On Optimal Reinsurance Arrangement

Yisheng Bu, Ph.D.

Abstract:

The purpose of this paper is to develop a theoretical framework within which the optimal reinsurance arrangement for catastrophic risks is explored and derived. In the model, it is assumed that the insurer values the stability of its underwriting results in purchasing reinsurance protection. The optimal solutions to the model are obtained through numerical simulation and intend to provide justifications and explanations for the profile of reinsurance purchase that has been observed in practice.

Keywords: Catastrophic Risk, Excess-of-Loss Reinsurance, Optimality, Contingent Capital Calls

1. INTRODUCTION

Optimal reinsurance arrangement has been extensively studied in a series of papers from various perspectives. Borch (1961) examined risk sharing between insurers through quota-share reinsurance arrangements. Some of the recent studies have focused on the pricing and optimal design of excess-of-loss reinsurance contracts. Cummins et al. (1999) developed a pricing methodology for index-based catastrophe loss contracts. Gajek and Zagrodny (2004) derived optimal forms for stop-loss contracts when the insurer attempts to minimize the probability of ruin.

The purpose of this paper is to develop a theoretical framework within which the optimal reinsurance arrangement for catastrophic risks is explored and derived. In the model, it is assumed that the insurer values the stability of its underwriting results in purchasing reinsurance protection. The optimal solutions to the model are obtained through numerical simulation and intend to provide justifications and explanations for the profile of reinsurance purchase that has been observed in practice. From over 4,000 catastrophe reinsurance layers transacted during the period 1970-1998, Froot (2001) observed that: (i) reinsurance contracts had been more often used to cover lower catastrophic risk layers (which have higher probability to be penetrated) rather than more severe but lower-probability layers; and (ii) reinsurance contracts had been priced in such a way that higher reinsurance layers had higher ratios of premium to expected losses.

The rest of the paper is organized as follows: The next section sets up a model of reinsurance purchase from an insurer's standpoint and derives the optimality conditions. Section 3 numerically solves the model, specifies the simulation methodology and discusses the results. Section 4 assumes a discrete loss distribution and derives the analytical solutions for the optimal design of reinsurance contract. Section 5 suggests possible ways to modeling the reinsurer's behavior and endogenizing the rule of reinsurance pricing. Section 6 concludes.

2. THE MODEL

This section introduces a simple model in which the reinsurance-pricing rule is exogenously given in deriving the system of optimal solutions for the insurer, while Section 5 discusses modeling of the reinsurer's behavior and choices. Specifically, the model makes the following simplifying assumptions:

(i) The reinsurance market consists of one insurer and one reinsurer;

(ii) The reinsurer sets its own pricing rule which may be a function of its own cost of capital;

(iii) The insurer has perfect information about the reinsurance pricing rule, and chooses the reinsurance layer for full coverage; and

(iv) The insurer and reinsurer have access to the same information on the underlying loss distribution.

The Reinsurer. The reinsurer underwrites an excess-of-loss contract i for catastrophic risks (shortened as "cat" hereafter) and assumes a certain portion of cat losses arising from the underlying insurance contracts. The reinsurance layer is defined by $[a, b]$, where a denotes the insurer's retention and b the retention plus limit. Cat losses occur with a continuous distribution function $F(x)$, where $x \in [0, \infty)$. For the reinsurer, the expected value and variance of loss payment from underwriting contract i is given, respectively, as

$$E[L_R^i(x_i; a_i, b_i)] = \int_{a_i}^{b_i} (x_i - a_i) dF(x_i) + (b_i - a_i) \int_{b_i}^{\infty} dF(x_i) \tag{1}$$

and

$$Var[L_R^i(x_i;a_i,b_i)] = (E[L_R^i])^2 \int_0^{a_i} dF(x_i) + \int_{a_i}^{b_i} (x_i - a_i - E[L_R^i])^2 \, dF(x_i) + (b_i - a_i - E[L_R^i])^2 \int_{b_i}^{\infty} dF(x_i) \quad (2)$$

It is further assumed that loss payments under a marginal reinsurance contract is stochastically independent of those under all other reinsurance contracts in the existing portfolio held by the reinsurer. Naturally, the assumption of stochastic independence among risks in the reinsurer's portfolio may not be realistic[1], as a cat event may impact on many of the risk exposures under different contracts covered by the reinsurer. This assumption, however, will simplify the following exposition and simulation but not change the nature of the results to be derived. Based on the Capital Asset Pricing Model, the reinsurance pricing formula can be formulated as:

$$Z^i(x_i;a_i,b_i) = E[L_R^i(x_i;a_i,b_i)] + \gamma_R \cdot Var[L_R^i(x_i;a_i,b_i)], \quad (3)$$

