The Casualty Actuarial Society *Forum*
Summer 2000 Edition
Including the Dynamic Financial Analysis Call Papers

To CAS Members:

This is the Summer 2000 Edition of the Casualty Actuarial Society *Forum*. It contains five Dynamic Financial Analysis Papers, and one additional paper.

The Casualty Actuarial Society *Forum* is a nonrefereed journal printed by the Casualty Actuarial Society. The viewpoints published herein do not necessarily reflect those of the Casualty Actuarial Society.

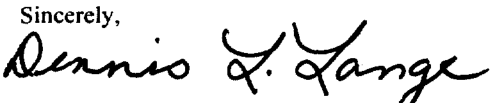
The CAS *Forum* is edited by the CAS Committee for the Casualty Actuarial Society *Forum*. Members of the committee invite all interested persons to submit papers on topics of interest to the actuarial community. Articles need not be written by a member of the CAS, but the paper's content must be relevant to the interests of the CAS membership. Members of the Committee for the Casualty Actuarial Society *Forum* request that the following procedures be followed when submitting an article for publication in the *Forum*:

1. Authors should submit a camera-ready original paper and two copies.
2. Authors should not number their pages.
3. All exhibits, tables, charts, and graphs should be in original format and camera-ready.
4. Authors should avoid using gray-shaded graphs, tables, or exhibits. Text and exhibits should be in solid black and white.
5. Authors should submit an electronic file of their paper using a popular word processing software (e.g., Microsoft Word, WordPerfect) for inclusion on the CAS Web Site.

The CAS *Forum* is printed periodically based on the number of call paper programs and articles submitted. The committee publishes two to four editions during each calendar year.

All comments or questions may be directed to the Committee for the Casualty Actuarial Society *Forum*.

Sincerely,



Dennis L. Lange, CAS *Forum* Chairperson

The 2000 CAS Dynamic Financial Analysis Call Papers
Presented at the
2000 Dynamic Financial Analysis Seminar
July 17-18, 2000
Marriott Marquis
New York, New York

The Summer 2000 Edition of the CAS *Forum* is a cooperative effort of the CAS *Forum* Committee and the CAS Committee on Dynamic Financial Analysis.

The CAS Dynamic Financial Analysis Committee presents for discussion five papers prepared in response to its Call for 2000 Dynamic Financial Analysis Papers.

This Forum includes papers that will be discussed by the authors at the 2000 CAS Dynamic Financial Analysis Seminar, July 17-18, in New York City.

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Dynamic Financial Analysis Call Papers

Additional Paper

Strategic Asset Allocation for Multi-Line Insurers Using Dynamic Financial Analysis

Allan M. Kaufman, FCAS, MAAA and
Thomas A. Ryan, FCAS, MAAA

STRATEGIC ASSET ALLOCATION FOR MULTI-LINE INSURERS USING DYNAMIC FINANCIAL ANALYSIS

ABSTRACT

The capital base of property casualty insurers includes an increasing proportion of equities relative to fixed income securities. This paper analyzes the risk/reward attributes of various fixed income/equity asset allocation alternatives using dynamic financial analysis (DFA) and demonstrates that a typical company could improve its returns without significantly reducing its financial security by further increasing its proportion of equities.

I. STRATEGY UNDER CONSIDERATION

Asset allocation and asset management are increasingly vital components of property casualty (p/c) insurance company operations. Facing a prolonged soft market, investment income produced by the traditional asset mix that is heavily weighted in bonds may not be adequate to ensure profitability and growth. In order to survive in the current financial industry environment, p/c insurer management's attention may have to shift to increasingly include asset management. Key questions for p/c insurers are whether their current portfolio mix and expected returns are adequate and how can they be improved without significant increase in risk to the enterprise.

We performed an analysis to review the impact of varying levels of equity or stock holdings in a multi-line insurer asset portfolio on insurer financial performance. The projected risks and rewards of increased equity holdings were reviewed to draw conclusions regarding the appropriateness of this strategy. In order to test the robustness of the results of our analysis, we also varied assumptions regarding initial insurer financial strength and profitability.

Background

Over the last 5 years, the equity or stock markets, as represented by the Standard & Poors 500 Index, have risen over 220%, an average annual increase of over 25%. While these recent results may not alone make a compelling argument for investment in these markets, expanding the time horizon to the last 40 years shows that equities have produced an average return of well over 8.5%. This is greater than the 6.9% compounded annual return over the same period of time on bonds, which make up the bulk of insurer assets.

The property-casualty insurance industry has historically avoided carrying a large portfolio of stocks, concentrating its assets more in bonds. Table 1 below shows the average amount of equities, expressed as a percentage of total assets, invested assets, and surplus held by the p/c industry over the past ten years. The increase shown in Table 1 for the recent years can be traced more to gains from the current bull market rather than an intentional reallocation of asset mix.

Table 1
Common Stock Holdings of the Property-Casualty Industry on a Consolidated Basis

Year	As a Percentage of Total Assets	As a Percentage of Inv. Assets	As a Percentage of Surplus
1988	12.8%	15.3%	52.0%
1989	13.9	16.4	54.5
1990	12.5	14.8	50.1
1991	13.8	16.1	52.3
1992	13.5	15.9	52.7
1993	13.6	15.8	50.2
1994	14.3	16.5	52.1
1995	16.0	18.4	53.2
1996	17.0	19.4	53.2
1997	19.9	22.6	56.1

Source: Best's Aggregates and Averages consolidated p/c industry annual statement. Common stock holdings include investments in non-p/c insurer affiliates of approximately 4% of total assets, 4.5% of invested assets, and 12.5% of surplus for the most recent years.

It is no secret that stocks, in the long run, achieve greater returns than bonds. It is also no secret that stock returns are more volatile than the return on bonds. For example, in 1987, the value of several stock market indices plunged more than 20% in one day. The large variation in stock returns is one of the major reasons property-casualty insurers have created company investment policies that limit investments in stocks. Insurers must be able to depend on stable investments to cover large unexpected claims.

Another major reason why p/c insurers have limited the amount of equities in their portfolio is statutory regulation. Currently, more than half of all states limit equity holdings to a fixed percentage of assets or surplus[1]. Other reasons for limiting equity holdings include concern over risk-based capital requirements and rating agency reviews. Equity holdings greater than 50% of surplus tend to raise concerns during rating reviews.

The aversion to large equity holdings of p/c insurers in the United States is not shared by non-life counterparts in the United Kingdom (U.K.). Large U.K. insurers commonly hold equities in amounts well over 125% of surplus. UK companies that write predominately non-life insurance but have some life insurance business, have equity to surplus ratios that sometimes exceed 200%. These companies have shown that p/c insurers can survive and even benefit from large equity holdings.

II. ANALYTICAL PROCESS

Overview

To determine the impact on property-casualty insurer financial results of increasing equity holdings, we used Dynamic Financial Analysis (DFA). Specifically, our analysis involved using Milliman & Robertson's FINANS model. The model simulates insurer balance sheet, cash flow and income statement results in future years based upon a reasonably wide variety of stochastically generated economic and insurance business scenarios.

The model was initially run for a base scenario that used current insurance industry financial statistics and asset allocation. We next ran the model using different asset allocations, produced by varying the amount of equity holdings, and determined how selected risk and reward criteria changed as a result.

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[1] *The state limits were provided by the National Association of Insurance Commissioners (NAIC). According to the NAIC, several large states, such as Florida and New York, limit the maximum investment in common stock to only 10% of assets.*

Model Input and Key Assumptions

Our plan was to model a typical U.S. multi-line insurer. Instead of using actual company data, we chose to use consolidated p/c industry data, such as balance sheets and income statements compiled in Best's Aggregates and Averages. Our assumption was that the U.S. p/c insurance industry as a whole would be representative of a large multi-line insurer. Absolute values for financial statistics such as reserves and surplus were scaled down but relative values were maintained. We hoped the results would not only provide information concerning our target audience, multi-line insurers, but might also reveal information on the p/c market as a whole.

A disadvantage of using this approach is that the data, while appropriate on average, may not be realistic when compared to actual company data. It may also contain biases in terms of the mix of business, with large casualty reserves predominating. To address these concerns, we decided to vary several model inputs such as the reserve-to-surplus ratio and loss ratios. We hoped that by testing several different variations, we could understand how our results were applicable to insurers with different financial characteristics.

Some of the important assumptions used in our model include the following:

- Annual premium growth was assumed to be 3%.
- Standard industry tax rates were used assuming the Schedule P Composite IRS reporting patterns.
- Capital gains were realized on a staggered basis over 4 years.
- All bonds were held at amortized cost.
- Loss ratios were projected on a calendar year basis using a regression of historic loss ratios and interest rates. The average loss ratio for the model-generated scenarios was approximately 74% with a high loss ratio of 94% and a low of 54%.
- A simplified catastrophe model was used that projected cats up to a maximum of $25 billion and with an average return period of 20 years.
- Interest rates, gross domestic product, inflation, and other economic variables were projected using a proprietary economic scenario generator.

- All lines of business data were consolidated into one line that used a single aggregate loss payout pattern.
- No explicit provisions were made for reinsurance coverage. As we employed consolidated net industry data (which includes U.S. reinsurer data), our implicit assumption is that data is gross of domestic reinsurance but net of foreign cessions.

A complete technical discussion describing the model is outside the scope of this paper but is available from the authors upon request.

As of year-end 1997, the domestic P/C industry was holding 23% of invested assets in common stocks, 69% in bonds, 5% in cash and 3% in other. For our analysis, we assumed all bonds are taxable and that the asset allocation will not change over time. We selected four alternative scenarios of various holding levels that are summarized in Table 2 below. For each scenario, the change (increase or decrease) in stock holdings was offset by a corresponding change (decrease or increase) in bond holdings so that the total asset amount would remain constant.

Table 2
Selected Alternative Invested Asset Allocations

Scenario	% Stocks	% Bonds	% Other	% Total
1	15%	77%	8%	100%
2	30%	62%	8%	100%
3	40%	52%	8%	100%
4	50%	42%	8%	100%

Interpretation of Results

The DFA model simulates an unlimited number of different economic and business scenarios from the selected starting point, in this case year-end 1997. (We define a scenario to be a unique set of economic and operating assumptions on the basis of which projections are made.) After testing the model, it became clear that a model run consisting of 1,000 scenarios was adequate to ensure convergence of results. We also selected a 5-year projection period as a reasonable time for p/c industry business plans.

Therefore, each model run consisted of a total of 5,000 scenarios (1,000 scenarios per year for 5 years).

To review and present these results in a clear, comprehensive, and meaningful manner, we decided to graphically compare selected significant risk and reward measures. The graphic presentation would hopefully allow us to quickly and clearly deduce how the change in asset mix was affecting our risk and reward measures without the noise of the results from an overwhelming number of scenarios.

The resulting risk and reward measures for each of the selected portfolios were analyzed for comparative dominance. A portfolio of assets "dominates" another when it offers greater reward for the same or lower risk. "Efficient" portfolios offer the maximum return for a given risk measure. A set of efficient portfolios over a range of values for a given risk measure form a graph known as the efficient frontier.

We were looking to see if the current asset portfolio of multi-line insurers, and that of the industry, is (a) not efficient and lies below the efficient frontier, or (b) makes an unduly pessimistic risk/reward tradeoff.

Risk/Reward Measures

There are a wide variety of financial risk and reward measures available. Our goal in selecting measures for our analysis was to select those most critical to our proposed audience – senior company management. Also, we did not want to use too many measures that would become confusing without adding value. If it were possible to draw similar conclusions from viewing only 3 risk/reward comparisons as opposed to viewing 5 or 6, for the sake of efficiency, we determined to use only 3. By limiting the number of risk/reward measure comparisons presented, we hoped to increase the impact of their message. However, we would also verify that all the measures, whether presented or not, allowed for the same conclusions to be drawn. If alternative measures were not consistent with those first presented, the alternative measures would be presented and we would explain the differences.

As our single reward measure, we selected the median tax-adjusted return on equity (ROE) as the single most important/commonly used reward measure. The tax-adjusted

ROE relates all underwriting and investment income (net income) and a tax-adjusted portion of unrealized capital gains to the prior year surplus adjusted for deferred taxes on unrealized gains. While based on statutory results, the adjustments result in a measure that is perhaps closer to market value GAAP ROE than statutory measurements.

For the risk criteria, we selected the following 4 measures:

- Probability that ROE was less than 10% in any one year;
- Probability that surplus decreases by 10% or more in any one year;
- Probability that surplus decreases by 25% or more in one year;
- Probability that risk-based capital (RBC) is greater than surplus.

The probability that each of the risk measures occurs was determined by examining how many times the risk thresholds were crossed within the 1,000 model generated scenarios. For example, if out of the 1,000 scenarios (each with 5 years of results), the ROE dropped below 10% exactly 10 times, our probability of failure for this risk measure would be 10/5,000 or 0.2%. If, for a particular scenario, the ROE dropped below 10% in 3 of the 5 years, we counted this as 3 failures, not 1.

Consistent with our selection process for the reward measure, we sought to limit the risk measures to a manageable number that would present important results and still allow for comparisons and conclusions.

Testing Robustness of Results

The initial model input was based upon p/c industry financial results at year-end 1997 as discussed above. These data implies a certain average level of profitability and financial position. In order to sensitivity test our results for companies not at average levels, but at more adverse positions, we ran the model with lower initial levels of surplus and, separately, with a higher projected loss ratio. (We did not run the model with higher surplus and lower loss ratios as, presumably, the results would be the same or better than the base case.) Our assumption was that any additional risk introduced by the increased holdings of equities might be more pronounced on the results for companies with lower profitability or in an initially weaker financial position.

We examined the loss and LAE reserve-to-policyholder surplus ratios for large multi-line writers. We found a wide range that varied from under 1.0 to over 3.0. Our base case, the industry on a consolidated basis, had a reserve-to-surplus ratio of 1.18. To determine what the financial risks and rewards would be for the full spectrum of multi-line insurers, we selected three alternative initial reserve-to-surplus ratios (1.5, 2.5, and 3.0). Initial surplus at year-end 1997, either capital or unassigned funds, was reduced until the selected reserve/surplus ratio was achieved.

To test the effect of insurer profitability on our results, we changed our loss ratio projections by increasing them 10% on average to about 84%. We used an initial reserve-to-surplus ratio of 2.5 for our tests of the impact of higher loss ratios.

III. RESULTS AND CONCLUSIONS

A table of results from the model is provided as Exhibit 1 and summary graphs are provided as Exhibits 2 through 5. Each risk/reward measure is graphed on a separate exhibit, with all scenarios presented on each graph for comparison. Exhibits 6 and 7 provide individual graphs of the probability of a 25% surplus drop and the probability that RBC is greater than surplus for the 1.5 reserve-to-surplus scenario. These exhibits are presented to clearly highlight the observed movement of portfolios toward an efficient frontier. Analysis of the results is presented below.

Base Case (Reserve-to-Surplus =1.18)

Our initial model runs used industry values from year-end 1997, a 1.18 reserve-to-surplus ratio. The current industry asset portfolio has 23% of invested assets in common stocks. For this asset portfolio, the operating assumptions produced a median ROE of 9.54% for the next five years, as shown in the first panel of Exhibit 1. The risk that the ROE would be under 10% was high, as would be expected with a median ROE of 9.54%. The other risk measures produced negligible values for this portfolio, i.e., a "very safe" position, if a 9.54% ROE were considered acceptable.

When equity holdings in the portfolio were increased, median ROE increased as expected but with very little increase in the other risk measures. We examined several other risk measures, and found them all to be consistently minimal.

We tested other reserve-to-surplus scenarios to test the extent to which the low risk measures were due to the low reserve-to-surplus ratio representative of the current status of the p/c industry.

Reserve-to-Surplus = 1.5

For the second set of tests, initial surplus was reduced to produce an overall reserve-to-surplus ratio of 1.5. The results for these tests are provided in the second panel of Exhibit 1 and graphically on Exhibits 2 through 5. For the current asset portfolio, a median ROE of 11.71% was projected, more than 200 basis points higher than ROE for the same asset portfolio in the base case. The higher ROE follows because surplus is reduced more than earnings, assuming that assets are earning the same return.

With reduced surplus, all risk measures increased to the levels shown in Exhibit 1. The risk of a 25% surplus drop and RBC greater than surplus are minimal in absolute terms but are revealing in relative terms for each portfolio. Graphing the results shown above for the risk of surplus decrease and the risk of RBC greater than surplus against ROE separately on Exhibits 6 and 7 revealed a pattern of movement toward an efficient frontier. The current industry asset portfolio (23% stocks) appears to be just on or slightly below the frontier. The alternative portfolio with 15% in stocks is definitely off the frontier and is dominated by the higher equity portfolios.

Surprisingly, the probability of an ROE less than 10% is strictly decreasing in risk as equity holdings are increased, with no inflection point or movement to an efficient frontier. These results indicate that increasing stock holdings provides higher returns that more than offset the variability or risk in results.

Reserve-to-Surplus = 2.5

For the third set of tests, initial surplus was reduced to produce an overall reserve-to-surplus ratio of 2.5. The results for these tests are provided in the third panel in Exhibit 1 and graphically on Exhibits 2 through 5. For the base asset portfolio, a median ROE of 15.18% was projected, again higher than the previous tests run with higher surplus. The reduced surplus resulted in risk measures that increased significantly to the levels shown in Exhibit 1. The risk of surplus decrease, either 10% or 25%, became significant as did the risk of RBC greater than surplus. However, the risk of ROE falling below 10% declined. This occurred as the surplus, the denominator in our ROE equation, was decreased, therefore raising the median ROE.

Another observation from the graphs is that the inflection point, representing the start of the efficient frontier, was at a portfolio with a higher percentage of stocks. All the portfolios with 30% or less in stocks appear to be inefficient.

Reserve-to-Surplus = 3.0

For the fourth set of tests, initial surplus was reduced to produce an overall reserve-to-surplus ratio of 3.0. The results for these tests are provided in the fourth panel of Exhibit 1 and graphically on Exhibits 2 through 5. For the base asset portfolio, a median ROE of 17.45% was projected, slightly higher than the same portfolio in the previous tests. Again, all risk measures increased as shown below, except for the probability of ROE falling to less than 10%, which again declined from previous results.

Higher Loss Ratio

For the final set of tests, initial surplus was changed to produce an overall reserve-to-surplus ratio of 2.5 and the loss ratio equation was adjusted to produce an average loss ratio approximately 10% higher than those previously projected. The results for these tests are provided in the last panel of Exhibit 1 and graphically on Exhibits 2 through 5. For the base portfolio, a median ROE of 12.93% was projected, lower than the prior tests using the same reserve-to-surplus ratio but with unadjusted loss ratios. The decrease in median ROE is not strictly inversely correlated with the increase in loss ratio due to modeled policyholder dividends and federal income taxes.

When compared to the results of tests run with the same reserve-to-surplus ratio, the results for the higher loss ratio tests had greater risk, even for the probability of ROE decreasing below 10%. These results were consistent with our expectations.

General Conclusions

For the property-casualty industry as a whole the reserve-to-surplus ratio is low by historical standards. From this starting point, increased equity holdings in the consolidated asset portfolio of the industry will increase ROE but have little affect on the risk measures presented, as risk is negligible.

For individual companies the benefit of changes in asset allocation will depend on the level of surplus and loss ratio expectations.

Some Limitations and Other Considerations

For this analysis, results were reviewed on an annual basis. If the analysis was performed on a quarterly basis, we would expect the results to be more volatile.

As with any model, there are many factors not considered in this analysis that would need to be considered by a company actually implementing a revised investment strategy. The model identifies the direction of possible change and thus the potential benefit justifying the value of further analysis by the company.

