0	0.0	0.0	100	25.0	35.0	35.0	40.0
5%	95%	150.0	100.0	25.0	0.0	0.0	100	25.0	75.0	75.0	80.0
5%	100%	200.0	100.0	25.0	0.0	0.0	100	25.0	125.0	125.0	130.0
	Expected Values:	70.0	100.0	25.0	0.0	0.0	100.0	25.0	-5.0	12.5	14.0

Sensitivity of Capital to Changes in Share

Exhibit 6
Sheet 2

Scenario 2

Share Multiple	200.0%
Prov Premium	200.0
Prov Ceding %	25.0%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	135.0	135.0	135.0	270.0	125.0	135.0
Prem Leverage	1.481	1.481	1.481	0.741	1.600	1.481
Loss Leverage	1.037	1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	50.0	200.0	50.0	0.0	0.0	200	50.0	-100.0	0.0	0.0
20%	30%	90.0	200.0	50.0	0.0	0.0	200	50.0	-60.0	0.0	0.0
25%	55%	110.0	200.0	50.0	0.0	0.0	200	50.0	-40.0	0.0	0.0
15%	70%	130.0	200.0	50.0	0.0	0.0	200	50.0	-20.0	0.0	0.0
10%	80%	150.0	200.0	50.0	0.0	0.0	200	50.0	0.0	0.0	10.0
5%	85%	180.0	200.0	50.0	0.0	0.0	200	50.0	30.0	30.0	40.0
5%	90%	220.0	200.0	50.0	0.0	0.0	200	50.0	70.0	70.0	80.0
5%	95%	300.0	200.0	50.0	0.0	0.0	200	50.0	150.0	150.0	160.0
5%	100%	400.0	200.0	50.0	0.0	0.0	200	50.0	250.0	250.0	260.0
	Expected Values:	140.0	200.0	50.0	0.0	0.0	200.0	50.0	-10.0	25.0	28.0

Sensitivity of Capital to Changes in Share

Exhibit 6
Sheet 3

Scenario 3

Share Multiple	50.0%
Prov Premium	50.0
Prov Ceding %	25.0%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	33.8	33.8	33.8	16.9	31.3	33.8
Prem Leverage	1.481	1.481	1.481	2.963	1.600	1.481
Loss Leverage	1.037	1.037				
Scale Constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	12.5	50.0	12.5	0.0	0.0	50	12.5	-25.0	0.0	0.0
20%	30%	22.5	50.0	12.5	0.0	0.0	50	12.5	-15.0	0.0	0.0
25%	55%	27.5	50.0	12.5	0.0	0.0	50	12.5	-10.0	0.0	0.0
15%	70%	32.5	50.0	12.5	0.0	0.0	50	12.5	-5.0	0.0	0.0
10%	80%	37.5	50.0	12.5	0.0	0.0	50	12.5	0.0	0.0	2.5
5%	85%	45.0	50.0	12.5	0.0	0.0	50	12.5	7.5	7.5	10.0
5%	90%	55.0	50.0	12.5	0.0	0.0	50	12.5	17.5	17.5	20.0
5%	95%	75.0	50.0	12.5	0.0	0.0	50	12.5	37.5	37.5	40.0
5%	100%	100.0	50.0	12.5	0.0	0.0	50	12.5	62.5	62.5	65.0
	Expected Values:	35.0	50.0	12.5	0.0	0.0	50.0	12.5	-2.5	6.3	7.0

Capital by Layer

Exhibit 7
Sheet 1

Scenario 1

Base Case	Combined
Prov Premium	100.0
Prov Ceding %	25.0%

Results by Method	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	67.5	67.5	67.5	67.5	62.5	67.5
Prem Leverage	1.481	1.481	1.481	1.481	1.600	1.481
Loss Leverage		1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	UW Loss Excess of Expectation
10%	10%	25.0	100.0	25.0	0.0	0.0	100	25.0	-50.0	0.0	0.0
20%	30%	45.0	100.0	25.0	0.0	0.0	100	25.0	-30.0	0.0	0.0
25%	55%	55.0	100.0	25.0	0.0	0.0	100	25.0	-20.0	0.0	0.0
15%	70%	65.0	100.0	25.0	0.0	0.0	100	25.0	-10.0	0.0	0.0
10%	80%	75.0	100.0	25.0	0.0	0.0	100	25.0	0.0	0.0	5.0
5%	85%	90.0	100.0	25.0	0.0	0.0	100	25.0	15.0	15.0	20.0
5%	90%	110.0	100.0	25.0	0.0	0.0	100	25.0	35.0	35.0	40.0
5%	95%	150.0	100.0	25.0	0.0	0.0	100	25.0	75.0	75.0	80.0
5%	100%	200.0	100.0	25.0	0.0	0.0	100	25.0	125.0	125.0	130.0
	Expected Values:	70.0	100.0	25.0	0.0	0.0	100.0	25.0	-5.0	12.5	14.0

Capital by Layer

Exhibit 7
Sheet 2

Scenario 2

Per Occ Layer	1
Prov Premium	68.9
Prov Ceding %	25.0%

Results by Method						
	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	46.5	46.5	19.3	5.5	15.8	19.3
Prem Leverage	1.481	1.481	3.578	12.539	4.362	3.581
Loss Leverage	1.037	1.037				
Scale constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	25.0	68.9	17.2	0.0	0.0	69	17.2	-26.7	0.0	0.0
20%	30%	40.0	68.9	17.2	0.0	0.0	69	17.2	-11.7	0.0	0.0
25%	55%	45.0	68.9	17.2	0.0	0.0	69	17.2	-6.7	0.0	0.0
15%	70%	50.0	68.9	17.2	0.0	0.0	69	17.2	-1.7	0.0	1.8
10%	80%	55.0	68.9	17.2	0.0	0.0	69	17.2	3.3	3.3	6.8
5%	85%	60.0	68.9	17.2	0.0	0.0	69	17.2	8.3	8.3	11.8
5%	90%	65.0	68.9	17.2	0.0	0.0	69	17.2	13.3	13.3	16.8
5%	95%	70.0	68.9	17.2	0.0	0.0	69	17.2	18.3	18.3	21.8
5%	100%	75.0	68.9	17.2	0.0	0.0	69	17.2	23.3	23.3	26.8
	Expected Values:	48.3	68.9	17.2	0.0	0.0	68.9	17.2	-3.4	3.5	4.8

Capital by Layer

Exhibit 7
Sheet 3

Scenario 3

Per Occ Layer	2
Prov Premium	31.1
Prov Ceding %	25.0%

Results by Method

	Fixed Prem Leverage	Fixed Loss Leverage	Stnd Dev UW Loss	Variance UW Loss	LSCC	DSCC
Capital	21.0	21.0	50.4	37.6	46.7	48.3
Prem Leverage	1.481	1.481	0.617	0.827	0.665	0.644
Loss Leverage	1.037	1.037				
Scale Constant			122%	1.48%		

Probability	Cumulative Probability	Loss	Provisional Premium	Provisional Commission	Premium Adjustment	Commission Adjustment	Final Premium	Final Commission	UW Loss	Capped UW Loss	Capped UW Loss Xs of Expectation
10%	10%	0.0	31.1	7.8	0.0	0.0	31	7.8	-23.3	0.0	0.0
20%	30%	5.0	31.1	7.8	0.0	0.0	31	7.8	-18.3	0.0	0.0
25%	55%	10.0	31.1	7.8	0.0	0.0	31	7.8	-13.3	0.0	0.0
15%	70%	15.0	31.1	7.8	0.0	0.0	31	7.8	-8.3	0.0	0.0
10%	80%	20.0	31.1	7.8	0.0	0.0	31	7.8	-3.3	0.0	0.0
5%	85%	30.0	31.1	7.8	0.0	0.0	31	7.8	6.7	6.7	8.3
5%	90%	45.0	31.1	7.8	0.0	0.0	31	7.8	21.7	21.7	23.3
5%	95%	80.0	31.1	7.8	0.0	0.0	31	7.8	56.7	56.7	58.3
5%	100%	125.0	31.1	7.8	0.0	0.0	31	7.8	101.7	101.7	103.3
	Expected Values:	21.8	31.1	7.8	0.0	0.0	31.1	7.8	-1.6	9.3	9.7

Stochastic Excess-of-Loss Pricing within a Financial Framework

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Abstract

This paper is aimed at the practicing actuary to introduce the theory of extreme values and a financial framework to price excess-of-loss reinsurance treaties. We introduce the reader to extreme value theory via the classical central limit theorem. Two key results in extreme value theory are presented and illustrated with concrete examples. The discussion then moves on to collective risk models, considerations in modeling tail events, and measures of risk. All these concepts are brought together with the modeling of actual losses. In the last section of the paper all previous elements are brought together with a financial framework for the pricing of a layer of reinsurance. The cash flows between the insurance company and its equity holders are modeled.

Keywords. Collective Risk Model, Experience Rating, Extreme Event Modeling, Extreme Values, IRR, Large Loss and Extreme Event Loading, Monte Carlo Valuation, Reinsurance Excess (Non-Proportional), Risk Pricing and Valuation Models, Simulation, Tail-Value-at-Risk.

1 Introduction

The main goal of this paper is to give the practicing actuary some tools (such as extreme value theory, collective risk models, risk measures, and a cash flow model) for the pricing of excess-of-loss reinsurance treaties. In particular, we have in mind the pricing of high layers of reinsurance where empirical data is scarce and reliance on a mathematical model of the tail of the loss distribution is necessary.

We introduce extreme value theory through the central limit theorem. The central limit theorem tells us that the limiting distribution of the sample mean is a normal distribution. The analogous result from extreme value theory is that the limiting distribution of the sample maximum is an extreme value distribution.[1] There are three distinct families of extreme value distributions: the Fréchet, Weibull, and Gumbel

[1]We are not being very precise but the gist of the result is correct.

distributions. But these three families can be represented as a one parameter family of distributions.

The next result in extreme value theory is the key result for pricing excess-of-loss reinsurance treaties. This result shows that under certain circumstances the limiting distribution of the excess portion of a loss approaches the generalized Pareto distribution (as the threshold increases). This result provides the theoretical underpinnings for using the generalized Pareto distribution in reinsurance excess-of-loss pricing.

At this point we have a good theoretical model. The rest of the paper is a hands-on approach to pricing an excess-of-loss treaty within a financial framework. In Section 3 we introduce the collective risk model together with the underlying data necessary for pricing. We guide the reader with the adjustments necessary to get the data ready for use in modeling the tail of the distribution of losses. We discuss graphical techniques and the estimation of the parameters for both loss and claim count distributions.

In Section 4 we introduce the collective risk model [4, 26] and various measures to quantify risk: standard deviation or variance, value at risk, tail value at risk, expected policyholder deficit, and probability of ruin. We also discuss the concept of rented capital and incorporate that into our cash flow model.

In the last section we bring everything together to determine the price of a reinsurance layer. Our methodology revolves around the concept of the implied equity flows [10]. The equity flows represent the transfer of money between the insurance company and its equity holders.[2] Our cash flow model is comprehensive. It includes all relevant components of cash flow for an insurance company: underwriting operations, investment activity, assets (both income and non-income producing), and taxes. Our model does not take a simplistic view of taxes where most actuaries in the past have calculated them as a straight percentage applied to the results of each calendar year. Instead we compute the taxable income according to the Internal Revenue Service tax code.

In Appendix B we provide a full set of exhibits showing all the components of the cash flows and the implied equity flows.

[2] One should not interpret this sentence literally. In most situations these transfers do not actually occur between the equity holders and the company. Rather they occur virtually between the company's surplus account and the various business units that require capital to guard against unexpected events from their operations.

2 Extreme Value Theory

The investigation of extreme events has a long history. Hydrologists, studying floods, were probably the first ones to develop methods of analysis and prediction. The book *Statistics of Extremes* [13] was the first one devoted exclusively to extreme values and is considered now a classic. The author stresses the importance of graphical methods over tedious computations and has illustrated the book with over 90 graphs. Since its publication in 1958 extreme value theory has grown tremendously and there are many deep and relevant results, but for our purposes we will mention only two of them. Both results tell us about the limiting behavior of certain events.

The first result (the Fischer-Tippett theorem) is the analog of the well known central limit theorem. Here the extreme value distributions play the same fundamental role as the normal distribution does in the central limit theorem. The second result (the Pickands and Balkema & de Haan theorem) shows that events above a high enough threshold behave as if they were sampled from a generalized Pareto distribution. This result is directly applicable to excess-of-loss reinsurance modeling and pricing. Two well known modern references with applications in insurance and finance are the books by Embrechts et. al. [8] and Reiss & Thomas [25].

In this section we also introduce a powerful graphical technique: the quantile-quantile (or QQ-) plot [5, chapter 6]. In many situations we need to compare two distributions. For example, is the empirical distribution of losses compatible with the gamma distribution? A quantile-quantile plot will help us answer that question.

2.1 Distribution of normalized sums

Actuaries are well aware of the central limit theorem [7, 15]; namely, if the random variables $X_1, \ldots, X_n$ form a random sample of size n from a unknown distribution with mean μ and variance σ^2 ($0 < \sigma^2 < \infty$), then the distribution of the statistic $(X_1 + X_2 + \cdots + X_n)/n$ will approximately be a normal distribution with mean μ and variance σ^2/n.

An equivalent way to think about the central limit theorem and to introduce extreme value theory is as follows: consider a sequence of random variables $X_1, X_2, X_3, \ldots$ from an unknown distribution with mean μ and finite variance ($0 < \sigma^2 < \infty$). Let $S_n = \sum_{i=1}^{n} X_i$ (for $n = 1, 2, \ldots$) be the sequence of partial sums. Then the central limit theorem says that if we normalize this sequence of partial sums

$$\frac{S_n - b_n}{a_n} \qquad \text{with } a_n = n, \text{ and } b_n = n\mu, \tag{1}$$

then the limiting distribution is a normal distribution.

2.1.1 Understanding QQ-plots

Before proceeding with extreme value theory let us introduce a powerful graphical technique known as a *quantile-quantile plot* or QQ-plot which will help us assess whether a data set is consistent with a known distribution. For this graphical display we will plot the quantiles of one distribution function against the quantiles of another distribution function.

The quantile function Q is the generalized inverse function[3] of the cumulative distribution function F;

$$Q(p) = F^{\leftarrow}(p) \qquad \text{for } p \in (0,1) \tag{2}$$

where the generalized inverse function $F^{\leftarrow}$ is defined as[4] (see [8, page 130])

$$F^{\leftarrow}(p) = \inf\left\{x \in \mathbb{R} \colon F(x) \geq p\right\}, \quad 0 < p < 1. \tag{3}$$

The quantity $x_p = F^{\leftarrow}(p)$ defines the pth quantile of the distribution function F.

Suppose that our data set consists of the points $x_1, x_2, \ldots, x_n$. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ denote our data sorted in increasing order.[5] We also use the convention [5, page 11] that $x_{(i)}$ is the $p_i = (i - 0.5)/n$ quantile.

To check if the distribution of our empirical data is consistent with the distribution function F we plot the points $(Q(p_i), x_{(i)})$; that is, the quantiles of F against the quantiles of our data set.

If the empirical distribution is a good approximation of the theoretical distribution, then all the points would lie very close to the line $y = x$; departures form this line give us information on how the empirical distribution differs from the theoretical

[3] We denote generalized inverse function with a left arrow as a superscript ($F^{\leftarrow}$) instead of the more traditional -1 superscript (F^{-1}). We cannot use the traditional definition of inverse function because some of our cumulative distribution functions are not one-to-one mappings.

[4] Ignoring some technicalities, the operator inf selects the smallest member of a set.

[5] Some authors would denote this sequence using double subscripts: $x_{n,n} \leq x_{n-1,n} \leq \cdots \leq x_{1,n}$.

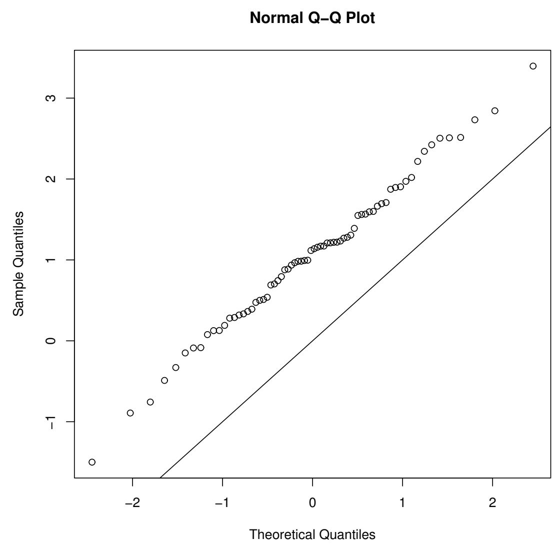

Figure 1: QQ-plot of normal distribution $N(0, 1)$ against $N(1, 1)$. The solid line is $y = x$.

distribution. Figure 1 shows the QQ-plot of a normal distribution $N(1, 1)$ with $\mu = 1$ and $\sigma^2 = 1$ against the standard normal distribution $N(0, 1)$, $(\mu = 0, \sigma^2 = 1)$. Note that the points on the graph do not follow the line $y = x$. Rather they follow the line $y = x + 1$. This configuration tells us that we have mis-specified the location parameter. In Figure 2 we have the QQ-plot for a normal distribution with variance equal to 2 against the standard normal distribution. In this case we have mis-specified the variance. This can be readily seen from the graph because the points follow a straight line with slope different from one.

2.1.2 Visualizing the central limit theorem

To visualize the central limit theorem consider a sequence of random numbers from an unknown distribution: $X_1, X_2, X_3, \ldots$. For $n = 1, 2, 3, \ldots$ compute the mean statistic μ_n of the first n terms; that is,

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} X_i. \tag{4}$$

The central limit theorem tells us that for large enough n the distribution of the mean statistic μ_n is very close to a normal distribution. How can we check that the distribution of μ_n is indeed very close to a normal distribution? Let us draw many

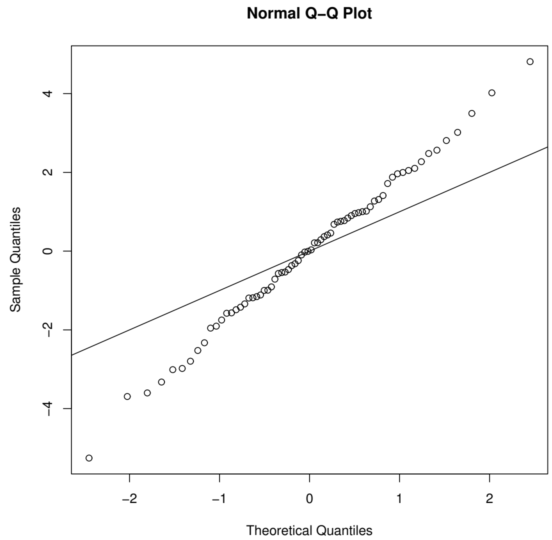

Figure 2: QQ-plot of normal distribution $N(0, 1)$ against $N(0, 2)$. The solid line is $y = x$.

samples of size n, compute μ_n, and look at the distribution of the mean μ_n.

For an example, take $n = 25$ and calculate 200 means

$$\mu_{25}^{(1)}, \mu_{25}^{(2)}, \mu_{25}^{(3)}, \ldots, \mu_{25}^{(200)} \tag{5}$$

How can we check that the distribution of these sampled means really follows a normal distribution?

We can calculate various numerical summaries: the mean, variance, skewness, kurtosis and others. But relying on numerical summaries alone can be misleading. Rather we should use graphical methods. To assess if our data come from a normal distribution we will show two graphs (see Figure 3). For the first one we will plot the cumulative density function of the sample mean μ_{25} along with the theoretical cumulative density function for the normal distribution. For the second graph we will plot the quantiles of the distribution of μ_{25} against the quantiles of the normal distribution. In this particular example the choice $n = 25$ is large enough so that the central limit theorem applies. Other underlying distributions might require a larger value of n.

Figure 4 shows how the central limit theorem applies to any underlying distribution. For this figure we have chosen three underlying distributions: uniform, gamma, and log-normal. The first row of the display shows the underlying distribution's prob-

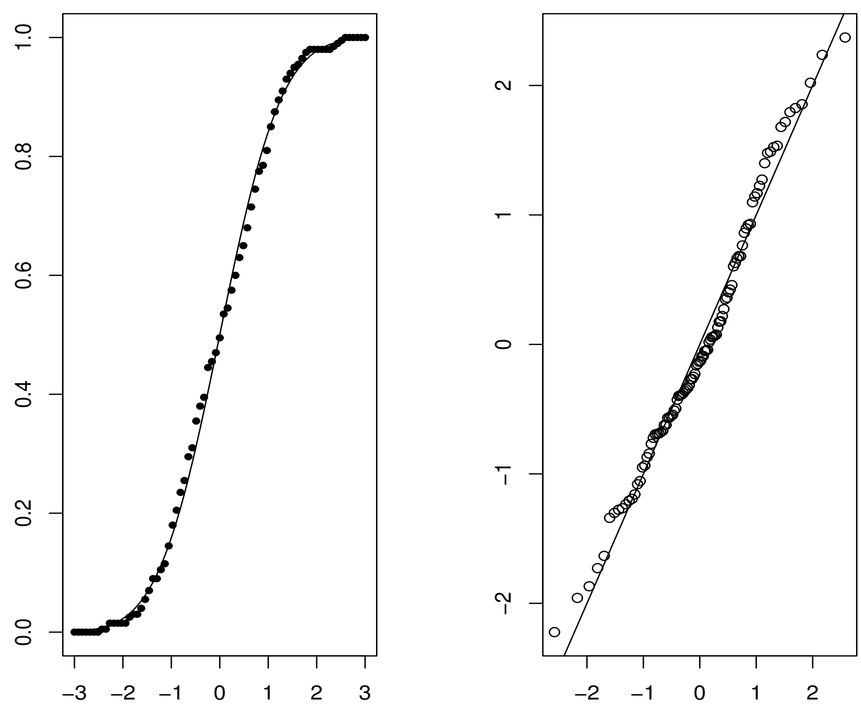

Figure 3: Cumulative density function and quantile-quantile plots for the distribution of the mean μ_{25}.

