Paid Loss Development of Fixed Size Claims

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Abstract: *This paper considers a simple context in which we can quantify the impact of the payment schedule on paid loss development. To isolate the effect of the payment schedule, we restrict to the special case when all claims have the same incurred loss. We consider three simple periodic payment schedules: (1) a uniform payment schedule (2) an escalated (discounted) payment schedule and (3) a schedule that allows a single, fixed proportional adjustment to the payment amount. The paper defines a mathematical model for paid loss development and presents numeric examples to illustrate the sensitivity of paid loss development to the different schedules.*

It is apparent that the payment schedule influences paid loss development. In general a faster (slower) schedule will make losses develop faster (slower). While the direct nature of that relationship is apparent, it is not so apparent how to quantify it. This paper quantifies it in some very particular cases.

Let $S(t)$ denote a survival function on the time interval $(0, b)$.[2] We regard $S(t)$ as a distribution of closure times and let $F(t) = 1 - S(t)$ be the corresponding cumulative distribution function [CDF]. In effect, all claims are assumed to close on or before time b.

We are interested in a related CDF, which we denote by $\tilde{F}(t)$ to emphasize its relation with $F(t)$, which models the paid loss development as a function of time. More precisely, $\tilde{F}(t)$ is the proportion of total loss paid by time t, i.e. the proportion paid out during $(0, t)$ (without any discount adjustment). $\tilde{F}(t)$ is the reciprocal of the paid to ultimate loss development factor and we will refer to $\tilde{F}(t)$ as the paid loss development divisor [PLDD].[3]

In this note we make two basic assumptions on the size and the payment pattern of each claim:

- The same (undiscounted) amount is paid out on all claims.
- Payments are made continuously from a common the time of loss, $t = 0$ to claim closure.

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[1] The author expresses his thanks to Greg Engl, also of NCCI, who reviewed many versions of this paper, pointed out some serious errors, and made numerous suggestions for improvements.

[2] We are most interested in the case when $b < \infty$ is finite, although most of what we say applies to the case $b = \infty$. We are, however, admittedly rather cavalier about making whatever assumptions are needed to assure that all improper integrals exist and are finite.

[3] Gillam and Couret [4] consider the reciprocal of the loss development factor and call it the loss development divisor.

We consider first the "flat" case when all payments are of the same amount. We then weaken that assumption in a couple ways: first we allow the payments to vary at a constant rate of inflation—this is called a case of "COLA". Second we allow a single fixed proportional change in the payment amount, applicable during the unit of time just prior to claim closure--called a case of "Step". Some simple numeric examples are followed through the three cases. We begin the discussion with a general model for paid loss development.

Notation and Setup

With S, F, $\widetilde{F}$ and b as above, we also let $f(t) = \frac{dF}{dt}$ be the probability density function [PDF], $h(t) = \frac{f(t)}{S(t)}$ the hazard rate function, $CV = \frac{\sigma}{\mu}$ the coefficient of variation, and T the random variable that gives the "time" of closure t. We use those same letter symbols and "transparent" notation to specify the relationship between these functions. For example $\widetilde{h}_\alpha(t)$ denotes the hazard rate function of the PLDD $\widetilde{F}_\alpha(t)$ that corresponds to the claim survival function $S_\alpha(t)$ and $\widetilde{T}_\alpha$ the random variable with CDF $\widetilde{F}_\alpha(t)$.

In each of the cases we consider, the complete payment pattern of a claim is completely determined by the claim duration. So we make the assumption that for any time t, 0<t<b, all claims with duration t have the same pre-determined and differentiable payment pattern. We can capture this mathematically by defining the function

$G(x,t)$=amount paid through time t on a claim, conditional upon claim duration=x.

Then define

$g(x,t)$=partial derivative of $G(x,t)$ with respect to t.

We may interpret $g(x,t)$ as the rate of payment at time t on any claim of duration x. Both $G(x,t)$ and $g(x,t)$ are defined for x,t in $(0,b)$. Note that for $t>x$ we have $g(x,t) = 0$ and $G(x,x) = G(x,t) = G(x,b)$= the ultimate incurred on any claim of duration x. In this paper we only consider the case when all claims have the same ultimate incurred cost. So without any real loss of generality we further make the assumption throughout the rest of this paper that $G(x,b) = 1$ for all x (see [1] for a consideration of the more general case).

As noted, we refer to the case when the rate of payment $g(x,t)$ does not vary with time t as the "flat case". The "COLA case" means the rate corresponds to a fixed rate of inflation or discount and the "step case" provides for a one-time change in the rate $g(x,t)$—the precise meaning of those assumptions provided in their respective sections of the paper. We consider the cumulative payment for such a claim distribution in which all claims occur at time t=0 and conforming to these assumptions (sort of an accident instant, as opposed to an accident year). The only "stochastic" ingredient in this model is claim duration, for which the distribution $F(t)$ is specified. Under these assumptions, $F(t)$ determines not just closures but all payments. There is a well-defined expected cumulative paid loss $P(t)$ at any time t, from t=0 to ultimate paid at t=b. Indeed, we have:

$$
\begin{aligned}
P(t) &= \int_0^t \int_0^b g(x,y) f(x)\,dx\,dy = \int_0^b f(x) \int_0^t g(x,y)\,dy\,dx = \int_0^b f(x) G(x,t)\,dx \\
&= \int_0^t f(x) G(x,t)\,dx + \int_t^b f(x) G(x,t)\,dx \\
&= \int_0^t f(x)\,dx + \int_t^b f(x) G(x,t)\,dx \\
&= F(t) + \int_t^b f(x) G(x,t)\,dx
\end{aligned}
$$

since $G(x,t) = 1$ for $t>x$. In particular, the expected ultimate loss per claim is normalized by our assumptions:

$$P(b) = \int_0^b f(x)\,dx = F(b) = 1$$

The (expected) ultimate paid loss development factor from time t is:

$$\lambda(t) = \frac{P(b)}{P(t)} = \frac{1}{P(t)}$$

and the inverse provides the PLDD on $(0,b)$ that is the focus of this study:

(*) $$\widetilde{F}(t) = P(t) = F(t) + \int_t^b f(x) G(x,t)\,dx$$

For the PDF of the PLDD, we have, by the fundamental theorem of calculus:

(**) $$\widetilde{f}(t) = \frac{d}{dt}\left(\int_0^t \int_0^b g(x,y) f(x)\,dx\,dy \right) = \int_0^b g(x,t) f(x)\,dx = \int_t^b g(x,t) f(x)\,dx$$

since $g(x,t) = 0$ for $t>x$.