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Acknowledgements

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Summary of Results

Projections using Original 1.18 Reserve-to-Surplus Ratio

Stocks as a % of Inv. Assets	Median ROE	Prob. of 10% Surplus Decrease	Prob. of 25% Surplus Decrease	Prob. of ROE <10%	Prob. Of RBC > Surplus
15%	8.75%	1.84%	0.16%	55.04%	0.02%
23	9.54	1.68	0.08	51.25	0.02
30	10.38	1.72	0.08	47.45	0.02
40	11.45	2.24	0.12	43.46	0.00
50	12.49	2.78	0.12	40.86	0.00

Projections using 1.5 Reserve-to-Surplus Ratio

Stocks as a % of Inv. Assets	Median ROE	Prob. of 10% Surplus Decrease	Prob. of 25% Surplus Decrease	Prob. Of ROE <10%	Prob. Of RBC > Surplus
15%	10.84%	3.62%	0.54%	48.09%	0.08%
23	11.71	3.62	0.46	45.13	0.04
30	12.63	3.64	0.48	42.32	0.06
40	13.91	3.92	0.60	39.26	0.10
50	15.10	4.48	0.78	37.12	0.16

Projections using 2.5 Reserve-to-Surplus Ratio

Stocks as a % of Inv. Assets	Median ROE	Prob. of 10% Surplus Decrease	Prob. of 25% Surplus Decrease	Prob. Of ROE <10%	Prob. of RBC > Surplus
15%	13.58%	9.53%	3.04%	40.16%	2.68%
23	15.18	9.13	2.78	37.48	2.28
30	16.50	8.78	2.60	34.83	1.98
40	18.34	9.21	2.50	32.97	2.12
50	19.81	9.67	2.76	31.77	2.62

Projections using 3.0 Reserve-to-Surplus Ratio

Stocks as a % of Inv. Assets	Median ROE	Prob. of 10% Surplus Decrease	Prob. of 25% Surplus Decrease	Prob. Of ROE <10%	Prob. of RBC > Surplus
15%	15.67%	12.21%	4.81%	38.12%	6.73%
23	17.45	11.35	4.29	34.99	5.47
30	18.96	11.11	4.15	33.03	5.49
40	21.07	11.27	4.13	31.31	5.63
50	22.76	11.85	4.19	30.33	6.01

Projections using 2.5 Reserve to Surplus Ratio With Increased Projected Loss Ratio

Stocks as a % Inv. of Assets	Median ROE	Prob. of 10% Surplus Decrease	Prob. of 25% Surplus Decrease	Prob. Of ROE <10%	Prob. of RBC > Surplus
15%	11.41%	13.88%	4.77%	45.79%	4.84%
23	12.93	12.53	4.23	42.62	4.10
30	14.41	12.22	4.23	39.98	3.96
40	16.04	12.11	4.23	37.14	4.16
50	17.79	12.81	4.29	35.84	4.70

Exhibit 2

ROE vs. Probability Surplus Drops 10% or More in One Year

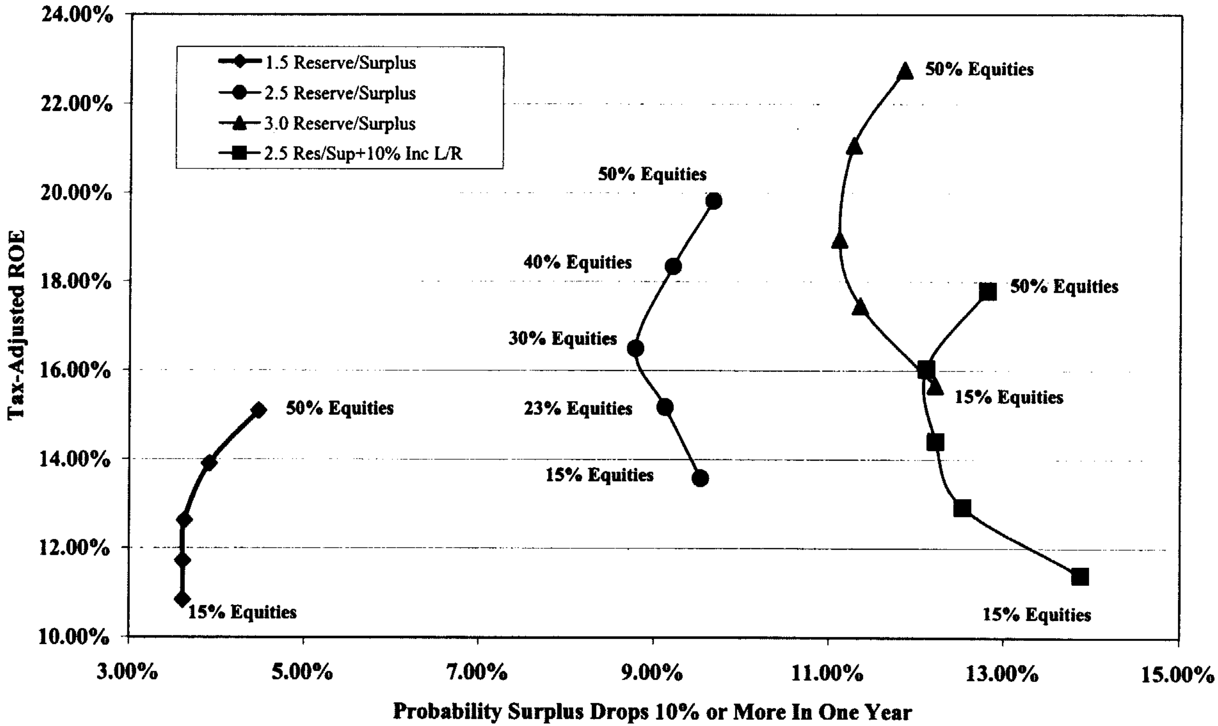

Exhibit 3

ROE vs. Probability Surplus Drops 25% or More in One Year

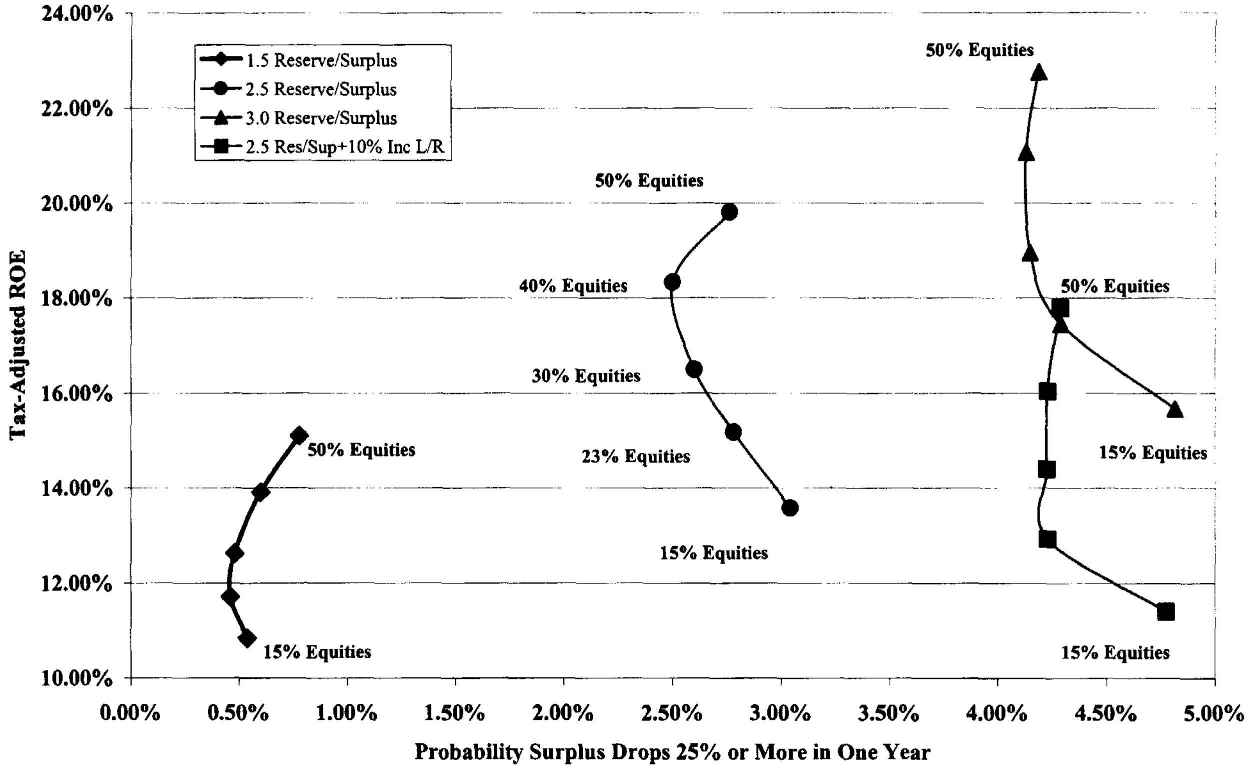

Exhibit 4

ROE vs. Probability RBC > Surplus

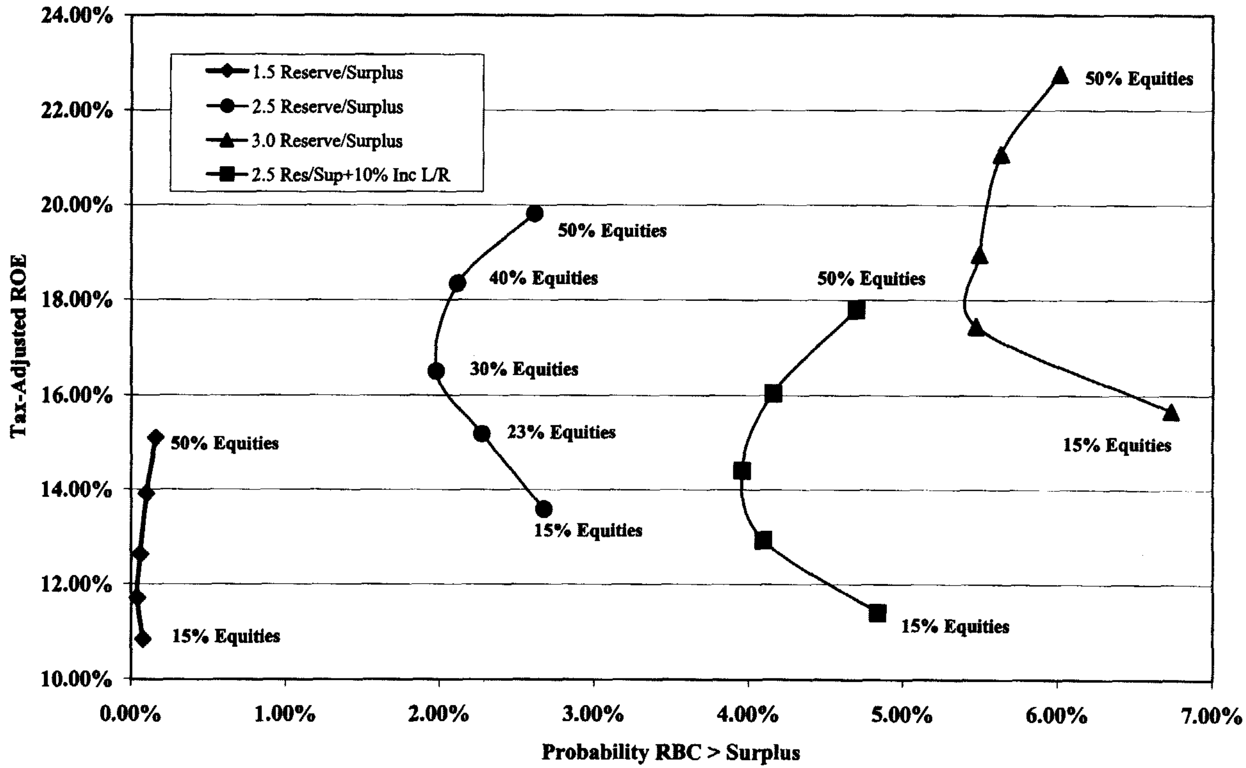

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Exhibit 5

ROE vs. Probability of ROE < 10%

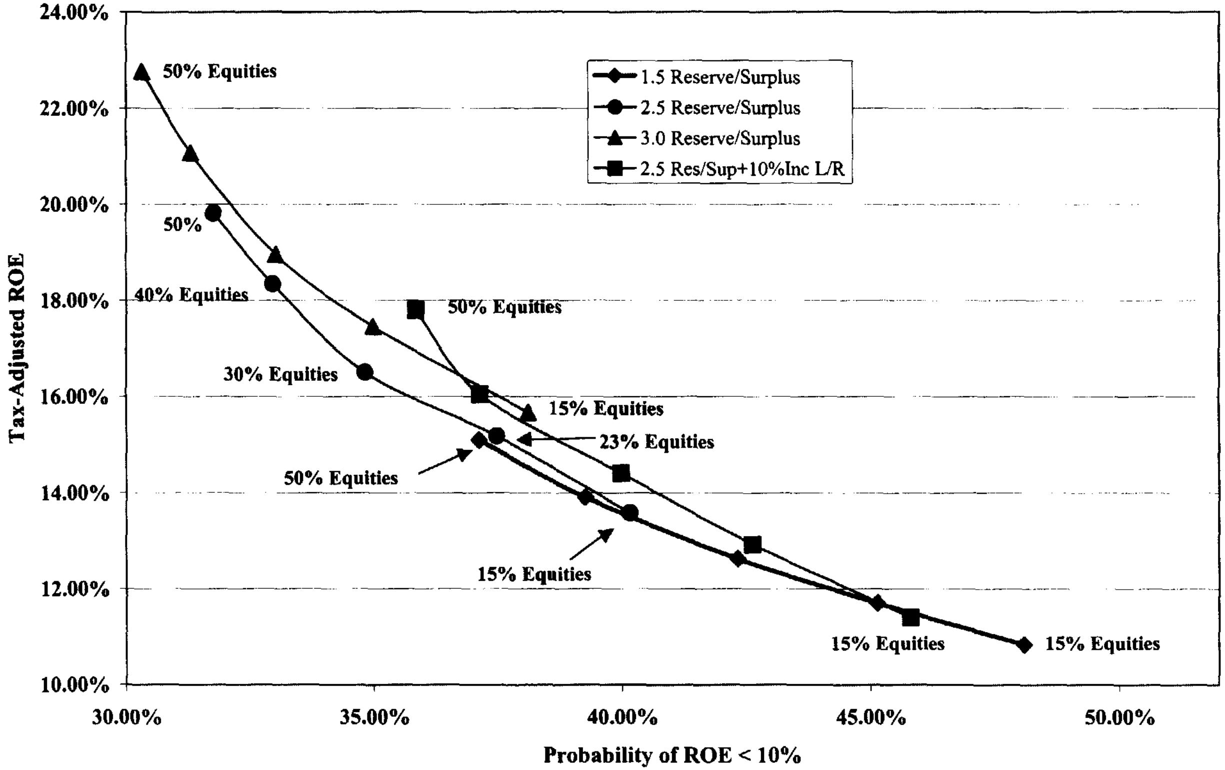

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Exhibit 6
Projection Using 1.5 Reserve-To-Surplus Ratio
ROE vs. Probability Surplus Drops 25% or More in One Year

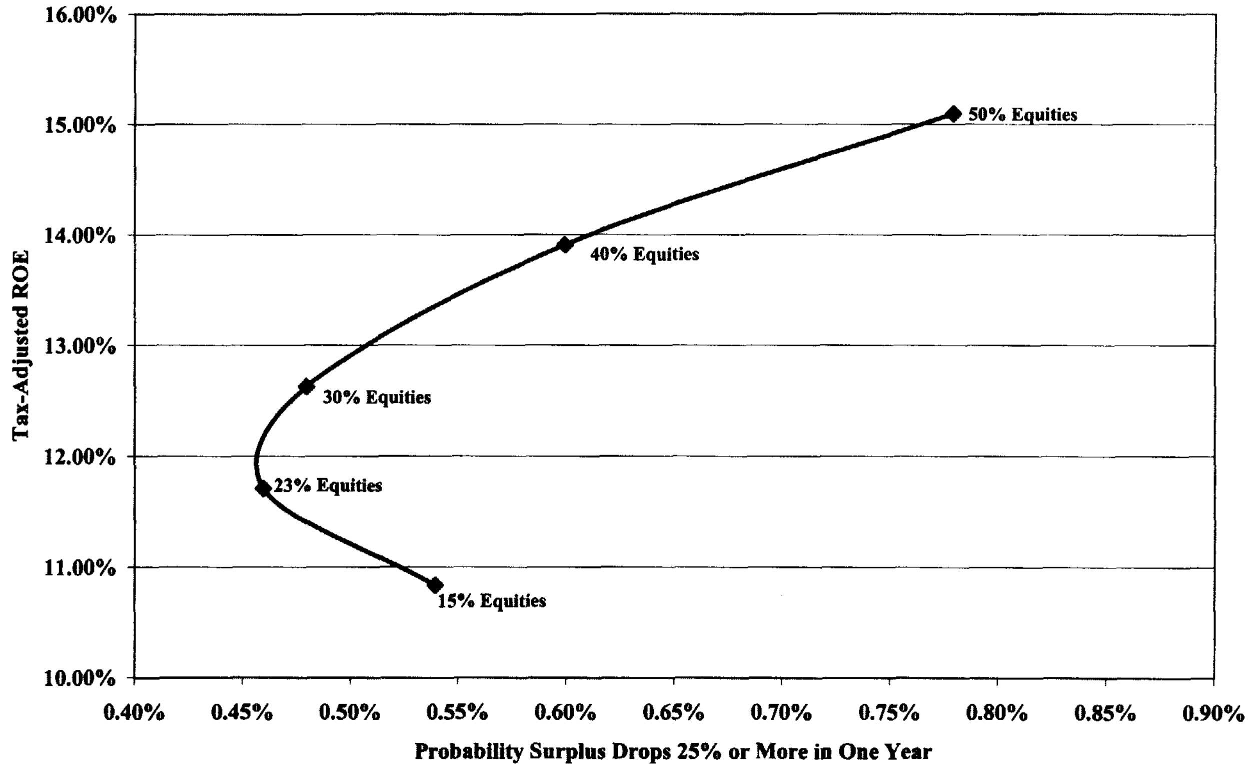

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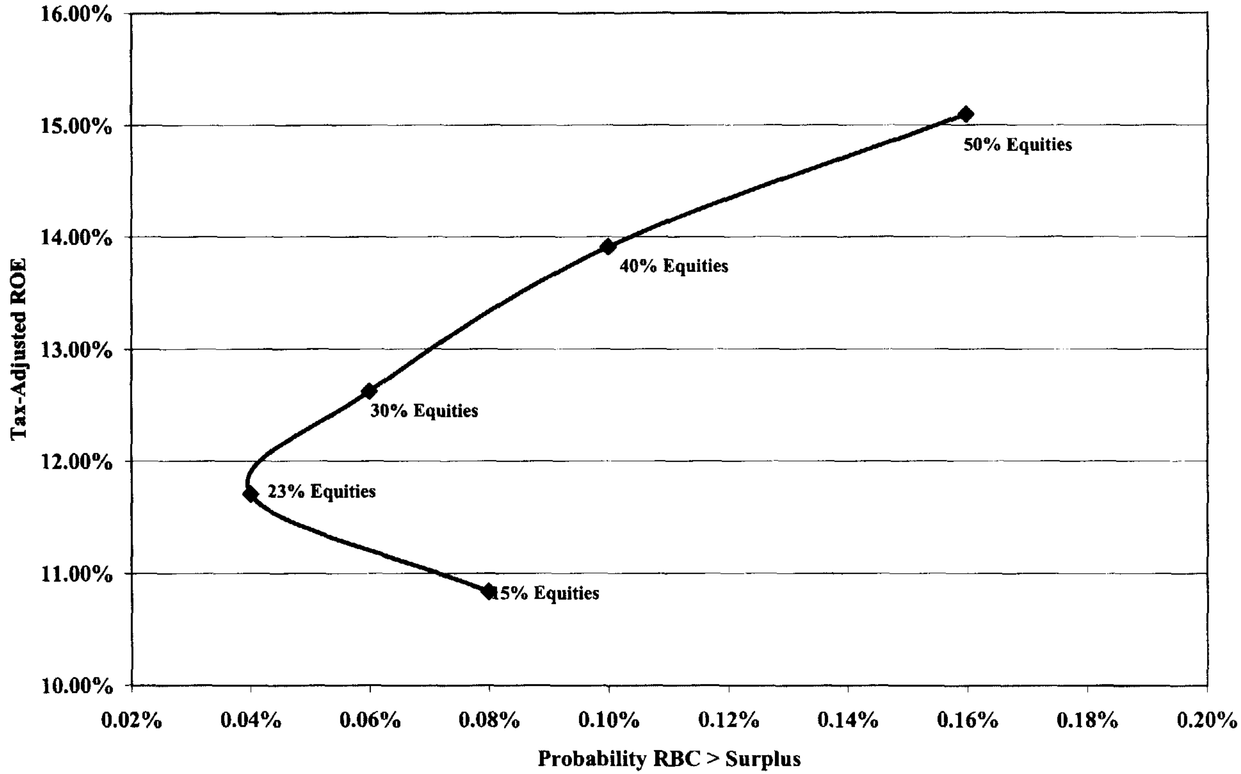
Exhibit 7
Projection Using 1.5 Reserve-To-Surplus Ratio
ROE vs. Probability RBC > Surplus



A Cost/Benefit Analysis of Alternative Investment Strategies Using Dynamic Financial Analysis Tools

Gerald S. Kirschner, FCAS, MAAA

A Cost/Benefit Analysis of Alternative Investment Strategies Using Dynamic Financial Analysis Tools

By

Gerald S. Kirschner, FCAS, MAAA

Abstract

The focus of the 2000 Call for Papers, as put forth by the Casualty Actuarial Society's Committee on Dynamic Financial Analysis, is the "evaluation of strategic alternatives and presentations of conclusions." This paper presents such a study. The paper is laid out in much the same way the analysis was performed, as a journey of discovery, in which one set of conclusions would lead to another set of questions and so on and so forth. The journey is by no means complete. Many more questions are still to be asked and many more conclusions are yet to be drawn. However, one must recognize that most, if not all DFA analyses start by overcoming small hurdles on the way to addressing larger ones. That is what is presented here; a beginning, an analysis on a small scale that has laid the basic framework for more thorough and complex analyses down the road.

Acknowledgement

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Introduction

For the past several years, there has been an annual call for papers on Dynamic Financial Analysis (DFA).[1] The topics of these calls have progressed through many of the key elements in the creation and use of models appropriate for DFA. The current topic focuses on the use of a DFA model in order to achieve an objective – namely the use of a model to evaluate strategic alternatives and to develop information that can be presented as a series of conclusions and strategic recommendations. Previous topics have focused on more elemental aspects of the modeling process – designing a model, parameterizing it, etc.[2]

Since the presentation of results, not model description or model parameterization is the subject of the 2000 Call for Papers, this paper will not go into much detail on the underlying model itself. Readers interested in learning more about these aspects of DFA model development and usage are encouraged to review submissions from previous DFA Calls that do focus on the more technical aspects of dynamic financial modeling. Instead, this paper will follow the trail of

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[1] Dynamic financial analysis is defined by the Casualty Actuarial Society's Dynamic Financial Analysis Committee as "a systematic approach to financial modeling in which financial results are projected under a variety of possible scenarios, showing how outcomes might be affected by changing internal and/or external conditions." Furthermore, the Actuarial Standards Board defines a "scenario" as a set of economic, demographic, and operating assumptions on the basis of which projections are made." In the context of this paper, a scenario can be thought of as one possible combination of external economic conditions and random selections from a variety of statistical distributions that describe the variability inherent in certain aspects of the company's operations.

[2] The topics of the prior years' Call Papers are as follows:

- 1996: Papers that describe DFA models that have been put to use at property-casualty insurers;
- 1997: Papers that identify and explain the variables that should be incorporated into a DFA model;
- 1998: Papers that discuss the applications and uses for DFA models;
- 1999: Papers that discuss the parameterization of DFA models.

discovery that ultimately led to the framing of the question to be answered and the structuring and presentation of information in response to that question. As such, the paper is organized in a series of steps that build one upon the other. A brief overview of the steps is provided to assist the reader in following the discussion:

Step 1) Frame a question suitable for analysis using the entity's DFA model;

Step 2) Identify one or more key measurement values, or "metrics" that will be used to decide if one strategy[3] is "better" than another strategy;

Step 3) Analyze the environment in which the company operates in order to gain or improve the company's understanding of the importance of internal and external influences on the key metric or metrics;

Step 4) Establish the base strategy against which alternative strategies are to be compared;

Step 5) Postulate a series of alternative strategies against which the base strategy will be compared;

Step 6) Evaluate the model's results for the base strategy versus the alternative strategies. Compare results in the context of both a "return" metric and a "risk" metric;[4]

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[3] A strategy, in the context of this paper, can be thought of as a series of management decisions that are made with the goal of achieving certain objectives that are desirable to senior management.

[4] A "return" metric is one that seeks to maximize something of positive value or minimize something of negative value to the organization. Common examples of return metrics might include operating income or surplus growth. A "risk" metric is a measurement of the volatility associated with each strategy. A traditional statistical risk metric is the standard deviation or the variance of the return metric's observed values around the mean value. Anther risk metric might be the number of times an observation falls below a minimally acceptable threshold value.

Step 7) Refine the alternative strategies, discarding the ones that are not appealing, and adjusting the ones that are appealing in an effort to develop an "efficient frontier" of alternative strategies;

Step 8) Run the model using the refined strategies. Analyze results and develop final conclusions and recommendations.

The "Client"

The "client" for this project was the investment department of a large multi-line insurance company. The client wanted to understand how it could reposition the company's asset portfolio so as to increase the likelihood of raising the entity's net worth[5] while minimizing the potential of running afoul of the various capital adequacy tests that exist in the insurance marketplace.