where γ_R ($\gamma_R > 0$) is the price of risk determined by the reinsurer's existing portfolio, and mathematically, it can be expressed as $\gamma_R = (Z_R^{(i)} - E[L_R^{(i)}]) / Var[L_R^{(i)}]$, where (i) refers to all risks excluding contract i. As stated in Borch (1982), one advantage of this formulation is that it ensures the additive property of reinsurance contacts so that the price of risk will not be altered by the addition of stochastically independent risks. There are several issues that are worthy of comments. First, the risk load as specified in (3) does not explicitly take into account parameter uncertainty associated with the underlying loss distribution, nor is it directly modeled as a function of the "down-side" variance that may seem to be the more reasonable and appropriate one than the total variance. Nevertheless, the formulation in (3) has been supported by many empirical findings on reinsurance pricing (for instance, Kreps and Major 2001, Lane 2004). Second, , Kreps (2004) suggested a probability-weighted average of the deviations of loss from its expected value multiplied by a "riskiness leverage ratio" as a more general form for risk load. The riskiness leverage ratio can be a function of

[1] The correlation among the risk exposures in a reinsurer's portfolio was analyzed via copula approach in Venter (2003).

higher moments of loss function. As pricing reinsurance contracts is not the main focus of this paper, further studies should explore optimal reinsurance arrangements using other forms of risk load specifications. For the simplicity of exposition, the subscript i will be dropped from all following mathematical expressions.

The Insurer. The insurer knows about the reinsurer's pricing rule, and purchases the optimal reinsurance layer, or makes the optimal choices about a and b. By choosing a and b, the insurer attempts to minimize the sum of the reinsurance premium the insurer pays for reinsurance coverage and the expected loss payment net of reinsurance recovery. Besides, the objective function of the insurer also includes a penalty term for the variation of net loss payment. The penalty for loss variations is assumed to be a function of the variance of net loss payment. Note that this paper abstracts from the consideration of probability ruin in deriving optimal reinsurance arrangements. To summarize, the insurer attempts to minimize the following objective function subject to the budget constraint (denoted by B) on reinsurance purchase,

$$\underset{a,b}{MIN}: Z + E[L_S(x;a,b)] + \gamma_S \cdot Var[L_S(x;a,b)] \qquad (4)$$

$$s.t.\ Z \le B,$$

where

$$E[L_S(x;a,b)] = \int_0^a x dF(x) + a\int_a^b dF(x) + \int_b^\infty (x-b+a) dF(x),$$

$$\begin{aligned} Var[L_S(x;a,b)] = &\int_0^a (x-E[L_S])^2 dF(x) + (a-E[L_S])^2 \int_a^b dF(x) \\ &+ \int_b^\infty (x-b+a-E[L_S])^2 dF(x) \end{aligned},$$

and γ_S ($\gamma_S > 0$) measures the extent to which the insurer values the stability of its underwriting results, or the degree of its risk aversion. Since $E[L_R] + E[L_S] = \int_0^\infty x dF(x)$, for a given loss distribution, the amount of gross insurance premium received under the underlying insurance contacts can be fixed if the insurance contracts are priced so that the expected loss ratio remains roughly constant over time. As such, the problem stated in (4) would be equivalent to a problem of maximizing the expected net income minus some function of its variance to account for the associated uncertainty.

Optimal Conditions. Maximizing (4) with respect to a and b yields the following first-order conditions

$$\begin{aligned}&[1-\int_a^\infty dF(x)]\int_b^\infty\{\gamma_R(b-a-E[L_R])-\gamma_S(x-b+a-E[L_S])\}dF(x)+\\&\quad[1-\int_a^\infty dF(x)]\int_a^b\{\gamma_R(x-a-E[L_R])-\gamma_S(a-E[L_S])\}dF(x)\qquad,\\&\quad-\int_a^\infty dF(x)\int_0^a\{\gamma_R(-E[L_R])-\gamma_S(x-E[L_S])\}dF(x)=\lambda\cdot(\partial Z/\partial a)\end{aligned}\tag{5}$$

$$\begin{aligned}&[1-\int_b^\infty dF(x)]\int_b^\infty\{\gamma_R(b-a-E[L_R])-\gamma_S(x-b+a-E[L_S])\}dF(x)\\&\quad-\int_b^\infty dF(x)\int_a^b\{\gamma_R(x-a-E[L_R])-\gamma_S(a-E[L_S])\}dF(x)\\&\quad-\int_b^\infty dF(x)\int_0^a\{(\gamma_R(-E[L_R])-\gamma_S(x-E[L_S])\}dF(x)=\lambda\cdot(\partial Z/\partial b)\end{aligned}\tag{6}$$

and

$$Z\le B,\lambda\ge 0\ \text{ c.s.,}\tag{7}$$

where

$$\frac{\partial Z}{\partial a}=-\int_a^\infty dF(x)-2\gamma_R(1-\int_a^\infty dF(x))\cdot E(L_R)$$

and

$$\frac{\partial Z}{\partial b}=\int_b^\infty dF(x)+2\gamma_R\cdot\int_b^\infty(b-a-E(L_R))dF(x)\,.$$