Findings-Flat Case

In this section we assume a constant payment pattern. With the above notation, the following proposition documents some basic relationships between the duration density and the PLDD density:

Proposition 1: *Assume the "flat case" holds, then for* $t \in (0,b)$

i) $\tilde{F}(t) = F(t) + t\int_t^b \frac{f(x)}{x}dx = t\left(\frac{1}{b} + \int_t^b \frac{F(x)}{x^2}dx\right)$

ii) $\tilde{f}(t) = \int_t^b \frac{f(x)}{x}dx = \frac{1}{b} + \int_t^b \frac{F(x)}{x^2}dx - \frac{F(t)}{t} = \frac{S(t)-\tilde{S}(t)}{t} = \frac{\tilde{F}(t)-F(t)}{t}$

iii) $\tilde{h}(t) = \dfrac{\frac{S(t)}{\tilde{S}(t)} - 1}{t}$

iv) $E(\tilde{T}^k) = \frac{E(T^k)}{k+1} \quad k = 1,2,...$

v) $\tilde{S}(t) = t\int_t^b \frac{S(x)}{x^2}dx$

Proof: By our assumptions on the payment pattern, and using the above notation, we have:

$$G(x,t) = \begin{cases} \frac{t}{x} & 0 \le t \le x \\ 1 & x \le t \end{cases}$$

From that we confirm that:

$$g(x,t) = \begin{cases} \frac{1}{x} & 0 \le t \le x \\ 0 & x \le t \end{cases}$$

does not vary with t. The above equations (*) (**) show that in this flat case:

$$\tilde{F}(t) = F(t) + \int_t^b f(x)G(x,t)dx = F(t) + t\int_t^b \frac{f(x)}{x}dx$$

$$\tilde{f}(t) = \int_t^b g(x,t)f(x)dx = \int_t^b \frac{f(x)}{x}dx$$

Integration by parts gives:

$$\int_t^b \frac{f(x)}{x}dx = \int_t^b u dv \qquad u = x^{-1} \quad dv = f(x)dx \quad du = -x^{-2}dx \quad v = F(x)$$

$$= uv\Big]_t^b - \int_t^b v du = \frac{F(x)}{x}\Bigg]_t^b + \int_t^b \frac{F(x)}{x^2}dx = \frac{1}{b} - \frac{F(t)}{t} + \int_t^b \frac{F(x)}{x^2}dx$$

and we find that:

$$\widetilde{F}(t) = F(t) + t\int_t^b \frac{f(x)}{x}dx$$

$$= F(t) + t\left(\frac{1}{b} - \frac{F(t)}{t} + \int_t^b \frac{F(x)}{x^2}dx\right) = t\left(\frac{1}{b} + \int_t^b \frac{F(x)}{x^2}dx\right)$$

proving (i) and the first two equations in (ii). For the rest of (ii) and (iii) we observe that:

$$\widetilde{S}(t) = 1 - \widetilde{F}(t) = 1 - F(t) - t\int_t^b \frac{f(x)}{x}dx = S(t) - t\widetilde{f}(t).$$

Integration by parts also gives (c.f [2]):

$$E(T^k) = k\int_0^b t^{k-1}S(t)dt$$

And applying this to T and $\widetilde{T}$:

$$E(\widetilde{T}^k) = k\int_0^b t^{k-1}\widetilde{S}(t)dt = k\int_0^b t^{k-1}S(t) - t^k\widetilde{f}(t)dt = E(T^k) - kE(\widetilde{T}^k)$$

and (iv) holds. For (v), substitute $1-S$ for F in (ii):

$$S(t) - \widetilde{S}(t) = t\left(\frac{1}{b} + \int_t^b \frac{F(x)}{x^2}dx - \frac{F(t)}{t}\right) = t\left(\frac{1}{b} + \int_t^b \frac{1-S(x)}{x^2}dx - \frac{1-S(t)}{t}\right)$$

$$= t\left(\frac{1}{b} + \left[\frac{-1}{x}\right]_t^b - \int_t^b \frac{S(x)}{x^2}dx - \frac{1-S(t)}{t}\right) = t\left(-\int_t^b \frac{S(x)}{x^2}dx + \frac{S(t)}{t}\right)$$

$$\Rightarrow \widetilde{S}(t) = t\int_t^b \frac{S(x)}{x^2}dx$$

This completes the proof of Proposition 1.

Now we clearly have that the PDF $\widetilde{f}(t)$ is decreasing, indeed $\frac{d\widetilde{f}}{dt} = -\frac{f(t)}{t} \le 0$ and so the mode of the PLDD $\widetilde{F}(t)$ is 0. From the following Corollary, we see that the shift from $F(t)$ to $\widetilde{F}(t)$ shrinks the mean and increases the coefficient of variation, but the effect on the variance depends on value of CV ($\widetilde{\sigma} > \sigma \Leftrightarrow CV < \frac{1}{2}\sqrt{\frac{1}{2}}$).

Corollary 1.1:

i) $\tilde{\mu} = \frac{\mu}{2}$

ii) $\tilde{\sigma}^2 = \frac{\sigma^2}{3} + \frac{\mu^2}{12} = \frac{\sigma^2}{3}\left(1 + \frac{1}{4CV^2}\right)$

iii) $\tilde{CV} = \sqrt{\frac{4}{3}CV^2 + \frac{1}{3}}$

Proof: The proof is clear from the general observation that $\sigma^2 + \mu^2 = E(T^2)$ and Proposition 1 (iv).

In the WC work that motivated this, pension cases emerge as those that take longer to close and it is natural to try and use that as a way to isolate them. This leads us to consider what happens when there is a delay period to closure that applies to all pension claims, i.e. when $f(t) = 0$ for $t \in (0,a)$ where $0 \le a < b$. This is readily accommodated, as indicated in:

Corollary 1.2: *Suppose $f(t) = 0$ for $t \in (0,a)$ where $0 \le a < b$ then*

$$\tilde{F}(t) = \frac{t}{a}\tilde{F}(a) \quad \text{for } t \in (0,a).$$

Proof: Under these assumptions, Proposition 1 (i) implies that.

$$\tilde{F}(a) = F(a) + a\int_a^b \frac{f(x)}{x}dx = a\int_a^b \frac{f(x)}{x}dx$$

but then for $t \in (0,a)$:

$$\tilde{F}(t) = t\int_a^b \frac{f(x)}{x}dx = \left(\frac{t}{a}\right)\left(a\int_a^b \frac{f(x)}{x}dx\right) = \left(\frac{t}{a}\right)\tilde{F}(a).$$

Probably the most useful family of distributions defined on a finite interval is the class of Beta densities on (0,1). Recall that the Beta distribution is a two-parameter, α, β, distribution that is usually defined in terms of its PDF:

$$f(\alpha,\beta;x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} \quad x \in (0,1), \alpha > 0, \beta > 0$$

where B and Γ denote the usual Beta and Gamma functions (c.f. [1], [3]). The CDF of the Beta density is:

$$\mathrm{B}(\alpha,\beta;t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t x^{\alpha-1}(1-x)^{\beta-1}dx \quad t \in (0,1), \alpha > 0, \beta > 0$$

and we have:

Corollary 1.3: *For* $\alpha > 1, \beta > 0$ *let* $f(t) = f(\alpha, \beta; t), F(t) = \mathrm{B}(\alpha, \beta; t)$ *be the PDF and CDF of a beta density on* $(0,1)$*, as just defined, then:*

$$\tilde{f}(t) = \frac{\alpha+\beta-1}{\alpha-1}(1-\mathrm{B}(\alpha-1,\beta;t))$$
$$\tilde{F}(t) = \tilde{\mathrm{B}}(\alpha,\beta;t) = \mathrm{B}(\alpha,\beta;t) + \frac{(\alpha+\beta-1)t}{\alpha-1}(1-\mathrm{B}(\alpha-1,\beta;t)) \quad 0 < t < 1.$$

Proof: The proof is a straightforward application of Proposition 1. For the PDF:

$$\begin{aligned}
\tilde{f}(t) &= \int_t^1 \frac{f(x)}{x}dx = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\int_t^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{x}dx \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\int_t^1 x^{(\alpha-1)-1}(1-x)^{\beta-1}dx \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\left(\int_0^1 x^{(\alpha-1)-1}(1-x)^{\beta-1}dx - \int_0^t x^{(\alpha-1)-1}(1-x)^{\beta-1}dx\right) \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\left(\frac{\Gamma(\alpha-1)\Gamma(\beta)}{\Gamma(\alpha-1+\beta)} - \int_0^t x^{(\alpha-1)-1}(1-x)^{\beta-1}dx\right) \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\left(\frac{\Gamma(\alpha-1)\Gamma(\beta)}{\Gamma(\alpha-1+\beta)}\right)(1-\mathrm{B}(\alpha-1,\beta;t)) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-1)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}(1-\mathrm{B}(\alpha-1,\beta;t)) = \frac{\alpha+\beta-1}{\alpha-1}(1-\mathrm{B}(\alpha-1,\beta;t))
\end{aligned}$$

And for the CDF, Proposition 1 gives:

$$\tilde{F}(t) = F(t) + t\tilde{f}(t) = \mathrm{B}(\alpha,\beta;t) + \frac{(\alpha+\beta-1)t}{\alpha-1}(1-\mathrm{B}(\alpha-1,\beta;t))$$

as claimed.

We next consider some more specific examples:

Example 1: Consider the case when $S(t) = \dfrac{b-t}{b}, b < \infty,$ is a DeMoivre survival curve. In this case (because we will be returning to these examples by number, we specify them with a subscript):

$$F_1(b;t) = F(t) = \frac{t}{b} \quad f(t) = \frac{1}{b} \quad \tilde{F}(t) = \frac{t}{b}\left(1-\ln\left(\frac{t}{b}\right)\right) \quad 0 \le t \le b$$

Example 2: It is easy to generalize the first example in a couple of ways. Let $0 < a \le b < \infty$ where we let a represent a potentially earlier time at which all claims close and pick $\varphi > 0$. Then consider the case when claim closure has the CDF:

$$F_2(\varphi,a,b;t) = F(t) = \begin{cases} \left(\dfrac{t}{a}\right)^{\varphi} & t \le a \\ 1 & a \le t \le b \end{cases}$$

When $\varphi = 1$, we readily see from Example 1 that

$$\tilde{F}(t) = \begin{cases} \dfrac{t}{a}\left(1 - \ln\left(\dfrac{t}{a}\right)\right) & t \le a \\ 1 & a \le t \le b \end{cases}$$

When $\varphi \ne 1$, we let the reader verify that:

$$\tilde{F}(t) = \begin{cases} \dfrac{1}{\varphi - 1}\left(\dfrac{\varphi t}{a} - \left(\dfrac{t}{a}\right)^{\varphi}\right) & t \le a \\ 1 & a \le t \le b \end{cases}$$

Example 3: Consider the case when fewer claims close over time according to a linear pattern:

$$f_3(t) = \begin{cases} \dfrac{2(a-t)}{a^2} & t \le a \\ 0 & a \le t \le b < \infty \end{cases}$$

We leave to the reader the straightforward verification that $f_3(t)$ is indeed a PDF on $[0,b]$ and that:

$$F_3(a,b;t) = \begin{cases} \dfrac{t}{a}\left(2 - \dfrac{t}{a}\right) & t \le a \\ 1 & a \le t \le b \end{cases}$$

Then Proposition 1 implies:

$$\tilde{F}_3(t) = \begin{cases} \dfrac{t}{a}\left(\dfrac{t}{a} - 2\ln\left(\dfrac{t}{a}\right)\right) & t \le a \\ 1 & a \le t \le b \end{cases}$$

While we are primarily interested in the case of finite support, it may be useful to consider a couple of examples when $b = \infty$.

Example 4: Consider the case of a single parameter Pareto (c.f. [6], p 584):

$$F_4(\alpha,\theta;t) = F(t) = 1 - \left(\frac{\theta}{t}\right)^{\alpha} \qquad f_4(\alpha,\theta;t) = f(t) = \frac{\alpha\theta^{\alpha}}{t^{\alpha+1}} \quad for\ t > \theta.$$

It is natural to extend the definition of the PDF to assign $f(t) = 0$ for $t \in (0,\theta]$. For $t \ge \theta$, Proposition 1 gives:

$$\tilde{F}(t) = F(t) + t\int_t^{\infty} \frac{f(x)}{x}dx = 1 - \left(\frac{\theta}{t}\right)^{\alpha} + t\int_t^{\infty} \frac{\alpha\theta^{\alpha}}{x^{\alpha+2}}dx = 1 - \left(\frac{\theta}{t}\right)^{\alpha} + \alpha\theta^{\alpha} t \frac{1}{(\alpha+1)t^{\alpha+1}}$$

$$= 1 - \frac{\left(\frac{\theta}{t}\right)^{\alpha}}{\alpha+1}.$$

And by Corollary 1.2 we have:

$$\tilde{F}_4(\alpha,\theta;t) = \begin{cases} \dfrac{\alpha t}{(\alpha+1)\theta} & 0 < t \le \theta \\ 1 - \dfrac{\left(\frac{\theta}{t}\right)^{\alpha}}{\alpha+1} = F_4(\alpha,\vartheta;t) & \theta \le t, \vartheta = \theta(\alpha+1)^{-\frac{1}{\alpha}} \end{cases}$$

For the final example, we recall the following integration formula (c.f. [6] page 570):

$$E_1(t) = \int_t^{\infty} \frac{e^{-u}}{u}du = -\gamma - \ln(t) - \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n \cdot n!}$$

Where $\gamma = 0.577215...$ is Euler's constant.

