The Casualty Actuarial Society *Forum*

Fall 2003 Edition

Including the 2003 Reserves Call Papers and Discussions of the 2002 ARIA Prize Paper

To CAS Members:

This is the Fall 2003 Edition of the Casualty Actuarial Society *Forum*. It contains 10 Reserves Call Papers, six discussions of the 2002 ARIA Prize Paper, and five additional papers.

The *Forum* is a nonrefereed journal printed by the Casualty Actuarial Society. The CAS is not responsible for statements or opinions expressed in the papers in this publication. See the Guide for Submission to CAS *Forum* (www.casact.org/aboutcas/forum.htm) for more information.

The CAS *Forum* is edited by the CAS Committee for the Casualty Actuarial Society *Forum*. Members of the committee invite all interested persons to submit papers on topics of interest to the actuarial community. Articles need not be written by a member of the CAS, but the paper's content must be relevant to the interests of the CAS membership. Members of the Committee for the Casualty Actuarial Society *Forum* request that the following procedures be followed when submitting an article for publication in the *Forum*:

1. Authors should submit a camera-ready original paper and two copies.
2. Authors should not number their pages.
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5. Authors should submit an electronic file of their paper using a popular word processing software (e.g., Microsoft Word and WordPerfect) for inclusion on the CAS Web Site.

The CAS *Forum* is printed periodically based on the number of call paper programs and articles submitted. The committee publishes two to four editions during each calendar year.

All comments or questions may be directed to the Committee for the Casualty Actuarial Society *Forum*.

Sincerely,

Joseph A. Smalley

Joseph A. Smalley, CAS *Forum* Chairperson

The 2003 CAS Reserves Call Papers
Presented at the
2003 CAS/AAA/CCA Casualty Loss Reserve Seminar
September 8-9, 2003
Chicago Marriott Downtown
Chicago, IL
and

Discussions of the 2002 ARIA Prize Paper
Presented at the 2003 CAS Spring Meeting
May 18-21, 2003
Marco Island Marriott Resort, Golf Club & Spa
Marco Island, FL

The Fall 2003 Edition of the CAS *Forum* is a cooperative effort between the Committee for the CAS *Forum* and the CAS Committee on Reserves.

The CAS Committee on Reserves present for discussion 10 papers prepared in response to their 2003 call for papers. This *Forum* includes papers that will be discussed by the authors at the 2003 CAS/AAA/CCA Casualty Loss Reserve Seminar, September 8-9, 2003, in Chicago, IL.

This *Forum* also includes papers discussing the 2002 American Risk and Insurance Association (ARIA) Prize Paper, "Capital Allocation for Insurance Companies," by Stewart C. Myers and James A. Read Jr. The Myers-Read paper, which was published in the December 2001 issue of the *Journal of Risk and Insurance*, stimulated a great deal of comment within the CAS—so much that a special session was scheduled to follow the presentation of the paper itself at the 2003 CAS Spring Meeting in Marco Island, Florida.

Authors were invited to submit a discussion of the Myers-Read paper for presentation at the session. Six discussions were submitted and presented. The discussions were provided to the ARIA Prize authors, who presented a rejoinder at the end of the session.

Reserving Call Papers

Discussions of the 2002 ARIA Prize Paper

Additional Papers

Probabilistic Framework for Evaluating Materiality and Variability in Loss Reserve Estimates

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I. Abstract

A reserve[1] point estimate is usually presented without explicit quantitative reference to the variability associated with it. The literature provides little guidance on how to go about providing such quantitative representations. In this paper the authors present a new function, called the *coefficient of estimation*, as a measure of the placement of a reserve point estimate on the continuum of reserve estimates defined by the underlying aggregate loss distribution. The authors further use this coefficient to discuss six commonly used reserving terms to illustrate how the variability inherent in a point reserve estimate may be quantified. The authors also illustrate these ideas with six different demonstrations for each of two lines of business, including tables and charts depicting underlying aggregate loss distributions. The authors conclude the paper with a series of observations to amplify some of the salient issues as well as set some boundaries for the usefulness of the proposed coefficient of estimation.

[1] In this paper "reserve" refers only to a loss or loss adjustment expense reserve.

II. Background

Currently available guidance for the actuary who is analyzing loss and loss adjustment expense reserves provides references to various elements of reserves that suggest a stochastic approach to reserving. Yet a number of these terms are left undefined or are not defined in a manner that suggests probabilistic quantification. For example, Principle 3 of the Statement of Principles Regarding Property and Casualty Loss and Loss Adjustment Expense Reserves (Reserving Principles) states:

> *The uncertainty inherent in the estimation of required provisions for unpaid losses or loss adjustment expenses implies that a range of reserves can be actuarially sound.*

Principle 4 states:

> *The most appropriate reserve within a range of actuarially sound estimates depends on both the relative likelihood of estimates within the range and the financial reporting context in which the reserve will be presented.*

Other references can be found in Actuarial Standard of Practice No. 36 (ASOP 36) *Statements of Actuarial Opinion Regarding Property/Casualty Loss and Loss Adjustment Expense Reserves*. Section 3.6.5 states:

> *The potential variation in the actual amount that will be needed to pay unpaid claims gives rise to uncertainty in the reserve estimates. An adverse deviation occurs when such a variation results in paid amounts higher than provided for in the reserves.*

Section 2.6 of ASOP 36 also provides a definition of *expected value estimate*:

> *An estimate of the mean value of an unknown quantity where the mean value represents a probability-weighted average of the quantity over the range of all possible values.*

The December 2001 American Academy of Actuaries Property and Casualty Practice Note discusses *materiality* as follows:

> *Requiring the use of professional judgment and placing importance on intended purpose both emphasize the role of qualitative considerations in evaluating materiality. Actuaries will naturally also focus on quantitative considerations related to judgments on materiality. No formula can be developed that will substitute for professional judgment by providing a materiality level for each situation.*

The above citations are but a few that illustrate the fact that important elements of reserves that need to be addressed by the actuary are presented in a manner that suggests their stochastic nature, yet fall short of providing the tools to define the elements of the reserves reflecting their probabilistic nature.

It is clear that commonly used actuarial terms such as *best estimate, range of reasonableness, confidence interval, provision for uncertainty (risk margin), reasonableness,* and *materiality* can be found throughout the actuarial principles, standards, and other literature. And it is also clear that these terms have a stochastic element to them although none of the above referenced documents includes any suggestion as to how the stochastic element may be indicated or quantified.

The lack of rigorous definition of these terms has, in the authors' opinion, led to the use of numerous caveats in actuarial work products. These caveats offer the reader little insight into the actual degree of confidence the actuary places in the estimate. For example, a typical caveat in an actuarial report that contains reserve estimates states: "The ultimate value of the liability for future development, when all losses are reported and settled, may vary, perhaps significantly, from the estimates in this report." The reader of this caveat, although duly warned that there will be variation in the actual results from that which was estimated by the actuary, has little understanding of the amount of variability present in the estimates. Such is the state of the art today.

In this paper, the authors propose to discuss some of the more commonly used terms in a way that associates a probability statement with each. It is our hope that these concepts, if used by actuaries in estimating and communicating reserve estimates, will lead to a greater understanding of the variability associated with loss reserve estimates.

III. Foundational Framework: Aggregate Loss Distributions

When a reserve point estimate is put forth, there is always the implicit understanding that a specific estimate is but one of a number of plausible alternative reserve estimates, each of which is actuarially sound.[2] Taken a step further, it is also implicit that there is an underlying distribution of reserve estimates that contains the set of all such reserve estimates along with their associated probabilities. A major premise of this paper is that until and unless the actuary identifies and makes use of such distributions, it is not possible to communicate meaningfully and completely about reserve estimates, their expected degree of adequacy, and their inherent variability.

Although the construction of such distributions is beyond the scope of this paper, we will discuss the subject very briefly in order to complete the foundational work for this paper. Generally speaking, such distributions exist in one of two ways: as *assumed* distributions (expressed in closed form, using parameters suggested by the raw data utilized in deriving a reserve estimate) and as *empirical* (or deterministic) distributions.

Assumed distributions require the actuary to select the type of distribution function (i.e., the "shape" of the distribution) as well as its parameters. There is currently substantial literature in this area and thus the subject requires little discussion beyond acknowledging the availability of such distributions.[3]

Empirical distributions, on the other hand, are those that arise naturally from considering all the available data and compiling all possible outcomes contemplated by various actuarial methodologies. One example is the set of all possible outcomes produced by

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[2] "The uncertainty inherent in the estimation of required provisions for unpaid losses or loss adjustment expenses implies that a range of reserves can be actuarially sound." Lines 121-122 of *Statement of Principles Regarding Property And Casualty Loss And Loss Adjustment Expense Reserves* as adopted by the Casualty Actuarial Society in May 1988.

[3] See, for example: Heckman & Myers, PCAS Vol. LXX (1983); Hewitt & Lefkowitz, PCAS Vol. LXVI (1979); Klugman & Rahykha, PCAS Vol. LXXX (1993); and Hayne, PCAS Vol. LXXVI (1989).

every possible combination of link ratios calculated from an incurred loss development array. This calculation, when carried out completely, delivers a specific distribution of outcomes of ultimate losses, along with associated frequencies of occurrence, from which specific probabilities of adequacy can be derived and associated with various reserve estimates.[4]

For our purposes, we assume that at time **t** the actuary has constructed—for a well-defined cohort of claims—a relevant distribution of outcomes, either assumed or empirical, from which a cumulative frequency distribution of outcomes is constructed. In turn, the cumulative frequency distribution of outcomes may be used to associate a probability that the final value, when it becomes known, when all claims are finally settled, will be less than or equal to the proposed reserve estimate.

This distribution, however constructed at time **t**, in turn yields a mean reserve value, $\mathfrak{R}$, a median, **M**, along with a standard deviation, σ. And since $\mathfrak{R}$, **M**, and σ are all identified at time **t**, we will designate them as $\mathfrak{R}_t$, $\mathbf{M}_t$, and σ_t, respectively. The entire discussion of commonly used actuarial terms builds on these values. Also, we will make use of some familiar probability notations as follows:

$\mathbf{P_t(X)}$: The probability at time **t** that the ultimate value will fall in the interval $(-\infty, X)$.

$\mathbf{P_t(A,B)}$: The probability at time **t** that the ultimate value will fall in the interval **(A,B)**. Note that $P_t(A,B) = P_t(B) - P_t(A)$.

IV. The Basic Idea: The Coefficient of Estimation

The basic idea advanced in this paper is that a reserve estimate, in order to be meaningfully and completely presented, needs to be associated with a statement that

[4] The time needed to complete such constructions may render such distributions impossible to produce. However, various approximation algorithms can be useful in compressing the problem to the point where the construction of such distributions is perfectly possible.

gives the user an idea of the probability of adequacy of the proposed estimate. We call that proposed probability statement the *coefficient of estimation* (denoted by **CE**) that is associated with the reserve estimate.

The coefficient of estimation, **CE** at time **t**, denoted by $\mathbf{CE_t}$, of a point estimate of reserves, also calculated at time **t**, denoted by $\mathbf{X_t}$, is defined by:

$$CE_t(X_t) = 100(P_t(X_t) / P_t(\Re_t))$$

Some immediately obvious properties of **CE**:

1. The domain of **CE** is $(-\infty, +\infty)$.

2. The range of **CE** is $[0, 100/ P_t(\Re_t)]$.

3. $CE_t(\Re_t) = 100$.

4. **Limit** $CE_{\Re \to M,\, X \to \infty}(X) = 200$. Note that the limit of **200** may be approached from above or below.

5. $CE_t(X_t) = \alpha(P_t(X_t))$, where α is a scalar given by $100/P_t(\Re_t)$. Thus the shape of the graph of **CE** is identical to the shape of the graph of the underlying cumulative frequency distribution of outcomes. This phenomenon is illustrated in the charts described in Section VI.

6. For the great majority of distributions of aggregate insurance losses, because they are usually skewed to the left, one can expect $\Re$ to be greater than **M**. One can expect this relationship to hold in all but some very special situations where one is dealing with a great predominance of small claims with only occasional claims of significant values.

7. The **CE** function has an associated inverse function, which is denoted by $\mathbf{CE^{-1}}$. In other words, given a coefficient of estimation **a** one can identify a unique estimate **e** such that $\mathbf{CE^{-1}(e) = a}$.

We now propose that a point estimate of reserves, **R**, derived at time **t**, denoted by $\mathbf{R_t}$, be presented along with its associated coefficient of estimation, $\mathbf{CE_t(R_t)}$. Thus the full reserve estimate statement would appear as $\mathbf{(R_t, CE_t(R_t))}$.

Because reserve estimates are subject to uncertainty, it follows that one would want to allow leeway with respect to values of $\mathbf{CE_t}$ that are near $\mathbf{CE_t(R_t)}$. In other words one would like to find a way to accord values that are in the neighborhood of $\mathbf{R_t}$ substantially the same meaning as $\mathbf{R_t}$. This idea in turn gives rise to a number of related concepts, including but not limited to: *range of reasonableness, confidence intervals, best estimate, materiality, reserve strengthening, and risk margin*. Given the basic framework suggested by $\mathbf{(R_t,CE_t(R_t))}$, we can proceed to make the extensions such that each term is framed probabilistically, and more specifically, in terms of $\mathbf{CE_t}$.

V. Discussion of Selected Reserving Terminology

In the remainder of this paper we discuss six commonly used terms placing each of them in a probabilistic context using the foundational framework set forth above. In Section VI all of these terms are illustrated with live examples.

A. **Best Estimate**. This term, essentially undefined in the literature, is often used to label the actuary's final selection of a point estimate of reserves. A little reflection suggests a number of possible meanings:

    1. It is the actuary's selected point estimate, in a literal sense, given all the quantitative and qualitative information as well as relying on his education, experience, and judgment.

2. It is the actuary's selected point estimate, among several options of plausible outcomes.

3. It is the mean value of reasonable estimates derived by actuary, each of which is equally plausible.

4. It is the weighted mean value of reasonable estimates derived by the actuary, thus recognizing the respective likelihood of each of a number of plausible outcomes.

5. It is the mean of an underlying distribution of outcomes.

And there are others. While we do not propose to suggest than any of these meanings is the proper one, because different actuaries are likely to use the term to mean different things, we do propose that the actuary, in addition to using the term *best estimate*, attach to it the associated coefficient of estimation. Thus a point estimate, when described as a best estimate will have two dimensions to it: one is the proposed "standard" meaning implied by its coefficient of estimation (and therefore having to identify and use the underlying and implied distribution of outcomes) and the other is the colloquial meaning that the actuary intends.

This convention is a special case of the general proposition advanced in this paper, that a reserve estimate be set forth as a pair of values, $(\mathbf{R_t},\mathbf{CE_t}(\mathbf{R_t}))$. Note that, if the actuary chose not to use the term *best estimate*, the general convention would also have relevance—as the point estimate is simply the actuary's selection given the data and the circumstances at time **t**.

A subtle implication of the $(\mathbf{R_t},\mathbf{CE_t}(\mathbf{R_t}))$ convention is the implicit requirement that is placed on the actuary in the event the selected reserve has a coefficient of estimation that is significantly different from **100**. The actuary would have to make the case, much more directly than heretofore required, as to why his estimate should be so different from the indicated reserve $\mathfrak{R}$. This duty applies regardless of whether the coefficient of estimation is much higher or much lower than **100**. One collateral issue in this discussion is the level at which the difference between the coefficient of estimation and **100** is significant. It is not possible to set hard and fast rules for such standards. However, the use of some proportion of the standard deviation might be useful. For example, one may use the following coefficients of estimation to establish the standard of significance for further investigation: $\mathbf{CE_t(\Re_t - 0.5\,\sigma_t)}$ and $\mathbf{CE_t(\Re_t + 0.5\,\sigma_t)}$. And in any event, whatever the actuary uses as the standard for significance, it should be disclosed so that there is no mystery as to what is operating.

Similarly, we need to point out that the mere fact of a reserve estimate having a coefficient of estimation equal to **100** is not, by itself, dispositive. The actuary still has the professional duty to identify for himself, and certainly include such demonstration in his work papers, the rationale for the selection of the underlying frequency distribution of outcomes.

B. **Range of Reasonableness**. A use of this concept is generally on the order of a *range of reasonableness* being defined to be within ± 5% (or some other increment) of the selected reserve estimate". The immediate problem with such statements is that the invoked degree of tolerance means different things depending on the shape of the distribution of outcomes. This concept, in reality, does little more than introduce a bit of speculation in the communication process as no real information is imparted to the user as to the underlying variability of the reserve estimate.

A way to eliminate the problem associated with the use of this terminology is to actually identify the coefficient of estimation associated with the endpoints of the specified degree of tolerance. In symbols, suppose that one is advancing an estimate, $\mathbf{R_t}$, then, given a tolerance amount δ_t, an amount that may be defined absolutely or as a proportion of $\mathbf{R_t}$, one can calculate the following coefficients of estimation: $\mathbf{CE_t(R_t - \delta_t)}$ and $\mathbf{CE_t(R_t + \delta_t)}$.

When one states that the ultimate value is within δ_t of the reserve estimate $\mathbf{R_t}$, one can immediately append a coefficient of estimation to this statement, using the underlying distribution. In this manner, the user can have a sense of the significance of the indicated tolerance δ_t. For example, for a personal line of business, such as private passenger automobile liability, where the loss distribution can be expected to be compact and to exhibit a high degree of central tendency, a tolerance of δ_t, yielding a change in the coefficient of estimation, say $\mathbf{\Delta CE}$, would almost certainly yield a smaller change in the coefficient of estimation if the line of business is commercial auto liability. This is because the commercial auto liability loss distribution can be expected to be much flatter (*i.e.*, inherently more disperse, more skewed) than the private passenger automobile liability distribution.[5] The user can now view the meaning of a reserve coupled with a statement of tolerance through the prism of the associated coefficients of estimation.

We should also note that although this discussion is couched in terms of the interval $(\mathbf{R_t}-\delta_t,\mathbf{R_t}+\delta_t)$, in reality the real concern is with the degree of adequacy of the two endpoints – and therefore the coefficients of estimation for the two endpoints. For example, the regulatory authorities can be expected to be more concerned with $\mathbf{CE_t(R_t}-\delta_t)$. On the other hand, the IRS can be expected to be more concerned with $\mathbf{CE_t(R_t}+\delta_t)$.

This discussion would not be complete if the possibility of the converse of this proposition were not considered. If the degree of tolerance is stated as a probabilistic tolerance – that is a tolerance in the coefficient of estimation, π_t, around $\mathbf{CE_t(R_t)}$, then, once again using the underlying distribution, one can calculate the absolute amount of the range that corresponds to the suggested tolerance in the coefficient of estimation, π_t. In other words, one can identify an amount δ_t' such that:

[5] This phenomenon is illustrated in the Section VI.

$$CE_t(R_t+\delta_t') - CE_t(R_t-\delta_t') = 2\pi_t$$

The idea of some voluntarily[6] identified boundaries, or, as more commonly known, a range of reasonableness, whether set in absolute terms or in probabilistic terms, is immediately placed in context such that a user may be able to appreciate in concrete terms the significance of the suggested range of reasonableness.

In concluding this section, it should be noted that no quantitative definition of the *range of reasonableness* is provided. After the preceding discussion, it is obvious that no such definition is possible. What we created is the framework in which such language may be used meaningfully. In other words, given a numerical tolerance in the reserve estimate, one is able to produce a corresponding probabilistic statement. All three items (the reserve estimate, the numerical tolerance, plus the associated coefficient of estimation associated with the numerical tolerance) are required elements that need to be present in order to be fully credible in the use of the *range of reasonableness.*

Until such time as the Actuarial Standards Board (ASB) may adopt a uniform benchmark for what constitutes a *range of reasonableness*, the concept is destined to remain a function of the training, experience, and judgment of the individual actuary as well as the facts and circumstances of the case under consideration. In other words no two actuaries need adopt the same standard in order for this concept to operate. However, what we have done in this paper is to identify the three elements of the statement that need to be present in order to be able to view consistently various statements about the *range of reasonableness*.

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[6] The term "voluntarily" is used to indicate that it is a choice of the presenting actuary to employ such boundaries. It is not required *per se* by any principle or standard. However, what we are suggesting in this paper is that if the actuary chooses to go down this voluntary path, then he has the obligation to follow through with a complete presentation of these boundaries and their probabilistic significance.

C. **Confidence Interval**. The idea of a *confidence interval* is an extension (or a generalization) of the concept of *range of reasonableness*. In the preceding section we identified the three elements necessary in order to be able to use the language of a *range of reasonableness*. Thus in the affirmative case where a reserve estimate, $\mathbf{R}_t$, is advanced, a numerical tolerance, δ_t, is selected, one can identify the coefficient of estimation of the resultant endpoints, given by $\mathbf{CE}_t(\mathbf{R}_t\text{-}\delta_t)$ and $\mathbf{CE}_t(\mathbf{R}_t\text{+}\delta_t)$. The confidence interval concept is identical in all respects except that the connection to $\mathbf{R}_t$ is eliminated. In other words, the confidence interval can refer to *any* interval. Thus given any two numerical values, at a time **t**, such as $\mathbf{A}_t$ and $\mathbf{B}_t$, one is able to calculate the coefficient of estimation for each of the endpoints of the interval $(\mathbf{A}_t,\mathbf{B}_t)$, based on the underlying distribution, yielding $\mathbf{CE}_t(\mathbf{A}_t)$ and $\mathbf{CE}_t(\mathbf{B}_t)$. In other words, the range of outcomes implied by $\mathbf{A}_t$ and $\mathbf{B}_t$ is now associated with the respective coefficients of estimation and thus yielding valuable insight as to the significance of the interval $(\mathbf{A}_t,\mathbf{B}_t)$.

Note that, as in the case of the range of reasonableness, the converse of this proposition is also possible. Given two coefficients of estimation, one can calculate the corresponding interval with endpoints having the given coefficients of estimation.

D. **Materiality**. As noted earlier, the December 2001 American Academy of Actuaries Property and Casualty Practice Note discusses *materiality* as follows:

> *Requiring the use of professional judgment and placing importance on intended purpose both emphasize the role of qualitative considerations in evaluating materiality. Actuaries will naturally also focus on quantitative considerations related* to judgments on materiality. *No formula can be developed that will substitute for professional judgment by providing a materiality level for each situation.*

While this statement is reasonable in that it leaves the determination of materiality to the actuary, there is no guidance as to the elements that need to be present in order to make a coherent statement about materiality. We propose to fill this gap in the following paragraphs.

First, the idea of materiality is a comparative concept. That is, the difference between *two* quantities is the object of materiality discussions. For example, given a reserve estimate, $\mathbf{R}_t$, then an alternate reserve estimate, $\mathbf{R}'_t$, is materially different from $\mathbf{R}_t$ if and only if the difference between the two estimates is more than a specified benchmark.

Second, materiality has to be set against some benchmark. The practice note does not provide guidance on this point. While this is fine, as the selection of the benchmark is left to the judgment of the actuary, the suggestion advanced in this paper is that such a benchmark needs to be disclosed as part of the actuary's statement on materiality – along with the rationale for such selection. The practice note affords the actuary great latitude, both qualitatively and quantitatively, in selecting such a standard. Alternatively, a benchmark may be stated in terms of probabilistic increments – pertaining to the coefficient of estimation. In other words, a benchmark may be stated as the maximum difference in the coefficients of estimation of the two amounts being compared.

One way to illustrate how these concepts can be pulled together is to recognize that we have two immediate *a priori* amounts to be compared: $\mathbf{R}_t$ and $\mathbf{R}'_t$, and then we also note that we have σ_t to form the foundation of an objective benchmark. For example, the actuary can set his benchmark as the difference between $\mathbf{CE}_t(\mathfrak{R}_t + 0.5\sigma_t)$ and $\mathbf{CE}_t(\mathfrak{R}_t)$. Thus, within this framework, the difference between two estimates would be material if and only if:

$$|\mathbf{CE}_t(\mathbf{R}_t)-\mathbf{CE}_t(\mathbf{R}'_t)| > (\mathbf{CE}_t(\mathfrak{R}_t + 0.5\sigma_t) - \mathbf{CE}_t(\mathfrak{R}_t)).$$

We also need to point out that one need not go to such lengths as to calculate complicated standards such as $(\mathbf{CE}_t(\mathfrak{R}_t + 0.5\sigma_t) - \mathbf{CE}_t(\mathfrak{R}_t))$. Any other standard that is appropriate, in the judgment of the actuary, may be used provided the actuary identifies the rationale for such selection.

Another interesting possibility for identifying the standard of materiality is to set it as a function of the company's surplus – say some fraction, β, of surplus, **S**, denoted by **βS**. In that case the corresponding probabilistic condition for materiality would be set as: $|\mathbf{CE_t(R_t)\text{-}CE_t(R'_t)}| > (\mathbf{CE_t(\Re_t + \beta S) - CE_t(\Re_t)})$.

Yet another way that the materiality standard may be set is in terms of solvency standards. That is, selecting an increment that maintains the company's quantitative elements of solvency as may be set in the IRIS tests (such as maintaining a maximum premium-to-surplus ratio). Note that in the examples advanced here the full latitude afforded the actuary by the practice note is fully preserved. What these ideas advance is the manner in which the actuary may state his judgment as to materiality using the underlying loss distribution.

E. **Provision for uncertainty (risk margin).** ASOP 36 defines *risk margin* as: *An amount that recognizes uncertainty; also known as a provision for uncertainty.* Note that this definition provides a very wide berth for the actuary to set any risk margin he deems appropriate. Once again, while this is fine as far as it goes, in this paper we break down this statement such that the actuary is still free to set his own standard for the appropriateness of a specific risk margin, yet is able to produce a coherent statement of the meaning and basis for his selection.

First, given that the risk margin is an *amount*, we begin by searching for the types of bases that may be used to arrive at such an amount – which we may designate as the risk margin. The most obvious and natural benchmark to examine is a measure of dispersion of the underlying loss distribution. One measure of dispersion we have identified in this discussion is σ_t. Another element of establishing a basis for a *risk margin* is the size of the surplus of the company – in that any risk margin that is built into $\mathbf{R_t}$ is an amount that serves to directly reduce the otherwise available surplus. And this observation notes the obvious linkage between the size of the surplus (either on a pre- or post-risk margin basis) and the adequacy of reserves (including any risk margins that may be used). These are complicated relationships and any light one can shed on the issue in communicating them to the user has to be helpful.

The concept of a risk margin is similar to the idea of converting *materiality* into a probabilistic statement. Thus, when an actuary adds a risk margin, in fact he is increasing the probability of adequacy of his otherwise applicable estimate. Using our adopted notation, if an indicated reserve $\underline{\mathbf{R}}_t$ (set before any risk margin is added) is increased by some risk margin, $\Delta\underline{\mathbf{R}}_t$, then we can identify a change in the coefficients of estimation of the two alternative estimates: $|\mathbf{CE}_t(\underline{\mathbf{R}}_t+\Delta\underline{\mathbf{R}}_t)-\mathbf{CE}_t(\underline{\mathbf{R}}\mathbf{t})|$. The risk margin is now stated in probabilistic terms. Once this amount is given, one can see the extent to which the risk margin is significant – by making use of the characteristics of the underlying loss distribution. For example, if adding a risk margin causes the coefficient of estimation to increase from **88** to **90**, one can question whether the addition of the risk margin to the otherwise applicable estimate is significant. On the other hand, if the increase is from **88** to **98**, one may view $\Delta\underline{\mathbf{R}}_t$ as a legitimate candidate to be designated as the risk margin. We should note that at this point the linkage between *materiality* and *risk margin* is clear. In other words, a *risk margin* is material if it exceeds some benchmark that is selected and motivated by the actuary.

The discussion is concluded by noting an implicit condition that should be observed whenever an actuary makes use of the terminology "*risk margin*". Saying that a *risk margin* is added to an otherwise indicated reserve estimate that merely brings $\mathbf{R}_t$ closer to $\mathfrak{R}_t$ may be inadvertently misleading. In this case the coefficient of estimation of the final reserve, $\mathbf{R}_t(=\underline{\mathbf{R}}_t+\Delta\underline{\mathbf{R}}_t)$ inclusive of a risk margin, is simply raised closer to **100**, the condition under which the proposed estimate is simply approaching the mean of the underlying loss distribution. In this case it is clear that a true risk margin is not provided – in spite of using the terminology of risk margins. At least it is not obvious how such a statement can be meaningful. Using our notation: if an indicated reserve $\underline{\mathbf{R}}_t$ is increased by some risk margin, $\Delta\underline{\mathbf{R}}_t$, then, absent some very unusual conditions, which should be fully explained, one should be able to expect that $\mathbf{CE}_t(\mathbf{R}_t) > \mathbf{100}$. If this is not part of the outcome of adding a risk margin, additional explanation and rationale needs to be provided by the actuary.

F. **Reserve Strengthening**. This language is often used in actuarial reports. Its meaning has never been established in the actuarial literature. One common usage occurs in connection with strengthening case loss reserves. That is generally understood to mean that the case loss reserves are now established to be closer to the ultimate settlement values than is historically indicated. This is often used to justify a lower-than-indicated aggregate reserve. In this paper when we refer to reserve strengthening, we are talking about strengthening of the total reserve (the sum of case reserves and IBNR reserves) in relation to what might have been done normally.

The basic idea of (the total) reserve strengthening simply suggests that the total carried reserve is materially closer to the ultimate value than would be the case had the otherwise indicated reserve been carried. Note here that there is no concept of the passage of time anywhere in the reserve strengthening idea. It is an instantaneous concept.

Thus for our purposes, we begin by identifying some indicated reserve, denoted by $\underline{\mathbf{R}}_t$. This reserve is arrived at by using a particular set of assumptions and methods (denoted by **<u>A&Ms</u>**), that are consistent or identical to the assumptions and methods used in the past. The actuary, then, for good and sufficient reason, determines that a different set of assumptions and methods is more appropriate (denoted by **A&Ms**) is more appropriate. And in so doing, if applying **A&Ms** yields a higher reserve than the reserve produced using **<u>A&Ms</u>**, we can now say that the reserves are strengthened. We can set this condition probabilistically by noting that the reserve are strengthened if and only if:

$$CE_t(R_t|A\&Ms) > CE_t(\underline{R_t}|\underline{A\&Ms})$$

We should note here that **A&Ms** are those used in the prior period. In other words, if the actuary continues with the same **A&Ms** as in the past, then the idea of reserve strengthening cannot be meaningful. Also note this is not introducing an element of time in our construction. Time here is used to simply identify and anchor the assumptions and methods that form the baseline.

With just these six illustrations, it is now possible to appreciate that practically any of the "soft" language that may be used to represent reserve estimates may be converted to a probabilistic basis. While that is not an end unto itself, the use of probabilistic representations makes it possible to harden the representations that actuaries make in connection with the presentation of loss reserve estimates.

VI. A Demonstration

This section contains a number of simple demonstrations of the concepts advanced in this paper. For our purposes, we are given two sets of raw data, one set is for line of business A (commercial automobile liability) and one for line of business B (private passenger automobile liability), as of a specific time **t**, from which we are able to construct two loss distributions, one for each line of business.[7] The following tables and charts are included at the end of this paper:

1. Tables A1 and B1 contain a compressed form of the cumulative frequency distributions for lines of business A and B, respectively.[8]

2. Tables A2 and B2, contain a compressed form of the coefficients of estimation associated with each of the significant outcomes in the underlying loss distributions for lines of business A and B, respectively.

[7] From this point forward, we will omit the reference to **t**, as all valuations and associated statements are as of time **t**.
[8] The full distribution using the intervals shown in Table A requires 15 pages to set forth completely.

3. Charts A1 and B1 show graphs of the cumulative frequency distributions set forth in Tables A1 and B1, for lines of business A and B, respectively.

4. Charts A2 and B2 show the graphs of the frequency distributions that underlie the graphs shown in Charts A1 and B1, for lines of business A and B, respectively.

5. Charts A3 and B3 show the graphs of the coefficients of estimation shown in Tables A2 and B2 for lines of business A and B, respectively.

The key parameters of the underlying loss distributions are calculated to be:

$$\mathfrak{R}(A) = \$3{,}486{,}577 \qquad \sigma(A) = \$1{,}754{,}637$$

$$\mathfrak{R}(B) = \$7{,}148{,}286 \qquad \sigma(B) = \$899{,}038$$

For the rest of this section, we will erect a number of scenarios and discuss the application of the concepts advanced in this paper to each scenario as appropriate, in turn illustrating the application of the particular facts to one of the terms discussed above.

Scenario 1. Best Estimate.

Line A. In this scenario suppose the selected point estimate of reserves for line of business A is **$3,000,000**. The reporting actuary calls this his best estimate. Our first observation, drawing on the values in Table A2, page 1, is that **CE($3,000,000) = 79.8**. Note that the coefficient of estimation of the mean of the distribution is **100**. That is **CE($3,486,577) = 100**. Thus even though the **$3,000,000** estimate is **$486,577** away from the mean of the underlying distribution (giving a preliminary and unconfirmed indication of a reserve deficiency), this amount represents a significant deviation from the mean of the distribution. The actuary then would endeavor to provide the rationale for departing from the mean of the distribution to the extent that he has. We should also note that the final representation of the reserve estimate is **($3,000,000 ; 79.8)**

Line B. In this scenario suppose the selected point estimate of reserves for line of business B is **$7,100,000**. The actuary calls this his best estimate. According to Table B2, **CE($7,100,000) = 97.4**. Note that the coefficient of estimation for the mean of the distribution is **100**. That is **CE($7,148,286) = 100**. The estimate of **$7,100,000** is **$48,286** from the mean of the underlying distribution (giving a preliminary and unconfirmed indication of an appropriate reserve selection – not redundant and not inadequate). Since the proximity of the point estimate to the mean of the loss distribution is not necessarily dispositive of the condition of the loss reserves, the actuary has the obligation to review the contemporaneous facts on operations to satisfy himself that there is nothing in the environment that would serve as a counter-indicator to the **$7,100,000** estimate. Assuming that the search turns up no significant counter indicators that would discredit the indicated estimate, the actuary would represent the statement of the reserve estimate as **($7,100,000 ; 97.4)**.

Scenario 2. Range of Reasonableness.

Suppose the reserving actuary has provided a voluntary range of reasonableness that each of his estimates has a range of reasonableness of **10%**. Now we review the significance of this statement as discussed above:

Line A. For this line of business the range of reasonableness represents **10%** of **$3,000,000**, or **$300,000**. Thus the range of reasonableness is **($2,700,000 ; $3,300,000)**. We note that the coefficients of estimation of the endpoints are as follows: **CE(2,700,000) = 68.0** and **CE(%3,300,000) = 92.0**. The interesting outcome here is that the distribution is substantially symmetrical about the **$3,000,000** estimate, in that the **CE** values at the boundaries are also symmetrical about the **CE** of the estimate (i.e., **79.8** is almost exactly halfway between **68** and **92**). These **CE**'s also indicate that the **10%** range of reasonableness is a fairly narrow range given the spread of the distribution of the **CE**'s. In other words the bulk of the expected outcomes remains outside the indicated range of reasonableness.

<u>Line B</u>. For this line of business the range of reasonableness represents **10%** of **$7,100,000**, or **$710,000**. Thus the range of reasonableness is **($6,390,000 ; $7,810,000)**. We note that the coefficients of estimation of the endpoints are as follows: **CE($6,390,000) = 39.5** and **CE($7,810,000) = 147.3**. Thus, the **CE** of the original estimate, at **97.4**, extends to cover the interval of **CE**'s consisting of **(39.5 ; 147.3)**. The indication is that the distribution is somewhat symmetrical about the selected estimate. More specifically, the **CE** of the point estimate, at **97.4**, is **57.8** points greater than the **CE** of the lower bound of the range of reasonableness and **49.9** points less than the **CE** of the upper bound of the range of reasonableness. Finally, the range of **10%** appears to cover the vast bulk of the distribution of possible outcomes.

It is noteworthy that the **10%** range of reasonableness covers a band of **CE**'s that spans **24.0** points **(= 92.0 - 68.0)** for line A while the same **10%** range of reasonableness spans **107.8** points **(= 147.3 – 39.5)** for line B. The reason for this difference is that the distribution for line A is much flatter than the distribution for line B. In evaluating these observations, it is useful to recall that the range of outcomes for the **CE** function is **200**.

Scenario 3. Confidence Interval.

<u>Line A</u>. For this scenario, suppose the actuary has calculated an interval of possible outcomes but did not select a point estimate.[9] The interval in the instant case is given as **($2,800,000 ; $3,800,000)**. We calculate the **CE**'s for these values: **CE($2,800,000) = 71.4** and **CE(3,800,000) = 110.8**. The spread of **CE**'s that corresponds to this confidence interval is **39.4** points **(= 110.8 – 71.4)**. The **$3,000,000** estimate is within the interval – but is near the low end. The final reserve statement by the reviewing actuary may well contain a remark to point out the flatness of the distribution and that

[9] This situation arises often in the case of one actuary reviewing the work of another, such as the actuary for an audit firm. Here the actuary calculates a range and tests the estimate of the audit client against the interval he has derived.

the bulk of the possible outcomes remain outside the indicated confidence interval. Even though it is obvious that the selected point estimate is within the confidence interval, the value of the **CE**'s in this case is to assist the actuary in finding out just how much of the distribution is actually covered by interval of coefficients of estimation in relation to the coefficient of estimation of the point estimate of the reserve being tested.

Line B. For this scenario, we are told that an actuary has calculated an interval of possible outcomes but did not select a point estimate. The interval in the instant case is given as **($7,000,000 ; $8,000,000)**. We calculate the **CE**'s for these values: **CE($7,000,000) = 91.3** and **CE($8,000,000) = 157.1**. The spread of **CE**'s that corresponds to this confidence interval is **65.8** points **(= 157.1 – 91.3)**. The **$7,100,000** estimate is within the interval – but is near the low end. However, the **CE**, even for the lower boundary of the confidence interval is in the neighborhood of the mean of the distribution so that the reviewing actuary could easily accept this value without reservation. The opining actuary may well include a comment in his opinion to express the high degree of comfort that is indicated by the selected point estimate of the reserves under review. Once again, even though it is obvious that the selected point estimate is within the confidence interval, the value of the **CE**'s in this case is to assist the actuary in finding out just how much of the distribution is actually covered by the interval of coefficients of estimation in relation to the coefficient of estimation of the point estimate of the reserve being tested.

Scenario 4. Materiality.

Line A. For this scenario, suppose the actuary has estimated the reserve at **$4,000,000**. The question arises as to the materiality of the difference between this estimate and the carried reserve at **$3,000,000**. The respective **CE**'s are: **CE($3,000,000) = 79.8** and **CE($4,000,000) = 117.3**. The reviewing actuary decides to use the materiality threshold as half the standard deviation. In this case that amount is **$877,319**. Following the construction from earlier in this paper, the **CE** spread that is implied by this standard is **|CE($3,486,577) - CE($4,363,896)| = |100.0 – 128.4| = 28.4** points.[10] On the other hand, the absolute value of the difference between **CE($3,000,000)** of **79.8** and **CE($4,000,000)** of **117.3,** is **37.5** points. Accordingly, since **37.5 > 28.4**, one is able to conclude that the difference is material.

Line B. For this scenario, suppose the actuary has estimated the reserve at **$7,500,000**. The question arises as to the materiality of the difference between this estimate and the carried reserve at **$7,100,000**. The respective **CE**'s are: **CE($7,100,000) = 97.4** and **CE($7,500,000) = 128.9**. The reviewing actuary again decides to use the materiality threshold as half the standard deviation. In this case that amount is **$449,519**. Following the construction from earlier in this paper, the **CE** spread that is implied by this standard is **|CE($7,148,286)-CE($7,597,805)|=|100.0–133.1| = 33.1** points. On the other hand, the absolute value of the difference between **CE($7,100,000)** of **97.4** and **CE($7,500,000)** of **128.9,** is **31.5** points. Accordingly, since **33.1 > 31.5**, one is able to conclude that the difference is not material.

Even if a different standard for materiality is used, such as a percentage of surplus, the mechanics illustrated above are applicable.

Scenario 5. Risk Margin.

Line A. In this scenario the actuary would like to consider adding a risk margin to his reserve estimate. The standard the actuary selects that the risk margin must meet in order to be considered material is **25%** of σ. The question is what is the amount that corresponds to this additional potential risk margin. **25%** of σ for this line of business is **$438,659**. Next we calculate the spread in **CE**'s that is represented by the difference between the **CE** of the mean of the distribution and the **CE** of the proposed higher value (mean of the distribution plus the proposed risk margin of **25%** of σ). Thus the spread is given by **|CE($3,486,577) – CE($3,925,236)| = |100.0 – 115.0| = 15.0** points. We already know that the **CE** of the original estimate is given by **CE($3,000,000) = 79.8**. Thus we are looking for that amount which, when added to **$3,000,000** will yield a **CE** of

[10] $4,363,896 = the mean + one half the standard deviation = $3,486,577 + $877,319.

94.8 (=79.8 + 15.0). Consulting Table A2, and locating the cell with the coefficient of estimation that is closest to **94.8**, yields a total reserve of **$3,382,405**[11], which in turn yields a risk margin of **$382,405 (= $3,382,405 – $3,000,000)**.

Line B. In this scenario the actuary again would like to consider adding a risk margin to his reserve estimate. Once again the standard the actuary selects that the risk margin must meet in order to be considered material is **25%** of σ. The question then is what is the amount that corresponds to this additional potential risk margin. **25%** of σ for this line of business is **$224,260**. Next we calculate the spread in **CE**'s that is represented by the difference between the **CE** of the mean of the distribution and the **CE** of the proposed higher value (mean of the distribution plus the proposed risk margin of **25%** of σ). Thus the spread is given by **|CE($7,148,286) – CE($7,372,546)|=|100.0-119.2| = 19.2** points. We already know that the **CE** of the original estimate is given by **CE($7,100,000) = 97.4**. Thus we are looking for that amount which, when added to $7,100,000 will yield a **CE** of **116.6 (=97.4+19.2)**. Consulting Table B2 yield a total reserve of **$7,317,595**, which in turn yields a risk margin of **$217,595 (= $7,317,595 – $7,100,000)**.

We should note that the difference in the spread of the distributions is showing up rather remarkably in these examples. For example, using the same standard of materiality of **25%** of σ, the amount of risk margin for line A, **$382,405**, is equal to **13%** of the otherwise selected point estimate, while the amount of risk margin for line B, **$217,595**, is equal to **3%** of the otherwise selected estimate. Clearly the shape of the distribution is a significant variable in interpreting the reserve estimates as well as collateral issues related to them, such as risk margins.

Scenario 6. Reserve Strengthening.

For this scenario, suppose the actuary, having arrived at the estimates in Scenario 1, using assumptions and methods that were used the last time reserves were set, **A&Ms,**

[11] Once the appropriate cell is located, we simply use the midpoint of the corresponding interval.

is considering an alternative set of assumptions and methods, **A&Ms**. He has done the work and the new estimates are given as **$3,100,000** for line A and **$7,500,000** for line B. While it is clear that the new reserve estimates are higher than the original estimates, it is not clear that either one represents a reserve strengthening. Let us now consider if these new reserve estimates represent a strengthening.

Line A. We begin by noting that **CE($3,100,000|A&Ms) = 84.1**. Note that for the original reserve **CE($3,000,000|A&Ms) = 79.8**. The **4.3** point increase in **CE** does not suggest that this is a true strengthening. We can also invoke a standard of materiality which could be used to identify the increase in reserves as a strengthening or not. For our illustrative purposes we shall use the standard of **25%** of σ. This standard implies that a change in **CE** of less than **15** points is not material (See Scenario 5 for the derivation). Thus, using that standard we can conclude that the increase in reserves in this case is not material.

Line B. Once again we begin by noting that **CE($7,500,000|A&Ms) = 128.9**. And again note that for the original reserve **CE($7,100,000|A&Ms) = 97.4**. The increase in **CE** due to the revision in assumptions and methods is **31.5** points. Following the same standard of materiality of **0.25**σ yields a spread in **CE** of **19.2** points as the requirement to meet before we can pronounce a change to be material. In the instant case, the proposed change in reserves due to the revised methods and assumptions is **31.5** points, which is greater than the threshold standard of **19.2** points, and hence we are able to conclude that the proposed change in reserves would represent a strengthening.

VII. Concluding Remarks

The authors believe that the concept of the coefficient of estimation is useful in improving the clarity of statements made about a reserve estimate. The clarity is made possible because the actuary is using a fixed reference point (i.e., a landscape) against which various reserving statements and/or comparative statements are made. Having described and illustrated a process for bringing such clarity, we must conclude this paper with a series of remarks that need to be considered as an actuary uses this tool:

A. Emphasis on **t**. The reader will note the insistence on mentioning **t** at every point of the construction. This is an essential point of emphasis as the condition of reserves can be assessed only contemporaneously. All other statements about a reserve that make use of later development are made at a later time are statements about the runoff.

B. Uncertainty. Even though the coefficient of estimation is a useful tool – in that it gives both the actuary and a user an opportunity to understand the texture of the underlying probabilities and the associated uncertainty, using the coefficient of estimation does not eliminate the inherent uncertainty of reserve estimates.

C. Distribution Choices. The authors acknowledge that no two actuaries need select the same underlying distribution for a line of business. However, whichever distribution is used by the actuary, he needs to identify the rationale for such choice.

D. Standard of Materiality. We need to emphasize again that no two actuaries will necessarily come up with the same standard of materiality. While the actuary has this freedom to select a standard of materiality, the obvious consequent duty is that the actuary needs to make an appropriate disclosure whenever he changes the standard of materiality.

E. Convolutions. Even though the discussion above dealt with a single line of business, all observations and methods are equally applicable to a convolution distribution of two of more underlying loss distributions.

F. The Opining Actuary. The actuary who actually opines on the reasonableness of a given reserve is now in a position to actually set that reserve in the framework of the historically indicated reserve and the **CE** associated with that distribution.

G. The Reviewing Actuary. The constructions described in this paper make it possible to more clearly delineate the work of an actuary in constructing a reserve estimate and associated statements and the work of a actuary charged with reviewing the work of another.

H. Direct and Net Reserves. All constructions and observations apply equally to both direct and net experience. The underlying distribution for the direct case, although not necessarily so, can be expected to be different from the underlying distribution for the net case.

I. Reinsurance. All constructions and observations apply equally to reinsurance experience. We should note, however, that in the case of reinsurance applications the distributions can be expected to exhibit greater skew.

J. Adequacy. A high **CE**, by itself, does not necessarily imply that a high level of adequacy may be attached to the associated reserve estimate. Over time, the claims situation may change so that adequacy can be measured only against what is known at time **t**. The converse is true in the case of a low **CE**. These comments represent a special case of the general condition that actuaries should not rely exclusively on the size of the associated **CE** in evaluating the instantaneous adequacy that can be attached to a reserve estimate.

The authors believe that careful application of the coefficient of estimation can help illuminate the difficult task of making statements about reserve estimates. Perhaps over time it will be possible to identify benchmarks by line of business as well as other materiality benchmarks. Such benchmarks can emerge by company, by line of business, and/or by industry segment or in total. All such developments are capable of advancing casualty actuarial practice such that users of reserve estimates may be able to place greater reliance on the work of the actuary.

* * *

Table A1, page 1
Cumulative Frequency Distribution of IBNR
Commerical Auto Liability

Interval		Cumulative	Interval		Cumulative
>	≤	Frequency	>	≤	Frequency
1,255,810	1,281,586	5.1%	2,183,778	2,209,555	24.9%
1,281,586	1,307,363	5.4%	2,209,555	2,235,332	25.5%
1,307,363	1,333,140	6.1%	2,235,332	2,261,109	26.2%
1,333,140	1,358,917	6.4%	2,261,109	2,286,886	26.8%
1,358,917	1,384,694	7.0%	2,286,886	2,312,663	27.5%
1,384,694	1,410,471	7.4%	2,312,663	2,338,440	28.5%
1,410,471	1,436,248	7.8%	2,338,440	2,364,216	29.2%
1,436,248	1,462,025	8.3%	2,364,216	2,389,993	29.9%
1,462,025	1,487,802	8.7%	2,389,993	2,415,770	30.9%
1,487,802	1,513,579	9.2%	2,415,770	2,441,547	31.5%
1,513,579	1,539,356	9.6%	2,441,547	2,467,324	32.1%
1,539,356	1,565,132	10.2%	2,467,324	2,493,101	32.8%
1,565,132	1,590,909	10.7%	2,493,101	2,518,878	33.4%
1,590,909	1,616,686	11.1%	2,518,878	2,544,655	34.0%
1,616,686	1,642,463	11.7%	2,544,655	2,570,432	34.7%
1,642,463	1,668,240	12.2%	2,570,432	2,596,209	35.5%
1,668,240	1,694,017	12.7%	2,596,209	2,621,986	36.1%
1,694,017	1,719,794	13.2%	2,621,986	2,647,762	36.8%
1,719,794	1,745,571	13.8%	2,647,762	2,673,539	37.5%
1,745,571	1,771,348	14.3%	2,673,539	2,699,316	38.2%
1,771,348	1,797,125	14.9%	2,699,316	2,725,093	39.6%
1,797,125	1,822,901	15.6%	2,725,093	2,750,870	40.2%
1,822,901	1,848,678	16.2%	2,750,870	2,776,647	40.8%
1,848,678	1,874,455	16.9%	2,776,647	2,802,424	41.6%
1,874,455	1,900,232	17.5%	2,802,424	2,828,201	42.3%
1,900,232	1,926,009	18.1%	2,828,201	2,853,978	42.9%
1,926,009	1,951,786	18.7%	2,853,978	2,879,755	43.5%
1,951,786	1,977,563	19.3%	2,879,755	2,905,531	44.1%
1,977,563	2,003,340	19.9%	2,905,531	2,931,308	44.7%
2,003,340	2,029,117	20.5%	2,931,308	2,957,085	45.3%
2,029,117	2,054,894	21.0%	2,957,085	2,982,862	45.9%
2,054,894	2,080,671	21.6%	2,982,862	3,008,639	46.5%
2,080,671	2,106,447	22.3%	3,008,639	3,034,416	47.2%
2,106,447	2,132,224	23.0%	3,034,416	3,060,193	47.8%
2,132,224	2,158,001	23.6%	3,060,193	3,085,970	48.3%
2,158,001	2,183,778	24.2%	3,085,970	3,111,747	49.0%

Table A1, page 2					
Cumulative Frequency Distribution of IBNR Commerical Auto Liability					
Interval		Cumulative Frequency	Interval		Cumulative Frequency
>	≤		>	≤	
3,111,747	3,137,524	49.6%	4,039,715	4,065,492	69.3%
3,137,524	3,163,301	50.2%	4,065,492	4,091,269	69.7%
3,163,301	3,189,077	50.8%	4,091,269	4,117,046	70.1%
3,189,077	3,214,854	51.3%	4,117,046	4,142,823	70.6%
3,214,854	3,240,631	51.9%	4,142,823	4,168,600	71.2%
3,240,631	3,266,408	52.5%	4,168,600	4,194,377	71.7%
3,266,408	3,292,185	53.1%	4,194,377	4,220,154	72.1%
3,292,185	3,317,962	53.6%	4,220,154	4,245,931	72.6%
3,317,962	3,343,739	54.2%	4,245,931	4,271,707	73.0%
3,343,739	3,369,516	54.9%	4,271,707	4,297,484	73.4%
3,369,516	3,395,293	55.4%	4,297,484	4,323,261	73.7%
3,395,293	3,421,070	56.4%	4,323,261	4,349,038	74.1%
3,421,070	3,446,847	56.9%	4,349,038	4,374,815	74.5%
3,446,847	3,472,623	57.5%	4,374,815	4,400,592	74.8%
3,472,623	3,498,400	58.3%	4,400,592	4,426,369	75.2%
3,498,400	3,524,177	58.9%	4,426,369	4,452,146	75.5%
3,524,177	3,549,954	59.4%	4,452,146	4,477,923	75.9%
3,549,954	3,575,731	59.9%	4,477,923	4,503,700	76.4%
3,575,731	3,601,508	60.5%	4,503,700	4,529,477	76.7%
3,601,508	3,627,285	61.1%	4,529,477	4,555,253	77.1%
3,627,285	3,653,062	61.6%	4,555,253	4,581,030	77.6%
3,653,062	3,678,839	62.1%	4,581,030	4,606,807	77.9%
3,678,839	3,704,616	62.6%	4,606,807	4,632,584	78.2%
3,704,616	3,730,392	63.1%	4,632,584	4,658,361	78.6%
3,730,392	3,756,169	63.6%	4,658,361	4,684,138	78.9%
3,756,169	3,781,946	64.1%	4,684,138	4,709,915	79.2%
3,781,946	3,807,723	64.6%	4,709,915	4,735,692	79.5%
3,807,723	3,833,500	65.1%	4,735,692	4,761,469	79.8%
3,833,500	3,859,277	65.6%	4,761,469	4,787,246	80.1%
3,859,277	3,885,054	66.1%	4,787,246	4,813,022	80.3%
3,885,054	3,910,831	66.6%	4,813,022	4,838,799	80.6%
3,910,831	3,936,608	67.0%	4,838,799	4,864,576	80.9%
3,936,608	3,962,385	67.5%	4,864,576	4,890,353	81.5%
3,962,385	3,988,162	68.0%	4,890,353	4,916,130	81.8%
3,988,162	4,013,938	68.4%	4,916,130	4,941,907	82.1%
4,013,938	4,039,715	68.8%	4,941,907	4,967,684	82.4%
Mean = 3,486,577			Standard Deviation = 1,754,637		

Table A2, page 1					
Table of Coefficients of Estimation					
Commerical Auto Liability					
Interval		Coeff. Of	Interval		Coeff. Of
>	≤	Estimation	>	≤	Estimation
1,255,810	1,281,586	8.7	2,183,778	2,209,555	42.7
1,281,586	1,307,363	9.3	2,209,555	2,235,332	43.8
1,307,363	1,333,140	10.4	2,235,332	2,261,109	44.9
1,333,140	1,358,917	11.0	2,261,109	2,286,886	46.0
1,358,917	1,384,694	12.0	2,286,886	2,312,663	47.2
1,384,694	1,410,471	12.7	2,312,663	2,338,440	48.9
1,410,471	1,436,248	13.4	2,338,440	2,364,216	50.1
1,436,248	1,462,025	14.2	2,364,216	2,389,993	51.3
1,462,025	1,487,802	15.0	2,389,993	2,415,770	53.0
1,487,802	1,513,579	15.8	2,415,770	2,441,547	54.0
1,513,579	1,539,356	16.6	2,441,547	2,467,324	55.1
1,539,356	1,565,132	17.5	2,467,324	2,493,101	56.2
1,565,132	1,590,909	18.3	2,493,101	2,518,878	57.3
1,590,909	1,616,686	19.1	2,518,878	2,544,655	58.4
1,616,686	1,642,463	20.0	2,544,655	2,570,432	59.5
1,642,463	1,668,240	20.9	2,570,432	2,596,209	60.8
1,668,240	1,694,017	21.7	2,596,209	2,621,986	61.9
1,694,017	1,719,794	22.7	2,621,986	2,647,762	63.1
1,719,794	1,745,571	23.6	2,647,762	2,673,539	64.4
1,745,571	1,771,348	24.6	2,673,539	2,699,316	65.5
1,771,348	1,797,125	25.5	2,699,316	2,725,093	68.0
1,797,125	1,822,901	26.7	2,725,093	2,750,870	69.0
1,822,901	1,848,678	27.7	2,750,870	2,776,647	70.1
1,848,678	1,874,455	28.9	2,776,647	2,802,424	71.4
1,874,455	1,900,232	30.0	2,802,424	2,828,201	72.5
1,900,232	1,926,009	31.0	2,828,201	2,853,978	73.6
1,926,009	1,951,786	32.1	2,853,978	2,879,755	74.6
1,951,786	1,977,563	33.1	2,879,755	2,905,531	75.7
1,977,563	2,003,340	34.1	2,905,531	2,931,308	76.7
2,003,340	2,029,117	35.1	2,931,308	2,957,085	77.8
2,029,117	2,054,894	36.1	2,957,085	2,982,862	78.8
2,054,894	2,080,671	37.1	2,982,862	3,008,639	79.8
2,080,671	2,106,447	38.2	3,008,639	3,034,416	80.9
2,106,447	2,132,224	39.4	3,034,416	3,060,193	81.9
2,132,224	2,158,001	40.4	3,060,193	3,085,970	82.9
2,158,001	2,183,778	41.5	3,085,970	3,111,747	84.1

Table A2, page 2					
Table of Coefficients of Estimation					
Commerical Auto Liability					
Interval		**Coeff. Of Estimation**	**Interval**		**Coeff. Of Estimation**
>	≤		>	≤	
3,111,747	3,137,524	85.1	4,039,715	4,065,492	118.8
3,137,524	3,163,301	86.1	4,065,492	4,091,269	119.5
3,163,301	3,189,077	87.1	4,091,269	4,117,046	120.3
3,189,077	3,214,854	88.1	4,117,046	4,142,823	121.1
3,214,854	3,240,631	89.1	4,142,823	4,168,600	122.1
3,240,631	3,266,408	90.1	4,168,600	4,194,377	122.9
3,266,408	3,292,185	91.0	4,194,377	4,220,154	123.6
3,292,185	3,317,962	92.0	4,220,154	4,245,931	124.6
3,317,962	3,343,739	93.0	4,245,931	4,271,707	125.2
3,343,739	3,369,516	94.1	4,271,707	4,297,484	125.9
3,369,516	3,395,293	95.0	4,297,484	4,323,261	126.5
3,395,293	3,421,070	96.7	4,323,261	4,349,038	127.1
3,421,070	3,446,847	97.6	4,349,038	4,374,815	127.8
3,446,847	3,472,623	98.6	4,374,815	4,400,592	128.4
3,472,623	3,498,400	100.0	4,400,592	4,426,369	129.0
3,498,400	3,524,177	101.0	4,426,369	4,452,146	129.6
3,524,177	3,549,954	101.9	4,452,146	4,477,923	130.2
3,549,954	3,575,731	102.7	4,477,923	4,503,700	131.1
3,575,731	3,601,508	103.8	4,503,700	4,529,477	131.6
3,601,508	3,627,285	104.7	4,529,477	4,555,253	132.2
3,627,285	3,653,062	105.6	4,555,253	4,581,030	133.1
3,653,062	3,678,839	106.6	4,581,030	4,606,807	133.6
3,678,839	3,704,616	107.4	4,606,807	4,632,584	134.2
3,704,616	3,730,392	108.3	4,632,584	4,658,361	134.8
3,730,392	3,756,169	109.2	4,658,361	4,684,138	135.3
3,756,169	3,781,946	110.0	4,684,138	4,709,915	135.8
3,781,946	3,807,723	110.8	4,709,915	4,735,692	136.3
3,807,723	3,833,500	111.7	4,735,692	4,761,469	136.8
3,833,500	3,859,277	112.5	4,761,469	4,787,246	137.3
3,859,277	3,885,054	113.3	4,787,246	4,813,022	137.8
3,885,054	3,910,831	114.2	4,813,022	4,838,799	138.3
3,910,831	3,936,608	115.0	4,838,799	4,864,576	138.8
3,936,608	3,962,385	115.8	4,864,576	4,890,353	139.8
3,962,385	3,988,162	116.6	4,890,353	4,916,130	140.3
3,988,162	4,013,938	117.3	4,916,130	4,941,907	140.9
4,013,938	4,039,715	118.1	4,941,907	4,967,684	141.3

Chart A1

Commercial Auto Liability
Cumulative Frequency Distribution of IBNR

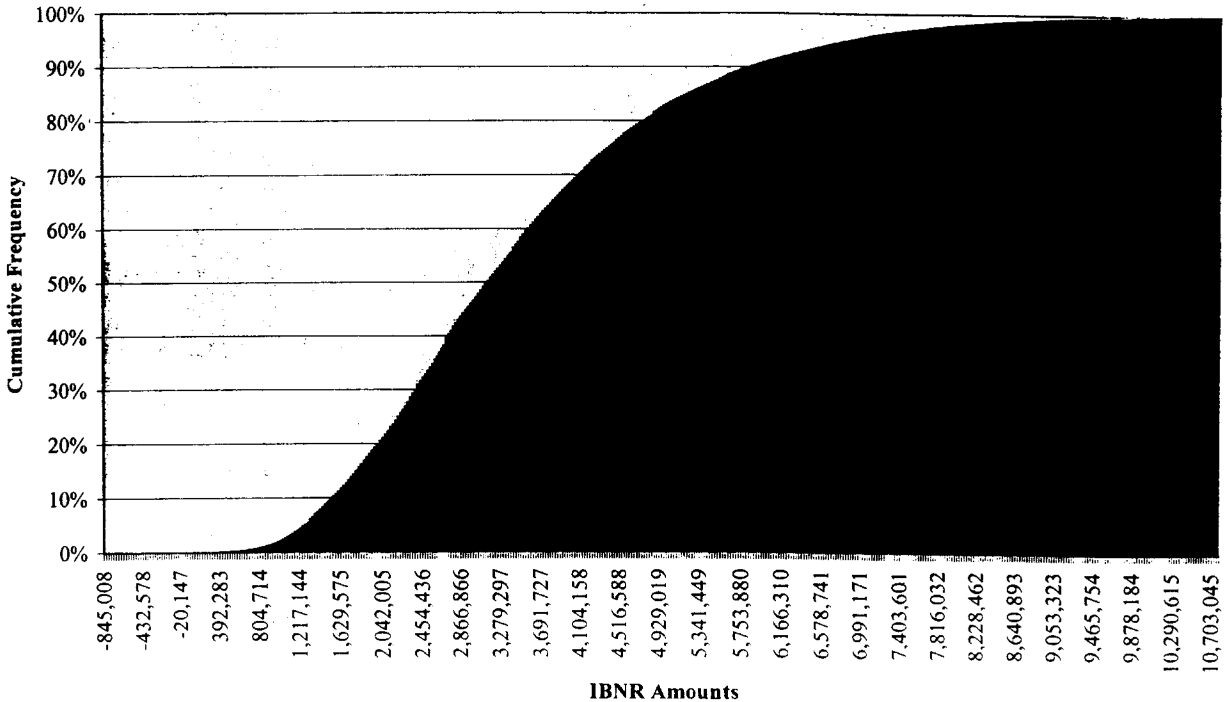


Chart A2

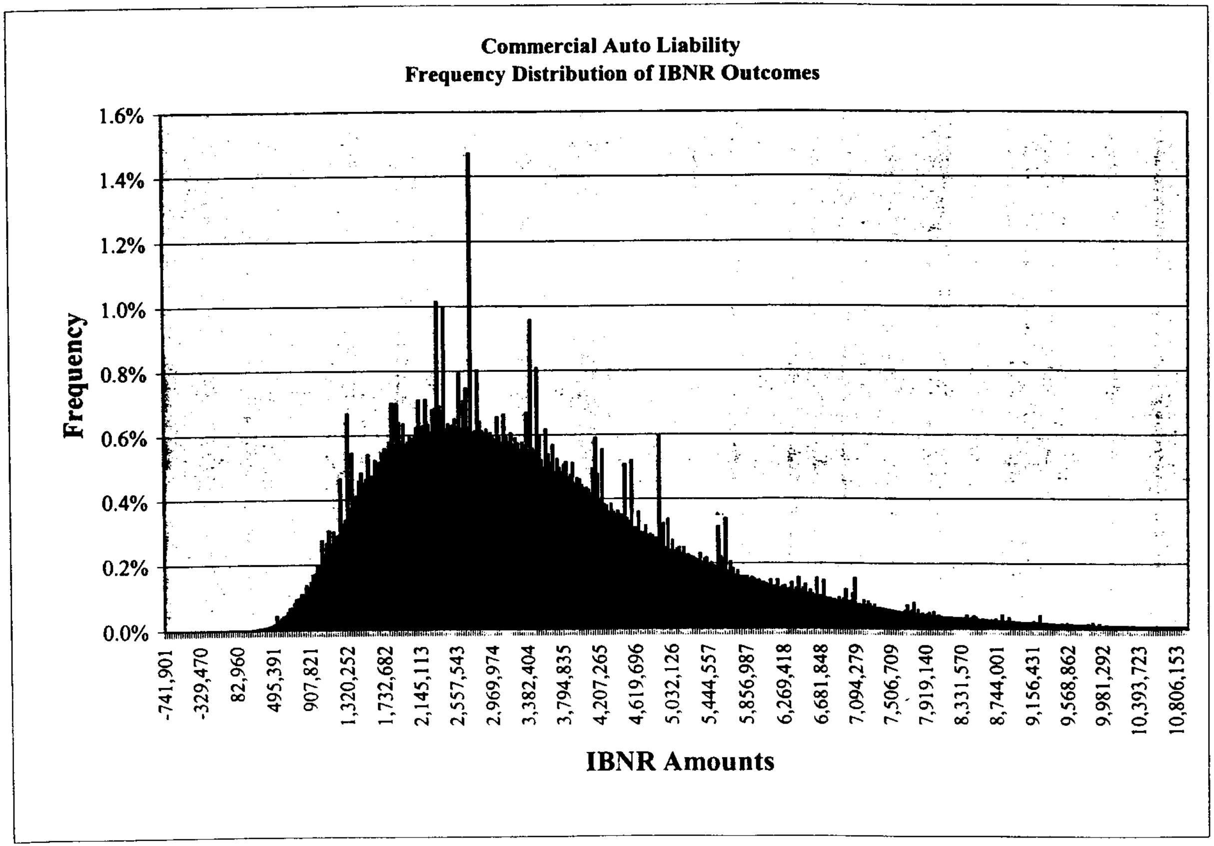

Commercial Auto Liability
Frequency Distribution of IBNR Outcomes
Frequency
1.6%
1.4%
1.2%
1.0%
0.8%
0.6%
0.4%
0.2%
0.0%
-741,901
-329,470
82,960
495,391
907,821
1,320,252
1,732,682
2,145,113
2,557,543
2,969,974
3,382,404
3,794,835
4,207,265
4,619,696
5,032,126
5,444,557
5,856,987
6,269,418
6,681,848
7,094,279
7,506,709
7,919,140
8,331,570
8,744,001
9,156,431
9,568,862
9,981,292
10,393,723
10,806,153
IBNR Amounts

Chart A3

Commercial Auto Liability
Graphic Representation of the Coefficients of Estimation

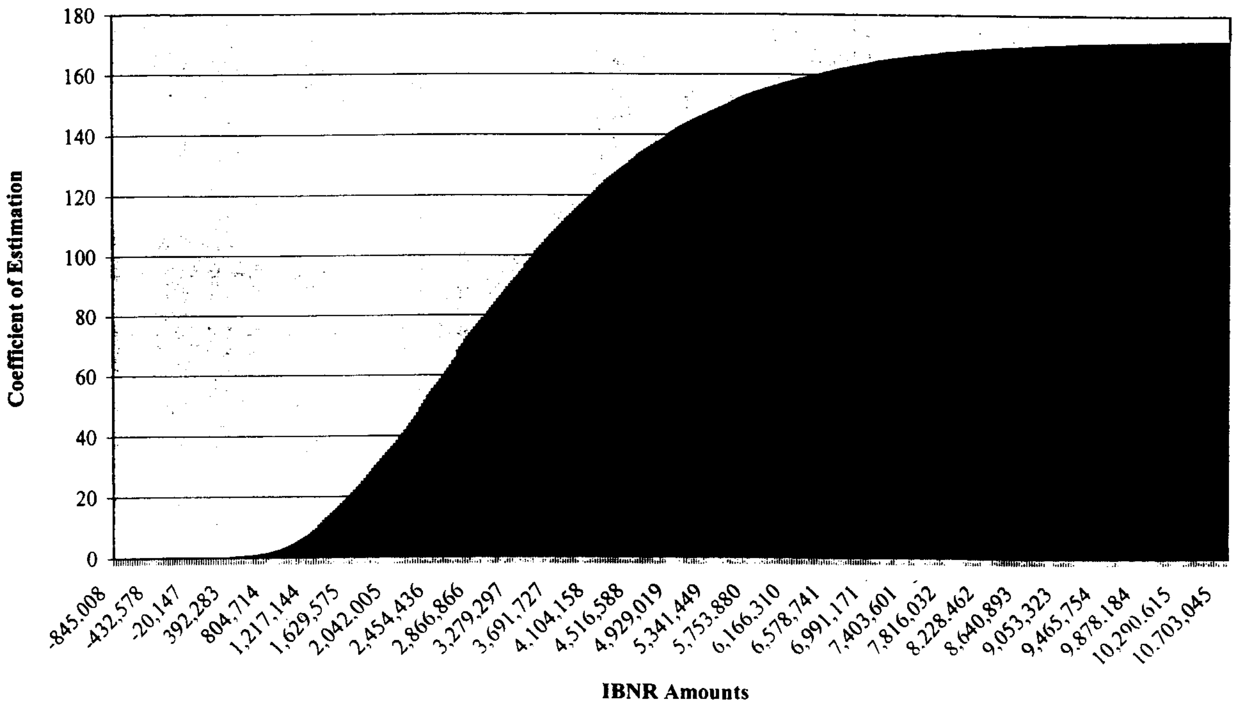


Table B1					
Cumulative Frequency Distribution Private Passenger Auto Liability					
Interval		**Cumulative Frequency**	**Interval**		**Cumulative Frequency**
>	≤		>	≤	
5,757,171	5,803,751	4.8%	7,434,044	7,480,624	68.0%
5,803,750	5,850,330	5.7%	7,480,624	7,527,204	69.3%
5,850,330	5,896,910	6.4%	7,527,204	7,573,784	70.5%
5,896,910	5,943,490	7.2%	7,573,783	7,620,363	71.6%
5,943,490	5,990,070	8.0%	7,620,363	7,666,943	73.9%
5,990,070	6,036,650	8.8%	7,666,943	7,713,523	75.9%
6,036,650	6,083,230	9.7%	7,713,523	7,760,103	77.3%
6,083,229	6,129,809	11.5%	7,760,103	7,806,683	78.4%
6,129,809	6,176,389	12.6%	7,806,682	7,853,262	79.2%
6,176,389	6,222,969	14.3%	7,853,262	7,899,842	80.3%
6,222,969	6,269,549	15.8%	7,899,842	7,946,422	81.2%
6,269,549	6,316,129	18.6%	7,946,422	7,993,002	82.0%
6,316,128	6,362,708	19.8%	7,993,002	8,039,582	84.5%
6,362,708	6,409,288	21.3%	8,039,582	8,086,162	86.0%
6,409,288	6,455,868	22.5%	8,086,161	8,132,741	86.8%
6,455,868	6,502,448	25.4%	8,132,741	8,179,321	87.4%
6,502,448	6,549,028	27.3%	8,179,321	8,225,901	88.1%
6,549,027	6,595,607	29.7%	8,225,901	8,272,481	88.7%
6,595,607	6,642,187	31.4%	8,272,481	8,319,061	89.2%
6,642,187	6,688,767	33.1%	8,319,060	8,365,640	89.7%
6,688,767	6,735,347	34.7%	8,365,640	8,412,220	90.3%
6,735,347	6,781,927	36.3%	8,412,220	8,458,800	91.6%
6,781,927	6,828,507	37.8%	8,458,800	8,505,380	92.4%
6,828,506	6,875,086	42.6%	8,505,380	8,551,960	92.8%
6,875,086	6,921,666	44.6%	8,551,960	8,598,540	93.2%
6,921,666	6,968,246	47.4%	8,598,539	8,645,119	93.9%
6,968,246	7,014,826	49.1%	8,645,119	8,691,699	94.2%
7,014,826	7,061,406	50.7%	8,691,699	8,738,279	94.6%
7,061,405	7,107,985	52.4%	8,738,279	8,784,859	95.3%
7,107,985	7,154,565	53.8%	8,784,859	8,831,439	95.6%
7,154,565	7,201,145	55.3%	8,831,438	8,878,018	96.1%
7,201,145	7,247,725	56.9%	8,878,018	8,924,598	96.3%
7,247,725	7,294,305	59.5%	8,924,598	8,971,178	96.5%
7,294,305	7,340,885	61.5%	8,971,178	9,017,758	96.8%
7,340,884	7,387,464	64.1%	9,017,758	9,064,338	97.0%
7,387,464	7,434,044	65.4%	9,064,337	9,110,917	97.1%
Mean = 7,148,286			Standard Deviation = 899,038		

Table B2					
Table of Coefficients of Estimation					
Private Passenger Auto Liability					
Interval		**Coeff. Of**	**Interval**		**Coeff. Of**
>	≤	**Estimation**	>	≤	**Estimation**
5,757,171	5,803,751	9.0	7,434,044	7,480,624	126.4
5,803,750	5,850,330	10.6	7,480,624	7,527,204	128.9
5,850,330	5,896,910	11.8	7,527,204	7,573,784	131.1
5,896,910	5,943,490	13.3	7,573,783	7,620,363	133.1
5,943,490	5,990,070	14.9	7,620,363	7,666,943	137.4
5,990,070	6,036,650	16.4	7,666,943	7,713,523	141.1
6,036,650	6,083,230	18.1	7,713,523	7,760,103	143.7
6,083,229	6,129,809	21.5	7,760,103	7,806,683	145.7
6,129,809	6,176,389	23.5	7,806,682	7,853,262	147.3
6,176,389	6,222,969	26.5	7,853,262	7,899,842	149.2
6,222,969	6,269,549	29.4	7,899,842	7,946,422	150.9
6,269,549	6,316,129	34.5	7,946,422	7,993,002	152.5
6,316,128	6,362,708	36.8	7,993,002	8,039,582	157.1
6,362,708	6,409,288	39.5	8,039,582	8,086,162	159.8
6,409,288	6,455,868	41.8	8,086,161	8,132,741	161.3
6,455,868	6,502,448	47.2	8,132,741	8,179,321	162.5
6,502,448	6,549,028	50.7	8,179,321	8,225,901	163.8
6,549,027	6,595,607	55.1	8,225,901	8,272,481	164.9
6,595,607	6,642,187	58.4	8,272,481	8,319,061	165.8
6,642,187	6,688,767	61.6	8,319,060	8,365,640	166.8
6,688,767	6,735,347	64.4	8,365,640	8,412,220	167.8
6,735,347	6,781,927	67.5	8,412,220	8,458,800	170.2
6,781,927	6,828,507	70.2	8,458,800	8,505,380	171.7
6,828,506	6,875,086	79.2	8,505,380	8,551,960	172.4
6,875,086	6,921,666	82.8	8,551,960	8,598,540	173.2
6,921,666	6,968,246	88.2	8,598,539	8,645,119	174.5
6,968,246	7,014,826	91.3	8,645,119	8,691,699	175.1
7,014,826	7,061,406	94.2	8,691,699	8,738,279	175.8
7,061,405	7,107,985	97.4	8,738,279	8,784,859	177.1
7,107,985	7,154,565	100.0	8,784,859	8,831,439	177.8
7,154,565	7,201,145	102.8	8,831,438	8,878,018	178.6
7,201,145	7,247,725	105.7	8,878,018	8,924,598	179.0
7,247,725	7,294,305	110.6	8,924,598	8,971,178	179.5
7,294,305	7,340,885	114.3	8,971,178	9,017,758	179.8
7,340,884	7,387,464	119.2	9,017,758	9,064,338	180.2
7,387,464	7,434,044	121.6	9,064,337	9,110,917	180.6

Chart B1

Private Passenger Auto Liability
Cumulative Frequency Distribution of IBNR Outcomes

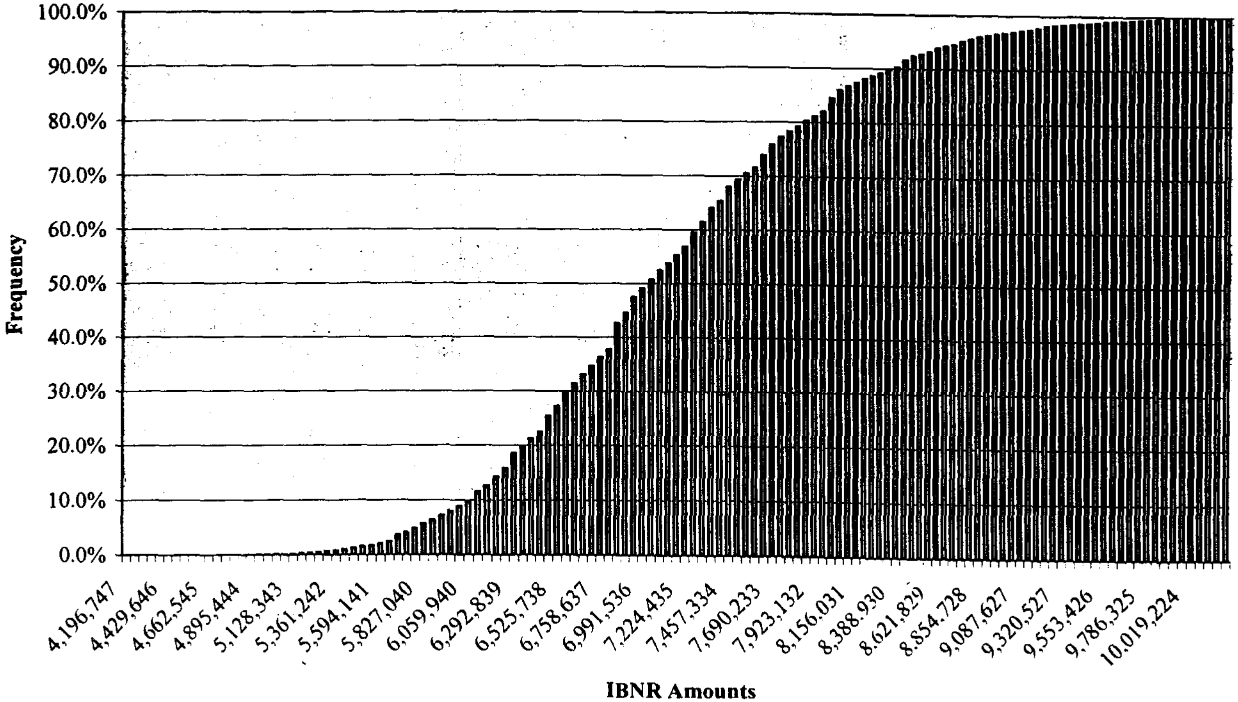


Chart B2

Private Passenger Auto Liability
Frequency Distribution of IBNR Outcomes

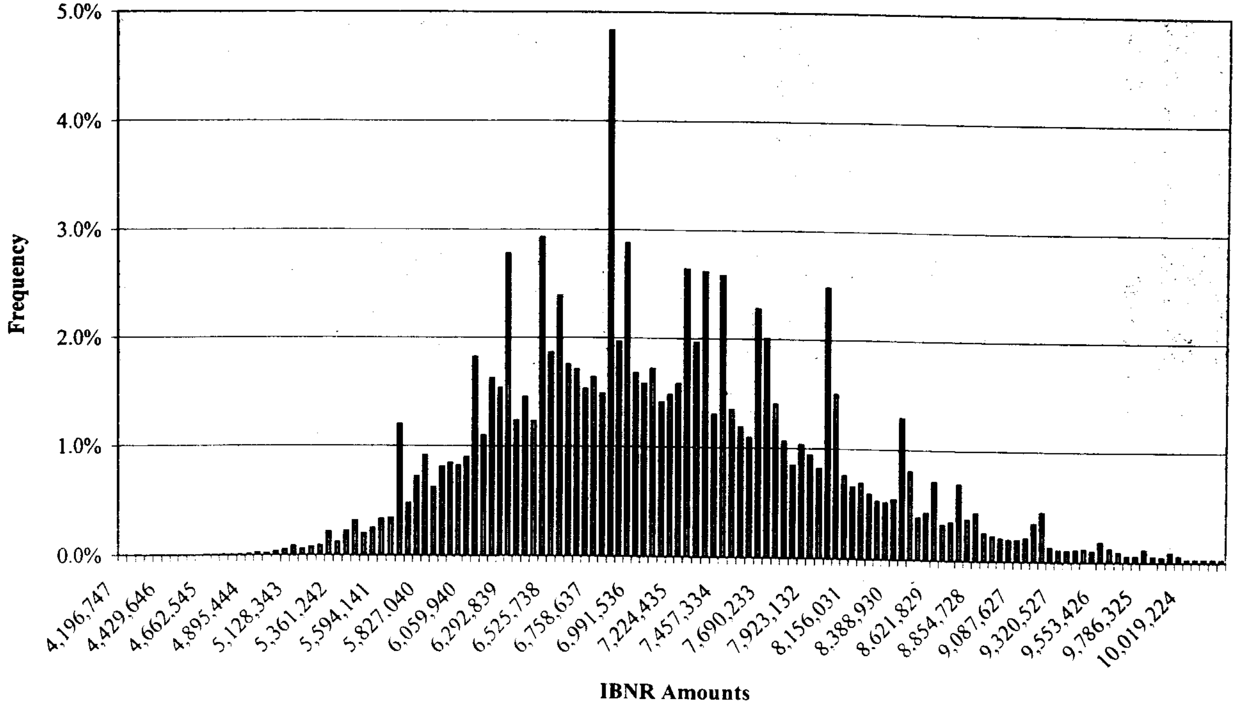
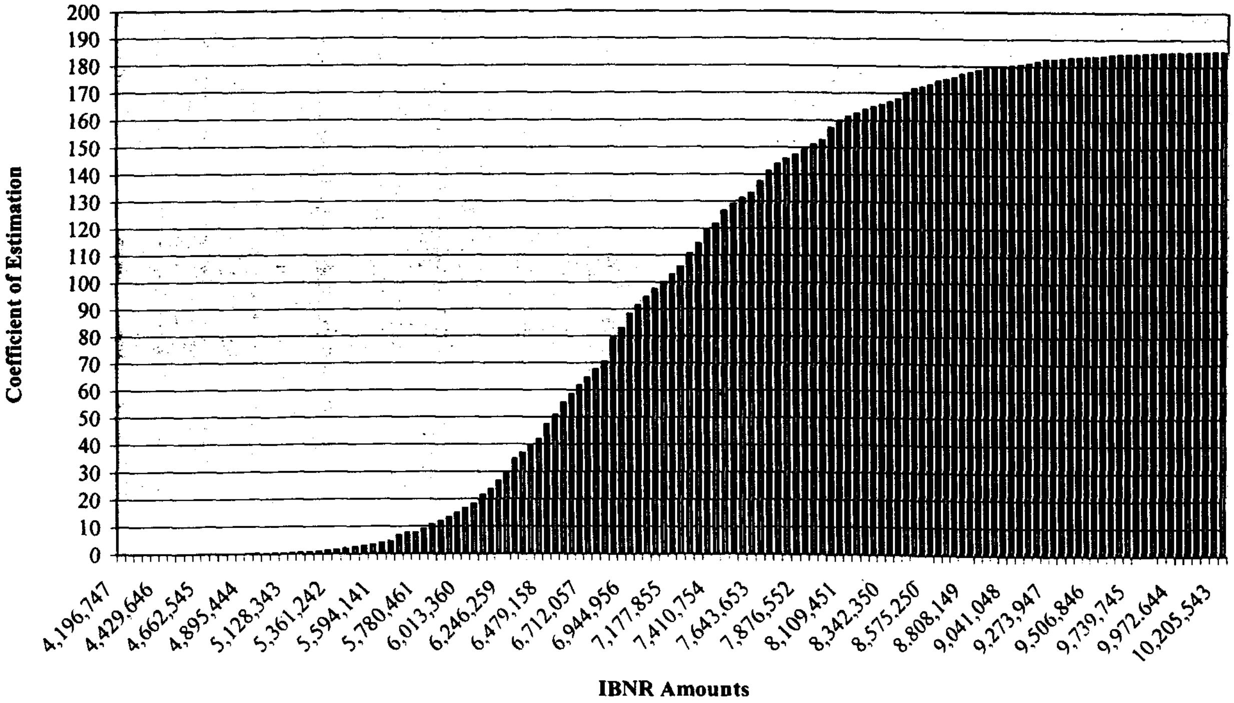


Chart B3

Private Passenger Auto Liability
Graphic Representation of the Coefficient of Estimation



LDF Curve-Fitting and Stochastic Reserving: A Maximum Likelihood Approach

David R. Clark, FCAS, MAAA

LDF Curve-Fitting and Stochastic Reserving: A Maximum Likelihood Approach

or

How to Increase Reserve Variability with Less Data

David R. Clark
American Re-Insurance

2003 Reserves Call Paper Program

Abstract

An application of Maximum Likelihood Estimation (MLE) theory is demonstrated for modeling the distribution of loss development based on data available in the common triangle format. This model is used to estimate future loss emergence, and the variability around that estimate. The value of using an exposure base to supplement the data in a development triangle is demonstrated as a means of reducing variability. Practical issues concerning estimation error and extrapolation are also discussed.

The author gratefully acknowledges the help and encouragement of the following people:
Dick Currie, Jeff Davis, Leigh Halliwell, Don Mango, Dave Spiegler, and Chuck Thayer.

Introduction

Many papers have been written on the topic of statistical modeling of the loss reserving process. The present paper will focus on one such model, making use of the theory of maximum likelihood estimation (MLE) along with the common Loss Development Factor and Cape Cod techniques. After a review of the underlying theory, the bulk of this paper is devoted to a practical example showing how to make use of the techniques and how to interpret the output.

Before beginning a discussion of a formal model of loss reserving, it is worth re-stating the objectives in creating such a model.

The primary objective is to provide a tool that describes the loss emergence (either reporting or payment) phenomenon in simple mathematical terms as a guide to selecting amounts for carried reserves. Given the complexity of the insurance business, it should never be expected that a model will replace a knowledgeable analyst, but the model can become one key indication to assist them in selecting the reserve.

A secondary objective is to provide a means of estimating the range of possible outcomes around the "expected" reserve. The range of reserves is due to both random "process" variance, and the uncertainty in the estimate of the expected value.

From these objectives, we see that a statistical loss reserving model has two key elements:

- The expected amount of loss to emerge in some time period
- The distribution of actual emergence around the expected value

These two elements of our model will be described in detail in the first two sections of this paper. The full paper is outlined as follows:

Section 1:	Expected Loss Emergence
Section 2:	The Distribution of Actual Loss Emergence and Maximum Likelihood
Section 3:	Key Assumptions of the Model
Section 4:	A Practical Example
Section 5:	Comments and Conclusion

The practical example includes a demonstration of the reduction in variability possible from the use of an exposure base in the Cape Cod reserving method. Extensions of the model for estimating variability of the prospective loss projection or of discounted reserves are discussed more briefly.

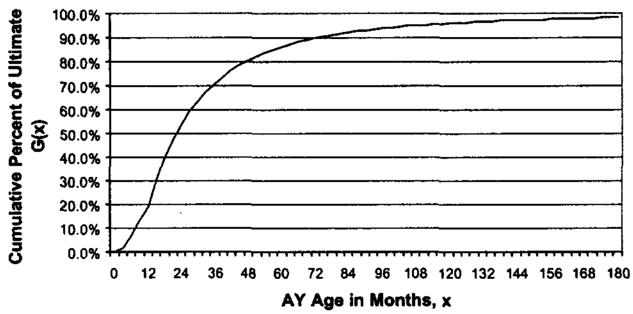
Most of the material presented in this paper makes use of maximum likelihood theory that has already been described more rigorously elsewhere. The mathematics presented here is sufficient for the reader to reproduce the calculations in the examples given, but the focus will be on practical issues rather than on the statistical theory itself.

Section 1: Expected Loss Emergence

Our model will estimate the expected amount of loss to emerge based on a) an estimate of the ultimate loss by year, and b) an estimate of the pattern of loss emergence.

For the expected emergence pattern, we need a pattern that moves from 0 to 100% as time moves from 0 to 8. For our model, we will assume that this pattern is described using the form of a cumulative distribution function[1] (CDF), since a library of such curves is readily available.

$$G(x) = 1/LDF_x \quad = \text{cumulative \% reported (or paid) as of time x}$$



We will assume that the time index "x" represents the time from the "average" accident date to the evaluation date. The details for approximating different exposure periods (e.g., accident year versus policy year) are given in Appendix B.

For convenience, the model will include two familiar curve forms: Weibull and Loglogistic. Each of these curve forms can be parameterized with a scale θ and a shape ω ("warp"). The Loglogistic curve is familiar to many actuaries under the name "inverse

[1] We are using the form of the distribution function, but do not mean to imply any probabilistic model. The paper by Weissner [9] makes the report lag itself the random variable. By contrast, the loss dollars will be the random variable in our application.

power" (see Sherman[2] [8]), and will be considered the benchmark result. The Weibull will generally provide a smaller "tail" factor than the Loglogistic.

The Loglogistic curve has the form:

$$G(x \mid \omega,\theta) = \frac{x^{\omega}}{x^{\omega}+\theta^{\omega}} \qquad\qquad LDF_x = 1+\theta^{\omega}\cdot x^{-\omega}$$

The Weibull curve has the form:

$$G(x \mid \omega,\theta) = 1-\exp\left(-(x/\theta)^{\omega}\right)$$

In using these curve forms, we are assuming that the expected loss emergence will move from 0% to 100% in a strictly increasing pattern. The model will still work if some actual points show decreasing losses, but if there is real expected negative development (e.g., lines of business with significant salvage recoveries) then a different model should be used.

There are several advantages to using parameterized curves to describe the expected emergence pattern. First, the estimation problem is simplified because we only need to estimate the two parameters. Second, we can use data that is not strictly from a triangle with evenly spaced evaluation dates – such as the frequent case in which the latest diagonal is only nine months from the second latest diagonal. Third, the final indicated pattern is a smooth curve and does not follow every random movement in the historical age-to-age factors.

The next step in estimating the amount of loss emergence by period is to apply the emergence pattern G(x), to an estimate of the ultimate loss by accident year.

Our model will base the estimate of the ultimate loss by year on one of two methods: either the LDF or the Cape Cod method. The LDF method assumes that the ultimate loss

[2] Sherman actually applies the inverse power curve to the link ratios between ages. Our model will apply this curve to the age-to-ultimate pattern.

amount in each accident year is independent of the losses in other years. The Cape Cod method assumes that there is a known relationship between the amount of ultimate loss expected in each of the years in the historical period, and that this relationship is identified by an exposure base. The exposure base is usually onlevel premium, but can be any other index (such as sales or payroll), which is reasonably assumed to be proportional to expected loss.

The expected loss for a given period will be denoted:

$\mu_{AY;x,y}$ = expected incremental loss dollars in accident year AY between ages x and y

Then the two methods for the expected loss emergence are:

Method #1: "Cape Cod"

$$\mu_{AY;x,y} = \text{Premium}_{AY} \cdot ELR \cdot [G(y \mid \omega,\theta) - G(x \mid \omega,\theta)]$$

Three parameters: ELR, ω, θ

Method #2: "LDF"

$$\mu_{AY;x,y} = ULT_{AY} \cdot [G(y \mid \omega,\theta) - G(x \mid \omega,\theta)]$$

n+2 Parameters: n Accident Years (one ULT for each AY) + ω, θ

While both of these methods are available for use in estimating reserves, Method #1 will generally be preferred. Because we are working with data summarized into annual blocks as a development triangle, there will be relatively few data points included in the model (one data point for each "cell" in the triangle). There is a real problem with overparameterization when the LDF method is used.

For example, if we have a triangle for ten accident years then we have provided the model with 55 data points. The Cape Cod method requires estimation of 3 parameters, but the LDF method requires estimation of 12 parameters.

The Cape Cod method may have somewhat higher process variance estimated, but will usually produce a significantly smaller estimation error. This is the value of the information in the exposure base provided by the user[3]. In short: *the more information that we can give to the model, the smaller the reserve variability due to estimation error.*

The fact that variance can be reduced by incorporating more information into a reserve analysis is, of course, the point of our ironic subtitle: How to Increase Reserve Variability with Less Data. The point is obvious, but also easy to overlook. The reduction in variability is important even to those who do not explicitly calculate reserve ranges because it still guides us towards better estimation methods: lower variance implies a better reserve estimate.

[3] Halliwell [2] provides additional arguments for the use of an exposure index. See especially pages 441 - 443.

Section 2: The Distribution of Actual Loss Emergence and Maximum Likelihood

Having defined the model for the expected loss emergence, we need to estimate the "best" parameters for that model and, as a secondary goal, estimate the variance around the expected value. Both of these steps will be accomplished making use of maximum likelihood theory.

The variance will be estimated in two pieces: process variance (the "random" amount) and parameter variance (the uncertainty in our estimator).

2.1 Process Variance

The curve $G(x \mid \omega, \theta)$ represents the expected loss emergence pattern. The actual loss emergence will have a distribution around this expectation.

We assume that the loss in any period has a constant ratio of variance/mean[4]:

$$\frac{Variance}{Mean} = \sigma^2 \approx \frac{1}{n-p} \cdot \sum_{AY,t}^{n} \frac{\left(c_{AY,t} - \mu_{AY,t}\right)^2}{\mu_{AY,t}}$$

where p = # of parameters

$c_{AY,t}$ = actual incremental loss emergence

$\mu_{AY,t}$ = expected incremental loss emergence

(this is recognized as being equivalent to a chi-square error term)

For estimating the parameters of our model, we will further assume that the actual incremental loss emergence "c" follows an over-dispersed Poisson distribution. That is, the loss dollars will be a Poisson random variable times a scaling factor equal to σ^2.

[4] This assumption will be tested by analysis of residuals in our example.

Standard Poisson: $\Pr(x) = \dfrac{\lambda^x \cdot e^{-\lambda}}{x!}$ $\qquad E[x] = Var(x) = \lambda$

Actual Loss: $c = x \cdot \sigma^2$ $\qquad \Pr(c) = \dfrac{\lambda^{c/\sigma^2} \cdot e^{-\lambda}}{(c/\sigma^2)!}$ $\qquad E[c] = \lambda \cdot \sigma^2 = \mu$

$$Var(c) = \lambda \cdot \sigma^4 = \mu \cdot \sigma^2$$

The "over-dispersed Poisson" sounds strange when it is first encountered, but it quickly proves to have some key advantages. First, inclusion of the scaling factor allows us to match the first and second moments of any distribution, which gives the model a high degree of flexibility. Second, maximum likelihood estimation exactly produces the LDF and Cape Cod estimates of ultimate, so the results can be presented in a format familiar to reserving actuaries.

The fact that the distribution of ultimate reserves is approximated by a discretized curve should not be cause for concern. The scale factor σ^2 is generally small compared to the mean, so little precision is lost. Also, the use of a discrete distribution allows for a mass point at zero, representing the cases in which no change in loss is seen in a given development increment.

Finally, we should remember that this maximum likelihood method is intended to produce the mean and variance of the distribution of reserves. Having estimated those two numbers, we are still free to switch to a different distribution form when the results are used in other applications.

2.2 The Likelihood Function – Finding the "Best" Parameters

The likelihood function is:

$$\text{Likelihood} = \prod_i \Pr(c_i) = \prod_i \frac{\lambda_i^{c_i/\sigma^2} \cdot e^{-\lambda_i}}{(c_i/\sigma^2)!} = \prod_i \frac{(\mu_i/\sigma^2)^{c_i/\sigma^2} \cdot e^{-\mu_i/\sigma^2}}{(c_i/\sigma^2)!}$$

This can be maximized using the logarithm of the likelihood function:

$$\text{LogLikelihood} = \sum_i (c_i/\sigma^2)\cdot\ln(\mu_i/\sigma^2) - \mu_i/\sigma^2 - \ln((c_i/\sigma^2)!)$$

Which is equivalent to maximizing:

$$\boxed{\ell = \sum_i c_i \cdot \ln(\mu_i) - \mu_i} \qquad \text{if } \sigma^2 \text{ is assumed to be known}$$

Maximum likelihood estimators of the parameters are found by setting the first derivatives of the loglikelihood function ℓ equal to zero:

$$\frac{\partial \ell}{\partial ELR} = \frac{\partial \ell}{\partial \theta} = \frac{\partial \ell}{\partial \omega} = 0$$

For "Model #1: Cape Cod", the loglikelihood function becomes:

$$\ell = \sum_{i,t} (c_{i,t} \cdot \ln(ELR \cdot P_i \cdot [G(x_t) - G(x_{t-1})]) - ELR \cdot P_i \cdot [G(x_t) - G(x_{t-1})])$$

where $c_{i,t}$ = actual loss in accident year i, development period t

P_i = Premium for accident year i

x_{t-1} = beginning age for development period t

x_t = ending age for development period t

$$\frac{\partial \ell}{\partial ELR} = \sum_{i,t}\left(\frac{c_{i,t}}{ELR} - P_i \cdot [G(x_t) - G(x_{t-1})]\right)$$

For $\dfrac{\partial \ell}{\partial ELR} = 0, \qquad ELR = \dfrac{\sum_{i,t} c_{i,t}}{\sum_{i,t} P_i \cdot [G(x_t) - G(x_{t-1})]}$

The MLE estimate for ELR is therefore equivalent to the "Cape Cod" Ultimate. It can be set based on θ and ω, and so reduce the problem to be solved to two parameters instead of three.

For "Model #2: LDF", the loglikelihood function becomes:

$$\ell = \sum_{i,t} \left(c_{i,t} \cdot \ln(ULT_i \cdot [G(x_t) - G(x_{t-1})]) - ULT_i \cdot [G(x_t) - G(x_{t-1})] \right)$$

$$\frac{\partial \ell}{\partial ULT_i} = \sum_t \left(\frac{c_{i,t}}{ULT_i} - [G(x_t) - G(x_{t-1})] \right)$$

For $\dfrac{\partial \ell}{\partial ULT_i} = 0, \qquad ULT_i = \dfrac{\sum_t c_{i,t}}{\sum_t [G(x_t) - G(x_{t-1})]}$

The MLE estimate for each ULT_i is therefore equivalent to the "LDF Ultimate"[5]. It can also be set based on θ and ω, and to again reduce the problem to be solved to two parameters instead of $n+2$.

A final comment worth noting is that the maximum loglikelihood function never takes the logarithm of the actual incremental development $c_{i,t}$. The model will work even if some of these amounts are zero or negative.

[5] See Mack [5], Appendix A, for a further discussion of this relationship.

2.3 Parameter Variance[6]

The second step is to find the variance in the estimate of the parameters. This is done based on the Rao-Cramer approximation, using the second derivative information matrix I, and is commonly called the "Delta Method" (c.f. Klugman, et al [3], page 67).

The second derivative information matrix for the "Cape Cod Method" is 3x3 and assumes the same ELR for all accident years:

$$I = \begin{bmatrix} \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial ELR^2} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial ELR \, \partial \omega} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial ELR \, \partial ?} \\ \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial \omega \, \partial ELR} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial \omega^2} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial \omega \, \partial ?} \\ \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial ? \, \partial ELR} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial ? \, \partial \omega} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial ?^2} \end{bmatrix}$$

The covariance matrix is calculated using the inverse of the Information matrix:

$$\Sigma = \begin{bmatrix} Var(ELR) & Cov(ELR,\omega) & Cov(ELR,\theta) \\ Cov(\omega, ELR) & Var(\omega) & Cov(\omega,\theta) \\ Cov(\theta, ELR) & Cov(\theta,\omega) & Var(\theta) \end{bmatrix} \geq -\sigma^2 \cdot I^{-1}$$

The scale factor σ^2 is again estimated as above:

$$\sigma^2 \approx \frac{1}{n-p} \cdot \sum_{AY,t}^{n} \frac{\left(c_{AY,t} - \hat{\mu}_{AY,t}\right)^2}{\hat{\mu}_{AY,t}}$$

The second derivative matrix for "LDF Method" is (n+2)x(n+2) and assumes that there is a different ULT for each accident year. The information matrix, I, is given as:

[6] To be precise, we are calculating the variance in the estimator of the parameter; the parameter itself does not have any variance. Nonetheless, we will retain the term "parameter variance" as shorthand.

$$\left[\begin{array}{cccc|cc}
\sum_t \frac{\partial^2 \ell_{1,t}}{\partial ULT_1^2} & 0 & \cdots & 0 & \sum_t \frac{\partial^2 \ell_{1,t}}{\partial ULT_1 \partial \omega} & \sum_t \frac{\partial^2 \ell_{1,t}}{\partial ULT_1 \partial ?} \\
0 & \sum_t \frac{\partial^2 \ell_{2,t}}{\partial ULT_2^2} & \cdots & 0 & \sum_t \frac{\partial^2 \ell_{2,t}}{\partial ULT_2 \partial \omega} & \sum_t \frac{\partial^2 \ell_{2,t}}{\partial ULT_2 \partial ?} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \sum_t \frac{\partial^2 \ell_{n,t}}{\partial ULT_n^2} & \sum_t \frac{\partial^2 \ell_{n,t}}{\partial ULT_n \partial \omega} & \sum_t \frac{\partial^2 \ell_{n,t}}{\partial ULT_n \partial ?} \\
\hline
\sum_t \frac{\partial^2 \ell_{1,t}}{\partial \omega \partial ULT_1} & \sum_t \frac{\partial^2 \ell_{2,t}}{\partial \omega \partial ULT_2} & \cdots & \sum_t \frac{\partial^2 \ell_{n,t}}{\partial \omega \partial ULT_n} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial \omega^2} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial \omega \partial ?} \\
\sum_t \frac{\partial^2 \ell_{1,t}}{\partial ? \partial ULT_1} & \sum_t \frac{\partial^2 \ell_{2,t}}{\partial ? \partial ULT_2} & \cdots & \sum_t \frac{\partial^2 \ell_{n,t}}{\partial ? \partial ULT_n} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial ? \partial \omega} & \sum_{y,t} \frac{\partial^2 \ell_{y,t}}{\partial ?^2}
\end{array}\right]$$

The covariance matrix Σ is again calculated using the inverse of the Information matrix, but for the LDF Method this matrix is larger.

2.4 The Variance of the Reserves

The final step is to estimate the variance in the reserves. The variance is broken into two pieces: the process variances and the estimation error (loosely "parameter variance"). For an estimate of loss reserves R for a given period $\mu_{AY;x,y}$, or group of periods $\sum \mu_{AY;x,y}$, the process variance is given by:

Process Variance of R: $\qquad \sigma^2 \cdot \sum \mu_{AY;x,y}$

The estimation error makes use of the covariance matrix Σ calculated above:

Parameter Variance of R: $\qquad Var(E[R]) \;=\; (\partial R)' \cdot \Sigma \cdot (\partial R)$

where

$$\partial R \;=\; \left\langle \frac{\partial R}{\partial ELR}, \frac{\partial R}{\partial \theta}, \frac{\partial R}{\partial \omega} \right\rangle \qquad \text{or} \qquad \partial R \;=\; \left\langle \left\{ \frac{\partial R}{\partial ULT_i} \right\}_{i=1}^{n}, \frac{\partial R}{\partial \theta}, \frac{\partial R}{\partial \omega} \right\rangle$$

The future reserve R, under the Cape Cod method is given by:

$$\text{Reserve:} \qquad R \;=\; \sum \text{Premium}_i \cdot ELR \cdot \left(G(y_i) - G(x_i)\right)$$

The derivatives needed are then easily calculated:

$$\frac{\partial R}{\partial ELR} \;=\; \sum \text{Premium}_i \cdot \left(G(y_i) - G(x_i)\right)$$

$$\frac{\partial R}{\partial \theta} \;=\; \sum \text{Premium}_i \cdot ELR \cdot \left(\frac{\partial G(y_i)}{\partial \theta} - \frac{\partial G(x_i)}{\partial \theta}\right)$$

$$\frac{\partial R}{\partial \omega} \;=\; \sum \text{Premium}_i \cdot ELR \cdot \left(\frac{\partial G(y_i)}{\partial \omega} - \frac{\partial G(x_i)}{\partial \omega}\right)$$

For the LDF Method, let $\text{Premium}_i = 1$ and $ELR = ULT_i$.

All of the mathematics needed for the estimate of the process and parameter variance is provided in Appendix A. For the two curve forms used, all of the derivatives are calculated analytically, without the need for numerical approximations.

Section 3: Key Assumptions of this Model

- Incremental losses are independent and identically distributed (iid)

  The assumption that all observed points are independent and identically distributed is the famous "iid" of classical statistics. In introductory textbooks this is often illustrated by the problem of estimating the proportion of red and black balls in an urn based on having "randomly" selected a sample from the urn. The "independence" assumption is that the balls are shaken up after each draw, so that we do not always pull out the same ball each time. The "identically distributed" assumption is that we are always taking the sample from the same urn.

  The "independence" assumption in the reserving context is that one period does not affect the surrounding periods. This is a tenuous assumption but will be tested using residual analysis. There may in fact be positive correlation if all periods are equally impacted by a change in loss inflation. There may also be negative correlation if a large settlement in one period replaces a stream of payments in later periods.

  The "identically distributed" assumption is also difficult to justify on first principles. We are assuming that the emergence pattern is the same for all accident years; which is clearly a gross simplification from even a rudimentary understanding of insurance phenomenon. Different risks and mix of business would have been written in each historical period, and subject to different claims handling and settlement strategies. Nonetheless, a parsimonious model requires this simplification.

- The Variance/Mean Scale Parameter σ^2 is fixed and known

  In rigorous maximum likelihood theory, the variance/mean scale parameter σ^2 should be estimated simultaneously with the other model parameters, and the variance around its estimate included in our covariance matrix.

Unfortunately, including the scale parameter in the curve-fitting procedure leads to mathematics that quickly becomes intractable. Treating the scale parameter as fixed and known is an approximation made for convenience in the calculation, and the results are sometimes called "quasi-likelihood estimators". McCullough & Nelder [7] give support for the approximation that we are using.

In effect, we are ignoring the variance on the variance.

In classical statistics, we usually relax this assumption (e.g., in hypothesis testing) by using the Student-T distribution instead of the Normal distribution. Rodney Kreps' paper [4] provides additional discussion on how reserve ranges could increase when this additional source of variability is considered.

- Variance estimates are based on an approximation to the Rao-Cramer lower bound.

The estimate of variance based on the information matrix is only exact when we are using linear functions. In the case of non-linear functions, including our model, the variance estimate is a Rao-Cramer lower bound.

Technically, the Rao-Cramer lower bound is based on the true expected values of the second derivative matrix. Since we are using approximations that plug in the estimated values of the parameters, the result is sometimes called the "observed" information matrix rather than the "expected" information matrix. Again, this is a limitation common to many statistical models and is due to the fact that we do not know the true parameters.

All of the key assumptions listed above need to be kept in mind by the user of a stochastic reserving model. In general, they imply that there is potential for more variability in future loss emergence than the model itself produces.

Such limitations should not lead the user, or any of the recipients of the output, to disregard the results. We simply want to be clear about what sources of variability we are able to measure and what sources cannot be measured. That is a distinction that should not be lost.

Section 4: A Practical Example

4.1 The LDF Method

For the first part of this example, we will use the "LDF Method" (referred to above as "Method 2"). The improvements in the model by moving to the Cape Cod method will be apparent as the numbers are calculated.

The triangle used in this example is taken from the 1993 Thomas Mack paper [6]. The accident years have been added to make the display appear more familiar.

	12	24	36	48	60	72	84	96	108	120
1991	357,848	1,124,788	1,735,330	2,182,708	2,745,596	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463
1992	352,118	1,236,139	2,170,033	3,353,322	3,799,067	4,120,063	4,647,867	4,914,039	5,339,085	
1993	290,507	1,292,306	2,218,525	3,235,179	3,985,995	4,132,918	4,628,910	4,909,315		
1994	310,608	1,418,858	2,195,047	3,757,447	4,029,929	4,381,982	4,588,268			
1995	443,160	1,136,350	2,128,333	2,897,821	3,402,672	3,873,311				
1996	396,132	1,333,217	2,180,715	2,985,752	3,691,712					
1997	440,832	1,288,463	2,419,861	3,483,130						
1998	359,480	1,421,128	2,864,498							
1999	376,686	1,363,294								
2000	344,014									

The incremental triangle, calculated by taking differences between cells in each accident year, is given by:

	12	24	36	48	60	72	84	96	108	120
1991	357,848	766,940	610,542	447,378	562,888	574,398	146,342	139,950	227,229	67,948
1992	352,118	884,021	933,894	1,183,289	445,745	320,996	527,804	266,172	425,046	
1993	290,507	1,001,799	926,219	1,016,654	750,816	146,923	495,992	280,405		
1994	310,608	1,108,250	776,189	1,562,400	272,482	352,053	206,286			
1995	443,160	693,190	991,983	769,488	504,851	470,639				
1996	396,132	937,085	847,498	805,037	705,960					
1997	440,832	847,631	1,131,398	1,063,269						
1998	359,480	1,061,648	1,443,370							
1999	376,686	986,608								
2000	344,014									

This incremental triangle is actually better arranged as a table of values, rather than in the familiar triangular format (see Table 1.1). In the tabular format, the column labeled "Increment" is the value that we will be approximating with the expression

$$\mu_{AY;x,y} = ULT_{AY} \cdot [G(y \mid \omega,\theta) - G(x \mid \omega,\theta)].$$

The x and y values are the "From" and "To" dates.

Before calculating the fitted values, it is worth showing the flexibility in this format. First, if we have only the latest three evaluations of the triangle, we can still use this method directly.

The original triangle becomes:

	12	24	36	48	60	72	84	96	108	120
1991								3,606,286	3,833,515	3,901,463
1992							4,647,867	4,914,039	5,339,085	
1993						4,132,918	4,628,910	4,909,315		
1994					4,029,929	4,381,982	4,588,268			
1995				2,897,821	3,402,672	3,873,311				
1996			2,180,715	2,985,752	3,691,712					
1997		1,288,463	2,419,861	3,483,130						
1998	359,480	1,421,128	2,864,498							
1999	376,686	1,363,294								
2000	344,014									

and the incremental triangle is:

	12	24	36	48	60	72	84	96	108	120
1991								3,606,286	227,229	67,948
1992							4,647,867	266,172	425,046	
1993						4,132,918	495,992	280,405		
1994					4,029,929	352,053	206,286			
1995				2,897,821	504,851	470,639				
1996			2,180,715	805,037	705,960					
1997		1,288,463	1,131,398	1,063,269						
1998	359,480	1,061,648	1,443,370							
1999	376,686	986,608								
2000	344,014									

The tabular format then collapses from 55 rows down to 27 rows, as shown in Table 1.2.

Another common difficulty in working with development triangles is the use of irregular evaluation periods. For example, we may have accident years evaluated at each year-end

- producing ages 12, 24, 36, etc – but the most recent diagonal is only available as of the end of the third quarter (ages 9, 21, 33, etc). This is put into the tabular format by simply changing the evaluation age fields ("Diag Age") as shown in Table 1.3.

Returning to the original triangle, we calculate the fitted values for a set of parameters ULT_{AY}, ω, θ and the MLE term to be maximized.

Fitted Value: $\mu_{AY;x,y} = ULT_{AY} \cdot [G(y \mid \omega,\theta) - G(x \mid \omega,\theta)]$

MLE Term: $c_{AY;x,y} \cdot \ln(\mu_{AY;x,y}) - \mu_{AY;x,y}$

In Table 1.4, these numbers are shown as additional columns. These values also have the desired unbiased property that the sum of the actual incremental dollars $c_{AY;x,y}$ equals the sum of the fitted values $\hat{\mu}_{AY;x,y}$.

The fitted parameters for the Loglogistic growth curve are:

ω	1.434294
θ	48.6249

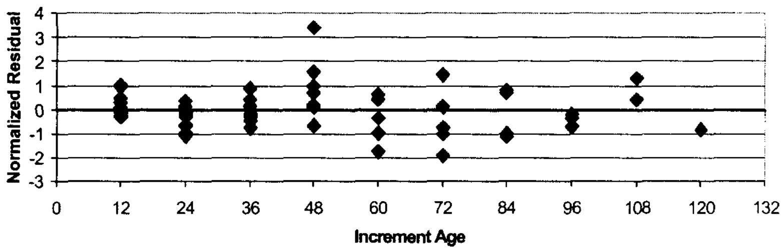
The fitted parameters are found by iteration, which can easily be accomplished in the statistics capabilities of most software packages. Once the data has been arranged in the tabular format, the curve-fitting can even be done in a spreadsheet.

The scale parameter σ^2 is also easily calculated. We recall that the form of this calculation is the same as a Chi-Square statistic, with 43 degrees of freedom (55 data points minus 12 parameters). The resulting σ^2 is 65,029. This scale factor may be thought of as the size of the discrete intervals for the over-dispersed Poisson, but is better thought of simply as the process variance-to-mean ratio. As such, we can calculate the process variance of the total reserve, or any sub-segment of the reserve, by just multiplying by 65,029.

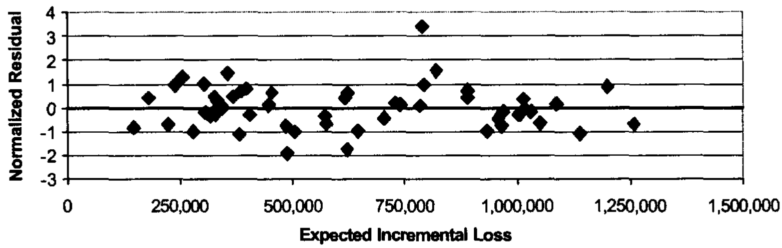
The scale factor σ^2 is also useful for a review of the model residuals (error terms).

$$\text{Normalized Residual:} \qquad r_{AY;x,y} = \frac{\left(c_{AY;x,y} - \hat{\mu}_{AY;x,y}\right)}{\sqrt{\sigma^2 \cdot \hat{\mu}_{AY;x,y}}}$$

The residuals can be plotted in various ways in order to test the assumptions in the model. The graph below shows the residuals plotted against the increment of loss emergence. We would hope that the residuals would be randomly scattered around the zero line for all of the ages, and that the amount of variability would be roughly constant. The graph below tells us that the curve form is perhaps not perfect for the early 12 and 24 points, but the pattern is not enough to reject the model outright.



A second residual plot of the residuals against the expected loss in each increment (the fitted values) is shown below. This graph is useful as a check on the assumption that the variance/mean ratio is constant. If the variance/mean ratio were not constant, then we would expect to see the residuals much closer to the zero line at one end of the graph.



The residuals can also be plotted against the accident year, the calendar year of emergence (to test diagonal effects), or any other variable of interest. The desired outcome is always that the residuals appear to be randomly scattered around the zero line. Any noticeable pattern or autocorrelation is an indication that the some of the model assumptions are incorrect.

Having solved for the parameters ω and θ, and the derived ultimates by year, we can estimate the needed reserves.

Accident Year	Reported Losses	Age at 12/31/2000	Average Age (x)	Growth Function	Fitted LDF	Ultimate Losses	Estimated Reserves
1991	3,901,463	120	114	77.24%	1.2946	5,050,867	1,149,404
1992	5,339,085	108	102	74.32%	1.3456	7,184,079	1,844,994
1993	4,909,315	96	90	70.75%	1.4135	6,939,399	2,030,084
1994	4,588,268	84	78	66.32%	1.5077	6,917,862	2,329,594
1995	3,873,311	72	66	60.78%	1.6452	6,372,348	2,499,037
1996	3,691,712	60	54	53.75%	1.8604	6,867,980	3,176,268
1997	3,483,130	48	42	44.77%	2.2338	7,780,515	4,297,385
1998	2,864,498	36	30	33.34%	2.9991	8,590,793	5,726,295
1999	1,363,294	24	18	19.38%	5.1593	7,033,659	5,670,365
2000	344,014	12	6	4.74%	21.1073	7,261,205	6,917,191
Total	34,358,090					69,998,708	35,640,618

From this initial calculation, we can quickly see the impact of the extrapolated "tail" factor. Our loss development data only includes ten years of development (out to age 120 months), but the growth curve extrapolates the losses to full ultimate. From this data, the Loglogistic curve estimates that only 77.24% of ultimate loss has emerged as of ten years.

Extrapolation should always be used cautiously. For practical purposes, we may want to rely on the extrapolation only out to some finite point – an additional ten years say.

Accident Year	Reported Losses	Age at 12/31/2000	Average Age (x)	Growth Function	Fitted LDF	Truncated LDF	Losses at 240 mo	Estimated Reserves
		240	234	90.50%	1.1050	1.0000		
1991	3,901,463	120	114	77.24%	1.2946	1.1716	4,570,810	669,347
1992	5,339,085	108	102	74.32%	1.3456	1.2177	6,501,273	1,162,188
1993	4,909,315	96	90	70.75%	1.4135	1.2792	6,279,848	1,370,533
1994	4,588,268	84	78	66.32%	1.5077	1.3644	6,260,358	1,672,090
1995	3,873,311	72	66	60.78%	1.6452	1.4888	5,766,692	1,893,381
1996	3,691,712	60	54	53.75%	1.8604	1.6836	6,215,217	2,523,505
1997	3,483,130	48	42	44.77%	2.2338	2.0215	7,041,021	3,557,891
1998	2,864,498	36	30	33.34%	2.9991	2.7140	7,774,286	4,909,788
1999	1,363,294	24	18	19.38%	5.1593	4.6689	6,365,149	5,001,855
2000	344,014	12	6	4.74%	21.1073	19.1012	6,571,068	6,227,054
Total	34,358,090						63,345,723	28,987,633

As noted above, the process variance for the estimated reserve of 28,987,633 is found by multiplying by the variance-to-mean ratio of 65,029. The process standard deviation around our reserve is therefore 1,372,966 for a coefficient of variation (CV = SD/mean) of about 4.7%.

As an alternative to truncating the tail factor at a selected point, such as age 240, we could make use of a growth curve that typically has a lighter "tail". The mathematics for the Weibull curve is provided for this purpose. An example including a fit of the Weibull curve is shown below.

Accident Year	Reported Losses	Age at 12/31/2000	Average Age (x)	Growth Function	Weibull LDF	Ultimate Losses	Estimated Reserves
1991	3,901,463	120	114	95.01%	1.0525	4,106,189	204,726
1992	5,339,085	108	102	92.54%	1.0806	5,769,409	430,324
1993	4,909,315	96	90	89.00%	1.1237	5,516,376	607,061
1994	4,588,268	84	78	84.01%	1.1904	5,461,745	873,477
1995	3,873,311	72	66	77.14%	1.2963	5,020,847	1,147,536
1996	3,691,712	60	54	67.95%	1.4717	5,433,242	1,741,530
1997	3,483,130	48	42	56.01%	1.7853	6,218,284	2,735,154
1998	2,864,498	36	30	41.19%	2.4277	6,954,204	4,089,706
1999	1,363,294	24	18	23.94%	4.1764	5,693,693	4,330,399
2000	344,014	12	6	6.37%	15.6937	5,398,863	5,054,849
Total	34,358,090					55,572,851	21,214,761

The fitted Weibull parameters θ and ω are 48.88453 and 1.296906, respectively. The lower "tail" factor of 1.0525 (instead of 1.2946 for the Loglogistic) may be more in line with the actuary's expectation for casualty business. The difference between the two curve forms also highlights the danger in relying on a purely mechanical extrapolation formula. The selection of a truncation point is an effective way of reducing the reliance on the extrapolation when the thicker-tailed Loglogistic is used.

The next step is our estimate of the parameter variance.

The parameter variance calculation is more involved than what was needed for process variance. As discussed in Section 2.3, we need to first evaluate the Information Matrix, which contains the second derivatives with respect to all of the model parameters, and so is a 12x12 matrix. The mathematics for all of these calculations is given in Appendix A, and is not difficult to program in most software. For purposes of this example, we will simply show the resulting variances:

Accident Year	Reported Losses	Estimated Reserves	Process Std Dev	CV	Parameter Std Dev	CV	Total Std Dev	CV
1991	3,901,463	669,347	208,631	31.2%	158,088	23.6%	261,761	39.1%
1992	5,339,085	1,162,188	274,911	23.7%	257,205	22.1%	376,471	32.4%
1993	4,909,315	1,370,533	298,537	21.8%	298,628	21.8%	422,260	30.8%
1994	4,588,268	1,672,090	329,749	19.7%	356,827	21.3%	485,860	29.1%
1995	3,873,311	1,893,381	350,891	18.5%	401,416	21.2%	533,160	28.2%
1996	3,691,712	2,523,505	405,094	16.1%	518,226	20.5%	657,768	26.1%
1997	3,483,130	3,557,891	481,005	13.5%	704,523	19.8%	853,064	24.0%
1998	2,864,498	4,909,788	565,047	11.5%	968,806	19.7%	1,121,545	22.8%
1999	1,363,294	5,001,855	570,321	11.4%	1,227,880	24.5%	1,353,867	27.1%
2000	344,014	6,227,054	636,348	10.2%	2,838,890	45.6%	2,909,336	46.7%
Total	34,358,090	28,987,633	1,372,966	4.7%	4,688,826	16.2%	4,885,707	16.9%

From this table, one conclusion should be readily apparent: the parameter variance component is much more significant than the process variance. The chief reason for this is that we have overparameterization of our model; that is, the available 55 data points are really not sufficient to estimate the 12 parameters of the model. The 1994 Zehnwirth paper ([10], p. 512f) gives a helpful discussion of the dangers of overparameterization.

The main problem is that we are estimating the ultimate loss for each accident year independently from the ultimate losses in the other accident years. In effect, we are saying that knowing the ultimate loss for accident year 1999 provides no information about the ultimate loss for accident year 2000. As such, our model is fitting to what may just be "noise" in the differences from one year to the next.

This conclusion is unsettling, because it indicates a high level of uncertainty not just in our maximum likelihood model, but in the chain-ladder LDF method in general.

4.2 The Cape Cod Method

A natural alternative to the LDF Method is the Cape Cod method. In order to move on to this method, we need to supplement the loss development triangle with an exposure base that is believed to be proportional to ultimate expected losses by accident year. A natural candidate for the exposure base is onlevel premium – premium that has been adjusted to a common level of rate per exposure.

Unadjusted historical premium could be used for this exposure base, but the impact of the market cycle on premium is likely to distort the results. We prefer onlevel premium so that the assumption of a constant expected loss ratio (ELR) across all accident years is reasonable.

A further refinement would include an adjustment for loss trend net of exposure trend, so that all years are at the same cost level as well as rate level.

There may be other candidates for the exposure index: sometimes the original loss projections by year are available; the use of estimated claim counts has also been suggested. In practice, even a judgmentally selected index may be used.

For the example in the Mack paper, no exposure base was supplied. For this exercise, we will use a simplifying assumption that premium was \$10,000,000 in 1991 and increased by \$400,000 each subsequent year.

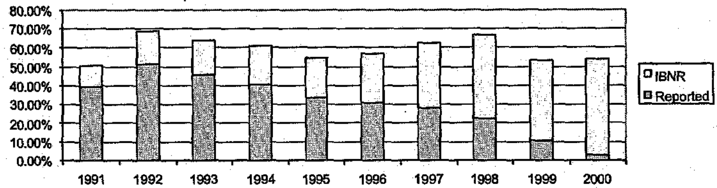
The tabular format of our loss data is shown in Table 2.1. This is very similar to the format used for the LDF Method but instead of the "AY Total" column (latest diagonal), we display the onlevel premium for each accident year. The expected ultimate loss by year is calculated as the ELR multiplied by the onlevel premium.

Accident Year	Onlevel Premium	Age at 12/31/2000	Average Age (x)	Growth Function	Premium x Growth Func	Reported Losses	Ultimate Loss Ratio
1991	10,000,000	120	114	77.76%	7,775,733	3,901,463	50.17%
1992	10,400,000	108	102	74.85%	7,784,279	5,339,085	68.59%
1993	10,800,000	96	90	71.29%	7,699,022	4,909,315	63.77%
1994	11,200,000	84	78	66.87%	7,489,209	4,588,268	61.27%
1995	11,600,000	72	66	61.31%	7,112,024	3,873,311	54.46%
1996	12,000,000	60	54	54.24%	6,508,439	3,691,712	56.72%
1997	12,400,000	48	42	45.17%	5,600,712	3,483,130	62.19%
1998	12,800,000	36	30	33.60%	4,301,252	2,864,498	66.60%
1999	13,200,000	24	18	19.46%	2,568,496	1,363,294	53.08%
2000	13,600,000	12	6	4.69%	638,334	344,014	53.89%
Total	118,000,000				57,477,500	34,358,090	59.78%

The Loglogistic parameters are again solved for iteratively in order to maximize the value of the log-likelihood function in Table 2.1. The resulting parameters are similar to those produced by the LDF method.

ω	1.447634
θ	48.0205

One check that should be made on the data before we proceed with the reserve estimate is a quick test on the assumption that the ELR is constant over all accident years. This is best done with a graph of the estimated ultimate loss ratios:



From this graph, the ultimate loss ratios by year do not appear to be following a strong autocorrelation pattern, or other unexplained trends. If we had observed an increasing or decreasing pattern, then there could be a concern of bias introduced in our reserve estimate.

The following calculation shows the method of estimating reserves out to the 240 month evaluation point. As in the LDF method, this truncation point is used in order avoid undue reliance on a mechanical extrapolation formula.

The Cape Cod method works much like the more familiar Bornhuetter-Ferguson formula. Estimated reserves are calculated as a percent of the premium and the calculated expected loss ratio (ELR).

Accident Year	Onlevel Premium	Age at 12/31/2000	Average Age (x)	Growth Function	90.83% minus Growth Func	Premium x ELR	Estimated Reserves
		240	234	90.83%			
1991	10,000,000	120	114	77.76%	13.07%	5,977,659	781,218
1992	10,400,000	108	102	74.85%	15.98%	6,216,765	993,281
1993	10,800,000	96	90	71.29%	19.54%	6,455,872	1,261,416
1994	11,200,000	84	78	66.87%	23.96%	6,694,978	1,604,006
1995	11,600,000	72	66	61.31%	29.52%	6,934,085	2,046,646
1996	12,000,000	60	54	54.24%	36.59%	7,173,191	2,624,620
1997	12,400,000	48	42	45.17%	45.66%	7,412,297	3,384,400
1998	12,800,000	36	30	33.60%	57.22%	7,651,404	4,378,344
1999	13,200,000	24	18	19.46%	71.37%	7,890,510	5,631,298
2000	13,600,000	12	6	4.69%	86.13%	8,129,616	7,002,255
Total	118,000,000					70,536,377	29,707,484

For the variance calculation, we again begin with the process variance/mean ratio, which follows the chi-square formula. The sum of chi-square values is divided by 52 (55 data points minus 3 parameters), resulting in a σ^2 of 61,577. This turns out to be less than the 65,029 calculated for the LDF method because there we divided by 43 (55 data points minus 12 parameters).

The covariance matrix is estimated from the second derivative Information Matrix, and results in the following:

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{ELR} & \omega & \theta \end{array} \\
\begin{array}{c} \text{ELR} \\ \omega \\ \theta \end{array} &
\begin{pmatrix}
0.002421 & -0.002997 & 0.242396 \\
-0.002997 & 0.007853 & -0.401000 \\
0.242396 & -0.401000 & 33.021994
\end{pmatrix}
\end{array}
$$

The standard deviation of our reserve estimate is calculated in the following table.

Accident Year	Reported Losses	Estimated Reserves	Process Std Dev	CV	Parameter Std Dev	CV	Total Std Dev	CV
1991	3,901,463	781,218	219,329	28.1%	158,913	20.3%	270,848	34.7%
1992	5,339,085	993,281	247,312	24.9%	192,103	19.3%	313,156	31.5%
1993	4,909,315	1,261,416	278,701	22.1%	229,523	18.2%	361,047	28.6%
1994	4,588,268	1,604,006	314,277	19.6%	270,790	16.9%	414,846	25.9%
1995	3,873,311	2,046,646	355,002	17.3%	314,629	15.4%	474,360	23.2%
1996	3,691,712	2,624,620	402,015	15.3%	358,200	13.6%	538,445	20.5%
1997	3,483,130	3,384,400	456,510	13.5%	396,353	11.7%	604,563	17.9%
1998	2,864,498	4,378,344	519,235	11.9%	421,934	9.6%	669,054	15.3%
1999	1,363,294	5,631,298	588,862	10.5%	430,873	7.7%	729,664	13.0%
2000	344,014	7,002,255	656,641	9.4%	439,441	6.3%	790,118	11.3%
Total	34,358,090	29,707,484	1,352,515	4.6%	3,143,967	10.6%	3,422,547	11.5%

In the earlier LDF example, the standard deviation on the overall reserve was 4,885,707 and this reduces to 3,422,547 when we switch to the Cape Cod method. The reduction is primarily seen in the more recent years 1999 and 2000, but is generally true for the full loss history. The reduction in the variance (the standard deviations squared) is even more extreme – the overall variance in reserves is cut in half.

This conclusion is at first surprising, since the two methods are very familiar to most actuaries. The difference is that we are making use of more information in the Cape Cod method, namely the onlevel premium by year, and this information allows us to make a significantly better estimate of the reserve.

4.3 Other Calculations Possible with this Model

Once the maximum likelihood calculations have been done, there are some other uses for the statistics besides the variance of the overall reserve. We will briefly look at three of these uses.

4.3.1 Variance of the Prospective Losses

Reserve reviews always focus on losses that have already occurred, but there is an intimate connection to the forecast of losses for the prospective period. The variability estimates from the Cape Cod method help us make this connection.

If the prospective period is estimated to include 14,000,000 in premium, we have a ready estimate of expected loss as 8,369,200 based on our 59.78% ELR. The process variance is calculated using the variance/mean multiplier 61,577, producing a CV of 8.6%.

The parameter variance is also readily calculated using the covariance matrix from the earlier calculation.

$$\begin{array}{cc} & \begin{array}{ccc} \text{ELR} & \omega & \theta \end{array} \\ \begin{array}{c} \text{ELR} \\ \omega \\ \theta \end{array} & \begin{pmatrix} 0.002421 & -0.002997 & 0.242396 \\ -0.002997 & 0.007853 & -0.401000 \\ 0.242396 & -0.401000 & 33.021994 \end{pmatrix} \end{array}$$

The .002421 variance on the ELR translates to a standard deviation of 4.92% (by taking the square root) around our estimated ELR of 59.78%. Combined with the process variance, we have a total CV of 11.9%.

The CV from this estimate can then be compared to numbers produced by other prospective pricing tools.

4.3.2 Calendar Year Development

The stochastic reserving model can also be used to estimate development or payment for the next calendar year period beyond the latest diagonal. An example, using the LDF method is shown below.

Accident Year	Reported Losses	Age at 12/31/2000	Growth Function	Age at 12/31/2001	Growth Function	Estimated Ultimate	Est. 12 month Development
1991	3,901,463	120	77.24%	132	79.67%	5,050,867	122,450
1992	5,339,085	108	74.32%	120	77.24%	7,184,079	210,145
1993	4,909,315	96	70.75%	108	74.32%	6,939,399	247,928
1994	4,588,268	84	66.32%	96	70.75%	6,917,862	305,811
1995	3,873,311	72	60.78%	84	66.32%	6,372,348	353,146
1996	3,691,712	60	53.75%	72	60.78%	6,867,980	482,859
1997	3,483,130	48	44.77%	60	53.75%	7,780,515	699,093
1998	2,864,498	36	33.34%	48	44.77%	8,590,793	981,372
1999	1,363,294	24	19.38%	36	33.34%	7,033,659	981,996
2000	344,014	12	4.74%	24	19.38%	7,261,205	1,063,384
Total	34,358,090					69,998,708	5,448,182

The estimated development for the next 12-month calendar period is calculated by the difference in the growth functions at the two evaluation ages times the estimated ultimate losses. The standard deviation around this estimated development is:

Accident Year	Reported Losses	Est. 12 month Development	Process Std Dev	CV	Parameter Std Dev	CV	Total Std Dev	CV
1991	3,901,463	122,450	89,234	72.9%	24,632	20.1%	92,572	75.6%
1992	5,339,085	210,145	116,900	55.6%	37,767	18.0%	122,849	58.5%
1993	4,909,315	247,928	126,974	51.2%	42,716	17.2%	133,967	54.0%
1994	4,588,268	305,811	141,020	46.1%	50,260	16.4%	149,708	49.0%
1995	3,873,311	353,146	151,541	42.9%	57,208	16.2%	161,980	45.9%
1996	3,691,712	482,859	177,200	36.7%	74,987	15.5%	192,413	39.8%
1997	3,483,130	699,093	213,217	30.5%	106,043	15.2%	238,131	34.1%
1998	2,864,498	981,372	252,621	25.7%	158,978	16.2%	298,482	30.4%
1999	1,363,294	981,996	252,702	25.7%	225,920	23.0%	338,966	34.5%
2000	344,014	1,063,384	262,965	24.7%	480,861	45.2%	548,068	51.5%
Total	34,358,090	5,448,182	595,223	10.9%	635,609	11.7%	870,798	16.0%

A major reason for calculating the 12-month development is that the estimate is testable within a relatively short timeframe. If we project 5,448,182 of development, along with a standard deviation of 870,798, then one year later we can compare the actual development and see if it was within the forecast range.

4.3.3 Variability in Discounted Reserves

The mathematics for calculating the variability around discounted reserves follows directly from the payout pattern, model parameters and covariance matrix already calculated. The details are provided in Appendix C. This calculation is, of course, only appropriate if the analysis is being performed on paid data.

For the Cape Cod calculation of reserves, along with the 240 month truncation point, the discounted reserve using a 6.0% rate is provided below.

