A Method of Implementing Myers-Read Capital Allocation in Simulation

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Abstract
In this paper, we show an especially simple way to produce Myers-Read capital allocations in simulations, by using the Ruhm-Mango-Kreps (RMK) conditional risk algorithm. The algorithm uses only weighted averages. In particular, it does not require any calculus, even though the Myers-Read formula is a differential equation. This is possible because the Myers-Read method is additive, and the Ruhm-Mango theorem guarantees that any additive allocation method can be reproduced by RMK. While Myers-Read capital allocation is based on probability of ruin, the RMK algorithm can easily be adapted to alternative risk measures of the user's choice.

Introduction
Myers and Read (2001) describe a method for allocating an insurer's total capital to individual lines of business, according to the equation:

$$S_k = L_k \times (dS / dL_k) ,$$

where S = total capital, S_k = capital for line k, and L_k = expected losses for line k. The formula is based on the assumption that total capital is determined by a fixed probability-of-ruin constraint, so that the expression (dS / dL_k) represents how much capital would have to increase, in response to an increase in expected loss volume[1]. An appealing feature of the theory is that the capital allocations so derived will sum to the total capital:

$$\Sigma \, S_k = S$$

Although it is probably most natural to think of surplus allocation in terms of lines of business, this concept could be extended. For instance, one could use the same formula to allocate capital to loss layers within a line. The "sum-of-the-parts-equals-the-whole" additive property remains intact[2], allowing a consistent allocation of capital down to almost any level.

Ruhm and Mango (2003) describe a method of calculating risk charges based on conditional probabilities and total portfolio risk[3]. Total portfolio risk charge is calculated from any risk measure that the user specifies, and is then allocated to all components of the portfolio based on conditional probabilities. Like Myers-Read allocations, conditional risk charges are also additive: the individual risk charges sum to the portfolio risk charge. As an added benefit, the Ruhm-Mango-Kreps ("RMK") algorithm is extremely simple to implement in a simulation, once the risk measure has been selected by the user.

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[1] This assumes that the shape of the loss distribution remains unchanged – see Mildenhall (2002).

[2] Again, subject to the conditions described in Mildenhall (2002).

[3] An alternative derivation of the same algorithm was discovered independently by Kreps (2003).

Myers-Read and RMK both allocate portfolio risk to components, just expressed in different forms (capital vs. risk charge). Since both are also additive, it is natural to wonder if there might be a connection. One difference between the methods is that the Myers-Read method specifies probability-of-ruin as the standard for setting capital, while RMK allows any risk measure to be used.

As it turns out, any additive allocation method can be reproduced by conditional risk charges. The only differences among additive methods are choice of risk measure and scale[4]. This means that Myers-Read capital allocation can be implemented using the RMK algorithm, by choosing probability of ruin as the risk measure. As will be shown below, capital allocations produced by RMK in simulation equal those derived by applying the Myers-Read differential equation.

An Example of the RMK algorithm applied to Myers-Read
A simple example will be used to demonstrate how to implement the method. The simplified conditions in the example are only present to make the example as transparent as possible; the method will work just as well for any number of risks and any dependence structure.

Consider a portfolio of 2 independent risks, both distributed normally and parameterized as follows:

$$R_1 \sim N(100, 900)$$
$$R_2 \sim N(200, 1600)$$

The total portfolio is thus also normal:

$$R_1 + R_2 \sim N(300, 2500)$$

The standard deviation of the total portfolio is 50. The capital requirement is set at 100. This is two standard deviations, which produces 97.725% confidence level (probability of ruin = 2.275%). Total required funding is the sum of expected loss and required capital:

$$\text{Total Funding} = \text{Total Expected Loss} + \text{Capital} = 400$$

The problem is to allocate the 100 of capital to the individual risks. First, we will apply the Myers-Read formula to obtain the result. (A reader who is less interested in the details of this calculation can skip to the next page, where the result is found.) From above,

$$S_k = L_k \times (dS / dL_k) ,$$

where $L_k = \mu_k = E[R_k]$. To calculate dS / dL_k, we first express S, the capital requirement, as a function of the components. This was set to be two times the standard deviation of

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[4] See Ruhm-Mango (2003). This statement (the Ruhm-Mango Theorem), its proof and the general risk pricing formulas are the main theoretical results of the paper. Technically, the theorem applies to any additive method that unambiguously prices all derivatives ("additive-complete" methods).

the total portfolio. The total portfolio's variance is the sum of the individual variances, so we have:

$$S = 2(\sigma_1{}^2 + \sigma_2{}^2)^{1/2}$$

Differentiating with respect to L_k:

$$dS / dL_k = (dS / d\sigma_k) (d\sigma_k / dL_k)$$
$$dS / dL_k = (2) (1/2) (\sigma_1{}^2 + \sigma_2{}^2)^{-1/2} (2\sigma_k) (d\sigma_k / dL_k)$$
$$dS / dL_k = (2) (S^{-1}) (2\sigma_k) (d\sigma_k / dL_k)$$
$$dS / dL_k = (2) (1/100) (2\sigma_k) (d\sigma_k / dL_k)$$