For simplicity, the measure of net worth that will be used through the remainder of this paper is **economic net worth**. Economic net worth differs from statutory surplus in the following ways:

- All invested assets are marked to market;
- Uncollected premium is recorded at its present value;
- Loss and loss adjustment expense reserves are recorded at the present value of the "actual" future loss and loss adjustment expense cashflows.[6] Note that

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[5] "Net worth" is a generic description of the value metric that the company wanted to maximize. The specific composition of the value metric is not relevant to the discussion at hand. It could be any number of things, including but not limited to statutory surplus, GAAP equity, economic net worth (all assets at their market values, all liabilities at their present values), or some combination of income and equity elements.

[6] The ability to accurately predict the amount and timing of actual future cashflows is not possible in real life. It is, however, possible in the modeled environment through the use of assumptions about the future. A real life calculation of this economic balance sheet item would be a "best estimate" at any given valuation date. In the modeled world, no such uncertainty exists. Through the assumptions included in the model, the modeled world removes uncertainty that exists in the real world.

from this point on, it is assumed that any discussion of losses or loss cashflows includes loss adjustment expenses as well as losses;

- The unearned premium reserve is recorded at the present value of the "actual" future loss outflows that will arise from that portion of in-force policies still to be earned.

The Question to be Addressed

The company began its research by thinking about the areas of operation within an insurance organization that could most easily be altered in search of improved economic net worth. The company concluded that there are really only three areas that would be both sizable enough and substantially controllable enough to warrant consideration:

- Asset mix – the company could modify how it reinvested available cashflow. Available cashflow is the net new money the company has collected during the year. It is composed primarily of cash from maturing and prepaying bonds, investment income collected, and net cash from underwriting. This is also referred to as a "new money" reinvestment approach. The new money approach can be contrasted against a "portfolio rebalancing" approach, in which the entire portfolio is restructured at the end of each year so that the relative percentage of assets within each asset class matches a targeted overall asset distribution.
- Volume of new and renewal writings – the company could decide to write more or less business in the coming years.
- Profitability of new and renewal writings – to the extent that the company is not a price taker in any given market, the company could endeavor to increase the amount of premium received for policies written.

Of these three, the company focused only on the first one in this analysis. The latter two are to be the subjects of future analyses.

Quantifying Relative Influences of Different Areas of Variability

Establishing the "Fully Deterministic State"

To validate or disprove the thesis with regards to the major drivers of economic net worth, the company established what will be referred to as the results associated with a "**fully deterministic state**." The fully deterministic state uses as inputs:

- A static set of economic assumptions that were derived from a combination of current economic conditions and long term historical averages;
- A static set of underwriting assumptions, including the amounts of premium to be written, the loss and expense ratios that will be experienced on the premium writings, the and the timing of the payout of claims;
- A static estimate of the liability for unpaid claims as of the model start date (time T_0) and the timing of the future payment of the unpaid claims obligations;
- A static asset reinvestment strategy that defines how the model purchases new assets over the projection horizon (times T_1 through T_5).

Establishing the "Stochastic Base Case"

The "**stochastic base case**" differs from the fully deterministic state in six areas. These are the areas of variability or randomness within the model. The six variable elements were interest rates, inflation rates, stock market returns, the adequacy of time T_0 loss reserves, the loss ratio on future writings and the speed with which loss reserves and claims on future writings were paid out. Underlying the stochastic base case is a series of just over eight hundred different "iterations", or alternative projections of the company's financial performance in times T_1 through T_5.

Economic variability (i.e. variability in interest rates, inflation rates, and stock market returns) was derived by looking back into history. It was assumed that history would provide a sufficiently robust range of economic conditions to reasonably predict the range of possible future economic conditions. The historical dataset used consists of monthly observations from January 1926 to November 1998. A series of 800+ "annual rates of change" was computed for each economic variable. For example, the first rate of change for long-term interest rates is the yield on a long-term bond issued in January 1927 divided by the yield on a long-term bond issued in January 1926. The second rate of change is the yield on a long-term bond issued in February 1927 divided by the yield on a long-term bond issued in February 1926. The final rate of change is the yield on a long-term bond issued in November 1998 divided by the yield on a long-term bond issued in November 1997.

The first iteration of the model would apply the January 1927/January 1926 rate of change to the actually observed economic conditions at time T_0 to develop the projected economic conditions at time T_1. The first iteration would then apply the January 1928/January 1927 rate of change to the projected time T_1 economic conditions to develop the projected economic conditions at time T_2. The January 1929/January 1928 rate of change would be used to project time T_3 conditions, the January 1930/January 1929 rate of change would be used to project time T_4 conditions and the January 1931/January 1930 rate of change would be used to project time T_5 conditions. The second iteration would use the rates of change between February 1926 and February 1927 to go from time T_0 to time T_1, and the rate of change between February 1927 and February 1928 to go from time T_1 to time T_2, etc., etc., etc.[7]

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[7] Two major benefits are achieved by using historical economic information. One is from the standpoint of the model builder/user. If future economic scenarios are generated by a model (see Ahlgrim, et al., "Parameterizing Interest Rate Models," Casualty Actuarial Society Forum, Summer 1999, pp. 1-50 for a description of different types of economic scenario generation models), appropriate cause and effect relationships must be established between the key economic drivers. This is one of the more difficult and contentious areas of model parameterization. By using historical data, there is no need to establish causal relationships. It is enough to know that in year A, interest rates moved by X%, while inflation rates moved by Y%, and the stock market moved by Z%. The underlying causal relationships become superfluous because all that is needed is the actual

The three non-economic sources of variability (loss ratios, loss reserve adequacy, and payout pattern randomness) were modeled through a more traditional process. Here, historical results were examined to develop the parameters of lognormal distributions that could be used to describe the observed variability. A series of random numbers were generated. These random numbers in turn were used to produce random values from each of the lognormal distributions. Each lognormal distribution was assumed to be independent of the others.

Figure 1 displays the difference between the company's economic net worth at the end of time T_5 under the fully deterministic state versus the values produced by "turning on" the variability and volatility in interest rates, stock market returns, inflation rates, loss ratios, loss reserve estimates and payout patterns.

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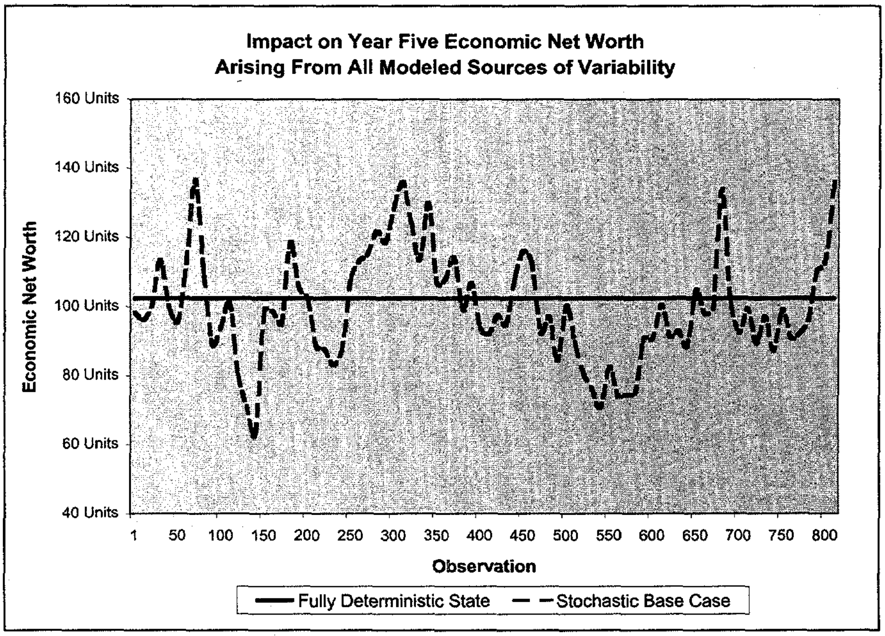
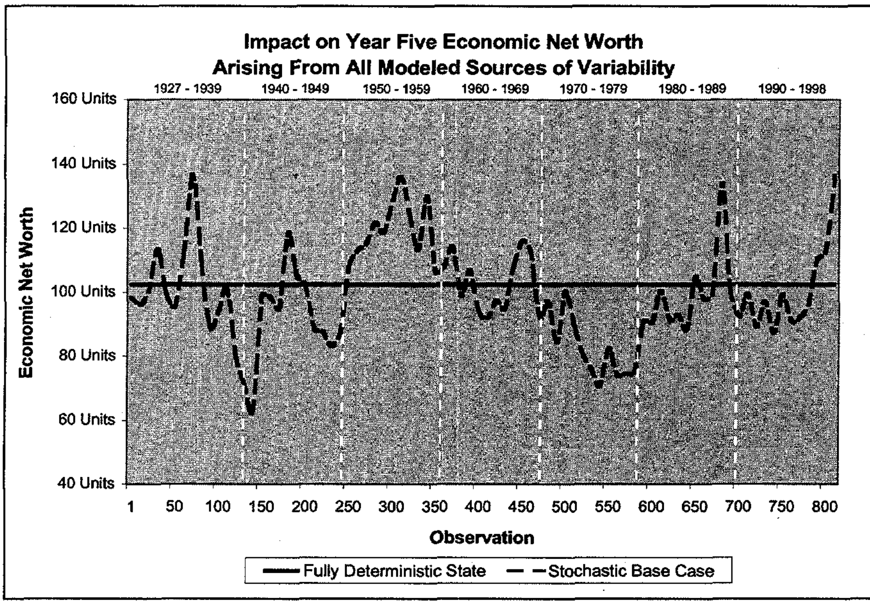
observations. The second benefit is from the standpoint of results presentation. Results can now be presented in the context of history. For example, the impact of a recurrence of the 1970's stagflation can be prefaced by a comment such as, "Now suppose we were to try this business plan while the economy goes through a crisis similar to what was endured in the late 1970s..."

Figure 1



As noted in the explanation of how the economic variability was created, if the progression of data points for the stochastic base case in Figure 1 seem as if they are based on some underlying time series, it is because they are. Observation one reflects the economic conditions between January 1926 and January 1931, observation two reflects the economic conditions between February 1926 and February 1931, and so on and so forth. By retaining the time series concept in the graphical display, it is possible to focus an audience's attention on the economic conditions in one period or another, a capability that would not exist if the results were sorted from low to high. Figure 2 redisplays the results from Figure 1, but this time with the historical context that underlies the economic conditions also displayed.

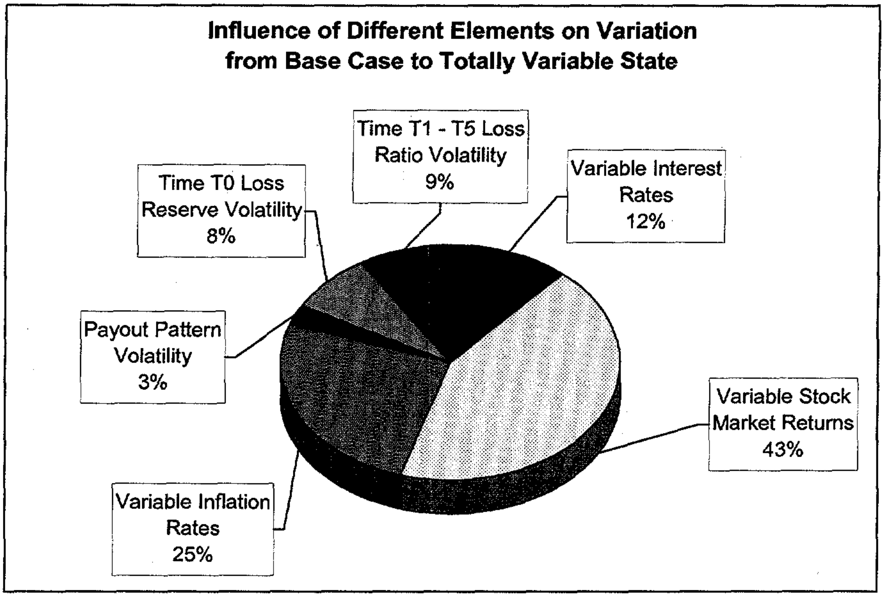
Figure 2



To understand how much of difference in observed net worth values could be attributed to the six volatile elements, the company regressed the difference between the deterministic state value and the stochastic base case values with regards to changes in the values of the volatile elements. The regression's R^2 value indicated that approximately 84% of the total variability was attributable to changes in the values of the six volatile elements.

To quantify the relative influence of each of the six volatile elements, the analysis compared the relative level of variation in the time T_5 net worth that was caused by making each individual element volatile. The pie chart in Figure 3 shows the relative influence of each of the six volatile elements in producing the change from the base case values to the volatile values. This supports the company's *a priori* hypothesis that the relative significance of the external environment is greater than that caused by noise in the company's loss reserve, loss payout, or future loss ratio assumptions.

Figure 3



Asset Mix Alternatives

The company analyzed the relative influence of asset mix by turning "on" the volatility in all modeled elements with volatility provisions, namely interest rates, inflation rates, stock market returns, loss ratios, loss reserve adequacy and loss payout speed. The asset strategy used in this simulation was a "status quo" one, i.e. one in which the mix of new asset purchases was the same as the mix of assets at time T_0. As noted earlier, this is the "stochastic base case." The simulation model was then run six additional times, altering the asset reinvestment strategy in each of the subsequent runs. The different asset reinvestment strategies tested are shown in Table 1.

Table 1

	Status Quo	Alternate 1	Alternate 2	Alternate 3	Alternate 4	Alternate 5	Alternate 6
Government bonds	6%	100%					
Corporate bonds	60%		100%				
High yield bonds	2%			100%			
Tax-exempt bonds	14%				100%		
Cash	5%					100%	
Common stock	13%						100%

It was assumed that the "government" bond was a ten-year Treasury bond and that bonds purchased in the other bond classes would have approximately a ten year average life and a seven year duration.[8] The interest rate applied to cash balances was assumed to be equal to the simulated interest rate for a one-year Treasury bond. The total return of common stocks was assumed to be equal to the simulated return of the overall stock market index. Table 2 shows the modeling parameters used to establish yield relationships between the different bond classes. The company's investment department established these parameters. They represent the

[8] The weighted average life of a bond is calculated as $\sum_{t=1}^{n} CF_t * t \Big/ \sum_{t=1}^{n} CF_t$

The duration of a bond is calculated as $\sum_{t=1}^{n} \frac{t * CF_t}{(1+y)^t} \Big/ \sum_{t=1}^{n} \frac{CF_t}{(1+y)^t}$

where t = year of the cash flow (i.e. year 1, year 2, etc.)

CF_t = cash flow in year t

n = number of years to maturity

y = yield to maturity

company's expectations about average current and future yields available in the bond market.

Table 2

Bond	Coupon premium = assumed additional yield over the yield on a 10 year Treasury bill
Government bonds	No premium. The interest rate simulation process derives the yield.
Corporate bonds	+125 basis point premium
High yield bonds	+350 basis point premium
Tax-exempt bonds	-100 basis point premium
All Cash (a proxy for continuous reinvestment in short-term government bonds – bonds with maturities of 6 months to 1 year)	No premium over the yield on a 1-year Treasury bill. The interest rate simulation process derives the yield on a 1 year Treasury bill.

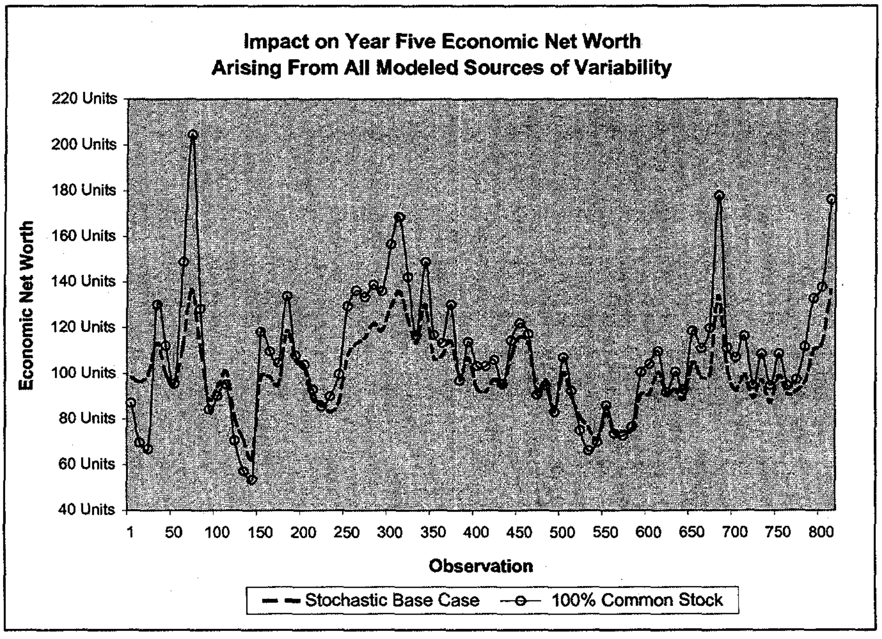
Table 3 shows the time T_5 economic net worths' mean values and standard deviations under the status quo and the six alternative reinvestment scenarios:

Table 3

	Base Status Quo	Alt. 1: All Govt. Bonds	Alt 2: All Corp. Bonds	Alt 3: All High Yield Bonds	Alt 4: All Tax Exempt Bonds	Alt 5: All Cash	Alt 6: All Common Stock
Economic Net Worth Mean Value	100.0	97.2	98.6	100.6	98.3	97.9	108.0
Percent increase over Status Quo	N/A	-2.8%	-1.4%	0.6%	-1.7%	-2.1%	8.0%
Standard Deviation	15.6	14.0	14.1	14.2	14.6	14.3	27.3
Percent increase over Status Quo	N/A	-10.6%	-9.9%	-8.8%	-6.6%	-8.5%	74.6%

From this perspective, it appears that the alternative of reinvesting all new money in common stocks is the preferred alternative. The average economic net worth increases by 8.0% and, while the standard deviation of the results increases by 74.6%, it is only when the stock market experiences a substantial downturn that the economic net worth under the common stock reinvestment strategy falls below the base strategy. A graphical comparison of the status quo versus the "100% common stock" strategy is shown in Figure 4.

Figure 4

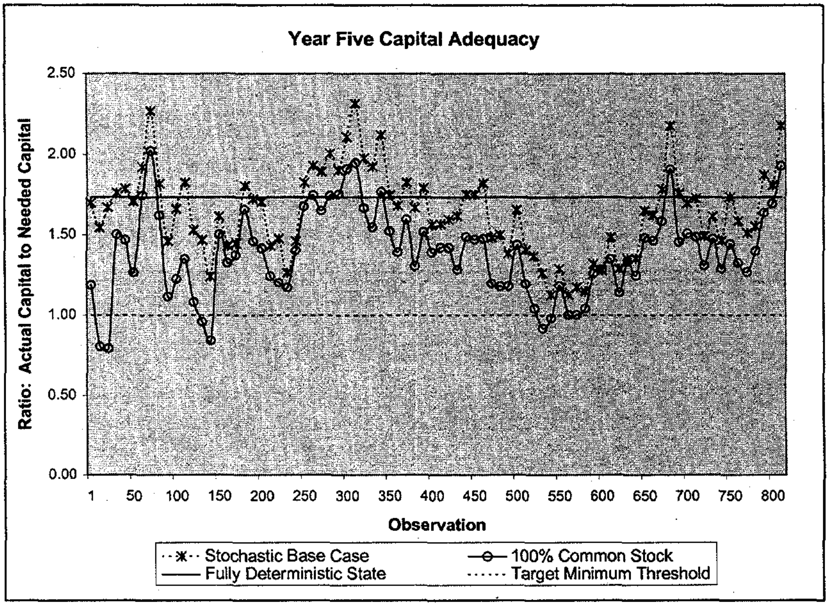


The Other Consideration: Risk Measurements

The desire to grow the company's net worth to the greatest degree possible is only half of the story. The other half of the story is the potential drop in asset values inherent in pursuing a more volatile investment strategy. The more volatile the investment strategy, the greater the potential swings in both economic net worth and statutory surplus. As the company pursues its quest for enhanced net worth, the company must remain cognizant of how it is being viewed by the outside world. The company, through analyses of peer groups and through conversations with the different rating agencies, has developed a "targeted minimum capital" metric against which statutory surplus can be compared. (This threshold does not have to be equal to the level of capital needed to avoid regulatory oversight, i.e. twice the company's Authorized Control Level. It can be something of the company's choosing.) It is the company's objective to never have statutory surplus fall below this threshold.

The chart in Figure 5 shows how the fully deterministic state, the stochastic base case and the "100% common stock" reinvestment strategies fare versus the threshold. As can be seen, the fully deterministic state is substantially in excess of the threshold. However, when the last seventy years of economic history are overlaid upon the current reinvestment strategy (the stochastic base case), instead of the deterministic economic conditions, it can be seen that the current reinvestment strategy at times places the company close to the threshold. If the strategy of reinvesting all new money into common stocks were to be followed and history were to repeat itself, there are several instances of stock market declines that would place the company below the targeted minimum threshold.

Figure 5



As indicated in the pie chart in Figure 3, the three major drivers of variation between the fully deterministic state and the stochastic base case were variations in common stock returns, interest rates, and inflation rates. To verify that the major cause of the situations in which the 100% common stock reinvestment strategy runs into trouble is due to stock market volatility and not interest rate or inflation rate volatility, a series of regression equations were developed. Each equation compares the held/needed capital ratio under the 100% common stock reinvestment strategy to an element in the economic environment. These can be seen in the graphs in Figure 6, Figure 7, and Figure 8. The significance, or lack thereof, of the regression equations displayed in Figures 6, 7, and 8 support the *a priori* expectation that neither interest rate volatility nor inflation rate volatility are major influences on the capital adequacy of the 100% common stock strategy.