In equations (5) and (6), $\int_a^b dF(x)$ and $\int_b^\infty dF(x)$ are the exceedence and exhaustion probabilities, respectively. Complicated at first glance, equations (5) and (6) virtually state that by choosing the reinsurance coverage, the insurer attempts to achieve the optimal balance between the reduction in the cost of loss variation because of reinsurance coverage and the price for shifting such variation to the reinsurer. Many of the terms in (5) and (6) describe the deviations of loss payment from the expected values in each interval for the insurer or the reinsurer. For instance, $b - a - E[L_R]$ is the amount of loss payment by the reinsurer and $x - b + a - E[L_S]$ is the amount of loss payment by the insurer when the layer limit is exceeded, in excess of their respective expected loss payment. To the extent that the parameters γ_R and γ_S measure the cost of reinsurance and insurance capital, respectively, the terms multiplied by these two parameters should be interpreted as the cost of such deviations.

Note that $\partial Z / \partial a < 0$ and $\partial Z / \partial b > 0$, which are quite intuitive in that higher reinsurance coverage demands higher price. However, it is not obvious how the reinsurance premium responds to the retention while holding the layer limit constant. Substituting $a + l$ for b in equations (3), and differentiating the resulting equation with respect to a gives

$$\partial Z / \partial a = 2\gamma_R \int_a^{a+l} (-x + a + E(L_R)) dF(x) - \int_a^{a+l} dF(x) ,$$

where l denotes the layer limit. The sign of $\partial Z / \partial a$ is uncertain, which depends on the layer boundaries and γ_R.

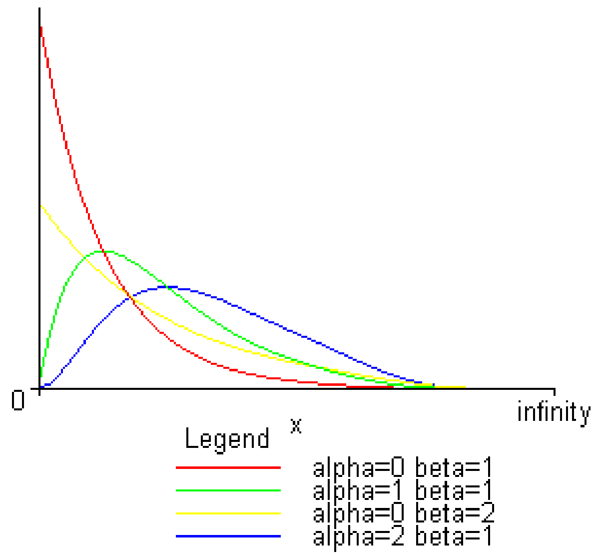
3. NUMERICAL SIMULATION

Methodology

While it is difficult to obtain the closed form solutions to the equations (5) and (6), the optimal values of a and b can be numerically solved for through simulation. In the simulation, it is assume that the insurer has no budget constraint on purchasing reinsurance. For illustrative purposes, the cat loss is assumed to be described by a Gamma distribution with the following probability density function:

$$f(x) = \frac{x^{\alpha} e^{(-x/\beta)}}{\alpha! \beta^{(\alpha+1)}}. \tag{8}$$

Figure 1. Gamma Distribution



In Figure 1, the probability density curves are plotted for several combinations of α and β. As the benchmark case in the simulation, $\alpha = 1$ and $\beta = 1$, and (8) then is simplified to $f(x) = x \cdot e^{-x}$ with mean and variance both equal to 2. For the benchmark case, the reinsurance premium and the value of the insurer's objective function, as functions of a and b, are plotted respectively in two three-dimensional figures (see Figures 2 and 3). In plotting the two figures, $\gamma_R = 2$ and $\gamma_S = 2$ are assumed.

To obtain the numerical solutions to equations (5) and (6), the following simulation procedures are used:

(i) Specify the values of γ_R and γ_S and choose the initial values of a and b (which are denoted by a_0 and b_0, respectively);

Figure 2. Reinsurance Premium as a Function of a and b (assuming $\gamma_R = 2$, $\gamma_S = 2$