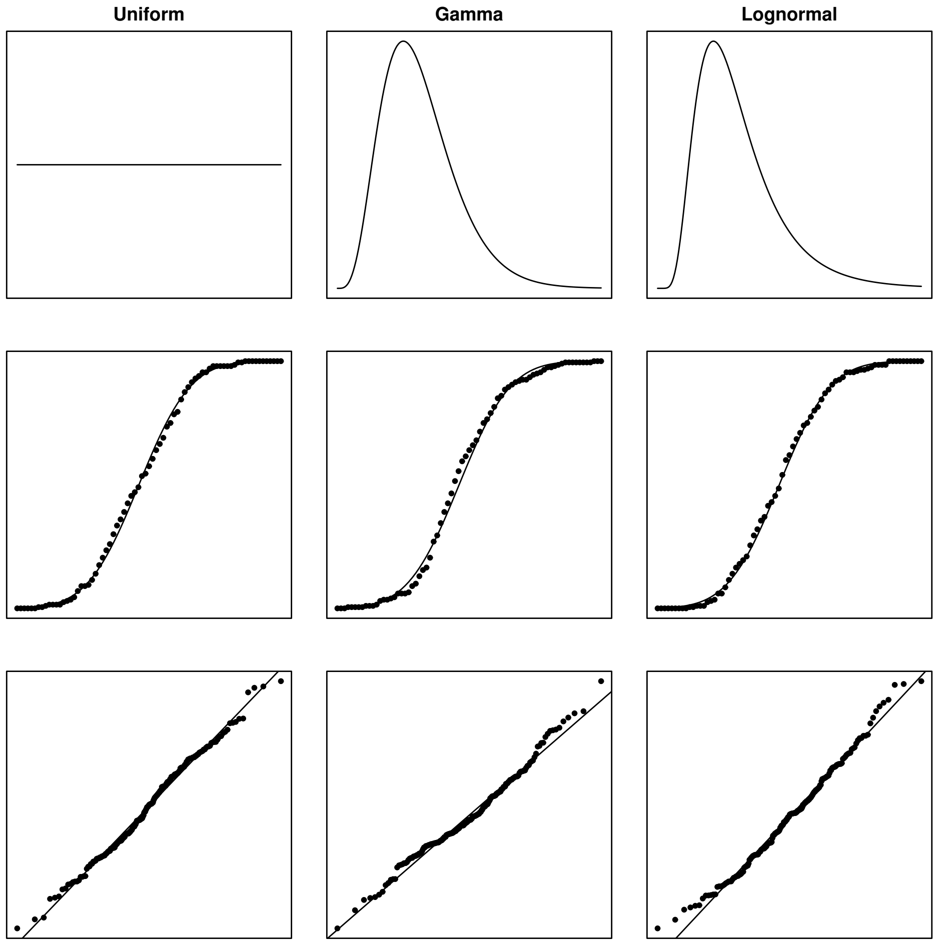

Figure 4: Visualizing the central limit theorem. Top row: underlying density function. Middle row: CDF-plot. (Only 75 of the 200 points were plotted.) Bottom row: QQ-plot.

ability density function. The second row shows the cumulative density function of the mean (dots) along with the cumulative density function for the normal distribution (solid line). Even though the curve does seem to approximate the normal curve fairly close on all three displays of the middle row it is hard for our visual system to distinguish differences from the two curves. The last row of the display shows the QQ-plots. Here it is much easier for us to see that our data (in all three cases) does fall fairly close to the line $y = x$.

Regardless of the underlying distribution (as long it satisfies some mild conditions) the distribution of the mean of a sample follows a normal distribution.

2.1.3 Does the central limit theorem apply to maxima?

While actuaries are interested in the mean severity of claims, they also want to know how large an individual loss might be. Hence, the following question arises naturally: if we replace the mean of a sample with another statistic, say the maximum of the sample, is the limiting distribution (if it exists) still the normal distribution?

As before, consider a sequence of random numbers from an unknown distribution: $X_1, X_2, X_3, \ldots$. For $n = 1, 2, 3, \ldots$ compute the maximum statistic of the first n terms:

$$M_n = \max(X_1, X_2, \ldots, X_n). \tag{6}$$

For n large, does the distribution of M_n converge to a normal distribution?

Using the same experimental procedure as for the mean statistic take $n = 25$ and calculate 200 maxima:

$$M_{25}^{(1)}, M_{25}^{(2)}, M_{25}^{(3)}, \ldots, M_{25}^{(200)}. \tag{7}$$

In Figure 5 we have displayed the cumulative distribution function of the maximum statistic (transformed to have mean zero and unit variance). We have also plotted the standard normal distribution. While we can see that both sets of data do not agree it is hard to know if the departures we see are significant. Our eyes have a hard time distinguishing differences between curved lines. The quantile-quantile plot provides a more powerful graphical technique because we are looking for discrepancies between a straight line and the data. Figure 6 shows clearly that the distribution of the maximum does not follow a normal distribution. If it did the data would fall approximately on a straight line. Rather the points form a concave line. At the upper right-hand corner the data are below the straight line. This implies that the distribution of the maximum is thicker tailed than the normal distribution. The region below the straight line corresponds to points where $Q_t(p) > Q_e(p)$; that is, for a given value of $p \in (0, 1)$ the pth quantile of the theoretical distribution (in our case the normal distribution) is greater than the pth quantile of the distribution. One could argue that our choice of $n = 25$ random numbers is not large enough to show (in our example) that the distribution of the maximum statistic converges to a normal distribution. We performed the same experiment with $n = 100, 1000,$ and 10000 and we still reached the same conclusion: for our example, the distribution of the maximum statistic does not converge to the normal distribution.

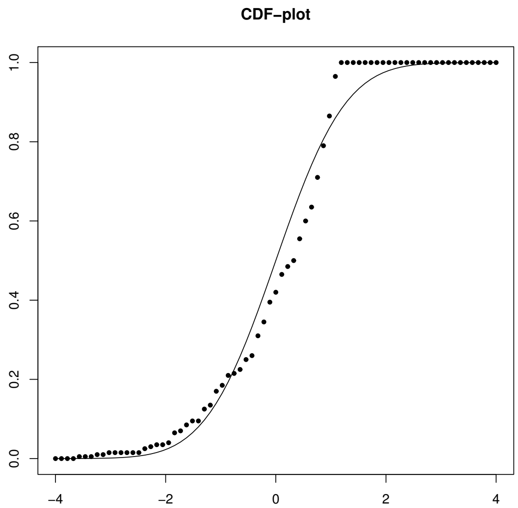

Figure 5: CDF-plot of maximum statistic. The solid line is the standard normal distribution $N(0,1)$.

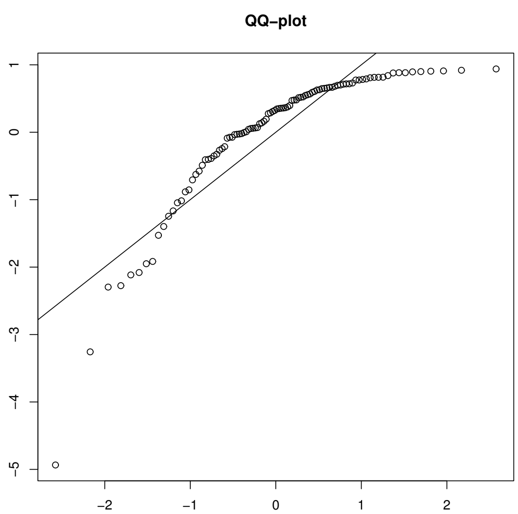

Figure 6: QQ-plot of maximum statistic. The solid line is $y = x$.

2.2 Distribution of normalized maxima

The extreme value theory result analogous to the central limit theorem specifies the form of the limiting distribution for normalized maxima. In place of the partial sums S_n we have the maximum $M_n = \max(X_1, X_2, \ldots, X_n)$.

We know that the distribution of maxima do not follow a normal distribution (see Figure 6). It turns out that the distribution of maxima converges to one of three distributions known as the *extreme value distributions.* The following theorem by Fischer and Tippett [11] explicitly states these three distributions.

Theorem 1 (Fischer-Tippett). *Let X_n be a sequence of independent and identically distributed random variables and let $M_n = \max(X_1, X_2, \ldots, X_n)$ be the maximum of the first n terms. If there exists constants $a_n > 0$ and b_n and some non-degenerate distribution function H such that*[6]

$$\frac{M_n - b_n}{a_n} \xrightarrow{d} H, \tag{8}$$

then H belongs to one of the three standard extreme value distributions:

$$\textit{Fréchet:} \qquad \Phi_\alpha(x) = \begin{cases} 0, & x \le 0, \quad \alpha > 0, \\ \exp(e^{-x^{-\alpha}}), & x > 0, \quad \alpha > 0, \end{cases} \tag{9}$$

$$\textit{Weibull:} \qquad \Psi_\alpha(x) = \begin{cases} \exp(-(-x^\alpha)), & \textit{if } x \le 0 \textit{ and } \alpha > 0, \\ 1, & \textit{if } x > 0 \textit{ and } \alpha > 0, \end{cases} \tag{10}$$

$$\textit{Gumbel:} \qquad \Lambda(x) \;= \exp(-e^{-x}), \quad \textit{if } x \in \mathbb{R}. \tag{11}$$

A distribution F is said to belong to the *maximum domain of attraction* of the extreme value distribution H if $M_n = \max(X_1, \ldots, X_n)$ satisfies equation (8), where the X_i's are random variables with distribution F.

The Fréchet, Weibull and Gumbel distributions can be written in terms of a one parameter ξ family:

$$H_\xi(x) = \begin{cases} \exp(-(1 + \xi\, x)^{-1/\xi}), & \text{if } \xi \neq 0 \\ \exp(-e^{-x}), & \text{if } \xi = 0 \end{cases} \tag{12}$$

[6]The notation $\xrightarrow{\text{d}}$ refers to convergence in distributions.

Table 1: Maximum likelihood estimates (and their standard errors) for the generalized extreme value distribution $H_\xi([x-\mu]/\sigma)$.

Parameter	Uniform		Gamma		Log-normal	
location (μ)	0.965	(0.003)	9.538	(0.192)	6.521	(0.151)
scale (σ)	0.033	(0.003)	1.707	(0.139)	1.326	(0.111)
shape (ξ)	-0.932	(0.005)	-0.011	(0.073)	0.028	(0.079)

where x is such that $1+\xi\,x>0$. This representation is obtained from the Fréchet distribution by setting $\xi=\alpha^{-1}$, from the Weibull distribution by setting $\xi=-\alpha^{-1}$ and by interpreting the Gumbel distribution as the limit case for $\xi=0$.

We visualize the Fischer-Tippett theorem using the same three underlying distributions (uniform, gamma, log-normal) we used for the central limit theorem. For each underlying distribution we have collected 100 maxima. Each maximum is taken over 25 points chosen at random from the distributions. Table 1 shows the maximum likelihood estimates for each distribution and in Figure 7 we show the corresponding QQ-plots. Note that the shape parameter for the maxima sampled from the uniform distribution is negative. This implies that the uniform distribution is in the maximum domain of attraction of the Weibull distribution. Similarly the shape parameters for the gamma and log-normal are not statistically different from zero. Hence these distributions are in the maximum domain of attraction of the Gumbel distribution.

Distributions that belong to the maximum domain of attraction of the Fréchet distribution include Pareto, Burr, and log-gamma. They are usually categorized as heavy-tailed distributions. Other distributions that actuaries are familiar with include the normal, gamma, exponential, log-normal and Benktander type-I and type-II (see [8, pages 153–7]). These distributions are not as heavy-tailed as the previous examples. They belong to the maximum domain of attraction of the Gumbel distribution. These are medium-tailed distributions. Examples of distributions belonging to the maximum domain of attraction of the Weibull distribution include the beta and uniform distributions. These we shall call thin-tailed distributions.

2.3 Distribution of exceedances

We have seen that the distribution of the maximum does not follow the normal distribution. Rather it follows one of the extreme value distributions: Fréchet, Weibull or Gumbel.

While reinsurance actuaries are interested in the maximum single loss over a given

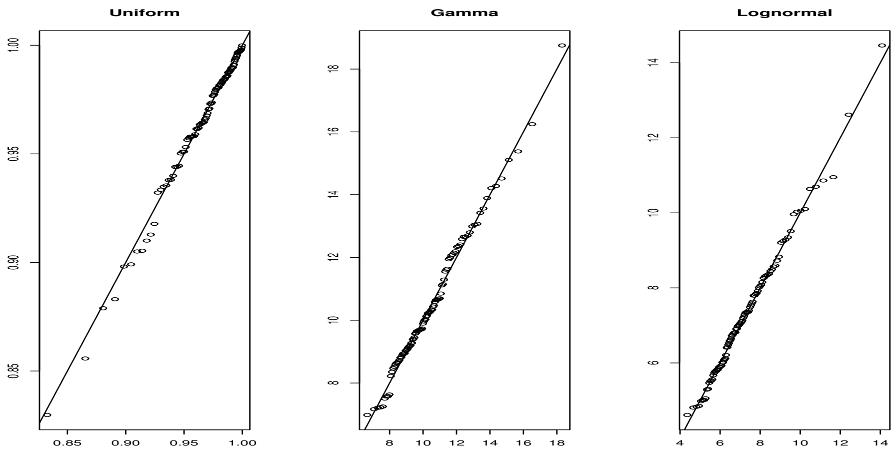

Figure 7: QQ-plots comparing sampled maxima against the fitted generalized extreme value distribution $H_\xi([x-\mu]/\sigma)$.

time period this information is not the area of focus when pricing a contract. The excess of loss reinsurance actuary is concerned about *any* loss that exceeds a predetermined threshold (or attachment point). Suppose that $X_1, X_2, \ldots, X_n$ represent the ground-up losses over a given period. Let u be the predetermined threshold and let

$$Y = [X - u | X \geq u] \tag{13}$$

be the excess of X over u given that the ground-up loss exceeds the threshold. The pricing actuary is interested in the distribution of the exceedances; that is, in the conditional distribution of $Y = X - u$ given that X exceeds the threshold u.

Let F denote the distribution of the random variable X,

$$F(x) = \text{Prob}\,(X < x)\,, \tag{14}$$

and let F_u denote the conditional distribution of the exceedance $Y = X - u$ given that X exceeds the threshold u:[7]

$$F_u(y) = \frac{F(y+u) - F(u)}{1 - F(u)}. \tag{15}$$

Just like the mean statistic converges in distribution to the normal distribution and the maximum statistic converges in distribution to one of the extreme value

[7]The distribution function F_u is also known as the exceedance distribution function, the conditional distribution function, or in reinsurance the excess-of-loss distribution function.

distributions, the exceedances converge in distribution to the generalized Pareto distribution (provided we choose a high enough threshold). The following theorem due to Pickands [24] and Balkema & de Haan [2] shows the result.

Theorem 2 (Pickands, Balkema & de Haan). *For a large class of underlying distribution functions F the conditional excess distribution function $F_u(y)$, for u large, is well approximated by the generalized Pareto distribution $G_{\xi,\sigma}(y)$ where*

$$G_{\xi,\sigma}(y) = \begin{cases} 1 - \left(1 + \frac{\xi}{\sigma} y\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - \exp(-y/\sigma) & \text{if } \xi = 0 \end{cases} \tag{16}$$

for $y \in [0, (x_F - u)]$ if $\xi \geq 0$ and $y \in [0, -\sigma/\xi]$ if $\xi < 0$.

The point x_F denotes the rightmost point of the distribution function F (which could be finite or infinite).

The class of underlying distribution functions for which the above theorem applies includes most of the standard distribution functions used by actuaries: Pareto, gamma, log-normal, and others (see [16]).

2.3.1 Peaks over threshold method

In order to apply the above theorem we have to choose a threshold. But how do we choose a good threshold? The theorem tells us that if we pick a high enough threshold our data should behave like data that comes from the generalized Pareto distribution. The question is, what characteristics does the generalized Pareto distribution have that we could check against our data? One such characteristic is the mean excess function. The mean excess function for the generalized Pareto distribution $G_{\sigma,\xi}(x)$ is a straight line with positive slope:

$$e(u) = \frac{\sigma + \xi u}{1 - \xi} \tag{17}$$

where $\sigma + \xi u > 0$. The mean excess function[8] for various standard distributions can be found on Table 2.

[8] We use Landau's notation where $o(1)$ stands for an unspecified function of u whose limit is zero as $u \to \infty$.

Table 2: Mean excess functions for some standard distributions.

Distribution	Mean excess function
Pareto	$\frac{\kappa+u}{\alpha-1}, \quad \alpha > 1$
Burr	$\frac{u}{\alpha\tau-1}(1+o(1)), \quad \alpha\tau > 1$
Log-gamma	$\frac{u}{\alpha-1}(1+o(1)), \quad \alpha > 1$
Log-normal	$\frac{\sigma^2 u}{\ln u-\mu}(1+o(1))$
Benktander-type-I	$\frac{u}{\alpha+2\beta\ln u}$
Benktander-type-II	$\frac{u^{1-\beta}}{\alpha}$
Weibull	$\frac{u^{1-\tau}}{c\tau}(1+o(1))$
Exponential	λ^{-1}
Gamma	$\beta^{-1}\left(1+\frac{\alpha-1}{\beta u}+o\left(\frac{1}{u}\right)\right)$
Truncated Normal	$u^{-1}(1+o(1))$

The empirical mean excess function for a sample of data points X_i is given by

$$e_n(u) = \frac{\sum_{i=1}^{n} \max(0, X_i - u)}{\sum_{i=1}^{n} 1_{X_i>u}} \tag{18}$$

where $1_{X>u}$ is the indicator function with value 1 if $X > u$ and zero otherwise.

Figure 8 shows a sample of 2860 general liability losses. Note that most of the losses are very small (say below 500) but there are a few extremely large losses.[9] Figure 9 shows the empirical mean excess plot for these data. Since the mean excess function for the generalized Pareto distribution is a straight line with positive slope, we are looking for the threshold points from which the mean excess plot follows a straight line. There are two regions where the plotted points seem to follow a straight line with positive slope. The first one is from thresholds between 600 and 1000 and the second is between 1000 and 2500. Of course, the second region has very few data points. Based on a threshold $u = 600$ we can fit a generalized Pareto distribution (GPD) (see Figure 10) and check the goodness-of-fit against the data (see Figure 11 for a QQ-plot).

[9]The losses have been scaled so that the largest loss has a value of 10,000.

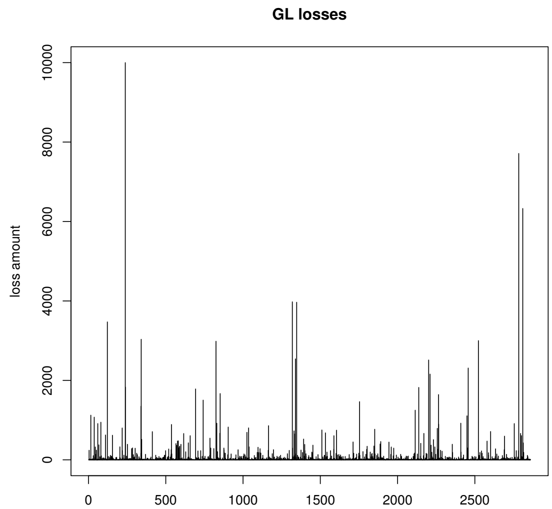

Figure 8: General liability losses. The losses have been normalized so that the maximum loss has a value of 10,000.

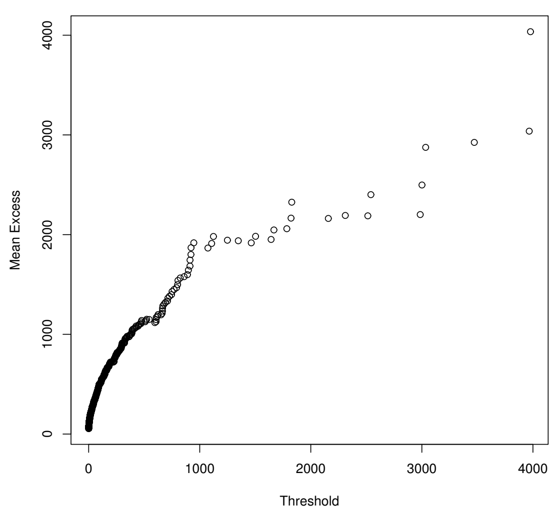

Figure 9: Mean excess plot for general liability losses.

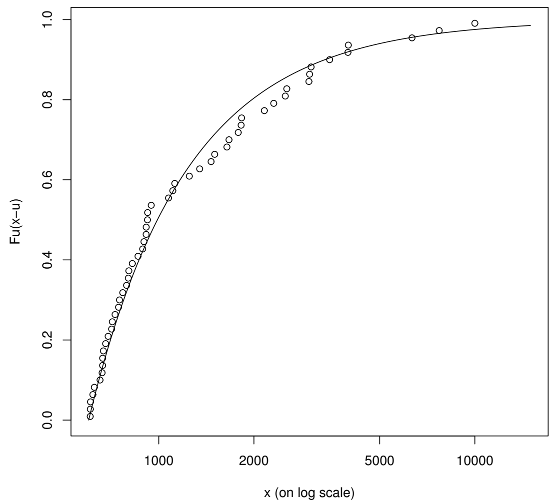

Figure 10: GPD fit (threshold $u = 600$) to the general liability data. The maximum likelihood parameter estimates are $\xi = 0.7871648$ and $\sigma = 423.0858245$

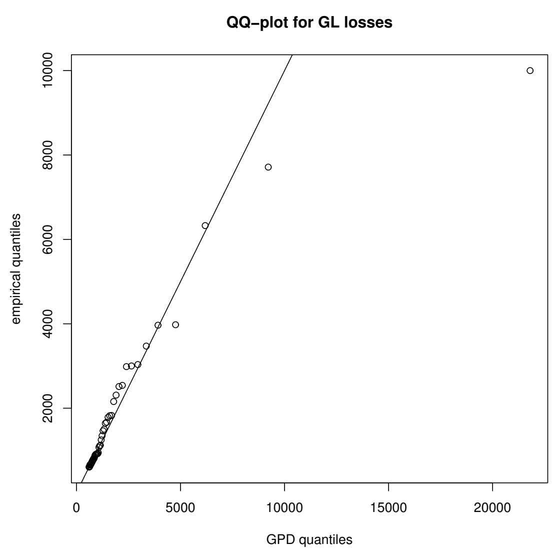

Figure 11: Quantile-quantile plot for general liability losses. The maximum likelihood parameters for the GPD fit are: $u = 600$, $\xi = 0.7871648$ and $\sigma = 423.0858245$.

2.4 Quantile estimation

Estimates of quantiles are important for the actuary and it is easy to calculate them with the generalized Pareto distribution function. To estimate the tail above a threshold u start by re-writing the conditional probability function F_u as follows:

$$F(x) = \text{Prob}\left(X \leq x\right) = \left(1 - \text{Prob}\left(X \leq u\right)\right) F_u(x-u) + \text{Prob}\left(X \leq u\right). \tag{19}$$

From the previous section we know that for large enough threshold u we can approximate $F_u(x-u)$ with the generalized Pareto distribution $G_{\xi,\sigma}(x-u)$. Also using the empirical data we can estimate Prob$(X \leq u)$ with the empirical cumulative density function $F_n(u)$:

$$F_n(u) = \frac{n - N_u}{n} \tag{20}$$

where n is the number of points in the sample and N_u is the number of points in the sample that exceed the threshold u.