Accident Year	Estimated Reserves	Discounted Reserves	Process Std Dev	C.V.	Parameter Std Dev	C.V.	Total Std Dev	C.V.
1991	781,218	632,995	179,807	28.4%	125,961	19.9%	219,538	34.7%
1992	993,281	796,674	201,069	25.2%	149,689	18.8%	250,670	31.5%
1993	1,261,416	1,003,816	225,216	22.4%	175,899	17.5%	285,767	28.5%
1994	1,604,006	1,269,446	252,987	19.9%	204,084	16.1%	325,043	25.6%
1995	2,046,646	1,614,650	285,275	17.7%	232,952	14.4%	368,305	22.8%
1996	2,624,620	2,068,611	323,114	15.6%	259,904	12.6%	414,672	20.0%
1997	3,384,400	2,669,559	367,518	13.8%	280,605	10.5%	462,394	17.3%
1998	4,378,344	3,459,057	418,912	12.1%	289,876	8.4%	509,427	14.7%
1999	5,631,298	4,449,320	475,291	10.7%	286,857	6.4%	555,147	12.5%
2000	7,002,255	5,490,513	526,186	9.6%	284,582	5.2%	598,213	10.9%
Total	29,707,484	23,454,641	1,089,311	4.6%	2,198,224	9.4%	2,453,322	10.5%

From Section 4.2 above, we saw that the full-value reserve of 29,707,486 had a CV of 11.5%. The discounted reserve of 23,454,641 has a CV of 10.5%. The smaller CV for the discounted reserve is because the "tail" of the payout curve has the greatest parameter variance and also receives the deepest discount.

Section 5: Comments and Conclusion

5.1 Comments

Having worked through an example of stochastic reserving, a few practical comments are in order.

1) Abandon your triangles!

The maximum likelihood model works most logically from the tabular format of data as shown in tables 1.1 and 2.1. It is possible to first create the more familiar triangular format and then build the table, but there is no need for that intermediate step. All that is really needed is a consistent aggregation of losses evaluated at more than one date; we can skip the step of creating the triangle altogether.

2) The CV Goes with the Mean

The question of the use of the standard deviation or CV from the MLE is common. If we select a carried reserve other than the maximum likelihood estimate, then can we still use the CV from the model?

The short answer is "no". The estimate of the standard deviation in this model is very explicitly the standard deviation around the maximum likelihood estimate. If you do not trust the expected reserve from the MLE model, then there is even less reason to trust the standard deviation.

The more practical answer is an equivocal "yes". The final carried reserve is a selection, based on many factors including the use of a statistical model. No purely mechanical model should be the basis for setting the reserve, because it cannot take into account all of the characteristics of the underlying loss phenomenon. The standard deviation or CV around the selected reserve must therefore also be a selection, and a reasonable basis for that selection is the output of the MLE model.

The selection of a reserve range also needs to include consideration about changes in mix of business and the process of settling claims. These types of considerations might better be labeled "model variance", since by definition they are factors outside of the assumptions of the model.

3) Other Curve Forms

This paper has applied the method of maximum likelihood using growth curves that follow the Loglogistic and Weibull curve forms. These curves are useful in that they smoothly move from 0% to 100%, they often closely match the empirical data, and the first and second derivatives are calculable without the need for numerical approximations. However, the method in general is not limited to these forms and a larger library of curves can be investigated.

In this paper the Loglogistic and Weibull curves were applied to the average evaluation age, rather than the age from inception of the historical policy period. This was done for practical purposes, and is one way of improving the fit at immature ages. When evaluation ages fall within the period being developed (that is the period is not yet fully earned), then a further annualizing adjustment is needed. The formulas for this adjustment are given in Appendix B.

5.2 Conclusion

The method of maximum likelihood is a very useful technique for estimating both the expected development pattern and the variance around the estimated reserve. The use of the over-dispersed Poisson distribution is a convenient link to the LDF and Cape Cod estimates already common among reserving actuaries.

The chief result that we observe in working on practical examples is that the "parameter variance" component is generally larger than the "process variance" – most of the uncertainty in the estimated reserve is related to our inability to reliably estimate the expected reserve, not to random events. As such, our most pressing need is not for more sophisticated models, but for more complete data. Supplementing the standard loss development triangle with accident year exposure information is a good step in that direction.

Table 1.1
Original Triangle in Tabular Format

AY	From	To	Increment	Diag Age	AY Total
1991	0	12	357,848	120	3,901,463
1991	12	24	766,940	120	3,901,463
1991	24	36	610,542	120	3,901,463
1991	36	48	447,378	120	3,901,463
1991	48	60	562,888	120	3,901,463
1991	60	72	574,398	120	3,901,463
1991	72	84	146,342	120	3,901,463
1991	84	96	139,950	120	3,901,463
1991	96	108	227,229	120	3,901,463
1991	108	120	67,948	120	3,901,463
1992	0	12	352,118	108	5,339,085
1992	12	24	884,021	108	5,339,085
1992	24	36	933,894	108	5,339,085
1992	36	48	1,183,289	108	5,339,085
1992	48	60	445,745	108	5,339,085
1992	60	72	320,996	108	5,339,085
1992	72	84	527,804	108	5,339,085
1992	84	96	266,172	108	5,339,085
1992	96	108	425,046	108	5,339,085
1993	0	12	290,507	96	4,909,315
1993	12	24	1,001,799	96	4,909,315
1993	24	36	926,219	96	4,909,315
1993	36	48	1,016,654	96	4,909,315
1993	48	60	750,816	96	4,909,315
1993	60	72	146,923	96	4,909,315
1993	72	84	495,992	96	4,909,315
1993	84	96	280,405	96	4,909,315
1994	0	12	310,608	84	4,588,268
1994	12	24	1,108,250	84	4,588,268
1994	24	36	776,189	84	4,588,268
1994	36	48	1,562,400	84	4,588,268
1994	48	60	272,482	84	4,588,268
1994	60	72	352,053	84	4,588,268
1994	72	84	206,286	84	4,588,268
1995	0	12	443,160	72	3,873,311
1995	12	24	693,190	72	3,873,311
1995	24	36	991,983	72	3,873,311
1995	36	48	769,488	72	3,873,311
1995	48	60	504,851	72	3,873,311
1995	60	72	470,639	72	3,873,311
1996	0	12	396,132	60	3,691,712
1996	12	24	937,085	60	3,691,712
1996	24	36	847,498	60	3,691,712
1996	36	48	805,037	60	3,691,712
1996	48	60	705,960	60	3,691,712
1997	0	12	440,832	48	3,483,130
1997	12	24	847,631	48	3,483,130
1997	24	36	1,131,398	48	3,483,130
1997	36	48	1,063,269	48	3,483,130
1998	0	12	359,480	36	2,864,498
1998	12	24	1,061,648	36	2,864,498
1998	24	36	1,443,370	36	2,864,498
1999	0	12	376,686	24	1,363,294
1999	12	24	986,608	24	1,363,294
2000	0	12	344,014	12	344,014

Table 1.2
Triangle Collapsed for Latest Three Diagonals

AY	From	To	Increment	Diag Age	AY Total
1991	0	96	3,606,286	120	3,901,463
1991	96	108	227,229	120	3,901,463
1991	108	120	67,948	120	3,901,463
1992	0	84	4,647,867	108	5,339,085
1992	84	96	266,172	108	5,339,085
1992	96	108	425,046	108	5,339,085
1993	0	72	4,132,918	96	4,909,315
1993	72	84	495,992	96	4,909,315
1993	84	96	280,405	96	4,909,315
1994	0	60	4,029,929	84	4,588,268
1994	60	72	352,053	84	4,588,268
1994	72	84	206,286	84	4,588,268
1995	0	48	2,897,821	72	3,873,311
1995	48	60	504,851	72	3,873,311
1995	60	72	470,639	72	3,873,311
1996	0	36	2,180,715	60	3,691,712
1996	36	48	805,037	60	3,691,712
1996	48	60	705,960	60	3,691,712
1997	0	24	1,288,463	48	3,483,130
1997	24	36	1,131,398	48	3,483,130
1997	36	48	1,063,269	48	3,483,130
1998	0	12	359,480	36	2,864,498
1998	12	24	1,061,648	36	2,864,498
1998	24	36	1,443,370	36	2,864,498
1999	0	12	376,686	24	1,363,294
1999	12	24	986,608	24	1,363,294
2000	0	12	344,014	12	344,014

Table 1.3
Latest Diagonal Representing only 9 Months of Development

AY	From	To	Increment	Diag Age	AY Total
1991	0	12	357,848	117	3,901,463
1991	12	24	766,940	117	3,901,463
1991	24	36	610,542	117	3,901,463
1991	36	48	447,378	117	3,901,463
1991	48	60	562,888	117	3,901,463
1991	60	72	574,398	117	3,901,463
1991	72	84	146,342	117	3,901,463
1991	84	96	139,950	117	3,901,463
1991	96	108	227,229	117	3,901,463
1991	108	117	67,948	117	3,901,463
1992	0	12	352,118	105	5,339,085
1992	12	24	884,021	105	5,339,085
1992	24	36	933,894	105	5,339,085
1992	36	48	1,183,289	105	5,339,085
1992	48	60	445,745	105	5,339,085
1992	60	72	320,996	105	5,339,085
1992	72	84	527,804	105	5,339,085
1992	84	96	266,172	105	5,339,085
1992	96	105	425,046	105	5,339,085
1993	0	12	290,507	93	4,909,315
1993	12	24	1,001,799	93	4,909,315
1993	24	36	926,219	93	4,909,315
1993	36	48	1,016,654	93	4,909,315
1993	48	60	750,816	93	4,909,315
1993	60	72	146,923	93	4,909,315
1993	72	84	495,992	93	4,909,315
1993	84	93	280,405	93	4,909,315
1994	0	12	310,608	81	4,588,268
1994	12	24	1,108,250	81	4,588,268
1994	24	36	776,189	81	4,588,268
1994	36	48	1,562,400	81	4,588,268
1994	48	60	272,482	81	4,588,268
1994	60	72	352,053	81	4,588,268
1994	72	81	206,286	81	4,588,268
1995	0	12	443,160	69	3,873,311
1995	12	24	693,190	69	3,873,311
1995	24	36	991,983	69	3,873,311
1995	36	48	769,488	69	3,873,311
1995	48	60	504,851	69	3,873,311
1995	60	69	470,639	69	3,873,311
1996	0	12	396,132	57	3,691,712
1996	12	24	937,085	57	3,691,712
1996	24	36	847,498	57	3,691,712
1996	36	48	805,037	57	3,691,712
1996	48	57	705,960	57	3,691,712
1997	0	12	440,832	45	3,483,130
1997	12	24	847,631	45	3,483,130
1997	24	36	1,131,398	45	3,483,130
1997	36	45	1,063,269	45	3,483,130
1998	0	12	359,480	33	2,864,498
1998	12	24	1,061,648	33	2,864,498
1998	24	33	1,443,370	33	2,864,498
1999	0	12	376,686	21	1,363,294
1999	12	21	986,608	21	1,363,294
2000	0	9	344,014	9	344,014

Table 1.4
Original Triangle along with Fitted Values – LDF Method

AY	From	To	Increment	Diag Age	AY Total	Est. ULT	Fitted	MLE Term	Chi-Square
1991	0	12	357,848	120	3,901,463	5,050,868	239,295	4,192,814	58,734
1991	12	24	766,940	120	3,901,463	5,050,868	739,686	9,624,727	1,004
1991	24	36	610,542	120	3,901,463	5,050,868	705,171	7,516,507	12,698
1991	36	48	447,378	120	3,901,463	5,050,868	576,987	5,357,739	29,114
1991	48	60	562,888	120	3,901,463	5,050,868	453,829	6,878,055	26,208
1991	60	72	574,398	120	3,901,463	5,050,868	355,106	6,985,799	135,422
1991	72	84	146,342	120	3,901,463	5,050,868	279,911	1,555,543	63,737
1991	84	96	139,950	120	3,901,463	5,050,868	223,278	1,500,370	31,098
1991	96	108	227,229	120	3,901,463	5,050,868	180,455	2,569,751	12,124
1991	108	120	67,948	120	3,901,463	5,050,868	147,745	661,056	43,099
1992	0	12	352,118	108	5,339,085	7,184,081	340,360	4,144,834	406
1992	12	24	884,021	108	5,339,085	7,184,081	1,052,089	11,206,001	26,848
1992	24	36	933,894	108	5,339,085	7,184,081	1,002,997	11,902,020	4,761
1992	36	48	1,183,289	108	5,339,085	7,184,081	820,675	15,293,216	160,220
1992	48	60	445,745	108	5,339,085	7,184,081	645,502	5,317,578	61,817
1992	60	72	320,996	108	5,339,085	7,184,081	505,083	3,710,390	67,094
1992	72	84	527,804	108	5,339,085	7,184,081	398,131	6,407,657	42,235
1992	84	96	266,172	108	5,339,085	7,184,081	317,579	3,054,416	8,321
1992	96	108	425,046	108	5,339,085	7,184,081	256,670	5,037,510	110,456
1993	0	12	290,507	96	4,909,315	6,939,401	328,768	3,361,574	4,453
1993	12	24	1,001,799	96	4,909,315	6,939,401	1,016,256	12,840,263	206
1993	24	36	926,219	96	4,909,315	6,939,401	968,836	11,798,028	1,875
1993	36	48	1,016,654	96	4,909,315	6,939,401	792,724	13,016,722	63,256
1993	48	60	750,816	96	4,909,315	6,939,401	623,517	9,394,719	25,990
1993	60	72	146,923	96	4,909,315	6,939,401	487,881	1,436,491	238,280
1993	72	84	495,992	96	4,909,315	6,939,401	384,571	5,993,828	32,282
1993	84	96	280,405	96	4,909,315	6,939,401	306,763	3,235,826	2,265
1994	0	12	310,608	84	4,588,268	6,917,864	327,748	3,616,974	896
1994	12	24	1,108,250	84	4,588,268	6,917,864	1,013,102	14,312,364	8,936
1994	24	36	776,189	84	4,588,268	6,917,864	965,829	9,730,631	37,236
1994	36	48	1,562,400	84	4,588,268	6,917,864	790,264	20,427,319	754,424
1994	48	60	272,482	84	4,588,268	6,917,864	621,582	3,013,334	196,065
1994	60	72	352,053	84	4,588,268	6,917,864	486,366	4,123,668	37,092
1994	72	84	206,286	84	4,588,268	6,917,864	383,377	2,268,795	81,803
1995	0	12	443,160	72	3,873,311	6,372,350	301,903	5,289,828	66,093
1995	12	24	693,190	72	3,873,311	6,372,350	933,213	8,595,646	61,734
1995	24	36	991,983	72	3,873,311	6,372,350	889,668	12,699,114	11,767
1995	36	48	769,488	72	3,873,311	6,372,350	727,947	9,658,589	2,371
1995	48	60	504,851	72	3,873,311	6,372,350	572,566	6,120,690	8,008
1995	60	72	470,639	72	3,873,311	6,372,350	448,014	5,676,214	1,143
1996	0	12	396,132	60	3,691,712	6,867,982	325,384	4,702,625	15,382
1996	12	24	937,085	60	3,691,712	6,867,982	1,005,797	11,945,927	4,694
1996	24	36	847,498	60	3,691,712	6,867,982	958,865	10,714,153	12,935
1996	36	48	805,037	60	3,691,712	6,867,982	784,566	10,142,109	534
1996	48	60	705,960	60	3,691,712	6,867,982	617,100	8,795,314	12,796
1997	0	12	440,832	48	3,483,130	7,780,518	368,618	5,281,753	14,147
1997	12	24	847,631	48	3,483,130	7,780,518	1,139,436	10,681,663	74,730
1997	24	36	1,131,398	48	3,483,130	7,780,518	1,086,268	14,638,194	1,875
1997	36	48	1,063,269	48	3,483,130	7,780,518	888,809	13,675,465	34,244
1998	0	12	359,480	36	2,864,498	8,590,795	407,006	4,236,247	5,550
1998	12	24	1,061,648	36	2,864,498	8,590,795	1,258,098	13,652,867	30,675
1998	24	36	1,443,370	36	2,864,498	8,590,795	1,199,393	19,003,928	49,629
1999	0	12	376,686	24	1,363,294	7,033,660	333,234	4,456,931	5,666
1999	12	24	986,608	24	1,363,294	7,033,660	1,030,060	12,629,654	1,833
2000	0	12	344,014	12	344,014	7,261,202	344,014	4,041,627	0
			34,358,090				34,358,090		2,796,260

Table 2.1
Original Triangle along with Fitted Values – Cape Cod Method

AY	From	To	Increment	Diag Age	Premium	Est. ULT	Fitted	MLE Term	Chi-Square
1991	0	12	357,848	120	10,000,000	5,977,659	280,569	4,208,482	21,285
1991	12	24	766,940	120	10,000,000	5,977,659	882,582	9,617,292	15,152
1991	24	36	610,542	120	10,000,000	5,977,659	845,554	7,486,969	65,319
1991	36	48	447,378	120	10,000,000	5,977,659	691,227	5,324,318	86,024
1991	48	60	562,888	120	10,000,000	5,977,659	542,171	6,889,829	792
1991	60	72	574,398	120	10,000,000	5,977,659	422,833	7,018,339	54,329
1991	72	84	146,342	120	10,000,000	5,977,659	332,202	1,528,317	103,985
1991	84	96	139,950	120	10,000,000	5,977,659	264,171	1,483,014	58,412
1991	96	108	227,229	120	10,000,000	5,977,659	212,900	2,574,877	964
1991	108	120	67,948	120	10,000,000	5,977,659	173,860	646,001	64,519
1992	0	12	352,118	108	10,400,000	6,216,765	291,792	4,139,189	12,472
1992	12	24	884,021	108	10,400,000	6,216,765	917,885	11,219,571	1,249
1992	24	36	933,894	108	10,400,000	6,216,765	879,376	11,902,801	3,380
1992	36	48	1,183,289	108	10,400,000	6,216,765	718,876	15,238,302	300,023
1992	48	60	445,745	108	10,400,000	6,216,765	563,858	5,338,946	24,742
1992	60	72	320,996	108	10,400,000	6,216,765	439,746	3,731,261	32,068
1992	72	84	527,804	108	10,400,000	6,216,765	345,490	6,385,446	96,207
1992	84	96	266,172	108	10,400,000	6,216,765	274,738	3,058,687	267
1992	96	108	425,046	108	10,400,000	6,216,765	221,416	5,009,964	187,273
1993	0	12	290,507	96	10,800,000	6,455,872	303,015	3,363,630	516
1993	12	24	1,001,799	96	10,800,000	6,455,872	953,188	12,839,147	2,479
1993	24	36	926,219	96	10,800,000	6,455,872	913,198	11,798,887	186
1993	36	48	1,016,654	96	10,800,000	6,455,872	746,525	13,001,875	97,746
1993	48	60	750,816	96	10,800,000	6,455,872	585,545	9,385,515	46,648
1993	60	72	146,923	96	10,800,000	6,455,872	456,660	1,457,996	210,084
1993	72	84	495,992	96	10,800,000	6,455,872	358,778	5,985,187	52,477
1993	84	96	280,405	96	10,800,000	6,455,872	285,305	3,236,950	84
1994	0	12	310,608	84	11,200,000	6,694,978	314,238	3,617,409	42
1994	12	24	1,108,250	84	11,200,000	6,694,978	988,491	14,309,720	14,509
1994	24	36	776,189	84	11,200,000	6,694,978	947,020	9,734,175	30,816
1994	36	48	1,562,400	84	11,200,000	6,694,978	774,174	20,411,270	802,533
1994	48	60	272,482	84	11,200,000	6,694,978	607,232	3,021,320	184,538
1994	60	72	352,053	84	11,200,000	6,694,978	473,573	4,127,077	31,182
1994	72	84	206,286	84	11,200,000	6,694,978	372,066	2,273,929	73,866
1995	0	12	443,160	72	11,600,000	6,934,085	325,460	5,299,568	42,565
1995	12	24	693,190	72	11,600,000	6,934,085	1,023,795	8,569,280	106,759
1995	24	36	991,983	72	11,600,000	6,934,085	980,842	12,704,721	127
1995	36	48	769,488	72	11,600,000	6,934,085	801,823	9,659,092	1,304
1995	48	60	504,851	72	11,600,000	6,934,085	628,919	6,111,729	24,475
1995	60	72	470,639	72	11,600,000	6,934,085	490,486	5,676,368	803
1996	0	12	396,132	60	12,000,000	7,173,191	336,683	4,704,848	10,497
1996	12	24	937,085	60	12,000,000	7,173,191	1,059,098	11,941,015	14,056
1996	24	36	847,498	60	12,000,000	7,173,191	1,014,664	10,706,291	27,541
1996	36	48	805,037	60	12,000,000	7,173,191	829,472	10,142,011	720
1996	48	60	705,960	60	12,000,000	7,173,191	650,606	8,799,134	4,710
1997	0	12	440,832	48	12,400,000	7,412,297	347,906	5,276,973	24,821
1997	12	24	847,631	48	12,400,000	7,412,297	1,094,401	10,692,516	55,643
1997	24	36	1,131,398	48	12,400,000	7,412,297	1,048,487	14,635,924	6,556
1997	36	48	1,063,269	48	12,400,000	7,412,297	857,121	13,668,552	49,581
1998	0	12	359,480	36	12,800,000	7,651,404	359,129	4,239,137	0
1998	12	24	1,061,648	36	12,800,000	7,651,404	1,129,704	13,666,979	4,100
1998	24	36	1,443,370	36	12,800,000	7,651,404	1,082,309	18,972,750	120,451
1999	0	12	376,686	24	13,200,000	7,890,510	370,351	4,459,595	108
1999	12	24	986,608	24	13,200,000	7,890,510	1,165,008	12,616,168	27,319
2000	0	12	344,014	12	13,600,000	8,129,616	381,574	4,039,715	3,697
			34,358,090				34,358,090		3,202,001

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See Chapter 6 on Log-linear Models, and the over-dispersed Poisson.

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Appendix A: Derivatives of the Loglikelihood Function

The loglikelihood function for the over-dispersed Poisson is proportional to

$$\ell = \sum_i c_i \cdot \ln(\mu_i) - \mu_i$$

$$\text{where } \mu_{i,t} = ELR \cdot P_i \cdot [G(x_t \mid \omega,\theta) - G(x_{t-1} \mid \omega,\theta)]$$

as described in section 2.2 of this paper. The derivatives below are then used to complete the Information Matrix needed in the parameter variance calculation.

The derivatives of the exact loglikelihood function would require dividing all of these numbers by the constant scale factor σ^2, but it is easier to omit that here and apply it to the final covariance matrix at the end.

$$\frac{\partial^2 \ell}{\partial ELR^2} = \sum_{i,t}\left(\frac{-c_{i,t}}{ELR^2}\right)$$

$$\frac{\partial^2 \ell}{\partial ELR\,\partial\omega} = -\sum_{i,t} P_i \cdot \left[\frac{\partial G(x_t)}{\partial\omega} - \frac{\partial G(x_{t-1})}{\partial\omega}\right]$$

$$\frac{\partial^2 \ell}{\partial ELR\,\partial\theta} = -\sum_{i,t} P_i \cdot \left[\frac{\partial G(x_t)}{\partial\theta} - \frac{\partial G(x_{t-1})}{\partial\theta}\right]$$

$$\frac{\partial \ell}{\partial\omega} = \sum_{i,t}\left\{\left[\frac{c_{i,t}}{G(x_t) - G(x_{t-1})} - ELR \cdot P_i\right] \cdot \left[\frac{\partial G(x_t)}{\partial\omega} - \frac{\partial G(x_{t-1})}{\partial\omega}\right]\right\}$$

$$\frac{\partial^2 \ell}{\partial\omega^2} = \sum_{i,t}\left\{\left[\frac{-c_{i,t}}{(G(x_t) - G(x_{t-1}))^2}\right] \cdot \left[\frac{\partial G(x_t)}{\partial\omega} - \frac{\partial G(x_{t-1})}{\partial\omega}\right]^2 + \left[\frac{c_{i,t}}{G(x_t) - G(x_{t-1})} - ELR \cdot P_i\right] \cdot \left[\frac{\partial^2 G(x_t)}{\partial\omega^2} - \frac{\partial^2 G(x_{t-1})}{\partial\omega^2}\right]\right\}$$

$$\frac{\partial^2 \ell}{\partial\omega\,\partial\theta} = \sum_{i,t}\left\{\left[\frac{-c_{i,t}}{(G(x_t)-G(x_{t-1}))^2}\right]\cdot\left[\frac{\partial G(x_t)}{\partial\omega}-\frac{\partial G(x_{t-1})}{\partial\omega}\right]\cdot\left[\frac{\partial G(x_t)}{\partial\theta}-\frac{\partial G(x_{t-1})}{\partial\theta}\right]+\right.$$

$$\left.\left[\frac{c_{i,t}}{G(x_t)-G(x_{t-1})}-ELR\cdot P_i\right]\cdot\left[\frac{\partial^2 G(x_t)}{\partial\omega\,\partial\theta}-\frac{\partial^2 G(x_{t-1})}{\partial\omega\partial\theta}\right]\right\}$$

$$\frac{\partial \ell}{\partial\theta} = \sum_{i,t}\left\{\left[\frac{c_{i,t}}{G(x_t)-G(x_{t-1})}-ELR\cdot P_i\right]\cdot\left[\frac{\partial G(x_t)}{\partial\theta}-\frac{\partial G(x_{t-1})}{\partial\theta}\right]\right\}$$

$$\frac{\partial^2 \ell}{\partial\theta^2} = \sum_{i,t}\left\{\left[\frac{-c_{i,t}}{(G(x_t)-G(x_{t-1}))^2}\right]\cdot\left[\frac{\partial G(x_t)}{\partial\theta}-\frac{\partial G(x_{t-1})}{\partial\theta}\right]^2+\right.$$

$$\left.\left[\frac{c_{i,t}}{G(x_t)-G(x_{t-1})}-ELR\cdot P_i\right]\cdot\left[\frac{\partial^2 G(x_t)}{\partial\theta^2}-\frac{\partial^2 G(x_{t-1})}{\partial\theta^2}\right]\right\}$$

For the LDF Method, these same formulas apply but replacing:

$ELR \rightarrow ULT_i$ and $P_i \rightarrow 1$.

Weibull Distribution

$$G(x) \;=\; F(x) \;=\; 1 \;-\; \exp\left[-(x/\theta)^{\omega}\right]$$

$$f(x) \;=\; \frac{\omega}{x}\cdot\left(\frac{x}{\theta}\right)^{\omega}\cdot\exp\left[-(x/\theta)^{\omega}\right]$$

$$E[x^{k}] \;=\; \theta^{k}\cdot\Gamma(1+k/\omega)$$

θ is approximately the 63.2%-tile $=1-\exp[-1]$, $LDF_{\theta}\approx 1.582$

$$\frac{\partial G(x)}{\partial\omega} \;=\; \exp\left[-\left(\tfrac{x}{\theta}\right)^{\omega}\right]\cdot\left(\frac{x}{\theta}\right)^{\omega}\cdot\ln\left(\tfrac{x}{\theta}\right)$$

$$\frac{\partial G(x)}{\partial\theta} \;=\; \exp\left[-\left(\tfrac{x}{\theta}\right)^{\omega}\right]\cdot\left(\frac{x}{\theta}\right)^{\omega}\cdot\left(\frac{-\omega}{\theta}\right)$$

$$\frac{\partial^{2} G(x)}{\partial\omega^{2}} \;=\; \exp\left[-\left(\tfrac{x}{\theta}\right)^{\omega}\right]\cdot\left(\frac{x}{\theta}\right)^{\omega}\cdot\ln\left(\tfrac{x}{\theta}\right)^{2}\cdot\left[1-\left(\frac{x}{\theta}\right)^{\omega}\right]$$

$$\frac{\partial^{2} G(x)}{\partial\omega\partial\theta} \;=\; \exp\left[-\left(\tfrac{x}{\theta}\right)^{\omega}\right]\cdot\left(\frac{x}{\theta}\right)^{\omega}\cdot\left(\frac{-1}{\theta}\right)\cdot\left\{1+\omega\cdot\ln\left(\tfrac{x}{\theta}\right)\cdot\left[1-\left(\frac{x}{\theta}\right)^{\omega}\right]\right\}$$

$$\frac{\partial^{2} G(x)}{\partial\theta^{2}} \;=\; \exp\left[-\left(\tfrac{x}{\theta}\right)^{\omega}\right]\cdot\left(\frac{x}{\theta}\right)^{\omega}\cdot\left(\frac{\omega}{\theta^{2}}\right)\cdot\left\{1+\omega\cdot\left[1-\left(\frac{x}{\theta}\right)^{\omega}\right]\right\}$$

Loglogistic Distribution (for "inverse power" LDFs)

$$G(x) = F(x) = \frac{x^{\omega}}{x^{\omega}+\theta^{\omega}} = 1-\left(\frac{1}{1+(x/\theta)^{\omega}}\right)$$

$$f(x) = \frac{\omega}{x}\cdot\left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{\theta^{\omega}}{x^{\omega}+\theta^{\omega}}\right)$$

$$E[x^k] = \theta^k\cdot\Gamma(1+k/\omega)\cdot\Gamma(1-k/\omega)$$

θ is the median of the distribution $\quad LDF_{\theta} = 2.000$

$$\frac{\partial G(x)}{\partial \omega} = \left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{\theta^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\ln\left(\tfrac{x}{\theta}\right)$$

$$\frac{\partial G(x)}{\partial \theta} = \left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{\theta^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{-\omega}{\theta}\right)$$

$$\frac{\partial^2 G(x)}{\partial \omega^2} = \left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{\theta^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\ln\left(\tfrac{x}{\theta}\right)^2\cdot\left[1-2\cdot\left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\right]$$

$$\frac{\partial^2 G(x)}{\partial \omega\partial\theta} = \left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{\theta^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{-1}{\theta}\right)\cdot\left\{1+\omega\cdot\ln\left(\tfrac{x}{\theta}\right)\cdot\left[1-2\cdot\left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\right]\right\}$$

$$\frac{\partial^2 G(x)}{\partial \theta^2} = \left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{\theta^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\cdot\left(\frac{\omega}{\theta^2}\right)\cdot\left\{1+\omega\cdot\left[1-2\cdot\left(\frac{x^{\omega}}{x^{\omega}+\theta^{\omega}}\right)\right]\right\}$$

Appendix B: Adjustments for Different Exposure Periods

The percent of ultimate curve is assumed to be a function of the average accident date of the period being developed to ultimate.

$G^*(x \mid \omega, \theta)$ = cumulative percent of ultimate as of average date x

Further, we will assume that this is the percent of ultimate for the portion of the period that has already been <u>earned</u>. For example, if we are 9 months into an accident year, then the quantity $G^*(4.5 \mid \omega, \theta)$ represents the cumulative percent of ultimate of the 9-month period only. The loss development factor $LDF_9^* = 1 / G^*(4.5 \mid \omega, \theta)$ is the adjustment needed to calculate the ultimate loss dollars for the 9-month period (before annualizing).

In order to estimate the cumulative percent of ultimate for the full accident year, we also need to multiply by a scaling factor representing the portion of the accident year that has been earned.

The AY cumulative percent of ultimate as of 9 months is

$$G_{AY}(9 \mid \omega, \theta) \quad = \quad \left(\frac{9}{12}\right) \cdot G^*(4.5 \mid \omega, \theta)$$

We find therefore that we need to make two calculations:

1) Calculate the percent of the period that is exposed; *Expos(t)*
2) Calculate the average accident date given the age from inception t; *AvgAge(t)*

These functions can be easily calculated for accident year or policy year periods.

1) Calculate the percent of the period that is exposed: $Expos(t)$

For accident years (AY):

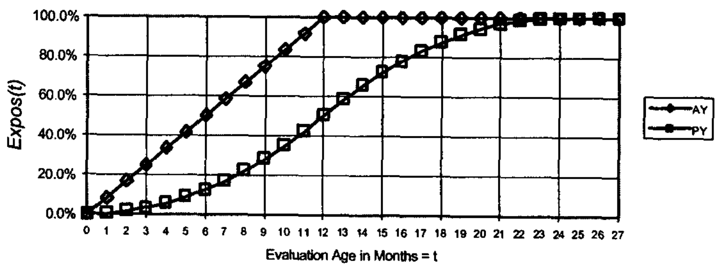
$$Expos(t) = \begin{cases} t/12 & t \le 12 \\ 1 & t > 12 \end{cases}$$

or

For policy years (PY):

$$Expos(t) = \begin{cases} \tfrac{1}{2}\cdot(t/12)^2 & t \le 12 \\ 1-\tfrac{1}{2}\cdot\max(2-t/12,0)^2 & t > 12 \end{cases}$$

Cumulative Percent of Exposure Expos(t)



2) Calculate the average accident date of the period that is earned: $AvgAge(t)$

For accident years (AY):

$$AvgAge(t) = \begin{cases} t/2 & t \leq 12 \\ t-6 & t > 12 \end{cases} \qquad \text{or} \qquad AvgAge(t) = \max(t-6, t/2)$$

For policy years (PY):

$$AvgAge(t) = \begin{cases} t/3 & t \leq 12 \\ \dfrac{(t-12) + \frac{1}{3}\cdot(24-t)\cdot\left(1 - Expos(t)\right)}{Expos(t)} & t > 12 \end{cases}$$

The final cumulative percent of ultimate curve, including annualization, is given by:

$$\boxed{G_{AY\,or\,PY}(t \mid \omega, \theta) = Expos(t) \cdot G^{*}\left(AvgAge(t) \mid \omega, \theta\right)}$$

Appendix C: Variance in Discounted Reserves

The maximum likelihood estimation model allows for the estimation of variance of discounted reserves as well as the variance of the full-value reserves. These calculations are a bit more tedious, and so are given just in this appendix.

Calculation of Discounted Reserve

We begin by recalling that the reserve is estimated as a sum of portions of all the historical accident years, and is calculated as:

$$\text{Reserve:} \qquad R = \sum_{AY} \mu_{AY;x,y} = \sum_{AY} ULT_{AY} \cdot (G(y) - G(x))$$

This expression can be expanded as the sum of individual increments.

$$R = \sum_{AY} \sum_{k=1}^{y-x} ULT_{AY} \cdot (G(x+k) - G(x+k-1))$$

To be even more precise, we could write this as a continuous function.

$$R = \sum_{AY} ULT_{AY} \cdot \int_x^y g(t)\,dt \qquad \text{where } g(t) = \frac{\partial G(t)}{\partial t}$$

The value of the discounted reserve R_d would then be written as follows.

$$R_d = \sum_{AY} ULT_{AY} \cdot \int_x^y v^{t-x} \cdot g(t)\,dt \qquad \text{where } v = \frac{1}{1+i}$$

For purposes of this paper, we will assume that the discount rate i is constant. There is also some debate as to what this rate should be (cost of capital?, market yield?), but we will avoid that discussion here.

An interesting note on this expression is seen in the case of $x=0$ and $y=\infty$, in which the form of the discounted loss at time zero is directly related to the moment generating function of the growth curve.

$$\int_0^\infty v^t \cdot g(y)\,dt = \int_0^\infty e^{-t\cdot\ln(1+i)} \cdot g(t)\,dt = MGF(-\ln(1+i))$$

Unfortunately, for the Loglogistic and Weibull growth curves, the moment generating function is intractable and so does not simplify our calculation. For practical purposes we will use the incremental approximation instead.

$$R_d \approx \sum_{AY}\sum_{k=1}^{y-x} ULT_{AY} \cdot v^{k-1/2} \cdot (G(x+k) - G(x+k-1))$$

The variance can then be calculated for the discounted reserve in two pieces: the process variance and the parameter variance.

Process Variance

The process variance component is actually trivial to calculate. We already know that the variance of the full value reserve is estimated by multiplying by the scale factor σ^2. We then need to recall that the variance for some random variable times a constant is given by $Var(v^k \cdot R) = v^{2k} \cdot Var(R)$.

The process variance of the discounted reserve is therefore:

$$Var(R_d) \approx \sigma^2 \cdot \sum_{AY}\sum_{k=1}^{y-x} ULT_{AY} \cdot v^{2k-1} \cdot (G(x+k) - G(x+k-1))$$

Parameter Variance

The parameter variance again makes use of the covariance matrix of the model parameters Σ. The formula is then given below.

$$Var(E[R_d]) = (\partial R_d)' \cdot \Sigma \cdot (\partial R_d)$$

where

$$\partial R_d = \left\langle \frac{\partial R_d}{\partial ELR}, \frac{\partial R_d}{\partial \omega}, \frac{\partial R_d}{\partial \theta} \right\rangle \qquad \text{for the Cape Cod method}$$

or

$$\partial R_d = \left\langle \left\{ \frac{\partial R_d}{\partial ULT_{AY}} \right\}_{AY=1}^{n}, \frac{\partial R_d}{\partial \omega}, \frac{\partial R_d}{\partial \theta} \right\rangle \qquad \text{for the LDF method}$$

In order to calculate the derivatives of the discounted reserves, we make use of the same mathematical expressions as for the full value reserves. That is,

$$\frac{\partial R}{\partial \omega} = \sum_{AY,x} \frac{\partial \mu_{AY,x}}{\partial \omega} \qquad \text{becomes} \qquad \frac{\partial R_d}{\partial \omega} = \sum_{AY,x} v_{AY,x} \cdot \frac{\partial \mu_{AY,x}}{\partial \omega}$$

The calculation is similar to the variance calculation for the full value reserve, but now it is expanded for each increment so that the time dimension is included. The complexity of the calculations does not change, but the number of times they are performed greatly increases.

The combination of the process and parameter variances is simple addition, the same as for the full value reserves, since we make the assumption that the two sources of variance are independent.

4

Estimating ULAE Liabilities: Rediscovering and Expanding Kittel's Approach

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ESTIMATING ULAE LIABILITIES:

REDISCOVERING AND EXPANDING KITTEL'S APPROACH

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ABSTRACT

Existing actuarial literature provides guidance on the use of dollar and count-based methods for the estimation of ULAE liabilities. Traditional dollar-based methods are based on widely available, and usually audited, company financial data, while count-based methods rely on relatively detailed information regarding the number and cost of various claim-handling activities and events. In the case of fast reporting, slow paying lines of business, traditional dollar-based methods may not produce the best estimate of ULAE liabilities, since the familiar "50/50" assumption does not apply. On the other hand, the application of count-based methods is sometimes impractical. For example, the detailed claim count and activity cost data used in the structural methods can be quite difficult to compile and verify – especially to "outside" actuaries. This article describes a generalization to a familiar ULAE liability estimation approach, which attempts to duplicate some of the benefits of the structural methods, while relying exclusively on aggregate loss data.

INTRODUCTION[1]

The need for a refinement to the traditional ULAE reserving methods surfaced while we were evaluating the liabilities of a mid-size, single-state workers compensation insurer. The long duration of claim payments, as well as rapid expansion since the company had started operations a few years earlier, made the traditional paid-to-paid ratio approach inappropriate. Also, conversations with management, together with our knowledge of the specific characteristics of workers compensation claims handling, made it clear that the usual 50/50 assumption (half of the ULAE is incurred in opening claims, the other half in closing claims) did not apply. Count-based approaches to estimating ULAE liabilities, although perhaps conceptually appropriate for accurately modeling the dynamics of the organization in question, were not practical due to the unavailability of accurate claim count, or refined transaction and expense information.

In the subsequent sections, we present a brief survey of several established approaches to ULAE liability estimation, directing readers interested in the details of each method to relevant literature. We then present a description of a generalized dollar-based methodology, and its relationship to some of the traditional approaches. Our generalized dollar-based method is anticipated in a little-noted formula in an article by Kittel [4], and thus we refer to the method as "generalized Kittel" method or "generalized" method or formula, for short. We show how to expand this generalized formula to allow its application as a count-based method, and we also outline several conceptual refinements that would incorporate various reserving refinements

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1 We are indebted to Jon Michelson for building a spreadsheet model to assess the relative accuracy of the generalized method.

suggested by other authors, as well as a simplified application of the method. Finally, we discuss practical complications such as errors in the estimations of parameters, and suggest areas for further research and improvement.

We have included illustrations of some of the concepts presented in this article. The attached exhibits display an actual application of the generalized method in comparison to the traditional method and Kittel's method. The Appendix sets forth the detailed derivation of the generalized formula to estimate an ULAE-to-loss ratio.

Throughout this article, we use the term "losses" to refer to "losses and allocated loss adjustment expenses"[2].

BRIEF SURVEY OF TRADITIONAL METHODS OF ESTIMATING ULAE LIABILITIES

We begin our discussion by briefly surveying the actuarial literature regarding methodologies for the estimation of ULAE liabilities. In our particular case, several of the methods surveyed were inappropriate, either due to the lack of detailed historical information, or to the specific characteristics of the company in question. Ideally, the actuary would have access to sufficient

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[2] Throughout this paper, we refer to ULAE as the traditional categorization of general overhead expenses associated with the claims-handling process, and particularly including the costs of investigating, handling, paying and resolving claims. Several issues associated with the 1998 change in loss adjustment expense categories are discussed in the Practical Difficulties section of this paper.

data to employ both dollar-based and count-based methods, and make a choice of methodology based on which approach is likely to produce the best estimate.

Dollar-based versus count-based methods – We first describe several dollar-based methods (Classical, Kittel Refined, and Mango-Allen Smoothing), followed by a description of count-based methods. These two broad classes of methods differ significantly in the amount of data and calculations required, and are based on fundamentally different assumptions. In the case of the dollar-based methods, a fundamental assumption is that ULAE expenditures track with loss dollars. Most importantly, this assumption means that the general timing of ULAE expenditures (or of specified portions of ULAE expenditures) follows the timing of the reporting or payment of loss dollars. In addition, this assumption implies that a $1,000 claim requires 10 times as much ULAE resources as a claim with losses of $100. By contrast, count-based methods incorporate fundamental assumption that the same kind of transaction costs the same amount of ULAE, regardless of claim size. However, because these count-based methods typically include some parameter to reflect the cost of ongoing management and maintenance of claims, they also imply that a claim that stays open longer will cost proportionately more than a quick-closing claim, at least with respect to some component of ULAE.

In practice, these seemingly divergent assumptions may not affect the results of the methods quite as severely as it might seem at first glance. Since the methods are being used for an entire population of claims, they need to be correct only for the "average" claim being reported, handled, paid, or closed during a time period – not for each individual claim. In other cases, the gulf can be bridged by stratifying the claims data and types of transactions and making assumptions about the relative ULAE resources required in the various sub-populations. In every case, it is a useful exercise for the actuary to reflect upon the assumptions underlying a selected method, and the implications of those assumptions regarding the underlying ULAE process and resources, as well as the implications for the results of the reserving method.

We also describe several triangle-based ULAE projection methodologies towards the end of this section.

Classical Paid-to-Paid Ratio Method – By reviewing the ratios of calendar year paid ULAE to calendar year paid losses, the actuary estimates an ULAE-to-loss ratio. To reflect the assumption that half of ULAE is incurred when new claims are set up, and the remaining half is spent closing them[3], this ratio is applied to the incurred but not reported (IBNR) loss reserves, plus half of case reserves. This method has several implicit assumptions, including (a) the specific company's ULAE-to-loss relationships have achieved a steady state (so that the ratio of paid ULAE to paid losses provides a reasonable approximation of the relationship of ultimate ULAE to ultimate losses); (b) that the relative volume and cost of future claims-management activity on

[3] In descriptions of the classical method, the concepts of "closing" a claim, and "paying" a claim seem to be used interchangeably, implying that the descriptions were written in the context of claim types for which the only payment occurs at closing. Given the use of paid loss dollars and case reserve dollars to apply the classical method, it would be more accurate to describe the classical assumption as "half the ULAE is spent with the *payment* of claims."

not-yet-reported claims and reported-but-not-yet-closed claims, respectively, will be proportional to the dollars of IBNR reserves and case reserves.[4]

As described in Kittel's article, the Classical Paid-to-Paid Ratio Method can lead to inaccurate results whenever the volume of losses is growing – since the paid-to-paid ratios will be overstated due to the mismatch between ULAE and losses paid. As mentioned above, the company in question had been expanding rapidly since its incorporation. That led to the material overstatement of ULAE reserves by purely mechanical application of this methodology. Also, we believed that the 50/50 assumption did not describe this company's application of resources to the various stages in the life cycle of its claims[5].

Kittel's Refinement to the Classical Method – A refinement to the Classical Method, detailed in Kittel's paper, explicitly recognizes the fact that ULAE is incurred as claims are reported, even if no loss payments are made. That is, ULAE payments for a specific calendar year would not be expected to track loss payments perfectly, because actual ULAE is related to both the reporting and the payment of losses. In contrast, the Classical Method, by assuming a steady state, makes the implicit simplifying assumption that paid losses are approximately equal to reported losses, and thus that the two quantities can be used interchangeably. To derive the indicated ULAE ratio

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[4] Another imprecision with the usual description and frequent application of the classical method is the equating of "IBNR" reserves with reserves on not-yet-reported claims. In practice, IBNR reserve dollars typically include not only provision for not-yet-reported claims (IBNYR), but also provision for development on known cases (IBNER, or Incurred But Not Enough Reported). A more correct application of the classical method is to apply the full ULAE-to-loss ratio to the IBNYR reserve, and to apply half of that ratio to the sum of case reserves and IBNER.

[5] Kittel notes that inflation also can create distortions in the classical method.

under Kittel's refined method, the actuary reviews several years' ratios of calendar year paid ULAE to the average between paid and reported losses for that year. Conceptually, Kittel's use of the ratio of ULAE to the average of paid losses and reported losses derives directly from the assumption that half of a claim's ULAE is expended when a loss is reported, half when it is paid. As in the classical method, the actuary's selected ULAE-to-loss ratio is applied fully to the IBNR reserve and half the ratio is applied to the case reserve dollars to obtain the estimate of unpaid ULAE. Although the Kittel refinement addresses the distortion in the Classical Method associated with a growing company, it maintains the traditional "50/50" assumption regarding ULAE expenditures. Therefore, it does not allow for the particular allocation of ULAE cost between opening, maintaining and closing claims exhibited by the company in question.

While Kittel's paper typically is associated with the refined formula described in the preceding paragraph, the paper also shows a brief outline of a potential generalization. In this generalization, the cost of ULAE is described as the sum of incurred losses multiplied by an "opening factor", paid losses multiplied by a "closing factor", and mean loss reserves, multiplied by an "open factor."

With the "opening factor" set at 50%, the "closing factor" set at 50%, and the "open factor" set at 0%, this formula simplifies to the familiar Kittel formula. As specifically presented by Kittel, this generalization suffers from the same incorrect equating of paid losses with closed claims, and of incurred losses with the ultimate cost of reported claims. Nonetheless, as described in the following pages, it provides the core of the approach elaborated in our paper.

Mango-Allen Smoothing Adjustment – Mango and Allen [5] provide a general discussion of Kittel's refinement to the classical method. They specifically suggest a possible variation on the application of the formula when the actuary is working with a line of business where the actual historical calendar period paid losses are volatile, perhaps due to the random timing associated with the reporting or settling of large claims. In this case, Mango and Allen suggest replacing the actual calendar period losses with "expected" losses for those historical calendar periods, which can be estimated by applying a selected reporting and payment pattern to a set of accident year estimated ultimate losses. We expect that this type of adjustment would most likely be necessary in a line of business with a relatively small number of claims of widely varying sizes.

Early count-based methods – Skurnick [7] summarizes an early (1967) proposal for a count-based method by R.E.Brian in the Insurance Accounting and Statistical Association Proceedings. Brian suggested breaking the ULAE process into five kinds of transactions: transactions associated with setting up new claims, maintaining outstanding claims, making a single payment, closing a claim, and reopening a claim. To estimate the future ULAE effort required for a set of claims, the actuary projects the future numbers of each type of transaction. Brian estimated that each of these transactions would bear a similar cost, and suggested estimating the cost per transaction using ratios of historical ULAE expenditures to the number of claim transactions occurring during the same calendar periods. Conceptually, this approach is based on the assumption that all kinds of claim transactions require similar ULAE resources and expenditures. However, this weakness could be remedied by refining the formula to allow for different cost levels for the different types of transactions. The need to forecast the numbers of future transactions is a considerable practical difficulty in the application of this approach. For our particular client situation, reliable claim count and claim transaction data were not available. Thus, we were unable to consider this and other count-based methods.

Wendy Johnson Method – This count-based method, presented in Johnson's 1989 paper [3], follows a line similar to Brian's. Johnson's specific example suggests using the reporting and maintenance as the key transactions. Johnson, like Brian, then projects the future number of newly reported claims, as well as the number of claims that will be in a pending status each year – and thus will have required maintenance during the year. Also like Brian, Johnson estimates the cost of each transaction by comparing historical aggregate ULAE expenditures to the number of transactions occurring in the same time historical period.

Johnson introduces a clever innovation by allowing for an explicit differential in the amount of ULAE resource or cost required for different types of claim transactions. Johnson's specific example assumes, based on qualitative input, that the process of opening a claim costs $*x*, and the process of maintaining existing claims costs additional $*x*.

Alternative weights, as well as additional transaction types, could be introduced directly into Johnson's formula (for example, our model assumes that the cost of closing a claim is in addition to the cost of maintenance.) The benefit of Johnson's innovation is that it requires only that the actuary estimate the *relative* amount of resources required for each transaction type, and does not require that the actuary perform detailed time-and-motion studies to calculate the *actual* cash cost of each transaction type.

The mechanics of the Johnson method involve estimating the ULAE cost per claim activity by calculating weighted claim counts based on historical data[6] and comparing those weighted claim counts to the total ULAE costs in the same historical period. The estimate of unpaid ULAE is obtained by projecting the number of, and the ULAE cost associated with, weighted claim counts at each subsequent year-end, related only to claims occurring prior to the reserve evaluation date.

Rahardjo and Mango-Allen: costs varying over time – Whereas Johnson introduces the concept that opening a claim requires a different quantity of resources, Rahardjo [6] and Mango-Allen [5] focus on the situation in which the annual (or quarterly) cost of maintaining and managing a claim varies over the life of the open claim. Mango and Allen, for example, introduce the concept that claims (liability claims, from the context of the paper) which are still open after a long period of time are likely to be complex claims requiring more claim adjuster time, and from a more senior (and probably more highly paid) adjuster. Their paper also introduces a specific inflation adjustment. The final reserve indication is likely to be quite sensitive to the magnitude of the parameters used, as the reader of the Mango-Allen paper will realize after working with the illustrative parameters presented. In addition, the estimates will be affected by parameters not explicitly considered in the articles, such as Mango and Allen’s implicit assumption that equal amounts of ULAE resources are required to open, close, and handle one average claim for a year.

Spalla: quantifying transaction costs – Spalla describes that manual time-and-motion studies no longer are necessary to determine the cost of various claim-related activities and transactions.

[6] For example, by adding newly opened and open claim counts at each evaluation, after multiplying the counts by the relative ULAE effort.

Rather, because so many of these activities are computer-supported, modern claim department management systems are equipped to track the amount of time spent on individual claim activities, by level of employee. By combining the individual claim management activities into somewhat more macroscopic transactions, it is feasible to calculate the average cost of each type of claim transaction. These average costs, loaded for overhead and other costs not captured by the computerized tracking systems, can be applied within analytical frameworks as described by Rahardjo and Mango-Allen. Another benefit of working with the underlying cost data that Spalla describes is that it allows for more detailed analysis of the claim activity costs. Using the detailed information, the actuary can determine which types of claim, which types of claim transactions and which stages of the claim life cycle have relatively similar (or relatively different) costs. The insight gained allows the actuary to treat those transactions with different costs (e.g., opening a workers compensation medical only claim versus opening a lost-time claim) separately for ULAE reserving purposes. Spalla describes her method of loading unmeasured costs on top of the costs specifically measured. We suggest that the actuary using Spalla's method consider an equally important additional step as a "reality check": if the selected costs per transaction were applied to the numbers of transactions that were undertaken last year, would the result match that period's actual total ULAE expenditures?

While Spalla describes determining the actual cost of various transactions, the process she describes could be effectively used to quantify the *relative* amount of cost per transaction, as compared to the cost of other kinds of claim transactions. This relativity is less subject to annual change, versus the dollar cost per transaction or per activity. With such relativities in hand, the general approaches described in Rahardjo and Mango-Allen could be used, but now with some quantitative basis for the magnitudes of the parameters.

Triangle Projection Methods – In the paid loss development method, losses paid to date for a particular accident year are used as the basis for estimating unpaid losses. Similar approaches can be applied to the paid ULAE that has been reported to date for a particular accident year. The projection of paid ULAE to ultimate ULAE can use selected parameters based on historical observations regarding the amount of paid ULAE reported before or after a comparable age of maturity. The literature describes three methods for quantifying the parameters to project paid-to-date ULAE to ultimate.

In much the same kind of methodology as used with paid losses in the paid loss development method, an accident year by evaluation year triangle of paid ULAE can be used to calculate development factors from evaluation point to ultimate. Note, however, that the construction of paid ULAE triangles relies on the manner in which ULAE payments are allocated to accident year – since "actual" ULAE by accident year is not observable, at least not for all categories of ULAE expenditures. This allocation of ULAE payments is typically based on the pattern of claim payments, which *can* be observed. The accident year triangles of ULAE may be distorted if either the method of allocating calendar year ULAE to accident years changes over time, or if the loss payment patterns change.

The triangle projection method also raises a philosophical question in cases where a company is using a simplistic ULAE allocation method (e.g., the historical 50/50 rule) that does not mirror the actual distribution of expenses. In this case, the triangle projection method may produce a good estimate of the ULAE dollars that will be reported on future Schedule Ps for the current and prior accident years. However, this estimate may not accurately estimate the actual ULAE expenditures that will be required to handle and settle claims for the same period. Which one of the two is the actuary really trying to accomplish? We believe that the answer should be that the actuary is estimating a reserve for the actual future expenditures, but one might also argue that the objective is to predict the future values on Schedule P. This question may arise less frequently now that carriers are allowed to allocate ULAE payments to accident years based on their own analysis.

Slifka [8] and Kittel [4] each describe methodologies that project ultimate or unpaid ULAE based on historical ULAE payments. Slifka suggests using a time-and-motion study to estimate the claim department's allocation of resources between current accident year claims and prior accident year claims. This relationship between the "cost" of current year's claim management activities and prior years' claim management activities can be used then to estimate the future payment activity. Let us assume for example that such a study suggests that 60% of the current accident year's ULAE remains unpaid, 15% of the prior accident year's ULAE remains unpaid, and 5% of the second prior accident year's ULAE remains unpaid as of December 31. Then the total unpaid ULAE at this evaluation date is estimated as 80% (60% + 15% + 5%) of a typical calendar year's ULAE payment. Although this approach presumes a steady state, it can be refined it to reflect volume growth as well as the effects of inflation.

A third approach we have used in some real-life applications is to construct paid ULAE triangles, not by using the actual historical allocations of ULAE to accident year, but by restating those allocations using current time-and-motion studies, and/or relationships to loss payment patterns. For example, let us assume that these studies suggest that half of ULAE is paid at the time a claim is reported, and half is paid in proportion to claim payments. Then historical calendar year ULAE can be assigned to accident year-calendar year cohorts: half according to the distribution of reported claims across current accident year, prior accident year, second prior accident year, and so on; and half according to the distribution of paid losses – as indicated by an appropriate accident year loss payment pattern (e.g., 10% to the current accident year, 15% to the prior, and so on). Once the ULAE triangle is constructed, traditional triangle projections can be applied.

THE GENERALIZED APPROACH

During the course of our client assignment, we set out to define a procedure to estimate ULAE liabilities which would recognize this company's rapid growth, and be consistent with our understanding of the patterns of the company's ULAE expenditures over the life of a claim. The objectives were (a) to reproduce the key concepts behind the Johnson method, while using commonly available – and usually reliable – aggregate payment and reserve data; and (b) to develop an extension to the Kittel refinement which could allow for alternatives to the traditionally-assumed "half and half" pattern of ULAE expenditures over the life cycle of the claim. The generalized method described in this section accomplishes both of these objectives.

Indeed, the reader will recognize the roots of the generalized method in both Johnson's and Kittel's methods. As done by Wendy Johnson, our approach employs the concept of "weighted" claims, by which claims "use up" different amounts of ULAE at each different stage of their life cycle, from opening to closing. Therefore, newly opened, open and newly closed claims should be given different weights when determining the "loss basis" to which ULAE payments during a past or future calendar period would be related. However, because we believed that handling costlier claims warrants and requires relatively more resources than handling smaller claims, we sought to use claim dollars rather than claim counts.

Following this thought process, we defined our *loss basis* for a particular time period as a weighted average of the ultimate cost of claims reported during the period, the ultimate cost of claims closed during the period, and losses paid during the period. Note that by "ultimate cost of claims reported" we mean the reported amounts as well as any future development on known claims. Analogously, we define "ultimate cost of claims closed" as the final cost of claims that are currently closed, that is, we include in that amount any future payments made after the closing of the claim[7].

Kittel in fact introduced a weighted average of this sort, but Kittel's average includes only incurred losses and paid losses, and Kittel's weights are fixed at 50/50. By comparison, the generalized method introduces a third loss measure that allows distinguishing the cost of

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[7] As discussed in a subsequent section, this approach assumes that there is no additional cost associated with reopening or "reclosing" a reopened claim. The formulas do provide, however, for the cost of maintaining reopened claims.

maintenance from the cost of closing (an important distinction for workers compensation), and allows the flexibility of selecting weights appropriate to the company and segment of business.

In the following paragraphs, we present the explicit definition of the loss basis, and how it is used to calculate the projected ultimate ULAE, as well as the estimated ULAE liability.

Let U_1, U_2 and U_3 be such that $U_1 + U_2 + U_3 = 100\%$, and U_1, U_2 and U_3 are defined as follows:

- U_1 is the percentage of ultimate ULAE spent opening claims,
- U_2 is the percentage of ultimate ULAE spent maintaining claims, and
- U_3 is the percentage of ultimate ULAE spent closing claims.

In the course of a loss and loss adjustment expense reserve review, it would be appropriate to determine reasonable ranges for U_1, U_2 and U_3, and test the sensitivity of the final result (unpaid ULAE) to variations within those ranges. Several considerations for the selection of U_1, U_2 and U_3 are discussed in a subsequent section.

Also, for a particular time period T, let us define:

- R as the ultimate cost of claims that have been reported during the period T,
- C as the ultimate cost of claims that have been closed during the period T, and
- P as the losses paid during the period T,

following the definition of "ultimate cost" of reported and closed claims stated in a preceding paragraph.

Conceptually, the time period *T* could represent activity occurring between t_1 and t_2 related to a particular accident year, or activity occurring between t_1 and t_2 related to all accident years, where t_1 and t_2 are selected points in time.

The use of the aggregate claim dollar values *R*, *C*, and *P* as driving values in the generalized method reveals an assumption that the expenditure of ULAE resources is proportional to the dollars of losses being handled. This assumption is in contrast to Wendy Johnson's assumption that ULAE costs are independent of the claim size and nature. More specifically, the generalized method is based on the following assumptions:

- ULAE amounts spent opening claims are proportional to the ultimate cost of claims being reported,
- ULAE amounts spent maintaining claims are proportional to payments made, and
- ULAE amounts spent closing claims are proportional to the ultimate cost of claims being closed.

The appropriateness and sensitivity of these assumptions warrant further analysis, both as a matter of general research, and for a particular application of either method. We concluded that the dollar proportionality was an assumption that would produce reasonable indications for our particular application.

From the preceding definitions and assumptions, the total amount spent on ULAE during a time period *T* would be described by the relationship:

$$M = (R \times U_1 \times W) + (P \times U_2 \times W) + (C \times U_3 \times W),$$

where *W* represent the ratio of ultimate ULAE to ultimate losses (*L*), and *M* represents the total ULAE expenditures during time period *T*.

We can now define our loss basis *B* for the time period *T* as:

$$B = (U_1 \times R) + (U_2 \times P) + (U_3 \times C).$$

By simple algebra:

$$M = B \times W \text{ , and } W = M / B .$$

Each component of the loss basis *B* can be understood conceptually as the value of the claims underlying the ULAE payments. Thus,

- $U_1 \times R$ represents the loss basis for ULAE spent setting up new claims,
- $U_2 \times P$ represents the basis for ULAE spent maintaining open claims, and
- $U_3 \times C$ represents the basis for ULAE spent closing existing claims.

A more detailed algebraic and intuitive description of the derivation of the loss basis *B* and the ULAE ratio *W* can be found in the Appendix.

In practice, companies typically observe, measure and report *M*, the ULAE payments during a period, such as a calendar year[8]. Once U_1, U_2 and U_3 are estimated or selected, the loss basis *B* can be calculated from loss amounts *R*, *P*, and *C* (defined in a preceding paragraph) that can typically be determined from data and calculations underlying an actuarial loss reserve analysis. In particular, *M* and *B* can be calculated for historical calendar periods. By computing the ratio $W = M / B$, where both *M* and *B* are expressed on a calendar-year basis, we obtain ratios of ULAE to loss by calendar year. We can then select an overall ratio of ULAE to loss, which we will name W^*, to be used in estimating future ULAE expenditures.

Based on the concepts and notation defined above, we could estimate the ultimate ULAE (*U*) for a group of accident years as the product between W^* and L, where *L* represents the independently estimated ultimate losses for the same group of accident years, and W^* represents the selected ultimate ULAE to loss ratio. That is, $U = W^* \times L$.

This representation of ultimate ULAE suggests different ways to estimate a reserve for unpaid ULAE for a group of accident years. As discussed below, these approaches will produce different results, and may be appropriate for use under different circumstances.

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[8] It must be noted that the 1998 revisions to the statutory classification of loss adjustment expenses create a practical difficulty for the application of this and other ULAE reserving methods, as discussed in a subsequent section of this article.

A possible method (not the one we prefer, as described below) is to estimate unpaid ULAE from the estimate of ultimate ULAE ($V = W^{*} \times L$), reduced by the amount of ULAE already paid (M). That is, we can compute:

$$\text{Unpaid ULAE} = \left(W^{*} \times L\right) - M \ .$$

In many situations, this method presents both practical and conceptual difficulties. From a practical perspective, it may be difficult to quantify the historical paid ULAE that corresponds only to the accident year losses represented by L. And, conceptually, this approach has some similarities to, and shares the potential distortions of, an expected loss ratio approach to unpaid losses, in which unpaid losses are estimated as the product of a pre-set expected loss ratio and premium (expected ultimate), reduced for actual paid amounts. As a period matures, the reserve estimate can become increasingly distorted if actual paid losses do not approach the pre-set expected ultimate.

Another method, which we prefer, is analogous to the Bornhuetter-Ferguson loss reserving method, in that an *a priori* provision of unpaid ULAE is computed. Using the notation introduced in this article, we would calculate:

$$\text{Unpaid ULAE} = W^{*} \times \{L - B\}$$

To understand the derivation of this estimate, let

- $R(t)$ be the ultimate cost of claims known as of time t,
- $C(t)$ be the ultimate cost of claims closed as of time t,
- $P(t)$ be the total amount paid as of time t.

If L, $R(t)$, $C(t)$ and $P(t)$ relate to a specific group of accident years, then we could express ULAE liabilities on these accident years at time t as:

$$\text{Unpaid ULAE } = W^* \times \{U_1 \times [L - R(t)] + U_2 \times [L - P(t)] + U_3 \times [L - C(t)]\}.$$

Each component of this formula represents a provision for the expenses associated with:

- opening claims not yet reported,
- making payments on currently active claims and on those claims that will be reported in the future, and
- closing “unclosed” claims, i.e., closing claims open at time t and closing those claims that will be reported/opened in the future.

Rearranging the terms in the equation above, we obtain:

$$\text{Unpaid ULAE} = W^* \times \{L \times (U_1 + U_2 + U_3) - [U_1 \times R(t) + U_2 \times P(t) + U_3 \times C(t)]\}$$

$$= W^* \times \{L - [(U_1 \times R(t)) + (U_2 \times P(t)) + (U_3 \times C(t))]\}$$

$$= W^* \times \{L - B\},$$

As noted above, this methodology implies that the amount of ULAE paid to date and the ULAE liability are not directly related, except to the extent that these payments influence the selection of the ratio W^*. The reader may recall that a similar assumption is the basis behind the popular Bornhuetter-Ferguson reserving approach (a thorough discussion of the Bornhuetter Ferguson loss reserving approach, as well as of the expected loss ratio method, can be found in the 1972 *Proceedings* article by Bornhuetter and Ferguson [1].)

A third possible approach implied by the definition of B would be analogous to the loss development reserving method. ULAE liabilities could be estimated as:

$$\text{Unpaid ULAE} = M \times \left(\frac{L}{B} - 1 \right).$$

Such approach, which warrants further investigation, would imply that ULAE liabilities are proportional to paid amounts reported to date. Aside from the practical difficulty of establishing the ULAE amounts paid that correspond to accidents occurring during a particular period, this methodology, similarly to the paid loss development approach, may be overly responsive to random fluctuations in ULAE emergence.

Readers interested in the comparison of the three corresponding loss reserving methodologies are directed to a study note by Brosius [2].

The foregoing discussion presents the generalized approach based on relating historical ULAE payments and estimated ULAE reserves to loss amounts. This approach and notation can easily be adapted to using claim counts or transaction counts. For example, if the analyst believes that ULAE is best described as being related to the number of claims reported, the number of claims open at any point during the period, and the number of claims closing during the period, then our "loss basis" – using lower case notation to differentiate from the standard generalized method – is:

$$b = (v_1 \times r) + (v_2 \times o) + (v_3 \times c)$$

where *r* represent reported claims, *o* are open claim counts, and *c*, closed claims. We find it most convenient in this formula to describe v_1, v_2 and v_3 as being estimates of the relative cost of handling the reporting of a claim, managing an open claim for one year (or portion thereof), and closing a claim, respectively. As in Johnson's paper, discussed earlier, it is not necessary to predetermine the actual cost of these various activities, just their relative magnitudes – Johnson, for example, assumes $v_1 = 2$, $v_2 = 1$ and $v_3 = 0$. We now select w^*, representing the cost of an activity having v = 1, and estimated from historical data as $w = M / b$, where M still represents ULAE payments.

After the analyst selects a value of $w*$ for use in projecting future costs (perhaps a series of $w*_i$'s, reflecting explicit future inflation adjustments), the ULAE reserve can be estimated as

$$\text{Unpaid ULAE} = \sum_i w_i * \times [(v_1 \times r_i) + (v_2 \times o_i) + (v_3 \times c_i)],$$

where:

- r_i represents the number of claims to be reported in each calendar year *i*,
- o_i is the number of open claims at the end of calendar year *i*,
- c_i represents the claims to be closed during calendar year *i*, and
- *i* represents the series of future calendar year-ends until all claims are closed.

In each case, only claims occurring on or before the date of evaluation of ULAE liabilities should be considered. The reader will note that, according to the formula above, a claim that stays open for several years is counted multiple times in the summation. This is consistent with the assumption that ULAE is incurred each year such a claim stays open.

It is relatively straightforward to see that this formulation of ULAE based on claim counts is equivalent to that presented by Wendy Johnson. It could be adapted to recognize the Rahardjo and Mango-Allen concepts of costs varying over time by stratifying the claims activities more finely than just reporting, open, and closing, i.e., by having more than three categories of claims activities.

U_1, U_2 AND U_3

No doubt the reader will have identified by now that there is no convenient handbook providing the values of U_1, U_2 and U_3 for a particular category of business. Certainly, we expect that the values could vary significantly from carrier to carrier, and between coverages. For example, a litigation-intense liability book of business might have a strong concentration of activity close to the time of claim settlement and payment, versus a large front-end cost for workers compensation.

We have found it feasible to develop a range of values for U_1, U_2 and U_3 for a particular company and line of business by interviewing claims personnel. The resulting ranges can be used to test the consistency of the resulting ULAE ratios, as well as to assess the sensitivity of the ULAE ratios to different choices of U_1, U_2 and U_3 within the range suggested by the interview process. Time and motion studies as described by Spalla could be used to develop an empirical basis for the parameters needed.

An interesting research project would be to develop a series of benchmark values of U_1, U_2 and U_3 (and v_1, v_2 and v_3) by line of business, market segment or carrier characteristics.

A SIMPLIFICATION

We realize that in many cases, the estimation of *R* and *C*, that is, the ultimate cost of reported and closed claims, may not be a trivial exercise.

As defined in the beginning of this article, the ultimate cost of claims reported as of a certain date represents the total payments that will ultimately be made in connection with all claims known to the carrier as of that date. Another way of thinking about these costs is as the ultimate for the accident period ending on that date, reduced for the pure IBNR amounts, which represent the ultimate cost of not-yet-reported claims. Analogously, the ultimate cost of closed claims as of a certain evaluation point represents the final cost of claims that are closed as of the evaluation date, including any subsequent payments.

Although an actuary familiar with the reserving and claims handling practices of the specific company would normally be able to produce accurate estimates of *R* and *C*, the necessary detailed information may not be available, or the additional effort may not be justified. To address situations like these, we explored a simplification of the generalized methodology that does not require the estimation of *R* or *C*.

First, we used the estimated ultimate losses for the *accident* year as a proxy for the ultimate cost of claims reported in the *calendar* year. This calendar year amount can be expressed exactly as the sum of the corresponding *accident* year ultimate and the pure IBNR at year-end, reduced by the amount of pure IBNR at the beginning of the year. The actuary can evaluate the error inherent in using the suggested approximation after considering the difference in exposures between accident years as well as the characteristics of the coverage being analyzed, and then make judgmental adjustments as necessary. For example, given the minimal delay in the reporting of workers compensation claims, we can often assume the pure IBNR component of the ultimate is not likely to vary much from one year to the next. Therefore, the accident year ultimate would be a reasonable approximation of the true value of the parameter R.

Secondly, if no particular additional effort is required to close an existing claim (as is the case in the example presented in a subsequent section), we can assume that U_3 equals zero. This assumption may be inappropriate for some lines of business. For example, a significant portion of the cost of handling an employment practices liability claim will be incurred in connection with its settlement.

If the particular coverage allows making the assumption that U_3 equals zero, then $U_1 + U_2$ should equal 1.0, and we can approximate the loss basis B for each calendar year as

$$\hat{B} = (U_1 \times A) + (U_2 \times P),$$

where A represents the ultimate losses for the corresponding accident year, and compute the observed W values as $M / \hat{B}$ for each year. After reviewing those observed ULAE ratios, the actuary will select an appropriate ratio W^* for use in estimating ULAE liabilities.

An overall estimate of pure IBNR as of the evaluation date can be obtained (perhaps by analyzing claim reporting patterns and ultimate severities), which could then be deducted from *L* to obtain an estimation of the ultimate cost of claims reported to date, which we denote *R*. Unpaid ULAE can be then calculated according to the formulas presented above, as

Unpaid ULAE = $W^* \times \{L - [(U_1 \times R) + (U_2 \times P)]\}$,

which can also be expressed as:

Unpaid ULAE = $W^* \times [U_1 \times (L - R) + U_2 \times (L - P)]$.

AN EXAMPLE

To illustrate our approach, we included an example of the application of the traditional, Kittel and generalized methodologies in the evaluation of ULAE liabilities for a workers compensation book of business.

This sample insurance company began operations in 1997, and over the course of its 6 years of operations, paid ULAE has averaged 18% of paid losses (as seen in Exhibit B). Following a review of the paid-to-paid ratios by year, the traditional method might lead the analyst to select 16% as the ratio of ULAE to loss for use in establishing a ULAE reserve. This would be appropriate if ULAE payments were proportional to paid losses for a particular calendar year. We have found, however, that for workers compensation, ULAE expenditures are concentrated more heavily towards the front end of the claim than are the loss payments. Consider a hypothetical extreme, in which all ULAE is incurred at the moment the claim occurs, with the amount of the ULAE being proportional to the size of the claim. In this hypothetical case, the appropriate relationship to examine would be the ratio of ULAE to ultimate losses for an accident period[9]. Furthermore, the growth experienced by this company will cause the indicated ULAE liability using the traditional methodology to be overstated.

Interviews with management of this company's claims department, and examination of the flows of work and allocation of resources in the claims department suggested that approximately 60%

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[9] The reader will recognize elements of the suggested simplification of the generalized method in the discussion of this extreme case.

to 70% of the work for a claim is concentrated at the time the claim is reported, and 30% to 40% of the work is spread over the remaining life of the claim. For this company, no particular extra degree of effort is associated with closing the claims. Since ULAE expenditures are heavier at the beginning of a claim's life cycle, it should come as no surprise to the reader that the standard Kittel method (shown in Exhibit C) indications of unpaid ULAE are overstated.

Applying the generalized method, and setting U_1 equal to a value in the range 60% to 70%, U_2 in the range 40% to 30%, and U_3 equal to zero, the observed ULAE to loss ratios range between 8% and 11% for the various years, as can be seen in Exhibits D and E. The selected ratio in this illustration is 10%. While this selection was based on the company's total history, rather than the individual accident periods, we note that, for individual periods, the ULAE ratios implied by this method behave much more regularly than if the traditional paid-to-paid ratios are used. This behavior provides some support for the reasonableness of the selected values of U_1, U_2 and U_3. The reader will note the significant difference between this ratio and the ratios indicated by the traditional and Kittel methods.

Given the selected W^* ratio of 10%, Exhibits D and E display the application of the three alternative ULAE reserving formulas derived from the generalized method ("expected loss", "Bornhuetter-Ferguson" and "development" methods) which we describe in a preceding section.

The simplified version of the generalized method is shown in Exhibit F. In this case, we chose to present a likely range for estimated pure IBNR to be used in computing the ULAE reserve.

PRACTICAL DIFFICULTIES

Inconsistencies in the reporting of claim adjustment expenses would create obvious difficulties to the application of ULAE reserve estimation methodologies. With the 1998 change in statutory rules requiring the classification and reporting of “Other Adjusting Expenses”, some insurers may no longer capture traditional ULAE, or a consistent history of such expense payments may not be available. The methods described in this paper could be applied to traditional ULAE, to the new “Other Adjusting” expenses, to the individual component activities and expenses that comprise these broader categories of loss adjustment expense, or to historical loss adjustment expenses reclassified to approximate the current Defense and Cost Containment and Other Expense definitions. These methods could also be applied to the whole, or components of, ALAE or its statutory replacement, “Defense and Cost Containment” expenses, although likely using different weighting parameters.

Furthermore, as noted in a preceding section, estimation of R and C, that is, the ultimate cost of reported and closed claims, may not be trivial . The simplification shown in this article is only one of the many approaches that an actuary could take to sidestep that difficulty.

As noted earlier in this paper, the generalized methodology is consistent with the assumption that the claims adjusting activities associated with reopening and “reclosing” a claim have no cost. An alternative approach, which we have not used in practice, is to assume that the ultimate cost of closed claims C equals the sum of total amounts paid on closed claims as of the evaluation date (noted here as $\hat{C}$). An approximated loss basis $\hat{B}$ can be expressed as:

$$\hat{B} = (U_1 \times R) + (U_2 \times P) + (U_3 \times \hat{C}).$$

Under this approach, the cost of "reclosing" a claim is assumed to be equal to the cost of closing a claim of the same size. However, this alternative approach would still fail to capture the cost of reopening claims.

In cases where reopenings of claims are more than negligible, and the ULAE cost of such reopenings (and subsequent "reclosings") is not immaterial, the actuary could obtain a separate provision for the cost of future claims handling activities relating to claims that are closed as of the evaluation of ULAE liabilities. This provision could perhaps be based on a study of the frequency of reopenings and average cost in ULAE of handling the reopened claims.

As noted by Kittel, loss inflation can cause material distortions in the projection of future ULAE payments. We have not attempted to measure the relative accuracy of the generalized method (as compared to other dollar-based methods) in an inflationary environment. Two other issues that warrant further investigation are: the effect of reopened claims on the accuracy of the estimates of unpaid ULAE, and how to modify the approach to properly reflect the change over time in the quantity or cost of resources dedicated to the handling of a claim, as that claim ages.

As mentioned before, the actuary may introduce a measure of estimation error for parameters U_1, U_2 and U_3, and obtain an associated range of reasonable ULAE liability estimates.

As with any reserving methodology, the practicing actuary should carefully examine the explicit and implicit assumptions of the generalized method, as well as the potential effect of external issues when estimating ULAE liabilities, and customize the approach accordingly.

KITTEL’S REFINED APPROACH AS A SPECIAL CASE OF THE GENERALIZED APPROACH

We can quite easily prove that the approach described in this article is simply a generalization of the familiar Kittel refined method described in a preceding section of this article. Indeed, each of the assumptions in Kittel’s refined approach can be translated to assumptions about the parameters of the generalized approach.

For example, Kittel’s refined method implicitly assumes no future case reserve development or reopened claims. In other words, the estimated IBNR reserves amount to “pure IBNR” only. The Kittel approach also assumes implicitly that all payments associated with a claim occur at closing. Therefore, according to Kittel’s implicit assumptions, and using the notation described in the preceding section, R equals reported losses and $P = C$ equal paid losses. Furthermore, by selecting $U_1 = U_3 = 50\%$, and $U_2 = 0\%$, the two approaches are algebraically equivalent.

Table 1 shows the equivalence between the refined and generalized methods, given the assumptions for the refined approach.

Table 1 – Equivalence of Kittel's refined method and generalized approach

Kittel's assumptions and calculations	Adapted to generalized approach notation
• There are no partial payments or reopened claims	• P = C = paid losses
• 50% of ULAE is spent opening claims, and 50% is spent closing claims	• U_1 = 50%, U_2 = 0%, U_3 = 50%
• W = paid ULAE / [50% x (paid loss + reported loss)]	• $W = M / B =$ $= M / (R \times U_1 + P \times U_2 + C \times U_3) =$ $= M / [50\% \times (R + C)] =$ = paid ULAE / [50% x (paid loss + reported loss)]
• Unpaid ULAE = $= W^* \times [\text{IBNR} + 50\% \times \text{case reserves}]$	• Unpaid ULAE $= W^* \times (L - B) =$ $= W^* \times [L - 50\% \times (R + C)] =$ $= W^* \times [L - R + 50\% \times (R - C)] =$ $= W^* \times [\text{IBNR} + 50\% \times \text{case reserves}]$

Where W^* is the selected ULAE-to-loss ratio, based on observed W's.

XYZ INSURANCE COMPANY
REVIEW OF ULAE RESERVES AS OF 12/31/2002
($000's)

EXHIBIT A.1 -- INPUT PARAMETERS

Calendar Year	**Cal. Year Paid ULAE**	**Cal. Year Paid Loss & ALAE**	**Cal. Year Reported Loss & ALAE**	**Est. Ultimate Loss & ALAE on Claims Reported in Cal. Year**
(1)	(2)	(3)	(4)	(5)
1997	$1,978	$4,590	$19,534	$27,200
1998	4,820	14,600	57,125	76,700
1999	8,558	38,390	85,521	106,900
2000	12,039	58,297	128,672	154,300
2001	13,143	86,074	145,070	163,100
2002	15,286	105,466	163,626	176,400
Total	$55,824	$307,417	$599,547	$704,600

Notes:

(2), (3), (4) As shown in XYZ's 2002 and prior years' Annual Statement.
(5) Estimated in year-end 2002 actuarial analysis.

XYZ INSURANCE COMPANY
REVIEW OF ULAE RESERVES AS OF 12/31/2002
($000's)

EXHIBIT A.2 -- INPUT PARAMETERS

Accident Year	**Ultimate Loss & ALAE**	**IBNR Loss & ALAE at 12/31/2002**	**Reported Loss & ALAE at 12/31/2002**
(1)	(2)	(3)	(4)
1997	$28,600	$257	$28,343
1998	79,200	1,742	77,458
1999	108,400	5,095	103,305
2000	156,700	16,140	140,560
2001	163,400	34,477	128,923
2002	177,100	56,141	120,959
Total	$713,400	$113,853	$599,547

Notes:

(2), (3) Estimated in year-end 2002 actuarial analysis.
(4) As shown in XYZ's 2002 Annual Statement.

XYZ INSURANCE COMPANY
REVIEW OF ULAE RESERVES AS OF 12/31/2002
($000's)

EXHIBIT B -- APPLICATION OF TRADITIONAL METHOD

Calendar Year	**Cal. Year Paid ULAE**	**Cal. Year Paid Loss & ALAE**	**Paid-to-Paid ULAE Ratio**
(1)	(2)	(3)	(4)
1997	$1,978	$4,590	0.431
1998	4,820	14,600	0.330
1999	8,558	38,390	0.223
2000	12,039	58,297	0.207
2001	13,143	86,074	0.153
2002	15,286	105,466	0.145
Total	$55,824	$307,417	0.182

(5) Selected ULAE ratio	0.160
(6) Case reserve	$292,130
(7) IBNR	$113,853
(8) Indicated ULAE Reserve	$41,587

Notes:

(2), (3), (6) As shown in XYZ's 2002 and prior years' Annual Statement.
(4) Equals (2) / (3).
(5) Judgmentally selected.
(7) Estimated in year-end 2002 actuarial analysis.
(8) Equals (5) x [(7) + 50% x (6)].

XYZ INSURANCE COMPANY
REVIEW OF ULAE RESERVES AS OF 12/31/2002
($000's)

EXHIBIT C -- APPLICATION OF KITTEL METHOD

Calendar Year	Cal. Year Paid ULAE	Cal. Year Paid Loss & ALAE	Cal. Year Reported Loss & ALAE	ULAE Ratio
(1)	(2)	(3)	(4)	(5)
1997	$1,978	$4,590	$19,534	0.164
1998	4,820	14,600	57,125	0.134
1999	8,558	38,390	85,521	0.138
2000	12,039	58,297	128,672	0.129
2001	13,143	86,074	145,070	0.114
2002	15,286	105,466	163,626	0.114
Total	$55,824	$307,417	$599,547	0.123

(6) Selected ULAE ratio	0.115
(7) Case reserve	$292,130
(8) IBNR	$113,853
(9) Indicated ULAE Reserve	$29,891

Notes:

(2), (3), (4), (7) As shown in XYZ's 2002 and prior years' Annual Statement.
(5) Equals (2) / { 50% x [(3) + (4)] }.
(6) Judgmentally selected.
(8) Estimated in year-end 2002 actuarial analysis.
(9) Equals (6) x [(8) + 50% x (7)].

XYZ INSURANCE COMPANY
REVIEW OF ULAE RESERVES AS OF 12/31/2002
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EXHIBIT D -- APPLICATION OF GENERALIZED METHOD
USING 60/40 ASSUMPTION

Calendar Year	**Cal. Year Paid ULAE**	**Est. Ultimate Loss & ALAE on Claims Reported in Cal. Year**	**Cal. Year Paid Loss & ALAE**	**Loss Basis**	**ULAE Ratio**
(1)	(2)	(3)	(4)	(5)	(6)
1997	$1,978	$27,200	$4,590	$18,156	0.109
1998	4,820	76,700	14,600	51,860	0.093
1999	8,558	106,900	38,390	79,496	0.108
2000	12,039	154,300	58,297	115,899	0.104
2001	13,143	163,100	86,074	132,290	0.099
2002	15,286	176,400	105,466	148,026	0.103
Total	$55,824	$704,600	$307,417	$545,727	0.102

(7) Selected ULAE ratio	0.100
(8) Ultimate loss and LAE	$713,400
(9) Indicated ULAE Reserve	
(a) Using "expected loss" method	$15,516
(b) Using "Bornhuetter - Ferguson" method	$16,767
(c) Using "development" method	$17,152

Notes:

(2), (4), (8) As shown in XYZ's 2002 and prior years' Annual Statement.
(3) Estimated in year-end 2002 actuarial analysis.
(5) Equals 60% x (3) + 40% x (4).
(6) Equals (2) / (5).
(7) Judgmentally selected.
(8) Estimated in year-end 2002 actuarial analysis.
(9a) Equals (7) x (8) - [Total (2)].
(9b) Equals (7) x { (8) - [Total (5)] }.
(9c) Equals { (8) / [Total (5)] - 1.0 } x [Total (2)].

XYZ INSURANCE COMPANY
REVIEW OF ULAE RESERVES AS OF 12/31/2002
($000's)

EXHIBIT E -- APPLICATION OF GENERALIZED METHOD
USING 70/30 ASSUMPTION

Calendar Year	Cal. Year Paid ULAE	Est. Ultimate Loss & ALAE on Claims Reported in Cal. Year	Cal. Year Paid Loss & ALAE	Loss Basis	ULAE Ratio
(1)	(2)	(3)	(4)	(5)	(6)
1997	$1,978	$27,200	$4,590	$20,417	0.097
1998	4,820	76,700	14,600	58,070	0.083
1999	8,558	106,900	38,390	86,347	0.099
2000	12,039	154,300	58,297	125,499	0.096
2001	13,143	163,100	86,074	139,992	0.094
2002	15,286	176,400	105,466	155,120	0.099
Total	$55,824	$704,600	$307,417	$585,445	0.095

(7) Selected ULAE ratio	0.100
(8) Ultimate loss and LAE	$713,400
(9) Indicated ULAE Reserve	
(a) Using "expected loss" method	$15,516
(b) Using "Bornhuetter - Ferguson" method	$12,795
(c) Using "development" method	$12,201

Notes:

(2), (4), (8) As shown in XYZ's 2002 and prior years' Annual Statement.
(3) Estimated in year-end 2002 actuarial analysis.
(5) Equals 70% x (3) + 30% x (4).
(6) Equals (2) / (5).
(7) Judgmentally selected.
(8) Estimated in year-end 2002 actuarial analysis.
(9a) Equals (7) x (8) - [Total (2)].
(9b) Equals (7) x { (8) - [Total (5)] }.
(9c) Equals { (8) / [Total (5)] - 1.0 } x [Total (2)].

XYZ INSURANCE COMPANY
REVIEW OF ULAE RESERVES AS OF 12/31/2002
($000's)

EXHIBIT F -- APPLICATION OF SIMPLIFIED GENERALIZED METHOD
USING 60/40 ASSUMPTION

Calendar Year	**Cal. Year Paid ULAE**	**Acc. Year Ultimate Loss & ALAE**	**Cal. Year Paid Loss & ALAE**	**Loss Basis**	**ULAE Ratio**
(1)	(2)	(3)	(4)	(5)	(6)
1997	$1,978	$28,600	$4,590	$18,996	0.104
1998	4,820	79,200	14,600	53,360	0.090
1999	8,558	108,400	38,390	80,396	0.106
2000	12,039	156,700	58,297	117,339	0.103
2001	13,143	163,400	86,074	132,470	0.099
2002	15,286	177,100	105,466	148,446	0.103
Total	$55,824	$713,400	$307,417	$551,007	0.101

(7)	Selected ULAE ratio	0.100
(8)	Ultimate loss and LAE	$713,400
(9)	Estimated pure IBNR based on	
	(a) Pure IBNR amounts to 4% of latest accident year ultimate	$7,084
	(b) Pure IBNR amounts to 6% of latest accident year ultimate	$10,626
(10)	Indicated ULAE Reserve	
	(a) If pure IBNR amounts to 4% of latest accident year ultimate	$16,664
	(b) If pure IBNR amounts to 6% of latest accident year ultimate	$16,877

Notes:

(2), (4), (8) As shown in XYZ's 2002 and prior years' Annual Statement.
(3) Estimated in year-end 2002 actuarial analysis.
(5) Equals 60% x (3) + 40% x (4).
(6) Equals (2) / (5).
(7) Judgmentally selected.
(8) Estimated in year-end 2002 actuarial analysis.
(9a), (9b) Based on claims reporting pattern and severity analysis in year-end 2002 actuarial analysis.
(10a) Equals (7) x {60% x 9(a) + 40% x [(8) - [Total (4)]]}.
(10b) Equals (7) x {60% x 9(b) + 40% x [(8) - [Total (4)]]}

Note that (8) - [Total (4)] represents total unpaid loss and ALAE as of the evaluation date.

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APPENDIX: DERIVATION OF THE FORMULAS FOR LOSS BASIS *B* AND ULAE RATIO *W*

Over the life of a cohort of claims corresponding to a particular accident year, let us assume that *L* dollars will be spent on losses. That is, the ultimate losses for claims occurred during that accident year amount to *L* dollars. Using the notation described in this paper, the insurer will spend *W* times *L* dollars on ULAE during the life of these claims, as follows:

- $U_1 \times W \times L$ dollars are spent on the initial opening and set up of claims,
- $U_2 \times W \times L$ dollars are spent on the ongoing maintenance and payment of claims, and
- $U_3 \times W \times L$ dollars are spent closing claims.

If *t* is some point in time after the start of the accident year *y*, the amount of ULAE spent cumulatively through time *t*, or *M*(*y*,*t*) is the sum of:

- $U(1, y, t) = U_1 \times W \times R(y, t)$, where *R*(*y*, *t*)is the ultimate cost of accident year *y* claims reported by time *t*,
- $U(2, y, t) = U_2 \times W \times P(y, t)$, where *P*(*y*, *t*) are the loss dollars paid by time *t*, in connection with accident year *y* claims, and
- $U(3, y, t) = U_3 \times W \times C(y, t)$, where *C*(*y*, *t*) is the ultimate cost of accident year *y* claims closed on or before *t*.

The reader will note that each component of total ULAE payments to date is assumed to be proportional to the ultimate cost of claims reported or closed, or payments made, respectively.

Naturally, as *t* grows, *R*(*y*, *t*), *P*(*y*, *t*) and *C*(*y*, *t*) all approach the ultimate losses for year *y*, or *L*(*y*), and *U*(1,*y*,*t*) + *U*(2,*y*,*t*) + *U*(3,*y*,*t*) approaches the ultimate ULAE for *y*.

By summing across all past accident years, we can express the total ULAE paid on or before time *t*, in connection with claims occurring on or before time *t*, or *M*(*t*), as the sum of:

- $\sum_{y \le t} U(1, y, t) = U_1 \times W \times \sum_{y \le t} R(y, t) = U_1 \times W \times R(t);$
- $\sum_{y \le t} U(2, y, t) = U_2 \times W \times \sum_{y \le t} P(y, t) = U_2 \times W \times P(t);$ and
- $\sum_{y \le t} U(3, y, t) = U_3 \times W \times \sum_{y \le t} C(y, t) = U_3 \times W \times C(t).$

That is, $M(t) = W \times [U_1 \times R(t) + U_2 \times P(t) + U_3 \times C(t)]$, or, ULAE paid to date is proportional to a weighted average of the ultimate cost of reported and closed claims, and loss payments to date.

Finally, if *s* and *t* are January 1st and December 31st of a specific year, the amount of ULAE paid during that year can be described as the difference between *M*(*t*) and *M*(*s*). We can now derive a formula for *W* as:

$$W = \frac{[M(t) - M(s)]}{U_1 \times [R(t) - R(s)] + U_2 \times [P(t) - P(s)] + U_3 \times [C(t) - C(s)]}.$$

That is, the actuary can obtain indications of the ratio of ULAE to losses by observing the ratio of ULAE paid during a certain period to the weighted average of the ultimate cost of claims reported, ultimate cost of claims closed and loss amounts paid in that same period.

The reader will recognize the formula for $W = M/B$, where $M = M(t) - M(s)$, $R = R(t) - R(s)$, $P = P(t) - P(s)$, and $C = C(t) - C(s)$.

Typically, the actuary would calculate observed W's for several calendar years, and select the appropriate W^* to be applied for reserving purposes, based on his or her knowledge of any special company circumstances.

The reader will note that this procedure is not restricted to accident years, and could have been as easily applied to accident quarters, or inception-to-date losses, for example.

Measurement of Reserve Variability

Roger M. Hayne, FCAS, MAAA

MEASUREMENT OF RESERVE VARIABILITY

By

Roger M. Hayne, FCAS, MAAA

Abstract

Actuaries and others have long been trying to quantify the uncertainty in reserve estimates. Attempts to address this question have led to the development of stochastic reserving methods as well as the framing of some traditional reserving methods in a stochastic setting. Stochastic methods give insight into the volatility of the forecasts or parameters for a single model and do not necessarily provide an estimate of the distribution of reserves. This paper looks at various sources of uncertainty in projections and tries to give the reader a framework in which to view different attempts to measure the distribution of reserves. Finally the author presents an approach that attempts to at least recognize the issue of model uncertainty and to see its influence on the measurement of reserve uncertainty.

Biography

Roger Hayne is a Fellow of the Casualty Actuarial Society, a Member of the American Academy of Actuaries and a Consulting Actuary in the Pasadena, California office of Milliman USA. He holds a Ph.D. in mathematics from the University of California and joined Milliman in 1977. Roger has been involved in reserve estimation for a wide range of property and liability coverages with emphasis on exposures with longer tails and in situations where full data may not be readily available. The winner of the 1995 Dorweiler Prize, he long has had interest in reserve variability and has authored several papers on the topic that have appeared in both the *Proceedings* and in the *Forum*.

MEASUREMENT OF RESERVE VARIABILITY

1. *Introduction*

Traditional actuarial methodologies, though not necessarily stochastically based are robust and when used as intended tend to be a holistic approach to estimating reserves. In the end the actuary using such approaches may develop a "gut feel" for the uncertainty in his or her estimates, but may not necessarily be able to quantify that "gut feel."

Conversely, more modern stochastic methods bring with them quantification of the volatility of their forecasts, but usually conditioned on a specific set of assumptions and often based on a single set of data (for example the paid loss triangle).

In this paper we will review various aspects of uncertainty. We will finish by presenting an approach that combines holistic aspects of the traditional approach with estimates of uncertainty in those estimates.

2. *Reserves Are Uncertain?*

If you reference an insurer's financial statement you will find a single number identified as liability for losses and another for loss adjustment expenses. There is nothing uncertain about that, it is a number printed in a financial statement. So why should we be talking about uncertainty in reserves at all?

The reason is that the number booked is an estimate of the actual liabilities. Accounting guidance tells us it must be "management's best estimate" of the amount that will be paid in the future on covered claims. We note that the guidance does not say that the reserve is an estimate of the expected or average value, it does not say that the reserve is an estimate of the mode (most likely) value, nor does it say that the reserve is an estimate of the median or middle value. The guidance only states that the reserve is management's best estimate of the amount that eventually will be paid.

The accounting guidance does not provide us with a quantitative or statistical framework to assist in setting the reserves. Actuarial guidance is similarly vague. Our statement of principles talks of actuarially sound reserves as "a provision, based on estimates derived from reasonable assumptions and appropriate actuarial methods, for the unpaid amount required to settle all claims, whether reported or not, for which liability exists."[1] That statement further comments that "[t]he uncertainty inherent in the estimation of required provision for unpaid loss or loss adjustment expenses implies that a range of reserves can be actuarially sound,"[2] and "[t]he most appropriate reserve within a range of actuarially sound estimates depends on both the relative likelihood of estimates within the range and the financial reporting context in which the reserve will be presented."[3]

The message that these references seem to give is that if there is greater uncertainty involved in estimating future payments, then there is likely the need for some sort of margin recognizing that uncertainty when setting the reserves. Of course, there is no mention as to the number to which the "margin" should be added.

At this point, a cynic may say that this brief discussion alone proves that the definition of reserves is itself uncertain, but we will leave that discussion to another day and another forum. A reader interested in this discussion is strongly encouraged to read Rodney Kreps' excellent paper[4] addressing this topic. In addition to a most lucid and informative review of these concepts Kreps advances a reasoned and logical answer to the question "if reserves are uncertain what is the correct amount to book in a financial statement?" Briefly he suggests minimizing the penalty of getting the reserves wrong over the entire distribution of reserves.

We hope, however, that we have made the point that there is little formal guidance as to the statistical quantity to be booked as reserves. Nevertheless, to talk in terms of statistical notions, we must know the distribution of reserves and how to estimate that distribution. That is the topic of this paper.

3. A Look at Traditional Methods

Traditional actuarial methods are generally ad-hoc, and are not originally based on specific statistical models. Probably the oldest of these traditional methods is the development factor or link ratio method. It is fairly easy to explain and has been the subject of much literature. It was not originally grounded in mathematical or statistical theory; though there is some recent work to set it into a statistical framework. In addition, it is known to be quite volatile, particularly for less mature exposure periods.

Another traditional approach is the Bornhuetter-Ferguson[5] method. Rather than being multiplicative and leveraged for less mature exposure periods, this method is additive and tends to be more stable. However, the method needs both an estimate of the loss emergence or development (as does the development factor method) but as well as an á-priori estimate of ultimate losses for each exposure year. This latter requirement can be overcome using a variant approach sometimes called the Stanard-Bühlmann or Cape Cod method. In this variant, one estimates the initial "seed" by using an approach equivalent to the development factor projection method. As with the development factor method, this method was largely developed on an ad-hoc basis.

Conceptually similar to the Bornhuetter-Ferguson method is the Frequency/Severity method presented by Berquist and Sherman.[6] Again, the method is ad-hoc and is not based on a specific statistical model. Here the focus is on incremental average cost per claim with separate selections for claim counts and trends in the incremental averages. It exhibits some of the stability of the Bornhuetter-Ferguson method for less mature exposure periods, and does not require an á-priori estimate of ultimate losses. It does exhibit some volatility due to the forecasts of ultimate claim counts, and in the selection of trends for both current leveling and forecasting into the future.

We see a common thread in these and other traditional reserving methods. The methods are generally ad-hoc, and were originally constructed without reference to an underlying statistical model; thus there is no direct way to quantify the uncertainty in their projections.

This shortcoming has been recognized by most practitioners using the traditional approaches. Rather than relying on an underlying statistical assumption to gauge the uncertainty in forecasts, practitioners using traditional techniques usually consider a range of different methods applied to different groupings of data. If the various methods tend to give reasonably consistent results, then the practitioner might get a sense of comfort with the forecasts.

If however, the estimates from the methods diverge then the practitioner might want to dig more deeply into the underlying data and situation to see whether the assumptions underlying one or another method are violated. In the end, by use of several different methods and looking into the operations underlying the data, even without specific quantification, the practitioner of traditional methods can develop a qualitative "feel" for the uncertainty. This is a significant benefit of traditional approaches that seems to be lacking from more recent statistically based methods.

This traditional approach has stood the test of time. Although not statistically sophisticated, it is a very powerful and robust approach. The variety of traditional methods has the added advantage of taking several different data elements into account including paid losses, incurred losses, open claim counts, closed claim counts, etc. By intentionally using a variety of methods the actuary has the ability to test a variety of hypotheses that may affect the final outcome of his or her projections.

4. Moving on From the Traditional

The traditional qualitative "feel" just described is often what is meant by the degree of reserve uncertainty. Though it is quite valuable to the actuary estimating reserves, it is at best subjective and difficult (impossible?) to quantify. If we wish to put numbers around this uncertainty, we probably should first specify what we are trying to measure. In this case, the author believes that the holy grail of reserve uncertainty is the distribution of the amount and timing of future payments for a particular book of policies. If we knew that distribution, we could then speak intelligently about its mean, variance, skewness, and any other characteristic we could think of. We could specifically calculate particular probability levels (amounts not to be exceeded a specific proportion of the time), as well as estimate the least painful amount to be booked given Rodney Kreps' approach.

We cannot overemphasize the importance of understanding what we seek. In particular, this holy grail can be thought of as answering the fundamental question:

> Given current knowledge, what is the distribution of possible future payments (possible reserve numbers)?

Whenever we are presented with an attempt of assigning probabilities to reserves, we should ask ourselves to what extent those probability estimates answer this fundamental question. To better frame this discussion, we will consider the basic sources of uncertainty in most statistical estimates.

5. Sources of Uncertainty

We can identify at least three sources of uncertainty that may arise in estimating the distribution of reserves:

1. Process uncertainty, the fundamental uncertainty due to the presence of randomness even when all other aspects of the distribution are known,

2. Parameter uncertainty, the uncertainty that arises due to unknown parameters of statistical models for the distribution, even if the selection of those models is perfectly correct, and

3. Model or specification uncertainty, the uncertainty that arises if the specified distributions or underlying models are unknown.

Some authors separate model and specification uncertainty; having the former relate to whether the model selected is actually the correct one for the process under review, and the latter dealing with whether the actual distributions selected in the model are correct. For example, is a gamma or lognormal the right distribution to use? For ease of discussion, we combine both here.

At this point, a brief discussion of each of these sources of uncertainty may help in understanding them and their import in the question of estimating the distribution of reserves. Suppose we throw a fair six-sided die. In this statement of the problem, the entire process generating uncertainty is known. We know that we can only observe one of six outcomes, each with equal likelihood. Even with this perfect knowledge, we do not know the outcome of the next roll of the die for certain. This is an example of process uncertainty.

The very existence of insurance depends on process uncertainty and the risk averse individual's reaction to that uncertainty. The law of large numbers implies that process uncertainty regarding the average cost per insured can be reduced to a negligible level if there are a sufficiently large number of independent insureds. A risk averse individual will pay more than his or her own expected costs if the payment amount is certain, the consequences are uncertain and there is a significant potential financial impact.

In the case of the die if it is thrown a large number of times and the result from each throw is recorded, then the sum of all throws will be rather close to 3.5 times the number of throws. Otherwise said, the average from a large number of throws will be close to the expected value of 3.5.

Alternatively, if we now do not assume that the die is fair, then there will be added uncertainty regarding the final outcome. In this case, the underlying model is the same as in the first example, the generation of numbers between one and six depends on which side lands up. However, we now lack the luxury of knowing the probability of each of the six outcomes. The model and the distributions are known with certainty, but we are uncertain about the parameters of the distribution; hence, an example of parameter uncertainty.

An example of the third source of uncertainty (model uncertainty) would arise if we try to model a series of numbers between one and six by assuming they came from the throw of a single die that may or may not be fair. If we have a more complex process, this may not be sufficient. For example, we could be observing the throw of one of many dice, each with different probabilities attached to each number. The particular die selected for a particular throw could be chosen as a function of prior throws. Here, no simple, single weighted die would be the correct model.

When evaluating methods that claim to measure uncertainty in reserves, the reader should ask which of these sources are considered, and in what way. Given the complexity of property and casualty insurance processes it is unlikely that all will or even can be completely addressed.

6. A Relatively Simple Example

Thomas Mack[7] has addressed uncertainty in development factor (chain ladder) forecasts, and has developed some fairly simple formulae based on some fairly broad, and possibly reasonable assumptions. In particular, if we are willing to assume that the development factor method is actually correct (assuming away the third source of uncertainty) and that there is a certain structure to the variance of payments at each age then Mack derives fairly simple formulae for the standard error of reserves, both by exposure year and in total. The reader is referred to the full paper, but we will attempt to provide a brief summary here.

Let C_{ij} denote cumulative payments for exposure year i at age j with I accident years and I stages of development. Mack makes the following assumptions:

1. There are age-to-age development factors f_j such that $E(C_{ij+1}| C_{i1}, C_{i2},\ldots, C_{ij})=f_jC_{ij}, 1 \le i \le I, 1 \le j \le I-1$

2. $\{C_{i1}, C_{i2},\ldots, C_{iI}\}, \{C_{j1}, C_{j2},\ldots, C_{jI}\}$ independent for $i \ne j$, and

3. There are constants σ_k such that $Var(C_{ik+1}|C_{i1}, C_{i2},\ldots, C_{ik})= C_{ik}\,\sigma_k^2, 1 \le k \le I-1$

Under the first two assumptions Mack shows that the following are unbiased estimators of the development factors f_k:

(5.1) $$\hat{f}_k = \sum_{j=1}^{I-k} C_{j,k+1} \Big/ \sum_{j=1}^{I-k} C_{j,k}$$

These are simply the volume-weighted averages of the development factors in a particular column. More importantly however, is the estimate of the variance or the reserve forecasts from the development factor method. For this set

(5.2) $$\hat{\sigma}_k^2 = \begin{cases} \dfrac{1}{I-k-1} \sum_{i=1}^{I-k} C_{ik} \left(\dfrac{C_{ik+1}}{C_{ik}} - \hat{f}_k \right)^2, 1 \le k \le I-2 \\ \min\left(\hat{\sigma}_{I-2}^4 \big/ \hat{\sigma}_{I-3}^2, \min\left(\hat{\sigma}_{I-2}^2, \hat{\sigma}_{I-3}^2 \right) \right), k = I-1 \end{cases}$$

Mack shows that the $\hat{\sigma}_k^2$ values are unbiased estimators for $1 \le k \le I$-2. He faced the practical problem of having only one development factor from the I – 1st age to the I^{th} age and relied on a general pattern for the variances for that factor. This problem does not exist if one is willing to assume that the data presented are fully mature, thus leading one to conclude no variance in the last factor or so.

Now taking estimates of future payments from the development factor model, that is

(5.3) $$\hat{C}_{ik} = \begin{cases} C_{iI+1-i} \hat{f}_{I+1-i} \ldots \hat{f}_{k-1}, k > I+1-i \\ C_{iI+1-i}, k = I+1-i \end{cases}$$

Mack shows that the mean squared error of the reserve forecast for one exposure year can be estimated by

(5.4) $$\widehat{mse}\left(\hat{R}_i\right) = \hat{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2}\left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}}\right)$$

He further shows that the total reserve for all exposure years combined can be estimated by

(5.5) $$\widehat{mse}\left(\hat{R}\right) = \sum_{i=2}^{I}\left\{\left(\text{s.e.}\left(\hat{R}_i\right)\right)^2 + \hat{C}_{iI}\left(\sum_{j=i+1}^{I}\hat{C}_{jI}\right)\sum_{k=I+1-i}^{I-1}\frac{2\hat{\sigma}_k^2/\hat{f}_k^2}{\sum_{n=1}^{I-1} C_{nk}}\right\}$$

Although these formulae are a bit complicated, they are in closed form and do provide estimates of the error of development factor forecasts. One may be tempted to say that our job is done, but before we jump to that conclusion we will look at a relatively simple example.

Consider the data set shown in Exhibit 1. These hypothetical data are based on personal automobile bodily injury coverage, net of reinsurance for a rather homogeneous database. The data have been disguised, though they retain the salient features of the actual experience. The accident dates shown are real, so more than ten years later we now virtually know the ultimate losses by accident year.

Applying this approach to the paid and incurred triangles separately, and recalling that the difference between the ultimate projections and the amounts to date ("reserve" in Mack's analysis) based on an incurred triangle is actually combined provisions for incurred but not reported claims, for additional development on known claims, and claims in transit, we obtain the estimates in Table 1:

Table 1
Standard Error of Reserve Estimates

	Paid Method	Incurred Method
Case Reserve		$96,917
Estimated Future	$358,453	90,580
Total Reserve	$358,453	$187,497
s.e.(Estimate)	$41,639	$13,524

To assist in comparing these results, Exhibit 2 shows two normal distributions with means and standard deviations equal to the expected total reserve and standard error estimates respectively. As can be seen, the two distributions actually have little in common.

Obviously something is happening. The two data sets, paid and incurred development triangles, though from the same data source are telling two very different stories. What then does this tell us about the distribution of reserves?

It is likely that one or both of the paid and incurred development triangles do not satisfy Mack's hypotheses; thus, the differences are most likely due to model or specification uncertainty. This simple example highlights the importance of the third area of uncertainty. Moreover it highlights the likelihood that model or specification uncertainty can overwhelm both parameter and process uncertainty when trying to measure uncertainty in reserves rather than uncertainty in the projections of one particular model. In this case, and in many actual reserving applications, model or specification uncertainty is probably the largest single source of variability in reserve estimates, and often the source most difficult, or even impossible to quantify.

This is a key point to remember when reviewing statistically based methods applied to actuarial problems. Most statistically based methods we have seen to date deal with a single statistical model, and in most cases consider only one data set (for example a paid loss development triangle). Therefore their results would apply to the projections of a particular method and not necessarily to the final distribution of reserves.

The actuary should also be aware of what statistical element is being considered by a particular stochastic method. For example, does the distribution apply to expected forecasts for a method or to the forecasts themselves?

7. An Alternative

Rather than approaching the problem of estimating the distribution of reserves from the view of one model, we could consider the reserve distribution from a micro level. In its most simple formulation, we can assume that there is a number N of open and IBNR claims, all of which are statistically independent, and have the same probability distribution, say with mean μ and variance σ^2. Then the distribution of reserves will have mean and variance:

(6.1)
$$\begin{aligned} \mathrm{E}(R) &= N\mu \\ \mathrm{Var}(R) &= N\sigma^2 \end{aligned}$$

If the distribution for the claims X_i is known then the resulting reserve distribution will only exhibit process uncertainty. For some distributions of claim sizes, the distribution of reserves will be known and have a closed form. One simple example, though unrealistic for property and casualty reserves occurs when the claims all are drawn from the same normal distribution. In this case reserves will be normally distributed with known mean and variance.

There are few situations in property and casualty reserve applications when the number of claims is known with certainty. If the number of claims N is also random, is independent from the claim size distribution, and has mean λ and variance τ^2, then the reserves will have the following mean and variance

$$\text{(6.2)} \qquad \begin{aligned} E(R) &= \lambda\mu \\ \operatorname{Var}(R) &= \lambda\sigma^2 + \mu^2\tau^2 \end{aligned}$$

Often in collective risk applications, the random variable N is assumed to have a Poisson distribution, in which case $\tau^2 = \lambda$, and we have

$$\text{(6.3)} \qquad \operatorname{Var}(R) = \lambda\left(\sigma^2 + \mu^2\right)$$

With a Poisson claim count distribution, we see that the variance of the average reserve is:

$$\text{(6.4)} \qquad \begin{aligned} \operatorname{Var}\left(\frac{R}{\lambda}\right) &= \frac{\lambda\left(\sigma^2 + \mu^2\right)}{\lambda^2} \\ &= \frac{\sigma^2 + \mu^2}{\lambda} \end{aligned}$$

This variance approaches zero as λ becomes arbitrarily large. Otherwise said, in the case that claim counts have a Poisson distribution, process uncertainty inherent in the average reserve will effectively disappear as the expected number of claims gets large.

A benefit of this model for estimating the distribution of reserves is that it allows us to specifically incorporate both parameter and process uncertainty and allows us to quantify the effects of each. As with most other models, estimating model or specification uncertainty is more difficult, and may not be able to be done in general reserving situations.

Heckman and Meyers[8] outline an approach that can be used to incorporate parameter uncertainty into this classical collective risk model.

The work by Heckman and Meyers referenced here presented a fundamental advance in the use of the collective risk model, essentially solving the problem of calculating aggregate distributions for collective risk models with quite weak restrictions on the claim size distribution. The solution is sufficiently straight-forward to be able to be easily programmed, in fact a copy of such a program is included as an exhibit to the paper. Their solution applies to a generalized collective risk model that includes the potential for "contagion" (the possibility that an external event could affect the frequency of claims across lines of insurance or years of coverage) and for "mixing" (the possibility of an external event could affect the size of all claims).

We will adopt their notation here. To this end, we assume that χ and β are two random variables with

$$\begin{aligned} &E(\chi)=1, \operatorname{Var}(\chi)=c \\ &E(1/\beta)=1, \operatorname{Var}(1/\beta)=b \end{aligned} \tag{6.5}$$

We will use χ and β to incorporate uncertainty into our collective risk model. We then consider the algorithm for generating one observation of aggregate reserves:

1. Randomly select a value for χ,
2. Randomly select the number of claims N from a Poisson distribution with expected value $\chi\lambda$,
3. Randomly select a value for β,
4. Randomly select N claims from the claim size distribution, and
5. Add the values of the N claims and divide the result by β.

Heckman and Meyers call the c parameter the "contagion" parameter and the b the "mixing" parameter. Under the assumptions that the claim count and claim size distributions are independent, and that the claim selections in step 4 are independent of each other and of the random variables χ and β, then we can calculate the expected value and variance of the aggregate reserves. These values are:

$$\text{(6.6)} \qquad \begin{aligned} E(R) &= \lambda\mu \\ \operatorname{Var}(R) &= \lambda\left(\mu^2+\sigma^2\right)(1+b)+\lambda^2\mu^2(b+c+bc) \end{aligned}$$

We see that in this formulation of the problem, the variance of the average expected reserve does not approach 0 with a large number of expected claims unless $b = c = 0$ (or in the trivial case the expected losses are zero). In the case that $b = c = 0$, the formula reduces to the case without parameter uncertainty.

The alternative approach to estimating the distribution of reserves we present here uses traditional methods in an attempt to estimate the parameters c and b.

The algorithm presented by Heckman and Meyers actually allows for the combination of aggregate loss distributions for several lines of insurance, each with its own contagion parameter c but with a global mixing parameter b. In the approach we present here, we will take advantage of this feature and have different contagion parameters for each accident year, as well as a single global mixing parameter reflecting uncertainty that affects the reserves for all accident years at once. Examples of this global uncertainty would be estimated future inflation, court decisions, and so forth.

We will first take a traditional approach to estimating reserves by accident year for the data contained in Exhibit 1. A detailed review of the data would lead the actuary to conclude that there are many changes occurring in the historical data. There appears to have been changes in the rates at which claims are being closed. There also appear to be changes in relative reserve adequacy over time. As with traditional approaches we applied a variety of different methods to both the actual data and to data adjusted for the estimated effects of changing rates of claim closure and of relative reserve adequacy. For this we used methods outlined in the paper by Berquist and Sherman.[9]

Exhibit 3 shows the reserve estimates by accident year for each method used. The development methods simply apply the usual development method to the indicated data set. By "Adjusted Incurred" we mean historical incurred losses adjusted to reflect current relative reserve adequacy. By "Severity" we mean the incremental average cost projection method described in Berquist and Sherman. The "Hindsight" method is an iterative approach that makes use of historical average future costs per open and IBNR claim to derive estimates of ultimate losses.

The bottom portion of Exhibit 3 shows the weights we assigned to each of the methods. These weights reflect our subjective view of the applicability of the particular method for a particular accident year. We will use the variation in estimates from the various methods to gauge the uncertainty of reserve estimates by accident year. In fact we will use the standard deviations in the last column of Exhibit 3 to estimate the contagion parameters for each accident year. For this we consider formula (6.6) and set $b = 0$. This then gives us the following variance estimate for reserves for accident year i:

(6.7) $$\mathrm{Var}(R_i) = \lambda_i\left(\mu_i^2 + \sigma_i^2\right) + c_i\lambda_i^2\mu_i^2$$

Here, λ_i denotes the expected number of open and IBNR claims for accident year i while μ_i and σ_i^2 are the mean and variance, respectively, of the reserves for a single claim for accident year i. We note that the two terms in this sum can be interpreted as the variance without parameter uncertainty and the contribution of parameter uncertainty to the total variance.

We note that though the derivation does not make sense, the formulae developed by Heckman and Meyers allow the contagion parameter c to be negative. In that case, the claim counts will have a binomial distribution with mean greater than its variance. In the case of a positive c value, the claims will have a negative binomial distribution with variance greater than the mean. As we have seen above, in the case of $c = 0$ claims will have a Poisson distribution.

Our analysis of the Exhibit 1 data provided us with estimates of the number of open and IBNR claims, and hence estimates of the values for μ_i. For sake of illustration we assumed that the open claims for each accident year will each have lognormal distributions with the same coefficient of variation. More sophisticated analysis of open and IBNR claims for older accident years may provide more accurate estimates of these distributions. In any case, the standard deviations for individual claims based on these distributions are also shown in Exhibit 4.

The column titled "Aggregate Process Standard Deviation" is the standard deviation implied by a collective risk model with no parameter uncertainty and a Poisson claim count distribution as described above. We can then solve equation (6.7) for c_i to obtain estimates of the contagion parameters by accident year implied by our analysis, and claim count and size distributions. That is what was done in the last column of Exhibit 4.

We now turn our attention to the mixing parameter b. In the modeling, the β random variable uniformly affects all claims in a particular iteration. In our model here we will use it as an overlay to reflect global uncertainty in the forecasts. To measure this uncertainty, we compare estimated ultimate severities against an expected smooth transition from one year to the next. Since there is volatility in the percentage of paid claims, we elected to measure this global uncertainty by reviewing the severity per ultimate claim as opposed to the average ultimate claim with payment (selected in the reserve analysis due to the fact that there is 0 probability of a lognormal claim having 0 payment).

Exhibit 5 shows our estimation of the mixing parameter b. Here, we compare the selected severities (per ultimate reported claim) to averages based on an exponential fit through all data points. We assume that observations of $1/\beta$ are ratios of the actual severity over the fitted severity. The estimate for the mixing parameter b is then the variance of the observed $1/\beta$ values.

We now have sufficient information to derive an estimate of the distribution of reserves for this sample problem. We used the algorithm discussed in Heckman and Meyers to estimate the distribution of aggregate reserves, both with parameter uncertainty (non-zero values for the c_i and b parameters) and without such parameter uncertainty. Exhibit 6 graphically compares these two distributions. As can be seen, parameter uncertainty is substantial in this case.

One striking observation from this analysis is the dispersion of the reserve distribution in this case. The distribution has a standard deviation of \$39 million on total reserves of \$202 million, for a coefficient of variation of more than 19%. The 90th and 95th percentiles for this aggregate distribution are \$250 million and about \$278 million, respectively, or 24% and 38% above the expected amount. This is a far cry from the "plus or minus 10%" that is sometimes cited in ranges for reserves. These results simply reflect the substantial uncertainty inherent in the reserve forecasts, in this case, due largely to the changes that have been occurring in the historical experience.

It is also likely that we have missed sources of uncertainty in the above analysis. We have identified methods we believed to be appropriate, and gave them subjective weights based on their relative strengths and weaknesses and the situations occurring in the experience. These weightings are subjective and potentially volatile, adding to the model or specification uncertainty, and probably not directly accounted for in the analysis.

In addition, we assumed that all random quantities are independent from one another. We attempted to take some potential correlation into account by the use of the contagion and mixing parameters. This is a crude approach at best. There has been some recent work in calculating aggregate distributions where there is some form of correlation among some of the distributions. Examples of this can be found in Wang[10] and Dhaene, et.al[11,12] The inclusion of correlation between years, should such correlation exist, would be an obvious refinement to the approach we have outlined here.

As noted above, the accident years are real for the data. As such, all accident years are now virtually completely closed. The current data would imply a December 31, 1991 reserve of approximately $170 million, outside of the "plus or minus 10%" range and at an approximate 19% probability level given the analysis discussed above. Although in hindsight our methodology was not as accurate as we would like the answer does not appear to be unreasonable given the volatility of the estimates.

8. Conclusion

We recognize this approach is far from perfect. The traditional approaches are very robust and provide the actuary with a substantial amount of valuable information, which may not be present in the application of a single statistically based approach. There is obviously more work to be done to make that information rigorous and quantifiable.

Work by Mack and others have gone a long way to putting the chain ladder or development factor approaches on statistical footing. Other traditional methods can probably also benefit from such a rigorous approach. For example, one might think of a simplified version of the incremental severity method, presented by Berquist and Sherman to be formulated by a statistical model with parameters representing an inherent trend and on-level averages for each age of development. Nonlinear statistical approaches may be helpful in gaining statistical insight to the properties of that traditional technique. A similar approach may also prove beneficial in gaining additional understanding into the Stanard-Bühlman or Bornhuetter-Ferguson approaches.

If we work with a variety of forecast methods, which is a fundamental characteristic of the traditional approach to estimating reserves, then we should also understand the correlation of results among the various methods. This understanding would also help us to better estimate the distribution of reserves.

There obviously remains much yet to be done.

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[1] "Statement of Principles Regarding Property and Casualty Loss and Loss Adjustment Expense Reserves," Casualty Actuarial Society, 1988, p. 59.

[2] Ibid.

[3] Ibid.

[4] Kreps, R.E., "Management's Best Estimate of Loss Reserves," *Casualty Actuarial Society Forum*, Fall 2002, pp. 247-258.

[5] Bornhuetter, R.L., and Ferguson, R.E., "The Actuary and IBNR," *Proceedings of the Casualty Actuarial Society*, LIX, 1972, pp. 181-195.

[6] Berquist, J.R., and Sherman, R.E., "Loss Reserve Adequacy Testing: A Comprehensive, Systematic Approach," *Proceedings of the Casualty Actuarial Society*, LXXIV, 1977, pp. 123-184.

[7] Mack, T., "Distribution-free Calculation of the Standard Error of Chain Ladder Reserve Estimates," *ASTIN Bulletin*, Vol. 23, No. 2, November 1993, pp. 213-226

[8] Heckman, P.E., and Meyers, G.G., he Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions," *Proceedings of the Casualty Actuarial Society*, LXX, 1983, pp. 22-61, addendum in LXXI, 1984, pp. 49-66.

[9] Berquist, J.R., and Sherman, R.E., op.cit.

[10] Wang, S, "Aggregation of Correlated Risk Portfolios: Models and Algorithms," *Proceedings of the Casualty Actuarial Society*, LXXXV, 1998, pp. 848-939.

[11] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., and Vyncke, D., "The concept of comonotonicity in actuarial science and finance: theory (review)," *Insurance: Mathematics & Economics*, Volume 31, Number 1 (20 August 2002), pp. 3-34.

[12] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., and Vyncke, D., "The concept of comonotonicity in actuarial science and finance: applications (review)," *Insurance: Mathematics & Economics*, Volume 31, Number 2 (18 October 2002), pp. 133-162.

EXAMPLE PRIVATE PASSENGER AUTO BODILY INJURY LIABILITY DATA

Cumulative Paid Losses

Accident	Months of Development																	
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	$267	$1,975	$4,587	$7,375	$10,661	$15,232	$17,888	$18,541	$18,937	$19,130	$19,189	$19,209	$19,234	$19,234	$19,246	$19,246	$19,246	$19,246
1975	310	2,809	5,686	9,386	14,884	20,654	22,017	22,529	22,772	22,821	23,042	23,060	23,127	23,127	23,127	23,127	23,159	
1976	370	2,744	7,281	13,287	19,773	23,888	25,174	25,819	26,049	26,180	26,268	26,364	26,371	26,379	26,397	26,397		
1977	577	3,877	9,612	16,962	23,764	26,712	28,393	29,656	29,839	29,944	29,997	29,999	29,999	30,049	30,049			
1978	509	4,518	12,067	21,218	27,194	29,617	30,854	31,240	31,598	31,889	32,002	31,947	31,965	31,986				
1979	630	5,763	16,372	24,105	29,091	32,531	33,878	34,185	34,290	34,420	34,479	34,498	34,524					
1980	1,078	8,066	17,518	26,091	31,807	33,883	34,820	35,482	35,607	35,937	35,957	35,962						
1981	1,646	9,378	18,034	26,652	31,253	33,376	34,287	34,985	35,122	35,161	35,172							
1982	1,754	11,256	20,624	27,857	31,360	33,331	34,061	34,227	34,317	34,378								
1983	1,997	10,628	21,015	29,014	33,788	36,329	37,446	37,571	37,681									
1984	2,164	11,538	21,549	29,167	34,440	36,528	36,950	37,099										
1985	1,922	10,939	21,357	28,488	32,982	35,330	36,059											
1986	1,962	13,053	27,869	38,560	44,461	45,988												
1987	2,329	18,086	38,099	51,953	58,029													
1988	3,343	24,806	52,054	66,203														
1989	3,847	34,171	59,232															
1990	6,090	33,392																
1991	5,451																	

Claims Closed with Payment

Accident	Months of Development																	
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	268	607	858	1,090	1,333	1,743	2,000	2,076	2,113	2,129	2,137	2,141	2,143	2,143	2,145	2,145	2,145	2,145
1975	294	691	913	1,195	1,620	2,076	2,234	2,293	2,320	2,331	2,339	2,341	2,343	2,343	2,343	2,343	2,344	
1976	283	642	961	1,407	1,994	2,375	2,504	2,549	2,580	2,590	2,596	2,600	2,602	2,603	2,603	2,603		
1977	274	707	1,176	1,688	2,295	2,545	2,689	2,777	2,809	2,817	2,824	2,825	2,825	2,826	2,826			
1978	269	658	1,228	1,819	2,217	2,475	2,613	2,671	2,691	2,706	2,710	2,711	2,714	2,717				
1979	249	771	1,581	2,101	2,528	2,816	2,930	2,961	2,973	2,979	2,986	2,988	2,992					
1980	305	1,107	1,713	2,316	2,748	2,942	3,025	3,049	3,063	3,077	3,079	3,080						
1981	343	1,042	1,608	2,260	2,596	2,734	2,801	2,835	2,854	2,859	2,860							
1982	350	1,242	1,922	2,407	2,661	2,834	2,887	2,902	2,911	2,915								
1983	428	1,257	1,841	2,345	2,683	2,853	2,908	2,920	2,925									
1984	291	1,004	1,577	2,054	2,406	2,583	2,622	2,636										
1985	303	1,001	1,575	2,080	2,444	2,586	2,617											
1986	318	1,055	1,906	2,524	2,874	2,958												
1987	343	1,438	2,384	3,172	3,559													
1988	391	1,671	3,082	3,771														
1989	433	1,941	3,241															
1990	533	1,923																
1991	339																	

EXAMPLE PRIVATE PASSENGER AUTO BODILY INJURY LIABILITY DATA

Cumulative Reported Claims

Accident Year	Months of Development 12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	1,912	2,854	3,350	3,945	4,057	4,104	4,149	4,155	4,164	4,167	4,169	4,169	4,169	4,170	4,170	4,170	4,170	4,170
1975	2,219	3,302	3,915	4,462	4,618	4,673	4,696	4,704	4,708	4,711	4,712	4,716	4,716	4,716	4,716	4,716	4,717	
1976	2,347	3,702	4,278	4,768	4,915	4,983	5,003	5,007	5,012	5,012	5,013	5,014	5,015	5,015	5,015	5,015		
1977	2,983	4,346	5,055	5,696	5,818	5,861	5,884	5,892	5,896	5,897	5,900	5,900	5,900	5,900	5,900			
1978	2,538	3,906	4,633	5,123	5,242	5,275	5,286	5,292	5,298	5,302	5,304	5,304	5,306	5,306				
1979	3,548	5,190	5,779	6,206	6,313	6,329	6,339	6,343	6,347	6,347	6,348	6,348	6,348					
1980	4,583	6,106	6,656	7,032	7,128	7,139	7,147	7,150	7,151	7,153	7,154	7,154						
1981	4,430	5,967	6,510	6,775	6,854	6,873	6,883	6,889	6,892	6,894	6,895							
1982	4,408	5,849	6,264	6,526	6,571	6,589	6,594	6,596	6,600	6,602								
1983	4,861	6,437	6,869	7,134	7,196	7,205	7,211	7,212	7,214									
1984	4,229	5,645	6,053	6,419	6,506	6,523	6,529	6,531										
1985	3,727	4,830	5,321	5,717	5,777	5,798	5,802											
1986	3,561	5,045	5,656	6,040	6,096	6,111												
1987	4,259	6,049	6,767	7,206	7,282													
1988	4,424	6,700	7,548	8,105														
1989	5,005	7,407	8,287															
1990	4,889	7,314																
1991	4,044																	

Outstanding Claims

Accident Year	Months of Development 12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	1,381	1,336	1,462	1,660	1,406	772	406	191	98	57	23	13	3	4	0	0	0	0
1975	1,289	1,727	1,730	1,913	1,310	649	358	167	73	30	9	6	4	2	2	1	1	
1976	1,605	1,977	1,947	1,709	1,006	540	268	166	79	48	32	18	14	10	10	7		
1977	2,101	2,159	2,050	1,735	988	582	332	139	66	38	27	21	21	8	3			
1978	1,955	1,943	1,817	1,384	830	460	193	93	56	31	15	9	7	2				
1979	2,259	2,025	1,548	1,273	752	340	150	68	36	24	18	13	4					
1980	2,815	1,991	1,558	1,107	540	228	88	55	28	14	8	6						
1981	2,408	1,973	1,605	954	480	228	115	52	27	15	11							
1982	2,388	1,835	1,280	819	354	163	67	44	21	10								
1983	2,641	1,765	1,082	663	335	134	62	34	18									
1984	2,417	1,654	896	677	284	90	42	15										
1985	1,924	1,202	941	610	268	98	55											
1986	1,810	1,591	956	648	202	94												
1987	2,273	1,792	1,059	626	242													
1988	2,403	1,966	1,166	693														
1989	2,471	2,009	1,142															
1990	2,642	2,007																
1991	2,366																	

EXAMPLE PRIVATE PASSENGER AUTO BODILY INJURY LIABILITY DATA

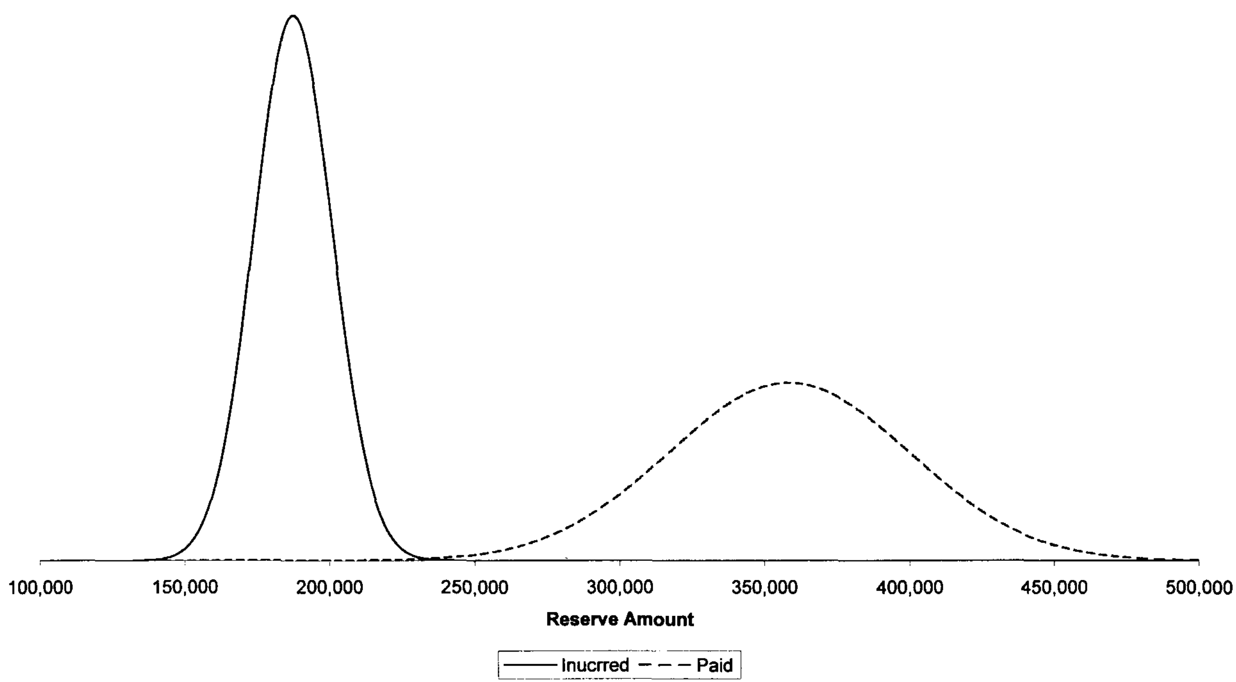
Outstanding Losses

Accident Year	Months of Development 12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216
1974	$5,275	$8,867	$12,476	$11,919	$8,966	$5,367	$3,281	$1,524	$667	$348	$123	$82	$18	$40	$0	$0	$0	$0
1975	6,617	11,306	13,773	14,386	10,593	4,234	2,110	1,051	436	353	93	101	10	5	5	3	3	
1976	7,658	11,064	13,655	13,352	7,592	4,064	1,895	1,003	683	384	216	102	93	57	50	33		
1977	8,735	14,318	14,897	12,978	7,741	4,355	2,132	910	498	323	176	99	101	32	14			
1978	8,722	15,070	15,257	11,189	5,959	3,473	1,531	942	547	286	177	61	67	7				
1979	9,349	16,470	14,320	10,574	6,561	2,864	1,328	784	424	212	146	113	38					
1980	11,145	16,351	14,636	11,273	5,159	2,588	1,290	573	405	134	81	54						
1981	10,933	15,012	14,728	9,067	5,107	2,456	1,400	584	269	120	93							
1982	13,323	16,218	12,676	6,290	3,355	1,407	613	398	192	111								
1983	13,899	16,958	12,414	7,700	4,112	1,637	576	426	331									
1984	14,272	15,806	10,156	8,005	3,604	791	379	159										
1985	13,901	15,384	12,539	7,911	3,809	1,404	827											
1986	15,952	22,799	16,016	8,964	2,929	1,321												
1987	22,772	24,146	18,397	8,376	3,373													
1988	25,216	26,947	17,950	8,610														
1989	24,981	30,574	19,621															
1990	30,389	34,128																
1991	28,194																	

Accident Year	Earned Exposures
1974	11,000
1975	11,000
1976	11,000
1977	12,000
1978	12,000
1979	12,000
1980	12,000
1981	12,000
1982	11,000
1983	11,000
1984	11,000
1985	11,000
1986	12,000
1987	13,000
1988	14,000
1989	14,000
1990	14,000
1991	13,000

Exhibit 2

Distributions Based on Mack's Method



EXAMPLE RESERVE FORECASTS

Accident Year	Reserve Estimates by Ultimate Forecast Method: Incurred Development	Paid: Development	Paid: Severity	Paid: Pure Premium	Paid: Hindsight	Adjusted Incurred	Paid Adjusted for Claim Closing Changes: Development	Paid Adjusted for Claim Closing Changes: Severity	Paid Adjusted for Claim Closing Changes: Pure Premium	Paid Adjusted for Claim Closing Changes: Hindsight	Weighted Average	Weighted Standard Deviation
1974	$0	$0	$0	$0		$0	$0	$0	$0		0	0
1975	3	0	0	0		3	0	0	0		0	0
1976	33	0	0	0		33	21	0	0		11	14
1977	5	0	0	0		8	24	0	0		5	8
1978	-15	10	9	10		7	26	0	0		6	11
1979	-10	35	34	33		-35	28	0	0		11	24
1980	-7	54	55	50		-29	61	33	31		31	30
1981	-37	49	73	75		-20	77	47	49		39	41
1982	-41	107	136	131		-58	100	79	75		66	70
1983	114	275	297	297		-68	200	176	172		156	126
1984	-161	416	394	446		-135	352	318	351		181	258
1985	403	761	713	812		130	692	702	779		567	248
1986	744	2,143	1,760	1,909	$1,687	394	1,936	1,842	1,950	$675	1,357	637
1987	2,335	6,847	5,583	5,128	5,128	2,348	6,000	5,790	5,220	2,301	4,260	1,620
1988	8,371	19,768	16,246	13,451	14,428	10,391	17,352	16,433	13,399	8,001	12,866	3,525
1989	25,787	44,631	36,887	29,232	32,199	26,048	39,241	36,431	28,512	19,174	30,212	6,428
1990	60,211	83,760	73,987	61,846	62,974	55,734	79,667	70,246	57,192	43,286	62,516	10,198
1991	83,093	130,907	95,283	95,185	78,616	79,573	154,268	87,625	84,688	72,157	90,014	19,166
Total											202,298	

Selected Weights

Accident Year	Incurred Development	Paid Development	Paid Severity	Paid Pure Premium	Paid Hindsight	Adjusted Incurred	Adj. Development	Adj. Severity	Adj. Pure Premium	Adj. Hindsight
1974	1	1	1	1		1	1	1	1	
1975	0	1	1	1		0	1	1	1	
1976	1	1	1	1		1	1	1	1	
1977	1	1	1	1		1	1	1	1	
1978	1	1	1	1		1	1	1	1	
1979	1	1	1	1		1	1	1	1	
1980	1	1	1	1		1	1	1	1	
1981	1	1	1	1		1	1	1	1	
1982	1	1	1	1		1	1	1	1	
1983	3	1	2	2		3	1	2	2	
1984	3	1	2	2		3	1	2	2	
1985	3	1	2	2		3	1	2	2	
1986	3	1	2	2	2	3	1	2	2	2
1987	3	1	2	2	2	3	1	2	2	2
1988	3	1	2	2	2	3	1	2	2	2
1989	3	1	2	2	2	3	1	2	2	2
1990	3	1	2	2	2	3	1	2	2	2
1991	3	1	2	2	2	3	1	2	2	2

Exhibit 4

ESTIMATION OF CONTAGION PARAMETERS BY ACCIDENT YEAR

Accident Year	Estimated Reserve	Paid Claim Count Estimates Ultimate	Closed	Open & IBNR	Average Reserve	Single Claim Standard Deviation	Aggregate Process Standard Deviation	Estimated Total Standard Deviation	Implied *c* Value
1974	$0	2,145	2,145	$0					
1975	0	2,344	2,344	0					
1976	11	2,604	2,603	1	$11,000	$16,251	$20	$14	-1.477
1977	5	2,827	2,826	1	5,000	7,387	9	8	-0.858
1978	6	2,718	2,717	1	6,000	8,864	11	11	0.092
1979	11	2,994	2,992	2	5,500	8,126	14	24	3.172
1980	31	3,083	3,080	3	10,333	15,266	32	30	-0.097
1981	39	2,865	2,860	5	7,800	11,523	31	41	0.473
1982	66	2,922	2,915	7	9,429	13,929	45	70	0.669
1983	156	2,938	2,925	13	12,000	17,728	77	126	0.405
1984	181	2,658	2,636	22	8,227	12,155	69	258	1.882
1985	567	2,661	2,617	44	12,886	19,038	152	248	0.120
1986	1,357	3,064	2,958	106	12,802	18,913	235	637	0.190
1987	4,260	3,889	3,559	330	12,909	19,072	418	1,620	0.135
1988	12,866	4,697	3,771	926	13,894	20,527	754	3,525	0.072
1989	30,212	5,135	3,241	1,894	15,951	23,566	1,238	6,428	0.044
1990	62,516	5,270	1,923	3,347	18,678	27,595	1,928	10,198	0.026
1991	90,014	4,410	339	4,071	22,111	32,666	2,517	19,166	0.045

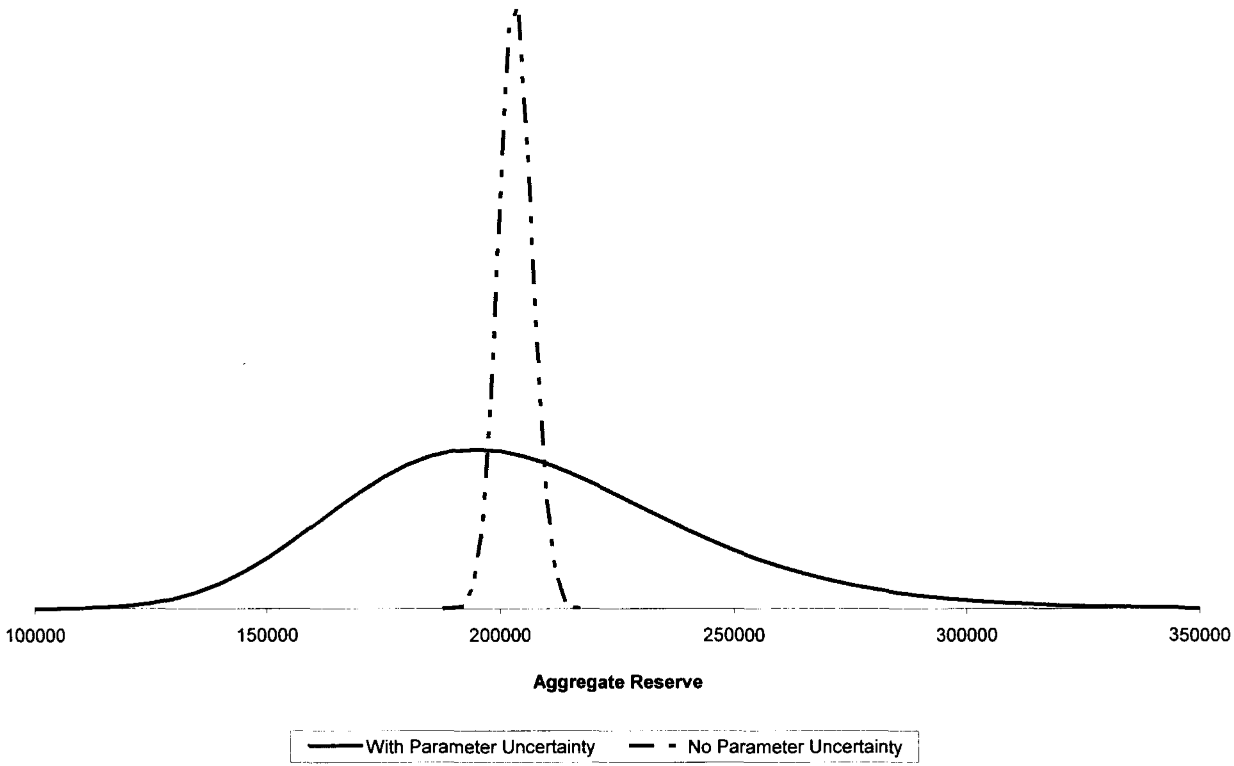
Exhibit 5

ESTIMATE OF MIXING PARAMETER

Accident Year	Estimated Ultimate Losses	Estimated Ultimate Reported Claims	Indicated Severity	Smoothed Severity	Estimate of 1/ß
1974	19,246	4,170	4,615	4,165	1.108
1975	23,159	4,717	4,910	4,388	1.119
1976	26,408	5,016	5,265	4,623	1.139
1977	30,054	5,901	5,093	4,870	1.046
1978	31,992	5,307	6,028	5,130	1.175
1979	34,535	6,349	5,439	5,404	1.007
1980	35,993	7,155	5,030	5,693	0.884
1981	35,211	6,897	5,105	5,997	0.851
1982	34,444	6,605	5,215	6,317	0.825
1983	37,837	7,219	5,241	6,655	0.788
1984	37,280	6,539	5,701	7,011	0.813
1985	36,626	5,812	6,302	7,385	0.853
1986	47,345	6,130	7,723	7,780	0.993
1987	62,289	7,327	8,501	8,196	1.037
1988	79,069	8,256	9,577	8,634	1.109
1989	89,444	9,017	9,919	9,095	1.091
1990	95,908	8,931	10,739	9,581	1.121
1991	95,465	7,829	12,194	10,093	1.208
Variance (estimate of *b*)					0.019

Exhibit 6

ESTIMATED DISTRIBUTION OF RESERVES



Reserving for Asbestos Liabilities

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Reserving for Asbestos Liabilities

Abstract

Significant uncertainties surround the ultimate costs of asbestos liabilities. The goal of the present work is to provide the actuary with the necessary framework to perform a rigorous analysis of such liabilities. It should be noted that while there is no algorithm that guarantees success, there is a proper approach to the problem.