$a \in (0, b), b \in (0,10)$**)**

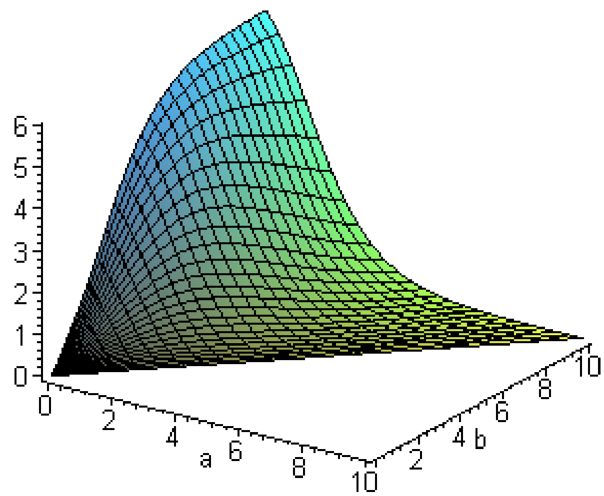
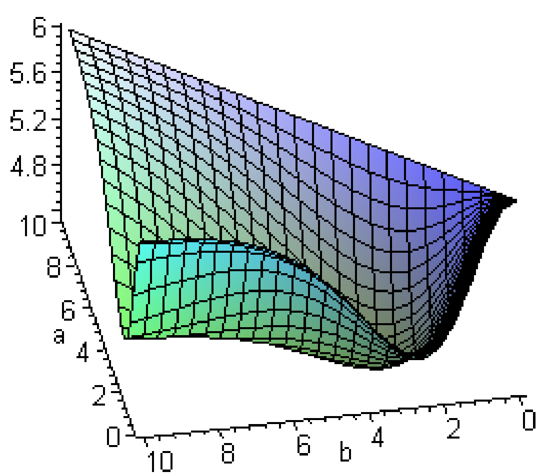


Figure 3. The Value of the Insurer's Objective Function (assuming $\gamma_R = 2$, $\gamma_S = 2$,

$a \in (0, b), b \in (0,10)$**)**



(ii) Hold b_0 constant, and with a_0 as the starting value, apply the Newton's iteration method to find the "optimal" value of a (denoted by a_1) that satisfies Equation (5);

(iii) Holding a_1 constant, and with b_0 as the starting value, apply the Newton's iteration method to find the "optimal" value of b (denoted by b_1) that satisfies Equation (6); and

(iv) Repeat (ii) and (iii) a number of times (usually 50 times would be sufficient) until the differences between a_t and a_{t+1}, and between b_t and b_{t+1} are sufficiently small. Then a_{t+1} and b_{t+1} are the optimal solutions to (5) and (6) (denoted by a^* and b^*).

Results

Table 1. Numerical Simulation Results with $\gamma_R = 2$, $\gamma_S = 2$

Parameter Values, Expected Value and Variance of Underlying Loss Distribution

(1) alpha	(2) beta	(3) E(x)	(4) Var(x)
0	1	1	1
0	2	2	4
1	1	2	2
2	1	3	3

Simulation Results

(5) a	(6) b	(7) limit	(8) Z	(9) E(Lr)	(10) Obj	(11) ROL	(12) Z/E(Lr)
1.018	2.611	1.594	0.806	0.288	2.180	0.506	2.798
2.035	5.222	3.187	2.648	0.576	6.719	0.831	4.597
1.805	3.813	2.008	1.531	0.497	4.367	0.762	3.078
2.631	4.982	2.351	2.218	0.670	6.556	0.943	3.311

The numerical simulation results are summarized in Table 1. The optimal reinsurance layers are obtained through simulation for four sets of α and β values. In the table, the first two columns are the parameter values of α and β. Columns (3) and (4) are the expected value and variance of the cat loss distribution. For the cases where $\beta = 1$, the loss distribution is equi-dispersed, and over-dispersed when $\beta > 1$. The simulated values of the layer boundaries and policy limit are given in columns (5)-(7), the rate-on-line (ROL) and ratio of premium to expected loss in columns (11) and (12). Note that in the simulation, the budget constraint and the probability of ruin have been ignored, and the values of α and β are chosen arbitrarily so that the implied probability distribution of cat losses does not mirror the ones forecasted by engineering cat models in reality.

As the simulation results show, for all the four cases as presented in Table 1, the insurer who aims to stabilize its book of business should optimally use reinsurance protection against risks of moderate sizes, but leave the most severe loss scenarios uncovered or self-insured. This result justifies the aggregate profile of reinsurance purchases observed in Froot (2001)[2]. Also, as observed from the simulation results, the insurer's retention is set to be comparable with the expected value of ground-up cat losses under the underlying insurance contracts that are covered by the reinsurance treaty. As compared with the benchmark case ($\alpha = 1, \beta = 1$), the insurer, at the optimum, should purchase higher retention and higher limit for the case $\alpha = 0$, $\beta = 2$,the distribution which has the same expected loss but is more dispersed. The optimal layer in the latter case also has higher ROL and higher ratio of premium to expected loss.

It may be helpful to look at how the optimal choices of the insurer change while varying the parameters of the loss distribution. Table 1 reports the simulation results for the cases with different values of α ($\alpha = 0, 1, 2$), while holding β constant at 1. For the density function specified in (8), the value of α determines the shape of the distribution; the coefficient of variation decreases with the value of α, even though the loss distribution remains equi-dispersed ($Var(x)/E(x) = 1$). The optimal choices of the reinsurance layer can be very sensitive to the chosen values of the model parameters. With higher values of α, events of higher severity occur with larger probabilities(see Figure 1), and the insurer should have more protection (as shown by higher limit of reinsurance layer) against more severe events. On the other hand, the reinsurer would demand higher ROL and ratio of premium to expected loss for worse cat loss scenarios. Similar conclusions can be drawn by comparing the simulation results for different values of β; for instance, comparing the results between the cases with $\beta = 1$ and $\beta = 2$ (while $\alpha = 0$). To the extent that higher