If we let $\widehat{F(x)}$ be our approximation to $F(x)$, then for $x \geq u$ we can estimate the tail of the distribution $F(x)$ with

$$\widehat{F(x)} = \left(1 - F_n(u)\right) G_{\xi,\sigma}(x-u) + F_n(u). \tag{21}$$

It is easy to show that $\widehat{F(x)}$ is also a generalized Pareto distribution function with the same ξ parameter but different σ and u parameters. In fact,

$$\widehat{F(x)} = G_{\xi,\tilde{\sigma}}(x - \tilde{u}) \tag{22}$$

where $\tilde{\sigma} = \sigma(1 - F_n(u))^{\xi}$ and $\tilde{u} = u - [\sigma(1 - (1 - F_n(u))^{\xi})/\xi]$. Appendix A shows the derivation of these new parameters.

From equation (21) we can solve for x to obtain our quantile estimator. Let n be the total number of data points and N_u be the number of observations that exceed the threshold u. Then the pth quantile is given by solving the equation

$$p = \left(1 - \frac{n - N_u}{n}\right)\left\{1 - \left(1 + \frac{\xi}{\sigma}(x_p - u)\right)^{-1/\xi}\right\} + \frac{n - N_u}{n} \tag{23}$$

in terms of x_p. This yields the estimator

$$\widehat{x_p} = u + \frac{\sigma}{\xi}\left[\left(\frac{n}{N_u}(1-p)\right)^{-\xi} - 1\right]. \tag{24}$$

2.5 Risk Premium

Once we have estimated the generalized Pareto distribution for our data it is easy to calculate the risk premium (expected losses) in any given layer in excess of our threshold. Let (r, R) (with $R > r > u$) denote the excess-of-loss layer $(R - r)$ xs r. The risk premium in this layer is

$$P = \int_r^R (x-r)\, f_u(x-u)\, dx + (R-r)\,(1 - F_u(R-u)) \tag{25}$$

where $f_u(x-u)$ is the density of the fitted generalized Pareto model. Notice that the price for any layer above the threshold depends only on the excess distribution F_u.

Let

$$G_{\xi,\sigma}(x-u) = \begin{cases} 1 - \left(1 + \frac{\xi}{\sigma}(x-u)\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{x-u}{\sigma}\right) & \text{if } \xi = 0 \end{cases} \tag{26}$$

be the generalized Pareto distribution function including a location parameter u. Using the Pickands and Balkema & de Haan Theorem we can approximate $F_u(x-u)$ with $G_{\xi,\sigma}(x-u)$ and so any questions about a particular layer of reinsurance can be answered by calculating the appropriate moments using the estimated generalized Pareto distribution function.

Calculating the integral (25) to determine the risk premium we have the following explicit formula

$$P = \begin{cases} \frac{\sigma}{\xi}\left[\left(1 + \frac{\xi}{\sigma}(R-u)\right)^{1-1/\xi} - \left(1 + \frac{\xi}{\sigma}(r-u)\right)^{1-1/\xi}\right] & \text{if } \xi \neq 0 \\ \sigma\left[\exp\left(-\frac{r-u}{\sigma}\right) - \exp\left(-\frac{R-u}{\sigma}\right)\right] & \text{if } \xi = 0 \end{cases} \tag{27}$$

3 Collective Risk Models

We shall look at the aggregate losses from a portfolio of risks. Let S_n denote the sum of n individual claim amounts $(X_1, X_2, \ldots, X_n)$, where n is a random number and the claim amounts X_i's are independent and identically distributed random variables.

That is, S_n follows a collective risk model

$$S_n = X_1 + X_2 + \cdots + X_n, \qquad \text{for } n = 0, 1, 2, \ldots \tag{28}$$

with $S_0 = 0$.

In this paper, we focus on experience rating, rather than exposure rating. The next example will be used throughout the paper to illustrate the concepts.

Illustration

Consider pricing an excess-of-loss reinsurance treaty. The treaty covers a small auto liability portfolio with a retention of 3 million, a limit of 12 million, and an annual aggregate deductible of 3 million for accident year 2005. The cedant has provided the following data on large losses

Table 3: Large losses by accident year (I).

Accident Year	1995	1996	1997	1998	1999
Incurred Losses	692,351	902,742	2,314,953	3,183,920	1,168,803
	767,671	2,037,328	702,022	535,590	1,178,212
	1,274,118	1,232,477	1,023,062	742,667	3,722,663
	1,280,334	822,814	3,579,147	922,728	1,830,560
	779,054	684,503	656,957	923,000	509,205
	525,584		1,796,454	831,689	930,300
	1,101,540		589,947	1,622,289	
	980,171		530,295	4,291,141	
	1,268,650		750,693		
	807,076		531,515		
			1,624,021		
			765,879		

Table 4: Large losses by accident year (II).

Accident Year	2000	2001	2002	2003	2004
Incurred Losses	1,172,325	531,500	870,000	1,297,600	851,259
	1,978,249	630,741	592,600	502,776	1,530,050
	512,380	811,327	1,759,111	2,050,000	1,750,000
	1,441,546	989,497	1,856,305	2,350,000	
	925,617	502,603	750,503	9,510,500	
	774,997	566,382			
	1,102,500	1,118,255			
	608,446				
	2,130,454				
	526,483				
	1,600,942				
	1,547,415				

For simplicity, we assume that the loss data are not censored. That is, the given losses are from ground-up losses without capping at the underlying policy limits. This assumption is not crucial and we will discuss briefly in the sections below how one adjusts the modeling procedure when such an assumption is removed. □

3.1 Loss severity distributions

Estimates of the loss severity distribution play an important role in high excess-of-loss reinsurance layers where relevant empirical losses are scarce. It is particularly important that the selected loss distribution fits well the historical large losses and less relevant in explaining the small losses. In practice, when looking at the historical claims one usually ignores the small losses and analyzes only those losses that exceed a threshold.

Before historical losses can be used in any rating procedure they have to be projected to their ultimate values. This is often done by applying loss development factors. Recall that the loss development factors obtained from the usual accident-year triangle analysis contain two parts: development for known claims and development for unreported cases. For loss severity distribution fitting purpose, we need the development factors for known claims.

In addition, because of the well-recognized differences in development between large losses and small losses, we recommend using loss development factors on known claims derived from large claims only. Depending on the treatment for loss adjustment expenses in the reinsurance treaty one must determine whether to include such expenses in the loss data. In this paper, we shall assume that expenses are included in the losses and from this point forward we shall refer to the sum of the two as losses. In addition to the projection to their ultimate values, losses should also be trended to reflect the changes between the experience period and the coverage period. For example loss severity trends may include monetary inflation, increases in jury awards, and increases in medical expenses.

Illustration

In our auto liability example, assume for simplicity that we have a constant inflation of 3% (per annum) and the following loss development factors for known claims:

Table 5: LDF for known claims by accident year.

Year	LDF for known claims
1995	1.001
1996	1.002
1997	1.003
1998	1.009
1999	1.024
2000	1.044
2001	1.050
2002	1.081
2003	1.108
2004	1.172

Losses should be best divided into paid and outstanding and be adjusted/inflated accordingly. Our constant inflation assumption simplifies this process. Depending on the data, certain losses might be closed and should not be developed further. In our simplified illustration, we shall assume that all losses are subject to further development. Then, for example, the first claim in the amount of $692,351$ in accident year 1995 would be developed and inflated to its ultimate value of

$$692,351 \cdot 1.001 \cdot 1.03^{(2005-1995)} = 931,392 \qquad (29)$$

In summary, we have the following indexed claim history:

Table 6: Indexed historical large losses.

Accident Year	1995	1996	1997	1998	1999
Indexed	931,392	1,180,229	2,940,118	3,950,126	1,428,916
Incurred	1,032,717	2,663,567	891,607	664,479	1,440,418
Losses	1,714,020	1,611,319	1,299,345	921,389	4,551,127
	1,722,382	1,075,733	4,545,715	1,144,781	2,237,944
	1,048,030	894,907	834,372	1,145,119	622,527
	707,047		2,281,596	1,031,834	1,137,335
	1,481,858		749,265	2,012,690	
	1,318,585		673,504	5,323,798	
	1,706,664		953,422		
	1,085,727		675,053		
			2,062,597		
			972,709		

Accident Year	2000	2001	2002	2003	2004
Indexed	1,418,819	628,304	1,027,807	1,525,982	1,027,644
Incurred	2,394,198	745,620	700,090	591,266	1,847,084
Losses	620,114	959,097	2,078,191	2,410,806	2,112,608
	1,744,647	1,169,718	2,193,015	2,763,607	
	1,120,238	594,144	886,635	11,184,378	
	937,949	669,540			
	1,334,313	1,321,927			
	736,378				
	2,578,405				
	637,182				
	1,937,558				
	1,872,776				

□

If the losses are censored at the underlying policy limit, without knowing exactly how large the ground-up losses are, then one conservative adjustment is to assign such losses at the appropriate policy limits for the current underwriting standards. For example, suppose that a risk had a policy limit of one million and it generated a loss that was capped at the policy limit. Furthermore, assume that the current underwriting standards would give this risk a policy limit of 1.5 million. Then the *as-if* loss for this risk would be the full policy limit of 1.5 million. On the other hand, if one knows the exact size of the ground-up loss, then one should index the ground-up loss as above and limit it at the appropriate policy limit if necessary.

After the historical losses have been adjusted to an *as-if* basis but before we start the model fitting, it is important to explore the data further to gain better understanding. One way to do so is to plot the empirical mean excess function (18). An upward trend in the mean excess plot suggests a heavy tailed behavior, a horizontal line would be exponentially distributed, and thin tailed distribution usually gives a downward trended mean excess plot (see Table 2).

Illustration

In our auto liability data, we have the following mean excess plot:

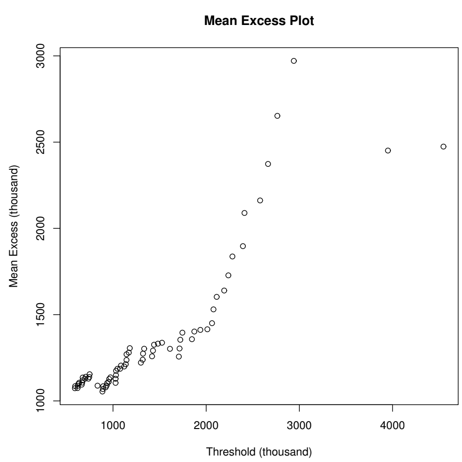

Figure 12: Mean excess loss plot.

The plot shows an upward trend, which suggests that the tail is heavier than an exponentially distributed function. The points above a threshold of 2,000,000 seem to follow a straight line (ignoring the last couple of points which are the average of very few observations). This suggests that a generalized Pareto fit with a threshold of 2 million should provide a good fit. □

Commonly used loss severity distributions in reinsurance pricing include Pareto, log-normal, log-gamma, exponential, gamma, transformed beta, and others. Pareto distributions are particularly popular. Actuaries have recently been applying extreme value theory in estimating the tails of the loss severity distributions [8, 19, 16, 20]. It is particularly useful in pricing high excess of loss layers. The theory suggests that the excess losses above a high threshold are asymptotically distributed according to a generalized Pareto distribution. The loss severity distributions commonly used in reinsurance pricing belong to the class of functions to which the Pickands and Balkema & de Haan theorem 2 applies; showing that the excess loss above a high threshold can be well approximated by the generalized Pareto distribution. This theorem provides the theoretical underpinnings for the popularity of the Pareto distribution in the reinsurance industry when pricing high excess-of-loss layers.

There are various methods of estimating the parameters of the loss severity distributions: method of moments, percentile matching, maximum likelihood, and least squares are among them. The method of moments and percentile matching are easy to implement and convenient but lack of the desirable optimality properties of maximum likelihood and least squares estimators. Maximum likelihood in essence seeks to find the parameters that give the maximum probability to the observed data. Maximum likelihood estimators are asymptotically unbiased and have minimum variance. Unfortunately, it can be heavily biased for small samples. The least squares method seeks to find the parameters estimates that produce the minimum distance between the observed data and the fitted data. The least squares method can be applied more generally than maximum likelihood. However, it is not readily applicable to censored data and is generally considered to have less desirable optimality properties than maximum likelihood. In our model, we will estimate our parameters using maximum likelihood.

Recall that maximum likelihood method selects the parameters θ's which maximize the likelihood function:

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) \tag{30}$$

or equivalently the log-likelihood function:

$$l(\theta) = ln(L(\theta)) = \sum_{i=1}^{n} ln(f(x_i|\theta)) \tag{31}$$

where $f(x_i|\theta)$ is the probability density function evaluated at x_i given θ.

Illustration

We continue with the auto liability example. To fit a generalized Pareto distribution to the indexed excess loss data, first recall that generalized Pareto distribution probability density function is of the form

$$f_{\xi,\sigma}(x) = \frac{1}{\sigma}\left(1 + \frac{\xi}{\sigma}x\right)^{-(\frac{1}{\xi}+1)} \tag{32}$$

and the log-likelihood function is

$$l(\xi,\sigma) = n\left(\frac{1}{\xi}+1\right)\ln\sigma - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n}\ln\left(1+\frac{\xi}{\sigma}x_i\right). \tag{33}$$

With the selected threshold of $2,000,000$, we plug in the excess loss (indexed loss - threshold) values and obtain maximum likelihood estimators of $\xi = 0.66784$ and $\sigma = 591,059.8$.

The graph below shows the cumulative density functions for the generalized Pareto as well as the adjusted empirical distributions (adjusted for inflation and loss development).

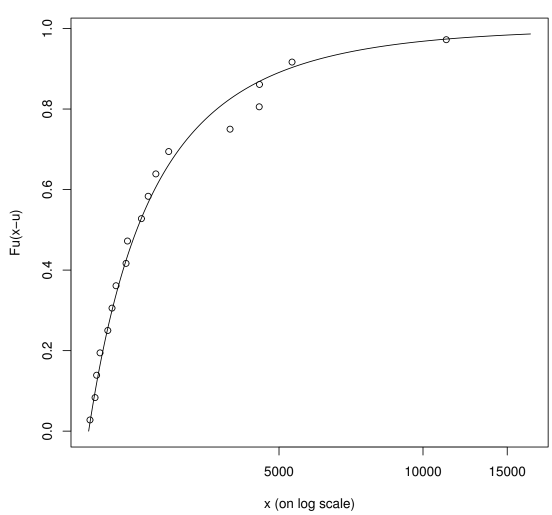

Figure 13: Generalized Pareto cumulative density function.

□

In dealing with censored data, one adjusts the probability function by assigning a mass density at the censor point c (see [17]):

$$\tilde{F}(x) = \begin{cases} F(x), & \text{if } x < c, \\ 1, & \text{if } x \geq c, \end{cases} \tag{34}$$

3.2 Claim frequency distribution

Similar to the loss severity, the claim frequency also needs to be trended and developed to ultimate. One needs to estimate for the not yet reported claims and the possible trends. To estimate the not yet reported claims is easier: one develops the claim number triangle to ultimate. To estimate the possible trends, on the other hand, is certainly not an easy task. Court decisions may influence the liability frequencies; an amendment in the governing law can change the reporting of the WC claims. Both legal and social factors need to be considered when identifying the trends. One further adjustment to the historical frequency is to reflect the historical portfolio sizes. This can be done by comparing the historical exposure sizes to the treaty year exposure size.

Illustration

Continuing with our auto liability example, suppose that we have the following exposure information and claim number development factors:

Table 7: Historical exposure size and claim frequency LDF by accident year.

Year	No. of Exposures	Claim Freq Dev. Factor
1995	21,157,000	1.007
1996	19,739,000	1.007
1997	19,448,000	1.007
1998	19,696,000	1.022
1999	19,406,000	1.030
2000	19,543,000	1.037
2001	19,379,000	1.073
2002	21,186,000	1.197
2003	24,425,000	1.467
2004	27,990,000	2.379
2005	28,000,000	

Then, for example, there is one claim reported in 1996 above the chosen threshold of 2,000,000, with the adjustment for unreported claims and exposure sizes, we get

$$1 \cdot 1.007 \cdot 28,000,000/19,739,000 = 1.43 \tag{35}$$

That is, assuming no other trends are necessary, we expect to see 1.43 claims above 2 million if year 1996 experience were to happen again with the 2005 exposure size. The following table summarizes the combined adjustments:

Table 8: Indexed claim experience.

Year	No. of claims	No. of *as-if* claims
1995	0	0
1996	1	1.43
1997	4	5.80
1998	3	4.36
1999	2	2.97
2000	2	2.97
2001	0	0
2002	2	3.16
2003	3	5.04
2004	1	2.38

□

To model the claim frequency distribution, we consider three choices of claim frequency distributions: Poisson, negative binomial, and binomial.

- Poisson

  The Poisson distribution is often used in reinsurance pricing for its simplicity. It has a great advantage: the sum of two independent Poisson variables also follows a Poisson distribution. Another advantage is that if the number of claims in a fixed time period follows a Poisson distribution, then

  1. the number of claims above a fixed retention is also Poisson distributed.
  2. the claim number for a subinterval is also Poisson distributed.

The first advantage is particularly useful in excess-of-loss reinsurance pricing because it provides the theoretical background for assuming that loss events are Poisson distribution while adjusting retentions when fitting the distributions. The second advantage works well, for example, when one removes certain benefits from the current plan. The assumption of Poisson needs not be changed if the frequency distribution under the current plan follows a Poisson distribution.

One disadvantage is that the assumption that the rate at which the claims occur is constant over time. This in particular is not applicable in certain sections of reinsurance (for example, earthquake) where the probability of another loss occurring is a lot higher given that one has already occurred.

- Negative binomial

  The negative binomial is a generalization of the Poisson distribution by mixing a Poisson distribution with a gamma mixing distribution. That is, by assuming the parameter λ of the Poisson distribution to be gamma distributed, the resulting distribution is negative binomial. This is particularly useful when the practitioner incorporates parameter uncertainly into the Poisson parameterization.

  The disadvantage of this distribution lies in the difficulty in solving for its maximum likelihood estimators; there is no closed form for them.

- Binomial

  The process of having claims from m independent risks with each risk having probability q of having a claim follows a binomial distribution with parameters m and q. This distribution has finite support $0, 1, 2, \ldots, m$. That is, at most m claims can happen in the specific period of time. This makes the Binomial distribution less popular for reinsurance pricing.

In estimating the parameters of the claim frequency distribution one can use the same methods as for the loss severity distribution; namely, the method of moments, maximum likelihood, least squares, etc. as discussed in Section 3.1. In general, because of the much smaller volatility involved in the claim frequency versus the loss severity, there is less concern in estimating frequency distribution. In our model, we use the method of moments—due to its simplicity—to estimate the parameters of the distribution.

Illustration

In the auto liability example, the average of the sample frequencies is 2.812 and its variance is 3.821. Since the sample variance is larger than the sample mean, we select the negative binomial as the claim frequency distribution. The parameters s, p of a negative binomial distribution are such that[10]

$$\mu = \frac{s(1-p)}{p} \quad and \quad \sigma^2 = \frac{s(1-p)}{p^2} \tag{36}$$

We input the sample mean and sample variance into the equations and solve for s and p in the system of equations with the restriction that s must be an integer. The approximate solutions are $s = 8$ and $p = 0.73993$. This approximation is unbiased but slightly underestimates the variance. □

3.3 Aggregate loss distribution

As stated earlier in this section, we model the aggregate loss distribution from a collective risk theory point of view. The aggregate loss S_n for a specific period of time (usually one calendar year) is the sum of n individual claim amounts (from ground-up):

$$S_n = X_1 + X_2 + \cdots + X_n, \qquad n = 0, 1, 2, \ldots \tag{37}$$

with $S_0 = 0$. n is a random number following the selected claim frequency distribution and the X_is are independent, identically distributed, and follow the selected loss severity distribution. It is also assumed that n and X_i's are independent.

For the annual aggregate layer losses under the reinsurance program, we modify the above formula by applying the reinsurance coverage:

$$\bar{X}_i = \min(l, \max(X_i - r, 0)), \quad \text{for } i = 1, 2, \ldots, n \tag{38}$$

$$\bar{S}_n = \min(L, \max((\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_n) - D, 0)) \tag{39}$$

for limit l, retention r, aggregate limit L, and aggregate deductible D.

In general insurance practice, there are four ways in computing the aggregate loss distribution from the selected claim frequency and the loss severity distributions:

1. Method of moments

[10] We are using the following parametrization of the negative binomial density function: $f(k) = \binom{s+k-1}{k} p^s (1-p)^k$.

This method assumes a selected aggregate loss distribution whose parameters are estimated by the empirical moments of the aggregate losses. Such moments are derived algebraically from the moments of the claim frequency and loss severity distributions:

$$\mathrm{E}(\bar{S}_n) = \mathrm{E}(n) \cdot \mathrm{E}(\bar{X}) \tag{40}$$

$$\mathrm{var}(\bar{S}_n) = \mathrm{E}(n) \cdot \mathrm{var}(\bar{X}) + \mathrm{E}(\bar{X})^2 \cdot \mathrm{var}(n) \tag{41}$$

It has the advantage of simplicity and ease of calculation. However, its main disadvantage is inaccuracy. In general, the fitted loss distribution does not model the true aggregate losses well.

2. Monte Carlo simulation

   This method calculates the aggregate loss distribution directly from simulating the claim frequency and loss severity distributions. First, one samples from the frequency distribution to determine the number of claims n in the period. Then, we pick n claims from the severity distribution at random. The sum of the n random claim amounts (adjusted for the reinsurance coverage in place) gives one outcome for the aggregate losses. We repeat many times this sampling procedure to estimate the distribution of the aggregate losses.

   This method provides easy and accurate aggregate distributions. However, some argue that it takes considerable computing time. For further details see [12].

3. Recursive method

   In general, this method requires a discretization of the loss severity distribution and a selection of a large enough number of points for the claim frequency distribution. It involves inverting the Laplace transform of the aggregate loss distribution (for example, see [14]). Panjer [23] gave a direct recursive formula for a particular family of claim frequency distributions that does not involve the Laplace transformation (see also [26])

   The recursive method is fast and accurate most of the time. The disadvantage is the requirement of discretization of the loss severity distribution. There are two methods for carrying out the discretization of a continuous distribution function: the midpoint method and the unchanged expectation method. With both methods one loses information. For further details see [6].

4. Fast Fourier Transform

   The fast Fourier transformation inverts the characteristic function of the aggregate loss distribution, a procedure similar to the recursive method. It also requires a discretization of the loss severity distribution. See [3, 26] for further details.

   The advantage of this method lies in its efficiency and speed. However, the computation tends to be complicated.

We use Monte Carlo simulation to compute the aggregate loss distribution in our model. It is simple and intuitive.