Example 5: Consider the case when claim closures follow an exponential density, so here again $b = \infty$. In this case, we have:

$$F_5(\theta;t) = F(t) = 1 - e^{-\frac{t}{\theta}}$$

$$f_5(\theta;t) = f(t) = \frac{e^{-\frac{t}{\theta}}}{\theta}$$

Then from Proposition 1 we have:

$$\tilde{f}_5(\theta;t) = \tilde{f}(t) = \int_t^{\infty} \frac{f(x)}{x}dx = \int_t^{\infty} \frac{e^{-\frac{x}{\theta}}}{\theta x}dx = \frac{1}{\theta}\int_t^{\infty} \frac{e^{-\frac{x}{\theta}}}{\frac{x}{\theta}}\frac{dx}{\theta} = \frac{1}{\theta}\int_{\frac{t}{\theta}}^{\infty} \frac{e^{-u}}{u}du = \frac{E_1\left(\frac{t}{\theta}\right)}{\theta}$$

$$\tilde{F}_5(\theta;t) = \tilde{F}(t) = F(t) + t\tilde{f}(t) = 1 - e^{-\frac{t}{\theta}} + \left(\frac{t}{\theta}\right)E_1\left(\frac{t}{\theta}\right)$$

Findings-COLA Case

In this section we replace the assumption of a flat payment rate with the assumption that payments are subject to a constant proportional adjustment equal to $1+\delta$ per unit of time. Payments are still assumed to be made continuously over the interval from the time of loss, $t=0$, to claim closure. Since we have considered the case $\delta=0$ in the previous section, we will assume throughout this section that $\delta \neq 0$. It is more convenient to express findings in terms of the force of interest $\gamma=\ln(1+\delta)\neq 0$ (c.f. [5]).

Under these "COLA" case assumptions, it is again straightforward—just a bit messier--to determine the PLDD, again denoted $\widetilde{F}(t)$, on the time interval $(0,b)$. The basic result is:

Proposition 2: *Assume the "COLA case" holds, then for* $t\in(0,b)$

i) $\widetilde{F}(t)=F(t)+\left(e^{\gamma t}-1\right)\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx$

ii) $\widetilde{f}(t)=\gamma e^{\gamma t}\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx=\frac{\gamma(S(t)-\widetilde{S}(t))}{1-e^{-\gamma t}}$

iii) $\widetilde{h}(t)=\dfrac{\gamma\left(\frac{S(t)}{\widetilde{S}(t)}-1\right)}{1-e^{-\gamma t}}$

Proof: Observe that for any claim with closure at x, the amount paid to time $t\leq x$ is in constant proportion to the continuous annuity (c.f. [5]):

$$\int_0^t (1+\delta)^w\,dw=\int_0^t e^{\gamma w}\,dw=\frac{e^{\gamma t}-1}{\gamma}$$

By our assumptions on the payment pattern, then, for any claim with closure at x, we have:

$$G(x,t)=\begin{cases}\dfrac{e^{\gamma t}-1}{e^{\gamma x}-1} & 0\leq t\leq x\\ 1 & x\leq t\end{cases}$$

We again employ the earlier formula (*) for $\widetilde{F}(t)$:

$$\widetilde{F}(t)=P(t)=F(t)+\int_t^b f(x)G(x,t)dx=F(t)+\left(e^{\gamma}-1\right)\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx$$

And for the PDF's we have:

$$g(x,t)=\begin{cases}\dfrac{\gamma e^{\gamma t}}{e^{\gamma x}-1} & 0\le t\le x\\ 0 & x\le t\end{cases}$$

and by (**):

$$\tilde{f}(t)=\int_t^b g(x,t)f(x)dx=\gamma e^{\gamma t}\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx$$

The second equation in (ii) now follows from:

$$\tilde{S}(t)=1-\tilde{F}(t)=1-F(t)-\left(e^{\gamma t}-1\right)\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx = S(t)-\left(\frac{e^{\gamma t}-1}{\gamma e^{\gamma t}}\right)\tilde{f}(t).$$

Finally, we have:

$$\tilde{h}(t)=\frac{\tilde{f}(t)}{\tilde{S}(t)}=\frac{\left(S(t)-\tilde{S}(t)\right)\left(\dfrac{\gamma}{1-e^{-\gamma t}}\right)}{\tilde{S}(t)}=\frac{\gamma\left(\dfrac{S(t)}{\tilde{S}(t)}-1\right)}{1-e^{-\gamma t}}.$$

This completes the proof of Proposition 2.

Corollary 2.1: $\tilde{\mu}-\mu=\dfrac{M_{\tilde{T}}(-\gamma)-1}{\gamma}$ *where* $M_{\tilde{T}}(z)$ *is the moment generating function of the PLDD*

Proof: We have:

$$\tilde{S}(t)-S(t)=-\left(\frac{e^{\gamma t}-1}{\gamma e^{\gamma t}}\right)\tilde{f}(t)=\frac{-1+e^{-\gamma t}}{\gamma}\tilde{f}(t)$$

and so:

$$\tilde{\mu}-\mu=\int_0^b \tilde{S}(t)-S(t)dt=\int_0^b\left(\frac{e^{-\gamma t}-1}{\gamma}\right)\tilde{f}(t)dt=\frac{\int_0^b e^{-\gamma t}\tilde{f}(t)dt-1}{\gamma}=\frac{E\left(e^{-\gamma\tilde{T}}\right)-1}{\gamma}$$

and the result follows.

In the COLA case, we may regard the PLDD $\tilde{F}(t)=\tilde{F}(\gamma,t)$ as a function of γ, and we have:

$$
\begin{aligned}
\frac{\partial \widetilde{F}}{\partial \gamma} &= \frac{\partial}{\partial \gamma}\left(F(t) + \left(e^{\gamma t}-1\right)\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx \right) = \left(e^{\gamma t}-1\right)\frac{\partial}{\partial \gamma}\left(\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx \right) + \left(\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx \right)\frac{\partial}{\partial \gamma}\left(e^{\gamma t}-1\right) \\
&= \left(e^{\gamma t}-1\right)\left(\int_t^b \frac{\partial}{\partial \gamma}\left(\frac{f(x)}{e^{\gamma x}-1} \right)dx \right) + te^{\gamma t}\left(\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx \right) = \left(1-e^{\gamma t}\right)\int_t^b \frac{xe^{\gamma x}}{\left(e^{\gamma x}-1\right)^2} f(x)dx + te^{\gamma t}\left(\int_t^b \frac{f(x)}{e^{\gamma x}-1}dx \right) \\
&= \int_t^b \frac{xe^{\gamma x} - te^{\gamma t} + (t-x)e^{\gamma(t+x)}}{\left(e^{\gamma x}-1\right)^2} f(x)dx.
\end{aligned}
$$

This can be used to formally prove what is intuitively rather evident, namely that $\widetilde{F}(t) = \widetilde{F}(\gamma, t)$ is a decreasing function of γ. Indeed, for any $\alpha > 0$:

$$
e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2} + \ldots + \frac{\alpha n}{n!} + \ldots > 1 + \alpha
$$