---

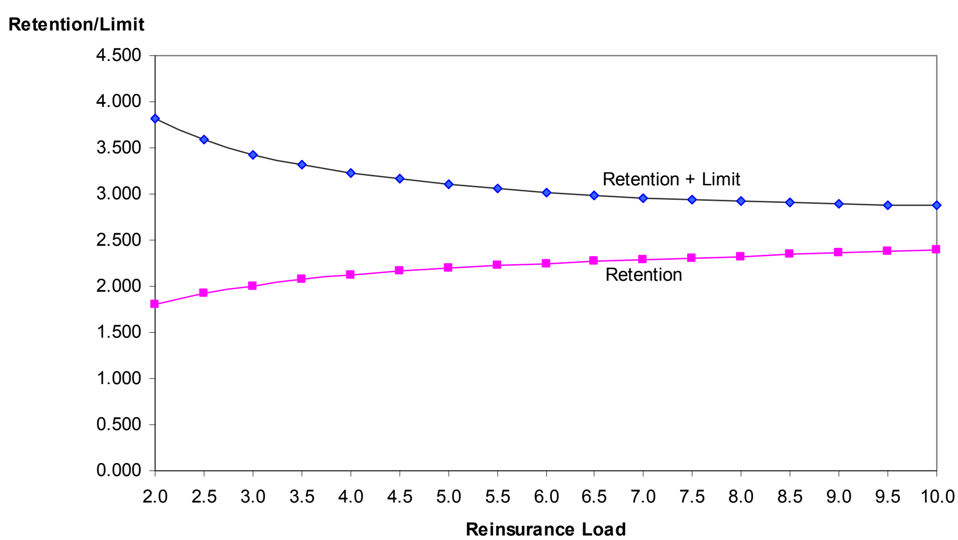
[2] As observed in Froot (2001, p. 536), "reinsurance coverage as a fraction of exposure is high at first (after some small initial retention) and then declines markedly with the size of the event, falling to a level of less than 30% for events of only about \$8 billion (the author's note: \$8 billion refers to the industry-wide loss).

reinsurance layers are more vulnerable to prediction errors from engineering models, parameter uncertainty may well explain high prices for low-probability layers as noted in Froot (2001).

Discussion

Varying the value of γ_R. Varying the value of γ_R between 2 and 10, Figure 4 plots the optimal values of a and b. The figure shows that when the price per unit of risk charged by the reinsurer increases relative to that by the insurer, or equivalently, as γ_R / γ_S increases, less reinsurance coverage will be purchased in terms of lower b and higher a and thus lower policy limit.

Figure 4. Retention and Limit As Reinsurance Load Changes (assuming $\gamma_S = 2$)



4. DISCRETE LOSS DISTRIBUTION: AN EXAMPLE

Table 2. Discrete Loss Distribution

	Scenario 1	Scenario 2	Scenario 3
Total Cat Loss	s1	s2	s3
Probability	f1	f2	f3

This section examines the optimal reinsurance arrangement when the loss distribution is discrete. Assume that there are a finite number of states for cat losses. Table 2 gives a simple discrete distribution of cat losses, where one of the following states of the world could occur: little or no occurrence (Scenario 1), moderate (Scenario 2), and most severe (Scenario 3). For the loss distribution given in Table 2, it is assumed that $0 \le s_1 < s_2 < s_3$, $f_1 \ge f_2 \ge f_3 \ge 0$ and $f_1 + f_2 + f_3 = 1$. For the simplicity of illustration, further set $s_1 = 0$ (no cat loss) and write $f_1 = 1 - f_2 - f_3$. There could be three choices regarding the sizes of retention and limit relative to loss severities: (i) $0 \le a \le s_2 \le b \le s_3$, (ii) $0 \le a \le b \le s_2$, and (iii) $s_2 \le a \le b \le s_3$. For this discrete distribution, it can be mathematically shown that

1. the optimal solutions always come from (i), or at the optimum,

   $0 < a^* < s_2 < b^* < s_3$. Specifically, the optimal reinsurance layer boundaries are

   given by $a^* = \dfrac{s_2 \gamma_R}{\gamma_R + \gamma_S}$ and $b^* = a^* + \dfrac{s_3 \gamma_S}{\gamma_R + \gamma_S} = \dfrac{s_2 \gamma_R + s_3 \gamma_S}{\gamma_R + \gamma_S}$.

2. the layer limit is independent of the probability with which each event occurs,

   and satisfies that $l^* = b^* - a^* = \dfrac{s_2 \gamma_R + s_3 \gamma_S}{\gamma_R + \gamma_S}$,

3. The minimum (optimal) value of the insurer's value function is equal to the rate on line of the reinsurance contract.

The proof is provided in the Appendix.