Illustration

Assume that we have fitted a generalized Pareto distribution as our loss severity distribution with parameters $\xi = 0.66784$, $\sigma = 591,059.8$, and threshold of $2,000,000$ and a negative binomial as our claim frequency distribution with parameters $s = 8$ and $p = 0.73993$. Let's also assume that, in one random iteration, the negative binomial distribution produces 4 claims and we generate the following 4 claims from the generalized Pareto distribution:

Table 9: Sample GPD generated losses.

Loss (from ground up)
2,590,062
3,107,208
2,874,384
7,800,324

Keep in mind that the generalized Pareto generates excess loss above the threshold. To convert excess loss to ground up loss one adds back the threshold.

This represents one possible annual outcome. To evaluate the reinsurance recovery, we apply the coverage: 12 million excess of 3 million with annual aggregate deductible of 3 million. The table bellow shows the result.

Table 10: Sample layer losses.

Loss	Layer Loss
2,590,062	0
3,107,208	107,208
2,874,384	0
7,800,324	4,800,324

The sum of all layer losses is $4,907,532$ and the reinsurance recovery is $1,907,532$. Thousands of iterations are generated to derive all possible outcomes and they give us the aggregate loss distribution for the reinsurance coverage. The following graph shows the result of $5,000$ iterations:

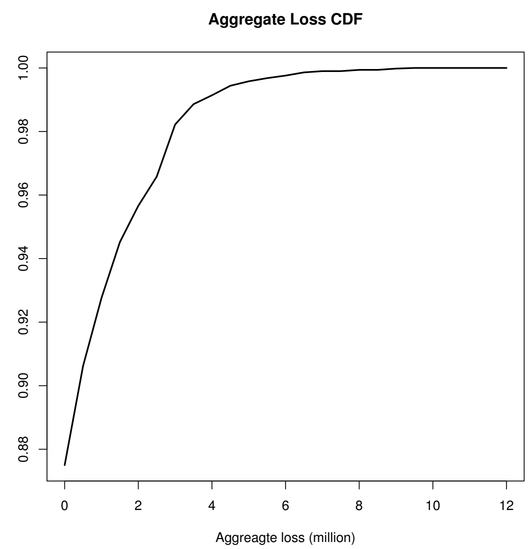

Figure 14: Generated aggregate loss distribution.

The resulting aggregate loss distribution is as expected highly skewed, with 78.1% probability of no losses and a mean of \$1,108,974. □

4 Risk Loads and Capital Requirements

An essential job of actuaries is to quantify risks. In this section, we will introduce various risk measures and capital requirements that could be incorporated in our pricing model. We would like to stress that quantifying risks is a complex undertaking. So far no single risk measure can fulfill all of the properties that actuaries would like to have.

For example, the risk measures standard deviation or variance are symmetric. They do not differentiate between losses and gains. In practice, actuaries and management teams are concerned with the management of potential losses. Another risk measure, value at risk, is concerned with potential losses above a threshold. Unfortunately, this measure does not tell us anything about how severe losses could be if they exceed the threshold. Probability of ruin is another risk measure that actuaries have spent considerable time studying. Here the actuary would set the capital requirements of a company so that the probability of ruin is acceptably small. Similar to value at risk, this measure provides no information about the severity of ruin.

4.1 Risk Measures

Various risk measures have appeared in the actuarial literature: standard deviation, variance, probability of ruin, value at risk (VaR), expected policyholder deficit (EPD), and tail VaR are among them.

- Standard deviation or variance

  Standard deviation and variance methods equate more volatility in the loss distribution with more riskiness. These methods set a risk load directly proportional to the standard deviation or variance. They are popular for their simplicity and mathematical tractability. However, they ignore the distinction between the upside and downside risks, which is critical for proper pricing especially when the loss distribution is highly skewed.

- Probability of ruin

  Probability of ruin focuses on the theoretical ruin threshold, the point where the liabilities are greater than the assets. For a probability ϵ, it seeks the capital amount such that

  $$\mathrm{Prob}(X \leq \mathrm{capital} + \mathrm{E}(X)) = 1 - \epsilon \tag{42}$$

  Probability of ruin is easy to understand and to compute. Unfortunately, it considers only the probability of ruin and lacks the consideration of loss severity when ruin occurs.

- Value at risk (VaR)

Value at Risk is generally defined as the capital necessary, in most cases, to cover the losses from a portfolio over a specified holding period. The VaR is defined as the smallest value that is greater than a predetermined percentile of the loss distribution. That is, for a selected probability α,

$$\mathrm{VaR}_\alpha = \inf\{x|\,\mathrm{Prob}(X \leq x) > \alpha\} \tag{43}$$

Similar to the probability of ruin risk measure, VaR is easy to understand and to compute but lacks the consideration of loss severity.

- Policyholder deficit (EPD)

  Expected Policyholder Deficit is the expected value of the difference between the amount the insurer is obligated to pay the claimant and the actual amount paid by the insurer, provided that the former is greater. Mathematically, it is

$$\int_{\mathrm{capital}+\mathrm{E}(X)}^{\infty} (x - \mathrm{capital} - \mathrm{E}(X)) \cdot f(x)\,dx \tag{44}$$

  where $\mathrm{E}(X)$ is the expected loss and capital refers to the excess of assets over liabilities. When considering the EPD as the risk measure, one usually uses the ratio of the EPD to the expected loss (called the EPD ratio) to adjust to the scale of different risk element sizes. That is, an EPD ratio of ϵ would set capital to be the amount such that

$$\frac{\int_{\mathrm{capital}+\mathrm{E}(X)}^{\infty} (x - \mathrm{capital} - \mathrm{E}(X)) \cdot f(x)\,dx}{\mathrm{E}(X)} = \epsilon \tag{45}$$

  Expected policyholder deficit considers the severity as well as the probability of the deficit. However, it is highly sensitive to extreme events.

- Tail value at risk (Tail VaR)

  Tail Value at Risk is also called tail conditional expectation or expected shortfall and it is the conditional expected value of losses:

$$\mathrm{TailVaR}_\alpha(X) = \mathrm{E}\left(X|X \geq \mathrm{VaR}_\alpha(X)\right) \tag{46}$$

  That is, for a selected probability α, TailVaR at α is the expected value of those losses greater than or equal to $\mathrm{VaR}_\alpha(X)$. Unlike VaR, TailVaR considers the loss severity. It is also less sensitive to extreme losses than the expected policyholder deficit measure.

Among the risk measures stated so far in this section, TailVaR is the only *coherent* risk measure in the sense discussed in the paper *Coherent Measures of Risk* [1]. A risk measure is said to be coherent if it satisfies four axioms: sub-additivity, monotonicity, positive homogeneity, and translation invariance.

The sub-additivity axiom ensures that the merging of two portfolios of risks does not create extra risk. Monotonicity says that if portfolio X always generates losses smaller than portfolio Y, then the risk measure for X should not be larger than that of Y. Positive homogeneity tells us that merging two *identical* portfolios doubles the risk measure. Finally, translation invariance says that if we add a constant to all losses of a portfolio, then the risk measure of the portfolio should also increase by the same constant (see [1, 21]). For our model we have selected Tail VaR at 99% as our risk measure.

In addition to the risk measures mentioned above, there are many other risk measures such as CAPM (see [9]), marginal cost of capital with and without the application of game theory [18, 22], and worse conditional expectation [1] (this is also a coherent risk measure). One should also take them into consideration when selecting a risk measure.

4.2 Capital Requirements

For insurance companies operating in the United States, the capital requirements are heavily regulated by the NAIC risk-based capital standards. Reinsurance companies (especially non-US reinsurers), on the other hand, are usually not as heavily regulated as primary insurance companies are with respect to capital requirements. In our model we will simplify matters and not consider how the NAIC risk based capital requirements would be affected in the pricing of a single excess-of-loss treaty. Rather we will take the position that we are evaluating the treaty on a stand-alone-basis. Moreover, the capital requirements are directly tied to the selected risk measure.

One main reason to purchase reinsurance is to mitigate large losses and to reduce the volatility of the underwriting results. As a result of the reinsurance purchase, the amount of capital required to guard against unexpected losses and high volatility is reduced. We consider the reduction in capital as *rented* capital from the reinsurer. That is, the reduction in required capital from before reinsurance to after reinsurance is the amount of capital that reinsurer provides. The ceding insurer must pay a fee for renting this capital from the reinsurer. The amount of rented capital depends on the reinsurance coverage and can be computed directly from the simulation results.

Illustration

The simulation results from our auto liability example are as follows:

Table 11: Aggregate loss distribution statistics before and after reinsurance.

Percentile	Gross	Net
5.0 %	0	0
10.0 %	2,034,094	2,034,094
25.0 %	4,105,924	4,105,924
50.0 %	7,578,306	7,459,913
75.0 %	13,198,430	12,000,000
90.0 %	20,148,295	17,003,454
92.0 %	22,175,644	18,257,892
98.0 %	37,985,818	27,054,490
99.0 %	51,212,932	38,456,724
99.5 %	76,985,851	62,732,939
99.8 %	143,818,339	122,821,239
100.0 %	904,128,288	882,361,851

The following graph makes the reinsurance effect clearer:

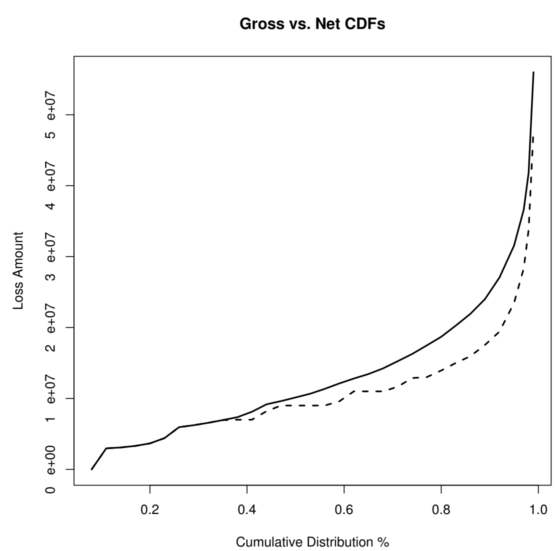

Figure 15: Gross versus net loss distributions.

With the chosen risk measure of Tail VaR at 99%, we first find the VaR at 99% and then compute the conditional expected value of losses given that they are larger than or equal to the 99% VaR. In our simulation we have to take the average of all losses greater than or equal to the 99% VaR value. The following table shows the VaR values and the Tail VaR values at 99% on a gross and net bases.

Table 12: Tail VaR calculation.

	Gross	Net
$\text{VaR}_{99\%}$	$51,212,932$	$38,456,724$
$\text{TailVaR}_{99\%}$	$128,583,553$	$115,354,489$

Therefore, we have a capital reduction of

$$128,583,553 - 115,354,489 = 13,229,064$$

and this is the amount of *rented* capital that will be incorporated in the premium calculation. □

5 Reinsurance IRR Pricing Model

Our pricing methodology follows closely the paper *Financial Pricing Model for Property and Casualty Insurance Products: Modeling the Equity Flows* [10]. Readers interested in the reasoning and intuitions of the details should refer to the paper.

The goal of IRR pricing is to generate the equity flows (net cash flows) associated with the treaty being priced. The amount of premium is an unknown that must be solved for so that the IRR on the resulting equity flows is equal to the pricing target. In theory an iterative process is used to solve for the premium. In practice the premium is found by running the goal seek algorithm in Excel©.

With the objective of generating the equity flow, the model is designed to calculate the cash flows necessary for the calculation of the equity flow. These cash flows are:

- U/W cash flows
- Investment income cash flows
- Federal income tax flows ("+" denotes a refund; "−" denotes a payment)

- Asset flows

The equity flow is then calculated via the basic relation:[11]

$$\text{Equity Flow} = \text{U/W Flow} + \text{Investment Income Flow} \\ + \text{Tax Flow} - \text{Asset Flow} + \text{DTA Flow}. \quad (47)$$

We use the convention that a positive equity flow denotes a flow of cash from the insurer to the equityholders, and a minus a payment by the equityholders to the insurer.

5.1 Illustration

Recall the illustrative excess-of-loss treaty from Section 3. It is assumed to be effective Jan 1, 2005.

Certain treaty characteristics serve as inputs to the model. These characteristics consist of the following *costs* for the layer being priced:

- amount of expected ultimate loss for the layer ($1,108,974 as detailed in Section 3),
- brokerage expenses as a percentage of base premium (10%),
- LAE as a percentage of base premium (3%),

and the following collection/payment *patterns*:

- premium collection pattern (assumed to be 100% at treaty inception)
- loss payment pattern:

Table 13: Loss payment pattern.

Year	Loss payment	Year	Loss payment
2005	22.2%	2010	4.7%
2006	29.3	2011	4.3
2007	15.9	2012	3.7
2008	7.9	2013	3.5
2009	5.8	2014	2.7

[11] Properly speaking, the change in the deferred tax asset (DTA) is not a cash flow, if by cash one means cash equivalents. Since assets include DTA, the change in assets is also not a cash flow. But, then item (assets − DTA) consists of cash equivalents and hence $\Delta(\text{assets} - \text{DTA})$ is a cash flow. We have simply expressed $\Delta(\text{assets} - \text{DTA})$ as the difference Δassets minus ΔDTA.

The model uses annual valuations. With the exception of the written premium and UEPR which incept on Jan 1, our simplifying assumption is that all accounting and cash flow activity occur at year end.

The model also requires certain *parameter* inputs consisting of:

- investment rate of return on invested assets (5.5%),
- effective tax rate for both investments and U/W income (assumed to be 35% for both),
- surplus assumptions (specifics discussed in Section 4.2 above),
- the target return on capital (12%), and
- IRS loss & LAE reserve discount factors:

Table 14: Internal Revenue Service discount factors.

Year	Discount Factor	Year	Discount Factor	Year	Discount Factor
2005	0.7410	2010	0.7583	2015	0.8805
2006	0.7367	2011	0.7554	2016	0.9221
2007	0.7438	2012	0.7823	2017	0.9766
2008	0.7040	2013	0.8117	2018	0.9766
2009	0.7264	2014	0.8441	beyond	0.9766

5.1.1 Assets

Required Surplus

Surplus is held only for the policy term in our illustration. It exists to cover unforeseen contingencies and is determined to maintain an acceptable level of risk. As discussed above in section 4.2 we used a rented capital approach using a 99% TVaR level of risk to give a surplus need of $13,229,064.This surplus we assume is held for the first year only. This assumption reflects the fact that new business writings pose a greater risk than business in reserve run-off which has capital embedded in reserves to support unforeseen contingencies.

Total Reserve

The total reserve at any point in time is the sum of the unearned premium reserve and the held loss & LAE reserves. We assume no reserve deficiency and so as losses pay out the held loss reserves are taken down dollar for dollar.

Required Assets

The amount of assets the insurance company needs to support the policy is equal to total reserves as defined above, plus the required:

$$\text{Required Assets} = \text{Total Reserves} + \text{Required Surplus} \tag{48}$$

Illustration

On Jan 1, 2005 the UEPR is equal to the WP which is $3,044,605. The contract is yet fully unearned and the loss reserves are $0. With surplus of $13,229,064 put up at treaty inception the total assets are $16,273,669. By year end the UEPR is $0, loss reserves are equal to $862,782 (ultimate losses less paid losses of $246,192), and surplus is $0 for total assets of $862,782.

Table 15: Asset calculation.

	UEPR	Held Reserve	Surplus Capital	Held Asset
1/ 1/2005	3,044,605	0	13,229,064	16,273,669
12/31/2005	0	862,782	0	862,782
12/31/2006	0	537,964	0	537,964
12/31/2007	0	361,694	0	361,694
12/31/2008	0	274,218	0	274,218
12/31/2009	0	209,898	0	209,898
12/31/2010	0	157,776	0	157,776
12/31/2011	0	110,090	0	110,090
12/31/2012	0	69,058	0	69,058
12/31/2013	0	30,244	0	30,244
12/31/2014	0	0	0	0

□

Income Producing Assets

Not all of the assets held by the company to support the policy generate investment income. Both the premium receivable (if any) and the deferred tax asset are non-income producing assets:

$$\text{Income Producing Assets} = \text{Required Assets} - \text{Premium Receivable} - \text{DTA} \tag{49}$$

In our formulation weve assumed all premium is collected up front and consequently the premium receivable asset is zero. The calculation of DTA is discussed below.

Investment Income

The Investment Income earned over the year is simply calculated as the product of the annual effective investment rate of return times the amount of income producing assets held at the beginning of the year:

$$\text{Invest Inc}_{\text{@time } T} = \text{Annual Invest ROR} \cdot \text{Investible Assets}_{\text{@time } T-1} \tag{50}$$

Table 16: Investment income calculation.

	Held Asset	Non-Income Producing	Income Producing	Investment Income
1/ 1/2005	16,273,669	0	16,273,669	0
12/31/2005	862,782	28,656	834,125	895,052
12/31/2006	537,964	17,134	520,830	45,877
12/31/2007	361,694	4,028	357,666	28,646
12/31/2008	274,218	8,307	265,911	19,672
12/31/2009	209,898	6,756	203,142	14,625
12/31/2010	157,776	3,919	153,857	11,173
12/31/2011	110,090	4,163	105,927	8,462
12/31/2012	69,058	3,269	65,789	5,826
12/31/2013	30,244	1,994	28,251	3,618
12/31/2014	0	0	0	1,554

5.1.2 Taxes

IRS Discounted Reserves

The IRS Discounted Reserves are calculated by multiplying the company's Held Reserves by a discount factor. The discount factor varies by line of business, accident year, and by age of the accident year. Our basic formula for IRS discounted reserves is thus

$$\text{IRS Discounted Reserves} = \text{IRS Discount Factor} \cdot \text{Held Reseves} \tag{51}$$

Taxable U/W Income

The IRS defines the taxable U/W income earned over an accounting year as

$$\text{Written Premium} - 0.8 \cdot \Delta\text{UEPR} - \text{Paid Expenses} - [\text{Paid Losses} + \Delta\text{IRS Disc Reserves}] \quad (52)$$

where all activity is over the relevant accounting year. In our illustration the treaty is effective Jan 1 and so the change in the UEPR is identically zero. Table 17 shows the computation of the tax on U/W income.

Tax on Investment Income

This tax is simply calculated as 35% of earned investment income for the year.

Total Tax

The total federal income tax paid each year is equal to the sum of the yearly tax on U/W income and the yearly tax on investment income.

Table 18: Tax calculation.

	Tax on UW Inc	Tax on Inv Inc	Tax Total
1/ 1/2005			
12/31/2005	617,167	313,268	930,436
12/31/2006	−28,656	16,057	−12,599
12/31/2007	−17,134	10,026	−7,108
12/31/2008	−4,028	6,885	2,857
12/31/2009	−8,307	5,119	−3,188
12/31/2010	−6,756	3,910	−2,845
12/31/2011	−3,919	2,962	−957
12/31/2012	−4,163	2,039	−2,124
12/31/2013	−3,269	1,266	−2,003
12/31/2014	−1,994	544	−1,450

Deferred Tax Asset

There are two components to the DTA: the portion due to the Revenue Offset; and the portion due to IRS Discounting of Loss & LAE Reserves.

The DTA due to the Revenue Offset is equal to

$$35\% \cdot 20\% \cdot \text{UEPR} \quad (53)$$

	WP	Expenses	Held Reserve	Paid Loss	IRS Disc Factors	IRS Disc Reserves	Chg in Disc Reserves	Taxable UW Income
1/ 1/2005	3,044,605	395,799	0	0				
12/31/2005			862,782	246,192	0.7410	639,279	639,279	1,763,335
12/31/2006			537,964	324,817	0.7367	396,336	−242,943	−81,874
12/31/2007			361,694	176,270	0.7438	269,021	−127,315	−48,955
12/31/2008			274,218	87,476	0.7040	193,054	−75,968	−11,508
12/31/2009			209,898	64,320	0.7264	152,468	−40,586	−23,735
12/31/2010			157,776	52,122	0.7583	119,648	−32,820	−19,301
12/31/2011			110,090	47,686	0.7554	83,160	−36,488	−11,198
12/31/2012			69,058	41,032	0.7823	54,022	−29,137	−11,895
12/31/2013			30,244	38,814	0.8117	24,548	−29,474	−9,340
12/31/2014			0	30,244	0.8441	0	−24,548	−5,696

For our illustration the year end UEPR is identically equal to zero.

The DTA due to IRS Discounting at the end of Accounting Year T is equal to

$$35\% \cdot [(\text{Held Loss Reserve}_{\text{at time } T} - \text{IRS Loss Reserve}_{\text{at time } T}) \\ - (\text{Held Loss Reserve}_{\text{at time } T+1} - \text{IRS Loss Reserve}_{\text{at time } T+1})] \quad (54)$$

The amount in each square bracket is the amount that reverses in the year (which is all that is statutorily recognized).

Illustration

At year end 2005 the held loss reserve is \$862,782 while the IRS discounted reserve is \$639,279. If the full DTA were recognized it would be $(862,782 - 639,279) \cdot 35\% = 78,226$. But only the amount that reverses in one year is recognized. That is, the fully recognized DTA at year end 2006 would be $(537,964 - 396,336) \cdot 35\% = 49,570$. Thus the amount that reverses during 2006 is \$28,656 and this is the amount of DTA at year end 2005. □

Table 19: Deferred tax asset calculation.

	Held Reserve	IRS Disc Reserves	DTA (Reserve Disc)
1/ 1/2005			
12/31/2005	862,782	639,279	28,656
12/31/2006	537,964	396,336	17,134
12/31/2007	361,694	269,021	4,028
12/31/2008	274,218	193,054	8,307
12/31/2009	209,898	152,468	6,756
12/31/2010	157,776	119,648	3,919
12/31/2011	110,090	83,160	4,163
12/31/2012	69,058	54,022	3,269
12/31/2013	30,244	24,548	1,994
12/31/2014	0	0	0

5.1.3 Cash Flows

The relevant cash flows for determining the Equity Flow are described below.

U/W Cash Flow

This item is defined as

$$\text{U/W Cash Flow} = \text{WP} - \text{Paid Expenses} - \text{Paid Loss} \qquad (55)$$

Investment Income Flow

This item is defined as the yearly investment income earned. The calculation is described above.

Tax Flow

The Tax Cash Flow is defined at the negative (to denote a flow from the company) of the federal income taxes paid that year. The calculation of this flow item is described above.

DTA Flow

The DTA Flow is defined as the change in the DTA asset over a year.

Asset Flow

The asset flow is defined as the change in the required assets. The composition and calculation of the required assets are described above.