Figure 6

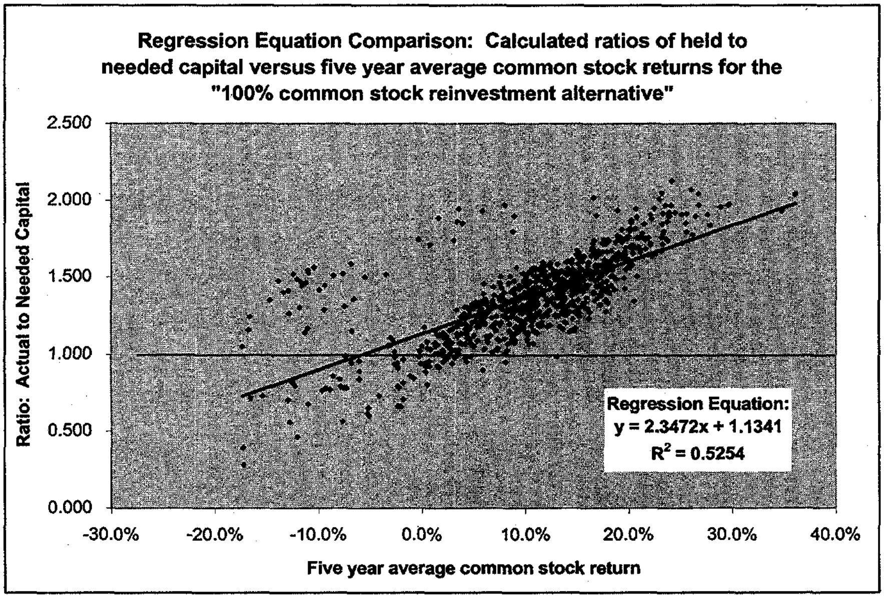


Figure 7

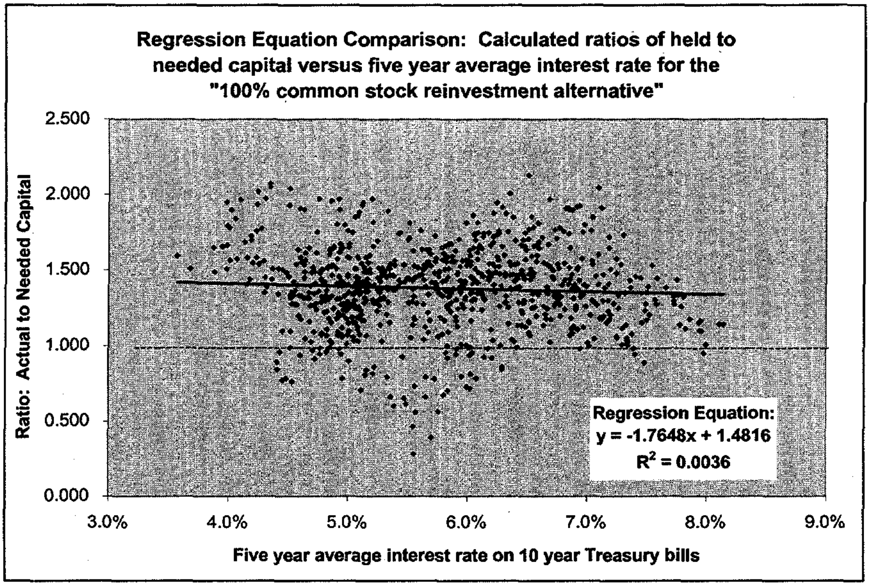
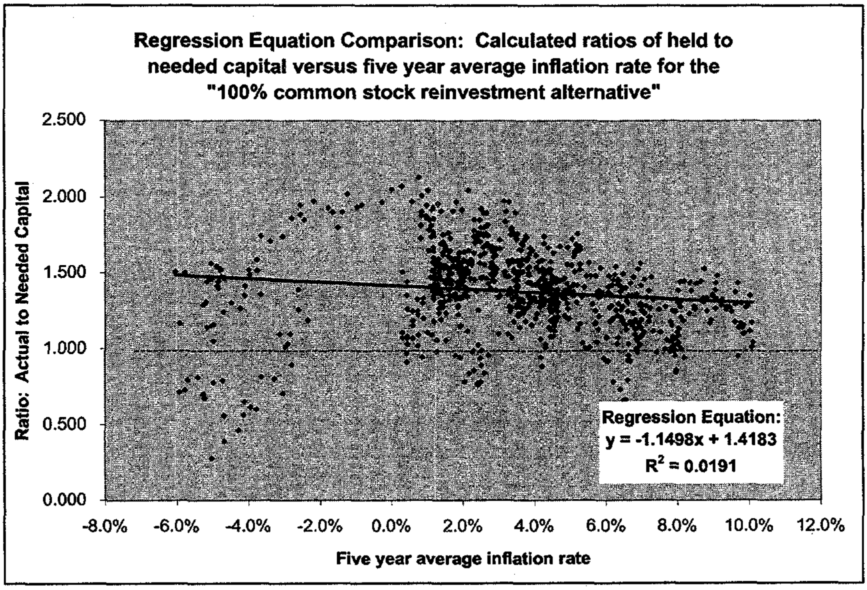


Figure 8



Clearly, then, the strategy of reinvesting all free cash flow in common stock leaves the company exposed to the possibility of an impaired capital base should common stock performance falter. The regression analysis in Figure 6 indicates that, on the average, a sustained common stock return of –5% will lead to an undesirably low level of held statutory capital.

Asset Mix Alternatives Revisited

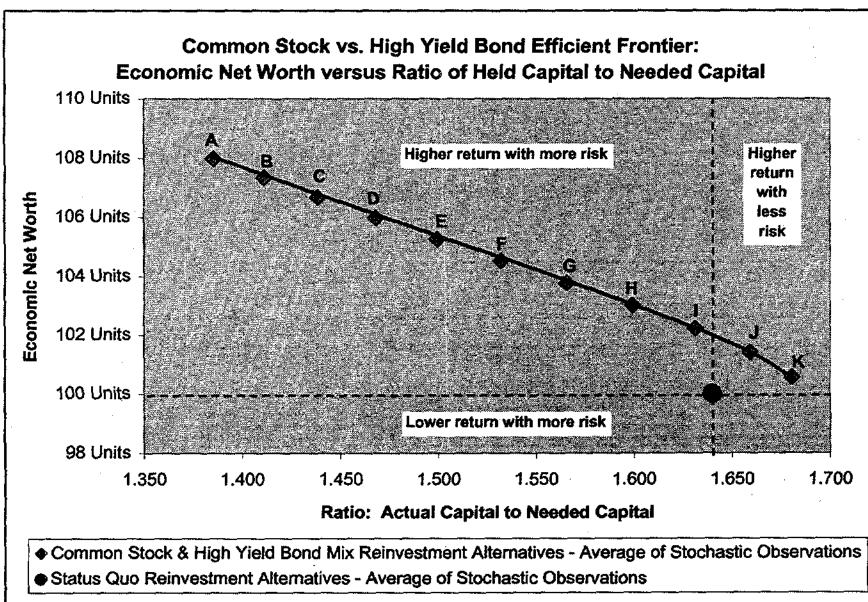
After seeing the potential downside risk of moving completely towards a strategy of common stock reinvestment, the company revisited the information in Table 3. This information is redisplayed in Table 4. The company's desire was to find an asset allocation strategy that enhanced future net worth beyond what the current "status quo" strategy would produce, yet would not result in quite so much downside risk exposure as was produced by the "100% common stock" strategy.

Table 4

	Base Status Quo	Alt. 1: All Govt. Bonds	Alt 2: All Corp. Bonds	Alt 3: All High Yield Bonds	Alt 4: All Tax Exempt Bonds	Alt 5: All Cash	Alt 6: All Common Stock
Economic Net Worth Mean Value	100.0	97.2	98.6	100.6	98.3	97.9	108.0
Percent increase over Base	N/A	-2.8%	-1.4%	0.6%	-1.7%	-2.1%	8.0%
Standard Deviation	15.6	14.0	14.1	14.2	14.6	14.3	27.3
% of simulated scenarios that fall below capital threshold	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	7.9%

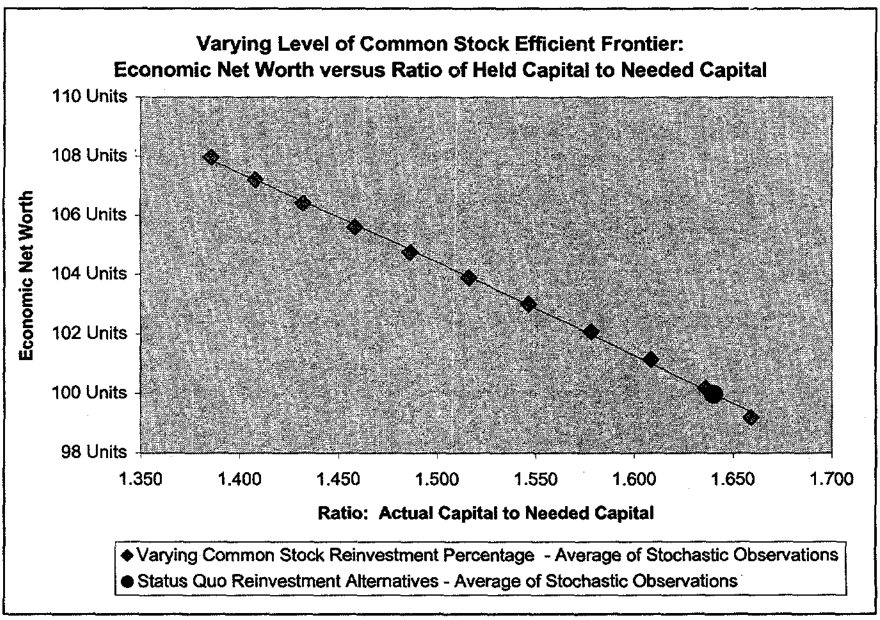
Upon reviewing the data, it was observed that the only two asset types producing higher economic net worth were high yield bonds and common stocks. Furthermore, high yield bonds produced the higher economic net worth with less volatility than the status quo reinvestment strategy. In theory, then, an efficient frontier could be established that ranged from reinvesting entirely in high yield bonds to reinvesting entirely in common stocks. Every point on the frontier would have a greater economic net worth than status quo reinvestment strategy. At least some of the points would also have less volatility than the status quo reinvestment strategy. Figures 9 shows graphically how the status quo reinvestment strategy compares to this efficient frontier. As can be seen in Figure 9, the reinvestment strategy underlying points "J" and "K" (J = 10% common stock, 90% high yield bonds, K = 100% high yield bonds) produce a higher economic net worth with less risk than the status quo strategy. Points "A" through "I" produce still higher economic net worth, but require more capital to support the higher level of risk.

Figure 9



The only problem with these results is that none of the strategies on the efficient frontier could realistically be implemented. Both the state insurance department and the company have limits on the amount of high yield bonds and common stocks that can be held. Since Table 4 indicates that no other asset class outperforms the status quo, the best the company can do is adjust the weighting of common stocks relative to the other assets in the status quo portfolio. Another way of looking at this is to think that the status quo portfolio ***is already*** an efficient frontier portfolio. The only thing the company can do is decide to move further up the risk curve in order to achieve a higher net worth expectation. Of the asset portfolios available to the company, there is no portfolio that will produce a higher net worth at a lower risk. Figure 10 shows the risk/return tradeoffs that are available to the company by varying the level of common stocks relative to the asset mix of the rest of the current portfolio.

Figure 10

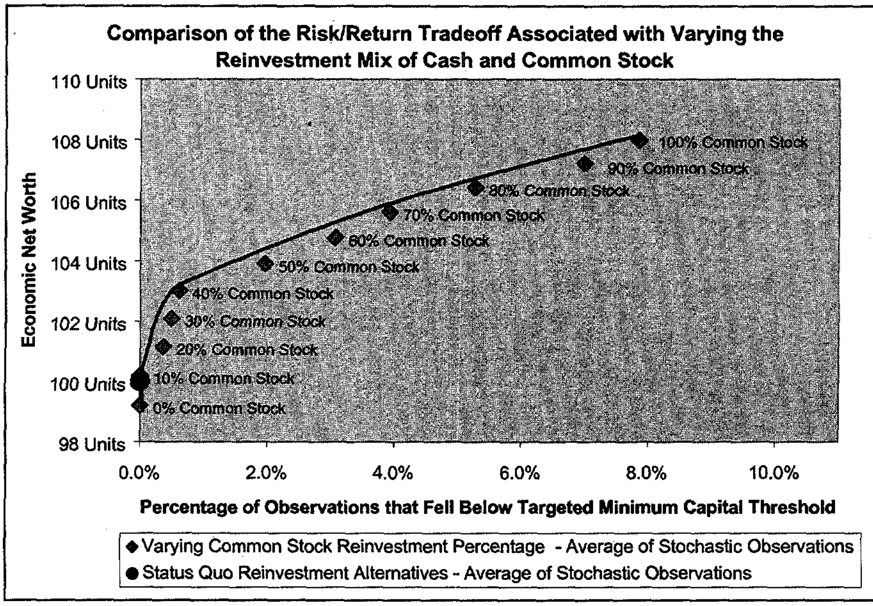


The question remains, though, "Which mix?" The data in Table 5 was prepared to help the company answer this question. Table 5 shows the same information that was displayed in Figure 10, but adds information about the number of times the company would fall below the target floor. Figure 11 displays this additional information in a graphical format.

Table 5

% Common Stock	Average economic net worth	Average ratio of statutory capital to capital threshold	% of simulated scenarios that fall below capital threshold
0%	99.2	1.659	0.0%
10%	100.2	1.636	0.0%
20%	101.2	1.608	0.4%
30%	102.1	1.578	0.5%
40%	103.0	1.547	0.6%
50%	103.9	1.516	2.0%
60%	104.8	1.487	3.1%
70%	105.6	1.459	3.9%
80%	106.4	1.432	5.3%
90%	107.2	1.408	7.0%
100%	108.0	1.386	7.9%

Figure 11



Comparison of the Risk/Return Tradeoff Associated with Varying the Reinvestment Mix of Cash and Common Stock
110 Units
108 Units
106 Units
104 Units
102 Units
100 Units
98 Units
Economic Net Worth
0.0%
2.0%
4.0%
6.0%
8.0%
10.0%
Percentage of Observations that Fell Below Targeted Minimum Capital Threshold
100% Common Stock
90% Common Stock
80% Common Stock
70% Common Stock
60% Common Stock
50% Common Stock
40% Common Stock
30% Common Stock
20% Common Stock
10% Common Stock
0% Common Stock
Varying Common Stock Reinvestment Percentage - Average of Stochastic Observations
Status Quo Reinvestment Alternatives - Average of Stochastic Observations

Conclusions and Recommendations to Management

From the previous charts and graphs and tables, it appears that, based on the risk and reward measurements used, a reinvestment strategy that mimics the current investment mix is an efficient option, albeit a conservative one. Senior management has expressed an interest in taking on more investment risk. From the results shown earlier in Figure 5 and Table 4 it was clear that, while the 100% common stock alternative might be the most advantageous in terms of long-term growth of economic net worth, too much volatility and risk accompanied this alternative. Finally, from the information in Figure 11 and Table 5, the basis for a recommendation appeared. The recommendation, which is still before senior management, is to move the level of unaffiliated common stock holdings from the eight percent level that it is at today towards a position somewhere in the range of twenty to thirty percent.

This recommendation to increase the company's common stock holdings is not a new revelation. Both Feldblum [3] and Noris [7] drew similar conclusions more than a decade ago. So one must ask, "Why has the percentage of assets invested in common stocks by insurance companies not grown more significantly, despite articles and recommendations to the contrary?"[9] Feldblum suggests a few reasons,

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[9] In 1988, according to A.M Best's 1989 Aggregates and Averages, unaffiliated common stocks accounted for 10% of the insurance industry's invested assets. In 1998, according to A.M. Best's 1999 Aggregates and Averages, unaffiliated common stocks accounted for 18% of total invested assets. Based on the performance of the stock market between 1988 and 1998, this actually represents a reduction in common stocks as a percentage of total invested assets. Suppose one assumes that the industry stock portfolio had returns similar to that of the broad S&P 500 index between 1988 and 1998. If the industry had just held onto all the stock owned at December 1988 and reinvested all dividends, the value of the industry's common stock holdings would have been 61% greater than it actually was at December 1998. If one were to adjust the total invested assets at December 1998 for this difference, it can be seen that unaffiliated common stocks would have grown to almost 27% of total invested assets. Of course, this is the value of 20/20 hindsight. Who knew in 1988 how well the stock market was going to do for the next ten years?

with the bottom line being that there are many more considerations that enter into the investment decision than just maximizing the growth of net worth. For example, two of the considerations Feldblum identifies are stability of statutory financial results and tax considerations.[10] To that, one might also add the maximization of current income.

How does owning bonds increase the stability of statutory financial results? Since bonds are recorded in the Annual Statement at amortized cost instead of at market value, changes in the underlying market value of owned bonds are not reflected in property-casualty financial statements. Except for bonds that are classified as being below investment grade, the only time the difference between market and book value becomes evident is when bonds are sold.[11] Common stocks, on the other hand, are recorded at market value. Any changes in the market value of stocks are immediately reflected in the company's surplus.

What is the influence of tax considerations? Once again, Annual Statement rules play a role. Statutory accounting does not require the establishment of a deferred tax asset (or liability) for unrealized gains (or losses) in a company's common stock portfolio. When a company tries to capture gains in a stock portfolio, then, the conversion of unrealized gains to realized gains triggers a previously unrecognized tax cost. A company that has a highly appreciated stock portfolio may

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[10] Feldblum [3] p.122. Statutory financial statement stability: Feldblum notes that "insurers do not want to add investment risks to the fluctuations of the insurance underwriting cycle...Common stocks must be reported at their market value on Annual Statements, so their [reported] values fluctuate more than those of bonds...Were bonds reported on the Annual Statement at their market values, instead of amortized values, their actual riskiness would be apparent, and insurers would invest more heavily in common stocks." Tax considerations: Feldblum comments that "federal income tax laws influence financial portfolios. Tax law changes affect asset holdings in ways that asset/liability matching theory does not recognize."

[11] Below investment grade bonds are carried at the lesser of market or book value. Therefore, changes in the market value of these bonds could appear in the financial statements without the bonds being sold.

be unwilling to take advantage of the appreciation because of the tax bite that will accompany selling the stock. This leaves the company in a position of having a highly valued asset, the value of which can't be touched unless the company either has losses from other operations that can be used to offset the capital gains realization or is willing to accept the loss of value arising from capital gains taxes on the asset sale.

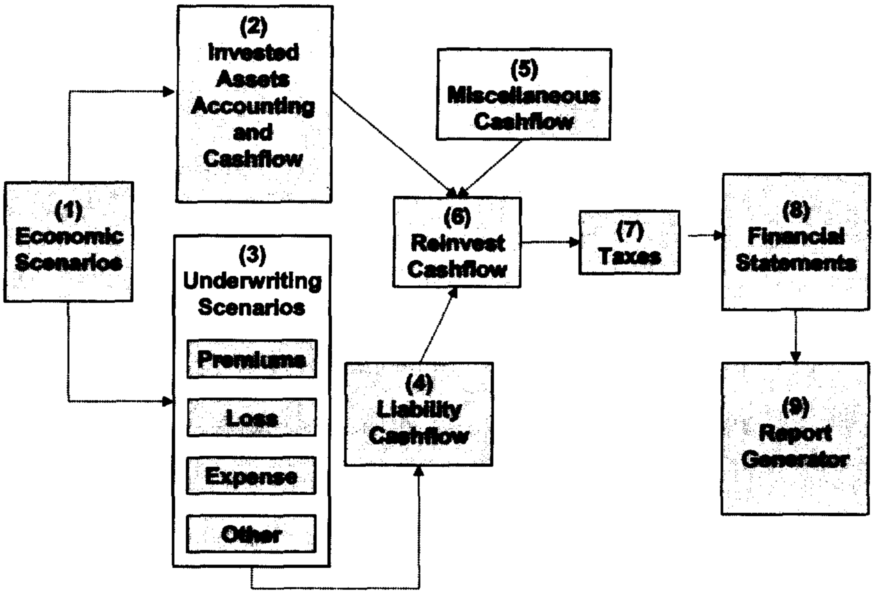
How does maximization of current income play a role? Property-casualty insurance companies expect a steady premium inflow that, along with investment income, can be used to pay current claims. Investing more heavily in common stocks will reduce a company's investment income inflow compared to making a similar investment in bonds. If a company does not have sufficiently large cash inflows from insurance operations and other sources to pay current claims, the company is forced into either short-term borrowing or the forced sale of assets. If the economic environment is not favorable for either of these actions, it could have a detrimental impact on the company's financial position. Since owning a higher percentage of common stocks reduces investment income, a company that invests more heavily in common stocks is assuming additional risk of having insufficient cash inflow to cover cash outflows.

Ultimately, senior management must decide which of these considerations is most important. Investing more heavily in common stocks will reduce current income and increase financial statement volatility. The act of accessing any gains that may be achieved on common stocks triggers tax consequences that the company may or may not want to incur. If senior management concludes that the maximization of net worth is worth incurring these costs, then the recommendation has merit. Otherwise, the analysis will have proven to be educational and informative but not sufficient to justify any action at this time.

Model Structure Overview

The model has nine basic sections, organized as shown in Figure A-1:

Figure A - 1



The structure and functionality of the model is similar in nature to those described in the following papers:

- D'Arcy, et al., "Using the Public Access DFA Model: A Case Study," Casualty Actuarial Forum, Casualty Actuarial Society, Summer 1998, pp. 53-118;
- Hodes, et al., "The Financial Modeling of Property/Casualty Insurance Companies," Casualty Actuarial Forum, Casualty Actuarial Society, Spring 1996, pp. 3-88;

- Kirschner & Scheel, "The Mechanics of a Stochastic Corporate Financial Model," Proceedings of The Casualty Actuarial Society 1998, Casualty Actuarial Society, Volume LXXXV, pp. 404-454;
- Witcraft, "Profitability Targets: DFA Provides Probability Estimates," Casualty Actuarial Forum, Casualty Actuarial Society, Summer 1998, pp. 273-302.

The following is a brief description of each section of the model.

1. Economic Scenario Generator

The economic scenarios used in this modeling exercise were rolling five-year observations taken from actual United States economic history. The economic variables captured in this way include the one year constant maturity US Treasury bill (a proxy for a short term risk free interest rate), the ten year constant maturity US Treasury bill (a proxy for a long term risk free interest rate), a stock market total return index that is proxied by the S&P 500, general inflation based on the overall Consumer Price Index, and medical inflation based on the medical component of the Consumer Price Index. The rolling five-year observations were used to develop economic projections for the model's five year time horizon. For example, the first simulation takes as its economic scenario the economic history from January 1926 through December 1930. The economic scenario for the first projection year is the twelve-month change in each index between January 1926 and December 1926. The economic scenario for the second projection year is the twelve month change in each index between January 1927 and December 1927, and so on and so forth for the third, fourth and fifth projection years. The second simulation takes as its economic scenario the economic history from February 1926 through January 1931. By using historical data as the basis for economic scenarios, the company avoids the problems inherent in a theoretical economic scenario generation process, namely the parameterization of the theoretical model, including the parameterization of internal correlations and interrelationships between the different economic variables.

2. Invested Asset Accounting and Cashflow

The economic scenario determines what happens to the fixed income assets and common stock holdings over the course of a projection year. Changes in interest rates cause a greater or lesser level of prepayments in each of the modeled bond classes, with rising interest rates resulting in less prepayment than falling interest rates. Changes in interest rates do not affect the investment income produced by each bond class – the investment income is a function of the coupon rate available at the time the bond was purchased.[12] The total return of the economic scenario's stock market index determines the market value change of all common stocks that were in the company's portfolio at the start of the projection year.