The optimal solutions satisfy $0 < a^* < s_2 < b^* < s_3$, provided that the ratio of γ_R to γ_S does not take extreme values. This implies that it is advisable for the insurer to purchase some reinsurance protection against both moderate and most severe cat loss scenarios rather than against any one particular scenario only, even though the coverage for any one of the scenarios in the latter case may be larger than that in the former case. Further, the insurer has more reinsurance protection against cat losses when $0 \le a \le s_2 \le b \le s_3$. As shown in the Appendix, the optimal reinsurance arrangement for (i) has the highest layer limit as compared with the other two choices. It is also observed that the comparative static results, $\partial b^* / \partial \gamma_R < 0$ and $\partial a^* / \partial \gamma_R > 0$, are consistent with the simulation results obtained for the continuous loss distribution (see Figure 4).

At the optimum, $Obj_{\min} = Z^* /(b^* - a^*)$, or in words, the objective function has its minimum value equal to the ratio of reinsurance premium to the layer limit, or the "rate on line". As compared, the simulation results for the continuous loss distribution (see Table 1) do not imply such a relationship between the two elements.

However, it is not intuitively clear why the optimal layer limit is independent of the occurrence probability of each cat loss scenario.

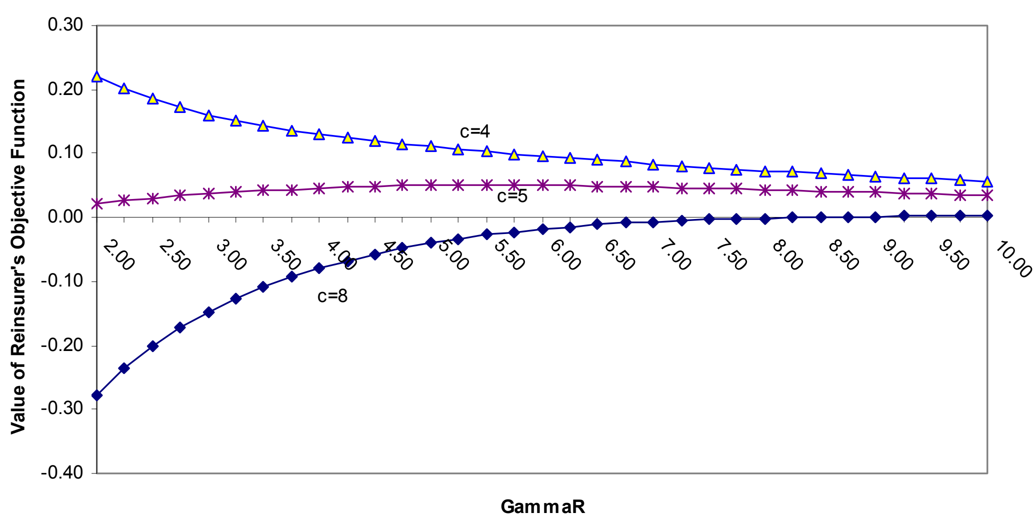
5. THE VALUE OF γ_R AND CONTINGENT CAPITAL CALLS

Mango (2004) introduced a capital consumption methodology for pricing reinsurance contracts, which in essence uses the value of potential capital usage as the risk load. Such potential access to surplus account is called contingent capital calls in that paper and other relevant studies. The discussion in the previous sections has been focused on the situation where the insurer makes its optimal decisions on reinsurance purchase subject to the pricing rule of the reinsurer who has been assumed not to consequently respond to the optimal choices of the insurer. In other words, the reinsurance pricing rule has been assumed exogenously given and fixed. Since reinsurers are also profit maximizing firms just like insurers, it is reasonable to assume that the reinsurer attempts to maximize the firm's expected net income after adjusting for the capital costs in the unprofitable states. For instance, using the methodology proposed in Mango (2004), the objective function of the reinsurer is formulated as:

$$\underset{\gamma_R}{MAX}: \gamma_R \cdot Var[L_R] - \int_{a+z}^{b} g(x-(a+Z))dF(x) - g(b-(a+Z)) \cdot \int_{b}^{\infty} dF(x), \qquad (9)$$

where Z ($Z < b - a$) is a function of a and b as formulated in (3), and the function $g(\cdot)$ is the capital call charge function and convex so that $g'(\cdot) > 0$, $g''(\cdot) \geq 0$. The condition that $g''(\cdot) \geq 0$ requires nondecreasing marginal cost of capital calls. The reinsurer chooses the value of γ_R to maximize the problem in (9), or maximize the risk load minus the cost of contingent capital calls.