Equity Flow

To compute the Equity Flow at each year we use the cash flow definition:

$$\begin{aligned}\text{Equity Flow} = {} & -\text{Asset Flow} + \text{U/W Flow} \\ & + \text{Investment Income Flow} + \text{FIT Flow} + \text{DTA Flow} \qquad (56)\end{aligned}$$

Recall that we use the convention that a positive equity flow denotes a flow of cash from the insurer to the equityholders, and a negative a payment by the equityholders to the insurer. The relevant cash flows for our illustration are summarized in Table 20.

The IRR on the resulting equity flows is 12%. The premium of $3,044,605 was iteratively determined with this goal.

	Cash UW	Inv Income	Tax	DTA	Asset	Equity Flow
1/ 1/2005	2,648,806	0	16,273,669	0	0	−13,624,863
12/31/2005	−246,192	895,052	−15,410,887	−930,436	28,656	15,157,968
12/31/2006	−324,817	45,877	−324,817	12,599	−11,522	46,954
12/31/2007	−176,270	28,646	−176,270	7,108	−13,107	22,648
12/31/2008	−87,476	19,672	−87,476	−2,857	4,279	21,094
12/31/2009	−64,320	14,625	−64,320	3,188	−1,552	16,262
12/31/2010	−52,122	11,173	−52,122	2,845	−2,836	11,182
12/31/2011	−47,686	8,462	−47,686	957	244	9,663
12/31/2012	−41,032	5,826	−41,032	2,124	−894	7,056
12/31/2013	−38,814	3,618	−38,814	2,003	−1,275	4,346
12/31/2014	−30,244	1,554	−30,244	1,450	−1,994	1,010

6 Summary

The pricing of high layers of reinsurance is a difficult task primarily because of the nature of extreme events. The practicing actuary requires a diverse toolbox to tackle this pricing problem. As part of the toolbox he needs well grounded statistical methods for analyzing the data at hand, a good understanding of the modeling techniques and risk assessment, and a comprehensive pricing model that does not sweep under the rug many of the regulatory, tax, and business constraints of the insurance company.

In the past very large losses would be labeled as *outlier observations*, rationalized as extremely improbable, and sometimes even removed from the data set. For the reinsurance actuary these observations are likely to be the most important observations in the data set.

In this paper we have introduced results from a well grounded statistical theory to deal with extreme events. The first result tells us that the distribution of the maximum of a sample converges to one of the three extreme value distributions. This result is analogous to the central limit theorem. The second result shows that the distribution of excess losses converges to the generalized Pareto distribution as the threshold increases. This is the key result for pricing very high layers of reinsurance. We also introduce the peaks over threshold method from extreme value theory and a powerful graphical technique, the QQ-plot, to assess distributional assumptions.

The paper also provides a hands-on approach to loss modeling. We present the collective risk model and use it to calculate the aggregate loss distribution for the example that is carried throughout the paper. We also introduced various measures to quantify risk and our treatment of capital requirements. Our discussions on collective risk models and risk measures are by no means complete but the framework we have laid should provide the practicing actuary with a foundation that can be put to practice.

Finally, the cash flow model (IRR pricing model) brings everything together to determine the price of a reinsurance layer. It is designed to calculate the equity flows; that is, the cash flows between the company and its equity holders. This pricing model is comprehensive: it includes all relevant components of cash flow for an insurance company to derive the final price given the risk premium and other parameters.

Appendix A

In this appendix, we will show that the distribution $\widehat{F(x)}$ in equation (21) in section 2.4 is a generalized Pareto distribution by deriving the associate parameters.

$$
\begin{aligned}
\widehat{F(x)} &= (1 - F_n(u))\, G_{\xi,\sigma}(x-u) + F_n(u) \\
&= (1 - F_n(u)) \left(1 - \left(1 + \frac{\xi}{\sigma}(x-u) \right)^{-1/\xi} \right) + F_n(u) \\
&= 1 - (1 - F_n(u)) \left(1 + \frac{\xi}{\sigma}(x-u) \right)^{-1/\xi} \\
&= 1 - \left((1 - F_n(u))^{-\xi} \left(1 + \frac{\xi}{\sigma}(x-u) \right) \right)^{-1/\xi} \\
&= 1 - \left((1 - F_n(u))^{-\xi} + \frac{\xi}{(1-F_n(u))^{\xi} \cdot \sigma}(x-u) \right)^{-1/\xi} \\
&= 1 - \left(1 + \frac{\xi}{(1-F_n(u))^{\xi}\, \sigma} \left(x - \left\{ u - \frac{\sigma}{\xi} \left[1 - (1 - F_n(u))^{\xi} \right] \right\} \right) \right)^{-1/\xi} \\
&= 1 - \left(1 + \frac{\xi}{\tilde{\sigma}}(x - \tilde{u}) \right)^{-1/\xi} \\
&= G_{\xi,\tilde{\sigma}}(x - \tilde{u})
\end{aligned}
$$

where $\tilde{\sigma} = \sigma(1 - F_n(u))^{\xi}$ and $\tilde{u} = u - [\sigma(1 - (1 - F_n(u))^{\xi})/\xi]$.

Appendix B *IRR cash flow model exhibits*

	WP (1)	UEPR (2)	LAE (3)	Brokerage (4)	Paid Loss (5)	Nominal Reserve (6)
11/ 1/2005	3,044,605	3,044,605	91,338	304,461		
12/31/2005					246,192	862,782
12/31/2006					324,817	537,964
12/31/2007					176,270	361,694
12/31/2008					87,476	274,218
12/31/2009					64,320	209,898
12/31/2010					52,122	157,776
12/31/2011					47,686	110,090
12/31/2012					41,032	69,058
12/31/2013					38,814	30,244
12/31/2014					30,244	0

$(3) = (1) \cdot 3.0\%$
$(4) = (1) \cdot 10.0\%$

	Held Reserve (7)	Surplus Capital (8)	Held Asset (9)	Non-Income Producing (10)	Income Producing (11)
1/ 1/2005	0	13,229,064	16,273,669	0	16,273,669
12/31/2005	862,782	0	862,782	28,656	834,125
12/31/2006	537,964	0	537,964	17,134	520,830
12/31/2007	361,694	0	361,694	4,028	357,666
12/31/2008	274,218	0	274,218	8,307	265,911
12/31/2009	209,898	0	209,898	6,756	203,142
12/31/2010	157,776	0	157,776	3,919	153,857
12/31/2011	110,090	0	110,090	4,163	105,927
12/31/2012	69,058	0	69,058	3,269	65,789
12/31/2013	30,244	0	30,244	1,994	28,251
12/31/2014	0	0	0	0	0

$(7) = 100\% \cdot (6)$
$(9) = (2) + (7) + (8)$
$(10) = (21)$
$(11) = (9) - (10)$

	Investment Income (12)	IRS Disc Factors (13)	Taxable UW income (14)	Tax Paid UW (15)	Taxable Invest (16)
1/ 1/2005	0				
12/31/2005	895,052	0.7410	1,763,335	617,167	895,052
12/31/2006	45,877	0.7367	−81,874	−28,656	45,877
12/31/2007	28,646	0.7438	−48,955	−17,134	28,646
12/31/2008	19,672	0.7040	−11,508	−4,028	19,672
12/31/2009	14,625	0.7264	−23,735	−8,307	14,625
12/31/2010	11,173	0.7583	−19,301	−6,756	11,173
12/31/2011	8,462	0.7554	−11,198	−3,919	8,462
12/31/2012	5,826	0.7823	−11,895	−4,163	5,826
12/31/2013	3,618	0.8117	−9,340	−3,269	3,618
12/31/2014	1,554	0.8441	−5,696	−1,994	1,554

$(12)_t = (11)_{t-1} \cdot 5.5\%$
$(14)_t = (1)_t - 80\% \cdot ((2)_t - (2)_{t-1}) - (3)_t - (4)_t - (5)_t - ((7)_t \cdot (13)_t - (7)_{t-1} \cdot (13)_{t-1})$
$(15) = (14) \cdot 35\%$
$(16) = (12)$

	Tax Paid Inv Inc (17)	Total Tax Paid (18)	DTA Revenue Offset (19)	DTA Reserve Disc (20)	Total DTA (21)
1/ 1/2005		0	0	0	0
12/31/2005	313,268	930,436	0	28,656	28,656
12/31/2006	16,057	−12,599	0	17,134	17,134
12/31/2007	10,026	−7,108	0	4,028	4,028
12/31/2008	6,885	2,857	0	8,307	8,307
12/31/2009	5,119	−3,188	0	6,756	6,756
12/31/2010	3,910	−2,845	0	3,919	3,919
12/31/2011	2,962	−957	0	4,163	4,163
12/31/2012	2,039	−2,124	0	3,269	3,269
12/31/2013	1,266	−2,003	0	1,994	1,994
12/31/2014	544	−1,450	0	0	0

$(17) = (16) \cdot 35\%$
$(18) = (15) + (17)$
$(19) = 20\% \cdot (2) \cdot 35\%$
$(20)_t = (((7)_t - (7)_{t-1}) - ((7)_t \cdot (13)_t - (7)_{t-1} \cdot (13)_{t-1}) \cdot 35\%$
$(21) = (19) + (20)$

	Cash UW (22)	Investment Income (23)	Asset (24)	Tax (25)	DTA (26)	Equity Flow (27)
1/ 1/2005	2,648,806	0	16,273,669	0	0	−13,624,863
12/31/2005	−246,192	895,052	−15,410,887	−930,436	28,656	15,157,968
12/31/2006	−324,817	45,877	−324,817	12,599	−11,522	46,954
12/31/2007	−176,270	28,646	−176,270	7,108	−13,107	22,648
12/31/2008	−87,476	19,672	−87,476	−2,857	4,279	21,094
12/31/2009	−64,320	14,625	−64,320	3,188	−1,552	16,262
12/31/2010	−52,122	11,173	−52,122	2,845	−2,836	11,182
12/31/2011	−47,686	8,462	−47,686	957	244	9,663
12/31/2012	−41,032	5,826	−41,032	2,124	−894	7,056
12/31/2013	−38,814	3,618	−38,814	2,003	−1,275	4,346
12/31/2014	−30,244	1,554	−30,244	1,450	−1,994	1,010

$(22) = (1) - (3) - (4) - (5)$
$(23) = (16)$
$(24)_t = (9)_t - (9)_{t-1}$
$(25) = -(18)$
$(26)_t = (21)_t - (21)_{t-1}$.
$(27) = -(24) + (22) + (23) + (25) + (26)$

Appendix C *Distribution Functions*

In Table 21 (on the next page) we present some common distribution functions and their parametrizations.

Name	CDF F or density f	Parameters
Exponential	$F(x) = 1 - e^{-\lambda x}$	$\lambda > 0$
Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	$\alpha, \beta > 0$
Weibull	$F(x) = 1 - e^{-cx^\tau}$	$c > 0, \tau > 0$
Log-normal	$f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-(\ln x - \mu)^2/(2\sigma^2)}$	$\mu \in \mathbb{R}, \sigma > 0$
Pareto	$F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^\alpha$	$\alpha, \kappa > 0$
Benktander-type-I	$F(x) = 1 - (1 + 2(\beta/\alpha)\ln x) e^{-\beta(\ln x)^2 - (\alpha+1)\ln x}$	$\alpha, \beta > 0$
Benktander-type-II	$F(x) 1 - e^{\alpha/\beta} x^{-(1-\beta)} e^{-\alpha x^\beta/\beta}$	$\alpha > 0, 0 < \beta < 1$
Log-gamma	$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\ln x)^{\beta-1} x^{-\alpha-1}$	$\alpha, \beta > 0$

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Reinsurance Applications for the RMK Framework

David R. Clark

Abstract

Recent work by Ruhm, Mango and Kreps, known as the RMK Framework, has proven to be a great advance in the theory of risk. The RMK Framework is a way of viewing an allocation problem that focuses on the scenarios of greatest concern and the probability that those scenarios take place. This paper avoids the mathematical details of the model, but instead gives three applications for the RMK Framework, using non-technical language to explain the basic concept.

Keywords. Risk Theory, RMK Framework, Reinsurance

1. INTRODUCTION

Over the last few years, a significant advance has taken place in the theory of risk. The idea has centered around papers by Ruhm/Mango [4], Mango [3] and Kreps [2], and so is becoming known as the RMK Framework.[1]

While these papers have given the underlying theory, widespread acceptance is still slow in coming. The purpose of the present paper is to demonstrate the RMK Framework in a couple of familiar reinsurance applications to illustrate its appeal to the more general audience.

The RMK Framework is not a single method, but rather a framework for viewing the risk/reward problem that gives rise to a family of methods which share consistent mathematical properties. While mathematical elegance and flexibility make RMK very appealing to "technical" actuaries, they actually raise suspicion outside actuarial circles – aren't we once again picking the answer we want and then covering our tracks with complicated formulas?

The surprising answer is that RMK is very much in line with the way insurance management already thinks about its business, and it can be presented in a very transparent fashion.

The key idea is that we concentrate on the scenarios in which the company as a whole could lose money, and then ask which business segments contributed to that loss. This idea will be illustrated using three examples:

[1] The spark of the idea can be traced back even earlier to Halliwell [1], especially "Appendix E – The Allocation Problem", pages 346-348.

1. Allocation of aggregate stop-loss cost to line of business
2. Allocation of profit commission to policy year (the deficit carry forward problem)
3. Allocation of target profit loads by line of business

The reader seeking a more rigorous mathematical treatment of the RMK Framework is advised to read the original papers. Here we are just illustrating the approach, with the hope that seeing its results in practice will be more convincing than mathematical proofs.

2. EXAMPLE #1: ALLOCATION OF AGGREGATE STOP-LOSS COST TO LINE OF BUSINESS

The first problem that we will review deals with how an insurance company should allocate its ceded premium for reinsurance that applies across multiple lines of business.

In this example, you work for a small insurance company that writes three lines of business. You have purchased reinsurance that protects your overall loss ratio. The reinsurer will cover 20% points of loss ratio in excess of a gross 80% loss ratio (that is, the ceding company will be back on the hook for paying losses above a 100% loss ratio). The cost of this cover is 4% of gross premium.

The profile of the business is as follows:

	Subject Premium	ELR	Coef. of Variation (CV)
Line A	1,250	80.0%	.500
Line B	1,875	80.0%	.500
Line C	2,150	69.8%	1.000
All Lines	5,275	75.8%	.438

We make the additional simplifying assumption that losses for the three lines of business come from independent lognormal distributions, though this is not necessary in practice.

How should the 4% reinsurance charge be allocated to line of business? The simplest approach would be to charge each line of business the same 4%. However, the managers for each line immediately begin arguing about why their line should get less than the 4% charge.

The managers for Lines A and B insist that the charge should be proportional to the variance of their loss distributions, leading to something less than 4% for them. The manager for Line C objects, noting that her ELR is well below the 80% attachment point of the reinsurance, and therefore should be charged less than the other lines.

Who is right? We can answer this question by first posing a different question: What would a scenario look like in which the overall 80% attachment point is pierced – which line(s) of business would have caused it?

We can think of several situations in which the reinsurance would be triggered based on the 80% attachment point being pierced. Obviously, any one line could have an extremely bad year, causing the overall loss ratio to be above 80% even if the other two lines of business were better than expected. There could also be various combinations in which two lines of business were a little worse than expected, but still cause the 80% attachment point to be hit.

As the actuary, we can list out many possible loss scenarios in which the reinsurance is triggered. Further, for each of these scenarios, we can compare each line's actual loss ratio to the 80% attachment point to see how much it contributed to the overall loss. Given a loss distribution for each line of business (and our independence assumption), it is also easy to assign relative probabilities to each of these scenarios. A reasonable allocation scheme will simply be a probability-weighted average of all the scenarios.

This thought process is what we have been calling the "RMK Framework." For ease of illustration, it is best thought of using a simulation model. The steps are as follows:

1. Simulate losses for each line of business.
2. For each line of business, calculate the difference between the actual loss and the 80% attachment point.
3. For all lines combined, calculate the difference between the actual loss and the 80% attachment point. Store this scenario if the answer is positive.
4. Repeat steps 1-3 many times.

5. For each of the scenarios in which overall losses were above 80%, cap the total loss at 20% of the total all-lines premium (this is the reinsurer's limit). Lower the contributions from the individual lines proportionately when the cap applies.[2]

6. Average all of the simulated scenarios.

This procedure is shown on Exhibit 1a. In this example, only twenty scenarios have been generated, though a realistic calculation would require many more simulations.

A great advantage of this method is that we can bring the simulated scenarios back to the line of business managers and defend the allocation by pointing to the scenarios that caused the reinsurance to be triggered.

In fact, we can note several advantages of this way of framing the allocation problem:

- It is easy to explain to the business managers.
- It works directly with a simulation model that may have been created already for other purposes. In fact, if we had created a dependence or correlation structure between the lines of business, the method would still be applied with no changes.
- The answer does not depend on whether two of the lines of business are grouped together or are kept separate.[3]

After discussing Exhibit 1a with company management, a number of refinements or alternatives could be proposed.

One reaction may be that under some scenarios we actually allocate a negative dollar amount to some lines of business. This may in fact be very reasonable, since we are then saying that a "good" line is subsidizing a "bad" line of business; there is no theoretical reason to disallow negatives. However, that may not be acceptable on a practical basis given that it would create potential difficulties in explaining negative ceded premium to external audiences. To illustrate the flexibility in the RMK Framework, we can modify the method so that the charge is allocated in proportion to total loss dollars, eliminating the negative allocations. This is shown on Exhibit 1b.

This flexibility is a strength in viewing RMK as a decision-making framework and not as a

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[2] In each example, the factor that accomplishes this reduction is labeled *L(x)*, in order to be consistent with Kreps' notation for risk measures.

[3] This characteristic is the "additive" in Kreps' "Additive Co-Measures" label.

rigid allocation method.

3. EXAMPLE #2: ALLOCATION OF PROFIT COMMISSION TO POLICY YEAR

For our second example, we assume that you are now a reinsurance actuary pricing an excess-of-loss treaty that includes a profit commission that is calculated on a three-year block. The effective date for the third year is coming up shortly, and you need to know the expected profit commission under the proposed terms. The difficulty is that the first two years are still very immature and, while they appear to be profitable, the results are far from certain. The question is how to estimate the value of this uncertain carry forward of results from prior years.

We are faced with the problem of estimating the overall expected profit commission for the three-year block and then also the allocation problem of assigning the expected commission to the individual policy years.

The profit commission formula is calculated as follows.

$$\text{Profit Commission} = (\text{Reinsurance Premium} - \text{Expense} - \text{Actual Loss}) \cdot \text{Profit\%}$$

where Expense = 20% of Reinsurance Premium
Profit % = 35%

As in the example for the aggregate stop-loss reinsurance program, we begin by simulating a number of loss scenarios. For the profit commission problem, however, we are simulating losses for the same business but for three different policy periods. We could potentially complicate this model by simulating only unpaid losses for the first two years, and also by building in some year-to-year correlation structure. Such complication would not change the way we will be performing the allocation, but it would change the numbers in the scenarios that we examine.

For each of the simulated scenarios, we calculate a profit or loss for each policy year by comparing the actual loss with the available funding premium (reinsurance premium net of the 20% expense allowance). For scenarios in which the three-year block produces a profit, we multiply each year by the 35% profit-sharing amount. For scenarios in which the three-year block does not produce a profit, we do not include a commission payment.

By taking an average over all of the simulated scenarios, we then have an expected profit commission for the three-year block and also the contribution from each of the three policy periods. Exhibit 2 shows the numbers for a sample of simulated values.

4. EXAMPLE #3: ALLOCATION OF TARGET PROFIT LOADS BY LINE OF BUSINESS

Finally, we turn to the application that was the basis for developing the RMK Framework in its original context: the question of setting profit loads for individual lines of business (or product types).

While it is generally acknowledged that profit loads should be based on the risk inherent in the business written, there has not been much of a consensus on how to define that "risk."

From a stockholders perspective, the risk that matters most is the risk that losses will eat into the capital invested in the company (i.e. that capital will be "consumed"). We will therefore begin with this question – in what scenarios do actual losses exceed the pure premiums actually collected, such that our company loses value?

Following the same example used for the basket aggregate application, we will assume that our company writes three lines of business with the expected losses given in Exhibit 3a. We will also add the information that $2 million of capital is invested in the company.[4]

For each loss scenario generated via simulation, we can readily observe how much capital is taken, and which line(s) of business are most responsible for causing the loss. The capital consumed by each line of business is simply the difference between its actual loss and its expected loss (or pure premium) within a given scenario. In cases where the total loss exceeds the available capital, we simply reduce all lines proportionally. In Exhibit 3a, the factor that accomplishes this reduction is labeled the "Riskiness Leverage Ratio" or $L(x)$, following Kreps' notation.

By averaging together all of the simulated scenarios, we can produce an "expected" amount of capital that is consumed. This could alternatively be described as the stockholder's expected downside result. It is reasonable to allocate our target profit loads proportionally to each line's contribution to this amount.

As stated previously, other risk measures can be used as variations within the RMK

[4] It may be noted that the amount of capital in the company acts in a manner similar to the limit that the reinsurer provided in the stop-loss example. Once this amount is exhausted, the stockholder is no longer responsible for additional loss payments.

Framework. The stockholders may be interested, for example, in minimizing the variance of the company's results; and setting an overall profit load as a percent of this variance. The allocation scheme then simply changes the Riskiness Leverage Ratio, *L(x)*, to be proportional to the difference between actual and expected results for each scenario. Exhibit 3b shows the results with this change. The resulting allocation is equivalent to setting profit loads in proportion to the covariances of losses by line of business.[5]

5. RESULTS AND DISCUSSION

The RMK Framework is a very clear way of addressing an allocation problem. In addition to its useful mathematical properties, the chief advantage is that it allows decision making to take place with the most significant loss scenarios given the closest consideration.

This paper has deliberately been restricted to simplified examples, but the framework can easily be adapted to larger simulation models and to include risks other than nominal value losses. It should also be clear that the RMK Framework does not itself depend on a particular correlation structure among the variables being simulated; it works with the simulated output regardless of the complexity of the model generating the simulations.

All of the examples in this paper have assumed that a simulation model is used to generate the loss scenarios being reviewed. This also does not need to be the case. The same theory can be applied if a finite number of loss scenarios are selected by the business managers, with subjective weights assigned to each scenario.

6. CONCLUSIONS

The Ruhm/Mango/Kreps (RMK) Framework has been shown to be a very useful way of addressing a variety of insurance allocation problems. This paper has not established any new mathematical theory, but has attempted to show that the RMK Framework is intuitive and transparent for use by actuarial and non-actuarial decision makers.

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[5] This is not the only situation in which RMK is equivalent to a covariance allocation. For example, if the losses are modeled using a multivariate normal distribution, then any choice of risk-measure r(x) will equal the covariance allocation. The full theory on necessary conditions for the two methods to produce equivalent results has not yet been worked out.