Then for all $x \geq t > 0$:

$$
\begin{aligned}
& e^{\gamma t} > 1 + \gamma t \Rightarrow \gamma\left(x - t + \frac{t}{1-e^{\gamma t}} \right) + 1 > 0 \\
& 1 - e^{\gamma t} < 0 \Rightarrow \gamma\left(x - t + \frac{t}{1-e^{\gamma t}} \right)\left(1-e^{\gamma t}\right) + \left(1-e^{\gamma t}\right) < 0 \\
& \Rightarrow \gamma x + 1 + \gamma t e^{\gamma t} - \gamma x e^{\gamma t} - e^{\gamma t} = \gamma\left(x - t + \frac{t}{1-e^{\gamma t}} \right)\left(1-e^{\gamma t}\right) + \left(1-e^{\gamma t}\right) < 0
\end{aligned}
$$

Fix t and consider the function:

$$
h(x) = xe^{\gamma x} - te^{\gamma t} + (t-x)e^{\gamma(t+x)} \qquad x \geq t.
$$

Observe that $h(x)$ has the same sign as the integrand in the expression for $\frac{\partial \widetilde{F}}{\partial \gamma}$. But we have:

$$
\frac{dh}{dx} = \gamma x + 1 + \gamma t e^{\gamma t} - \gamma x e^{\gamma t} - e^{\gamma t} < 0
$$

$$
h(t) = 0 \Rightarrow h(x) < 0 \quad \text{for } x \geq t
$$

and it follows that $\frac{\partial \widetilde{F}}{\partial \gamma} \leq 0$.

Observe that:

$$
\widetilde{F}(t) \geq F(t) \Rightarrow \widetilde{S}(t) \leq S(t) \Rightarrow \widetilde{\mu} = \int_0^b \widetilde{S}(t)dt \leq \int_0^b S(t)dt = \mu
$$

with equality only when $\mu = 0$. Combining these observations with the flat case, we have:

Corollary 2.2: *Assume* $\mu > 0$

$$\frac{\mu}{2} < \tilde{\mu} < \mu \Leftrightarrow \gamma > 0$$

$$\frac{\mu}{2} = \tilde{\mu} \qquad \Leftrightarrow \gamma = 0$$

$$0 < \tilde{\mu} < \frac{\mu}{2} \Leftrightarrow \gamma < 0$$

Proof: Indeed, consider the function $g(\gamma) = \tilde{\mu} = \int_0^b \tilde{S}(t)dt$. The above shows that $\tilde{F}(t) = \tilde{F}(\gamma, t)$ is a strictly decreasing function of γ when $\mu > 0$. But then $\tilde{S} = 1 - \tilde{F}$ is a strictly increasing function of γ. It follows that $g(\gamma)$ is monotonic increasing when $\mu > 0$. But we know from the flat case that $g(0) = \frac{\mu}{2}$ and Corollary 2.2 follows.

Note that as $\gamma \to \infty$ the PLDD distribution reflects payments more concentrated at time of closure, making that distribution approximate the distribution of closures, and we would expect $\lim_{\gamma \to \infty} \tilde{\mu} = \mu$. On the other hand, as $\gamma \to -\infty$ the PLDD $\tilde{F}(t)$ reflects payments becoming concentrated at time 0, suggesting that $\lim_{\gamma \to -\infty} \tilde{\mu} = 0$. More formally, we have:

Corollary 2.3: $\lim_{\gamma \to \infty} \tilde{\mu} = \mu$ *and* $\lim_{\gamma \to -\infty} \tilde{\mu} = 0$.

Proof: First assume $b < \infty$. Notice that for $0<t<x$:

$$\lim_{\gamma \to \infty} G(x,t) = \lim_{\gamma \to \infty} \frac{e^{\gamma t} - 1}{e^{\gamma x} - 1} = \lim_{\gamma \to \infty} \frac{e^{\gamma t}}{e^{\gamma x}} = \lim_{\gamma \to \infty} e^{\gamma (t-x)} = 0$$

Now $\tilde{S}(t)$ is continuous on $(0,b)$ and with $\tilde{S}(0) = 1, \tilde{S}(b) = 0$ it follows that $\tilde{S}(t)$ is uniformly continuous on the finite interval $[0,b]$. But then by Proposition 2 (i):

$$t > 0 \Rightarrow \lim_{\gamma \to \infty} \tilde{F}(t) = F(t) \Rightarrow \lim_{\gamma \to \infty} \tilde{S}(t) = S(t)$$

$$\Rightarrow \lim_{\gamma \to \infty} \tilde{\mu} = \lim_{\gamma \to \infty} \int_0^b \tilde{S}(t)dt = \int_0^b \left(\lim_{\gamma \to \infty} \tilde{S}(t) \right) dt = \int_0^b S(t)dt = \mu$$

Proposition 2 (i) also implies that

$$t > 0 \Rightarrow \lim_{\gamma \to -\infty} \tilde{F}(t) = F(t) + \lim_{\gamma \to -\infty} \left(e^{\gamma t} - 1\right) \int_t^b \frac{f(x)}{e^{\gamma x} - 1} dx$$

$$= F(t) + \lim_{\gamma \to -\infty} (-1) \int_t^b \frac{f(x)}{(-1)} dx = F(t) + \int_t^b f(x) dx = 1.$$

Whence:

$$\lim_{\gamma \to -\infty} \tilde{\mu} = \lim_{\gamma \to -\infty} \int_0^b \tilde{S}(t)dt = \int_0^b \left(\lim_{\gamma \to -\infty} \tilde{S}(t) \right) dt = \int_0^b 0\, dt = 0$$

For the case $b = \infty$, consider the "restriction" of the density $f(t)$ to $f_a(t) = \frac{f(t)}{F(a)}$ on the finite interval $[0,a]$. Note that Proposition 2 (i) implies that $\tilde{f}_a(t) = \frac{\tilde{f}(t)}{F(a)}$ which in turn implies that $\lim_{a\to\infty} \tilde{f}_a(t) = \tilde{f}(t)$. From the finite case we have for any $a>0$:

$$\lim_{\gamma\to\infty} \tilde{\mu}_a = \mu_a$$

Notice that:

$$\lim_{a\to\infty} \mu_a = \lim_{a\to\infty} \int_0^a xfa(x)dx = \lim_{a\to\infty} \int_0^a x\frac{f(x)}{F(a)}dx = \lim_{a\to\infty} \frac{1}{F(a)}\int_0^a xf(x)dx$$

$$= \lim_{a\to\infty} \frac{1}{F(a)} \lim_{a\to\infty} \int_0^a xf(x)dx = 1\int_0^\infty xf(x)dx = \mu$$

and by the same argument:

$$\lim_{a\to\infty} \tilde{\mu}_a = \tilde{\mu}.$$

Consider the mean of the restriction $g_2(a) = \mu_a$ as defining a function of a. It is intuitively clear that $g_2(a)$ is non-decreasing (adding larger observations cannot lower the mean); to verify this formally, note that