Figure 5. The Choice of γ_R and the Value of the Reinsurer's Objective Function



For a given reinsurance layer, a higher value of γ_R necessarily implies a higher ratio of reinsurance premium to expected loss. Figure 4 graphically shows that simply raising the value of γ_R would influence reinsurance purchase by increasing the insurer's retention and lowering the policy limit. As a results, it may well be the case that the reinsurer's objective function as specified in (9) is non-monotonic in γ_R. The optimal value of γ_R may be a function of the parameters of the underlying cat loss distribution and of the cost function of capital calls. For instance, the cost of access to surplus account is assumed to take the following functional form

$$g(\varepsilon) = \varepsilon + c \cdot \varepsilon^2, \tag{10}$$

where ε $(\varepsilon \geq 0)$ is the amount of capital calls and c $(c > 0)$ is the rate at which the marginal cost of capital calls increases. With higher values of c, the reinsurer would find it more costly to underwrite more severe cat events. By assuming (10), Figure 5 graphs the trajectories of the value of (9) for $\gamma \in [2,10]$ for $c = 4$, 5 and 8, respectively. For the case of $c = 5$, the value of (9) is maximized when γ_R is around 6.25, while for $c = 4$ (or $c = 8$), smaller (or larger) values of γ_R always yield higher values of (9) when γ_R is within the stated range. Comparing the three curves in the figure would show: when the marginal cost of capital calls increases relatively faster for the reinsurer, the reinsurer sets higher γ_R and the insurer tends to purchase reinsurance protection for moderate losses only and leave higher layers uncovered.

6. CONCLUDING REMARKS

This paper has examined the optimal reinsurance arrangement for cat risks when the insurer values the stability of its underwriting results, subject to the reinsurance-pricing rule set by the reinsurer. In the model, the optimal solutions for reinsurance coverage purchase are obtained through numerical simulation, and the analytical solutions derived for the case when the loss distribution is discrete. Using the model results, the aggregate profile of reinsurance purchase observed for industry practice in previous studies is explained and justified.

As Froot and Posner (2000) stated, the risk pricing for cat reinsurance contracts is largely determined by the reinsurer. The first author in his 2001 paper further found some evidence implying that reinsurers possess certain market power in the reinsurance market. The general equilibrium model of reinsurance market was studied in Borch (1962), in which reinsurance capital market was assumed to be perfectly competitive and the pricing of quota share contracts were examined. Future research should develop a conceptual framework in which the reinsurer's behavior is systematically modeled and analytical solutions can be derived, and focus on the empirical measurement and determination of the cost of reinsurance capital in industry practice.

7. REFERENCES

[1] Borch, Karl, 1961, Some Elements of a Theory of Reinsurance, Journal of Insurance 28(3), 35-43.

[2] Borch, Karl, 1962, Equilibrium in a Reinsurance Market, Econometrica 30(3), 424-444.

[3] Borch, Karl, 1982, Additive Insurance Premium: A Note, Journal of Finance 37(5), 1295-1298.

[4] Cummins, David J., Christopher M. Lewis, and Richard D. Phillips, 1999, Pricing Excess-of-Loss Reinsurance Contracts against Catastrophic Risk, in Kenneth A. Froot, ed.: The Financing of Catastrophe Risk (University of Chicago Press, Chicago, Ill.).

[5] Froot, Kenneth A., 2001, The Market for Catastrophe Risk: A Clinical Examination, Journal of Financial Economics 60, 529-571.

[6] Froot, Kenneth A., and Steven Posner, 2000. Issues in the Pricing of Catastrophe Risk. Special Rport, March McLennan Securities Corporation.

[7] Gajek, Leslaw, and Dariusz Zagrodny, 2004, Reinsurance Arrangements Maximizing Insurer's Survival Probability, Journal of Risk and Insurance 71(3), 421-435.

[8] Kreps, Rodney, 2004, A Risk Class with Additive Co-measures, unpublished manuscript.

[9] Kreps, Rodney and John Major, 2001, Catastrophe Risk Pricing in the Traditional Reinsurance Market, presentation at the Bond Market Association Risk-Linked Securities Conference, March 21-23, Aventura, FL.

[10] Lane, Morton N., 2000, Pricing Risk Transfer Functions, ASTIN Bulletin 30(2), 259-293.

[11] Mango, Donald F., 2003, Capital Consumption: An Alternative Methodology for Pricing Reinsurance, Forums of the Casualty Actuarial Society, 351-379.

[12] Venter, Gary G., 2001, Measuring Value in Reinsurance, Forums of the Casualty Actuarial Society, 179-199.

[13] Venter, Gary G., 2003, Quantifying Correlated Reinsurance Exposures with Copulas, Forums of the Casualty Actuarial Society, 215-229.