Exhibit 1a **Allocation of Excess Loss to LOB for a "Basket Aggregate"**

	Total Loss by Line of Business (simulation results)				Loss Compared to Attachment Point (AP)				20.0% excess of 80.0%*	Capped Portion**	Allocated Excess Loss			
Scenario	Line A	Line B	Line C	Total	Line A	Line B	Line C	Total			Line A	Line B	Line C	Total
Formula:	X_1	X_2	X_3	X	X_1-AP_1	X_2-AP_2	X_3-AP_3	X-AP	r(x)	L(x)	$[X_i$-$AP_i] \cdot L(x)$			
1	428	1,926	2,254	4,608	-572	426	534	388	**388**	1.000	-572	426	534	388
2	516	773	1,267	2,556	-484	-727	-453	-1,664	**0**	0.000	0	0	0	0
3	1,277	657	532	2,466	277	-843	-1,188	-1,754	**0**	0.000	0	0	0	0
4	1,182	2,525	822	4,529	182	1,025	-898	309	**309**	1.000	182	1,025	-898	309
5	825	781	1,822	3,428	-175	-719	102	-792	**0**	0.000	0	0	0	0
6	1,564	1,075	307	2,945	564	-425	-1,413	-1,275	**0**	0.000	0	0	0	0
7	1,009	904	685	2,598	9	-596	-1,035	-1,622	**0**	0.000	0	0	0	0
8	1,162	1,449	1,699	4,310	162	-51	-21	90	**90**	1.000	162	-51	-21	90
9	513	2,486	1,278	4,277	-487	986	-442	57	**57**	1.000	-487	986	-442	57
10	742	1,444	6,813	8,998	-258	-56	5,093	4,778	**1,055**	0.221	-57	-12	1,124	1,055
11	992	730	945	2,666	-8	-770	-775	-1,554	**0**	0.000	0	0	0	0
12	1,298	756	2,482	4,535	298	-744	762	315	**315**	1.000	298	-744	762	315
13	796	1,597	488	2,882	-204	97	-1,232	-1,338	**0**	0.000	0	0	0	0
14	566	1,648	930	3,144	-434	148	-790	-1,076	**0**	0.000	0	0	0	0
15	647	1,259	674	2,580	-353	-241	-1,046	-1,640	**0**	0.000	0	0	0	0
16	1,242	2,054	4,197	7,494	242	554	2,477	3,274	**1,055**	0.322	78	179	798	1,055
17	1,214	864	433	2,510	214	-636	-1,287	-1,710	**0**	0.000	0	0	0	0
18	916	836	2,134	3,887	-84	-664	414	-333	**0**	0.000	0	0	0	0
19	1,545	679	1,379	3,602	545	-821	-341	-618	**0**	0.000	0	0	0	0
20	382	2,148	1,826	4,356	-618	648	106	136	**136**	1.000	-618	648	106	136
Average	941	1,329	1,648	3,919	-59	-171	-72	-301	**170**		-51	123	98	**170**

* r(x) = MIN[MAX(X-AP,0),20.0%xPremium]
** L(x) = r(x)/[X-AP]

	Line A	Line B	Line C	Total
Premium	1,250	1,875	2,150	5,275
ELR	80.0%	80.0%	69.8%	75.8%
Mean	1,000	1,500	1,500	4,000
Std Dev	500	750	1,500	
C.V.	0.500	0.500	1.000	
LN Mu	6.7961835	7.20164861	6.9666468	
LN Sigma	0.47238073	0.47238073	0.83255461	

Aggregate Limit: 20.0%
Attachment Point (AP): 80.0%

Exhibit 1b **Allocation of Excess Loss to LOB for a "Basket Aggregate"**

Removing Potential for Allocation of Negative Values

	Total Loss by Line of Business (simulation results)				Loss Compared to Attachment Point of 0				20.0% excess of 80.0%*	Percent of Total Loss**	Allocated Excess Loss			
Scenario	Line A	Line B	Line C	Total	Line A	Line B	Line C	Total			Line A	Line B	Line C	Total
Formula:	X_1	X_2	X_3	X	X_1-0	X_2-0	X_3-0	X-0	r(x)	L(x)		$[X_i-0] \cdot L(x)$		
1	428	1,926	2,254	4,608	428	1,926	2,254	4,608	**388**	8.4%	36	162	190	388
2	516	773	1,267	2,556	516	773	1,267	2,556	**0**	0.0%	0	0	0	0
3	1,277	657	532	2,466	1,277	657	532	2,466	**0**	0.0%	0	0	0	0
4	1,182	2,525	822	4,529	1,182	2,525	822	4,529	**309**	6.8%	81	172	56	309
5	825	781	1,822	3,428	825	781	1,822	3,428	**0**	0.0%	0	0	0	0
6	1,564	1,075	307	2,945	1,564	1,075	307	2,945	**0**	0.0%	0	0	0	0
7	1,009	904	685	2,598	1,009	904	685	2,598	**0**	0.0%	0	0	0	0
8	1,162	1,449	1,699	4,310	1,162	1,449	1,699	4,310	**90**	2.1%	24	30	36	90
9	513	2,486	1,278	4,277	513	2,486	1,278	4,277	**57**	1.3%	7	33	17	57
10	742	1,444	6,813	8,998	742	1,444	6,813	8,998	**1,055**	11.7%	87	169	799	1,055
11	992	730	945	2,666	992	730	945	2,666	**0**	0.0%	0	0	0	0
12	1,298	756	2,482	4,535	1,298	756	2,482	4,535	**315**	6.9%	90	52	172	315
13	796	1,597	488	2,882	796	1,597	488	2,882	**0**	0.0%	0	0	0	0
14	566	1,648	930	3,144	566	1,648	930	3,144	**0**	0.0%	0	0	0	0
15	647	1,259	674	2,580	647	1,259	674	2,580	**0**	0.0%	0	0	0	0
16	1,242	2,054	4,197	7,494	1,242	2,054	4,197	7,494	**1,055**	14.1%	175	289	591	1,055
17	1,214	864	433	2,510	1,214	864	433	2,510	**0**	0.0%	0	0	0	0
18	916	836	2,134	3,887	916	836	2,134	3,887	**0**	0.0%	0	0	0	0
19	1,545	679	1,379	3,602	1,545	679	1,379	3,602	**0**	0.0%	0	0	0	0
20	382	2,148	1,826	4,356	382	2,148	1,826	4,356	**136**	3.1%	12	67	57	136
Average	941	1,329	1,648	3,919	941	1,329	1,648	3,919	**170**		26	49	96	**170**

* r(x) = MIN[MAX(X-80.0%xPremium,0),20.0%xPremium]

** L(x) = r(x)/[X-0]

	Line A	Line B	Line C	Total
Premium	1,250	1,875	2,150	5,275
ELR	80.0%	80.0%	69.8%	75.8%
Mean	1,000	1,500	1,500	4,000
Std Dev	500	750	1,500	
C.V.	0.500	0.500	1.000	
LN Mu	6.7961835	7.20164861	6.9666468	
LN Sigma	0.47238073	0.47238073	0.83255461	

Aggregate Limit:	20.0%
Attachment Point (AP):	80.0%

Exhibit 2 **Allocation of 3-Year-Block Profit Commission to Year**

	Total Loss by Policy Year (simulation results)				Profit or Loss						Allocated Profit Commission			
Scenario	2002	2003	2004	3 Yr Total	2002	2003	2004	3 Yr Total	Profit Comm*	PC %**	2002	2003	2004	3 Yr Total
Formula:	X_1	X_2	X_3	X	P_1-E_1-X_1	P_2-E_2-X_2	P_3-E_3-X_3	P-E-X	r(x)	L(x)		$[P_i$-E_i-$X_i]$ · L(x)		
1	989	845	1,451	3,285	51	299	-192	157	**55**	35.00%	18	105	-67	55
2	1,013	897	1,610	3,519	27	247	-351	-77	**0**	0.00%	0	0	0	0
3	1,987	956	1,641	4,584	-947	188	-382	-1,142	**0**	0.00%	0	0	0	0
4	554	1,096	427	2,076	486	48	832	1,366	**478**	35.00%	170	17	291	478
5	1,719	790	2,150	4,658	-679	354	-891	-1,216	**0**	0.00%	0	0	0	0
6	1,409	524	646	2,579	-369	620	612	864	**302**	35.00%	-129	217	214	302
7	969	881	2,165	4,014	71	263	-906	-572	**0**	0.00%	0	0	0	0
8	611	546	851	2,008	429	598	407	1,434	**502**	35.00%	150	209	143	502
9	660	756	855	2,271	380	388	403	1,171	**410**	35.00%	133	136	141	410
10	1,362	1,109	895	3,366	-322	35	363	76	**27**	35.00%	-113	12	127	27
11	1,552	1,105	1,210	3,866	-512	39	49	-424	**0**	0.00%	0	0	0	0
12	684	1,166	538	2,388	356	-22	721	1,055	**369**	35.00%	124	-8	252	369
13	1,193	1,111	927	3,232	-153	33	331	211	**74**	35.00%	-54	11	116	74
14	1,473	619	899	2,992	-433	525	359	451	**158**	35.00%	-152	184	126	158
15	613	1,081	1,369	3,064	427	63	-111	378	**132**	35.00%	149	22	-39	132
16	625	1,368	1,647	3,640	415	-224	-388	-197	**0**	0.00%	0	0	0	0
17	813	934	648	2,395	227	210	610	1,047	**367**	35.00%	80	73	214	367
18	555	1,326	2,289	4,171	485	-182	-1,031	-728	**0**	0.00%	0	0	0	0
19	882	2,001	561	3,444	158	-857	698	-2	**0**	0.00%	0	0	0	0
20	717	416	650	1,783	323	728	608	1,659	**581**	35.00%	113	255	213	581
Average	1,019	976	1,171	3,167	21	168	87	276	**173**		25	62	87	**173**

* r(x) = MAX(P-E-X,0)x35.0%
** L(x) = r(x)/[P-E-X]

	2002	2003	2004	3 Yr Total
Premium	1,300	1,430	1,573	4,303
ELR	76.9%	74.1%	71.4%	71.2%
Mean Loss	1,000	1,060	1,124	3,062
Std Dev	500	530	562	
C.V.	0.500	0.500	0.500	
LN Mu	6.7961835	6.85445241	6.91272132	
LN Sigma	0.47238073	0.47238073	0.47238073	

Expense %	20.00%
% Returned	35.00%

Exhibit 3a **Capital Consumption**

	Total Loss by Line of Business (simulation results)				Loss Compared to Expected Loss					Riskiness	Capital Consumed			
Scenario	Line A	Line B	Line C	Total	Line A	Line B	Line C	Total	Capital Consm'd*	Leverage Ratio**	Line A	Line B	Line C	Total
Formula:	X_1	X_2	X_3	X	X_1-μ_1	X_2-μ_2	X_3-μ_3	X-μ	r(x)	L(x)		[X_i-μ_i] · L(x)		
1	682	1,246	1,013	2,941	-318	-254	-487	-1,059	**0**	0.000	0	0	0	0
2	435	1,555	1,641	3,631	-565	55	141	-369	**0**	0.000	0	0	0	0
3	415	2,840	643	3,898	-585	1,340	-857	-102	**0**	0.000	0	0	0	0
4	1,624	813	1,237	3,674	624	-687	-263	-326	**0**	0.000	0	0	0	0
5	443	569	1,125	2,137	-557	-931	-375	-1,863	**0**	0.000	0	0	0	0
6	2,174	1,033	1,760	4,967	1,174	-467	260	967	**967**	1.000	1,174	-467	260	967
7	841	953	1,169	2,963	-159	-547	-331	-1,037	**0**	0.000	0	0	0	0
8	994	1,924	1,191	4,109	-6	424	-309	109	**109**	1.000	-6	424	-309	109
9	705	1,006	2,003	3,714	-295	-494	503	-286	**0**	0.000	0	0	0	0
10	564	1,194	1,374	3,132	-436	-306	-126	-868	**0**	0.000	0	0	0	0
11	524	857	2,007	3,389	-476	-643	507	-611	**0**	0.000	0	0	0	0
12	1,087	1,101	1,952	4,140	87	-399	452	140	**140**	1.000	87	-399	452	140
13	1,310	2,462	3,231	7,002	310	962	1,731	3,002	**2,000**	0.666	206	641	1,153	2,000
14	384	958	854	2,196	-616	-542	-646	-1,804	**0**	0.000	0	0	0	0
15	750	3,653	1,086	5,489	-250	2,153	-414	1,489	**1,489**	1.000	-250	2,153	-414	1,489
16	442	1,883	1,060	3,386	-558	383	-440	-614	**0**	0.000	0	0	0	0
17	1,619	908	754	3,281	619	-592	-746	-719	**0**	0.000	0	0	0	0
18	582	1,982	1,854	4,418	-418	482	354	418	**418**	1.000	-418	482	354	418
19	1,091	1,069	3,851	6,011	91	-431	2,351	2,011	**2,000**	0.995	90	-428	2,338	2,000
20	1,808	2,564	501	4,873	808	1,064	-999	873	**873**	1.000	808	1,064	-999	873
Average	924	1,528	1,515	3,968	-76	28	15	-32	**400**		85	173	142	**400**
Premium	1,250	1,875	2,150	5,275										
ELR	80.0%	80.0%	69.8%	75.8%										
Mean (μ_i)	1,000	1,500	1,500	4,000			Available Surplus:	2,000						
Std Dev	500	750	1,500											
C.V.	0.500	0.500	1.000											
LN Mu	6.7961835	7.20164861	6.9666468											
LN Sigma	0.47238073	0.47238073	0.83255461											

* r(x) = MIN[MAX(X-μ,0), 2,000]
** L(x) = r(x)/[X-μ]

Exhibit 3b **Capital Consumption (Risk Measure = Variance)**

	Total Loss by Line of Business (simulation results)				Loss Compared to Expected Loss				Risk Measure*	Riskiness Leverage Ratio**	Allocation of Risk Measure			
Scenario	Line A	Line B	Line C	Total	Line A	Line B	Line C	Total	(variance)		Line A	Line B	Line C	Total
Formula:	X_1	X_2	X_3	X	X_1-μ_1	X_2-μ_2	X_3-μ_3	X-μ	r(x)	L(x)		[X_i-μ_i] · L(x)		
1	682	1,246	1,013	2,941	-318	-254	-487	-1,059	**112,097**	-105.9	33,626	26,931	51,540	112,097
2	435	1,555	1,641	3,631	-565	55	141	-369	**13,606**	-36.9	20,825	-2,015	-5,204	13,606
3	415	2,840	643	3,898	-585	1,340	-857	-102	**1,036**	-10.2	5,949	-13,637	8,725	1,036
4	1,624	813	1,237	3,674	624	-687	-263	-326	**10,626**	-32.6	-20,345	22,395	8,576	10,626
5	443	569	1,125	2,137	-557	-931	-375	-1,863	**346,977**	-186.3	103,691	173,470	69,816	346,977
6	2,174	1,033	1,760	4,967	1,174	-467	260	967	**93,599**	96.7	113,617	-45,174	25,156	93,599
7	841	953	1,169	2,963	-159	-547	-331	-1,037	**107,592**	-103.7	16,468	56,768	34,356	107,592
8	994	1,924	1,191	4,109	-6	424	-309	109	**1,187**	10.9	-65	4,616	-3,365	1,187
9	705	1,006	2,003	3,714	-295	-494	503	-286	**8,174**	-28.6	8,431	14,131	-14,388	8,174
10	564	1,194	1,374	3,132	-436	-306	-126	-868	**75,398**	-86.8	37,855	26,605	10,938	75,398
11	524	857	2,007	3,389	-476	-643	507	-611	**37,321**	-61.1	29,056	39,254	-30,989	37,321
12	1,087	1,101	1,952	4,140	87	-399	452	140	**1,970**	14.0	1,227	-5,600	6,343	1,970
13	1,310	2,462	3,231	7,002	310	962	1,731	3,002	**901,251**	300.2	92,958	288,719	519,574	901,251
14	384	958	854	2,196	-616	-542	-646	-1,804	**325,314**	-180.4	111,016	97,826	116,471	325,314
15	750	3,653	1,086	5,489	-250	2,153	-414	1,489	**221,579**	148.9	-37,259	320,471	-61,633	221,579
16	442	1,883	1,060	3,386	-558	383	-440	-614	**37,725**	-61.4	34,250	-23,528	27,004	37,725
17	1,619	908	754	3,281	619	-592	-746	-719	**51,683**	-71.9	-44,503	42,525	53,661	51,683
18	582	1,982	1,854	4,418	-418	482	354	418	**17,452**	41.8	-17,468	20,137	14,782	17,452
19	1,091	1,069	3,851	6,011	91	-431	2,351	2,011	**404,318**	201.1	18,225	-86,618	472,710	404,318
20	1,808	2,564	501	4,873	808	1,064	-999	873	**76,262**	87.3	70,579	92,920	-87,237	76,262
Average	924	1,528	1,515	3,968	-76	28	15	-32	**142,258**		28,907	52,510	60,842	**142,258**
											20.32%	36.91%	42.77%	

* r(x) = 10.00%x(X-μ)^2

** L(x) = r(x)/[X-μ]

Note: This is now equivalent to Covariance.

	Line A	Line B	Line C	Total
Premium	1,250	1,875	2,150	5,275
ELR	80.0%	80.0%	69.8%	75.8%
Mean (μ_i)	1,000	1,500	1,500	4,000
Std Dev	500	750	1,500	
C.V.	0.500	0.500	1.000	
LN Mu	6.7961835	7.20164861	6.9666468	
LN Sigma	0.47238073	0.47238073	0.83255461	

Risk as % of Variance: 10.00%

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Supplementary Material

An Excel spreadsheet containing each of the exhibits in this paper is available upon request from the author.

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Abbreviations and notations

RMK, for Ruhm/Mango/Kreps

X_i random variable for losses in Line or Period *i*

$\mu_i = E[X_i]$ the expected value of X_i

$X = \sum X_i$ random variable for the sum of all loss categories in the portfolio

$r(x)$ a function of the total loss in the portfolio, representing the quantity to be allocated

$L(x)$ "Leverage" ratio: a multiplier ensuring the allocation balances to the correct overall amount

Biography of Author

Dave Clark is Vice President & Actuary with American Re-Insurance. His past papers include "LDF Curve-Fitting and Stochastic Reserving: A Maximum Likelihood Approach", which won the 2003 prize for the Reserves Call Paper; and "Insurance Applications of Bivariate Distributions", co-authored with David Homer, which won the 2004 Dorweiler Prize.

Bridging Minimum Bias and Maximum Likelihood Methods through Weighted Equation

Noriszura Ismail and Abdul Aziz Jemain

Abstract
In classification ratemaking, the multiplicative and additive models derived by actuaries are based on two common methods; minimum bias and maximum likelihood. These models are already considered as established and standard, particularly in automobile and general liability insurance. This paper aims to identify the relationship between both methods by rewriting the equations of both minimum bias and maximum likelihood as a weighted equation. The weighted equation is in the form of a weighted difference between observed and fitted rates. The advantage of having the weighted equation is that the solution can be solved using regression model. Compared to the classical method introduced by Bailey and Simon (1960), the regression model provides an improved and simplified programming algorithm. In addition, the parameter estimates could also be rewritten as a weighted solution; for multiplicative model the solution can be written in the form of a weighted proportion whereas for additive model, the form is of a weighted difference. In this paper, the weighted equation will be applied on three types of classification ratemaking data; ship damage incidents data of McCullagh and Nelder (1989), data from Bailey and Simon (1960) on Canadian private automobile liability insurance and UK private car motor insurance data from Coutts (1984).

Keywords: Classification ratemaking; Minimum bias; Maximum likelihood; Multiplicative; Additive.

1. INTRODUCTION

In order to determine pure premium rates in casualty insurance, actuaries have to fulfil two requirements. First, they have to ensure that the insurer will receive premiums at a level adequate to cover losses and expenses. Next, they have to allocate premiums "fairly" between insureds, i.e., high risk insured should pay higher premium. For the first requirement, actuaries are required to adjust the overall level of premiums, taking into account short-term economic effects such as inflation, and other external factors such as government regulation, that can be dealt with minimum statistical analysis. However, for the second requirement, the relative premium levels need to be determined. At this stage, statistical modelling and actuarial judgement are important and actuaries can achieve this by using classification ratemaking.

The goal of classification ratemaking is to group homogeneous risks and charge each group a premium to commensurate with the expected average loss. Failure to achieve this goal may lead to adverse selection to insureds and economic losses to insurers. The risks may be categorized according to rating factors; for instance in auto insurance, driver's gender, claim experience, and location, or vehicle's make, capacity and year, can be considered as rating factors.

Among the pioneer studies of classification ratemaking, Bailey and Simon (1960) compared the systematic bias of multiplicative and additive models. Following their work, a few studies focusing and debating on additive and multiplicative models were published. Bailey (1963) compared multiplicative and additive models by producing two statistical criteria, namely the minimum chi-squares and the average absolute difference. Freifelder (1986) predicted the pattern of over and under estimation of multiplicative and additive models if true models are misspecified. Jee (1989) compared the predictive accuracy of multiplicative and additive models, and Holler *et al.* (1999) compared the initial values sensitivity of multiplicative and additive models.

In addition, researchers of classification ratemaking also suggested various statistical procedures to estimate the model parameters. Bailey and Simon (1960) suggested the minimum chi-squares, Bailey (1963) used the zero bias, Jung (1968) produced a heuristic method for minimum modified chi-squares, Ajne (1975) suggested the method of moments, Chamberlain (1980) used the weighted least squares, Coutts (1984) produced the method of orthogonal weighted least squares with logit transformation, Harrington (1986) suggested the maximum likelihood method for models with functional form, and Brockmann and Wright (1992) used the generalized linear models with Poisson error structure for claim frequency and Gamma error structure for claim severity. With the development of computing packages in the recent years, various statistical packages were also suggested and used, including GLIM by Brown (1988) and SAS by Holler *et al.* (1999) and Mildenhall (1999).

Based on the literature review, most researchers studied classification ratemaking in terms of two main perspectives; the models of multiplicative vs. additive, and the methods of minimum bias vs. maximum likelihood; using a variety of criteria, namely biasness, interaction terms, goodness of fit, initial value sensitivity and prediction accuracy. This paper differs such that it tries to bridge both methods via a weighted equation. This author believes that the weighted equation makes understanding the similarities and differences between both methods an easier task.

The objective of this paper is to bridge minimum bias and maximum likelihood methods by rewriting their equations as a weighted equation. The weighted equation can be written in the form of a weighted difference between observed and fitted rates. The advantage of having the weighted equation is that the solution can be solved using regression model. Compared to the classical method introduced by Bailey and Simon (1960), the regression model provides an improved and simplified programming algorithm. In addition, the parameter estimates could also be rewritten as a weighted solution; for multiplicative model it is in the form of a weighted proportion whereas for additive model, the form is of a weighted difference. In this paper, the weighted equation will be applied on three types of classification ratemaking data; ship damage incidents data of McCullagh and Nelder (1989), data from Bailey and Simon (1960) on Canadian private automobile liability insurance and UK private car motor insurance data from Coutts (1984).