This module quantifies the cashflow arising from invested assets, before any forced asset sales might occur. The cashflow is comprised of investment income received, bond maturities and prepayments, less investment expenses paid.

3. Underwriting Scenarios

The Underwriting Scenario Module is divided into a series of line of business groupings. The functionality of the Underwriting Scenario Module is identical within each grouping. The grouping process allows the modeler to specify different characteristics for each line of business grouping. For each line of business grouping, the underwriting module takes as input information on a series of initial conditions and a series of anticipated future actions. The initial conditions include unearned and uncollected premium at time T_0, indicated and held loss reserves at time T_0, and unpaid underwriting expenses and policyholder dividends at time T_0. The anticipated future actions include projected premium writings during times T_1

[12] Bonds are grouped according to both type (government, municipal, etc.) and purchase year. Bonds purchased in one of the projection years are assumed to have coupon rates commensurate with the risk free long term interest rate in effect for that projection year, plus a user-specified risk spread. The coupon rates of bonds purchased before the model's "start date" are already known.

through T_5 and the loss, expense, and dividend ratios associated with the premium writings.

Additional inputs to the Underwriting Scenario Module include patterns for premium earning, premium collection, loss payout, expense payout, and dividend payout. These patterns are used to produce the necessary income statement and cash flow statement accounts from the initial conditions and anticipated future actions.

Lastly, information is included on key variability parameters. For each line of business grouping, the variability parameters allow the model to

- Randomly vary the profitability of future business by varying the user-entered expected loss ratio;
- Randomly vary the indicated time T_0 loss reserves (to simulate the uncertainty inherent in the time T_0 "best estimate" loss reserve indication);
- Randomly adjust loss payment speed;
- Randomly generate catastrophe losses;
- Quantify the effects of unanticipted inflation on loss payments.[13]

4. Underwriting Cashflow

The results from the line of business groupings within the Underwriting Scenarios Module are combined into one underwriting cashflow projection.

[13] Robert Butsic, in his 1981 paper [1, pp. 58-102] describes how inflation can impact losses. The model assumes that the loss reserves and the target loss ratio entered by the modeler include an implicit level of future inflation in the loss estimate, i.e. the expected future inflation rate. The model uses the techniques described by Butsic to adjust the projected payment levels by the difference between the "actual" inflation rate produced by the economic scenario generator and the modeler's expected inflation rate.

5. Miscellaneous Cashflow

This module quantifies all cashflows that are not otherwise accounted for in the Invested Asset or Underwriting Modules. Examples of items that might be captured here include capital infusions or payments of dividends to stockholders, payment of fixed expenses that are not directly related to either investment or underwriting activities, receipt of miscellaneous asset receivables or payment of miscellaneous asset payables. The specific assumptions used in the modeling exercise described in this paper are not material to the overall results and conclusions. What is relevant is that the model has the capacity to address miscellaneous items of this nature.

6. Cashflow Reinvestment

This module combines the cashflows from the Invested Asset Module, the Underwriting Module, and the Miscellaneous Module into a net cashflow for each time period being projected. Depending on the way the modeler has specified the asset reinvestment process should take place (either rebalancing the entire portfolio to a specified distribution or just reinvesting net cash flow), and whether or not the net cash flow is positive or negative, a series of reinvestment activities are triggered. The reinvestment activities could involve the sale of some or all existing assets in an asset class. This could be done to force turnover within an asset class, such as might exist in a company with an active strategy of realizing capital gains on a stock market portfolio, in which case the proceeds might be reinvested back into that asset class. Alternatively, the sale of assets could be done if the modeler has set limits on how much of a particular asset class the company can hold, and the asset sale is being done to bring the holdings within the desired limitation. In this case, the proceeds would be reinvested in a different asset class. Lastly, the reinvestment activity could trigger the purchase of new assets within an asset class. If the new assets to be purchased are bonds, the model uses the "actual" risk-free interest rate developed by the Economic Scenario Generator as the basis for determining the coupon rates the newly purchased bonds will pay in the future.

7. Taxes

After the asset reinvestment is completed, the model goes in a quantification of federal income taxes. The Tax Module captures information from the Underwriting Module about tax-discounted loss reserves. It captures information from the Invested Asset Accounting and Cashflow Module and the Asset Reinvestment Module that allows it to quantify the portion of investment income that arises from tax-free bonds and stock dividends. It also captures realized capital gain information from the Asset Reinvestment Module. All this information is used to produce the company's tax liability in each projection period.

The tax calculation in the model is a simplification of the actual tax calculation a company would have to follow. It includes a number of the provisions from the Tax Reform Act of 1986, including:

- Discounting of loss reserves using discount factors provided by the Internal Revenue Service;
- Unearned premium reserve revenue offset, whereby twenty percent of the change in the unearned premium reserve is added to statutory net income;
- Proration of investment income from tax-exempt bonds, whereby fifteen percent of tax exempt bonds' investment income is included in taxable income;
- Proration of the "dividends received deduction" on stock dividends. The model assumes that 59.5% of all stock dividends received are tax exempt. This is the net result of exempting from tax considerations 70% of all stock dividends, but then adding back fifteen percent of exempted amount;
- Calculation of an Alternative Minimum Tax.

It does not, however, include tax carryforwards or carrybacks. The model assumes that a tax loss results in a "rebate check" being issued to the company from the

Internal Revenue Service, instead of having a loss carryforward that can be used to offset future tax payments.

8. Financial Statement Module

This module rolls all the information produced by the other modules into a series of financial statements and associated risk measurements. This module produces balance sheets, income statements, and cash flow statements over the projection horizon. It also contains calculations of desired risk measurements, such as the NAIC Risk-Based Capital calculation or the Standard & Poor's Capital Adequacy test.

9. Report Generator

This module produces output reports that display statistical and graphical information for selected metrics. Information captured and displayed includes the specific values for each iteration as well as statistics such as mean, standard deviation and various percentiles. Graphical displays of results, either in the form of distributions for one particular time period or as time series over multiple time periods are also displayed.

References

[1] Butsic, Robert "The Effect of Inflation of Losses and Premiums for Property-Liability Insurers," Casualty Actuarial Society 1981 Discussion Paper Program, Casualty Actuarial Society, 1981, pp. 58-102.

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[3] Feldblum, Sholom "Asset Liability Matching for Property Casualty Insurers," Casualty Actuarial Society 1989 Discussion Paper Program, Casualty Actuarial Society, April 1989.

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[5] Ibbotson Associates, Stocks, Bonds, Bills and Inflation, 1999 Yearbook, Chicago: Ibbotson Associates, 1999.

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Capital Adequacy and Allocation Using Dynamic Financial Analysis

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Capital Adequacy and Allocation Using Dynamic Financial Analysis

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Abstract

This paper will discuss the use of a Dynamic Financial Analysis (DFA) model to assist a client company in determining the total capital required to support its underwriting activities, and the portion of that total required capital allocated to each operating division. It will discuss issues related to risk measures, capital adequacy standards, and allocation techniques. Most importantly, it will cover the presentation of findings to the Company's Board of Management.

Acknowledgements

We thank the following for their support and efforts in this project: Chris Madsen, Michael Belfatti, Jeremy Pardoe, Shuh-Ren Tzeng, Avi Farah, Jen Ehrenfeld, Tom Weist, Sean McDermott, Stacey Gleeson, and Dave Spiegler.

1. Introduction

Dynamic financial analysis or "DFA" models can help insurers with many critical strategic issues and decisions. Examples include:

- Assessing alternative reinsurance programs;
- Evaluating capital structure, adequacy and allocation;
- Determining optimal asset allocation; and
- Providing a more accurate base for allocation of corporate-level reinsurance costs or investment income to operating divisions.

This paper will discuss the use of a DFA model[1] to assist a client company (the "Company") in determining the total capital required to support its underwriting activities, and the portion of that total required capital allocated to each operating division. Equally important, it will cover the presentation of findings to the Company's Board of Management (the "Board").

The first step in the DFA study was the parameterization of the DFA model for the Company. Their own reserve, planning, and investment information was used to fit loss distributions, expected payment patterns, premium levels, expenses, and reserve runoff distributions. Asset holdings and detailed representations of their reinsurance programs were also input. Once parameterized, the model generated thousands of iterations of Company results, producing as output distributions of the company results.

The next step was for the Company to decide on a risk measure (e.g., probability of ruin) and a standard for that risk measure (e.g., 1 in 100 years or 1%) for determining required capital. There is no industry consensus for risk measure. Therefore, many alternative risk measures were calculated using the detailed output distribution of company results, including: probability of ruin (either on a statutory or GAAP basis); variance or standard deviation of surplus; expected policyholder deficit; and expected annual default loss rate on surplus [6]. **Section 2** covers the evaluation of alternative risk measures, and the determination of required capital.

Given a total required capital amount, the next issue was allocation to the operating divisions. Conceptually, the desire was to allocate based on the relative contribution of each division to the overall risk of the company. Given a selected risk measure, this became an issue of determining each division's contribution to the total risk measure value. This meant "decomposing" an overall risk measure based on some aggregate distribution for the whole company (e.g., probability of ruin as derived from the distribution of surplus) into the component contributions. Any attempt to decompose an aggregate distribution into its component distributions will quickly run into order dependency issues (see [5] and [9]). To overcome these issues, and arrive at as "fair" an allocation as possible, techniques from game theory were employed. **Section 3** details the allocation approach.

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[1] The model used is ARMS, American Re's proprietary DFA model. Details of the ARMS system can be found in the **Appendix**.

After the technical analysis was completed, the initial presentation to the Board was prepared. The audience consisted of seasoned professionals with different backgrounds and varying familiarity levels with DFA and probability. The choices made as to what to present and how to present it form the basis of **Section 4**.

As a result of the initial presentation, the Board selected several of its members to take a deeper look into the DFA study. Each of these members met with the DFA study team for individual intensive reviews. These reviews are highlighted in **Section 5**.

Because the material was so new, and the study so exhaustive, a substantial presentation binder was also included, with an executive summary, graphs, financial exhibits, and extensive backup detail. The choice of binder material is discussed in **Section 6**.

2. Risk Measures and Required Capital

The choice of risk measure for capital determination is more complex than it may initially appear. There are many valid possibilities, each with its own strengths and weaknesses. The actuarial community has also not converged on a consensus "best" measure, adding to the confusion. To top it off, even if a risk measure is chosen, there is no consensus standard for the "correct" level—should required capital be pegged to a 1% probability of ruin? And over what time horizon–one year?[2]

The actuarial literature describes many viable measures of risks, including:

- Probability of Ruin
- Variance or Standard Deviation of Surplus
- Expected Policyholder Deficit
- Expected Default Loss Rate on Surplus

Each has its merits and weaknesses.

Probability of Ruin

Probability of ruin (exhaustion of surplus) has several advantages. The concept is readily explainable to non-technical audiences (likelihood of bankruptcy). It is also easy to calculate using the distribution of policyholder surplus. It has support from regulators and rating agencies with their focus on company solvency and claims-paying ability. It also translates fairly well to a capital market framework, being roughly comparable to Value-at-Risk (VaR).

However, probability of ruin has weaknesses as well. It is essentially a binary measure (solvent/insolvent), ignoring what Philbrick calls "gradations of solvency" [10]. It also

[2] This quandary is not limited to the actuarial and insurance communities. The very same dilemmas exist in capital market risk management—what Value-at-Risk (VaR) threshold should a company manage to, and over what time horizon?

implicitly associates "risk" with a single percentile of the surplus distribution. This can be problematic when considering the marginal impact of changes in the portfolio—changes that do not impact the selected percentile (e.g., 99^{th}) have not "added any risk" according to this measure.

Variance or Standard Deviation of Surplus
Variance and standard deviation are well-known statistical parameters of distributions. They are well known within the capital market world through the work of Harry Markowitz [7]. They are also convenient as shorthand for characterizing the dispersion of a distribution in a single number.

However, they do not add much beyond probability of ruin[3]. They also can give a distorted notion of variability for skewed distributions.

Expected Policyholder Deficit
Expected Policyholder Deficit or "EPD" [2] provides a better indicator of safety for a large organization than probability of ruin, since the measure reflects the whole tail of the distribution rather than a single percentile. It also has rating agency support[4].

EPD is however more complex to explain to non-technical audiences, and more difficult to calculate. It also uses expected loss as its "base," expressing the target deficit as a percentage of expected loss. From the policyholder perspective (the original focus of EPD [2]), this is appropriate, since they are concerned with expected insurer "defaults" (deficits) as a percentage of their expected loss payments (their "asset"). However, from a capital adequacy perspective, expected loss may not be the most relevant base. Finally, EPD is difficult to translate to capital market risk measures, although it has a parallel in so-called "Conditional Value-at-Risk" [11].

Expected Default Loss Rate on Suplus
Expected Default Loss Rate on Surplus (EDLR), first proposed by Mango [6], takes the severity of ruin focus from EPD one step further by explicitly associating various default percentages with required risk premiums[5]. It also uses the deficit like EPD, but expresses it as a percentage of the surplus itself. This has the advantage of making capital market comparisons very straightforward—see [6]. This ease of comparability also makes explanation to non-technical audiences easy.

EDLR has the disadvantage of not being well known. Also, many are uncomfortable with its utility focus. Even though utility theory is a cornerstone of modern economics, its lack

[3] In fact, if the functional form of the distribution is known, they add nothing. If the distance between the mean and a given percentile is known for the normal distribution, it's variance and standard deviation are also known.
[4] For instance, A.M. Best associates certain Best's Capital Adequacy Ratio (BCAR) values to EPD measures.
[5] The risk premium standards are based on the company utility profile. See Halliwell [3] for an excellent exposition on the insurance applications of utility theory.

of "units" or other real world ties causes concern among some users. For instance, how would one go about parameterizing one's company utility curve?

Risk Measure Standards

Even if a risk measure is chosen, the battle is only half over. A standard must be selected for determination of required capital. This apparently straightforward question in fact has several difficult dimensions that must be considered:

- *On what basis should capital adequacy be assessed—economic, GAAP, statutory?*
  Probability of ruin for example is quite different on an economic versus accounting basis. Economic "ruin"—zero net present value of future payment streams—will be much harder to reach than accounting ruin. Also, a company with positive economic value can be insolvent on an accounting basis.
- *What is the "right" probability standard?*
  Should it be 1%, 0.4%, 0.1%? Companies face the same issue in catastrophe modeling when trying to define their "capacity" in a given geographical region, and set their reinsurance retention.
- *What is the "right" time horizon?*
  One year? Two years? Five years? As the time horizon increases, the spread of variability increases, which means the probability of ruin increases, but so does the forecast error.

Framing the Capital Adequacy Question for Presentation

There are really two questions a client can be asking regarding capital adequacy:

- What is the safety level of my current capital?
- What is my capital redundancy/(deficiency) for other safety levels?

The safety level of current capital was expressed using all the available risk measures. This effectively drove home the point that "required capital" is not yet a firm concept with a single, definitive value. It also made clear the effects of the differing focuses and assumptions underlying the various risk measures. Table 1 below shows an example of the Safety Level of Current Capital exhibit (all were done using the same time horizon—e.g., the distribution of surplus one year in the future):

Table 1
Example Safety Level of Current Capital Table

Risk Measure	Level Implied by Current Capital
Probability of Ruin	1 in 200 years or 0.5%
EPD	1.2% of Expected Loss
EDLR	2% of Capital

For assessing how redundant or deficient the current capital is when compared against other target values of the risk measures, exhibits like Table 2 below were used (assume current capital = $1,100):

Table 2
Example Table for Capital Redundancy/(Deficiency)

Risk Measure	Capital Need	Excess/ (Deficit) Capital
1 in 100 probability of ruin	$ 800	$300
1 in 250 probability of ruin	$1,000	$100
1 in 500 probability of ruin	$1,400	($300)
2% EPD	$ 900	$200
1% EPD	$1,200	($100)
0.5% EPD	$1,700	($600)
2.0% EDLR	$1,000	($100)
1.0% EDLR	$2,000	($900)
0.5% EDLR	$3,000	($1,900)

Risk and Safety Trade-off
If all the company cared about was safety, they could simply increase capital until the required safety level was achieved. In most cases, unfortunately, increasing capital without any change in business activity or the investment asset mix will decrease the Company's profitability. Output from the DFA must show this trade-off in a simple and direct manner. The Board of Management needs to see the impact of increasing or decreasing capital. **Exhibit 1** is an example of the type of graph used to demonstrate this. This graph shows the trade-off between risk and reward for different levels of capital. The graph shows the 50th percentile ROE versus the Safety level (here 1-EPD %) for different levels of capital. As expected, removing capital increases the ROE but decreases the safety level. This chart has been found to be an effective tool for communicating the critical trade-off issue for overall capital.

3. Capital Allocation

The capital allocation to a division should be based as much as possible on the relative contribution of the divisions to the overall company total risk. The company requires a certain amount of capital to function. That capital is needed because of business written by the divisions. Each division enjoys the benefit of additional underwriting capacity – beyond what it could write as a standalone entity – from its "membership" in the company. However, that combined capital figure needs to be supported with returns. How much of the capital support burden should each division bear? An immediate answer is to allocate capital to division in proportion to the division's contribution to the total company risk measure.

One way to estimate a division's contribution to the total risk measure would be to determine its marginal impact – how much does the addition of that division to the rest of the company change the total risk measure? A simple technique to determine the marginal impacts is to "swap in and out" each division—subtract each division in turn from the total company and determine the resulting total risk measure. The marginal impact is the difference between the total company risk measure and the [total company – division] risk measure.

However, for most popular actuarial risk measures – variance, standard deviation, ruin probability, expected policyholder deficit – the sum of these marginal impacts will not equal the total risk measure. Computationally there is no issue; the allocation percentages are relative measures, so each division is allocated in proportion to its marginal impact as a percentage of the sum of the marginal impacts. But is there something else occurring here which merits deeper attention?

The short answer is yes. We must consider **additivity**, **order dependency**, and **stability**. These concepts are known within game theory[6] and the study of "cooperative games with transferable utilities." Cooperative games with transferable utilities have the following characteristics:

- Participants or "players" have something to share – either a benefit (e.g., bonus pool) or penalty (e.g., taxes);
- The item to be shared is valued the same by all participants (e.g., money);
- The item must be allocated to the players;
- The opportunity to share results from the cooperation of all players;
- Individual players are free to engage in negotiations, bargaining and coalitions; and
- Players have conflicting objectives, each wanting the most benefit or least penalty.

One of the primary goals of the study of a cooperative game is the determination of a fair allocation scheme for dividing the benefit or penalty. Any valid allocation scheme should

[6] For a fuller discussion of the insurance parallels with game theory, see Lemaire [4] or Mango [5]. An abridged discussion follows here.

first and foremost be **additive**: the sum of all players' allocations must equal the total amount to be allocated. Many popular actuarial risk measures are not additive for purposes of allocation [5]. For example, stand-alone Expected Policyholder Deficit violates this criterion—the sum of the individual capital allocations is greater than the required total.

In many allocation schemes, a player's marginal impact determines the amount of benefit or penalty allocated; however, the marginal impact **depends on the player's order of entry** into the coalition. It is important for an allocation scheme to smooth the effects of order dependency as much as possible[7].

The allocation scheme must also not systematically punish or reward certain players on a basis not reflected in the risk measure. In short, they should be fair and impartial. Otherwise, there would be incentives for the punished player or players to break apart from the group and form a faction. In such an instance, the coalition is referred to as unstable. A fair allocation scheme will result in a **stable coalition**.

These desirable characteristics of additivity, order independence and stability can all be found in an allocation scheme based on the **Shapley value**. It is named after Lloyd Shapley, one of the early leaders in the field of game theory. The Shapley value is an allocation scheme that is:

- Additive;
- Order independent; and
- Stable.

The Shapley value is the average of marginal impacts taken over all possible entrance orders. For example, consider a company with three divisions A, B, and C. The Shapley value for division A would be:

[Marginal impact of A being added to an empty company +
Marginal impact of A being added to division B +
Marginal impact of A being added to division C +
Marginal impact of A being added to divisions B & C] / 4

For a small number of divisions, this calculation is not too burdensome. However, as the number of divisions increases, the number of permutations grows geometrically. Is there any way the process can be simplified?

It turns out that for the risk measure of variance (applied to any variable such as net income, losses or other), the Shapley value reduces to

Shapley value = Var[division] + Cov[Rest of Company, division],

When compared to the formula for marginal variance,

[7] See Philbrick [5] or Mango [2] for discussion of this phenomenon.

Marginal variance = Var[division] + 2 x Cov[Rest of Company, division],

the Shapley value splits the co-variance evenly among divisions.

Using the Shapley value and a risk measure of variance makes the calculation manageable. Each division's Shapley value is the division's variance plus the co-variance with the remaining divisions. This is an extremely desirable quality, as we can now get all the information we need from only one run.

Specifically, the allocated capital for the Company was based on each division's variance of statutory net income.

4. The Initial Board Presentation

The original results were presented during a two-hour meeting with the Board of Management. The DFA team focused first on capital adequacy, then capital allocation. An exhibit similar to Table 1 showed the implied safety level of current capital using the different risk measures (see Section 2). An exhibit similar to Table 2 showed the additional capital needed to achieve various target safety levels.

Next came the simulated GAAP and SAP financial statements. **Exhibit 2** shows the layout of the GAAP Balance sheet and Income Statement. Median values are shown, along with standard deviations. Standard deviation was selected as a simple measure of variability. With so many figures on the page, it was important to convey variability in the simplest manner possible. As mentioned before, standard deviation is effective at conveying variability in a single number. This audience was not particularly statistically inclined, so very little was lost in making this simplifying decision.

The balance of the presentation was spent on the allocation of capital among the major divisions of the Company – see **Exhibit 3**. Allocation output should be displayed not only as absolute amounts of allocated capital, but also as percentages of the total. These percentages will often draw a great deal of attention. In this case, some of the Board members represented individual divisions. One cannot expect to present allocation percentages representing relative risk contributions without digging more deeply into the basis of risk measurement. This presentation was no exception, and issues raised in Section 3 were discussed in some detail, including variance of net income as a risk measure, order dependency, covariance, and fairness of allocations.

Dialogue at the Board level of this nature is one of the real benefits of a DFA study. By framing the implications of these issues, DFA facilitates the discussion by grounding it in measurable quantities. Without the DFA study, the discussions would be anecdotal at best.

In addition to the allocated capital, expected return on that capital was also displayed. This Return on Risk-Adjusted Capital or "RORAC" raised still more engaging discussion. Here, not only are divisional differences in risk reflected, but also market reward. Few more politically sensitive measurements can be conceived.