The keys to a rigorous analysis of asbestos liabilities can be summarized as follows:

- Effective knowledge gathering regarding the liabilities of the risk entity under investigation via thorough, open, and constant communication with those responsible for disposing of those liabilities;
- A commitment to keeping abreast of the global issues in the asbestos litigation;
- The application of actuarial skills, judgment and creativity in designing a flexible and transparent model with well documented assumptions and well communicated interpretation of results.

Acknowledgments

The authors would like to thank Mark Shapland for many useful conversations on the topic of asbestos reserving and for providing immensely valuable assistance in the design and implementation of several asbestos reserving models. We also thank Don Mango for reading early drafts of this paper and offering advice that improved it greatly. In addition, Bill Rowland, David Ostrowski, and Peter Cooper were intimately involved in the development of some of the ideas and methods discussed in Appendix III. Several anonymous attorneys and claims professionals were also instrumental in the development of this work and in the writing of the text. Any errors or omissions that remain are the responsibility of the authors.

1 Introduction

Significant uncertainties surround the ultimate costs of asbestos liabilities. Billions of dollars are in the balance. The actions of many insureds, the actual or potential harm to claimants and the legal environment have resulted in staggering asbestos litigation costs. To make matters worse for the actuary, the unique combination of insurance coverage, length of exposure and disease latency issues makes the quantification of asbestos liabilities of insurance and reinsurance companies extremely difficult.

The goal of the present work is to provide the actuary with a framework to perform a rigorous study of asbestos liabilities. However, it should be noted that there is no cookbook recipe for success in this arena. As in all endeavors surrounding the valuation of contingent liabilities, the quality and quantity of available data can be the determining factor in the design and thoroughness of the analysis to be performed. Furthermore, the specific nature of the risk bearer under review (e.g. primary insurance, assumed quota share reinsurance, direct excess insurance, retrocessional reinsurance, characteristics of the (re)insureds) suggests that valuation approaches may need to vary significantly between risk bearing entities.

While it is true that there is no asbestos valuation algorithm that guarantees success, one can say that there is a proper approach to the problem – namely, modeling as much of the exposure as possible at the level of the insured defendant, and modeling the remainder in sensibly defined groups. However, the only way to carry out a truly useful analysis of a specific insurer is to turn to the claims personnel for the necessary details. In fact, the defining theme of this paper is that the only solution to the problem presented by the complexities of the asbestos challenge is the sharing of knowledge between the claims settling function and the actuarial function. A rigorous study of asbestos liabilities requires the analyst to become intimately familiar with the details of the liabilities in question. To that end, there is no substitute for thorough communication with those responsible for discharging the liabilities.

It is the hope of the authors that the present work will spur discussion amongst actuaries and lead to the publication of more papers on this important valuation topic. There are a number of ways to handle the complex asbestos valuation problems, and this paper addresses only a few of the possible ways. We hope that other actuaries will come forward and discuss the tools they have developed to address the valuation of asbestos liabilities.

Reserving for asbestos liabilities is complicated by some rather unique circumstances. The goal of this paper is not to provide an exhaustive description of these unique aspects. Many, if not most, have been discussed in great detail elsewhere (see, for example, [AAA], [CD], [RAND], as well as the session handouts from several recent Casualty Loss Reserve Seminars on the topic of asbestos).

In fact, the environment has been changing so rapidly that any attempt to add to this literature with an exhaustive treatment on this topic would prove futile, as it risks either being redundant or quickly outdated. Our goal here is to make the reader aware of the main issues that must be considered when conducting an asbestos valuation study.

So, what is it that makes asbestos so different, not to mention so difficult? In a nutshell, the answer is:

- the nature of asbestos diseases,
- the legal environment, and
- data.

These are discussed in more detail in Appendix I. The actuary must become comfortable enough with the qualitative issues to arrive at reasonable methods for using the available data.

If an insurance company has reliable data on asbestos payments or reserves, it isn't amenable to triangular analysis. Among the reasons for this are

- the policies were issued years ago, and the company may have no record of them;
- the asbestos "cause of loss" occurred over a period of years – hence the concept of accident year doesn't apply;
- asbestos payments made to an insured cannot be tied to one policy, so policy year is not an appropriate concept.

As background, the defining themes of all rigorous studies of (re)insurance asbestos liabilities are

1. Analyze known sources of liability:
   a. Analyze the liability as close to the source as possible;
   b. Quantify as much of the qualitative facts and opinions held by those responsible with discharging the liability as is possible;
   c. Recognize correlations and dependencies where they exist – even if they cannot be determined with any sense of certainty;
   d. Check all results and assumptions for reasonability *ad nauseum*;
   e. Produce reasonable ranges of aggregate liabilities based upon reasonable assumptions regarding the individual liabilities;
   f. Focus on gross liabilities;
2. Analyze unknown sources of liability ("pure" IBNR).

Points 1.a. and 1.e. are the defining characteristics of a *ground up* analysis, which is the preferred method (provided this is feasible). There is no pre-packaged program for a ground up analysis. The determination of how it will be performed is driven by:

- the available data
- the amount of time during which the valuation study is to be performed
- the available qualitative information
- the nature of the liability
- other factors unique to the risk bearer

Reserving for asbestos losses is best done at the gross level. A net of reinsurance approach poses a substantial risk that the true liability will be understated. Probably the most important reasons for this are:

- The age of the policies in question
- The need to understand the coverage allocation to the various years
- Changes in reinsurance programs over the years
- The large number of reinsurer insolvencies in recent decades, which suggests that a significant portion of an insurer's reinsurance recoveries may be uncollectible
- Many companies may have exhausted their reinsurance limits
- Many solvent reinsurers are insisting on more extensive documentation, making it difficult for cedants to collect.

Once one determines the indicated range of gross liabilities, one can then analyze the reinsurance structure to arrive at the indicated range of net liabilities.

The most important feature of a rigorous analysis is the presence of open and constant communication between the actuary and those responsible for discharging the liabilities. The staff members handling the claims know much more than the actuary about the specifics of the liabilities and the actuary needs to find a way to facilitate the transfer of that knowledge in order to build an appropriate valuation model. Efficiency and effectiveness require the actuary to make simplifying assumptions in building a model - assumptions that may be wrong on an account by account basis but that are, in the aggregate, a reasonable reflection of reality.

The appropriate abstraction from detail in developing an efficient model is critical and involves a bit of tightrope walking. After all, a risk bearing entity does not need an actuary to calculate likely dispositions of specific known claims. That is not where actuarial skills are most needed, and there are other more qualified providers of claim settlement services. The actuary's value added lies in the ability to produce a reasonable range of the total aggregate liability faced by the risk bearing entity. The best way for the actuary to add value is by exercising creativity and judgment in a reasonable fashion at each step of the process and in assuring the process is transparent.

The goal of this paper is to sketch the general framework in which an analysis of asbestos liabilities should be performed. This paper will not attempt to exhaustively discuss the various details that affect the ultimate liability faced by a risk bearing entity. It is imperative that any actuary conducting an asbestos valuation spend considerable time and effort discussing the details of the insureds' liabilities with the claims department and, if possible, with the attorneys engaged by the claims department. While we, the authors, are neither attorneys nor claims professionals and we claim no expertise in the field of law or the disposition of claims, our extensive discussions with the experts in those fields underlie the development of the valuation approach.

2 Analysis of Known Gross Liabilities

This section focuses on quantifying the liabilities from known insureds. Appendix II provides a more detailed discussion of these topics.

As noted earlier, there is no single "right" way to perform an analysis of asbestos liabilities. The central valuation concept is to estimate the total indemnity and legal expense costs for each insured entity, and then apply the insurance coverages to arrive at the insurance company's share of the liability.[1] Therefore, one must obtain as much information as is possible from those handling the claims.

In many companies, the claims department periodically reviews pending and potential asbestos claims to provide company management with a range of possible outcomes. Such a review is highly sensitive, as it requires claims personnel to opine on the probable and possible ultimate liabilities of active claims. Were this information to fall into the hands of the insureds (or of the other insurers responding to the asbestos claims), it could weaken the insurer's position in settlement negotiations. In light of this, many claims departments refuse to offer any written opinion on anything other than the currently held case reserves.

Those who are responsible for disposing of the liabilities have as good an indication as anyone as to the likely ultimate costs of the various pending and potential claims. However, most claims professionals and attorneys are not comfortable enough with the concepts of mean, median, mode, probability distributions and correlations to be able to meaningfully combine their expertise and knowledge regarding the individual accounts to produce a reasonable aggregate cost distribution. It is in this regard that the actuary can add value to the reserving process.

The purely actuarial part of a ground up analysis need not be unduly sophisticated. The complicating factors are generally legal issues.

Once an estimate of the insurance company's share of each insured's liability is available, the actuary must appropriately combine this information to arrive at aggregate liabilities. The crudest way to do this is to arrive at low, medium and high estimates for each insured, and then sum them to arrive at low, medium, and high estimates for the portfolio of known insureds. The biggest drawbacks to this approach are:

1. There is no recognition of dependencies between insureds
2. If the low, medium and high estimates for each insured are being independently produced by several different people (rather than derived by statistical modeling techniques), it may be inappropriate to simply sum these figures to determine the range of outcomes. For example, if the claims professionals managing the accounts periodically produce these estimates without clear guidance, then there will be an unacceptably high level of subjectivity. Some of the claims personnel may view the high estimate as representing a true "worst-case" scenario akin to a 95^{th} or 99^{th} percentile on the distribution of (unknown) possible outcomes.

[1] The allocation of the insured's costs to the policy years in the coverage block and correct information regarding the insurance coverage provided by the insurer are arguably the most important components of a ground up exposure model.

Others may view it as a realistic high cost outcome akin to a 70^{th} or 75^{th} percentile. Some of the claims professionals may view the medium estimate as the most likely outcome, and others may feel that is the role of the low estimate.

Before trying to sum these estimates, it is therefore incumbent upon the actuary to make sure he or she understands what these values represent. The only way to do that is to spend time communicating with the claims department. This can be difficult, as the claims department and the actuarial department have very different tasks within an insurance company, and hence have different viewpoints and specialized jargons. When using claims department estimates in deriving a reasonable range of total liabilities, the actuary ideally would statistically model a sample of cases to "calibrate" the claims department case estimates.

Whether a sample of the cases or the universe of claims is being statistically modeled, the goal of the actuary in this part of the analysis is to build a reasonably realistic model with many "moving parts" – the parts being the insured's liabilities and insurance coverages. In this modeling process there is no substitute for obtaining a fundamental understanding of the claims settling process and gaining the trust of the claims department.

It should be noted that the claims department may have obtained modeled estimates of future liabilities for some of the insureds. There are econometric firms that produce such models, and in fact some firms specialize in not only modeling the liabilities and cash flows at the insured level, but also model the allocation of these liabilities to the years in the coverage block. If these are available, then by all means the actuary should make use of them, but should bear in mind that the models could be biased in favor of the insured (if the insured paid for the study) or in favor of a particular insurer or reinsurer. Frequently, however, these models are more complete than what is discussed below, as they are only developed for insureds whose involvement in this litigation is significant.

2.1 Direct Exposure

The actuary needs to find a way to take the information provided by the claims department and produce a reasonable range of liabilities stemming from known insureds. The bulk of the liabilities – and a large portion of the uncertainties surrounding them – is usually due to a relatively small percentage of the insureds. These insureds deserve close scrutiny. One way to do this is to build a frequency and severity model (ideally stochastic) that estimates the number of future claims that will be filed against the insured and estimates the average cost per claim for the pending and future claims. The data behind the model will be thin, so a large amount of judgment is required. The assumptions underlying the model should be informed by general knowledge of the mechanics of asbestos litigation and by the knowledge obtained from the claims department. Key assumptions should be peer reviewed by claims personnel and/or attorneys.

Once the model is developed and the total liabilities of an insured have been estimated, the liabilities need to be allocated to the relevant insurance policies. This is not a trivial matter. Not all companies allocate liabilities in the same manner. Some of this is driven by legal decisions, and some of it is a matter of practice. It is not uncommon for U.S. primary companies to allocate liabilities based on "time on risk" – this is effectively (in most instances) a uniform allocation across the coverage block. The London excess market demands that liabilities be allocated based on dates of actual exposure, as best as can be determined (referred to as a "bell curve" allocation). It is imperative that the model reflect the allocation methodology employed by the insurer under investigation.

It is probably not feasible (or desirable) to build individual models for the remaining accounts. Instead, the actuary should review the nature of the insured exposure, the attachment points and limits of the exposed policies, and discuss likely outcomes with the claims department. Aggregate analyses of these accounts, either all together or in obvious groupings, will suffice. It is probably desirable to perform a policy limits analysis as discussed in [CD]. In that paper, the authors suggest policy limits analyses of selected representative accounts, which is then extrapolated to arrive at the total IBNR provision. This is a reasonable approach to take in analyzing large groupings of accounts that do not comprise the bulk of the asbestos liability. The actuary should request that the claims department produce point estimates of ultimate liabilities for each of these accounts to be used as a starting point for this analysis.

It is also advisable to perform benchmark analyses (discussed below) on this group of claims to test the results for reasonableness. Benchmark analyses can be performed quickly, and can sometimes signal unreasonable IBNR provisions or areas that require more attention.

2.2 Assumed Exposure

Assumed reinsurance is usually more difficult to analyze than the primary insured liability. If the assumed exposure is made up entirely of quota share contracts, it may be possible to perform an analysis as described in the previous section - provided the necessary data is available. In most cases, however, this level of detailed analysis will not be possible.

The actuary should analyze recent paid and reserve activity by cedant (tying as much of this as possible to the named insureds) and should obtain information regarding the reinsurance contracts exposed to asbestos liabilities. In particular, the actuary should identify every ceding company that has already ceded asbestos liabilities to the assuming company. Every assumed reinsurance contract with these entities should be examined for possible asbestos exposure. A database containing the named insureds, ceding companies, direct policy details and reinsurance contract provisions would be immensely helpful, but can be difficult to develop. Ceding companies may not want to share any information other than the details of specific reinsurance claims being presented to the reinsurer.

A reinsurance company typically will assume losses from several ceding companies who have common insureds. For example, suppose Company A issued the primary cover to Insured Z, and Company B issued excess cover attaching at the per occurrence limits of the Company A policies. Further suppose that both Companies A and B purchased some form of reinsurance from Reinsurer X. In this case, Reinsurer X may very well know more than Company B does about the actions brought against Insured Z. Clearly Reinsurer X cannot share this information with Company B, but can use this information to arrive at appropriate reserve estimates.

Alternatively, Reinsurer X may have reinsured Company A and issued retrocessional cover to Reinsurer Y, who also reinsured Company A. Data contained in the assumed claims file for Company A can be used to assist in the development of IBNR related to the contracts issued to Reinsurer Y.

In the absence of enough data to perform a ground up analysis, the actuary must find a way to make use of all available information to devise a top down analysis. This can be exhaustive (and at times, even frustrating), as it involves analyzing the information contained in the claims files of each cedant in a quest for commonalities (e.g. the same named insureds).

Given the lack of data, the following top-down model uses the available information to develop the Low, Medium and High IBNR for a given cedant, by adjusting the carried assumed case reserves for each cedant to provide for future development on cases known to the cedants and future asbestos liabilities emanating from insureds of which the cedants are not yet aware (or for which the cedants have not yet made provisions). The adjustments reflect six considerations:

(1) the ratio of ceded IBNR recorded by the cedant in its Annual Statement relative to the cedant's ceded case reserves,

(2) the speed with which the cedant reports claims to its reinsurers,

(3) the quality and reliability of the information the cedant provides its reinsurers,

(4) the recent level of claims activity experienced by the cedant,

(5) the nature of the exposure being ceded, and

(6) the perceived inadequacy of the asbestos reserves of the U.S. insurance industry.

The IBNR is then calculated as follows:

$$\text{IBNR} = \begin{matrix}\text{Case}\\ \text{Reserves}\\ \text{Carried by}\\ \text{Assuming}\\ \text{Entity}\end{matrix} \;\text{X}\; \begin{matrix}\text{Ratio of}\\ \text{Ceded}\\ \text{IBNR to}\\ \text{Ceded}\\ \text{Case}\\ \text{Reserves}\end{matrix} \;\text{X}\; \begin{matrix}\text{Reserve}\\ \text{Factor}\end{matrix} \;\text{X}\; \begin{matrix}\text{Leverage}\\ \text{Factor}\end{matrix} \;\text{X}\; \begin{matrix}\text{Inadequacy}\\ \text{Multiplier}\end{matrix}$$

Appendix II discusses the derivation and the purpose of each of the above factors. The goal of the approach is to overcome some of the data deficiencies when developing the assumed reinsurance IBNR reserve.

3 Analysis of Unknown Gross Liabilities

The previous section was devoted to modeling asbestos liabilities from known sources of exposure. One must also recognize the substantial liability related to truly unknown sources – what the authors prefer to call "pure" IBNR.

One possibility is to examine recent emergence of new defendants (new to the insurer), and make assumptions regarding

- the expected number of new defendants during the next several years

- the number of years during which new defendants will be named
- the future liabilities associated with these new defendants.

If the data to perform such an analysis is available, then it certainly should be done. We are aware of innovative techniques for doing so, and it is our hope that the developers of such methods will publish them and add to the literature on this topic. This is basically a frequency and severity approach. If the data is available, use it to arrive at emergence patterns of new defendants (it is advisable to group these defendants by the nature of the exposure), and also arrive at projected liabilities.

Appendix III discusses a few other methods that can be used.

4 Benchmark Reserving Methods

Certain "benchmark" methods have been developed to perform tests of the adequacy of asbestos reserves. It must be emphasized that these methods are extremely crude, and rely heavily on actuarial judgment, much more so than standard reserving methodologies. They are all highly leveraged. These methods also rely on an estimate of industry parameters (e.g. AM Best's estimate of US insurance industry ultimate asbestos losses), together with company specific parameters. Unfortunately, the data is extremely thin (and volatile), and the actuary must rely heavily on qualitative information.

These tests are not really appropriate for arriving at needed reserves, but they can be used to arrive at a generally wide range of reasonable reserves. They can also be used to determine when one needs to investigate further. The methods are

- *The survival ratio method*
- *The market share method,*
- *The loss "development" method.*

A survival ratio is the number of years that current reserves will suffice ("survive") if average future payments equal average current payments. For example, suppose an insurer has $6M in asbestos reserves. Further suppose that recent asbestos payments have averaged $1M per year. Then the survival ratio of this company is 6, indicating that reserves are adequate to pay $1M per year for 6 years. The actuary can use this method to arrive at a reasonable range of indicated asbestos liabilities by multiplying estimated average future annual asbestos payments by an estimate of the number of years that such payments will be made. The result is indicated total asbestos liabilities. Indicated IBNR is then calculated by subtracting case reserves.

The market share method uses the insurance company's "market share" of the asbestos arena to estimate asbestos liabilities. The market share can be based on premium or on paid losses. One problem with this is that it is very difficult to determine a particular company's market share of the GL (and marine and aviation) policies sold to asbestos defendants. It is possible to determine the company's market share of total industry premium by line by year, but most companies are not exposed to asbestos losses for all years in which they wrote such policies.

Several published studies are available that estimate the US insurance industry's ultimate net asbestos liabilities. These can be used to calculate implied industry paid and incurred loss development factors. One can then adjust these factors to reflect the nature of an insurance company's asbestos exposure, and then apply them to the company's paid to date and incurred to date liabilities to arrive at estimates of ultimate liabilities.

These bulk methods are discussed in more depth in Appendix IV.

5 Ceded Reinsurance

Thus far, we have stressed the need to analyze gross liabilities. But, what an insurance company really cares about is its net liability.

At this stage, we have, at the very least, point estimates of ultimate liabilities for every policy known to be exposed to asbestos losses. The reinsurance department should use this information to calculate the resulting cessions and net liabilities – or provide the actuary with the detail necessary to do so. The ratio of net to gross liabilities can then be used as a starting point for determining the retained portion of the pure IBNR. Special care and attention should be given to any issues regarding reinsurance collectibility and the erosion of reinsurance cover by other sources of loss.

6 Summary

There is no single 'right' way to perform an analysis of asbestos liabilities. The actuary must gather qualitative information from those handling the claims, and must then use all available skills, judgment and creativity to analyze the specific challenges posed by the risk bearing entity under investigation. Issues of materiality, time, costs, and available resources must be considered. In addition, the nature of the risks assumed by the (re)insurer as well as its claims settling philosophy must be taken into account.

However, there is a single unifying theme to every rigorous actuarial analysis of asbestos liabilities. This theme can be summarized as follows:

- Effective knowledge gathering regarding the liabilities of the risk entity under investigation via thorough, open, and constant communication with those responsible for disposing of those liabilities;
- A commitment to keeping abreast of the global issues in the asbestos litigation;
- The application of actuarial skills, judgment and creativity in designing a flexible and transparent model with well documented assumptions and well communicated interpretation of results.

7 Afterword

This work is but one example of the authors' vision of the value an actuary brings to the user of the actuarial work product. In some settings, the actuary is presented with a large quantity of reliable data and a well tested and well accepted actuarial tool box. In other settings, the quantity and/or reliability or credibility of the data specific to the liability being studied may not be optimal, but data from a larger class (of which the entity being analyzed is a member) are readily available, and the existence of the actuarial toolbox is undisputed.

The valuation of asbestos liabilities is a high profile example of another common setting: very little credible data and very few widely accepted actuarial tools or methods, but an abundance of qualitative facts and well educated opinions and reasonable assumptions. In many of these settings, an actuary is not consulted, and some actuaries may not even realize that the problem is amenable to an actuarial approach. The desire to work with cold hard data may lead some to avoid the challenges posed by lack of traditional data.

The reality is that in ALL actuarial projects a considerable amount of judgment is exercised, and what is judgment if not the application of well informed opinions and reasonable assumptions? In the absence of data and well defined and accepted tools, the challenge is to learn as much as possible from the experts (in this case, the claims handlers and those responsible for collecting reinsurance) and to make as much use as possible of their expertise by transforming the expertise into an actuarial model.

One of the side benefits of this approach is that it helps these experts to test their assumptions: Do the perfectly reasonable assumptions regarding individual liabilities support or contradict the experts' opinions as to the aggregate liabilities? Do some of the reasonable looking assumptions contradict each other or contradict known facts? The modeling also provides the opportunity to document the assumptions and assess their continued applicability in the light of emerging experience.

The documentation and validation of modeling assumptions aids in the communication process and provides management with the requisite insight into the derivation of the actuarial liabilities prior to booking a specific reserve position. The areas where traditional loss reserve valuation techniques are not appropriate are most indicative of where actuaries can add immense value to the consumers of their work product.

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Appendix I Unique Aspects of Asbestos Liabilities

As mentioned in the Introduction, a rigorous study of asbestos liabilities requires the analyst to become intimately familiar with the details of the liabilities in question. To that end, there is no substitute for thorough communication with those responsible for discharging the liabilities.

The Nature of Asbestos Diseases

Asbestos is an incredibly deadly substance. The sad reality is that some of the major defendants knowingly unleashed this toxic substance upon society. There are widely publicized "smoking gun" documents that have been said to show that some defendants knew that their products would lead to the deaths of thousands of their employees. In a widely publicized letter dated September 12, 1966, E.A. Martin, Director of Purchases at Bendix, writes to Noel Hendry of the Canadian Johns-Manville plant in Asbestos, Quebec:

> "Just to be sure that you have a copy, an article that appeared in Chemical Week magazine is inclosed (sic).
>
> So that you'll know that Asbestos is not the only contaminate (sic), a second article from O.P. & D Reporter assess a share of the blame on trees.
>
> My answer to the problem is: If you have enjoyed a good life while working with asbestos products why not die from it. There's got to be some cause."

It has been estimated that more than 100 million people in the United States were exposed to asbestos in the workplace during the 20th century ([AAA]). Not everyone exposed to asbestos will become ill, but some will. A small percentage of these people will develop a deadly and painful cancer known as mesothelioma. Other cancers (of the lungs, throat, larynx, esophagus, stomach, colon, and lymphoid) may also result. Asbestosis (a slowly progressing, sometimes fatal pulmonary disease) and pleural injuries may also result. All of these diseases have long latency periods – from 10 to 40 years depending on the disease. This means that there could very well be people developing mesothelioma as of this writing that were last exposed to significant levels of asbestos in the 1960s.

Many epidemiological studies have been performed on the topic of asbestos related diseases, their incidence and latency periods. The AAA monograph contains references to many of them. Each one of these studies indicates that mesothelioma victims make up a minority of those who become ill due to asbestos exposure. As we shall discuss later, the long latency periods of these diseases causes considerable difficulty in quantifying the insurance liabilities related to the use of asbestos.

The Legal Environment

The legal environment surrounding the disposition of asbestos liabilities is perhaps the biggest complicating factor in their analysis[1]. One hint of the complexity of this environment is provided by a quick glance at the specializations of the attorneys involved:

- The plaintiffs' bar
- Defense attorneys
- Coverage attorneys representing the defendants in pursuit of insurance recoveries
- Coverage attorneys representing insurance companies
- Opposing parties in disputes involving the related reinsurance recoveries
- Those specializing in asbestos related Chapter 11 proceedings.

It is extremely difficult for a defendant to arrive at a reasonable estimate of its total asbestos liabilities, and this is one of the reasons that so many of them have pursued the remedy of Chapter 11 reorganization. Econometric firms have entire practice groups dedicated to modeling the asbestos liabilities of defendants, the investment community and ratings agencies perform their own analyses of these liabilities, and there is a burgeoning business of estimating the liabilities to future unknown defendants in the world of Chapter 11 proceedings. This particular "level" of the asbestos litigation is heavily dependant on that area of the law that affects plaintiffs and defendants. This law differs from state to state, and the federal courts have their own unique law as well.

The typical asbestos claimant was exposed to asbestos over a number of years, and, most likely, the asbestos did not come from one source. Many asbestos claimants are members of a large group being represented by the same law firm, who is demanding payment from many companies. The list of defendant companies is growing, with attorneys recently filing claims against companies with only minimal involvement in the manufacture or distribution of asbestos, especially since many of the large asbestos defendants have filed for bankruptcy.

Developments over the last few years have led to what some consider a crisis. There are several good references that discuss this in detail (e.g., [AAA], [RAND], [R], [P]). Several years ago the federal courts instituted procedures to try to make this litigation manageable. One of the unintended consequences of this has been the increase in filings in state courts. One can argue that many of these cases belong in the federal courts. Many states have made changes to their laws or their procedures to deal with this litigation. This has led to an increase in filings in those states that have not done so. In some states it has been permissible for an attorney to represent several plaintiffs in an action against several defendants wherein only one of the plaintiffs is now or ever has been a resident of the state. Furthermore, in some states all that is required for a plaintiff to prevail is a showing of exposure to asbestos and the existence of a lung x-ray

[1] It is prudent at this time to note the case of *Borel v. Fibreboard,* (1973), in which the Fifth Circuit U.S. Court of Appeals ruling effectively shifted asbestos awards from the workers' compensation system to the court system.

indicating that there *may* be scarring of the lung tissue – no actual injury or impairment is required. Some plaintiffs' attorneys who only represent cancer victims claim that these actions are harming their current and future clients. If a company is required to pay a settlement to every person exposed to their asbestos containing products with a shadow on a lung x-ray, that company may not be around to pay the damages due to those who develop cancer in the future (recall the long latency periods involved).

More than 50 companies have declared bankruptcy or filed for Chapter 11 reorganization claiming that they are doing so due to the magnitude of their asbestos liabilities. This has had an enormous impact on the U.S. economy, and on the evolution of the asbestos litigation. As more companies file for Chapter 11 reorganization, plaintiffs are forced to look elsewhere for damage awards. This has led to a wave of new defendants, many of whom were only peripherally involved in the manufacture or distribution of asbestos products. For example, there has been a recent increase in suits against companies who make products with encapsulated asbestos, such as fireproof doors. Another recent target class of defendants is manufacturers and distributors of gaskets, brakes and other friction products – including "mom and pop" auto parts distributors.

For the actuary, however, it gets even more complicated. There is no single algorithm that can be applied to determine the resulting liability of the insurance companies in all cases. In fact, one cannot even say that there is one algorithm that applies for each state. There are a few theories that can be used to determine how asbestos losses are allocated to insurance policies. These have been expounded upon elsewhere. The key is to understand how to apply them.

In practice, the indemnity and legal expenses borne by the insured are allocated to a coverage block. Either by agreement between the insured and the insurance companies or as the result of a court ruling in a declaratory judgment (DJ) action, a determination is made as to which primary and excess policies will respond to the insured liabilities – and how the liabilities will be shared amongst the entities. This is very dependant upon

- the history of the insured (when did they manufacture or distribute asbestos containing products? when was asbestos in use at their facilities?),
- the financial health of the insured,
- the financial health of the insurers
- the claims settling practices of the insurers and
- the amount of coverage available.

Another important question that needs to be addressed is how the claims will be classified. Are they products/completed operations claims, or are they premises and operations claims? This is usually referred to as products vs. non-products. Almost all CGL policies issued after 1986 contain asbestos exclusions that hold up in court. Many of the exposed policies contain aggregate limits for products claims, but only occurrence limits for non-products claims. Therefore, if the claims are considered to be products liability claims, then the total indemnity costs involved are limited to the products aggregate.

Let's consider a very simple example. Suppose an insured manufactured and distributed an asbestos containing insulation product between 1962 and 1971 and that 1000 insulation installers have filed suit against the insured for asbestos related injuries caused by exposure to the insured's product. For illustrative purposes, let's assume we can spread the liabilities of these suits evenly across the 10 years that the insured manufactured the product (it would not be unusual to allocate the liabilities in other ways depending on the specific facts of the case – some insurers strenuously object to uniform allocations). If the insured has paid $100M in indemnity and $150M in legal expenses to dispose of these claims then each policy year would be allocated $10M of indemnity and $15M of legal expense. Assuming policy limits of $5 million per occurrence and $5 million annual aggregate with legal expenses paid in addition to policy limits, the insurance coverage would be $50M for indemnity payments and up to $150M for legal expense[1].

If we make a slight change to the above example, the situation can change dramatically. Suppose that the insured is engaged in a high temperature industry and that 1000 of its employees have filed suit against the insured due to injuries sustained as a result of exposure to asbestos containing insulation. The employees do not have the ability to positively identify the manufacturer of the insulation to which they were exposed, and over the course of their employment may have been exposed to asbestos containing products purchased from many different manufacturers and/or distributors. These claims could be considered premises claims. The primary policies probably do not contain aggregate limits for the premises and operations hazard. We now have to decide a very important question: how many occurrences are there per policy? Again, this is decided either through a negotiated agreement between the insured and the insurers, or through a DJ action. The following are common approaches. Others are possible, as well.

- Each claimant is considered to constitute a separate occurrence, with the losses spread evenly over the coverage block. Total insured losses would probably be $100 million indemnity, $150 million for expense.
- Each claimant is considered to constitute a separate occurrence. Losses are spread over the coverage block based on dates of employment and actual liabilities incurred by the insured. Total insured losses would likely be $100 million of indemnity and $150 million for expense.
- Each physical location is considered to constitute an occurrence. For example, if there were 3 plants operating during the entire 10 year coverage block, and a 4th plant in operation during 5 years of the coverage block – say 1967 to 1971 – then there would be 3 occurrences from 1962 to 1966, and 4 occurrences from 1967 to 1971. Total insured losses could be $85.5 million of indemnity and $150 million of expense.

In these examples, the primary insurers' liability is much greater than it would be if the losses were classified as products claims.

[1] The exact amount of legal expense covered by the policies is dependant upon the timing of the indemnity payments – a policy that pays costs in addition will not respond to legal expenses after the indemnity payments have exhausted policy limits.

Let's put one more wrinkle into this to show why it truly is a legal issue. Suppose the insured had a $500,000 SIR for each of the years in the coverage block. Note that this interpretation of occurrence in the first and second premises/operations cases above probably eliminates all insured losses. There are cases wherein the courts have ruled that considering each plaintiff to constitute a separate occurrence with the insured responsible for one SIR per year per occurrence is against public policy, as it "eviscerates" the insurance coverage. In other words, the insured would never recover any of the loss from its insurers.

Possible approaches to this situation would be to declare one occurrence per policy year with the insured responsible for one SIR per year (this would be strenuously objected to by the excess insurers), or to declare each claimant is an occurrence, with the insured's liability restricted to one SIR per year in the aggregate. Other approaches are possible.

In addition to the uncertainties in reserve evaluation resulting from liability issues involving individual claimants, disputed coverage issues complicate the evaluation of an insurer's liability to the insured. As can be expected when large sums of money are involved, insureds' coverage attorneys carefully review all policies to assess the possibility of and/or extent of insurance coverage. This additional contingency further complicates the valuation/reserving process and tends to lead to a greater variability in possible outcomes from the "best estimate" reserve, i.e. a wider range of possible values. The following are of particular importance:

- *Reclassification of claims.* Many insureds, having exhausted their products liability limits, are going back to their insurers and claiming that a large portion of the losses they have paid are non-products in nature. Any policy wherein the insured has exhausted products limits but has available non-products coverage is exposed to this contingency. In addition to the impact this has on primary insurers who are already deeply involved in the asbestos claims process, reclassification of claims to premises/operations could lead to excess carriers (and reinsurers) suddenly finding themselves with exposure they didn't contemplate.
- *Hunting for all available insurance.* Asbestos plaintiffs, and the insurance attorneys of the defendants, are pursuing recoveries from other types of policies. It seems that any company who issued any insurance to the large asbestos defendants may find themselves faced with an asbestos claim at some time in the future. In addition, insurers who insured companies who had or have any relationship to asbestos, no matter how minor, may well receive notice of asbestos claims. The following scenario is not uncommon: Company A purchased Company B in 1990. By 2001, Company B's asbestos payments have exhausted all of their available insurance coverage (and are, in fact, significantly worse than anyone anticipated when Company A purchased Company B). In an attempt to maximize coverage, Company A files claims against all of its 1986 and prior liability policies, despite the fact that when those policies were issued Companies A and B had no relationship.

Data

There are significant data issues involved in any analysis of asbestos liabilities. As mentioned previously, many commercial liability policies issued after 1986 contain an asbestos exclusion. This means that a large proportion of policies with exposure to asbestos losses were issued before insurance companies had computerized systems. Many insurance companies have no idea what their true asbestos exposure is because, e.g., they do not know who purchased a CGL policy from them in 1950. If they insured any of the big defendants in the asbestos arena, they would know by now, but there is a good chance that many companies who think they have no asbestos exposure, did, in fact, issue policies to companies who are just now being named in asbestos lawsuits.

For ground-up analyses, it is best to obtain as much information as possible at the insured level. In particular, it is desirable to have

- A history of annual payments made by the insured (indemnity and legal expenses separately);
- The total number of claims filed against the insured;
- The number of outstanding claims;
- The number of claims settled;
- The number of claims dismissed.

It is also desirable to obtain a coverage chart for each insured – that is a schedule of all available insurance and how prior payments have been allocated to the coverage. Obtaining all the data desired is not always possible. For one thing, the claims department may only have data at the insurer level. It is also true that the defendant may only have historical data going back a few years.

Appendix II Modeling Liabilities from Known Sources

Direct Exposure

It is most likely not cost effective to model the liabilities stemming from each known account. The important decision is to determine which accounts will be individually modeled.

For example, one could build a stochastic model for that subset of the known asbestos insureds that has been identified as requiring close scrutiny. The inputs will be subjective, and must be tested for reasonability – mainly by asking for the opinions of those handling the claims. The key variables could be:

- The total number of future claims
- The first year that new claims will be filed (if the insured is in Chapter 11)
- The claims filing pattern
- The number of claims closed each year
- The average indemnity cost of closed claims
- The average legal expense of closed claims
- The number of occurrences per policy for exposures other than product liability
- A methodology to allocate liabilities to policy year (this is a key assumption!)

The discussion below assumes stochasticity, but the stochastic routines need not be overly sophisticated.

It is very important to model indemnity costs and legal expenses separately. One could assume as a default that primary policies cover defense costs in addition to limits and that excess policies consider legal expenses to be subject to the policy limits, but this is not always true. In the default situation, it is not unusual for defense costs to be the major driver of the primary policies' liabilities.

The total number of future claims. The claims department and/or outside counsel representing the insurer in coverage matters should have historical and current data on the insured. The actuary must use the qualitative information about the nature of the alleged exposure to arrive at estimates of the likely number of future claims. For example

- Is the exposure products liability, non-products liability, or both? Is there any marine, aviation, or railroad exposure (these are handled differently than 'typical' CGL exposures)?
- Is the classification a matter of debate?
- In what state are the actions being brought?
- Who are the plaintiff attorneys?
- Is the insured a traditional defendant, a recent target defendant, or a peripheral defendant with limited exposure?

A reasonable way to approach this problem is to obtain some general information about the insured, and then suggest plausible percentiles of the distribution of future claims. It is doubtful that a claims handler or attorney will answer a question such as "How many future claims do you think there will be?" or "What do you envision as a worst case scenario for this insured?" They are much more likely to respond to something like this: "This insured has only been named in asbestos suits for the last five years, and I see the main exposure stems from their manufacture of brake linings. We know the plaintiffs' bar is targeting manufacturers of friction products, so there is a high likelihood that this insured will be named in many more suits. The latest data indicates that there have been a total of 8000 claimants, with 6500 still pending. There have been very few dismissals, and the insured has changed their defense strategy from vigorously defending every claim to settling those claims that have a high probability of being decided against them. Considering the states in which the suits are being filed and the success of the plaintiffs thus far, it seems to me that it is reasonable to expect another 20,000 claims. It also seems that there could be as many as another 50,000 claims, but the probability of that many claims is roughly 5%."

Phrasing the question as a statement begins a dialog. If those knowledgeable about the litigation involving the insured find the assumptions unreasonable, then a conversation will ensue that allows the actuary to gain a much better understanding of the exposure.

One can implement the assumptions outlined in the above example with a negative binomial distribution with parameters $n = 2.0002$ and $p = 0.9999$. This distribution has an expected value of 20,000 and 50,000 is close to the 95th percentile. The low percentiles for this distribution might be too low - this needs to be verified by the claims professionals.

The first year that new claims will be filed (if the assured is in Chapter 11). If the insured has recently filed for Chapter 11 protection, then a temporary restraining order is in place, blocking the filing of any suits until the reorganization plan is approved. If this is the case, then the year in which claims will again be filed should be a variable of the model. A discrete distribution, with the years and associated probabilities judgmentally selected can be used for this purpose.

The claims filing pattern. There a few obvious ways to model a filing pattern. One of the keys is the year in which the last claim will be filed against the insured - actually, the last year in which a claim that would trigger insurance coverage would be filed. For example, if all of the insured's 1986 and subsequent policies contain asbestos exclusions then it would be safe to assume that any claims filed after the period 2025 to 2035 (due to latency periods, depending on diseases suffered by the plaintiffs) would not trigger insured claims. A judgmentally selected discrete distribution for the final year in which claims will be made will suffice.

The final year in which claims are filed and the total number of future claims should be correlated – more years of claims filing should, on average, lead to more claims. One way to do this would be to select a distribution for the number of claims filed in each year, taking care that the resulting distribution of total future claims is in agreement with the assumptions arrived at earlier. Another way would be to arrive at an expected filing pattern that is used as a baseline to be adjusted given the number of total claims filed and the number of years in which they will be filed. In this case one should introduce random variation into the expected filing pattern.

The claim closure pattern. For each insured a claim closure pattern is needed. This can be based on data specific to the insured, the insurer or on industry data. A discrete distribution should suffice. Note that we need to model the closing pattern of the pending claims as well as that of the future claims.

The average indemnity cost of closed claims. The points discussed above all relate to frequency. We now address severity. The most elementary approach is to select a baseline average indemnity cost per closed claim and apply annual inflation factors so as to have average indemnity amounts per closed claim per year. If this is done, then random variation should be introduced. A random walk process is fairly easy to use for this process.

A more sophisticated approach would be to explicitly determine expected distributions of disease type (including those that will be closed without payment), with associated average indemnity costs. There has been a recent explosion of claims filed by those suffering from non-malignant injuries. These claims are usually settled for much less than those of victims suffering from mesothelioma or other cancers. Appropriate assumptions can be made regarding the likely future disease mix and related liabilities.

If an insured has already completed the Chapter 11 reorganization plan, then it will probably have a schedule of benefits paid based upon disease type. The 524(g) trusts established by the bankruptcy courts to dispose of these liabilities usually have stringent rules regarding who is indemnified and the amount of indemnification they receive.

The average legal expense of closed claims. Legal expenses will be incurred by the defendant whether a claim is dismissed or not. Similar to the discussion above, it is desirable to model the average liabilities per closed claim per year.

At this stage, one has a model that produces

- Total number of future claims filed against the insured
- The year in which these claims will be closed
- The associated costs of these claims.

We now turn our attention to the insurer.

A methodology to allocate losses to policy year. The issue of how various insurance policies respond to asbestos claims is fundamental to the modeling process. There is no single "right" way to allocate the losses. The specific details of the insured and the insurer are the driving factors. A coverage block is determined, either through a DJ action or through agreement of all interested parties. It is not uncommon for U.S. primary companies to allocate liabilities based on "time on risk" – this is effectively (in most instances) a uniform allocation across the coverage block. The London excess market demands that liabilities be allocated based on dates of actual exposure, as best as can be determined (referred to as a "bell curve" allocation). In some states, court rulings require the *entire* block of primary coverage be exhausted before *any* excess policy will respond. This is referred to as *horizontal allocation* or *filling the* bathtub. It is imperative that the model reflects the allocation methodology employed by the insurer under investigation.

For example, suppose the following:

- There are $10M in products liabilities to be allocated to the period 1967 to 1986
- From 1967 to 1972 the insured purchased primary insurance with per occurrence limits of $250,000
- From 1973 to 1978 the primary limits were $500,000
- From 1979 to 1986 the limits were $1M.
- There have been no other products liability claims filed against these policies
- None of the policies contain an SIR
- None of the primary policies cover legal fees in defense of claims.

Let us assume that the liabilities are allocated uniformly across the coverage block ($500,000 per year). The 1967 to 1972 primary policies will only pay $250,000 each, for a total of $1.5M. This leaves $8.5M for the remaining 14 years. The 1973 to 1978 primary policies will each exhaust, paying a total of $3M, leaving $5.5M for the 1979 to 1986 policies. Each of these will pay $687,500, for a total of $5.5M.

Now suppose that the $10M is made up of $4M of legal expenses and $6M of indemnity, and that each of the primary policies are "costs-in-addition". Then the liabilities could be allocated to the policies as follows:

Policy	Per Occurrence Limit	Indemnity	Legal Expenses
1967	250,000	250,000	166,667
1968	250,000	250,000	166,667
1969	250,000	250,000	166,667
1970	250,000	250,000	166,667
1971	250,000	250,000	166,667
1972	250,000	250,000	166,667
1973	500,000	321,429	214,286
1974	500,000	321,429	214,286
1975	500,000	321,429	214,286
1976	500,000	321,429	214,286
1977	500,000	321,429	214,286
1978	500,000	321,429	214,286
1979	1,000,000	321,429	214,286
1980	1,000,000	321,429	214,286
1981	1,000,000	321,429	214,286
1982	1,000,000	321,429	214,286
1983	1,000,000	321,429	214,286
1984	1,000,000	321,429	214,286
1985	1,000,000	321,429	214,286
1986	1,000,000	321,429	214,286
Total	12,500,000	6,000,000	4,000,000

Now let us assume a few years have gone by, and the primary policies are all exhausted. Further suppose there is $50M in indemnity and $75M in legal expenses to be allocated to the following excess policies:

- From 1967 to 1972, there was one layer of excess coverage, $750,000 xs $250,000, and these policies cover costs in addition
- From 1973 to 1978 there were three layers of excess coverage - $500,000 xs $500,000, $1.5M xs $1M and $2.5M xs $2.5M. All of the policies cover legal expenses within policy limits
- From 1979 to 1986 there were three layers of excess coverage: $4M xs $1M (costs inclusive), $5M xs $5M (costs excluded) and $15M xs $10M (costs excluded).
- Assume there were no other claims eroding the available limits.

Then the liabilities could be allocated as follows. (This would be the allocation if the decision had been made that each layer of coverage in the coverage block must exhaust before the next layer responds. So the $4M xs $1M policies in the 1979 to 1986 period would exhaust before the $1.5M xs $1M policies in the 1973 to 1978 period would respond).

Policy Year	1st Excess		2nd Excess		Total	
	Indemnity	Legal Expenses	Indemnity	Legal Expenses	Indemnity	Legal Expenses
1967	750,000	1,125,000	N/A	N/A	750,000	1,125,000
1968	750,000	1,125,000	N/A	N/A	750,000	1,125,000
1969	750,000	1,125,000	N/A	N/A	750,000	1,125,000
1970	750,000	1,125,000	N/A	N/A	750,000	1,125,000
1971	750,000	1,125,000	N/A	N/A	750,000	1,125,000
1972	750,000	1,125,000	N/A	N/A	750,000	1,125,000
1973	200,000	300,000	600,000	900,000	800,000	1,200,000
1974	200,000	300,000	600,000	900,000	800,000	1,200,000
1975	200,000	300,000	600,000	900,000	800,000	1,200,000
1976	200,000	300,000	600,000	900,000	800,000	1,200,000
1977	200,000	300,000	600,000	900,000	800,000	1,200,000
1978	200,000	300,000	600,000	900,000	800,000	1,200,000
1979	1,600,000	2,400,000	3,487,500	0	5,087,500	2,400,000
1980	1,600,000	2,400,000	3,487,500	0	5,087,500	2,400,000
1981	1,600,000	2,400,000	3,487,500	0	5,087,500	2,400,000
1982	1,600,000	2,400,000	3,487,500	0	5,087,500	2,400,000
1983	1,600,000	2,400,000	3,487,500	0	5,087,500	2,400,000
1984	1,600,000	2,400,000	3,487,500	0	5,087,500	2,400,000
1985	1,600,000	2,400,000	3,487,500	0	5,087,500	2,400,000
1986	1,600,000	2,400,000	3,487,500	0	5,087,500	2,400,000
Total	18,500,000	27,750,000	31,500,000	5,400,000	50,000,000	33,150,000

Note that this allocation leaves $41.85M in legal expenses paid by the insured. The insured might argue that the total excess limits available in a year – regardless of how many layers make up the total limits - should determine the allocation. The table below shows this allocation, which leaves only $32.85M in legal expenses unfunded.

	1st Excess		2nd Excess		3rd Excess		Total	
Policy Year	Indemnity	Legal Expenses	Indemnity	Legal Expenses	Indemnity	Legal Expenses	Indemnity	Legal Expenses
1967	750,000	1,125,000	N/A	N/A	N/A	N/A	750,000	1,125,000
1968	750,000	1,125,000	N/A	N/A	N/A	N/A	750,000	1,125,000
1969	750,000	1,125,000	N/A	N/A	N/A	N/A	750,000	1,125,000
1970	750,000	1,125,000	N/A	N/A	N/A	N/A	750,000	1,125,000
1971	750,000	1,125,000	N/A	N/A	N/A	N/A	750,000	1,125,000
1972	750,000	1,125,000	N/A	N/A	N/A	N/A	750,000	1,125,000
1973	200,000	300,000	600,000	900,000	1,000,000	1,500,000	1,800,000	2,700,000
1974	200,000	300,000	600,000	900,000	1,000,000	1,500,000	1,800,000	2,700,000
1975	200,000	300,000	600,000	900,000	1,000,000	1,500,000	1,800,000	2,700,000
1976	200,000	300,000	600,000	900,000	1,000,000	1,500,000	1,800,000	2,700,000
1977	200,000	300,000	600,000	900,000	1,000,000	1,500,000	1,800,000	2,700,000
1978	200,000	300,000	600,000	900,000	1,000,000	1,500,000	1,800,000	2,700,000
1979	1,600,000	2,400,000	2,737,500	0	0	0	4,337,500	2,400,000
1980	1,600,000	2,400,000	2,737,500	0	0	0	4,337,500	2,400,000
1981	1,600,000	2,400,000	2,737,500	0	0	0	4,337,500	2,400,000
1982	1,600,000	2,400,000	2,737,500	0	0	0	4,337,500	2,400,000
1983	1,600,000	2,400,000	2,737,500	0	0	0	4,337,500	2,400,000
1984	1,600,000	2,400,000	2,737,500	0	0	0	4,337,500	2,400,000
1985	1,600,000	2,400,000	2,737,500	0	0	0	4,337,500	2,400,000
1986	1,600,000	2,400,000	2,737,500	0	0	0	4,337,500	2,400,000
Total	18,500,000	27,750,000	25,500,000	5,400,000	6,000,000	9,000,000	50,000,000	42,150,000

The situation can get much more complicated. If some of the policies contain SIRs, or if the treatment of legal expenses are significantly different between policies, or if there were other products liability claims that impacted available limits, then the allocation would be different.

The number of occurrences for exposures other than product liability. The policyholder only cares about number of occurrences in so far as it affects collection of insurance proceeds. If the claims are clearly products liability claims, then this is rarely an issue – there is almost always an aggregate limit for products liability claims.

However, if it is unclear how the claims should be classified, then the insured will try to find a way to avoid the products aggregate. If the insured has a large amount of excess coverage available, and legal expenses are covered by these policies, then the insured may not aggressively pursue this point. If the total products aggregate limits are woefully inadequate to fund the insured's liability, and there are no aggregate limits for the premises and operations hazard, then the insured may very well argue that a portion (perhaps 100%) of the claims are non-products in nature. Primary carriers have a vested interest in arguing for a products/completed operations classification. Excess carriers have a vested interest in arguing for a non-products classification – provided the underlying cover is not exhausted.

Each of the account specific models produces a distribution of possible insured liabilities for each insured. These results must now be aggregated. Clearly, these accounts are not independent of one another. For example,

- They may, on occasion, be named as codefendants;
- They may be in the same or similar industries;
- They may have common corporate ancestors;
- They may have the same legal representation;
- The various state and federal laws and court rulings affect them all – though not all in the same way.

The model that aggregates the results of the individual account models should reflect the correlation among them. This is not a simple matter, especially since there is no data upon which to base the correlations.

Perhaps the best solution is to use several different correlation coefficients and review the sensitivity of the results, and perhaps the best way to reflect the implicit dependencies is to recognize that there is some correlation between the number of claims filed against one insured and those filed against another insured. There is also some correlation between the liabilities incurred by one insured and those incurred by another. These are not exact relationships, and determining them precisely is impossible. The important thing is to recognize that dependencies exist and find a reasonable (and creative) way to reflect them.

Assumed Exposure

Assumed reinsurance is usually more difficult to analyze than the primary insured liability. If the assumed exposure is made up essentially of quota share contracts, it may be possible to perform an analysis as described in the previous section - provided the necessary data is available. In most cases, however, this level of detailed analysis will not be possible.

In the absence of enough data to perform a ground up analysis, the actuary must find a way to make use of all available information to devise a top down analysis. The following top-down approach can be used. Begin by adjusting the carried assumed case reserves for each cedant. These adjustments are intended to provide for future development on cases known to the cedants and future asbestos liabilities emanating from insureds of which the cedants are not yet aware (or for which the cedants have not yet made provisions). The adjustments reflect six considerations:

(1) the ratio of ceded IBNR recorded by the cedant in its Annual Statement relative to the cedant's ceded case reserves,

(2) the speed with which the cedant reports claims to its reinsurers,

(3) the quality and reliability of the information the cedant provides its reinsurers,

(4) the recent level of claims activity experienced by the cedant,

(5) the nature of the exposure being ceded, and

(6) the perceived inadequacy of the asbestos reserves of the U.S. insurance industry.

The first step of the procedure is to calculate the cedants' ratios of ceded IBNR to ceded case reserves, as recorded on Note 29 of the Annual Statement. The task here is somewhat complicated by the way in which insurers record liabilities, especially for those that are part of a large underwriting group or which are no longer filing an Annual Statement. There will also be some cedants whose ratios appeared unrealistic or for whom data is not available.

The second step is the calculation of a "reserve factor" to adjust the case reserves for the speed with which the cedant reports claims to the reinsurer, as well as the quality and/or reliability of the data. Total asbestos case reserves for assumed liabilities from cedants who are slow to report are less adequate – relative to ultimate liabilities – than total asbestos case reserves from cedants that report losses quickly. Furthermore, prudence and conservatism require one to assume that total asbestos case reserves for assumed liabilities from cedants with a history of poor data quality and/or reliability are less adequate – relative to ultimate liabilities – than total asbestos case reserves from cedants known for providing good, reliable data.

The third step is the calculation of a "leverage factor" in recognition of two aspects of the cedant – the typical risk being ceded (primary, excess, or retrocessional) and the level of activity currently being reported by the cedant. Excess and retrocessional losses will, on average, be reported to the cedant later than primary losses will. Such losses are reported even later to the reinsurers assuming them. However, the amount of adjustment necessary should be tempered by the amount of recent claim activity experienced by the cedant.

The reserve and leverage factors can be determined by interviewing assumed reinsurance claims professionals and asking them to score the cedants in the four categories discussed above – speed of reporting, type of risk ceded, level of recent claim activity and quality of data. The selected factors will be based on actuarial judgment; a review of the reasonability of implied results is extremely important. The actuary should search the business, investment and trade presses for announcements regarding significant settlements, reserve increases and other actions that have been taken during the current year, as this information will not be reflected in the most recent Annual Statement, and could have a significant bearing on the assumed liabilities.

At this stage we have the following data for each cedant:

- case reserves carried by the assuming company,
- the cedant's ratio of ceded asbestos IBNR to ceded asbestos case reserves from Note 29 of the Annual Statement,
- a reserve factor and
- a leverage factor.

It would seem natural to multiply the four numbers to arrive at the IBNR related to the cedant. The implicit assumption underlying the procedure thus far is that carried asbestos reserves are a reasonable reflection of the ultimate expected liabilities. However, it is widely believed that the carried asbestos reserves for the U.S. insurance industry are inadequate – i.e. the implicit assumption is flawed. To overcome this deficiency in the reported asbestos liabilities, we should rely on other expert assessments of the total liabilities.

Several firms and research groups publish separate studies of asbestos liabilities. Frequently, these studies estimate that the ratio of net unfunded liability to net carried reserves. This information can be used to select "inadequacy multipliers" and arrive at Low, Medium and High estimates of IBNR. The multipliers should be chosen to reflect the industry reserve inadequacy, but should also recognize that some of the inadequacy is already reflected by the leverage factor, and, possibly, by the reserve factor and the carried reserves of the assuming company (if the assuming company has conservative reserving practices, it may be a matter of practice that the carried assumed case reserves from a particular cedant are higher than those reported by the cedant).

The Low, Medium and High IBNR for a given cedant is then computed by performing the following calculation:

IBNR =	Case Reserves Carried by Assuming Entity	X	Ratio of Ceded IBNR to Ceded Case Reserves	X	Reserve Factor	X	Leverage Factor	X	Inadequacy Multiplier

It is important to note another assumption implicit in this methodology: the assuming entity is aware of all of the cedants from whom it is assuming asbestos liabilities, but substantial uncertainty surrounds the original sources of the liability (that is, there are possibly many "unknown" original insured defendants).

Appendix III Modeling Liabilities from Unknown Sources

Judgmental Selection

One of the drawbacks of a rigorous exposure based analysis of asbestos liabilities is that by its very nature it exclusively considers known sources of exposure. After the analysis of known defendants is completed, one must then add a provision for liabilities emanating from defendants who are not known to the (re)insurer. A portion of this "pure" IBNR liability could be estimated by performing an exhaustive policy audit, comparing all known GL, aviation, and marine policies to the universe of all known asbestos defendants. This is discussed below. Assuming the expenditure of time and resources presented by such a project were worthwhile (not to mention the critical data issues raised), this approach does not provide a complete solution to the problem since many, if not most, of these defendants have not yet been named in any asbestos litigation.

Given the above data and analysis issues, sometimes all that can be done is to set up a reasonable provision for this pure IBNR. Determining a reasonable provision is not easy: there is no widely accepted methodology for arriving at this IBNR provision – a large amount of actuarial judgment is required.

Consider the following table, which is based on data contained in the September 2002 RAND report entitled Asbestos Litigation Costs and Compensation: An Interim Report (This table reflects the total asbestos litigation universe, *not* just the insurance industry).

	1982	2000
Number of Claimants	21,000	600,000
Number of defendants	300	At least 6,000
Total paid liabilities	$1B	$54B
Bankruptcies	3	60
Estimated Future Liabilities	$38B	$145B - $210B

Note, in particular, the explosive growth in named defendants from 1982 to 2000. There is considerable uncertainty as to how many more defendants will eventually be named in the asbestos litigation, and this is the heart of the challenge one faces in trying to estimate pure IBNR.

Consider also these additional facts:

I. From the RAND report:

1. Estimates of the number of people who will file claims in the future vary widely, but they are all extremely high. All accounts agree that, at best, only about half the final number of claimants have come forward. At worst, only one-fifth of all claimants have filed claims to date.
2. Annual Claims Filings Have Risen Sharply in the last few years.
3. Analysts' projections of the numbers of future claims and their likely costs also vary dramatically. Analysts at Tillinghast-Towers Perrin project an ultimate total of 1 million claims, costing defendants and insurers $200 billion ([AB]). Analysts

at Milliman project a total of 1.1 million claims, but they estimate that the total liabilities of asbestos personal injury claims will reach $265 billion ([BMR]).

4. The Manville Trust commissioned a deliberately high-side estimate designed to set an upper boundary on what would happen if everything turned out to be as bad as it could get. The estimate was 3 million total claimants, which means the process is only about one-fifth finished ([A]).
5. RAND estimates that defendants and insurers have spent $54 billion through the end of 2000 to compensate the 600,000 claimants who have come forward. Thus, these projections imply that we have seen only about half of the claims and roughly one-fourth to one-fifth of the eventual liabilities.
6. RAND estimates (thru 12/00) US insurers have paid $22B, non-US insurers have paid $8B - $12B (half of which are London), and the remainder has been paid by the defendants.
7. Bankruptcies are causing the plaintiffs bar to seek out new defendants.

II. Tillinghast-Towers Perrin projects an approximate 50/50 split of ultimate insured losses between U.S. and non-U.S insurers ([AB]).

III. Milliman projects a 70/30 split of ultimate insured losses between U.S. and non-U.S insurers ([BMR]).

IV. U.S. Net Insurance Industry (A.M. Best) as of 12/2000:

Cumulative Paid Loss and Expense	$21.6B
Stated Reserves	$10.3B
Incurred Liability @ 12/00	$31.9B
Unfunded Liability	$33.1B
Total	$65.0B

V. The Faculty and Institute of Actuaries projects a $30B - $60B total liability for non-US insurers.

VI. Average severities are decreasing and the plaintiffs bar claims that they are getting less money for their clients.

VII. U.S. Insurance Industry 2001 Note 29 (formerly Note 27) of US Annual Statement:

	2001	2000	1999	1998	1997	1996
Gross Reserves	$23.49B	$19.29B	$19.02B	$19.80B	$18.67B	$18.94B
CY Gross Paid	$3.43B	$3.54B	$5.07B	$2.35B	$2.25B	
Gross Incurred	$7.64B	$3.80B	$4.29B	$3.61B	$1.98B	
Incurred / Beginning Reserves	40%	20%	22%	19%	10%	
Gross IBNR	$12.86B					

1997 – 2001 Incurred Losses and Expenses	21,317,086,618
As a % of 1997 beginning reserves	113%

Note that the table above shows that US insurance industry gross paid losses and expenses plus increases in gross reserves in the period 1997 to 2001 are greater than the gross reserves held at the beginning of the period – by $2.4B!

In summary, an exposure based modeling approach projects the number of new claimants for the known defendants, but provides no information regarding liabilities emanating from unknown defendants. There is no way to know how many defendants there are likely to be in the future. Some analysts suggest that the majority of U.S. based companies will eventually become part of the litigation.

In the absence of the data necessary to project the emergence of new defendants and the associated costs, there are few options other than to judgmentally select a factor to apply to the IBNR, or to the total liability, resulting from the modeling process. The modeling process provides rigorously produced estimates of liabilities stemming from known asbestos defendants, taking into account the policy attachment points, limits, and prior consumption, as well as the nature of the asbestos exposure of each individual assured. It is reasonable to assume that the unknown defendants will have similar policy characteristics, but that the nature of the exposure may be different from that of the known defendants.

The resulting "pure" IBNR is very subjective, but must be driven by a desire to make a reasonable but not overly burdensome provision for this unknowable liability. For example, using a rather crude analysis, it can be shown that a "pure" IBNR provision of 50% of the IBNER is equivalent to assuming 2,000 to 6,000 future defendants.

IBNR based on "probable" future insureds[1]

It is possible to obtain lists of known asbestos defendants. It is also possible, though expensive and labor intensive, to compare such a list against the policy records of an insurer to determine if any of these known defendants are possible sources of IBNR. If the insured has enough past experience with asbestos liabilities, then there are two methods for estimating the IBNR stemming from these insureds. Both methods use historical experience to select a base line for some year, say 2002, which can then be trended into the future. A reporting pattern and a trend must therefore be selected.

It is unlikely that any of the insureds that have yet to bring claim will suffer losses as large as those of the larger well-known defendants. Therefore, it is reasonable to exclude this experience from the data.

Account Based Method

This method assumes that the insured has an estimate of annual ultimate ground-up losses for each known account. Projected annual ultimate ground-up losses are bucketed to layers, with total losses by layer calculated for each report year. The burn rate for a layer in a report year is computed by dividing the total losses in the layer by the product of the number of accounts and the width of the layer. For example, suppose the following

[1] The authors are indebted to Peter Cooper, Dave Ostrowski, and Bill Rowland for many valuable discussions on the topics contained in this section.

- The best estimate projected liabilities for report year 1995 in the layer $0 to $500,000 is $6,000,000
- There were 20 accounts with losses reported in 1995.

Then the burn rate for the $0 to $500,000 layer for report year 1995 is $6,000,000 / (20 * $500,000) = 60%.

Suppose further that

- The best estimate projected loss for the last 10 years of experience in the layer $0 to $500,000 is $20,000,000
- There were 250 accounts with losses reported during that time.

Then the burn rate for the layer $0 to $500,000 is $20,000,000 / (250 * 500,000) = 16%.

As an example, the table below illustrates the calculations for a given report year. For simplicity's sake, assume that ABC Asbestos Co, Insulations R Us and Acme Widgets each purchased a total of $40M of limits.

	Account:	ABC Asbestos Co	Insulations R Us	Acme Widgets	TOTAL	Burn Rate
	Annual Loss:	$35,000,000	$6,000,000	$48,000	$41,048,000	
Layer						
$0	$500,000	$500,000	$500,000	$48,000	$1,048,000	69.87%
$500,000	$1,000,000	$500,000	$500,000	$0	$1,000,000	66.67%
$1,000,000	$5,000,000	$4,000,000	$4,000,000	$0	$8,000,000	66.67%
$5,000,000	$10,000,000	$5,000,000	$1,000,000	$0	$6,000,000	40.00%
$10,000,000	$20,000,000	$10,000,000	$0	$0	$10,000,000	33.33%
$20,000,000	$30,000,000	$10,000,000	$0	$0	$10,000,000	33.33%
$30,000,000	$40,000,000	$5,000,000	$0	$0	$5,000,000	16.67%

Ideally, one would want a reasonable range of burn rates. It is probably best to do this for each report year and then judgmentally select the lower and upper bounds of the burn rates to be used for each layer.

The final step is to apply the selected burn rates to the policies in question. This is done by

(1) Distributing the exposure of each potentially exposed policy to the relevant layers

(2) Computing the total potential exposure for each layer

(3) Applying the burn rates to the total potential exposure of each layer.

For example, suppose the identified insureds have a total of $500M in limits for the layer $4M xs $1M. The table below shows an example of the calculation of the lower bound of the expected IBNR from these insureds for the layer $4M xs $1M. The table assumes a 20 year reporting period, with a -2% annual trend in report year losses.

Lower limit of Burn Rate for Layer: 26% Exposure in Layer: $500,000,000			
Year	% Reported	Trend Factor	Projected Ultimate Losses
1	10.4%	0.980	$13,200,299
2	9.4%	0.960	$11,717,663
3	8.5%	0.941	$10,424,130
4	7.7%	0.922	$9,291,765
5	7.1%	0.904	$8,297,446
6	6.4%	0.886	$7,421,905
7	5.9%	0.868	$6,648,983
8	5.4%	0.851	$5,965,055
9	4.9%	0.834	$5,358,567
10	4.5%	0.817	$4,819,680
11	4.2%	0.801	$4,339,976
12	3.8%	0.785	$3,912,226
13	3.5%	0.769	$3,530,194
14	3.3%	0.754	$3,188,488
15	3.0%	0.739	$2,882,427
16	2.8%	0.724	$2,607,937
17	2.6%	0.709	$2,361,460
18	2.4%	0.695	$2,139,885
19	2.2%	0.681	$1,940,481
20	2.0%	0.668	$1,760,847
		TOTAL	$111,809,413

It is probable that there would have been no accounts during the historical period with annual ultimate ground-up losses piercing layers above a certain threshold – say $40,000,000 for example. It hardly seems prudent to select burn rates of 0.0% for these layers. Therefore, burn rates for the higher layers should be extrapolated from the burn rates for the lower layers.

The total IBNR provision from the potential insureds is given by adding the IBNR of the individual layers.

Policy Based Method[1]

In [H], Haidu arrives at projected report year ultimate losses by applying a loss cost factor (he calls it "Ultimate Percent of Exposure") to the potentially exposed policy limits, but he doesn't tell us how he arrived at the policy limits. In many settings, one may be confronted by a wide assortment of attachment points and coverage amounts, making the use of a single factor for all policies problematic. Therefore, if one were to do this, one would need to 'normalize' the exposure, so that sensible results would result

[1] Bill Rowland provided the inspiration for this method

from *both* the application of this factor to the 'normalized' $45M layer share attaching at $300M and also to the 'normalized' $10M layer attaching at $1M. The normalization procedure used by this method is to multiply the layer share by the probability that a loss actually pierces the layer. This product will be referred to as the adjusted layer share:

- Adjusted Layer Share = Layer Share * Prob(Ground-Up Loss > Attachment Point).

So, the loss cost factor should be based on the ratio of report year direct losses to total report year adjusted exposure (sum of adjusted layer shares). All that remains is to compute the probability of piercing a layer. The ground up liabilities from the model of known asbestos accounts can be used to compute empirical cost distributions. Interpolation is used for attachment points not in the historical data.

The low and high adjusted layer shares are computed for each policy, and summed by report year, leading to low and high report year adjusted exposure ("low" and "high" refer to the lower and upper bounds of a reasonable range of ultimate liabilities – remember, we are assuming the existence of an account based model that produces such projections). The low and high losses for each report year are then divided by their respective adjusted exposures to arrive at the loss cost factors.

The table below contains an example of calculating a loss cost factor for a given report year.

	Policy Att Point	Pr(GUL > AP)	Policy Limit	Adjusted Exposure	Policy Liabilities
Policy 1	$1,000,000	40.0%	$500,000	$200,000	$2,500
Policy 2	$1,000,000	40.0%	$1,500,000	$600,000	$2,500
Policy 2	$5,000,000	30.0%	$4,000,000	$1,200,000	$0
Policy 2	$10,000,000	20.0%	$15,000,000	$3,000,000	$0
Policy 2	$15,000,000	10.0%	$5,000,000	$500,000	$5,000,000
Policy 2	$20,000,000	5.0%	$5,000,000	$250,000	$0
Policy 2	$25,000,000	1.0%	$25,000,000	$250,000	$1,000
			TOTAL	$6,000,000	$5,006,000
			LOSS COST FACTOR		0.834

As with the Account Based Method, the resulting cost factors must be adjusted for trend by application of the decay factors. The table below calculates the trended projected loss cost factor for a given report year, and assumes a 20 year reporting period, with a -2% annual trend in report year losses. It is important to note that these would be calculated for each report year. Judgment would be applied to select trended projected loss cost factors to be used for the identified policies.

The result of applying these two methods is a set of ranges of asbestos pure IBNR (depending on the decay rates and reporting patterns used). Judgment must be used to select a reasonable range of pure IBNR for the policies analyzed. It is highly likely that additional defendants will be named in future asbestos litigation and there also exists the potential of re-openings of closed accounts. Therefore, an addition "truly unknown" IBNR provision should be added to the above estimates.

Example of Calculating Trended Projected Loss Cost Factor

Year	% Reported	Trend Factor	Baseline Loss Cost Factor	Product
1	10.4%	0.980	0.834	0.085
2	9.4%	0.960	0.834	0.075
3	8.5%	0.941	0.834	0.067
4	7.7%	0.922	0.834	0.060
5	7.1%	0.904	0.834	0.053
6	6.4%	0.886	0.834	0.048
7	5.9%	0.868	0.834	0.043
8	5.4%	0.851	0.834	0.038
9	4.9%	0.834	0.834	0.034
10	4.5%	0.817	0.834	0.031
11	4.2%	0.801	0.834	0.028
12	3.8%	0.785	0.834	0.025
13	3.5%	0.769	0.834	0.023
14	3.3%	0.754	0.834	0.020
15	3.0%	0.739	0.834	0.018
16	2.8%	0.724	0.834	0.017
17	2.6%	0.709	0.834	0.015
18	2.4%	0.695	0.834	0.014
19	2.2%	0.681	0.834	0.012
20	2.0%	0.668	0.834	0.011
			Trended Projected Loss Cost Factor	**0.717**

The selected trended projected loss cost factors would then be applied to the adjusted potential exposure of the identified policies. For example, if the insurer had discovered named defendants with the following policies, and chose to use the trended projected loss cost factor calculated in the table above, then the IBNR from these insured would be 0.717 x $147,000,000 = $105,399,000.

Number of Policies	Policy Attachment Point	Pr (GUL > AP)	Policy Limit	Adjusted Exposure
25	$500,000	60.00%	$500,000	$7,500,000
50	$1,000,000	40.00%	$1,500,000	$30,000,000
45	$5,000,000	30.00%	$5,000,000	$67,500,000
12	$10,000,000	20.00%	$10,000,000	$24,000,000
72	$25,000,000	1.00%	$25,000,000	$18,000,000
			TOTAL	$147,000,000

Appendix IV Bulk Reserving Methods

The Survival Ratio Method

A survival ratio is the number of years that current reserves will suffice ("survive") if average future payments equal average current payments. For example, suppose an insurer has $6M in asbestos reserves. Further suppose that recent asbestos payments have averaged $1M per year. Then the survival ratio of this company is 6, indicating that reserves are adequate to pay $1M per year for 6 years.

The actuary can use this method to arrive at a reasonable range of indicated asbestos liabilities as follows.

- Use historical asbestos paid loss data to arrive at an average annual asbestos paid loss amount. This average loss amount should be adjusted to remove the effects of any larger than average payments, or the effects of years in which payment activity was unusual (e.g. due to changes in claims or litigation practices).
- Estimate the number of years into the future that such payments will be made
- Multiply the two estimates to arrive at indicated asbestos liabilities.

Suppose Company A's paid asbestos liabilities are given by the table below. The opining actuary has learned that deteriorating results during the mid 1990s led to the hiring of a latent claims specialist in 1998. This caused a slow down in payments during 1998, followed by a "catch-up" period during 1999. The actuary also discovered that there was one large gross payment of $3M in 2001, of which $2.5M was ceded to various reinsurance contracts. The claims specialist is of the opinion that such a payment is highly unlikely in the future, and that the company is aggressively settling claims with those insureds that present the most significant exposure to the asbestos loss. Policy buybacks are being pursued on all claims, with limited success.[1]

Year	Gross Paid Asbestos Losses	Net Paid Asbestos Losses	Net to Gross Ratio
1996 and prior	$8.00M	$6.50M	0.81
1997	$2.00M	$1.50M	0.75
1998	$0.50M	$0.40M	0.80
1999	$4.50M	$3.50M	0.78
2000	$1.40M	$1.15M	0.82
2001	$4.00M	$1.30M	0.33
5 year average	$2.48M	$1.57M	0.63
"high/low average"	$2.47M	$1.32M	0.50

Armed with this information the actuary creates the table below. Conversations ensue with the reinsurance department, wherein it is determined that there should be no reinsurance collection issues in the future. In recognition of the company's focus on

[1] Insurers frequently try to obtain agreements from their insureds that they will file no additional asbestos claims. Such an agreement is called a "policy buy-back". Insureds usually refuse to enter such agreements, but those with limited asbestos exposure, or with other pressures to obtain payment from their insurers sometimes will do so.

asbestos and their aggressive claims practices, the actuary selects $1M to $1.5M as a reasonable range for the average annual loss amount. Company A reinsures all GL exposure above $500K per occurrence. The actuary selects .77 to .85 as a reasonable range of the ratio of net to gross liabilities, producing a range of $770,000 to $1.25M for the average annual loss amount.

Year	**Modified Gross Paid Losses**	**Modified Net Paid Losses**	**Net to Gross Ratio**
1997	$2.00M	$1.50M	0.75
Average for 1998 & 1999	$2.50M	$1.95M	0.78
2000	$1.40M	$1.15M	0.82
2001	$1.00M	$0.80M	0.80
5 yr average	$1.88M	$1.47M	0.78
3 yr average	$1.63M	$1.30M	0.80
Selected Average	**$1.0 to $1.5M**	**$.77M to $1.25M**	**.77 to .85**

The only thing left to do is to arrive at a number of years for future claims payments. A.M. Best has begun to use a discounted survival ratio of 12 (meaning the ratio of discounted asbestos reserves to current average payments is 12). This implies that the undiscounted survival ratio is higher than 12. Let's say it is 15 (meaning 15 is "in the middle" of a reasonable range of survival ratios). It is the opining actuary's opinion that Company A will settle all of its asbestos claims a few years before the industry does, and that sometime in the next 5 to 10 years there will be a noticeable downward trend in their asbestos payments.

	Low Estimate	**High Estimate**
Average Annual Gross Paid Losses	$1.00M	$1.50M
Average Annual Net Paid Losses	$0.77M	$1.25M
Selected Survival Ratio	8	15
Indicated Gross Asbestos Liability	$8.00M	$22.50M
Indicated Net Asbestos Liability	$6.16M	$18.75M

The difficulties, advantages, and disadvantages of this method are clearly explained in [AMB].

The Market Share Method

The market share method uses the insurance company's "market share" of the asbestos arena to estimate asbestos liabilities. The market share can be based on premium or on paid losses.

Year	Industry Net Paid Asbestos Losses
1995	$1,297M
1996	$1,146M*
1997	$972M
1998	$1,038M
1999	$1,595M*
2000	$1,350M
Cumulative Net Paid Losses through 12/00	$21.6 Billion
*excludes unusual Fibreboard payments (AM Best Special Report 5/7/01)	

The table to the right shows P&C industry net paid asbestos losses from 1995 to 2000 (unfortunately, we do not have historical gross paid asbestos losses). Let us continue with our previous example. Company A's "market share" of net asbestos payments from 1997 to 2000 averaged 0.1244%, with a weighted average of 0.1322%. Company A's cumulative asbestos losses through December 2000 net paid asbestos losses as of December 2000 were $13M, so we can see that Company A's market share of cumulative net paid is 0.0604% ($13M / $21.6B). The recent market shares (with the exception of 1998) are rather high, but this has been partially explained. In light of the discussion in the section on survival ratios, a reasonable range for Company A's market share of future asbestos liabilities could be 0.070% to 0.10%.

The table below shows estimates of ultimate net asbestos liabilities for the US P&C industry from 3 different sources. These numbers indicate that future industry liabilities (for calendar years 2001 and subsequent) are between $33B and $48B. Applying the range of market shares from the preceding paragraph leads to asbestos liabilities (@ 12/2000) for Company A of between $8.75M and $29M. Now subtract net payments for calendar year 2001 of $1.3M to arrive at the range $7.45M to $27.7M.

Estimates of Ultimate Net Asbestos Liabilities for the US P&C Insurance Industry

AM Best (May 7, 2001)	$65 Billion
Tillinghast (3rd quarter 2001) http://www.towers.com/towers/services_products/Tillinghast/sizing_up_asbestos.pdf	$55 to $65 Billion
Milliman USA (3rd quarter 2001) http://www.bestreview.com/2001-09/pc_asbestos.html	$70 Billion

Another method referred to as the "market share method" relies on premium instead of losses. One problem with this is that it is very difficult to determine a particular company's market share of the GL policies sold to asbestos defendants. It is possible to determine the company's market share of total industry GL premium by year, but most companies are not exposed to asbestos losses for all years in which they wrote GL policies.

"Loss Development" Method

According to the numbers above, the remaining asbestos liabilities for the US P&C industry are between 1.546 and 2.24 times cumulative paid losses as of December 2000. Assuming that Company A's asbestos liabilities will pay out, on average, in a manner similar to those of the industry leads to ultimate liabilities of between \$20M and \$29M. One can then adjust this range based on the nature of the company's asbestos exposure. This could be called a "paid loss development" method.

As mentioned before, the opining actuary believes that Company A will settle all of its asbestos claims a few years before the industry does, and that sometime in the next 5 to 10 years there will be a noticeable downward trend in their asbestos payments. Therefore, it would be reasonable to conclude that the range of \$20M to \$29M is too high.

The methods discussed above yield the following results.

	Indicated Gross Asbestos Liabilities		**Indicated Net Asbestos Liabilities**	
Method	**Low**	**High**	**Low**	**High**
Survival Ratios	\$8.00M	\$22.50M	\$6.16M	\$18.75M
Market Share	N/A	N/A	\$7.45M	\$27.7M
Loss Development	N/A	N/A	\$20.00M	\$29.00M

It would not be unreasonable for the actuary to select a range of \$7M to \$20M for Company A's ultimate asbestos liabilities.

Ideally one should obtain gross industry paid and incurred to date data so one can apply the Market Share and Loss Development Methods to Company A's gross losses. Both Tillinghast and Milliman have published estimates of the US P&C industry's gross asbestos liabilities, so such an analysis could be performed if one could obtain the needed industry data.

Estimation and Application of Ranges of Reasonable Estimates

Charles L. McClenahan, FCAS, ASA, MAAA

Estimation and Application of Ranges of Reasonable Estimates

Charles L. McClenahan

INTRODUCTION

Until about 30 years ago, the term "range of reasonable estimates" was not generally applied to the loss[1] reserving process. While reserving actuaries were often asked, usually by management, to assess the range around the reserve values, more often than not the actuary could get away with "plus or minus five percent" as a range. The question "five percent of what?" went largely unasked. The result was a general agreement that the carried reserves were within five percent of those needed so long as the five percent could be applied, as required, to unpaid losses or ultimate losses or company assets or industry assets or GDP.

In his 1973 review of David Skurnick's paper *A Survey of Loss Reserving Methods*, Robert Anker describes three ranges: the "absolute range," which is the range from the lowest indication of any method to the highest indication of any method; the "likely range," representing the range from the lowest selected value of any method to the highest selected value of any method; and the "best estimate range."[2] I believe the development of the concept that would become the "range of reasonable estimates" started with the Anker review.

[1] The term "loss" is used herein for simplicity and should be interpreted as "loss and/or loss adjustment expense"

[2] *Proceedings* of the Casualty Actuarial Society Vol. LX, p. 59

In 1988 the CAS Board adopted the *Statement of Principles Regarding Property and Casualty Loss and Loss Adjustment Expense Reserves* (Statement of Principles) which included the following two principles:

3. *The uncertainty inherent in the estimation of required provisions for unpaid losses or loss adjustment expenses implies that a range of reserves can be actuarially sound. The true value of the liability for losses or loss adjustment expenses at any accounting date can be known only when all attendant claims have been settled.*
4. *The most appropriate reserve within a range of actuarially sound estimates depends on both the relative likelihood of estimates within the range and the financial reporting context in which the reserve will be presented.*

With the adoption of the statutory *Statement of Actuarial Opinion* on loss reserves, the Committee on Property and Liability Financial Reporting of the American Academy of Actuaries promulgated the interpretation *that a reserve makes a "reasonable provision" if it is within the range of reasonable estimates of the actual outstanding loss and loss adjustment expense obligations*, where the *range of reasonable estimates is a range of estimates that would be produced by alternative sets of assumptions that the actuary judges to be reasonable, considering all information reviewed by the actuary*.[3]

In 2000, the Actuarial Standards Board adopted ASOP No. 36 – *Statements of Actuarial Opinion Regarding Property/Casualty Loss and Loss Adjustment Expense Reserves*

[3] Property and Casualty Practice Note 1994-2, p.28.

wherein "range of reasonable estimates" is described *as a range of estimates that could be produced by appropriate actuarial methods or alternative sets of assumptions that the actuary judges to be reasonable*.

This paper will discuss the concept of a range of reasonable estimates, will describe some methods for determining ranges, will demonstrate a sound basis for the aggregation of ranges from individual line of business (or other subdivision) ranges, and will recommend a basis for the application of the range to individual loss reserving decisions.

RANGE OF REASONABLE ESTIMATES

It is unfortunate that the language of actuarial loss reserving has produced "reasonable" as the primary modifier of "estimate." Not only does it inexorably lead to the implication that all estimates outside the range of reasonable estimates are unreasonable, it also lends itself to circular definition as in ASOP No. 36, Section 3.6.4 as quoted above. The Statement of Principles language, combining *reasonable* assumptions with *appropriate* methodology to produce *actuarially sound* estimates would have been preferable.

Whatever the language, it is clear that the range arises from the uncertainty associated with the problem of estimating future loss payments and that the purpose of the range is to reflect not only the process variance but the parameter variance as well. This is clear from both the *Statement of Principles*, which seems to deal primarily with the process variance, and from the ASOP No. 36 language focusing on methods and assumptions.

While likelihood is a consideration cited in Principle 4, a sound range will not necessarily contain the most likely result. As an example, suppose that as of a reserving date an actuary estimates that there is a .01 probability of a $1 million IBNR loss on a policy, with the probability of a $0 IBNR being .99. The actuary might reasonably reserve to the expectation of $10,000 or might add some risk margin and reserve to $20,000 or $50,000. But it would not be reasonable to reserve at $0, even though it represents both the mode and the median of the loss distribution. The concept of actuarial soundness demands that we discard the answer we expect to be precisely accurate 99 times out of a hundred and adopt instead a reserve which we expect will always be wrong!

Carrying on with our example we note that the range of reasonable estimates might include neither of the only two possible outcomes. This is an important distinction. The range of reasonable estimates is not intended to include all, or perhaps even a majority of the *possible* values.

FINANCIAL CONDITION AND THE RANGE OF REASONABLE ESTIMATES

Although the Statement of Principles clearly indicates that the selected value within the range of reasonable estimates may depend upon the financial condition of the company[4] the range itself may depend upon such condition as well. Again considering our example above, in the context of a billion dollar surplus, the range might be from $0 to $20,000 with the midpoint representing the expectation and the range encompassing 99% of the

[4] Principle 4 "The most appropriate reserve within a range of actuarially sound estimates depends on both the relative likelihood of estimates within the range and the financial reporting context in which the reserve will be presented."

probability. But in the context of a million dollar surplus, that is, where the million dollar loss could render the company insolvent, $0 would probably not be a reasonable estimate for the liability. In that instance, it would be hard to contend that anything less than the expectation of $10,000 could be considered reasonable.

The fact that the bottom of the range will tend to increase with the materiality of the estimate is easily understood if we recall that the provision for uncertainty in the carried reserves should reflect not only the uncertainty of the individual reserve value, but the impact of that uncertainty upon financial condition as well. Where surplus is low, the provision for uncertainty will tend to increase and, since the range is intended to include only those values which the actuary believes would represent reasonable reserves, the range will increase as well.

METHODS FOR ESTIMATING RANGES

Assumed Allowable Deviations

As long as there have been actuarial estimates of loss reserves, there have been CEOs asking for some quantification of the accuracy of those estimates. As mentioned above, for years, actuaries were able to respond to such requests with the assurance that the reserves were "within plus or minus five percent." Regulators and the IRS also used percentage benchmarks, typically five percent of carried reserves. With this history, it is not surprising that the earliest quantifications of ranges of reasonable estimates tended to be percentages of reserves.

Unfortunately, the method does not work very well. The inherent differences between lines, for example commercial property (with high but reasonably ascertainable losses) and excess workers' compensation, require different assumed allowable percentage deviations. Calculation of the appropriate deviations by line is tantamount to calculation of the range of reasonable estimates. In addition, the requirements of the actuarial standards of practice are such that there must be a demonstrable and documented basis for a material assumption.

Alternative Methods

One common approach to the establishment of a range of reasonable estimates is for the actuary to apply multiple methods to the same line of business and to use the results to estimate the range. In applying this method, the actuary must be careful to discard any results which are inconsistent with the other indications. If the paid loss development method and the incurred loss development method are producing indications which are materially different, the difference may not constitute a range but an unexplained difference. It is also important that each method and related assumptions be individually reasonable.

The actuary using this method should also compare the results line-to-line. If the incurred loss development method always produces the low indication, it is likely that there has been a change in the underlying development – perhaps a decrease in case reserve adequacy.

Finally, this method benefits from the application of multiple and independent methods. The addition of a frequency-severity method to a set of loss development and Bornhuetter-Ferguson indications adds additional information to the process and produces a more representative range.

Alternative Assumptions

Occasionally I have seen situations in which an actuary varies the assumptions, as opposed to the methods, to establish the range. For example, the actuary will pick the highest and lowest reasonable incremental development factors at each age and use the highest to generate the high end of the range and the lowest to generate the low end. This approach produces ranges which are too wide. The probability that each age-to-age development for each accident year will be at the low end of the observed history is too low to make the resultant indication reasonable. There is, however, a method which the actuary can use to vary the assumptions and produce information which is useful in the establishment of a range of reasonable estimates. This method is the *method of convolutions*.

Method of Convolutions

The general availability of powerful computing resources has made it possible to apply techniques which would have been impossibly time-consuming in earlier times. The method of convolutions as applied to the loss development methodology is simply the application of each combination of the observed age-to-age factors to the current data and then the combination of the resultant individual year indications to produce a large number of indications arising out of the observed history.[5]

As an example, consider the hypothetical incurred loss development data in Table 1:

Table 1

Accident	Case Incurred by Age				
Year	12	24	36	48	60
1998	$1,503,839	$2,490,404	$4,266,948	$6,144,355	$6,266,584
1999	1,535,773	3,028,897	4,874,340	7,348,570	
2000	1,989,915	3,574,304	5,790,811		
2001	1,660,687	3,031,952			
2002	2,224,336				

Which give rise to the development factors in Table 2:

Table 2

Accident	Incremental Development Factors			
Year	12-24	24-36	36-48	48-60
1998	1.656	1.713	1.440	1.020
1999	1.972	1.609	1.508	
2000	1.796	1.620		
2001	1.826			

For purposes of illustration, we assume that all claims are settled by age 60. Our observed history then gives rise to 4! or 24 different combinations of development factors for the 2001 year, 3! or 6 combinations for the 2000 year, 2 for the 1999 year and 1 for the 1998 year. Combining these indications produces 24 x 6 x 2 x 1 = 288 convolutions which can be sorted into a surrogate cumulative aggregate IBNR distribution.

[5] To the best of my knowledge the first documented construction of a convolution distribution of reserve outcomes was carried out by C. K. Stan Khury *circa* 1992.

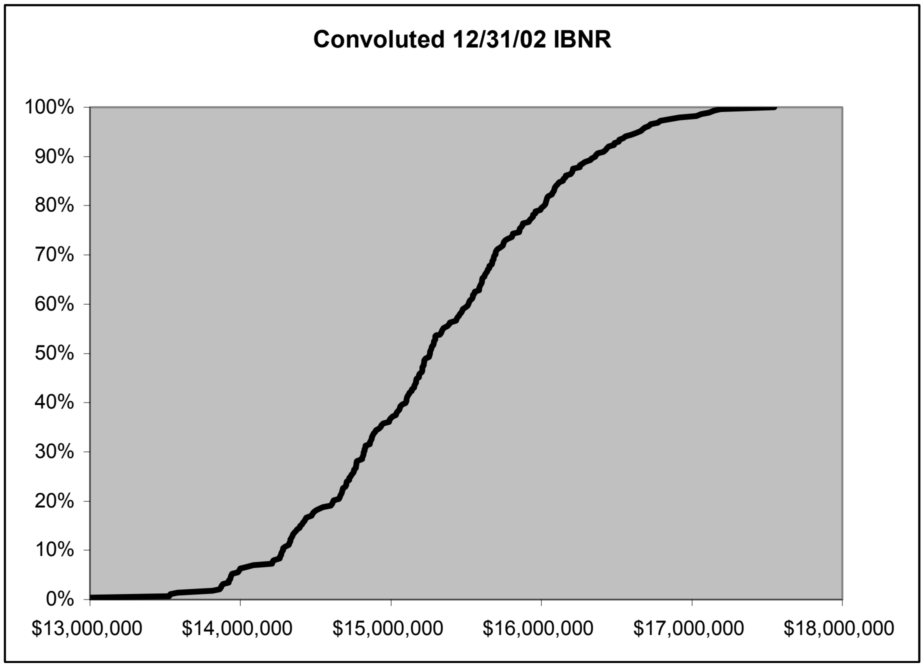


Figure 1

Continuing with our example, let's assume that we have established our "best estimate" IBNR using the unweighted average of the observed factors as shown in Table 3:

Table 3

Accident Year	Incurred Losses	Average Factor	Ultimate Factor	Indicated IBNR
1998	$6,266,584	1.000	1.000	$0
1999	7,348,570	1.020	1.020	146,971
2000	5,790,811	1.474	1.503	2,912,778
2001	3,031,952	1.648	2.477	4,478,193
2002	2,224,336	1.813	4.491	7,765,157
Total				$15,303,099

Plotting this estimate against our convolutions we see that it falls at approximately the 54th percentile, about what we would expect for a lognormal distribution.[6]

[6] The distribution of the product of independent normally-distributed random variables is lognormal.

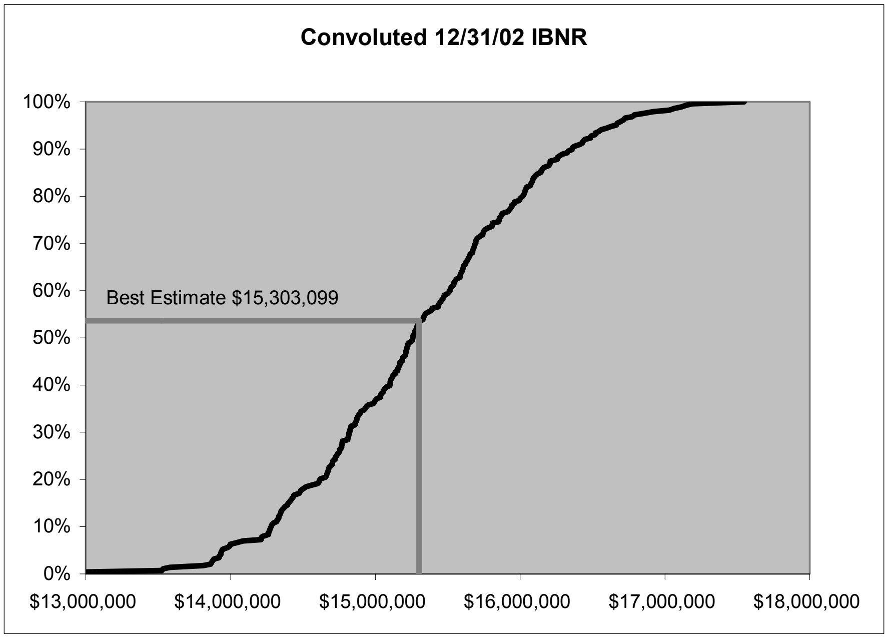


Figure 2

In a similar manner we can plot the extent of whatever we may consider a reasonable range. In this case, with the small triangle, we might select an 80% range from 10% to 90% as follows:

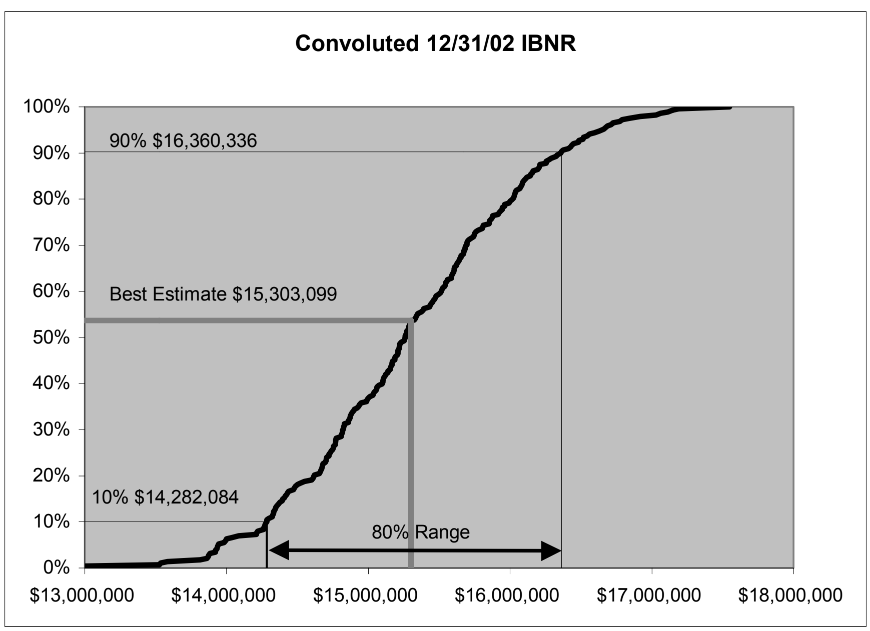


Figure 3

It is preferable that the method of convolutions be applied to several methods, not just a single method as in the above example. The convolutions from multiple methods can be combined into a single distribution of estimates.

The number of convolutions tends to get out of hand quickly. The number of individual estimates for a k-by-k development factor triangle is $\prod_{1}^{k} k!$ which is a manageable 288 for the 4-by-4 triangle of our example, but becomes 5,056,584,744,960,000 for an 8-by-8 triangle. Even the fastest of personal computers can take a while to calculate 5 quadrillion values. In such cases, it sacrifices little to limit each convolution to the youngest four by four triangle, with the 5^{th} value to ultimate being assumed as the product of the average observed 5^{th} and subsequent incremental factors.

	Age							
Year	**1-2**	**2-3**	**3-4**	**4-5**	**5-6**	**6-7**	**7-8**	**8-Ult**
1					Factor	Factor	Factor	Factor
2				Factor	Factor	Factor	Factor	CNV 1x1
3			**Factor**	**Factor**	**Factor**	**Factor**	CONVOLUTED 2x2	
4		Factor	**Factor**	**Factor**	**Factor**	CONVOLUTED 3x3		
5	Factor	Factor	**Factor**	**Factor**	CONVOLUTED 4x4			
6	Factor	Factor	**Factor**	CONVOLUTED 4x4				Avg. 8-Ult
7	Factor	Factor	**CONVOLUTED 4x4**				Avg. 7-Ult	
8	Factor	CONVOLUTED 4x4				Avg. 6-Ult.		
9	CONVOLUTED 4x4				Avg. 5-Ult			

This cuts the number of convolutions for the 8-by-8 triangle to 1!×2!×3!×4! ×4! ×4! ×4! ×4! = 95,551,488.

AGGREGATION OF RANGES

The combination of individual line of business or line and year ranges into an actuarially sound aggregate range of reasonable estimates requires some consideration. Recall that the range is of estimates, not possibilities, and only *reasonable* estimates are included within the range, the lows and highs of the individual years cannot be added to generate the range for the line and the lows and highs of the ranges for the individual lines cannot be added to generate the range for the aggregate reserve. The individual lows and highs represent the extremes of the actuary's reasonable estimates and while the low or high might be reasonable for a single year within a single line, it would not be reasonable to reserve to the sum of the lows or the sum of the highs.

If we posit a situation where we have four lines being reserved and four open accident years within each line and we assume that for each year within each line the proper reserve is either the low or the high, each with 50% probability, the chance that either the sum of the lows or the sum of the highs will be the proper reserve is $.5^{16}$ or 0.001526%. The actual distributions of estimates are such that the probabilities of the lows or highs being the proper reserves are well below 50%.

Viewed differently, suppose we were asked to estimate the range of reasonable estimates for the number of heads in the toss of ten true coins. We might select from 3 to 7 heads as our range knowing (or at least being able to determine) that we would expect this result about 89% of the time. But in a toss of 100 true coins the range from 30 to 70 heads would not be the range of reasonable estimates, comprising as it does 99.99678%

of the expected results. The range most consistent with the individual ranges of 3 to 7 would be from 42 to 58 heads out of the toss of 100 true coins which results in an expectation of about 91%. Note that adding the individual highs and lows for the years and or lines of business is the equivalent of adopting the 30 to 70 range.

How then can we combine the individual line and/or year ranges into a reasonable aggregate range? If we make the assumption that individual estimates are independent the solution is straightforward. Knowing that if x and y are independent random variables with variances $V(x)$ and $V(y)$ that $V(x+y) = V(x) + V(y)$[7] and assuming that our individual ranges represent k standard deviations (of the individual distributions of estimates) in width, then the width of the aggregate range is the square root of the sum of the squares of the widths of the individual estimates. The placement of the aggregate best estimate, the sum of the individual best estimates, is then determined by weighting the position within the range of each best estimate by the ratio of that best estimate to the total of the best estimates. An example of this process is shown in Table 4:

[7] See, for example Brunk, H.D. *An Introduction to Mathematical Statistics*, Blaisdall, 1965, p.91

Table 4

Line [1]	Accident Year [2]	Total Needed Reserves ($000) Low [3]	Best Estimate [4]	High [5]	Range Width [5]-[3] [6]	Square of Width $[6]^2$ [7]	Calculated Width √[7] [8]
Auto BI	1999	$450	$500	$600	$150	22,500	
	2000	2,700	3,000	3,500	800	640,000	
	2001	6,000	7,000	7,500	1,500	2,250,000	
	2002	9,000	11,000	14,000	5,000	25,000,000	
	Total		$21,500			27,912,500	$5,283
Auto PD	1999	$90	$100	$115	$25	625	
	2000	1,400	1,500	1,650	250	62,500	
	2001	2,800	3,000	3,300	500	250,000	
	2002	6,800	7,500	8,400	1,600	2,560,000	
	Total		$12,100			2,873,125	$1,695
Total	Total		$33,600			30,785,625	$5,548

Line [1]	Accident Year [2]	Best Estimate Weight [4]/Sum[4] [9]	Best Est. Position in Range {[4]-[3]}/{[5]-[3]} [10]	Weighted Position in Range [9]×[10] [11]	Calculated Position in Range [11]/[9] [12]	Calculated Low [4]-{[12]×[8]} [13]	Calculated High [8]+[13] [14]
Auto BI	1999	1.488%	0.3333	0.004960			
	2000	8.929%	0.3750	0.033482			
	2001	20.833%	0.6667	0.138889			
	2002	32.738%	0.4000	0.130952			
	Total	63.988%		0.308284	0.481783	$18,955	$24,238
Auto PD	1999	0.298%	0.4000	0.001190			
	2000	4.464%	0.4000	0.017857			
	2001	8.929%	0.4000	0.035714			
	2002	22.321%	0.4375	0.097656			
	Total	36.012%		0.152418	0.423244	$11,383	$13,078
Total	Total	100.000%		0.460702	0.460702	$31,044	$36,592

In our simple example in Table 4, the total range of reasonable estimates around the aggregate best estimate of $33,600 is from $31,044 to $36,592.

We know that the individual estimates are not strictly independent. Where traditional loss development methodology is used, incremental development assumptions affect multiple years producing some correlation between estimated ultimate losses for years. Court decisions, regulatory climate and economic conditions impact ultimate losses for multiple lines. For the most part, however, these are outweighed by the independent stochastic nature of the observed frequencies and severities which form the basis for the projections and the fact that it is the unpaid losses which are the subject of the range. Given the computational difficulties introduced in any attempt to measure and reflect the covariance matrix in the aggregation of the ranges, the assumption of independence seems a reasonable approach.

APPLICATION OF RANGES

Having established a basis for the determination of actuarially sound ranges of reasonable estimates, it is natural to turn to the application of those estimates. In order to examine the question of how to apply the concept of a range of reasonable estimates it is important to understand that while each reserve value within the range is presumed to be reasonable, that is meets the ASOP requirement that it *could be produced by appropriate actuarial methods or alternative sets of assumptions that the actuary judges to be reasonable*, not all values within the range are qualitatively equal. The low and high values represent the demarcation between presumably sound and presumably unsound reserves, and the actuary establishing a reserve should adhere to the requirements of the *Statement of Principles* and should consider *both the relative likelihood of estimates within the range and the financial reporting context in which the reserve will be presented*.

The proper application of the range will depend to a great extent upon the "ownership" of the estimate. If the actuary is opining upon the reasonableness of a carried reserve which the company has already established without knowing the results of the opining actuary's analysis, and that reserve is within the opining actuary's range of reasonable estimates it is deemed reasonable. We refer to such a reserve as being "untutored."

If, however, the company knows the results of the opining actuary's analysis before establishing the reserve and then selects a reserve at the low end of the opining actuary's range, the company no longer "owns" the estimate. In such a case, if the reserve is not one which the opining actuary would have established in accordance with the *Statement of Principles*, it does not represent a reasonable reserve. To allow the low end of the range to serve as a target reserve is to subjugate the opining actuary's best estimate to his or her view of what reserve might be established by a hypothetical actuary using different methods and applying different assumptions.

Monoline Insurance & Financial Guaranty Reserving

James P. McNichols, ACAS, MAAA

MONOLINE INSURANCE

& FINANCIAL GUARANTY RESERVING

James P. McNichols, ACAS, MAAA

Abstract

'Twas brillig, and the slithy toves
Did gyre and gimble in the wabe;
All mimsy were the borogoves,
And mome raths outgrave

\- *Jabberwocky*, Lewis Carroll (1872)

Mr. Carroll's penultimate foray into language and verse that beautifully skates the thin ice between comprehensibility and nonsense had a certain relevance in my early days in the financial guaranty business. This was all I could think of during my first financial guaranty credit underwriting committee meeting. The thesis and content of the credit risk/return debate seemed vaguely within reach but the tenor and rules were entirely alien. It was soon evident that understanding this business model would not just be a matter of deciphering similar functions and concepts by transitive conversion. It was clear that an entirely different arena was in play with foreign registers and constraints.

1. INTRODUCTION

This paper describes a practical approach to reserving financial guaranty risks. It is intended as a primer for property/casualty actuaries in the basic risk principles and business models of financial guaranty insurance. It is requisite to review the underwriting and pricing theory and other practices of this trade. An additional goal is to highlight several areas that will likely benefit from the application of traditional and alternative actuarial techniques.

2. BACKGROUND

A. Insurance

The financial guaranty industry began in Milwaukee, Wisconsin in 1971 when MGIC Investment Corp. convinced an Alaska municipality to purchase an insurance guaranty policy from a highly rated insurer to "wrap" (i.e. guarantee) the principal and interest on its first ever debt issue ($650,000) of general obligation bonds for a medical arts building and an adjacent sewage treatment facility. The incentive for the local government was to reduce its overall borrowing costs. They were right. It did.

A small number of credit insurers emerged that would provide an indemnity against the default risk of investment grade rated public finance debt issuance. They became known as monoline financial guaranty ("F/G") insurers since they only underwrote this unique risk (and in some jurisdictions were precluded from underwriting anything else). The operating thesis was that given sufficient security from existing revenue flows and considering the taxing authority available in support of many public finance debt issues, no municipal bond as defined would ever ultimately fail to pay interest or principal. Rather, a debt restructuring would likely be negotiated and any potential insurance loss would simply be limited to the *cost of carry* (i.e., bridge financing during the negotiation phase).

The financial guaranty industry has since grown into a major source of credit enhancement. Financial guarantee insurance provides investors with guaranteed payment of timely interest and ultimate principal in the event that a debt issuer is unable to meet its financial obligations. The insurance guarantee is irrevocable and unconditional (and waives all defenses, including fraud) and results in the guarantor stepping into the shoes of the issuer in that it guarantees payments in accordance with the original transaction schedule on a timely basis. In the event the issuer fails to pay the coupon and/or principal on a timely basis the investor has recourse to the F/G insurer who will pay the timely interest and/or ultimate principal in accordance with the terms of the affected bond. This is a significant departure from the P&C business whereby a claim is made and negotiations begin as to what extent the claim is deemed valid. In F/G insurance you pay the investor now and argue with the issuer later. Absent that type of insurer performance, (known as a "capital market" standard), investors would have no incentive to buy "wrapped" bonds.

The established primary financial guarantors are rated AAA (or their equivalent) by each of Standard & Poors, Moodys and Fitch[1] and, by virtue of the guarantee, securities they wrap inherit their AAA rating.

[1] Standard & Poors, The McGraw-Hill Companies.
Moodys Investors Service.
Fitch IBCA, Duff & Phelps, a subsidiary of Fimalac.

Such AAA ratings provide the issuer with reduced borrowing costs (as the pricing benefits outweigh the cost of the guarantee) and better marketability of the bonds. As a general rule, monolines target roughly 2/3rds of the available spread as the required insurance premium. Investors benefit from enhanced security and liquidity of the insured bonds. They also benefit from the credit monitoring expertise of the guarantor and the comfort that the insurer is sharing the risk by lending its credit quality to the issue.

The most important strengths of the primary monoline insurers are their ratings. As a consequence, they work closely with the rating agencies to preserve them. Capital adequacy and solvency obviously play a key role in the rating agencies' credit assessments. In addition, rating agencies require that all potential transactions be of investment grade quality (i.e., at least BBB- or equivalent) before any insurance wrap is considered. Therefore, each transaction generally receives a "shadow" (non-public) rating by at least two of the three major rating agencies and, thus, a full deal rating agency review.

One of the more noteworthy regulations for the monolines is the New York Financial Guaranty Insurance Law (Article 69). The law establishes, amongst other things, the single risk limits applicable to all obligations issued by a single entity and backed by a single revenue source. Such limits are specific to the type of insured obligation (for example, municipal ("Muni") or structured-finance ("S-F") bonds (i.e. ABS, CMBS, CDO, etc...)). The limits compare the insured net par outstanding (for S-F) or average annual debt service (for Muni), as applicable, for a single risk to the insurer's qualified statutory capital, which is defined as the insurer's policyholders' surplus and contingency reserves.

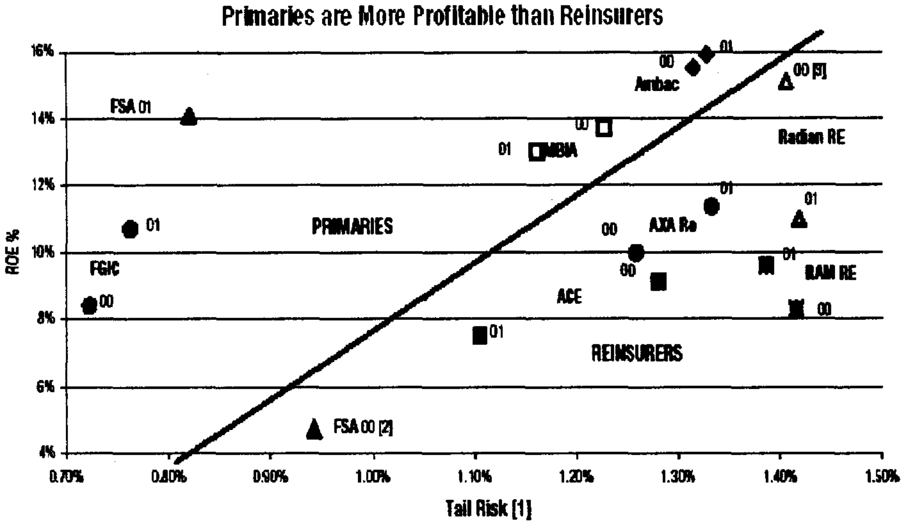
B. Reinsurance

Once the monoline insurance market began to mature, the primaries had a need for reliable and committed sources of reinsurance. Through simple quota share treaty support they could effectively leverage their capital bases. A small number of AAA monoline reinsurers emerged. These were basically passive, low cost operations that followed the fortunes of the primary insurers and embraced the concept of underwriting the underwriter.

Over time, however, the relationship between primary and reinsurer has changed and their interests became misaligned. F/G insurers had used reinsurance for risk management and portfolio shaping purposes. Currently, the F/G reinsurers are viewed as one possible option from several alternatives to effect capital and risk management solutions, putting the established reinsurers at a competitive disadvantage.

Graph 1 below demonstrates the dichotomy that currently exists in the relationship between the primary insurers and F/G reinsurers. As the primary insurers increased their capital leverage, at expense levels less than they charge, the reinsurers margin of safety was directly eroded. This results in a bi-modal distribution whereby the insurers systemically retain a better risk/return distribution.

Graph 1



1) Tail risk defined as loss at 99.9 percentile as a percentage of adjusted net par.
2) FSA's weak 2000 ROE reflected restructuring charges following the acquisition by Dexia.

Moody's[2] has recently published a monograph on the state of the F/G reinsurance market which provides an excellent overview of the risk/return thesis and other key issues affecting this business segment.

3. DIFFERENCES FROM PROPERTY & CASUALTY INSURANCE

The following highlights and explains several key areas. Throughout this paper the terms guarantor, insurer, monoline, F/G insurer, and the primary are all used interchangeably to reference a primary monoline financial guaranty insurance company.

[2] Moody's Investors Service "The End of the Monoline Financial Guaranty Reinsurance Sector?" (Special Comment December 2002)

A. Written and Earned Premiums

Muni risk exposures have relatively long terms (i.e. tenors) until final maturity. Most Muni bonds have final maturities that extend 20 to 30 years. Insurance premiums in the Muni area are in the form of non-refundable, upfront premiums, meaning that the full amount of the premium is paid at the time of the issuance of the guaranteed bonds. Under regulatory and GAAP constraints, the written premiums that have been paid become "earned" or recognized over a long time, according to a specific risk amortization schedule. The purpose of this accounting is to link the premiums paid to the average life of the "wrapped" obligation in order to provide for the fiscal stability of the F/G primary insurance company. A portfolio of Muni bonds will typically demonstrate aggregate straight-line amortization characteristics as the mixture of means tends to distribute uniformly across the book. Consequently, an in-force portfolio with average maturity of 20 years will have an average life of roughly 10 years (or one-half the legal term).

The total portfolio of pre-paid Muni deals results in a large unearned premium reserve (UEPR) which is recognized as earned premium over time as these long tenor obligations amortize. Changes in growth rate and earnings rate of the UEPR are critical estimates for the management of these books of public finance bonds. The UEPR is recognized as hard capital (i.e. cash or cash equivalent) for rating agency capital adequacy modeling since there are no conditions to its recognition except the passage of time. The actual recognition of the UEPR in reality is faster than the estimated accrual largely due to the incidence of bond refinancings during periods of lower interest rates.

If premiums are not paid in full at the beginning of the transaction, then they pay in installments (e.g., monthly or quarterly in arrears) over the life of the insured credit obligation. This is the typical method of payment for S-F deals. S-F deals usually have much shorter tenors, typically ranging from 3 to 12 years. While this money has not yet been received by the F/G primary insurance company, it represents a contractual annuity-like stream of money that will become paid in capital over time.

There is some risk in the F/G market that these future written premiums will not materialize. To mitigate this risk in structured finance deals, the flow of funds from the assets may be arranged so that the payment of premiums will come out of the available cash once payments to bondholders and other priority claims are made. In other words, the risk premium is obtained from siphoning off a portion from the available cash flow within the structured "waterfall" of payments.

B. Adjusted Gross Premium ("AGP")

The present value of the future installment premiums is an important statistic and when added to earned premium to date results in AGP for a given origination year. That is, cumulative premium earned to date plus the present value of future installment premium equals AGP. The estimated total AGP for an in-force risk portfolio contributes to the balance sheet capital strength. It is considered a highly secured receivable and almost the entire amount is contributed as soft capital in rating analyst capital adequacy models. Subtracting from AGP the present value of expected underwriting and operating costs, as well as the estimated ultimate loss costs, results in an estimate of the economic value added. Typically, F/G underwriters are subject to budgeted amounts of expected AGP production per year. It is an efficient yardstick of deal production since it directly impacts growth in future earnings.

C. Adjusted Book Value

The stated Book Value ("BV") of an F/G insurer equals Capital & Surplus.

Adjusted Book Value (ABV) = BV + (PV of Future Installments) + UEPR.

It is growth in ABV that Market rating analyst's view as a credible proxy for growth in future earnings.

For mature portfolios the annuity-like earnings stream that derives from the in-force portfolio yields a stable growth in earnings pattern. Thus, it is not uncommon for mature F/G insurers to predict in advance up to 90% of subsequent period earned income. This type of stability in earnings growth promotes high relative multiples of the market value of equity over the book value of equity for publicly traded insurers.

D. Principal and Interest

All debt obligations are denominated in terms of principal (Par) and interest (Coupon) payments. There is usually a set schedule for the amortization of the debt but in several areas such as Asset Backed Securities ("ABS") the amortization schedule is variable and depends upon pre-payment levels, actual default experience and realized excess spread amounts within the structure. For ABS, an expected principal and interest (P&I)

schedule is established at inceptions and revised as appropriate if material volatility is observed.

Par Outstanding is the most common denominator used when disclosing notional risk exposure amounts or calculating capital charges. Principal & Interest ("P&I") is more often the reference numerator when calculating the relative leverage implicit in the portfolio.

E. Leverage

Total P&I divided by Total Hard Capital equals Leverage.

For example assume a monoline insurer with $15 billion par outstanding exposure, split $10 billion Muni and $5 billion S-F risk, and total interest obligations equal to $7 billion, (thus P&I equals $22 billion). If the insurer holds hard capital of $200 million then it retains a book that is Leveraged 110 to 1 (i.e. P&I / Hard Capital = $22 billion/ $200 million = 110).

Monolines are able to operate at much higher leverage amounts than many other financial markets owing to the fundamentally low-risk nature of their insured portfolio as well as the limited liquidity requirements they face. A typical book of Muni risks will run at leverage levels of 175 to 225 times hard capital and S-F books at 125 to 150 times. High leverage can be assumed because of the low credit risk assumed.

Table 1 below summarizes the Operating Leverage Statistics of the four largest established Primary Insurers as of Sept. 30, 2002

Table 1

	Qualified Statutory Capital	Debt Service Insured Ratio
Ambac	$3,597,000,000	146
MBIA	$5,326,000,000	143
FSA	$1,728,000,000	204
FGIC	$2,094,000,000	153
Weighted Average		153

Source: Bank of America Securities, Research Brief, Bond Insurance Monthly, January 2003

The risk/return strategies among the primaries have diverged since the business diversified away from its Muni origin in the late 1980's. At that time they all had similar risk portfolios at similar levels of leverage.

This highly leveraged capital model is not unique to financial guarantors. Nonlife insurance products are, in effect, derivatives (swaps and put options) that can accumulate risk to the seller in a highly leveraged manner. The guarantor leveraged capital model is also similar to catastrophe-exposed homeowners' insurers that do not buy catastrophe reinsurance or purchase reinsurance from companies facing similar risks.

F. Risk Amortization

Tracking the amortization of the in-force par risk is important to monoline insurers for a few reasons. First, it allows the insurer to monitor premium payments and forecast future embedded economic value. Secondly, it determines the premium earnings rate for GAAP income purposes. Also, it provides a credible input into the estimation of the likely loss emergence pattern.

P&C insurance companies book premium received and earned in that underwriting period, but tail losses (and specifically latent loss liabilities) can emerge at distant future dates with little predictability. However, in the F/G business, as the credit obligation decreases with time, we observe an unbiased estimator of decreased loss potential which absolutely terminates (i.e., no tail risk exists) at final maturity. As such, demographic sorts by asset class of the average life statistics on F/G risk portfolios provides excellent surrogate "*a priori*" indicators of loss emergence probability. Herein lies the concept of **predictive latency**. As the observations from a given origination year increase with the passage of time, we obtain improved knowledge of the remaining loss potential. It partially relates to the increased credibility that derives from observing actual experience to date. However, it is different from latent P&C risks where tail risk predominates the uncertainty associated with estimates of the remaining unreported loss. Conversely, F/G risk falls away precipitously as the issues mature. The ultimate performance of the portfolio of structured debt obligations becomes more and more certain as the par risk outstanding unwinds.

Based on current information and prior knowledge, Philbrick's[3] approach would expect the credibility attached to the current observations to increase with:

- Increasing number of observations (i.e. the par risk continues to burn off);

[3] Philbrick, Stephen W. An Examination of Credibility Concepts. PCAS LXVIII, 1981

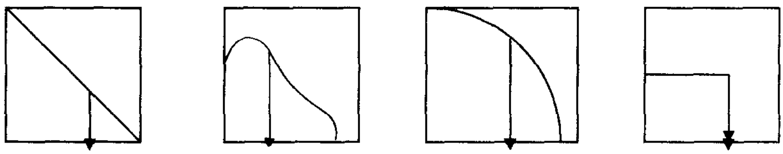
- Decreasing process variance (i.e., the remaining probable losses are more closely bunched together than at time = zero); and
- Increasing variance of the hypothetical means (i.e., the remaining probable losses by product type produce means that are farther apart than at time = zero.)

G. Outstanding Average Life

Typical examples of risk amortization patterns and their corresponding average life estimates are provided in Figure 1 below.

Figure 1

Muni	Consumer ABS	Mortgage ABS	Bullet Amort.



Term = 20 yrs	Term = 5 yrs	Term = 25 yrs	Term = 3 yrs
Avg Life = 12 yrs	Avg Life = 2 yrs	Avg Life = 15 yrs	Avg Life = 3 yrs

Average life = Sum {(par payments) x (time index)} / Sum {(par payments)}.

This par weighted index of the undiscounted midpoint of the risk amortization period is an important statistic. The present value of average life yields risk duration.

Average Outstanding Life =

$$\frac{\text{Sum }\{(\text{remaining par payments}) \times (\text{time index})\}}{\text{Sum }\{(\text{remaining par payments})\}}.$$

The present value of average outstanding life is analogous to the concept of curtate expectation from life contingencies (except that rate q(x) is replaced by a risk amortization rate). That is, given the observed performance of the credit to date, we actually have better information regarding its loss propensity over the remaining life than we did at risk inception. For example, given that you have survived to age 45, your curtate expectation for future longevity is reset to 40 more years. This risk-adjusted life expectancy estimate of 85 years exceeds the original life expectancy of say, 75 years established at birth (time = zero). Also, the confidence in the curtate expectation has increased. Similarly, in F/G risk, given the structure has performed as expected to the current observation point (i.e., survived), the confidence associated with the remaining expected default (i.e., death) potential has increased relative to that expected at inception. This is the inference of predictive latency.

H. Loss Payment Acceleration

In the event of a default on a F/G obligation, monolines are required only to pay timely interest and ultimate principal. That is to say, the F/G insurer is only required to pay interest and amortization payments on the defaulted obligation as they come due. New York insurance law does not permit the company to guarantee obligations that accelerate in the event of default. Article 69 of New York's Insurance Law regulates "financial guaranty insurance," which is defined in section 6901(a), as insurance

> where a loss is payable upon failure of any obligor on or issuer of any debt instrument or other monetary obligation (including equity securities guarantied under a surety bond, insurance policy or indemnity contract) *to pay when due to be paid by the obligor or scheduled at the time insured to be received by the holder* of the obligation, principal, interest, premium, dividend or purchase price of or on, or other amounts due or payable with respect to, such instrument or obligation, when such failure is the result of a financial default or insolvency or, provided that such payment source is investment grade, any other failure to make payment, regardless of whether such obligation is incurred directly or as guarantor by or on behalf of another obligor that has also defaulted.

This prohibition against guaranteeing accelerating obligations is very significant for F/G insurers since the leverage present in their capital structure limits their ability to cover large losses on short notice. That is, monoline insurers are not geared for unpredictable liquidity calls.

I. Credit Default Swaps

Accounting standard SFAS 133 defines a derivative thus:

A derivative instrument is a financial instrument or other contract with all three of the following characteristics:

a. It has (1) one or more underlyings and (2) one or more notional amounts or payment provisions or both. Those terms determine the amount of the settlement or settlements... and in some cases, whether or not a settlement is required.

b. It requires no initial net investment or an initial net investment that is smaller than would be required for other types of contracts that would be expected to have a similar response to changes in market factors.

c. Its terms require or permit net settlement, it can readily be settled net by a means outside the contract, or it provides for delivery of an asset that puts the recipient in a position not substantially different from net settlement.

There are several general types of derivatives which include forwards, futures, options, swaps, caps, floors and collars. It is the interest rate, currency and credit default swap categories which F/G insurers have entered.

In a swap, both parties exchange recurring payments with the idea of exchanging one stream of payments for another. The credit default risk inherent in collateralized debt obligation (pools of corporate bonds or loans) transactions is often swapped through an International Swaps & Derivatives Association (ISDA) contract. This has become an area of investor focus, as has the underlying accounting for these transactions. In general, credit default swaps and the guarantees on collateralized debt obligations are considered derivative instruments per SFAS 133 for accounting purposes. As such, they must be marked to market, with the resulting economic gain or loss flowing through net income.

J. Mark to Market ("MTM") Accounting for Financial Instruments

MTM is an accounting method that relates to how traders calculate their trading gains and losses (the amount calculated) and how these gains and losses are reported (characterized) on a trader's income statement. MTM refers to the procedure F/G insurers follow at quarterly close, when they mark all open derivative positions to market prices evaluated at the last day of the close period. In effect a sale is imputed of all open positions (long and short positions). MTM is sort of like the "accrual method of accounting" in the sense that the "economic" reality (in deference to the cash reality) is reported on the income statement in the form of "realized" and "unrealized" gains and losses.

It is understandable that the monolines view the MTM income adjustment as temporary. Indeed, many MTM adjustments caused by widening market spreads on performing S-F credits "zero out" when the guarantee expires. Why then, F/G insurers argue, do they need to introduce volatility to the loss reserves and premium earnings where it does not in fact exist unless there is a permanent impairment in value? They assert that if the structures perform, then the interim mark provides a simple proxy for current market pricing and yields artificial profits as the deals mature. The monoline insurers do not view the underwriting risk any differently than if the risk had been executed as an F/G insurance policy. Consequently, they hold the open positions to maturity and thus any interim "imputed" adjustment is not particularly relevant to potential ultimate losses.

The primary insurers also assert that the mark-to-market should not be viewed as a consensus market measure of the required loss reserves on those policies. The capital market presumption with which the primary insurers do not agree is that changes in surrogate index market spreads across a portfolio of such trades provides an efficient predictive estimator for the risk adjusted capital charge implicit in these structured pools of largely corporate credit risk. As will be discussed further in the reserving section, the events that precede default on any credit enhanced bond are likely non-random and highly correlated. Suffice it to say that, at best, this would be an inefficient estimator of any such risk charge. At the discrete level (case specific) the individual MTM adjustment as described is not a credible estimate for expected case specific reserve liability. In the event that an S-F deal becomes distressed to a near loss likelihood, the best estimate of future liability depends upon the outcomes of several dependent, non-random events.

For example, given an S-F pool of corporate debt that is sufficiently distressed by a prolonged period of elevated corporate defaults, there are usually at least three parties that would prefer to remedy the debt issuance rather than force declaration of a default. These are the debt issuer, the investment banker/broker and the F/G insurer. In the case of the debt issuer it is clear that having to claim under the F/G insurance policy will impair its subsequent costs of borrowing. The investment banker that brokered the deal seeks to avoid impairment to its reputation from having structured a deal that failed. The insurer has an obligation to pay timely interest and ultimate principal but is concerned about whether investors who purchase its wrapped paper may demand a higher spread if it becomes known that it has recently underwritten some defaulted credit. Consequently, a whole myriad of workout proposals may be tabled and agreed in advance of declaring any default. These economic forces converge such that the case specific claims process for most F/G insurance is dependent and non-random.