To finish this, we have to find $(d\sigma_k / dL_k)$, which is not difficult. Following Mildenhall (2002), we must assume that the individual lines increase or decrease by a scale factor, which means the coefficient of variation is constant. For instance, if line 1 (R_1) doubles in volume of expected loss (L_k), its standard deviation (σ_k) also doubles. From above:

$$L_1 = 100, \sigma_1 = 30$$
$$L_2 = 200, \sigma_2 = 40$$

Then:

$$\sigma_1 = 0.30 \ L_1, \text{ so } d\sigma_1 / dL_1 = 0.30$$
$$\sigma_2 = 0.20 \ L_2, \text{ so } d\sigma_2 / dL_2 = 0.20$$

Substituting these coefficients and standard deviations into the formula above yields:

$$dS / dL_1 = (2) (1/100) (60) (.30) = 0.36$$
$$dS / dL_1 = (2) (1/100) (80) (.20) = 0.32$$

Finally,

$$S_1 = L_1 \times (dS / dL_1) = (100)(0.36) = 36$$
$$S_2 = L_2 \times (dS / dL_2) = (200)(0.32) = 64$$

These sum to the total capital of 100, as expected.

Next, the example is simulated and the RMK algorithm is applied. For this case, the two normals were each simulated using one hundred points, the center-points of all the unit percentiles. These were cross-combined, to produce 10,000 sample points for the sum. Each sample point is equally likely in this case, however the procedure is just as easily applied to simulations for which probabilities differ by iteration.

Exhibit 1 shows the results of the simulation. For clarity, the iterations have been sorted by total loss, and only the most significant rows are shown. The first five columns show the iteration number, the simulated losses for the two risks, the total losses and the

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percentile for the total. The rightmost column, "Risk Discount Function," contains the risk measure (probability of ruin for Myers-Read) in the form of a discount function. This column is the center of the algorithm.

As discussed above, the ruin point should fall at about the 97.725% level, between iterations 9772 and 9773[5]. We want to take a small sample of the distribution around this point – for this example, we'll use a 0.50% interval on either side. This sample, consisting of iterations 9723 through 9822, is shown in bold in the exhibit. The probability-of-ruin discount function is very simple: it is one for points in the sample (i.e., at or near the ruin point), and zero everywhere else.

The capital allocation can now be calculated by taking weighted averages, using the risk discount function as the weights. This produces funding amounts, which equal expected losses plus capital allocations. For example, the expected value of the "Risk 1" column is 100 (the expected loss for Risk 1), while the weighted expected value is 135.64, for a capital requirement of $135.64 - 100.00 = 35.64$. The results, as shown in the exhibit, approximate those produced by formula (with minor differences due to simulation).

As this example shows, implementing the algorithm is fairly easy. One simply has to add an additional column for the risk discount function, chooses a small sample space around the ruin point, and calculate weighted averages.

Other Applications and Risk Measures
The RMK algorithm allows one to specify a risk measure and allocate total capital or total risk charge (more generally, total risk) accordingly. Moreover, the allocation does not have to be limited to lines of business – the method can also be applied to sources of risk in general, such as investment-related risks in dynamic financial analysis (DFA) applications and business risks in the enterprise risk management (ERM) contexts. In short, it is a general method for decomposing overall risk into its components, by source of risk.

The risk discount function for the probability-of-ruin measure, given in the example above, explicitly shows the measure's abruptness: it values entirely on the point-of-ruin, and discounts all other points with a factor of zero. For instance, if the outer tail severity were to change, the indicated capital, and corresponding risk charge, would not change.

As an alternative, one could weight the entire tail with ones, reflecting all points of undesirable outcomes. If a dollar-value cutoff point is used, rather than a percentile, then frequency can be reflected. As a further refinement, smaller weights can be used for desirable "upside" outcomes, with larger, surcharging weights used for more severe, less desirable outcomes. Such a risk function would allow all parts of the distribution to be reflected. At the extreme, a continuous function could be used: both Black-Scholes option prices and CAPM market prices can be modeled in this way[6].

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[5] Due to simulation approximation error, the ruin point actually falls at iteration 9780.
[6] See Ruhm (2003) and Ruhm-Mango (2003), respectively, for these and their risk discount functions.

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Exhibit 1: Simulation Results with Myers-Read Capital Allocation

Iteration	Risk 1	Risk 2	Total	% ile	Risk Discount Function
1	22.72	96.97	119.69	0.01%	0.0
2	34.90	96.97	131.86	0.02%	0.0
9722	122.66	272.48	395.14	97.22%	0.0
9723	**137.61**	**257.58**	**395.19**	**97.23%**	**1.0**
9724	**165.10**	**230.22**	**395.32**	**97.24%**	**1.0**
9821	**165.10**	**238.96**	**404.07**	**98.21%**	**1.0**
9822	**101.13**	**303.03**	**404.16**	**98.22%**	**1.0**
9823	125.79	278.40	404.19	98.23%	0.0
10000	177.28	303.03	480.31	100.00%	0.0
Expected	100.00	200.00	300.00		100.0
Funds = Wtd Exp'd	135.64	263.78	399.42		
Capital	35.64	63.78	99.42		
Exact Values, formula	36.00	64.00	100.00		