Presenters must always be cognizant of the familiarity level of their audience with the material. When presenting new material, it is critical to provide comparable context with more familiar terminology or concepts. In the case of the risk measures, this meant providing familiar counterparts such as Premium to Surplus ratio. Risk is a multi-dimensional phenomenon that can only be appreciated and understood in pieces. The right side of Exhibit 3 shows all of these more familiar risk measures:

Asset Needed Ratio = [Allocated Capital + Premium] / Expected Losses

Premium to Surplus Ratio = Premium / Allocated Capital

Loss Percentage = Divisional share of Total Expected Loss

Loss Ratio = Expected Losses / Premium

5. Follow-Up Meetings

Subsequent to this were several one-on-one follow-up meetings with selected Board members (representing different operating divisions of the Company) whose charges were to:

- Increase their understanding of the DFA model, its parameterization and output;
- Dig more deeply into certain issues raised in the initial presentation; and
- Address certain division-specific concerns.

These meetings provided a more focused and interactive forum for the DFA study team to provide details behind the study. Among the items raised in these sessions:

- **Possible Error in Risk Measure Calculation**
  The capital adequacy results for one of the risk measures "did not feel right" to some of the Board members. Their intuitions turned out to be correct, and a calculation error was uncovered as a result of further review. This kind of fresh perspective can often uncover anomalous results that those performing the study miss due to their intimate involvement[8].

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[8] Actuaries in general are so technically focused they often underestimate the value of input from those less technically inclined. However, what these others may lack in technical expertise can be more than made up for in business sense. This business sense is most often expressed intuitively. Such hunches and feelings are to be ignored at one's own peril.

- **Details Behind the 20 Worst Scenarios**
  The Board was also interested in the drivers behind the 20 worst scenarios. Subsequent research revealed (not surprisingly) that the most severe scenarios resulted from the compounded effect of two or more of the following occurring in the same time period:
  - Major natural catastrophe
  - Adverse reserve development
  - Casualty line loss ratio deterioration
  - Asbestos and Environmental reserve deterioration
  - Unusually low investment returns

- **Concern over the Probability of Achieving a Target ROE**
  Board members were also uncomfortable with the estimated probability of achieving a target ROE (they felt the probability was too high!). Further review revealed another calculation error. This sort of feedback cycle is critical to properly evaluating the results of a complex study like this.

- **Splitting Runoff Capital from Ongoing Capital**
  This issue was raised as part of a discussion of the practical implications of capital allocation. Should ongoing business be allocated all the investment returns (from reserves as well as premium funds), but also all the capital? Or should separate "Runoff" versus "Ongoing" capital amounts (and asset pools) be maintained? In response to the request, a new allocation was generated with the divisional capital amounts for ongoing business only. All input reserve categories were aggregated into the "Runoff" division. The resulting familiar risk measures (e.g., Premium to Surplus ratios) were more in line with expectations.

6. The Reference Binder

The presentation of results for a study of this magnitude requires significant backup material, in addition to that covered in the presentation itself. Typically, senior management members will have varying levels of familiarity with DFA, probability, simulation, and correlation. It is important to provide supporting material in one location where attendees can make notes, seek more detail, and refer back in the coming weeks. To support those needs, a detailed reference binder (300 pages of detailed exhibits and explanations) was prepared.

The binder had the following sections:

1. Executive Summary
2. Introduction to the DFA model
3. Overview of Findings
4. Economic Modeling
5. Asset Modeling

6. Liability Modeling
7. Reinsurance Modeling
8. Risk Measures and Capital Adequacy

The Executive Summary section covered the actual presentation material discussed in Section 4. The other portions of the binder will be discussed here.

2. Introduction to the DFA Model
Comfort comes with familiarity. For senior management of an insurance company today, many of the concepts underlying a typical DFA model may be unfamiliar. The results of such a model can therefore make management uncomfortable, and rightly so. Comfort will come slowly over time, as they grow conversant in the new terminology, and become confident the model is accurately modeling the behavior of their company.

The DFA model introduction (see the **Appendix**) pictorially displays the flow of information through the model. This is followed by brief, bullet point descriptions of each major model component. It was important to build knowledge and comfort slowly, in stages, starting from high-level overview descriptions like this. The role of pictures cannot be underestimated. Pictures can provide a structural framework around which the detailed flesh of the model is later built.

3. Overview of Findings
Attempts at distilling the voluminous output of the study to a limited, manageable number of exhibits proved extremely difficult. This section contained:

- GAAP and SAP Balance Sheets and Income Statements
- Plots of the projected distributions over the next three years of Stockholder's Equity, ROE and Net Income
- Profitability vs. Safety plots (similar to Exhibit 1)
- Summaries of important input statistics

4. Economic Modeling
This provides detailed background on the technical foundation and parameterization of the Global Economic Module—the economic scenario generation portion of the DFA model. The economic scenarios provide a consistent integrated framework that drives both asset valuation and liability trends.

5. – 7. Asset Modeling, Liability Modeling, Reinsurance Modeling
These sections discussed the parameterization of the Company, including the data issues and shortcuts that an ambitious timeframe necessitated.

For Asset modeling, the Company's actual asset portfolio was input in asset class detail. The DFA model has advanced asset capabilities, making it worthwhile to enter the assets in such detail. Sophisticated asset modeling adds to the total risk picture by "setting in motion" pieces that are static in many other models.

Liabilities were modeled at the detail level dictated by many constraints, including available supporting data (e.g., reserve studies) and per-risk reinsurance covers requiring individual claim level simulation. The binder covered category definitions, data gathering, development and trend factor selection, loss curve fitting, and reconciliation with the Company's business plan.

The Reinsurance portion lists the in-force covers that were modeled, and shows the "reinsurance map"—the graphical reinsurance coverage depiction tool. An example map is shown in **Exhibit 4**. Reinsurance covers were modeled in extensive detail, including ceded premium. Results were produced on gross, ceded and net bases.

8. Risk Measures and Capital Adequacy
This section is similar to Section 2 of the paper, discussing many possible risk measures, their relative advantages and disadvantages, and issues related to selecting a risk measure.

7. Conclusion

DFA models are the actuarial equivalent of advanced experimental apparatus. Like our counterparts in physics (though on a lesser scale), actuaries can use DFA models to pose and answer hypothetical questions that previously could not even have been asked. This paper addresses many such questions. It is therefore not surprising that many of these issues have yet to be fully and satisfactorily resolved. We must be careful when presenting our DFA studies not to oversell it. Focus on the strengths of the models, the questions they allow us to answer, but be open to criticisms, because there is much that is unanswered. Breakthroughs can come from unexpected places. It is because of this that we must strive in our communications to simplify our results, and translate them so they may reach the widest possible audience.

DFA *system* development has progressed fairly rapidly within the industry. What is lagging behind is widespread understanding and comfort with the issues DFA raises. DFA systems can produce so many answers, there may not be enough people who know the right questions to ask. The business leaders of our industry are looking to the CAS membership to be the bridge between the science of DFA and the art of business management and strategy. They need guidance on the best risk measures, the interpretation of the levels of those risk measures, the incorporation into planning, the practical meaning of using distributions in place of static values, and the details behind the challenges of parameterization. Clearly continued sharing of all aspects of our research efforts – such as this Call Paper Program – will lead us all closer to those goals.

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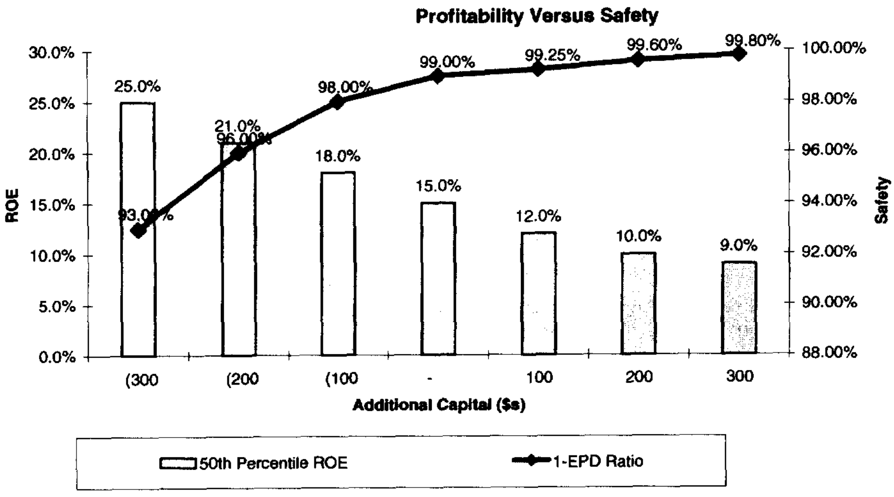
Exhibit 1
Example of 50th Percentile ROE and Safety Trade-off Graph



This exhibit shows an example of the ROE versus Safety trade-off graph. The 50th percentile of ROE (left y-axis) and the Safety measure (right y-axis) are shown for different levels of additional capital (x-axis).

When capital is removed, the ROE improves but the Safety score deteriorates.

When capital is added, the ROE deteriorates but the Safety score improves.

Exhibit 2
Summary Financials

XYZ Corporation
GAAP Income Statement

Line Item	*50th Percentile*				*Standard Deviation*		
	1998	1999	2000	2001	1999	2000	2001
0 Gross Premiums Written							
1 Ceded Premiums Written							
2 Net Premiums Written							
3 Change in Unearned Premiums							
4 Premiums Earned - Net							
5 Losses Incurred							
6 Commissions							
7 ULAE							
8 Overhead							
9 Brokerage							
10 Total Losses and Expenses							
11 **Underwriting Gain/ (Loss)**							
12 Investment Income - Taxable							
13 Investment Income - Non-Taxable							
14 Capital Gains (Losses)							
15 Investment Expenses							
16 Income From Subsidiaries							
17 Total Investment Income							
18 Total Misc.							
19 **EBIT**							
20 Interest Expenses							
21 **EBT**							
22 Taxes - Cap Gains							
23 Taxes - Ordinary							
24 Taxes - Total							
25 Operating Earnings							
26 Divs. #1							
27 Divs. #2							
28 **Net Income**							

XYZ Corporation
GAAP Balance Sheet

Line Item	*50th Percentile*				*Standard Deviation*		
	1998	1999	2000	2001	1999	2000	2001
0 Investments - Tradable Bonds							
1 Investments - Non-Tradable Bonds							
2 Investments - Preferred Stock							
3 Investments - Tradable Equities							
4 Investments - Non-Tradable Equities							
5 Investments - Other							
6 Cash							
7 Total Investments And Cash							
8 Accrued Investment Income							
9 Premium and Other Receivables							
10 Deferred Acquisition Costs							
11 Reinsurance Recoverables							
12 Other Assets							
13 **Total Assets**							
14 Loss and ALAE Reserves							
15 Unearned Premium Reserves							
16 Total Reserves							
17 Loss Balances Payable							
18 Funds held under reins treaties							
19 Senior debt							
20 Senior Notes							
21 Other Liabilities							
22 Total Liabilities							
23 Special Debt							
24 Common Stock							
25 Paid-In Capital							
26 Retained Earnings							
27 Other Income							
28 Total Stockholder's Equity							
29 Preliminary Liabs Etc.							
30 Misc.							
31 **Total Liabilities**							

The financials display each accounting line item's median and standard deviation by year. For less statistically sophisticated audiences, standard deviation is a familiar measure that adequately conveys differences in variability for presentation purposes.

Exhibit 3
Capital Allocation

Capital Allocation

Economic Basis / Excluding Runoff / Inv Inc Allocated / 3 Year Run
Major Divisions

Division Label	*Capital*		*Reward*			*Risk*			*Other Statistics*		
	% of Total Company Capital	Capital Allocation	Div. ROE	Contribution to Company Bottom Line	Contribution to Company Bottom Line	Asset Needed Ratio	Premium/ Surplus Ratio	Loss Percentage	Losses	Premiums	Loss Ratio
Div. #1	16.4%	$ 16	12%	$ 2.5	25%	3.1	0.8	15%	$ 11	$ 14	78%
Div. #2	1.0%	1	24%	0.3	3%	2.0	4.7	4%	3	5	66%
Div. #3	55.8%	56	4%	3.0	30%	3.5	1.1	51%	36	60	60%
Div. #4	16.2%	16	5%	1.0	10%	3.3	1.0	15%	11	16	66%
Div. #5	10.5%	11	24%	3.1	31%	2.9	1.5	14%	10	16	62%
		$ 100		$ 10.0	100%			100.0%			

Capital allocations and Reward measures are displayed as both absolute amounts and percentages of the total. Some interesting risk measures are then shown:

Asset Needed Ratio = [Allocated Capital + Premium] / Expected Losses

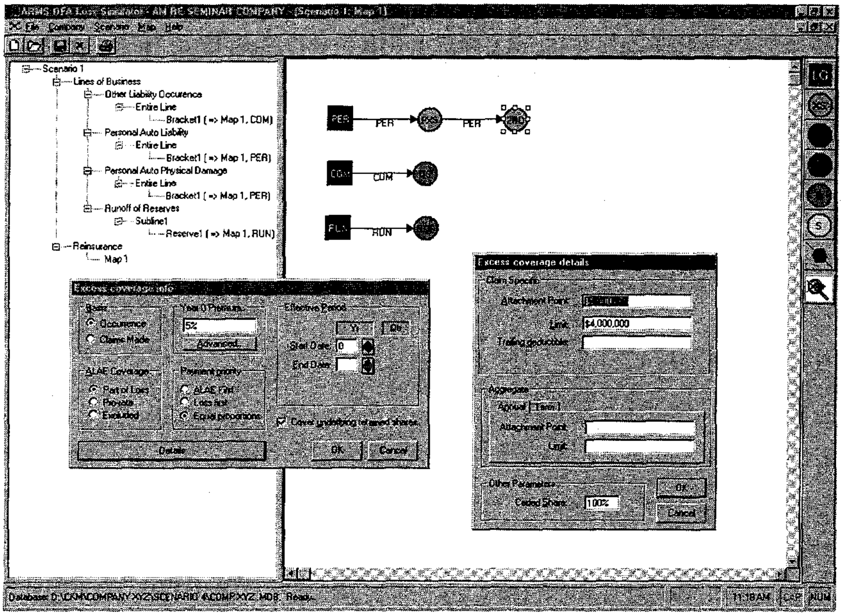
Premium to Surplus Ratio = Premium / Allocated Capital

Loss Percentage = Divisional share of Total Expected Loss

Loss Ratio = Expected Losses / Premium

<u>***Exhibit 4***</u>

<u>***Example of Reinsurance Structuring***</u>



The DFA model uses a graphical "coverage map" to depict reinsurance programs. The palette on the right has objects representing subject losses (squares) and various types of covers. In the above illustration, an excess cover and two quota shares have been added. A second excess cover is in the process of being added, and a few of the screens "behind" the excess cover are displayed. The graphical map, once completed, serves as very effective documentation that the reinsurance program has been correctly depicted in the model.

Appendix
Introduction to ARMS, American Re's DFA Model

ARMS is American Re-Insurance Company's DFA model. It integrates assets and liabilities across economic scenarios. It also provides detailed modeling capabilities for insurance liabilities and reinsurance. The system is also used to assist both Munich Re[9] and American Re-Insurance Company clients in evaluating and setting up efficient re-insurance and investment structures. The structure of the system is laid out in Figure 1.

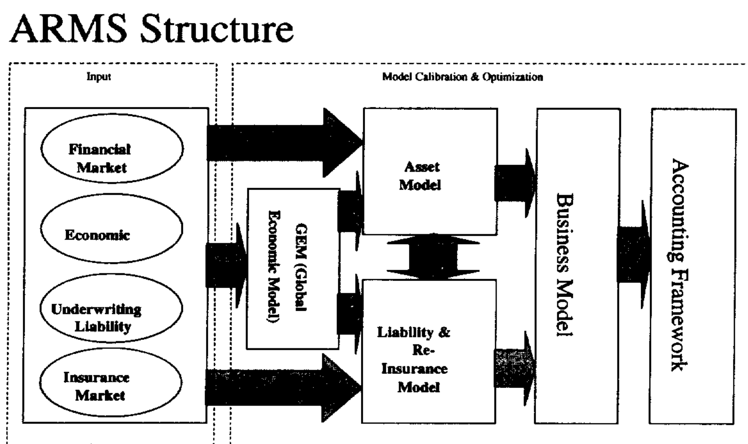


Figure 1. American Re-Insurance Company's Risk Management System (ARMS) is an integrated compilation of models. Historical data from financial and economic markets, underwriting decision processes, and insurance market trends are inputs to the system (left). Output includes balance sheet and income statements, and illustrative charts and reports.

ARMS is composed of several integrated modules which handle different aspects of the simulation.

The **Global Economic Module** or **GEM** generates plausible time series outcomes of future economies based on user specifications and parameter settings. The user specifications are inputs reflecting the current economic environment and expectations for long-term median trends. The parameter settings are referred to as calibration parameters and those are set via the **Constraint Evaluator System**[10].

[9] American Re-Insurance Company is a member of the Munich Re Group.
[10] See Berger and Madsen [1] for details behind the calibration of the GEM.

Each of the economic time series scenarios are fed to the **Asset Module** as well as the **Liability and Re-insurance Module**. Economic scenarios integrate the simulation of liabilities and assets, ensuring internally consistent simulations. For example, inflation parameters from the economic model influence the trend in the prospective loss severity distributions. Similarly, the prospective premium trend can also be tied to inflation. Any discounting for future pricing purposes is based on output from the economic model.

The **Accounting Framework** refers not only to accounting but also to tax implications. There are several advantages to separating this functionality. They include the facilitation of operating in a multi-country (and therefor multi-regulatory) environment.

Wrapped around all this functionality is a non-convex optimization engine – the driving force behind the **Constraint Evaluator System**. Since each of these models must be calibrated in one form or another, access to a non-convex optimization system minimizes traditional trial and error attempts to ensure the reasonability of results. Ideally, we want to back-test the models with historical data and ensure optimal performance before we start modeling prospectively.

The Cost of Financing Insurance—Version 1.0

Glenn G. Meyers, FCAS, MAAA

The Cost of Financing Insurance – Version 1.0

By
Glenn Meyers
Insurance Services Office, Inc.

This paper provides an illustrative analysis of how to use DFA to determine profitability targets for the various underwriting divisions of an insurance company.

The ABC Insurance Company is a sizeable commercial lines insurance company. Its goal is to obtain an above-average return on equity by setting profitability targets for each of its underwriting divisions that reflect the cost of capital needed to support each division's contribution to the overall underwriting risk.

While ABC's management recognizes the important role played by regulators and rating agencies in determining an insurer's capital, it feels that controlling the insurer's risk, as measured by its statistical distribution of outcomes, provides a meaningful yardstick that can be used to set profitability targets.

In addition, ABC's management wants to take the following considerations as input into its decisions.

- How long must capital be held?
- How much investment income is generated by the insurance operation?
- How closely correlated are the losses in the various lines of insurance?
- What is the effect of reinsurance?
- What is the effect of hedging losses with options on an insurance index?

The cost of financing an insurance company is defined to be the combined cost of capital, reinsurance and options on a catastrophe index. The ABC Insurance Company wants to allocate its cost of financing back to its individual underwriting divisions. The results will be expressed as a target combined ratio for each underwriting division.

Author's Note

This "paper" is not a physical paper. It is a web which is intended to be read on a browser. (I used Internet Explorer when designing this web, but it should be readable on other browsers.) To view the web, point your browser to:

http://www.casact.org/pubs/forum/00sforum/meyers/index.htm

A Dynamic Financial Analysis Application Linked to Corporate Strategy

Elizabeth R. Wiesner, FCAS, MAAA and
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A Dynamic Financial Analysis Application Linked to Corporate Strategy

Authors
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Abstract - In this paper the authors describe how to link the technical aspects of the Dynamic Financial Analysis (DFA) modeling process with the ultimate purpose of that process, the enlightenment of senior management for the purposes of strategic thinking. The authors desire to enlighten both the model user and the senior executive by describing the elements that connect the merits of a rigorous quantitative analysis to fundamental strategic issues. A case study is described for a workers compensation carrier relative to its corporate vision. This is intended to be a non-technical paper. The technical aspects of the modeling process are described only to the extent they are useful in describing the management education process.

Overview

The financial services market is going through significant change. With increasing frequency, companies are changing the way they operate and offer products. For insurance companies, many of the changes are not only good, but also necessary. However, change for the wrong reason can be destructive to companies. Companies can spend years going down a strategic path only to find out that a strategy did not achieve the intended objectives or its objectives ended up being inconsistent with the company's long term vision. Companies need to heed the warning, "be careful what you wish for".

It is important for companies to choose strategies that are consistent with their long-term vision for many reasons. First, this helps profit centers, executive management, business units and employees stay focused on common objectives. Second, in today's market, most companies have multiple strategies being developed simultaneously. These strategies should support one another, and not work against each other. Finally, executive management needs to select strategies that have a high likelihood of achieving the desired business objectives. It can be demoralizing for a workforce to achieve a strategy to find out that, while the strategy achieved the expected results, those are not the results the company needs to achieve its vision.

This is where a Dynamic Financial Analysis (DFA) approach can help in strategic planning. DFA is a tool that can help companies select strategies that are consistent with their corporate vision. For purposes of the paper, a vision is a simple statement from senior management which defines an intended future state of the company. For example, a vision may involve being a financially stable leader in the personal lines market. A corporate vision can include specific smaller visions relating to the financial, product or distributional aspects of the whole organization. A strategy is a major management initiative that helps achieve the vision, like expanding into other states. Usually, several strategies come together to achieve the vision.

The goal of this paper is to describe the use of DFA in corporate strategy. We do this by use of a case study where we show how a workers compensation carrier writing in one state uses DFA in selecting among potential strategic initiatives. This includes an overview of the process to align the DFA process to the corporate vision, running the model, and communicating results to executive management in a meaningful way. The vision, data, results and conclusions are modified, but that does not affect the intended message of our example.

This paper has a few themes:

- When using DFA, it is important not to get swept up in the technology. Like all technology, DFA is the tool, and not the objective. DFA processes or communications that lose this focus also have the possibility of losing the interest and support from executive management.
- DFA is not a crystal ball. Sometimes executive management will look for a tool that can predict what the company's precise return on equity will be in five years, for example. That is not the purpose of this DFA application. DFA is a tool to help educate management on the comparative strengths and weaknesses of business options. This is one of the biggest communication challenges; keeping the audience focused on the comparisons and patterns, and not on specific projections.

- Effective communication starts at the beginning of the DFA process, and is not something that is done after "all the numbers are run." Decisions made throughout the process impact the quality and understanding of the communication of the results.

Getting Started

The case study company writes workers compensation insurance in one state. Over 90% of the premium for this company comes from small employers, with less than $50,000 in premium. The five-year vision for the company includes several aspects on which two are focused in this paper. The first is financial superiority. The second is being an industry leader in the core competencies of the company: medical and disability management.

Executive management is considering several strategic initiatives to help achieve the vision. For simplicity, this paper will only look at three of them:

1. Business as usual; just try and take what is already being done and do it better
2. Expansion into other states concentrating on the small account expertise
3. Diversification through writing other lines of business, specifically disability related lines.