K. SURVEILLANCE

As indicated earlier, the monolines only consider underwriting credit risks that are of investment grade quality. At inception, the probability of default on Muni and S-F bonds is very low and in fact in most cases the cumulative chance of loss is less than 1 in 100. However, some deals do underperform and the stress can trip performance triggers within the structure. This migration in credit quality is cause for concern to the primary monoline. These insurers have surveillance professionals whose job it is to monitor the on-going performance of each credit. Although the specific scales vary, a credit impairment hierarchy exists to segment the portfolio as follows:

1. Performing credits with little or no need to actively monitor.
2. Performing credits with complex triggers that necessitate active monitoring.
3. Underperforming credits but with sound structure and active monitoring. These are called Caution List Credits.
4. Underperforming credits with a distressed structure and active remediation status. These are called Watch List Credits.
5. Distressed credits in which a default is imminent and/or losses are probable and estimable. These are called Loss List Credits.

4. UNDERWRITING

Credit risk is the common exposure throughout the monoline business and the entire range of financial guaranty products. However, in Muni and across S-F transactions it manifests in differing ways. The underwriting resources in this market typically come from a banking credit and/or capital markets trading background. As such the credit risk structuring rules and risk selection criteria derive from understanding the risks and designing or structuring the mitigants to each discrete risk under consideration. The following summarizes the key factors by type of product.

A. Municipal Bonds

These can be either general obligation bonds ("GO") (i.e. municipalities backed by the tax raising ability of the local government) or revenue bonds (where P&I is paid from cash flows of a specific project or site such as a highway toll, sewage plant, hospital, school board, etc.). Some of the larger debt issuers include California, NY, and their local governments and agencies. Average life is usually greater than 15 years but there is a low risk of default and high recovery upon default. All risks are investment grade (unless subsequent credit migration to BB+/Ba1 or lower which would result in immediate placement on the surveillance watch list).

The major types of Credit Risks include:

1. State obligor or municipality (function of tax paying ability of residents).

2. Revenue bond (function of volume or usage at a specific site).

The Surveillance Monitoring includes:

1. S&P Rating, Moody's Rating, capital charge, internal rating.
2. Single name exposure as a percent of capital base. Exposure could also be monitored by state, type, rating, term.

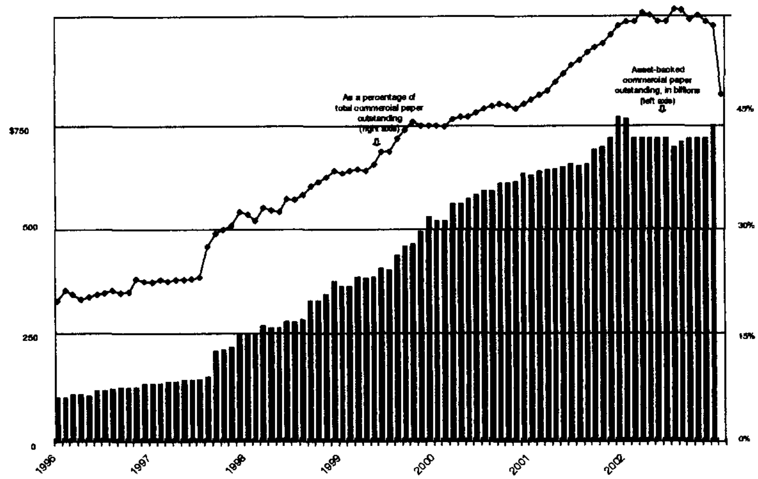
B. Asset Backed Securities

Generally defined as a financial guaranty of P&I obligations (bonds) backed by pools of illiquid assets such as credit card loans, residential mortgages, auto loans, equipment leases (including aircraft), small business loans, timeshare loans, etc.

In theory, the credit risk of the loan originator/loan servicer is structured out of the deal; in practice, the transition to a replacement servicer is not always smooth and some decline in asset value during transition to a replacement servicer is possible. This is generally a US-based business, but is expanding to Europe, Australia, Japan, South Korea and Latin America.

Graph 3 below summarizes the size of the market in Asset-backed commercial paper as compared to total commercial paper outstanding.

Graph 3



Wall Street Journal, December 20, 2002

C. Collateralized Debt Obligations

Financial guaranty of debt obligations (bonds) backed by a diversified pool of corporate loans or corporate bonds (which individually may be either investment grade or non-investment grade). Issuers include both investment management firms seeking to grow assets under management, normally through capital market issuances; and financial institutions seeking to hedge their corporate exposures and/or to lower required bank capital allocated to such exposures, normally through a "synthetic" transaction evidencing the risk transfer through a credit default swap. Assets may also include ABS bonds, catastrophe (P&C risk) bonds, other Collateralized Debt Obligation ("CDO") debt, venture capital loans or private equity, and emerging market corporate or sovereign debt. These pools of securities are not likely to contain muni bonds, since their tax-free lower yields do not provide sufficient rate arbitrage. These instruments function like a leveraged version of an institutionally financed debt mutual fund. Through diversification, over-collateralization, subordination and cash trapping triggers embedded within the structure of the excess cash, investment grade ratings of the CDO debt are possible, even if the underlying collateral is below investment grade. There is a wide array of associated risks and other issues which include:

1. Single name risk within the CDO, although there is no loss payment until the first loss protection is eroded; depending on the structure, the deductible could cover numerous individual defaults.
2. Asset manager could be a bank (originator) or a portfolio manager – there is no direct risk other than a performance risk. Assets are held by a collateral manager or trustee.
3. Some trading of individual names is possible so the risk portfolio will change dynamically and reporting lag is variable.
4. CDO debt is rated. Each asset within the CDO is rated or shadow rated by at least one rating agency.
5. Assets within the CDO are monitored by rating (cash flow structure) or by price and liquidity (market value structures.
6. Aggregates are managed by industry and by geography to avoid concentration risks.

These credit types are monitored by name of CDO, the single name obligors within each pool, capital charge, type/rating.

D. Project Finance/Infrastructure Finance

Financial guaranty of P&I on debt used to finance essential infrastructure projects in the areas of power generation, highway toll roads, water treatment, etc. This may include quasi-utility supported type obligations. Typically structured to be non-recourse or limited recourse to a corporate sponsor but not near the same degree of isolation from bankruptcy risk of the sponsor as is implicit in ABS deals.

A matrix of credit risks relate to this guaranty including corporate risks/entities – off-take purchasers (customer of project), suppliers of raw materials, maintenance company, developer during construction, insurance company for insurance proceeds, etc. Extensive structuring makes these deals much more akin to ABS but implicitly Project Finance exposure is single risk so typically it often is grouped in Muni risk terms together with the banking/ legal/ sovereign risks.

E. Future Flow

Financial guaranty of P&I on financial-based flows of debt obligations backed by future cash receipts collected offshore which result from the sale, typically of a homogeneous export commodity (e.g. oil, copper and gas) or certain financial transactions (airline ticket receivables, credit card receivables, wire remittances etc.). Sponsor/servicer is typically a local blue chip corporate in a near-investment grade sovereign country, which can use future flow structuring to achieve an investment grade-rates transaction, which a monoline in turn can enhance to AAA. Transactions employ offshore, bankruptcy remote special purpose entities ("SPE's"). This eliminates sovereign interference. Purchasers or financial counterparties sign irrevocable payment instructions, agreeing to pay US dollars directly to the offshore trustee. The structures are designed to permit debt issued at a higher rating level than that of the country in which the issuer is located. That is, the intent is to pierce the "sovereign ceiling" of the country rating through a structured credit.

There is performance risk on the sponsor rather than a direct credit risk. In other words, even if the sponsor is bankrupt, so long as it continues to sell products, cash will be generated to service the debt. Offshore purchaser of the exported product is under a long term contract.

F. Other Products

There are several emerging product areas which include:

- Sub-prime credit card receivables
- CDOs with municipal collateral
- Alternative student loans
- Business owner/operator loans
- Various types of leases
- Trade receivables
- Structured liquidity guarantees
- Structured investment vehicles

5. PRICING

The pricing of F/G products is not actuarially derived but rather based on capturing the majority of the available spread between the yield the issuer must pay with and without a surety wrap. In the ABS market it is estimated that roughly 1/3 of all transactions are wrapped by AAA monolines. Investors view surety wraps as appropriate for volatile collateral or that without a long performance history. Investors must also be careful to factor in early call risk that is often deemed to be low but is not nonexistent.

Monoline pricing constraints are clearly different from P&C since the monoline's highest priority is maintenance of its AAA ratings. Subject to this 3rd party constraint F/G insurers seek to maximize profit and optimize return on equity (ROE). Thus, the pricing paradigm for F/G insurers focuses on incremental risk capital requirements and the associated ROE. The business is ultimately a function of risk management (i.e. underwriting) and capital management.

Capital charges are attempts to measure transaction risk *within the context of a portfolio*. Consequently, the sum of the individual capital charges is not a reasonable proxy for the resulting capital allocation on the total risk portfolio. As used by Standard and Poor's[4] in the capital adequacy testing of bond insurers, capital charges forecast the level of losses that would be expected in a worst-case scenario. These worst-case scenario losses (net of reinsurance) are one input in the capital adequacy model. The other major inputs include new business growth, premiums written, net income, premiums earned, operating

[4] S&P Bond Insurance Book 2002, Understanding the Bond Insurance Capital Adequacy Model, pp 34-41.

expenses, investment income, asset sales, policyholders surplus, contingency reserves, asset carrying value, and dividends to holding company.

The primary output of the model is the ending statutory capital that in turn yields the margin of safety ratio. A margin of safety of 1.25 times signifies that ending capital (i.e. in a hypothetical wind-down scenario) exceeded losses by 25%. Stated another way, losses could have been 25% larger without driving the statutory capital below zero. The stated minimum margin of safety for 'AAA' rated bond insurers is 1.25 times and 1.00 times for 'AA' rated insurers.

In order to calculate a deal specific "return on equity" estimate, monoline insurers have developed an elegant shortcut to running the entire stress model each time a new transaction enters the existing risk portfolio. Rather, they begin with the capital charge but adjust it for the offsets provided by income flows and claims paying ability. The algebra reduces to an interaction among the debt service, cap charge, and risk leverage. The derivation of this formula as well as other credit risk and market risk pricing concepts are not the focus of this reserving paper. A subsequent pricing paper may provide analyses of the theory and practice of portfolio credit models and review actuarial approaches that apply.

There are several areas on the structured finance side that benefit from the application of traditional actuarial methods. In particular, consumer ABS products involve numerous cash flow and asset value distributions. Data availability and credibility are usually high.

Structuring depends heavily on time series analyses of historical pool performance. These mean regressive wave indications are used as a reference when setting the critical values of deal performance triggers to be embedded into the structure. The goal is to create a structure that demonstrates that the deal could withstand some multiple of the expected stress levels and still hold up under such pressure. These protection multiples often dictate the rating agency viewpoint. In the example we are about to review the letter ratings are determined as follows.

AAA	3.75 or greater times expected
AA	3 .00- 3.75 times expected
A	2.50 – 3.00 times expected
BBB	2 .00 – 2.50 times expected

Protection multiples and letter ratings are directly related but vary by asset class. Capital charge and letter ratings are inversely related. Higher ratings yield lower capital charges. A lower capital charge benefits the ROE estimate and improves the chances that the deal can be approved by the credit underwriting committee.

The following, Table 2, provides an example of calculating the protection multiple on a hypothetical pool of consumer ABS loans.

Seller/Servicer Bank - Consumer Receivables Securitization Pools — Table 2

Calculation of Coverage Multiples by Issue and on a Cross-Collateralized Aggregate Portfolio Basis

Evaluated @12/31/02

Issue	Age	Expected Future Losses on Original Par	Expected Losses on Unamortized Par	Breakeven on Unamortized	Actuarial Coverage Multiple	Corresponding Letter Rating
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1999-B	45	0.6%	4.0%	19.76%	4.94	AAA
1999-C	42	0.7%	4.7%	18.59%	3.98	AAA
1999-D	39	0.7%	3.5%	17.22%	4.92	AAA
2000-A	36	1.0%	3.8%	17.45%	4.54	AAA
2000-B	33	1.1%	3.8%	16.81%	4.43	AAA
2000-C	30	1.4%	4.1%	16.61%	4.03	AAA
2000-D	27	1.7%	4.1%	16.36%	3.95	AAA
2001-A	24	2.1%	4.5%	16.60%	3.72	AA
2001-B	21	2.4%	4.4%	16.50%	3.71	AA
2001-C	18	2.9%	5.0%	18.09%	3.62	AA
2001-D	15	3.5%	4.9%	18.19%	3.74	AA
2002-A	12	4.3%	5.5%	18.59%	3.37	AA
2002-B	9	5.4%	6.7%	18.79%	2.82	A
2002-C	6	6.2%	6.9%	17.73%	2.57	A
2002-D	3	7.0%	7.4%	16.21%	2.20	BBB
				Cross-Collateralized Portfolio =>	3.20	AA

Notes:

(1) Outstanding in-force Securitizations @12/31/02.

(2) number of months since the term securitization incepted.

(3) = Exhibit 2, Sheet 1, [Col. (6) - Col. (4)].

(4) = (3) / {Exhibit 2, Sheet 2, Col. (6)}.

(5) from Exhibit 2, Sheet 1, column (8).

(6) = (5) / (4).

Exhibit 3 provides historical default frequency and loss severity amounts, expressed as a function of original par, in Sheet 3. Traditional actuarial development approaches are applied including the curve fitting steps from Sheet 2. Sheet 1 summarizes the ultimate' estimates. This core frequency and severity analysis is basic but produces key assumptions for the calculation of the protection multiple.

Applying the summary portfolio statistics from Exhibit 2 on the seasoned pool performance to date allows the calculation of the protection multiples. All calculations and formulae are provided in the notes to the exhibits.

The progression toward higher letter ratings as each deal matures is to be expected. This is a critical differentiator from P&C risk in that the risk of loss is rapidly diminished as performing deals mature toward their average life. In addition, ABS structures often have minimum levels of credit enhancement which grow rapidly (as a % of par outstanding) as par declines. Of course, if you could cross-collateralize the individual issues into one collateralized bond obligation, then your protection multiple would be greater than that for any newly issued individual bond. The cross allows gains to inure to the benefit of losses across bond deals and offers a significant and measurable amount of additional security.

6. RESERVING

A. Historical

In the early years of the F/G industry, GAAP accounting prohibited mono-line insurers from establishing IBNR reserves, otherwise known as unallocated or non-specific reserves. The rationale was fairly straightforward and relied on the observation that once a municipal bond went into default, it would become a known "discrete" event in the financial markets and the F/G insurer would simply establish an appropriate case reserve estimate based on current information.[5]

Since market inception in the early 1970's average credit default rates on investment grade rated municipal bonds have been extremely low; in fact, lower than the default rate on AAA rated corporate bonds. General obligation and essential service bonds have been particularly safe investments. Compared to corporate bond experience, rated municipal bond defaults have been much less common and recoveries in the event of default have been much higher.

A recent Moody's default study indicates that out of 28,000 municipal issuers rated over the past 30 years, only 18 (0.06%) have defaulted on their public debt obligations, compared to 819 (11.7%) defaults out of 7,000 rated single corporate issuers (Note:

[5] McKnight, M.B.,"Reserving for Financial Guaranty Products," Casualty Actuarial Society *Forum*, Fall 2001, 256-269

Monoline insurers do not underwrite default risk coverage to individual corporations, with the exception of regulated utilities, but rather to structured pools of corporate loans and debt).

A main tenet of the early market reserve treatment was that there could not be any "pure" IBNR claim; therefore, there is no requirement to establish an unpaid liability provision for that which has not occurred. There can be future development on known claims, but these reserve movements would be reflected in future periods by adjusting the case reserve as information improved on the expected recovery rate.

This approach assumed discrete loss emergence when in fact loss emergence on financial guaranty risk derives from a continuous process. On a portfolio basis, at T = O it is expected that losses will occur. *A priori*, however it is unknown which individual bonds will produce losses. At any point, after inception, socio-economic and dynamic market forces are in play, and each guarantee has a loss propensity that fluctuates in a process not unlike the movements of mark-to-market estimates on a basket of highly liquid currency options, for example.

Surveillance monitors the risk of loss on all deals and highlights those that have tripped performance triggers or have had their subordination (deductible) levels materially eroded. These transactions are placed on caution lists and considerable internal resources monitor the performance of the underlying credit. If it further migrates to a watch list, remediation activity is considered. This is the inflection point whereby the expected loss outcome ceases to be determined by independent and/or fortuitous events. Negotiations incept in a partisan or tripartite manner to attempt to reasonably avoid incurring losses. This defines a biased, non-random variable that will not likely improve any estimate of true mean loss by type of product.

Clearly, at inception higher rated credits (i.e. AA/AAA) are less likely to require loss payments than those starting at lower ratings (i.e. BBB). Nevertheless, independent and covariant forces of inflation, tax rates, interest rates, unemployment, etc., conspire to produce losses in all guarantee types. The frequency and severity characteristics vary widely by product type but the losses are embedded within the in-force book at time = 0, in other words, inception of the origination year.

For many years the F/G insurers were predominantly underwriting municipal bonds, insuring general obligation and project-specific financings for municipalities. The Muni guarantee business had minimal losses and was profitable for many years because municipalities rarely default and almost never repudiate their debts. Since the monolines were rarely required to pay bond interest payments, and typically only for brief periods of time, the business was inherently low risk and had limited liquidity requirements. In other words the early underwriting of F/G insurance on GO and essential service bonds was equivalent to "zero loss" underwriting.

The IBNR (or general) reserves were established as a function of new debt service (i.e. P&I) underwritten and the average rate was around 2 to 4 basis points on total P&I. This level had been established based on a study of historic bond defaults experienced by the F/G insurers and the composition of their portfolio.

Table 3 below summarizes the Loss Reserve Positions of the four largest established Primary Insurers @ Sept. 30, 2002.

Table 3

	Unallocated Loss Reserve	Net Par Outstanding	Reserve as % of Net Par Outstanding
Ambac	$120,000,000	$354,017,000,000	0.034% or 3.4 bps
MBIA	$283,000,000	$483,374,000,000	0.059% or 5.9 bps
FSA	$108,000,000	$257,932,000,000	0.042% or 4.2 bps
FGIC	$23,000,000	$181,535,000,000	0.013% or 1.3 bps
Weighted Average			0.042% or 4.2 bps

Source: Bank of America Securities, Research Brief, Bond Insurance Monthly, January 2003

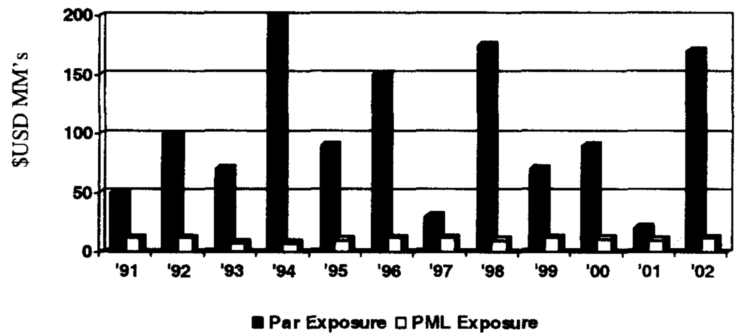
Due to saturation of market penetration in the basic types of Muni bonds, the monolines expanded into non-taxpayer supported, project based, public finance transactions like hospitals, stadiums, and toll roads which suffer from similar risks to those incurred in private enterprise. Unlike traditional municipal guarantees that rely on a city's or state's taxing authority, tax-exempt project finance relies solely on a project's cash flows and its long-term operating performance to meet its obligations.

Consequently the mix of business was changing dramatically and viewing notional Par Outstanding as the common denominator of the risk metric was becoming no longer valid. The better measure of loss value at risk could be derived from the Adjusted Gross Premium.

Graph 2 below demonstrates an incongruity by using Par O/S as proxy common denominator in any measure of value at risk. Monoline insurers underwrite to different leverage targets that are themselves ever changing as a result of differing business strategies in dynamic markets. That is, each origination year defines a unique mixture of mean loss propensities. The notional par insured amounts rise and fall dramatically from year to year. Better estimators of capital at risk are available and earned premium will likely better reflect changes in underlying risk.

Graph 2

Par O/S vs. Value at Risk



B. Recent Developments

In the mid to late 90's the F/G monoline insurers expanded rapidly into domestic and global structured finance guarantees on asset classes including sub-prime home equity mortgages, manufactured housing finance, aircraft leases and equipment trusts, bonds backed by hotel taxes, commercial mortgage-backed securities (CMBS), credit card receivables, auto loans, rental fleets, health care equipment financings, student loans, investor-owned utilities, credit default swaps, collateralized debt obligations backed by high yield and investment grade bonds (CDOs), synthetic CDOs (portfolios of credit-default swaps that are then securitized and guaranteed), emerging market CDOs, and other project finance.

The F/G insurers today hold a markedly different book than that retained in the early Muni era. Corporations and consumers, the underlying borrowers of structured finance portfolios, are more likely than cities to default on their obligations and do, in fact, repudiate their debts in the bankruptcy process. Corporations also, of course, have no ability to access taxpayer funds to repay their liabilities.

The result of all of this is that whilst the concept of zero loss underwriting may still be valid for a few traditional classes of Muni bonds, the F/G insurers have gravitated to an in-force risk portfolio that contains higher potential default frequency and loss severity characteristics with more uncertain correlations than those observed in the past. Recently, the largest monoline insurer altered its longstanding reserving methodology and moved to an earned premium based metric.

C. Basic Actuarial Approach

An actuarial postulate that losses exists at time = 0 within the in-force book of a portfolio of financial guaranty risks is the same as that applied on a book of mortality risk on a pool of insured lives. The only difference is the relative credibility assigned to the hypothetical means[6]. In life insurance, mortality tables can be applied to determine, with minimal mean estimation error, how many deaths (defaults) the pool will experience in

[6] Hypothetical mean refers to the average frequency, average severity, or average aggregate claim amount (i.e. pure premium) of an individual combination of risk characteristics. Philbrick [1981]

subsequent periods. In neither case, can we indicate with any certainty which individual risks will incur a loss.

In the case of mortality risk, the credibility associated with the mean frequency and severity estimates is relatively high whereas, for financial guaranty, the confidence around the mean frequency and severity estimates is relatively low. As such, the unpaid loss reserves in life insurance can reasonably be selected at the conditional expectation (i.e., the 50^{th} percentile from the cumulative distribution function).

Due to the greater relative uncertainty associated with the estimates of mean frequency and severity in financial guaranty products, the variance by asset type hypothetical means produces lower credibility in the estimated aggregate loss distribution. The more skewed form of the financial guaranty loss distribution produces an expected value of the process variance that is significantly higher than its mortality risk counterpart. Therefore it is more prudent when establishing the expected losses to book at higher relative confidence levels. This type of reserve risk loading for parameter uncertainty is common to all risk classes that require actuarial estimates of unpaid liabilities.

Table 4 below provides an informal force-ranking of the relative credibility under various insurance risks underwritten by large P&C multi-lines that also assume financial guaranty risk and the associated reserving methods.

Table 4

Risk Type	Aggregate Loss Distribution Credibility Ranking	Reserving Methods
Primary Workers Comp.	Extremely High	LDF
Life Insurance	Very High	Mortality Tables
Personal Automobile	High	LDF
Commercial Liability	Medium	LDF, B-F
Umbrella Liability	Medium	LDF, B-F
XS Property	Low/Medium	B-F, S-B
XS Umbrella Liability	Low/Medium	B-F, S-B
Financial Guaranty	Low	S-B
XS Casualty Reins.	Very Low	ELR, S-B
Wind & Quake Cats	Extremely Low	ELR

Today, low credibility risk portfolios, such as excess casualty reinsurance and hurricane & earthquake cats, have a widely accepted methodology for IBNR reserves. This is a portfolio-wide Bayesian approach. The reserves have been established on the basis that the portfolio of risk will incur a long-term mean level of losses. In recent years, GAAP accounting has accepted the practice of establishing unpaid liability reserves for the traditional mono-line insurers. However, in the current movement toward accounting transparency (largely affecting life products, pensions and investments) there is a renewed debate as to which actuarial method and analysis will best apply. Bayesian approaches deal with this "credibility debate" directly through mathematical modeling. Accounting methods do not want to work with uncertainty but rather seek a point estimate.

Financial guaranty premium is earned in lock-step with the par amortization and via capital market mechanisms it tends to self-correct for arbitrage from credit spreads and leverage. Capital market risk pricing is typically efficient thereby producing a premium stream that inherently reflects the imputed market risk. Given sufficient prior knowledge and substantial technical and computational resources, we would construct a predictive distribution for aggregate claims during each subsequent period, based upon prior aggregate claim parameters. An innovative alternative that does not explicitly require prior information to calculate the credibility, and does not require as many resources, has been suggested by Bühlmann.

Appropriately determined mathematical models are extremely good descriptors of size-of-loss distributions. They are often more convenient than the actual or empirical distributions when changes are necessary, for example, to predict future conditions. Bayesian methods can be used to introduce subjective ideas about the model. That is, actuaries are encouraged to introduce any sound a priori beliefs into the inference.

A reserve estimation technique that overcomes some of the problems with the Bornhuetter-Ferguson method was independently derived by James Stanard[7] and Hans Bühlmann[8]. Like the LDF and B-F methods, Stanard-Bühlmann uses an aggregate loss emergence pattern that is estimated via the amortization of the risk obligation. The key innovation is that the initial expected loss ratio across the book is estimated from the composite industry loss experience, instead of being arbitrarily selected based upon informed management judgment.

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[7] Weissner, Edward W. "Evaluation of IBNR on a low frequency book where the emergence pattern is incomplete". *Casualty Loss Reserve Seminar Transcript*, 1981.
[8] Bühlmann, H., *Mathematical methods in risk theory. New York: Springer-Verlag.*

A clear advantage of the S-B technique over the ELR method is that as actual losses emerge, the portfolio reserve estimate adapts to yield the credibility weighted mix of the mean losses and the prior expectation. The portfolio reserve level will gradually rise and fall with time driven by the underlying risk characteristics influencing the loss emergence pattern. This provides a natural mechanism that determines when and to what extent accrued reserves for maturing origination years may be released to pay losses or to income in the absence of losses. The S-B determined IBNR reserve provision may be viewed as a rolling annuity provision whose aggregate accrual rate tracks with the inherent risk of the book. It is as close to a fair value estimate of the unpaid liabilities you may hope to obtain, given the shortcomings of the data and the imposed constraints of biased, dependant, and non-random claims events.

i) **Analysis**

In risk portfolios like excess property/casualty reinsurance and financial guaranty, the observed loss ratio from several successive years observations may be zero but other non-zero results may occur that vary widely to pure loss ratios as high as 100% or more.

Stanard and Bühlmann argued that by establishing an in-force portfolio reserve that mimics the inherent industry composite ratio over several years, the *a priori* reserve estimate strikes the appropriate balance between stability and responsiveness. As the risks amortize and actual losses emerge the portfolio reserve level is self-adjusting according to the barometer of current conditions. This strikes the appropriate balance between Stat and GAAP accounting pressures. The balance sheet (stability) and income statement (responsiveness) are stated with minimal accounting distortion driven by the absence or presence of sporadic individual loss events.

ii) Initial Expected Loss Ratio

Table 5 below summarizes the ultimate loss estimates (000's omitted) from the Annual Statement- Schedule P, results from the financial guaranty insurers for the 1990's.

Table 5

	Earned Premium		Ultimate Loss&ALAE		Ult. Loss & ALAE Ratio	
	Direct	Ceded	Direct	Ceded	Direct	Ceded
1990	$ 275,805	$ 51,863	$ 30,112	$ 1,991	10.9%	3.8%
1991	455,560	115,294	33,509	11,845	7.4%	10.3%
1992	582,146	166,810	108,725	55,236	18.7%	33.1%
1993	807,661	195,584	4,690	283	0.6%	0.1%
1994	698,865	162,916	165,910	94,603	23.7%	58.1%
1995	594,420	144,338	18,156	5,365	3.1%	3.7%
1996	725,974	168,923	288	131	0.0%	0.1%
1997	826,034	182,550	33,123	2,395	4.0%	1.3%
1998	1,071,590	228,864	480,756	68,423	44.9%	29.9%
1999	1,244,612	297,198	74,956	36,818	6.0%	12.4%
Total	$ 7,282,667	$ 1,714,340	$ 950,225	$ 277,090	**13.0%**	**16.2%**

Source: Annual Statement for the year 2000, Schedule P – Part 1 – Summary for MBIA, Ambac, FSA, FGIC.

It is not unexpected that the aggregate 10 year ceded ratio would exceed its corresponding direct ratio (here by roughly 1/4). Whether this is a function of adverse selection or excessive ceding commission is problematic. That an industry-wide portfolio of reinsurance bears a higher loss ratio than its direct portfolio is not entirely surprising.

The mean ten year observed ceded loss ratio of 16.2% derives from a continuous loss distribution with a large coefficient of variation ("CV" = Std. Dev/Mean). This is reasonable since we are dealing with extremely low frequency/high severity exposures.

Since we have only ten observations, the sample error associated with the 16.2% mean estimate is also relatively high. In a primary worker's compensation comparison, the expected error around the mean loss ratio estimate is relatively low. As such, selecting the 50^{th} percentile fitted ratio as a proxy for the true mean ratio is reasonable; however, for financial guaranty risk it is more prudent to select an *a priori* ratio at higher confidence levels.

The sample loss ratio data were drawn from an industry with initial conditions largely insuring lower risk municipal bonds during a strong prolonged growth economy. This would tend to produce actual loss ratios lower than that embedded within the current in-force book.

For the reasons stated above, the initial expected loss ratio for current market risk portfolios should probably be set a level greater than historical average of 12% to 16% of AGP.

iii) Loss Emergence Pattern

For the Stanard-Bühlmann method the "percent of ultimate" pattern is assumed to remain relatively stable within product type. Stable "percentages of ultimate" is the assumption that we use to determine the outstanding losses. It is not necessarily the assumption we use to determine the pattern[9].

It has been demonstrated that the potential default frequency and loss severity characteristics of the traditional Muni exposure and S-F exposure are likely different. As such, each origination year will possess a unique aggregate loss emergence pattern that derives from the composite mix of these two basic business exposures. While it is tempting to bifurcate the analysis, there exists no credible basis from which to determine whether neither, either or both risk types will contribute to actual loss. Consequently, the loss emergence pattern is constructed as a hybrid of both. After all, mixture of means is not an encumbrance to this approach. We will not attempt to apportion IBNR back to type of product.

In the absence of any credible loss development history (like schedule P or other historical average loss development metric) one could establish the loss emergence pattern to be concurrent with the amortization of the par outstanding exposure.

This provides a fairly latent pattern that would expect very little if any loss emergence in the early years. The resulting approach would be more akin to an ELR method in that almost all of the accrued IBNR would remain as reserve in the early years and in the absence of any observed loss activity in later years large chunks of IBNR reserve would be released to income. A major shortcoming is that it lacks an objective mechanism whereby IBNR is accrued and subsequently released to pay losses or to income in the absence of expected loss payments.

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[9] Feldblum, Sholom, "The Stanard-Bühlmann Reserving Procedure – A Practitioner's Guide"

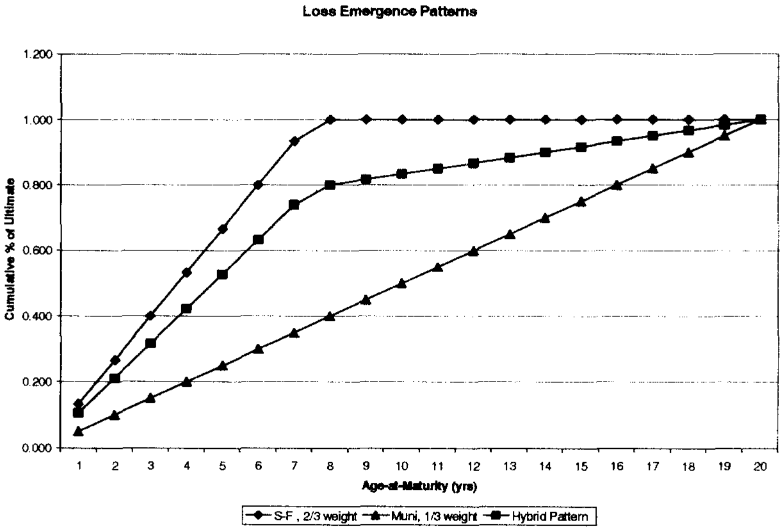
Analysis of the risk demographic by origination year demonstrates that while the proportionate mix of Muni vs. S-F may fluctuate from year to year the average outstanding life parameter within each product type remains fairly stable between origination years. The S-F segment will typically have an aggregate average life of 5 to 10 years and the muni book with 15 to 20 years.

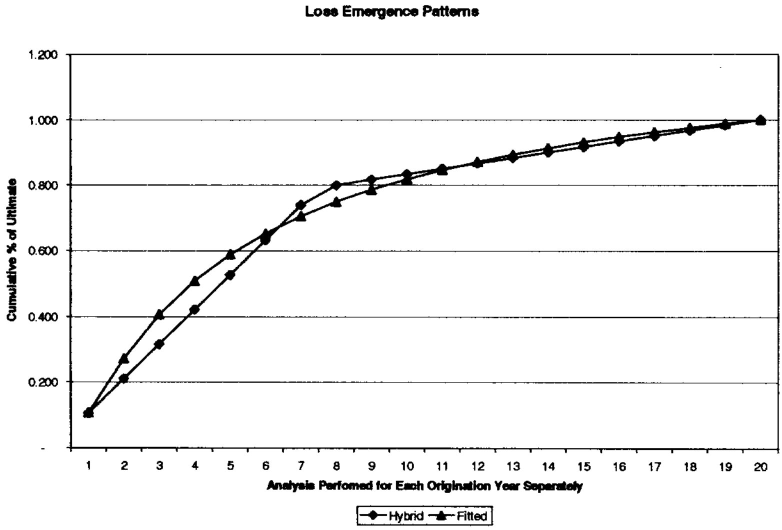
Based on discussions with surveillance and credit officers from various monolines and rating agency analysts there appears an emerging consensus that the loss emergence for S-F classes tends to be front-loaded. For example, in consumer ABS there are clear warning signs sooner rather than later in those instances whereby the credit is underperforming. Early underperformance does not necessarily predict that incurred loss will result. The structure of the deal may often mitigate an actual loss event. Conversely, if S-F deals perform more or less as expected in the early stages, the protection multiples usually increase with time and the loss propensity drops off precipitously. Similarly, municipal default statistics demonstrate a propensity toward increased relative defaults in the early years and less in the later years. This has an intuitive appeal in that once a municipality has geared its revenue flows to meet its debt borrowing obligations and these are performing as expected, it becomes increasingly unlikely that the existing debt burden cannot be adequately serviced in the future from the same revenue base.

Accordingly it appears reasonable to estimate the loss emergence pattern by reflecting the proportionate mix of Muni vs. S-F. This results in an expectation that loss activity will emerge sooner than that indicated by the scheduled par amortization schedule.

Graph 3 showing (1) the composite pattern and (2) the fitted pattern used in the example.

<u>Graph 3</u>





An expected pattern is calculated as the par weighted average of the S-F and Muni books. The individual plots reflect the separate amortization tendencies toward a target outstanding life parameter. The hybrid pattern is fitted to an inverse power curve to produce a more continuous emergence pattern.

iv) Stanard-Bühlmann IBNR Estimate

Table 6 below summarises the calculations required to obtain a Stanard-Bühlmann estimate of IBNR for a hypothetical F/G risk portfolio.

Projected IBNR Reserve Analysis @12/31/10 Table 6

Calculation of Stanard-Bühlmann (S-B) IBNR Estimate

Origin. Year	Earned Premium	Incurred Losses	Incurred Loss Lag	S-B IBNR Estimate	IBNR via LDF Method	IBNR via ELR Method	S-B based Ultimate Loss Ratio	LDF based Ultimate Loss Ratio	ELR based Ultimate Loss Ratio
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
2001	$100,000,000	$20,000,000	1.000	$0	$0	$5,000,000	20.0%	20.0%	25.0%
2002	$180,000,000	$5,000,000	0.900	$4,500,000	$555,556	$40,000,000	5.3%	3.1%	25.0%
2003	$240,000,000	$20,000,000	0.800	$12,000,000	$5,000,000	$40,000,000	13.3%	10.4%	25.0%
2004	$294,000,000	$25,000,000	0.700	$22,050,000	$10,714,286	$48,500,000	16.0%	12.1%	25.0%
2005	$327,000,000	$0	0.600	$32,700,000	$0	$81,750,000	10.0%	0.0%	25.0%
2006	$341,000,000	$125,000,000	0.500	$42,625,000	$125,000,000	($39,750,000)	49.2%	73.3%	25.0%
2007	$300,300,000	$0	0.400	$45,045,000	$0	$75,075,000	15.0%	0.0%	25.0%
2008	$247,750,000	$15,000,000	0.300	$43,356,250	$35,000,000	$46,937,500	23.6%	20.2%	25.0%
2009	$181,680,000	$0	0.200	$36,336,000	$0	$45,420,000	20.0%	0.0%	25.0%
2010	$99,920,000	$0	0.100	$22,482,000	$0	$24,980,000	22.5%	0.0%	25.0%
Total	$2,311,650,000	$210,000,000		$261,094,250	$176,269,842	$367,912,500	20.4%	16.7%	25.0%

Notes:

(2) Cumulative premium earned on insurance policies and structured credit derivatives from Exhibit 4.

(3) from Exh. 4.

(4) assumed for simplicity to emerge 10% each year.

(5) = [(2) x 0.25] x [1 - (4)]. Initial Expected Loss Ratio assumed = 25%.

(6) = [(3) / (4)] – (3).

(7) = [(2) x 0.25] - (3).

(8) = [(3) + (5)] / (2).

(9) = [(3) + (6)] / (2).

(10) = [(3) + (7)] / (2).

At any given evaluation point, the S-B method will strike a balance between the inelastic ELR method and the highly elastic LDF method. However, the more meaningful advantages of the S-B method for F/G are demonstrated when we review the estimates on individual origination years and the overall portfolio over successive evaluation intervals.

All the requisite information to construct the table above is provided in Exhibit 7. Also all of the hypothetical data supporting the following discussion may be found in Exhibits 4 through 7 depending upon the specific scenario.

Exhibit 4 – Assumes no losses are ever reported.

Exhibit 5 – Assumes that reported losses always emerge as expected.

Exhibit 6 – Assumes that reported losses are observed at three times the expected case.

Exhibit 7 - Assumes hypothetical sparse and erratic reported losses.

Otherwise given for each scenario;

1. A 25% industry-wide *a priori* expected loss ratio.
2. A 10 year linear emergence pattern. This is for the sake of simplicity but any inferences derived are valid for other curve-linear emergence patterns.
3. Expected (over the life) notional premium for the first origination year equal to $100 MM. Premium growth for successive origination years at 100%, 50%, 40%, 30%, 25%, and 10% thereafter.

Each exhibit tracks ten years of the following key statistics for each origination year separately and for the overall risk portfolio combined:

A. Estimated Ultimate Loss Ratio

B. Cumulative Incurred Loss

C. Cumulative Earned Premium

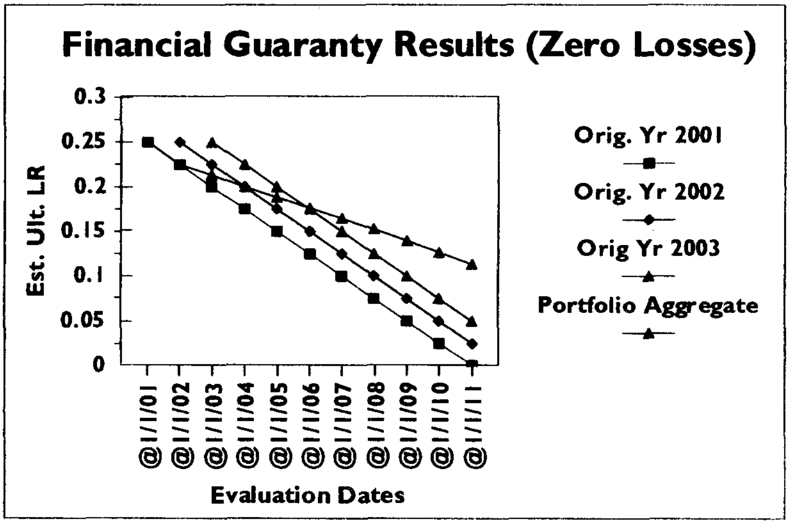
D. Reported Loss Ratio

E. Cumulative Estimated IBNR

F. Expected Emergence of Reported Losses

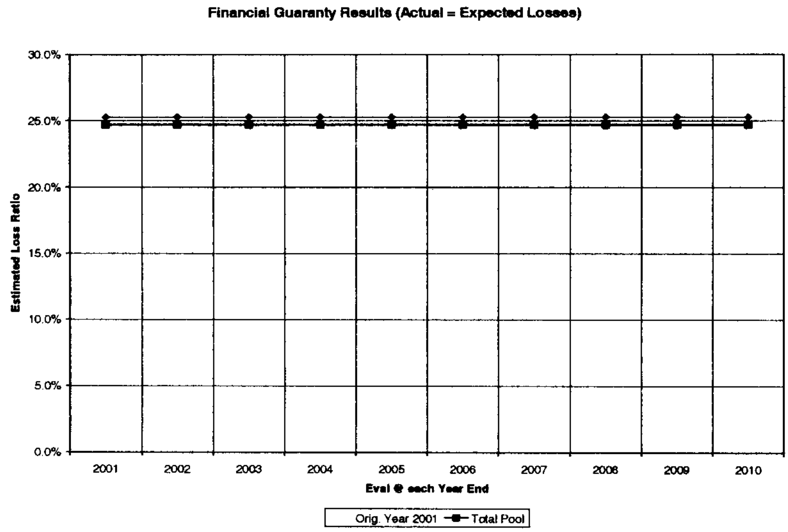
Graph 4 below plots the movements in estimated ultimate loss ratios for the early origination years and the overall portfolio assuming no losses are ever reported.

Graph 4

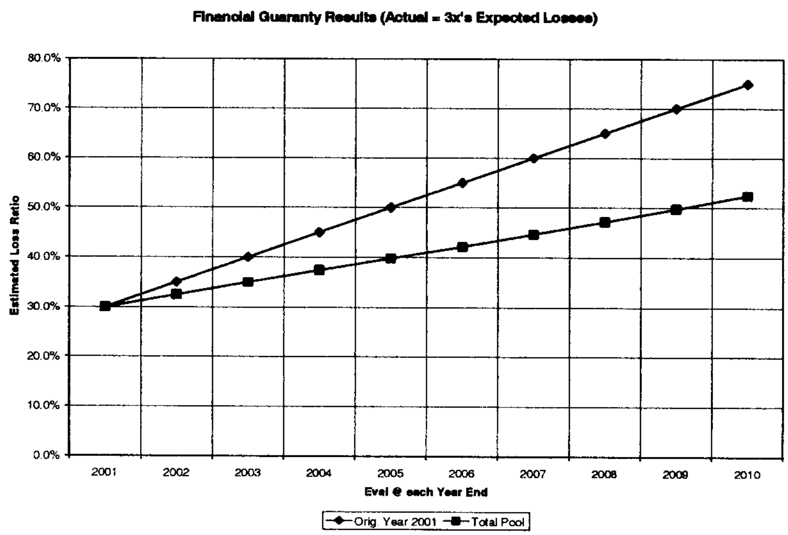


The next two graphs, 5 and 6, plot the first origination year and aggregate portfolio estimates assuming losses always emerge as expected and at 3 times the expected rate, respectively.

Graph 5

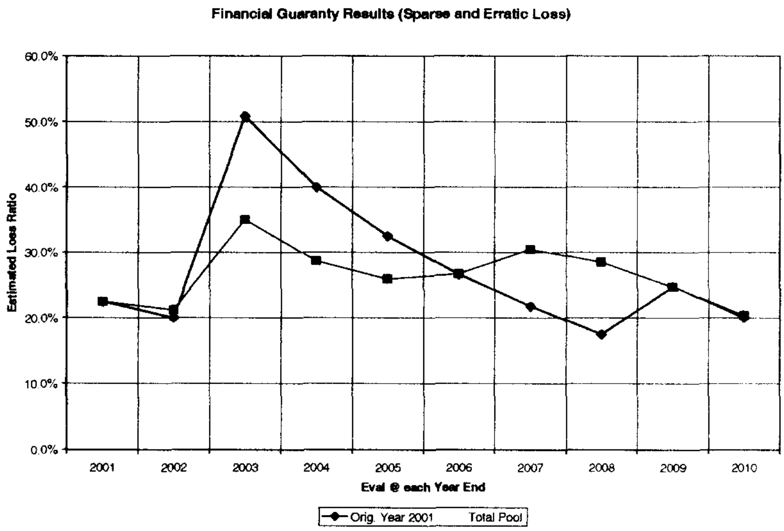


Graph 6



A cursory review of the graphs for each of these three scenarios yields an intuitive result. That is, if the industry wide *a priori* loss ratio is materially in error, the resulting portfolio ratios will gravitate toward the true mean. Conversely, if actual losses emerge as expected, then the 25% estimate level persists. This would encourage periodic review of the base case aggregate loss ratio but as we will see in the next chart, it does not necessitate constant tinkering based only upon the absence or presence of a few claims.

Graph 7



Graph 7 above with sparse and erratic default events is instructive. While the individual origination year loss ratio indications may fluctuate over time the overall portfolio results will move gradually toward the long term mean loss level. As such the portfolio reserve levels will likely remain within a reasonable range and not overreact to reported events nor be too inertial to disregard zero loss activity.

It may be tempting to posit an accelerated earning of a portion of the future guaranteed premium on an individual origination year when reported losses spike in advance of the "expected" loss emergence. However, there are at least two good reasons not to take that approach.

1. This is largely an installment premium business and one would be accelerating the earning of premiums that have not yet been received.

2. Even if this was a prepaid premium business, by accelerating premium recognition to smooth the loss ratio from spike events would presuppose knowledge about the remaining loss experience which does not creditably exist.

In other words, using IBNR and premium that relates to the subsequent risk emergence period to shore up near-term results implicitly presumes that subsequent loss experience will be more favorable than initially assumed. Clearly, this would not be valid and in the event that the subsequent loss activity was adverse, it would create even more volatile swings in subsequent financial reporting.

7. SUMMARY & CONCLUSIONS

The reserve practice endorsed for the financial guaranty industry is inherently structured as a portfolio wide Bayesian approach. The two critical assumptions (*a priori* loss ratio and loss emergence pattern) need be revised only to the extent that credible suppositions and observations derive from the prevailing market based conditions.

A few aspects of F/G insurance enhance the applicability of the Stanard-Bühlmann IBNR reserve method.

1. The absence of any liability tail risk after maturity.
2. Installment premiums, AGP measures and the gradual recognition of earned premium and annuity type IBNR accrual rate.
3. The effect of predictive latency and its corollary: increased credibility in pure premium estimates as the portfolios mature.

Acknowledgment

Several colleagues and business partners were very helpful in discussing and reviewing this paper at different stages including David Stevens, Wynne Morriss, Peter Giacone, Rick Kastellec, Rick Lines, Tom Currie, Melodie Wakefield, Sean Symons, Sylvia Tavares and Tom Weidman. Thank you also to Sally Browne (Editing), Sarah Carr (Analytics), and Lorraine King (Research).

Exhibits

1. ABS example: Calculation of Coverage Multiples
2. ABS example: Summary Sample ABS Portfolio Statistics
3. ABS example: Estimated Ultimate Defaults and Losses
4. S-B IBNR example: No Losses
5. S-B IBNR example: Actual Losses = Expected
6. S-B IBNR example: Actual Losses = 3 x's Expected
7. S-B IBNR example: Hypothetical Sparse & Erratic Losses

Reserve Method Definitions

Expected Loss Ratio ("ELR") Method. This technique assumes that the estimated ultimate losses are equal to the product of the earned premium and an initial expected loss ratio (IELR). It has the advantage of simplicity and stability but it ignores actual results as they emerge.

Loss Development Factor ("LDF") Method. This method is a common reserving method in which ultimate losses are estimated by applying loss development factors to those losses which already emerged. The development factors are based on historical reporting patterns of the company or composite industry experience or some other credibility weighted average.

Bornhuetter-Ferguson ("B-F") Technique. The B-F method is commonly used when loss experience is relatively immature and /or lacks sufficient credibility for the application of other methods. The B-F method is essentially a blend of the two methods described above. It combines the two methods by splitting expected losses into two pieces- namely expected reported and expected unreported. Estimated ultimate losses are then derived by adding the actual reported losses to the expected unreported losses. Two parameters need to be determined in order to apply this method - the IELR and the expected reporting pattern. This method is described in the proceedings of the Casualty Actuarial Society, Volume LIX, 1972 ("The Actuary and IBNR" by R.L. Bornhuetter and R.E. Ferguson).

Stanard-Bühlmann ("S-B") Technique. An estimation method which overcomes some of the problems with the LDF method and the B-F technique was independently derived by James Stanard and by Hans Bühlmann (internal Swiss Re publication). As with the LDF method and B-F technique, the Stanard-Bühlmann technique uses an aggregate known loss lag pattern which may be estimated via the LDF method. The key innovation is that the ultimate expected loss ratio for all years combined is estimated from a composite loss experience measure, instead of being selected arbitrarily.

Seller/Servicer Bank - Consumer Receiveables Securitization Pools

Exhibit 1

Calculation of Coverage Multiples by Issue and on a Cross-Collateralized Aggregate Portfolio Basis
Evaluated @12/31/02

Issue	Age	Expected Future Losses on Original	Expected Losses on Unamortized	Breakeven on Unamortized	Actuarial Coverage Multiple	Corresponding Letter Rating
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1999-B	45	0.6%	4.0%	19.76%	4.94	AAA
1999-C	42	0.7%	4.7%	18.59%	3.98	AAA
1999-D	39	0.7%	3.5%	17.22%	4.92	AAA
2000-A	36	1.0%	3.8%	17.45%	4.54	AAA
2000-B	33	1.1%	3.8%	16.81%	4.43	AAA
2000-C	30	1.4%	4.1%	16.61%	4.03	AAA
2000-D	27	1.7%	4.1%	16.36%	3.95	AAA
2001-A	24	2.1%	4.5%	16.60%	3.72	AA
2001-B	21	2.4%	4.4%	16.50%	3.71	AA
2001-C	18	2.9%	5.0%	18.09%	3.62	AA
2001-D	15	3.5%	4.9%	18.19%	3.74	AA
2002-A	12	4.3%	5.5%	18.59%	3.37	AA
2002-B	9	5.4%	6.7%	18.79%	2.82	A
2002-C	6	6.2%	6.9%	17.73%	2.57	A
2002-D	3	7.0%	7.4%	16.21%	2.20	BBB
				Cross Collaterized Portfolio Factor =>	3.20	AA

Notes:

(1) Outstanding in-force Securitizations @12/31/02.

(2) number of months since the term securitization incepted.

(3) = Exhibit 2, Sheet 1, [Col. (6) - Col. (4)].

(4) = (3) / {Exhibit 2, Sheet 2, Col. (6)}.

(5) from Exhibit 2, Sheet 1, column (8).

(6) = (5) / (4).

Seller/Servicer Bank - Consumer Receivables Securitization Pools
Expected Defaults, Losses and Breakeven on Unamortized
Evaluated @12/31/02

Exhibit 2
Sheet 1

Issue	Age-at-Maturity	Inc'd Defaults as % of Original	Inc'd Losses as % of Original	Actuarial Estimate Lifetime Cumulative Defaults	Actuarial Estimate Lifetime Cumulative Losses	Actuarial Estimate Percent of Ultimate Losses	Breakeven on Unamortized
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1999-B	45	12.7%	6.5%	13.7%	7.1%	91.5%	19.76%
1999-C	42	12.1%	6.1%	13.3%	6.8%	89.7%	18.59%
1999-D	39	10.0%	6.1%	11.2%	6.8%	89.7%	17.22%
2000-A	36	10.5%	6.3%	12.1%	7.3%	86.3%	17.45%
2000-B	33	10.2%	6.0%	12.2%	7.1%	84.5%	16.81%
2000-C	30	11.5%	5.4%	14.2%	6.8%	79.4%	16.61%
2000-D	27	9.6%	5.3%	12.7%	7.0%	75.7%	16.36%
2001-A	24	7.7%	4.7%	11.4%	6.8%	69.1%	16.60%
2001-B	21	8.2%	3.9%	13.0%	6.3%	61.9%	16.50%
2001-C	18	7.3%	3.8%	12.9%	6.7%	56.7%	18.09%
2001-D	15	6.5%	3.3%	13.4%	6.8%	48.5%	18.19%
2002-A	12	5.2%	2.6%	13.6%	6.9%	37.7%	18.59%
2002-B	9	3.7%	2.1%	13.7%	7.5%	28.0%	18.79%
2002-C	6	2.6%	1.2%	14.1%	7.4%	16.2%	17.73%
2002-D	3	1.2%	0.4%	14.2%	7.4%	5.4%	16.21%

Notes:

(1) Outstanding in-force Securitizations @12/31/02. Insurai
(2) number of months since the term securitization incepted.
(3) from Exhibit 3, Sheet 1a, Column (3).
(4) from Exhibit 3, Sheet 1b, Column (3).
(5) from Exhibit 3, Sheet 1a, Column (8).
(6) from Exhibit 3, Sheet 1b, Column (8).
(7) = (4) / (6).
(8) Provided by Seller/Servicer Bank.

Seller/Servicer Bank - Consumer Receivables Securitization Pools
Summary Portfolio Statistics
Evaluated @12/31/02

Exhibit 2
Sheet 2

Issue	Age at Maturity	Initial Par	Outstanding Par	Receivable Pool Factor	Outstanding Collateral	Spread Account Cash Balance	O-C	Total Subord.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1999-B	45	327,000	44,591	15.0%	49,050	10.0%	10.0%	20.0%
1999-C	42	363,000	49,500	15.0%	54,450	10.0%	10.0%	20.0%
1999-D	39	393,000	72,778	20.0%	78,600	8.0%	10.0%	18.0%
2000-A	36	416,000	104,000	26.0%	108,160	4.0%	10.0%	14.0%
2000-B	33	443,000	122,352	29.0%	128,470	5.0%	10.0%	15.0%
2000-C	30	471,000	152,514	34.0%	160,140	5.0%	10.0%	15.0%
2000-D	27	515,000	201,095	41.0%	211,150	5.0%	9.5%	14.5%
2001-A	24	567,000	253,800	47.0%	266,490	5.0%	9.5%	14.5%
2001-B	21	637,000	327,600	54.0%	343,980	5.0%	9.0%	14.0%
2001-C	18	688,000	380,038	58.0%	399,040	5.0%	9.0%	14.0%
2001-D	15	737,000	505,371	72.0%	530,640	5.0%	7.3%	12.3%
2002-A	12	798,000	598,500	78.0%	622,440	4.0%	5.3%	9.3%
2002-B	9	848,000	666,874	81.0%	686,880	3.0%	6.3%	9.3%
2002-C	6	857,000	756,176	90.0%	771,300	2.0%	3.7%	5.7%
2002-D	3	976,000	918,020	95.0%	927,200	1.0%	1.3%	2.3%
Total		9,036,000	5,153,210		5,337,990			

Notes:

(1) Outstanding in-force Securitizations @12/31/02. Insurance risk terminates when pool factor decreases below 10%.
(2) number of months since the term securitization incepted.
(3) Provided by the Seller/Servicer.
(4) Provided by the Seller/Servicer.
(5) Provided by the Seller/Servicer.
(6) = (3) x (5).
(7) Provided by the Seller/Servicer.
(8) Provided by the Seller/Servicer.
(9) = (7) + (8).

Seller/Servicer Bank - Consumer Receivables Securitisation Pools
Estimated Ultimate Cumulative Defaults as a Percent of Initial Par
Evaluated @12/31/02

Exhibit 3
Sheet 1a

Issue	Initial Par	Reported Defaults	Age-at-Maturity (mos.)	Unreported Defaults Factor	LDF Method Estimated Ultimate	S-B Method Estimated Ultimate	Selected Estimated Ultimate
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1997-A	100,000	13.18%	45	1.08	14.2%	14.2%	14.2%
1997-B	112,000	12.33%	45	1.08	13.3%	13.4%	13.3%
1997-C	119,000	12.75%	45	1.08	13.8%	13.8%	13.8%
1997-D	124,000	11.05%	45	1.08	11.9%	12.1%	12.0%
1997-A	140,000	11.26%	45	1.08	12.2%	12.3%	12.2%
1997-B	160,000	10.68%	45	1.08	11.5%	11.7%	11.6%
1997-C	175,000	11.16%	45	1.08	12.1%	12.2%	12.1%
1997-D	185,000	11.05%	45	1.08	11.9%	12.1%	12.0%
1998-A	185,000	11.93%	45	1.08	12.9%	13.0%	12.9%
1998-B	213,000	12.74%	45	1.08	13.8%	13.8%	13.8%
1998-C	229,000	13.68%	45	1.08	14.8%	14.7%	14.7%
1998-D	256,000	12.49%	45	1.08	13.5%	13.5%	13.5%
1999-A	291,000	12.36%	45	1.08	13.3%	13.4%	13.4%
1999-B	327,000	12.70%	45	1.08	13.7%	13.7%	13.7%
1999-C	363,000	12.12%	42	1.09	13.2%	13.3%	13.3%
1999-D	393,000	9.95%	39	1.11	11.0%	11.3%	11.2%
2000-A	416,000	10.52%	36	1.14	12.0%	12.3%	12.1%
2000-B	443,000	10.22%	33	1.18	12.1%	12.4%	12.2%
2000-C	471,000	11.54%	30	1.24	14.3%	14.2%	14.2%
2000-D	515,000	9.59%	27	1.31	12.6%	12.9%	12.7%
2001-A	567,000	7.67%	24	1.43	11.0%	11.9%	11.4%
2001-B	637,000	8.15%	21	1.57	12.8%	13.2%	13.0%
2001-C	688,000	7.26%	18	1.73	12.5%	13.2%	12.9%
2001-D	737,000	6.46%	15	2.04	13.2%	13.6%	13.4%
2002-A	798,000	5.17%	12	2.60	13.4%	13.8%	13.6%
2002-B	848,000	3.67%	9	3.51	12.9%	13.7%	13.7%
2002-C	857,000	2.59%	6	5.62	14.5%	14.1%	14.1%
2002-D	976,000	1.19%	3	14.04	16.7%	14.2%	14.2%

Notes:

(1) Provided by Seller/Servicer Bank
(2) Provided by Seller/Servicer Bank
(3) From Development Triangle, Exhibit 3, Sheet 3a.
(4) From Development Triangle, Exhibit 3, Sheet 3a.
(5) From Development Triangle, Exhibit 3, Sheet 3a.
(6) = (3) x (5).
(7) = {[(1 - (1/ Col. (5))] x 14%} + Col (3).
(8) Based on Cols. (6) & (7).

S-B a priori= 14.00%

Seller/ Servicer Bank - Consumer Receivables Securitisations Pools
Estimated Ultimate Cumulative Losses as a Percent of Initial Par
Evaluated @12/31/02

Exhibit 3
Sheet 1b

Issue	Initial Par	Reported Losses	Age-at-Maturity (mos.)	Unreported Loss Factor	LDF Method Estimated Ultimate	S-B Method Estimated Ultimate	Selected Estimated Ultimate
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1997-A	100,000	6.1%	45	1.09	6.6%	6.7%	6.7%
1997-B	112,000	6.0%	45	1.09	6.5%	6.6%	6.5%
1997-C	119,000	5.3%	45	1.09	5.7%	5.9%	5.8%
1997-D	124,000	6.2%	45	1.09	6.7%	6.8%	6.7%
1997-A	140,000	7.0%	45	1.09	7.6%	7.6%	7.6%
1997-B	160,000	6.1%	45	1.09	6.7%	6.7%	6.7%
1997-C	175,000	4.6%	45	1.09	5.0%	5.2%	5.1%
1997-D	185,000	5.7%	45	1.09	6.2%	6.3%	6.3%
1998-A	185,000	6.3%	45	1.09	6.8%	6.9%	6.8%
1998-B	213,000	4.9%	45	1.09	5.3%	5.5%	5.4%
1998-C	229,000	6.2%	45	1.09	6.7%	6.8%	6.7%
1998-D	256,000	5.2%	45	1.09	5.7%	5.8%	5.7%
1999-A	291,000	5.8%	45	1.09	6.3%	6.4%	6.4%
1999-B	327,000	6.5%	45	1.09	7.1%	7.1%	7.1%
1999-C	363,000	6.1%	42	1.10	6.7%	6.8%	6.8%
1999-D	393,000	6.1%	39	1.12	6.8%	6.9%	6.8%
2000-A	416,000	6.3%	36	1.15	7.3%	7.3%	7.3%
2000-B	443,000	6.0%	33	1.19	7.1%	7.2%	7.1%
2000-C	471,000	5.4%	30	1.25	6.8%	6.9%	6.8%
2000-D	515,000	5.3%	27	1.31	7.0%	7.1%	7.0%
2001-A	567,000	4.7%	24	1.41	6.6%	6.9%	6.8%
2001-B	637,000	3.9%	21	1.54	6.1%	6.6%	6.3%
2001-C	688,000	3.8%	18	1.69	6.5%	6.9%	6.7%
2001-D	737,000	3.3%	15	2.03	6.6%	7.1%	6.8%
2002-A	798,000	2.6%	12	2.54	6.7%	7.2%	6.9%
2002-B	848,000	2.1%	9	3.48	7.4%	7.5%	7.5%
2002-C	857,000	1.2%	6	5.57	6.7%	7.4%	7.4%
2002-D	976,000	0.4%	3	14.48	5.5%	7.4%	7.4%

Notes:

(1) Provided by Seller/Servicer Bank
(2) Provided by Seller/Servicer Bank
(3) From Development Triangle , Exhibit 3, Sheet 3b.
(4) From Development Triangle , Exhibit 3, Sheet 3b.
(5) From Development Triangle , Exhibit 3, Sheet 3b.
(6) = (3) x (5).
(7) = {[(1 - (1/ Col. (5))] x 7.5%} + Col (3).   S-B a priori= 7.50%
(8) Based on Cols. (6) & (7).

Seller/Servicer Bank
Consumer Receivables Securitization Pools
Cumulative Defaults as a Percent of Initial Par
Evaluated @ 12/31/02

Exhibit 3
Sheet 2a

Exponential Power Curve Fitting Analysis Detail

Regression Output:

Constant		-1.631
Std Err of Y Est		0.317
R Squared		0.923
No. of Observations		13
Degrees of Freedom		11
X Coefficient(s)	-1.750	
Std Err of Coef.	0.153	

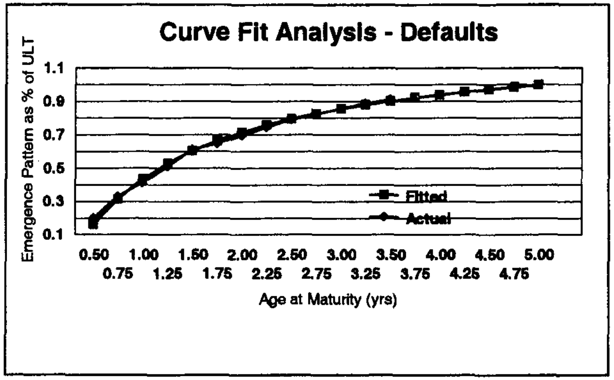
Transformation Formulae

$$y = e^{a x^{b}}$$

$$\ln(y) = a x^{b}$$

$$\ln(\ln(y)) = \ln(a) + b \ln(x)$$

$$Y = A + b X$$



Time(t)	LOGe(t)	Actual Ln(ln(f))	Actual LDF(f)	Fitted LDF(f)	Fitted CDF	Fitted % of ULT	Actual % of ULT
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.50	-0.693	-0.808	1.562	1.932	6.128	16.3%	20.2%
0.75	-0.288	-1.290	1.317	1.382	3.172	31.5%	33.1%
1.00	0.000	-1.393	1.282	1.216	2.295	43.6%	41.3%
1.25	0.223	-1.759	1.188	1.142	1.887	53.0%	50.9%
1.50	0.405	-2.506	1.085	1.101	1.653	60.5%	61.4%
1.75	0.560	-2.304	1.105	1.076	1.501	66.6%	64.9%
2.00	0.693	-2.462	1.089	1.060	1.395	71.7%	69.8%
2.25	0.811	-2.721	1.068	1.048	1.316	76.0%	74.6%
2.50	0.916	-3.123	1.045	1.040	1.255	79.7%	79.3%
2.75	1.012	-3.342	1.036	1.034	1.207	82.9%	82.7%
3.00	1.099	-3.625	1.027	1.029	1.167	85.7%	85.9%
3.25	1.179	-4.143	1.016	1.025	1.134	88.2%	89.0%
3.50	1.253	-4.276	1.014	1.022	1.106	90.4%	91.1%
3.75				1.020	1.082	92.4%	
4.00				1.017	1.062	94.2%	
4.25				1.016	1.043	95.8%	
4.50				1.014	1.027	97.3%	
4.75				1.013	1.013	98.7%	
5.00				1.012	1.000	100.0%	

Notes:

(1) evaluation age-at-maturity (yrs).
(2) = LOGe (col (1)). Independent regression variable.
(3) = LOGe(LOGe (col (4))). Dependent regression variable.
(4) Par weighted average LDF from Exhibit 3, Sheet 3a.
(5) = e^{e^ (-1.631) x col(1)^(-1.750)}.
(6) Reverse cumulative product of Col (5).
(7) Inverse of Col (6).
(8) Inverse of the reverse cumulative product of Col (4).

Seller/Servicer Bank
Consumer Receivables Securitization Pools
Cumulative Losses as a Percent of Initial Par
Evaluated @12/31/02

Exhibit 3
Sheet 2b

Exponential Power Curve Fitting Analysis Detail

Transformation Formulae

$$y = e^{a x^{b}}$$

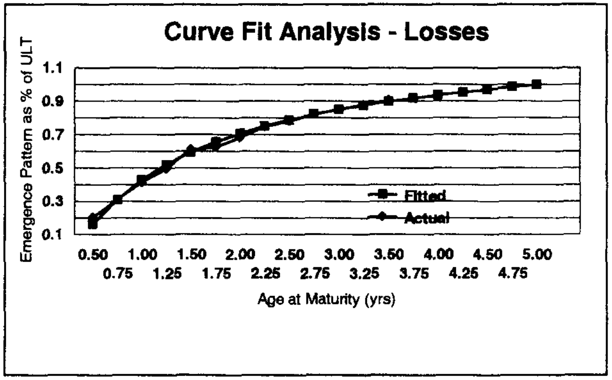
$$\ln(y) = a x^{b}$$

$$\ln(\ln(y)) = \ln(a) + b \ln(x)$$

$$Y = A + b\ X$$

Regression Output:

Constant		-1.629
Std Err of Y Est		0.358
R Squared		0.900
No. of Observations		13
Degrees of Freedom		11
X Coefficient(s)	-1.717	
Std Err of Coef.	0.173	



Time(t)	LOGe(t)	Actual Ln(ln((f))	Actual LDF(f)	Fitted LDF(f)	Fitted CDF	Fitted % of ULT	Actual % of ULT
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.50	-0.693	-0.885	1.511	1.906	6.136	16.3%	20.6%
0.75	-0.288	-1.160	1.368	1.379	3.220	31.1%	31.3%
1.00	0.000	-1.465	1.260	1.217	2.334	42.8%	41.4%
1.25	0.223	-1.679	1.205	1.143	1.919	52.1%	49.4%
1.50	0.405	-2.707	1.069	1.103	1.678	59.6%	61.5%
1.75	0.560	-2.073	1.134	1.078	1.522	65.7%	62.4%
2.00	0.693	-2.351	1.100	1.061	1.412	70.8%	68.3%
2.25	0.811	-2.963	1.053	1.050	1.330	75.2%	74.9%
2.50	0.916	-2.963	1.053	1.042	1.267	78.9%	78.1%
2.75	1.012	-3.428	1.033	1.035	1.217	82.2%	82.4%
3.00	1.099	-3.589	1.028	1.030	1.175	85.1%	85.3%
3.25	1.179	-3.874	1.021	1.026	1.141	87.7%	88.1%
3.50	1.253	-4.349	1.013	1.023	1.112	90.0%	90.8%
3.75				1.020	1.087	92.0%	
4.00				1.018	1.065	93.9%	
4.25				1.016	1.046	95.6%	
4.50				1.015	1.029	97.2%	
4.75				1.014	1.014	98.7%	
5.00				1.012	1.000	100.0%	

Notes:
(1) evaluation age-at-maturity (yrs).
(2) = LOGe (col (1)). Independent regression variable.
(3) = LOGe(LOGe (col (4))). Dependent regression variable.
(4) Par weighted average LDF from Exhibit 3, Sheet 3b.
(5) = e^[e^ (-1.629) x col(1)^(-1.717)].
(6) Reverse cumulative product of Col (5).
(7) Inverse of Col (6).
(8) Inverse of the reverse cumulative product of Col (4).

Seller/Servicer Bank - Consumer Receivables Securitization Pools
Cumulative Defaults as a Percent of Initial Par
Evaluated @12/31/02

Exhibit 3
Sheet 3a

Issue	Initial Par	@ 3 Mos.	@ 6 Mos.	@ 9 Mos.	@ 12 Mos.	@ 15 Mos.	@ 18 Mos.	@ 21 Mos.	@ 24 Mos.	@ 27 Mos.	@ 30 Mos.	@ 33 Mos.	@ 36 Mos.	@ 39 Mos.	@ 42 Mos.	@ 45 Mos.
1997-A	100,000	0.98	2.47	3.33	4.82	6.55	7.49	8.40	9.86	11.19	11.32	12.28	12.57	13.06	13.08	13.18
1997-B	112,000	1.05	2.40	4.20	5.48	5.95	7.09	7.84	8.51	9.45	10.11	10.99	11.71	11.90	12.00	12.33
1997-C	119,000	1.15	2.43	3.73	4.74	6.45	7.31	8.28	9.17	10.63	11.01	11.28	11.64	11.97	12.41	12.75
1997-D	124,000	0.90	2.11	3.45	4.56	5.79	7.73	8.12	8.60	9.37	9.64	10.37	10.54	10.93	11.02	11.05
1997-A	140,000	1.03	2.30	4.20	4.71	6.35	7.69	8.05	9.20	10.16	10.20	10.57	10.67	10.97	11.20	11.26
1997-B	160,000	0.88	2.60	3.65	5.12	5.98	7.91	7.52	7.86	8.91	9.05	9.53	10.03	10.19	10.57	10.68
1997-C	175,000	1.13	2.76	3.38	5.07	5.82	7.06	7.51	8.20	9.11	9.69	10.01	10.24	10.75	10.90	11.16
1997-D	185,000	1.23	2.91	3.85	4.62	5.95	7.74	8.16	8.40	8.99	9.99	10.31	10.80	10.93	10.94	11.05
1998-A	185,000	1.18	2.25	3.63	4.85	6.54	7.55	8.40	9.63	10.33	10.52	10.62	11.18	11.48	11.77	11.93
1998-B	213,000	1.21	2.97	3.47	4.83	6.24	7.52	8.46	8.99	9.97	10.54	11.37	12.07	12.36	12.51	12.74
1998-C	229,000	1.06	2.31	4.00	5.13	6.66	7.07	8.30	9.72	10.97	11.72	12.34	12.77	13.37	13.49	13.68
1998-D	256,000	1.12	2.40	3.38	4.81	6.70	7.48	8.26	8.76	10.18	11.12	11.68	11.88	12.00	12.47	12.49
1999-A	291,000	0.75	2.97	3.55	5.17	6.43	7.06	7.52	8.75	9.64	10.54	11.53	11.79	11.89	12.12	12.36
1999-B	327,000	0.95	2.24	4.18	5.06	6.26	7.26	7.96	9.53	10.26	11.10	11.33	12.01	12.43	12.57	12.70
1999-C	363,000	0.86	2.98	3.25	4.95	6.07	7.20	8.43	9.90	10.48	10.85	11.34	11.69	12.11	12.12	
1999-D	393,000	0.87	2.24	4.16	5.47	5.90	7.09	7.75	7.88	8.38	8.90	9.20	9.70	9.95		
2000-A	416,000	1.16	2.19	3.66	4.53	6.57	7.32	8.16	8.51	8.97	9.96	10.43	10.52			
2000-B	443,000	1.05	2.19	4.19	4.95	6.54	7.86	7.96	9.48	9.66	10.06	10.22				
2000-C	471,000	0.94	2.14	3.64	4.61	5.78	7.26	8.14	9.57	10.25	11.54					
2000-D	515,000	0.92	2.20	3.69	4.73	6.17	7.16	7.89	8.52	9.59						
2001-A	567,000	0.98	2.92	3.31	5.11	5.85	7.37	7.53	7.67							
2001-B	637,000	1.24	2.20	3.74	4.81	6.09	7.64	8.15								
2001-C	688,000	1.09	2.70	3.36	4.80	6.52	7.26									
2001-D	737,000	0.87	2.18	4.22	4.71	6.46										
2002-A	798,000	0.93	2.33	4.01	5.17											
2002-B	848,000	1.11	2.61	3.67												
2002-C	857,000	0.92	2.59													
2002-D	976,000	1.19														

Seller/Servicer Bank - Consumer Receivables
Cumulative Defaults - Age to Age Development Factor Analysis

Issue	Initial Par	6/3	9/6	12/9	15/12	18/15	21/18	24/21	27/24	30/27	33/30	36/33	39/36	42/39	45/42	45 to Ult
1997-A	100,000	2.51	1.35	1.45	1.36	1.14	1.12	1.17	1.14	1.01	1.08	1.02	1.04	1.00	1.01	
1997-B	112,000	2.27	1.75	1.30	1.09	1.19	1.11	1.09	1.11	1.07	1.09	1.07	1.02	1.01	1.03	
1997-C	119,000	2.12	1.54	1.27	1.36	1.13	1.13	1.11	1.16	1.04	1.02	1.03	1.03	1.04	1.03	
1997-D	124,000	2.33	1.64	1.32	1.27	1.33	1.05	1.06	1.09	1.03	1.08	1.02	1.04	1.01	1.00	
1997-A	140,000	2.24	1.82	1.12	1.35	1.21	1.05	1.14	1.10	1.00	1.04	1.01	1.03	1.02	1.01	
1997-B	160,000	2.95	1.40	1.40	1.17	1.32	0.95	1.04	1.13	1.02	1.05	1.05	1.02	1.04	1.01	
1997-C	175,000	2.45	1.23	1.50	1.15	1.21	1.06	1.09	1.11	1.06	1.03	1.02	1.05	1.01	1.02	
1997-D	185,000	2.37	1.32	1.20	1.29	1.30	1.05	1.03	1.07	1.11	1.03	1.05	1.01	1.00	1.01	
1998-A	185,000	1.91	1.61	1.33	1.35	1.16	1.11	1.15	1.07	1.02	1.01	1.05	1.08	1.03	1.01	
1998-B	213,000	2.46	1.17	1.39	1.29	1.21	1.12	1.06	1.11	1.06	1.08	1.06	1.02	1.01	1.02	
1998-C	229,000	2.18	1.73	1.28	1.30	1.06	1.17	1.17	1.13	1.07	1.05	1.03	1.05	1.01	1.01	
1998-D	256,000	2.14	1.41	1.42	1.39	1.12	1.10	1.06	1.16	1.09	1.05	1.02	1.01	1.04	1.00	
1999-A	291,000	3.96	1.19	1.46	1.24	1.10	1.07	1.16	1.10	1.09	1.09	1.02	1.01	1.02	1.02	
1999-B	327,000	2.36	1.86	1.21	1.24	1.16	1.10	1.20	1.08	1.08	1.02	1.06	1.04	1.01	1.01	
1999-C	363,000	3.46	1.09	1.52	1.23	1.18	1.17	1.17	1.06	1.04	1.05	1.03	1.04	1.00		
1999-D	393,000	2.56	1.86	1.31	1.08	1.20	1.09	1.02	1.06	1.06	1.03	1.05	1.03			
2000-A	416,000	1.88	1.67	1.24	1.45	1.11	1.11	1.04	1.05	1.11	1.05	1.01				
2000-B	443,000	2.09	1.91	1.18	1.32	1.20	1.01	1.19	1.02	1.04	1.02					
2000-C	471,000	2.28	1.70	1.27	1.25	1.26	1.12	1.18	1.07	1.13						
2000-D	515,000	2.39	1.68	1.28	1.30	1.16	1.10	1.08	1.13							
2001-A	567,000	2.98	1.13	1.54	1.15	1.26	1.02	1.02								
2001-B	637,000	1.77	1.70	1.29	1.27	1.25	1.07									
2001-C	688,000	2.48	1.25	1.43	1.36	1.11										
2001-D	737,000	2.50	1.93	1.12	1.37											
2002-A	798,000	2.51	1.73	1.29												
2002-B	848,000	2.36	1.40													
2002-C	857,000	2.80														
2002-D	976,000															
Linear	Average	2.456	1.542	1.325	1.276	1.191	1.087	1.106	1.098	1.059	1.049	1.036	1.027	1.016	1.014	
Weighted	Average	2.415	1.514	1.317	1.272	1.188	1.086	1.107	1.098	1.059	1.049	1.036	1.028	1.016	1.014	
Par Weight	Average	2.472	1.562	1.317	1.282	1.188	1.085	1.105	1.089	1.068	1.045	1.036	1.027	1.016	1.014	
Fitted	Average		1.932	1.382	1.216	1.142	1.101	1.076	1.060	1.048	1.040	1.034	1.029	1.025	1.022	1.082
Selected	Age/Age	2.500	1.600	1.350	1.275	1.180	1.100	1.100	1.090	1.060	1.045	1.035	1.030	1.015	1.012	1.080
Selected	Age/Ult	14.037	5.615	3.509	2.599	2.089	1.728	1.571	1.428	1.310	1.236	1.183	1.143	1.109	1.098	1.080
Emergence	Pattern	7.1%	17.8%	28.5%	38.5%	49.0%	57.9%	63.7%	70.0%	76.3%	80.9%	84.6%	87.5%	90.1%	91.5%	92.6%

Seller/Servicer Bank - Consumer Receivables Securitisation Pools
Cumulative Losses as a Percent of Initial Par
Evaluated @12/31/02

Exhibit 3
Sheet 3b

Issue	Initial Par	@ 3 Mos.	@ 6 Mos.	@ 9 Mos.	@ 12 Mos.	@ 15 Mos.	@ 18 Mos.	@ 21 Mos.	@ 24 Mos.	@ 27 Mos	@ 30 Mos.	@ 33 Mos.	@ 36 Mos.	@ 39 Mos.	@ 42 Mos	@ 45 Mos
1997-A	100,000	0.50	1.45	1.75	2.66	3.26	3.60	3.89	4.60	4.74	5.32	5.55	5.75	5.86	6.05	6.09
1997-B	112,000	0.53	1.26	1.68	2.67	3.23	3.65	3.99	4.23	4.77	5.33	5.51	5.57	5.67	5.79	5.95
1997-C	119,000	0.61	1.27	1.95	2.33	3.07	3.80	3.87	4.56	4.70	4.78	4.80	5.00	5.14	5.24	5.26
1997-D	124,000	0.48	1.03	1.67	2.50	2.96	3.74	3.91	4.27	4.74	5.05	5.41	5.76	6.02	6.10	6.16
1997-A	140,000	0.48	1.41	2.06	2.28	3.28	3.89	3.94	4.74	5.16	5.73	6.27	6.71	6.94	6.97	7.01
1997-B	160,000	0.48	1.02	1.73	2.49	3.00	3.84	3.79	4.49	5.15	5.25	5.56	5.80	5.96	6.03	6.12
1997-C	175,000	0.53	1.03	1.80	2.43	3.34	3.98	3.76	3.78	3.96	4.28	4.34	4.39	4.39	4.43	4.56
1997-D	185,000	0.59	1.27	1.79	2.35	3.37	3.60	4.21	4.40	5.07	5.17	5.25	5.30	5.48	5.68	5.74
1998-A	185,000	0.60	1.05	1.86	2.65	3.24	3.88	4.25	4.51	5.25	5.52	5.75	6.07	6.17	6.26	6.27
1998-B	213,000	0.52	1.21	1.86	2.32	3.29	3.61	3.96	4.04	4.34	4.49	4.52	4.55	4.71	4.80	4.88
1998-C	229,000	0.60	1.17	1.77	2.72	3.24	3.80	4.22	4.94	5.03	5.33	5.70	5.77	5.98	6.08	6.15
1998-D	256,000	0.41	1.37	2.01	2.67	3.32	3.53	3.87	3.94	4.11	4.50	4.78	4.82	4.95	5.13	5.21
1999-A	291,000	0.50	1.29	1.83	2.75	3.12	3.99	3.76	4.10	4.65	4.72	5.08	5.44	5.72	5.73	5.81
1999-B	327,000	0.40	1.27	1.92	2.70	3.09	3.72	4.14	4.75	5.07	5.65	6.01	6.23	6.27	6.40	6.48
1999-C	363,000	0.48	1.43	1.92	2.45	3.27	3.92	3.84	4.28	4.89	5.06	5.54	5.73	5.89	6.12	
1999-D	393,000	0.60	1.23	1.89	2.39	3.05	3.82	4.12	4.95	5.46	5.69	5.80	5.88	6.08		
2000-A	416,000	0.54	1.15	1.79	2.34	3.21	3.54	4.14	4.73	5.33	5.60	6.03	6.34			
2000-B	443,000	0.61	1.39	2.06	2.56	2.90	3.93	4.10	4.88	5.39	5.69	5.95				
2000-C	471,000	0.53	1.45	1.68	2.64	2.93	3.96	3.92	4.77	5.39	5.44					
2000-D	515,000	0.52	1.49	2.00	2.34	2.94	3.86	4.22	4.92	5.30						
2001-A	567,000	0.38	1.38	1.90	2.46	3.33	3.68	4.22	4.69							
2001-B	637,000	0.41	1.25	1.71	2.73	3.31	3.68	3.94								
2001-C	688,000	0.38	1.03	1.90	2.41	3.07	3.81									
2001-D	737,000	0.43	1.44	1.89	2.51	3.25										
2002-A	798,000	0.54	1.01	1.77	2.64											
2002-B	848,000	0.38	1.24	2.12												
2002-C	857,000	0.41	1.20													
2002-D	976,000	0.38														

Seller/Servicer Bank - Consumer Receivables Securitisation Pools
Cumulative Losses - Age to Age Development Factor Analysis

Issue	Initial Par	6/3	9/6	12/9	15/12	18/15	21/18	24/21	27/24	30/27	33/30	36/33	39/36	42/39	45/42	45 to Ult
1997-A	100,000	2.93	1.21	1.52	1.23	1.10	1.08	1.18	1.03	1.12	1.04	1.04	1.02	1.03	1.01	
1997-B	112,000	2.37	1.33	1.59	1.21	1.13	1.09	1.06	1.13	1.12	1.03	1.01	1.02	1.02	1.03	
1997-C	119,000	2.09	1.53	1.19	1.32	1.24	1.02	1.18	1.03	1.02	1.00	1.04	1.03	1.02	1.00	
1997-D	124,000	2.15	1.61	1.50	1.18	1.26	1.04	1.09	1.11	1.07	1.07	1.06	1.04	1.01	1.01	
1997 A	140,000	2.95	1.46	1.11	1.44	1.19	1.01	1.20	1.09	1.11	1.09	1.07	1.03	1.00	1.01	
1997 B	160,000	2.13	1.69	1.44	1.20	1.28	0.99	1.19	1.15	1.02	1.06	1.04	1.03	1.01	1.01	
1997-C	175,000	1.95	1.75	1.35	1.38	1.19	0.95	1.01	1.05	1.08	1.01	1.01	1.00	1.01	1.03	
1997 D	185,000	2.14	1.41	1.31	1.44	1.07	1.17	1.05	1.15	1.02	1.02	1.01	1.03	1.04	1.01	
1998-A	185,000	1.75	1.78	1.42	1.22	1.20	1.09	1.06	1.16	1.05	1.04	1.05	1.02	1.01	1.00	
1998-B	213,000	2.33	1.53	1.25	1.42	1.10	1.10	1.02	1.08	1.03	1.01	1.01	1.04	1.02	1.02	
1998-C	229,000	1.96	1.51	1.54	1.19	1.17	1.11	1.17	1.02	1.06	1.07	1.01	1.04	1.02	1.01	
1998-D	256,000	3.31	1.46	1.33	1.24	1.06	1.10	1.02	1.04	1.10	1.06	1.01	1.03	1.04	1.02	
1999-A	291,000	2.59	1.41	1.50	1.14	1.28	0.94	1.09	1.14	1.01	1.08	1.07	1.05	1.00	1.01	
1999 B	327,000	3.17	1.51	1.40	1.15	1.20	1.11	1.15	1.07	1.11	1.06	1.04	1.01	1.02	1.01	
1999-C	363,000	2.98	1.35	1.28	1.33	1.20	0.98	1.11	1.14	1.04	1.09	1.03	1.03	1.04		
1999-D	393,000	2.05	1.53	1.26	1.28	1.25	1.08	1.20	1.10	1.04	1.02	1.01	1.03			
2000-A	416,000	2.12	1.56	1.31	1.37	1.10	1.17	1.14	1.13	1.05	1.08	1.05				
2000-B	443,000	2.27	1.48	1.24	1.13	1.36	1.04	1.19	1.11	1.06	1.04					
2000-C	471,000	2.72	1.16	1.57	1.11	1.35	0.99	1.22	1.13	1.01						
2000-D	515,000	2.86	1.35	1.17	1.26	1.31	1.09	1.17	1.08							
2001-A	567,000	3.66	1.37	1.29	1.35	1.11	1.15	1.11								
2001-B	637,000	3.02	1.37	1.60	1.21	1.11	1.07									
2001 C	688,000	2.73	1.83	1.27	1.27	1.24										
2001-D	737,000	3.33	1.31	1.33	1.29											
2002-A	798,000	1.87	1.75	1.49												
2002-B	848,000	3.26	1.72													
2002-C	857,000	2.91														
2002-D	976,000															
Linear	Average	2.578	1.500	1.370	1.265	1.196	1.063	1.124	1.096	1.059	1.049	1.034	1.028	1.020	1.013	
Weighted	Average	2.513	1.482	1.363	1.261	1.192	1.061	1.124	1.096	1.058	1.050	1.035	1.028	1.020	1.012	
Par Weight	Average	2.713	1.511	1.368	1.260	1.205	1.069	1.134	1.100	1.053	1.053	1.033	1.028	1.021	1.013	
Fitted	Average		1.906	1.379	1.217	1.143	1.103	1.078	1.061	1.050	1.042	1.035	1.030	1.026	1.023	1.087
Selected	Age/Age	2.600	1.600	1.370	1.250	1.200	1.100	1.090	1.075	1.055	1.045	1.033	1.030	1.020	1.010	1.087
Selected	Age/Ult	14.481	5.570	3.481	2.541	2.033	1.694	1.540	1.413	1.314	1.246	1.192	1.154	1.120	1.098	1.088
Emergence	Pattern	6.9%	18.0%	28.7%	39.4%	49.2%	59.0%	64.9%	70.8%	76.1%	80.3%	83.9%	86.7%	89.3%	91.0%	92.0%

S-B IBNR Example
Scenario: No Losses

Exhibit 4
Sheet 1

Estimated Ultimate Loss Ratio

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	22.5%	20.0%	17.5%	15.0%	12.5%	10.0%	7.5%	5.0%	2.5%	0.0%
2002		22.5%	20.0%	17.5%	15.0%	12.5%	10.0%	7.5%	5.0%	2.5%
2003			22.5%	20.0%	17.5%	15.0%	12.5%	10.0%	7.5%	5.0%
2004				22.5%	20.0%	17.5%	15.0%	12.5%	10.0%	7.5%
2005					22.5%	20.0%	17.5%	15.0%	12.5%	10.0%
2006						22.5%	20.0%	17.5%	15.0%	12.5%
2007							22.5%	20.0%	17.5%	15.0%
2008								22.5%	20.0%	17.5%
2009									22.5%	20.0%
2010										22.5%
Aggregate Portfolio=>	22.5%	21.3%	20.0%	18.8%	17.6%	16.4%	15.2%	13.9%	12.6%	11.3%

Cumulative Incurred Loss

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2002		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2003			0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2004				0.000	0.000	0.000	0.000	0.000	0.000	0.000
2005					0.000	0.000	0.000	0.000	0.000	0.000
2006						0.000	0.000	0.000	0.000	0.000
2007							0.000	0.000	0.000	0.000
2008								0.000	0.000	0.000
2009									0.000	0.000
2010										0.000

Cumulative Earned Premium

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10	Ultimate Earned Premium
2001	10.00	20.00	30.00	40.00	50.00	60.00	70.00	80.00	90.00	100.00	100.0
2002		20.00	40.00	60.00	80.00	100.00	120.00	140.00	160.00	180.00	200.0
2003			30.00	60.00	90.00	120.00	150.00	180.00	210.00	240.00	300.0
2004				42.00	84.00	126.00	168.00	210.00	252.00	294.00	420.0
2005					54.60	109.20	163.80	218.40	273.00	327.60	546.0
2006						68.25	136.50	204.75	273.00	341.25	682.5
2007							75.08	150.15	225.23	300.30	750.8
2008								82.58	165.17	247.75	825.8
2009									90.84	181.68	908.4
2010										99.92	999.2

Reported Loss Ratio

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
2002		0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
2003			0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
2004				0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
2005					0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
2006						0.0%	0.0%	0.0%	0.0%	0.0%
2007							0.0%	0.0%	0.0%	0.0%
2008								0.0%	0.0%	0.0%
2009									0.0%	0.0%
2010										0.0%

Cumulative Estimated IBNR

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	2.250	4.000	5.250	6.000	6.250	6.000	5.250	4.000	2.250	0.000
2002		4.500	8.000	10.500	12.000	12.500	12.000	10.500	8.000	4.500
2003			6.750	12.000	15.750	18.000	18.750	18.000	15.750	12.000
2004				9.450	16.800	22.050	25.200	26.250	25.200	22.050
2005					12.285	21.840	28.665	32.760	34.125	32.760
2006						15.356	27.300	35.831	40.950	42.656
2007							16.892	30.030	39.414	45.045
2008								18.581	33.033	43.356
2009									20.439	36.336
2010										22.483

Expected Emergence of Reported Losses

Cumulative Loss Emergence ==>	Yr 1	Yr 2	Yr 3	Yr 4	Yr 5	Yr 6	Yr 7	Yr 8	Yr 9	Yr 10
	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.250	1.000	2.250	4.000	6.250	9.000	12.250	16.000	20.250	25.000
2002		0.500	2.000	4.500	8.000	12.500	18.000	24.500	32.000	40.500
2003			0.750	3.000	6.750	12.000	18.750	27.000	36.750	48.000
2004				1.050	4.200	9.450	16.800	26.250	37.800	51.450
2005					1.365	5.460	12.285	21.840	34.125	49.140
2006						1.706	6.825	15.356	27.300	42.656
2007							1.877	7.508	16.892	30.030
2008								2.065	8.258	18.581
2009									2.271	9.084
2010										2.498

S-B IBNR Example
Scenario: Actual Losses = Expected

Exhibit 5
Sheet 1

Estimated Ultimate Loss Ratio

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%
2002		25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%
2003			25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%
2004				25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%
2005					25.0%	25.0%	25.0%	25.0%	25.0%	25.0%
2006						25.0%	25.0%	25.0%	25.0%	25.0%
2007							25.0%	25.0%	25.0%	25.0%
2008								25.0%	25.0%	25.0%
2009									25.0%	25.0%
2010										25.0%
Aggregate Portfolio=>	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%	25.0%

Cumulative Incurred Loss

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.250	1.000	2.250	4.000	6.250	9.000	12.250	16.000	20.250	25.000
2002		0.500	2.000	4.500	8.000	12.500	18.000	24.500	32.000	40.500
2003			0.750	3.000	6.750	12.000	18.750	27.000	36.750	48.000
2004				1.050	4.200	9.450	16.800	26.250	37.800	51.450
2005					1.365	5.460	12.285	21.840	34.125	49.140
2006						1.706	6.825	15.356	27.300	42.656
2007							1.877	7.508	16.892	30.030
2008								2.065	8.258	18.581
2009									2.271	9.084
2010										2.498

Cumulative Earned Premium

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10	Ultimate Earned Premium
2001	10.00	20.00	30.00	40.00	50.00	60.00	70.00	80.00	90.00	100.00	100.0
2002		20.00	40.00	60.00	80.00	100.00	120.00	140.00	160.00	180.00	200.0
2003			30.00	60.00	90.00	120.00	150.00	180.00	210.00	240.00	300.0
2004				42.00	84.00	126.00	168.00	210.00	252.00	294.00	420.0
2005					54.60	109.20	163.80	218.40	273.00	327.60	546.0
2006						68.25	136.50	204.75	273.00	341.25	682.5
2007							75.08	150.15	225.23	300.30	750.8
2008								82.58	165.17	247.75	825.8
2009									90.84	181.68	908.4
2010										99.92	999.2

S-B IBNR Example
Scenario: Actual Losses = Expected

Exhibit 5
Sheet 2

Reported Loss Ratio

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	2.5%	5.0%	7.5%	10.0%	12.5%	15.0%	17.5%	20.0%	22.5%	25.0%
2002		2.5%	5.0%	7.5%	10.0%	12.5%	15.0%	17.5%	20.0%	22.5%
2003			2.5%	5.0%	7.5%	10.0%	12.5%	15.0%	17.5%	20.0%
2004				2.5%	5.0%	7.5%	10.0%	12.5%	15.0%	17.5%
2005					2.5%	5.0%	7.5%	10.0%	12.5%	15.0%
2006						2.5%	5.0%	7.5%	10.0%	12.5%
2007							2.5%	5.0%	7.5%	10.0%
2008								2.5%	5.0%	7.5%
2009									2.5%	5.0%
2010										2.5%

Cumulative Estimated IBNR

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	2.250	4.000	5.250	6.000	6.250	6.000	5.250	4.000	2.250	0.000
2002		4.500	8.000	10.500	12.000	12.500	12.000	10.500	8.000	4.500
2003			6.750	12.000	15.750	18.000	18.750	18.000	15.750	12.000
2004				9.450	16.800	22.050	25.200	26.250	25.200	22.050
2005					12.285	21.840	28.665	32.760	34.125	32.760
2006						15.356	27.300	35.831	40.950	42.656
2007							16.892	30.030	39.414	45.045
2008								18.581	33.033	43.356
2009									20.439	36.336
2010										22.483

Expected Emergence of Reported Losses

Cumulative Loss Emergence ==>	Yr 1	Yr 2	Yr 3	Yr 4	Yr 5	Yr 6	Yr 7	Yr 8	Yr 9	Yr 10
	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.250	1.000	2.250	4.000	6.250	9.000	12.250	16.000	20.250	25.000
2002		0.500	2.000	4.500	8.000	12.500	18.000	24.500	32.000	40.500
2003			0.750	3.000	6.750	12.000	18.750	27.000	36.750	48.000
2004				1.050	4.200	9.450	16.800	26.250	37.800	51.450
2005					1.365	5.460	12.285	21.840	34.125	49.140
2006						1.706	6.825	15.356	27.300	42.656
2007							1.877	7.508	16.892	30.030
2008								2.065	8.258	18.581
2009									2.271	9.084
2010										2.498

S-B IBNR Example
Scenario: Actual Losses = 3 x's Expected

Exhibit 6
Sheet 1

Estimated Ultimate Loss Ratio

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	30.0%	35.0%	40.0%	45.0%	50.0%	55.0%	60.0%	65.0%	70.0%	75.0%
2002		30.0%	35.0%	40.0%	45.0%	50.0%	55.0%	60.0%	65.0%	70.0%
2003			30.0%	35.0%	40.0%	45.0%	50.0%	55.0%	60.0%	65.0%
2004				30.0%	35.0%	40.0%	45.0%	50.0%	55.0%	60.0%
2005					30.0%	35.0%	40.0%	45.0%	50.0%	55.0%
2006						30.0%	35.0%	40.0%	45.0%	50.0%
2007							30.0%	35.0%	40.0%	45.0%
2008								30.0%	35.0%	40.0%
2009									30.0%	35.0%
2010										30.0%
Aggregate Portfolio=>	30.0%	32.5%	35.0%	37.4%	39.8%	42.2%	44.6%	47.2%	49.8%	52.4%

Cumulative Incurred Loss

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.750	3.000	6.750	12.000	18.750	27.000	36.750	48.000	60.750	75.000
2002		1.500	6.000	13.500	24.000	37.500	54.000	73.500	96.000	121.500
2003			2.250	9.000	20.250	36.000	56.250	81.000	110.250	144.000
2004				3.150	12.600	28.350	50.400	78.750	113.400	154.350
2005					4.095	16.380	36.855	65.520	102.375	147.420
2006						5.119	20.475	46.069	81.900	127.969
2007							5.631	22.523	50.676	90.090
2008								6.194	24.775	55.743
2009									6.813	27.252
2010										7.494

Cumulative Earned Premium

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10	Ultimate Earned Premium
2001	10.00	20.00	30.00	40.00	50.00	60.00	70.00	80.00	90.00	100.00	100.0
2002		20.00	40.00	60.00	80.00	100.00	120.00	140.00	160.00	180.00	200.0
2003			30.00	60.00	90.00	120.00	150.00	180.00	210.00	240.00	300.0
2004				42.00	84.00	126.00	168.00	210.00	252.00	294.00	420.0
2005					54.60	109.20	163.80	218.40	273.00	327.60	546.0
2006						68.25	136.50	204.75	273.00	341.25	682.5
2007							75.08	150.15	225.23	300.30	750.8
2008								82.58	165.17	247.75	825.8
2009									90.84	181.68	908.4
2010										99.92	999.2

S-B IBNR Example
Scenario: Actual Losses = 3 x's Expected

Exhibit 6
Sheet 2

Reported Loss Ratio

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	7.5%	15.0%	22.5%	30.0%	37.5%	45.0%	52.5%	60.0%	67.5%	75.0%
2002		7.5%	15.0%	22.5%	30.0%	37.5%	45.0%	52.5%	60.0%	67.5%
2003			7.5%	15.0%	22.5%	30.0%	37.5%	45.0%	52.5%	60.0%
2004				7.5%	15.0%	22.5%	30.0%	37.5%	45.0%	52.5%
2005					7.5%	15.0%	22.5%	30.0%	37.5%	45.0%
2006						7.5%	15.0%	22.5%	30.0%	37.5%
2007							7.5%	15.0%	22.5%	30.0%
2008								7.5%	15.0%	22.5%
2009									7.5%	15.0%
2010										7.5%

Cumulative Estimated IBNR

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	2.250	4.000	5.250	6.000	6.250	6.000	5.250	4.000	2.250	0.000
2002		4.500	8.000	10.500	12.000	12.500	12.000	10.500	8.000	4.500
2003			6.750	12.000	15.750	18.000	18.750	18.000	15.750	12.000
2004				9.450	16.800	22.050	25.200	26.250	25.200	22.050
2005					12.285	21.840	28.665	32.760	34.125	32.760
2006						15.356	27.300	35.831	40.950	42.656
2007							16.892	30.030	39.414	45.045
2008								18.581	33.033	43.356
2009									20.439	36.336
2010										22.483

Expected Emergence of Reported Losses

Cumulative Loss Emergence ==>	Yr 1	Yr 2	Yr 3	Yr 4	Yr 5	Yr 6	Yr 7	Yr 8	Yr 9	Yr 10
	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.250	1.000	2.250	4.000	6.250	9.000	12.250	16.000	20.250	25.000
2002		0.500	2.000	4.500	8.000	12.500	18.000	24.500	32.000	40.500
2003			0.750	3.000	6.750	12.000	18.750	27.000	36.750	48.000
2004				1.050	4.200	9.450	16.800	26.250	37.800	51.450
2005					1.365	5.460	12.285	21.840	34.125	49.140
2006						1.706	6.825	15.356	27.300	42.656
2007							1.877	7.508	16.892	30.030
2008								2.065	8.258	18.581
2009									2.271	9.084
2010										2.498

Estimated Ultimate Loss Ratio

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	22.5%	20.0%	50.8%	40.0%	32.5%	26.7%	21.8%	17.5%	24.7%	20.0%
2002		22.5%	20.0%	17.5%	21.3%	17.5%	14.2%	11.1%	8.1%	5.3%
2003			39.2%	36.7%	34.2%	31.7%	25.8%	21.1%	17.0%	13.3%
2004				22.5%	20.0%	37.3%	29.9%	24.4%	19.9%	16.0%
2005					22.5%	20.0%	17.5%	15.0%	12.5%	10.0%
2006						22.5%	74.9%	78.6%	60.8%	49.1%
2007							22.5%	20.0%	17.5%	15.0%
2008								22.5%	29.1%	23.6%
2009									22.5%	20.0%
2010										22.5%
Aggregate Portfolio=>	22.5%	21.3%	35.0%	28.7%	26.0%	26.7%	30.5%	28.5%	24.7%	20.4%

Cumulative Incurred Loss

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.000	0.000	10.000	10.000	10.000	10.000	10.000	10.000	20.000	20.000
2002		0.000	0.000	0.000	5.000	5.000	5.000	5.000	5.000	5.000
2003			5.000	10.000	15.000	20.000	20.000	20.000	20.000	20.000
2004				0.000	0.000	25.000	25.000	25.000	25.000	25.000
2005					0.000	0.000	0.000	0.000	0.000	0.000
2006						0.000	75.000	125.000	125.000	125.000
2007							0.000	0.000	0.000	0.000
2008								0.000	15.000	15.000
2009									0.000	0.000
2010										0.000

Cumulative Earned Premium

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10	Ultimate Earned Premium
2001	10.00	20.00	30.00	40.00	50.00	60.00	70.00	80.00	90.00	100.00	100.0
2002		20.00	40.00	60.00	80.00	100.00	120.00	140.00	160.00	180.00	200.0
2003			30.00	60.00	90.00	120.00	150.00	180.00	210.00	240.00	300.0
2004				42.00	84.00	126.00	168.00	210.00	252.00	294.00	420.0
2005					54.60	109.20	163.80	218.40	273.00	327.60	546.0
2006						68.25	136.50	204.75	273.00	341.25	682.5
2007							75.08	150.15	225.23	300.30	750.8
2008								82.58	165.17	247.75	825.8
2009									90.84	181.68	908.4
2010										99.92	999.2

Reported Loss Ratio

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.0%	0.0%	33.3%	25.0%	20.0%	16.7%	14.3%	12.5%	22.2%	20.0%
2002		0.0%	0 0%	0.0%	6.3%	5.0%	4.2%	3.6%	3.1%	2.8%
2003			16.7%	16.7%	16.7%	16.7%	13.3%	11.1%	9.5%	8 3%
2004				0 0%	0.0%	19.8%	14.9%	11.9%	9.9%	8 5%
2005					0.0%	0.0%	0.0%	0.0%	0 0%	0.0%
2006						0.0%	54.9%	61.1%	45.8%	36.6%
2007							0 0%	0.0%	0.0%	0.0%
2008								0.0%	9.1%	6 1%
2009									0.0%	0.0%
2010										0.0%

Cumulative Estimated IBNR

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	2.250	4.000	5.250	6.000	6.250	6.000	5.250	4 000	2.250	0.000
2002		4.500	8.000	10.500	12 000	12.500	12.000	10.500	8.000	4.500
2003			6.750	12.000	15.750	18.000	18.750	18.000	15.750	12.000
2004				9.450	16.800	22.050	25.200	26.250	25.200	22.050
2005					12.285	21.840	28.665	32.760	34.125	32.760
2006						15.356	27.300	35.831	40.950	42 656
2007							16.892	30 030	39.414	45.045
2008								18.581	33.033	43 356
2009									20.439	36.336
2010										22.483

Expected Emergence of Reported Losses

Cumulative Loss Emergence ==>	Yr 1	Yr 2	Yr 3	Yr 4	Yr 5	Yr 6	Yr 7	Yr 8	Yr 9	Yr 10
	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%

Orig. Yr.	Evaluated @ 12/01	Evaluated @ 12/02	Evaluated @ 12/03	Evaluated @ 12/04	Evaluated @ 12/05	Evaluated @ 12/06	Evaluated @ 12/07	Evaluated @ 12/08	Evaluated @ 12/09	Evaluated @ 12/10
2001	0.250	1.000	2.250	4.000	6.250	9.000	12.250	16.000	20.250	25.000
2002		0.500	2.000	4.500	8.000	12.500	18.000	24.500	32.000	40.500
2003			0.750	3.000	6 750	12.000	18.750	27.000	36.750	48.000
2004				1.050	4 200	9.450	16.800	26.250	37 800	51.450
2005					1.365	5.460	12.285	21.840	34.125	49.140
2006						1.706	6.825	15.356	27.300	42.656
2007							1.877	7.508	16.892	30.030
2008								2.065	8.258	18.581
2009									2.271	9.084
2010										2 498

A Statistical Simulation Approach for Estimating the Reserve for Uncollectible Reinsurance

Nicholas H. Pastor, FCAS, MAAA

A Statistical Simulation Approach for Estimating the Reserve for Uncollectible Reinsurance

Nicholas H. Pastor, FCAS, MAAA

It is important to note that the ideas presented in this paper are purely the author's own. Depending on the actual circumstances, it is possible that other approaches for estimating the uncollectible reinsurance reserve may be more appropriate than that presented herein.

Abstract

Recent insolvencies and catastrophic events have heightened concern in the insurance industry over the risk of uncollectible reinsurance. The current approach for estimating reserves for this line in the Annual Statement is relatively unscientific and, as a result, may not reflect the company's true reinsurance recoverability risk.

The objective of this paper is to introduce a statistical approach for estimating this reserve that considers more specifically the risks of the company's reinsurers and the potential for correlations between reinsurer failures within a given period as well as over time.

The paper will describe the basic framework for this model, including:

1. Defining the data required
2. Setting up the basic structure of the model
3. Consideration of the timing of future payments and potential offsets
4. Consideration of correlations
5. Potential applications

Introduction

The potential for uncollectible reinsurance has always been a major concern for both insurers and reinsurers[1]. For some companies, reinsurance recoveries represent one of the largest assets on their balance sheet (or contra-liability under statutory accounting principles). Some carriers have gone insolvent over the years as a result of an inability to collect reinsurance recoveries (usually because the reinsurer has gone insolvent as well). In times of financial difficulty for the industry, whether due to the market cycle, the general economy, or catastrophic events, the ripple effect of reinsurer insolvencies is often felt throughout the industry for years. As a result of these concerns, security considerations are often (and should be) the largest factor for a ceding company when purchasing reinsurance, both in selecting a reinsurer and in negotiating the terms of the reinsurance contract.

Given all of the above factors, it is surprising that the process for estimating uncollectible reinsurance recoveries in a company's financial statements utilized by regulators in the U.S. is fairly unscientific and often does not receive thorough scrutiny. This may be in part due to the fact that standard actuarial opinion wording requires mention of such amounts, but does not require sign off by the opining actuary. Recently, many countries throughout the world have

[1] As of December 31, 2002, reported reinsurance recoverables amounted to nearly 80% of reported surplus.

been reviewing their accounting and actuarial requirements, and some of these countries have now required that the reserve for uncollectible reinsurance be a part of the actuarial sign off requirements.

Without delving into the question as to whether such signoff is appropriate or should be required, this paper will discuss an alternative methodology for estimating this reserve. The paper will first discuss in broad terms some potential factors that may influence reinsurer default as well as describe the current approach for estimating the reserve and some of its weaknesses. An alternative methodology will then be described. The alternative method described uses a statistical simulation approach that more specifically considers the potential failure of the company's reinsurers and the potential for correlations between reinsurer failures within a given period as well as over time. The key parameters of the model and the data required will be introduced and the general structure of the model will be described. Key issues such as offset and correlations will also be discussed in more detail. Finally, a simple example will be presented, followed by a discussion of potential applications and areas where more research could be undertaken.

Potential Factors Influencing Reinsurer Default

Failure to recover amounts due from reinsurers can result from a number of factors. The most common factors causing defaults have been:

1) Disputes between the cedant and the reinsurer, and
2) An inability to pay due to financial difficulties for the reinsurer.

The financial difficulties of the reinsurer may have been caused by poor economic conditions, poor insurance market conditions, exposure to cumulative causation claims, or catastrophic events. Unfortunately, factors which contribute to a particular reinsurer's financial difficulty will most likely also negatively influence many of the insurance company's other reinsurers, as well as the insurance company itself. Further, these factors are likely to have lingering effects for a number of years, which may influence reinsurance collections in future years.

This paper is primarily intended to address uncollectible reinsurance arising from the inability to pay due to reinsurer financial difficulty. Disputes between insurer and reinsurer are typically distinct events and need to be evaluated based on the unique conditions of the particular dispute. While the incidence of disputes often increases during times of financial stress, it is frequently the case that the potential for uncollectibility will depend primarily on the specific issues of the dispute.

Current Regulatory Approach for Estimating the Reserve

For Statutory Annual Statements, the currently required methodology used in the U.S. for estimating uncollectible reinsurance is outlined in the NAIC Instructions to the Annual Statement. The uncollectible reinsurance reserves (referred to as the Provision for Reinsurance) are calculated in Schedule F of the Statement and can be broadly separated into two components: overdue authorized reinsurance and unauthorized reinsurance. Authorized reinsurers include most U.S. reinsurers meeting certain conditions (e.g. non-affiliates and/or not in liquidation), certain pools and associations, and Lloyd's of London. Unauthorized reinsurers include other foreign reinsurers, affiliates, certain pools and associations, and reinsurers in liquidation.

The provision for overdue authorized reinsurance is calculated as 20% of amounts due and unpaid over 90 days plus 20% of disputed amounts. An additional penalty can be applied for reinsurers where at least 20% of all recoveries are overdue. In this case, the total provision is the maximum of the amount calculated using the formulas above, or 20% of the net unsecured recoverables[2]. The provision for unauthorized reinsurance is calculated as the sum of the net unsecured recoverables, 20% of amounts due over 90 days, and 20% of amounts in dispute. However, this sum is limited to the total reinsurance recoverable from the particular reinsurer. The provision for unauthorized reinsurance can be quite significant. Its existence often results in a significant competitive disadvantage for foreign reinsurers operating in the U.S., as reinsurance contract provisions often require any unauthorized reinsurer to post collateral for the full amount of outstanding losses, IBNR, and unearned premiums in order for the cedant to avoid this penalty. Some of the largest international reinsurers have developed or acquired U.S. operations in part to avoid the penalty.

The current regulatory approach does consider many factors that likely have a significant influence on reinsurance recoveries, such as security, disputes, and late payments. In addition, it also encourages desirable behaviors among insurers and reinsurers, such as collateral requirements and increased pressure to recover and pay amounts due. However, this approach also has a number of weaknesses and limitations. Some of the key limitations are:

- Timing of recoveries – The current approach does not consider the potentially significant differences in the reinsurance recoverability risk between reinsurers based on the expected timing of the recoveries. Expected timing in recoveries between reinsurers may differ significantly based on the lines of business written, the limits and attachment points, and the terms of the contracts. All things being equal, recoveries that are due sooner (e.g. for a Property reinsurer) are less risky than recoveries due a number of years in the future (e.g. for an excess Workers' Compensation reinsurer). This is because any number of negative events could influence the reinsurer's financial condition in the future, before the long-term recovery is due. Also, in situations where a reinsurer may already be facing financial difficulties, short-term recoveries may be available from current assets, but there will be no guarantee that assets will be available to make payments in future years. Such considerations can also make a cedant pursue a commutation of recoverables, providing certainty of cash flow, at the expense of reinsurance coverage in the future.

- No reflection of relative financial strength – The blanket 20% provision for overdue or disputed amounts and the flat penalty for certain unsecured recoveries does not make any distinction between the financial stability of a cedant's specific reinsurers, other than the general theory that foreign reinsurers are generally not as strong as U.S reinsurers and that weaker reinsurers are more likely to have amounts overdue or in dispute. While these premises are generally accepted in the industry, more specific measures of financial

[2] "Net unsecured recoverables" is the total recoverables for paid and unpaid loss and LAE, unearned premiums, and contingent commissions less offsets (which include funds held under reinsurance treaties, letters of credit, ceded balances payable, and other balances).

strength are available and could be used to more precisely estimate the uncollectibility risk.

- Overdue balances and disputed amounts may not reflect the ability to pay – Further to the points above, if overdue balances and amounts in disputes are due from an otherwise financially strong reinsurer, the expected recovery may still be 100%; however, the timing of such recovery may just be delayed. While the timing issue can have a critical impact on the recoverability risk, some disputes and delays may just be caused by a reinsurer being contentious in their settlement practices.

- Correlation between reinsurers and over time – As discussed, many of the factors that influence the recoverability risk for a specific reinsurer will also influence many of the other reinsurers in the industry. As a result, the provision for uncollectible reinsurance at any point in time will be a function of the relative strength of the insurance industry as well as other factors, such as the prevailing economic and interest rate environment. Further, the relative risk will likely change over time and will also be a function of the specific reinsurers, type of business reinsured, and the timing of future recoveries. The correlation between reinsurers at a point in time and over a longer period of time could have a significant effect on the expected non-collections for an individual company.

- Foreign bias – The current provision for uncollectible reinsurance has been accused of having a bias against foreign reinsurers, as collateral requirements on unauthorized reinsurance can be costly and potentially restrictive, along with creating additional frictional costs. While various reasons can be put forth supporting this practice, the effects can limit foreign reinsurers ability to compete in the U.S. market, which may further weaken their relative financial standing against U.S. reinsurers

Proposed Approach

The proposed approach involves estimating the timing and amount of expected cash flows from each individual reinsurer and simulating expected failure rates (i.e. the percent of the reinsurance recovery *not* received) against each cash flow. The sum of the failure rates times the cash flows equals the uncollectible reinsurance reserve.

The failure rates at each point in time are based on the likelihood of default for each given reinsurer, i.e. financially unstable reinsurers would likely have higher expected failure rates. Further, the simulation model would include an "industry effect" to reflect the potential correlation of failure rates between reinsurers. The application of this industry factor would increase the expected failure rates for each reinsurer in a poor environment and reduce the failure rates in a favorable environment. In addition, the correlation of failure rates over time would be reflected in both the industry effect and at the individual reinsurer level (i.e. a poor industry in one time period is unlikely to immediately become a favorable industry in the next period, and a reinsurer who defaulted in one period is likely to also default in future periods).

Note that a best estimate of the reserve could be estimated without using simulation by simply multiplying the expected failure rates by the expected recoveries. However, use of the approach detailed here will help give the company a better understanding of the underlying risk and can also be used in a number of different applications.

Data Required

The following information would be required to build the model:

- Details of reinsurance program participants and their shares of each contract
- Current outstanding balances and amounts payable by reinsurer
- Expected recoveries and future premium payments for each reinsurance contract
- Expected payment patterns and timing of future premium payments by contract
- Potential funds available to offset uncollectible recoveries by reinsurer (premiums payable, funds held, letters of credit, etc.)
- Expected failure rates by reinsurer
- Correlation coefficients for the failure rates between reinsurers and across time periods

The expected failure rates and correlation coefficients would clearly be the most difficult and judgmental data items to determine and such a discussion is beyond the scope of this paper. The author is not aware of any studies performed or methodologies developed to estimate failure rates specifically for reinsurers. However, a number of methods and techniques have been used to analyze factors such as bond default rates in a variety of industries. One potential approach is to use the results of one of these techniques to measure default rates by bond rating (such as Moody's or S&P) and utilize these as the expected failure rating for each reinsurer. Further analysis could be performed to try and relate these bond ratings to more commonly accepted insurer ratings, such as A.M. Best ratings. Of course, this leaves the difficult issue of foreign reinsurers, where no published rating may exist. An approach in such a case could be to assign a rating based on surplus level or some other financial measure.

The estimate of the correlation between reinsurers and between time periods would also be difficult to determine. One potential approach for estimating the reinsurer correlation could be to gather historic information on reinsurer failure rates and to simulate industry failure rates (assuming various correlations) against actual failures. Estimating the correlations between time periods could be done by gathering data on insurance underwriting cycles and measuring the correlation in underwriting results over different periods of time.

Estimating the Timing of Recoveries

The first step in building the model is to estimate the amount and timing of expected recoveries by reinsurer. Depending upon the complexity of the company's reinsurance program this can be a very time-consuming and data-intensive step. That being said, cedants already need to perform a significant part of this step to complete Schedule F. The only additional effort that is required beyond what is needed to produce Schedule F is to determine the timing of the expected recoveries.

To estimate the timing, loss payment patterns can be applied to the expected recoveries by reinsurer. The payment patterns should be applied at a detailed enough level to produce accurate overall recovery estimates for each reinsurer. If the data is not too unwieldy, this can be done at the contract level for each reinsurer. Or, if the reinsurance program is extremely complicated, this can be done at a more summarized level (e.g. line of business, type of contract, etc.).

An example of the output from this step for a single reinsurer is shown in the following table:

Table 1
Reinsurer A – Timing of Expected Recoveries by Calendar Period
(All figures in 000's)

Contract Type	Period =>	Curr O/S Balance	Cal Yr 2003	Cal Yr 2004	Cal Yr 2005	Cal Yr 2006	Cal Yr 2007	Cal Yr 2008+
Q/S – Work Comp		$1,000	$2,000	$1,000	$750	$500	$250	$250
Q/S – Genl. Liab		0	400	0	0	0	0	0
Q/S – Property		0	2,000	600	50	0	0	0
XS – CAT Property		0	0	0	0	0	0	0
XS – All Casualty		2,000	3,000	2,000	1,500	1,000	750	500
Finite – Whole Account		1,000	4,000	2,000	1,500	750	400	250
Total		$4,000	$11,400	$5,600	$3,800	$2,250	$1,400	$1,000

For simplicity, the recoveries in the chart above have been summarized by contract type. In addition, all recoveries expected to be made after five years have been grouped. This is another potential simplification that can be made to the model. If the expected recoveries beyond a certain point in time are small, the impact may be minimal. In such a case, in the actual application of the model it could reasonably be assumed that a reinsurer who had defaulted on obligations prior to this point would also default on all subsequent obligations.

In the above table, the timing of recoveries would be determined based on the date the payments would be due from the reinsurer (i.e. the date of recovery assuming no defaults or slow-paying reinsurers). Further adjustments can be made to the above figures to reflect slow-paying reinsurers or, alternatively, this could be handled as part of the actual modeling. This is discussed in more detail below.

Offsets

The second step in setting up the model is to determine the potential offsets that could be applied to uncollectible balances at each point in time. The three most commonly used offsets to uncollectible balances are: 1) funds withheld under reinsurance treaties, 2) ceded balances payable, and 3) letters of credit or other allowable forms of collateral. In addition, if the cedant also assumes business from the same reinsurer, then amounts payable under such a treaty could also be used as an offset.

Once again, the majority of this information is already collected at the reinsurer level in order to complete Schedule F. However, an additional factor that could be considered within the model is the possibility for premiums due on reinsurance contracts written in the future being used to offset current uncollected balances. Of course, in such a case the future reinsurance contracts will also likely produce losses that would also have similar collection problems. For now, we will focus solely on balances due and offsets on contracts written previously.

The following table shows an example of the output from this step.

Table 2
Potential Offsets by Reinsurer at 12/31/02
(All figures in 000's)

Reinsurer	**Offset Item =>**	**Funds Held**	**Letters of Credit**	**Ceded Bal. Payable**	**Assumed Balances**		**Total**
Reinsurer A		$7,000	$5,000	$2,500	$0		$14,500
Reinsurer B		2,000	0	300	650		2,950
Reinsurer C		0	3,500	0	3,300		6,800
Reinsurer D		3,650	3,650	0	0		7,300
Total		$12,650	$12,150	$2,800	$3,950		$31,550

An additional consideration that can be incorporated into the model is the timing and availability of the various offsets, and the priority in which they may be applied. While the full amount of the potential offsets could be available to offset uncollectible balances in the coming year, some of these amounts may be paid to the reinsurer before the reinsurer defaults, hence reducing the available offsets at the time of default. Further, the priority in which the offsets are applied may also impact the available amount of offsets at any point in time.

For example, a reinsurance contract may specify that letters of credit are to be drawn down for any overdue balance before any other offset can be applied, and balances due the reinsurer can not be delayed as long as capacity exists on the LOC. In such a case, the ceding company may have to continue to pay balances due to a reinsurer in default as long as they are able to draw down on the LOC. Subsequently, these balances may be greatly reduced or eliminated at the point that the LOC is finally depleted and future uncollectibles may not be able to be offset by such funds. In this example, the timing and priority of offsets would have had a significant impact on the net uncollectible balance.

The following table shows the available offsets for a single reinsurer over time, assuming that recoveries are paid as due (i.e. as recoveries are made, collateral requirements are reduced). The structure of this chart is designed to be consistent with recovery timing chart above.

Table 3
Availability of Offsets for Reinsurer A
(All figures in 000's)

Offset Item	Period =>	Current Offsets	Yr End 2003	Yr End 2004	Cal Yr 2005	Yr End 2006	Yr End 2007
Funds Withheld		$7,000	$2,700	$1,400	$800	$400	$200
Letters of Credit		5,000	0	0	0	0	0
Ceded Balance Payable		2,500	0	0	0	0	0
Assumed Balances		0	0	0	0	0	0
Total Available Offsets		$14,500	$2,700	$1,400	$800	$400	$200

Simulating Failure Rates

The failure rates can reflect various different scenarios of reinsurer default, both full and partial. A full default would reflect an insolvency situation while partial defaults could reflect negotiated settlements (e.g. with a financially troubled reinsurer or a reinsurer in receivership). Slow-paying reinsurers could also be considered (i.e. payments defaulted in one year could potentially be recovered in a subsequent year).

The simulation of failure rates can be done in a fairly straightforward fashion. A simple approach is to create a uniform distribution for each reinsurer, with various points on the distribution corresponding to specific failure rates. For example, assume a specific reinsurer is expected to have a 5% probability of defaulting on 25% of their obligations, a 5% probability of defaulting on 50% of their obligations, and a 5% probability of total default. In this case, a random number could be simulated from a uniform distribution where a value between 0 and 0.05 would correspond to a 100% default, a value between 0.05 and 0.10 would correspond to a 50% default, and a value between 0.10 and 0.15 would correspond to a 25% default. An alternative could also be to create a continuous distribution of failure rates, though it would probably be necessary for this distribution to have a certain amount of probability mass at a level that corresponded to a full recovery.

The correlation between reinsurers can be modeled by first simulating an industry effect to reflect whether the environment was favorable or adverse for reinsurer solvency in that year. The simplest approach would be to simulate a random number from a uniform distribution. This random number could then be used to adjust the expected failure rates for each reinsurer. If we assumed that the adjustment varied between +/- 100%, then you could simply multiply the initial failure rates by 2 times the random number (i.e. if the randomly generated number was 0.57, the failure rates would be multiplied by 2 x 0.57, or 1.14). If the correlation between reinsurers was assumed to be lower, then the factor could be adjusted (e.g. this factor could be weighted with another random number).

The correlation across time periods is modeled in both the industry effect and at the individual reinsurer level. For the industry effect, the random number for a given period can be weighted with industry effect from the prior period. For example, if the Year 1 industry effect was 0.57 and the random number generator for Year 2 produced a factor of 0.49, these two numbers could be weighted together (using the year-over-year correlation coefficient as the weight) to produce the industry factor for Year 2.

One potential approach is to require that the default percentage in one year be at least as high as the default rate in the prior year (e.g. if the default rate in Year 1 was 25%, then the default rate in the following year would be 25% or higher). This would be consistent with insolvency and negotiated settlement scenarios. The downside of this approach is that it would not allow for a scenario where the reinsurer could recover from financial difficulties.

Another approach could be to weight the random number generated for a given year with the number generated for the previous year. At the individual reinsurer level, it is likely that the weight assigned to the prior year should be relatively high, since a reinsurer who fails to pay in one year is likely to fail to pay in the subsequent year as well. This approach could implicitly reflect slow-paying reinsurers as well as default. For example, if the default rate was 50% in Year 1 but reduced to 25% or 0% in a subsequent year, applying the reduced rate to the cumulative outstanding balance in Year 2 would allow partial or full recovery of the uncollected amounts from Year 1.

Simple Example

For this example, we will work off of the data shown above. In this case, we have assumed that the insurer has four reinsurers.

For our first step, assume we have determined the expected recoveries by calendar year period and reinsurer. This is shown in the chart below:

Table 4
Expected Recoveries by Reinsurer
(All figures in 000's)

Reinsurer	Period =>	Curr O/S Balance	Cal Yr 2003	Cal Yr 2004	Cal Yr 2005	Cal Yr 2006	Cal Yr 2007	Cal Yr 2008+
Reinsurer A		$4,000	$11,400	$5,600	$3,800	$2,250	$1,400	$1,000
Reinsurer B		550	1,000	700	700	0	0	0
Reinsurer C		10,000	4,000	1,000	0	0	0	0
Reinsurer D		800	5,000	3,000	1,000	500	500	0
Total Exp. Recoveries		$15,350	$21,400	$10,300	$5,500	$2,750	$1,900	$1,000

Note that the actual payments in the model will be specified by both reinsurer and reinsurance contract in order to allow any offsets to be applied appropriately to the specific recoveries that they support.

The potential (current) offsets by reinsurer are shown in the following chart:

Table 5
Potential Offsets by Reinsurer at 12/31/02
(All figures in 000's)

Reinsurer	**Offset Item =>**	**Funds Held**	**Letters of Credit**	**Ceded Bal. Payable**	**Assumed Balances**		**Total**
Reinsurer A		$7,000	$5,000	$2,500	$0		$14,500
Reinsurer B		2,000	0	300	650		2,950
Reinsurer C		0	3,500	0	3,300		6,800
Reinsurer D		3,650	3,650	0	0		7,300
Total		$12,650	$12,150	$2,800	$3,950		$31,550

For this example, we will use A.M. Best rating as an evaluator of each reinsurer's financial condition, and assume that each rating level relates to a specified failure distribution. The distribution will include three possible outcomes, 1) full recovery, 2) 50% failure, and 3) 100% failure.

The following chart shows each reinsurer's A.M. Best rating, the total outstanding recoveries (summarized from the chart above), the total current offsets, and the assumed probabilities of failure in a given year.

Table 6
Recoverables and Failure Rates by Reinsurer at 12/31/02
(All figures in 000's)

Reinsurer	A.M. Best Rating	Total O/S Recoveries	Total Curr. Offsets	Prob. Of 50% Failure	Prob. of 100% Failure
Reinsurer A	A-	$29,450	$14,500	1.5%	0.6%
Reinsurer B	B+	2,950	2,950	6.0%	1.5%
Reinsurer C	A-	15,000	6,800	1.5%	0.6%
Reinsurer D	A	10,800	7,300	0.6%	0.2%
Total		$58,200	$31,550		

We have assumed that Reinsurer B is fully collateralized, but the other three reinsurers only have collateral available to support certain contracts. In each case, the collateral is reduced in proportion to the remaining outstanding recoveries for the specific contract at the end of each calendar year.

Our correlation assumptions will be that each reinsurer's failure rate is 50% correlated with the industry factor, the industry factor is 50% correlated with the prior year's industry factor, and each reinsurer's failure probability in a given year is 80% correlated with their failure probability in the prior year.

Our failure rate simulations can then be performed. The first factors to be simulated are the industry effect factors. This process is shown below:

Table 7
Industry Effect Factors

Simulation Element	Period =>	Cal Yr 2003	Cal Yr 2004	Cal Yr 2005	Cal Yr 2006	Cal Yr 2007	Cal Yr 2008+
(1) Uniform Random Number		0.19	0.41	0.23	0.34	0.80	0.43
(2) Base Effect 2 x (1)		0.38	0.82	0.46	0.68	1.60	0.86
(3) Correlated Effect = [Avg (2), Prior Year Effect]		0.38	0.60	0.53	0.60	1.10	0.93

The reinsurer failure rates are then adjusted in the first year for the industry effects, as shown below.

Reinsurer	(1) Init. Prob. 50% Failure	(2) Init. Prob. of 100% Failure	(3) CY 2003 Industry Effect	(4) Uniform Random Number	(5)* Adj. Prob. Of 50% Failure	(6)* Adj. Prob. Of 100% Failure
Reinsurer A	1.5%	0.6%	0.38	0.67	1.29%	0.52%
Reinsurer B	6.0%	1.5%	0.38	0.82	6.06%	1.52%
Reinsurer C	1.5%	0.6%	0.38	0.43	0.93%	0.37%
Reinsurer D	0.6%	0.2%	0.38	0.56	0.45%	0.15%

* Failure rates are adjusted by the following factor: Average [(3), 2 * (4)]

Random numbers can then be generated to simulate whether any failures occur in the first year. The failure percentage (50% or 100%) is then applied to the balance due and reduced by any offsets.

For subsequent calendar years, the failure rates for each reinsurer can be adjusted in a similar fashion. Then, for each reinsurer, the random number generated to determine whether failure occurs is correlated with the prior year's number. This is done in a similar fashion to the manner that the industry effect was adjusted. For Reinsurer A, the first chart below shows the adjusted failure rates over each calendar year period. The second chart shows the calculation of the correlated random number for each period.

Table 8
Reinsurer A – Adjusted Failure Rates

Adjusted Failure Rates	Period =>	Cal Yr 2003	Cal Yr 2004	Cal Yr 2005	Cal Yr 2006	Cal Yr 2007	Cal Yr 2008+
(1) Init. Prob. of 50% Failure		1.5%	1.5%	1.5%	1.5%	1.5%	1.5%
(2) Init. Prob. of 100% Failure		0.6%	0.6%	0.6%	0.6%	0.6%	0.6%
(3) Industry Effect		0.38	0.60	0.53	0.60	1.10	0.93
(4) Uniform Random Number		0.67	0.49	0.51	0.42	0.38	0.68
(5) Adj. Prob. of 50% Failure		1.29%	1.19%	1.16%	1.08%	1.39%	1.72%
(6) Adj. Prob. of 100% Failure		0.52%	0.47%	0.46%	0.43%	0.56%	0.69%

Table 9
Reinsurer A – Correlated Random Numbers

Simulation Element	Period =>	Cal Yr 2003	Cal Yr 2004	Cal Yr 2005	Cal Yr 2006	Cal Yr 2007	Cal Yr 2008+
(1) Uniform Random Number		0.019	0.005	0.241	0.482	0.170	0.496
(2) Correlated Number = .20*(1) + .80*Prior Year		0.019	0.016	0.061	0.145	0.150	0.219

In this example, Reinsurer A would default on 50% of their obligations in calendar year 2004, since the correlated number of 0.016 is greater than the 100% failure rate of 0.0047, but still less than the sum of the 100% and 50% failure rates (0.0119+0.0047=0.0166). However, these amounts would subsequently be recovered in calendar year 2005.

Hence, for Reinsurer A, the simulated bad debt amounts for iteration 1 would be as follows:

Table 10
Reinsurer A – Bad Debt (in 000's)

1

	Period =>	Cal Yr 2003	Cal Yr 2004	Cal Yr 2005	Cal Yr 2006	Cal Yr 2007	Cal Yr 2008+
(1) Correlated Number		0.019	0.016	0.061	0.145	0.150	0.219
(2) Failure %		0%	50%	0%	0%	0%	0%
(3) O/S Balance = Prior Year (9)		4,000	0	100	0	0	0
(4) Cal Year Recoverables		11,400	5,600	3,800	2,250	1,400	1,000
(5) Total Due = (3)+(4)		15,400	5,600	3,900	2,250	1,400	1,000
(6) Amount Defaulted = (2)*(5)		0	2,800	0	0	0	0
(7) Available Offset		14,500	2,700	1,400	800	400	200
(8) Remaining Offsets*		14,500	2,700	0	800	400	200
(9) Net Default = Max[(6)–(8),0]		0	100	0	0	0	0
(10) Amount Recovered = (5) – (9)		15,400	5,500	3,900	2,250	1,400	1,000
(11) Ending Balance = (5) – (10)		0	100	0	0	0	0
(12) Bad Debt =11–Prior Year (11)		0	100	(100)	0	0	0
(13) Cumulative Bad Debt		0	100	0	0	0	0

* Remaining offsets are reduced by offsets applied in the prior year

The result for this iteration is that no bad debt reserve would be needed for Reinsurer A, though there would be an interruption in their expected cash flow pattern. Similar calculations would be performed for each reinsurer.

The following chart shows sample output from a complete simulation.

Table 10
Estimated Bad Debt Amounts by Reinsurer
(in 000's)

Reinsurer	Total Recoveries	Current Offset	Current Unsecured Recoveries		Estimated Bad Debt
Reinsurer A	$29,450	$14,500	$14,950		$280
Reinsurer B	2,950	2,950	0		10
Reinsurer C	15,000	6,800	8,200		155
Reinsurer D	10,800	7,300	3,500		30
Total	$58,200	$31,550	$26,650		$475

Table 11
Distribution of Overall Bad Debt Amounts
(All figures in 000's)

Simulation Element	Mean	50%	60%	70%	80%	90%
Cumulative Bad Debt Amount	475	35	96	175	860	2,500

The confidence levels in the above chart are based on the results of all simulations for all reinsurers.