$$g_2(a) = \mu_a = \int_0^a \frac{xf(x)}{F(a)}dx = \frac{1}{F(a)}\int_0^a xf(x)dx$$

$$\Rightarrow \frac{dg_2}{da} = \frac{1}{F(a)}af(a) + \int_0^a xf(x)dx(-1)F(a)^{-2}f(a) = \frac{f(a)}{F(a)}\left(a - \frac{\int_0^a xf(x)dx}{F(a)}\right)$$

$$= \frac{f(a)}{F(a)}\left(a - \frac{\alpha\int_0^a f(x)dx}{F(a)}\right) \quad \text{for some } \alpha \in [0,a]$$

$$= \frac{f(a)}{F(a)}(a-\alpha) \geq 0.$$

Define the function $g(\gamma,a) = \tilde{\mu}_a$ and $g_1(\gamma) = \lim_{a\to\infty} g(\gamma,a) = \tilde{\mu}$. As in the proof of Corollary 2.2, $g(\gamma,a)$ is a non-decreasing function of γ and from we have just noted $g(\gamma,a)$ is also a non-decreasing function of a:

$$\frac{\partial g}{\partial \gamma} \geq 0 \quad \frac{\partial g}{\partial a} \geq 0.$$

Now we clearly have $g(\gamma,a) \leq \mu$ for all γ and a. So, by way of contradiction, suppose $\mu > \lim_{\gamma\to\infty} \tilde{\mu} = \lim_{\gamma\to\infty} g_1(\gamma) = \lim_{\gamma\to\infty}\lim_{a\to\infty} g(\gamma,a)$. That would imply that there is $\varepsilon > 0$ such that $g(\gamma,a) \leq \mu - \frac{\varepsilon}{2}$ for all γ and a. But then

$$\lim_{a\to\infty} \mu_a = \mu \Rightarrow \exists a_0 \text{ such that } \left|\mu - \mu_{a_0}\right| < \frac{\varepsilon}{4}$$

and by the finite case:

$$\lim_{\gamma\to\infty} g(\gamma,a_0) = \lim_{\gamma\to\infty} \tilde{\mu}_{a_0} = \mu_{a_0} \quad \Rightarrow \exists \gamma_0 \text{ such that } \left|g(\gamma_0,a_0) - \mu_{a_0}\right| < \frac{\varepsilon}{4}.$$

But then we have:

$$\left|g(\gamma_0,a_0) - \mu\right| = \left|g(\gamma_0,a_0) - \mu_{a_0} + \mu_{a_0} - \mu\right| \leq \left|g(\gamma_0,a_0) - \mu_{a_0}\right| + \left|\mu_{a_0} - \mu\right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

$$\Rightarrow g(\gamma_0,a_0) > \mu - \frac{\varepsilon}{2}.$$

This contradiction shows that $\lim_{\gamma\to\infty} g(\gamma) = \mu$. Finally, suppose that $\lim_{\gamma\to-\infty} g(\gamma) = \alpha > 0$. This implies that:

$$\exists \gamma_0 \text{ such that } g(\gamma) > \frac{\alpha}{2} \quad \text{for all } \gamma \geq \gamma_0$$

$$\text{and } \mu = \int_0^\infty S(t)dt \Rightarrow \exists a_0 \text{ such that } \int_{a_0}^\infty S(t)dt < \frac{\alpha}{6}$$

Also

$$\lim_{\gamma\to-\infty} \tilde{S}\left(\frac{\alpha}{6}\right) = 0 \Rightarrow \exists \gamma_1 \text{ such that } \tilde{S}\left(\frac{\alpha}{6}\right) < \frac{\alpha}{6a_0} \quad \text{for all } \gamma \geq \gamma_0$$

$$\Rightarrow \tilde{S}(t) < \frac{\alpha}{6a_0} \quad \text{for } t \in \left[\frac{\alpha}{6}, a_0\right] \quad \text{and } \gamma \geq \gamma_0$$

Selecting $\gamma > Max(\gamma_0,\gamma_1)$, we find that:

$$\frac{\alpha}{2} < g(\gamma) = \int_0^\infty \tilde{S}(t)dt = \int_0^{\frac{\alpha}{6}} \tilde{S}(t)dt + \int_{\frac{\alpha}{6}}^{a_0} \tilde{S}(t)dt + \int_{a_0}^\infty \tilde{S}(t)dt$$

$$\leq \frac{\alpha}{6} + \frac{\alpha}{6a_0}\left(a_0 - \frac{\alpha}{6}\right) + \int_{a_0}^\infty S(t)dt \leq \frac{\alpha}{6} + \frac{\alpha}{6} + \frac{\alpha}{6} = \frac{\alpha}{2}$$

This contradiction shows that $\lim_{\gamma\to-\infty} g(\gamma) = 0$. This completes the proof of Corollary 2.3.

Example 1 (COLA case): Recall that here $S(t)$ is the DeMoivre survival curve $F_1(t)=\frac{t}{b}, f_1(t)=\frac{1}{b}, b<\infty$ and so Proposition 2 gives:

$$\tilde{F}_1(t) = F_1(t) + \left(e^{\gamma t}-1\right)\int_t^b \frac{f_1(x)}{e^{\gamma x}-1}dx = \frac{t}{b} + \frac{e^{\gamma t}-1}{b}\int_t^b \frac{dx}{e^{\gamma x}-1}.$$

Notice that setting

$$g(x) = \ln\left(\frac{e^{\gamma x}-1}{e^{\gamma x}}\right) = \ln\left(e^{\gamma x}-1\right) - \gamma x \quad \Rightarrow \frac{dg}{dx} = \frac{\gamma e^{\gamma x}}{e^{\gamma x}-1} - \gamma = \frac{\gamma e^{\gamma x} - \gamma(e^{\gamma x}-1)}{e^{\gamma x}-1} = \frac{\gamma}{e^{\gamma x}-1}$$

and we have the formula:

$$\begin{aligned}\tilde{F}_1(t) &= \frac{t}{b} + \frac{e^{\gamma t}-1}{\gamma b}\int_t^b \frac{\gamma dx}{e^{\gamma x}-1} = \frac{t}{b} + \frac{e^{\gamma t}-1}{\gamma b}\left(g(b)-g(t)\right) \\ &= \frac{t}{b} + \frac{e^{\gamma t}-1}{\gamma b}\left(\ln\left(\frac{e^{\gamma b}-1}{e^{\gamma t}-1}\right) - \gamma(b-t)\right)\end{aligned}$$