Why DFA?

Why did we use DFA for this strategic exercise? An improvement to traditional strategic planning tools, DFA provides a basis for measuring and analyzing the financial aspects of the corporate strategy. No other tool has the ability to do as rigorous an analysis of the underlying risk factors as that offered by DFA.

There are several steps to the DFA process when analyzing strategic initiatives.

- Selecting a model appropriate for the company
- Understanding the business implications of the objectives of the corporate vision
- Selecting business measurements that are consistent with the corporate objectives
- Running and analyzing the model
- Communicating the results

Selecting a Model

The company first determined which DFA model attributes serve management's analytical needs. Among the initial considerations was whether standardized (off-the-shelf) or customized (specifically built) model attributes would best serve the purpose at hand. More specifically, another important question was which basic risk factors are fundamental to the company's existence and which factors are immaterial. Within those considerations, the company further examined which factors were most appropriately modeled as stochastically generated variables and which could be driven by user-selected, static scenarios. The final consideration was what data is available to carefully parameterize both exogenous (mostly economic) and endogenous (mostly operational) variables.

For the application described here, management ultimately favors a high degree of customization. The necessary complexity of multi-line, multi-state models contain features not needed for this company. Some models do not have the focus on risk factors or business relationships appropriate for a company with all of its business in workers compensation. Then, after a careful internal risk assessment exercise, economic variables including interest rates, inflation, and unemployment are cited as the critical external risk drivers, while pricing, and underwriting versus growth plans are key internal risk drivers, in addition to potential reserve misstatements. The exogenous variables (economic and market condition metrics) are deemed to be best generated by stochastic processes, while business growth is input as user generated data.

Understanding the objectives of the vision
When analyzing objectives for use with a DFA tool, it is important to get to the right level of detail. A high level objective, like financial superiority, is good for a vision. It is too vague, though, for effective use with DFA. Thus, we need to break the high level objective into smaller goals that lend themselves to measurement. These goals should be company specific and consistent with the company vision and philosophy. If there is not a pre-existing knowledge of the company philosophy, discussions with executive management before beginning the DFA process may be of value. By getting an idea of what management is looking for, these pre-process interviews can also help with communicating the results at the end of the process.

As an example of visions and philosophies, one company may care most about underwriting integrity, so their financial superiority vision could be equated to a goal of underwriting profitability. Another company may place more value on overall return with less emphasis on the particular source of the return. For that company the same financial superiority vision could be best described by a goal relating to minimum returns to shareholders. Selecting goals consistent with the company vision and philosophy will aid in any communication plan to executive management. If goals are inconsistent with the vision and philosophy, the DFA process will not answer management's primary question; will this strategic initiative help us meet our vision? Further, it helps to keep the number of goals to a manageable few. Selection of the few, most important goals brings focus to the strategic process and avoids overwhelming executive management with a multitude of figures, many of which do not have significant impact on the decision making process.

It is also important to distinguish between goals and tools. An example is diversification. On the surface, diversification sounds like a good goal. However, diversification is really not an end goal. Rather diversification is a tool to achieve other objectives, such as stability and longevity. To better understand this distinction, consider the three strategies under consideration in this paper. If diversification is a goal, the first strategy, business as usual, is automatically eliminated as a good option. However, as a DFA model can show, there are other ways to achieve stability and longevity. Recognizing this distinction also helps with the communication to executive management. With this mono-line, mono-state company, the executive team hears from regulators and rating agencies that diversification is a necessity for the company to achieve financial stability. DFA can be a powerful tool to show that diversification is not the only way to achieve financial stability. This is an important message to communicate to executive management so that the right goals are driving the selection of strategic initiatives.

With this project, we translated the financial superiority vision to three goals; an acceptable long-term return on equity, stability in returns, and longevity of the company. The other portion of the vision considered in this paper, being an industry leader in medical and disability management, has a qualitative impact on the process. One aspect of being a leader implies that the company must excel at meeting the product needs of customers. This helped us select among possible strategic initiatives and narrow the possibilities to the three in this example.

Exhibit 1 shows the relationship between the vision, goals and business measures for the example company.

Selecting Business Measures

Selecting business measures in advance of running the model helps assure the measures are consistent with the corporate goals. A DFA model can produce every value on an income statement and balance sheet, plus other measures important for managing the business, such as average rate, frequency, severity, or accident year loss ratios, just to name a few. As examples, if a corporate goal is primarily concerned with underwriting results, appropriate business measures could be accident year loss ratio or combined ratio. If a goal relates to the shareholders receiving a minimum level of annual income from a subsidiary, then using a business measure of dollar of net income may be more helpful.

Just as it is helpful to select only a few goals to keep communications focused and manageable, it is also helpful to do the same with the business measures. Looking at more than six to eight business measures may serve to unnecessarily complicate the message to executive management. Earlier we discussed the three corporate goals that tied to the corporate vision of financial superiority. Those goals were return on equity, stability and longevity. To keep the example simple for this paper, we selected four business measures related to the goals. For each business measure, there is also a desired result that is consistent with the corporate goals.

Business Measure	Tie to Corporate Goal	Performance Standard
Return on equity (net income/surplus)	Long term return on equity	Long term target of 10%, never less than 7%
Growth in direct written premium	Longevity requires this to be stable or increasing	Minimum of 5% a year
Growth in surplus	Longevity requires this to be stable or increasing	Target of 5% a year, minimum of 3% a year
Reserve to surplus ratio, or "reserve leverage"	Keeping level of liabilities consistent when compared to surplus gives stability	Minimal variation, even in pessimistic scenario

Running and Analyzing the Model

<u>Strategies</u>

The basic application of the DFA model to our case involved three strategic actions. They were described earlier, but are repeated below:

1. Business as usual, try and take what is already being done and do it better

2. Expand into other states concentrating on the small account expertise
3. Diversification through writing other disability lines of business.

Scenarios

In practice a continuum of scenarios is desirable. For purposes of the case study in this paper, we'll simplify those by defining two basic scenarios:

1. Expected level
2. Pessimistic level

Within this DFA model, a scenario represents a version of the stochastic trial runs defined by varying critical distribution parameters of the random processes in the model. The selection of scenarios should be carried out in consideration of the goals of the DFA process. For example, the goal of financial stability appears to demand at least one if not many adverse scenarios. The number and degree of these demands that the term financial stability be defined quantitatively. With a sufficient number of adverse scenarios a continuum of potential adverse scenarios (possibly expressed using graphs) can be observed.

Each of these scenarios was generated by varying the parameters that the stochastically generated variables used. This included interest rates, inflation rates, stock market performance, etc. For example, a mean value of 5.0% for short term interest rates was used as an expected level scenario, while a 6.0% mean with a higher variability component was deemed an appropriate parameter for pessimistic results. Specifically for the case study, the expected level scenario assumed the underwriting cycle remained soft (intensely competitive prices) for years and had not yet started to turn; that interest, medical inflation, general inflation, unemployment and duration were all consistent with recent history; and, that stock returns were steady and current reserve levels adequate. The pessimistic scenario assumed the depths of the soft market had not yet been fully realized; that interest, medical inflation, general inflation, unemployment and duration were all increasing; that stock returns were poor; and, that current reserve levels adequate. As stated earlier, many more scenarios are used in practice to try and isolate the impact of changes in certain variables.

For each combination of the three strategies with the two scenarios, a set of 1,000 stochastic trials was generated. This appeared to be a sufficient number of stochastic simulations for the initial runs based upon the observed convergence of the metrics under analysis using several random seeds.

Range of Results

To directly analyze the impact of each of the three strategies, each set of runs is compared using the same set of random numbers. The same process is used under each of the two scenarios. With this process available, a range of results is offered through observing the resulting distribution of the selected financial performance measure over the 1,000 trials. Basic descriptive statistics are typically used, including the mean, the standard deviation, the coefficient of variation (CV), skewness, and selected percentiles. For example, a key measure under analysis is the mean and CV of the ROE statistic over 1,000 trials. Based on the above described check on the sufficient number of trials needed, one should be careful not to place too much reliance on the outcomes in the most extreme percentiles. In these cases the parameter and model risk elements tend to render such observations highly uncertain.

Naturally, when using scenarios the user is required to interpret the results with respect to the deemed likelihood of each set of parameters. Therefore, it is important that the ultimate end users in senior management be educated as to the meaning of each generated financial distribution. For example, when viewing the variability of ROE, the management audience should understand that these probabilities are not absolute, but contingent upon the occurrence of the provided risk factor scenarios. We found that it is this presentation style using a hybrid of DFA modeling techniques (specified "what-if's" combined with stochastic analytics) which senior management often finds most meaningful.

Predictors

Using the above procedures the model user is positioned to quantify the impact of the company's fundamental risk drivers. Each run (set of 1,000 trials) provides a table of data from which financial results can be related to their underlying risk drivers. Specifically, correlation analysis can be performed on such tables. Results tend to be more meaningful when certain outliers are removed and the core results are examined. This process may result in a quantifiable basis for ranking the relative strength of independent and combined risk drivers. As an example, a workers compensation writer may find that ROE is driven foremost by variations in economic conditions (which may best be represented by unemployment rates). However, high unemployment combined with a hardening in the pricing market may result in a sufficient hedge to ROE.

Unusual Results/Extreme Outcomes

Despite the limited predictive quality of extreme observations, outliers provide useful insight to the analytic process. A typical DFA application requires the ability to "drill down" to understand the specific factors underlying an unusual outcome. To facilitate this, random number regeneration helps the model user re-examine a specific trial yielding such outlying results. While the result precision involved in this stage of the analysis may be statistically weak, the user can gain great insight as to which risk drivers are material and to what relative degree. As an example, for a workers compensation company, the user may discover that the 10 most adverse ROE results may coincide most with the highest unemployment rates through their effect on higher claim frequency.

Results of the Case Study

Exhibit 2 contains the projected direct written premium under the expected scenario for the business as usual strategy. The exhibit shows a portion of the simulations along with the summary statistics. There are similar sheets for each combination of business measure, scenario and strategic initiative.

To begin the analysis, the data is first summarized into a manageable format. Exhibit 3 shows a sample summary for the surplus measure. This exhibit selects a few key statistics and combines them in one location. For this analysis, the company is not only interested in an expected surplus, but also in stability and in maintaining the growth in surplus above a minimum level.

Thus, the summarization uses the mean, growth in mean, CV, the 10% confidence level and the 90% confidence level. Similar exhibits are done for each of the business measures.

The direct written premium and surplus increase steadily each year under each strategy and scenario. Surplus growth in the pessimistic scenarios is below minimum targets. The ROE dips below the long-term target in the multi-state expansion scenario. Reserve leverage stays well below industry norms and decreases each year in the expected scenarios and stays flat in the pessimistic scenarios. As expected, the CV increases as the analysis goes further into the future. For many business measures, the CV is also greater for the two expansion strategies than for the business as usual strategy. This is a little surprising. After five years the expectation was the expansion scenarios would start to stabilize the results. It could be that the strategies are not mature enough to have the intended effect on the company to start bringing the stability, and a look at seven or more years in the future would start to show that stability. Also as expected, the CV is greater for most business measures in the pessimistic scenario.

A random number regeneration allows a look at specific simulations for patterns in results. Starting with the direct written premium, the first question is why does the premium increase in every combination of strategy and scenario? The individual simulations also predominantly show this pattern. A group of simulations is identified that result in increasing premium with both good underwriting results and poor underwriting results. In the expected scenarios, increasing premium comes from a hardening of the market, increasing rates, and the expansion efforts involved with Strategies (2) and (3). In the pessimistic scenarios, increasing premium results from an underlying increase in costs associated with increasing inflation.

For the case study, another question is what happens to cause the surplus to decrease or stay flat in certain simulations? Here we take a little more structured approach than in the direct written premium review. The starting point is selection of variables that relate to the surplus level, like accident year loss ratio, investment income, average rate and change in average losses. Again using random number regeneration, we capture these new stats for all simulations below the 10th percentile surplus level and also for a group of simulations near the mean surplus levels. See Exhibit 4 as an example of this new information. The low surplus levels are highly correlated with poor underwriting results, driven by a continuing soft market, rate levels declining 1-10% each year for the next five years, and loss levels increasing 6% or more per year during the same time period. The simulations near the mean show a more stable underwriting return. Similar searches on low ROE and high reserve leverage simulations show the same relationship to the continuing and extreme soft market.

There is also a search for predictor variables. Most of the needed information is already part of the reviews above. For this situation, we search the simulations for large increases in surplus and ROE in the early years, then smaller increases or declining surplus or ROE in the later years. After identifying these simulations, we look through patterns in the other statistics from 2-3 years prior to the decline. One noticeable pattern is in the reserve leverage ratio (R/S). This company historically has a low reserve to surplus level, safely below 2.0. But for simulations where surplus starts to decline, the R/S ratio reaches or exceeds 2.25 a couple of years prior. There is also a review of the data in the reverse order; when the R/S ratio increases to 2.25, does it always follow that surplus starts to drop in later years? While this correlation is not perfect, it is common enough that it is a good early warning sign for the company. Exhibit 5 shows a sample of this data.

The previous paragraphs show a sample of the searches through the data. In practice, for each new pattern derived from the data, several new questions arise. These searches provide valuable insight into the relationships in data, and are a useful part of the communication on the results.

Some Lessons Learned

There were several interesting results in the data for this case study. The first was that the expansion strategies did not bring as much stability as desired. By reviewing individual simulations and by trying different investment strategies, it is found that conservative reserve and surplus positions as well as certain investment strategies have a larger impact on stability for this company than expansion. A future project might be to look at projections further out to see if the stability from expansion takes hold beyond the model's five year horizon.

While the expansion strategies do not have a major impact on the stability goal, they are necessary for longevity. The company does not achieve the desired growth and spread without the expansion alternatives.

The overall returns fall during multi-state expansion and stay steady with multi-line expansion. This appears to be consistent with management's intuition as it takes a company time to reach ultimate profitability goals in new regions or with new lines of business. While most of senior management easily understands this concept, the DFA process adds a new level of clarity to the issue.

The final surprise result from the analysis deals with the R/S ratio. The company's strong historical reserve and capital position leave it with a low R/S ratio compared to historical industry norms for workers compensation. The DFA process shows, though, that to keep stability in results while lacking a large spread of risk, it is best if the company keeps a reserve leverage ratio much lower than traditional benchmarks.

Communicating Results

Executive Management

Put it before them briefly so they will read it, clearly so they will appreciate it, picturesquely so they will remember it, and, above all, accurately so they will be guided by its light. - Joseph Pulitzer

The purpose of communication to senior management is to relay an understanding of how well the strategic options correlate to the corporate vision. The quality of the DFA process and results are minimized if the communication fails to weave the corporate goals and visions into the process. For strategic alternatives, the messages relate to trends and relationships, and not precise predictions. Showing more ranges, changes in values and comparisons rather than actual projected numbers accomplishes this.

In addition to the above purposes of the communication, a different style of communication may be necessary than the style used for communication to technical staff. Knowing the preferences and background of the audience allows the communicator to communicate in the preferred style of the audience. Most people have a mix of visual and verbal learning behaviors. Weaving visual representations into the communication relays messages in powerful ways.

Stability:
Exhibit 6, Sheets 1 & 2, show samples of visual ways to communicate the stability of the strategies.

Exhibit 3 is a basic summary of surplus for the three strategies. Numerically, the increase in CV is easy to see under the multi-state expansion. This would imply less stability.

But is the difference in CV significant? Exhibit 6, Sheet 1 is a scatter plot that graphs each of the 1,000 simulated surplus levels five years out for the strategies of business as usual and multi-state expansion. The top, middle and bottom lines on the graphs represent the 90th percentile, mean and 10th percentile, respectively. Showing the graphs side by side brings out some comparisons. First, it visually shows the variance of the results around the mean. Comparing the spread between the 10th and 90th percentile is easy to do visually. Second, the overall dispersion of results is easier to appreciate with each of the simulations plotted. Last, changes in mean between strategies and scenarios are easy to see. Adding notes to the graph of causes of extreme results brings in the lessons learned from reviews of the simulations. Similar scatter plots are done, but not shown, for the other business measures and strategy/scenario combinations.

Sheet 2 is an alternative way to express the same message. The graph shows the mean and percentile points for the same surplus projections as in Sheet 1. This graph is less visual for some people, but does allow more strategy/scenario combinations on one page.

In practice, a project reviewing strategies would have many more strategic options, including combinations of multiple strategies. In our example, we may not want to just look at multi-state and multi-line individually, but at the combination of the two. Summarizing the strategies for executive management as to which ones best meet the corporate goals can be done through efficient frontier graphs. Exhibit 7, Sheet 1, is an example where many strategies are compared for the impact on surplus. Comparing the volume of surplus for the risk involved aids in the selection of strategies with the best risk/reward trade-off. Creating the efficient frontier for the different business measures indicates which strategies correlate best with the corporate goals and vision. Repeating the process for the pessimistic scenario shows whether potential adverse conditions change the results of the risk/reward trade-off. Exhibit 7, Sheet 2, summarizes the results of this exercise.

Target ROE:
A graph similar to Exhibit 6, Sheet 2, also shows how well the strategy/scenario combinations achieve the target and minimum ROE goals. See Exhibit 8.

Longevity:
To achieve longevity a company must have a way to protect and even grow the business. The two measures for the case study are direct written premium and surplus. Exhibit 9,

Sheets 1 and 2, display the growth in these numbers for the various strategy/scenario combinations. While the business-as-usual strategy combined with the expected level scenario achieves the long-term growth goals, the pessimistic scenario indicated more difficulty in achieving target growth. Both expansion scenarios clearly help achieve longevity.

Pulling it all Together:
The DFA process creates a vast amount of information and numbers. Effective communication depends on summarizing the information down to a manageable volume and keeping focus on the issues most important to executive management. The goal is not to demonstrate the amount of knowledge the DFA practitioner has, but to demonstrate how effectively different strategies align with the corporate vision and goals. The following are some highlights that the case study company found useful in the communication process:

1. *Keep communications brief and focused. Eliminate measures or information that are secondary to the primary objectives.*
2. *Throughout the communication relate how the strategies align with the corporate vision and goals. Include any relevant assumptions about the strategies.*
3. *Select the three or four most important results of the process and include in a brief executive summary. Supporting graphs, scatter plots or efficient frontier exhibits can be part of an appendix. Even in the appendix, be focused on what is included. Not every piece of data is important.*
4. *When discussing strategies, avoid projections of specific numbers. Keep communications geared to trends and patterns.*
5. *Do not discuss the DFA process at length. Overviews of DFA may be useful at a time other than when the results of a process are presented. Again, DFA is not the goal, it is a tool.*
6. *Support the DFA analysis with other information in the company, from the budget process, planning sessions, product development work, or any other relevant research.*
7. *For technical concepts, divide and conquer. One-on-one meetings with executives to go over results prior to a general presentation gives each executive a chance to ask their own unique questions and a chance for the actuary to prepare additional information to answer the questions. This process greatly increases the amount of communication time, but has a much greater success rate in having recommendations approved.*

<u>Other Communications</u>
In addition to the needs of senior management, the results of our application have profound implication on many tactical issues involving the company's operations. For example, the ratemaking unit can utilize the basic capital variability results to refine rate of return calculations. Capital allocations to various product groups may also be employed from the DFA findings so as to refine pricing techniques of individual products.

<u>Final Results</u>
The DFA process uncovered useful information for the case study company in regards to strategic planning. The most surprising result is that stability for this company is better

achieved, in the near future, through conservative surplus and reserve levels, reserve leverage ratios below industry norms, and specific investment strategies, rather than through expansion. While the expansion strategies do not increase stability, they also do not have a significant negative impact. Variance around expected results remain at an appropriate risk level for this company. Adverse results develop from unlikely scenarios related to an extremely soft market continuing for an additional five years.

The expansion options are a big piece of ensuring longevity for the company. As the company moves forward with expansion strategies, the DFA process becomes part of managing expectations of executive management. The general expectation appears to be for returns to drop slightly until the multi-state and multi-line initiatives are in place for a couple years.

The process also had an unanticipated benefit on other projects in the company. New targets for the R/S ratio become part of the company's internal rate of return analysis and affect the target underwriting ratios. There is also a desire to more closely manage the ratio. As an example, the company can use the target reserve leverage ratio to help in managing dividends from surplus to the parent company.

Future Enhancements to Process

From this initial application many future possibilities exist. Some of the enhancements the DFA team for this company is considering include the following:

- Deeper analysis of the model's outlier results to better evaluate the validity of potentially extreme results.
- More rigorous statistical analysis of underlying risk factors to present a more quantifiable basis for correlation effects.
- More rigorous statistical analysis of whether the difference in variance among scenarios is relevant.
- Training more of the company staff to utilize the modeling applications for basic operational needs (ratemaking, etc). This will provide a more comprehensive basis for total company financial analysis at all levels of the company.
- General management training on how all the financial measures relate to one another and to business decisions.
- Building in rating agency metrics. This may assist with the goal of financial superiority.
- Evaluating potential acquisition targets for the company to consider. This will help evaluate the financial superiority goals of an expanded enterprise.

Summary

The process of creating a DFA model and using it for specific business applications worked a little differently than the company expected when it started the project. The biggest surprise was the amount of information produced by the model. Managing the data and sorting through it to develop effective communication is a challenge. While trying to meet that challenge the company learned a few lessons, the most important of which are the themes of the paper, mentioned early on and summarized again below:

- DFA is the tool, and not the objective.

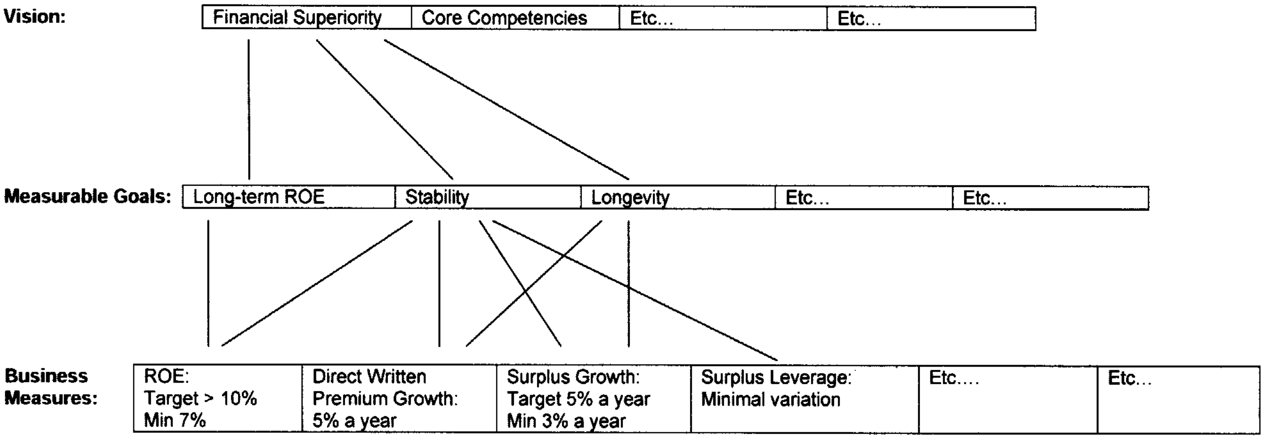
- DFA is not a crystal ball.
- Effective communication starts at the beginning of the DFA process.