Potential Applications

There are a variety of potential applications for this model. Areas of use could include loss reserving and analysis of relative risk between reinsurers. Both of these areas could also impact pricing and reinsurer selection, as well as affect commutation decisions and negotiations. The results of such a model could also be incorporated into a company's corporate risk or DFA analyses.

- Reserving – Currently the statutory provisions for uncollectible reinsurance are driven by formula (though a company can have some impact on the provision if they choose to write off certain recoverables sooner than the statutory provisions would require). GAAP rules are not as strict, as they only require a company to book their best estimate of provision. The model presented here could be used as a means to determine a best estimate as well as test the reasonability of the statutory provision, in addition to giving a company greater insight as to the potential (ultimate) impact on the company's balance sheet. Also, some countries now require insurers to estimate the confidence level in their booked reserves, including the potential for bad debt. In some cases companies are required to book additional reserves or carry capital to bring their overall funding level to a certain confidence level. A model such as this would be needed as part of this process.

- Reinsurer risk analysis – Creditworthiness is often the most critical factor considered by an insurer when purchasing reinsurance. Pricing and terms can be significantly different for reinsurers based on the perceived collectibility risk for each assuming company. As discussed previously, collateral requirements are often driven by the perceived collection risk for certain reinsurers implicit in Schedule F. Analysis of this risk could be crucial to the pricing process and a derivation of this model could be relatively easily incorporated into a company model for pricing outward reinsurance.

- Commutations - A company with an active commutation program could use the results of this model to help target reinsurers for commutation. Reinsurers could be classified by collection-risk level. "High-risk" reinsurers and reinsurance contracts could potentially be commuted before collection problems arise. The negotiation process for commutations could also be significantly affected, with a company's target price being adjusted for the future uncollectibility risk.

- Corporate risk and DFA analysis - This model could be incorporated as part of an analysis of a company's overall underwriting and credit risk. As discussed at the beginning of the paper, the reinsurance asset is often one of the largest and potentially most uncertain items on a company's balance sheet. Further, the risk for this asset is likely to be significantly correlated with many of the other major risks for an insurance company (i.e. catastrophe risk, underwriting or market cycle risk, and other timing risks). Consideration of the uncollectibility risk in conjunction with these other risks can be a critical component of a company's corporate risk model.

Areas for Future Research

This paper provides a basic framework for a model to estimate uncollectible reinsurance. A number of areas could be considered to help further refine the results and enhance its applicability within a company. Some of these areas include:

- Reinsurer Failure rates – Further research could be done to investigate actual historical reinsurer failure rates and the leading indicators or metrics that could be used to help predict such failures. One potential study could involve collecting historical financial ratings and other financial data for reinsurers and attempt to measure whether reinsurer failures could have been predicted from such data. Such a study could also be used to help refine a company's approach for selecting reinsurers, and even give regulators an additional tool for identifying potentially troubled companies so they can take corrective action.

- Correlations – Further analysis can be undertaken to estimate the correlation effects in the model, both between reinsurers and over time. Historical reinsurer failures could be analyzed against various underlying insurance industry measures, such as industry combined ratios, operating ratios, and premium growth or decline. The correlation of the failures to such measures could then be determined. In addition, another area of consideration with regard to correlations is the extent to which recoveries in specific lines of business may be subject to greater uncollectibility risk (for example, catastrophes versus excess casualty). In the two lines mentioned, one argument is that catastrophe events may be more likely to cause reinsurer failure, and should be subject to greater risk. However, the alternative argument is that catastrophes are short-tail, so even in such a scenario it is the long-tail recoveries that will ultimately not be recovered (since the funds needed to pay such recoveries will be depleted by the catastrophe recoveries).

- Time series effects – The time series aspect of the model presented here relates to the potential correlation of failures over time. The model shown here treats the "industry effect" and the correlation in a particular reinsurer's failure rate over time essentially as a random walk, i.e. a poor year in the industry is likely followed by another relatively poor year. The same is true for an individual reinsurer, where default in one year is more likely to be followed by default in the following. The reinsurer-specific correlation over time is likely indisputable. A company who defaulted on reinsurance recoveries and/or suffered financial stress is likely not expected to re-emerge from these difficulties, and previous defaulted amounts are typically not fully recovered. However, the industry as a whole is subject to market cycles. Various DFA models have attempted to measure such an effect in their financial projections. A similar approach could be used to capture this effect in this model.

- Recovery size effect/disputes –In the basic framework of the current model, the expected recoveries by reinsurer are considered only as a point estimate. A technical enhancement to the model could also allow the recoveries themselves to be introduced as a random variable, one that would be heavily correlated with the "industry effect" factor. Further research could be undertaken to estimate the extent that adverse treaty experience also impacts failure rates. From a general standpoint, this relationship is obvious, as poor industry results would likely cause both poor company results and higher reinsurer failures. However, analysis could also be performed to measure whether poor performance on specific treaties may influence uncollectbility by resulting in increased disputes with reinsurers.

Summary

Reinsurance recoverables can be a very significant factor in influencing the financial health of an insurance organization. The Annual Statement method for estimating recoveries does not consider a number of factors that often have significant effects on the risk of non-collection. The simulation methodology presented herein attempts to more specifically consider the risks of the company's reinsurers and the potential for correlations between reinsurer failures.

The results of this model can be used as a means for testing the reasonability of current provisions, as well as helping to identify areas of risk in a company's portfolio. Results can also be utilized during the reinsurance purchasing and selection process, the cedant management and commutation process, and other risk analysis and DFA-type initiatives.

Loss Reserve Estimates: A Statistical Approach for Determining "Reasonableness"

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LOSS RESERVE ESTIMATES:
A STATISTICAL APPROACH FOR DETERMINING "REASONABLENESS"

ABSTRACT

With the NAIC's adoption of the Accounting Practices and Procedures Manual, the statutory accounting practices for the P&C insurance industry have now been codified in a series of Statements of Statutory Accounting Principles (SSAP's). Within the SSAP's, various terms such as "Management's Best Estimate," "Ranges of Reserve Estimates" and "Best Estimate by Line" have been defined. In addition, the Actuarial Standard of Practice (ASOP) No. 36, adopted by the Actuarial Standards Board in March 2000, provides definitions for terms such as "Risk Margin," "Determination of Reasonable Provision" and "Range of Reasonable Reserve Estimates." While they are both well designed and a definite improvement, these new principles and standards of practice provide only broad guidance to the actuary on what is "reasonable." This broad guidance is based on the principle that "reasonable" assumptions and models lead to "reasonable" estimates. Unfortunately, this broad guidance can leave the low end of a range of "reasonable" reserves open to an interpretation which could lead to unintended consequences in practice. This paper will review some current actuarial practices and examine how they relate to the question of what is "reasonable" from a statistical perspective. Moreover, it will review and further develop some statistical concepts and principles that actuaries can add to their repertoire when developing ranges of liability estimates and then evaluating the "reasonableness" of management's best estimate of reserves within those ranges. It is hoped that the Actuarial Standards Board and others will consider adopting a more definitive definition of "reasonableness" in order to help avoid the unintended consequences of allowing the reserves to get "too low" in practice.

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LOSS RESERVE ESTIMATES:
A STATISTICAL APPROACH FOR DETERMINING "REASONABLENESS"

"The work of science is to substitute facts for appearances and demonstrations for impressions."
- John Ruskin

1. INTRODUCTION

The work of the actuary in developing loss liability estimates is a relatively scientific process, yet it is guided by some very subjective terms like "reasonable." The purpose of this paper is to develop a more definitive framework for the term "reasonable reserve estimate" based on statistical principles that the actuary can use when developing ranges of liability estimates and then evaluating management's best estimate of reserves within those ranges. Along the way, it will show how the current broad guidelines could be "misinterpreted." The first step in developing any set of principles is to start with a solid foundation, so this paper will begin by reviewing some "codified" terms and their definitions, defining some terms for use in this paper, and reviewing various statistical measures of risk. Next, it will examine some of the current practices for determining "reasonableness" and suggest a framework for defining "reasonableness" more precisely. Then various risk concepts will be reviewed and, more importantly, how they relate to the question of "reasonableness." Once all of these definitions and concepts are outlined, some general models for calculating ranges will be examined and some practical applications will be reviewed to see how these principles might be applied in practice. Finally, the paper will conclude by suggesting some areas for further research and an overview of the findings.

2. DEFINITION OF TERMS

Throughout this paper, unless noted otherwise, loss reserves are intended to include both loss and allocated loss adjustment expense reserves.[1] The SSAP's and ASOP No. 36 contain some definitions related to the term "reasonable." From the SSAP's we have the following:

Management's Best Estimate – Management's best estimate of its liabilities is to be recorded. This amount may or may not equal the actuary's best estimate.

[1] While many of the principles and analyses in this paper might also apply to unallocated loss adjustment expense reserves, they have been kept outside the scope of the discussion.

Ranges of Reserve Estimates – When management believes no estimate is better than any other within the range, management should accrue the midpoint.[2] If a range can't be determined, management should accrue the best estimate. Management's range may or may not equal the actuary's range.

Best Estimate by Line – Management should accrue its best estimate by line of business and in the aggregate. Recognized redundancies in one line of business cannot be used to offset recognized deficiencies in another line of business.[3]

From ASOP No. 36, we have the following:

Risk Margin – An amount that recognizes uncertainty; also known as a *provision for uncertainty*.

Determination of Reasonable Provision – When the stated reserve amount is within the actuary's range of reasonable reserve estimates, the actuary should issue a statement of actuarial opinion that the stated reserve amount makes a reasonable provision for the liabilities.

Range of Reasonable Reserve Estimates – The actuary may determine a range of reasonable reserve estimates that reflects the uncertainties associated with analyzing the reserves. A *range of reasonable estimates* is a range of estimates that could be produced by appropriate actuarial methods or alternative sets of assumptions that the actuary judges to be reasonable. The actuary may include risk margins in a range of reasonable estimates, but is not required to do so. A range of reasonable reserves, however, usually does not represent the range of all possible outcomes.

These definitions provide the actuary with only broad guidance on what is "reasonable." For example, is any reserve "reasonable," as long as it falls within any range of reserves based on any set of assumptions and models as long as those assumptions and models[4] are deemed reasonable by a competent actuary?[5] Of course any set of assumptions and models deemed reasonable by the actuary must also stand up to peer review scrutiny, but does this imply that two actuaries can create a quorum for determining reasonableness? Should the actuary's judgment about the assumptions

[2] Statutory guidance was silent on this point before the SSAP's, however when no estimate is better than any other within a range GAAP accounting standards state that the lowest estimate in the range should be accrued.

[3] Definitions of GAAP accounting terms may also be useful, but differences between GAAP and Statutory accounting principles are beyond the scope of this paper.

[4] Throughout this paper, the terms "method" and "model" are used interchangeably. However, preference is given to the term "model" to emphasize the need to think about actuarial reserve calculations as a model of the underlying process that is generating the claims rather than simply as a process for making calculations.

[5] A competent actuary could be defined as someone who is trained in the application of generally accepted actuarial methods and assumptions, but, interestingly, this creates a circular logic for determining "reasonableness."

and models be the only criteria for reasonableness, or do we need additional context to put these questions in perspective? In essence, these terms seem to imply a "reasonable person" standard much like you would find in a legal context.

This paper will argue that a statistical approach *should be added to* the "reasonable person" standard so that a more informed judgment, by both actuaries and users of the actuarial work product, about whether a stated reserve is "reasonable" or not can be made. In order to develop this approach, some basic definitions are offered. Consider the following:

- **Reserve** – an amount carried in the liability section of a risk-bearing entity's balance sheet for claims incurred prior to a given accounting date.
- **Liability** – the actual amount that is owed and will ultimately be paid by a risk-bearing entity for claims incurred prior to a given accounting date.[6]
- **Loss Liability** – the *value* of all estimated future claim payments.
- **Risk** (*from the "risk-bearers" point of view*) – the uncertainty[7] (deviations from expected) in both timing and amount of the future claim payment stream.[8]

3. MEASURES OF RISK

From statistics, actuaries often use a variety of measures that help define risk. These measures could include: variance, standard deviation, kurtosis, average absolute deviation, Value at Risk, Tail Value at Risk, *etc.* which are measures of dispersion. Other measures that help to define aspects of the distribution that might be useful in determining "reasonableness" could include: mean, mode, median, *etc.* The choice for measure of risk will also be important when considering the "reasonableness" and "materiality" of the reserves in relation to the capital position.

[6] The Statement of Principles Regarding Property and Casualty Loss and Loss Adjustment Expense Reserves define **Loss Reserve** as "a provision for its related liability." While reserves and liabilities are sometimes used interchangeably, they are given separate definitions in this paper, and used differently throughout, to help clarify the concepts discussed.

[7] In section 3.6.1 of ASOP No. 36, sources of uncertainty are described and include the following: random chance; erratic historical development data; past and future changes in operations; changes in the external environment; changes in data, trends, development patterns and payment patterns; the emergence of unusual types or sizes of claims; shifts in types of reported claims or reporting patterns; and changes in claim frequency or severity.

[8] If the loss liabilities are discounted, this would add an additional source of uncertainty to the expected value of the future payment stream. For purposes of the paper, "interest rate risk" will be ignored and liabilities are assumed to be undiscounted.

For insurance risks, actuaries often discuss the need to consider both "process" and "parameter" risk since both of these are part of the risk-bearer's burden.

> **Process Risk** – the randomness of future outcomes given a *known* distribution of possible outcomes.
>
> **Parameter Risk** – the potential error in the estimated parameters used to describe the distribution of possible outcomes, assuming the process generating the outcomes is *known*.

Statistically, both of these can be measured and used to calculate the distribution of possible outcomes. However, these calculations assume that the process that is generating the outcomes is known and the only requirement is to estimate the parameters of that process. Thus, for the purpose of describing a range of possible liability outcomes an additional type of risk could be defined as:

> **Model Risk** – a measure of the effect (*i.e.*, forecast error) given the model ("process") used to estimate the distribution of possible outcomes is incorrect or incomplete.[9]

While some models will allow us to capture the most salient characteristics of a set of data, the fact remains that no model is ever completely "correct" or "complete." [10]

Consider an example from gambling. In the game of Roulette, the casino knows exactly what the distribution of numbers and colors are on the roulette wheel, so determining the payouts (odds) involves only the process risk for the game since the parameters are certain (assuming a fair game). If we were to change the game so that the casino did not know the exact distribution of numbers and colors, then the casino could only determine appropriate payouts by continuous sampling of the outcomes.[11] In this case the casino, like the insurance risk-bearer, does not know the exact

[9] In common vernacular, actuaries and statisticians generally use the term "parameter risk" to include both parameter risk and model risk as defined in this paper. The two risks are separated here in order to distinguish the portion that is readily measurable (assuming a given model) from the portion that is not. They are also separated to emphasize the fact that all models used by actuaries make assumptions about the claim process that are critical to the estimates they produce.

[10] Model Risk could also be further divided into: i) model selection uncertainty, and ii) model specification risk. **Model Selection Uncertainty** is where you choose one model from a set of candidate models and forecast as if the chosen model was the "correct" one, when in fact there may be a variety of quite plausible models. **Model Misspecification Risk** is the contribution to forecast error from the fact that none of the candidate models is actually correct.

[11] If the numbers and colors could also change over time, this would make the example more "real" in terms of its applicability to insurance, but the point about "process" and "parameter" risk does not change.

parameters of the game, so excluding the "parameter" risk from their payouts could lead to potential bankruptcy and, at a minimum, less profit than was expected.

So far this example implicitly assumes that the game still resembles a game of roulette, except that the numbers on the wheel are not known in advance. If we were to change the game even more such that the casino did not know how the outcomes are produced, then the casino would also be forced to guess at the process used to create outcomes when they are estimating the odds from their continuous sampling. The observed outcomes may resemble the outcomes from one or more mathematical distributions, which can be used to estimate the parameters, but the actual process that is generating outcomes is still unknown. Again, the casino, like the insurance risk-bearer, would need to add in an additional "risk load" in order to include "model" risk and be properly compensated.[12]

Returning to the insurance world, if there were no risk there would be no need for insurance. Even if there were no parameter or model risk, the insurance risk-bearer would still have some chance of insolvency. Failing to recognize parameter and model risk increases the danger of insolvency.

Before moving on to look at how these various types of risk relate to the reasonableness of reserves, note that standard statistical techniques (and terminology) are already available and, hence, do not need to be reinvented. For example, standard deviation and standard error have slightly different formulae and different meanings. Standard deviation describes a characteristic of a known distribution and includes only "process" risk, while standard error is an estimate of that characteristic of the underlying distribution based on sample data and includes both "process" and "parameter" risk. Unfortunately, calculating model risk may not be possible.[13] While model risk is implied with the common definition of parameter risk and, therefore, implied to be included in

[12] Returning to the earlier definition of **Loss Liabilities**, this analogy would imply that all 3 types of risk (*i.e.*, process, parameter and model risk) should be included as part of the calculated expected value. Alternatively, some or all of these types of risk could be included in **Risk Margin** as defined under ASOP No. 36.

[13] In fact, some sources of model uncertainty can be estimated in some circumstances. For example, if selecting from a sufficiently flexible group of models that the bulk of the information in the data about the future has been captured, then one may estimate *Model Selection Uncertainty* from the data. Of course there are other sources of model uncertainty (*e.g.*, *Model Misspecification Risk*) that must still be included judgmentally.

standard error calculations, it would seem more prudent to include a separate measure or loading for model risk.

4. HOW DO WE DEFINE REASONABLE?

In accordance with the SSAP's and ASOP's, the actuary must opine on the reasonableness of management's reserves, but the definition of what constitutes "reasonable" simply refers to a range. Thus, the actuary, and management, needs to consider a range of estimates, but there seems to be no defined process for determining what is "reasonable" within this range or whether the range itself is "reasonable."[14] A range of estimates, by itself, creates several problems that need to be overcome in order to determine "reasonableness":

- A range (arbitrary or otherwise) can be misleading to the layperson – it can give the impression that any number in that range is equally likely.[15]
- A range can also give a false sense of security to the layperson – it gives the impression that as long as the carried reserve is "within the range" anything is reasonable (and therefore in compliance) as long as it can be justified by other means.
- There is currently no specific guidance on how to consistently determine a range within the actuarial community (*e.g.*, +/- X%, +/- $X, using various estimates, *etc.*).[16]
- A range, in and of itself, therefore has insufficient meaning without some other context to help define it.

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[14] One of the few places where more specific guidance is found is in SSAP 55, which states, in part, "when no estimate within a range is better than any other, the midpoint of the range should be accrued."

[15] Another gambling example might be useful here. Let's start with a game of chance where you wager a certain amount ($X) and in return you receive the dollar amount for the number that turns up on a roll of a fair die, plus $10. The range of possible outcomes is $11 to $16 and expected value is $13.50, so a fair wager is $13.50. A higher wager would be "good" for the house (they would gain over time), while a lower wager would be "bad" for the house (they would lose over time). Converting this to an insurance example, suppose an actuary was to tell management that the expected value of the liability estimate is $13.5 million, but the estimated range is $11 to $16 million and that each value in that range is equally likely to occur. What values in that range are "reasonable" for management to accrue?

[16] This statement does not imply that there has been no discussion about how to calculate ranges within the actuarial community. Quite the contrary, there have been numerous valuable contributions on this topic from authors of papers, editorials in the Actuarial Review, committee research, *etc.* The point is that the current guidelines simply say that a range *may* be used and that it *could* be calculated in a certain way, but the actuary is *not required* to create one.

Rather than simply saying that the actuary should calculate a range of liability estimates, it is the contention of this paper that actuaries should generally focus on calculating a distribution of possible outcomes such that the carried reserves would be sufficient to cover all future estimated claim payments at least X% of the time.[17] Once we define a "reasonable" range of the distribution based on probabilities, it can be translated into a range of liabilities that correspond to these probabilities. For example, telling management the liability estimate is $100 +/- $20 lacks sufficient meaning because of the reasons noted above. Contrast this to telling management the liability estimate shows they need $100 in order to have sufficient reserves at least 50% of the time and if they would like to increase the probability of having sufficient reserves to 75% they will need $120 in reserves. The second approach will be much more meaningful to management and other users of actuarial reports.[18]

Using a probability range to define a range of reasonable liabilities has the advantage of using the "risk" inherent in the data to define the range instead of a simple constant percentage. For example, if we were to define "reasonable" as a probability range of 50-75%, then the corresponding range of reasonable reserves might be $97-115 for a line of business with a relatively consistent claim payment stream, while the corresponding range of reasonable reserves might be $90-150 for a line of business with a more volatile claim payment stream. Contrast this with the common approach of using the estimated liabilities +/- X% for each line of business.

Table 1: Comparison of "Reasonable" Reserve Ranges by Method

	Relatively Stable LOB			More Volatile LOB		
Method	Low	Expected	High	Low	Expected	High
Expected +/- 20%	80	100	120	80	100	120
50^{th} to 75^{th} Percentile	97	100	115	90	100	150

17 Conversely, we could also define the probability range such that the carried reserves would be insufficient to cover all future expected claim payments at most (1-X)% of the time, although this approach has less intuitive appeal.

18 Continuing the simple example from Footnote 15, the actuary could advise management that reserves of a least $13.5 million was required in order to insure at least a 50% probability that they were sufficient and that $14.75 million would be required in order to insure at least a 75% probability that they would be sufficient. This would give the range some "reasonability" context that management could use to set reserves.

Using a probability range will also add context to other statistical measures. For example, as most liability distributions are skewed to the right, the mean will usually represent a value that is greater than the 50th percentile and can be used to help illustrate how the potential for the actual outcome to be worse than expected is greater than the potential to come in better than expected. Some actuaries have argued that the mode or the median could also be considered when describing what is "reasonable" in this context but, like the mean, discussing these as part of a probability range will complete and tie these various measures together.

The argument for using the mode as the "reasonable" reserve is that it has the highest probability of actually occurring. However, since liability distributions are usually skewed to the right (as illustrated in Graph 1), the mode would generally be less than the 50th percentile. In the context of liability distributions, the mode is the least desirable option for the low end of the range. Looking at the median (50th percentile), this would appear to be a logical low end to a range of "reasonable" reserves, but care must be exercised when selecting reserves by line of business compared to the aggregate reserves for all lines combined.

When reserves are selected by line of business and then simply added together to arrive at the total for all lines of business combined, this process is the same as assuming 100% correlation between lines. Generally, there is some level of independence between lines (*i.e.*, less than 100% correlation) which means that the total of selected individual medians (or modes) will be less than the median (or mode) of the aggregate for all lines combined. This concept is illustrated in Graph 2. Thus, if the median (or mode) is considered to be a "reasonable" low end for a range of reserves, then the medians (or modes) for the individual lines of business will need to be adjusted so they sum to the median (or mode) for the aggregate of all lines.[19] Using the expected value as the low end of the "reasonable" range of reserves will avoid this problem.[20]

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[19] These "adjustments" by line also seem consistent with the SSAP definition of **Best Estimates by Line** which implies consistency by line and in the aggregate.

[20] While acknowledging the usefulness of mode and median, and that it is a matter for the industry to define, the remainder of the paper will focus on the estimated expected value as the low end of a reasonable range. Indeed, section 3.6.3 of ASOP No. 36 states, in part, that "[o]ther statistical values such as the mode... or the median...may not be appropriate measures... such as when the expected value estimates can be significantly greater than these other measures."

The concept of a "reserve margin" is often discussed in terms of a prudent excess over the expected value.[21] This definition of reserve margin is consistent with using probability ranges for reserves. For example, if the carried reserve is greater than the expected value, then the reserve margin is the difference between the carried reserve and the expected value.[22] However, nothing in this paper should be construed as implying that a carried reserve margin is not reasonable. On the contrary, recognition of "process," "parameter" and "model" risk would imply that having a reserve margin is not only reasonable, but prudent.

At the high end of the range, considerations related to materiality[23] of the reserve compared to the resulting surplus come into play. One way to tie materiality to the probability range of liabilities would be to use dynamic risk modeling to estimate how liability outcomes relate to the probabilities of insolvency. Consider the following tables:[24]

Table 2: Comparison of "Reasonable" Reserve Ranges with Probabilities of Insolvency

"Low" Reserve Risk

		Corresponding Surplus Depending on Situation[25]					
Loss Reserves		Situation A		Situation B		Situation C	
Amount	Prob. Of Sufficiency	Amount	Prob. Of Insolvency	Amount	Prob. Of Insolvency	Amount	Prob. Of Insolvency
100	50%	80	40%	120	15%	160	1%
110	75%	70	40%	110	15%	150	1%
120	90%	60	40%	100	15%	140	1%

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21 Further distinctions between the "actual reserve margin" (determined after all claims incurred prior to a given accounting date are settled) and the "estimated reserve margin" (using the estimated expected value) could also be examined. However, since the scope of this paper involves estimated liabilities all references to reserve margins will imply estimated margins.

22 A negative reserve margin could also be defined as the difference between the carried reserve and the expected value.

23 ASOP No. 36 provides some guidance for evaluating Materiality – *In evaluating materiality within the context of a reserve opinion, the actuary should consider the purposes and intended uses for which the actuary prepared the statement of actuarial opinion.*

24 The numbers in these tables are purely hypothetical and designed for illustration purposes only.

25 If all else were equal, increasing the amount of the carried reserves will directly decrease the amount of surplus (Surplus = Assets – Liabilities) and the probability of insolvency wouldn't necessarily change. However, in practice, if the higher outcome actually occurs then the possibility that surplus could be eroded due to such things as insufficient rates, non-recoverable reinsurance, *etc.* would normally increase somewhat.

"Medium" Reserve Risk

		Corresponding Surplus Depending on Situation					
Loss Reserves		Situation A		Situation B		Situation C	
Amount	Prob. Of Sufficiency	Amount	Prob. Of Insolvency	Amount	Prob. Of Insolvency	Amount	Prob. Of Insolvency
100	50%	80	60%	120	40%	160	10%
120	75%	60	60%	100	40%	140	10%
140	90%	40	60%	80	40%	120	10%

"High" Reserve Risk

		Corresponding Surplus Depending on Situation					
Loss Reserves		Situation A[26]		Situation B		Situation C	
Amount	Prob. Of Sufficiency	Amount	Prob. Of Insolvency	Amount	Prob. Of Insolvency	Amount	Prob. Of Insolvency
100	50%	80	80%	120	50%	160	20%
150	75%	30	80%	70	50%	110	20%
200	90%	-20	80%	20	50%	60	20%

The relationship between reserve risk and the risk of insolvency is a complex issue. As illustrated in the tables above, there is a very strong interrelationship between how well an insurance enterprise is capitalized and the magnitude of the reserve risk. For example, if two companies have the same distribution of loss liabilities but Company A has only half the surplus as Company C, the range of reasonable reserves is the same for both companies even though the probability of insolvency for Company A is significantly higher. Alternatively, if Companies A and C both change their mix of business over time in such a manner that it increases their reserve risk (from, say, "low" to "high" risk), then the probability of insolvency will also increase for both but not to the same degree.

Of course, insolvency risk also depends on several other types of risk such as asset default risk, interest rate risk, reinsurance risk, catastrophe risk, *etc.* However, when all else is equal, the probability of insolvency decreases as the amount of surplus increases.

Interestingly, statistical analysis using ruin theory shows that pricing and reserving to the expected value every year, without any margin for risk loading, will eventually lead to insolvency with

[26] A particularly interesting example in these tables is the "high" risk situation A. In theory, the probability of insolvency wouldn't change if the company booked reserves of 200 instead of 100 even though the balance sheet would show negative surplus. Conversely, there would be pressure to book less than 100 to give the false impression that the company is more secure than it actually is.

probability of 100%.[27] This suggests that a prudent lower bound to the "reasonable" probability range for reserves should be *at least* the expected value, *if not higher.*

From the tables and discussion above, we might assume that a probability range from the expected value to 90% is "reasonable" so that every company can recognize the impact of reserve risk on their balance sheet and be properly compensated for risk in their pricing. Since market considerations related to "perceived" undercapitalization and the distortion of earnings that occur when a company strengthens their reserve position within this range put a natural economic barrier on the high end of the range, it seems like most regulators would be mainly concerned with keeping carried reserves above the low end of the range. Alternatively then, we might consider any carried reserves above the expected value to be "reasonable."[28]

Relating the concept of materiality to a probability range of liabilities could also prove useful in other related areas such as discussions of risk based capital and other solvency measures. For example, in a recent paper by Herbers [14] the viewpoints of different users of Statements of Actuarial Opinion are considered and a variety of sources for defining materiality are identified. Among all the different interests identified, the common goal among them is to make sure that risk is adequately disclosed. Conversely, the differences seem to be related to what level of risk needs to be disclosed. In order to satisfy the needs of all different users of actuarial opinions, the author suggests using the:

> **Principle of Greatest Common Interest** – the "largest amount" considered "reasonable" when a variety of constituents share a common goal or interest, such that all common goals or interests are met; and the
>
> **Principle of Least Common Interest** – the "smallest amount" considered "reasonable" when a variety of constituents share a common goal or interest, such that all common goals or interests are met.

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[27] For example, see: Beard, Robert E., Pentikäinen, T. and Pesonen, E., "Risk Theory," Chapman & Hall, 1984, 3rd Edition.

[28] In order to help identify strong reserve positions, categories for subsets above the expected value could also be added. For example, the range from the expected value to 75% could be "reasonable and prudent" and the range above 75% could be "reasonable and conservative."

These principles could be used separately or in conjunction with each other, depending on which goal or interest is being considered. For example, at the low end of a probability range the principle of greatest common interest would imply using the highest minimum such that the requirements of all constituents are met. For materiality, the principle of least common interest would imply using the least amount of surplus change considered "reasonable" by all constituents concerned with materiality.

5. OTHER RISK CONCEPTS, ASSUMPTIONS AND CONSIDERATIONS

Before discussing the practical aspects of actually calculating these probability distributions, it is important to review other risk concepts, assumptions and considerations that will be relevant to the discussion. For example, covariance becomes important, both by year and LOB:[29]

- **Concept 1:** For each (accident, policy or report) year, the coefficient of variation (standard error as a percentage of estimated liabilities) should be the largest for the oldest (earliest) year and will, generally, get smaller when compared to more and more recent years.
- **Concept 2:** For each (accident, policy or report) year, the standard error (on a dollar basis) should be the smallest for the oldest (earliest) year and will, generally, get larger when compared to more and more recent years.[30] To visualize this, remember that the liabilities for the oldest year represent the future payments in the tail only, while the liabilities for the most current year represent many more years of future payments including the tail. Even if payments from one year to the next are completely independent, the sum of many standard errors will be larger than the sum of fewer standard errors.
- **Concept 3:** The coefficient of variation (standard error as a percentage of estimated liabilities) should be smaller for all (accident, policy or report) years combined than for any individual year.

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[29] These covariance standard error concepts assume that the underlying exposures are relatively stable from year to year – *i.e.*, no radical changes. In practice, random changes do occur from one year to the next which could cause the actual standard errors to deviate from these concepts somewhat. In other words, these concepts will generally hold true, but should not be considered hard and fast rules in every case.

[30] For example, the total reserves for 1990 might be 100 with a standard error of 100 (coefficient of variation is 100%), while the total reserves for 2000 might be 1,000 with a standard error of 300 (coefficient of variation is 30%).

- **Concept 4:** The standard error (on a dollar basis) should be larger for all (accident, policy or report) years combined than for any individual year.
- **Concept 5:** The standard error should be smaller for all lines of business combined than the sum of the individual lines of business – on both a dollar basis and as a percentage of total liabilities (*i.e.*, coefficient of variation).
- **Concept 6:** In theory, it seems reasonable to allocate any overall "reserve margin" (selected by management) based on the standard error by line after adjusting for covariances between lines.

To simplify the calculations, claim payments by period are often assumed to be normally distributed in many of the commonly used models for to estimating liabilities. This can be a useful assumption when working through fictitious examples, but the actuary must be very careful when using these assumptions with real data:

- **Assumption 1:** For lines of business with small payment sizes (*e.g.*, Auto Physical Damage) this might be a reasonable simplifying assumption.[31]
- **Assumption 2:** For most lines of business, the distribution of individual payments, or payments grouped by incremental period, is skewed toward larger values. Thus, it would be better to model the claim payment stream using a Lognormal, Gamma, Pareto, Burr or some other skewed distribution function that seems to fit the observed values.
- **Assumption 3:** Estimating the distribution of loss liabilities (in total or by accident or payment period) assuming that the claims are normally distributed could produce misleading results for management whenever the actual claims are not normally distributed. The relevance of this distortion compared to the cost of improving the estimates needs to be considered.
- **Assumption 4:** Estimating the standard error in the claim payments assuming a normal distribution and then simulating the total loss distribution using a lognormal distribution (or some other skewed distribution) is marginally better, but it will require much greater skill and

[31] Even though using the normal distribution might be a reasonable simplifying assumption, the actuary must still exercise caution. For example, for some combinations of mean and standard error (*e.g.*, low mean, high standard error) the calculated range could include negative values.

care than using an assumption based on parameters assuming a lognormal (or some other skewed distribution) and testing to see how well this fits the actual data.

Since the projection of incurred losses does not directly measure the variability of the future payment stream, its usefulness in determining liability distributions should be considered:

- **Consideration 1:** The "extra" information in the case reserves is generally believed to add value by giving a "better" estimate of the expected mean. The exceptions to this are well documented in the actuarial literature. However, does this "extra" information really change the estimate of the *expected value* of the payment stream (by year), or does it give a better "credibility adjusted" estimate of the *likely outcome* (by year) as the additional information comes to light and leave the expected value of the payments unchanged?[32]
- **Consideration 2:** Consider two identical books of business with two different insurance companies.[33] They are identical except that one company sets up case reserves on the claims and the other does not. The estimates of the total liabilities (IBNR vs. case plus IBNR) are identical. Will the deviations of actual from the expected value of the future claim payments be any different?
- **Consideration 3:** Since measuring the variations in the incurred claims does not directly measure the variations in the payment stream, should risk measures based on incurred claims be used to quantify risk for management? With consistent levels of case reserves, the variations in the incurred claims might be more stable and might converge more quickly towards the actual outcome, but would this measure mask some of the true volatility? On

[32] The appropriate question here is whether the case reserve information can be used "optimally" in the sense that an appropriate credibility-weighted estimate is produced from the paid data and the case reserves. Let us assume that there is a small amount of information in the case reserves, but the additional information it contains about the payments requires the use of a model (at a minimum, you'll need to work out the mean, variance and covariance of the forecasts given the case reserves). That is, if A is some forecast of payments (whether an individual forecast or some total), P is the set of past payments, C is the set of past case reserves, and let's say we want the distribution of A, f[A | P,C]. Then if there is no parameter uncertainty (and ignore all the kinds of model uncertainty), it is true that f[A | P,C] must have a smaller (or no larger) standard error than f[A | P]. However, the moment you look at a predictive distribution, this is no longer true, because you have additional parameter uncertainty (and model uncertainties will compound the problem). For case reserves to help you forecast, any additional information would have to be larger than the additional predictive uncertainty the larger model introduces.

[33] This thought exercise also applies to the same book of business before and after the addition of case reserves to the claim settlement process.

the other hand, with case reserve strengthening or weakening, the variations in incurred claims may be less stable than for paid claims and could possibly overestimate volatility.

6. MODELS FOR CALCULATING RANGES

Historically, the problem of quantifying a probability distribution for a defined group of claim payments has been solved using "collective risk theory."[34] Actuaries have built many sophisticated models based on this theory, but it is important to remember that each of these models make assumptions about the processes that are driving claims and their settlement values. Some of the models make more simplifying assumptions than others, but none of them can ever completely capture all of the dynamics driving claims and their settlement values. In other words, none of them can ever completely eliminate "model risk."

For example, consider this thought exercise. Do claim adjusters base their individual claim payments on the cumulative value of past payments for each claim? No, they base each incremental payment on the circumstances at the time.[35] Thus, claim payments are not generally correlated to the cumulative payments to date. However, a convenient simplifying assumption is made when using models based on link ratios that the cumulative payments are correlated, but this creates a bias whereby "unusually" low cumulative values tend to under-predict the ultimate and "unusually" high cumulative values tend to over-predict the ultimate. Every actuary recognizes this bias (either implicitly or explicitly) and quite often the Bornhuetter-Ferguson model and informed judgment are used to adjust for this bias.

In fact, Venter [24] has shown that models based on link ratios often fail to be good predictors when you test the underlying assumptions. The chain ladder model (*i.e.*, weighted average of all link ratios) is actually a form of regression through the origin. Venter showed that quite often a better

[34] There are a number of good books on the subject, including, but not limited to, Buhlmann, "Mathematical Models in Risk Theory"; Gerber, "An Introduction to Mathematical Risk Theory"; and Seal, "Survival Probabilities".

[35] A possible exception to this might be cases involving annuity type claims, but even here if the circumstances change then the future claim payments could change or stop altogether. Quite often, claim adjusters make one payment on a claim and not multiple payments. When evaluating that payment, similar cases are considered at that time. It might be the timing of when the payments on similar type cases are made that matters more, but this still implies that the timing of when the payment is made is more significant than the cumulative history of other payments.

predictor is an average plus a constant (*i.e.*, slope not through the origin) or perhaps just a constant term.

A range of estimates using models based on link ratios should necessarily exclude using link ratio models when the assumptions underlying the models aren't strictly met – *i.e.*, they fail tests of their predictive value as described by Venter. In other words, if you have "bad" estimates, they are "bad" estimates and shouldn't enter into the determination of the "reasonable" range.[36] In the discussions that follow, all estimates using link ratio models are assumed to pass these tests.

Models based on incremental payments get around this "limitation" of the link ratio models and also have the advantage of more directly measuring the fluctuations in the timing and amount of the future claim payment stream. On the other hand, incremental models are less well known (or at least seem to be used in practice and discussed less often) and can be more difficult to apply for certain data sets. As always, the practicing actuary needs to be familiar with the advantages and disadvantages of each model used to estimate liabilities.

For purposes of this paper, the models used to calculate liability ranges will be grouped into four general categories: multiple projection models, statistics from link ratio models, incremental models, and simulation models.

A. Multiple Projection Models

In this category, the actuary uses multiple models and possibly various assumptions for each model to come up with a variety of possible estimates. Usually this involves models based on link ratios (at least in part) and it is assumed that these various estimates are a good proxy for the variation of the expected outcomes. This is inconsistent with the process underlying the concepts set forth in this paper in several important respects:

[36] While common sense and various sections of ASOP No. 36 would seem to imply this type of testing of the assumptions in a loss estimation model, the Actuarial Standards Board may wish to consider adding language to more directly address this issue.

- The projected estimates produce a range, but it does ***not*** provide a measure of the density of the distribution for the purpose of producing a probability function – it simply produces a range of estimates for the mean, but only to the extent that the actuary varies the models and assumptions.[37]
- The "distribution" of the projected means is a distribution of the models and assumptions used, ***not*** a distribution of the expected future claim payments.[38]
- While models based on link ratios are often assumed to be estimating the expected value of the reserves, in point of fact they only produce a single point estimate and there is no statistical process for determining if this point estimate is close to the expected value of the distribution of possible outcomes or not.
- Since there are no statistical measures for these models, any overall distribution for all lines of business combined will be based on the addition of the individual ranges by line of business with judgmental adjustments for covariance, if any.

While there are serious statistical limitations and drawbacks to using multiple projections to determine a liability range, we must recognize that producing any range is better than no range at all. Also, data limitations may prevent the use of more advanced models. Unfortunately, multiple projections don't provide a true probability range based on statistics, so the more sophisticated models described later would normally need to be used in practice or appropriate caveats will need to be included in the actuarial report.

Unfortunately, a strict interpretation of the guidelines in ASOP No. 36 would generally lead the actuary to use this model to create a "reasonable" range. In addition, one may wonder how often the tests outlined by Venter are actually being used to remove estimates that fail these tests from these "reasonable" ranges in practice. Given these limitations, therefore, it would seem

37 Perhaps a better description for a range of estimates of the mean is "scenario testing."

38 With enough estimates a nice bar chart showing the number of estimates that fall into selected intervals can be produced. However, while it may look rather like a probability distribution, it is just a bar chart that looks like a histogram and it wasn't generated by any random process. It was generated by the principle that underlies all scientific investigation: If something is quite reasonable, it can be justified in a lot of different ways. But if something is almost unreasonable, then it can be justified in only a limited number of ways, often only one.

prudent for the actuarial profession to consider adding language similar to the following to ASOP No. 36:

"Whenever a range of expected values is produced as the range of reasonable estimates, and the actuary has no further means of producing a reasonable distribution of possible outcomes, then the midpoint of the range of expected values should be used as the minimum acceptable reserve."

This would add language to ASOP No. 36 which is consistent with the definition used in the SSAP's for "Ranges of Reserve Estimates."

B. *Statistics from Link Ratio Models*

In this category, the models described by either Mack [16, 17] or Murphy [20] and others, can be used by the actuary to calculate the standard error in the payment stream using the variation in the link ratios. The actuary can use the standard error to calculate the distribution of the liabilities using the cumulative normal distribution or use logs to get a skewed distribution. These models are better than using Multiple Projections, but they are still inconsistent with some of the concepts set forth in this paper:

- The expected value used in these models is still based on multiple models and is subject to most of the same limitations described above for multiple projections.
- The standard error calculations in these models often assume that the distribution of the link ratios is normally distributed and is constant by (accident) year – this violates three concepts: 1) link ratios are a measure of the cumulative claim payment variations not the incremental variations (definition of risk), 2) the claim payments are usually not normally distributed (Assumption 2), and 3) the standard errors should not be constant across (accident) years (Concept 1).
- The standard error values from these models provide a process for calculating an overall probability distribution for all lines of business combined. However, this will require making assumptions about the covariances between lines or assuming independence among lines. Further research is needed to develop additional formulas for calculating the covariances between lines of business.

Using statistics from link ratio models is a significant improvement over ranges based on multiple projections since the variations in the underlying data are more directly modeled and used in the results. In other words, it is focused on calculating a distribution of possible outcomes given an estimate of the expected value. For these models, it would also seem reasonable to apply the language suggested above for ASOP No. 36 to the expected value portion of the calculations.

If data limitations prevent the use of models based on incremental values, then this model will need to be used. Otherwise, incremental models would normally be preferable.

C. *Incremental Models*

Models based on the incremental values of claims paid from one period to the next have been under development for quite some time.[39] These models generally overcome the "limitations" of using cumulative values and have the advantage of modeling calendar year inflation (along the diagonal) using a separate parameter(s). They also generally comply with the concepts set forth in this paper, with only a few exceptions:

- Several of the models in general use assume that the distribution of incremental claims is lognormal. The actual distribution of incremental payments may or may not be lognormal, but this is a significant improvement over models that assume normality and generally this provides a good fit to the actual data. Other skewed distributions are also used, but they generally add complexity to the formulations.
- Like for the other categories, when adding liability estimates for individual lines of business the correlations between lines will need to be considered when they are combined. Recent papers by Brehm [6] and Kirschner, et. al. [15] are good examples of how incremental models can be correlated and combined. Research in this area is ongoing.
- An added bonus is that some of these models allow the actuary to thoroughly test the model parameters and assumptions to see if they are supported by the data. They also allow the actuary to compare various goodness of fit statistics to evaluate the reasonableness of

[39] A brief sampling from the actuarial literature could include papers by Finger [11], Hachemeister [12], Zehnwirth [3, 28], England [9, 10] and Verrall [9, 10] to name but a few.

different models and/or different model parameters. Essentially, they allow the actuary to tailor the model parameters to fit the characteristics of the data.

For the purpose of calculating a distribution of possible outcomes, incremental models are a significant improvement over models based on link ratios since they are focused on directly calculating the distribution and then the expected value is determined from the distribution itself. The main limitation to these models seems to be only when some data issues are present.[40]

D. *Simulation Models*

Because of the complex interactions between claims, reinsurance, surplus, *etc.*, a dynamic risk model may be needed in order to more fully test the reasonableness of the range of liabilities. Models from all of the previous three categories can be used to create such a risk model, but in order to evaluate them we need one more concept:

- **Concept 7:** Whenever simulated data is created, it should exhibit the same statistical properties as the real data. In other words, the simulated data should be statistically indistinguishable from real data.

Unfortunately, simulation models based on link ratios tend to be the least useful since they quite often exhibit statistical properties not found in the real data being modeled. Whenever link ratios are shown to be worse predictors than a constant, or link ratios plus a constant, data simulated using link ratios *will be distinguishable* from real data. While this problem may not invalidate the conclusions from a liability simulation study, it will certainly reduce the reliability of the results.[41]

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[40] A good example is when separate data for Salvage & Subrogation is not available. In this case, when the "tail" of the loss development pattern contains a significant amount of negative values they cannot be modeled using logs.

[41] While taken out of context, the following quote is still relevant. "The bone of contention will be whether a model, to be of any use, must be 'essentially' realistic, or whether an admittedly unrealistic model may have its purposes. I hold that, so long as we don't forget the unrealistic assumptions we have made, we are free to make what models we will and then see what insight, if any, they yield."; Gene Callahan, "Choice and Preference", Ludwig von Mises Institute, www.mises.org, article posted Feb. 20, 2003.

This problem with "link ratio simulations" is usually overcome with models based on incremental values. It can also be overcome with ground-up simulations using separate parameters for claim frequency, severity, closure rates, *etc.* As with any model, the key is to make sure the model and model parameters are a close reflection of reality.[42]

7. PRACTICAL CONSIDERATIONS

Up to this point, the discussion has been mainly focused on theoretical and philosophical issues related to using probability ranges. Before the paper is concluded, it will also be useful to focus on some considerations of using probability ranges in practice.

A. Are Reasonable Assumptions Enough?

Some actuaries may find themselves not agreeing with the conclusion that the phrase "a reasonable range" is meaningless without some other context. Their reaction may be that context is provided by the phrase, "that could be produced by appropriate actuarial models or alternative sets of assumptions that the actuary judges to be reasonable." In other words, the sentence, "The reasonable range is from $A to $B" must make sense in light of reasonable statements about the history of cost drivers (such as premium, exposure, and benefit changes) and about the history of loss development (such as age-to-age factors or severity trend rates).

Turning to what is "reasonable" under the definition in ASOP No. 36, it seems safe to say that "reasonableness" is determined by the actuarial culture. By talking to other actuaries, attending conferences, talking with clients, reading the newspapers, and reading some of the actuarial literature, we maintain a culture that reflects actuarial expertise. Assumptions and statements that are consistent with this culture are necessarily reasonable, even if we personally disagree with them. Assumptions and statements that would be considered misleading in the context of

[42] Actually, there is a very real sense in which "unrealistic" models are to be preferred when forecasting. A model should tend to under-parameterize somewhat, if one wants a minimum mean square prediction error forecast – one should, for example, tend to over-smooth rather than fully fit all changes in trend, ***even where you know for certain*** there is a change. Often a substantial reduction in the effect of parameter uncertainty on the variance of the forecast comes at the price of a smaller increase in (squared) bias.

that culture are usually unreasonable – but one exception is statements that are well argued and supported with data, because that is how the culture is changed over time.

The author would certainly agree that culture is an appropriate context for our guidelines, but the use of probability ranges will add a new dimension to the guidelines. For example, even if every actuary in the world were to agree that all of the assumptions and models used to develop the range $A to $B are reasonable, we are still left with the question, from a solvency point of view at least, of "What makes selecting $A as the final reserve any more or less 'reasonable' than $B or any other number in between?"[43] Without any further guidance do we, as a profession, have any basis for selecting one number in the range over another?

What if two or three actuaries with appropriate training and experience estimate that a given liability has an expected value of $100 million[44] but the range of expected values is $70 to $140 million based on the information and support their conclusion with reasonable models and assumptions. Is $70 million a reasonable estimate? Based on current standards, unless there are assumptions that are "unreasonable," or data they have overlooked, or a mistake in their work, then the $70 million must be considered reasonable since it is "within the reasonable range" as currently described in our guidelines.

On the other hand, what if those same actuaries develop a distribution of possible outcomes with an expected value of $100 million and the end points of the range noted above correspond to the 25th and 80th percentiles, respectively. If there is only a 25% chance that $70 million is sufficient to cover all future claims, then is it still a "reasonable" estimate? It is not up to the author alone to determine at what percentile an estimate changes from reasonable to unreasonable, but it sure seems like it should be much closer to the expected value (or higher) than the 25th percentile. Since no model can ever remove all of the subjectiveness from the estimation process, setting an absolute percentile that the actuary cannot go below may not be a good idea. But theoretically at least, the expected value seems to be a logical minimum for a reasonableness standard.

[43] More or less adequate is a different question than where to draw the line on "reasonableness."
[44] As with previous examples, the time value of money is being ignored to simplify the discussion.

A standard that is less than the expected value would be akin to recommending to a casino that they set the odds at something less than in their favor.[45] While some constituents may consider a percentage lower than the expected value to be a reasonable lower bound, the principle of greatest common interest would suggest that other interested parties, such as stockholders, policyholders and solvency regulators, who would likely insist on at least an expected value standard as the minimum for the reasonable probability range.

Stated differently, the current guidelines seem to be saying that as long as the actuary can document the reasonableness of the models and assumptions used to arrive at a "possible outcome" then, ipso facto, that "possible outcome" is reasonable. Rather than *only* reviewing the reasonableness of the underlying models and assumptions, in and of themselves, the contention of this paper is that the actuary *also* needs to look at the reasonableness of that "possible outcome" in relation to all other possible outcomes. In other words, no matter how reasonable a given model and assumptions are, is that "possible outcome" reasonable if it is less than the expected value given a reasonable distribution of possible outcomes?

Turning to Statements of Actuarial Opinion, how should the actuary respond to the example described above if management wishes to book $70 million? Some actuaries may say "I can't find a way to shoot down the 'optimistic' assumptions that resulted in an estimate of $70 million as being unreasonable, I just think there is a lot of uncertainty." Should the actuary then give a "clean" opinion because management made a good case, but unless something changes, include a sentence in the "risks" section of the opinion that there is a 75% chance this will prove to be inadequate? Or, should the actuary give a qualified opinion? This will need to be answered by the actuarial profession and other constituents that are the intended audiences for the actuarial work product. On the other hand, if management does book the expected value, at what point does the actuary need to report the high end of the liability range in the "risk" section of the opinion?[46]

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[45] Actually, the casino would not want to set their odds at less than the expected value, plus a risk margin based on the process risk.

[46] This point has been debated among actuaries for at least 25 years as attested by the following quote from Bailey, Robert A. *The Actuarial Dilemma*, The Actuarial Review, Volume 5, No. 1, January, 1978, p. 7. "Loss reserving is about as actuarial as any work can be because it involves an estimation of an unknown quantity which is subject to future contingencies (inflation,

It is hoped that clarifications to the standards of practice will provide answers to these questions. In addition, the Committee on Property-Liability Financial Reporting may wish to define "risk" for purposes of a Statement of Actuarial Opinion in relation to the range of possible liability outcomes. For example, it could be "recommended that, if possible, the actuary disclose the 95th percentile for their estimated range of possible liabilities."

Another problem with the current definition of a "reasonable range" is the way it is implemented in practice. In theory, if actuary A says that the liability is $X, and actuary B finds that this is in the reasonable range as measured by ASOP No. 9 (Documentation), ASOP No. 36 (Reserves) and the CAS principles, then actuary B should give a clean opinion. That is, actuary A, who presumably knows the situation better, is to be believed unless there is a problem. In practice, insurance companies can use the existence of the "reasonable range" as currently defined to create space to manage earnings. Using a "probability standard," actuary A would then be required to report where they believe $X is with respect to the probability distribution of possible outcomes. In addition, actuary A could also be required to treat any material change in this percentage from one year to the next as a change in assumptions.[47]

It is easy to see how well-intentioned experienced actuaries could follow the standards of practice to the letter and end up signing a clean opinion on reserves that have a "high" probability of being deficient. In addition, in practice some of the modeling deficiencies described in the previous section could be compounding this issue by distorting the quality of the actuary's calculated range.

The wording in the ASOP's was worked out by actuaries who were familiar with mathematical models and yet decided that such models did not provide the solution. It may be safe to surmise they were concerned that mathematical models alone do not create a wide enough "safe harbor"

court settlements, *etc.*) based on past experience and informed judgment. But if estimating the value of unpaid claims is actuarial, certainly the appraisal of the degree of uncertainty associated with that estimate is at the very core of actuarial work. What could be closer to the theory of risk? If we succeed in avoiding the appraisal of the uncertainty in loss reserves, by simply stating that in our opinion the reserves are 'reasonable,' which means, I suppose, that the reserves have a 50% likelihood of being adequate, don't we leave a vacuum to be filled by some other profession?"

47 Materiality for these purposes will need to be related to the concept of materiality in other contexts noted earlier in the paper. For example, a "material change" could be defined as "an increase or decrease of more than 10 percentage points in the probability that the carried reserves are adequate."

for actuarial practice. Yet, given the questions raised by looking at probability ranges, one has to wonder if we might have inadvertently created a "safe harbor" that is potentially too wide at the low end? While there are many references to "uncertainty" in the ASOP's, additional guidance on what should be disclosed at the high end of the range also seems appropriate.

B. The Evolution of Information

It can be said that a range of reasonable reserves is a function of evidence, not just possible outcomes. For example, if the only information about a block of business is that it was priced to produce an 80% loss ratio, then the only reasonable liability estimate one can make is 80% of earned premium. The range widens and shifts as, and only as, other evidence emerges showing that other outcomes are reasonable (and perhaps that 80% is no longer reasonable).[48]

For a new block of business, the only evidence for setting reserves is the pricing documentation used to produce the rates (let's call this anecdotal evidence). As this block of business is observed over time, more and more evidence (let's call this physical evidence) emerges about how it is performing relative to the initial estimates and to any new updated pricing estimates (more anecdotal evidence). However, even if an 80% loss ratio is reasonable throughout this entire process that does not mean that other outcomes are not possible at every point along the way. As time passes, the physical evidence leads us toward the actual outcome and less weight is given to the anecdotal evidence, but in general 100% weight is not given to the physical evidence until all claims are closed.[49] While the physical evidence is leading toward the actual outcome for each year, statistically the *a priori expected* outcome may not be moving or may be moving in the opposite direction from the actual outcome (See Graph 3).

This discussion can be summarized using one of the questions noted earlier in the paper. Namely, does this "extra" evidence really change the estimate of the *expected value* of the payment stream (by year), or does it give a better "credibility adjusted" estimate of the *likely outcome* (by year) as the additional evidence comes to light and leave the expected value of the payments

[48] This does not mean that there is no range to start with. Quite the contrary, historical data or other anecdotal evidence could be used to calculate a reasonable *a priori* estimate of the range.

[49] A nice feature of the Bornhuetter-Ferguson model is that it shifts the weight over time using a nice mathematical (Bayesian) process.

unchanged? While the earlier question was aimed at the merits of determining risk using paid claims vs. incurred claims, it is equally relevant here.

This question, in turn, leads us to the realization that reserves are accounting fictions – they are estimates of liabilities, not the liabilities themselves.[50] Thus, we might also look to the accounting profession for some additional principles that might be relevant. For example:

At the high end of the range, according to a general principle of accounting, a liability should not be recorded for an "event" that has not yet occurred. It is a settled issue that an "event" is the claim itself, but how far does it go to include the conditions under which the claim will be settled? For example, if inflation (CPI) has historically been about 3% and the data for a line of business is consistent with the CPI, it seems reasonable to estimate the high end of the range assuming inflation of 3% in the future. Would the high end of the range only increase if inflation actually increased above 3%? Or, is it reasonable to assume that inflation could increase above 3% and include that possibility as part of the reasonable range? Another area where these questions are relevant is with emerging theories of law or legislated changes that are allowing new claims to be filed which were not anticipated in years past. A good example here is newly emerging legal theories of asbestos liability that were not known years ago.

At the low end of the range, according to a general principle of accounting, a business should not record a profit on a particular activity until it has data to support the estimation of that profit. Accordingly, the low end of the range should be selected in order to produce zero profit in the period if there is insufficient data to establish that a profit has been earned. Recording a liability any less than $X would create the incorrect impression that the business was known to be profitable. This principle seems consistent with keeping the minimum probability for the reasonable reserve range at the expected value or above.

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[50] As noted earlier, this "realization" is already recognized in the Statement of Principles definition of **Loss Reserves** as "a provision for its related liability."

C. Who is the Audience?

While it is the contention of this paper that a probability range should be used to determine what is "reasonable," we must also recognize that precisely defining what a "reasonable" probability range is may depend on the audience and, if possible, the audience should define what is "reasonable" to them. For example, solvency regulations and organizations concerned mainly about solvency (*e.g.*, state regulators, A. M. Best's, S&P, *etc.*) may feel that prudence would require a range with a minimum corresponding to the expected value and a maximum of, say, 85 or 90%. Other regulatory bodies might define the "reasonable" probability range differently (*e.g.*, the IRS might consider a range from 50% to 75% to be reasonable for tax considerations and the NAIC might have different ranges for statutory reserves compared to rate filing regulations). However, all of these different constituencies could use a probability range as a consistent starting point or perhaps even agree on a consistent lower bound to the probability range.

The *principles of least (greatest) common interest* apply when there are multiple parties that have an interest in a certain outcome. This is almost always true of actuarial reports, which means that there can be conflicting goals from the different audiences. It is easy to identify direct users of the report (*e.g.*, management, the Board of Directors, regulators, *etc.*), but it is not always clear who might indirectly use or benefit from the report (*e.g.*, stockholders, policyholders, consumer groups, *etc.*).[51]

We should also recognize that these two principles have the potential to cause ranges from two difference audiences to not intersect (*e.g.*, the high end of the range for one party is below the low end of the range for another party). If this should occur, it is hoped this approach to determining "reasonableness" will provide both parties with a method for working out their differences. Alternatively, it could be used to more clearly define difference between accounting standards used for different audiences (*e.g.*, GAAP vs. Statutory vs. Tax Accounting rules).

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[51] The *principles of least (greatest) common interest* are ***not*** intended to suggest that the actuary should attempt to identify all possible users of their work. This would be an onerous requirement. What it does suggest is that the actuary should not be able to select an end point for their liability range that is acceptable to one of the users of their work when it would clearly not be acceptable to other ***readily identifiable*** users of their work.

The final phrase in ASOP No. 36's definition of the range of reasonable reserves is "A range of reasonable reserves, however, usually does not represent the range of all possible outcomes." While the use of a probability range is not in conflict with this statement, the example discussed in Section 7.A. shows that it is subject to interpretation. In that example, it could be used to simply state that the range from $70 to $140 million does not include all possible outcomes. However, under a probability range approach it would be used to say "Of course outcomes less than $100 million are possible, but they are not reasonable since the probabilities that they are insufficient are too high. On the other hand, there is a 20% chance that outcomes above $140 million are also possible and the 20% probability may be too low given model risk that is incalculable or other unforeseen events."

Given the wide range of possible audiences for an actuarial work product, it seems prudent to err on the side of including more information rather than less. While in some cases this could increase the actuary's exposure to malpractice, in most cases this exposure should be reduced. For example, if the unexpected happens (let's say payments end up equaling $200 million in the example from Section 7.A. and the company ends up in bankruptcy), the actuary may be *exposed* to a claim of malpractice no matter what they said.[52] If the actuary simply told management the range ends at $140 million then there will be some explaining to do. But, if the actuary provided management with a probability range and also noted that there was a 5% chance that it could reach $200 million, then management will be in a much better position to make a decision on what reserves to carry and will not be able to say that this outcome was unforeseeable.

Using a probability range for liabilities, there seems to be two main reasons that actuarial malpractice could occur (excluding other potential reasons, like fraud):

1) If the actuarial models, assumptions and/or calculations used to create the overall expected outcome (within the distribution of possible outcomes) are faulty, or

[52] Being exposed to a claim of malpractice and actually being guilty of malpractice are a far cry from each other. Within the actuarial profession, the possible reasons for being guilty of malpractice have been the subject of considerable debate and are the purview of the Actuarial Board for Counseling and Discipline (ABCD). It is hoped that a statistical approach for determining reasonableness will help bring additional focus to the debate.

2) If the distribution of possible outcomes is "correct" given fully tested models and assumptions, but the actuary failed to alert the proper authorities that management was booking an amount that was less than the "reasonable" minimum, whatever percentage that turns out to be.

It doesn't seem right that getting the distribution of possible outcomes "correct," but years later finding out that the actual outcome is higher than the expected value, would be grounds for malpractice in and of itself. However, the public perception of getting it right and actually getting it right are two different things (especially in the hands of a skilled attorney). How much longer can the actuarial profession risk telling our constituents what is expected and not also telling them what is possible?

4. *When Does Insolvency Occur?*

The previous discussions about how probability ranges for liabilities are related to materiality can naturally lead to the question: "When is an insurer insolvent?" Does an insurer become insolvent when their surplus was actually inadequate or when a regulator finds out about it?

For instance, suppose a "clean" loss reserve opinion is given on the company described in Section 4 as "medium" risk in scenario A (*i.e.*, carried reserves of $100 million, surplus of $80 million and probability of insolvency is 60%). Years later it turns out that the paid losses for claims represented by those reserves are likely to exceed $200 million. Was the company actually insolvent when the opinion was given? Or, does it become insolvent when the "higher than expected" claim payments indicate that the likely outcome will exceed $180 million? What if subsequent years improve such that cash flow never becomes an issue? What if subsequent years get worse?

At one extreme it could be reasoned that the insolvency actually took place when the clean opinion was given or even as early as when the business was written that resulted in the eventual insolvency. The rationale for this view rests on the assumption that insolvency is a technical condition not a human discovery of that condition. This would also be distinguished from actions taken by management and/or regulators in response to their discoveries.

At the other extreme, it could be reasoned that insolvency doesn't take place until the insurer reaches the point where it can't meet current cash flow needs. Unfortunately, at this extreme the identified liabilities will usually far exceed the current assets. It's not surprising then that regulators have set solvency requirements, via Risk-Based Capital requirements, so that they can take action before the insurer gets into cash flow difficulties. Therefore, a more reasonable alternate extreme might be that the insolvency has taken place at the time the information becomes available to value the company's surplus below RBC standards.

While both of these extremes are useful in framing the discussion, both of them rest on the assumption that future liabilities are known (or knowable with a very high degree of certainty). Until the liabilities are completely run-off no actuary can tell exactly what they will be. At either point in time (original valuation date or retroactive discovery date), two different actuaries will have two (or more) different estimates of what the liabilities are. If one estimate indicates that liabilities exceed assets and the other one doesn't, which one is right? The answer is neither of them is right.

If liabilities are viewed as a distribution of possible outcomes, instead of an actuary's best estimate or even a range of best estimates, at any point in time there is some probability that the future liability payments will exceed current assets (or more accurately future assets). So, from this perspective, the question becomes how high must this probability become in order for insolvency to occur or regulatory action to be triggered? Perhaps the added perspective of probability ranges will prove useful to actuaries and regulators as they continue to fine tune and improve the RBC formulas.

8. AREAS FOR FUTURE RESEARCH AND ANALYSIS

Throughout the paper, several areas for future research have been identified (or at least hinted at). For easy reference, they are summarized below:

- One of the suppositions in this paper is that measures of reserve risk should be based primarily on paid data, although some potential information from incurred data was also discussed. Research of measures of risk based on paid claims vs. incurred claims would be necessary to reach any definitive conclusions. Research papers to develop models that quantify the predictive value of case reserves and credibility weight that information with estimates based on paid data would also be a valuable addition to our literature.
- Various models for calculating probability ranges are discussed in the paper along with advantages and disadvantages of each. A research project involving retrospective testing of various models used to calculate ranges would yield insights into how significant these advantages and disadvantage are. To accomplish this, the author suggests a "blind" test with old data from multiple companies and multiple lines of business. The data should be at least 10 years old so that the final results are already known, but the tests should be run using only the triangles that would have been known 10 (or more) years ago.
- Continuing research on covariance calculation methods is a significant feature of any model used to calculate probability ranges of liabilities for an entire company.
- Further research on the relationship between reserve risk and insolvency risk could lead to additional insights on how to define a "reasonable" probability range. It might also lead to some RBC insights or triggers for when a company should consider increasing its capitalization or have enough "extra" capital before paying dividends.
- Research on the quantification of "model" risk would be a welcome addition which could help move this from a judgmental to a calculated amount. Even when calculated amounts aren't a possibility, it would help improve informed judgment.
- Research on the differences between measures of reserve risk based on quarterly data vs. annual data should be performed in order to help guide actuaries when dealing with issues related to quarterly vs. annual accounting statements.

9. CONCLUSIONS

This paper started by reviewing some of the professional standards for determining the "reasonableness" of loss reserves and proceeded to examine how various statistical concepts might be used in conjunction with current standards. The main conclusions of this analysis are that using a probability range has the following benefits:

- Users of actuarial liability estimates based on probability ranges will get much more information for risk evaluation and decision-making,
- The width of the dollar range will be directly related to the potential volatility (uncertainty) of the actual data,
- The concept of materiality can be more directly related to the uncertainty of the estimates,
- Risk-Based Capital calculations could be related to the probability "level" of the reserves,
- Both ends of the "reasonable" range of reserves will be related to the probability distribution of possible outcomes *in addition to* the "reasonableness" of the underlying assumptions,
- The concept of a "prudent reserve margin" could be related to a portion of the probability range and will then be directly related to the uncertainty of the estimates, and
- The users of actuarial liability estimates would have the opportunity to give more specific input on what they consider "reasonable."

In order to implement the advantages of the statistical approach, the actuarial profession should consider adding wording similar to the following to ASOP No. 36:

> *"Whenever the actuary can produce a reasonable distribution of possible outcomes, a reasonable reserve estimate should not be less than the expected value of that distribution."*

Essentially, this paper is NOT proposing that we eliminate the "what a reasonable person might do" standard and replace it with probabilities. What it is suggesting is that we can improve the "reasonable person" concept by adding some additional context. There must be no illusions here. Adding a probability measure to the "reasonable person" standard will not provide a magic solution to define the exact number where the minimum "reasonable" reserves should be. Calculating the mean of the distribution is no less difficult. However, adding "probability standards" can make the "reasonable person" standards more meaningful.

In addition, the ASOP No. 36 definition of **Risk Margin** could be improved by adding wording similar to the following:

> *"A risk margin should include an amount(s) to reflect 'process,' 'parameter' and 'model' risk. Whenever possible, it should be statistically calculated, otherwise a judgmental amount can be included."* [53]

Other issues mentioned in the paper that should also be addressed in our standards include: 1) the need to consider language to more directly require testing of the assumptions for different models, 2) a more definitive solution for how to consistently disclose the relative reserve risk, and 3) a more precise definition of "material change" as it relates to reserve risk.

Finally, we must not forget that calculating a distribution of possible outcomes is not always possible. In that event, adding wording similar to the following to ASOP No. 36, as suggested earlier in the paper, would be consistent with the SSAP's:

> *"Whenever a range of expected values is produced as the range of reasonable estimates, and the actuary has no further means of producing a reasonable distribution of possible outcomes, then the midpoint of the range of expected values should be used as the minimum acceptable reserve."*

In closing, ask yourself the following question: "WHAT IF you knew the EXACT distribution of possible liability outcomes, would you feel comfortable giving a clean opinion to a company that wanted to carry less than the expected value on their books?" As a profession we want the outside world to rely on our "actuarial judgment" to determine what is "reasonable." Will your answer give the public added confidence in the profession? Doesn't it make sense to strengthen our standards in order to increase public confidence?

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[53] Definitions or "process," "parameter," and "model" risk consistent with the definitions in this paper may also need to be added.

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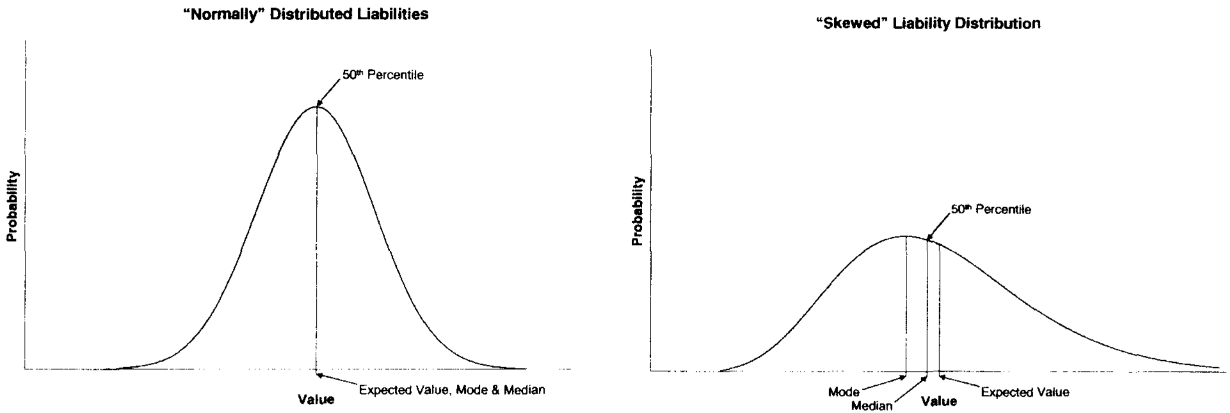
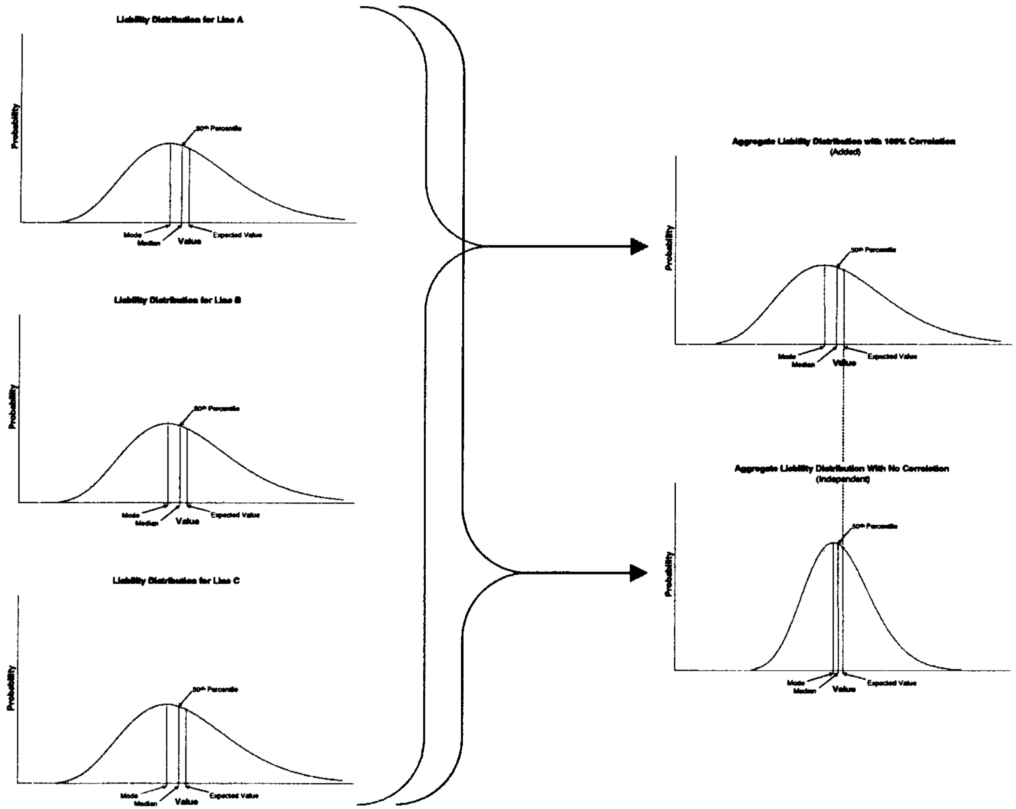
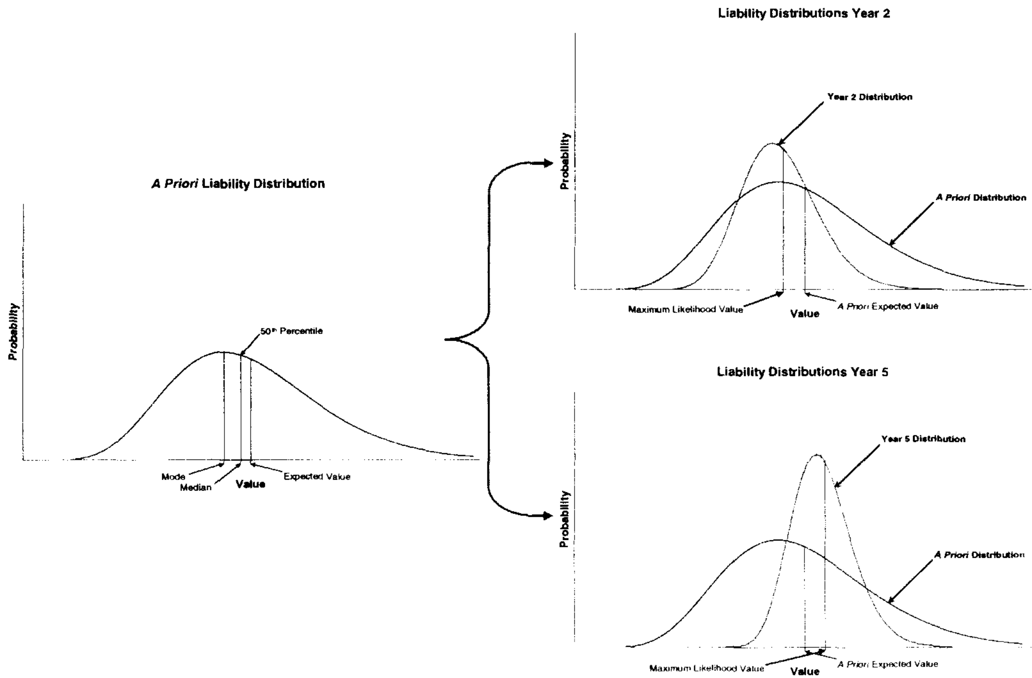
Graph 1: Comparison of "Normal" vs. "Skewed" Liability Distributions



Graph 2: Comparison of Aggregate Liability Distributions



Graph 3: Comparison of *A Priori* vs. Credibility Adjusted Liability Distributions



Prior to the incidence of a cohort of claims, their distribution and expected value can be estimated (the *a priori* distribution). Once the claims have occurred and the settlement process begins, the new estimates of their ultimate value will gradually become more certain over time until all claims are completely settled and their value is known with 100% certainty. As an example illustrated here, in year 2 the claims paid to date are less than anticipated in the first two years and, therefore, the remaining expected value plus the current paid to date results in a new distribution with a total expected value (maximum likelihood) which is less than the *a prior* expected value. Continuing the example, in year 5 the claims paid to date are now greater than anticipated in the first five years and, therefore, the remaining expected value plus the current paid to date results in a new distribution with a total expected value (maximum likelihood) which is greater than the *a prior* expected value. At both 2 years and 5 years, the remaining uncertainty is getting smaller.

A Generic Claims Reserving Model: A Fundamental Risk Analysis

Graciela Vera

A GENERIC CLAIMS RESERVING MODEL

A FUNDAMENTAL RISK ANALYSIS

By Graciela Vera

In their diversity, insurance risks often require very different types of claims reserving models to describe them and to estimate the necessary reserves. The tractability of the chain ladder has contributed to its popularity. A brief analysis is given of its statistical basis and its implied limitations, of which the most important is its propensity to underestimate the reserves. This paper proposes a new paradigm for actuarial risk analysis where the reserving estimate is just one of the many results it delivers. The generic claims reserving model is consistent with the claims development process, and from it a rich family of claims reserving models can be constructed. Without loss of generality, the variance is simply defined to be a function of the mean. No particular consideration is given to estimation procedures, as those will depend on the selected model structure.

1. INTRODUCTION

A typical claims array of a book of business is cross-referenced by underwriting year and delay period, and is additionally described by payment year defined as underwriting year plus delay period minus one. As an example of annualised claims data consider the following array:

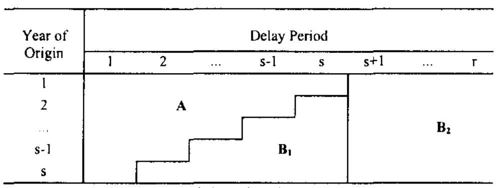


Fig.1 Example of a claims development array, A represents known claims, and B_1 and B_2 projected claims.

The array is divided into three regions: **A** for the cells with known claims. **A** typically consists of the upper left triangular region of the array. The last diagonal of **A** is s. It corresponds to the last payment year for which claims data are known. The triangular array right below **A** is denoted by $\mathbf{B}_1$ and the cells with unknown claims for delay periods beyond the last with know claims by $\mathbf{B}_2$.

Variations in the shape of the data array represented by **A** are usual, and are well within the scope of the models discussed in this paper. They are normally the result of data exclusions from the latest delay periods, from the latest origin years or from the earliest payment periods. Truncation of the data in the underwriting year direction is a consequence of cessation of business or of changes in the underwritten risk.

The models that concern this paper have the capacity to predict future claims beyond delay period s. This is a distinctive advantage over models such as the chain ladder (Zehnwirth (1989)), or other models derived from it.

A very large variety of models has been developed to predict future claims for the lower triangle of the claims array $\mathbf{B}_1$. The chain ladder, being one of them, has been quoted and referenced frequently. Kremer (1982) proves the connection between the chain ladder and a two-way analysis of variance and the results are discussed and applied as the basis of further work by other authors. However, the use of the analysis of variance to provide a statistical justification to the chain ladder gives a hint of the limitations of this method and related models. The most important is that it excludes the tail factor beyond delay period s. This is acknowledged by Zehnwirth (1989), to which close reference is made in order to set the framework for the method this paper proposes. For consistency with the rest of the paper and without altering the conclusions that can be derived from it, the exposition of Zehnwirth (1989) made with reference to accident year, is summarised below with reference to underwriting year.

1.1 THE CHAIN LADDER METHOD

Denote the incremental paid claims in development year j and underwriting year w by $y^{\bullet}_{w,j}$. Then the cumulative claim amounts and development factors for underwriting year w at development year j can be defined by $c^{\bullet}_{w,j} = \sum_{h=1}^{j} y^{\bullet}_{w,h}$ and $D_{w,j} = \left(c^{\bullet}_{w,j-1}\right)^{-1} c^{\bullet}_{w,j}$. Zehnwirth (1989) summarises the chain ladder assumptions as follows:

ASSUMPTION 1: Each underwriting year has the same development factor, with an estimate defined as $\hat{D}_j = \left(\sum_{w=1}^{s-j+1} c^{\bullet}_{w,j-1}\right)^{-1} \sum_{w=1}^{s-j+1} c^{\bullet}_{w,j}, \forall j = 2,..,s$. Then, projections of $c^{\bullet}_{w,j}$ for $w = 2,...s$ and $j = s-w+2,...,s$ are $\hat{c}^{\bullet}_{w,s} = c^{\bullet}_{w,s-w+1} \prod_{k=s-w+2}^{s} \hat{D}_k$. For consistency let $\hat{c}^{\bullet}_{1,s} = c^{\bullet}_{1,s}$

ASSUMPTION 2: Each underwriting year has a parameter representing its level. For underwriting year w this is $c^{\bullet}_{w,s-w+1}$. The final underwriting year is represented by $c^{\bullet}_{s,1}$. If all underwriting years were homogeneous, the level estimate $c^{\bullet}_{s,1}$ would be given by $\frac{1}{s}\sum_{w=1}^{s} c^{\bullet}_{w,1}$.

To establish the statistical basis of the chain ladder, let $\hat{p}^{\bullet}_j = \left(\sum_{w=1}^{s-j+1} c^{\bullet}_{w,j} - \sum_{w=1}^{s-j+1} c^{\bullet}_{w,j-1}\right)\left(\sum_{w=1}^{s-j+1} \hat{c}^{\bullet}_{w,s}\right)^{-1}$, hence $\hat{p}^{\bullet}_s = \left(c^{\bullet}_{1,s} - c^{\bullet}_{1,s-1}\right)\left(c^{\bullet}_{1,s}\right)^{-1}$. From assumptions 1 and $\sum_{w=1}^{s} \hat{c}^{\bullet}_{w,s} = \left(\prod_{k=j+1}^{s} \hat{D}_k\right)\sum_{w=1}^{s} c^{\bullet}_{w,j}$, $\hat{p}^{\bullet}_j = \left(1-\hat{D}_j^{-1}\right)\left(\prod_{k=j+1}^{s} \hat{D}_k\right)^{-1}$ and

$\hat{p}_s^{\bullet}=\left(1-\hat{D}_s^{-1}\right)$, then $\hat{D}_j=\left(1-\hat{p}_j^{\bullet}\prod_{k=j+1}^{s}\hat{D}_k\right)^{-1}, j=2,..,s-1$ and $\hat{D}_s=\left(1-\hat{p}_s^{\bullet}\right)^{-1}$. Given the expression for $\hat{p}_j^{\bullet}$, $\sum_{j=1}^{s}\hat{p}_j^{\bullet}\sum_{w=1}^{s-j+1}\hat{c}_{w,s}^{\bullet}=\sum_{j=1}^{s}\left(\sum_{w=1}^{s-j+1}c_{w,j}^{\bullet}-\sum_{w=1}^{s-j+1}c_{w,j-1}^{\bullet}\right)=\sum_{j=1}^{s}\left(\sum_{w=1}^{s-j+1}y_{w,j}^{\bullet}\right)$ and the following can be concluded:

1. Based on the marginal estimates, the equality $\left(\sum_{j=1}^{s}\sum_{w=1}^{s-j+1}\hat{p}_j^{\bullet}\hat{c}_{w,s}^{\bullet}=\sum_{j=1}^{s}\sum_{w=1}^{s-j+1}y_{w,j}^{\bullet}\right)$ suggests the stochastic model proposed by Kremer (1982): $y_{w,j}^{\bullet}=\alpha_w^{'}\beta_j^{'}e_{w,j}^{'}$ where $\alpha_w^{'}$ and $\beta_j^{'}$ are the parameters for underwriting year w and development year j, and $e_{wj}^{'}$ the random error term. This model can be re-stated as a two-way analysis of variance $y_{w,j}^{\bullet\bullet}=\ln y_{w,j}^{\bullet}=\mu+\alpha_w+\beta_j+e_{w,j}$ such that $e_{w,j}\sim N\left(0,\sigma^2\right),\sum_{w=1}^{s}\alpha_w=\sum_{j=1}^{s}\beta_j=0$.

2. From assumptions 1 and 2, $\hat{c}_{1,s}^{\bullet}=c_{1,s}^{\bullet}=\sum_{h=1}^{s}y_{1,h}^{\bullet}$. Hence, $\left(\hat{c}_{1,s}^{\bullet}\sum_{j=1}^{s}\hat{p}_j^{\bullet}=\sum_{j=1}^{s}y_{1,j}^{\bullet}\right)$ implies $\left(\sum_{j=1}^{s}\hat{p}_j^{\bullet}=1\right)$.

3. Since $\sum_{w=1}^{s-j+1}\hat{c}_{w,j-1}^{\bullet}\equiv\left(\sum_{k=1}^{j-1}\exp\left(\beta_k\right)\right)\left(\sum_{w=1}^{s-j+1}\exp\left(\mu+\alpha_w\right)\right)$, the equivalent relationships emerging from points 1 and 2 are: $\hat{p}_j^{\bullet}\equiv\exp\left(\beta_j\right)\left(\sum_{k=1}^{s}\exp\left(\beta_k\right)\right)^{-1}$ and $\hat{D}_j\equiv1+\exp\left(\beta_j\right)\left(\sum_{k=1}^{j-1}\exp\left(\beta_k\right)\right)^{-1}$. The former gives a clear interpretation to $\hat{p}_j^{\bullet}$ as the percentage of the total claim amount to be paid in development year j. The definition of $\hat{p}_j^{\bullet}$ is restricted to the first s development years for which data exists. Hence, for $j>s$, $\hat{p}_j^{\bullet}=0$.

Conclusions 1 and 2 show that the chain ladder understates the reserves for runoff triangles that at development year s are not fully developed. It is usual for practitioners who use the chain ladder to value their reserves to apply industry benchmarks in order to estimate future losses for the periods within the region $\mathbf{B}_2$ on the array. Particularly when they are generated from portfolios similar in claims experience and composition to those to which they are applied, benchmarks can certainly be a very useful source of reference to achieve safe reserves. Industry benchmarks could contain trends not yet apparent in the company's data. Nevertheless, in the context of conclusion 3, where the chain ladder is interpreted as the product of two marginal functions: the total claim amount and the percentage cash flow, the use of benchmarks shows that the chain ladder's implied assumptions are frequently adopted in error, and stresses the need for a more coherent reserving approach.

To construct the generic model the relationship $\left(\sum_{j=1}^{s}\sum_{w=2}^{s-j+1}\hat{p}_j^{\bullet}\hat{c}_{w,s}^{\bullet}=\sum_{j=1}^{s}\sum_{w=1}^{s-j+1}y_{w,j}^{\bullet}\right)$ should not be dismissed altogether, though more careful formulations of the marginal estimates $\hat{p}_j^{\bullet}$ and $\hat{c}_{w,s}^{\bullet}$ are required to ensure that $\hat{p}_j^{\bullet}$ is defined for all the possible periods of exposure, and $\hat{c}_{w,s}^{\bullet}$ is replaced by an estimate independent of s.

This paper is organised as follows. In section 2.1 the empirical data are defined. In sections 2.2 to 2.4 the components of the generic model are developed and the properties of the underlying function to the percentage cash flow are addressed. The mean square errors are derived in section 3 preceded by a version of the incremental claims reserving model. Behind the form of the generic model derived in section 2 is the assumption that the data of interest represents a single class of actuarial risk. This is relaxed in section 4, and a generic reserving model for claims data containing more than one class of actuarial risks is constructed.

2. A GENERIC MODEL

2.1 THE CUMULATIVE CLAIMS PROCESS

Let C_w^* be the ultimate loss incurred in underwriting year w during its entire settlement period.

For simplicity of notation, it will be assumed that claims are reported at regular periods, although in practice this needs not be a limitation of the models discussed. Let $t_0 = 0$, and consider the claims process for underwriting year w, reported at times $t_1, t_2, ..., t_e$, such that $0 < t_1 < t_2 < ... < t_e$, and t_e is the time when the ultimate settlement is made. The number of partial or total claim payments by time t is defined by

$$N_w^*(t) = \begin{cases} 0 & t = t_0 \\ N_w^*(t_j) & t_{j-1} < t \le t_j, 0 < t_j < t_e \\ N_w & t_e \le t \end{cases}$$

where N_w is the total or ultimate number of claims from underwriting year w. Let the i_{th} incremental claim settlement for underwriting year w paid during time period $(t_{j-1}, t_j]$ be denoted by $(X_{w,i,j}^*) = \{i \in [1, ..., N_w^*(t_j) - N_w^*(t_{j-1})], j \in [1, e]\}$. Then the ultimate claim amount for underwriting year w is given by $C_w^* = \sum_{j=1}^{e} \sum_{i=1}^{[N_w^*(t_j) - N_w^*(t_{j-1})]} X_{w,i,j}^*$. Hence the aggregate claim amount for underwriting year w and delay period j and the corresponding percentage cash flow denoted respectively by $Y^*(w, j)$ and $P^*(w, j)$ can be defined as

$$Y^*(w,j) = \begin{cases} \sum_{k=1}^{j} \sum_{i=1}^{[N_w^*(t_k) - N_w^*(t_{k-1})]} X_{w,i,k}^* & 1 \le t_j < t_e \\ C_w^* & t_j \ge t_e \end{cases}$$
$$P^*(w,j) = \begin{cases} Y^*(w,j) C_w^{*-1} & 0 < t_j < t_e \\ 1 & t_j \ge t_e \end{cases} \quad (2.1)$$

Equation (2.1) shows that it is justifiable to represent the cumulative percentage cash flow by a continuous function, and this should be an integral. Its properties are defined below.

2.2 THE PERCENTAGE CASH FLOW

For theoretical purposes, the claims process is assumed to be continuous. We define a continuous function $P(w,j)$ for the percentage cash flow and a function C_w for the ultimate claim amount. Hence, for a random variable $Y(w,j)$

$$E\left(Y(w,j)\right) = C_w P(w,j) \tag{2.2}$$

Let the future percentage cash flow at time j be represented by $S(w,j) = 1 - P(w,j)$. Estimates of $S(w,j)$ can be easily obtained for regions $\mathbf{B_1}$ and $\mathbf{B_2}$ of the array in figure 1.

If the underlying function of integral $\Pi(w,t)$ is denoted by $\pi(w,t)$ then

$$\pi(w,t) = \lim_{\partial t \to 0}\left(\frac{\Pi(w,t+\partial t) - \Pi(w,t)}{\partial t}\right) = \left(\frac{\partial \Pi(w,z)}{\partial z}\right)_{z=t}$$

and the properties of $\pi(w,t)$ are as follows:

i. $\pi(w,t) \geq 0 \quad \forall t$

ii. $P(w,t_j) = \int\limits_{z=0}^{z=t_j} \pi(w,z)dz$

iii. $P(w,t_e) = \int\limits_{z=0}^{z=t_e} \pi(w,z)dz = \int\limits_{z=0}^{z=\infty} \pi(w,z)dz = 1$

iv. $S(w,t_j) = \int\limits_{z=t_j}^{z=\infty} \pi(w,z)dz$

v. Let $p(w,j)$ denote the percentage cash flow during the time interval $\left(t_{j-1}, t_j\right]$ recorded at the end of development period j.

$$p(w,j) = \int\limits_{z=t_{j-1}}^{t_j} \pi(w,z)dz$$

Individual incurred claims could be negative as a consequence of the recording process of paid and outstanding claims. Incurred claims data, as the total of paid and outstanding claims, is subject to fluctuations in both data sets. Their fluctuations and adjustments often result in negative incremental incurred claims entries. However, a reserving model with a systematic component defined as (2.2) could deal with negative incremental adjustments, since those are normally corrections of earlier entries.

When the evaluation of integral $\Pi(w,t)$ cannot be obtained in terms of known functions, $p(w,j)$ and $P(w,j)$ need to be approximated. Numerical integration techniques can be used for this purpose. Newton-Cotes, Euler, Runge-Kutta and Simpson's Rule are computationally intensive. For simpler methods consider the following. If $\delta = \frac{x_2 - x_1}{K}$ and $\mathfrak{A}$ represents the area under the curve $\tilde{f}(.)$, this can be approximated from below by $\mathfrak{A}_L(x_1, x_2) = \sum_{j=0}^{K-1} \tilde{f}(x_1 + j\delta)\delta$ and from above by $\mathfrak{A}_U(x_1, x_2) = \sum_{j=1}^{K} \tilde{f}(x_1 + j\delta)\delta$, such that $\mathfrak{A}_L \le \mathfrak{A} \le \mathfrak{A}_U$. The trapezoidal rule approximation between the interval (x_1, x_2) can be defined by

$$\mathfrak{A}_{tr}(x_1, x_2) = \delta\left(\frac{\tilde{f}(x_1)}{2} + \sum_{j=1}^{K-1} \tilde{f}(x_1 + j\delta) + \frac{\tilde{f}(x_2)}{2} \right)$$

Clearly as $K \to \infty$, $\mathfrak{A}_L, \mathfrak{A}_U$ and $\mathfrak{A}_{tr}$ tend to $\mathfrak{A}$, but $\mathfrak{A}_{tr}$ does so with greater accuracy than either $\mathfrak{A}_L$ or $\mathfrak{A}_U$.

If the area under curve $\pi(w,t)$ is segmented at discrete consecutive periods, such that $\delta = 1$, the above suggest three alternative approximations for $p(w,j)$ and $P(w,j)$:

$$\begin{aligned} P(w,j) &\approx \sum_{k=0}^{j-1} \pi(w,k) \\ p(w,j) &\approx \pi(w,j-1) \end{aligned} \qquad (2.3)$$

$$\begin{aligned} P(w,j) &\approx \sum_{k=1}^{j} \pi(w,k) \\ p(w,j) &\approx \pi(w,j) \end{aligned} \qquad (2.4)$$

$$\begin{aligned} P(w,j) &\approx \begin{cases} \dfrac{\pi(w,j)}{2} & j = 1 \\ \displaystyle\sum_{k=1}^{j-1} \pi(w,k) + \dfrac{\pi(w,j)}{2} & j > 1 \end{cases} \\ p(w,j) &\approx \frac{\pi(w,j-1) + \pi(w,j)}{2} \end{aligned} \qquad (2.5)$$

Against the more accurate estimate of $P(w,j)$ and $p(w,j)$ that method (2.5) could possibly produce, is the neater construction of the cumulative and incremental claims reserving models that could be achieved by considering either of methods (2.3) and (2.4). The consequences of underestimating the percentage cash flow are more serious in underwriting years with claims known for only a few delay periods, as their proportion of the overall reserves is greater than the proportion represented by underwriting years with similar exposure and claims at a more advanced stage in their development. When method (2.4) is adopted,

$$S(w,j) = \int_j^\infty \pi(w,z)dz \approx \sum_{k=j+1}^{\infty} \pi(w,k) \quad \forall j \ge 0 \qquad (2.6)$$

In cases where $P(w,j)$ and $p(w,j)$ need to be approximated, as the losses develop, the characteristics of the underwritten risk should be extracted from $\Pi(w,j)$. Those that could be made immediately available are briefly described in the next section.

2.3 CHARACTERISTICS OF THE ACTUARIAL RISK

The aggregation of claims data for the purpose of reserving assumes that the data in each claims array broadly follows a similar development. This assumption can and should be assessed by extracting and comparing the information contained in $\Pi(w,t)$.

From the expression of the cumulative percentage cash flow the hazard rate and its integral can be obtained:

$$h(w,t) = \frac{\left(\frac{\partial \Pi(w,z)}{\partial z}\right)_{z=t}}{1-\Pi(w,t)} = -\left(\frac{\partial\left(\log\left(1-\Pi(w,z)\right)\right)}{\partial z}\right)_{z=t} \quad (2.7)$$

$$H(w,t) = -\ln\left(1-\Pi(w,t)\right) \quad (2.8)$$

Kurtosis and skewness, as measures of shape, will also be available when the values of the reserving model parameters permit estimating the necessary moments.

Estimates of $S(w,t)$ and of future claims for the regions $\mathbf{B_1}$ and $\mathbf{B_2}$ of the array in figure 1, are essential for the various analytical tasks of a claims portfolio, such as the evaluation of the solvency status, the assessment of the underwriting volume versus IBNR projections by class, the assessment of reinsurance cover, the calculation of future premiums, etc. There are significant advantages when function $\pi(w,t)$ is a probability density function. In this case the claims reserving model makes available all the descriptive statistics of the density function as characteristics of the percentage cash flow function. From section 2.2

$$\Pi(w,t) = \int_{z=0}^{z=t} \pi(w,z)dz \quad (2.9)$$

If (2.9) is cumulative density function, $\pi(w,t) = \Pr[z=t]$. When $\Pi(w,t) = \Pr[z \leq \Pi^{-}(\alpha)] = \alpha$, then $t = \Pi^{-}(\alpha) = \Pi^{-}(\Pi(w,t))$. Hence for a given percentage cash flow α and inverse function $\Pi^{-}(\alpha)$, it is possible to calculate the corresponding value of t. In the same way, an inverse function for $S(w,t) = 1 - P(w,t) = 1-\alpha$ can be defined, and the value of t for a given estimate of future losses $C_w S(w,t)$ can be determined. The behaviour of the right tail of the probability distribution function is particularly relevant and important to the calculation of reserves and to subsequent analysis of the claims portfolio. Therefore, the selected distribution function should be a suitable description of the actuarial risk's percentage cash flow.

In portfolios on run-off, where all aspects of asset management are drastically simplified as the flow of premiums reduces, the importance of having readily available IBNR projections is more apparent, since those are essential to unable a company to formulate and update a coherent commutation strategy

2.4 ESTIMATION TECHNIQUES AND THE GENERIC MODEL

Decisions on estimation techniques are determined by the overall structure of the model. If the percentage cash flow function can be linearized, generalized linear models could be considered (McCullagh and Nelder (1989)). For more complex claims reserving models, simulation techniques would be more useful.

Equation (2.2) gives the first part of the cumulative claims reserving model. The selection of the variance function should depend on the data. When cumulative claims reserving models are used, particular attention is required to the possible presence of serial correlation. In such cases this should be explained by a serial correlation matrix, say $\Gamma_w(\rho)$ with some parameter ρ. The literature available to help the practitioner explore and select the most suitable variance function is extensive.

The mean square errors for incremental claims reserving model for a single actuarial risk class are derived in section 3. To ensure that these can be immediately applied to different types of functions for $\pi(w,t)$, it is necessary to make the definition of $p(w,j)$ and C_w slightly more explicit. The final expression for the incremental claims reserving model is given by equation (3.1).

3. GENERAL INCREMENTAL CLAIMS RESERVING MODEL AND PREDICTED MEAN SQUARE ERRORS

Along the lines of Renshaw (1994), in this section the mean prediction errors for a hierarchical generic model for one actuarial risk are derived. In a claims array such as the one illustrated in Fig. 1, projected claims fall in the region defined by set $B=\bigcup_{i=1}^{2} B_i$. Renshaw (1994) assumes mutual independence between individual predictions in B, and independence between past and future claims. The main implication of these assumptions is that the results from Renshaw (1994) derived for non-hierarchical models can be easily extended for hierarchical models. To illustrate this, the mean square errors are derived in sections 3.1 to 3.4.

For our purposes we classify probability distributions functions into those with or without a normalising function. An all-encompassing definition could be achieved by the following: $\pi(w,t)=g_w(.)G(w,\beta_w,t)$ with $g_w^{-1}(.)=\int_0^\infty G(w,\beta_w,t)dt$. $g_w(.)$ is a normalising function, independent of t and possibly dependent on β_w, where β_w is a vector parameter $\beta_w=\left(\beta_{w_1},..,\beta_{w_k}\right)$. When the distribution does not have a normalising function $g_w(.)=1$ and $\pi(w,t)=G(w,\beta_w,t)$. Hence, if $f(j,\beta_w)=\int_{t=j-1}^{t=j} G(w,\beta_w,t)dt$ then $p(w,j)=f(j,\beta_w)g_w(.)$ and $C_w=\exp\left(\lambda_w\right)g_w^{-1}(.)$, where λ_w and β_w are the model parameters.

Hence, if $y(w,j)$ denotes the random variable representing the incremental claims data, then

$$\begin{aligned} E(y(w,j)) &= \mu(w,j) \\ Var(y(w,j)) &= \frac{\gamma}{\varpi_w} V(\mu(w,j)) \end{aligned} \tag{3.1}$$

ϖ_w are prior weights. The mean can be alternatively expressed as

$$\mu(w,j) = \mu(j,\lambda_w,\beta_w) = \exp(\lambda_w) f(j,\beta_w) \tag{3.2}$$

Let $z(w,j)$ be the unknown future incremental claims in development year j. The estimates of individual predictions, future losses for underwriting year w alone and for a book of business of u underwriting years with know claims up to payment year s are respectively:

$$E(z(w,j)) = \exp(\lambda_w) \int_{j-1}^{j} G(w,\beta_w,t)dt$$

$$E\left(\sum_{j=s-w+2}^{\infty} z(w,j)\right) = \exp(\lambda_w) \sum_{j=s-w+2}^{\infty} \int_{j-1}^{j} G(w,\beta_w,t)dt = \exp(\lambda_w) \int_{s-w+1}^{\infty} G(w,\beta_w,t)dt$$

$$E\left(\sum_{w=1}^{u} \sum_{j=s-w+2}^{\infty} z(w,j)\right) = \sum_{w=1}^{u} \exp(\lambda_w) \int_{s-w+1}^{\infty} G(w,\beta_w,t)dt$$

It is evident that when function $\pi(w,t)$ includes a normalising function, it is better to exclude it from the estimation procedures. In general, a simpler model tends to converge more rapidly, and the calculations of the mean square errors associated with it are also more transparent. By defining $\beta_w = (\beta_{w_1},\ldots,\beta_{w_k})$ the percentage cash flow function is allowed to change with underwriting year w, and the claims reserving models, as well as the prediction errors, become immediately suitable for hierarchical structures.

In the sections that follow the prediction errors are estimated up to delay period ζ.

3.1 INDIVIDUAL PREDICTIONS AND ERRORS

For future incremental claims in development year j

$$\begin{aligned} E(z(w,j)) &= \mu(j,\lambda_w,\beta_w) \\ Var(z(w,j)) &= \gamma V(\mu(j,\lambda_w,\beta_w)) \end{aligned} \tag{3.3}$$

For simplicity of notation denote $\mu_{wj} = \mu(j,\lambda_w,\beta_w)$, and let $\hat{\mu}_{wj} = \mu(j,\hat{\lambda}_w,\hat{\beta}_w)$ be the predictor of $z(w,j)$, such that $\hat{\lambda}_w$ and $\hat{\beta}_w$ are the parameter estimates. Then the first order Taylor series approximation of $\hat{\mu}_{wj}$ is

$$\hat{\mu}_{wj} \approx \mu_{wj}\left(1+\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_{w_k}}\left(\ln\left(f\left(j,\beta_w\right)\right)\right)\right) \tag{3.4}$$

For $(w,j)\in B$ the mean square error associated with the predictor is given by $E\left(\left(z(w,j)-\hat{E}\left(z(w,j)\right)\right)^2\right)$. However

$$\left(z(w,j)-\hat{E}\left(z(w,j)\right)\right)^2=\left(z(w,j)-\hat{\mu}_{wj}\right)^2=\left(\left(z(w,j)-\mu_{wj}\right)-\left(\hat{\mu}_{wj}-\mu_{wj}\right)\right)^2 \tag{3.5}$$

The expectations of the two square terms on the right hand side of (3.5) are

$$E\left(\left(z(w,j)-\mu_{wj}\right)^2\right)=Var\left(z(w,j)\right)$$

$$E\left(\left(\hat{\mu}_{wj}-\mu_{wj}\right)^2\right)\approx\mu_{wj}^2E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_{w_k}}\left(\ln\left(f\left(j,\beta_w\right)\right)\right)\right)^2\right) \tag{3.6}$$

Since $\hat{\mu}_{wj}$ is generated by past claims, owing to the assumptions of independence between past and future claims

$$E\left(\left(z(w,j)-\mu_{wj}\right)\left(\hat{\mu}_{wj}-\mu_{wj}\right)\right)=0$$

Hence

$$E\left(\left(z(w,j)-\hat{\mu}_{wj}\right)^2\right)\approx Var\left(z(w,j)\right)+\mu_{wj}^2E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_{w_k}}\left(\ln\left(f\left(j,\beta_w\right)\right)\right)\right)^2\right) \tag{3.7}$$

3. 2 PREDICTED ROW TOTALS AND THEIR MEAN SQUARE ERRORS

Let $\Im_w\subset B$, such that $\Im_1=\left(\bigcup_{j=s+1}^{\zeta}B_{2,(1,j)}\right)$ and $\Im_w=\bigcup\left(\bigcup_{j=s+2-w}^{s}B_{1,(w,j)}\right)\left(\bigcup_{j=s+1}^{\zeta}B_{2,(w,j)}\right), w>1$, represent in the matrix in Fig 1 the periods in underwriting year w for which claims are yet unknown. For all, ζ is the maximum projection period. For $(w,j)\in\Im_w$ the unknown total claim amount, the mean and the mean square error associated with the predictor can be defined as $Z_w=\sum_{(w,j)\in\Im_w}z(w,j)$, and

$$E\left(Z_w\right)=\sum_{(w,j)\in\Im_w}E\left(z(w,j)\right)=\sum_{(w,j)\in\Im_w}\mu_{wj} \tag{3.8}$$

$$E\left(\left(Z_w-\hat{E}\left(Z_w\right)\right)^2\right)=E\left(\left(\sum_{(w,j)\in\Im_w}\left(z(w,j)-\hat{\mu}_{wj}\right)\right)^2\right) \tag{3.9}$$

The right hand side of (3.9) gives rise to two types of terms:

i. When $i\neq j$:

$$\left(z(w,i)-\hat{\mu}_{wi}\right)\left(z(w,j)-\hat{\mu}_{wj}\right)=\left(z(w,i)-\mu_{wi}\right)\left(z(w,j)-\mu_{wj}\right)+\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(\hat{\mu}_{wj}-\mu_{wj}\right)$$
$$-\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(z(w,j)-\mu_{wj}\right)-\left(\hat{\mu}_{wj}-\mu_{wj}\right)\left(z(w,i)-\mu_{wi}\right) \quad (3.10)$$

From the assumptions of independence between $z(w,i)$ and $z(w,j)$ and between past and future claims

$$E\left(\left(z(w,i)-\mu_{wi}\right)\left(z(w,j)-\mu_{wj}\right)\right)=\operatorname{cov}\left(z(w,i),z(w,j)\right)=0$$

$$E\left(\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(z(w,j)-\mu_{wj}\right)\right)=E\left(\left(\hat{\mu}_{wj}-\mu_{wj}\right)\left(z(w,i)-\mu_{wi}\right)\right)=0$$

Finally, from equation (3.4)

$$E\left(\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(\hat{\mu}_{wj}-\mu_{wj}\right)\right)\approx \mu_{wj}\mu_{wi}Var(\lambda_w)$$
$$+\mu_{wj}\mu_{wi}E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\left(\frac{\partial}{\beta_{w_k}}\left(\ln\left(f(i,\beta_w)*f(j,\beta_w)\right)\right)\right)\right)\right)$$
$$+\mu_{wj}\mu_{wi}E\left(\left(\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\beta_{w_k}}\left(\ln\left(f(i,\beta_w)\right)\right)\right)\left(\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\beta_{w_k}}\left(\ln\left(f(j,\beta_w)\right)\right)\right)\right)$$

ii. When $i=j$:

$$\sum_{(w,j)\in\Im_w}E\left(\left(z(w,j)-\hat{\mu}_{wj}\right)^2\right)\approx\sum_{(w,j)\in\Im_w}Var\left(z(w,j)\right)$$
$$+\sum_{(w,j)\in\Im_w}\mu_{wj}^2E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_k}\left(\ln\left(f(j,\beta_w)\right)\right)\right)^2\right) \quad (3.11)$$

Hence, the only expectations contributing to the mean square error (3.9) are (3.11) and

$$2\sum_{\substack{(w,j)\in\Im_w\\ i<j}}E\left(\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(\hat{\mu}_{wj}-\mu_{wj}\right)\right)$$

3.3 PREDICTED PAYMENT YEAR TOTALS AND THEIR MEAN SQUARE ERRORS

For a book of business of u underwriting years, let $\wp_\tau=\bigcup\left(\bigcup_{w=u}^{\tau-s+1}B_{1,(w,\tau-w+1)}\right)\left(\bigcup_{w=\tau-s}^{1}B_{2,(w,\tau-w+1)}\right), \wp_\tau\subset B$ represent in the matrix in Fig. 1 the periods in payment year τ, and let the unknown total claim amount for payment year τ be defined as $Z_{++\tau}=\sum_{(w,j)\in\wp_\tau}z(w,j)$. Then

$$E(Z_{++\tau})=\sum_{(w,j)\in\wp_\tau}E\left(z(w,j)\right)=\sum_{(w,j)\in\wp_\tau}\mu_{wj} \quad (3.12)$$

$$E\left(\left(Z_{++\tau}-\hat{E}(Z_{++\tau})\right)^2\right)=E\left(\left(\sum_{(w,j)\in\wp_\tau}\left(z(w,j)-\hat{\mu}_{wj}\right)\right)^2\right) \quad (3.13)$$

i. From (3.7) when $w_1 = w_2 = w$ the square terms of (3.13) are

$$E\left(\sum_{(w,j)\in\wp_r}\left(z(w,j)-\hat{\mu}_{wj}\right)^2\right) \approx \sum_{(w,j)\in\wp_r} Var\left(z(w,j)\right) + \sum_{(w,j)\in\wp_r} \mu_{wj}^2 E\left(\left(\left(\hat{\lambda}_w - \lambda_w\right) + \sum_k \left(\hat{\beta}_{w_k} - \beta_{w_k}\right)\frac{\partial}{\partial \beta_{w_k}}\left(\ln\left(f\left(j,\beta_w\right)\right)\right)\right)^2\right) \tag{3.14}$$

ii. For $w_1 \neq w_2$ the combined terms in the right hand side summation of (3.13) can be written as follows:

$$\begin{aligned}\left(z(w_1,i)-\hat{\mu}_{w_1 i}\right)\left(z(w_2,j)-\hat{\mu}_{w_2 j}\right) &= \left(z(w_1,i)-\mu_{w_1 i}\right)\left(z(w_2,j)-\mu_{w_2 j}\right)+\left(\hat{\mu}_{w_1 i}-\mu_{w_1 i}\right)\left(\hat{\mu}_{w_2 j}-\mu_{w_2 j}\right)\\ &-\left(\hat{\mu}_{w_1 i}-\mu_{w_1 i}\right)\left(z(w_2,j)-\mu_{w_2 j}\right)-\left(\hat{\mu}_{w_1 j}-\mu_{w_1 j}\right)\left(z(w_2,i)-\mu_{w_2 i}\right)\end{aligned} \tag{3.15}$$

From the assumption of independence between $z(w_1,i)$ and $z(w_2,j)$ and between past and future claims

$$E\left(\left(z(w_1,i)-\mu_{w_1 i}\right)\left(z(w_2,j)-\mu_{w_2 j}\right)\right) = \operatorname{cov}\left(z(w_1,i), z(w_2,j)\right) = 0$$

$$E\left(\left(\hat{\mu}_{w_1 i}-\mu_{w_1 i}\right)\left(z(w_2,j)-\mu_{w_2 j}\right)\right) = E\left(\left(\hat{\mu}_{w_2 j}-\mu_{w_2 j}\right)\left(z(w_1,i)-\mu_{w_1 i}\right)\right) = 0$$

Finally, from equation (3.4)

$$\begin{aligned} E\left(\left(\hat{\mu}_{w_1 i}-\mu_{w_1 i}\right)\left(\hat{\mu}_{w_2 j}-\mu_{w_2 j}\right)\right) &\approx \mu_{w_1 i}\mu_{w_2 j} Cov\left(\lambda_{w_1}\lambda_{w_2}\right)\\ &+\mu_{w_1 i}\mu_{w_2 j} E\left(\left(\left(\hat{\lambda}_{w_1}-\lambda_{w_1}\right)\sum_k\left(\hat{\beta}_{w_{2k}}-\beta_{w_{2k}}\right)\frac{\partial}{\partial\beta_{w_{2k}}}\left(\ln\left(f\left(j,\beta_{w_2}\right)\right)\right)+\left(\hat{\lambda}_{w_2}-\lambda_{w_2}\right)\sum_k\left(\hat{\beta}_{w_{1k}}-\beta_{w_{1k}}\right)\frac{\partial}{\partial\beta_{w_{1k}}}\left(\ln\left(f\left(i,\beta_{w_1}\right)\right)\right)\right)\right)\\ &+\mu_{w_1 i}\mu_{w_2 j} E\left(\left(\sum_k\left(\hat{\beta}_{w_{1k}}-\beta_{w_{1k}}\right)\frac{\partial}{\partial\beta_{w_{1k}}}\left(\ln\left(f\left(i,\beta_{w_1}\right)\right)\right)\right)\left(\sum_k\left(\hat{\beta}_{w_{2k}}-\beta_{w_{2k}}\right)\frac{\partial}{\partial\beta_{w_{2k}}}\left(\ln\left(f\left(j,\beta_{w_2}\right)\right)\right)\right)\right)\end{aligned}$$

Hence, the only expectations contributing to the mean square error (3.13) are (3.14) and

$$2\sum_{\substack{(w,j)\in\wp_r\\ i<j}} E\left(\left(\hat{\mu}_{w_1 i}-\mu_{w_1 i}\right)\left(\hat{\mu}_{w_2 j}-\mu_{w_2 j}\right)\right)$$

3.4 PREDICTED OVERALL TOTAL AND ITS MEAN SQUARE ERROR

For $(w,j)\in B$ the unknown total claim amount can be defined as $Z = \sum_{(w,j)\in B} z(w,j)$ and

$$E(Z) = \sum_{(w,j)\in B} E\left(z(w,j)\right) = \sum_{(w,j)\in B}\mu_{wj} \tag{3.16}$$

$$E\left(\left(Z-\hat{E}(Z)\right)^2\right) = E\left(\left(\sum_{(w,j)\in B}\left(z(w,j)-\hat{\mu}_{wj}\right)\right)^2\right) \tag{3.17}$$

Equation (3.17) is equal to

$$\sum_{(w,j)\in B} E\left(\left(z(w,j)-\hat{\mu}_{wj}\right)^2\right)+2\sum_{\substack{(w_1,i)\in B\\(w_2,j)\in B\\(w_1,i)\neq(w_2,j)}} E\left(\left(z(w_1,i)-\hat{\mu}_{w_1i}\right)\left(z(w_2,j)-\hat{\mu}_{w_2j}\right)\right) \tag{3.18}$$

$(w_1,j),(w_2,j)$ in (3.18) represent all the distinct combinations of paired elements in set B The left-hand term of (3.18) can be obtained from (3.7) and the right hand term from the results of (3.10) and (3.15).

4. CLAIMS RESERVING MODELS IN THE PRESENCE OF RISK DISTORTIONS

The generic models defined in sections 2 and 3 assume that a claims portfolio can be segmented into distinctive data sets, such that within each set there is a single underlying claims process. This assumption cannot be readily extended to a reinsurance claims portfolio, which generally contains contracts underwriting more than one type of actuarial risk, or reflect distortions resulting from portfolio transfers or excess of loss policies with different limits. Although the generic model would not be appropriate in such cases, it can still be used to construct more complex claims reserving models. As an example of the simplest possible case, consider a reinsurance book of business underwriting two distinct and independent actuarial risk groups. If the development of the losses emerging from each can be assumed to have a hierarchical structure, with equivalent notation to (2.2), the systematic component of the reserving model would be given by

$$E\left(Y(w,j)\right)=C_{1w}P_1(w,j)+C_{2w}P_2(w,j) \tag{4.1}$$

where $P_1(w,j)$ and $P_2(w,j)$ are the percentage cash flow functions for each actuarial risk group, and C_{1w} and C_{2w} their ultimate claim amount functions. Equation (4.1) can be re-expressed as

$$E\left(Y(w,j)\right)=\left(C_{1w}+C_{2w}\right)\left(\frac{C_{1w}}{\left(C_{1w}+C_{2w}\right)}P_1(w,j)+\frac{C_{2w}}{\left(C_{1w}+C_{2w}\right)}P_2(w,j)\right) \tag{4.2}$$

And the ultimate claim amount, the percentage cash flow and the hazard rate functions for the contract for underwriting year w and development period j are:

$$C_w=\sum_{k=1}^{2}C_{kw} \tag{4.3}$$

$$P(w,j)=\sum_{k=1}^{2}\vee_{w_k}P_k(w,j) \tag{4.4}$$

$$h(w,j)=\sum_{k=1}^{2}\wedge_{w_k}\left(\frac{\left.\frac{\partial P_k(w,z)}{\partial z}\right|_{z=j}}{\left(1-P_k(w,j)\right)}\right) \tag{4.5}$$

where $\vee_{w_i}$ and $\wedge_{w_i}$ are weights defined as

$$\vee_{w_i}=\frac{C_{iw}}{\sum_{k=1}^{2}C_{kw}}$$

$$\wedge_{w_i}=\frac{C_{iw}\left(1-P_i(w,j)\right)}{\sum_{k=1}^{2}C_{kw}\left(1-P_k(w,j)\right)}$$

Hence, consistently with (2.2)

$$E\left(Y(w,j)\right)=C_wP(w,j)$$

The weights for the percentage cash flow and the hazard rate functions are intuitively obvious. The model can be easily generalized for a contract with n types of actuarial risks by replacing 2 by n in the above equations. When the percentage cash flow functions of individual risks all satisfy the criteria given in section 2.2, in general so will $P(w,j)$.

The estimation method selected to model reinsurance reserves for claims emerging from different types of actuarial risks will depend on the formulation of the claims reserving model. Equation (4.1) already excludes generalized linear modeling techniques.

5. CONCLUSION

Sections 2.2 and 2.3 show that the generic model brings to light and suggests innumerable types of claims reserving models: incremental and cumulative, hierarchical and non-hierarchical. When $\pi(w,j)$ is a probability density function the reserving models provide a sound statistical basis for the calculation of reserves and for subsequent portfolio analyses. Once the parameters of the reserving models have been estimated, when $\pi(w,j)$ is a standard probability density function, values of $S(w,j)$ and $P(w,j)$ would be readily available from most statistical packages. Hence, a limited amount of programming would be required to obtain the projected losses needed for portfolio modeling.

The variance for the reserving model has been simply defined to be a function of the mean. The form that the function takes will depend on the data.

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Review of "Capital Allocation for Insurance Companies" by Stewart C. Myers and James R. Read Jr.

Paul J. Kneuer, FCAS, MAAA

Review of Capital Allocation for Insurance Companies

ARIA Paper by Stewart C. Myers and James R. Read, Jr.

PCAS Review by Paul J. Kneuer, FCAS

The CAS must thank Doctors Myers and Read for their intriguing article. They have developed a practical algorithm for a previously subjective problem. Regulators often require a way to measure at least the indirect cost of an insurer's Surplus in ratemaking. This article offers a well-defined solution, together with a theoretical and philosophical explanation. There are practical problems with any approach to pricing administration in a largely free economy and with the most common theoretical context for administered rate regulation. But these issues are outside of the author's scope.

The Authors' Proposal

Profit targets or premium levels for regulated insurance products often reflect the amount of Surplus that an insurer commits to support the business under review and the cost of committing that Surplus. The authors suggest the following algorithm to appropriately reflect the cost of committing Surplus to a particular insurance product:

1. Compute the total expected default value of an insurer or of a group of insurers. This would be for the entire industry in an administered pricing state, such as Massachusetts.
2. Compute the insurers' marginal default value in respect of each product segment (the partial derivatives of the overall default value with respect to an increase in the amount of expected losses for each product.)

The authors show the novel and intriguing result that when the quantity of (expected losses x marginal default value) for each product is summed over all products, the result is equal to the overall expected default value. This is a surprising result. There are diversification benefits in combining risky but partly uncorrelated ventures, so the marginal cost of adding more of a product is generally less than the average cost. The by-line costs usually do not "add up." (This is the financial root of all insurance.) The new contribution here is to multiply these marginal values by the current amount of expected losses in each product category. Since these results do "add up", they can be used as an allocation base.

3. Allocate the overall Surplus among products in proportion to (marginal default value x expected losses.)

In the Myers-Cohn pricing approach commonly used in Massachusetts, regulators recognize that the allocated Surplus earns investment profits in addition to operating returns and that these investment profits are currently subject to two rounds of taxation: once paid by insurers corporately, and then paid again by the owners of the insurers. Regulated rates must allow a provision for the cost of this second taxation, or else they are confiscatory.

4. Load the premiums by a pre-tax provision of (Allocated Surplus x Return on Assets x Time Factor x Tax Rate). A sample calculation is shown on page two of the following exhibit, assuming a one-year maturity and that the authors' algorithm provided a 50% surplus-to-expected-loss ratio.

Application

Massachusetts rate regulation has been a very fertile ground for analyzing the capital structure and profit requirements for insurance companies. This paper is another important contribution to that history. The authors give an objective and consistent measure of the amount of Surplus which is subject to this double taxation. The authors show that marginal default values **can** be an allocation base for Surplus in rate regulation calculations. They argue that it **should** be, for well-founded reasons. But they do not, and cannot, show that it **must** be.

The authors provide two strong arguments for using their marginal default values. First, marginal default values have the high merit that they "add up". However, allocations based on premiums, losses, expenses, historic profit provisions, aggregate amount of limits provided, or policy counts also add up and have some plausible arguments as allocation bases. Second, and what is more important, the authors consider an environment where all insurance is sold on a "retail" base, subject to guarantee funds. Regulators can view the marginal contribution to default risk as the true cost to society of providing coverage, and thus a gauge of the fair price to charge to insureds. Not argued by the authors, but in a simpler view, regulators might also feel that committing more Surplus to a product provides a higher quality of insurance protection and therefore merits a proportionally higher profit.

Reassuringly, marginal default risks and capital commitments certainly move together. Products that contribute more to the potential default of a company, or to a larger default if it should happen, are greater commitments of insurers, and do merit higher rewards from regulators. While other allocation bases are possible, this reviewer feels that the approach suggested by the authors is a very reasonable one, and one that regulators should strongly consider, when an allocation is required.

A recent trend among U.S and other insurers has been the movement to off-shore domiciles. In these situations, many insurers are exempt from income tax. They are, instead, subject to excise tax which can be as little as 1% of premiums for the risk-bearer, when exposure is sent off-shore in the form of reinsurance. Understanding the cost of an off-shore company's commitments does not require a Surplus allocation.

But at least today, most primary insurance provided in the United Sates is written by U.S.-based companies who are subject to income tax on their investment returns. Moreover, regardless of tax status and domicile, insurers' Surplus has other costs. For example, insurers cannot freely invest in the asset mix which they view as optimal. They also find difficulty in moving capital in and out of their companies without regulatory approval and delay and rating penalties.

Critique

The authors are concerned that inefficient allocations of capital will result in inaccurate regulated prices, and thus insurers will "push the wrong product". However there are many other factors which assuredly do results in regulated insurers "pushing the wrong product".

For example, insurers use different distribution systems which have very different costs. For some products, there are companies that have operating costs differences of more than 10% of premiums. This is a multiple of the difference regulators would find between various product lines from the cost of double taxation on different amounts of allocated capital.

In addition to distribution channels, insurers have differences in time horizons, growth plans, product offerings and their ownerships' risk perspectives that result in different plans and aspirations. These companies will not view the same product in the same way. On a practical level, may a regulator take legal notice of the diversification benefits of exposures outside of the state or country?

Another concern with the authors' algorithm is that companies with different product mixes or levels of investment risk generally have different marginal default values for the same product. This can cause a destructive incentive. For example, using the authors' algorithm, if a company finds Automobile insurance to be insufficiently attractive at the approved pricing, by writing a much larger amount of credit derivatives, Catastrophe reinsurance or excess D&O or by investing in Iraqi bonds, the marginal default value for Automobile falls and it becomes more attractive.

A final concern with the authors' algorithm is circularity. Marginal default values depend on the current mix of business. But a company using this approach to choose the business they write does not yet know its mix of business. Thus, the profit depends on the allocation, which depends on the mix of business, which depends on the profit. (See Dan Gogol's review of Rodney Kreps' article in PCAS LXXVII.) Including a history of regulation, at times misdirected regulation, results in a knottier problem. A regulator cannot develop appropriate pricing in the future, unless we know that the regulated prices in the past resulted in the "appropriate" mix of business

Requiring the same rate for all companies for all risks guarantees that most companies face at best inefficient incentives. This is especially true when regulation also distorts price-setting to meet unrelated social objectives, as in Massachusetts Automobile insurance.

CAPM in Insurance

The key underlying assumption in applying a Surplus allocation in rate regulation is not the authors'. The regulatory model in Massachusetts and other states is that insurers should only earn profits consistent with those earned on other investments of comparable risk. This is based on the U.S. Supreme Court's Hope Natural Gas decision, which is usually interpreted to mean the profit margins which would be calculated using the Capital Asset Pricing Model (CAPM) method. That is, only the systematic, non-diversifiable risks within an insurance portfolio will (and should) earn a return above Treasuries, and that return is the same as a stock with equal correlation to the overall market would earn. MBA students are traditionally taught that a dynamite factory is a systematically less risky investment than a diversified, but leveraged, stock portfolio.

CAPM argues that the capital market cannot allow the dynamite factory to earn a higher return than an investment with a comparable beta factor.

A recent empirical challenge to the use of CAPM in insurance pricing is the market for Catastrophe bonds. These bonds earn significant excess returns on their total risk, even though that risk is seen as diversifiable. Catastrophe bonds have a low beta: some authors (example: Kenneth Froot) suggest that catastrophe insurance risk has a zero beta. However, Catastrophe bonds actually earn returns very significantly above equivalent Treasury bonds (see the recent work of Morton Lane). Contradicting the dynamite factory analogy, investors require an extra reward for the clear, but diversifiable, chance of a catastrophic loss of their investments.

We see that the CAPM results do not currently hold for insurance investments. This is only possible in the long term if the essential assumptions of CAPM do not apply to insurance investments. Do they? Insurers may not borrow or short sell without limit. Capital does not move freely into and out of insurance companies. Insureds and insurers and their investors and regulators all have different time horizons. Insurance contracts are not transferable or divisible. Several of the key assumptions underlying CAPM rate regulation clearly do not apply, although regulators persist in applying them out of respect for precedent or for lack of a practical alternative.

One alternative possible now for regulators is to model a notional portfolio representing a mix of Catastrophe bonds and Treasury bonds that matches the degree of total risk of an insurance product. The excess return on this notional portfolio would be an appropriate profit provisions to include in rates. This provides an objective measure of return without relying on the CAPM's assumptions, which we can see are violated both in theory and by market results. The alternative would however provide a non-confiscatory return "of equivalent risk" as required by Hope. A sample calculation is shown on page three of the exhibit.

A more direct analysis, that was also impossible in the past, is to look at the long-term loss ratio for the same product in the many states that now have vibrant and effective competitive market. This would provide a direct benchmark of the relationship between price and risk that Hope says insurers should earn. This comparison would also allow regulators to ignore differences of expense and product mix that insureds do not care about. Insureds only value what they receive as expected loss recoveries, defense and cost containment expenses, loss control services (and perhaps premium taxes, as a surrogate for sales tax that they might pay to replace their damaged property). This different approach could be viewed as "demand side" regulation.

The Author's Derivations

The key result - - that marginal default values "add up" - - is nicely proven for the case of two products and a fixed asset return. The authors generalize to multiple products and variable assets. This would be a stronger development with two additions. First, it would also be nice to see that the "adding up" works going from two products to three. Then, by "nesting" the definition of products as A, A+B, A+B+C, etc. we could induce that the conclusion holds for any number and ordering of products. Second, the authors briefly move from an unknown asset return to all asset returns. An illustration or two would make this more persuasive, for example, two products with respectively positive and negative correlation with assets.

Finally, I very much enjoyed the authors' description of a U-shaped profitability curve reflecting the changing contribution of operating and capital costs as the number of different insurance products increases. An opposite conclusion has been drawn for other industries. Operating costs and relative pricing often move in opposite directions with unit volumes, but don't stabilize in the middle.

For example, McDonald's has the highest market share among restaurants and can spread its general costs over famously its billions and billions of transactions. Conversely, each reader's neighborhood favorite has a particular format that reflects local tastes and circumstances better than any national firm can. This allows effective product differentiation and prices much higher than McDonald's. A smaller national chain can hope for neither benefit and is typically less profitable than either extreme.

Insurance has a similar problem but, as the authors note, the correlation between risks adds a dimension. Insurers can pursue a competitive advantage in three ways, not just two:

1. High volume produces low per-unit operating expenses.
2. Specialization allows better customer responsiveness and higher pricing levels.

3. Diversification allows lower overall variability and a relatively lower cost of risk per unit.

Unfortunately, these tactics are essentially contradictory.

- High volume or a specialized focus prevents diversification of risk.
- Specialization or diversification prevents economies of scale
- High volume or diversification prevents customer responsiveness.

While a two-dimensional market space allows two stable and optimal solutions, a three-dimensional space allows three polar optima, and perhaps also three hybrids, each balancing two of the poles at a time. This is a much more challenging problem, and may help explain the instability seen in insurance markets.

"I liked white better," I said.

"White!" he sneered. "It serves as a beginning. White cloth may be dyed. The white page can be overwritten; and the white light can be broken."

"In which case it is no longer white," said I. "And he that breaks a thing to find out what it is has left the path of wisdom."

Gandolf, recounting a conversation with Saruman:

The Fellowship of the Ring, J.R.R. Tolkien, Book Two, Chapter 2.

Exhibit – Page One

Simplifying Assumptions

A. Cat Bond, Priced at Market

• Face amount:	$100,000,000 Limit
• Term:	One year
• Dividends:	None
• Current price:	$94,000,000
• Estimated frequency of loss:	0.001 per year
• All losses are total limits	
• Estimated beta:	0

B. Capital Market Results

• Expected return on total market	7%
• Risk-free interest rate (1-year Treasury):	2%

C. Regulated Automobile Liability Policies

• Limit:	$1,000,000
• Estimated frequency of loss:	0.01 per year per policy
• Average severity:	$100,000 (including ALAE)
• Standard deviation of severity:	$100,000
• Frequency and severity are independent	
• Estimated beta:	20%
• Average duration:	1 year

D. Industry Totals

• Number of cars:	5,000,000
• Expected losses:	$5,000,000,000
• Expected number of claims:	50,000
• Standard deviation of number of claim:	15,000
• Average costs (including Taxes, Commissions, ULAE):	30% of GWP
• Premium tax only:	5% of GWP

Exhibit – Page Two

CAPM Pricing Model

• Expected industry losses:	$5,000,000,000
• Beta of losses:	20%
• Risk-adjusted discount rate:	3% = 2% +20% x (7%-2%)
• Discounted losses:	$4,854,000,000 = (Losses/1.03)
• Allocated Surplus:	$2,427,000,000 (Assumed 50% per authors' model)
• Interest on Surplus:	$48,544,000 (2% risk-free rate)
• Tax on interest:	$16,019,000 (33%)
• Losses plus tax allowance:	$4,870,000,000 (Discounted losses + tax on interest)
• Gross Premiums:	$4,870,000,000 / (1-30%) = $6,957,000,000
• Rate per car:	$1,391.

Exhibit – Page Three

Pricing via Reference to Cat Bond Returns

Massachusetts Industry Totals

- Expected losses: $5,000,000,000
- Standard deviation of losses: $5,400,000,000 (simulation results)

Cat Bond Pricing

- Expected losses: $1,000,000 (1% of $100,000,000)
- Promised return: $6,000,000 ($100 Mn - $94 Mn)
- Expected return: $5,000,000 ($6 Mn - $5 Mn)
- Risk-free return: $1,880,000 (2% of $94 Mn current price)
- Excess return: $3,120,000
- Standard deviation of losses: $9,950,000 (via binomial model)
- Excess return / SD: 31.36%

Required Rates

- Expected losses: $5,000,000,000
- Discounted losses: $4,902,000,000 (discounted at 2% risk-free)
- Allowed return: $1,693,440,000 (31.36% of $5.4 Bn)
- Losses plus risk charge: $6,595,000,000 ($4.902 Bn + $1.639 Bn)
- Gross premiums (Supply side): $6,595,000,000 / (1-30%) = $9,422,000,000
- Rate per car: $1,884.
- Gross premiums (Demand side): $6,595,000,000 / (1-5%) = $6,942,000,000
- Rate per car: $1,388.

The Economics of Capital Allocation

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The Economics of Capital Allocation

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Abstract

On the surface, capital allocation sounds contradictory to the stated purpose of insurance, which is diversifying risk. In spite of that, it is commonly used as a tool by insurers to manage their underwriting risk. This paper examines the economics underlying how insurers might use capital allocation when capital is scarce and has a price. Starting from a risk-based capital framework, the paper establishes strategies for increasing the insurer's expected return on capital. It then derives capital allocation methods that are consistent with these economic strategies.

1. Introduction

The practice of allocating capital for ratemaking has long been controversial. As McClenahan [1990] said several years ago, "In essence, the method treats a multiline national company with $100 million of capital – *$1 million of which is allocated to California private passenger automobile – identically* with the California private passenger automobile carrier capitalized at $1 million." In spite of this criticism, many insurance company executives continue the practice. Their reasons for doing so are understandable. The executives are answerable to the insurance company's investors who demand a competitive return on their investment. The executive's job is to direct the managers of the various lines of business within the insurance company to achieving this goal. A seemingly straightforward way of doing this is to establish a yardstick for each line of business based on its return on allocated capital.

The problem with this management strategy lies in the particular methodology for allocating capital. That is to say, the devil is in the details. As I will show in an example below, some ways of allocating capital can lead to decisions on the part of the individual line managers that do not benefit the insurance company as a whole.

The purpose of this paper is to give some ways of allocating capital that lead to sound economic decisions. By "sound economic decisions," I mean decisions that increase the insurer's expected return on its capital investment.

In writing this paper, I do not want to imply that allocating capital is necessary. It is possible to devise a pricing methodology that is economically sound without allocating capital. But for those who choose to allocate capital, I hope to provide some useful economic advice.

An insurer operates by making pricing and underwriting decisions on the insurance policies it writes. These decisions will be evaluated according to their impact on the insurer's cost of capital. Thus our first step is to establish a yardstick for the insurer's cost of capital.

2. The Insurer's Cost of Capital

Let X be a random variable representing an insurer's loss for a particular book of business. Let $C(X)$ equal the capital needed to support the book of business with random loss X. We assume that C satisfies the following two axioms:

1. Subadditivity – For random losses X and Y, $C(X+Y)$ $C(X) + C(Y)$.
2. Positive homogeneity – For all constants k 0, $C(k \cdot X) = k \cdot C(X)$.