Example 2 (COLA case): Recall that

$$F_2(\varphi,a,b;t) = F(t) = \begin{cases}\left(\frac{t}{a}\right)^{\varphi} & t \le a \\ 1 & a \le t \le b\end{cases}$$

then $\tilde{F}_2(t) \ge F_2(t) = 1$ for $t \ge a$ and for $t \le a$:

$$\tilde{F}_2(t) = F_2(t) + \left(e^{\gamma t}-1\right)\int_t^a \frac{f_2(x)}{e^{\gamma x}-1}dx = \left(\frac{t}{a}\right)^{\varphi} + \frac{\varphi\left(e^{\gamma t}-1\right)}{a^{\varphi}}\int_t^a \frac{x^{\varphi-1}}{e^{\gamma x}-1}dx.$$

Example 3 (COLA case): Recall that

$$f_3(t) = \begin{cases}\frac{2(a-t)}{a^2} & t \le a \\ 0 & a \le t \le b < \infty\end{cases}$$

$$F_3(a,b;t) = \begin{cases}\frac{t(2a-t)}{a^2} & t \le a \\ 1 & a \le t \le b\end{cases}$$

then $\tilde{F}_3(t) \ge F_3(t) = 1$ for $t \ge a$ and for $t \le a$:

$$\tilde{F}_3(t) = F_3(t) + \left(e^{\gamma t}-1\right)\int_t^a \frac{f_3(x)}{e^{\gamma x}-1}dx = F_3(t) + \frac{2\left(e^{\gamma t}-1\right)}{a^2}\int_t^a \frac{a-x}{e^{\gamma x}-1}dx$$

$$= \frac{1}{a^2}\left(t(2a-t) + 2a\left(\frac{e^{\gamma t}-1}{\gamma}\right)\int_t^a \frac{\gamma dx}{e^{\gamma x}-1} - 2\left(\frac{e^{\gamma t}-1}{\gamma^2}\right)\int_t^a \frac{\gamma^2 x}{e^{\gamma x}-1}dx\right)$$

The function $\operatorname{dilog}(x) = \int_1^x \frac{\ln(t)}{1-t}\,dt$ is useful for evaluating $\tilde{F}_3(t)$ because of the integral formula:

$$\int \frac{x}{e^x - 1}\,dx = -\operatorname{dilog}\left(e^x\right) - \frac{x^2}{2}.$$

Combining this formula with what was observed in Example 1(COLA), it can be verified that:

$$\tilde{F}_3(t) = \frac{1}{a^2}\left(t(2a-t) + 2a\left(\frac{e^{\gamma t}-1}{\gamma}\right)\int_t^a \frac{\gamma dx}{e^{\gamma x}-1} - 2\left(\frac{e^{\gamma t}-1}{\gamma^2}\right)\int_t^a \frac{\gamma^2 x}{e^{\gamma x}-1}dx \right)$$

$$= \frac{1}{a^2}\left(\begin{array}{l} t(2a-t) + 2a\left(\dfrac{e^{\gamma t}-1}{\gamma}\right)\left(\ln\left(\dfrac{e^{\gamma a}-1}{e^{\gamma t}-1}\right) - \gamma(a-t)\right) \\ \qquad + 2\left(\dfrac{e^{\gamma t}-1}{\gamma^2}\right)\left(\operatorname{dilog}\left(e^{\gamma a}\right) - \operatorname{dilog}\left(e^{\gamma t}\right) + \dfrac{\gamma^2}{2}\left(a^2 - t^2\right)\right) \end{array} \right)$$

The following table provides values of the tail factor $\lambda = \tilde{F}_3(N)^{-1}$ at various values of δ, N and a; it provides some quantification of the sensitivity of the tail factor to inflation:

COLA Case			
δ	N	a	$\tilde{F}_3(N)^{-1}$
-0.05	10	40	1.222
-0.05	20	40	1.032
-0.05	30	40	1.002
0	10	40	1.323
0	20	40	1.06
0	30	40	1.006
+0.05	10	40	1.431
+0.05	20	40	1.094
+0.05	30	40	1.011

Findings-Step Case

In this section we replace the assumption of a flat payment pattern with the assumption that payments are at a constant rate β during the last unit of time (i.e. the interval $(x-1,x)$ prior to closure at x), and otherwise at the constant rate α. We also require that the ratio $\rho = \dfrac{\beta}{\alpha}$ is the same for all claims. Payments are still assumed to be made continuously over the interval from the time of loss, $t = 0$, to claim closure. Obviously, the "flat case" is just the special case $\alpha = \beta$ of this "step case". We assume b>1 in this step case section, as otherwise this would reduce to the flat case.

Under these assumptions, it is again straightforward—but messier still--to determine the PLDD $\tilde{F}(t)$, on the time interval $(0, b)$. Indeed, by our assumptions on the payment pattern, for any claim with closure at $x \le 1$, payments are at the rate $\beta = \beta_x = \dfrac{1}{x}$ and the payment pattern implies

$$G(x,t) = \begin{cases} \dfrac{t}{x} & t \le x \\ 1 & x \le t \end{cases}$$

while for $x \ge 1$:

$$1 = \int_0^{x-1} \alpha_x + \int_{x-1}^{x} \beta_x = \alpha_x(x-1) + \rho\alpha_x \Rightarrow \alpha_x = \frac{1}{x+\rho-1}$$

and we find that:

$$G(x,t) = \begin{cases} \alpha_x t = \dfrac{t}{x+\rho-1} & t \le x-1 \\ \alpha_x(x-1) + \beta_x(t-x+1) = 1 - \rho\left(\dfrac{x-t}{x+\rho-1}\right) & x-1 \le t \le x \\ 1 & x \le t \end{cases}$$

A straightforward verification, again using equation (*) and the fact that $f(x)=0$ for $x>b$, yields:

$$\tilde{F}(t) = \begin{cases} F(t+1) + \displaystyle\int_t^1 \frac{(t-x)f(x)}{x}dx + \rho\int_1^{t+1} \frac{(t-x)f(x)}{x+\rho-1}dx + t\int_{t+1}^{b} \frac{f(x)}{x+\rho-1}dx & t \le 1 \\ F(t+1) + \rho\displaystyle\int_t^{t+1} \frac{(t-x)f(x)}{x+\rho-1}dx + t\int_{t+1}^{b} \frac{f(x)}{x+\rho-1}dx & 1 \le t \end{cases}$$

or:

$$\tilde{F}(t) = F(t+1) + \delta(t)\int_t^1 \frac{(t-x)f(x)}{x}dx + \rho\int_{Max(1,t)}^{t+1} \frac{(t-x)f(x)}{x+\rho-1}dx + t\int_{t+1}^{b} \frac{f(x)}{x+\rho-1}dx$$

where δ is the characteristic function of the interval (0,1), i.e. $\delta(t)=1$ on the interval (0,1) and is 0 elsewhere. Note that the function $\tilde{F}(t)$ is continuous, even though δ is not. Note too that the last integral in the formula vanishes when $t>b-1$ and in that case the upper limit of the middle integral can be shifted down to b—this observation is helpful when the functional form of $f(t)$ only behaves on $(0,b)$.