Appendix

This appendix provides proof for the three statements made in Section 4 for the discrete distribution. First, solve the maximization problem for each of the three cases (i) $0 \le a \le s_2 \le b \le s_3$, (ii) $0 \le a \le b \le s_2$, and (iii) $s_2 \le a \le b \le s_3$. For instance, for case (i), the maximization problem can be written as

$$\underset{a,b}{MIN}: Z + E[L_S(x;a,b)] + \gamma_S \cdot Var[L_S(x;a,b)]$$

$$s.t.\ \ 0 \le a \le s_2 \le b \le s_3,$$

where

$$\begin{aligned} Z = {} & f_2(s_2 - a) + f_3(b-a) + \gamma_R\{(1 - f_2 - f_3)(f_2(s_2 - a) + f_3(b-a))^2 \\ & + f_2(s_2 - a - f_2(s_2 - a) - f_3(b-a))^2 + f_3(b - a - f_2(s_2 - a) - f_3(b-a))^2\}, \end{aligned}$$

$$E[L_S(x;a,b)] = f_2 a + f_3(s_3 - b + a),$$

and

$$\begin{aligned} Var[L_S(x;a,b)] = {} & (1 - f_2 - f_3)(f_2 a + f_3(s_3 - b + a))^2 \\ & + f_2(a - f_2 a - f_3(s_3 - b + a))^2 + f_3(s_3 - b + a - f_2 a - f_3(s_3 - b + a))^2. \end{aligned}$$

The optimal solutions for each case are given below:

Case 1: $$a^* = \frac{s_2\gamma_R}{\gamma_R + \gamma_S},\ b^* = a^* + \frac{s_3\gamma_S}{\gamma_R + \gamma_S} = \frac{s_2\gamma_R + s_3\gamma_S}{\gamma_R + \gamma_S}, \tag{A.1}$$

$$Z^* = (b^* - a^*)\cdot((f_2 s_2 + f_3 s_3) + \frac{\gamma_R\gamma_S}{\gamma_R + \gamma_S}\cdot(f_2 s_2^2 + f_3 s_3^2 - (f_2 s_2 + f_3 s_3)^2)),\ \text{and}$$

$$Obj_{\min} = Z^* / (b^* - a^*)$$
$$= (f_2 s_2 + f_3 s_3) + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} \cdot (f_2 s_2^2 + f_3 s_3^2 - (f_2 s_2 + f_3 s_3)^2)) ;$$

Case 2: $b^* = a^* + \dfrac{\gamma_S}{\gamma_R + \gamma_S} \cdot \dfrac{f_2 s_2 + f_3 s_3}{f_2 + f_3}$, (A.2)

$$Z^* = (b^* - a^*) \cdot (f_2 + f_3 + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} \cdot (f_2 s_2 + f_3 s_3) \cdot (1 - f_2 - f_3)) \text{, and}$$

$$Obj_{\min} = f_2 s_2 + f_3 s_3 + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} (f_2 s_2^2 + f_3 s_3^2 - (f_2 s_2 + f_3 s_3)^2) + M ,$$

where $M = \dfrac{\gamma_S^2}{\gamma_R + \gamma_S} \cdot \dfrac{f_2 f_3}{f_2 + f_3} \cdot (s_3 - s_2)^2 > 0$;

Case 3: $b^* = a^* + \dfrac{\gamma_S}{\gamma_R + \gamma_S} \cdot \dfrac{s_3 - f_2 s_2 - f_3 s_3}{1 - f_3}$, (A.3)

$$Z^* = (b^* - a^*) \cdot (f_3 + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} \cdot f_3 \cdot (s_3 - f_2 s_2 - f_3 s_3)) \text{, and}$$

$$Obj_{\min} = f_2 s_2 + f_3 s_3 + \frac{\gamma_R \gamma_S}{\gamma_R + \gamma_S} (f_2 s_2^2 + f_3 s_3^2 - (f_2 s_2 + f_3 s_3)^2) + N ,$$

where $N = \dfrac{\gamma_S^2}{\gamma_R + \gamma_S} \cdot \dfrac{f_2 s_2^2 \cdot (1 - f_2 - f_3)}{1 - f_3} > 0$.

Obviously, $Obj_{\min}$ obtained in case (i) has the lowest value among all three cases. Therefore, the insurer's objective function has its global minimum value when $0 < a^* < s_2 < b^* < s_3$. Note that for the cases $0 \le a \le b \le s_2$ and $s_2 \le a \le b \le s_3$, there exist multiple solutions for a and b, as only the layer limit ($b - a$), but not the limit boundaries (b, a) individually, matters for the insurer's value function. It is also easy to observe that the optimal layer limit for case (i) is larger than those for the other two cases.

Biography of the Author

Yisheng Bu is a Senior Actuarial Analyst at Liberty Mutual Group, Boston. He earned his Ph.D. in 2002 from Indiana University - Bloomington, major in Economics and minor in Finance. His areas of interest are risk management, reinsurance and macroeconomic aspects of insurance. He has been a member of the CAS Research Working Party on Quantifying Variability in Reserve Estimates, and participated in the writing of a survey paper "The Analysis and Estimation of Loss & ALAE Variability: A Summary Report". He also has an article forthcoming in the Journal of Development Studies, and several other working papers, which are all on possible anomalies in the capital accumulation process in certain developing countries.

Liberty Mutual Group
175 Berkeley Street, S3B
Boston, MA 02116 USA
Phone: 617 654 4894
Email: yisheng.bu@libertymutual.com