The subadditivity axiom means that when you pool books of business, you do not need more total capital. In fact, an efficient pooling of risk should result in needing less total capital.

Here are some examples of capital formulas that satisfy these two axioms.

<u>Example 1 – The Standard Deviation Capital Formula</u>

$$C(X) \equiv T \cdot \sigma_X$$

Generally T is in the 2 to 3 range.

$C(X)$ satisfies the two axioms above.

Proof:

$$C(X+Y) = T \cdot \sigma_{X+Y} = T \cdot \sqrt{\sigma_X^2 + 2 \cdot \rho \cdot \sigma_X \cdot \sigma_Y + \sigma_Y^2} \le T \cdot \sqrt{\sigma_X^2 + 2 \cdot \sigma_X \cdot \sigma_Y + \sigma_Y^2}$$
$$= T \cdot (\sigma_X + \sigma_Y) = C(X) + C(Y)$$

$$C(k \cdot X) = T \cdot \sqrt{Var[k \cdot X]} = T \cdot \sqrt{k^2 \cdot Var[X]} = k \cdot C(X)$$

Example 2 – Capital derived from coherent measures of risk

Let $\rho(X)$ be a "measure of risk" that we assign to a random loss X. If $\rho(X)$ satisfies the following axioms:

1. Subadditivity – For all random losses X and Y,
$$\rho(X+Y) \le \rho(X) + \rho(Y).$$
2. Monotonicity – If $X \le Y$ for each scenario, then,
$$\rho(X) \le \rho(Y).$$
3. Positive Homogeneity – For all k 0 and random losses X,
$$\rho(k \cdot X) = k \cdot \rho(X).$$
4. Translation Invariance – For all random losses X and constants a,
$$\rho(X+a) = \rho(X) + a$$

then $\rho(X)$ is called a coherent measure of risk. These measures of risk were originated by Artzner, Delbaen, Eber and Heath [1999]. I have previously written about these measures in Meyers [2001] and Meyers [2002].

Here are some examples of coherent measures of risk.

- Let X take its values on a finite set of scenarios. Then $\rho(X) \equiv Max\{X\}$ is a coherent measure of risk.
- For a given percentile, α, let the Value-at-Risk, $VaR_\alpha(X)$, be defined as the α^{th} percentile of X. Meyers [2001] demonstrates that $VaR_\alpha(X)$ is not a coherent measure of risk; but the Tail Value-at-Risk,
$$TVaR_\alpha(X) = E\left[X \mid X > VaR(X)\right]$$
is a coherent measure of risk. One would choose α according to their aversion to risk.

Let $\rho(X)$ be a coherent measure of risk. If we normalize $\rho(X)$ so that $\rho(0) = 0$, we can use $\rho(X)$ to denote the value of the assets held by an insurer to support its random losses X.

Let's assume that the policyholders supply $E(X)$ through their premiums. Then using a capital of

$$C(X) = \rho(X) - E[X]$$

provides the insurer with sufficient assets to support its random loss X. Also, $C(X)$ satisfies the two axioms listed at the beginning of this section.

Example 3 – Transformed probability formulas

Let g, mapping [0,1] to [0,1], be a nondecreasing, concave up, continuous function. Let $F(x)$ be the cumulative distribution function of X. Let U be a random variable with the cumulative distribution $g(F(u))$. Then according to Theorem 3 of Wang, Young and Panjer [1997],

$$\rho(X) \equiv E[U]$$

is a coherent measure of risk. Measures of this form have the additional property of being *co-monotonic additive*; i.e., if $(X_i - Y_i)\cdot(X_j - Y_j)$ 0 for all scenarios i and j, then $\rho(X + Y) = \rho(X) + \rho(Y)$.

Meyers [2002] gives an example of a coherent measure or risk that is not co-monotonic additive.

- If $g(u) = Max\{0, (u - \alpha)\}/(1 - \alpha)$ for $\alpha \in [0,1]$, then $E[U] = TVaR_\alpha(X)$.

- If $g(u) = \Phi(\Phi^{-1}(u) - \lambda)$, where Φ is the cumulative distribution function for the standard normal distribution, then $E[U]$ is called the Wang Transform. λ increases with risk aversion.

As these examples show, there is a good supply of capital formulas that reflect varying degrees of risk aversion.

The final step in calculating the insurer's cost of capital is to determine the expected rate of return on its capital investment. This rate of return is determined by examining the rate of return obtained by other investments of comparable risk. Security analysts have been doing this for years and I have no special insight to offer. In this paper, I take this expected rate of return as a given. Unless the insurer makes major changes in its operations, I think it is reasonable to assume that it is constant.

Once we decide how to calculate the insurer's overall cost of capital, the next step is to analyze how insurer pricing and underwriting decisions affect this overall cost of capital. This leads us to consider the insurer's marginal cost of capital.

3. The Insurer's Marginal Cost of Capital

Suppose the insurer is currently maintaining a book of business with random loss X, capital requirement C, and expected profit P. Suppose further that it is considering adding a new set of insurance policies with random loss ΔX, and expected profit ΔP, to its book of business. Define the marginal capital for the new policies as

$\Delta C = C(X + \Delta X) - C(X)$.

Proposition 1

An insurer will increase its return on capital if and only if the new business' return on marginal capital is greater than the insurer's overall return on existing capital.

Proof:

$$\frac{P+\Delta P}{C+\Delta C} > \frac{P}{C}$$

$$\Leftrightarrow PC + \Delta P \cdot C > CP + \Delta C \cdot P$$

$$\Leftrightarrow \Delta P \cdot C > \Delta C \cdot P$$

$$\Leftrightarrow \frac{\Delta P}{\Delta C} > \frac{P}{C}$$

The opposite of Proposition 1 should also be clear. An insurer will increase its return on capital by dropping business if and only if the marginal capital of the dropped business is less than the insurer's overall return on existing capital.

This proposition provides an analytic method for insurers to increase their return on capital. For a given insurance policy an insurer can follow this underwriting strategy.

Underwriting Strategy #1

1. Observe the premium that the market allows for this insurance policy.
2. Calculate the expected profit that the insurer can obtain by writing the insurance policy.
3. Calculate the marginal capital needed if it were to write the insurance policy.
4. If the expected return on the marginal capital exceeds the insurer's current expected return on capital, the insurer should increase its capital and write the new insurance policy.

At this point, I would like to introduce a fairly lengthy example.

Example 4

- An insurer writes two independent lines of business. The amount of business it writes in each line is a decision variable, and we will quantify these amounts of business by their expected claim counts ν_1 and ν_2. The amount of each claim is set equal to one.

- Let N_i be the random number of claims for line i. Each N_i will have a mixed Poisson distribution. Let χ_i be a random variable with a mean = 1 and variance = $c_i > 0$. Given χ_i, the i^{th} distribution is a Poisson with mean $\chi_i \cdot \nu_i$. Unconditionally, the mean of each claim count distribution is ν_i and the variance of each claim count distribution is:

$$E_{\chi_i}\left[Var[N_i \mid \chi_i]\right] + Var_{\chi_i}\left[E[N_i \mid \chi_i]\right] = E_{\chi_i}\left[\nu_i \cdot \chi_i\right] + Var_{\chi_i}\left[\nu_i \cdot \chi_i\right] = \nu_i + c_i \cdot \nu_i^2 .$$

  Note that the variance of the loss ratio

$$Var\left[\frac{N_i}{\nu_i}\right] = \frac{1}{\nu_i} + c_i$$

  decreases as the exposure increases but, as we often observe in real life, this variance is always greater than zero – no matter how much exposure the insurer writes. You can think of the random variable χ_i as an analogue to changing economic conditions.

- The insurer determines its capital by taking a multiple of two times the standard deviation of its total losses; i.e.,

$$C = 2 \cdot \sqrt{v_1 + c_1 \cdot v_1^2 + v_2 + c_2 \cdot v_2^2}\,. \tag{1}$$

- The insurer's expected profit is proportional to the expected claim counts, i.e.

$$P = r_1 \cdot v_1 + r_2 \cdot v_2. \tag{2}$$

- If the insurer makes a small change in its exposure for line i, its return on marginal capital is closely approximated by:

$$\left.\frac{\Delta P}{\Delta C}\right|_{\text{Line } i} \approx \left.\frac{dP}{dC}\right|_{\text{Line } i} \equiv \frac{\dfrac{\partial P}{\partial v_i}}{\dfrac{\partial C}{\partial v_i}} = \frac{r_i}{2 \cdot \dfrac{1 + 2 \cdot c_i \cdot v_i}{C}}. \tag{3}$$

The formulas above are easily programmed into a spreadsheet. Let's plug some numbers into these formulas. Set:

Line i	r_i	c_i
1	5%	0.02
2	2%	0.01

In this example, I view these parameters as being beyond the control of the insurer.

Let's pause for a moment and digress on how this example relates to the real world. The r_i's and the c_i's describe the economic environment in which the insurer operates. The r_i's are backed out from the premium that the market will allow the insurer to charge. The c_i's are analogous to a measure of the inherent volatility of a line of insurance.

What the insurer can control is how much business it writes in each line of insurance. In this example, the control parameters are the v_i's. The quantities for Equations 1-3 for various v_i's are given in the following table.

Table 1

v_1	v_2	C	P	P/C	Line 1 dP/dC	Line 2 dP/dC
100.00	100.00	44.72	7.00	15.65%	22.36%	14.91%
117.69	64.15	44.72	7.17	16.03%	19.59%	19.59%
133.41	76.73	50.00	8.21	16.41%	19.73%	19.73%
163.44	100.76	60.00	10.19	16.98%	19.90%	19.90%
208.85	137.08	75.00	13.18	17.58%	20.04%	20.04%
285.03	198.02	100.00	18.21	18.21%	20.16%	20.16%
3,052.52	2,412.01	1,000.00	200.87	20.09%	20.31%	20.31%
30,747.90	24,568.32	10,000.00	2,028.76	20.29%	20.31%	20.31%
307,703.73	246,132.98	100,000.00	20,307.85	20.31%	20.31%	20.31%

There are several points that can be made with this table.

1. First let's consider the case $v_1 = v_2 = 100$. Note that the return on marginal capital for Line 1 is greater than the overall return, and the return on marginal capital for Line 2 is less than the overall return. By Proposition 1, the insurer can increase its overall rate of return by adding business in Line 1, and reducing business in Line 2.

2. Increasing v_1 to 117.69 and decreasing v_2 to 64.15 gives a higher return for the same amount of capital. These numbers can be derived by choosing v_1 and v_2 so the total profit, P, is maximized subject to a constraint on the capital, $C(X)$ I, using the method of Lagrange multipliers. This problem is solved in a more general setting in Equation 5.1 of Meyers [1991]. Here is the solution for this special case[1].

$$v_i^* = \frac{\dfrac{I \cdot r_i}{2 \cdot \lambda^*} - 1}{2 \cdot c_i}, \text{ where } \lambda^* = \frac{I}{2} \cdot \sqrt{\frac{\dfrac{r_1^2}{c_1} + \dfrac{r_2^2}{c_2}}{I^2 + \dfrac{1}{c_1} + \dfrac{1}{c_2}}}. \tag{4}$$

[1] Meyers [1991] had a variance constraint rather than a capital constraint. Since in this example our capital is a function of the variance, the solutions are equivalent. But the Lagrange multiplier, λ^*, for the capital constraint is $I/2$ times the Lagrange multiplier, λ^*, for the variance constraint. This change is cancelled out in the expression for v_i^*.

3. For the choice of ν_i's immediately above, the return on marginal capital for both lines is higher than the overall return on capital. This means that we can add exposure to both lines and increase the overall rate of return. Successive lines in Table 1 are calculated by first increasing the constraint, *I*, on the capital, *C*, and setting $\nu_1 = \nu_1^*$ and $\nu_2 = \nu_2^*$ using Equation 4.

4. When we use Equation 4 to choose the ν_i's, note that the returns on the marginal capitals for each line are equal. It turns out that that property of this example can be generalized.

Proposition 2

Suppose:

i. An insurer can write business in any of *n* lines of insurance.
ii. The amount of business in Line *i*, is quantified by an exposure amount e_i. The random loss, X_i, in Line *i*, does not decrease as e_i increases.
iii. The insurer's expected profit, $P(e_1,\ldots,e_n)$ is a differentiable function of each e_i.
iv. The insurer's total capital, $C_E(e_1,\ldots,e_n) \equiv C\left(\sum_{i=1}^{n} X_i \mid e_1,\ldots,e_n\right)$, is a differentiable function of each e_i.

The insurer wishes to choose exposure amounts, e_i^* for $i = 1, \ldots n$, in such a way as to maximize its expected profit, *P*, subject to a limitation, *I*, on its capital investment. Then for all lines, the return on marginal capitals:

$$\left.\frac{dP}{dC_E}\right|_{\text{Line } i} \equiv \left.\frac{\dfrac{\partial P}{\partial e_i}}{\dfrac{\partial C_E}{\partial e_i}}\right|_{e_i=e_i^*} \tag{5}$$

are all equal to each other.

I provide two proofs of this proposition.

Proof #1:

We solve for the exposures $\{e_i^*\}$ by the method of Lagrange multipliers. Set:

$$L = P(e_1,\ldots,e_n) + \lambda\cdot(I - C_E(e_1,\ldots,e_n)).$$

The method works by solving the $n + 1$ equations for $\{e_i^*\}$ and λ^*:

$$\left.\frac{\partial L}{\partial e_i}\right|_{e_i=e_i^*} = 0 \text{ for } i = 1,\text{K}\ ,n \text{ and } \left.\frac{\partial L}{\partial \lambda}\right|_{\lambda=\lambda^*} = 0\ .$$

The first n equations give:

$$\left.\frac{\partial L}{\partial e_i}\right|_{e_i=e_i^*} = \left.\frac{\partial P}{\partial e_i}\right|_{e_i=e_i^*} - \lambda^*\cdot\left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*} = 0 \Rightarrow \left.\frac{\dfrac{\partial P}{\partial e_i}}{\dfrac{\partial C_E}{\partial e_i}}\right|_{e_i=e_i^*} = \lambda^*, \tag{6}$$

which is what we need to prove. It turns out that the Lagrange multiplier, λ^*, is equal to the return on marginal capital that is common for all the lines of insurance.

Proof #2

Suppose we have chosen exposure amounts, e_i^* for $i = 1, \ldots n$, in such a way as to maximize expected profit, P, subject to a limitation, I, on the capital investment. Suppose further that for two lines i and j:

$$\left.\frac{dP}{dC_E}\right|_{\text{Line } i} > \left.\frac{dP}{dC_E}\right|_{\text{Line } j}$$

Then rewriting Equation 5 as a differential[2] we have:

$$dP = \left.\frac{\partial P}{\partial C_E}\right|_{Line\, i}\cdot\left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*}\cdot de_i \text{ and } dP = \left.\frac{\partial P}{\partial C_E}\right|_{Line\, j}\cdot\left.\frac{\partial C_E}{\partial e_j}\right|_{e_j=e_j^*}\cdot de_j \tag{7}$$

[2] I am using differential notation to express the approximate effect of a "small" change in one variable with a "small" change in another variable. If $y = f(x)$ the differential dy is equal to $f'(x)dx$.

Adjust the incremental exposures de_i and de_j so that:

$$dC_E = \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*} \cdot de_i = \left.\frac{\partial C_E}{\partial e_j}\right|_{e_j=e_j^*} \cdot de_j$$

Then reduce the exposure in Line j by de_j and increase the exposure in Line i by de_i without changing C_E. But according to Equation 7, doing so would increase P and lead to a contradiction.

The first proof provides the means, at least in principle, to explicitly solve for the optimal exposures, $\{e_i^*\}$. The second proof shows how to increase profitability when you are not at the optimal level of exposure. This is related to the following strategy which is a continuous analogue of Underwriting Strategy #1.

Underwriting Strategy #2

1. Observe the premium that the market allows for insurance policies in each line of insurance.
2. Calculate the insurer's profitability, P, as a function of its exposure, e_i, in each line of insurance i.
3. Calculate the insurer's needed capital, C_E, as a function of its exposure, e_i, in each line of insurance i.
4. If the expected return on marginal capital for line i (given by Equation 5), is greater than the insurer's current expected return on capital, increase the exposure in line i. Conversely, if the expected return on marginal capital for line i is less the insurer's current expected return on capital, decrease the exposure in line i. Adjust capital accordingly.

4. Allocating Capital

So far, we have not addressed the main topic of this paper – allocating capital. My reason for organizing the paper in this way was to emphasize the point that economically sound insurance decisions (defined in this paper as underwriting decisions that increase the insurer's expected return on capital) can be made without allocating capital.

Those who choose to allocate capital usually adopt an underwriting strategy similar to the following.

Underwriting Strategy #3

1. Establish a target rate of return for the insurance company.
2. Observe the premium that the market allows for a given insurance policy.
3. Calculate the expected profit that the insurer can obtain by writing this insurance policy.
4. Calculate the amount of capital that would be allocated to this insurance policy if it were written.
5. If the expected return on the allocated capital exceeds the insurer's target return on capital, the insurer should write the new insurance policy.

There are two differences between Underwriting Strategies #1 and #3. The first difference is the introduction of a "target" rate of return. According to Proposition 1, an insurer can increase its rate of return by adding policies where the market price allows an expected return on marginal capital greater than its current expected return on capital. As long as the insurer can raise the necessary capital, this is fine. But for a host of "practical" reasons there are limits on how much capital an insurer can raise. Thus, the insurer's board of directors will set the target rate of return based on what it feels is attainable with its scarce capital resources. Under these conditions, the insurer should be more selective and choose to underwrite the policies that yield the greatest expected return on marginal capital. Proposition 2 shows that when we can continuously adjust the exposure, this strategy of more selective underwriting leads to insurance policies each with an equally high expected return on marginal capital.

The second difference between Underwriting Strategy #1 and #3 is the substitution of the words "allocated capital" for the words "marginal capital." This is an important distinction because of the following proposition.

Proposition 3

The sum of the marginal capital for all exposures is less than or equal to the total capital.

Proof:

We prove the proposition when there are two distinct exposures. The general statement follows by induction.

The sum of the marginal capitals is equal to

$C(X+Y) - C(X) + C(X+Y) - C(Y) = 2 \cdot C(X+Y) - (C(X) + C(Y))$

$2 \cdot C(X+Y) - C(X+Y)$ (by the subadditivity axiom)

$= C(X+Y)$ which is the total capital.

Here is the general principle I use to allocate capital[3].

Back-Out Allocation Method

1. Establish an underwriting strategy in accordance with the economic principles described in Section 3 above.
2. Back out the method of allocating capital that is consistent with this strategy.

The only prior constraint on the method of allocating capital is that if the insurer properly executes Underwriting Strategy #3, it expects to achieve its target rate of return on its capital investment. That is to say, if A is the total allocated capital then:

$$P = r \cdot A. \tag{8}$$

I will first apply this general principle in the case where we can continuously adjust the exposure. The continuous analogue to Underwriting Strategy #3 is as follows.

[3] Gary Venter [2002] makes a similar tongue in cheek suggestion at the end of his article titled "Allocating Surplus – Not." But here, I am serious.

Underwriting Strategy #4

1. Establish a target rate of return for the insurance company.
2. Observe the premium that the market allows for insurance policies in each line of insurance.
3. Calculate the insurer's profitability, P, as a function of its exposure, e_i, in each line of insurance i.
4. Calculate the differential for the allocated capital, A_i, that accompanies a small change in exposure, e_i, in each line of insurance i.
5. If the expected return on allocated capital for line i is greater than the insurer's target rate of return on capital, increase the exposure in line i. Conversely, if the expected return on allocated capital for line i is less the insurer's target rate of return on capital, decrease the exposure in line i.

The question we now address is: What formula for allocating capital makes economic sense for this strategy? The answer is given by the following proposition.

Proposition 4

Assume the conditions of Proposition 2 hold and that the insurer has chosen exposure amounts, e_i^* for lines $i = 1, \ldots, n$, in such a way as to maximize its expected profit, P, subject to a limitation, I, on its capital investment. This strategy is equivalent to Underwriting Strategy #4 when allocating capital in proportion to the marginal capital.

Proof:

Using Equation 8 we have:

$$r \cdot \partial A\big|_{e_i=e_i^*} = \partial P\big|_{e_i=e_i^*} = \left.\frac{\partial P}{\partial C_E}\right|_{e_i=e_i^*} \cdot \partial C_E\big|_{e_i=e_i^*} = \left.\frac{\dfrac{\partial P}{\partial e_i}}{\dfrac{\partial C_E}{\partial e_i}}\right|_{e_i=e_i^*} \cdot \partial C_E\big|_{e_i=e_i^*}.$$

Since the insurer is maximizing its expected profit P, we can apply Equation 6 to the above with the result that:

$$r \cdot \partial A\big|_{e_i=e_i^*} = \lambda^* \cdot \partial C_E\big|_{e_i=e_i^*} \text{ or } \partial A\big|_{e_i=e_i^*} = \frac{\lambda^*}{r} \cdot \partial C_E\big|_{e_i=e_i^*}.$$

In words, this says that a small change in the exposure when $e_i = e_i^*$ causes the capital allocated to this change in exposure, $\partial A\big|_{e_i=e_i^*}$, to be equal to λ^*/r times the marginal capital, $\partial C_E\big|_{e_i=e_i^*}$. Furthermore, each small increment of exposure, $\partial e_i\big|_{e_i=e_i^*}$ adds $r \cdot \partial A\big|_{e_i=e_i^*}$ to the insurer's expected profit. Since the marginal expected profit is the same for all exposures in a given line of insurance the total expected profit, P, is equal to $r \cdot \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial A}{\partial e_i}\right|_{e_i=e_i^*} = r \cdot \frac{\lambda^*}{r} \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*}$, which is also equal to $r \cdot C_E$. It then follows that:

$$C_E = \frac{\lambda^*}{r} \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*}. \tag{9}$$

For reasons that will become clear in the next section, we call the ratio, λ^*/r, the ***heterogeneity multiplier***. So the allocated capital will be equal to marginal capital times the heterogeneity multiplier.

As a consequence of Proposition 3, the heterogeneity multiplier has a theoretical minimum of one.

I now illustrate the use of Proposition 4 by continuing Example 4. The results of applying the following equations are in Table 2 below. I think it will help your understanding of the results if you try to reproduce some of the numbers yourself.

Example 4 – Continued

- Using $e_i = v_i$ as our measure of exposure and Equation 1 we find that:

$$\partial C_{N}\big|_{v_i = v_i^*} = \frac{2 \cdot \left(1 + 2 \cdot v_i^* \cdot c_i\right)}{C_N} \cdot \partial v_i .$$

- The total marginal capital for line i is the sum over all the v_i's and is given by:

$$\partial C_{N}\big|_{v_i = v_i^*} \cdot v_i^* = \frac{2 \cdot \left(1 + 2 \cdot v_i^* \cdot c_i\right)}{C_N} \cdot v_i^* .$$

- In the various cases illustrated in Table 2, note that the total marginal capital over both lines is less than the total capital. This is predicted by Proposition 4.
- Set the target rate of return, r, equal to the maximum rate attainable subject to a given constraint on capital. This is equal to P/C in Table 1. Then use Equation 4 to calculate the Lagrange multiplier, λ^*. Finally, calculate the heterogeneity multiplier, λ^*/r.
- The total allocated capital for line i is the total marginal capital for line i times the heterogeneity multiplier. Note that the total allocated capital is equal to the original capital.

Table 2

C_E	ν_1^*	ν_2^*	Total Marginal Capital Line 1	Total Marginal Capital Line 2	Heterogeneity Multiplier	Total Allocated Capital Line 1	Total Allocated Capital Line 2
50.00	133.41	76.73	33.81	7.78	1.2021	40.65	9.35
60.00	163.44	100.76	41.07	10.13	1.1720	48.13	11.87
75.00	208.85	137.08	52.10	13.68	1.1403	59.40	15.60
100.00	285.03	198.02	70.69	19.65	1.1069	78.25	21.75
1,000.00	3,052.52	2,412.01	751.53	237.54	1.0110	759.84	240.16
10,000.00	30,747.90	24,568.32	7,569.61	2,419.32	1.0011	7,578.00	2,422.00
100,000.00	307,703.73	246,132.98	75,751.42	24,237.50	1.0001	75,759.81	24,240.19

Note that the heterogeneity multiplier approaches one as the capital constraint increases. Understanding the reason for this is not central to allocating capital, but I do regard it as an important curiosity. Let's look into this.

5. Allocating Capital with Homogeneous Loss Distributions

Suppose for a line of insurance i, the random losses, X_i, for the line are equal to a random number, U_i, times the exposure measure, e_i, for all possible values of e_i. Then, following Mildenhall [2002], the distribution of X_i is said to be ***homogeneous*** with respect to the exposure measure, e_i.

Lemma 1

Let the distribution of X_i be homogeneous with respect to the exposure measure, e_i. Then the sum of all marginal capital,

$$\sum_{i=1}^{n} e_i \cdot \frac{\partial C_E}{\partial e_i}$$

is equal to C_E.

Proof:

Since each X_i is homogeneous with respect to e_i we have:

$$C_E(X) = C_E\left(\sum_{i=1}^{n} X_i\right) = C_E\left(\sum_{i=1}^{n} e_i \cdot U_i\right).$$

Since the measure of capital, C_E, satisfies the positive homogeneity axiom, we can write:

$$C_E\left(\sum_{i=1}^{n} e_i \cdot U_i\right) = e_1 \cdot C_E\left(\sum_{i=1}^{n} \frac{e_i}{e_1} U_i\right) \equiv e_1 \cdot \tilde{C}_E\left(\frac{e_2}{e_1}, \mathrm{K}, \frac{e_n}{e_1}\right)$$

and similarly for $e_2, \ldots, e_n$.

The result follows from Lemma 2 in Mildenhall [2002].

Proposition 5

Assume the conditions of Proposition 2 hold and that the insurer has chosen exposure amounts, e_i^*, for lines $i = 1, \ldots, n$, in such a way as to maximize its expected profit, P, subject to a limitation, I, on its capital investment. Suppose further that the distribution of X_i is homogeneous with respect to the exposure measure, e_i. Then the heterogeneity multiplier is equal to one, $\partial A\big|_{e_i = e_i^*} = \partial C_E\big|_{e_i = e_i^*}$, and the total allocated capital is equal to the total marginal capital; i.e.,

$$C_E = \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i = e_i^*}.$$

Proof:

From Equation 9 we have that:

$$C_E = \frac{\lambda^*}{r} \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i = e_i^*}.$$

From Lemma 1 we have that:

$$C_E = \sum_{i=1}^{n} e_i^* \cdot \left.\frac{\partial C_E}{\partial e_i}\right|_{e_i = e_i^*}.$$

The conclusions follow.

Thus if the distribution of X_i is homogeneous with respect to e_i, the heterogeneity multiplier is equal to one and has no need to exist. But if the distribution of X_i is not homogeneous with respect to e_i, then we need the multiplier and hence the name "heterogeneity multiplier."

Example 4 – Continued

Recall in Example 4, the loss, N_i, in Line i has a mixed Poisson distribution. Let χ_i be a random variable with a mean = 1 and variance = $c_i > 0$. Given χ_i, the i^{th} distribution is a Poisson with mean $\chi_i \cdot \nu_i$. Now let's compare these distributions with distributions of the form $\chi_i \cdot \nu_i$, which are by definition homogeneous with respect to ν_i. As we compare the mixed and the homogeneous distributions for the same lines of insurance, we find that they closely approximate each other if ν_i is large. This explains why the heterogeneity multiplier is close to one for large values of ν_i.

Let's look at a pure homogeneous example.

Example 5

This example is the same as Example 4, with one key change. Instead of a loss of 1 per claim, the loss in line i is b_i per claim. For fixed ν_i's the b_i's are proportional to the expected loss and can serve as a measure of exposure. You can think of varying the b_i's by choosing a share of the loss whose claim sizes are much bigger than any b_i we may choose. This changes a number of the equations that describe Example 4. What follows are the equations that are analogous to those of Example 4.

- The capital is given by:

$$C_B = 2 \cdot \sqrt{b_1^2 \cdot \left(\nu_1 + c_1 \cdot \nu_1^2\right) + b_2^2 \cdot \left(\nu_2 + c_2 \cdot \nu_2^2\right)}. \qquad (1')$$

- The expected profit is given by:

$$P = r_1 \cdot b_1 \cdot \nu_1 + r_2 \cdot b_2 \cdot \nu_2. \qquad (2')$$

- Select the claim sizes, b_1 and b_2, so that the total profit, P, is maximized subject to a constraint on the capital, $C_B(X)$ I, using the method of Lagrange multipliers. The equations for the b_i's are given by:

$$b_i = \frac{r_i \cdot v_i \cdot I}{4 \cdot \lambda \cdot \left(v_i + c_i \cdot v_i^2\right)}, \text{ where } \lambda = \frac{1}{2}\sqrt{\frac{r_1^2 \cdot v_1^2}{v_1 + c_1 \cdot v_1^2} + \frac{r_2^2 \cdot v_2^2}{v_2 + c_2 \cdot v_2^2}}. \qquad (4')$$

Table 2′ gives sample calculations for various choices of v_1, v_2 and I that are comparable to those of Example 4.

Table 2′

C_B	v_1	v_2	λ^*	B_1	b_2	P	**Total Marginal Capital** **Line 1**	**Line 2**
100.00	250.00	250.00	0.1822	1.1436	0.7842	18.22	78.48	21.52
100.00	285.03	198.02	0.1823	1.0234	0.9203	18.23	80.00	20.00
1000.00	2500.00	2500.00	0.2006	1.2216	0.9585	200.63	761.12	238.88
1000.00	3052.52	2412.01	0.2009	1.0029	0.9909	200.87	762.02	237.98

Some observations:

- This table illustrates the results of Proposition 5. The heterogeneity multiplier λ^*/r (= $C_B \cdot \lambda^*/P$) is equal to one, and the total marginal capital is equal to C_B for all cases.
- The expected profit, P, varies with the choice of the v_i's, which remain fixed as you find the optimal b_i's.
- In two of the cases, I put in the same v_i's that maximized the expected profit in Example 4 (where the b_i's = 1). With those v_i's, the optimal b_i's did not equal one. This shows that the result of a capital allocation exercise depends upon the applicable exposure base.

Myers and Read [2001] prove a result that is similar to Proposition 5. Their use of the homogeneity assumption has generated some controversy. The justification for this assumption appears to follow from their statement: "The only requirement is frictionless financial markets and fixed state-contingent prices for all relevant outcomes."

Mildenhall [2002] illustrates that many commonly used actuarial loss models do not satisfy the homogeneity assumption. He further shows that the Myers/Read homogeneity assumption is both a necessary and sufficient condition to prove their analogue to Proposition 5.

Example 4 is one example where the homogeneity assumption is not met. I regard Proposition 4 as a generalization of Proposition 5 that applies when the homogeneity assumption is not met.

6. Allocating Capital with Discrete Exposure Changes

Strictly speaking, the continuity assumption underlying Propositions 4 and 5 is almost never met. For example, when an insurer increases its exposure in auto insurance, it typically writes an entire auto policy and increases its exposure by at least one car year. In cases like auto insurance, the discrete exposure environment is closely approximated by the continuous exposure environment, that is:

$$\left.\frac{\partial C_E}{\partial e_i}\right|_{e_i=e_i^*} \approx \frac{C_E\left(e_1^*,\mathrm{K}\,,e_i^*,\mathrm{K}\,,e_n^*\right)-C_E\left(e_1^*,\mathrm{K}\,,e_i^*-\Delta e_i,\mathrm{K}\,,e_n^*\right)}{\Delta e_i}. \tag{10}$$

If this approximation is good then you can estimate the heterogeneity multiplier and allocate capital as follows.

Gross-Up Allocation Method

1. Calculate the marginal capital required for each insurance policy in the current portfolio by calculating the capital needed when it is removed from the current portfolio and subtracting that from the current capital.
2. Calculate the heterogeneity multiplier by dividing the total required capital by the sum of the marginal capitals over all insurance policies.
3. The capital allocated to a given insurance policy is equal to its marginal capital times this heterogeneity multiplier.

To illustrate how well this can work, I calculated the heterogeneity multiplier by the gross-up method for the first line in Table 2 (Capital = 50) by dividing the total exposure by line into a varying number of insurance policies, with the following results.

Table 3

Number of Policies in Each Line	Gross-Up Heterogeneity Multiplier
(NA – Continuous)	1.2021
1000	1.2022
100	1.2034
10	1.2161
5	1.2320
1	1.5401

One can apply Underwriting Strategy #3 with the gross-up allocation method under any circumstance. Proposition 4 says that if you can continuously adjust the exposure, the strategy should lead to the optimal result. If Equation 10 provides a good approximation, the strategy should also get close to the optimal result with discrete exposures.

Quite often, insurers make bigger decisions such as adding or dropping entire lines of business. Consider the following example.

Example 6

- The insurer writes in two lines of insurance Line A and Line B. The insurer's only choices are to write all of Line A, all of Line B or both Lines A and B.
- There are no transaction costs and no interest is earned on invested assets.
- The "market" price provides an expected loss ratio of 60%. This means that the expected profit is equal to two thirds of the expected loss.
- The insurer must have capital equal to the maximum loss minus the expected loss.

- For losses payable in one year, there three possible scenarios.

Table 4

Scenario	Probability	Line A	Line B	Line A + Line B
1	2/39	60	135	195
2	7/39	150	45	195
3	30/39	0	0	0
Average Loss		30	15	45
Expected Profit		20	10	30
Required Capital		120	120	150
Marginal Capital		30	30	

- If the insurer writes Line A, it needs capital of 120 and has an expected profit of 20, which implies an expected return on its capital investment of 16.7%.

- If the insurer writes Line B, it needs capital of 120 and has an expected profit of 10, which implies an expected return on its capital investment of 8.3%.

- If the insurer writes both lines, it needs capital of 150 and has an expected profit of 30, which implies an expected return on its capital investment of 20%.

- Thus the best strategy is to write both lines since it yields the greatest return on capital.

- The return on marginal capital for Line A = 20/30 = 66.7%. The expected return on marginal capital for Line B is 10/30 = 33.3%. Both returns on marginal capital are higher than the 20% return on capital obtained by combining the two contracts.

Now let's apply Underwriting Strategy #3 using the gross-up method to allocate capital.

- The total capital is 150 and the sum of the marginal capitals is 60. Thus heterogeneity multiplier is equal to 2.5.

- The expected return on allocated capital for Line A is 20/(2.5·30) = 26.7%. The expected return on allocated capital for Line B is 10/(2.5·30) = 13.3%

- If the insurer followed Underwriting Strategy #3 it would write Line A since its expected return on allocated capital is higher than the 20% target. It would not write Line B since its expected return on allocated is below the 20% target. *This contradicts the fact that the insurer gets a higher expected ROE by writing both lines!*

This example gives a case where the gross-up capital allocation formula is not optimal. Note that if the insurer applies the back-out allocation method, it will allocate 100 to Line A, and 50 to Line B, yielding the target expected return on allocated capital of 20% for both lines.

7. Summary and Conclusions

Proposition 1 shows that if an insurer can obtain an expected rate of return on marginal capital on a given insurance policy that is greater than its current expected return on capital, then it can increase its rate of return by raising more capital and writing the policy.

If insurer capital is not a scarce resource, there is no need to allocate capital. But if there is a limit on the amount of capital that the insurer can raise, the insurer should be more selective in its underwriting and concentrate on the business that yields the greatest return on marginal capital.

If the insurer can make "small" adjustments in its exposure over time, Proposition 4 shows that the optimal result is obtained by:

1. Setting a high, but attainable target rate of return, r, on its capital.
2. Allocating capital in proportion to marginal capital using the gross-up allocation method.
3. Accepting only those policies for which the expected return on allocated capital is at least as high as r.

While I made use of Lagrange multipliers to illustrate this strategy, it is not necessary to resort to this mathematical technique. A simple trial and error analysis on the expected return on marginal capital for several lines of insurance should indicate what a realistic value of r should be. The proper execution of this strategy should incrementally move the insurer's expected return on capital toward the optimal result.

The strategy above should not be applied blindly for large scale underwriting decisions such as adding or dropping entire lines of insurance. Normally, the number of possible decisions on this scale is small, and so one can analyze each decision individually.

8. Additional Comments

For the interested reader, I would like to go a little beyond the scope of this paper with the following comments.

- The underwriting strategies discussed in this paper apply for all coherent measures of risk. I did not use the common coherent measures (such as the tail value-at-risk) in the examples because the ones I did use are more easily implemented on a spreadsheet. But for practical situations (especially if catastrophes are involved), I favor these other measures. See Meyers [2001] for an example where the choice of risk measure makes a noticeable difference in the ultimate conclusions.

- To keep the presentation as simple as possible, I ignored the time value of money. In practice, we should take it into account. Insurance policies covering natural disasters can be very risky; but once the policy has expired, the insurer can release some capital for other uses. For liability lines of insurance, the ultimate loss may not be known for some time. The insurer must hold capital until the loss is certain, and the cost of holding that capital must be considered in the underwriting strategy. Meyers [2001] discusses this also.

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A Note on the Myers and Read Capital Allocation Formula

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Abstract

The Myers and Read capital allocation formula is an important new actuarial result. In this paper, we give an overview of the Myers and Read result, explain its significance to actuaries, and provide a simple proof. Then we explain the assumption the allocation formula makes on the underlying families of loss distributions as expected losses by-line vary. We show that this assumption does not hold when insurers grow by writing more risks from a discrete group of insureds—as is typically the case.

Next, we discuss whether the inhomogeneity in a realistic portfolio of property casualty risks is material. We show how to decompose the relevant partial derivatives into homogeneous and inhomogeneous parts and examine the behaviour of each. We then apply the theory to some realistic examples. These clearly show that the lack of homogeneity is material. This failure will severely limit the practical application of the Myers and Read allocation formula.

1 Introduction

In an important paper for actuaries, Myers and Read (2001) showed how to allocate the expected policy holder deficit in a multi-line insurance company uniquely to each line. Their work can also be used to allocate surplus to each line. Previous work on the allocation problem, including Phillips et al. (1998) and Merton and Perold (2001), had concluded that such an allocation could be inappropriate and misleading. The Myers and Read result is, therefore, potentially a significant breakthrough, with obvious importance to actuaries.

Myers and Read repeatedly stress their result is independent of the distribution of losses by line and of any correlations between lines that may exist. They say their "proof requires no assumptions about the joint probability distributions of line-by-line losses and returns on the firm's portfolio of assets." However, while their result makes no assumptions about the *static* distribution of losses with fixed expected loss by line, their derivation does make an important assumption about how the *dynamic* distribution of losses changes shape with changing expected losses by line. This paper will explain the significance of the latter assumption. We will show it is a necessary and sufficient condition for the Myers and Read result to hold. Most importantly, we will show that the assumption does not hold when insurers grow through the assumption of risk from discrete insureds—as is typically the case.

For the convenience of readers not familiar with Myers and Read's work, we begin with an overview. Consider a simple insurance company which writes two lines of business. The losses from each line are represented by a random variables

X_1 and X_2, with means x_1 and x_2. Since the company can choose to write more or less of each line, we assume that the families $X_1(x_1)$ and $X_2(x_2)$, with varying means x_1 and x_2, are specified. For example, losses from line 1 may be normally distributed with mean x_1 and standard deviation 1000 and for line 2 be normally distributed with mean x_2 and coefficient of variation ν. Assume the company has capital k and total assets $x_1 + x_2 + k$. Also assume that interest rates are zero. (Myers and Read show how to convert from deterministic investment income to stochastic income. We focus on deterministic income and set it equal to zero for simplicity. Nothing of substance is lost in doing so.) Let

$$I(x_1, x_2, k) = \Pr(X_1 + X_2 > x_1 + x_2 + k)$$

be the probability of insolvency. Finally, assume that the company holds its probability of insolvency constant, by adjusting writings of each line and the amount of capital held. Let $K(x_1, x_2)$ satisfy

$$I(x_1, x_2, K(x_1, x_2)) = \text{constant}.$$

Then, under certain assumptions on the families $X_1(x_1)$ and $X_2(x_2)$ for varying x_1, x_2, but under *no assumptions on the distributions of losses given fixed* x_1 *and* x_2 we can prove

$$x_1 \frac{\partial K}{\partial x_1} + x_2 \frac{\partial K}{\partial x_2} = K. \tag{1}$$

This is obviously a very useful result: it tells the company that it should allocate capital at the rate $\partial K / \partial x_1$ to line 1 and $\partial K / \partial x_2$ to line 2, and that if it does so the total capital allocation will add up to actual capital! We prove Equation (1) in

Corollary 2, below. It is very similar to the actual Myers and Read result, which we prove in Corollary 1.

The main result of the paper, Proposition 1, states the assumptions on the families $X_i(x_i)$ required for Equation (1) to hold. We show that in most real-world situations these assumptions will, unfortunately, fail to hold. We also give a straight-forward proof of the Myers and Read "adds-up" result and we prove two related extensions. Finally we give several examples to illustrate the results.

The necessary distributional assumption highlights the difference between a continuous "representative insurer" approach, where each insurer assumes a share of a total market risk, and a discrete approach, where insurers assume risk from distinct and discrete individual insureds. The Myers and Read result requires a continuous view as we show in Proposition 1. Examples 4.4 and 4.5 show the result is not true in a discrete environment. Butsic (1999) used the representative insurer argument in his application of Myers and Read.

The rest of the paper is laid out as follows. In the next section we prove two technical lemmas. Section 3 states and proves the main Proposition. Section 4 gives several examples using the main result. Section 5 examines how the Myers and Read formula fails when losses are inhomogeneous and shows that in realistic examples the failure will be material.

2 Two Technical Lemmas

Lemma 1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function of n variables. Then*

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \cdots + x_n \frac{\partial f}{\partial x_n} = 0 \tag{2}$$

if and only if f is constant along rays from the origin.

Note: If f is constant on lines through the origin then f is called *homogeneous*. The lemma only requires f be constant along rays from the origin; along a line f can change as the line passes through the origin. The function $x \mapsto x/|x|$ is a good example of what can occur: it changes value from $+1$ to -1 at zero. If f is constant along rays from the origin, then in half spaces through the origin f can be expressed as a function of x_i/x_j, $i = 1, \dots, n$ when $x_j \neq 0$, for each j. In our applications of this lemma, the domain of f will be the positive quadrant, so there is no difference between lines through the origin and rays from the origin in the domain. I would like to thank Christopher Monsour for pointing this out.

Proof Sufficiency: if f is constant along rays through the origin, then by the note we can assume locally that $f(x_1, \dots, x_n) = \tilde{f}(x_1/x_n, \dots, x_{n-1}/x_n)$ for some function $\tilde{f}$ of $n-1$ variables. An easy calculation shows

$$\begin{aligned} x_1\frac{\partial f}{\partial x_1} + \cdots + x_n\frac{\partial f}{\partial x_n} &= \frac{x_1}{x_n}\tilde{f}_1 + \cdots + \frac{x_{n-1}}{x_n}\tilde{f}_{n-1} - x_n(\frac{x_1}{x_n^2}\tilde{f}_1 + \cdots + \frac{x_{n-1}}{x_n^2}\tilde{f}_{n-1}) \\ &= 0, \end{aligned}$$

where $\tilde{f}_i = \partial\tilde{f}(x_1, \dots, x_{n-1})/\partial x_i$.

Necessity: Let $\mathbf{v} = (x_1, \dots, x_n)$ be a differentiable curve, so $\mathbf{v} = \mathbf{v}(t) : \mathbb{R} \to \mathbb{R}^n$, with $d\mathbf{v}/dt = \mathbf{v}$. This means $\mathbf{v}$ is equal to its own tangent vector for each t. By separating variables it is easy to see that $\mathbf{v}$ is a line through the origin. (It has the form $e^t(k_1, \dots, k_n)$ for constants of integration k_i.) Then, by the chain-rule

$$\begin{aligned} \frac{d}{dt}f(\mathbf{v}(t)) &= x_1\frac{\partial f}{\partial x_1} + \cdots + x_n\frac{\partial f}{\partial x_n} \\ &= 0, \end{aligned}$$

by assumption, so the directional derivative of f along each half of any such line v is constant, i.e. f is constant along rays from the origin, as required. Since v never reaches the origin, we cannot assert that f is constant along lines through the origin. □

Lemma 2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function of n variables. Then,*

$$x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} = f \tag{3}$$

on a half-space where $x_1 > 0$ (resp. $x_1 < 0$) if and only if there exists a differentiable function $\tilde{f}$ so that $f(x_1, \ldots, x_n) = x_1 \tilde{f}(x_2/x_1, \ldots, x_n/x_1)$ on that half space, and similarly for $x_2, \ldots, x_n$.

Proof If $f(x_1, \ldots, x_n) = x_1 \tilde{f}(x_2/x_1, \ldots, x_n/x_1)$ then, using subscripts on $\tilde{f}$ to denote partial derivatives,

$$\begin{aligned} x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} &= \left(x_1 \tilde{f} - \sum_{j=2}^{n} x_j \tilde{f}_{j-1} \right) + \sum_{j=2}^{n} x_j \tilde{f}_{j-1} \\ &= f. \end{aligned}$$

The first sum comes from the partial derivative with respect to x_1 and the second sum comes from all the remaining partials.

On the other hand, suppose f satisfies Equation (3) and let $\tilde{f}(t, s_2, \ldots, s_n) = f(t, s_2 t, \ldots, s_n t)/t$ where $t > 0$ (resp. $t < 0$). We must show $\tilde{f}$ is independent of t. Differentiating

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{f(t, s_2 t, \ldots, s_n t)}{t} \right) &= -\frac{1}{t^2} f + \frac{1}{t} \left(\frac{\partial f}{\partial x_1} + \sum_{j=2}^{n} s_j \frac{\partial f}{\partial x_j} \right) \\ &= 0 \end{aligned}$$

and the result follows. □

3 Statement and Proof of Main Result

Before stating the proposition we need to define some more notation. We are modeling a multi-line insurance company. Losses from each line are modeled by random variables X_i, $i = 1, \ldots, n$, where X_i has mean x_i and distribution function F_i. We often regard x_i as a variable (but not a random variable), so each X_i is really a family of distributions indexed by x_i. Where necessary we emphasize this by writing $X_i(x_i)$. Changes in x_i correspond to increasing or decreasing volume in line i, since x_i is the *a priori* expected loss.

Assume that the company holds total assets equal to $x_1 + \cdots + x_n + k$, so in a very simplistic sense, k is the capital or surplus of the company.

Next, define the probability of insolvency function and the expected policyholder deficit function for a single line i as

$$I_i(x_i, k) = \Pr(X_i > x_i + k) = 1 - F_i(x_i + k) \tag{4}$$

and

$$D_i(x_i, k) = \int_{x_i+k}^{\infty} t - (x_i + k)\, dF_i(t). \tag{5}$$

In both of these equations x_i is performing double duty: it is the mean of X_i and in $x_i + k$ it determines where F_i is evaluated. To emphasize this we could write

$$I_i(x_i, k) = 1 - F_i(x_i + k; x_i). \tag{6}$$

Finally, let $X = X_1 + \cdots + X_n$ be the total losses with distribution function F. Define insolvency and deficit functions for the whole company as

$$I(x_1, \ldots, x_n, k) = \Pr(\sum X_i > \sum x_i + k) = 1 - F(x_1 + \cdots + x_n + k) \tag{7}$$

and

$$D(x_1,\ldots,x_n,k) = \int_{\sum t_i > \sum x_i + k} \cdots \int t_1 + \cdots + t_n - (x_1 + \cdots + x_n + k)\, dF(t_1,\ldots,t_n). \tag{8}$$

The following definition is key:

Definition 1 *A family of random variables $X(x)$ with $E(X(x)) \propto x$ is called* **homogeneous** *if there exists a single random variable U so that $X(x)/x$ has the same distribution as U for all x.*

Homogeneity is Myers and Read's only distributional assumption, and it means that losses come from a representative insurer. The requirement that U is independent of x is important—after all, any random variable can be written as $X = E(X)(X/E(X))$! An exponential variable X with mean x is a homogeneous family, since $X = xU$ where U has an exponential distribution with mean 1. However, a normal variable with mean x and standard deviation 1 is not homogeneous.

In order to compute expressions like $\partial I/\partial x$ we need to know how the family $X(x)$ changes shape with changes in x. We need to work with $X(x+\epsilon)$ as well as $X(x)$ because

$$\begin{aligned}
\frac{\partial I}{\partial x} &= -\frac{d}{dx}F(x+k;x) \\
&= -\lim_{\epsilon\to 0}\frac{F(x+k+\epsilon;x) - F(x+k;x)}{\epsilon} \\
&\quad -\lim_{\epsilon\to 0}\frac{F(x+k;x+\epsilon) - F(x+k;x)}{\epsilon}.
\end{aligned}$$

The partial derivative has a *static* part, where the mean of the underlying variable does not change, and a *dynamic* part, where the point of evaluation is fixed but the mean changes. This shows computing partial derivatives such as $\partial I/\partial x$ is inextricably linked to families of random variables.

With this notation we can now state our main result.

Proposition 1 *The following are equivalent.*

1. *For each* $i = 1, \ldots, n$, $X_i(x_i)$ *is a homogeneous family of random variables.*

2. *For each* $i = 1, \ldots, n$

$$x_i \frac{\partial I_i}{\partial x_i} + k \frac{\partial I_i}{\partial k} = 0. \tag{9}$$

3. *For each* $i = 1, \ldots, n$

$$x_i \frac{\partial D_i}{\partial x_i} + k \frac{\partial D_i}{\partial k} = D. \tag{10}$$

4. *We have equality*

$$x_1 \frac{\partial I}{\partial x_1} + \cdots + x_n \frac{\partial I}{\partial x_n} + k \frac{\partial I}{\partial k} = 0. \tag{11}$$

5. *We have equality*

$$x_1 \frac{\partial D}{\partial x_1} + \cdots + x_n \frac{\partial D}{\partial x_n} + k \frac{\partial D}{\partial k} = D. \tag{12}$$

The proposition says that each of the five statements holds if and only if all the other four hold. Put another way, if one of the five fails to hold then the other four will also fail. This means that we can construct simple one line examples and can use items 2 and 3 generalize to the multi-line case. This simplifies the mathematics of the examples.

Proof We shall prove (4) implies (2) implies (1) implies (4), and then (5) implies (3) implies (1) implies (5), which is enough to show all the statements are equivalent.

(4) implies (2): Set $x_j = 0$ for $j \neq i$ in Equation (11) to get Equation (9). This can also be seen geometrically using Lemma 1 which says I is constant along rays from the origin. Therefore I_i, which is a restriction of I, is also constant along such rays.

(2) implies (1): Lemma 1 applied to I_i shows there exists a function $\tilde{I}_i$ so that

$$I_i(x_i, k) = \tilde{I}_i(k/x_i).$$

Let $U_i = X_i/x_i$, then $\Pr(U_i > u) = \tilde{I}_i(u-1)$ is independent of x_i as required.

(1) implies (4): Assumption (1) implies that I is constant along rays from the origin, so the result follows from Lemma 1.

(5) implies (3): Set $x_j = 0$ for $j \neq i$ in Equation (12) to get Equation (10).

(3) implies (1): Let $U_i = X_i/x_i$. We have to show $\Pr(U_i > u)$ is independent of x_i. Let $x^+ = \max(x, 0)$. Then, notice that

$$\begin{aligned}
\frac{\partial D}{\partial k} &= \frac{\partial}{\partial k} E[(\sum x_i U_i - (\sum x_i + k))^+] && (13)\\
&= E[\frac{\partial}{\partial k}(\sum x_i U_i - (\sum x_i + k))^+] && (14)\\
&= E[-\mathbf{1}_{\{\sum x_i U_i > \sum x_i + k\}}] && (15)\\
&= -\Pr(\sum x_i U_i > \sum x_i + k) && (16)
\end{aligned}$$

is minus the probability of default. Next, use Lemma 2 to define $\tilde{D}_i$ so that $D_i(x_i, k) = x_i \tilde{D}_i(k/x_i)$. Therefore

$$\frac{\partial D_i}{\partial k} = \tilde{D}_i'(k/x_i)$$

and so

$$\Pr(U_i > u) = -\tilde{D}_i'(u - 1)$$

is independent of x_i as required.

(1) implies (5): Assumption (1) shows we can write D as

$$D(x_1, \ldots, x_n, k) = k\tilde{D}(x_1/k, \ldots, x_n/k)$$

so the result follows from Lemma 2. □

The results in Proposition 1 are clearly similar to Myers and Read's results but they are not exactly the same. We shall now explain how to derive their exact result and prove some other similar results. For simplicity we shall assume $n = 2$ and work with just x_1 and x_2 in the rest of the paper.

Myers and Read's "adds-up" result (their Equation A1-3) involves computing the marginal increase in surplus required to hold the default value constant, given a marginal increase in a particular line. We have been taking a slightly different approach: if we hold the surplus and default value constant, what decrease is needed in line 2 to offset an increase in line 1? However, it is easy to reconcile the two approaches. To do this, let κ_1 and κ_2 be the marginal surplus requirements for each line. Note that κ_1 and κ_2 are ratios whereas k is a dollar amount. Myers and

Read then use a capital amount $k = \kappa_1 x + \kappa_2 x_2$ and define the default value D_M (to distinguish from our D) as

$$D_M(x_1, x_2) := D(x_1, x_2, \kappa_1 x_1 + \kappa_2 x_2). \tag{17}$$

Myers and Read use the following notation in their Appendix 1. They write $\tilde{L}_a = L_a \tilde{R}_a$, where L_a corresponds to our x_1, $\tilde{R}_a$ to U_1 and $\tilde{L}_a$ to X_1. Thus $\tilde{L}_a = L_a \tilde{R}_a$ translates into our $X_1 = x_1 U_1$, i.e. the homogeneity assumption. The value L_a is the expected value of $\tilde{L}_a$ at time 0. We are ignoring the time value of money here by assuming an interest rate of zero. Myers and Read also work with a fixed interest rate and then integrate over all possible rates—an extra level of sophistication that need not concern us.

We can now prove their result.

Corollary 1 *(Myers and Read) Assume losses X_i form a homogeneous family for each i. Then default values "add-up" in that*

$$x_1 \frac{\partial D_M}{\partial x_1} + x_2 \frac{\partial D_M}{\partial x_2} = D_M. \tag{18}$$

Proof Computing using the chain-rule and then applying Proposition 1 item 5 in Equation (21) gives:

$$\begin{aligned} x_1 \frac{\partial D_M}{\partial x_1} + x_2 \frac{\partial D_M}{\partial x_2} &= x_1 (\frac{\partial D}{\partial x_1} + \kappa_1 \frac{\partial D}{\partial k}) + x_2 (\frac{\partial D}{\partial x_2} + \kappa_2 \frac{\partial D}{\partial k}) & (19) \\ &= x_1 \frac{\partial D}{\partial x_1} + x_2 \frac{\partial D}{\partial x_2} + (\kappa_1 x_1 + \kappa_2 x_2) \frac{\partial D}{\partial k} & (20) \\ &= D(x_1, x_2, \kappa_1 x + \kappa_2 x_2) & (21) \\ &= D_M(x_1, x_2) & (22) \end{aligned}$$

as required. □

Simple Proof Here is the simple, self-contained proof we promised in the introduction. Dividing through by x_1 in the definition of D, Equation (8), it is clear that $D_M(x_1, x_2) = x_1 \tilde{D}_M(x_2/x_1)$ for some function $\tilde{D}_M$. Thus

$$\begin{aligned} x_1 \frac{\partial D_M}{\partial x_1} + x_2 \frac{\partial D_M}{\partial x_2} &= x_1 \left(\tilde{D}_M - \frac{x_2}{x_1} \frac{\partial D_M}{\partial x_1} \right) + x_2 \frac{\partial D_M}{\partial x_2} \\ &= D_M \end{aligned}$$

which completes the proof. □

We now prove two more Myers and Read-like results which follow easily from Proposition 1. Using the implicit function theorem, Burkill and Burkill (1980), there is a function $K(x_1, x_2)$ so that $I(x_1, x_2, K(x_1, x_2)) = c$ is a constant.

Corollary 2 *Assume losses X_i form a homogeneous family for each i. Then surplus values defined by constant probability of default "add-up" in that*

$$x_1 \frac{\partial K}{\partial x_1} + x_2 \frac{\partial K}{\partial x_2} = K. \tag{23}$$

Proof Proposition 1 implies

$$x_1 \frac{\partial I}{\partial x_1} + x_2 \frac{\partial I}{\partial x_2} + k \frac{\partial I}{\partial k} = 0. \tag{24}$$

By the implicit function theorem

$$\frac{\partial K}{\partial x_1} = -\frac{\partial I}{\partial x_1} / \frac{\partial I}{\partial k} \tag{25}$$

and similarly for x_2. Rearranging Equation (24) and substituting Equation (25) gives

$$x_1 \frac{\partial K}{\partial x_1} + x_2 \frac{\partial K}{\partial x_2} = K, \tag{26}$$

so surplus values "add-up" just as Myers and Read's default values add-up. □

Next, use the implicit function theorem to define a function $L(x_1, x_2)$ so that $D(x_1, x_2, L(x_1, x_2)) = c$.

Corollary 3 *Assume losses X_i form a homogeneous family for each i. Then surplus values defined by constant expected policy holder deficit satisfy*

$$x_1 \frac{\partial L}{\partial x_1} + x_2 \frac{\partial L}{\partial x_2} = L + T \tag{27}$$

where $T = TVaR(x_1 + x_2 + L(x_1 + x_2))$ is the tail-value at risk beyond $x_1 + x_2 + L(x_1 + x_2)$.

Proof Using the implicit function theorem again, and dividing Proposition 1 item 5 by $-\partial D/\partial k$, we get

$$x_1 \frac{\partial L}{\partial x_1} + x_2 \frac{\partial L}{\partial x_2} = L - D/\frac{\partial D}{\partial k}. \tag{28}$$

Thus, by Equation (16)

$$x_1 \frac{\partial L}{\partial x_1} + x_2 \frac{\partial L}{\partial x_2} = L + T$$

where T is the tail-value at risk. □

4 Examples

By Proposition 1, we can give one-dimensional examples and know they will extend to the multivariate situation as expected. We make use of this simplification in several of the examples below.

4.1 Examples of Homogeneity

Homogeneous families can be made from a wide variety of continuous distributions. For example, varying the scale parameter θ and holding all other parameters constant for any of the distributions listed in Appendix A of Klugman, Panjer and Willmot (1998) which have a scale parameter θ, will produce a homogeneous family. This includes suitable parameterizations of the transformed beta, Burr, generalized Pareto, Pareto, transformed gamma, gamma, Weibull, exponential, and inverse Gaussian. By Proposition 1, sums of selected from such families will also be homogeneous. Also, trivially, if X is any distribution with mean 1 then xX is a homogeneous family as x varies.

For example if X has an exponential distribution with mean x, so $\Pr(X > t) = \exp(-t/x)$, then $X = xU$ where U has an exponential distribution with mean 1. This follows since

$$\Pr(X > t) = \exp(-t/x) = \Pr(U > t/x).$$

Here

$$I(x, k) = \Pr(X > x + k) = \exp(-k/x)/e$$

which clearly satisfies item 2 of Proposition 1.

4.2 Simple Example where Homogeneity Fails

It is easy to construct examples where the homogeneity assumption fails. All members of a homogeneous family have the same coefficient of variation, therefore a family with a non-constant coefficient of variation will not be homogeneous. For example, let X be normally distributed with mean x and constant standard deviation 1. Then X is not homogeneous. By definition $I(x,k) = 1 - \Phi(k)$ so

$$x\frac{\partial I}{\partial x} + k\frac{\partial I}{\partial k} = -k\phi(k) \neq 0,$$

where Φ and ϕ are the distribution and density for the standard normal.

If the reader is skeptical about using only one variable, he or she will find it easy to construct multivariate distribution examples using normal variables. For example, consider what Corollary 2 says when X_1 is distributed $N(x_1, 1)$ and X_2 is distributed $N(x_2, 1)$. $X_1 + X_2$ is distributed $N(x_1 + x_2, \sqrt{2})$, so

$$I(x_1, x_2, k) = 1 - \Phi(k/\sqrt{2}). \tag{29}$$

Thus $\partial K/\partial x_i = (\partial I/\partial x_i)/(\partial I/\partial k) = 0$ for $i = 1, 2$. Corollary 2 then reads $K = 0$, which is absurd! This shows the importance of the homogeneity assumption for the results derived from Proposition 1, including Myers and Read's allocation formula. This example can also be generalized to the case where X_1 and X_2 are correlated.

4.3 Homogeneity Fails with Constant Coefficient of Variation

It is less simple, but still possible, to construct examples where the coefficient of variation is a constant function of the mean, but which nevertheless fail to satisfy the homogeneity assumption.

For example let $X(x)$ be distributed as a gamma random variable with parameters $\alpha = 4x^2$, $\theta = 1/2$ shifted by $x(1-2x)$. Here we are using the Klugman, Panjer, Willmot parameterization so $f(t;\alpha,\theta) = (t/\theta)^\alpha e^{-t/\theta}/t\Gamma(\alpha)$. It is easy to check $X(x)$ has mean x, constant coefficient of variation 1 and skewness $1/x$, since the skewness of a gamma α, θ is $2/\sqrt{\alpha}$. I is given by the incomplete gamma function, $I(x,k) = \Gamma(4x^2, 4x^2 + 2k)$, which does not satisfy the assumptions of Lemma 1, so $X(x)$ is not homogeneous. The reason is clear: the family $X(x)$ changes shape with x and so cannot be homogeneous.

Taking this a step further, it is possible to construct a family all of whose higher cumulants (coefficient of variation, skewness, kurtosis, etc.) are independent of the mean, just as they would be for a homogeneous family, but which nevertheless fails to be homogeneous. To do this, let U be a lognormal random variable with $\ln(U)$ distributed as a standard normal. Let V be a random variable density function $f_V(x) = f_U(x)(1 + \sin(2\pi\log(x)))$, where f_U is the density of U. Then U and V have the same moments—see Feller (1971), Chapter VII.3. This type of trick is possible because the moments of a lognormal grow too quickly to ensure it is determined by its moments—see also Billingsley (1986) Section 30. Let $X(x)$ be a mixture of xU and xV with weights $p(x) = x/(x+1)$ and $1 - p(x)$. Then

$$I(x,k) := p(x)\Pr(U > 1 + k/x) + (1 - p(x))\Pr(V > 1 + k/x)$$

is not a function of k/x so the result follows from Lemma 1 and Proposition 1. Alternatively, writing $I_U(x,k) = \Pr(xU > x+k)$ and similarly for V one can compute directly

$$x\frac{\partial I}{\partial x} + k\frac{\partial I}{\partial k} = xp'(x)(I_U(x,k) - I_V(x,k)) \neq 0$$

since xU and xV are homogeneous, $xp'(x) > 0$ by construction, and $I_U - I_V \neq 0$. Thus $X(x)$ is not a homogeneous family.

4.4 Aggregate Distributions are Not Homogeneous

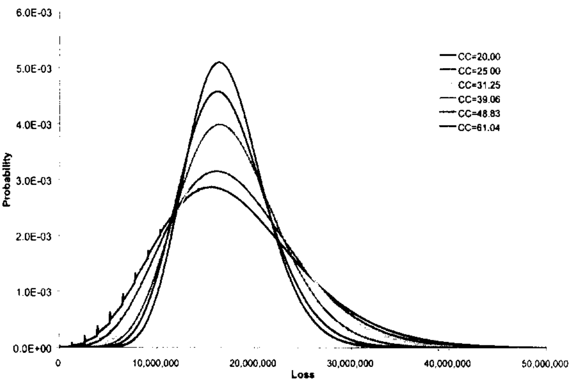
Example 4.1 shows a large number of continuous variables satisfy the homogeneity assumption. For our purposes, however, there is a very important class which does not: aggregate loss distributions.

Let $A = X_1 + \cdots X_N$ where the X_i are independent, identically distributed severities and N is a frequency distribution with mean n. Increasing expected losses in this model involves increasing n. Suppose N has contagion c, so, as suggested by Heckman and Meyers (1983), $\mathrm{Var}(N) = n(1+cn)$. Then

$$\mathrm{CV}(A)^2 = \frac{\mathrm{CV}(X)^2}{n} + \frac{1}{n} + c$$

is clearly not independent of n. Thus A does not satisfy the homogeneity assumption. Just as for Example 4.3, the aggregate loss distribution changes shape as n increases. This is illustrated in the figure below, which shows six aggregate loss distributions with the same severity distribution but different claim counts, indicated by "CC=20" for $n = 20$, and so forth. The individual densities have been scaled so that if the family were homogeneous then all the densities would be identical and only one line would appear in the plot.

If aggregate distributions can be approximated by various of the parametric distributions of Example 4.1, and if those distributions are homogeneous, does the result of this Example really matter? The answer is emphatically "yes". This example shows that in the real world, where insurers grow by adding discrete insureds, the "adds-up" results do not hold. The way the aggregate distribution changes shape forces parameters other than the scale parameter to change as the mean increases, and thus homogeneity is lost.



4.5 Compound Poisson Distributions are not Homogeneous

This example proves that various aggregate distributions can never be homogeneous families.

Proposition 2 *Let A be a compound Poisson aggregate distribution*

$$A = X_1 + \cdots + X_N \tag{30}$$

where N has a Poisson distribution with mean n and the X_i are independent and identically distributed. Then $A(n)$ is not a homogeneous family.

Proof The moment generating function of N is $M_N(t) = \exp(n(e^t - 1))$. The moment generating function of A is therefore $M_A(t) = \exp(n(M_X(t) - 1))$ where M_X is the moment generating function for severity X. If A is homogeneous with A distributed as nU for some fixed U, then $M_A(t) = M_U(nt)$. Thus

$$n(M_X(t) - 1) = \log(M_U(nt)).$$

Differentiating with respect to n shows

$$(M_X(t) - 1) = tM_U'(nt))/M_U(nt).$$

Therefore $M_U'(t)/M_U(t)$ must be a constant, since the left hand side is independent of n. Hence $M_U(t) = \exp(ct)$ for a constant c, and so $U = c$ is a degenerate distribution. But this is impossible unless N is constant or $X_i \equiv 0$. □

Corollary 4 *An aggregate distribution with frequency component N which is a mixture of Poisson distributions cannot be a homogeneous family.*

Proof Condition on the mixing parameter and apply Proposition 2. □

For example, the Corollary applies to negative binomial and Poisson-inverse Gaussian frequency distributions.

5 Is Inhomogeneity Material?

In this section we will show that the inhomogeneity inherent in a typical portfolio of property casualty risks is sufficiently large to invalidate the Myers and Read allocation formula. By Proposition 1, we can discuss inhomogeneity in the context of one random variable, rather than two or more, which simplifies the mathematics.

Let $X(x)$ be a smooth family of random variables with $\mathrm{E}(X(x)) = x$. Let $F(t,x) = \Pr(X(x) < t)$ be the distribution function of $X(x)$ and $f(t,x) = \partial F/\partial t$ be its density.

Recall that $X(x)$ is homogeneous (with respect to the mean) if there exists a random variable U so that

$$X(x) = xU \tag{31}$$

for all x. In this case, let F_U and f_U be the distribution and density functions of U.

Recall also that the expected default, with capital ratio κ, is defined as

$$D(x) = \int_{x(1+\kappa)}^{\infty} (t - x(1+\kappa)) f(t,x)dt. \tag{32}$$

Note that $x(1+\kappa)$ represents total assets: x from the loss and $x\kappa$ from allocated capital. In a more sophisticated model we could consider profit in the premium; here we simply assume this is subsumed into the constant κ.

By Proposition 1 we know

$$x\frac{\partial D}{\partial x} = D \tag{33}$$

if and only if $X(x)$ is a homogeneous family, which is then equivalent to the Myers-Read adds-up result.

5.1 Heuristics

A homogeneous family offers no diversification benefit as the mean increases. Property casualty insurance is based on diversification, and the inhomogeneity inherent in a portfolio of insurance risks means that the relative riskiness of the portfolio decreases as expected losses increase. Since a lower risk portfolio has a lower expected default, one would expect that

$$x\frac{\partial D}{\partial x} < D \tag{34}$$

for an inhomogeneous insurance portfolio.

Meyers (2003) introduces the heterogeneity multiplier, which is a constant λ defined so that

$$\lambda x\frac{\partial D}{\partial x} = D. \tag{35}$$

He shows that λ is typically greater than 1 (as expected). In further unpublished work, Meyers uses empirical data to estimate that λ is in the range 1.5 to 2.5, depending on the size of the company. This suggests that inhomogeneity is material.

If X is homogeneous then, for all $\kappa > 0$,

$$x\frac{\partial D}{\partial x} = D \geq 0. \tag{36}$$

However, intuitively, one would expect that for a large enough capital ratio κ it should be possible for the extra capital associated with writing more business to more than offset the extra risk. This would imply that

$$x\frac{\partial D}{\partial x} < 0 \tag{37}$$

should be possible for sufficiently large κ. This is another difference between homogeneous and inhomogeneous families.

5.2 Theory

In order to assess the impact of inhomogeneity, we will break the derivative $\partial D/\partial x$ into two pieces using a homogeneous approximation to the family $X(x)$. For a fixed x, define a new homogeneous family $Y(y)$ by

$$Y(y) = \frac{y}{x}X(x). \tag{38}$$

Let $G(t, y)$ and $g(t, y)$ be the distribution and density functions of Y. Note that $Y(x) = X(x)$ and that

$$g(t, y) = f(tx/y, x)x/y. \tag{39}$$

If X is already homogeneous, then clearly $Y(y) = X(y)$ for all y. Finally define $E_x(y)$ to be the expected default value of Y,

$$E_x(y) = \int_{y(1+\kappa)}^{\infty} (t - y(1+\kappa))g(t, y)dt. \tag{40}$$

The subscript x on E highlights the point x at which we have chosen to "homogenize" X. By definition $E_x(x) = D(x)$.

We can now compute

$$\begin{aligned}
\frac{\partial D}{\partial x} &= \lim_{\epsilon\to 0}\frac{D(x+\epsilon) - D(x)}{\epsilon} && (41)\\
&= \lim_{\epsilon\to 0}\frac{D(x+\epsilon) - E_x(x+\epsilon) + E_x(x+\epsilon) - D(x)}{\epsilon} && (42)\\
&= \lim_{\epsilon\to 0}\frac{D(x+\epsilon) - E_x(x+\epsilon)}{\epsilon} + \lim_{\epsilon\to 0}\frac{E_x(x+\epsilon) - D(x)}{\epsilon} && (43)\\
&= I(x) + \lim_{\epsilon\to 0}\frac{E_x(x+\epsilon) - E_x(x)}{\epsilon} && (44)\\
&= I(x) + \frac{D(x)}{x} && (45)
\end{aligned}$$

where $I(x)$ is defined by the first limit, and we have used the fact that Y is homogeneous to replace $\partial E_x/\partial x$ with $E_x(x)/x = D(x)/x$.

We can now prove the main lemma of this section.

Lemma 3 *With the above notation*

$$I(x) = \int_{x(1+\kappa)}^{\infty} (s - x(1+\kappa)) \left(\frac{\partial f}{\partial x} + \frac{s}{x}\frac{\partial f}{\partial t} + \frac{f(s,x)}{x} \right) ds. \tag{46}$$

Proof Substitute $s = tx/(x+\epsilon)$ in the limit defining I, swap the limit and integral (Lebesgue's dominated convergence theorem), and use the fact that the limit of a product (quotient) is the product (quotient) of the limits to get

$$I(x) = \int_{x(1+\kappa)}^{\infty} (s - x(1+\kappa)) \lim_{\epsilon \to 0} \left(\frac{(x+\epsilon) f(s + s\epsilon/x, x+\epsilon) - x f(s,x)}{(x+\epsilon)\epsilon} \right) ds. \tag{47}$$

Now add and subtract a term $(x+\epsilon) f(s + s\epsilon/x, x)$ in the limit, re-arrange and cancel. The result follows. □

We will call $I(x)$ the inhomogeneous derivative of D with respect to x. We will use the standard notation $f_1 = \partial f/\partial t$ and $f_2 = \partial f/\partial x$.

If X is homogeneous then

$$f(t,x) = f_U(t/x)/x \tag{48}$$

$$f_1(t,x) = f_U'(t/x)/x^2 \tag{49}$$

$$f_2(t,x) = -t f_U'/x^2 - f_U/x^2 \tag{50}$$

and so

$$f_2 + \frac{s}{x} f_1 + \frac{1}{x} f = 0. \tag{51}$$

Thus if X is homogeneous $I(x) = 0$ as expected.

The Lemma shows that

$$xI(x) = \int_{x(1+\kappa)}^{\infty} (s - x(1+\kappa))(xf_2 + sf_1)ds + D(x), \tag{52}$$

and so

$$x\frac{\partial D}{\partial x} = \int_{x(1+\kappa)}^{\infty} (s - x(1+\kappa))(xf_2 + sf_1)ds + 2D(x). \tag{53}$$

When X is homogeneous, the integral in I exactly cancels out the extra D term. In the tail of the distribution, we expect $f_1 < 0$, because the density will eventually be decreasing with t, and $f_2 > 0$ because for a given t the density $f(t, x)$ will increase as the mean x increases. The exact balance of these two terms depends on the degree of inhomogeneity.

5.3 Examples of Inhomogeneity

At this point we have developed enough general theory. For a realistic insurance portfolio we expect $x\partial D/\partial x < D$, and possibly that $x\partial D/\partial x < 0$. In order to test the magnitude of these effects we will use the following model.

Let $A(x)$ be an aggregate loss distribution with expected losses x, severity component $S(l)$ and frequency component N, so

$$A(x) = S(l)_1 + \cdots + S(l)_N. \tag{54}$$

We assume that $S(l) = \min(S, l)$ results from applying a limit l to a fixed unlimited severity S. In the example S is chosen to be reasonably close to ISO's Premises and Operations B curve. The frequency distribution N is negative binomial with claim count $n = x/\mathrm{E}(S(l))$ and contagion c, so $\mathrm{Var}(N) = n(1+cn)$. In

Table 1: Total Derivative $\partial D/\partial x$

$x \mid l$	1,000,000	5,000,000	10,000,000	100,000,000
10,000,000	0.28%	-0.42%	-0.37%	3.74%
50,000,000	0.39%	0.30%	0.13%	-0.40%
100,000,000	0.39%	0.37%	0.31%	-0.02%
1,000,000,000	0.39%	0.39%	0.39%	0.39%

the tables below $c = 0.15$, corresponding to an asymptotic coefficient of variation of $A(x)$ of 38.7%. The capital ratio $\kappa = 1$, so the expected loss to surplus ratio is 1 to 1.

In order to compute the necessary derivatives, we will approximate $A(x)$ with a shifted lognormal distribution, using the method of moments to match the mean, variance, and skewness. For large portfolios, the shifted lognormal is a very good
5 approximation to the true aggregate distribution. This can be seen by comparing the result of using FFTs to compute the true aggregate with the shifted lognormal approximation. Figure 1 shows that the approximation is quite spectacularly good, particularly in the relevant range beyond $2x$. Regardless of whether you believe this is a good approximation or not, the approximation has qualitatively the correct shape and behaviour as x changes.

Let $X(x)$ be the shifted lognormal approximation to $A(x)$. If $X(x)$ has parameters τ, μ, and σ, so $\ln(Y-\tau)$ is distributed $N(\mu, \sigma)$, then the homogeneous approximation $Y(y)$ to $X(x)$ has parameters $y/x\tau$, $\ln(y/x)+\mu$, and σ. Therefore we can compute I and D explicitly.

In each table, expected loss amounts x are shown vertically and different limits l are shown across the columns. Patterns in Table 1 are hard to see directly, and are

Table 2: Homogeneous Derivative = Expected Default Ratio = D/x

$x \| l$	1,000,000	5,000,000	10,000,000	100,000,000
10,000,000	0.97%	4.21%	8.12%	19.27%
50,000,000	0.49%	0.92%	1.49%	3.99%
100,000,000	0.44%	0.63%	0.87%	1.90%
1,000,000,000	0.40%	0.41%	0.43%	0.49%

Table 3: Inhomogeneous Derivative $I(x) = \partial D/\partial x - D/x$

$x \| l$	1,000,000	5,000,000	10,000,000	100,000,000
10,000,000	-0.69%	-4.64%	-8.49%	-15.53%
50,000,000	-0.10%	-0.62%	-1.36%	-4.39%
100,000,000	-0.05%	-0.26%	-0.56%	-1.92%
1,000,000,000	0.00%	-0.02%	-0.04%	-0.11%

Table 4: Heterogeneity Multiplier

$x \| l$	1,000,000	5,000,000	10,000,000	100,000,000
10,000,000	3.45	-9.96	-21.66	5.15
50,000,000	1.26	3.10	11.72	-9.97
100,000,000	1.12	1.72	2.80	-91.90
1,000,000,000	1.01	1.05	1.10	1.28

best understood as the sum of the homogeneous and inhomogeneous derivatives in Tables 2 and 3. In Table 2 we see that the homogeneous derivative (derivative of the homogeneous approximation to X) increases with the limit l and decreases with expected losses x. This makes sense: increasing the limit increases the riskiness of the portfolio and hence D. Increasing expected losses yields a diversification benefit and decreases D.

Table 3 shows that the inhomogeneous derivative increases with x, eventually tending to zero. This reflects the fact that the aggregate becomes very nearly homogeneous for large x, $x \gg l$. As l increases I decreases, reflecting the fact that the underlying distribution A is becoming more and more inhomogeneous.

Table 4 shows the heterogeneity multiplier, or ratio of the homogeneous derivative to the total derivative. In a reasonable range of x between 10M and 100M and smaller limits, this is of the same order of magnitude as in Meyers' study.

In practical applications, where the adds-up formula would be used in the context of allocating surplus between business units or lines of business, expected losses would be in the 10M to 100M range with limits of 1M to 10M. The Tables show that in such a range the lack of homogeneity in an insurance portfolio is material, and would mean the adds-up result would fail to hold by a substantial amount.

6 Conclusions

In this paper we have explained the importance of the homogeneity assumption in the derivation of Myers and Read's "adds-up" result. Proposition 1 shows the assumption is necessary as well as sufficient. We have used Proposition 1 to prove two other results in a similar vein, including one involving tail value at risk. Importantly, for practical applications, we have shown that most common families of aggregate distributions will never satisfy the homogeneity assumption. We have given several realistic examples to support the general theory. We conclude that, in a real-world situation, where insurers grow by adding individual risks from discrete insureds, the "adds-up" result will not hold.

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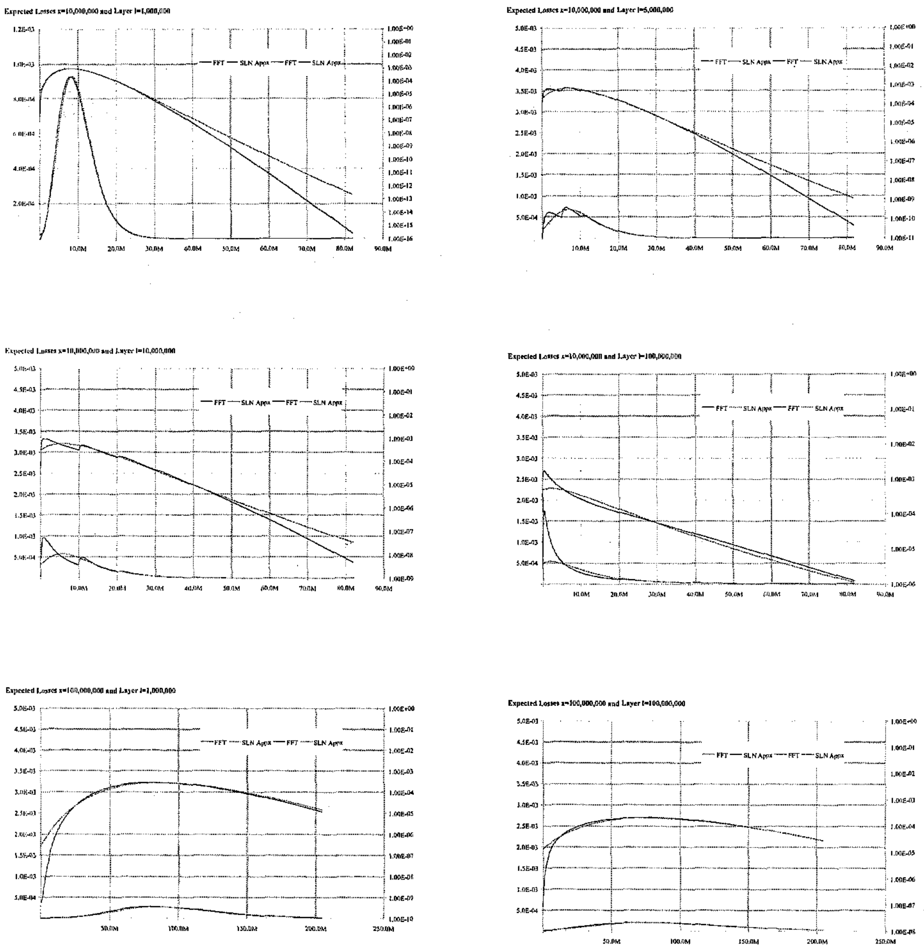
Figure 1
Comparison of Shifted Lognormal Approximation to Aggregate Distribution for Variety of Means x and Limits l



A Method of Implementing Myers-Read Capital Allocation in Simulation

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A Method of Implementing Myers-Read Capital Allocation in Simulation

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Abstract
In this paper, we show an especially simple way to produce Myers-Read capital allocations in simulations, by using the Ruhm-Mango-Kreps (RMK) conditional risk algorithm. The algorithm uses only weighted averages. In particular, it does not require any calculus, even though the Myers-Read formula is a differential equation. This is possible because the Myers-Read method is additive, and the Ruhm-Mango theorem guarantees that any additive allocation method can be reproduced by RMK. While Myers-Read capital allocation is based on probability of ruin, the RMK algorithm can easily be adapted to alternative risk measures of the user's choice.

Introduction
Myers and Read (2001) describe a method for allocating an insurer's total capital to individual lines of business, according to the equation:

$$S_k = L_k \text{ x } (dS / dL_k),$$

where S = total capital, S_k = capital for line k, and L_k = expected losses for line k. The formula is based on the assumption that total capital is determined by a fixed probability-of-ruin constraint, so that the expression (dS / dL_k) represents how much capital would have to increase, in response to an increase in expected loss volume[1]. An appealing feature of the theory is that the capital allocations so derived will sum to the total capital:

$$\Sigma S_k = S$$

Although it is probably most natural to think of surplus allocation in terms of lines of business, this concept could be extended. For instance, one could use the same formula to allocate capital to loss layers within a line. The "sum-of-the-parts-equals-the-whole" additive property remains intact[2], allowing a consistent allocation of capital down to almost any level.

Ruhm and Mango (2003) describe a method of calculating risk charges based on conditional probabilities and total portfolio risk[3]. Total portfolio risk charge is calculated from any risk measure that the user specifies, and is then allocated to all components of the portfolio based on conditional probabilities. Like Myers-Read allocations, conditional risk charges are also additive: the individual risk charges sum to the portfolio risk charge. As an added benefit, the Ruhm-Mango-Kreps ("RMK") algorithm is extremely simple to implement in a simulation, once the risk measure has been selected by the user.

[1] This assumes that the shape of the loss distribution remains unchanged – see Mildenhall (2002).
[2] Again, subject to the conditions described in Mildenhall (2002).
[3] An alternative derivation of the same algorithm was discovered independently by Kreps (2003).

Myers-Read and RMK both allocate portfolio risk to components, just expressed in different forms (capital vs. risk charge). Since both are also additive, it is natural to wonder if there might be a connection. One difference between the methods is that the Myers-Read method specifies probability-of-ruin as the standard for setting capital, while RMK allows any risk measure to be used.

As it turns out, any additive allocation method can be reproduced by conditional risk charges. The only differences among additive methods are choice of risk measure and scale[4]. This means that Myers-Read capital allocation can be implemented using the RMK algorithm, by choosing probability of ruin as the risk measure. As will be shown below, capital allocations produced by RMK in simulation equal those derived by applying the Myers-Read differential equation.

An Example of the RMK algorithm applied to Myers-Read

A simple example will be used to demonstrate how to implement the method. The simplified conditions in the example are only present to make the example as transparent as possible; the method will work just as well for any number of risks and any dependence structure.

Consider a portfolio of 2 independent risks, both distributed normally and parameterized as follows:

$$R_1 \sim N(100, 900)$$
$$R_2 \sim N(200, 1600)$$

The total portfolio is thus also normal:

$$R_1 + R_2 \sim N(300, 2500)$$

The standard deviation of the total portfolio is 50. The capital requirement is set at 100. This is two standard deviations, which produces 97.725% confidence level (probability of ruin = 2.275%). Total required funding is the sum of expected loss and required capital:

$$\text{Total Funding} = \text{Total Expected Loss} + \text{Capital} = 400$$

The problem is to allocate the 100 of capital to the individual risks. First, we will apply the Myers-Read formula to obtain the result. (A reader who is less interested in the details of this calculation can skip to the next page, where the result is found.) From above,

$$S_k = L_k \text{ x } (dS / dL_k)\,,$$

where $L_k = \mu_k = E[R_k]$. To calculate dS / dL_k, we first express S, the capital requirement, as a function of the components. This was set to be two times the standard deviation of

[4] See Ruhm-Mango (2003). This statement (the Ruhm-Mango Theorem), its proof and the general risk pricing formulas are the main theoretical results of the paper. Technically, the theorem applies to any additive method that unambiguously prices all derivatives ("additive-complete" methods).

the total portfolio. The total portfolio's variance is the sum of the individual variances, so we have:

$$S = 2(\sigma_1{}^2 + \sigma_2{}^2)^{1/2}$$

Differentiating with respect to L_k:

$$dS / dL_k = (dS / d\sigma_k)\ (d\sigma_k / dL_k)$$
$$dS / dL_k = (2)\ (1/2)\ (\sigma_1{}^2 + \sigma_2{}^2)^{-1/2}\ (2\sigma_k)\ (d\sigma_k / dL_k)$$
$$dS / dL_k = (2)\ (S^{-1})\ (2\sigma_k)\ (d\sigma_k / dL_k)$$
$$dS / dL_k = (2)\ (1/100)\ (2\sigma_k)\ (d\sigma_k / dL_k)$$

To finish this, we have to find ($d\sigma_k / dL_k$), which is not difficult. Following Mildenhall (2002), we must assume that the individual lines increase or decrease by a scale factor, which means the coefficient of variation is constant. For instance, if line 1 (R_1) doubles in volume of expected loss (L_k), its standard deviation (σ_k) also doubles. From above:

$$L_1 = 100,\ \sigma_1 = 30$$
$$L_2 = 200,\ \sigma_2 = 40$$

Then:

$$\sigma_1 = 0.30\ L_1,\ \text{so}\ d\sigma_1 / dL_1 = 0.30$$
$$\sigma_2 = 0.20\ L_2,\ \text{so}\ d\sigma_2 / dL_2 = 0.20$$

Substituting these coefficients and standard deviations into the formula above yields:

$$dS / dL_1 = (2)\ (1/100)\ (60)\ (.30) = 0.36$$
$$dS / dL_1 = (2)\ (1/100)\ (80)\ (.20) = 0.32$$

Finally,

$$S_1 = L_1 \times (dS / dL_1) = (100)(0.36) = 36$$
$$S_2 = L_2 \times (dS / dL_2) = (200)(0.32) = 64$$

These sum to the total capital of 100, as expected.

Next, the example is simulated and the RMK algorithm is applied. For this case, the two normals were each simulated using one hundred points, the center-points of all the unit percentiles. These were cross-combined, to produce 10,000 sample points for the sum. Each sample point is equally likely in this case, however the procedure is just as easily applied to simulations for which probabilities differ by iteration.

Exhibit 1 shows the results of the simulation. For clarity, the iterations have been sorted by total loss, and only the most significant rows are shown. The first five columns show the iteration number, the simulated losses for the two risks, the total losses and the percentile for the total. The rightmost column, "Risk Discount Function," contains the risk measure (probability of ruin for Myers-Read) in the form of a discount function. This column is the center of the algorithm.

As discussed above, the ruin point should fall at about the 97.725% level, between iterations 9772 and 9773[5]. We want to take a small sample of the distribution around this point – for this example, we'll use a 0.50% interval on either side. This sample, consisting of iterations 9723 through 9822, is shown in bold in the exhibit. The probability-of-ruin discount function is very simple: it is one for points in the sample (i.e., at or near the ruin point), and zero everywhere else.

The capital allocation can now be calculated by taking weighted averages, using the risk discount function as the weights. This produces funding amounts, which equal expected losses plus capital allocations. For example, the expected value of the "Risk 1" column is 100 (the expected loss for Risk 1), while the weighted expected value is 135.64, for a capital requirement of 135.64 – 100.00 = 35.64. The results, as shown in the exhibit, approximate those produced by formula (with minor differences due to simulation).

As this example shows, implementing the algorithm is fairly easy. One simply has to add an additional column for the risk discount function, chooses a small sample space around the ruin point, and calculate weighted averages.

Other Applications and Risk Measures

The RMK algorithm allows one to specify a risk measure and allocate total capital or total risk charge (more generally, total risk) accordingly. Moreover, the allocation does not have to be limited to lines of business – the method can also be applied to sources of risk in general, such as investment-related risks in dynamic financial analysis (DFA) applications and business risks in the enterprise risk management (ERM) contexts. In short, it is a general method for decomposing overall risk into its components, by source of risk.

The risk discount function for the probability-of-ruin measure, given in the example above, explicitly shows the measure's abruptness: it values entirely on the point-of-ruin, and discounts all other points with a factor of zero. For instance, if the outer tail severity were to change, the indicated capital, and corresponding risk charge, would not change.

As an alternative, one could weight the entire tail with ones, reflecting all points of undesirable outcomes. If a dollar-value cutoff point is used, rather than a percentile, then frequency can be reflected. As a further refinement, smaller weights can be used for desirable "upside" outcomes, with larger, surcharging weights used for more severe, less desirable outcomes. Such a risk function would allow all parts of the distribution to be reflected. At the extreme, a continuous function could be used: both Black-Scholes option prices and CAPM market prices can be modeled in this way[6].

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[5] Due to simulation approximation error, the ruin point actually falls at iteration 9780.

[6] See Ruhm (2003) and Ruhm-Mango (2003), respectively, for these and their risk discount functions.

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Exhibit 1: Simulation Results with Myers-Read Capital Allocation

Iteration	Risk 1	Risk 2	Total	% ile	Risk Discount Function
1	22.72	96.97	119.69	0.01%	0.0
2	34.90	96.97	131.86	0.02%	0.0
9722	122.66	272.48	395.14	97.22%	0.0
9723	**137.61**	**257.58**	**395.19**	**97.23%**	**1.0**
9724	**165.10**	**230.22**	**395.32**	**97.24%**	**1.0**
9821	**165.10**	**238.96**	**404.07**	**98.21%**	**1.0**
9822	**101.13**	**303.03**	**404.16**	**98.22%**	**1.0**
9823	125.79	278.40	404.19	98.23%	0.0
10000	177.28	303.03	480.31	100.00%	0.0
Expected	100.00	200.00	300.00		100.0
Funds = Wtd Exp'd	135.64	263.78	399.42		
Capital	35.64	63.78	99.42		
Exact Values, formula	36.00	64.00	100.00		

Discussion of "Capital Allocation for Insurance Companies" by Stewart C. Myers and James R. Read Jr.

Gary G. Venter, FCAS, MAAA

DISCUSSION OF

"CAPITAL ALLOCATION FOR INSURANCE COMPANIES"

Gary G Venter, Guy Carpenter Instrat

"Capital Allocation for Insurance Companies[1]" is a useful and insightful paper for casualty actuaries. However it does not provide the denominator for a return-on-capital ranking of business units that many actuaries have sought. It does provide the basis for an alternative framework for evaluating business unit profitability.

[1] Myers, Stewart C and Read, James A. 2001, "Capital Allocation for Insurance Companies," *Journal of Risk and Insurance*, 68:4, 545-580.

Discussion of "Capital Allocation for Insurance Companies"

My first introduction to the Myers-Read method was at a CAS session where Richard Derrig of the Massachusetts auto insurance bureau proclaimed "The capital allocation problem has finally been solved." Naturally I was glad to hear that, but as the session continued I began to suspect that he was talking about a different capital allocation problem than many actuaries had been addressing.

In the Massachusetts ratemaking scheme, insurers are permitted to incorporate into their rates a charge for the frictional costs of carrying capital. Since capital supports all lines of business, it is problematic how much of this cost can be attributed to any particular contract or even line of business. The Myers-Read approach does appear to provide an excellent methodology for this issue.

What a number of other actuaries have been seeking is a capital allocation to business unit in order to calculate the return on capital for each unit. This in turn would govern decisions as to which units to grow, shrink, revamp, drop, reward managers of, etc. I'll call this the problem of ranking profitability.

It now seems that the ranking problem and the allocation of frictional costs are distinct problems probably with different solutions. For instance, even in Massachusetts other elements of profitability are allowed into the ratemaking formula than just recovery of the frictional costs of carrying capital: carrying risk is rewarded within a CAPM framework over and above the frictional costs. This might be regarded by many actuaries as not much reward, but it illustrates that profit for bearing risk is not treated the same as recompense for frictional capital costs. In fact, the return for bearing risk is not even proportional to the allocated capital, indicating that the allocation is not intended to be the basis of a return calculation.

Nonetheless, I will argue later on that the risk pricing framework that Myers-Read presents does give a useful direction for solving the problem of ranking business units by profitability.

The remainder of this discussion has three main sections: the capital allocation problem, the Myers-Read solution, and an evaluation of applications and limitations.

THE CAPITAL ALLOCATION PROBLEM

Initial actuarial approaches to capital allocation tend to allocate using some risk measure. The chosen risk measure is used to quantify the risk of the overall firm and each business unit. Then these risk measurements are combined in an allocation method to spread the capital to business unit. A simple example would be allocating capital in proportion to the variance of each unit's operating results.

There are numerous risk measures and allocation methods that can be used in this schemata. For examples, see the papers presented at the CAS

DFA seminar of June 2001. Many of the allocation methods look at marginal impact – i.e., the increase in the risk measure of the overall firm due to a given business unit, either in total or from its last small increment of exposure. The idea is to charge each unit only for the increase in capital it generates for the firm as a whole. It usually turns out that the sum of these marginal capital contributions is less than the entire capital, so the rest has to be allocated somehow.

Often the solution presented is to allocate the remaining capital in proportion to the marginal capital. But this could lead to inappropriate conclusions about business unit profitability. This is analogous to the problem of fixed and marginal production costs for a manufacturer, as illustrated in the following example.

Suppose a company has invested a lot of money in making an assembly line to produce hand phones. This line can produce phones for $2 each, but to recoup the investment costs the company wants to charge wholesalers $8 each. But suppose that after a while there is an oversupply in the market, and it can only charge $5 for each phone. Since each one costs only $2 to make, it decides to keep using the assembly line and keep selling phones. But if it required the fixed costs to also be covered, it would shut down, giving up the $3 per phone profit.

A similar situation can arise in insurance. If a line is generating enough profit to cover its marginal costs, including the marginal costs of capital, but not enough to cover some allocated fixed charges, allocating fixed capital in proportion to marginal could shut it down when in fact it is contributing to the overall profitability of the firm. Of course if every line is in this situation, the firm is going to have to find some strategy to cover its fixed costs, such as growing like mad, or merging, etc. This is a different problem that should not be buried in the by-line profitability analysis.

It should also be noted that there are other additive approaches to allocation of capital that do not use marginal methods. A general class of such methods is outlined in Rodney Kreps' widely circulated working paper, *An Allocatable Generic Risk Load Formulation*, which shows how to create co-measures, analogous to covariance for the variance measure, that are totally additive across any partitions of an insurer's portfolio. A related procedure has been introduced by D. Tasche, *Risk contributions and performance measurement*, Zentrum Mathematik (SCA), U München, Feb. 2000, www-m4.mathematik.tu-muenchen.de/m4/pers/tasche/riskcon.pdf .

Co-measures can be defined for any risk measures that can be expressed as a conditional expectation, which most of them can be. Suppose a risk measure for risk X with mean m can be defined as:

$$R(X) = E[(X - am)g(x) \mid \text{condition}] \quad \text{for some value a and function g.}$$

Suppose that X is the sum of n portfolios X_i each with mean m_i. Then the co-measure for X_i is:

$$CoR(X_i) = E[(X_i - am_i)g(x) \mid \text{condition}]$$

Since expectations are additive, the sum of the CoR's of the n X_i's is R(X).

For example, define the measure excess tail value at risk by:

$XTVaR_q = E[X - m \mid X>x_q]$ where $F(x_q) = q$. Then

$Co\text{-} XTVaR_q = E[X_i - m_i \mid X>x_q]$

If capital is set by XTVaR, it would provide enough to cover losses above mean losses for the average of the years where losses exceeded the qth quantile. The capital allocated by Co-XTVaR to a line would be the line's average losses above its mean losses in those same adverse years. A constant loss would get no allocated capital in this procedure, for instance.

One issue this highlights is the arbitrary choice of risk measure. Does the company really know how much capital it needs, and how each business unit affects that? With arbitrary choices of risk measure and allocation method, unit managers are going to push for those that make them look better, and there will be no solid foundation to settle the matter.

This argues for some other approach to the ranking problem than allocating by risk measure. Some alternatives will be discussed in the evaluation section. But first the Myers-Read solution is addressed. It is at least able to avoid the problem of allocating fixed capital in proportion to marginal, simply because their marginal capital adds up to the total!

THE MYERS-READ SOLUTION

Robert Butsic in *Capital Allocation for Property-Liability Insurers: A Catastrophe Reinsurance Application*, www.casact.org/pubs/forum/99spforum/99spf001.pdf, provides an extensive discussion and application of Myers-Read (MR). Butsic provides a slightly different derivation of the allocation formula than do Myers and Read themselves. You can get the same result from slightly different sets of assumptions, so this is not one of those situations where if you accept the assumptions you must accept the result. The results and assumptions can be evaluated from various viewpoints, and so the question is, does the whole approach work well?

The method seeks to allocate the frictional costs of holding capital. What does that mean? Essentially frictional costs accrue just by a company holding capital, even if it doesn't put the capital at risk. The return for bearing risk is not a frictional cost, but a separate input into insurance pricing. Examples of frictional costs include taxation, agency costs, liquidity costs, and reduced investment opportunities, as detailed below.

In some countries, insurer investment income is subject to taxation, so tax is a frictional cost in those jurisdictions. But even on small islands where insurer investment income is not taxed, there are frictional costs of holding capital. Unless the insurer has really vast amounts of money, it often has to invest more conservatively than the capital owners themselves would want to, due to the interests of policyholders, regulators, and rating agencies. Thus the reduced investment income due to an insurer's reduced scope of investment alternatives is a frictional cost. There is also a liquidity penalty from insurers holding of capital, in that investors do not have direct access to the assets purchased. Further, there are agency costs associated with holding large pools of capital, i.e., an additional cost corresponding to the reluctance of investors to let someone else control their funds, especially if that agent can pay itself from the fund. All of these costs accrue to the insurer whether or not it bears any risk.

MR uses capital allocation to allocate the frictional costs to policyholders. Every policyholder gets charged the same percentage of its allocated capital for frictional costs. Thus it is really the frictional costs that are being allocated, and capital allocation is a way to represent that cost allocation.

A key element of the MR development is the value of the default put option. Assuming it is an entity with limited liability, an insurer does not pay losses once its capital is exhausted. So it can be said that the insurer holds an option to put the default costs to the policyholders. MR assumes a lognormal or normal distribution for the insurer's entire loss portfolio, so can use the Black-Scholes options pricing formula to compute D, the value of this put option. The distributional assumptions will be discussed further in the evaluation section.

Adding a little bit of exposure to any policy or business unit has the potential to slightly increase the value D of the default option for the firm as a whole. But adding a little more capital can bring D back to its original value, when expressed as a percentage of expected losses. The MR method essentially allocates this additional bit of capital to the additional exposure that generated it.

In other words, the default option value, as a percentage of expected losses, i.e., D/L, for the entire firm is held as a fixed target, and the last dollar of each policy is charged with the amount of extra capital needed to maintain that overall target option value. But any dollar could be considered the last, so the whole policy is charged at the per dollar cost of the last dollar of expected loss. The beauty of the method is that those marginal capital allocations add up to the entire capital of the firm.

In the MR development, the total capital requirement of the firm is never really specified, but it could be taken to be the amount of capital needed to get D/L to some target value. In practice, whatever D/L ratio the firm has can be taken to be the target. The allocation method then is based on the incremental marginal effect – the incremental dollar expected loss for a policy is charged with the amount of capital needed to keep the overall D/L ratio at its target. The typical problem of capital allocation by marginal methods – that fixed costs are allocated in proportion to marginal costs – is avoided because, unlike most marginal allocation approaches, the marginal capital amounts add up to the total capital of the firm with no proportional adjustment. This appears to be due to the additive nature of option prices.

The total capital is the sum of the individual capital charges, i.e., $\sum c_i L_i = cL$, where $c_i L_i$ is the capital for the ith policy with expected losses L_i, and cL is total capital. Thus each policy's (or business unit's) capital is proportional to its expected losses, and the capital allocation question becomes how to determine the proportionality factors c_i.

Formally, MR requires that the derivative of D with respect to L_i be equal to the target ratio D/L for every policy. Butsic shows that this condition follows from some standard capital market pricing assumptions. This requirement means that the change in the firm's overall default cost due to a small change in any policy's expected losses is D/L. Thus D/L does not change with an incremental change in the expected losses of any policy. How is this possible? Because increasing L_i by one unit increases capital by c_i units, and the c_i is found that will keep D/L constant. Thus the formal requirement that $\partial D/\partial L_i = D/L$ means that c_i is determined so that the change in c_iL_i, the policy's capital, due to a small change in L_i has to be the amount that keeps D/L constant.

The question then is, can allocation factors c_i be found to satisfy both conditions $\sum c_iL_i = cL$ and $\partial D/\partial L_i = D/L$? That is, can by-policy capital-to-expected-loss ratios be found so that any marginal increase in any policy's expected losses keeps D/L constant, while the marginal capital charges sum to the overall capital? The MR derivation says yes. Without going into the details of their derivation, the following reasoning shows why it is feasible.

In the MR setup, after expenses and frictional costs, assets are just expected losses plus capital, and so the Black-Scholes formula gives:

$$D = L[N(y+v) - (1+c)N(y)]$$

where v is the volatility of the company results, $y = -\ln(1+c)/v - v/2$ and $N(y)$ denotes the cumulative standard normal probability distribution.

Using this formula to expand the condition that $\partial D/\partial L_i = D/L$ requires the calculation of the partial derivative of D, and thus eventually c, w.r.t. L_i. Plugging in $\sum c_i L_i = cL$, the c derivative turns out to be $(c_i - c)/L$. This leads to an equation for each c_i in terms of c. Thus the two conditions required combine to give equations for c and all the c_i's. The derivation then consists of finding a convenient solution.

To show the resulting allocation formula, denote the coefficient of variation (CV) of total losses as k_L and the CV of losses for the ith policy or business unit by k_i. Also define the policy beta as $\beta_i = \rho_{iL} k_i/k_L$, where ρ_{iL} is the correlation coefficient between policy i and total losses. Myers-Read also considers correlation of assets and losses, but Butsic gives the following simplified version of the capital allocation formula, assuming that the loss-asset correlation is zero:

$$c_i = c + (\beta_i - 1)Z, \text{ where } Z = (1+c)n(y)/[v(1+k_L^{-2})N(y)]$$

Note that Z does not depend on i, so c_i is a linear function of β_i. Butsic provides a simple example of this formula. A company with three lines is assumed, with expect losses, CV's, and correlations as shown below. The total capital and its volatility are also givens. The rest of the table is calculated from those assumptions.

	line 1	line 2	line 3	total	volatilities
EL	500	400	100	1000	
CV	0.2	0.3	0.5	0.2119	0.2096
corr 1	1	0.75	0		
corr 2	0.75	1	0		
corr 3	0	0	1		
variance	10,000	14,400	2,500	44,900	
beta	0.8463	1.3029	0.5568		
capital	197.872	282.20	19.93	500	0.2209
assets				1500	0.0699
c:	0.3957	0.7055	0.1993	0.5	
- y:	1.9457807	**y+v:**	-1.7249		
N(y):	0.0258405	**N(y+v):**	0.042277		
n(y):	0.0600865	**1/n(y):**	16.64267		
Z:	0.6784		**D/L:**	0.0035159	

Changing the by-line expected losses in this example allows you to verify that if you add a dollar of expected losses to any of the lines, the overall D/L ratio is maintained by adding an amount to capital equal to the c_i ratio for that line.

Some aspects of the approach can be illuminated by varying some of the input assumptions.. The examples that follow keep the volatility of assets constant, even though assets vary, which seems reasonable.

First, consider what happens if the CV for line 3 is set to zero. In this case, the line becomes a supplier of capital, not a user, in that it cannot collect more than it's mean, but it can get less, in the event of default. Then the capital charge c_i for this line becomes –17%, and the negative sign appears appropriate, given that the only risk is on the downside. The size of the coefficient seems surprising, however, in that its default cost is only 0.3% (which is the same for the other lines as well), but it gets a 17% credit. Part of what is happening is that adding independent exposures to a company will increase the default cost, but will decrease the D/L ratio, as the company becomes more stable. Thus in this case, increasing line 3's expected losses by a dollar decreases the capital needed to maintain the company's overall D/L ratio by 17 cents. This is the incremental marginal impact. However if line 3 decides to go net entirely, leaving only lines 1 and 2, the company will actually need $19.50 in additional capital to keep the same default loss ratio. This is the entire marginal impact of the line, which will vary from the incremental marginal.

Another illustrative case is setting line 3's CV to 0.335. In this case, its needed capital is zero. Adding a dollar more of expected loss maintains the overall D/L ratio with no additional capital. The additional stability from its independent exposures exactly offsets its variability. Again the marginal impact is less than the overall: eliminating the line in this case would require $10.60 in additional capital for the other lines.

Evaluation: Limitations and Applications

The cost of the default option per dollar of expected loss seems to be a reasonable quantity to keep constant. If a policyholder increases this ratio by a change in exposure, that would reduce the value of the other policies, and so would be unfair to the other policyholders. Also the allocation principle that each dollar of expected loss be charged the frictional costs of the capital needed to maintain the target ratio also appears reasonable. And the fact that the marginal capital allocations add up to the total eliminates the problem of some other allocation methods that fixed costs are allocated using marginal costs. Thus all in all MR seems to be a good method of capital allocation. However, there are several issues that need to be addressed.

Lognormal Assumption

First of all, aggregate losses are assumed to be lognormally distributed. This is required only for the total company, not for individual lines or policies. This may or may not be reasonable depending on the company being analyzed. It is an assumption many actuaries are comfortable making, but should be evaluated for specific applications. It would be possible to extend the MR derivation to other distributions, but that would require an analogue of the Black-Scholes formula. That in turn would need a probability transform to a risk-neutral measure. That is not necessarily difficult to achieve. In many cases the problem is not of finding a transform, but of choosing among a number of possible candidates. Pricing papers for such situations often pick a transform with little justification for the choice. It would be interesting to see how the experts would handle this problem in the insurance pricing case.

Return on Allocated Capital

Second, it is clear from the Massachusetts auto context that the MR allocation was not intended to be the basis of a return-on-capital calculation, since other profit elements are added that are not proportional to the allocated capital. But would it be wrong to use the allocation for this? MR appears to be as good as any of the risk-measure allocations for coming up with a value for the capital required to support a line of business. But there is no theory to suggest that equalizing the return on this capital – or that from any other risk-measure's allocation – would produce appropriate by-line pricing. Butsic tested this for MR with a risk loading method, but didn't like the results. This could be a problem with the entire enterprise of allocating capital by a logical but arbitrary measure then pricing to equalize return on that capital.

Using Pricing Measures for Ranking by-Line Profit

MR is aimed at capital allocation for pricing. The pricing that results, including the costs of risk-bearing as well as the frictional costs, can be used for ranking by comparing it to the actual profitability realized. This could be put into a return-on-allocated-capital mode by reallocating capital by the combined risk-friction profit load in the model pricing. Shaun Wang suggested using pricing methods like this in *A Universal Framework For Pricing Financial And Insurance Risks*, ASTIN Bulletin, 2002, Volume 32, No. 2.

Carrying this out in practice would require a good theory of insurance pricing. Many actuaries are skeptical of CAPM because it does not take into account all sources of risk. However further financial research is refining the original CAPM assumptions and developing broader-based pricing formulas. For instance, company-specific risk needs to be added to CAPM pricing, as shown in Froot, Kenneth A. and Stein, Jeremy C., *A New Approach to Capital Budgeting for Financial Institutions,* Journal of Applied Corporate Finance, Summer 1998, Volume 11, Number 2, pp. 59-69. The estimation of beta itself is still an unresolved issue, with a new approach offered by Kaplan, Paul D. and Peterson, James D., *Full-Information Industry Betas* Financial Management 27 2 Summer 1998. Also other factors besides beta are needed to account for actual risk pricing, as discussed in Fama, Eugene F. and French, Kenneth R. *Multifactor Explanations of Asset Pricing Anomalies* Journal of Finance 51 1 March. Also, to have pricing that will account for the heavy tail of P&C losses, some method is needed to go beyond variance and covariance, such in as Wang's article above, or Kozik, Thomas J. and Larson, Aaron M. *The N-Moment Insurance CAPM,* Proceedings of the Casualty Actuarial Society LXXXVIII, 2001. Finally, the pricing of jump risk needs to be considered. Models for pricing the default risk of corporate bonds incorporate a risk element for the possibility of sudden jumps. The same degree of variability seems to be more expensive as a sudden jump than as a continuous movement, possibly because it is more difficult to hedge by replication. Large jumps are an element of some insurance risk, so need to be recognized in the pricing.

Some of the above elements of a risk pricing formula are being studied by the CAS Risk Premium Project, which is using MR for the frictional capital part of risk pricing. With a good understanding of the value of risk-bearing, insurers will be armed with better tools for comparing actual profitability to a risk-based target.

Other Methods for Ranking by-Line Profit

Return on capital allocated by risk measure and comparison to risk-based pricing are not the only alternatives for the profit ranking problem. Another is using pure marginal costs of capital without allocating fixed capital. This could be done with a risk measure for overall target company capital, or it could quantify the marginal cost of capital by the value of the financial guarantee provided by the firm to the customers of the business unit. This is an approach supported by the paper Merton, R. and Perold, A., *Theory of Risk Capital in Financial Firms,* Journal of Applied Corporate Finance, Fall 1993. The value of the financial guarantee could be priced as a put option, where the customers put all losses in excess of the net premium and investment income of the business unit to the overall firm (up to the assets of the firm – so its really the difference between two puts). This is the default put for the business unit as a separate entity with no capital, so it is a different order of magnitude than the default put for the whole firm that MR considers.

Another alternative method for ranking profitability is to create a model of a leveraged mutual investment fund that borrows enough money at the right interest rate and invests in the right way to have the same probability distribution of after-tax returns as does the insurer. The borrowing rate would be a key measure of the financial viability of the insurer. Then the marginal impact of each business unit on the borrowing rate can be found and used to rank the units.

These approaches are discussed further in my paper *Capital Allocation: An Opinionated Survey*, CAS Forum, to appear.

Time Frame

All the losses outstanding for an insurer would be affected by a default, so several accident or policy years share in the default risk. This complicates the capital allocation problem. The charge for frictional capital costs for a given policy year might consist of shares of a series of put options over several years, where the share could be based on the portion of policy reserves (loss plus unearned premium) represented. The more future years would have costlier options due to the time element in the options pricing formula. In fact, options in practice are priced by assuming even greater volatility for the longer-term options, using smile tables. This would further increase the prices of capital for the later year reserves, and so would tend to increase the proportion of capital allocated to the longer-tailed lines.

A similar method should work for pricing in the financial guarantee approach. The firm could be getting a sequence of call options and providing a sequence of put options, whose total prices could be compared.

For the hypothetical equivalent mutual fund, it would seem sufficient to look at the current annual risk including runoff risk for current liabilities. This would not be a totally prospective look at current strategies, but would still provide a valuable perspective on the financial status of the firm as it has been managed to date.

To Wrap up

The Myers-Read methodology appears to accomplish its aim – to allocate to insurance policies the frictional costs of holding capital. My chief concern in that regard is the time frame for loss payments, with the lognormal assumption a potential issue.

Actuaries would like to have a method of allocating capital in order to rank business units by profitability. Myers-Read seems no better or worse than a number of equally arbitrary but reasonable methods for doing that. In combination with a risk pricing methodology it does lead to an alternative route to that goal: rank by comparing actual profit to the value of the risk transfer provided.

Review of "Capital Allocation for Insurance Companies" by Stewart C. Myers and James R. Read Jr.
Practical Considerations for Implementing the Myers-Read Model

Kyle J. Vrieze, FCAS, MAAA, and
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Capital Allocation for Insurance Companies
Stewart Myers and James Read

Practical Considerations for Implementing the Myers-Read Model

A Review by
Kyle Vrieze and Paul Brehm

Introduction.
With their paper, "Capital Allocation for Insurance Companies," Stewart Myers and James Read have added a great contribution to the finance literature relating to the property and casualty insurance industry. On its face, and taking as given that capital allocation is necessary, the Myers-Read methodology is intuitively appealing and mathematically elegant.

Myers and Read propose a capital allocation methodology based on an options pricing framework. In their model, it is acknowledged that the insurance contract cannot provide a 100% guarantee of indemnification for loss; there is a certain level of default risk. Imagine that the insurance company could purchase an option to put any remaining liabilities to a third party in the event of a default, and that the cost of this put option was related to the volume of risky insurance liabilities as a cost per unit. Myers and Read propose to allocate all of the insurance company's capital to each of the lines of business of the insurance company, with each line receiving the amount of capital that would equalize the cost of that unit's default put option per unit of liabilities.

We have experimented with the Myers-Read (hereafter MR) proposal for capital allocation, creating examples or case studies with the characteristics of real-world insurance data. Our review of Myers-Read will accept the mathematical construct of their model for now and instead present some practical considerations actuaries will likely confront in trying to implement the MR capital allocation model. The balance of this paper presents a number of considerations, broadly grouped into the following five sections.

1. What are we trying to allocate, and why?
2. What would a real-world application look like?
3. How do you parameterize the MR model?
4. What about the asset side of MR?
5. How do you expand or contract the lines of business?

What are we trying to allocate, and why?
Let's step back for a moment and ask ourselves, "What are we trying to achieve with a capital allocation?" Gary Venter [10] said it very well: "Capital allocation is generally not an end in itself, but rather an intermediate step in a decision making process." The "end" decisions are typically about portfolio mix or business performance measurement

and management. There is a recent body of research that suggests actuaries can assist insurance companies in achieving these ends without the need to allocate capital at all.[1]

That said, and for the sake of argument, let's now assume that we have concluded that allocating our company's capital is an admirable task. But we must be clear about *what* we are allocating. Myers and Read state

> "…this paper clarifies how option pricing methods can be used to determine how much capital an insurance company should carry and how that capital requirement should be allocated. It is the convention of the insurance industry to refer to capital as surplus."

"Surplus" is a statutory accounting concept. Indeed, the Myers and Read model could be accused of being an algorithm largely, if only implicitly, written from a regulatory perspective, which might explain the slant. However, beyond regulatory purposes, there is little (no) need to allocate "surplus."

This is not a criticism of the MR model, but rather a clarification of its application. Rather than "surplus," we recommend that the practitioner first establish the amount of capital the company needs based on its risk profile.[2] It is this required level of capital that should be allocated to risk-bearing enterprises in the company. The MR model can do this without loss of generality. Throughout the remainder of this review, we will try to use the term "capital" to reflect the relevant value to be allocated. We will use "surplus" only when quoting or referring to passages in the Myers-Read paper.

What happens when "true" risk-related capital is less than or greater than the recorded level of GAAP or statutory capital? We believe this shortfall or redundancy should be attributed to executive management: to either raise additional capital or put any extra to good use, respectively. It should not be allocated to a risk-bearing activity, as that would tend to distort the very ends we set out to achieve. Allocating too much or too little capital would make it difficult to get an accurate gauge on the true economics of the business for management purposes.

What would a real-world example look like?

The examples used in Myers and Read's original paper, with their evenly sized liabilities, similarly-sized coefficients of variation (cv's), and a single asset class, are not very true-to-life for a typical insurance enterprise. In the discussion that follows, we will work with an example that conforms more to what the actuary might encounter in practice. Our example also has three lines of insurance liabilities, but they are diverse in size and in coefficient of variation. The largest line accounts for the large majority of liabilities, and is intended to represent relatively homogeneous commercial insurance business. The smallest line has a very large cv, and is intended to represent catastrophe business. The

[1] cf Mango [5]. As Don says, "Capital allocation is sufficient but not necessary."

[2] There are a variety of ways to do this, cf Lee and Ward [3].

final line contains more highly variable casualty business, and could represent such businesses as assumed reinsurance, excess and surplus lines, or medical malpractice. We also use two asset classes, intended to be representative of stocks and bonds. The following tables summarize the example:

Table 1, Liabilities:

	% of liabilities	Coeff. of Variation	Description
Line A	86%	5%	Represents standard commercial business
Line B	4%	130%	Cat risk
Line C	10%	30%	Long-tail, high-variance liability
Total			

Table 2, Assets:

	% of assets	Coeff. of Variation	Description
Asset 1	80%	3%	Bonds
Asset 2	20%	15%	Stocks

Correlations among lines and between liabilities and assets are shown in Table 3, along with the MR results for this example. Parameterization in general terms will be addressed, but for this particular model, note that the cat risk line is uncorrelated with the other lines, and the general commercial and high-variability casualty businesses are positively correlated. Note, too, the final allocation is shown on the far right in Table 3 as "surplus/liab."

Table 3:

	% of liabilities	CV	Liability Correlations			covariance w/liabilities	covariance w/assets	surplus /liab
			Line A	Line B	Line C			
Line A	86%	5%	1.0	0.0	0.4	0.003	-0.001	0.14
Line B	4%	130%	0.0	1.0	0.0	0.068	-0.007	2.81
Line C	10%	30%	0.4	0.0	1.0	0.014	-0.003	0.67
Liabilities	100%	8.1%	0.68	0.65	0.59	0.006	-0.001	0.30
Assets	130%	4.2%	-0.26	-0.13	-0.26		0.002	
surplus	30%							
asset/liab volatility	10.2%							
default value/liab	0.0184%							
Delta	-0.004							
Vega	0.017							

	Asset/Liab. Correlations		
	Line A	Line B	Line C
Asset 1	-0.2	-0.1	-0.2
Asset 2	-0.2	-0.1	-0.2

	% of assets	CV	Asset Correlations	
			asset 1	asset 2
Asset 1	80%	3%	1.0	0.2
Asset 2	20%	15%	0.2	1.0

This example can be modified somewhat to produce a scenario with a negative capital allocation. If the correlation between Lines A and B is changed to –0.8, values for Line B's CV between 5% and 66% will produce negative capital ratios for Line B. Here, Line B serves as a valuable hedge that the company is therefore willing to write at a marginal loss. In this case, at a CV in excess of 66%, the hedge value is offset by the capital requirement for Line B's own variability. At a CV less than 5%, Line B won't tend to "wiggle" far enough (in the opposite direction of Line A) to serve as a useful hedge.

Everywhere it appears in the paper, this example will use the MR calculations that arise from the joint lognormal assumption (Appendix 2 to the paper). The joint normal assumption is inappropriate in this case, particularly for the high CV line B, where the normal assumption would imply a reasonable probability for negative liability values. Even the lognormal distribution form may not adequately represent the skewness of a catastrophe risk line such as Line B. Furthermore, there may be segments of the investment portfolio for which the lognormal also may not fairly represent the skewness of returns, such as instruments exposed to significant credit default risk, or those with highly non-linear responses to interest rate changes. We mention this in the context of practical considerations confronting an actuary attempting to implement the MR model because there are a number of real world sources of risk to insurance companies that don't resemble a lognormal curve. Nevertheless, as the authors point out, the appropriate form of probability distribution is an empirical matter. Departing from the lognormal assumption only complicates the calculations; it does not invalidate the basic result.

Using the example in Table 3, we will discuss issues that arise because the allocation assigns all capital to the liability lines, without consideration of the capital consumed by the risky investment portfolio. We will also demonstrate a simple way to subdivide any liability line or asset class in order to produce greater granularity without disturbing the original total capital allocations. But first, we address issues associated with parameterizing the table.

How do you parameterize the MR model?

If the MR model is to be more than of academic interest, we need to be able to parameterize the model. Imagine that you need to fill out the Myers and Read's Table 2 (p. 28) or our Table 3 (above) for your company. The critical parameters that need to be estimated and entered are the standard deviation of the liabilities and the correlations between the classes of liabilities and between the liabilities and the assets.

The standard deviations of the liabilities (unpaid losses) are reasonably approachable. A number of methods exist for calculating these, Mack [4] and Zehnwirth [11] just to name two.

The Holy Grail is to derive the correlation matrix between the unpaid loss liabilities and between the liabilities and the assets. There is currently considerable interest in the search for the Grail. The CAS ERM Committee and the CAS RCM Committee have a joint call for papers in process now seeking insight into this very issue. The Reinsurance Committee has a similar call. The activity is indicative that no one has a clue[3].

One possibility for measuring the MR correlation matrix, at least for the liabilities, would be to start with a measurement of the variance of the entire unpaid loss estimate (σ^2), which is the sum of all of the variance-covariance matrix elements. If we then divided the unpaid loss inventory into two classes, and measured their respective variances (σ_1^2 and σ_2^2), we could calculate the implied covariance (σ_{12}) between the two classes as

$$\sigma_{12}= 0.5[\sigma^2 - (\sigma_1^2 + \sigma_2^2)]$$

By successively splitting aggregate data into classes accordingly, a variance-covariance matrix could be constructed. We're sorry to say that this is still just a thesis on our part; we haven't tried this at home ourselves.

The difficulty in trying to empirically parameterize the MR model is troublesome, but hardly unique. Many popular capital allocation schemes suffer the same fate. Even simple allocation rules such as proportionate standard deviation or variance rules require knowledge about the variance-covariance matrix.

For all the difficulty in expressing the correlation matrix, we have to wonder if it's worth the effort. Recent history, especially events such as September 11, suggest that the nature of the relationship between classes of hazards or between various hazards and other classes of risks may be far too complicated to capture in simple variance-covariance matrix. On the other hand, a conceptual model may still be of value even if it does not capture all the nuances of the real world (think Euclidian geometry, for example).

We alluded to the assumption of normal or lognormal risk distributions in the preceding section. In our experimentation, we relied on the lognormal assumption rather than using any empirical distributions. This was in large part because the math is just so much easier! We did find, however, that the MR model seems to under-allocate capital to highly skewed lines because of this simplifying assumption. Lines that are cat exposed and lines that are credit exposed (financial guarantees, surety) are a couple of examples that we found.

What about the asset side of Myers-Read?

The MR method produces an additive allocation of capital across liability lines. Assigning all costly capital to liability lines implies, correctly, that policyholders bear risk of default not only due to insurance risk, but also due to asset risk.

It is appropriate to expect policyholders to bear the entire cost of capital only if actual investment yields are used in the pricing of coverage. However, in practice, investment

[3] For papers on correlations in unpaid losses, see Brehm [1] or Meyers [7].

income is often accounted for in pricing exercises by calculating present values of expected cash flows using a new money Treasury yield curve. This practice is motivated in part by the perception that it would be unfair for insureds to pay a higher premium in order to cover the investment losses of the insurer; therefore it would be inappropriate to use higher, risky yields in pricing exercises. In such a situation, it might be desirable to determine an amount of capital attributable to risky investment activities, and allocate the remainder of capital to liability lines. Then insureds will neither receive the benefit, nor be charged for the capital requirement, of the risk borne by the investments function.

The performance of the investments area could be measured as the asset return in excess of the risk free value of the insurance float compared against the capital consumed by asset risk. Furthermore, the performance of different asset portfolios could be measured against the portfolios' capital consumption. This measurement could have application in asset allocation studies.

While the MR calculation method does not ascribe capital to asset classes, it could be used as the value function in a Shapley value calculation, which produces an additive, order-independent allocation of capital to each source of default risk in the insurance enterprise[4].

The Shapley value is related to game theory, and is used to calculate the relative power of individuals and coalitions in voting schemes and co-operative games. It is related to the "stand-alone surplus requirement" and "total surplus required for each line, given the other lines" that were discussed in the original MR paper. The idea is to use the MR process to calculate the required capital over all possible combinations of risk sources. In our example from Table 3, we have five sources of risk (three insurance lines and two asset classes). This gives us 2^5-1 = 31 possible combinations of risk sources (this excludes the null set, with no risk sources). We then calculate required capital, using the MR calculation, for each combination of risk sources. Specifically, for each excluded risk source in a given combination, we set the CV for the risk source to zero and adjust the capital ratio until the original default value is returned. This differs somewhat from the method employed by MR when they determined "stand-alone surplus requirement" and "surplus required by each line, given the other two lines". They set the liability value for the excluded line(s) to zero and adjusted the capital ratio until the original default value *as a percent of liabilities* resulted. Their process removed the liability entirely, while ours removes only the risk of the liability or asset.

Once the required capital is calculated for each combination of risk sources, the Shapley value can be computed. Let C represent any one of the 31 combinations of the five risk sources. Let |C| be the number of risk sources that combination C contains. For example, if C contains Line A, Line B and Asset 1 risk, but no Line C or Asset 2 risk, then |C| = 3. Let v(C) represent required capital corresponding to combination C and let N represent the set of all 31 combinations C. The Shapley value, $\phi_i(v)$, for risk source i is then given by the following equation:

[4] Cf Nealon and Yit [9]

$$\phi_i(v) = \sum_{\substack{C \subset N \\ i \in C}} \frac{(|C|-1)!(5-|C|)!}{5!}[v(C) - v(C-\{i\})]$$

The summation is over all the combinations in N that contain risk i. For each combination C, the difference between the capital for C and for the combination created by removing risk i from C is calculated. The weight assigned to this difference is proportional to the (|C|-1)! possible orders in which the |C|-1 risks could be added prior to adding risk i, times the (5-|C|)! orders in which the remaining risks could be added after risk i. All possible orders are contemplated, so the order dependence problem is solved, and the resulting sum of $\phi_i(v)$ for all i equals total required capital.

For our example, the possible combinations of risk sources and their associated MR volatility ratios and capital ratios are shown in Table 4:

Table 4

	Risk Combination					MR Volatility Ratio	Surplus Ratio
1	Line A	Line B	Line C	Asset 1	Asset 2	10.2%	30.0%
2	Line A		Line C	Asset 1	Asset 2	8.4%	23.6%
3	Line A	Line B		Asset 1	Asset 2	8.8%	25.0%
4	Line A	Line B	Line C		Asset 2	9.3%	26.5%
5	Line A	Line B	Line C	Asset 1		9.0%	25.4%
6	Line A			Asset 1	Asset 2	6.7%	17.8%
7	Line A	Line B			Asset 2	7.9%	21.8%
8	Line A	Line B	Line C			8.1%	22.2%
9	Line A		Line C		Asset 2	7.5%	20.1%
10	Line A		Line C	Asset 1		7.1%	19.0%
11	Line A	Line B		Asset 1		7.6%	20.7%
12	Line A	Line B				6.7%	17.8%
13	Line A		Line C			6.1%	15.8%
14	Line A			Asset 1		5.3%	13.3%
15	Line A				Asset 2	5.7%	14.5%
16	Line A					4.3%	10.2%
17		Line B	Line C	Asset 1	Asset 2	8.1%	22.4%
18			Line C	Asset 1	Asset 2	5.8%	14.6%
19		Line B		Asset 1	Asset 2	7.1%	18.9%
20		Line B	Line C		Asset 2	7.2%	19.3%
21		Line B	Line C	Asset 1		6.9%	18.2%
22				Asset 1	Asset 2	4.2%	9.9%
23		Line B			Asset 2	6.3%	16.2%
24		Line B	Line C			6.0%	15.4%
25			Line C		Asset 2	4.6%	11.2%
26			Line C	Asset 1		4.2%	9.9%
27		Line B		Asset 1		5.9%	15.2%
28		Line B				5.2%	12.9%
29			Line C			3.0%	6.6%
30				Asset 1		2.4%	5.0%
31					Asset 2	3.0%	6.6%

Obviously, this method is far more computationally intensive than the original MR calculation for any significant number n of liability lines or asset classes, since you are essentially performing 2^n-1 iterations of a numerical solution for the capital ratio in the MR default value function (unless there is a closed form solution for the capital ratio, but we didn't find one). However, PCs are up to the task (within reason), and an actuary who can translate an algorithm into an Excel macro would make short work of it.

The capital allocation resulting from the Shapley calculation is compared with the original MR allocation in Table 5.

Table 5

	MR Surplus Allocation	Shapley Surplus Allocation
Line A	0.12	0.08
Line B	0.11	0.09
Line C	0.07	0.05
Asset 1		0.04
Asset 2		0.05
Total	0.30	0.30

Differences in capital allocated to liability lines between the MR allocation and the Shapley value follow the marginal diversification characteristics of each liability line. Line A is the line with the highest correlation with liabilities (see Table 3), mainly because it accounts for the large majority of liabilities. At the margin, additional volume in Line A doesn't benefit much from diversification across the other lines, so it receives the largest capital allocation from the MR formula. Line B, the cat risk line, realizes a substantial diversification benefit at the margin because of its very high CV and its independence from the other liabilities (and lower correlation with assets). Its absolute capital allocation under MR is lower than Line A's.

The Shapley allocation looks at all possible combinations of the presence or absence of each risk source, which emphasizes total diversification impact rather than diversification at the margin. This has the effect of crediting Line A for the diversification benefits it imparts to the other lines and to the assets. As an example, consider the effect on the required capital ratio from adding Line B risk to the combination of all the other risk sources. This is the difference between the capital ratio for combination 1 in Table 4 (30.0%) and the capital ratio for combination 2 (23.6%), giving an increase of 6.4%. Now consider the impact of adding Line B risk to all the other risks *except Line A*. This is the difference in capital ratios for combination 17 (22.4%) and combination 18 (14.6%), an increase of 7.8%. The presence of Line A mitigates the capital requirement impact of adding Line B risk.

In further experimentation with the model, we found parameter sets that produced a negative allocation for a segment under MR, but a positive allocation using the Shapley allocation. We also found examples where a negative allocation was produced using both methods.

It could be argued that individual underwriting decisions are made at the margin; therefore the continuous marginal approach of the basic MR is a more appropriate capital allocation for use in pricing. If it is necessary to separate investment risk from the capital cost borne by policyholders, then perhaps the Shapley calculation could be used to determine the portion of capital attributable to investments, and the remainder could be allocated to liability lines in proportion to their original MR allocations. Unfortunately, this mars the elegance of the MR approach, which was a big part of what made it so appealing in the first place.

We promised that you could use this method to measure the performance of the investments function. In this example, the Shapley value assigns almost a third of the capital to the asset portfolio; so, for example, the investment function would realize a 29% pre-tax return on capital under this allocation by beating the risk free return on assets by two points: 1.3*.02/.09=29% (assets*spread/capital = investment ROE).

How do you expand the model for greater granularity?

Being in corporate America we take as axiomatic that it is necessary to reorganize the company almost every year. Wouldn't it be nice if the capital allocated to a block of business were impervious to regrouping the liabilities?

One of the attractive features of the M-R calculation is that the resulting capital allocation is additive, meaning that the individual capital requirements add up to the total enterprise capital. In practice, one is likely to frequently encounter the need to go a step beyond additivity to sub-additivity. For the method to work well in practice, there needs to be a way to split the capital allocated to a specified line into component parts of that line, without disturbing the rest of the allocation. For example, the split of line A's capital by region, by producer, or by product type may be of interest. Or it may be desirable to split the catastrophe line into the earthquake and hurricane perils.

This sub-additivity is easy to accomplish with the MR framework. One can expand any liability line into multiple segments, or combine different segments together. The new MR calculation will preserve the total volatility and default value, as well as the original capital allocations to the unaffected segments, under the constraint that the covariances of all the original liability segments with the total company liabilities and with assets are preserved.

Consider the case of expanding Line A into two similarly sized segments, Line A1 and Line A2, with weights, CVs and correlations as shown in Table 6. In this example, values were selected for the individual CVs of Lines A1 and A2, and the correlation coefficient was solved for in order to preserve the total CV for the combined Line A, following the method we describe in the parameterization section.