In the step case, we may regard the PLDD $\tilde{F}(t)=\tilde{F}(\rho,t)$ as a function of ρ and we have:

$$\frac{\partial\tilde{F}}{\partial\rho} = \frac{\partial}{\partial\rho}\left(F(t+1)+\delta(t)\int_t^1\frac{f(x)}{x}dx + \rho\int_{Max(1,t)}^{t+1}\frac{(t-x)f(x)}{x+\rho-1}dx + t\int_{t+1}^{b}\frac{f(x)}{x+\rho-1}dx\right)$$

$$= \rho\int_{Max(1,t)}^{t+1}\frac{\partial}{\partial\rho}\left(\frac{1}{x+\rho-1}\right)(t-x)f(x)dx + \int_{Max(1,t)}^{t+1}\frac{(t-x)f(x)}{x+\rho-1}dx + t\int_{t+1}^{b}\frac{\partial}{\partial\rho}\left(\frac{1}{x+\rho-1}\right)f(x)dx$$

$$= \int_{Max(1,t)}^{t+1}\left(\rho\ln(x+\rho-1)+\frac{1}{x+\rho-1}\right)(t-x)f(x)dx + t\int_{t+1}^{b}\ln(x+\rho-1)f(x)dx.$$

As the following examples illustrate, the integral form for $\tilde{F}(t)$ may be preferable to some closed form expressions, especially when there is access to decent numerical integration software. In the examples, we set $\hat{t} = Max(1,t) = t + \delta(t)(1-t)$.

Example 1 (Step case): Recall that here $F_1(t)=\frac{t}{b}$, $f_1(t)=\frac{1}{b}$, $b<\infty$, then for $t\le b-1$:

$$\tilde{F}_1(t) = \frac{1}{b}\left(\begin{array}{c} t+1+\delta(t)(t-1+t\ln t) \\ +\rho\left((t+\rho-1)\ln\left(\frac{t+\rho}{\hat{t}+\rho-1}\right)+\hat{t}-t-1\right) \\ +t\ln\left(\frac{b+\rho-1}{t+\rho}\right)\end{array}\right)$$

and for $t\ge b-1$:

$$\tilde{F}_1(t) = \frac{1}{b}\left(\begin{array}{c} t+1+\rho\hat{t}+\delta(t)(t-1+t\ln t) \\ +\rho\left((t+\rho-1)\ln\left(\frac{b+\rho-1}{\hat{t}+\rho-1}\right)\right)\end{array}\right)-\rho$$

Example 2 (Step case): Recall that

$$F_2(\varphi,a,b;t)=F(t)=\begin{cases}\left(\dfrac{t}{a}\right)^{\varphi} & t\le a\\ 1 & a\le t\le b\end{cases}$$

then $\tilde{F}_2(t)\ge F_2(t)=1$ for $t\ge a$ and for $t\le a$. When φ is a positive integer >1 and $t\le a-1$, we have:

$$\begin{aligned}\tilde{F}_2(t)=&\left(\frac{t+1}{a}\right)^{\varphi}+\frac{\delta(t)}{a^{\varphi}(\varphi-1)}\left(\varphi(t-1)-t^{\varphi}+1\right)\\&+\left(\frac{\varphi}{a^{\varphi}}\right)\sum_{i=1}^{\varphi-1}\frac{(1-\rho)^{\varphi-i-1}\left((\rho-1)(\rho+t)(t+1)^{i}-\rho(t+\rho-1)\hat{t}^{i}+ta^{i}\right)}{i}\\&+\left(\frac{\varphi}{a^{\varphi}}\right)\left(\rho(t+\rho-1)(1-\rho)^{\varphi}\ln\left(\frac{t+\rho}{\hat{t}+\rho-1}\right)+t(1-\rho)^{\varphi-1}\ln\left(\frac{a+\rho-1}{t+\rho}\right)\right)\end{aligned}$$

When φ is a positive integer and $t\ge a-1$, we have:

$$\begin{aligned}\tilde{F}_2(t)=&\left(\frac{t+1}{a}\right)^{\varphi}+\frac{\delta(t)}{a^{\varphi}(\varphi-1)}\left(\varphi(t-1)-t^{\varphi}+1\right)\\&+\left(\frac{\varphi\rho}{a^{\varphi}}\right)\sum_{i=1}^{\varphi-1}\frac{t(1-\rho)^{\varphi-i-1}\left(a^{i}-\hat{t}^{i}\right)}{i}\\&+\left(\frac{\varphi\rho}{a^{\varphi}}\right)\left(\frac{\hat{t}^{\varphi}-a^{\varphi}}{\varphi}\right)\\&+\left(\frac{\varphi\rho}{a^{\varphi}}\right)\left((t+\rho-1)(1-\rho)^{\varphi}\ln\left(\frac{a+\rho-1}{\hat{t}+\rho-1}\right)\right)\end{aligned}$$

Example 3 (Step case): Recall that

$$f_3(t)=\begin{cases}\dfrac{2(a-t)}{a^2} & t\le a\\ 0 & a\le t\le b<\infty\end{cases}$$

$$F_3(a,b;t)=\begin{cases}\dfrac{t(2a-t)}{a^2} & t\le a\\ 1 & a\le t\le b\end{cases}$$

then $\tilde{F}_3(t)\ge F_3(t)=1$ for $t\ge a$. When $t\le a-1$ we have:

$$\tilde{F}_3(t)=\frac{1}{a^2}\left(a^2-\hat{t}^2+2(1-\rho-a-t)(a-\hat{t})+2(\rho+t-1)(a+\rho-1)\ln\left(\frac{t+\rho}{\hat{t}+\rho-1}\right)\right)$$

and when $t\ge a-1$:

$$\widetilde{F}_3(t)=\left(\frac{t+1}{a}\right)\left(2-\left(\frac{t+1}{a}\right)\right)+\frac{2\delta(t)}{a^2}\left(\frac{1}{2}-a-t-at\ln t+(a+t)t-\frac{t^2}{2}\right)$$
$$+\frac{\rho}{a^2}\left(a^2-\hat{t}^2+2(1-\rho-a-t)(a-\hat{t})+2(\rho+t-1)(a+\rho-1)\ln\left(\frac{a+\rho-1}{\hat{t}+\rho-1}\right)\right)$$

The following table provides values of $\lambda=\widetilde{F}_3(N)^{-1}$ at various values of ρ, N and a; it provides some quantification of the sensitivity of the tail factor to a change of payment in the terminating year.

Step Case			
ρ	N	a	$\widetilde{F}_3(N)^{-1}$
1/2	10	40	1.307
1/2	20	40	1.056
1/2	30	40	1.005
1	10	40	1.323
1	20	40	1.06
1	30	40	1.006
2	10	40	1.354
2	20	40	1.068
2	30	40	1.008

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