Rewriting the equations of minimum bias and maximum likelihood as a weighted equation has its own advantages; the mathematical concept of the weighted equation is simpler, hence providing an easier understanding, particularly for insurance practitioners; the weighted equation allows the usage of regression model as an alternative programming algorithm to calculate the parameter estimates; the weighted equation provides a basic step to further understand the more complex distributions, primarily the distributions involving dispersion parameter; the weights of the parameter solution shows that each of multiplicative and additive models has similar solution; and finally, the weights of the parameter solution also shows that models with larger sample size and number of parameter have slower convergence.

2. CLASSIFICATION RATEMAKING

In casualty insurance, the risk premium, i.e., the premium excluding expenses, is equal to the product of claim frequency and severity. Classification ratemaking is the statistical procedure that classifies risks in claim frequency and severity models into groups of homogeneous risks, categorized by the rating factors. In this study, classification ratemaking is used to estimate claim frequency rates, expressed in terms of frequency per unit of exposure. For instance, the exposure unit used for auto insurance is based on a car-year unit.

Consider a regression model with n observations of claim frequency rates and p explanatory variables inclusive of intercept and dummy variables. Next, consider a data of frequency rates involving three rating factors, each respectively with three, two and three rating classes. Thus, this data has a total of $n = 18$ observed rates with $p = 6$ explanatory variables. In addition, let $\mathbf{c}$, $\mathbf{e}$ and $\mathbf{r}$ denote the vectors for claim counts, exposures and observed rates, respectively. Therefore, the observed rate for the *i*th rating class, $i = 1,2,\ldots,18$, is equivalent to $r_i = {c_i}/{e_i}$.

Furthermore, let $\mathbf{X}$ be the matrix of explanatory variables with the *i*th row equivalent to vector $\mathbf{x}_\mathbf{i}^\mathbf{T}$, and $\boldsymbol{\beta}$ be the vector of regression parameters. If x_{ij}, $i = 1,2,\ldots,18$, $j = 1,2,\ldots,6$, is the *ij*th element of matrix $\mathbf{X}$, the value for x_{ij} is either one or zero. Table 1 summarize the regression model for the data.

Table 1. Data summary

i	c_i	e_i	$r_i = {c_i}/{e_i}$	x_{i1}	x_{i2}	x_{i3}	x_{i4}	x_{i5}	x_{i6}
1	c_1	e_1	r_1	1	0	0	0	0	0
2	c_2	e_2	r_2	1	0	0	0	1	0
3	⋮	⋮	⋮	1	0	0	0	0	1
4				1	0	0	1	0	0
5				1	0	0	1	1	0
6				1	0	0	1	0	1
7				1	1	0	0	0	0
8				1	1	0	0	1	0
9				1	1	0	0	0	1
10				1	1	0	1	0	0
11				1	1	0	1	1	0
12				1	1	0	1	0	1
13				1	0	1	0	0	0
14				1	0	1	0	1	0
15				1	0	1	0	0	1
16				1	0	1	1	0	0
17				1	0	1	1	1	0
18	c_{18}	e_{18}	r_{18}	1	0	1	1	0	1

Moreover, let $\mathbf{f}$, a function of $\mathbf{X}$ and $\boldsymbol{\beta}$, denotes the vector for fitted rates. For a multiplicative model, the *i*th fitted rate is equivalent to

$$f_i = \exp(\mathbf{x}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta}),$$

which can also be written as

$$f_i = f_{i(-j)} \exp(\beta_j x_{ij}), \tag{1}$$

where $f_{i(-j)}$ is the *i*th multiplicative fitted rate without the *j*th effect. As for an additive model, the *i*th fitted rate is equal to

$$f_i = \mathbf{x}_\mathbf{i}^\mathbf{T}\boldsymbol{\beta},$$

which can also be written as

$$f_i = f_{i(-j)} + \beta_j x_{ij}, \tag{2}$$

where $f_{i(-j)}$ is the *i*th additive fitted rate without the *j*th effect. Thus, the objective of classification ratemaking is to have the fitted rates, f_i, be as close as possible to the observed rates, r_i, for all *i*.

3. MINIMUM BIAS

Bailey and Simon (1960) were among the pioneer researchers that consider bias in classification ratemaking and introduced the minimum bias method. They proposed a famous list of four criteria for an acceptable set of classification rates:

i. It should reproduce experience for each class and overall, i.e., be balanced for each class and overall.

ii. It should reflect the relative credibility of the various classes.

iii. It should provide minimum amount of departure from the raw data.

iv. It should produce a rate for each class of risks which is close enough to the experience so that the differences could reasonably be caused by chance.

3.1 Bailey Zero Bias

Bailey and Simon (1960) proposed a suitable test for Criterion (i) by calculating,

$$\frac{\sum_i e_i f_i}{\sum_i e_i r_i}, \tag{3}$$

for each *j* and total. A set of rates is balanced, i.e., zero bias, if equation (3) equals 1.00. Automatically, zero bias for each class implies zero bias overall.

From this test, Bailey (1963) derived a minimum bias model by setting the average difference between observed and fitted rates to be equal to zero. The zero bias equation for each *j* can be written in the form of a weighted difference between observed and fitted rates,

$$\sum_i w_i (r_i - f_i) = 0, \tag{4}$$

where w_i is equal to $e_i x_{ij}$.

Substituting (1) into (4), the zero bias equation for multiplicative model become

$$\sum_i e_i r_i x_{ij} = \sum_i e_i f_{i(-j)} \exp(\beta_j x_{ij}) x_{ij} .$$

Since x_{ij} is either one or zero, the solution for each *j* could be obtained and written in the form of a weighted proportion of observed over multiplicative fitted rates without the *j*th effect,

$$\exp(\beta_j) = \sum_i v_i \frac{r_i}{f_{i(-j)}} , \tag{5}$$

where v_i is the normalized weight of $\frac{z_i}{\sum_i z_i}$ and z_i is $e_i f_{i(-j)} x_{ij}$.

For additive model, the zero bias equation after substituting (2) into (4) is

$$\sum_i e_i (r_i - f_{i(-j)}) x_{ij} = \sum_i e_i (\beta_j x_{ij}) x_{ij} .$$

Again, since x_{ij} is either one or zero, the solution for each *j* could be obtained. However, for additive model, it is in term of a weighted difference between observed and additive fitted rates without the *j*th effect,

$$\beta_j = \sum_i v_i (r_i - f_{i(-j)}) , \tag{6}$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $e_i x_{ij}$.

3.2 Minimum Chi-Squares

Bailey and Simon (1960) also suggested the χ^2 statistics as an appropriate test for Criterion (iv),

$$\chi^2 = \sum_i \frac{e_i}{f_i} (r_i - f_i)^2 .$$

The same test is also suitable for Criterion (ii) and (iii) as well.

By minimizing the χ^2 statistics, another minimum bias model was derived. For each *j*, the minimum χ^2 equation could be written in the form of a weighted difference between observed and fitted rates,

$$\frac{\partial \chi^2}{\partial \beta_j} = \sum_i w_i (r_i - f_i) = 0, \tag{7}$$

where w_i is $\dfrac{e_i(r_i + f_i)}{f_i^2} \dfrac{\partial f_i}{\partial \beta_j}$.

For multiplicative model,

$$\frac{\partial f_i}{\partial \beta_j} = f_i x_{ij}, \tag{8}$$

whereas in additive model,

$$\frac{\partial f_i}{\partial \beta_j} = x_{ij}. \tag{9}$$

If multiplicative model is chosen, by substituting (1) and (8) into (7), the parameter solution is equivalent to

$$\exp(\beta_j) = \sum_i v_i \frac{r_i}{f_{i(-j)}}, \tag{10}$$

where v_i is $\dfrac{z_i}{\sum_i z_i}$ and z_i is $e_i(r_i + f_i) x_{ij}$.

For additive model, the parameter solution after substituting (2) and (9) into (7) is

$$\beta_j = \sum_i v_i (r_i - f_{i(-j)}), \tag{11}$$

where v_i is $\dfrac{z_i}{\sum_i z_i}$ and z_i is $\dfrac{e_i(r_i + f_i)}{f_i^2} x_{ij}$.

4. MAXIMUM LIKELIHOOD

Assume that the *i*th claim frequency count, $c_i = e_i r_i$, comes from a distribution whose probability density function is $g(c_i; f_i)$. A maximum likelihood method maximizes the likelihood function,

$$L = \prod_i g(c_i; f_i),$$

or equivalently, the log likelihood function,

$$\ell = \log L = \sum_i \log\left(g(c_i; f_i)\right).$$

Thus, the parameter solution can be obtained by setting $\frac{\partial \ell}{\partial \beta_j} = 0$ for each *j*.

4.1 Normal Distribution

If $c_i = e_i r_i$ is assumed to follow Normal distribution with mean $e_i f_i$, $g(c_i; f_i)$ can be written as

$$g(c_i; f_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}\left(e_i r_i - e_i f_i\right)^2\right\}.$$

Hence, the likelihood equation for each *j* is equivalent to

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (r_i - f_i) = 0, \tag{12}$$

where w_i is $e_i^{\,2} \frac{\partial f_i}{\partial \beta_j}$.

Assuming multiplicative model, the solution after substituting (1) and (8) into (12) is

$$\exp(\beta_j) = \sum_i v_i \left(\frac{r_i}{f_{i(-j)}}\right), \tag{13}$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $e_i^2 f_{i(-j)}^2 x_{ij}$.

For additive model, by substituting (2) and (9) into (12), the parameter solution is equivalent to

$$\beta_j = \sum_i v_i (r_i - f_{i(-j)}), \tag{14}$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $e_i^2 x_{ij}$.

4.2 Poisson Distribution

The same weighted equation could also be used to show that Poisson multiplicative is actually equivalent to zero bias multiplicative, derived by Bailey (1963). If $c_i = e_i r_i$ is assumed to have Poisson distribution with mean $e_i f_i$, the probability density function is

$$g(c_i; f_i) = \frac{\exp(-e_i f_i)(e_i f_i)^{e_i r_i}}{(e_i r_i)!}.$$

As a result, for each *j*, the likelihood equation is equal to

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (r_i - f_i) = 0, \tag{15}$$

where w_i is $\frac{e_i}{f_i} \frac{\partial f_i}{\partial \beta_j}$.

Substituting (1) and (8) into (15) for multiplicative model, the parameter solution can be written as

$$\exp(\beta_j) = \sum_i v_i \frac{r_i}{f_{i(-j)}},$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $e_i f_{i(-j)} x_{ij}$. This solution is equivalent to zero bias multiplicative shown by (5).

If additive model is chosen, by substituting (2) and (9) into (15), the parameter solution is equal to

$$\beta_j = \sum_i v_i (r_i - f_{i(-j)}), \tag{16}$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $\frac{e_i}{f_i} x_{ij}$.

4.3 Binomial Distribution

Assuming $c_i = e_i r_i$ comes from Binomial distribution with mean $e_i f_i$, $g(c_i; f_i)$ can be written as

$$g(c_i; f_i) = \binom{e_i}{c_i} f_i^{c_i} (1 - f_i)^{e_i - c_i} .$$

For each *j*, the likelihood equation is equivalent to

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (r_i - f_i) = 0 , \tag{17}$$

where w_i is $\dfrac{e_i}{f_i(1-f_i)} \dfrac{\partial f_i}{\partial \beta_j}$.

Using multiplicative model, the solution after substituting (1) and (8) into (17) is

$$\exp(\beta_j) = \sum_i v_i \left(\frac{r_i}{f_{i(-j)}} \right), \tag{18}$$

where v_i is $\dfrac{z_i}{\sum_i z_i}$ and z_i is $\dfrac{e_i f_{i(-j)}}{1 - f_i} x_{ij}$.

If additive model is chosen, by substituting (2) and (9) into (17), the solution can be written as

$$\beta_j = \sum_i v_i (r_i - f_{i(-j)}) , \tag{19}$$

where v_i is $\dfrac{z_i}{\sum_i z_i}$ and z_i is $\dfrac{e_i}{f_i(1-f_i)} x_{ij}$.

4.4 Negative Binomial Distribution

The advantage of using the weighted equation is that it can be used as an introductory step to understand the fitting procedure of a distribution with dispersion parameter. If the dependent variable, C_i, is a count with mean $E(C_i) = \mu_i$, a standard statistical procedure is to fit the data with Poisson distribution using multiplicative model. However, if overdispersion is detected in the data, i.e., $Var(C_i) > E(C_i)$, the parameter estimates for standard Poisson are still consistent, but inefficient. As an alternative, the standard overdispersion model is the Negative Binomial distribution with multiplicative model. If C_i

is distributed as Negative Binomial$(\mu_i;a)$, the probability density function is (Lawless, 1987),

$$g(c_i;\mu_i,a)=\frac{\Gamma(c_i+\frac{1}{a})}{c_i!\Gamma(\frac{1}{a})}\left(\frac{a\mu_i}{1+a\mu_i}\right)^{c_i}\left(\frac{1}{1+a\mu_i}\right)^{\frac{1}{a}},$$

and the mean and variance are

$$E(C_i)=\mu_i,$$

$$Var(C_i)=\mu_i(1+a\mu_i),$$

where a is the dispersion parameter. Since $a\geq 0$ and $\mu_i\geq 0$ for all *i*, the distribution allows for overdispersion.

For our classification ratemaking example, $c_i=e_i r_i$ and $\mu_i=e_i f_i$. Thus, the likelihood equation can also be written as

$$\frac{\partial\ell}{\partial\beta_j}=\sum_i w_i(r_i-f_i)=0, \qquad (20)$$

where w_i is $\frac{e_i}{f_i(1+ae_i f_i)}\frac{\partial f_i}{\partial\beta_j}$. Notice that the weight for Poisson (15) is a special case of the weight for Negative Binomial (20), when the dispersion parameter, a, is equal to zero.

Assuming multiplicative model, by substituting (1) and (8) into (20), the parameter solution is

$$\exp(\beta_j)=\sum_i v_i\left(\frac{r_i}{f_{i(-j)}}\right), \qquad (21)$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $\frac{e_i f_{i(-j)}}{1+ae_i f_i}x_{ij}$.

For additive model, the parameter solution after substituting (2) and (9) into (20) is equal to

$$\beta_j=\sum_i v_i(r_i-f_{i(-j)}), \qquad (22)$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $\frac{e_i}{f_i(1+ae_i f_i)}x_{ij}$.

4.5 Generalized Poisson Distribution

Another alternative for overdispersion is to use the Generalized Poisson distribution. The advantage of using Generalized Poisson distribution is that it can be used for both overdispersion, i.e., $Var(C_i) > E(C_i)$, as well as underdispersion, i.e., $Var(C_i) < E(C_i)$. If C_i is assumed to follow Generalized Poisson distribution, $g(c_i;\mu_i,a)$ can be written as (Wang and Famoye, 1997),

$$g(c_i;\mu_i,a) = \left(\frac{\mu_i}{1+a\mu_i}\right)^{c_i} \frac{(1+ac_i)^{c_i-1}}{c_i\,!} \exp\left(-\frac{\mu_i(1+ac_i)}{1+a\mu_i}\right),$$

with mean and variance,

$$E(C_i) = \mu_i\,,$$
$$Var(C_i) = \mu_i(1+a\mu_i)^2\,.$$

Since $a \geq 0$ or $a \leq 0$, the distribution allows for either overdispersion or underdispersion. Assuming $c_i = e_i r_i$ and $\mu_i = e_i f_i$, the likelihood equation is

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i w_i (r_i - f_i) = 0\,, \tag{23}$$

where w_i is $\dfrac{e_i}{f_i(1+ae_i f_i)^2}\dfrac{\partial f_i}{\partial \beta_j}$. Again, the weight for Poisson (15) is a special case of the weight for Generalized Poisson (23), when the dispersion parameter, a, is equal to zero.

Substituting (1) and (8) into (23) for multiplicative model, the parameter solution is

$$\exp(\beta_j) = \sum_i v_i \left(\frac{r_i}{f_{i(-j)}}\right), \tag{24}$$

where v_i is $\dfrac{z_i}{\sum_i z_i}$ and z_i is $\dfrac{e_i f_{i(-j)}}{(1+ae_i f_i)^2} x_{ij}$.

For additive model, by substituting (2) and (9) into (23), the parameter solution can be written as

$$\beta_j = \sum_i v_i (r_i - f_{i(-j)})\,, \tag{25}$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $\frac{e_i}{f_i(1+ae_if_i)^2}x_{ij}$.

5. OTHER MODELS

5.1 Least Squares

The same weighted equation could also be extended to other error functions as well. Define the sum squares error as (Brown, 1988),

$$S=\sum_i \frac{(e_ir_i-e_if_i)^2}{e_i}=\sum_i e_i(r_i-f_i)^2 .$$

So, the least squares equation can be written as

$$\frac{\partial S}{\partial \beta_j}=\sum_i w_i(r_i-f_i)=0\,, \tag{26}$$

where w_i is $e_i\frac{\partial f_i}{\partial \beta_j}$.

Substituting (1) and (8) into (26) for multiplicative model, the parameter solution is

$$\exp(\beta_j)=\sum_i v_i\frac{r_i}{f_{i(-j)}}\,, \tag{27}$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $e_if_{i(-j)}^2x_{ij}$.

Extending this equation to least squares with additive model, it can be shown that least squares additive is equivalent to zero bias additive, derived by Bailey (1963). The parameter solution after substituting (2) and (9) into (26) is equivalent to

$$\beta_j=\sum_i v_i(r_i-f_{(-j)})\,,$$

where $v_i=\frac{z_i}{\sum_i z_i}$ and $z_i=e_ix_{ij}$. This solution is equivalent to the zero bias additive shown by (6).

5.2 Minimum Modified Chi-Squares

If the function of error is a modified χ^2 statistics which is defined as,

$$\chi^2_{\text{mod}} = \sum_i \frac{e_i}{r_i}(r_i - f_i)^2 ,$$

the equation for minimum modified χ^2 is equivalent to

$$\frac{\partial \chi^2_{\text{mod}}}{\partial \beta_j} = \sum_i w_i (r_i - f_i) = 0 , \tag{28}$$

where w_i is $\frac{e_i}{r_i} \frac{\partial f_i}{\partial \beta_j}$.

For multiplicative model, by substituting (1) and (8) into (28), the parameter solution can be written as

$$\exp(\beta_j) = \sum_i v_i \frac{r_i}{f_{i(-j)}} , \tag{29}$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $z_i = \frac{e_i f^2_{i(-j)}}{r_i} x_{ij}$.

Substituting (2) and (9) into (28) for additive model, the parameter solution is

$$\beta_j = \sum_i v_i (r_i - f_{(-j)}) , \tag{30}$$

where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $\frac{e_i}{r_i} x_{ij}$.

Table 2 summarizes the weighted equations and parameter solutions for all of the models discussed above. From the table, the following conclusions can be made:

i. For additive models, the zero bias and least squares are equivalent.

ii. For multiplicative models, the zero bias and Poisson are equal.

iii. The weighted equation, which is in the form of a weighted difference between observed and fitted rates, show that all models are similar; each model is distinguished only by its weight.

iv. The weights in the parameter solutions show that each of multiplicative and additive models is expected to produce similar parameter estimates.

6. MODEL PROGRAMMING

6.1 Classical Method

The classical iterative method for finding parameter solutions was first introduced by Bailey and Simon (1960). This method solves the parameter individually for each *j*. In the first iteration, vector of initial values, $\boldsymbol{\beta}^{(0)}$, are needed to calculate the vector of next parameter estimates, $\boldsymbol{\beta}^{(1)}$. The process of iteration is then repeated until all solutions converge.

Table 2. Summary of weighted equations and parameter solutions

Models	w_i for weighted equation, $\sum_i w_i(r_i - f_i) = 0$	z_i for multiplicative parameter solutions, $\exp(\beta_j) = \sum_i v_i \frac{r_i}{f_{i(-j)}}$, $v_i = \frac{z_i}{\sum_i z_i}$	z_i for additive parameter solutions, $\beta_j = \sum_i v_i(r_i - f_{i(-j)})$, $v_i = \frac{z_i}{\sum_i z_i}$
Zero bias	$w_i = e_i x_{ij}$	$z_i = e_i f_{i(-j)} x_{ij}$	$z_i = e_i x_{ij}$
Poisson	$w_i = \frac{e_i}{f_i}\frac{\partial f_i}{\partial \beta_j}$	$z_i = e_i f_{i(-j)} x_{ij}$	$z_i = \frac{e_i}{f_i} x_{ij}$
Least Squares	$w_i = e_i \frac{\partial f_i}{\partial \beta_j}$	$z_i = e_i f_{i(-j)}^{\;2} x_{ij}$	$z_i = e_i x_{ij}$
Minimum χ^2	$w_i = \frac{e_i(r_i + f_i)}{f_i^2}\frac{\partial f_i}{\partial \beta_j}$	$z_i = e_i(r_i + f_i) x_{ij}$	$z_i = \frac{e_i(r_i + f_i)}{f_i^2} x_{ij}$
Normal	$w_i = e_i^{\;2}\frac{\partial f_i}{\partial \beta_j}$	$z_i = e_i^{\;2} f_{i(-j)}^{\;2} x_{ij}$	$z_i = e_i^{\;2} x_{ij}$
Binomial	$w_i = \frac{e_i}{f_i(1-f_i)}\frac{\partial f_i}{\partial \beta_j}$	$z_i = \frac{e_i f_{i(-j)}}{1-f_i} x_{ij}$	$z_i = \frac{e_i}{f_i(1-f_i)} x_{ij}$
Negative Binomial	$w_i = \frac{e_i}{f_i(1+ae_i f_i)}\frac{\partial f_i}{\partial \beta_j}$	$z_i = \frac{e_i f_{i(-j)}}{1+ae_i f_i} x_{ij}$	$z_i = \frac{e_i}{f_i(1+ae_i f_i)} x_{ij}$
Generalized Poisson	$w_i = \frac{e_i}{f_i(1+ae_i f_i)^2}\frac{\partial f_i}{\partial \beta_j}$	$z_i = \frac{e_i f_{i(-j)}}{(1+ae_i f_i)^2} x_{ij}$	$z_i = \frac{e_i}{f_i(1+ae_i f_i)^2} x_{ij}$
Minimum modified χ^2	$w_i = \frac{e_i}{r_i}\frac{\partial f_i}{\partial \beta_j}$	$z_i = \frac{e_i f_{i(-j)}^{\;2}}{r_i} x_{ij}$	$z_i = \frac{e_i}{r_i} x_{ij}$

An example for the parameter solution of zero bias multiplicative, $\exp(\beta_j) = \sum_i v_i \frac{r_i}{f_{i(-j)}}$, where v_i is $\frac{z_i}{\sum_i z_i}$ and z_i is $e_i f_{i(-j)} x_{ij}$, is discussed here.