Table 6:

	% of liabilities	coeff. of var.	correlations: Line A1	correlations: Line A2
Line A1	40%	7%	1.0	0.1962
Line A2	46%	6%	0.1962	1.0

One can verify that the CV of the sum of lines A1 and A2 is 5%, the same as for the combined Line A. The next problem is determining the correlations of A1 and A2 with Lines B and C, and asset classes 1 and 2. These correlations are constrained by the need to preserve the covariances of the original liability lines with each other and the asset classes. One way to accomplish this is to assume that the correlations with other lines and with asset classes are the same across subdivisions of Line A. Then Line A's original correlation coefficients may be divided by the ratio of the weighted average CV of the subdivisions of Line A (6.4651% in this example) to the original Line A CV of 5%. This adjustment will preserve the necessary covariances, but the resulting correlation matrix is certainly not the only one with this property. Using that adjustment, the new correlations and MR capital allocations are shown in Table 7 (the asset/asset data is unchanged from Table 3).

Table 7:

	% of liabilities	CV	Liability Correlations: Line A1	Line A2	Line B	Line C	covariance w/liabilities	covariance w/assets	surplus/ liab
Line A1	40%	7%	1.0	0.1962	0	0.30935	0.003	-0.001	0.15
Line A2	46%	6%	0.1962	1.0	0	0.30935	0.003	-0.001	0.13
Line B	4%	130%	0	0	1.0	0.0	0.068	-0.007	2.81
Line C	10%	30%	0.30935	0.30935	0.0	1.0	0.014	-0.003	0.67
Liabilities	100%	8.1%	0.53	0.53	0.65	0.59	0.006	-0.001	0.30
Assets	130%	4.2%	-0.20	-0.20	-0.13	-0.26		0.002	
surplus	30%								
asset/liab volatility	10.2%								
default value/liab	0.0184%								
Delta	-0.004								
Vega	0.017								

	Asset/Liab. Correlations: Line A1	Line A2	Line B	Line C
Asset 1	-0.15468	-0.15468	-0.1	-0.2
Asset 2	-0.15468	-0.15468	-0.1	-0.2

Note that the global values, such as the asset/liability volatility and default value, are unchanged from Table 3, as is the capital allocation to Lines B and C. The new capital allocation is once again additive. The capital to liability ratios for lines A1 and A2 are very similar to each other and to the original Line A, with a slightly higher capital requirement result for the sub-line with the larger CV.

We caution that this method of subdivision will give erroneous results if the uniform correlation assumption is violated. In other words, the fact that the capital allocation for the original segments is preserved does not imply that you have found the correct correlations for each of your subdivided lines. As an example, Table 8 shows the same subdivision of Line A into two segments, but this time Line A1 is strongly correlated with Line C, while Line A2 is negatively correlated with Line C. As in Table 7, the global values and the allocations for Lines B and C are unchanged, but the split of the original Line A's allocation between A1 and A2 is quite different.

Table 8:

	% of liabilities	CV	Liability Correlations: Line A1	Line A2	Line B	Line C	covariance w/liabilities	covariance w/assets	surplus/ liab
Line A1	40%	7%	1.0	0.1962	0	0.9	0.004	-0.001	0.20
Line A2	46%	6%	0.1962	1.0	0	-0.28958	0.001	-0.001	0.09
Line B	4%	130%	0	0	1.0	0.0	0.068	-0.007	2.81
Line C	10%	30%	0.9	-0.28958	0.0	1.0	0.014	-0.003	0.67
Liabilities	100%	8.1%	0.75	0.30	0.65	0.59	0.006	-0.001	0.30
Assets	130%	4.2%	-0.20	-0.20	-0.13	-0.26		0.002	
surplus	30%								
asset/liab volatility	10.2%								
default value/liab	0.0184%								
Delta	-0.004								
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Asset/Liab. Correlations	Line A1	Line A2	Line B	Line C
Asset 1	-0.15468	-0.15468	-0.1	-0.2
Asset 2	-0.15468	-0.15468	-0.1	-0.2

Conclusion.

We tried the MR model using fairly realistic insurance company examples. If you *have* to allocate capital, and you are comfortable with the explicit and implicit underlying mathematical assumptions, the MR model is an elegant, intuitively appealing, and tractable method of capital allocation. For this we thank the authors for their contribution.

We stop short of declaring victory in the search for the definitive capital allocation model. The MR model will be hard to parameterize for real-world application. It can yield strange answers when classes of liabilities are not homogenous, when they are highly skewed, or under certain conditions of covariance. Also, the MR model should be used to allocate economic (required) capital not statutory surplus. We further recommend separating the economic capital attributable to the underwriting function from that required for the investment function.

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Annuity Densities with Application to Tail Development

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Annuity Densities with Application to Tail Development[1]

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Abstract: *This paper considers the task of modeling "pension" claims whose durations may vary, but whose payment pattern is uniform and flat. We derive the aggregate payout pattern from the duration density and discuss and provide examples to show how this idea can be applied to calculating tail development factors.*

It is apparent that the claim duration influences paid loss development. In general a faster (slower) claim closure rate will make paid losses develop faster (slower). While the direct nature of that relationship is apparent, it is not so apparent how to quantify it. This paper quantifies it the case of "pension claims" with a constant periodic payment amount. More precisely, the paper considers continuous payment at a rate that is constant both over time and among claims.

Let $S(t)$ denote a survival function on the time interval $(0,b)$.[2] We regard $S(t)$ as a distribution of closure times and let $F(t) = 1 - S(t)$ be the corresponding cumulative distribution function [CDF]. In effect, all claims are assumed to close on or before time b.

We are interested in a related CDF, which we denote by $\widetilde{F}(t)$ to emphasize its relation with $F(t)$, which models the paid loss development as a function of time. More precisely, $\widetilde{F}(t)$ is the proportion of total loss paid by time t, i.e. the proportion paid out during $(0,t)$ (without any discount adjustment). $\widetilde{F}(t)$ is the reciprocal of the paid to ultimate loss development factor and we will refer to $\widetilde{F}(t)$ as the paid loss development divisor [PLDD].[3]

We are interested in claims whose payment schedule conforms to two very restrictive assumptions:

- All payments on all claims are of the same amount.
- Payments are made periodically at a common uniform time interval immediately following a common time of loss, $t = 0$, to claim closure.

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[1] The author expresses his thanks to Greg Engl, also of NCCI, who reviewed several versions of this paper, pointed out some serious errors, and made many suggestions for improvements.

[2] We are most interested in the case $b < \infty$, although most of what we say applies to the case $b = \infty$. We are, however, rather cavalier about making whatever assumptions are needed to assure that all improper integrals exist and are finite.

[3] Gillam and Couret [2] consider the reciprocal of the loss development factor and call it the loss development divisor.

We refer to these two assumptions in combination as describing the "pension case". We use a continuous model to represent this, translating this into the assumption that for every claim of duration x, the model assumes a continuous and constant payment rate of 1 over the interval $(0,x)$, and 0 elsewhere.

Consider the case when aggregate paid losses are followed over a series of N time units with $N < b$. The usual paid loss development patterns built from these N evaluations will not account for the "tail paid loss development" beyond the final evaluation at time $t = N$. With this notation, observe that this tail factor is just $\lambda = \widetilde{F}(N)^{-1}$.

Workers' Compensation [WC] provides a case in point, as some claims in that long-tail line remain open beyond the reporting window, even though that window has been expanding to near 20 years. It is reasonable to assume that payments beyond some valuation, say after 10 years, will be primarily made on pension like claims. Consequently, a model suited to such pension claims may be helpful in projecting the full payout pattern beyond 10 years. Indeed, suppose you have a collection of PLDD's that cover the portion of the loss "portfolio" that is expected to develop beyond 10 years. That is, for each type of claim you have a PLDD that is deemed appropriate, at least over the time frame beyond 10 years. The paper illustrates how to translate the mix of claims in the loss portfolio into a mixed distribution of those PLDD's (c.f. Corollary 1.3 below). That mixed distribution then provides an estimated tail factor. This approach to deriving a tail factor for WC losses is what motivated this paper.

Notation and Setup

With S, F, $\widetilde{F}$ and b as above, we also let $f(t) = \dfrac{dF}{dt}$ be the probability density function [PDF], $h(t) = \dfrac{f(t)}{S(t)}$ the hazard rate function, $CV = \dfrac{\sigma}{\mu}$ the coefficient of variation of claim duration, and T the random variable that gives the "time" of closure t and has the CDF $F(t)$. We use those same letter symbols and "transparent" notation to specify the relationship between these functions. For example $\widetilde{h}_\alpha(t)$ denotes the hazard rate function of the PLDD $\widetilde{F}_\alpha(t)$ that corresponds to the claim survival function $S_\alpha(t)$ and $\widetilde{T}_\alpha$ the random variable with CDF $\widetilde{F}_\alpha(t)$.

For pension claims, as described here, the entire payment schedule of a claim is completely determined by the claim duration. But for now we consider a somewhat more general situation. We make the assumption that for any time t, $0<t<b$, all claims with duration t have the same pre-determined and differentiable payment pattern. We can capture this mathematically by defining the function

$G(x,t)$=amount paid through time t on a claim, conditional upon claim duration=x.

Then define $g(x,t) = \frac{\partial G}{\partial t}$ = partial derivative of $G(x,t)$ with respect to t. We may interpret $g(x,t)$ as the rate of payment at time t on any claim of duration x. Both $G(x,t)$ and $g(x,t)$ are defined for x,t in $(0,b)$. Note that for $t>x$ we have $g(x,t) = 0$ and $G(x,x) = G(x,t) = G(x,b)$= the ultimate incurred on any claim of duration x. It is convenient to define the claim severity function:

$$\gamma(x) = G(x,x) = G(x,b) \quad x \in (0,b).$$

We consider the cumulative payment for such a claim distribution in which all claims occur at time t=0 and conform to these assumptions (sort of an accident instant, as opposed to an accident year). The only "stochastic" ingredient in this model is claim duration, for which the distribution $F(t)$ is specified. All payments are in effect determined by these assumptions and there is a well defined expected cumulative paid loss per claim $P(t)$ at any time t, from t=0 to ultimate paid at t=b. Indeed, we have:

$$\begin{aligned} P(t) &= \int_0^t\int_0^b g(x,y)f(x)dxdy = \int_0^b f(x)\int_0^t g(x,y)dydx = \int_0^b f(x)G(x,t)dx \\ &= \int_0^t f(x)G(x,t)dx + \int_t^b f(x)G(x,t)dx \\ &= \int_0^t f(x)\gamma(x)dx + \int_t^b f(x)G(x,t)dx \end{aligned}$$

since $G(x,t) = \gamma(x)$ for $x<t$. In particular, the expected ultimate loss per claim is just:

$$P(b) = \int_0^b f(x)\gamma(x)dx.$$

It is convenient to define yet another function of t:

$$\eta(t) = \int_0^t f(x)\gamma(x)dx$$

The (expected) ultimate paid loss development factor from time t is:

$$\lambda(t) = \frac{P(b)}{P(t)}$$

and the inverse provides the PLDD on $(0,b)$ that is the focus of this study:

$$\widetilde{F}(t) = \frac{P(t)}{P(b)}.$$

For the PDF of the PLDD, we have, by the fundamental theorem of calculus:

$$\eta(b)\widetilde{f}(t) = P(b)\widetilde{f}(t) = \frac{d}{dt}\left(\int_0^t\int_0^b g(x,y)f(x)dxdy\right) = \int_0^b g(x,t)f(x)dx = \int_t^b g(x,t)f(x)dx$$

since $g(x,t) = 0$ for $x<t$.

We have established most the following:

Proposition 1: *With the function G(x,t) defined as above, for* $t \in (0,b)$ *:*

i) $\widetilde{F}(t) = \dfrac{\eta(t) + \int_t^b f(x)G(x,t)dx}{\eta(b)}$

ii) $\widetilde{f}(t) = \dfrac{\int_t^b f(x)g(x,t)dx}{\eta(b)}$

iii) $\widetilde{S}(t) = \dfrac{\int_t^b f(x)(\gamma(x) - G(x,t))dx}{\eta(b)}$

Proof: All is clear except perhaps (iii):

$$\widetilde{S}(t) = 1 - \widetilde{F}(t) = \frac{\eta(b) - \eta(t) - \int_t^b f(x)G(x,t)dx}{\eta(b)} =$$

$$= \frac{\int_0^b f(x)\gamma(x)dx - \int_0^t f(x)\gamma(x)dx - \int_t^b f(x)G(x,t)dx}{\eta(b)}$$

$$= \frac{\int_t^b f(x)\gamma(x)dx - \int_t^b f(x)G(x,t)dx}{\eta(b)}$$

$$= \frac{\int_t^b f(x)(\gamma(x) - G(x,t))dx}{\eta(b)}$$

as required.

In the WC work that motivated this, the focus was on tail development. This in turn led to the consideration of pension cases. Since those cases take longer to resolve, it is natural to try and use that as a way to isolate them. This leads us to consider what happens when there is a delay period prior to closure, i.e. when $f(t) = 0$ for $t \in (0,a)$ where $0 \le a < b$. In that event we have:

Corollary 1.2: *Suppose* $f(t)=0$ *for* $t \in (0,a)$ *where* $0 \le a < b$, *then*

$$\widetilde{F}(t) = \frac{\int_a^b f(x)G(x,t)dx}{\eta(b)} \quad for \ t \in (0,a).$$

Proof: This is apparent from Proposition 1 (i), since by our assumption

$$\eta(t) = \int_0^t f(x)\gamma(x)dx = 0 \text{ for } t \in (0,a)$$

and the result follows.

This setup is convenient when the distribution of claim durations can be expressed as a mixture of simpler "component" densities. The following Corollary is actually a special case of a more general relationship between mixtures of losses and mixtures of PLDDs that in this pension case is just a simple calculation using Proposition 1:

Corollary 1.3: *Suppose* $F = \sum_{i=1}^{n} w_i F_i$, $\sum_{i=1}^{n} w_i = 1$ *is a weighted sum of CDF's on (0,b), then*

$$\widetilde{F} = \sum_{i=1}^{n} \widetilde{w}_i \widetilde{F}_i, \quad where \quad \widetilde{w}_i = \frac{w_i \eta_i(b)}{\eta(b)}, 1 \le i \le n.$$

Proof: This is a straightforward application of Proposition 1, noting that the same payment function $G = G_i$ applies to all the claims and so applies to each CDF F_i. More precisely:

$$\eta(b)\widetilde{F}(t) = \eta(t) + \int_t^b f(x)G(x,t)dx = \sum_{i=1}^{n} w_i \eta_i(t) + \int_t^b \sum_{i=1}^{n} w_i f_i(x)G(x,t)dx$$

$$= \sum_{i=1}^{n} w_i \eta_i(t) + \int_t^b \sum_{i=1}^{n} w_i f_i(x)G_i(x,t)dx = \sum_{i=1}^{n} w_i \left(\eta_i(t) + \int_t^b f_i(x)G_i(x,t)dx \right)$$

$$= \sum_{i=1}^{n} w_i \left(\eta_i(b)\widetilde{F}_i(t) \right)$$

and the result follows.

Recall that μ denotes the mean duration. More generally, define the higher moments of the distributions $F, \widetilde{F}$ as:

With the above notation, the following proposition documents some basic relationships between the duration density and the PLDD density:

Proposition 2: *In the pension case, for* $t \in (0, b)$

i) $\tilde{f}(t) = \dfrac{S(t)}{\mu}$

ii) $\tilde{F}(t) = \dfrac{\eta(t) + tS(t)}{\mu} = \dfrac{\eta(t)}{\mu} + t\tilde{f}(t)$

iii) $\gamma(t) = t$

iv) $\eta(t) = \int_0^t xf(x)dx$

v) $\tilde{h}(t) = \dfrac{S(t)}{\mu \tilde{S}(t)}$

vi) $\tilde{\mu}^{(k)} = \dfrac{\mu^{(k+1)}}{(k+1)\mu}$ *for* $k = 1,2,3,...$

Proof: By the pension case assumptions, we have:

$$g(x,t) = \frac{\partial G}{\partial t} = \begin{cases} 1 & 0 \le t \le x \\ 0 & x \le t \end{cases}$$

$$\Rightarrow G(x,t) = \begin{cases} t & 0 \le t \le x \\ x & x \le t. \end{cases}$$

We then find that:

$$\gamma(t) = G(t,t) = t$$

$$\Rightarrow \eta(t) = \int_0^t \gamma(x) f(x) dx = \int_0^t xf(x)dx \Rightarrow \eta(b) = \mu$$

which establishes (iii) and (iv). Note that from Proposition 1:

$$\tilde{F}(t) = \frac{\eta(t) + \int_t^b f(x)G(x,t)dx}{\eta(b)} = \frac{\eta(t) + t\int_t^b f(x)dx}{\mu} = \frac{\eta(t) + tS(t)}{\mu}$$

$$\tilde{f}(t) = \frac{\int_t^b f(x)g(x,t)dx}{\eta(b)} = \frac{\int_t^b f(x)dx}{\mu} = \frac{S(t)}{\mu}$$

proving (i) and (ii). For (v), observe that:

$$\widetilde{h}(t)=\frac{\widetilde{f}(t)}{\widetilde{S}(t)}=\frac{S(t)}{\mu\widetilde{S}(t)}$$

and for (vi), integration by parts also gives:

$$E(T^{k+1})=(k+1)\int_0^b x^k S(x)dx$$

and we have:

$$\widetilde{\mu}^{(k)}=\int_0^b x^k\widetilde{f}(x)dx=\int_0^b x^k\left(\frac{S(x)}{\mu}\right)dx=\frac{k+1}{(k+1)\mu}\int_0^b x^k S(x)dx=\frac{E(T^{k+1})}{(k+1)\mu}=\frac{\mu^{(k+1)}}{(k+1)\mu}$$

This completes the proof of Proposition 2.

Now we clearly have that the PDF $\widetilde{f}(t)$ is decreasing, indeed $\frac{d\widetilde{f}}{dt}=-\frac{f(t)}{\mu}\leq 0$ and so the mode of the PLDD $\widetilde{F}(t)$ is 0. From the following Corollary, we see how the shift from $F(t)$ to $\widetilde{F}(t)$ impacts the mean, in particular, we find that the shift increases the mean exactly when $\sigma > \mu$.

Corollary 2.1: $2\widetilde{\mu}=\mu+\sigma CV$

Proof: From Proposition 2 (vi):

$$2\widetilde{\mu}=\frac{\mu^{(2)}}{\mu}=\frac{\mu^2+\sigma^2}{\mu}=\mu+\sigma CV.$$

Corollary 2.2: *Suppose* $f(t)=0$ *for* $t\in(0,a)$ *where* $0\leq a<b$, *then*

$$\widetilde{F}(t)=\frac{t}{\mu}\quad for\ t\in(0,a).$$

Proof: Under these assumptions, Corollary 1.2 implies that for $t<a$:

$$\widetilde{F}(t)=\frac{\int_a^b f(x)G(x,t)dx}{\eta(b)}=\frac{\int_a^b f(x)tdx}{\mu}=\frac{t\int_a^b f(x)dx}{\mu}=\frac{t\int_0^b f(x)dx}{\mu}=\frac{t}{\mu}$$

as claimed.

Probably the most useful family of distributions defined on a finite interval is the class of Beta densities on (0,1). Recall that the Beta distribution is a two-parameter, α,β, distribution that is usually defined in terms of its PDF:

$$f(\alpha,\beta;x)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\quad x\in(0,1),\alpha>0,\beta>0$$

where B and Γ denote the usual Beta and Gamma functions (c.f.[1], [3]). The CDF of the Beta density is:

$$\mathrm{B}(\alpha,\beta;t)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t x^{\alpha-1}(1-x)^{\beta-1}dx \quad t\in(0,1),\alpha>0,\beta>0$$

and we have:

Corollary 2.3: *For* $\alpha>0,\beta>0$ *let* $f(t)=f(\alpha,\beta;t)$, $F(t)=\mathrm{B}(\alpha,\beta;t)$ *be the PDF and CDF of a beta density on* $(0,1)$, *as just defined, then:*

$$\tilde{f}(t)=\frac{\alpha+\beta}{\alpha}(1-\mathrm{B}(\alpha,\beta;t))=\frac{\alpha+\beta}{\alpha}(\mathrm{B}(\beta,\alpha;1-t))=\frac{\mathrm{B}(\beta,\alpha;1-t)}{\mu}$$

$$\tilde{F}(t)=\tilde{\mathrm{B}}(\alpha,\beta;t)=\mathrm{B}(\alpha+1,\beta;t)+\frac{(\alpha+\beta)t}{\alpha}\mathrm{B}(\beta,\alpha;1-t) \quad 0<t<1.$$

Proof: The proof is a straightforward application of Proposition 2. For the PDF, note that:

$$\tilde{f}(t)=\frac{1-F(t)}{\mu}=\frac{1-\mathrm{B}(\alpha,\beta;t)}{\mu}=\frac{1-\mathrm{B}(\alpha,\beta;t)}{\left(\dfrac{\alpha}{\alpha+\beta}\right)}=\frac{(\alpha+\beta)\mathrm{B}(\beta,\alpha;1-t)}{\alpha}=\frac{\mathrm{B}(\beta,\alpha;1-t)}{\mu}.$$

Note too that:

$$\begin{aligned}
\eta(t)=\int_0^t xf(x)dx&=\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\int_0^t xx^{\alpha-1}(1-x)^{\beta-1}dx\\
&=\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\int_0^t x^{(\alpha+1)-1}(1-x)^{\beta-1}dx\\
&=\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\left(\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}\right)\left(\frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}\right)\int_0^t x^{(\alpha+1)-1}(1-x)^{\beta-1}dx\\
&=\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}\right)\left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\right)\mathrm{B}(\alpha+1,\beta;t)\\
&=\left(\frac{\alpha}{\alpha+\beta}\right)\mathrm{B}(\alpha+1,\beta;t)=\mu\mathrm{B}(\alpha+1,\beta;t).
\end{aligned}$$

And so for the CDF:

$$\tilde{F}(t)=\frac{\eta(t)+tS(t)}{\mu}=\mathrm{B}(\alpha+1,\beta;t)+\frac{(\alpha+\beta)t\mathrm{B}(\beta,\alpha;1-t)}{\alpha}$$

as claimed.

We next consider some more specific examples:

Example 1: Consider the case when $S_1(t)=\dfrac{b-t}{b}$, $b<\infty$, is a DeMoivre survival curve. In this case:

$$F_1(b;t) = F_1(t) = \frac{t}{b} \quad f_1(t) = \frac{1}{b}.$$

Note that

$$\eta_1(t) = \int_0^t x f_1(x)dx = \frac{1}{b}\int_0^t x dx = \frac{1}{b}\left[\frac{x^2}{2}\right]_0^t = \frac{t^2}{2b} \Rightarrow \mu_1 = \eta_1(b) = \frac{b}{2}$$

from which we find:

$$\tilde{f}_1(t) = \frac{S_1(t)}{\mu_1} = \left(\frac{b-t}{b}\right)\left(\frac{2}{b}\right) = \frac{2(b-t)}{b^2}$$

and so the density of this PLDD decreases linearly with time. Whence:

$$\tilde{F}_1(t) = \frac{\eta_1(t) + tS_1(t)}{\mu_1} = \frac{2}{b}\left(\frac{t^2}{2b} + t\left(\frac{b-t}{b}\right)\right) = \frac{2}{b}\left(t - \frac{t^2}{2b}\right) = \frac{t}{b}\left(2 - \frac{t}{b}\right)$$

and finally:

$$\begin{aligned}\tilde{S}_1(t) &= 1 - \tilde{F}_1(t) = 1 - \frac{t}{b}\left(2 - \frac{t}{b}\right)\\ &= 1 - 2\left(\frac{t}{b}\right) + \left(\frac{t}{b}\right)^2 = \left(1 - \frac{t}{b}\right)^2 = \left(\frac{b-t}{b}\right)^2 = S_1(t)^2.\end{aligned}$$

Example 2: It is easy to generalize the first example, for $\varphi > 0$ let the claim closure have the CDF:

$$F_2(b;t) = F_2(t) = \begin{cases} \left(\frac{t}{b}\right)^\varphi & t \le b \\ 1 & t \ge b \end{cases}$$

then we have:

$$f_2(t) = \begin{cases} \frac{\varphi t^{\varphi-1}}{b^\varphi} & t \le b \\ 0 & t \ge b \end{cases}$$

$$\eta_2(t) = \int_0^t x f_2(x)dx = \frac{\varphi}{b^\varphi}\int_0^t x^\varphi dx = \frac{\varphi}{b^\varphi}\left[\frac{x^{\varphi+1}}{\varphi+1}\right]_0^t = \frac{\varphi t^{\varphi+1}}{(\varphi+1)b^\varphi}.$$

In particular, we have:

$$\mu_2 = \eta_2(b) = \frac{\varphi b}{\varphi + 1}$$

$$\tilde{f}_2(t) = \frac{S_2(t)}{\mu_2} = \frac{(\varphi+1)}{\varphi b}\left(1 - \left(\frac{t}{b}\right)^{\varphi}\right)$$

$$\tilde{F}_2(t) = \frac{\eta_2(t) + tS_2(t)}{\mu_2} = \frac{t}{\varphi b}\left(\varphi + 1 - \left(\frac{t}{b}\right)^{\varphi}\right).$$

Example 3: Consider the case when fewer claims close over time according to a linear pattern (like the PLDD of Example 1):

$$f_3(b;t) = f_3(t) = \begin{cases} \dfrac{2(b-t)}{b^2} & t \le b \\ 0 & b \le t. \end{cases}$$

By Example 1, $f_3(t)$ is indeed a PDF on $[0,b]$ and we have:

$$\eta_3(t) = \int_0^t x f_3(x)dx = \frac{2}{b^2}\int_0^t x(b-x)dx = \frac{2}{b^2}\left[\frac{bx^2}{2} - \frac{x^3}{3}\right]_0^t = \frac{2}{b^2}\left(\frac{bt^2}{2} - \frac{t^3}{3}\right) \Rightarrow \mu_3 = \frac{b}{3}.$$

But then from Example 1:

$$S_3(t) = \left(\frac{b-t}{b}\right)^2$$

$$\Rightarrow \tilde{f}_3(t) = \frac{S_3(t)}{\mu_3} = \frac{3}{b^3}(t-b)^2$$

$$\Rightarrow \tilde{S}_3(t) = \int_t^b \tilde{f}_3(x)dx = \left(\frac{b-t}{b}\right)^3$$

$$\Rightarrow \tilde{F}_3(b;t) = \tilde{F}_3(t) = 1 - \left(\frac{b-t}{b}\right)^3.$$

In the WC work that motivated this, we seek to find a 19th to ultimate paid loss development factor. One idea that we considered is to use a weighted sum (mixture) of PLDDs of the form $w\tilde{F}_3(b_1;t) + (1-w)\tilde{F}_3(b_2;t)$ (c.f. Corollary 1.3, extend to a common interval by setting the density of the shorter interval to vanish outside it's natural domain). This means that we are assuming all claims close after $Max(b_1, b_2)$ years and one part of the loss "portfolio" close by time $t = b_1$ and the complement by b_2. Empirical loss development factor data is used to fit a non-linear model in which the mixing weight variable w is a "parameter". When these simple functions are used with b_1, b_2 as selected constants, it is straightforward to set up the calculation so as to assure a closed form solution for the value of w that gives the best fit to the data.

Our experience to date of comparing this approach to alternative methods, suggests that the use of linear survival models for the claim duration distribution, while pedagogically and theoretically helpful, may be too simplistic for practical application. Although payment duration is effectively limited by the beneficiary's life-span, there may be no applicable limit to the incurred loss, especially when long term medical care may be covered. While we are primarily interested in the case of finite support, it may be useful to consider a couple of examples when $b = \infty$.

Example 4: Consider the case of a single parameter Pareto (c.f. [3], p 584):

$$F_4(\alpha,\theta;t) = F_4(t) = 1 - \left(\frac{\theta}{t}\right)^{\alpha} \qquad f_4(\alpha,\theta;t) = f_4(t) = \frac{\alpha\theta^{\alpha}}{t^{\alpha+1}} \quad for\ t > \theta.$$

It is natural to extend the definition of the PDF to assign $f_4(t) = 0$ for $t \in (0,\theta]$. Assume that $\alpha > 1$. In this case we have:

$$\mu_4 = \frac{\alpha\theta}{\alpha - 1} \quad \Rightarrow \tilde{f}_4(t) = \frac{S_4(t)}{\mu_4} = \begin{cases} \dfrac{\alpha - 1}{\alpha\theta} & t \le \theta \\ \dfrac{\left(\frac{\theta}{t}\right)^{\alpha}}{\frac{\alpha\theta}{\alpha - 1}} = \left(\dfrac{\alpha - 1}{\alpha\theta}\right)\left(\dfrac{\theta}{t}\right)^{\alpha} & t \ge \theta \end{cases}$$

and we find that:

$$\eta_4(t) = \int_0^t x f_4(x)dx = \begin{cases} 0 & t \le \theta \\ \alpha\theta^{\alpha}\displaystyle\int_{\theta}^{t}\frac{dx}{x^{\alpha}} = \left(\dfrac{\alpha\theta}{\alpha - 1}\right)\left(1 - \left(\dfrac{\theta}{t}\right)^{\alpha-1}\right) & t \ge \theta \end{cases}$$

$$\Rightarrow \tilde{F}_4(t) = \frac{\eta_4(t) + tS_4(t)}{\mu_4} = \begin{cases} \left(\dfrac{\alpha - 1}{\alpha\theta}\right)t & t \le \theta \\ 1 - \dfrac{\left(\frac{\theta}{t}\right)^{\alpha-1}}{\alpha} & t \ge \theta \end{cases}$$

$$\Rightarrow \tilde{S}_4(t) = \frac{\left(\frac{\theta}{t}\right)^{\alpha-1}}{\alpha} = \frac{S_4(t)^{\frac{\alpha-1}{\alpha}}}{\alpha} \quad \text{for } t \ge \theta.$$

Example 5: Consider the case when claim closures follow an exponential density, so here again $b = \infty$. In this case, we have:

$$F_5(\theta;t) = F_5(t) = 1 - e^{-\frac{t}{\theta}}$$

$$f_5(\theta;t) = f_5(t) = \frac{e^{-\frac{t}{\theta}}}{\theta}.$$

Then from Proposition 2 we have:

$$\tilde{f}_5(\theta;t) = \tilde{f}_5(t) = \frac{S_5(t)}{\mu_5} = \frac{e^{-\frac{t}{\theta}}}{\theta} = f_5(t)$$
$$\Rightarrow \tilde{F}_5(t) = F_5(t)$$

and we find that, in the pension case, an exponentially distributed duration has an exponentially distributed PLDD, with the same mean. This suggests that the use of an exponential density, or a mixed exponential (c.f. Corollary 1.3), to fit the PLDD may be quite reasonable when analyzing tail behavior of coverages for which the payments on long term claims become pension like.

Example 6: It is tempting to generalize Example 5, so consider the case when claim closures follow a Weibull density, and so here again $b = \infty$. In this case, we have:

$$F_6(\theta,\tau;t) = F_6(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^\tau}$$

$$f_6(\theta,\tau;t) = f_6(t) = \frac{\tau\left(\frac{t}{\theta}\right)^\tau e^{-\left(\frac{t}{\theta}\right)^\tau}}{t}.$$

$$\mu_6(\theta,\tau;t) = \mu_6 = \theta \cdot \Gamma\left(1 + \frac{1}{\tau}\right).$$

Then from Proposition 2 we have:

$$\tilde{f}_6(\theta,\tau;t) = \tilde{f}_6(t) = \frac{S_6(t)}{\mu_6} = \frac{e^{-\left(\frac{t}{\theta}\right)^\tau}}{\theta \cdot \Gamma\left(1 + \frac{1}{\tau}\right)}$$

and we find that, in the pension case, a Weibull distributed duration has a "transformed exponential" as PLDD.

Example 7: Finally, suppose a PLDD $\widetilde{F}_7(t)$ took the form of a Weibull density:

$$\widetilde{F}_7(\theta,\tau;t) = \widetilde{F}_7(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^\tau}$$

$$\widetilde{f}_7(\theta,\tau;t) = \widetilde{f}_7(t) = \frac{\tau\left(\frac{t}{\theta}\right)^\tau e^{-\left(\frac{t}{\theta}\right)^\tau}}{t}.$$

Then from Proposition 2 we would have:

$$S_7(t) = \mu_7 \cdot \widetilde{f}_6(t) = \frac{\mu_7 \cdot \tau\left(\frac{t}{\theta}\right)^\tau e^{-\left(\frac{t}{\theta}\right)^\tau}}{t}.$$

But then by L'Hôpital:

$$1 = \underset{t\to 0}{Lim}\, S_7(t) = \mu_7 \cdot \tau \underset{t\to 0}{Lim} \frac{\left(\frac{t}{\theta}\right)^\tau e^{-\left(\frac{t}{\theta}\right)^\tau}}{t}$$

$$= \mu_7 \cdot \tau \underset{t\to 0}{Lim} \frac{\left(\frac{t}{\theta}\right)^\tau e^{-\left(\frac{t}{\theta}\right)^\tau}}{t}$$

$$= \mu_7 \cdot \tau \underset{t\to 0}{Lim} \frac{\left(\frac{t}{\theta}\right)^\tau e^{-\left(\frac{t}{\theta}\right)^\tau}\left(-\left(\frac{\tau}{\theta}\right)\left(\frac{t}{\theta}\right)^{\tau-1}\right) + e^{-\left(\frac{t}{\theta}\right)^\tau}\left(\left(\frac{\tau}{\theta}\right)\left(\frac{t}{\theta}\right)^{\tau-1}\right)}{1}$$

$$= \frac{\mu_7 \cdot \tau^2}{\theta} \underset{t\to 0}{Lim} \frac{e^{-\left(\frac{t}{\theta}\right)^\tau}\left(\frac{t}{\theta}\right)^{\tau-1}\left(1-\left(\frac{t}{\theta}\right)^\tau\right)}{1} = \begin{cases} 0 & \tau > 1 \\ \frac{\mu_7}{\theta} & \tau = 1 \\ \infty & 0 < \tau < 1 \end{cases}$$

And it follows that the only Weibull density that can be a PLDD in the pension case is the exponential.

Application to Tail Development

We seek to fit Workers Compensation (WC) age-to-age paid LDFs to a PLDD distribution $\widetilde{F}(t)$. With an eye on Corollary 1.3 and recognizing that pension claims represent a small minority among all WC losses, we decide to consider models of the form:

$$\widetilde{F}(t) = w\widetilde{F}_\alpha(t) + (1-w)\widetilde{F}_\beta(t), \quad 0 \le w \le 1.$$

where $\widetilde{F}_\alpha(t)$ and $\widetilde{F}_\beta(t)$ are PLDD's of "known type." Because we focus on the tail, we only look at development beyond an 11-th report. Throughout this section, we consider as given a set of age-to-age paid loss development factors from an 11^{th} to a 19^{th} report:

$$\lambda_1 = 11^{th} \text{ to } 12^{th} \text{ paid loss LDF}$$
$$\lambda_2 = 12^{th} \text{ to } 13^{th} \text{ paid loss LDF}$$
$$\vdots$$
$$\lambda_8 = 18^{th} \text{ to } 19^{th} \text{ paid loss LDF.}$$

If we knew the true "tail factor" = 19th to ultimate paid loss LDF, we could readily combine that information with the λ_i to determine several "actual" values of $\widetilde{F}(t)$, namely for t=19, 18, ...,11. More precisely, let v^{-1} = tail factor, then the "true" PLDD would equal:

$$\begin{array}{ll} v & \text{at } t = 19 \\ v\lambda_8^{-1} & \text{at } t = 18 \\ \vdots & \\ v\prod_{i=k}^{8}\lambda_i^{-1} & \text{at } t = 10 + k \\ \vdots & \\ v\prod_{i=1}^{8}\lambda_i^{-1} & \text{at } t = 11. \end{array}$$

So defining $G(10+k) = \prod_{i=k}^{8}\lambda_i^{-1}$, we want:

$$vG(k) \approx \widetilde{F}(k) \quad k = 11, 12, \ldots, 19.$$

More precisely, we seek values of the "parameters" w and v, which minimize the weighted sum of squared differences:

$$D(w,v) = \sum_{k=11}^{19}(k-10)\left(\widetilde{F}(k) - vG(k)\right)^2 = \sum_{k=11}^{19}(k-10)\left(w\widetilde{F}_\alpha(k) + (1-w)\widetilde{F}_\beta(k) - vG(k)\right)^2.$$

Since the focus is on the tail, we opt to weight the sum heavier with increasing k. Setting the two partial derivatives $\frac{\partial D}{\partial w}$ and $\frac{\partial D}{\partial v}$ to 0 gives two equations in the two "unknowns" w and v, which are readily solved:

$$0 = \frac{\partial D}{\partial w} = \sum_{k=11}^{19} 2(k-10)\left(\widetilde{F}_\beta(k) + w\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right) - vG(k)\right)\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right)$$

$$\begin{aligned}
\Rightarrow 0 = \quad & \sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right)\widetilde{F}_\beta(k) \\
& + w\sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right)^2 \\
& - v\sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right)G(k) \\
& = a_0 + a_1 w - a_2 v
\end{aligned}$$

$$0 = \frac{\partial D}{\partial v} = \sum_{k=11}^{19} 2(k-10)\left(\widetilde{F}_\beta(k) + w\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right) - vG(k)\right)\left(-G(k)\right)$$

$$\begin{aligned}
\Rightarrow 0 = \quad & \sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\beta(k)G(k)\right) \\
& + w\sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right)G(k) \\
& - v\sum_{k=11}^{19} (k-10)G(k)^2 \\
& = a_3 + a_2 w - a_4 v
\end{aligned}$$

where:

$$\begin{aligned}
a_0 &= \sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right)\widetilde{F}_\beta(k) \\
a_1 &= \sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right)^2 \\
a_2 &= \sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\alpha(k) - \widetilde{F}_\beta(k)\right)G(k)
\end{aligned}$$

$$\begin{aligned}
a_3 &= \sum_{k=11}^{19} (k-10)\left(\widetilde{F}_\beta(k)G(k)\right) \\
a_4 &= \sum_{k=11}^{19} (k-10)G(k)^2
\end{aligned}$$

and which lead to the solution:

$$
\begin{aligned}
a_0 &= -a_1 w + a_2 v \\
-a_3 &= \ \ a_2 w - a_4 v \\
a_0 a_2 &= -a_1 a_2 w + a_2^{\,2} v \\
-a_1 a_3 &= \ \ a_1 a_2 w - a_1 a_4 v \\
a_0 a_2 - a_1 a_3 &= \left(a_2^{\,2} - a_1 a_4\right) v \\
\Rightarrow v &= \frac{a_0 a_2 - a_1 a_3}{a_2^{\,2} - a_1 a_4}
\end{aligned}
$$

$$
\begin{aligned}
a_0 a_4 &= -a_1 a_4 w + a_2 a_4 v \\
-a_2 a_3 &= \ \ a_2^{\,2} w - a_2 a_4 v \\
a_0 a_4 - a_2 a_3 &= \left(a_2^{\,2} - a_1 a_4\right) w \\
\Rightarrow w &= \frac{a_0 a_4 - a_2 a_3}{a_2^{\,2} - a_1 a_4}
\end{aligned}
$$

If this solution falls outside the square $[0,1] \times [0,1]$, it is necessary to inspect the edges and corners to determine the optimal choice for w and v.

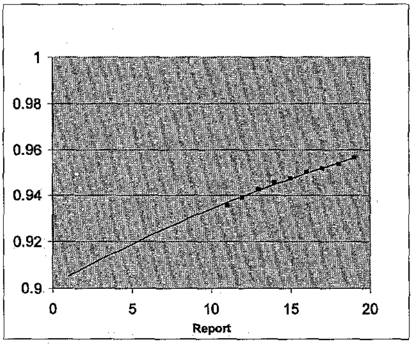
In this application, we break down $\widetilde{F}(t) = w\widetilde{F}_\alpha(t) + (1-w)\widetilde{F}_\beta(t)$ by selecting as one subset of claims those claims that close prior to an eleventh report. So we have:

$$\widetilde{F}_\alpha(k) = 1, \quad k = 11, 12, \ldots, 19.$$

There remains the selection of $\widetilde{F}_\beta$. Numeric examples are given which illustrate for $\widetilde{F}_\beta = \widetilde{F}_3$ and $\widetilde{F}_\beta = \widetilde{F}_6$, perhaps the two most attractive from the examples defined above. In the numeric examples below, we use the following age-to-age LDFs:

i	λ_i	k=10+i	$G(k)$
1	1.004808	11	.9779
2	1.003861	12	.9817
3	1.002915	13	.9855
4	1.001947	14	.9883
5	1.002930	15	.9903
6	1.001957	16	.9932
7	1.001961	17	.9951
8	1.002950	18	.9971
		19	1

Numeric Application 1: Since workers rarely start work much younger than age 20 and live beyond age 100, we select $\widetilde{F}_\beta(t) = \widetilde{F}_3(80;t) = 1 - \left(\frac{80-t}{80}\right)^3$. In the above notation, the solution occurs at w= 0.9016 and v=0.9565. We suggest that the tail factor be selected as $\frac{1}{\widetilde{F}(19)} = 1.046$. Charting the points $vG(k)$ together with the graph of the function $\widetilde{F}(t)$, the picture is:



In the next example we use an exponential. For that, it is convenient to note that the WC financial calls include open and closed indemnity claim counts. This provides the ability to estimate the conditional probability p_k of closure from report k to k+1, assuming a claim is open at report k. So suppose we have the eight probabilities $p_{11}, p_{12}, \ldots p_{18}$. We want to use that information to estimate the parameter $\hat{\theta}$ of $\widetilde{F}_5(\hat{\theta};t)$. For simplicity, suppose there were 100 claims open at report 11, then we would expect the following closure pattern:

$c_1 = 100p_{11}$ would close for some $t \in [11,12]$

$c_2 = 100p_{12}(1-p_{11})$ would close for some $t \in [12,13]$

$c_3 = 100p_{13}(1-p_{11})(1-p_{12})$ would close for some $t \in [13,14]$

$\vdots$

$c_k = 100p_k \prod_{j=11}^{k-1}(1-p_j)$ would close for some $t \in [k, k+1]$

$\vdots$

$c_8 = 100p_{18} \prod_{j=11}^{17}(1-p_j)$ would close for some $t \in [18,19]$

and $d = 100 - \sum_{j=11}^{18} c_j = 100\prod_{j=11}^{18}(1-p_j)$ would remain open at report 19.

It is convenient to simplify this still further and assume that the c_k claims all close at the midpoint of the time interval $= t_k = k + 10½$. Then it is easy to write out the maximum likelihood function for $\theta = \hat{\theta} + 11$. Indeed, we have c_k observed "failures" at $t = t_k$ and d observations "censored" at $t = 19 = s$.

$$L(\theta) = \prod_{i=1}^{8} \left(\frac{e^{-\frac{t_i}{\theta}}}{\theta} \right)^{c_i} \left(e^{-\frac{s}{\theta}} \right)^{d} = \theta^{-\sum_{i=1}^{8} c_i} e^{-\frac{\sum_{i=1}^{8} c_i t_i + ds}{\theta}}.$$

and the log-likelihood function is:

$$LL(\theta) = \log(L(\theta)) = \left(-\sum_{i=1}^{8} c_i \right) \log(\theta) - \frac{\sum_{i=1}^{8} c_i t_i + ds}{\theta}$$

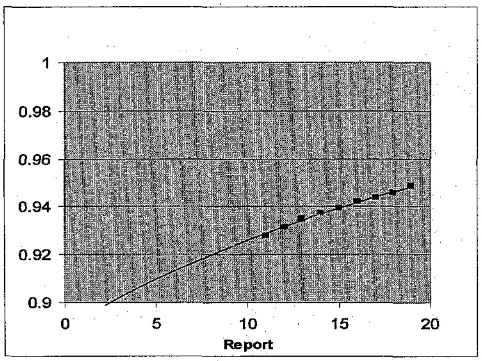
Setting $\frac{dLL}{d\theta} = 0$ to obtain the MLE estimator for θ. We obtain:

$$\theta = \frac{\sum_{i=1}^{8} c_i t_i + ds}{\sum_{i=1}^{8} c_i}$$

which is a useful rule of thumb for determining mean duration with such censored data—take the ratio that defines the weighted mean duration except only include the weight of non-censored observations in the denominator. For our purposes, this provides a simple way to estimate θ from the available WC data and so to specify $\widetilde{F}_{\beta}(t) = \widetilde{F}_5(\hat{\theta};t)$. For a specific numeric example, suppose:

k	p_k	c_k	t_k	*Censored?*
1	0.070	7	11.5	No
2	0.075	7	12.5	No
3	0.081	7	13.5	No
4	0.076	6	14.5	No
5	0.082	6	15.5	No
6	0.075	5	16.5	No
7	0.081	5	17.5	No
8	0.070	4	18.5	No
		53	19	Yes

In this example $\theta = 36$ and $\hat{\theta} = 25$. Then w=0.8895 and v=0.9485, $\frac{1}{\widetilde{F}(19)} = 1.054$ and the picture is:



We close with one more numeric application. The idea here is to use a model like $\widetilde{F}_3$ for $\widetilde{F}_\beta$ but to match the mean duration as we did with the exponential $\widetilde{F}_5$. Suppose, as before, we have used financial data to establish the mean duration $\hat{\theta}$ of claim conditional upon the claim being open at an 11th report. Unlike with an exponential, it is not so convenient to relate the conditional duration $\hat{\theta}$ for t>11 with the unconditional duration μ. Instead, we will get around this by generalizing $F_3(b;t)$ to allow for a deferral period of a=11 (c.f. Corollary 2.2).

So first generalize $F_3(b;t)$ to $F_3(a,b;t)$ as follows:

$$f_3(a,b;t) = f_3(t) = \begin{cases} 0 & t \le a \\ \dfrac{2(b-t)}{(b-a)^2} & a \le t \le b \\ 0 & b \le t \end{cases}$$

$$F_3(a,b;t) = F_3(t) = \begin{cases} 0 & t \le a \\ 1-\left(\dfrac{b-t}{b-a}\right)^2 & a \le t \le b \\ 1 & b \le t. \end{cases}$$

We find that for $a \le t \le b$:

$$\eta_3(a,b;t) = \int_0^t x f_3(x)dx = \frac{2}{(b-a)^2}\int_a^t x(b-x)dx$$

$$= \frac{2}{(b-a)^2}\left[\frac{bx^2}{2} - \frac{x^3}{3}\right]_a^t = \frac{2}{(b-a)^2}\left(\frac{b(t^2-a^2)}{2} - \frac{t^3-a^3}{3}\right)$$

$$= \frac{2(t-a)}{(b-a)^2}\left(\frac{b(t+a)}{2} - \frac{t^2+at+a^2}{3}\right)$$

$$= \frac{(t-a)}{(b-a)^2}\left(b(t+a) - \frac{2}{3}\left(t^2+at+a^2\right)\right)$$

and so when t=b:

$$\mu_3 = \mu_3(a,b) = \eta_3(b) = \frac{1}{b-a}\left(b(b+a) - \frac{2}{3}\left(b^2+ab+a^2\right)\right)$$

$$= \frac{1}{3(b-a)}\left(3b^2+3ab-2b^2-2ab-2a^2\right) = \frac{1}{3(b-a)}\left(b^2+ab-2a^2\right)$$

$$= \frac{(b-a)(b+2a)}{3(b-a)} = \frac{(b+2a)}{3} = a + \frac{b-a}{3}$$

which we could also have arrived at by recalling from the earlier example that the mean for the one parameter $F_3(b-a;t)$ case is just $\mu_3(b-a) = \dfrac{b-a}{3}$ and so:

$$\mu_3 = \mu_3(a,b) = a + \mu_3(b-a) = a + \frac{b-a}{3}.$$

In our numeric example we would calculate b via:

$$25 = \hat{\theta} = \frac{b-a}{3} = \frac{b-11}{3} \Rightarrow b = 75 + 11 = 86$$

which suggests using 86 as the maximum time duration, a selection that is at least consistent with earlier considerations. Finally, we have:

$$\widetilde{F}_3(a,b;t) = \frac{\eta_3(a,b;t) + tS_3(a,b;t)}{\mu_3(a,b)} = \frac{3}{b+2a}\left(\eta_3(a,b;t) + tS_3(a,b;t)\right)$$

$$= \frac{3}{b+2a}\left(\frac{(t-a)}{(b-a)^2}\left(b(t+a) - \frac{2}{3}\left(t^2+at+a^2\right)\right) + t\left(\frac{b-t}{b-a}\right)^2\right)$$

$$= \frac{3}{(b+2a)(b-a)^2}\left(b\left(t^2-a^2\right) - \frac{2}{3}\left(t^3-a^3\right) + t(t-b)^2\right)$$

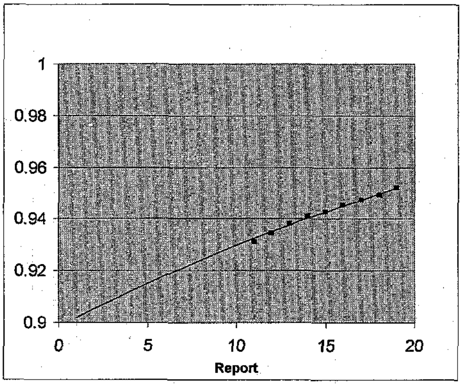
$$= \frac{3\left(b\left(t^2-a^2\right) + t(t-b)^2\right) - 2\left(t^3-a^3\right)}{(b+2a)(b-a)^2}$$

$$= \frac{3\left(t^3-bt^2+b^2t-a^2b\right) - 2\left(t^3-a^3\right)}{(b+2a)(b-a)^2} = \frac{t^3 - 3b\left(t^2-bt\right) + a^2(2a-3b)}{(b+2a)(b-a)^2}$$

When a=11, b=86 and $11 \le t \le 86$, we have the PLDD determined as the cubic polynomial:

$$\widetilde{F}_\beta(t) = \widetilde{F}_3(11,86;t) = \frac{t^3 - 258t^2 + 22188t - 28556}{607500}$$

Applying this to the above example, the best fit is for w=0.99028 and v=0.9520, $\frac{1}{\widetilde{F}(19)} = 1.051$ and the picture is:



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Financial Pricing Models for Property-Casualty Insurance Products: Implementation and Presentation

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Financial Pricing Models for Property-Casualty Insurance Products: Implementation and Presentation

by Sholom Feldblum and Neeza Thandi

INTRODUCTION

The pricing of the insurance policy in Feldblum and Thandi, [2002], "Modeling the Equity Flows." relies on setting the internal rate of return of the implied equity flows equal to the cost of equity capital. The performance measurement system in Feldblum and Thandi, [2002], "Income Recognition and Performance Measurement," and in Schirmacher and Feldblum, [2002], "Retrospective Analysis," relies on the economic value added in an NPV or an IRR accounting framework.

Insurance practitioners measure their performance by loss ratios and combined ratios. The equity flows and EVA performance measures are not always practical for judging the acceptability of an underwriting submission or for understanding the sensitivity of the policy premium to the pricing assumptions. To speak the language of its users, a pricing model must translate the target return on capital into target loss ratios or target combined ratios.

This is not an issue of education, as if the more sophisticated underwriter speaks of returns on capital and the naive underwriter speaks of loss ratios. Numerous factors affect the return on capital, including the surplus requirements, the investment yield, the reserve valuation rate, and the incidence of federal income taxes. Even a proficient pricing actuary can not judge the acceptability of a complex underwriting submission without the aid of computer software. It would be unrealistic to expect underwriters or sales personnel to judge coverage acceptability based on the implied equity flows alone.

This paper deals with implementation of a return on capital pricing model and with the presentation of pricing indications to insurance practitioners. The pricing model itself is documented in the companion papers in this series.

- Given the pricing assumptions, we show the loss ratio or combined ratio needed to achieve a target return on capital. The target loss ratio or combined ratio may be used as performance goals or plan objectives for underwriting units in the company. In addition, the target loss ratios may be used for actuarial rate filings.
- We show the sensitivity of the target loss ratio or combined ratio to the discretionary pricing assumptions. A keen understanding of this sensitivity is necessary for setting challenging but attainable performance objectives.

PRICING ASSUMPTIONS

There are several types of pricing assumptions discussed in this paper.

- Exogenous facts, such as federal income tax rates, premium tax rates, or state assessments; contractual items, such as agents' commissions; and general expenses items, such as overhead and salaries.
- Line of business data, such as cash flow patterns for premiums, losses, and expenses, and ratios of ALAE and ULAE to losses.
- Discretionary items selected by the company's management.

The discretionary assumptions are constrained by the financial and regulatory environments, but they also rely on business judgment. The pricing model uses four discretionary assumptions:

- the target return on capital
- the benchmark investment yield
- the surplus requirements
- the reserve valuation rate

Understanding the relationships between the pricing indications and these discretionary assumptions is essential for managing the underwriting operations of an insurer.

INSURANCE PRICING AND RATE FILINGS

Insurance rates are regulated in most states. Actuarial ratemaking varies by insurer, though the work can often be divided into two parts:

1. *Pricing:* The pricing model determines the target loss ratios or combined ratios, based on the line of business characteristics and the pricing assumptions.
2. *Rate Approval:* The statutory rate filing adjusts the premium rates to attain the targeted loss ratio during the coming policy period.[1]

The selection of the target combined ratio is sometimes expressed as the selection of the target underwriting profit margin, since the profit margin + the combined ratio = 100%.

Some states have formal procedures for selecting the target underwriting profit margin. Since 1982, Massachusetts has selected the target underwriting profit margin by a Myers/Cohn discounted cash flow analysis. Other states permit more leeway in selecting the underwriting profit margin.

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[1] Rate filing procedures are documented in the casualty actuarial literature; they are not discussed here. The standard reference for workers' compensation ratemaking is Feldblum [1992: WCR].

TARGET LOSS RATIOS

The determination of the indicated premium proceeds as follows.

BASE

A base dollar amount, such as $1,000, of losses plus ALAE is selected. The indication of the pricing model is the target loss ratio or combined ratio, any base dollar amount may be used.

LINE OF BUSINESS EXPENSES

- ULAE is added as a percentage loading onto losses plus ALAE, based on past company experience.
- Acquisition costs are loaded as a percentage of premiums, based either on contract terms (agents' commissions) or statute (premium taxes).
- General expenses are loaded as a percentage of premiums based on past experience.

CASH FLOW PATTERNS

Cash flow patterns are based on the line of business, the jurisdiction, and the policy characteristics. In workers' compensation, for instance, the loss payment pattern depends on the state compensation system and the size of the policy deductible. The premium collection pattern depends on the type of policy, such as first dollar prospectively priced policy versus a paid loss retrospectively rated policy.

DISCRETIONARY PRICING ASSUMPTIONS

The discretionary pricing assumptions are chosen by management. In most companies, these pricing assumptions are based on actuarial, financial, or investment department recommendations.

The pricing model solves for the policy premium that provides the target rate of return on the implied equity flows.

- The dollar amount of ultimate losses divided by the indicated premium is the target loss ratio.
- The target loss ratio plus the expense ratio is the target combined ratio.

PRICING RESULTS

Exhibit 1 illustrates the management output generated by the pricing model. This exhibit, like the sensitivity exhibits discussed further below, is geared to company management; the exhibits showing the derivation of the indicated premium and loss ratios are documented in Appendix B of Feldblum and Thandi, [2002], "Modeling the Equity Flows."

The exhibit has five sections:

1. Underwriting Assumptions
2. Finance Assumptions
3. Risk (Surplus) Assumptions
4. Pricing Results
5. Profitability

The exhibit shows simulated results for a first-dollar workers compensation block of business.[2]

Section I: Underwriting Assumptions

Subsection "A: Policy Costs" shows the ultimate loss plus ALAE of $1,000. The $1,000 loss is an arbitrary reference figure; all other costs are direct or indirect functions of the loss cost.

ULAE is loaded onto loss plus ALAE by a factor of 7.2% based on past company experience. It is shown separately from loss plus ALAE since ULAE has a faster payout pattern.

Acquisition and general expenses are treated as a percentage of premium (25.6%). Expected policyholder dividends are loaded as an additional expense item (5.7%). Since we are solving for the aggregate block of business target combined ratio, not for an individual policy premium, expense flattening procedures and premium discount factors by size of policy are not necessary.

Subsection "B: Cash Flow Patterns" shows the cash flow assumptions.

- The actual model uses the full cash flow patterns for losses and premiums.
- The exhibit shows the ratio of the discounted amount to the undiscounted amount as a proxy for the cash flow pattern. This ratio is used to convert undiscounted amounts to discounted amounts.

Workers' compensation is a long-tailed line of business; discounted losses are only three quarters of undiscounted losses. For many large account workers' compensation business, premiums are paid in installments.[3]

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2 "First dollar" means there is no deductible on the policy.

3 For workers' compensation, insurers may book the premium as it is billed, thereby delaying the incidence of state premium taxes and slightly reducing the capital requirements for the policy. The company being modeled here books all premium at the policy effective date, in compliance with IRS procedures. See

Exhibit 1

SUMMARY OF ASSUMPTIONS AND RESULTS FOR WORKERS' COMPENSATION

Fully Insured Policies

I. UNDERWRITING ASSUMPTIONS

A) Policy Costs	
Expense Ratio (as % WP)	25.6%
Dividend Ratio (as % WP)	5.7%
ULAE Ratio (as % of Loss&ALAE)	7.2%
Ultimate Loss & ALAE	1,000
B) Cash Flow Patterns	
Disc Loss&ALAE to Undisc	73.0%
Duration of Losses (in yrs)	4.3
Disc Premium to Undisc	95.3%
C) Average Effective Date	0.5
D) Level of Reserve Adequacy	
Held to Nominal Reserves	100.0%

II. FINANCE ASSUMPTIONS

A) Investment Rate of Return	
Assets Backing Liablities & Surplus	8.0%
B) Federal Income Taxes	
Marginal Income Tax Rate	35.0%
Tax on Investment Returns	35.0%
C) Target Return on Capital	
Post-Tax Return	**12.0%**

III. RISK (SURPLUS) ASSUMPTIONS

Reserve Leverage Ratio	0.0%
Premium Leverage Ratio	43.7%

IV. PRICING RESULTS

A) Premium		
Nominal Premium	1,374	
Discounted Premium	1,309	
B) Summary of Costs		
Disc Loss & LAE	784	
Disc Expense (incl PHR Dividends)	416	
Disc Taxes	67	
	1,267	
	Nominal	Discounted
C) Ratios	(% of Nominal Prem)	(% of Disc Prem)
Loss & ALAE Ratio	72.8%	55.8%
ULAE Ratio	5.2%	4.1%
Expense Ratio (incl PHR Dividends)	31.3%	31.8%
Combined Ratio	**109.2%**	91.6%

V. PROFITABILITY

A) Equity Charge	Nominal	Discounted
	107.48	42.56
B) IRR on Equity Flows		
Post-Tax IRR	**12.0%**	

Subsection "C: Average Effective Date," uses a mid-year (July 1) assumed effective date as a proxy for an even distribution of effective dates throughout the year. The effective date is relevant for pricing because the loss reserve discount factors applicable to the book of business depend on the accident year of the losses.

Subsection "D: Level of Reserve Adequacy" shows the expected loss reserve valuation rate. The 0% valuation rate implies that reserves will be carried at full value. For the effects of the reserve valuation rate on the indicated premium, see Feldblum and Thandi [2002], "Reserve Valuation Rates" and Feldblum and Thandi [2002] "Federal Income Taxes and the Cost of Holding Capital."

SECTION II: FINANCE ASSUMPTIONS

Subsection "A: Investment Rate of Return" is the pre-tax equivalent benchmark investment yield, expressed with semi-annual compounding.[4] The benchmark yield is based on the company's expected investment portfolio during the life of this block of business.

Some pricing actuaries use different expected investment yields on assets backing loss reserves and on assets backing surplus funds.

> *Illustration:* One might use an investment grade bond yield for assets backing loss reserves and a common stock yield for assets backing surplus.

The exhibits in this paper make no assumption about proper investment strategy for property-casualty insurance companies, and they use a single yield on all investable assets.

The benchmark investment yield is a discretionary assumption. For long-tailed lines of business, small changes in the investment yield assumption have relatively large effects on the target loss ratios; see the sensitivity exhibits below.

Subsection "B: Federal Income Taxes" shows the assumed marginal tax rate on underwriting income (35%) and the average tax rate on investment income (35%). In some circumstances, it may be proper to consider the effects of the alternative minimum income tax and the tax exemption of certain investment income. Because of the proration provision in the 1986 Tax Reform Act, few property-casualty insurance companies invest heavily in municipal bonds.

Feldblum [2002: SchP] for full explanation of the statutory and tax premium recognition requirements.

[4] The 7.3% yield in the exhibit is equivalent to a $1.0365^2 - 1 = 7.43\%$ effective annual yield. The investment yield on municipal bonds is converted to a pre-tax equivalent yield so that the 35% marginal tax rate may be applied to all investment income. The proration provision of the 1986 Tax Reform Act changed the effective tax rate paid by insurance companies on municipal bond income from 0% to 5.25%. A Z% municipal bond yield is converted to a $Z\% \times (1 - 5.25\%) / (1 - 35\%)$ pre-tax equivalent yield.

Subsection "C: Target Return on Capital," shows the target return on capital used as the hurdle rate for the NPV calculations and as the discount rate for the EVA calculations. In general, the target return on capital is the cost of equity capital for the company. The company may vary the target return on capital with the fluctuations of the insurance underwriting cycle, marketplace competition, regulatory prescriptions, or long term growth strategies. See the sensitivity exhibits below for the effects of this pricing parameter on the target loss ratio.

SECTION III. RISK (SURPLUS) ASSUMPTIONS

The pricing model derives the indicated premium by examining the return on invested capital. The invested capital is embedded in statutory reserves and in policyholders' surplus.

- The amount of capital embedded in reserves depends the loss payment pattern and the reserve valuation rate. This embedded capital is greatest for full value loss reserves in the long-tailed lines of business. In some instances, companies may hold less than full value loss reserves, particularly if full value loss reserves would make the necessary premiums uncompetitive. The pricing model determines the indicated premiums at any given reserve valuation rate (see Feldblum and Thandi [2002] "Reserve Valuation Rate."
- A minimum level of surplus is mandated by risk-based capital requirements. Most companies hold surplus equal to at least two times the minimum risk-based capital requirements.

The capital held in surplus is subject to management views on optimal capital structure. The allocation of this capital to line of business may be done in a variety of ways. The figures in the exhibit are based on three assumptions:[5]

a. Total company surplus is set at 200% of NAIC risk-based capital requirements, based on the average RBC ratios for companies receiving A ratings from A. M. Best's.
b. The allocation of this surplus by line of business is based on a risk analysis that equates the expected policyholder deficit ratio across lines.
c. Capital requirements are tied to premium writings, not to held reserves, to provide greater incentives for the EVA performance measurement system.

Surplus assumptions are widely debated; there is no consensus in the actuarial community. The assumptions in the pricing model should reflect the surplus carried by the company to support its insurance operations.

SECTION IV: PRICING RESULTS

Actuarial pricing has three distinct meanings.

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[5] See Feldblum and Thandi [2002] "Surplus Allocation" for the rationale of each assumption.

- *Rate Change:* The traditional loss ratio ratemaking procedure determines the required rate change to bring premiums to an adequate level for the future policy period.
- *Pure Premium:* The traditional pure premium ratemaking procedure determines the manual premium rate per unit of exposure.
- *Financial Pricing:* The financial pricing model determines the underwriting profit margin needed to achieve a given financial target, such as a target internal rate of return.

We deal here with financial pricing; we are not determining rate level changes or pure premiums. The output of the financial pricing model is the target underwriting profit margin.

We convert the target underwriting profit margin into two other target ratios that are commonly used by underwriters and actuaries: the target combined ratio and the target loss ratio. The target combined ratio is the complement of the target underwriting profit margin. The target loss ratio is the target combined ratio minus the expense ratio.

Illustration: If the target underwriting profit margin is –6.3%, and the expense ratio is 32.8%, the target combined ratio is 1 – (–6.3%) = 106.3%, and the target loss ratio is 106.3% – 32.8% = 73.5%.

The pricing model output shows the target loss ratio and combined ratio. It also shows several other summary figures, so that users more readily understand the costs of the policy.

Section IV shows nominal and discounted amounts of premiums, losses, and expenses. The nominal amounts are used to determine the target loss ratios and combined ratio. The discounted amounts show the relative sizes of the cost components of the premium. The discount rate is the benchmark investment yield, not the target return on capital.

Subsection "A: Premium" shows the nominal premium amount ($1,374) that provides an internal rate of return equal to the target return on capital of 12%, based on the implied equity flow pattern. This premium provides an net present value of zero at a hurdle rate equal to the target return on capital.

The discounted premium is based on the assumed premium collection pattern: 95.3% × $1,374 = $1,309.

Subsection "B: Summary of Costs" shows the present value of losses, expenses, and taxes at the benchmark investment yield.

- For ultimate losses plus ALAE of $1,000, the discounted amount is $730 (see section 1 of this exhibit).
- The ultimate ULAE is 7.2% × $1000 = $72. The ULAE discount factor is 75.0%, giving a discounted amount of $54. (The ULAE discount factor is not shown on the exhibit.)
- Total discounted losses plus LAE equal $730 + $54 = $784.

- Undiscounted expenses plus policyholder dividends are (25.6% + 5.7%) × $1,374 = $430.06. Most expenses are paid rapidly, and the discounted amount is $416.

The total discounted costs, consisting of losses, expenses, and taxes, is $1,267. The remaining $42.56 in the discounted premium of $1,309 is the net profit margin that covers the cost of holding capital.[6]

Subsection "C: Ratios" show the target loss ratio and combined ratio on both a nominal and discounted basis.

The target loss and ALAE ratio on a nominal basis equals 1000 / 1374 = 72.8%. The target ratio on a discounted basis equals the discounted losses and ALAE divided by the discount premium (not the nominal premium): $730 / $1309 = 55.8%.

The 72.8% target loss ratio transforms the return on capital indications to the more common industry benchmark. The sensitivity analyses below show the effects of the discretionary assumptions on the target loss ratio.

The ULAE ratio is converted from a percentage of losses to a percentage of premiums: 7.2% × 1000 / 1384 = 5.2%. Other expenses are taken from section 1 of the exhibit. The combined ratio is the sum of the loss plus ALAE ratio, the ULAE ratio, and the expense ratio.

INDUSTRY COMPARISON

The 109.2% target combined ratio is appropriate for a participating block of business issued by a mutual insurance company. The equivalent target combined ratio for non participating business issued by a stock insurance company is 109.3% / (1 – 5.7%) = 115.9%.

The company modeled in this exhibit is a large workers' compensation writer with an efficient distribution system, leading to the low 25.6% expense ratio. Many workers' compensation insurers have expense plus dividend ratios of 35% to 40%, giving target combined ratios of about 105% to 110%.

The discretionary assumptions used in this exhibit, such as the 12% cost of equity capital and the 7.3% benchmark investment yield, reflect industry norms. Actual premium to surplus ratios for workers' compensation carriers are about 1 to 1, necessitating more equityholder provided capital than the 43.7% surplus assumption in the exhibit. The use of a 1 to 1 premium to surplus ratio and industry average expense ratios would lead to a target combined ratio of about 100%. We examine the implications for industry profitability at the end of this paper.

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[6] Much of the analysis in the latter half of this chapter deals with the gross (pre-tax) profit margin. The federal income taxes consume 60%-80% of the gross profit margin on long-tailed lines of business. Careful analysis of tax implication is critical to optimal policy pricing.

Subsection "A: Equity Charge" shows the real profit margin in the indicated premium rates. The nominal underwriting profit margin is the undiscounted premiums minus losses and expenses, or –9.2% in this exhibit. This margin has no financial meaning, since it uses undiscounted figures and it takes no account of federal income taxes.

The present value of the policyholder premium ($1,309) covers three types of costs. (The numbers in parentheses are the values in this illustration.)

- The present value of the expected losses and expenses ($1,200).
- The present value of the expected federal income taxes ($67).
- The present value of the cost of holding capital ($42).

We differentiate between the gross profit margin and the net profit margin.

- The gross profit margin ($109) is the present value of the policyholder premium minus the present value of the expected losses, expenses, and policyholder dividends.
- The net profit margin ($42) is the gross profit margin minus the present value of the expected federal income taxes.

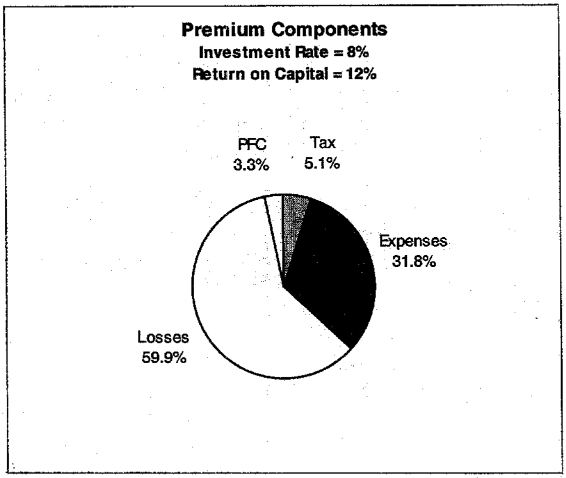
The federal income tax liability depends on the cost of holding capital, not the expected losses and expenses. If the policy premium just covered the expected losses and expenses and contained no margin for profit, and if there were no need for capital to support the insurance operations, the expected federal income tax liability would be zero. The discounted losses and expenses would just offset the premium, giving a zero profit in the first year and no tax liability. In subsequent years, the unwinding (amortization) of the interest discount in the tax-basis reserves would just offset the investment income on the assets supporting the reserves, leading to zero tax-basis profits and no tax liability.

In practice, capital is needed to support the insurance operations. Part of this capital is embedded in the undiscounted loss reserves and part of this capital is held as surplus. The capital is provided by the owners of the insurance enterprise, who are termed equityholders in this paper.

Equityholders demand the target return on capital for their funds, but the insurer invests the money at the benchmark investment yield. The policyholders make up the difference. This is the present value of the cost of holding capital, also termed the equity charge in this exhibit.

These funds are taxed twice at the corporate level: once as underwriting income when the premium is paid and once as investment income when the capital funds are invested. After being paid to the equityholders, the net income is taxed a third time at the personal level.

The net (discounted) profit margin, called the discounted equity charge in the exhiibit, equals the discounted premium minus the discounted losses, expenses, and taxes: $1,309 – $1,267



= $43 ($42.56). The exhibit also shows a nominal equity charge, which uses the undiscounted premium minus the discounted losses, expenses, and taxes. Some users of the pricing model are more comfortable with the nominal equity charge, though it has no financial meaning.

The effective tax rate about 60% is valid when reserves are held at full value. When reserves implicitly discounted, the effective tax rate is higher. If personal income taxes are included, the effective tax rate is between 70% and 80%; the exact figure varies with the tax position of the equityholders and with the form of the shareholder distribution by the insurance company.

Subsections B and C, which show the IRR on the equity flows and the economic value added, are used for performance measurement. For prospective policy pricing, the IRR is constrained to be 12% after-tax and the EVA is zero. Retrospective analysis based on actual results may show a higher or lower IRR and a positive or negative EVA.[7]

MARGINS

The accompanying pie chart shows the relative magnitudes of losses, expenses, taxes, and net profit margin in the policyholder premium.

- The provision for losses and expenses forms the bulk of the premium (91.6%); the gross profit provision is much smaller (8.3%). The insurance industry is about average among firms in the U.S.: supermarkets and department stores have profit margins of about 2%, whereas luxury goods have profit margins of 20% or higher.
- Taxes form the bulk of the gross profit margin; they are 5.1%/(5.1% + 3.2%) = 61.45% of the total. The IRS takes an even larger percentage if reserves are not held at full value.

[7] See Schirmacher and Feldblum [2003], "Retrospective Analysis."

The discretionary assumptions are selected by company management. They are based on financial and actuarial principles, they are not fixed attributes of the block of business. In the companion papers, we present the considerations in selecting each assumption.[8] To properly manage insurance operations and product pricing, one must understand the sensitivity of the target loss ratios to each discretionary assumption.

Illustration: The insurer uses a 12% after-tax return on capital, giving a target combined ratio of 109.2% for first-dollar workers' compensation business. A soft market currently prevails for workers' compensation, and the insurer's management believes that a combined ratio below 112% is not feasible. The company has several options:

A. The company may use a target combined ratio of 109.2% to measure underwriting performance, on the assumption that aggressive targets inspire better performance. This strategy is risky, since unrealistic targets may demoralize employees.
B. The company may reduce the size of its workers' compensation book of business, retaining better quality risks and shunning mediocre risks. In practice, all companies seek to retain better quality risks and shun mediocre risks. Stringent reunderwriting may improve the current year's combined ratio, but it is difficult to win back market share that has been lost during periods of stringent underwriting.[9]
C. The company may decide to accept a lower return on capital for its workers' compensation book of business. Companies rarely achieve all their targets. Astute management foregoes aggressive targets when they collide with business exigencies, as long as the results are still acceptable to the company's owners.

SENSITIVITY ANALYSES

We examine the sensitivity of the target loss ratio and target combined ratio to three of the discretionary pricing assumptions: (i) the benchmark investment yield, (ii) the target return on capital, and (iii) the surplus assumptions.

Target Return on Capital and Benchmark Investment Yield

Exhibit 2 shows the sensitivity of the target combined ratio to changes in (a) the target return on capital (along the vertical axis) and (b) the benchmark investment yield (along the horizontal axis) on both a nominal and discounted basis.

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[8] See particularly the papers on "Benchmark Investment Yield," "Target Return on Capital," "Surplus Requirements," and "Reserve Valuation Rates."

[9] See Feldblum [2002: UCBS]. Stringent underwriting works well if it is based on the long-term expected profitability of the risk. It is of less value if it seeks to weed out average risks during soft markets.

- The nominal basis uses undiscounted premiums, losses, and expenses. This is the combined ratio shown on the company's books.
- The discounted basis uses the ratio of discounted losses and expenses to discounted premiums, using the benchmark investment yield as the discount rate.

The sensitivity to the benchmark investment yield depends on the loss payment pattern for the line of business. The slower the loss payment pattern, the more sensitive is the target loss ratio to the benchmark investment yield.[10]

As the benchmark investment yield increases, the target combined ratio increases. The boxed number in the center of the exhibit, 109.2%, is the target combined ratio at an 8% benchmark investment yield and a 12% target return on capital. As one moves to the right along the row, the benchmark investment yield increases by 50 basis points per column. For a 9% benchmark investment yield (+100 basis points, or two columns to the right), the target combined ratio is 111.8%, or a 250 basis point increase. The increase in the target combined ratio is two and a half times the increase in the benchmark investment yield because the assets supporting the reserves are held for several years.

A higher target combined ratio means a lower indicated premium, since the premium is inversely proportional to the combined ratio. A higher benchmark investment yield means the company is earning more investment income on the insurance transactions, and a smaller underwriting profit margin is needed.

A common – but sometimes misleading – rule of thumb is that the investment yield times the lag between premium collection and loss payment offsets the underwriting income on a one to one basis, after suitable adjustment for expenses and non-investable assets. To avoid potential errors, any pricing rule must consider federal income taxes and changes in the implied equity flows; see below.

Changes in the target return on capital are reflected by movements up and down a column. Each step along a column is a 50 basis point change in the target return on capital. If the benchmark investment yield is 8% per annum, a 100 basis point increase in the target return on capital changes the target combined ratio from 109.2% to 107.8%. More profit is needed in the policyholder premium if equityholders demand a greater return on their invested capital.

For the illustration here, a 100 basis point change in the target return on capital necessitates a 140 basis point change in the combined ratio. This relationship depends on the time that the capital is held; if capital is held for a longer time period, a larger change in the combined ratio is needed. The federal income tax rate affects the relationship, since the combined ratio is on a pre-tax basis and the return on capital is on an after-tax basis.

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[10] The exhibits show the sensitivity for the workers' compensation illustration in Appendix B of Feldblum and Thandi, [2002], "Modeling the Equity Flows."

Exhibit 2

SENSITIVITY ANALYSIS

LINE OF BUSINESS: **Workers' Compensation Fully Insured**

TARGET NOMINAL COMBINED RATIO

Post-Tax ROC = 12.0% +	Investment Rate of Return = 8.0% +										
	-250 bp	-200 bp	-150 bp	-100 bp	-50 bp	0 bp	+ 50 bp	+ 100 bp	+ 150 bp	+ 200 bp	+250 bp
-250 bp	106.2%	107.5%	108.9%	110.2%	111.6%	113.0%	114.5%	116.0%	117.5%	119.1%	120.7%
-200 bp	105.6%	106.9%	108.2%	109.5%	110.8%	112.2%	113.6%	115.1%	116.6%	118.1%	119.7%
-150 bp	105.0%	106.2%	107.5%	108.8%	110.1%	111.4%	112.8%	114.2%	115.7%	117.2%	118.7%
-100 bp	104.4%	105.6%	106.8%	108.1%	109.4%	110.7%	112.0%	113.4%	114.8%	116.3%	117.7%
-50 bp	103.8%	105.0%	106.2%	107.4%	108.7%	109.9%	111.2%	112.6%	114.0%	115.4%	116.8%
0 bp	103.2%	104.4%	105.6%	106.7%	108.0%	109.2%	110.5%	111.8%	113.1%	114.5%	115.9%
50 bp	102.7%	103.8%	104.9%	106.1%	107.3%	108.5%	109.8%	111.0%	112.4%	113.7%	115.1%
100 bp	102.1%	103.2%	104.3%	105.5%	106.6%	107.8%	109.1%	110.3%	111.6%	112.9%	114.2%
150 bp	101.6%	102.6%	103.7%	104.9%	106.0%	107.2%	108.4%	109.6%	110.8%	112.1%	113.4%
200 bp	101.0%	102.1%	103.2%	104.2%	105.4%	106.5%	107.7%	108.9%	110.1%	111.3%	112.6%
250 bp	100.5%	101.5%	102.6%	103.7%	104.7%	105.9%	107.0%	108.2%	109.4%	110.6%	111.8%

SPREAD PRICING

Life actuaries often conceive of the profit margin in terms of a spread between the earned interest rate (adjusted for expenses and investment risks) and the credited interest rate. The pricing of annuity products is primarily the selection of the appropriate spread.

Illustration: The pricing actuary may target a 200 basis point spread for a variable annuity product. During the accumulation phase, the insurer credits the account balance with the earned interest rate minus 200 basis points. At the commencement of the payout phase, the expected periodic benefits are determined from the assumed interest rate, which is the current earned interest rate minus 200 basis points. During each subsequent year, the annuity benefits are adjusted by the ratio of that year's earned interest rate minus 200 basis points to the assumed interest rate. The actual achieved spread may vary with market demand for the annuity product and with competitive pressures for similar investment vehicles.

An analogous procedure suggested for property-casualty products would be to set a target spread between the return on capital and the benchmark investment yield. The target combined ratio would depend on the size of this spread.

This procedure presumes that if a 109% target combined ratio is appropriate for an 8% benchmark investment yield and a 12% target return on capital, it is also appropriate for a 10% benchmark investment yield and a 14% target return on capital and for a 6% benchmark investment yield and a 10% target return on capital. In other words, the target combined ratio depends on the spread between the benchmark investment yield and the target return on capital, not on the absolutes values of either figure. Since the benchmark investment yield and the target return on capital both vary with the risk-free interest rate, this simplifies the pricing.

Although a spread perspective is reasonable for universal life policies and for certain annuity products, it is misleading for property-casualty insurance products. The general rule is that

> *Spread pricing is appropriate for tax exempt products and for tax deferred products with long durations; common examples are life insurance policies, variable annuities, and pension products. It is also appropriate for pass-through financial intermediaries, such as mutual funds, investment houses, and certain depository institutions. It is incorrect for fully taxable products, such as property-casualty policies, and it may lead to severe under-pricing as interest rates increase.*

Spread pricing is most appropriate when customer funds are treated (for tax purposes) as deposits, not as revenue, and when the investment income earned by the financial intermediary is either tax deferred (or tax exempt) or passed through to the customer. These conditions are true for most financial institutions, such as life insurance companies, annuity writers, depository institutions, mutual funds, and investment houses. For property-casualty products, the policyholder premium is treated as revenue, not as a deposit, and all investment income is taxed at the corporate level before being passed to policyholders or equityholders.[11]

We examine the relationship between a constant spread and the target combined ratio, using both full value and discounted figures. We then explain the observed results, with emphasis on the federal income tax effects for property-casualty products.

Constant Spread and Full Value Targets

A constant spread is represented by a diagonal line running from the upper left to the lower right in Exhibit 2. The following pair of entries both represent a spread of 400 basis points.

- For a benchmark investment yield of 8% per annum and a target return on capital of 12% per annum, the target combined ratio is 109.2%.
- For a benchmark investment yield of 9% per annum and a target return on capital of 13% per annum, the target combined ratio is 110.3%.

With a constant spread of 400 basis points between the target return on capital and the benchmark investment yield, the target combined ratio increases by 100 basis points for a 100 basis point increase in the target assumptions.

Constant Spread and Discounted Targets

In Exhibit 2, the target combined ratio is expressed in nominal dollars. When interest rates increase, an insurer can afford to write business at a higher combined ratio. This does *not* mean that the insurer can afford to write the business at a lower premium rate, since the higher combined ratio is discounted at a higher rate.

Exhibit 3 shows the sensitivity of the discounted target combined ratio to the benchmark investment yield and the target return on capital.[12] For a constant spread of 400 basis points, a change in the pricing assumptions (return on capital and benchmark investment yield) is represented by a diagonal line through the center square.

- For a benchmark investment yield of 8% per annum and a target return on capital of 12% per annum, the target discounted combined ratio is 91.6%.
- For a benchmark investment yield of 9% per annum and a target return on capital of 13% per annum, the target discounted combined ratio is 91.2%.

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[11] SFAS 60, applicable to traditional life insurance products and to property-casualty products, treats premium as revenue. SFAS 97, applicable to universal life-type products and investment contracts, treats premium as a deposit.

[12] The discounted combined ratio uses discounted losses and expenses divided by discounted premiums.

Exhibit 3

SENSITIVITY ANALYSIS

LINE OF BUSINESS: **Workers' Compensation Fully Insured**

TARGET DISCOUNTED COMBINED RATIO

Post-Tax ROC = 12.0% +	Investment Rate of Return = 8.0% +										
	-250 bp	-200 bp	-150 bp	-100 bp	-50 bp	0 bp	+ 50 bp	+ 100 bp	+ 150 bp	+ 200 bp	+250 bp
-250 bp	93.0%	93.2%	93.5%	93.8%	94.2%	94.6%	95.0%	95.4%	95.9%	96.4%	97.0%
-200 bp	92.5%	92.7%	93.0%	93.3%	93.6%	93.9%	94.3%	94.8%	95.2%	95.7%	96.2%
-150 bp	92.0%	92.2%	92.4%	92.7%	93.0%	93.3%	93.7%	94.1%	94.6%	95.0%	95.5%
-100 bp	91.5%	91.7%	91.9%	92.2%	92.4%	92.8%	93.1%	93.5%	93.9%	94.4%	94.8%
-50 bp	91.0%	91.2%	91.4%	91.6%	91.9%	92.2%	92.5%	92.9%	93.3%	93.7%	94.2%
0 bp	90.5%	90.7%	90.9%	91.1%	91.4%	91.6%	92.0%	92.3%	92.7%	93.1%	93.5%
50 bp	90.1%	90.2%	90.4%	90.6%	90.8%	91.1%	91.4%	91.7%	92.1%	92.5%	92.9%
100 bp	89.6%	89.7%	89.9%	90.1%	90.3%	90.6%	90.9%	91.2%	91.5%	91.9%	92.3%
150 bp	89.2%	89.3%	89.4%	89.6%	89.8%	90.1%	90.3%	90.6%	90.9%	91.3%	91.7%
200 bp	88.7%	88.8%	89.0%	89.1%	89.3%	89.6%	89.8%	90.1%	90.4%	90.7%	91.1%
250 bp	88.3%	88.4%	88.5%	88.7%	88.9%	89.1%	89.3%	89.6%	89.9%	90.2%	90.5%

Even though the spread between the return on capital and the investment yield has not changed, the target discounted combined ratio declines by 40 basis points for a 100 basis point increase in the pricing assumptions. As the nominal investment yield increases, the insurance policy becomes less profitable. The decrease in profitability depends on the loss payment pattern of the line of business.

This effect stems from the federal income tax on investment income, and its magnitude varies directly with the taxability of the product. The policy reserves for life insurance and annuity products enjoy a tax deferred inside build-up, so this relationship is relatively weak. Similarly, mutual funds are not taxed at the corporate level; the investment income is passed through to investors. In contrast, the investment income of property-casualty insurers is fully taxed.

Spread Pricing and Federal Income Taxes

We explain the intuition by a pair of illustrations. For the illustrations, we assume a 200 basis point differential between the inflation rate and the risk-free interest rate and a constant spread of 400 basis points between the target return on capital and the risk-free interest rate.

Illustration A: An insurer writes commercial liability lines of business and invests in Treasury securities. The premium to surplus ratio is 1.5 to 1, and the expected loss ratio is 66.7%. We assume first that the inflation rate is 0% per annum, the investment yield (the risk-free interest rate) is 2% per annum, and the target return on capital is 6% per annum.

- The nominal pre-tax investment yield is 2% per annum.
- The nominal after-tax investment yield is 2% × (1 – 35%) = 1.30% per annum.
- The real after-tax investment yield is {[1 + 2% × (1 – 35%)] / 1.000} – 1 = 1.30% per annum. Since inflation is 0% per annum, the real yield equals the nominal yield.
- The cost of holding capital is the target return on capital minus the after-tax investment yield, or 6.00% – 1.30% = 4.70%. This is the extra return demanded by equityholders.
- The premium to surplus ratio for the policy is 1.5 to 1. The policy must provide a 4.70% / 1.5 = 3.13% after-tax return on premium or a 3.13% / (1 – 35%) = 4.82% pre-tax return on premium. This is the profit loading on a discounted basis.
- The premium is collected at policy inception. The profit loading valued at the inception of the year is 4.82%/1.02 = 4.73%.

Illustration B: The scenario remains the same, but the inflation rate is 10% per annum, the investment yield is 12% per annum, and the target return on capital is 16% per annum.

- The nominal pre-tax investment yield is 12% per annum.
- The nominal after-tax investment yield is 12% × (1 – 35%) = 7.80% per annum.
- The real after-tax investment yield is {[1 + 12% × (1 – 35%)] / 1.100} = –2.00% per annum.

- The cost of holding capital is 16.00% – 7.80% = 8.80%.
- With a 1.5 to 1 premium to surplus ratio, the insurance policy must provide an 8.80% / 1.5 = 5.87% after-tax return or a 5.87% / (1 – 35%) = 9.03% pre-tax return.
- The profit loading valued at the inception of the year is 9.03%/1.12 = 8.06%

In both scenarios, the profit loading is the percentage of the premium that funds the cost of holding capital. The remainder of the premium funds the discounted losses and expenses. This profit margin is on a discounted basis, not a nominal basis. The difference in the profit loading is 8.06% – 4.73% = 3.33%.

If the spread between the target return on capital and the benchmark investment yield is 400 basis points, the profit margin in the premium must be 333 basis points higher when the investment yield is 12% than when the investment yield is 2%. This difference stems from the effects of corporate income taxes on nominal returns.

SENSITIVITY: SURPLUS ASSUMPTIONS

The target combined ratio is sensitive to the surplus assumptions. The surplus assumptions for casualty pricing models are generally implemented as leverage ratios: a certain percentage of premium and a certain percentage of losses.

The surplus assumptions affect the indicated premium through the cost of holding capital. If the company requires more capital from its equityholders, the dollar cost of holding capital increases. The multiple layers of taxation – on underwriting income and on investment income – provide a multiplier effect. For each additional dollar paid to equityholders, the IRS takes about $1.50, and the policyholder must pay about $2.50.

Exhibit 4 shows the sensitivity of the target combined ratio to the premium leverage ratio for the workers' compensation book of business in Appendix B of Feldblum and Thandi, [2002], "Modeling the Equity Flows." The boxed target combined ratio in the middle of the exhibit corresponds to a premium leverage ratio of 43.7% and a benchmark investment yield of 8% per annum. Moving up or down a column changes the premium leverage ratio by increments of 500 basis points.

- Adding 500 basis points to the premium leverage ratio reduces the target combined ratio by 60 basis points, from 109.2% to 108.6%.
- Subtracting 500 basis points to the premium leverage ratio increases the target combined ratio by 60 basis points, from 109.2% to 109.8%.

Two factors mitigate this sensitivity:

Exhibit 4

SENSITIVITY ANALYSIS

LINE OF BUSINESS: **Workers' Compensation Fully Insured**

Post-Tax Return on Equity	12.0%

TARGET NOMINAL COMBINED RATIO

Premium Leverage Ratio = 43.7% +	Investment Rate of Return = 8.0% +										
	-250 bp	-200 bp	-150 bp	-100 bp	-50 bp	0 bp	+ 50 bp	+ 100 bp	+ 150 bp	+ 200 bp	+250 bp
-1500 bp	105.4%	106.4%	107.6%	108.7%	109.9%	111.0%	112.3%	113.5%	114.8%	116.1%	117.4%
-1000 bp	104.7%	105.8%	106.9%	108.0%	109.2%	110.4%	111.7%	112.9%	114.2%	115.6%	116.9%
-500 bp	103.9%	105.1%	106.2%	107.4%	108.6%	109.8%	111.1%	112.4%	113.7%	115.0%	116.4%
0 bp	103.2%	104.4%	105.6%	106.7%	108.0%	109.2%	110.5%	111.8%	113.1%	114.5%	115.9%
+500 bp	102.5%	103.7%	104.9%	106.1%	107.3%	108.6%	109.9%	111.2%	112.6%	114.0%	115.4%
+1000 bp	101.8%	103.0%	104.2%	105.5%	106.7%	108.0%	109.3%	110.7%	112.1%	113.5%	114.9%
+1500 bp	101.1%	102.3%	103.6%	104.8%	106.1%	107.4%	108.7%	110.1%	111.5%	113.0%	114.4%

- For the long-tailed lines of business, only part of the equityholder provided capital is contained in surplus. Additional capital is embedded in the statutory loss reserves and unearned premium reserves. If loss reserves are held on a full value basis, the capital embedded in workers' compensation loss reserves significantly exceeds the capital in the workers' compensation risk-based capital requirements.
- To facilitate performance measurement and incentive compensation systems, the exhibits in Appendix B of Feldblum and Thandi, [2002], "Modeling the Equity Flows," allocate capital based on written premium, not based on held reserves; Feldblum and Thandi [2002] "Surplus allocation." The capital in surplus is held for a single year, whereas the capital embedded in full value loss reserves is held as long as the claim remains unpaid.

The other three discretionary assumptions in the pricing model – the target return on capital, the benchmark investment yield, and the loss reserve valuation rate – affect the policy throughout its life. The surplus assumption has only a partial effect for one year.

The effects of the loss reserve valuation rate on the target loss and combined ratios is more intricate, partly because of the tax effects caused by implicit discounting of reserves. See Feldblum and Thandi [2002] "Federal Income Taxes and the Cost of Holding Capital," for a complete discussion of this topic.

Financial Pricing Models for Property-Casualty Insurance Products: Investment Yields

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Financial Pricing Models for Property-Casualty Insurance Products: Investment Yields

by Sholom Feldblum and Neeza Thandi

The investment yield used in a financial pricing model greatly affects the indicated premium. The proper investment yield depends on the target return on capital. If the target return on capital compensates for both investment risk and insurance risk, the investment yield should be the pre-tax yield expected to be earned during the lifetime of the block of business; if the target return on capital compensates for insurance risk but not for investment risk, the investment yield should be a risk-free rate.

The Massachusetts models – that is, the underwriting beta model of Fairley and the risk adjustment discounted cash flow Myers/Cohn model – sought to disentangle investment risk from insurance risk, so that state could mandate uniform premium rates for all companies. The goal of separating investment risk from insurance risk has proved quixotic (Kozik [1994]); we use a target return on capital that reflects all risk of the insurance enterprise.

We speak of a benchmark investment yield; one may also speak of this as the expected yield adjusted for default risk, other investment risks, investment expenses, and federal income taxes. We review the major considerations for selecting a benchmark investment yield.

EXPECTED DEFAULTS VS DEFAULT RISK

The expected defaults from an investment portfolio are subtracted from the gross yield to determine the expected yield. We distinguish three types of adjustments for default risk.

- The expected defaults must be subtracted from the stated yield to get the expected yield.
- The risk that actual defaults will be higher than expected defaults is reflected in the target return on capital.
- The "worse case" default scenario is reflected in the risk-based capital requirements.

Illustration: The gross yield on a portfolio of fixed-income securities is 8% per annum. The expected annual defaults are 0.2%, and the expected return on defaulted bonds is 40¢ on the dollar. The expected annual cost of default is 0.2% × 60% = 0.12%. The expected net yield on the bond portfolio is 7.88% per annum.[1]

INVESTMENT EXPENSES

Common practice is to subtract the insurance expenses from the yield of the security, and to use the net yield in the pricing model. This is in contrast to underwriting expenses, where the expenses are modeled explicitly. The rationale is that investment expenses are relatively uniform by type of security and do not vary with the insurance environment. If investment expenses are 0.20% for investment grade corporate bonds, the investment yield used in the pricing model is net of this investment expense.

We begin this discussion with the effects of federal income taxes on investment yields, since accurate consideration of the effective tax rates is important for policy pricing.

FEDERAL INCOME TAXES AND INVESTMENT YIELDS

The return on capital is the after-tax return; this is the pre-tax investment yield times the complement of the tax rate. Taxes may be applied to investment yields by two methods.

- The pricing actuary uses the gross investment yield and adjusts the tax rate on investment income by type of security.
- The pricing actuary uses the pre-tax equivalent yield and the full marginal tax rate (35%).

The latter approach avoids different tax rates in the pricing algorithms and makes the exhibits cleaner. The illustrations in this paper show conversion factors for several types of securities.

Federal income tax rates provide incentives for personal investors to hold stocks and for insurers (both life and property-casualty) to hold bonds.

- *Bonds:* The marginal tax rate on Treasury securities and corporate bonds is 35%. The marginal tax rate on municipal bonds is 5.25%, but the yield on municipal bonds is about 70% to 80% of the yield on comparable corporate bonds, and the after-tax yield on the two types of bonds is about equal, after adjusting for callability and liquidity.[2]
- *Stocks:* The marginal tax rate on dividends is 14.175% because of the dividends received deduction and the proration provision. The marginal tax rate on capital accumulation, assuming a ten year average holding period and a 12% average annual gain, is 25%.[3] Assuming a split of 15% dividends and 85% capital gains, the marginal tax rate on common stocks is 15% × 14.175% + 85% × 25% = 23.38%.

The marginal tax rates for high-tax bracket personal investors for bonds and stocks are:

- *Bonds:* The marginal tax rate on taxable bonds is about 32% to 39% before the tax amendments of 2003 and about 31% to 36% after the tax amendments. We use a 35% marginal tax rate here.[4] The marginal tax rate on municipal bonds is 0%.

- *Stocks:* The tax rate on shareholder dividends is 35%. Individual investors have a 20% tax rate on long-term capital gains; the marginal tax rate on capital accumulation, assuming a ten year average holding period and a 12% average annual gain, is 13.5%.[5] Assuming a split of 15% dividends and 85% capital gains, the marginal tax rate on common stocks is 15% × 35% + 85% × 13.5% = 16.73%.

Property-casualty insurers have a higher *relative* tax rate on stocks than individual investors have and a lower *relative* tax rate on bonds. Individual investors have a higher percentage of their investments in common stocks, and insurers have a higher proportion in bonds.[6]

Tax Rates

Property-casualty insurers have a slightly higher effective tax rate on investment income than other corporate investors have and most individual investors (if only federal income taxes are considered). The marginal tax rate on investment income for insurers is 35%, just as for other corporations, but insurers pay a higher tax rate on tax exempt investment income.

Personal investors have strong incentives – both tax and non-tax – to prefer common stocks to bonds; life insurers have equally strong incentives to prefer bonds to common stock. For property-casualty insurers, there is a tax incentive to hold bonds (especially municipal bonds, 1
as explained below); the non-tax incentives are not compelling for either bonds or stocks.

PRORATION

For personal taxpayers and non-insurance company corporate taxpayers, municipal bond interest income is exempt from federal income taxes. Insurance companies do not receive the full exemption: the proration provision of the 1986 Tax Reform Act adds 15% of tax-exempt municipal bond income to regular taxable income for insurance company taxpayers.[7]

For an insurance company taxpayer in the regular tax environment, corporate bond income is taxed at a 35% rate. Municipal bond income is taxed at a 15% × 35% = 5.25% rate.

We can compare investment vehicles two ways. Given the pre-tax yield, we compare after-tax yields by taxpayer and asset class, or given the after-tax yield, we compare the *pre-tax equivalent yield* by taxpayer and asset class.

- For corporate taxpayers other than insurance companies, the factor to adjust the municipal bond yield to the pre-tax equivalent yield is 1/(1 – 35%) = 153.85%.
- For insurance company taxpayers, the factor to adjust the municipal bond yield to the pre-tax equivalent yield is (1 – 5.25%)/(1 – 35%) = 145.77%.

Illustration: For corporate taxpayers other than insurance companies, a 5% municipal bond yield is equivalent to a 7.69% pre-tax equivalent yield. For insurance company taxpayers, a 5% municipal bond yield is equivalent to a 7.29% pre-tax equivalent yield.

Municipal bonds are exempt from federal income taxes and from state taxes of their domestic state. For example, a municipal bond issued by Massachusetts is exempt from federal income taxes and Massachusetts income taxes, but it is subject to state income taxes from other states. This would seem to provide a strong incentive for high tax brackets individual investors, whose combined federal plus state marginal tax rate may exceed 45% (before the 2003 tax amendments) to hold municipal bonds.

For several reasons, individuals do not invest heavily in municipal bonds. First, the lower tax rate and the tax deferral on long-term capital gains makes stocks a better investment for individuals than bonds, whether corporate bonds or municipal bonds. Second, stocks may be traded in units of several hundred dollars; bonds are traded in units of tens of thousands of dollars – beyond the reach of most individual investors. Third, bonds lack the pizzazz which stocks have. Large, unexpected gains in the bond markets are rare; large gains in the stock market occur frequently. Common stocks are a mix of investment and gambling; this enhances their appeal to individuals, though it somewhat dampens their appeal to corporations. Fourth, bonds are now packaged in *balanced funds* and sold to the average
8 individual investor. But municipal bonds appeal only to investors in tax brackets of about 32% or higher. Since balanced funds are geared to the average investor who may have a marginal tax rate of 30%, they rarely include municipal bonds. Fifth, only investment houses with a large clientele can offer well diversified municipal bond funds.[8] The greater appeal of municipal bonds for property-casualty insurers than for personal investors affects investment strategy.

Municipal Bonds

Before the Tax Reform Act of 1986, property-casualty insurers had a tax rate of approximately 46% on corporate bonds and 0% on municipal bonds. If corporate bonds were yielding 10% per annum, municipal bonds of similar investment grade (and absent other differences) could attract property-casualty insurers with rates as low as 5.4% per annum.

Life insurers, annuity writers, and pension plans, with tax deferral or exemption on their investment income, had little use for municipal bonds. Bonds (whether corporate or municipal) are not commonly held by non-insurance companies, since they serve no business purpose; for insurance companies, bonds are used to back reserves.[9] As long as the yield on municipal bonds is higher than the after-tax yield on corporate bonds, demand for municipal bonds by property-casualty insurers and high tax bracket personal investors is high.[10] Individual investors in the highest tax brackets faced marginal tax rates as high as 50% before the Reagan tax reductions of the early 1980's.[11] Until the past two decades, it was difficult for individuals to invest in bonds, since even small trades involved thousands of dollars. In contrast, stock trades of a few hundred dollars are simple, though high commission rates discourage them. The lower marginal tax rates on long-term capital gains and the trading difficulties led most individuals to invest in common stocks.

After 1986, the 35% tax rate faced by corporate taxpayers is again roughly the same as the personal tax rate faced by high income individuals. Property-casualty insurers lost 5.25% of the tax rate differential between corporate bonds and municipal bonds, rendering municipal bonds a slightly less efficient investment vehicle for them.[12]

Nevertheless, municipal bonds remain major components of insurance company investment portfolios.[13] Property-casualty insurers are the largest holders of municipal bonds, which constitute 46% of the aggregate industry portfolio. The current ratio of municipal bond yields to corporate bond yields is well above 68%, making municipal bonds a logical investment.[14]

Taxes and Invested Capital

If insurers held no capital – that is, if insurers held fair value reserves and no surplus, there were no underwriting (systematic) risk, and there were no costs of bankruptcy – the expected pre-tax income during the policy term would be zero, since the fair premium equals the present value of expected losses and expenses. In each year afterwards, the amortization of the interest discount in the reserves should offset the investment income, and pre-tax income would again be zero.[15]

But property-casualty insurers hold capital, either explicitly in surplus or implicitly in the gross unearned premium reserves and the full value loss reserves. The investment income on this capital is not offset by amortization of the interest discount in the loss reserves, and this investment income creates positive taxable income. Most investments are bonds and other fixed-income securities, common stock, and preferred stocks; state regulation prevents too much of the investment portfolio in other securities, such as venture capital. If the after-tax yield on municipal bonds for property-casualty insurance company taxpayers exceeds the after-tax yield on comparable corporate bonds, the insurer may hold part of its portfolio in municipal bonds. The maximum amount invested in municipal bonds is constrained by the alternative minimum income tax.[16]

For the insurance industry as a whole, the ratio of municipal bonds to capital is $237,079 million / $289,606 million = 81.86%. Some insurers do not hold municipal bonds, either because they prefer more aggressive investments, such as real estate, venture capital, high yield bonds, and common stocks, or because their have operating loss carryforwards that eliminate their expected taxable income. Other insurers hold enough municipal bonds that their alternative minimum taxable income is about 175% of their regular taxable income.

Table §.x: Bond Yields (June 2003)

	Volume	10 yr yield	20 yr yield

Treasury	$1.5 trillion	3.34%	4.00%
Corporate A-rated	$1.8 trillion	4.10%	5.70%
Municipal A-rated	$1.0 trillion	3.15%	4.40%

Table §.x: Composition of 2001 Bond Portfolio, Property-Casualty Insurers ($000,000)

	Percent of Total	*Statement Values*	*Interest Earned*
Treasuries	18.0%	$92,972	$5,606
Municipal Bonds	45.9%	$237,079	$9,559
Corporate Bonds	35.6%	$183,878	$16,696
Affiliates	0.5%	$2,583	$96
Total	100.0%	$516,511	$31,957

Source: Best's *Aggregates and Averages*, 2002.

In June 2003, the average yield on A-rated 20 year municipal bonds was 4.4%, which is 77% of the average yield on comparable grade corporate bonds (5.7%). For non-insurance company taxpayers, the pre-tax equivalent 20 year municipal bond yield is 4.4% / (1 – 35%) = 6.77%; for insurance company taxpayers, the pre-tax equivalent 20 year municipal bond yield is 4.4% × (1 – 5.25%) / (1 – 35%) = 6.41%.

Municipal bonds generally yield about 70% to 80% of the pre-tax yield on corporate bonds of comparable quality; from a pure tax analysis, they should yield 65% to 68.60%. Several other items affect the relative yields on corporate vs municipal securities:

- *Callability:* Nearly all municipal bonds are callable; few corporate bonds are now callable. When bond yields declined in the 1980's, many corporate bonds were called; investors began demanding higher call premiums, making the call option expensive. When interest rates continued to fall, corporate issuers saw little need to include call provisions. Municipal bonds continued to use call provisions, and their yields reflect this option.[17]
- *Liquidity:* Municipal bonds are less liquid than corporate bonds and much less liquid than Treasuries, perhaps lowering their market values and raising their yields.[18]
- *Tax legislation:* In 1986, the proration provision reduced the tax advantage of municipal bond for insurance company taxpayers (though bonds acquired before August 8, 1986, were grand-fathered). Investors may believe that the interest income on corporate bonds may receive some tax reduction from the increasingly Republican Congress, reducing the relative tax advantage of municipal bonds.

COMMON STOCK DIVIDENDS

Stockholder dividends received by corporate taxpayers are partially exempt from federal income tax, to avoid triple taxation of a single income flow.

- *Double* taxation of common stock dividends is the imposition of both corporate income taxes and personal income taxes on the same income. Dividends are paid from after-tax corporate earnings, but they are taxed again when they are received by the equityholders.
- *Triple* taxation of dividends received by one company from another is the imposition of two layers of corporate income tax and one layer of personal income tax on the same income.

Illustration: Company A earns $10 million, which it pays to its shareholders. Company B, which owns 1% of company A, pays its earnings to its shareholders, who have an average personal tax rate of 32%. Company A pays federal income taxes on its $10 million of earnings at a 35% rate; the remaining $6.5 million is paid to its owners, of which Company B receives $65,000. Were there no dividends received deduction, company B would pay federal income taxes on the $65,000 at the regular tax rate of 35%, or $6,500 × 35% = $2,275. The remaining money, $4,225, is paid to the owners of company B, who pay personal income taxes at a 32% rate, or $4,225 × 32% = $1,352. The net income received is $4,225 – $1,352 = $2,873. The total effective tax rate is 1 – $2,873/$10,000 = 71.27%.

Since such high marginal tax rates degrade economic efficiency, the Congress enacted the dividends received deduction, which exempts 70% of common stock dividends received from taxation. For non-insurance company corporate taxpayers, the effective tax rate on dividends from unaffiliated entities is 30% × 35% = 10.50%.[19] For insurance company taxpayers, the proration provision adds 15% of the tax-exempt income to regular taxable income; the tax rate on dividends from non-affiliated entities is *(30% × 35%) + (70% × 15% × 35%) = 14.175%.*

The factor to adjust the dividend yield to the tax equivalent yield is (1 – 14.175%)/(1 – 35%) = 132.04%. A 2% dividend yield is equivalent to a 2.64% pre-tax equivalent yield. The effective tax rate for the three layers of tax (two corporate and one personal) is 1 – (1 – 35%) × (1 – 14.175%) × (1 – 32%) = 62.07%.[20]

CAPITAL GAINS

Capital gains received by corporate taxpayers are subject to a 35% tax rate when they are realized; taxes are not paid on unrealized capital gains. The value of the tax deferral of the unrealized capital gains is inversely related to the stock turnover rates.[21] If the stocks are traded frequently, there is little value to the tax deferral. If the stocks are intended to be held indefinitely, the value of the tax deferral is large.[22]

The expected capital accumulation is the expected stock yield minus the expected dividend. For high income personal taxpayers, the tax rate on long-term capital gains of 20% is lower than the tax rate on regular income. Many common stocks now pay little dividends, and the average capital accumulation is about 1 to 1½ percentage points below the stock yield.[23]

Illustration – High Turnover: A stock portfolio worth $1,000 has an expected capital accumulation yield of 10% per annum and a turnover rate of 25% a year. To simplify, we assume that each stock is held for four years and then sold to realize the capital gains.[24] At the end of four years, the stock are worth $\$1,000 \times 1.100^4 = \$1,464.10$. The pre-tax income is \$464.10, and the after-tax income is $\$464.10 \times (1 - 35\%) = \$301.67$. The annual after-tax yield necessary to achieve this income is $(1 + \$301.67 / \$1000)^{¼} - 1) = 6.81\%$ per annum. The pre-tax equivalent yield is 6.81% / (1–35%) = 10.48% per annum.

Illustration – Low Turnover: With a turnover rate of 5% per annum, the stocks are held for 20 years (on average). At the end of twenty years, they are worth $\$1,000 \times 1.100^{20} = \$6,727.50$. The pre-tax income is \$5,727.50, and the after-tax income is $\$5,727.50 \times (1 - 35\%) = \$3,722.88$. The annual after-tax yield necessary to achieve this income is $(1 + \$3,722.88 / \$1000)^{0.05} - 1 = 8.07\%$ per annum. The pre-tax equivalent yield is 8.07% / (1–35%) = 12.42%.

The implications of the differential tax rates on common stocks are:

- Individual investors should prefer growth stocks with high capital accumulation rather than income paying stocks with high dividends.
- Non-insurance corporate taxpayers should prefer income paying stocks to growth stocks.
- Property-casualty insurers are between individual and corporate taxpayers.

A deferral of the tax liability increases the effective pre-tax equivalent yield for a given stated yield. The difference between the effective yield and the stated yield varies with the length of the tax deferral, or the holding period of the securities.

We write $E = f(Y, T, L)$, where E = pre-tax equivalent yield, Y = pre-tax stated yield, T = tax rate, L = length of the deferral period. The partial derivative of E with respect to each of the input variables is positive: $\partial E/\partial Y > 0$, $\partial E/\partial T > 0$, and $\partial E/\partial L > 0$.

- *Stated yield (Y):* The higher the stated yield, the higher is the pre-tax equivalent yield.
- *Tax rate (T):* The higher the tax rate, the greater is the gain from tax deferral.[25]
- *Length of deferral:* The longer the deferral, the greater is the gain from deferral.

Illustration: An insurer can invest \$100 million in bonds yielding 10% per annum or in common stocks that pay no dividends but that will increase in value by 10% per annum. To avoid issues of investment risk, we assume that both investments are risk-free.

One Year Holding Period: During the first year, the bond interest is \$10 million. The federal income tax liability is \$3.5 million, and the increase in statutory surplus is \$6.5 million. The after-tax investment yield is 6.5% per annum.

During the first year, the change in the stock value is +\$10 million. The deferred tax liability is \$3.5 million, so the change in the statutory balance sheet is a \$6.5 million increase in surplus. For a one-year holding period, the after-tax investment yield is 6.5% for the stocks.

Although the change in surplus is the same for the two investments in the first year, the cash flows are not the same. The stock scenario has an extra \$10 million in marketable securities, partly offset by a \$3.5 million non-cash deferred tax liability.[26]

For a multi-year scenario, the yields on the bonds and the common stocks are not identical. For the two year scenario, we assume the interest income is reinvested in the same bonds.

Two Year Holding Period: The bond portfolio pays interest of \$10 million the first year; \$3.5 million is paid to the U.S. Treasury; and \$6.5 million is reinvested in the bond portfolio. During the second year, the bond portfolio pays interest of \$106.5 million × 10% = \$10.65 million; \$10.65 million × 35% = \$3.7275 million is paid to the U.S. Treasury; and \$10.65 million × 65% = \$6.9225 million is reinvested in the bond portfolio. The totals bonds after two years is \$106.5 million + \$6.9225 = \$113.4225. This is a 6.5% annual yield, since $1.065^2 = 1.134225$. The pre-tax equivalent yield is 6.5% / (1 – 35%) = 10% per annum.

The common stock portfolio appreciates to \$110 million the first year. Nothing is paid to the U.S. Treasury, and the company sets up a \$3.5 million deferred tax liability on its balance sheet. During the second year, the common stock portfolio appreciates to \$121 million. Nothing is yet paid to the U.S. Treasury, and the company increases the deferred tax liability to \$21 million × 35% = \$7.35 million. The net common stock asset is \$121 million – \$7.35 million = \$113.65 million. This is a 6.607% annual yield, since $1.06607^2 = 1.1365$. The pre-tax equivalent yield is 6.607% / (1 – 35%) = 10.16%.

The effect of the tax deferral on the effective investment yield is small for a short holding period and greater for longer holding periods. Let y be the pre-tax yield, r be the tax rate, and h be the holding period in years. The pre-tax return after h years is $(1+y)^h - 1$; the after-tax return after h year is $((1+y)^h - 1) \times (1 - r)$; the equivalent after-tax yield is $(((1+y)^h - 1) \times (1 - r) + 1)^{1/h} - 1$; and the pre-tax equivalent yield is $((((1+y)^h - 1) \times (1 - r) + 1)^{1/h} - 1)/(1-r)$. For example, the pre-tax equivalent yield of a 30 year 8% annual bond to fund a pension liability with taxes deferred until its maturity is $((((1.08)^{40} - 1) \times (1 - 0.35) + 1)^{0.025} - 1)/(1-0.35) = 10.63\%$.

GROWTH VS INCOME STOCKS

We compare the marginal tax rates on growth stocks vs income stocks for personal investors, insurers, and non-insurance companies. Growth stocks, such as high-tech firms, provide little or no dividends but high capital accumulation. Income stocks, such as municipal utilities, pay regular dividends. For the illustration below, we assume the growth stock pays no dividends but has an average growth rate of 12% per annum, and the income stock pays an 8% dividend yield and grows at 4% per annum. We consider two turnover rates: 10% per annum and 20% per annum. We assume that investors are in a high tax bracket with a 35% marginal tax rate; this is reasonably accurate and it simplifies the comparison with corporate investors.[27]

Personal investors pay the full 35% tax rate on dividends and a 20% rate on realized capital gains. The 8% pre-tax dividend yield is an 8% × 65% = 5.20% after-tax yield. The after-tax yield from capital accumulation of 12% or 4% in a 5 year or 10 year horizon is

- 12% with 5 year holding period: $((1.12^5 - 1) \times 0.80 + 1)^{0.2} - 1 = 9.991\%$
- 12% with 10 year holding period: $((1.12^{10} - 1) \times 0.80 + 1)^{0.1} - 1 = 10.380\%$
- 4% with 5 year holding period: $((1.04^5 - 1) \times 0.80 + 1)^{0.2} - 1 = 3.248\%$
- 4% with 10 year holding period: $((1.04^{10} - 1) \times 0.80 + 1)^{0.1} - 1 = 3.305\%$

The corporate investor has a 35% marginal tax rate on capital gains; the after-tax yields are

- 12% with 5 year holding period: $((1.12^5 - 1) \times 0.65 + 1)^{0.2} - 1 = 8.382\%$
- 12% with 10 year holding period: $((1.12^{10} - 1) \times 0.65 + 1)^{0.1} - 1 = 9.007\%$
- 4% with 5 year holding period: $((1.04^5 - 1) \times 0.65 + 1)^{0.2} - 1 = 2.670\%$
- 4% with 10 year holding period: $((1.04^{10} - 1) \times 0.65 + 1)^{0.1} - 1 = 2.754\%$

The marginal tax rate on stockholder dividends is 10.50% for non-insurance companies and 14.175% for insurance companies; the 8% pre-tax yield is an 8% × 89.5% = 7.160% for non-insurance companies and 8% × 85.825% = 6.866% for insurance companies.

	Five year holding period			Ten year holding period		
	Personal	Insurer	Corporate	Personal	Insurer	Corporate
Growth	9.99%	8.38%	8.38%	10.38%	9.01%	9.01%
Income	8.45%	9.54%	9.83%	8.51%	9.62%	9.91%
Difference	1.54%	-1.15%	-1.45%	1.88%	-0.61%	-0.91%

A personal investor should prefer the growth stock, whose after-tax yield is 150 to 190 basis points higher. Corporate investors should prefer income stocks, whose after-tax yield is 60 to 115 basis points higher for insurers and 90 to 145 basis points higher for other companies.

ACCOUNTING ISSUES

Several accounting issues affect the investment yield used in a pricing model. We discuss below (i) asset allocation, (ii) the investment generation method, (iii) books yields vs new money yields, and (iv) the valuation of securities.

ASSET ALLOCATION

Some life insurance companies allocate assets to product line by means of segregated accounts. Property-casualty insurers do not allocate specific assets with segregated accounts, though they may use a *nominal* allocation of assets.

- *Actual asset allocation:* Specific assets are purchased to support specific product lines.
- *Nominal asset allocation:* The asset returns are ascribed to specific lines for pricing.

Illustration: Some insurers allocate fixed-income securities to back loss reserves and equities to back capital and surplus funds. The fixed-income securities may be further allocated by line of business so that the durations of the bonds match the payment pattern of the liabilities.

If assets are allocated by line of business, the pricing actuary may use a weighted average benchmark investment yield for each line. Alternatively, the pricing model may apply the appropriate investment yield to each class of investments.

Illustration: An insurer uses intermediate term fixed-income securities with an average yield of 8% per annum to back its workers' compensation loss reserves and common stocks with an average pre-tax equivalent yield of 12% per annum to back surplus funds. The reserves to surplus ratio for workers' compensation is three to one.

- The pricing actuary may use a weighted average investment yield of ¼ × (3 × 8% + 1 × 12%) = 9% applied to all the investable assets in the pricing model.
- The pricing model may apply the 8% investment yield to assets backing loss reserves and the 12% investment yield to assets backing surplus.

INVESTMENT GENERATION APPROACHES

Permanent life insurance products have premium inflows over multiple years. Life insurance actuaries often use investment year techniques to model the interest earned in each calendar year. Policyholder funds received in year X receive the year X investment yield; policyholder funds received in year X+1 receive the year X+1 investment yield; and so forth.

The investment generation method is useful for performance measurement, which tracks the actual investment yield. For prospective pricing, the current investment yield is generally the best estimate off future investment yields, and the investment generation method adds little.

For pricing a single policy, we assume that all premium is received at policy inception and earns the same investment yield. Only for performance measurement of a cohort of policies would the investment generation method be appropriate.

BOOK YIELDS AND NEW MONEY YIELDS

The investment yield is the market yield (the *new money rate*). Portfolio yields, book yields, and amortized yields are accounting constructs that have no place in a financial pricing model.

Illustration: A 10% coupon bond has five years remaining to maturity, a $1,000 par value, a $900 market value, and a $950 amortized value. Assume that the expected market value one year hence is $920, and the amortized value one year hence is $960.

- The coupon yield is $100 / $1,000 = 10%.
- The current yield is $100 / $950 = 10.53%.
- The amortized yield is ($100 + $10)/$950 = 11.58%.
- The market yield is ($100 + $20) / $900 = 13.33%.

Some analysts use average investment yields instead of new money investment yields. The IRS uses a 60 month rolling average of federal mid-term rates to set the discount rate for tax basis loss reserves. Similarly, Harwayne [1979], page 380, uses a five year average investment yield for determining the investment income on policyholder supplied funds.[28]

An average yield may be preferable if investment yields fluctuate widely; a new money rate is preferable if investment yields are more stable. Bond yields are relatively stable, and new money rates are proper. Stock returns fluctuate greatly. For stock returns we use an expected return based on a return factor model, such as the CAPM; we do not use the most recent actual return or even an average return over the past several years.

VALUATION OF SECURITIES

The implied equity flows depend on statutory accounting principles. For determining statutory capital requirements, assets are valued according to statutory rules.

For newly purchased assets, the statutory accounting value equals the market value, whether or not the assets are bought in the primary market or the secondary market, and whether or not the assets are purchased at par. The statutory amortization of fixed-income securities are not relevant to prospective policy pricing.[29]

The statutory valuation of bonds at amortized value and of stocks at market value affects the investment portfolio mix. Common stock values fluctuate with market movements, and the fluctuations directly affect statutory surplus. The market values of long term bond fluctuate with interest rate movements, but these fluctuations are not shown in statutory financial statements. Insurers who seek stable assets values on their balance sheets have a preference for bonds.

States may impose limits by asset class. Some states limit the investments in derivative securities or restrict the statement values of common stock investments to a percentage of policyholders' surplus. These restrictions do not affect the pricing actuary's work.

INVESTMENT STRATEGY

Insurers differ in their investment strategies. One insurer might invest in Treasury bonds and investment grade corporate bonds with an average investment yield of 7% per annum. A second insurer might invest in lower grade corporate bonds plus common stocks, real estate, and venture capital, for an average investment yield of 12% per annum.

Different investment yields give different indicated premiums, if all other parameters were kept the same. Higher investment yield generate a lower policy premium. But this is valid only if the other pricing assumptions are not changed. The target return on capital depends on both underwriting risk and investment risk. A change in the investment strategy means a change in the mix of securities by asset class, which is offset by a change in the target return on capital. A riskier investment strategy necessitates a higher target return on capital to induce investors to fund the insurance operations. If the pricing parameters are chosen correctly, the added investment return is just offset by the higher target return on capital.

The target return on capital compensates for both investment risk and underwriting risk; we have no method of disentangling the two types of risk. We have no objective measure of insurance risk, and we don't even have an accepted measure of investment risk, so we do not quantify the relationship between the target return on capital and these two types of risk.[30]

Illustration: An insurer investing in Treasury bonds and investment grade corporate bonds with an average investment yield of 7% per annum might have a target return on capital of 10% per annum, whereas an insurer investing in lower grade corporate bonds plus common stocks, real estate, and venture capital, for an average investment yield of 12% per annum, might have a 15% target return on capital.[31]

The fair value of the insurance liabilities does not depend on the assets held by the insurer.[32] Similarly, the price of the policy generating the liabilities does not depend on the assets held.

If two investment strategies with different expected returns are equally appropriate for the insurance company, the higher target return on capital should just offset the higher investment yield. If the investment strategy is not sound, the indicated premium rate may change.

Illustration: A workers' compensation carrier invests only in Treasury bills and cash equivalents. If the insurer has no sound rationale for its conservative investment portfolio, the return on capital demanded by its equityholders will not decline sufficiently to offset the lower investment yield, and the indicated premium rate will rise.

COMPANY VERSUS INDUSTRY YIELDS

The benchmark investment yield may be determined as a weighted average of the new money investment yields on either the company's targeted investment portfolio or the industry's aggregate investment portfolio. A company pricing its own business might use a company specific benchmark investment yield. An industry aggregate investment yield would be appropriate for a regulatory pricing model or for a rating bureau's pricing model.[33]

Some actuaries use risk-free interest rates for insurance pricing models, citing the Fairley formula and the Myers/Cohn discounted cash flow model, which assume that the insurer invests solely in Treasury securities; see Woll [1987] for an early example.[34] But these two models use a cost of capital equal to the risk-free interest rate adjusted for underwriting risk only, not for investment risk. This procedure was designed to meet the specifications set by Dr James Stone, the 1976 Massachusetts Commissioner of Insurance. This procedure is (perhaps) appropriate for state-made rates in a non-competitive environment such as Massachusetts; it is not appropriate for company pricing models in other states.[35]

In a return on capital pricing model, both investment risk and underwriting risk are reflected in the capital requirements and the cost of equity capital. The replacement of actual company expected yields with risk-free yields or risk adjusted yields is a duplicate offset and overstates the premium rate indications. If one wishes to use a risk-free investment yield, one must divide the risk of insurance operations into underwriting risk and investment risk.

ILLUSTRATION: BENCHMARK INVESTMENT YIELD

A company's investment portfolio consists of the following asset classes, with their expected yields and weights. The table shows three entries for each asset class:

- The weights in the target investment portfolio.
- The actual pre-tax yield.
- The pre-tax equivalent yield.

Asset Class		Percentage of Portfolio	Market Yield	Pre-Tax Equivalent
Fixed-Income Securities				
Five year Treasury notes		6%	5.0%	5.00%
Mortgage backed securities and CMO's		22%	6.3%	6.30%
Corporate and foreign bonds		32%	6.4%	6.40%
Municipal bonds		4%	4.32%	6.30%
Equities				
Common stocks	– Dividend yield	23%	2%	2.64%
	– Capital accumulation		10%	11.07%
Venture capital – Capital accumulation		5%	15%	18.21%
Real estate – Rental Income		8%	6%	6.00%
	– Capital Accumulation		5%	5.55%
Total (weighted average yields)		100%	8.30%	8.97%

The relatively high percentage of equities illustrates the difference between the risk-free rate and the company's investment yield. The expected new money pre-tax yields are shown in the column captioned *actual yields.* The equivalent pre-tax yields would provide the same after-tax income were these investments fully taxed at a 35% marginal rate.

Fixed-Income Securities: The company holds 64% of its investment portfolio in fixed-income securities; the industry average is between 75% and 80%.[36]

Treasuries: The company holds 6% of its investment portfolio in Treasury securities with maturities averaging about five years. Property-casualty insurers need liquid investments for catastrophe loss payments; routine claim payments are funded by premium inflows.

Mortgage-Backed Securities and Collateralized Mortgage Obligations are favored by some property-casualty insurers; they have little default risk, since the collateral is backed by federal agencies. Their yield depends on the prepayment risk: During the 1980's, the prepayment risk was high, and these securities had high yields. The low interest rates in the late 1990's and early 2000's reduces the prepayment risk.

Bonds: We have grouped corporate and foreign bonds together. In practice, we would use separate categories for investment grade corporate bonds, high yield corporate bonds, sovereign deby, and short term money market securities, since their yields and risks differ. The tax treatment is the same for all, so we do not subdivide them in this illustration.

Municipal bonds coupon income is exempt from federal income taxes. For non-insurance company taxpayers, the pre-tax equivalent yield is slightly above the yield on comparable corporate bonds; for insurance company taxpayers, the pre-tax equivalent yield is slightly lower.[37] The pre-tax equivalent yield here is 4.32% × (1 – 5.25%) / (1 – 35%) = 6.30%.

Equities: The company holds 36% of its investment portfolio in common stocks, venture capital, and investment real estate; the industry average is between 20% and 25%. The company's percentage partly reflects the rapid capital appreciation during the 1990's. State investment regulations sometimes prevent insurers from a high percentage of equities.

Common Stocks: The 12% yield is 2% dividends and 10% capital accumulation; the average turnover rate is 12.5% per annum. The pre-tax equivalent dividend yield is 2.0% × (1 – 14.175%) / (1 – 35%) = 2.64%. The pre-tax capital accumulation after eight years is 1.10^8 = 2.144, the tax is 1.144 × 35% = 0.400, and the after-tax accumulation factor is 2.144 – 0.400 = 1.743. The annual after-tax yield is $1.743^{(1/8)}$ = 1.072, and the pre-tax equivalent yield is (1.072 – 1) / (1 – 35%) = 0.1107, or 11.07%.

The pre-tax yield on common stocks is assumed here to be a 700 basis point premium over five year Treasury notes. The pre-tax equivalent yield is higher because of the dividends received deduction and the deferral of taxes on unrealized capital gains.

Venture Capital: The venture capital yield is all capital accumulation. Venture capital funds are invested for an average of twelve years per project before the gain (or loss) is realized.

The pre-tax yield after twelve years is 1.15^{12} = 5.350. The tax is 4.350 × 35% = 1.523, and the after-tax accumulation factor is 5.350 – 1.523 = 3.828. The annual after-tax yield is $3.828^{(1/12)}$ = 1.118. The pre-tax equivalent yield is (1.118 – 1) / (1 – 35%) = 0.1821, or 18.21%.

Investment Real Estate: The real estate is properties held as investments, not Home Office real estate.[38] The real estate is held for an average of 15 years before being sold.

The pre-tax yield after fifteen years is 1.05^{15} = 2.079. The tax is 1.079 × 35% = 0.378, and the after-tax accumulation factor is 2.079 – 0.378 = 1.701. The annual after-tax yield is $1.701^{(1/15)}$ = 1.036. The pre-tax equivalent yield is (1.036 – 1) / (1 – 35%) = 0.0555, or 5.55%.

We do not model the term structure of interest rates or the shape of the credit spread curve, and we do not assume any investments in derivative securities. These are important issues for investment strategy, but they are outside the scope of this paper.

The yields are net of investment expenses, and the yields on corporate bonds and municipal bonds are net of expected default rates. For instance, if the expected default rate on the corporate bonds in the portfolio is 0.5% per annum and the expected recovery upon default is 40¢ on the dollar, the gross yield is multiplied by *1 – 0.5% × (1 – 40%) = 99.7%.*

Expected default rates by investment grade and by type of fixed-income security are available from commercial investment houses.[39] The analysis of default rates and investment expenses would normally be done by the investment department, not by the pricing actuary.[40]

The weighted average pre-tax equivalent yield in this illustration is 8.973% per annum. This insurer has an aggressive investment portfolio, with a high percentage of its assets in equities (common stock, venture capital, and real estate). A more conservative investment strategy, with most assets in fixed-income securities, would show a pre-tax equivalent yield of about 6% to 8% per annum.

1 This does not mean that the bond portfolio is equivalent to a risk-free bond portfolio yielding 7.88% per annum. The actual bond portfolio has significant variance in its returns, which reduces the worth of these securities. The certainty equivalent of the bond portfolio is less than 7.88% per annum, assuming that investors are compensated for assuming default risk.

2 If the yield on municipal bonds is (1 – 35%) / (1 – 5.25%) = 68.60% of the yield on corporate bonds, the after tax yield is the same.

3 In ten years, one dollar accumulates to \$3.105848 at a 12% annual rate. The after-tax gain upon realization is 35% × (\$3.105848 – \$1) = \$1.368801. The after-tax investment yield needed to achieve this gain is $(\$1.368801 + \$1)^{0.1} - 1 = 9.0066\%$. This is a 25% tax rate on the pre-tax investment yield of 12%.

4 A more accurate (but complicated) analysis would include state income taxes. Municipal bonds are generally exempt from their domestic state's income tax, though the tax rules and the tax rates vary by state.

5 In ten years, one dollar accumulates to \$3.105848 at a 12% annual rate. The after-tax gain upon realization is 80% × (\$3.105848 – \$1) = \$1.684679. The after-tax investment yield needed to achieve this gain is $(\$1.684679 + \$1)^{0.1} - 1 = 10.3797\%$. This is a 13.5% tax rate on the pre-tax investment yield of 12%.

6 To be exact, we should include state income taxes and investment expenses in the analysis; see below.

7 More precisely, 15% of tax-exempt income is deducted from the loss reserves offset to taxable income. Municipal bonds bought before August 8, 1986, are *grandfathered* and exempt from the proration provision.

8 Vanguard has a municipal bond fund geared to high tax bracket investors; not all mutual fund families have municipal bond funds, though they generally have balanced funds.

9 Holding large cash reserves does not make sense for most corporations, since these funds incur double taxation; shareholders are better off receiving the funds as dividends or share repurchases. Companies in cyclical industries, such as auto manufacturing and airlines, sometimes hold large cash reserves.

10 The exposition here is simplified; we explain why property-casualty insurers provided the greatest demand for municipal bonds; we do not mean to say that municipal bonds were held only by property-casualty insurers.

11 In 1981, prior to Reagan's tax cuts, the top tax rate on investment (and most other) income was 70% (on taxable income over \$215,400 for married filing joint); the bottom rate was 14%. In 1982, after, or as a result of, Reagan's tax cuts, the top tax rate was 50% on income in excess of \$169,020; the bottom rate was 11%.

[12] Proration applies to life insurers as well, though since their bond holdings generally back policy reserves, the tax deferral makes municipal bonds inefficient investment vehicles. For example, suppose corporate bonds yield 10% per annum and equivalent grade municipal bonds yield 6.5% per annum. With no tax deferral and a tax rate of 35%, the two investments provide the same after-tax return. With a ten year tax deferral, the municipal bond still yields 6.5% after-tax. The corporate bonds yield $(1 + (1.1^{10} - 1) \times 65\%)^{0.1} - 1 = 7.37\%$ after-tax. The advantage of the tax deferral increases as the deferment lengthens.

[13] The incidence of the proration provision is difficult to judge. The Congress passed the proration provision to exact greater tax revenue from property-casualty insurers, who are the major purchasers of municipal bonds. Who actually pays the tax is unclear. If municipal bond yields do not change, insurers pay the tax. If municipal bond yields increase to offset the loss to insurers, the states pay the tax as a higher cost of debt capital.

[14] The 68.6% uses the proration provision for property-casualty insurers: $68.6\% = 65\%/(1 - 15\% \times 35\%)$.

[15] In practice, differences between tax basis and present value of liabilities creates minor taxable income.

[16] We are not implying that or capital funds should be invested in municipal bonds. Some insurers notionally allocate fixed-income securities to policyholder reserves and equities to capital and surplus funds. We are saying only that the maximum investment in municipal bonds depends on the capital held by the company; the more invested capital, the more taxable income, and more funds may be invested in municipal bonds.

[17] In theory, we could price the cost of the options embedded in municipal bonds, if we can make reasonable assumptions about interest rate volatility. In practice, the current interest rate environment in the United States of extremely low interest rates during a period of relative prosperity is unique. The other period of such low interest rates in the 1930's was a period of economic recession. Similarly, the low rates now in Japan reflect the recession in Japan.

[18] There is no accepted model for pricing liquidity; it is hard to know if it has much effect on yields. Most buyers of municipal bonds intend to hold them to maturity and do not need extra liquidity.

[19] A non-affiliated entity is an entity that is less than 20% owned by the taxpayer.

[20] Since the funds used to pay the tax on investment income come from policyholders, they are subject to tax on underwriting income. The effective tax rate is $1 - (1 - 35\%) \times (1 - 35\%) \times (1 - 14.175\%) \times (1 - 32\%) = 75.34\%$. For the commercial casualty lines of business, the effective tax rate on the profit margin in the policyholder premium is between 70% and 80% when both personal and corporate income taxes are included. This underscores the need to be cognizant of tax effects when pricing insurance products.

[21] Jeffrey [1995] examines the relation between the stock turnover rate and the effective tax rate.

[22] The turnover rate in the stock portfolio is the reciprocal of the average time that the stocks are held. If the annual turnover rate in the stock portfolio is $1/n$, the average stock is held for n years before being sold.

[23] To avoid the higher tax on dividends, corporations can use share repurchases. Alternatively, investors can produce virtual dividends by selling a fixed percentage of their shares each quarter. Either alternative produces a higher after-tax yield.