Let $f_{i(-j)}$ denotes the the *i*th row of $\mathbf{f_{(-j)}}$, the vector of multiplicative fitted rates without the *j*th effect. For multiplicative model, $\mathbf{f_{(-j)}} = \exp(\mathbf{X_{(-j)}} \boldsymbol{\beta}_{\mathbf{(-j)}})$, where $\mathbf{X_{(-j)}}$ denotes the matrix of explanatory variables without the *j*th column and $\boldsymbol{\beta}_{\mathbf{(-j)}}$ the vector of regression parameters without the *j*th row.

Moreover, let $\mathbf{x_j}$ denotes the vector equivalent to the *j*th column of matrix **X**. Thus, x_{ij} is equal to the *i*th row of vector $\mathbf{x_j}$. Further, let z_i, the *i*th row of vector $\mathbf{z}$, equal to the product of e_i, $f_{i(-j)}$ and x_{ij}. Therefore, v_i, the *i*th row of vector $\mathbf{v}$, is equivalent to the proportion of z_i over sum of z_i for all *i*.

For multiplicative models, the same programming can be used if z_i is written as

$$z_i = e_i^b r_i^d f_{i(-j)}{}^g (1 - f_i)^h (r_i + f_i)^k (1 + a e_i f_i)^l \; x_{ij} .$$

For example, in zero bias multiplicative, $a = 0, b = 1, d = 0, g = 1, h = 0, k = 0$, and $l = 0$. Similarly, z_i for additive model is of the form,

$$z_i = e_i^b r_i^d f_i^g (1 - f_i)^h (r_i + f_i)^k (1 + a e_i f_i)^l \; x_{ij} .$$

For instance, in zero bias additive, $a = 0, b = 1, d = 0, g = 0, h = 0, k = 0$, and $l = 0$.

Examples of S-PLUS programming for both multiplicative and additive models are shown in Appendix A. The same programming can be used since z_i can be written in a functional form of a, b, d, g, h, k and l. Note that for minimum modified χ^2, both multiplicative and additive models contain the observed rate, r_i, as the denominator in z_i. Thus, to avoid a "division by zero", it is suggested that a small constant is added to r_i in the programming.

6.2 Regression Model

In regression model, the estimates for $\beta_j, j = 1, 2, ..., p$, can be found by minimizing,

$$\sum_i w_i (r_i - f_i(\boldsymbol{\beta}))^2 ,$$

or equivalently, they are the solution of,

$$\sum_i w_i (r_i - f_i(\boldsymbol{\beta})) \frac{\partial f_i(\boldsymbol{\beta})}{\partial \beta_j} = 0 ,$$

for each *j*. This equation is similar to the weighted equation derived for classification rates discussed previously. Hence, the parameter solutions for classification ratemaking are allowed to be solved using a regression model.

By using Taylor series approximation, it can be shown that (Venables and Ripley, 1997),

$$\boldsymbol{\beta}^{(1)} = (\mathbf{Z}^{(0)T}\mathbf{W}^{(0)}\mathbf{Z}^{(0)})^{-1}\mathbf{Z}^{(0)T}\mathbf{W}^{(0)}(\mathbf{r} - \mathbf{s}^{(0)}) ,$$

where,

$\mathbf{Z}^{(0)} = (n \times p)$ matrix whose *ij*th element is equal to $\left.\dfrac{\partial f_i(\boldsymbol{\beta})}{\partial \beta_j}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}^{(0)}}$

$\mathbf{W}^{(0)} = (n \times n)$ diagonal matrix of weight, evaluated at $\boldsymbol{\beta} = \boldsymbol{\beta}^{(0)}$

$\mathbf{s}^{(0)}$ = vector where the *i*th row is equal to $f_i(\boldsymbol{\beta}^{(0)}) - \sum_{j=1}^{p} \beta_j^{(0)} z_{ij}^{(0)}$

In the first iteration, the vector of initial values, $\boldsymbol{\beta}^{(0)}$, are needed to calculate $\boldsymbol{\beta}^{(1)}$. The process of iteration is then repeated until the solution converges. Since the parameter estimates are represented by vector $\boldsymbol{\beta}$, the regression model solves them simultaneously, thus providing a faster convergence compared to the classical approach.

Consider an additive model where the *ij*th element of matrix $\mathbf{Z}^{(0)}$ is equal to $\left.\dfrac{\partial f_i(\boldsymbol{\beta})}{\partial \beta_j}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}^{(0)}} = x_{ij}$, which is free of $\boldsymbol{\beta}^{(0)}$. Since x_{ij} is the *ij*th element of matrix $\mathbf{X}$ and both matrices have the same dimension, $\mathbf{Z}^{(0)} = \mathbf{X}$ and $\mathbf{s}^{(0)} = \mathbf{f}(\boldsymbol{\beta}^{(0)}) - \mathbf{X}\boldsymbol{\beta}^{(0)} = \mathbf{0}$.

For example, the weighted equation for least squares (26) is equivalent to

$$\sum_i e_i (r_i - f_i) \frac{\partial f_i}{\partial \beta_j} = 0 .$$

Here, the *i*th diagonal element of matrix $\mathbf{W}^{(0)}$ is e_i, which is also free of $\boldsymbol{\beta}^{(0)}$. Therefore, for additive model, the vector of parameter estimates for least squares is

$$\boldsymbol{\beta}^{(0)} = \boldsymbol{\beta} = (\mathbf{X}^{T}\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^{T}\mathbf{W}\mathbf{r} ,$$

which is equivalent to the normal equation in linear regression model, thus allowing the solution to be solved without any iteration.

However, if multiplicative model is assumed, the *ij*th element of matrix $\mathbf{Z}^{(0)}$ is $\left.\frac{\partial f_i(\boldsymbol{\beta})}{\partial \beta_j}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}^{(0)}} = f_i(\boldsymbol{\beta}^{(0)})x_{ij}$, or equivalently $\mathbf{Z}^{(0)} = \mathbf{F}^{(0)}\mathbf{X}$, where $\mathbf{F}^{(0)}$ is the diagonal matrix whose *i*th diagonal element is $f_i(\boldsymbol{\beta}^{(0)})$. For this reason, vector $\mathbf{s}_{(0)}$ can also be written as

$$\mathbf{s}^{(0)} = \mathbf{f}(\boldsymbol{\beta}^{(0)}) - \mathbf{F}^{(0)}\mathbf{X}\boldsymbol{\beta}^{(0)}.$$

For all models discussed previously, the same programming can be used if the *i*th diagonal element of weight matrix is written as,

$$e_i^b r_i^d f_i^g (1-f_i)^h (r_i + f_i)^k (1+ae_i f_i)^l .$$

The simplest form is the weight for least squares whereby $a = 0, b = 1, d = 0, g = 0, h = 0, k = 0$ and $l = 0$.

Examples of S-PLUS programming for both multiplicative and additive models are shown in Appendix B. The same programming can be used since the weight can be written in a functional form of a, b, d, g, h, k and l. Note that for minimum modified χ^2, the weight, w_i, contain the observed rate, r_i, as the denominator. Thus, to avoid a "division by zero", it is suggested that a small constant is added to r_i in the programming.

7. EXAMPLES

Consider three types of classification ratemaking data; ship damage incidents data of McCullagh and Nelder (1989), data from Bailey and Simon (1960) on Canadian private automobile liability insurance, and UK private car motor insurance data from Coutts (1984). These data are also available and can be accessed from the Internet in the following websites; http://sunsite.univie.ac.at/statlib/datasets/ships for McCullagh and Nelder (1989) data, http://www.casact.org/library/astin/vol1no4/192.pdf for the data of Bailey and Simon (1960), and http://www.actuaries.org.uk/files/pdf/library/JIA-111/0087-0148.pdf for Coutts (1984) data.

For ship damage incidents data, the number of damage incidents and exposure for each class are available. The risk of damage was associated with three rating factors; ship type, year of construction and period of operation, involving a total of 40 classes, including 6 classes with zero exposure. For Canadian private automobile liability insurance data, the number of claims incurred and exposure for each class are available. Two rating factors are considered; class and merit ratings, involving a total of 20 classes. Finally, for UK private car motor insurance data, the incurred claim count and exposure for each class are available. Four rating factors are considered; coverage, vehicle age, vehicle group and policyholder age, involving a total of 120 classes.

Bailey and Simon (1960) also suggested the average absolute difference as a suitable test for Criterion (iii),

$$\frac{\sum_i e_i |r_i - f_i|}{\sum_i e_i r_i}.$$

Therefore, the χ^2 statistics, a test for Criterion (iv), and the average absolute difference, a test for Criterion (iii), will be calculated for all models. Table 3, Table 4 and Table 5 show the parameter estimates, χ^2 statistics and average absolute difference for the models discussed above.

Table 3. Parameters, χ^2 and absolute difference for ship data

Parameters & bias measures		Multiplicative models					
		Zero bias / Poisson	Least squares	Minimum χ^2	Normal	Binomial	Minimum modified χ^2
Intercept	$\exp(\beta_1)$	0.002	0.002	0.002	0.002	0.002	0.002
Ship type *B*	$\exp(\beta_2)$	0.581	0.563	0.568	0.588	0.581	0.593
Ship type *C*	$\exp(\beta_3)$	0.503	0.436	0.781	0.317	0.503	0.231
Ship type *D*	$\exp(\beta_4)$	0.927	1.087	1.113	0.926	0.927	0.652
Ship type *E*	$\exp(\beta_5)$	1.385	1.384	1.575	1.123	1.385	1.113
Const. yr 65-69	$\exp(\beta_6)$	2.008	2.071	2.040	2.038	2.008	1.938
Const. yr 70-74	$\exp(\beta_7)$	2.267	2.157	2.242	2.395	2.267	2.242
Const. yr 75-79	$\exp(\beta_8)$	1.574	1.368	1.584	1.767	1.573	1.576
Operation yr 75-79	$\exp(\beta_9)$	1.469	1.437	1.443	1.447	1.469	1.544
χ^2		42.275	45.211	36.393	59.567	42.277	85.18
absolute difference		0.187	0.194	0.209	0.165	0.187	0.169

Parameters (10^{-3}) & bias measures		Additive models					
		Zero bias/ Least squares	Poisson	Minimum χ^2	Normal	Binomial	Minimum Modified χ^2
Intercept	β_1	2.665	2.430	2.477	0.962	2.431	2.472
Ship type *B*	β_2	-1.821	-1.565	-1.619	-0.104	-1.566	-1.596
Ship type *C*	β_3	-2.149	-1.730	-1.026	-1.081	-1.731	-2.793
Ship type *D*	β_4	-0.376	-0.651	0.433	0.868	-0.650	-1.682
Ship type *E*	β_5	1.756	2.019	2.932	2.650	2.017	0.849
Const. yr 65-69	β_6	1.094	1.055	1.097	1.108	1.055	1.003
Const. yr 70-74	β_7	1.536	1.604	1.661	1.657	1.603	1.323
Const. yr 75-79	β_8	0.453	0.714	0.747	0.968	0.713	0.665
Operation yr 75-79	β_9	0.837	0.791	0.778	0.770	0.792	0.846
χ^2		41.063	40.042	35.591	50.044	40.040	71.436
absolute difference		0.168	0.160	0.178	0.179	0.160	0.154

Table 4. Parameters, χ^2 and absolute difference for Canadian data

Parameters & bias measures		Multiplicative models					
		Zero bias / Poisson	Least squares	Minimum χ^2	Normal	Binomial	Minimum modified χ^2
Intercept	$\exp(\beta_1)$	0.080	0.081	0.080	0.079	0.080	0.080
Class 2	$\exp(\beta_2)$	1.350	1.330	1.351	1.392	1.347	1.347
Class 3	$\exp(\beta_3)$	1.599	1.586	1.598	1.628	1.597	1.599
Class 4	$\exp(\beta_4)$	1.692	1.660	1.697	1.742	1.686	1.682
Class 5	$\exp(\beta_5)$	1.241	1.223	1.242	1.286	1.238	1.238
Merit rating X	$\exp(\beta_6)$	1.313	1.307	1.312	1.334	1.312	1.314
Merit rating Y	$\exp(\beta_7)$	1.427	1.405	1.428	1.483	1.423	1.423
Merit rating B	$\exp(\beta_8)$	1.637	1.611	1.640	1.705	1.632	1.633
χ^2		577.826	625.268	577.037	754.403	580.754	583.899
absolute difference		0.028	0.032	0.028	0.020	0.028	0.028

Parameters (10^{-2}) & bias measures		Additive models					
		Zero bias/ Least squares	Poisson	Minimum χ^2	Normal	Binomial	Minimum Modified χ^2
Intercept	β_1	7.878	7.877	7.876	7.875	7.877	7.878
Class 2	β_2	3.080	3.126	3.129	3.207	3.120	3.121
Class 3	β_3	5.296	5.242	5.248	5.081	5.252	5.232
Class 4	β_4	6.489	6.529	6.531	6.637	6.521	6.523
Class 5	β_5	2.100	2.167	2.174	2.323	2.158	2.152
Merit rating X	β_6	2.793	2.757	2.760	2.697	2.762	2.751
Merit rating Y	β_7	3.827	3.858	3.861	3.938	3.853	3.850
Merit rating B	β_8	5.884	5.878	5.881	5.896	5.879	5.870
χ^2		97.829	95.926	95.904	108.302	95.970	96.100
absolute difference		0.008	0.007	0.007	0.005	0.007	0.007

Table 5. Parameters, χ^2 and absolute difference for UK data

Parameters & bias measures		Multiplicative models					
		Zero bias / Poisson	Least squares	Minimum χ^2	Normal	Binomial	Minimum modified χ^2
Intercept	$\exp(\beta_1)$	0.243	0.241	0.249	0.242	0.242	0.227
Coverage N.Comp	$\exp(\beta_2)$	0.756	0.748	0.759	0.823	0.755	0.750
Veh. age 4-7	$\exp(\beta_3)$	0.804	0.817	0.803	0.827	0.806	0.806
Veh. age 8+	$\exp(\beta_4)$	0.643	0.649	0.642	0.618	0.645	0.647
Veh. group B	$\exp(\beta_5)$	1.139	1.123	1.133	1.133	1.137	1.151
Veh. group C	$\exp(\beta_6)$	1.238	1.218	1.231	1.268	1.235	1.251
Veh. group D	$\exp(\beta_7)$	1.605	1.574	1.600	1.614	1.600	1.615
P/H age 21-24	$\exp(\beta_8)$	0.846	0.838	0.841	0.813	0.843	0.864
P/H age 25-29	$\exp(\beta_9)$	0.639	0.647	0.631	0.622	0.641	0.667
P/H age 30-34	$\exp(\beta_{10})$	0.574	0.591	0.567	0.563	0.578	0.596
P/H age 35+	$\exp(\beta_{11})$	0.514	0.524	0.505	0.509	0.516	0.541
χ^2		107.049	109.491	106.487	122.232	107.290	112.873
absolute difference		0.063	0.065	0.063	0.063	0.063	0.064

Table 5, continued. Parameters, χ^2 and absolute difference for UK data

Parameters (10^{-2}) & bias measures		Additive models					
		Zero bias/ Least squares	Poisson	Minimum χ^2	Normal	Binomial	Minimum Modified χ^2
Intercept	β_1	21.864	21.387	21.732	14.444	21.458	20.643
Coverage N.Comp	β_2	-2.878	-2.500	-2.445	-1.451	-2.538	-2.603
Veh. age 4-7	β_3	-3.253	-3.397	-3.393	-2.773	-3.380	-3.391
Veh. age 8+	β_4	-5.443	-5.463	-5.504	-5.608	-5.457	-5.379
Veh. group B	β_5	1.208	1.269	1.271	1.144	1.261	1.266
Veh. group C	β_6	2.162	2.168	2.167	2.518	2.165	2.175
Veh. group D	β_7	6.241	6.193	6.273	6.557	6.204	6.020
P/H age 21-24	β_8	-1.883	-1.736	-1.864	4.035	-1.762	-1.413
P/H age 25-29	β_9	-5.767	-5.528	-5.789	0.933	-5.567	-4.962
P/H age 30-34	β_{10}	-7.096	-6.937	-7.206	-0.396	-6.967	-6.385
P/H age 35+	β_{11}	-8.437	-7.968	-8.282	-1.398	-8.035	-7.302
χ^2		133.859	127.935	127.252	234.493	127.902	133.542
absolute difference		0.072	0.072	0.072	0.077	0.072	0.074

Several conclusions can be made regarding the programming and results of parameter estimates from the fitted models:

i. The classical approach and regression model give equivalent parameter estimates.

ii. The regression model has faster convergence.

iii. The additive models are more sensitive to initial values.

iv. Each of multiplicative and additive models produced similar parameter estimates.

8. CONCLUSIONS

This paper bridged the minimum bias and maximum likelihood methods for both additive and multiplicative models via a weighted equation. The equations for both minimum bias and maximum likelihood can be rewritten as a weighted equation, in the form of a weighted difference between observed and fitted rates. The parameter estimates could also be rewritten as a weighted solution; for multiplicative model it is in the form of a weighted proportion whereas for additive model, the form is of a weighted difference.

Applying the weighted equation for maximum likelihood and minimum bias equations has several advantages; the weighted equation is mathematically and conceptually simpler, the weighted equation also allows the usage of regression model, and finally, the weighted equation provides an initial understanding of the fitting procedure for distribution with overdispersion parameter. In addition, the weights of the parameter solutions for both multiplicative and additive models show that they have similar estimates.

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APPENDIX A

S-Plus classical programming for multiplicative model

```
Classic.multi <- function(data,b,d,g,h,k,iter=200)
{
    X <- as.matrix(data[,-(1:2)])
    count <- as.vector(data[,1])
    exposure <- as.vector(data[,2])
    rate <- count/exposure
    parameter <- dim(X)[2]
    new.beta <- rep(c(0.5), dim(X)[2])
    for (i in 1:iter)
    {
        for (j in 1:parameter)
        {
            beta <- new.beta
            fitted <- as.vector(exp(X%*%log(beta)))
            fitted.noj <- as.vector(exp(X[,-j]%*%log(beta[-j])))
            z <- as.vector(exposure^b*(rate+0.5/exposure)^d*fitted.noj^g*
                (1-fitted)^h*(rate+fitted)^k*X[,j])
            v <- as.vector(z/sum(z))
            new.beta[j] <- as.vector(sum(v*(rate/fitted.noj)))
        }
        if (all(abs(new.beta-beta)<0.0000001))
            break
    }
    fitted <- as.vector(exp(X%*%log(beta)))
    chi.square <- sum((exposure*(rate-fitted)^2)/fitted)
    abs.difference <- sum(exposure*abs(rate-fitted))/sum(exposure*rate)
```

```
    list( beta=round(beta,4), chi.square=round(chi.square,3),
            absolute.difference=round(abs.difference,3))
}
```

S-Plus classical programming for additive model

```
Classic.add <- function(data,b,d,g,h,k,iter=300)
{
    X <- as.matrix(data[,-(1:2)])
    count <- as.vector(data[,1])
    exposure <- as.vector(data[,2])
    rate <- count/exposure
    parameter <- dim(X)[2]
    new.beta <- rep(c(0.000001), dim(X)[2])
    for (i in 1:iter)
    {
        for (j in 1:parameter)
        {
            beta <- new.beta
            fitted <- as.vector(X%*%beta)
            fitted.noj <- as.vector(X[,-j]%*%beta[-j])
            z <- as.vector(exposure^b*(rate+0.5/exposure)^d*fitted^g*
                (fitted*(1-fitted))^h*(rate+fitted)^k*X[,j])
            v <- as.vector(z/sum(z))
            new.beta[j] <- as.vector(sum(v*(rate-fitted.noj)))
        }
        if (all(abs(new.beta-beta)<0.0000001))      break
    }
    fitted <- as.vector(X%*%beta)
    chi.square <- sum((exposure*(rate-fitted)^2)/fitted)
    abs.difference <- sum(exposure*abs(rate-fitted))/sum(exposure*rate)
    list( beta=round(beta,6), chi.square=round(chi.square,3),
            absolute.difference=round(abs.difference,3))
}
```

APPENDIX B

S-Plus regression programming for multiplicative model

```
Regression.multi <- function(data,b,d,g,h,k,iter=20)
{
    X <- as.matrix(data[, -(1:2)])
    count <- as.vector(data[,1])
    exposure <- as.vector(data[,2])
    rate <- count/exposure
    new.beta <- rep(c(1),dim(X)[2])
    for (i in 1:iter)
    {
        beta <- new.beta
        fitted <- as.vector(exp(X%*%beta))
        Z <- diag(fitted)%*%X
```

```
        W <-    diag(exposure^b*(rate+0.5/exposure)^d*fitted^g*(1-fitted)^h*
                (rate+fitted)^k)
        r.s <- rate-fitted+as.vector(Z%*%beta)
        new.beta <- as.vector(solve(t(Z)%*%W%*%Z)%*%t(Z)%*%W%*%r.s)
        if (all(abs(new.beta-beta)<0.0000001))
            break
    }
    fitted <- as.vector(exp(X%*%beta))
    chi.square <- sum((exposure*(rate-fitted)^2)/fitted)
    abs.difference <- sum(exposure*abs(rate-fitted))/sum(exposure*rate)
    list( beta=round(exp(beta),4), chi.square=round(chi.square,3),
            absolute.difference=round(abs.difference,3))
}
```

S-Plus regression programming for additive model

```
Regression.add <- function(data,b,d,g,h,k,iter=20)
{
    X <- as.matrix(data[, -(1:2)])
    count <- as.vector(data[,1])
    exposure <- as.vector(data[,2])
    rate <- count/exposure
    new.beta <- rep(c(0.000001),dim(X)[2])
    for (i in 1:iter)
    {
        beta <- new.beta
        fitted <- as.vector(X%*%beta)
        W <- diag(exposure^b*(rate+0.5/exposure)^d*fitted^g*(fitted*(1-fitted))^h*
                    (rate+fitted)^k)
        r.s <- rate-fitted+as.vector(X%*%beta)
        new.beta <- as.vector(solve(t(X)%*%W%*%X)%*%t(X)%*%W%*%r.s)
        if (all(abs(new.beta-beta)<0.0000001))
            break
    }
    fitted <- as.vector(X%*%beta)
    chi.square <- sum((exposure*(rate-fitted)^2)/fitted)
    abs.difference <- sum(exposure*abs(rate-fitted))/sum(exposure*rate)
    list( beta=round(exp(beta),6), chi.square=round(chi.square,3),
            absolute.difference=round(abs.difference,3))
}
```

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