[24] A more accurate analysis would consider the distribution of stock holding periods. A 25% common stock turnover rate might be modeled as 25% of stocks have 1 year holding period; $(1 - 25\%) \times 25\% = 18.75\%$ have a 2 year holding period; $(1 - 25\%)^2 \times 25\% = 14.06\%$ have a 3 year holding period; and so forth.

[25] The higher the tax rate, the lower is the after-tax yield, but the higher is the pre-tax equivalent yield. One might suppose that the pre-tax equivalent yield must reach a maximum at some point, since when the tax rate is 100%, the investor gets no income, so the pre-tax equivalent yield must be zero. That is not correct; the pre-tax equivalent yield is not defined when the tax rate is 100%.

[26] The $10 million marketable security in the stock scenario is an investable asset, though it remains invested in the same security (the stock).

[27] The disparity between the growth stock and the income stock in this illustration is deliberately made stark, to show the effects of tax rates on investment strategy.

[28] "Five year average investment yields . . . is a reasonable reflection of the need for stability in considering that payment amounts are subject to substantial fluctuation and extend over long and fluctuating durations . . . A five year average investment return provides some stability . . ." Harwayne, Frank, "Restatement of the Consideration of Investment Income in Workers' Compensation Insurance Ratemaking," (Casualty Actuarial Society 1979 Discussion Paper Program), page 380.

[29] For retrospective analyses of profitability, statutory valuation rules are relevant. For the variance stemming from interest rate changes, the statutory valuation rules affect the economic value added. If bonds are held at amortized cost, the portfolio yield should be used instead of the market yield in each year.

[30] Some theorists have tried to quantify the systematic risk of underwriting using a CAPM-based approach (termed the Massachusetts method of calculating underwriting betas); see Fairley [1987], Kahana [1987], and Hill [1987]. This approach has not generally been successful; see Kozik [1994].

[31] We distinguish a change in investment strategy from a change in the investment yields. The effect of changes in asset class *returns* – investment *strategy* – are discussed below.

[32] Butsic uses this reasoning to argue that the loss reserve discount rate should not depend on the assets carried by the company.

[33] Some actuaries contend that the pricing assumptions should depend on industry figures; see Bault [19xx].

[34] See Fairley, William, "Investment Income and Profit Margins in Property-Liability Insurance: Theory and Empirical Results," *The Bell Journal of Economics* 10 (Spring 1979) pages 192-210, reprinted in J. David Cummins and Scott E. Harrington (eds.), *Fair Rate of Return in Property-Liability Insurance* (Boston: Kluwer*Nijhoff Publishing, 1987), pages 1-26, and Myers, Stewart and Richard Cohn, "A Discounted Cash Flow Approach to Property-Liability Insurance Rate Regulation," in J. David Cummins and Scott E. Harrington (eds.), *Fair Rate of Return in Property-Liability Insurance* (Boston: Kluwer*Nijhoff Publishing, 1987), pages 55-78.

[35] See Feldblum [discussion of D'Arcy and Dyer paper] for a full discussion of this issue.

[36] Life insurance companies hold larger percentages in fixed-income securities because of the need to perform asset adequacy analyses for many products; property-casualty companies do not have this restriction.

[37] We do not model any exemptions from state income taxes, since the statutes differ by jurisdiction.

[38] Real estate is more commonly held by life insurance companies than by property-casualty insurance companies, and it has been somewhat in disfavor after the early 1980's downturn in the real estate market. More recently, some property-casualty insurance companies are seeking higher yielding alternative investments, such as venture capital and real estate.

[39] Periodic studies are published by financial analysts, such as those by David Altman.

[40] A common assumption is that efficient markets lead to the same present value of investment yields from all fixed-income securities. If used too broadly, this assumption is not suitable for a pricing model. The yield on Treasury bills and notes is lower than the yields on investment grade corporate securities, even after adjusting for expected default rates. The demand for Treasury securities is particularly high because of their low risk (for overseas investors and certain institutional investors) and their use to fulfil margin requirements for derivative securities. Most insurance companies prefer higher yield corporate and mortgage backed securities, since the additional risk can be managed with sound investment strategy.

The Pricing of Commutations

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The Pricing of Commutations

by Sholom Feldblum

Pricing commutations is complex; one must consider cash flows, federal income taxes, deferred tax assets, capital requirements, the cost of holding capital, the required return on invested capital, and implied equity flows. Vincent F. Conner and Richard A. Olsen outline a pricing method in their 1991 *Proceedings* paper, "Commutation Pricing in the Post Tax-Reform Era," and Lee Steeneck has adapted their method for his CAS Exam 6 study note on commutations.[1] Conner and Olsen consider expected cash flows and taxes. This paper expands upon their method by considering the other pricing items listed above.

Some of Conner and Olsen's conclusions are counter-intuitive. The authors say on page 96:

> *In certain instances, the commutation price developed under this methodology can be negative. . . . In cases of reinsurance of long-tailed lines, such as workers' compensation, ... , negative commutation values can be expected frequently.*[2]

Presumably, a negative commutation price means that the primary insurer pays the reinsurer for the privilege of assuming back the loss liability. Before the commutation, the reinsurer has the reserve liability and the obligation to pay the claimant. By paying cash, the primary company gets to re-assume the liability and the obligation to pay the claimant.

A result this strange gives one pause. The Conner and Olsen paper deals with a complex issue – the handling of federal income taxes. Other items that must be considered are:

1. Conner and Olsen use after-tax interest rates. The better method is to use pre-tax interest rates for pre-tax revenues and expenditures and to explicitly model the federal income tax cash flows. This is particularly true for modeling the deferred tax assets stemming from IRS loss reserve discounting.
2. Conner and Olsen do not consider the statutory accounting requirements for long-tailed casualty full value loss reserves.
3. Conner and Olsen do not consider capital requirements for holding loss reserves, such as risk-based capital requirements and rating agency capital requirements.
4. Conner and Olsen do not consider the cost of holding capital, consisting of the double taxation of investment income on capital and surplus funds as well as any difference between the cost of equity capital and the company's investment yield.

Omitting the last three items understates the commutation price. This discussion provides a more complete analysis of commutation pricing in a post-RBC and post-codification era. It provides guidance for reinsurance actuaries pricing commutations, and it should help actuarial candidates understand the financial theory behind these transactions.

INVESTMENT YIELDS AND FEDERAL INCOME TAXES

Casualty actuaries are often berated for ignoring the federal income tax implications in their pricing analyses. Some actuaries, stung by this criticism, use after-tax interest rates, on the presumption that the lower discount rate accounts for the federal income tax liabilities.

The use of after-tax interest rate is as likely to obscure as to illuminate. It is not always a good proxy for the explicit modeling of taxes. Rather

a. Pre-tax cash flows should be discounted at pre-tax interest rates, and
b. The federal income tax cash flows should be modeled explicitly.

These principles apply to discounted cash flow pricing models [Myers and Cohn: 1987; Butsic and Lerwick, 1992]; Mahler [1985; 1998], internal rate of return models [Kahley and Halliwell, 1992], Robbin 1992, algorithm 7], and loss reserve discounting models [Butsic: 1988]. Butsic says:

> *We have demonstrated that the appropriate interest rate for reserve discounting under income taxation should be the same as that without taxes . . .* (page 177).
>
> *The discounting interest rate must be a pretax value* (page 183).

Similarly, Actuarial Standard of Practice No. 20, "Discounting of Property and Casualty Loss and Loss Adjustment Expense Reserves," paragraph 5.4.4, says:

> <u>*Effect of Income Taxes*</u> – *The actuary normally should use an interest rate or rates consistent with investment returns that are available before the payment of income taxes.*[3]

If used correctly, after-tax interest rates are not necessarily incorrect – but it is very hard to use them correctly. It is particularly hard to account for deferred tax assets and liabilities when using after-tax interest rates. Throughout this discussion, we use pre-tax interest rates and we explicitly model the federal income tax cash flows.

Conner and Olsen use an after-tax investment yield, and they back out the federal income tax credit stemming from loss reserve discounting. They note that this procedure is equivalent to using pre-tax discount rates if the IRS loss reserve discount factors are the same as the actuarial discount factors (page 92):

If the payment pattern and nominal interest rate used to determine the present value of the losses are identical to the factors used to develop the tax-basis discounted reserves, then the commutation price will equal the present value of the losses using the nominal interest rate.[4]

The apparent equivalence leaves out the effects of statutory accounting, capital requirements, and the cost of holding capital.

IDEALIZED ILLUSTRATION

If there were no surplus requirements and the insurer held fair value reserves, it could charge premiums equal to the present value of losses and expenses. The statutory requirement for full value loss reserves and the risk-based capital requirements necessitate equityholder funded capital to support the insurance business.

If there were no corporate income taxes, these regulatory requirements might have little effect on pricing. The equityholder provided capital is invested, and the equityholders receive the investment income.

> *Illustration:* Suppose that the owners (equityholders) of an insurance company provide $100 million of capital. In a world without taxes or insurance regulation, the insurance company invests this money in securities chosen by the equityholders and passes along the investment yield to them.

Federal income taxes change the analysis. The multiple layers of taxation give the IRS about 60% to 70% of the underwriting profit margin for long-tailed commercial lines of business.

This paper analyzes the cash flows and taxes related to claim commutations. We use a series of simple illustrations, so that the mathematics does not obscure the intuition. We then re-examine the illustrations in the Conner and Olsen paper and in the Steeneck study note.

STATUTORY ACCOUNTING

We begin with the implication of the statutory requirements for full value loss reserves.

SINGLE YEAR ILLUSTRATION

> *A loss is commuted on January 1, 20X2. The loss will be paid for $105,000 on December 31, 20X2. The risk-free interest rate is 5% per annum, and the federal income tax rate is 35%. No risk adjustments or risk loads are used.*

One might surmise that the appropriate commutation price is $100,000, or the present value of the loss payment. We examine the effects of federal income taxes to confirm Butsic's remarks on discount rates cited above.

We assume first that the insurer holds fair value reserves and that it does not need surplus to support the reserves. On January 1, 20X2, the insurer receives $100,000 in cash and sets up a loss reserve of $100,000. The two accounting entries offset each other, and there is no effect on income.

During the year, the $100,000 in cash is invested at the 5% per annum risk-free interest rate, yielding investment income of $5,000. On December 31, 20X2, the loss is paid for $105,000 and the loss reserve is taken down to zero. The additional incurred losses between January 1 and December 31 equals the paid loss plus the change in reserves, or $105,000 + ($0 – $100,000) = $5,000. The incurred loss offsets the investment income, and the taxable income during the year is zero.

We have not yet considered statutory accounting, capital requirements, and the cost of holding capital. This illustration puts all the cash flows into a single calendar year so that we can avoid the IRS loss reserve discount factors and their effect on the commutation price. The tax cash flows are crucial to commutation pricing, but we want to first examine the effects of statutory accounting and capital requirements without the complexities of the IRS discount factors.

NO CAPITAL REQUIREMENTS AND NO ADDITIONAL COST OF CAPITAL

We assume that the primary insurer must hold full value (undiscounted) loss reserves, but that it does not need to hold surplus to support the loss reserve. We initially assume that its equityholders are satisfied with a risk-free rate of return; we relax this assumption below. We examine the implications of a $100,000 commutation price.

On January 1, 20X2, the primary company receives $100,000 and it records a $105,000 statutory reserve to its books. The insurer is missing $5,000 of assets to back the statutory reserves. The shareholders of the insurance company contribute $5,000 on that date.

Investment income: The $105,000 is invested in risk-free securities yielding 5% per annum. During 20X2, the company earns 5% × $105,000 = $5,250.00 of interest income.

Incurred loss: The loss reserve is zero on December 31, 20X1 (before the commutation) and zero on December 31, 20X2, after the loss is paid.

- The statutory incurred loss in 20X2 is the paid loss plus the change in reserves, or $105,000 + ($0 – $0) = $105,000.
- The tax-basis incurred loss in 20X2 is the paid loss plus the change in discounted reserves, which is also 105,000 + ($0 – $0) = $105,000.

The components of the tax liability are as follows:

- The tax-basis underwriting income in 20X2 is the premium minus the paid loss, or \$100,000 – \$105,000 = –\$5,000.
- The investment income during 20X2 is \$105,000 × 5% = \$5,250.
- The net income to the company in 20X2 is \$5,250 – \$5,000 = \$250.
- The federal income tax on the net income is 35% × \$250 = \$87.50.

The net after-tax gain to the company in 20X2 is \$250 – \$87.50 = \$162.50. At the end of the year, the equityholders receive back their \$5,000 investment plus the net income of \$162.50. This is a return of \$162.50 / \$5,000 = 3.25%.

If there is no risk in the insurance operations, the shareholders expect a risk-free return of 5% per annum, not a return of 3.25%. The shareholders could get a 5% return by investing directly in risk-free securities instead of investing indirectly through the insurance company. The opportunity cost of the capital to the shareholders is 5% per annum, not 3.25% per annum.

For the primary insurance company to attract capital, it must provide a 5% rate of return to its shareholders. This means that the commutation price must be higher than \$100,000. We set the commutation price to be \$100,000 + *z*, and we solve for *z*. There are several changes to the example's cash flows and accounting entries.

A. The net underwriting income to the company during 20X2 is the earned premium minus the incurred losses, or \$100,000 + *z* – \$105,000 = *z* – \$5,000.
B. The investment income during 20X2 is \$5,250, since the company holds \$105,000 of assets to back the \$105,000 of reserves. Of this \$105,000, \$100,000+z is paid by the reinsurer and \$5,000–z is contributed by the equityholders of the primary insurer.
C. The cash received from the reinsurer is a pre-tax cash flow, and it is taxed in 20X2. The funds received from the equityholders are an after-tax cash flow, and they are not taxed.[5] The federal income taxes on underwriting income in 20X2 equals 35% × (\$100,000 + *z* – \$105,000) = 35% × (*z* – \$5,000) = 0.35z – \$1,750.
D. The \$105,000 of assets at the beginning of the year grow to \$110,250 by December 31, 20X2; the investment income is \$5,250. The company pays \$105,000 to the claimant and it pays taxes on the investment income of 35% × \$5,250 = \$1,837.50 to the U.S. Treasury. The total tax is \$1,837.50 + 0.35z – \$1,750 = 0.35z + \$87.50. The rest of the money is returned to the equityholders: this is \$110,250 – \$105,000 – \$87.50 – 0.35*z* = \$5,162.50 – 0.35*z*.
E. The company's shareholders expect a 5% annual return. This means that

$$(\$5,162.50 - 0.35z) / (\$5,000 - z) = 105\%$$
$$\$5,162.50 - 0.35z = \$5,250 - 1.05z$$
$$z = \$87.50 / 0.70 = \$125.00$$

The proper commutation price is not $100,000 but $100,125, or an increase of 0.125%. The difference is small, since the loss is paid one year after the commutation. As the subsequent illustrations show, the costs are substantial when we add the effects of statutory regulation and reserve requirements to long-tailed casualty lines of business.

The difference in the commutation price is in the right direction. The primary insurance company faces additional costs of double taxation on the cash contributed by equityholders.

INTUITION

We show the intuition for this result. The equityholders contribute $5,000 at the beginning of the year. The investment income is $250, and the tax on the investment income is 35% × $250 = $87.50. The reinsurer pays this tax by a profit margin in the commutation price. If the reinsurer did not pay this tax, the primary company would have no interest in the commutation. The commutation price is taxed as underwriting income, so the reinsurer must pay $87.50/(1–35%) = $134.62. The present value of this amount at policy inception is $134.62 / 1.05 = $128.21.

This is slightly too high, since the more that the reinsurer pays, the less must be contributed by the equityholders of the primary company. The money is not given to the equityholders until the end of the year. During the year, the money supports the full value loss reserves.

Since the reinsurer pays $100,125, the equityholder contribution is $4,875, not $5,000. The investment income is $4,875 × 5% = $243.75. The tax on the investment income is $243.75 × 35% = $85.31. To fund the double taxation on the investment income on the equityholder supplied funds, the margin in the commutation price must be $85.31 / (1–35%) = $131.25. This is the margin that would be needed at the end of the year. The margin needed at the beginning of the year, when the commutation is effected, is $131.25 / 1.05 = $125.00.

In multi-period illustrations, the money is paid to equityholders incrementally over the years. An algebraic formula for the commutation price becomes increasingly complex as the number of periods increases; spreadsheet pricing techniques are easier.

We examined the $100,125 commutation price indirectly by looking at the margin needed to fund the double taxation on the investment income on the equityholder provided capital. As a complementary perspective, we examine the direct cash flows for the commutation price.

The reinsurer pays $100,125, and the equityholders contribute $4,875.00. The underwriting income is $100,125 – $105,000.00 = –$4,875. The IRS takes –$4,875 × 35% = –$1,706.25. The underwriting income is negative, so the tax liability is negative. This is a tax refund, which is an offset against the federal income taxes on investment income.

The investment income earned during the year is 5% × \$105,000 = \$5,250.00. The IRS takes 35% × \$5,250 = \$1,837.50. The total tax liability is \$1,837.50 – \$1,706.25 = \$131.25.[6]

At the end of the year, the equityholders receive back their initial contribution plus the commutation price plus the investment income minus the loss payment minus the income taxes, or

$$\$4,875 + \$100,125 + \$5,250 - \$105,000 - \$131.25 = \$5,118.75.$$

The return on the invested capital is \$5,118.75 / \$4,875 – 1 = 5.00%.

Timing of the Equity Flows

Most of the cash flows and accounting flows for insurance transactions occur continuously. This is true for loss incurral and settlement, expenses, investment earnings, and premium earnings. Even if the cash flow for a particular event is discrete, such as the settlement of a loss, the expected cash flows are continuous.

An ideal pricing model would have continuous functions. Some life actuarial models use forces of mortality and interest. In practice, it is easier to work with discrete valuation dates, particularly for spreadsheet based applications.

The valuation dates and valuation periods reflect a compromise between accuracy and expediency. This discussion uses annual valuation periods and year-end valuation dates. More accurate results may be obtained with a quarterly pricing model.

When using discrete periods, some actuaries use present values of year end figures. For instance, Atkinson and Dallas [2000], argue that capital requirements are examined only at year end. At the beginning of the year, the company needs the present value of the year-end capital requirements, where the present value is calculated at the after-tax investment yield. Similarly, one might argue that the full value loss reserves need be held only at year end. At the beginning of the year, the company must hold the present value of the loss reserves.

This argument is not applicable to our illustration, for two reasons.

- Full value loss reserves are required at all times, not just at the end of the year. Insurers make quarterly disclosure of loss reserves to state regulators in Schedule X.
- We chose the January 1 commutation date simply to avoid the complexities of IRS loss reserve discounting. We change the illustration below to a commutation date of December 31, 20X1. On a December 31 commutation date, the insurer must surely hold full value loss reserves and risk-based capital requirements.

Reinsurance Regulation

If the reinsurer is subject to the same statutory accounting, capital requirements, and tax provisions as the primary insurer, the accounting for the reinsurer is the mirror image of the accounting for the primary company.[7] Just as the primary company incurs the cost of double taxation on equityholder supplied capital, the reinsurer saves the cost of double taxation on funds which its equityholders would otherwise have had to contribute. The additional premium for the commutation transfers the funds saved by the reinsurer to the primary company which now incurs the costs.

If the reinsurer is not domiciled in the U.S., and particularly if it is domiciled in a jurisdiction with less stringent reserve regulations and capital requirements, it may not be subject to the same costs as a U.S. domiciled primary insurance company. If the reinsurer is domiciled in the Bermudas, it does not save the costs of double taxation by a commutation. The reinsurer may not be willing to pay more than $100,000 for the commutation.

A commutation is feasible only if the benefits from the commutation outweigh any additional costs to the companies. These additional costs stem from statutory accounting, capital requirements, and federal income taxes, which may be incurred only by the primary company (or primarily by the primary company).

This should not be a surprise. Reinsurance with off-shore companies can be financially beneficial, even if there is no significant transfer of risk, as long as the reinsurance contract passes the SFAS 113 tests. Companies benefit from finite reinsurance transactions with off-shore reinsurers. Undoing the reinsurance with a commutation can be costly.

COST OF HOLDING CAPITAL

The illustration above assumes that shareholders are satisfied with a risk-free rate of return on their invested capital. Many actuaries assume that shareholders require a higher rate of return for capital supplied to insurance enterprises.[8]

For our illustrations in this discussion, we assume that investors in an insurance enterprise require a rate of return equal to 100 basis points less than the average return for publicly traded stock companies. In a CAPM perspective, this assumes that insurers investing in risk-free securities have a market beta of about 85% to 90%.

We assume the risk-free interest rate on Treasury bills is 5% per annum and the market risk premium is 8% per annum.[9] The return on capital demanded by insurance company stockholders is 5% + 8% – 1% = 12%. We solve for z using the following equation:

$$(\$5{,}162.50 - 0.35z) / (\$5{,}000 - z) = 112\%$$
$$\$5{,}162.50 - 0.35z = \$5{,}600.00 - 1.12z$$

$$z = \$437.50 / 0.77 = \$568.18$$

DOUBLE TAXATION AND THE COST OF HOLDING CAPITAL

If the equityholders are satisfied with a risk-free return, the double taxation of the equityholder provided capital is the only cost. If the equityholders demand a higher equity-type return, the additional premium is greater.

The tax on equityholder provided capital appears in two places.

- the tax on investment income on the equityholder provided capital
- the tax on underwriting income stemming from the commutation transaction.

Illustration: Suppose an insurer invests in Treasury bills yielding 5% per annum. The equityholders could obtain a 5% yield by investing in Treasury bills on their own. If they give their funds to the insurance company, the insurance company pays corporate taxes before remitting dividends to the equityholders. The yield received by the equityholders is (1 – 35%) × 5% = 3.25%. The difference of 5% – 3.25% = 1.75% must be paid by the policyholders.[10]

Were the policyholders to pay this amount directly to the equityholders, this would be the cost of double taxation. But this is not the actual cash flow. The policyholders pay this cost to the insurance company as part of the policy premium, which then remits these funds to the equityholders. This round-about flow of funds induces an additional layer of taxation on the underwriting profits. The cost to the policyholders is 1.75% / (1 – 35%) = 2.69% of the capital.

If the return on capital demanded by equityholders is 12% instead of 5%, the cost of holding capital is the required rate of return minus the after-tax investment yield, or 12% – (1 – 35%) × 5% = 8.75%.[11] Since the policyholders pay this though the policy premium, they subject the funds to an additional layer of taxation. The additional premium paid by the policyholders is 8.75% / (1 – 35%) = 13.46% of the capital provided by equityholders.

This cost in incurred at the end of the year. The policy premium is paid at the beginning of the year. The present value of this additional premium is 13.46% / 1.050 = 12.82%.[12]

CAPITAL REQUIREMENTS

State regulation imposes capital requirements that depend on the insurance operations. An increase in loss reserves causes an increase in the NAIC risk-based capital requirements.

The capital requirements depend on the line of business and the covariance adjustment, along with the capital philosophy of the insurer. Some insurers are satisfied with carrying capital only slightly in excess of the risk-based capital requirements; others set their target capital levels at a multiple of the risk-based capital requirements. Most well-rated property-casualty companies hold capital about twice their RBC requirements.

The marginal capital requirements from the commutation stem from the difference between the reserving risk charge (R_4) and the credit risk charge (R_3), which includes the charge for reinsurance recoverables.

The marginal effect of a change in an RBC charge on overall capital requirements is proportional to the size of that RBC charge in relation to the other RBC charges; see Feldblum [1996: RBC, pages 362-365]. For the average company, the reserving risk R_4 charge is about 10 times the size of the credit risk R_3 charge. Even for companies with large reinsurance recoverables, the remaining credit risk charge after half of the original amount has been transferred to the reserving risk charge is generally less than 10% of the final reserving risk charge (see the following paragraph). For the computations in this paper, we assume that the credit charge has 10% of the marginal effect as the reserving risk charge.

The 10% credit risk charge for reinsurance recoverables is split into two parts. Half the charge, or 5% of reinsurance recoverables, is transferred to the reserving risk category in the covariance adjustment. The other half remains in the credit risk category; it is equivalent to a ½ x 10% x 10% = 0.5% reserving risk charge. The combined charge is equivalent to a 5.5% reserving risk charge.

The reserving risk charge varies by line of business. We consider two scenarios: a commutation in a general liability line and a workers' compensation commutation.

The risk-based capital reserving risk charges for general liability, products liability, and medical malpractice are shown in the table below. The average reserving risk charge for these lines, weighted by industry premium volume, is 25.9% of held reserves.

Line of Business	Other Liability	Products Liability	Medical Malpractice
RBC Charge	52.0%	53.2%	56.5%
Interest Discount	83.2%	83.2%	80.8%
Reserving Risk Charge	26.46%	27.46%	26.45%
Unpaid Losses	$63,821 million	$10,423 million	$20,999 million
Claims-Made Percent			57.563%
Charge after C/M Offset			23.407%

The average reserving risk charge is

($63,821 x 26.464% + $10,423 x 27.462% + $20,999 x 23.407%) / $95,242 = 25.90%.

The workers' compensation reserving risk RBC percentage is 27.3% and the investment income offset is 87.2%. The reserving risk charge is [(1 + 27.3%) × 87.2%] – 1 = 11.0%.

The marginal effect of the reserving risk charge is reduced by the loss concentration factor and the covariance adjustment, reflecting the company's diversification.

- A monoline insurer gets no reduction from the loss concentration factor. A multi-line insurer with reserves split evenly among five different lines of business has a 24% reduction from the loss concentration factor.[13]
- The effect of the covariance adjustment depends primarily on the size of the written premium risk charge. It would be greatest for a property predominating insurer which writes a limited amount of workers' compensation or commercial liability business.

For most companies, the marginal effect of the reserving risk charge on overall capital requirements ranges from 60% to 80%, depending on their mix of business and the size of their other risk charges. A greater marginal effect increases the equityholder contribution and the commutation price.[14]

Few companies hold surplus just equal to their risk-based capital requirements. Most companies with Best's rating of A– or higher have risk-based capital ratios of 175% to 250%. For the illustrations here, we assume the company targets a risk-based capital ratio of 200%.

PRICING WITH FULL CAPITAL CONTRIBUTION

The average marginal risk-based capital charge from a commutation in general liability, products liability, or medical malpractice is (25.9% – 5.5%) × 60% × 200% = 24.48%. We round this to 25% to simplify the computations. The shareholders contribute 25% × $105,000 = $26,250 on January 1, 20X2, besides the $5,000 needed to fund the full value statutory loss reserves. The total contribution is $31,250.

If the commutation price equals the present value of the future loss payments, with no profit for the equityholders of the ceding company, the equityholders receive back their contribution at the end of the year plus the after-tax investment income, or

$$\$31{,}250 + 5\% \times \$31{,}250 \times 65\% = \$32{,}265.625.$$

This is a $32,265.625 / $31,250 – 1 = 3.25% return, which is inadequate.

For an adequate commutation price, the reinsurer pays $100,000+z. The company needs $131,250 of assets: $105,000 to back the statutory loss reserves and $26,250 as surplus capital. The equityholders provide $31,250 – *z*. We examine each cash flow:

A. The underwriting income in 20X2 is \$100,000 + z – \$105,000 = z – \$5,000. The tax on underwriting income is 35% × (z – \$5,000) = 35% × z – \$1,750.
B. The investment income is 5% × \$131,250 = \$6,562.50. The tax on investment income is \$6,562.50 × 35% = \$2,296.875.
C. On December 31, \$105,000 is paid to the claimant. The total tax paid to the U.S. Treasury is \$2,296.875 + 0.35z – \$1,750 = 0.35z + \$546.875. The rest of the money is returned to the equityholders; this \$131,250 – \$105,000 + \$6,562.50 – \$546.875 – 0.35z = \$32,265.625 – 0.35z.
D. The company's equityholders expect a 5% annual return. This means that

$$(\$32,265.625 - 0.35z) / (\$31,250 - z) = 105\%$$
$$\$32,265.625 - 0.35z = \$32,812.50 - 1.05z$$
$$z = \$546.875 / 0.70 = \$781.25$$

The RBC requirements raise the commutation price because they increase the tax liability of the primary company. The tax liability is proportional to the equityholders contribution, so the additional premium z is also roughly proportional to the equityholders contribution.

Illustration: If there are no surplus requirements, the equityholder contribution is \$4,875 and the margin in the commutation price is \$125. With a 25% reserving risk charge, the equityholder contribution is \$31,250 – \$781.25 = \$30,468.75 and the margin in the commutation price is \$781.25. As expected, \$4,875 : \$125 :: \$30,468.75 : \$781.25.

If equityholders require an equity-type return (vs. a risk-free return), the additional premium is

$$(\$32,265.625 - 0.35z) / (\$31,250 - z) = 112\%$$
$$\$32,265.625 - 0.35z = \$35,000 - 1.12z$$
$$z = \$2,734.375 / 0.77 = \$3,551.14$$

Workers' Compensation Commutations

Workers' compensation has a reserving risk charge of 11%. The additional charge from the commutation is 11% – 5.5% = 5.5%. The marginal capital requirements are 5.5% × 60% × 200% = 6.60%.

The equityholders contribute 6.6% × \$105,000 = \$6,930 on January 1, 20X2, to support the additional loss reserves, besides the capital embedded in the full value loss reserves.

- The after-tax investment income earned on the \$5,000 embedded in the full value loss reserves is 5% × \$5,000 × (1 – 35%) = \$162.50.
- The after-tax investment income earned on the \$6,930 capital requirement is 5% × \$6,930 × (1 – 35%) = \$225.225.

The offsetting of the investment income on the fair value (discounted) reserves against the incurred loss simplifies the calculations.

- The investment income on the fair value loss reserves is \$100,000 × 5% = \$5,000. The tax liability on this investment income is \$5,000 × 35% = \$1,750.
- The underwriting income during the year is \$100,000 + z – \$105,000 = z – \$5,000. The tax liability on the underwriting income is (z – \$5,000) × 35% = 0.35z – \$1,750.
- The net pre-tax income from these two pieces is \$5,000 + z – \$5,000 = z. The net after-tax income from these two pieces is 0.65z.

The equityholder provided capital on January 1 is \$5,000 + \$6,930 – z. The amount returned to the equityholders on December 31 is their capital contribution plus the after-tax net income of \$162.50 + \$225.225 + 0.65z.

If the equityholders require only a risk-free return, the equation is:

$$(\$5{,}000 + \$162.50 + \$6{,}930 + \$225.225 - z + 0.65z) / (\$5{,}000 + \$6{,}930 - z) = 105\%$$
$$(\$5{,}162.50 + \$7{,}155.225 - 0.35z) / (\$5{,}000 + \$6{,}930 - z) = 105\%$$
$$\$12{,}317.725 - 0.35z = \$12{,}526.50 - 1.05z$$
$$z = \$208.775 / 0.70 = \$298.25$$

If the equityholders require an "equity" return (versus a risk-free return), the equation is

$$(\$5{,}162.50 + \$7{,}155.225 - 0.35z) / (\$5{,}000 + \$6{,}930 - z) = 112\%$$
$$\$12{,}317.725 - 3.5z = \$13{,}361.60 - 1.12z$$
$$z = \$1{,}043.875 / 0.77 = \$1{,}355.68$$

For workers' compensation pension claims with tabular discounts, the commutation price is lower, since less capital is needed to back the discounted reserve. We discuss this issue further below.

The statutory line of business to which a claim is coded affects the commutation price. The commutation price has two components: (i) the present value of future losses and expenses, and (ii) a margin to cover the cost of holding capital.

- The present value of future losses and expenses, which forms the bulk of the commutation price, does not depend on the statutory line of business.[15]
- The margin to cover the cost of holding capital depends on the risk-based capital requirements for the line of business.

The margin for a commutation in medical malpractice or products liability is more than twice the margin for a workers' compensation commutation.[16]

TAX DISCOUNT FACTORS

The most complex part of the commutation calculations involves the IRS loss reserve discount factors. We continue the illustration with one modification: the premium for the commutation is paid on December 31 of the previous year.

> *A loss reserve is commuted on December 31, 20X2. The loss will be paid for $105,000 on December 31, 20X3. The risk-free interest rate is 5% per annum, and the federal income tax rate is 35%. No risk adjustments or risk loads are used. The IRS loss reserve discount factor is 1/1.05 = 95.238095%, which is the discount based on the payment pattern for this risk and the risk-free interest rate.*[17]

NO CAPITAL REQUIREMENTS AND NO ADDITIONAL COST OF CAPITAL

We assume first that the primary insurer does not need surplus to support the loss reserve and that its equityholders are satisfied with a risk-free rate of return. We examine the financial implications of a $100,000 commutation price.

When the commutation is done on December 31, 20X2, the primary company receives $100,000 and records a $105,000 statutory reserve on its books. The tax-basis reserve is 95.238095% × $105,000 = $100,000. The $100,000 of cash exactly offsets the $100,000 of
9 increased tax-basis incurred losses, so taxable income is not affected by the commutation.

The December 31, 20X2, the statutory loss reserve is $105,000, so the equityholders of the insurance company contribute $5,000 on that date. The cash is invested in risk-free securities yielding 5% per annum. During 20X3, the company earns $5,250 of interest income.

The incurred loss offset to tax-basis underwriting income in 20X3 is the paid loss plus the change in tax-basis reserves, or $105,000 + ($0 – $100,000) = $5,000. This is often referred to as the unwinding of the interest discount.[18] In practice, it is an offset to other underwriting (taxable) income. There is no other underwriting income in this illustration, so the total tax-basis underwriting income is –$5,000.

The taxable income in 20X3 is $5,250 – $5,000 = $250. The federal income tax is 35% × $250 = $87.50.

The after-tax income in 20X3 is $250 – $87.50 = $162.50. This is the reward to the shareholders for their contribution of $5,000 at the beginning of the year.

If there is no risk in the insurance operations, the shareholders expect a risk-free rate of return of 5% per annum, not a return of $162.50 / $5,000 = 3.25% per annum. The shareholders could get a 5% per annum return by investing directly in risk-free securities. This is the same as in the previous illustration.

For the primary insurance company to provide a 5% rate of return to its shareholders, the commutation price must be higher than \$100,000. Let the commutation price be \$100,000+z.

A. The taxable income on December 31, 20X2 is \$0 + z, and the tax is 35% × z.

B. The company holds assets of \$105,000 to fund the reserve. It has \$100,000 + (1 – 35%) × z from the commutation transaction, or \$100,000 + 65% × z. Shareholders contribute \$5,000 – 65% × z at the beginning of the year to fund the \$105,000 statutory reserve.

C. The company's shareholders expect a 5% annual return. The amount returned to the shareholders on December 31, 20X3 is \$5,162.50, so \$5,162.50 / (\$5,000 – 65% × z) = 105%. This means that

$$\$5{,}162.50 / (\$5{,}000 - 0.65z) = 105\%$$
$$\$5{,}162.50 = \$5{,}250 - 0.6825z$$
$$z = \$87.50 / 0.6825 = \$128.21$$

The proper commutation price is not \$100,000.00 (the present value of the future loss payments) or \$100,125.00 (the commutation price for an effective date of January 1) but \$100,128.21. This is an increase of 0.128% over the discounted loss payments. The commutation price rises by \$128.21 – \$125 = \$3.21 when the premium payment changes from January 1, 20X3, to December 31, 20X2. Since the premium payment is moved to the preceding calendar year, the tax on the premium payment is moved up one year, and the primary insurance company loses investment income on the tax payment. The present values of the pre-tax cash flows have not changed, but the after-tax cash flows are different.

The additional \$3.21 premium stems from two items.

1. Since the \$125 premium margin is moved up one year, the insurer pays taxes of 35% × \$125 = \$43.75 one year earlier. It loses the investment income of 5% × \$43.75 = \$2.1875. This investment income would be received at the end of the year. The present value of this investment income at the beginning of the year is \$2.1875 / 1.05 = \$2.0833.

2. The lost investment income must be compensated by additional premium, or the equityholders of the primary company would not agree to the commutation. If the reinsurance company were to pay the \$2.0833 directly to the equityholders of the primary company, this would be the total charge. But the reinsurance company pays the money indirectly through the commutation price. The premium received through the commutation price is taxed at the 35% corporate tax rate. To pay \$2.0833 to the equityholders fo the primary company, the additional premium must be

   \$2.0833 / (1 – 35%) = \$3.21.

The accounting entries clarify the implied equity flows. The reinsurer transfers $105,000 of loss reserves with a present value of $100,000 to the primary company, paying $100,128.21 in cash. The tax-basis discounted reserves are also $100,000.

The primary company receives $100,128.21 in cash and it increases its tax-basis reserves by $100,000.00, for an increase in taxable income of $128.21. The tax rate is 35%, so its tax liability is 35% × $128.21 = $44.87.

The primary company is left with $100,128.21 – $44.87 = $100,083.34 from the reinsurer. It must hold assets of $105,000 to back the statutory liabilities of $105,000. The primary company's shareholders contribute $105,000 – $100,128.21 + $44.87 = $4,916.66 to complete the funding of the statutory liabilities.

During the following year, the assets of $105,000 earn $5,250 of interest. The tax-basis reserves increase from $100,000 to $105,000, for a tax-basis incurred loss of $5,000. The net income is $250, and the federal income tax on this income is $250 × 35% = $87.50. The loss is paid for $105,000, and the remaining $5,250 – $87.50 = $5,162.50 is returned to the primary company's shareholders.

The return to the shareholders is $5,162.50 / $4,916.66 – 1 = 5.00%.[19]

COST OF HOLDING CAPITAL

If shareholders require an equity return instead of a risk-free return, the additional premium is

$$\$5{,}162.50 / (\$5{,}000 - 0.65z) = 112\%$$
$$\$5{,}162.50 = \$5{,}600 - 0.728z$$
$$z = \$437.50 / 0.728 = \$600.96.$$

As before, we have assumed a 8% market risk premium and an implicit beta of 87.5% for the property-casualty insurance industry, for a cost of capital of 5% × 87.5% × 8% = 12%.

CAPITAL REQUIREMENTS

We use the same assumptions for capital requirements as in the first illustration. The illustration below uses the 25% marginal effect of the reserving risk charge for the commercial liability lines of business (general liability, products liability, medical malpractice). A workers' compensation commutation would use the same formulas with 6.6% substituted for 25%.

The shareholders contribute 25% × $105,000 = $26,250 on December 31, 20X2, to support the additional loss reserves. On December 31, 20X3, the shareholders receive back this amount plus the after-tax investment income, or

$$\$26{,}250 + 5\% \times \$26{,}250 \times (1 - 35\%) = \$27{,}103.125.$$

If the shareholders require only a risk-free rate of return, the appropriate equation is:

$$(\$5{,}162.50 + \$27{,}103.125) / (\$5{,}000 + \$26{,}250 - 0.65z) = 105\%$$
$$\$32{,}265.625 = \$32{,}812.50 - 0.6825z$$
$$z = \$546.875 / 0.6825 = \$801.28$$

If the shareholders require an equity return, the appropriate equation is:

$$(\$5{,}162.50 + \$27{,}103.125) / (\$5{,}000 + \$26{,}250 - 0.65z) = 112\%$$
$$\$32{,}265.625 = \$35{,}000.00 - 0.728z$$
$$z = \$2{,}734.375 / 0.728 = \$3{,}756.01$$

The pricing method is straight-forward. The numerical result depends on the marginal effect of the reserving risk charge, ranging from 6% to 25%, and on the company's target return on capital, ranging from a 0% margin over the risk-free rate to a 7% margin over the risk-free rate. The price would also depend on any discounts in the statutory loss reserve, such as tabular discounts on workers' compensation claims.

DEFERRED TAX ASSETS AND LIABILITIES

The previous analysis is still incomplete, since we have not yet considered deferred tax assets and liabilities.

The deferred tax asset stemming from IRS loss reserve discounting has a material effect on the implied equity flows and the commutation price. The discussion here proceeds along the following path:

a. Accounting theory: current tax liability vs accrued taxes
b. Quantifying the deferred tax asset from IRS loss reserve discounting
c. One year illustration using the gross deferred tax asset
d. Statutory admissibility rules for deferred tax assets reversing over more than 12 months
e. Multi-year illustration using the admitted portion of the deferred tax asset

Current Taxes vs Accrued Taxes

The deferred tax asset stemming from IRS loss reserve discounting is unique to property-casualty insurance. To clarify the intuition, we use an example of deferred tax assets and liabilities stemming from realized capital gains and losses.

Illustration: An insurance company purchases 20,000 shares of common stock for $50 apiece on December 31, 20X2. By December 31, 20X3, each share of stock appreciates to $60. The company sells 10,000 shares at this price, and keeps the other 10,000 stocks.

- The realized capital gain is 10,000 × ($60 – $50) = $100,000. The tax liability on the realized capital gain is 35% × $100,000 = $35,000.
- The unrealized capital gain is 10,000 × ($60 – $50) = $100,000. The tax liability on the unrealized capital gain is zero, since capital gains are not taxed until they are realized.

On its balance sheet, the company shows common stocks at their market values. At December 31, 20X3, the remaining 10,000 shares are valued at $60 apiece. The income implied by the statutory balance sheet is the number of shares times the difference between the current price and the purchase price, or 10,000 × ($60 – $50) = $100,000.

If the company sells these shares, it must pay tax of $35,000 to the U.S. Treasury. Its after-tax income would be $65,000.

- The current tax liability is the tax on the realized gains, or $100,000 × 35% = $35,000.
- The accrued tax liability is the tax on the realized capital gains plus the deferred tax liability on the unrealized capital gains, for a total of $70,000.

Only realized capital gains flow through the statutory income statement. The unrealized capital gains and losses are direct charges and credits to surplus. For common stock gains and losses, there is no difference between statutory and taxable income.

Whether or not a balance sheet change is reflected in the statutory income statement is not relevant to deferred tax assets and liabilities. We compare taxable income to the *income implied by the statutory balance sheet*, not to actual statutory income; see SFAS 109.[20]

GAAP VS STATUTORY ACCOUNTING

Deferred tax assets and liabilities stem from timing differences in the realization of income. The unrealized capital gains or losses are recognized on the statutory balance sheet, but they are not recognized for taxable income until they are realized. When they are realized, the timing difference reverses: there is a gain or loss in future taxable income that is not reflected in the balance sheet changes of that accounting period.

- GAAP financial statements use the accrued tax basis of accounting. All deferred tax assets and liabilities that are expected to be realized in the future must be recognized on the balance sheet in the current accounting period.[21]
- Until 2001, statutory financial statements did not recognize deferred tax assets or liabilities. After codification of statutory accounting, the deferred tax liabilities and a

portion of the deferred tax assets are recognized on the statutory balance sheet; see SSAP No. 10, "Income Taxes."

Deferred tax assets and liabilities do not affect the cash flows of the company. However, they affect admitted assets and the implied equity flows, and thereby affect the commutation price.[22]

DEFERRED TAX ASSETS AND LOSS RESERVE DISCOUNTING

We illustrate the computation of the deferred tax asset stemming from loss reserve discounting and then examine the application to the commutation price.[23]

Illustration: On January 1, 20X2, a company writes a policy for a $12,000 premium and pays $2,000 in expenses. On December 31, 20X2, the company records a case reserve of $10,000, and it expects the loss to be paid on December 31, 20X3.

The risk-free interest rate is 5% per annum. For simplicity, we assume that the 60 month moving average of federal mid-term rates is the same as the current risk-free interest rate of 5% per annum, and that the IRS loss payment pattern matches the actual loss payment pattern for the block of business. The IRS loss reserve discount factor is 1/1.05 = 95.238095%.[24]

The statutory underwriting income is earned premium minus expenses minus incurred loss. In 20X2, this is $12,000 – $2,000 – $10,000 = $0. The *accrued* tax liability is the tax rate times the booked income, or $0 × 35% = $0.[25]

The tax basis incurred loss in 20X2 is 95.238095% × $10,000 = $9,523.81. The tax basis underwriting income in 20X2 is $12,000 – $2,000 – $9,523.81 = $476.19. The *current* tax liability is the actual amount owed to the IRS, or $476.19 × 35% = $166.67.

The timing difference between taxable income and statutory income is $166.67. The current tax liability is greater than the accrued tax liability, meaning that the company pays more tax than it would pay were the tax computed on the basis of its statutory balance sheet.

The timing difference reverses in 20X3. In 20X3, the statutory income is zero. The tax basis incurred loss is the paid loss plus the change in the discounted reserves, or

$10,000 + ($0 – $9,523.81) = $476.19.

The tax liability in 20X3 is –35% × $476.19 = –$166.67, or a tax refund of $166.67. In 20X2, the company holds a deferred tax asset of $166.67 on its statutory balance sheet.

DEFERRED TAX ASSETS AND COMMUTATION PRICING

One may conceive of the deferred tax asset as an "IOU" that is secured by the expectation of receiving a $166.67 tax refund in 20X3. It is not an investable asset, and it earns no investment income. It is an admitted asset, since it is expected to reverse within the next 12 months. We deal with the statutory admissibility constraints further below.

As an admitted asset, the deferred tax asset can back the loss reserves and it can back the risk-based capital needed to support the loss reserves. The deferred tax asset of $166.67 reduces the equityholder provided capital by $166.67.

We re-work the commutation illustration that we began earlier.

> *A loss reserve is commuted on December 31, 20X2. The loss will be paid for $105,000 on December 31, 20X3. The risk-free interest rate is 5% per annum, and the federal income tax rate is 35%. No risk adjustments or risk loads are used. The IRS loss reserve discount factor is 95.238095%, which is equal to the actual discount based on the true payment pattern for this risk and the current risk-free interest rate.*[26]

The changes from the previous calculation are as follows. On December 31, 20X2, the company holds a deferred tax asset of

$$35\% \times (\$105,000 - \$105,000 \times 95.238095\%) = \$1,750$$

The equityholder contribution on December 31, 20X2, is $1,750 lower. The deferred tax asset is not investable. The investment income in 20X3 is reduced by $1,750 × 5% = $87.50. The tax on investment income in 20X3 is reduced by $87.50 × 35% = $30.625.

The commutation is now priced in the same manner as before. We show the indicated commutation prices for four scenarios: a workers' compensation commutation with a 6.6% capital requirement vs. a commercial liability commutation with a 25% capital requirement and a 5% risk-free return to equityholders vs. a 12% "equity" return to equityholders.

The step-by-step documentation below explains the calculations for the commercial liability commutation with a 12% equity return. The computations for other scenarios are similar.

COMMERCIAL LIABILITY COMMUTATION WITH RISK-FREE RETURN: YEAR 20X2

- The commutation price is $100,000 + z.
- The undiscounted reserve is $105,000.
- The IRS loss reserve discount factor is 1/1.05 = 95.238095%.
- The tax basis loss reserve $105,000 × 1/1.05 = $100,000.
- The tax basis underwriting income in 20X2 is $100,000 + z – $100,000 = z.

- The tax on underwriting income in 20X2 is 35% × z.
- The deferred tax asset stemming from loss reserve discounting is 35% × ($105,000 – $100,000) = $1,750. The deferred tax asset does not depend on "z."
- The investment income in 20X2 is zero, and the tax on investment income in 20X2 is zero.
- The total tax paid in 20X2 is 35% × z.
- The capital requirement is 25% of the undiscounted reserves, or 25% × $105,000 = $26,250.00.
- The assets needed on December 31, 20X2, are $105,000 (reserves) + $26,250 (surplus) = $131,250.
- The cash received from the reinsurer is $100,000 + Z.
- The cash paid to the IRS is 35% × z.
- The net cash available is $100,000 + z – 35% × z = $100,000 + 65% × z.
- The non-cash asset (the DTA) is $1,750.
- The capital contribution needed from the equityholders is $131,250 – ($100,000 + 65% × z) – $1,750 = $29,500 – 65% × z.

YEAR 20X3

- The cash assets (investable assets) at the beginning of the year are $131,250 – $1,750 = $129,500.
- The investment yield is 5% per annum.
- The investment income in 20X3 is 5% × $129,500 = $6,475.00.
- The tax on investment income is 35% × $6,475 = $2,266.25.
- The tax basis incurred loss in 20X3 is the paid loss plus the change in reserves, or $105,000 + ($0 – $100,000) = $5,000.
- The tax basis underwriting income in 20X3 equals –$5,000.
- The tax liability on underwriting income in 20X3 is –$5,000 × 35% = –$1,750.
- The net taxes paid in 20X3 are $2,266.25 – $1,750 = $516.25.
- The cash available at the end of the year before payment of the loss is $129,500 + 6,475 – $516.25 = $135,458.75.
- The loss is paid for $105,000.
- The cash remaining after payment of the loss is $30,458.75.
- The required return on capital is 12% per annum.
- We solve for "z" as

$$\$30{,}458.75 / (\$29{,}500 - 65\% \times z) = 1.12$$
$$65\% \times z = \$2{,}304.6875$$
$$z = \$3{,}545.67$$

The addition to the commutation price depends on the capital requirements and the cost of equity capital. We show two choices for each of these dimensions:

- *Capital requirements:* 25% of reserves for commercial liability vs 6.6% of reserves for workers' compensation
- *Cost of equity capital:* 5% risk-free return vs 12% equity return

The value of $3,545.67 as the addition to the commutation price is the high end of the range. The four possibilities from the two dimensions listed above are shown in the matrix below, where the vertical axis represents the capital requirements and the horizontal axis represents the cost of equity capital.

Capital Requirement	*5% cost of equity capital*	*12% cost of equity capital*
Commercial Liability: 25%	$756.41	$3,545.67
Workers' Compensation: 6.6%	$261.03	$1,223.56

We summarize the results of the pricing analysis as follows.

- The commutation price is higher than the present value of the commuted reserves, where the present value is taken at the pre-tax interest rate.
- If there were no capital requirements, the commutation price would be the present value of the commuted reserves, except for variances between the IRS loss reserve discount factor and the actuarially determined discount factor for the commuted reserves. Over the long-term, the IRS loss reserve discount factors are not materially biased, and the variance should be small for the commutation of an entire block of business.
- When individual claims are commuted, the actual loss payment pattern is generally slower than the pattern assumed by the IRS for the entire line of business. The tax basis reserves are higher than fair value reserves (providing a benefit to the company holding the reserves), and the commutation price is lower.
- The required return on capital has a large effect on the indicated commutation price. Changing the required return on capital from a 5% risk-free rate to a 12% equity return causes about a five fold increase in the addition to the commutation price.
- The capital requirements also have a considerable effect on the commutation price. This capital requirement is the additional requirement from the commutation, or the difference in capital requirements between the reserving risk charge and the reinsurance charge. Changing the capital requirement from 6.6% of reserves to 25% of reserves causes about a three fold increase in the commutation price.

Deferred Tax Assets

Statutory Recognition of Deferred Tax Assets

All deferred tax liabilities are recognized on the statutory balance sheet. For most deferred tax assets, the admitted statutory portion equals the entire asset, and statutory accounting is the same as GAAP. In certain instances, only a portion of the deferred tax assets are recognized on the statutory balance sheet. This applies particularly to the deferred tax asset stemming from IRS loss reserve discounting for medium- and long-tailed lines of business.

SSAP No. 10, "Income Taxes," paragraph 10, says:

Gross DTAs shall be admitted in an amount equal to the sum of:

a *Federal income taxes paid in prior years that can be recovered through loss carrybacks for existing temporary differences that reverse by the end of the subsequent calendar year;*

b *The lesser of:*

  i *The amount of gross DTAs, after the application of paragraph 10 a., expected to be realized within one year of the balance sheet date; or*

  ii *Ten percent of statutory capital and surplus as required to be shown on the statutory balance sheet of the reporting entity for its most recently filed statement with the domiciliary state commissioner adjusted to exclude any net DTAs, EDP equipment and operating system software and any net positive goodwill; and*

c. *The amount of gross DTAs, after application of paragraphs 10 a. and 10 b., that can be offset against existing gross DTLs.*

A gross deferred tax asset is admissible if it will reverse within one year, as required by paragraph (a) and by paragraph (b.i).

The limitation of 10% of surplus in paragraph (b.ii) may affect companies with long-tailed lines of business, pre-paid annual policies, and high premium to surplus ratios.[27]

Illustration: An insurer with $100 million of surplus writes annual policies with effective dates spread evenly through the year. Its premium to surplus ratio is 2 to 1, and its reserves to surplus ratio is 4 to 1. Its average loss reserve discount factor is about 80%, and 30% of its deferred tax asset from loss reserve discount will reverse within 12 months. We work out its gross deferred tax asset and the portion admitted on the statutory balance sheet.

Annual premium is twice surplus, or $200 million. The unearned premium reserve is $200 million × 50% = $100 million, and the deferred tax asset stemming from revenue offset is $100 million × 20% × 35% = $7 million.

The reserves to surplus ratio is 4 to 1, so the held loss reserves are $400 million. The average loss reserve discount factor is 80%, so the discounted reserves are $400 million × 80% = $320 million. The gross deferred tax asset from IRS loss reserve discounting is ($400 million – $320 million) × 35% = $28 million. The portion admitted on the statutory balance sheet is $28 million × 30% = $8.4 million.

The total statutory deferred tax asset is $7 million + $8.4 million = $15.4 million. This is limited to 10% of statutory surplus, or $10 million, so an additional $5.4 million is not admitted.

Various changes in the scenario would mitigate the "10% of surplus" restriction.

- *Premium to surplus ratio:* The U.S. property-casualty insurance industry has a premium to surplus ratio of about 1 to 1. With this premium to surplus ratio, the 10% of surplus restriction has no effect in the illustration above.
- *Policy term:* The unearned premium reserve for six month policies and the deferred tax asset from revenue offset would be only half the size of those for annual policies.
- *Lines of business:* Property lines do not have material deferred tax assets from loss reserve discounting.
- *Effective dates:* Policies effective on January 1 have much lower unearned premium reserves and deferred tax assets at the end of the year.

The offsetting against existing gross deferred tax liabilities mentioned in paragraph (c) is relevant for companies with large unrealized capital gains from common stock holdings. The actuary should take this provision into account when quantifying the admitted portion of the deferred tax asset.

Common stock that has suffered an unrealized capital loss may be sold within the next 12 months to realize the tax benefits. A literal reading of the SSAP would permit the recognition of the deferred tax asset only if the company expects to realize the capital loss during the coming calendar year. In practice, most auditors do not require an explicit expectation to realize the loss in order to admit the deferred tax asset.

LOSS RESERVE DISCOUNTING

Statutory incurred losses are the paid losses plus the change in the undiscounted loss reserves. The tax basis incurred losses are the paid losses plus the change in the *discounted* loss reserves. The difference between statutory and tax basis incurred losses is a timing difference. The change in the deferred tax asset is 35% of this difference.

Illustration: A policy is issued on January 1, 20XX, for a premium of $1000 and expenses of $200. Losses of $800 are incurred in 20XX, of which half are paid in 20XX and half are paid in 20XX+1. The IRS loss reserve discount factor at the 12 month valuation is 90%. For simplicity, we assume that the companies earns no investment income.

- The statutory incurred losses in 20XX are $400 of paid losses plus $400 of loss reserve change = $800. Statutory income is $1000 – $200 – $800 = $0. The accrued taxes are 35% × $0 = $0.
- The taxable incurred losses in 20XX are $400 of paid losses plus $400 × 90% = $360 of change in tax basis (discounted) loss reserves = $760. Taxable income is $1000 – $200 – $760 = $40. The tax liability is 35% × $40 = $14

The difference between the income implied by the statutory balance sheet and taxable income is $0 – $14 = –$14. The gross deferred tax asset is $14.

The portion of the deferred tax asset that reverses within 12 months is admitted on the statutory balance sheet. We examine the statutory income and taxable income for 20XX+1.

- The statutory incurred losses in 20XX+1 are $400 of paid losses plus –$400 of loss reserve change = $0. There is no premium or expense in 20XX+1, so statutory income is $0. The accrued taxes are 35% × $0 = $0.
- The taxable incurred losses in 20XX+1 are $400 of paid losses plus –$360 of change in discounted loss reserves = $40. There is no premium or expense in 20XX+1, so taxable income is $0 – $40 = –$40. The tax liability is 35% × (–$40) = –$14.

The full difference between statutory and taxable income reverses in 20XX+1, so the full deferred tax asset of $14 is admitted on the statutory balance sheet.

TWELVE MONTH REVERSAL

We compute the admitted portion of the deferred tax asset from loss reserve discounting separately by line of business and accident year.

Illustration: For accident year 20XX in a given line of business, the loss reserve discount factors are Z_1 at December 31, 20YY, and Z_2 at December 31, 20YY+1. Let "R" be the held loss reserves at December 31, 20YY. Let "P" be the percentage of accident year 20XX reserves that will be paid during calendar year 20XX.

- At December 31, 20YY, the difference between statutory and taxable income for accident year 20XX is $R \times (1 - Z_1)$. The gross deferred tax asset is $35\% \times R \times (1 - Z_1)$.
- At December 31, 20YY+1, the difference between statutory and taxable income for accident year 20XX is $R \times (1 - P) \times (1 - Z_2)$. The gross deferred tax asset is $35\% \times R \times (1 - P) \times (1 - Z_2)$.
- The admitted portion of the deferred tax asset on the statutory balance sheet at December 31, 20YY is $35\% \times R \times [(1 - Z_1) - (1 - P) \times (1 - Z_2)]$.

The value of *P* depends on the company's estimated loss payment pattern, not the IRS loss payment pattern. The pattern should be based on actuarially justified discount factors.

ILLUSTRATION

Suppose accident year 20X5 incurred losses are $180,000, of which $15,000 is paid in 20X5. The paid loss development factors are 8.000 from 12 months to ultimate and 5.000 from 24 months to ultimate. The IRS loss reserve discount factors for accident year 20X5 are 77.8022% at 12 months and 78.7611% at 24 months. We determine the statutory and GAAP deferred tax assets on December 31, 20X5.

The accident year 20X5 loss reserves for statutory and GAAP balance sheets on December 31, 20X5 are $180,000 – $15,000 = $165,000. The discounted tax basis loss reserves are

$$\$165{,}000 \times 77.8022\% = \$128{,}373.63.$$

The difference between the GAAP loss reserves and the tax basis loss reserves is

$$\$165{,}000.00 - \$128{,}373.63 = \$36{,}626.37.$$

The addition to taxable income stemming from loss reserve discounting is $36,626.27 × 35% = $12,819.23. This is the deferred tax asset on the GAAP balance sheet.

The admitted portion of the deferred tax asset on the statutory balance sheet depends on the portion of the loss reserve that remains unpaid in one year's time. This is an actuarial estimate; it is not the IRS provision used in the loss reserve discounting calculation. We use the paid loss development factors to project the losses remaining unpaid at 24 months.

- At 12 months of development, 1/8.000 = 12.5% of incurred losses have been paid and 1 – 1/8.000 = 87.5% of incurred losses are still unpaid.
- At 24 months of development, 1/5.000 = 20.0% of incurred losses have been paid and 1 – 1/5.000 = 80.0% of incurred losses are still unpaid.

We expect 80.0% / 87.5% = 91.428571% of the December 31, 20X5, accident year 20X5 loss reserves to remain unpaid at December 31, 20X6. This amount is $165,000 × 91.4285714% = $150,857.14. The expected IRS discounted reserves at December 31, 20X6 equal this amount times the IRS loss reserve discount factor for accident year 20X5 at 24 months of development, or 78.7611%:

$150,857.14 × 78.7611% = $118,816.75.

IMPLICIT DISCOUNTING

Some companies implicitly discount reserves for long-tailed lines of business. Implicit discounting means that the company consciously holds less than full value loss reserves (for capital management purposes), not that the company mis-estimates the reserve indication.

One might be tempted to think that the amount of the implicit reserve discount should be taken into consideration when calculating the deferred tax asset. This is not correct. The deferred tax asset must be calculated as if the company held full value loss reserves.

Illustration: An insurer expects to pay a loss for $100,000 in three years. The IRS loss reserve discount factor for this line of business and accident year is 80% for the current valuation date and 85% for the valuation date 12 months hence.

- The (gross) deferred tax asset on the GAAP financial statements is 35% × $100,000 × (1 – 80%) = $7,000.
- The (net admitted) deferred tax asset on the statutory financial statements is 35% × $100,000 × (85% – 80%) = $1,750.

If the insurer implicitly discounts reserves at 5% per annum, its held reserves are $100,000 / 1.05^3 = $86,383.76, and its tax basis reserves are 80% × $100,000 / 1.05^3 = $69,107.01. Its expected held reserves one year hence are $100,000 / 1.05^2 = $90,702.95, and its expected tax basis reserves at that time are 85% × $100,000 / 1.05^2 = $77,097.51.

One might think that the gross (GAAP) and net admitted (statutory) deferred tax assets should be computed as follows:

- Gross (GAAP): 35% × ($100,000 – $69,107.01) = $10,812.55.
- Net admitted (statutory): 35% × ($77,097.51 – $69,107.01) = $2,796.68.

This is not correct. If the company shows a reserve of $86,383.76 on its statutory financial statements, it must treat that reserve at though it were a full value loss reserve for calculating the deferred tax asset. The appropriate calculations are as follows:

- The (gross) deferred tax asset on the GAAP financial statements is 35% × $86,383.76 × (1 – 80%) = $6,046.86.
- The (net admitted) deferred tax asset on the statutory financial statements is 35% × $86,383.76 × (85% – 80%) = $1,511.72.

DEFERRED TAX ASSETS: MULTIPLE PERIODS

Actual commutations have cash flows extending for many future years. To price the commutations, we determine the projected deferred tax assets at each future valuation date.

The deferred tax assets depend on the IRS loss reserve discount factors. These discount factors may be either industry factors or company specific factors. The industry factors are based on industry-wide loss payment patterns by line of business. They are promulgated by the Secretary of the Treasury each year. The company specific factors are based on the company's own loss payment patterns by line of business.

The loss reserve discount factors are determined for a specific accident year. Once the factors are determined, they are frozen (or "vintaged" in tax parlance).

> *Illustration:* By mid-December 20XX–1, all the required information is available to determine accident year 20XX loss reserve discount factors.[28] Between 11 and 16 loss reserve discount factors are computed, applicable to valuation dates of 12/31/20XX, 12/31/20XX+1, . . . , 12/31/20XX+15. These factors are applicable to accident year 20XX losses only, and they are not changed after the initial determination.

The calculation of the *gross* deferred tax assets stemming from loss reserve discounting is based solely on the loss reserve discount factors.[29] Just as the loss reserve discount factors are determined and frozen at the beginning of the accident year, the gross deferred tax asset as a percentage of the loss reserves at each valuation date is determined at the inception of the accident year.

In early 20XX, the company determines the loss reserve discount factors for accident year 20XX at each future valuation date. The deferred tax asset factor for each valuation date (as a percentage of the held reserves) is 35% of loss reserve discount at that valuation date.

Illustration: If the loss reserve discount factor for accident year 20XX at valuation date December 31, 20XX+2, is 75%, the (gross) deferred tax asset factor is 35% × (1 – 75%) =

8.75%. That is, the deferred tax asset for accident year 20XX on the December 31, 20XX+2, GAAP balance sheet is 8.75% of the held loss reserves.

The admitted portion of the deferred tax asset for the statutory balance sheet depends on the actuary's estimate of the loss liquidation pattern at the valuation date. These are statutory accounting numbers, not tax accounting numbers, and there is no "vintaging."

For pricing commutations, we must estimate the deferred tax asset as a percentage of the full value loss reserves at each future valuation date. Projecting loss liquidation patterns is a staple of casualty loss reserving, and the estimation of future deferred tax assets presents no unusual complications. We explain the estimation process by means of an illustration.

Illustration: An insurer commutes a workers' compensation permanent total disability claim with the following characteristics:

- Date of loss occurrence: July 1, 2000.
- Date of commutation: December 31, 2005.
- Annual indemnity benefits: $40,000 with no cost of living adjustments
- Annual medical benefits: None
- Life expectancy: 20 years
- Full value loss reserve: $800,000

The IRS loss reserve discount factors for accident year 2000 are shown below:

Tax Year	Valuation Date	IRS Loss Reserve Discount Factor	Tax Year	Valuation Date	IRS Loss Reserve Discount Factor
AY + 0	2000	0.819398	AY + 5	2005	0.675118
AY + 1	2001	0.807648	AY + 6	2006	0.661927
AY + 2	2002	0.802667	AY + 7	2007	0.670194
AY + 3	2003	0.761583	AY + 8	2008	0.703333
AY + 4	2004	0.710909	AY + 9	2009	0.739426

We examine three versions of this illustration:

- The company holds full value loss reserves with no tabular discount.
- The company holds reserves net of a tabular discount, and the tabular discount for all workers' compensation business in accident year 2000 is less than the IRS loss reserve discount for the company's workers' compensation business in this accident year. This would be true for the most recent five to ten accident years.
- The company holds reserves net of a tabular discount, and the tabular discount for all workers' compensation business in accident year 2000 is greater than the IRS loss

reserve discount for the company's workers' compensation business in this accident year. This is generally true only for old accident years, such as the Schedule P prior years row.

To ease the exposition, we initially assume that the probability of death in calendar year 2006 is insignificant. Once the intuition is clear, we relax this constraint.

Deferred Tax Asset at December 31, 2005

The full value loss reserve at 12/31/2005 is $800,000. The IRS loss reserve discount factor for accident year 2000 at 72 months is 0.675118. The tax basis reserve is $800,000 × 0.675118 = $540,094.

If we assume the probability of death in 2006 is 0%, the full value loss reserve at 12/31/2006 is $800,000 – $40,000 = $760,000.[30] The IRS loss reserve discount factor for accident year 2000 at 84 months is 0.661927. The tax basis reserve is $760,000 × 0.661927 = $503,065.

The loss reserve discount is $800,000 – $540,094 = $259,906 at 12/31/2005 and $760,000 – $503,065 = $256,935 at 12/31/2006. The change in the discount over the 12 months following 12/31/2005 is $259,906 – $256,935 = $2,159. The deferred tax asset admitted on the statutory balance sheet on 12/31/2005 is $2,159 × 35% = $756.

Deferred Tax Asset at December 31, 2007

The deferred tax asset in the illustration above is less than 0.1% of the full value loss reserve. The small size stems from the decline in the loss reserve discount factor from AY+5 to AY+6 and the slow payment pattern for this workers' compensation pension case.

If the commutation is effected on 12/31/2007 instead of 12/31/2005, the deferred tax asset is greater. We redo the computation with the new valuation date and without changing the remaining life expectancy of 20 years.

The full value loss reserve at 12/31/2007 is 20 × $40,000 = $800,000. The IRS loss reserve discount factor for accident year 2000 at 96 months is 0.670194. The tax basis reserve is $800,000 × 0.670194 = $536,155.

If we assume the probability of death in 2008 is 0%, the full value loss reserve at 12/31/2008 is $800,000 – $40,000 = $760,000. The IRS loss reserve discount factor for accident year 2000 at 108 months is 0.703333. The tax basis reserve is $760,000 × 0.703333 = $534,533.

The loss reserve discount is $800,000 – $536,155 = $263,845 at 12/31/2007 and $760,000 – $534,533 = $225,467 at 12/31/2008. The change in the discount over the 12 months following 12/31/2005 is $263,845 – $225,467 = $38,378. The deferred tax asset that is admitted on the statutory balance sheet on 12/31/2005 is $38,378 × 35% = $13,432.

The deferred tax asset is $13,432 / $756 = 17.8 times larger in 2007 than in 2005. The difference does not stem from any change in the full value loss reserve (which is $800,000 in both cases) or the tax basis loss reserve (which is about $535,000 to $540,000 in both cases). The difference stems from the change in the relative size of the current year's loss reserve discount factor and the next year's loss reserve discount factor.

Probability of Death

When the probability of death is considered, the deferred tax asset is computed with a mortality table. The explanation below shows the intuition; it is not an exact calculation. To keep the intuition clear, we assume that deaths occur only at year end.

We resume the illustration described above, with a current valuation date of December 31, 2005. We assume that the probability of the injured worker's death in 2006 is 1%.

On December 31, 2005, the injured worker's life expectancy is 20 years. There are two scenarios for the coming year.

- The worker dies on December 31, 2006, and his life expectancy on December 31, 2005 is 1 year.
- The worker does not die in 2006, and his life expectancy on December 31, 2005 is 1 year plus his life expectancy on December 31, 2006.

Let "Z" represent the conditional life expectancy on December 31, 2006, given that he is alive on that date. We determine "Z" as

$$20 \text{ years} = 1\% \times 1 \text{ year} + 99\% \times (1 \text{ year} + Z \text{ years})$$
$$19 \text{ years} = 99\% \times Z \text{ years}$$
$$Z = 19/0.99 = 19.191919 \text{ years}$$

The life expectancy for the this worker on December 31, 2006, based on the uncertainty present on December 31, 2006, is 1% × 0 years + 99% × 19.191919 years = 19.000 years.

This result is true for all scenarios. The probability of death during the coming year does not affect the expected life expectancy at the next valuation date. The probability of death during the coming year does not affect the deferred tax asset at the current valuation date.[31]

Tabular Discount

Workers' compensation claim may have tabular discounts, which affect the held reserve. The tabular discount reduces equityholder capital embedded in the reserves, thereby reducing the commutation price. It also reduces the reserving risk charge, further reducing the commutation price.

If the tabular discount is explicitly shown in the Annual Statement, the tabular discount does not affect the tax basis loss reserves, unless the held reserves are lower than the IRS discounted reserves.[32] Because the tabular discount changes the loss reserves on the statutory balance sheet but not the tax basis reserves, the deferred tax asset changes.

We explain the computations by means of the illustration begun above. Assume the discount rate for the tabular discount is 5% per annum. The reciprocal of the discount rate is 1/1.05 = 0.952381. The discounted reserve at December 31, 2005 is approximately

$$\$40{,}000 \times (1 - 0.952381^{20})/(1-0.952381) = \$523{,}413.04.$$ [33]

In this case, the statutory reserves are lower than the tax basis reserves of $540,094. There are two possible scenarios.

Scenario A: The permanent total disability cases are not the only workers' compensation loss reserves in this accident year. Other workers' compensation loss reserves are being held at full value. For all claims combined, the statutory held reserves are greater than the tax basis loss reserves. In this scenario, we use the tax basis reserve of $540,094 to determine the deferred tax asset or liability. We do not limit the IRS loss reserve discount, since the limit is offset by the full value loss reserves on other claims.

The *gross* deferred tax asset or liability is a deferred tax *liability* of ($540,094 – $523,413) × 35% = $5,838. This deferred tax liability would appear on the GAAP balance sheet. For the statutory balance sheet, the amount of the deferred tax asset or liability depends on the amount that reverses over the coming 12 months. In the illustration used here, this becomes a deferred tax asset, since the IRS loss reserve discount factors decline from 72 months to 84 months, whereas the pension discount factors increase from 72 months to 84 months. We show the appropriate calculation:

The IRS loss reserve discount factors are 67.5118% at 72 months and 66.1927% at 84 months. The tabular discount factors at these two dates are

- *72 months:* $(1 - 0.952381^{20})/[20 \times (1-0.952381)] = 65.4266\%$.
- *84 months:* $(1 - 0.952381^{19})/[19 \times (1-0.952381)] = 66.7873\%$.[34]

The expected change in the tax basis loss reserves over the coming 12 months is

$$\$800{,}000 \times 67.5118\% - \$760{,}000 \times 66.1927\% = \$37{,}030.$$

The tax basis incurred loss is $40,000 – $37,030 = $2,970.

The expected change in the statutory loss reserves over the coming 12 months is

$\$800,000 \times 65.4266\% - \$760,000 \times 66.7873\% = \$15,829$.

The statutory incurred loss is $40,000 – $15,829 = $24,171.

The statutory deferred tax asset is 35% × ($24,171 – $2,970) = $7,420.

Scenario B: The permanent total disability cases are the dominant portion of the workers' compensation loss reserves in this accident year, as is true for the Schedule P prior years row. For all claims combined, the statutory held reserves are lower than the tax basis loss reserves, so the tax basis reserve is set equal to the held reserve. There is no deferred tax asset on either the statutory or the GAAP balance sheet.

THE CONNOR AND OLSEN ILLUSTRATION

We redo the illustration in the Connor and Olsen paper, using the techniques outlined here. Because the Connor and Olsen technique does not consider risk-based capital requirements and the cost of holding capital, Connor and Olsen significantly overstate the proper commutation price. The inclusion of deferred tax assets in post-codification statutory accounting slightly reduces the proper commutation price, but it does not materially offset the bias in the Connor and Olsen technique.

Illustration: A company commutes a block of loss reserves with expected payments of $20,000 each year starting one year from now. The IRS loss reserve discount factors for this line of business and accident year are shown in the table below for the current valuation date and each of the five subsequent valuation dates. The investment yield is 8.5% per annum, and the federal income tax rate is 35%. (Connor and Olsen use a 34% tax rate, since their paper appeared in the late 1980's, when the rate was 34%, not 35%.) The figures are taken from Table 1 on page 86 of the Connor and Olsen paper.

Calendar Year	*Paid Loss*	*Year End Reserve*	*IRS Discount Factor*
1990	$0	$100,000	0.79812
1991	$20,000	$80,000	0.77935
1992	$20,000	$60,000	0.75561
1993	$20,000	$40,000	0.73577
1994	$20,000	$20,000	0.70271
1995	$20,000	$0	0.68950

PRICING ASSUMPTIONS

We add two items to price the commutation: the cost of equity capital and the capital requirements. We assume that equityholders demand a return 400 basis points above the investment yield, or 12.5% for this illustration.[35] For capital requirements, we choose low figures, to avoid any perception that the results are dependent on over-stated assumptions.[36] Specifically, we assume that the company holds 10% of written premium plus 15% of held reserves as its surplus.[37]

The commutation price depends somewhat on the effective date of the transaction. For simplicity, we assume that taxpayers remit the taxes at end of the year, so an effective date later in the year slightly raises the commutation price.[38] We use a December 31 date for the commutation, to match the Connor and Olsen illustration. Exhibits using a January 1 date for the commutation are included in the appendix.

The commutation price is the price which provides a return to equityholders commensurate with the cost of equity capital. If the primary company and the reinsurer are both subject to the same insurance regulation, and if both companies have the same cost of equity capital, then the savings gained by the reinsurance company are transferred to the primary company. The magnitude of the investment yield, the cost of equity capital, and the capital requirements do not affect this relationship.

Illustration: The primary company has a 12.5% cost of equity capital, and the pre-tax investment yield is 8.5% per annum. By accepting the commuted claims, the primary company must allocate supporting capital, and it needs additional profits to defray the cost of this capital. Conversely, the reinsurance company frees up an equal amount of capital, and it needs correspondingly less profit to cover its cost of capital.[39]

The commutation price is most easily determined by an iterative procedure.[40] We do not show the derivation of the commutation price by algebraic methods. Rather, we show that the indicated commutation price provides a 12.5% return on capital. Since a lower price would provide a lower return on capital and a higher price would provide a higher return on capital, the solution is unique.

The indicated commutation price is $87,962; in contrast, Connor and Olsen determine a commutation price of $79,437. The change in the tax rate from 34% to 35% has a minor effect on the commutation price. The major cause of the higher commutation price is the additional capital requirements stemming from holding additional reserves and the gap between the cost of equity capital and the company's investment yield.

The exhibits below show the sensitive of the commutation price to the pricing assumptions. The alternative illustration for the Connor and Olsen example uses capital requirements equal to 25% of premium and 20% of reserves. This gives about a one to one premium to surplus ratio.[41]

SURPLUS REQUIREMENTS

Surplus requirements have a material effect on the commutation price. Table 1 shows the commutation price for the illustration in the Connor and Olsen paper under varying surplus requirements (expressed as leverage ratios), ranging from 0% of held reserves and of written premium to 25% of held reserves and of written premium.

The reserving risk charge has a greater effect than the written premium risk charge, since the held reserves stay on the company's books for several years, whereas the written premium risk charge is in effect for a single year. The commutation price for the surplus requirements in the illustration above (10% written premium risk charge and 15% reserving risk charge) is shown in the boxed cell.

- As the written premium risk charge ranges from 0% to 25%, the commutation price ranges from $87,123 to $89,251, for a difference of $2,128.
- As the reserving risk charge ranges from 0% to 25%, the commutation price ranges from $84,219 to $90,457, for a difference of $6,238.

Table 1: Sensitivity Analysis – Surplus Requirements

Reserving Risk	*Written Premium Risk* 0%	5%	10%	15%	20%	25%
0%	83,416	83,816	84,219	84,627	85,038	85,454
5%	84,652	85,057	85,467	85,880	86,298	86,720
10%	85,887	86,299	86,715	87,134	87,558	87,986
15%	87,123	87,541	**87,962**	88,388	88,817	89,251
20%	88,359	88,782	89,210	89,641	90,077	90,517
25%	89,594	90,042	90,457	90,894	91,337	91,783

Table 2 shows the figures as percentages of the commutation price in the boxed cell. For example, the $89,641 commutation price with a 15% written premium risk charge and a 20% reserving risk charge is 1.91% higher than the $87,962 commutation price with a 10% written premium risk charge and a 15% reserving risk charge.

Table 2: Sensitivity Analysis – Surplus Requirements (Percentages)

Reserving Risk	Written Premium Risk 0%	5%	10%	15%	20%	25%
0%	-5.17%	-4.71%	-4.26%	-3.79%	-3.32%	-2.85%
5%	-3.76%	-3.30%	-2.84%	-2.37%	-1.89%	-1.41%
10%	-2.36%	-1.89%	-1.42%	-0.94%	-0.46%	0.03%
15%	-0.95%	-0.48%	**0.00%**	0.48%	0.97%	1.47%
20%	0.45%	0.93%	1.42%	1.91%	2.40%	2.90%
25%	1.86%	2.34%	2.84%	3.33%	3.84%	4.34%

TARGET RETURN ON CAPITAL AND BENCHMARK INVESTMENT YIELD

Table 3 below shows the sensitivity of the commutation price to the benchmark investment yield and the target return on capital; Table 4 shows the corresponding percentage changes. As the benchmark investment yield increases, the present value of the reserves decreases. Since the reserves in this illustration have a duration of about three years, a 100 basis point rise in the benchmark investment yield leads to about a 3% decline in the commutation price. The 3% change can be seen between adjacent columns along any row of the exhibit.

As the investment yield increases, the target return on capital increases as well. The higher target return on capital consumes about 45% of the reduction in the commutation price.

Illustration: A increase in the benchmark investment yield from 8.5% to 11.5% leads to a 9.20% reduction in the commutation price. If the 300 basis point increase in the investment yield is coupled with a 300 basis point increase in the target return on capital, the net reduction in the commutation price is only 5.14%, which is about 55% of 9.20%.

Table 3: Sensitivity Analysis – Yields and Returns

Return on Capital	Investment Yield 6.5%	7.5%	8.5%	9.5%	10.5%	11.5%
9.5%	90,177	87,319	84,467	81,619	78,777	75,940
11.0%	91,819	89,035	86,257	83,484	80,716	77,953
12.5%	93,381	90,669	**87,962**	85,260	82,562	79,870
14.0%	94,780	92,226	89,587	86,952	84,322	81,697
15.5%	96,291	93,711	89,137	88,566	86,000	83,439
17.0%	97,647	95,129	92,616	90,108	87,603	85,103

Table 4: Sensitivity Analysis – Yields and Returns (Percentages)

Return on Capital	Investment Yield 6.5%	7.5%	8.5%	9.5%	10.5%	11.5%
9.5%	2.52%	-0.73%	-3.97%	-7.21%	-10.44%	-13.67%
11.0%	4.38%	1.22%	-1.94%	-5.09%	-8.24%	-11.38%
12.5%	6.16%	3.08%	**0.00%**	-3.07%	-6.14%	-9.20%
14.0%	7.85%	4.85%	1.85%	-1.15%	-4.14%	-7.12%
15.5%	9.47%	6.52%	3.61%	0.69%	-2.23%	-5.14%
17.0%	11.01%	8.15%	5.29%	2.44%	-0.41%	-3.25%

STEENECK'S ILLUSTRATION

Lee Steeneck's CAS Exam 6 study note is the source from which most new casualty actuaries learn how to price commutations. Steeneck uses the Connor and Olsen method to price the commutation illustration in his study note. We reprice his illustration, using the method described in this discussion.

Illustration: A block of $1,000,000 in nominal reserves has a three year liquidation pattern, with the payment pattern shown below. The IRS loss reserve discount factors for this accident year at the current and two subsequent valuation dates are shown with the expected loss payments. The tax rate is 35%. The risk-free interest rate is 5% per annum.

Calendar Year	*Paid Loss*	*Year End Reserve*	*IRS Discount Factor*
1998	$0	$1,000,000	0.730
1999	$500,000	$500,000	0.723
2000	$300,000	$200,000	0.741
2001	$200,000	$0	——

To price the commutation, we must add surplus assumptions and the cost of equity capital. We use the same surplus assumptions as for the Connor and Olsen illustration: 10% of written premium plus 15% of held reserves. For the cost of equity capital, we choose 12% per annum. This gives a 700 basis point spread between the risk-free interest rate and the cost of equity capital, which is about right for the insurance industry.

The discounted reserves are

$$\$500,000 / 1.05 + \$300,000 / 1.05^2 + \$300,000 / 1.05^3 = \$921,066.84.$$

Steeneck arrives at a commutation price of $921,770. This is expected. If no account is taken of capital requirements or the cost of holding capital, the commutation price is about equal to the present value of the reserves, using a pre-tax discount rate.

For the revised pricing, we assume capital requirements equal to 10% of written premium and 15% of held reserves, along with a 12% target return on capital. The revised commutation price is $983,671, as shown in the exhibit at the end of this paper.[42]

Multi-Year Illustration

Connor and Olsen show a multi-year illustration as their final example. We re-price the same multi-year illustration using the methods in this discussion. The illustration assumes an 8% pre-tax investment yield. We use a 35% tax rate, a 12.5% cost of equity capital, and surplus requirements equal to 10% of written premium and 15% of held reserves.

The indicated commutation price of $20,717,770 is 16.70% higher than the price derived by Connor and Olsen of $17,753,000; see the pricing worksheets appended to this paper.

THE COMMUTATIONS MARKETPLACE

The commutations transacted in the reinsurance marketplace do not always take account of the cost of holding capital. For primary insurance contracts, differences between the indicated premium and the premium charged stem from market pressures or unusual attributes of the insured. In contrast, commutations are generally priced by actuaries or with actuarial advice. One might expect the transaction prices to closely reflect the indicated prices.

We categorize these reasons for these discrepancies into five groups: (i) financial knowledge, (ii) cost allocation, (iii) reinsurance advice, (iv) foreign reinsurers, and (v) accounting practice.

FINANCIAL KNOWLEDGE

The tax implications of claim commutations and the effects on capital requirements are not simple. Many claims department personnel handling commutation negotiations and actuaries aiding them are unaware of the multiple taxation effects and the other costs of holding capital.

Some actuaries estimate commutation prices from the discounted values of the future loss payments. Pricing "rules of thumb" – such as mark-ups over the cost of goods sold – are used in all industries; insurance is no exception.

One might wonder: "Wouldn't subsequent results show the gain or loss from the commutation? Wouldn't the claims personnel and actuaries learn from the subsequent profit measurement?"

Tax effects are rarely allocated to the particular operations that caused them. This is surely true for the double taxation effects on investment income from capital requirements, whether this capital is embedded in statutory reserves or it remains in policyholders' surplus.

To this day, many actuaries fail to incorporate double taxation costs and the costs of holding capital in their pricing analyses. These topics are not covered on the casualty actuarial syllabus, and many practicing actuaries find this subject perplexing.[43]

Business personnel learn the subjects by which they are measured, and they may avoid spending time on matters that do not affect their performance review. A claims examiner not measured by the return on capital could hardly be expected to have any interest in this subject.

COST ALLOCATION

Cost allocation issues are not simple. The double taxation on equityholders' capital is a real cash flow; it is money paid by the company to the U.S. Treasury. But this money is paid regardless of how the capital is used. Taxes are paid whether the capital is embedded in statutory reserves, forms part of risk-based capital requirements, or sits idly as excess capital.

Some analysts might say that the cost of holding capital is a sunk cost, not a marginal cost, and it should not be considered in pricing. Others say that the cost of holding capital is indeed a marginal cost, since if the capital were not tied up in the commutation transaction, it could be returned to shareholders, thereby avoiding the double taxation (and other costs of holding capital). In economic parlance, this is an opportunity cost, since if the capital were not tied up in the commutation transaction, it could be used to support other endeavors; see Feldblum and Thandi [2003A]. In practice, not all actuaries fully consider the opportunity costs of capital.

REINSURANCE ADVICE

It is sometimes heard in reinsurance circles that the commutation price may be lower than the discounted value of the future claim payments. This is the impression one gets from a superficial reading of the Conner and Olsen paper. In truth, Conner and Olsen say that the commutation price is less than the discounted value at the *after-tax* discount rate. It is approximately equal to the discounted value at the *pre-tax* discount rate, unless

- the IRS loss discount rate (the 60 month moving average of federal mid-term rates) differs material from the actuarial loss discount rate or
- the IRS loss payment pattern differs materially from the actuarial loss payment pattern.

Nonetheless, the impression from the paper is that the actuarial price may be well below the fully discounted value. Many practicing actuaries and claims personnel do not understand the reasoning of the authors. They sense that the result is not correct, but they can not specify exactly what is wrong.

We do not imply that reinsurance actuaries consciously distort their analyses to lower the indicated premiums their companies pay. Rather, all persons' judgments are affected by self-interest or the interests of companies and clients. A reinsurance actuary has no incentive to consider the effects of capital requirements and double taxation. It is human nature to omit these items from research papers and educational notes.[44]

FOREIGN REINSURERS

This discussion implicitly assumes that the reinsurer is subject to the same accounting, tax, and regulatory constraints as the ceding company, as is true for U.S. domiciled reinsurers. Alien reinsurers, particularly those domiciled in the Bermudas or the Cayman Islands, may face more lenient accounting, tax, and surplus requirements.[45]

If the reinsurer is not subject to U.S. accounting and tax constraints, the costs of holding capital for the primary company may be greater than the costs to the reinsurer. The disparity may be so great that it deters the commutation.

The Conner and Olsen method implicitly takes the viewpoint of a lightly regulated reinsurer. In contrast, this discussion is written from the viewpoint of a U.S. regulated primary company. The Conner and Olsen ambivalence point is lower for a lightly regulated reinsurer than for a U.S. domiciled primary insurer.

ACCOUNTING PRACTICE

This implicitly discussion assumes that companies hold full-value loss reserves, with the exception of tabular discounts on workers' compensation pension cases. Some insurers include the medical portions of these claims in the tabular discounts or implicitly discount the medical portions. Other long-term claims, such as medical malpractice claims, products liability claims, and general liability claims, may also be implicitly discounted.

We distinguish three types of discounting and examine their effects on capital requirements and federal income taxes.

	Capital Requirements	*Double Discounting*
Explicit Discounting – Non-tabular	no change	no effect
Explicit Discounting – Tabular	decreases	no effect
Implicit Discounting	decreases	increases present value of taxes

- ♦ Explicit non-tabular discounting does not affect NAIC risk-based capital requirements or the IRS loss reserve discounting procedure.
  - Explicit non-tabular discounts are removed from surplus and added to reserves when computing RBC requirements and adjusted surplus.
  - Explicit non-tabular discounts are added to reserves before computing the IRS discounted reserves.

- ♦ Explicit tabular discounting decreases the NAIC risk-based capital requirements, and it does not affect IRS loss reserve discounting.
  - Explicit tabular discounts are not removed from surplus and they are not added to reserves when computing RBC requirements and adjusted surplus.
  - Explicit non-tabular discounts are added to reserves before computing the IRS discounted reserves.

- ♦ Implicit discounting decreases the NAIC risk-based capital requirements, and it decreases the tax basis reserves, thereby raising the present value of the tax liability. Since the discount is not disclosed in the Annual Statement, it is not removed from surplus or added to reserves.

DISCOUNTING AND DOUBLE TAXATION

In the simplified one year illustration with which this paper begins, the commutation is effected on January 1, and the claim is settled by December 31. If the primary company reports discounted reserves on its statutory financial statements (with no disclosure), it holds assets equal to the discounted reserves. If the discounting is at the pre-tax investment yield, and if no additional capital is needed (that is, if the company needs no surplus), the shareholders need not contribute any capital when the commutation is effected. In this case, the commutation price is the discounted value of the future loss payments.

If the commutation involves cash flows in more than one tax year and the discounting is implicit, tax payments are speeded up, and the primary company suffers a tax loss from the commutation.

DISCOUNTING AND CAPITAL REQUIREMENTS

Explicit non-tabular discounts are removed from surplus and added to reserves before calculating the risk-based capital reserving risk charge. Neither the capital requirements nor the risk-based capital ratio are affected by explicit non-tabular discounts. Implicit discounts reduce the RBC capital requirements. Rating agencies which discern the discounting may lower the company's rating, so implicit discounting is not a panacea for capital constraints.

Explicit tabular discounts have the best of both worlds. They reduce capital requirements by more than the amount of the discount, are they are added to reserves for IRS loss reserve discounting. Explicit discounts are carefully prescribed by the NAIC accounting rules: They may be used only on the indemnity portion of workers' compensation permanent total disability and fatality cases and on long term disability health claims. They may not be applied to medical benefits or loss adjustment expenses; see Feldblum [2002: SchP].

CONCLUSION: INSURANCE PRICING, CAPITAL REQUIREMENTS, AND FEDERAL INCOME TAXES

In the past, casualty actuaries priced commutations and other products without considering capital requirements and federal income taxes. Common rationales were the following.

- Before the Tax Reform Act of 1986, most property-casualty insurance companies paid little federal income tax. When the volume of business was growing, whether because of increases in exposures or inflation, the underwriting losses occasioned by increasing amounts of pre-paid acquisition costs and full value loss reserves offset most of the taxable investment income. Three provisions of the 1986 Tax Reform Act substantially raised the tax liabilities for property-casualty insurance companies:

  - The revenue offset provision defers the tax deduction for pre-paid acquisition costs and spreads it over the policy term.
  - The loss reserve discounting provision allows an offset to taxable income only for the change in discounted reserves, not for the change in full value loss reserves.
  - The proration provision reduces the benefit of tax exempt investment income for insurance companies, effectively eliminating this investment vehicle for them.

  Before 1986, property-casualty insurance companies were in an enviable tax position. After 1986, they are in a "tax-plus" position, with an effective tax rate above that of most other industries.[46] Taxes are now a critical part of accurate pricing.

- Taxes are complex, and casualty actuaries lack the expertise to deal with them. This statement may be true, but the lesson is inverted. Since taxes are complex, actuaries must be sure to properly account for them in their pricing procedures. Just as ignorance of the law is no excuse for trespass, complexity of the law is no excuse for disregard.

- Before the advent of risk-based capital requirements in 1994, it was difficult to measure capital requirements for property-casualty insurance products. Actuaries developed theoretical models showing the amount of capital that *ought* to be held by an insurance company. These models may have been wonderful research, but they did not address the issue of the capital *required* to be held by insurance companies. After 1994, with the NAIC risk-based capital requirements and the similar rating agency capital requirements, the analysis of required capital can be included in actuarial ratemaking.[47]

This discussion deals with commutations, which are the reverse of retroactive reinsurance. The pricing of commutations is equally applicable to reinsurance pricing, whether prospective or retroactive and whether finite or standard.

[1] Vincent P. Connor and Richard A. Olsen, "Commutation Pricing in the Post Tax-Reform Era," *Proceedings of the Casualty Actuarial Society*, Volume 78 (1991), pages 81-109; Lee Steeneck, "Commutation of Claims," CAS Exam 6 study note, 1998.

[2] The full quotation is as follows: "In certain instances, the commutation price developed under this methodology can be negative. This can occur when there is a great mismatch between the payment profit / interest rate used to develop tax-basis discounted reserves and the payment profit / interest rate used to calculate the present value of the losses. Specifically, the tax-basis discounted reserves are substantially higher than the present value of the losses. This leads to the tax on the underwriting gain/loss becoming greater than the cost of not commuting. In cases of reinsurance of long-tailed lines, such as workers' compensation, where the overall industry average reinsurance payment profile is quite short relative to the actual payment profile, negative commutation values can be expected frequently. In these situations, commutations are not favored."

[3] The Standard of Practice adds: "The actuary may consider adjusting this rate if the amount of discount for tax purposes differs significantly from the amount of discount determined in accordance with this standard."

[4] Conner and Olsen use the term "nominal interest rate" to mean the pre-tax interest rate. See also page 94: "If the IRS payment profiles and interest rates equal the factors used to determine the present value of the losses, then the commutation price will equal the present value of the losses using the nominal interest rate."

[5] See also Atkinson and Dallas [2000], who have the same perspective.

[6] One might wonder: "There is $5,250 of investment income in this illustration. Were there no commutation, the IRS would receive 35% of the investment income, or 35% × $5,250 = $1,837.50. If the commutation is effected, the IRS receives only $131.25. Where did the rest of the tax liability go?" Answer: The reinsurer's cash payment is about $5,000 less than its reserve takedown, giving it an underwriting gain for this amount. It pays federal income taxes on this underwriting gain; the IRS has not lost any money.

The IRS loss reserve discounting procedure modifies the underwriting gain or loss of the reinsurer and the primary company by offsetting amounts. The total tax liability is not changed, unless one of the parties is not subject to U.S. federal income taxation.

[7] There are some differences, such as the loss reserve discount factors used by ceding and assuming companies for the same transaction; these are generally not material.

[8] Some analysts argue that a risk-free rate is appropriate if there is no systematic risk to the underwriting operations. More precisely, they argue that shareholders ought to be satisfied with a risk-free rate of return if the insurer invests only in Treasury securities and if the market value effects of the loss cash flows are not correlated with the cash flows of the overall securities markets.

This perspective, commonly associated with the pricing models of Fairley, Kahane, Hill, Myers, and Cohn, has been used in Massachusetts private passenger automobile and workers' compensation ratemaking. It has not been used in competitive insurance markets, nor has it found acceptance in the actuarial community. For discussion of this "underwriting beta" perspective, see Kozik [1994].

9 The exact magnitude of the market risk premium is unclear; figures between 7% and 9% are commonly used. Some analysts claim that the market risk premium has narrowed in recent years, resulting in the high P/E multiples common in the late 1990's. Other analysts believe that the high P/E multiples stem from the heady bull markets of the 1990's and will gradually subside to their longer term averages. The stock market decline in 2000-2002 supports the latter view.

10 Cf. Myers and Cohn [1987], who first drew attention to this topic.

11 See Atkinson and Dallas [2000], chapters 8 and 11, for a life insurance pricing example.

12 It may seem surprising that cost of holding capital is greater than the required return on capital. Half of this stems from the market risk premium of 7% per annum. The other half stems from the two layers of additional taxes: one on the underwriting income and one on the investment income. For each dollar of policy premium in excess of discounted losses and expenses in the casualty lines of business, the IRS takes between 60¢ and 80¢.

13 The loss concentration factor equals 70% + 30% x (reserves in largest line)/(total reserves). If the total reserves are split evenly among five lines of business, the loss concentration factor is 70% + 30% x 1/5 = 76%, giving a 24% reduction in the reserving risk charge.

14 In practice, companies do not always book full value loss reserves for long duration claims, which form the bulk of many commutations. Company booking practices vary with the stage of the underwriting cycle and the reserving philosophy of the company. The average implicit discount ranges from zero percent for the most conservative companies to about 20% for less conservative companies.

Implicit reserve discounts reduce capital requirements. For exact pricing, one should use the marginal effect of the reserving risk charge along with the implicit discount in the company's reserves. For the illustrations here, we assume full value loss reserves and we use the "60%" low end of the marginal effects.

15 The present value of the taxes on underwriting income does depend on the line of business, since the IRS loss reserve discount factors differ by line.

16 For the illustrations here, the relative sizes of the margins is $781.25 / $298.25 = 2.619 if the equityholders demand a risk-free rate and $3,551.14 / $1,355.68 = 2.619 if the equityholders demand an equity return. The relative sizes of the margin does *not* depend on the target return on capital.

17 Overall, the IRS loss reserve discount factors are not materially biased over the long run (though they be biased in the short run or for a specific block of business). Since we are using a simplified illustration, we use corresponding discount factors.

18 See, for instance, Conner and Olsen (page 86): "Thus, the present value of the tax benefit on the unwinding of the discount is calculated to be . . . ." and *passim* through much of the paper.

19 After all the mathematics, it pays to examine the actual rate of return by writing out the cash flows from the commutation. If the actual rate of return is lower than expected, the pricing calculations may not be correct.

20 In most scenarios, we add the direct charge or credit to surplus to the reported statutory income when making the comparison with taxable income. If future tax rate changes are anticipated, we use the expected tax rate when the timing difference are expected to reverse.

[21] If the reversal of the deferred tax asset or liability is uncertain, GAAP financial statements may use a "valuation allowance" to eliminate or reduce the deferred tax asset or liability. If the reversal is expected to occur several years in the future, some GAAP accountants use the present value of the deferred tax asset or liability (though this is not common GAAP practice). See the Financial Accounting Standards Board, Discussion Memorandum, an analysis of issues related to *Present Value-Based Measurements in Accounting* (December 1990) and White, Gerald I., Ashwinpaul C. Sondhi, and Dov Fried, *The Analysis and Use of Financial Statements*, 2nd edition (Wiley 1998).

[22] The financial community uses the term "free cash flow" instead of implied equity flow. Atkinson and Dallas [2000], chapter 11, use the term "distributable earnings" instead of implied equity flows.

The Actuarial Standards Board, "Actuarial Standard of Practice No. 19: Actuarial Appraisals" (October 1991), page 4, has the same view of distributable earnings: "5.2.1 Distributable Earnings – For insurance companies, statutory earnings form the basis for determining distributable earnings, since the availability of dividends to owners is constrained by the amount of accumulated earnings and minimum capital and surplus requirements, both of which must be determined on a statutory accounting basis. Distributable earnings consist of statutory earnings, adjusted as appropriate to allow for the retention of a portion thereof or the release of a portion of prior accumulated earnings therein, in recognition of minimum capital and surplus levels necessary to support existing business. . . . Economic value generally is determined as the present value of future cash flows. Statutory accounting determines the earnings available to the owner. Hence, while future earnings calculated according to generally accepted accounting principles (GAAP) will often be of interest to the user of an actuarial appraisal, as may other patterns of earnings, the discounted present-value calculations contemplated within the definition of actuarial appraisal in this standard should be developed in consideration of statutory earnings, rather than some other basis. . . . The actuary's report should include a discussion of factors, such as capital needs (whether perceived by the actuary or imposed by an external entity such as a regulator), that may cause the earnings available for shareholder or policyholder distribution to be different from statutory earnings."

[23] For a summary of U.S. tax law pertaining to property-casualty insurance companies, along with the post-codification statutory accounting rules pertaining to federal income taxes, see Appendix A of Feldblum and Thandi [2002], "Modeling the Equity Flows."

[24] For the multi-period illustrations below, we use actual IRS loss reserve discount factors.

[25] The booked income, or the book income, is the income shown on the company's accounting books. Booked income may be either statutory income or GAAP income. For computing deferred tax assets and liabilities, the booked income is not the income in the earnings statement (or income statement). Rather, it is the income implied by the balance sheet entries at the beginning of the year and the end of the year. In other words, the booked income used for computing deferred tax assets and liabilities is the income on the earnings statement plus direct credits to surplus minus direct charges to surplus. Credits and charges to surplus stemming from changes in the deferred tax assets and liabilities are not included in this computation.

[26] IRS loss reserve discount factors are computed to six decimal places. We show eight significant digits in this illustration to avoid rounding errors.

[27] This is one of the few occasions when the implied equity flows depend not just on the book of business being priced but on all operations of the company.

[28] The IRS promulgation of the industry factors is generally delayed until the summer months, if not later, though the formula for the factors is specified in the Internal Revenue Code.

[29] By gross deferred tax assets, we mean gross of statutory admissibility tests. The gross deferred tax assets are shown on GAAP financial statements.

[30] As noted earlier, using the actual probability of death does not change the result.

[31] The probabilities of death in each year do affect the tabular discount on the reserve. However, once the tabular discount has been determined, the probabilities of death do not affect the admitted portion of the deferred tax asset on the statutory balance sheet.

[32] The tabular discounts are disclosed in the notes to the financial statements; only non-tabular discounts are disclosed in Schedule P, Part 1.

[33] A more exact calculation would use actuarial present values, which include the mortality pattern.

[34] For simplicity, we estimate the discounted reserve using an annuity certain with a term equal to the claimant's life expectancy; the slight inaccuracy is not material.

[35] We assume a cost of equity capital for property-casualty insurers about 700 basis points above the Treasury bill rate. The average investment yield for property-casualty insurers is about 300 basis points above the Treasury bill rate. The 400 basis point spread is the difference between these two figures.

The commutation price depends on whether one uses the company's investment yield for discounting or the risk-free interest rate for discounting. Connor and Olsen uses the investment yield for discounting, so one must use the full market risk premium (times the equity beta for property-casualty insurance companies) to determine the equity return for investors.

[36] The exhibits at the end of this paper show the results for higher surplus assumptions as well.

[37] The premium to surplus ratio here is about 2 to 1, which accords with the (revised) Kenney rule of thumb. A lower premium to surplus ratio increases the commutation price.

[38] The assumption that taxes are paid at the end of the year is not actually correct, since corporate taxpayers pre-pay their taxes over the course of the year.

[39] The commutation actually causes a slight release of capital equal to the credit charge for reinsurance recoverables.

[40] This might have been a hindrance years ago; now it is no longer a concern. Most computer spreadsheets use iterative methods for all computations of the sort done here.

[41] Since the loss reserves are shown at undiscounted values on the statutory balance sheet, whereas much of the capital requirements in this illustration are needed in future years, we use the ratio of premium to discounted capital.

The overall industry premium to surplus ratio has been about one to one during the 1990's and the early years of the twenty-first century. However, much of the equity supports pricing risk, not reserving risk. A commutation does not have pricing risk, and a higher premium to surplus ratio is appropriate.

[42] Steeneck uses a risk-free rate instead of the company's investment yield. This does not materially change the capital requirements, but it does change the target return on capital expected by investors.

[43] This paper was stimulated by a fellow actuary's request to examine a proposed commutation price. The actuary knew the price was unreasonable, but he was stymied by the tax rationale provided by the reinsurer.

[44] The early 20th century social theorist Max Weber highlighted the difficulties inherent in objective analysis of social behavior; see especially Weber [1975: The Interpretation of Social Reality].

[45] Conner and Olsen mention that even if both the primary company and the reinsurer are domiciled in the U.S., differences in line of business coding for non-proportional reinsurance affects the IRS loss reserve discount factors and therefore the ambivalence point. The differences in the tax liabilities and capital requirements for U.S. vs off-shore reinsurers are even greater.

[46] Other industries pay no tax on tax-exempt investment income, and they enjoy various tax benefits for accelerated depreciation, investment tax credits, and similar items. The property-casualty insurance industry has a 5.25% effective tax rate on tax exempt investment income, and it has no tax benefits applicable to its operations.

[47] Cf Daykin, Pentikäinen, and Pesonen [1994] and Philbrick [2001].

Feldblum Example

Commuted on January 1 of Year 1

Price	105,959	0	20% of WP	Inv Yield	Tax
			25% of Reserves	5.0%	35%

1

105,000

Time	WP	Paid Loss	Nominal Reserve	Surplus	Held Asset	Non-income producing	Income producing	Inv Income	IRS Disc factors	Taxable UW income
0	105,959	0	105,000	47,442	152,442	0	152,442	0	1.000	
1	0	105,000	0	0	0	0	0	7,622	1.000	959

ROE	12.0%
NPV	0
IRR	12.00%

						COMPANY FLOWS					
Time	Tax paid UW	Taxable Inv Income	Tax paid Inv income	Total tax paid	DTA Disc Reserve	UW	Inv Inc	Asset	Tax	DTA	Equity Flow
0		0	0	0		105,959	0	152,442	0	0	-46,483
1	336	7,622	2,668	3,003	0	-105,000	7,622	-152,442	-3,003	0	52,060

Feldblum Example

Commuted on December 31 of Year 0

Price	106,096	0			
			20% of WP	Inv Yield	Tax
			25% of Reserves	5.0%	35%
			1		
		105,000			

Time	WP	Paid Loss	Nominal Reserve	Surplus	Held Asset	Non-income producing	Income producing	Inv Income	IRS Disc factors	Taxable UW income
0	106,096	0	105,000	47,469	152,469	1,750	150,719	0	0.952	6,096
1	0	105,000	0	0	0	0	0	7,536	1.000	-5,000

ROE	12.0%
NPV	0
IRR	12.00%

						COMPANY FLOWS					
Time	Tax paid UW	Taxable Inv Income	Tax paid Inv Income	Total tax paid	DTA Disc Reserve	UW	Inv Inc	Asset	Tax	DTA	Equity Flow
0	2,134	0	0	2,134	1,750	106,096	0	152,469	-2,134	1,750	-46,757
1	-1,750	7,536	2,638	888	0	-105,000	7,536	-152,469	-888	-1,750	52,368

Connor&Olsen Table 2 Example

Commuted on January 1 of Year 1

Price	89,978	0			
			20% of WP	Inv Yield	Tax
	79,854		25% of Reserves	8.5%	35%
			1		
		100,000			

Time	WP	Paid Loss	Nominal Reserve	Surplus	Held Asset	Non-income producing	Income producing	Inv Income	IRS Disc factors	Taxable UW income
0	89,978	0	100,000	42,996	142,996	0	142,996	0	0.788	
1	0	20,000	80,000	20,000	100,000	1,949	98,051	12,155	0.819	4,466
2	0	20,000	60,000	15,000	75,000	1,520	73,480	8,334	0.851	-5,569
3	0	20,000	40,000	10,000	50,000	1,054	48,946	6,246	0.886	-4,342
4	0	20,000	20,000	5,000	25,000	548	24,452	4,160	0.922	-3,011
5	0	20,000	0	0	0	0	0	2,078	0.922	-1,567

ROE	12.5%
NPV	0
IRR	12.50%

						COMPANY FLOWS					
Time	Tax paid UW	Taxable Inv income	Tax paid Inv income	Total tax paid	DTA Disc Reserve	UW	Inv Inc	Asset	Tax	DTA	Equity Flow
0		0	0	0		89,978	0	142,996	0	0	-53,017
1	1,563	12,155	4,254	5,817	1,949	-20,000	12,155	-42,996	-5,817	1,949	31,282
2	-1,949	8,334	2,917	968	1,520	-20,000	8,334	-25,000	-968	-429	11,937
3	-1,520	6,246	2,186	666	1,054	-20,000	6,246	-25,000	-666	-466	10,114
4	-1,054	4,160	1,456	402	548	-20,000	4,160	-25,000	-402	-505	8,253
5	-548	2,078	727	179	0	-20,000	2,078	-25,000	-179	-548	6,351

Connor&Olsen Table 2 Example
Commuted on December 31 of Year 0

Price	91,846	0			
			20% of WP	Inv Yield	Tax
	79,854		25% of Reserves	8.0%	35%
			1		
		100,000			

Time	WP	Paid Loss	Nominal Reserve	Surplus	Held Asset	Non-Income producing	Income producing	Inv Income	IRS Disc factors	Taxable UW income
0	91,846	0	100,000	43,369	143,369	2,236	141,133	0	0.799	11,992
1	0	20,000	80,000	20,000	100,000	1,855	98,145	11,291	0.828	-6,388
2	0	20,000	60,000	15,000	75,000	1,443	73,557	7,852	0.859	-5,299
3	0	20,000	40,000	10,000	50,000	999	49,001	5,885	0.892	-4,123
4	0	20,000	20,000	5,000	25,000	519	24,481	3,920	0.926	-2,853
5	0	20,000	0	0	0	0	0	1,959	0.926	-1,481

ROE	12.5%
NPV	0
IRR	12.50%

						COMPANY FLOWS					
Time	Tax paid UW	Taxable Inv income	Tax paid Inv income	Total tax paid	DTA Disc Reserve	UW	Inv Inc	Asset	Tax	DTA	Equity Flow
0	4,197	0	0	4,197	2,236	91,846	0	143,369	-4,197	2,236	-53,485
1	-2,236	11,291	3,952	1,716	1,855	-20,000	11,291	-43,369	-1,716	-381	32,563
2	-1,855	7,852	2,748	893	1,443	-20,000	7,852	-25,000	-893	-412	11,547
3	-1,443	5,885	2,060	616	999	-20,000	5,885	-25,000	-616	-445	9,824
4	-999	3,920	1,372	373	519	-20,000	3,920	-25,000	-373	-480	8,067
5	-519	1,959	685	167	0	-20,000	1,959	-25,000	-167	-519	6,273

Steeneck Example
Commuted on January 1 of Year 1

Price 974,956

921,770

10% of WP
15% of Reserves

Inv Yield 5.0%

Tax 35%

1,000,000

Time	WP	Paid Loss	Nominal Reserve	Surplus	Held Asset	Non-income producing	Income producing	Inv Income	IRS Disc factors	Taxable UW income
0	974,956	0	1,000,000	247,496	1,247,496	0	1,247,496	0	0.730	
1	0	500,000	500,000	75,000	575,000	30,393	544,607	62,375	0.723	113,690
2	0	300,000	200,000	30,000	230,000	18,164	211,836	27,230	0.741	-86,837
3	0	200,000	0	0	0	0	0	10,592	1.000	-51,897

ROE 12.0%

NPV 0
IRR 12.00%

						COMPANY FLOWS					
Time	Tax paid UW	Taxable Inv income	Tax paid Inv income	Total tax paid	DTA Disc Reserve	UW	Inv Inc	Asset	Tax	DTA	Equity Flow
0		0	0	0		974,956	0	1,247,496	0	0	-272,540
1	39,791	62,375	21,831	61,623	30,393	-500,000	62,375	-672,496	-61,623	30,393	203,641
2	-30,393	27,230	9,531	-20,862	18,164	-300,000	27,230	-345,000	20,862	-12,229	80,864
3	-18,164	10,592	3,707	-14,457	0	-200,000	10,592	-230,000	14,457	-18,164	36,885

Steeneck Example
Commuted on December 31 of Year 0

Price 983,671

921,770

10% of WP
15% of Reserves

Inv Yield 5.0%

Tax 35%

1,000,000

Time	WP	Paid Loss	Nominal Reserve	Surplus	Held Asset	Non-income producing	Income producing	Inv Income	IRS Disc factors	Taxable UW Income
0	983,671	0	1,000,000	248,367	1,248,367	45,945	1,202,422	0	0.730	253,676
1	0	500,000	500,000	75,000	575,000	30,393	544,607	60,121	0.723	-131,271
2	0	300,000	200,000	30,000	230,000	18,164	211,836	27,230	0.741	-86,837
3	0	200,000	0	0	0	0	0	10,592	1.000	-51,897

ROE 12.0%

NPV 0
IRR 12.00%

						COMPANY FLOWS					
Time	Tax paid UW	Taxable Inv Income	Tax paid Inv income	Total tax paid	DTA Disc Reserve	UW	Inv Inc	Asset	Tax	DTA	Equity Flow
0	88,786	0	0	88,786	45,945	983,671	0	1,248,367	-88,786	45,945	-307,538
1	-45,945	60,121	21,042	-24,902	30,393	-500,000	60,121	-673,367	24,902	-15,552	242,839
2	-30,393	27,230	9,531	-20,862	18,164	-300,000	27,230	-345,000	20,862	-12,229	80,864
3	-18,164	10,592	3,707	-14,457	0	-200,000	10,592	-230,000	14,457	-18,164	36,885

Connor&Olsen Exhibit Example
Commuted on June 30 of Year 1

Price				
20,717,770	0			10% of WP
17,753,000				15% of Reserves
				1

	24,501,000		
3,498,000	6,999,000	6,001,000	8,003,000

Time	WP	Acc Year 85 Paid Losses	Acc Year 86 Paid Losses	Acc Year 87 Paid Losses	Acc Year 88 Paid Losses	Acc Year 85 Nominal Reserve	Acc Year 86 Nominal Reserve	Acc Year 87 Nominal Reserve	Acc Year 88 Nominal Reserve	Surplus	Held Asset
0	20,717,770	-	-	-	-	3,498,000	6,999,000	6,001,000	8,003,000	5,746,927	30,247,927
1	0	330,000	556,000	449,000	736,000	3,168,000	6,443,000	5,552,000	7,267,000	3,364,500	25,794,500
2	0	594,000	1,111,000	816,000	1,011,000	2,574,000	5,332,000	4,736,000	6,256,000	2,834,700	21,732,700
3	0	528,000	1,000,000	816,000	920,000	2,046,000	4,332,000	3,920,000	5,336,000	2,345,100	17,979,100
4	0	396,000	889,000	735,000	920,000	1,650,000	3,443,000	3,185,000	4,416,000	1,904,100	14,598,100
5	0	330,000	667,000	653,000	828,000	1,320,000	2,776,000	2,532,000	3,588,000	1,532,400	11,748,400
6	0	264,000	556,000	490,000	736,000	1,056,000	2,220,000	2,042,000	2,852,000	1,225,500	9,395,500
7	0	198,000	444,000	408,000	552,000	858,000	1,776,000	1,634,000	2,300,000	985,200	7,553,200
8	0	198,000	333,000	327,000	460,000	660,000	1,443,000	1,307,000	1,840,000	787,500	6,037,500
9	0	198,000	333,000	245,000	368,000	462,000	1,110,000	1,062,000	1,472,000	615,900	4,721,900
10	0	132,000	333,000	245,000	276,000	330,000	777,000	817,000	1,196,000	468,000	3,588,000
11	0	132,000	222,000	245,000	276,000	198,000	555,000	572,000	920,000	336,750	2,581,750
12	0	66,000	222,000	163,000	276,000	132,000	333,000	409,000	644,000	227,700	1,745,700
13	0	66,000	111,000	163,000	184,000	66,000	222,000	246,000	460,000	149,100	1,143,100
14	0	66,000	111,000	82,000	184,000	0	111,000	164,000	276,000	82,650	633,650
15	0	-	111,000	82,000	92,000	0	0	82,000	184,000	39,900	305,900
16	0	-	-	82,000	92,000	0	0	0	92,000	13,800	105,800
17	0	-	-	-	92,000	0	0	0	0	0	0

Inv Yield 8.0%

Tax 35%

Time	Non-income producing	Income producing	Inv Income	Acc Year 85 IRS Disc factors	Acc Year 86 IRS Disc factors	Acc Year 87 IRS Disc factors	Acc Year 88 IRS Disc factors	Total Disc. Reserve	Taxable UW Income	Tax paid UW	Taxable Inv Income	Tax paid Inv Income
0	0	30,247,927	0	0.729	0.759	0.777	0.787	18,818,030		0	0	0
1	200,296	25,594,204	1,209,917	0.717	0.729	0.759	0.764	16,728,634	1,918,136	671,348	1,209,917	423,471
2	233,253	21,499,447	2,047,536	0.714	0.717	0.729	0.745	13,768,908	-572,274	-200,296	2,047,536	716,638
3	296,662	17,682,438	1,719,956	0.716	0.714	0.717	0.713	11,171,346	-666,437	-233,253	1,719,956	601,985
4	285,154	14,312,946	1,414,595	0.747	0.716	0.714	0.704	9,078,951	-847,605	-296,662	1,414,595	495,108
5	291,970	11,456,430	1,145,036	0.780	0.747	0.716	0.697	7,415,675	-814,724	-285,154	1,145,036	400,763
6	218,400	9,177,100	916,514	0.818	0.780	0.747	0.731	6,203,874	-834,199	-291,970	916,514	320,780
7	168,931	7,384,269	734,168	0.860	0.818	0.780	0.766	5,225,874	-624,000	-218,400	734,168	256,959
8	130,396	5,907,104	590,742	0.909	0.860	0.818	0.805	4,390,533	-482,659	-168,931	590,742	206,760
9	89,958	4,631,942	472,568	0.966	0.909	0.860	0.850	3,619,093	-372,560	-130,396	472,568	165,399
10	52,790	3,535,210	370,555	0.966	0.966	0.909	0.902	2,890,117	-257,024	-89,958	370,555	129,694
11	8,941	2,572,809	282,817	0.966	0.966	0.966	0.963	2,165,945	-150,828	-52,790	282,817	98,986
12	6,431	1,739,269	205,825	0.966	0.966	0.966	0.963	1,464,489	-25,544	-8,941	205,825	72,039
13	5,462	1,137,638	139,142	0.966	0.966	0.966	0.963	958,863	-18,373	-6,431	139,142	48,700
14	3,490	630,160	91,011	0.966	0.966	0.966	0.963	531,469	-15,606	-5,462	91,011	31,854
15	2,163	303,737	50,413	0.966	0.966	0.966	0.963	256,441	-9,973	-3,490	50,413	17,644
16	1,182	104,618	24,299	0.966	0.966	0.966	0.963	88,621	-6,180	-2,163	24,299	8,505
17	0	0	8,369	0.966	0.966	0.966	0.963	0	-3,379	-1,182	8,369	2,929

ROE	12.5%
NPV	0

							COMPANY FLOWS					
Time	Total tax paid	Acc Year 85 DTA Reserve Disc	Year 86 DTA Reserve Disc	Year 87 DTA Reserve Disc	Year 88 DTA Reserve Disc	Total DTA	UW	Inv Inc	Asset	Tax	DTA	Equity Flow
0	0						20,717,770	0	30,247,927	0	0	-9,530,157
1	1,094,819	55,965	83,805	19,079	41,447	200,296	-2,071,000	1,209,917	-4,453,427	-1,094,819	200,296	2,697,821
2	516,342	54,871	94,219	61,537	22,626	233,253	-3,532,000	2,047,536	-4,061,800	-516,342	32,957	2,093,952
3	368,731	56,837	92,385	69,250	78,190	296,662	-3,264,000	1,719,956	-3,753,600	-368,731	63,409	1,904,233
4	198,447	44,732	95,700	67,862	76,859	285,154	-2,940,000	1,414,595	-3,381,000	-198,447	-11,508	1,645,640
5	115,609	34,129	75,320	70,333	112,189	291,970	-2,478,000	1,145,036	-2,849,700	-115,609	6,816	1,407,943
6	28,810	25,344	57,399	55,328	80,329	218,400	-2,046,000	916,514	-2,352,900	-28,810	-73,570	1,121,034
7	38,559	20,959	42,625	42,260	63,086	168,931	-1,602,000	734,168	-1,842,300	-38,559	-49,470	886,440
8	37,829	15,609	35,250	31,366	48,172	130,396	-1,318,000	590,742	-1,515,700	-37,829	-38,534	712,078
9	35,003	1,578	26,251	25,940	36,189	89,958	-1,144,000	472,568	-1,315,600	-35,003	-40,438	568,728
10	39,736	1,578	2,655	19,320	29,236	52,790	-986,000	370,555	-1,133,900	-39,736	-37,169	441,551
11	46,196	789	2,655	1,949	3,547	8,941	-875,000	282,817	-1,006,250	-46,196	-43,849	324,021
12	63,098	789	1,327	1,949	2,365	6,431	-727,000	205,825	-836,050	-63,098	-2,510	249,267
13	42,269	789	1,327	981	2,365	5,462	-524,000	139,142	-602,600	-42,269	-969	174,504
14	26,392	0	1,327	981	1,182	3,490	-443,000	91,011	-509,450	-26,392	-1,972	129,098
15	14,154	0	0	981	1,182	2,163	-285,000	50,413	-327,750	-14,154	-1,327	77,681
16	6,342	0	0	0	1,182	1,182	-174,000	24,299	-200,100	-6,342	-981	43,077
17	1,747	0	0	0	0	0	-92,000	8,369	-105,800	-1,747	-1,182	19,240

Connor&Olsen Exhibit Example
Commuted on December 31 of Year 0

Price 17,476,775 0

17,753,000

20% of WP
25% of Reserves

		24,501,000								0	
		3,498,000	6,999,000	6,001,000	8,003,000						
Time	WP	Acc Year 85 Paid Losses	Acc Year 86 Paid Losses	Acc Year 87 Paid Losses	Acc Year 88 Paid Losses	Acc Year 85 Nominal Reserve	Acc Year 86 Nominal Reserve	Acc Year 87 Nominal Reserve	Acc Year 88 Nominal Reserve	Surplus	Held Asset
0	17,476,775	-	-	-	-	3,498,000	6,999,000	6,001,000	8,003,000	0	24,501,000
1	0	330,000	556,000	449,000	736,000	3,168,000	6,443,000	5,552,000	7,267,000	0	22,430,000
2	0	594,000	1,111,000	816,000	1,011,000	2,574,000	5,332,000	4,736,000	6,256,000	0	18,898,000
3	0	528,000	1,000,000	816,000	920,000	2,046,000	4,332,000	3,920,000	5,336,000	0	15,634,000
4	0	396,000	889,000	735,000	920,000	1,650,000	3,443,000	3,185,000	4,416,000	0	12,694,000
5	0	330,000	667,000	653,000	828,000	1,320,000	2,776,000	2,532,000	3,588,000	0	10,216,000
6	0	264,000	556,000	490,000	736,000	1,056,000	2,220,000	2,042,000	2,852,000	0	8,170,000
7	0	198,000	444,000	408,000	552,000	858,000	1,776,000	1,634,000	2,300,000	0	6,568,000
8	0	198,000	333,000	327,000	460,000	660,000	1,443,000	1,307,000	1,840,000	0	5,250,000
9	0	198,000	333,000	245,000	368,000	462,000	1,110,000	1,062,000	1,472,000	0	4,106,000
10	0	132,000	333,000	245,000	276,000	330,000	777,000	817,000	1,196,000	0	3,120,000
11	0	132,000	222,000	245,000	276,000	198,000	555,000	572,000	920,000	0	2,245,000
12	0	66,000	222,000	163,000	276,000	132,000	333,000	409,000	644,000	0	1,518,000
13	0	66,000	111,000	163,000	184,000	66,000	222,000	246,000	460,000	0	994,000
14	0	66,000	111,000	82,000	184,000	0	111,000	164,000	276,000	0	551,000
15	0	-	111,000	82,000	92,000	0	0	82,000	184,000	0	266,000
16	0	-	-	82,000	92,000	0	0	0	92,000	0	92,000
17	0	-	-	-	92,000	0	0	0	0	0	0

Inv Yield 8.0%

Tax 34%

Non-income producing	Income producing	Inv Income	Acc Year 85 IRS Disc factors	Acc Year 86 IRS Disc factors	Acc Year 87 IRS Disc factors	Acc Year 88 IRS Disc factors	Total Disc. Reserve	Taxable UW Income	Tax paid UW	Taxable Inv income	Tax paid Inv Income	Total tax paid
-6,255	24,507,255	0	0.729	0.759	0.777	0.787	18,818,030	-1,341,255	-456,027	0	0	-456,027
194,573	22,235,427	1,960,580	0.717	0.729	0.759	0.764	16,728,634	18,396	6,255	1,960,580	666,597	672,852
226,589	18,671,411	1,778,834	0.714	0.717	0.729	0.745	13,768,908	-572,274	-194,573	1,778,834	604,804	410,230
288,186	15,345,814	1,493,713	0.716	0.714	0.717	0.713	11,171,346	-666,437	-226,589	1,493,713	507,862	281,274
277,006	12,416,994	1,227,665	0.747	0.716	0.714	0.704	9,078,951	-847,605	-288,186	1,227,665	417,406	129,220
283,628	9,932,372	993,359	0.780	0.747	0.716	0.697	7,415,675	-814,724	-277,006	993,359	337,742	60,736
212,160	7,957,840	794,590	0.818	0.780	0.747	0.731	6,203,874	-834,199	-283,628	794,590	270,161	-13,467
164,104	6,403,896	636,627	0.860	0.818	0.780	0.766	5,225,874	-624,000	-212,160	636,627	216,453	4,293
126,671	5,123,329	512,312	0.909	0.860	0.818	0.805	4,390,533	-482,659	-164,104	512,312	174,186	10,082
87,388	4,018,612	409,866	0.966	0.909	0.860	0.850	3,619,093	-372,560	-126,671	409,866	139,355	12,684
51,281	3,068,719	321,489	0.966	0.966	0.909	0.902	2,890,117	-257,024	-87,388	321,489	109,306	21,918
8,685	2,236,315	245,497	0.966	0.966	0.966	0.963	2,165,945	-150,828	-51,281	245,497	83,469	32,188
6,247	1,511,753	178,905	0.966	0.966	0.966	0.963	1,464,489	-25,544	-8,685	178,905	60,828	52,143
5,306	988,694	120,940	0.966	0.966	0.966	0.963	958,863	-18,373	-6,247	120,940	41,120	34,873
3,391	547,609	79,096	0.966	0.966	0.966	0.963	531,469	-15,606	-5,306	79,096	26,892	21,586
2,101	263,899	43,809	0.966	0.966	0.966	0.963	256,441	-9,973	-3,391	43,809	14,895	11,504
1,149	90,851	21,112	0.966	0.966	0.966	0.963	88,621	-6,180	-2,101	21,112	7,178	5,077
0	0	7,268	0.966	0.966	0.966	0.963	0	-3,379	-1,149	7,268	2,471	1,322

ROE	8.0%
NPV	0
IRR	8.00%

Acc Year 85 DTA Reserve Disc	Year 86 DTA Reserve Disc	Year 87 DTA Reserve Disc	Year 88 DTA Reserve Disc	Total DTA	COMPANY FLOWS UW	Inv Inc	Asset	Tax	DTA	Equity Flow
17,899	-20,268	-321	-3,565	-6,255	17,476,775	0	24,501,000	456,027	-6,255	-6,574,453
54,366	81,411	18,534	40,263	194,573	-2,071,000	1,960,580	-2,071,000	-672,852	200,828	1,488,556
53,303	91,527	59,779	21,980	226,589	-3,532,000	1,778,834	-3,532,000	-410,230	32,016	1,400,619
55,213	89,745	67,271	75,956	288,186	-3,264,000	1,493,713	-3,264,000	-281,274	61,597	1,274,036
43,454	92,966	65,924	74,663	277,006	-2,940,000	1,227,665	-2,940,000	-129,220	-11,179	1,087,265
33,154	73,168	68,323	108,983	283,628	-2,478,000	993,359	-2,478,000	-60,736	6,621	939,245
24,620	55,759	53,748	78,033	212,160	-2,046,000	794,590	-2,046,000	13,467	-71,468	736,589
20,361	41,407	41,053	61,284	164,104	-1,602,000	636,627	-1,602,000	-4,293	-48,056	584,278
15,163	34,243	30,469	46,796	126,671	-1,318,000	512,312	-1,318,000	-10,082	-37,433	464,796
1,533	25,501	25,199	35,155	87,388	-1,144,000	409,866	-1,144,000	-12,684	-39,282	357,900
1,533	2,579	18,768	28,401	51,281	-986,000	321,489	-986,000	-21,918	-36,107	263,464
767	2,579	1,893	3,446	8,685	-875,000	245,497	-875,000	-32,188	-42,596	170,713
767	1,289	1,893	2,297	6,247	-727,000	178,905	-727,000	-52,143	-2,438	124,324
767	1,289	953	2,297	5,306	-524,000	120,940	-524,000	-34,873	-941	85,127
0	1,289	953	1,149	3,391	-443,000	79,096	-443,000	-21,586	-1,915	55,594
0	0	953	1,149	2,101	-285,000	43,809	-285,000	-11,504	-1,289	31,015
0	0	0	1,149	1,149	-174,000	21,112	-174,000	-5,077	-953	15,083
0	0	0	0	0	-92,000	7,268	-92,000	-1,322	-1,149	4,797

Evaluating Individual Unit Profitability via Value Impact

Gary G. Venter, FCAS, MAAA

Evaluating Individual Unit Profitability via Value Impact

One of the key problems for insurance company management is evaluating the profitability of individual units, such as lines of business, states, departments, and even contracts. Capital allocation, pricing theory, and other approaches have been proposed to do this. An alternative proposal is explored here, based on the units' contributions to a measure of company value.

Background

Merton and Perold (1993) propose a profitability target for a unit based on the cost of risk capital. For the unit this is the value of the financial guarantee the company provides to the customers of the unit. That is, if losses for the unit exceed its premium and the investment income on it, the customers have access to the full value of the company's capital to cover their losses. The value of their contingent claim then becomes the cost of risk capital for the unit, and thus is the minimum target profitability. Presumably this can be evaluated by standard financial methods, i.e., by using a risk-neutral valuation of the policyholders' option. This would involve transforming the loss probabilities into a heavier distribution so that the market value of a loss transfer would be the expected value of the deal under the transformed probabilities, including such transactions as this policyholder option.

Mango (2003) calls this approach capital consumption – the unit uses up capital of the firm. He proposes pricing the policyholders' option by mapping each loss event into a larger loss, and then pricing by the mean of the transformed loss. This mapping is equivalent to the risk-neutral valuation, as the transformed losses could be considered a new loss distribution whose mean is used to evaluate deals. It is also equivalent to the riskiness aversion function of Kreps (2003), which is a factor by size of loss applied to the loss size. Kreps uses these transforms to create co-measures for any risk measure that can be expressed as a transformed expected value. There is an analogy to utility theory here, as larger losses are less desirable and so are penalized more. Thus Mango calls the mapping function a utility function. However this is misleading terminology in that the mapping is applied to the loss itself, whereas in utility theory the transform is applied to the value of surplus before and after the loss. In any case, so far this approach is equivalent to Merton-Perold.

Mango carries this forward with a method of incorporating the time frame aspect. Some contracts pay out quickly, while others pay over many years. Mango discounts the adjusted losses at the risk-free rate to get a present value pricing of the option, which is similar to what is typically done under a risk-neutral valuation. In the Merton-Perold framework this would be expressed as evaluating a more complex option – an option which is exposed when losses from the unit exceed its premium plus investment income, and then continues to pay all claims after that. Merton-Perold did not explicitly consider this type of option, but it is a natural extension of their framework.

In a private conversation, Ken Froot pointed out that the mean profitability is not the right value to compare to the risk capital so defined, in that the profit is also a contingent claim – all the profit goes to the company, if it is positive. This claim could be evaluated by the risk-neutral valuation as well, and the values of the profit and guarantee compared. However Froot points out that the sum of the two claims is an ordinary forward contract – not a contingent claim at all. It represents the total position of the company in the unit's contracts. The present value of all the cash flows in the risk-neutral measure is the value of this position. This is a variation of the discounted cash flow (dcf) methodology. It discounts the transformed cash flows to get the economic value of the unit. The two separate options could be presented, however, to show the positive and negative components of this value.

Some Remaining Issues

The value of the unit so far has been considered with respect to the market as a whole, but this value could be different within a company. For instance, it might cost more for the company than the overall market to provide the financial guarantee, especially if the unit losses have a high correlation with the company's other losses. This relies on the fact that company specific risk does have a value impact, especially for insurers. See Major-Venter (2000) for a summary of literature on this issue.

Of course the actual form of the loss or probability transform is important, and that needs to be determined. Typically, proposed transforms are tested against known prices to find one that at works in some specific known realms. E.g., see Wang (2002). But there is a delicate issue here – the discounted cash flow method uses transforms of the specific loss payments, but the value impact of a given loss is really its entire present value, so that is what should be transformed. In Krep's framework, the leverage multiplier g(x) for a loss payment should depend on the whole loss x, not each payment.

Unit Contribution to Company Value

A way to address both of these issues is to look at the unit's contribution to overall company value. For this, company value can be measured by the transformed dcf method, where the transform is applied to the present values of the total company losses for each event. This is most clearly expressed in the riskiness-leverage function approach, where the x used in the g(x) function is the total company loss for the event. This will typically give a bigger percentage penalty to events that are large for the entire company, so if the unit losses are high in those events, they will get a greater hit than would the same sized unit loss in an event that is less significant overall. The ratio of transformed value to present value can then be applied to each payment of each claim for each unit that had a loss in that event. Thus each unit's loss transforms are based on the contributions of the unit's losses to the overall value of the company, and not just to the size of the individual payment for the unit itself. This is essentially a co-measure approach, where the measure is the value of the company under the transformed dcf methodology.

Some More Remaining Issues

One big detail: how do loss reserves get into this calculation? The simplest approach is to exclude them – take the value measure to be the transformed dcf for the company for a policy year. Then each unit's contribution to that year can be evaluated. The problem is that the company still has exposure to events that cause loss reserve increases, and the units that correlate most strongly with these increases essentially use more capital, and so their value would be overstated by just looking at individual policy years.

An alternative would be to model the events that cause reserve increases, such as increases in price levels, adverse court rulings, unintended coverage being extended back for several years, change in reserving philosophy, etc. The costs of such events could be modeled and transformed just like new loss events, but the modeling to do this gets more complex. The individual unit losses in these reserve-change events would have to be modeled to apply the loss transform. And since a reserve event can affect several years, the growth of the company and risks to that growth, as well as risks to the composition of the future company would also have to be modeled.

Another issue is what about other measures of company value? A more sophisticated value measurement would include systematic and non-systematic risk, and then measure the unit's contribution to these. The systematic-risk component itself can be evaluated with a transformed dcf approach, but the transform would measure the market impacts of an event, not just company impacts. The co-measure with the market would work for this value metric, but again the modeling would be more difficult.

Conclusion

Measuring unit profitability by the contribution to the economic value of the firm addresses many of the outstanding business unit management issues. However, doing it right appears to involve fairly intricate modeling of the company and even the market as a whole.

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