For this company DFA is a valuable and worthwhile tool. While the process takes longer than originally anticipated and communication can be more challenging than with other tools, the benefits more than offset the extra work. The application in this paper brought the company insights into the stability of results and predictor variables that are different from previous expectations. These two results alone are worth the time and effort of the DFA process. This company intends to use DFA as a standard tool for answering a wide range of business questions. With every DFA project there are new insights about the company, which is, of course, the ultimate goal.

Finally
The authors wish to acknowledge and thank Mr. Anthony Phillips for his hard work and analytical expertise in developing, running, and consistently fine-tuning the modeling approach. The authors would also like to thank Mr. Ron Schoen and Mr. Roosevelt Mosely for their time in reviewing the paper and providing editorial remarks.

Exhibit 1

Relationship Between Vision and Business Measures



Sample Simulations

Exhibit 2

Direct Written Premium, Business as Usual, Expected

Enter The Random Seed (Integer):	2				
Enter The Number of Simulations:	1000				
Enter Sheet Name:	Direct Total	Direct Total	Direct Total	Direct Total	Direct Total
Enter Cell Name:	AA16	AB16	AC16	AD16	AE16
Graph? (Yes or No)					
	126,341,507	132,330,622	139,211,814	146,985,715	155,553,088
Trial #	DWP 99	DWP 00	DWP 01	DWP 02	DWP 03
1	127,230,338	132,296,705	137,272,744	142,435,266	147,973,238
2	124,829,216	128,956,320	137,060,493	147,816,242	158,512,256
3	125,862,357	129,456,766	133,900,910	141,082,351	149,125,252
4	127,020,136	132,862,907	139,904,706	145,195,659	155,650,403
5	124,670,858	132,177,688	141,494,463	151,474,526	162,893,506
6	125,754,410	130,871,547	138,319,485	144,879,793	153,608,779
7	126,849,443	131,466,579	136,454,684	140,567,554	145,437,872
8	124,476,945	130,887,110	139,034,066	145,592,855	155,566,589
9	125,645,738	129,679,210	134,271,331	137,663,148	144,678,298
⋮					
998	126,933,054	131,692,250	136,415,683	141,901,811	148,744,156
999	124,341,829	130,749,555	137,033,943	145,092,472	153,325,637
1000	125,645,233	129,520,677	134,125,066	139,166,328	143,148,479

Business as usual, expected	DWP 99	DWP 00	DWP 01	DWP 02	DWP 03
Mean	126,341,507	132,330,622	139,211,814	146,985,715	155,553,088
Growth in Mean		4.7%	5.2%	5.6%	5.8%
Standard Deviation	1,357,377	2,929,611	4,801,207	6,789,422	8,884,551
CV	0.0107	0.0221	0.0345	0.0462	0.0571
Minimum	122,743,388	126,431,407	130,147,274	132,354,021	135,578,492
Maximum	130,401,785	142,557,369	157,008,937	170,888,519	185,047,902
2%	124,141,783	127,887,693	132,316,335	136,539,912	141,082,650
10%	124,793,295	129,150,099	133,713,573	138,818,993	144,486,369
25%	125,379,733	130,186,563	135,508,703	141,563,224	148,371,775
50%	126,078,667	131,744,594	138,255,209	145,974,212	154,909,572
75%	127,167,920	134,069,492	142,365,978	151,452,575	161,689,358
90%	128,431,801	136,576,310	145,886,771	156,070,388	167,246,278
95%	128,924,860	138,242,481	148,322,878	159,516,973	171,058,711

Exhibit 3

Surplus Levels (in millions)

		Mean					CV				
Strategy	Scenario	1999	2000	2001	2002	2003	1999	2000	2001	2002	2003
Business as Usual	Expected	396	420	442	465	496	1.3%	2.0%	2.6%	3.3%	4.4%
	Pessimistic	398	423	445	464	488	1.5%	2.4%	3.5%	5.0%	6.9%
Multi-state	Expected	396	419	440	459	480	1.3%	2.0%	2.6%	3.6%	5.3%
	Pessimistic	398	422	443	458	471	1.5%	2.4%	3.6%	5.4%	8.1%
Multi-line	Expected	396	419	441	464	493	1.3%	2.0%	2.6%	3.4%	4.5%
	Pessimistic	398	422	444	464	487	1.5%	2.4%	3.5%	5.1%	7.2%

		10% Confidence Level					90% Confidence Level				
Strategy	Scenario	1999	2000	2001	2002	2003	1999	2000	2001	2002	2003
Business as Usual	Expected	389	409	428	446	467	403	430	456	485	523
	Pessimistic	390	409	425	436	445	405	436	464	493	531
Multi-state	Expected	389	408	426	438	446	403	430	455	479	511
	Pessimistic	390	409	423	427	422	405	436	463	489	519
Multi-line	Expected	389	409	427	444	464	403	430	456	483	520
	Pessimistic	390	409	424	435	442	405	436	464	494	529

		Change in Mean				Change in Surplus at 10% Conf				Change in Surplus at 90% Conf			
		2000	2001	2002	2003	2000	2001	2002	2003	2000	2001	2002	2003
Business as Usual	Expected	6.1%	5.2%	5.2%	6.7%	5.1%	4.6%	4.2%	4.7%	6.7%	6.0%	6.4%	7.8%
	Pessimistic	6.3%	5.2%	4.3%	5.2%	4.9%	3.9%	2.6%	2.1%	7.7%	6.4%	6.3%	7.7%
Multi-state	Expected	5.8%	5.0%	4.3%	4.6%	4.9%	4.4%	2.6%	1.8%	6.7%	5.8%	5.3%	6.7%
	Pessimistic	6.0%	5.0%	3.4%	2.8%	4.9%	3.4%	0.9%	-1.2%	7.7%	6.2%	5.6%	6.1%
Multi-line	Expected	5.8%	5.3%	5.2%	6.3%	5.1%	4.4%	4.0%	4.5%	6.7%	6.0%	5.9%	7.7%
	Pessimistic	6.0%	5.2%	4.5%	5.0%	4.9%	3.7%	2.6%	1.6%	7.7%	6.4%	6.5%	7.1%

Yellow cells represent projected changes in surplus below target
Blue cells represent projected changes in surplus below minimum standard

Comparison of Surplus with Condition of Underwriting Market Exhibit 4

Trial No.	Surplus 2003	Acc Year Loss Ratio 2003	Change Avg Rate 2000	Change Avg Rate 2001	Change Avg Rate 2002	Change Avg Rate 2003	Change Loss Cost 2000	Change Loss Cost 2001	Change Loss Cost 2002	Change Loss Cost 2003
9	466,467	99.1%	-1.2%	-1.4%	-1.1%	-0.1%	-3.9%	14.5%	-5.0%	21.0%
34	491,470	85.8%	-1.5%	0.0%	1.0%	2.0%	-3.2%	3.5%	2.3%	14.0%
43	490,832	85.8%	-1.4%	0.0%	0.5%	0.6%	1.9%	-5.2%	-1.6%	15.0%
44	428,757	94.8%	-1.1%	-1.7%	-1.4%	-1.0%	11.6%	-2.7%	7.3%	6.5%
45	459,461	99.9%	-1.4%	-1.4%	-1.6%	-0.8%	4.4%	-1.5%	-1.7%	28.2%
46	463,170	96.1%	-1.6%	-1.2%	-1.9%	-1.0%	25.1%	-3.6%	17.6%	-0.5%
56	439,550	97.5%	-1.6%	-0.8%	-1.9%	-1.0%	17.4%	-2.4%	11.5%	-0.6%
62	493,034	71.7%	2.3%	2.8%	2.6%	3.1%	-10.3%	21.7%	-9.8%	9.9%
86	428,832	104.2%	-1.5%	-1.2%	-1.4%	-1.4%	19.0%	-14.6%	9.5%	18.6%
90	491,870	82.8%	-0.1%	1.0%	1.9%	1.2%	3.9%	-3.5%	12.3%	1.7%
104	451,591	102.4%	-1.2%	-1.6%	-1.5%	-1.4%	-13.7%	16.9%	7.4%	15.6%
109	494,884	71.7%	0.0%	0.6%	0.8%	0.7%	5.3%	2.0%	7.4%	-15.7%
114	460,749	97.1%	-1.1%	-1.2%	-1.6%	-1.3%	8.0%	14.8%	3.1%	8.5%
133	498,412	83.5%	-0.2%	2.4%	1.5%	-1.1%	2.0%	1.5%	14.5%	1.0%
168	498,495	77.1%	0.8%	0.5%	2.1%	3.0%	8.5%	14.7%	-4.4%	4.2%
170	495,213	77.9%	-0.1%	2.1%	2.8%	2.7%	13.0%	-14.4%	26.3%	-5.0%
172	461,859	94.6%	-1.1%	-1.1%	-1.4%	-1.6%	-8.1%	6.1%	15.5%	9.0%
175	463,269	94.5%	-1.4%	-0.7%	-1.6%	-1.8%	16.8%	-11.4%	18.9%	6.7%
189	498,742	75.4%	-1.5%	0.4%	0.7%	0.5%	17.8%	-8.7%	7.2%	-7.0%
199	494,068	85.3%	-1.5%	0.3%	0.7%	0.6%	-3.4%	6.8%	2.7%	3.7%
203	495,392	73.9%	-1.6%	0.2%	2.1%	1.0%	-8.6%	13.3%	21.0%	-20.9%
220	459,076	98.5%	-1.1%	-1.6%	-1.0%	0.0%	-12.9%	26.2%	2.7%	9.4%
222	492,024	78.3%	0.0%	0.7%	0.9%	0.4%	-15.8%	21.1%	2.3%	-8.5%
228	496,636	85.9%	0.4%	0.1%	2.5%	2.4%	22.1%	-1.8%	9.1%	1.4%
235	494,762	71.3%	-1.3%	0.2%	0.5%	0.5%	3.3%	8.1%	1.3%	-15.7%
241	494,462	71.7%	0.6%	0.7%	0.7%	0.4%	-13.2%	12.6%	-6.5%	2.1%
247	497,632	78.7%	0.6%	0.9%	0.5%	0.4%	2.6%	-5.7%	12.1%	0.1%
258	494,918	74.5%	-1.4%	0.1%	2.5%	2.8%	-8.8%	16.8%	-8.8%	1.3%
262	490,907	73.5%	0.1%	0.2%	0.5%	0.8%	8.7%	3.9%	-6.0%	-3.7%
269	457,042	103.3%	-1.5%	-1.2%	-0.3%	1.0%	2.4%	18.5%	-0.4%	19.8%
278	467,152	94.6%	-1.8%	-1.4%	-1.1%	-1.2%	21.6%	-4.4%	3.5%	15.7%
283	496,529	75.3%	0.2%	0.6%	2.0%	2.8%	5.9%	-4.5%	13.3%	-8.9%
Average	495,965	82.6%	-0.5%	0.0%	0.4%	0.6%	3.7%	4.7%	3.0%	3.9%
10 Percentile	467,157	72.1%								

Yellow Shading represents simulations with low surplus. These simulations have higher than average loss cost increases, and below average rate increases.
Non shaded simulations are those close to average surplus. These simulations have above average rate changes and below average loss cost increases.

Patterns of Decreasing ROE or Surplus Compared to R/S

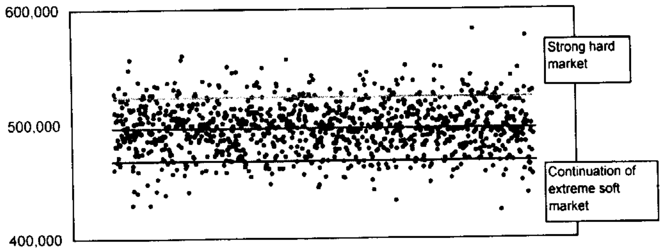
Exhibit 5

Trial No.	R/S 2000	R/S 2001	ROE 1999	ROE 2000	ROE 2001	ROE 2002	ROE 2003	Change Surplus 2000/1999	Change Surplus 2001/2000	Change Surplus 2002/2001	Change Surplus 2003/2002
7	2.12	1.97	8.1%	14.5%	11.7%	9.2%	4.8%	6.8%	7.1%	5.1%	2.6%
9	2.08	1.99	9.3%	14.9%	7.9%	9.9%	3.5%	7.6%	3.8%	5.5%	1.2%
10	2.14	2.03	8.2%	12.2%	9.4%	11.6%	9.1%	5.2%	5.3%	6.6%	6.3%
12	2.14	2.02	8.3%	11.6%	9.3%	9.7%	7.2%	4.6%	5.2%	5.5%	4.4%
25	2.13	2.00	9.8%	12.6%	9.0%	10.0%	9.1%	4.8%	5.0%	5.6%	5.9%
28	2.07	2.07	8.9%	16.3%	5.0%	11.9%	8.5%	8.7%	1.8%	7.0%	5.7%
37	2.12	2.02	11.5%	10.2%	9.3%	11.0%	8.2%	3.1%	5.1%	6.5%	5.4%
45	2.13	2.02	9.2%	12.7%	8.5%	8.0%	4.8%	5.6%	4.3%	3.8%	2.6%
46	2.01	1.88	11.6%	13.2%	10.0%	3.1%	4.9%	6.1%	5.9%	0.7%	2.4%
47	2.12	1.99	7.6%	13.6%	8.0%	6.8%	1.1%	6.8%	4.3%	3.4%	-0.8%
56	2.13	2.00	10.5%	8.6%	9.4%	2.0%	5.8%	2.4%	5.2%	-0.6%	3.3%
59	2.10	1.97	10.4%	11.3%	9.8%	10.4%	4.3%	4.3%	5.7%	6.2%	2.0%
71	2.14	2.08	9.4%	12.9%	9.4%	9.7%	6.7%	5.2%	5.3%	5.1%	3.9%
86	2.16	2.05	8.6%	8.0%	9.7%	7.3%	1.0%	1.8%	5.3%	3.3%	-0.7%
101	2.15	2.14	8.3%	11.3%	4.2%	7.6%	6.7%	4.6%	0.8%	3.6%	3.9%
104	2.05	1.90	9.5%	14.3%	9.0%	6.9%	1.4%	6.9%	5.1%	3.0%	-0.5%
114	2.06	1.97	9.0%	15.0%	8.4%	7.1%	3.4%	7.6%	4.2%	3.4%	1.3%
122	2.22	2.11	7.7%	9.0%	6.4%	8.7%	5.9%	2.1%	2.7%	4.5%	3.1%
138	2.05	1.90	9.2%	13.1%	11.3%	9.5%	2.5%	5.7%	7.1%	5.1%	0.5%
141	2.15	2.06	8.9%	12.5%	7.0%	9.0%	4.0%	5.3%	3.5%	4.7%	1.9%
150	2.20	2.10	8.2%	12.6%	8.2%	7.8%	7.2%	5.7%	4.0%	3.6%	4.3%
162	2.13	2.09	10.2%	10.2%	7.4%	10.7%	9.7%	3.3%	3.5%	6.0%	6.5%
169	2.15	2.09	6.2%	12.1%	5.5%	8.5%	8.5%	5.2%	2.3%	4.3%	5.6%
172	2.13	1.96	8.3%	12.8%	12.0%	7.4%	3.3%	5.7%	7.5%	3.3%	1.1%
175	2.11	1.95	8.1%	11.0%	11.9%	8.1%	4.1%	4.0%	7.5%	4.1%	2.0%
181	2.13	1.99	8.9%	13.2%	8.9%	9.7%	6.3%	6.1%	4.6%	5.7%	3.6%
192	2.16	2.03	11.1%	11.2%	9.1%	11.1%	4.3%	3.9%	4.8%	6.8%	2.0%
Average	1.92	1.81	10.2%	13.1%	9.6%	9.6%	9.8%	6.0%	5.4%	5.3%	6.6%
10th Percentile					7.5%	7.0%	6.4%				3.9%
90th Percentile	2.15	2.05	12.1%	15.1%				7.8%			

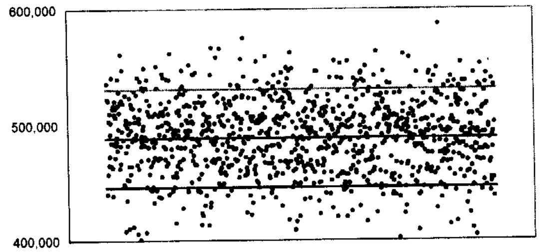
Surplus in 2003

Exhibit 6
Sheet 1

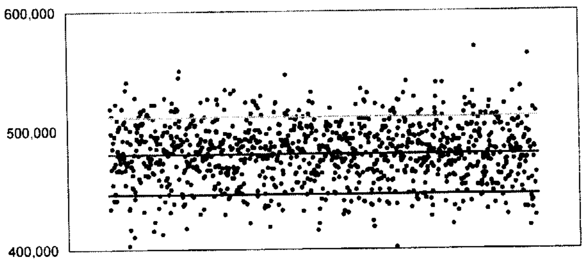
Expected Without Multi State



Pessimistic Without Multi State



Expected With Multi State



Pessimistic With Multi State

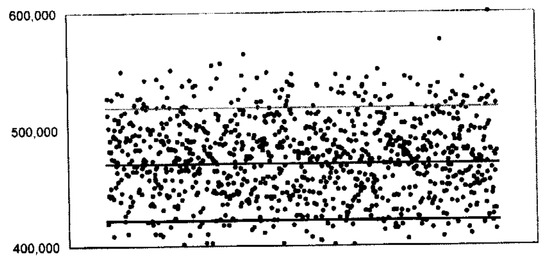


Exhibit 6
Sheet 2

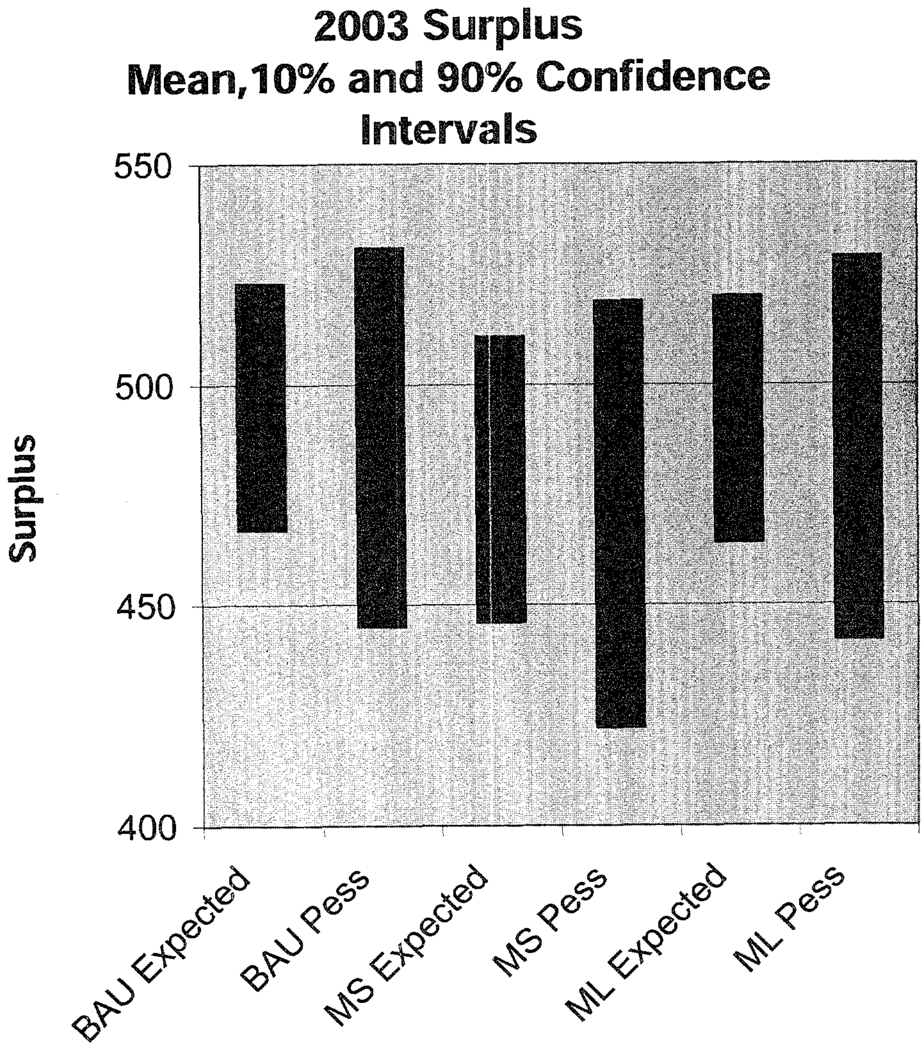
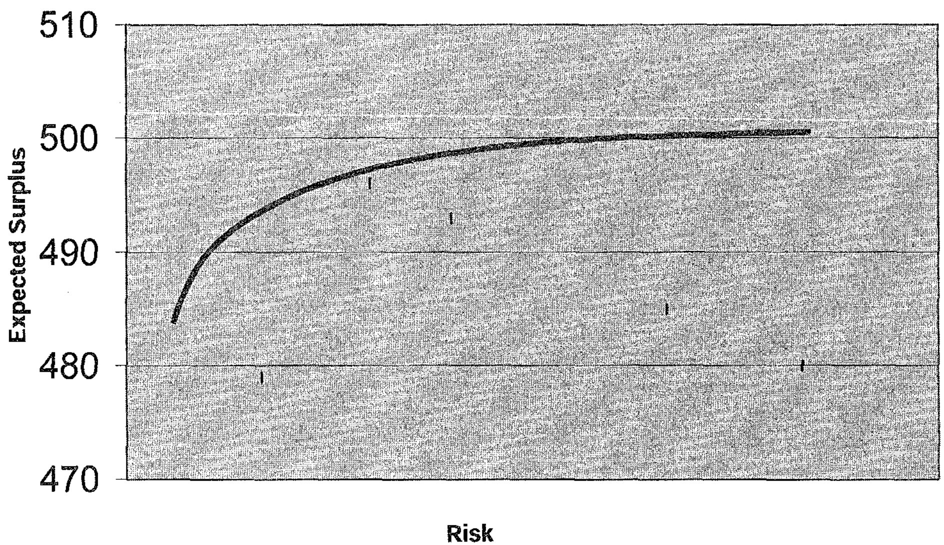


Exhibit 7.1
Sheet 1

Efficient Frontier for 2003 Surplus, Expected Scenario



Comparison of Efficient Frontier Results

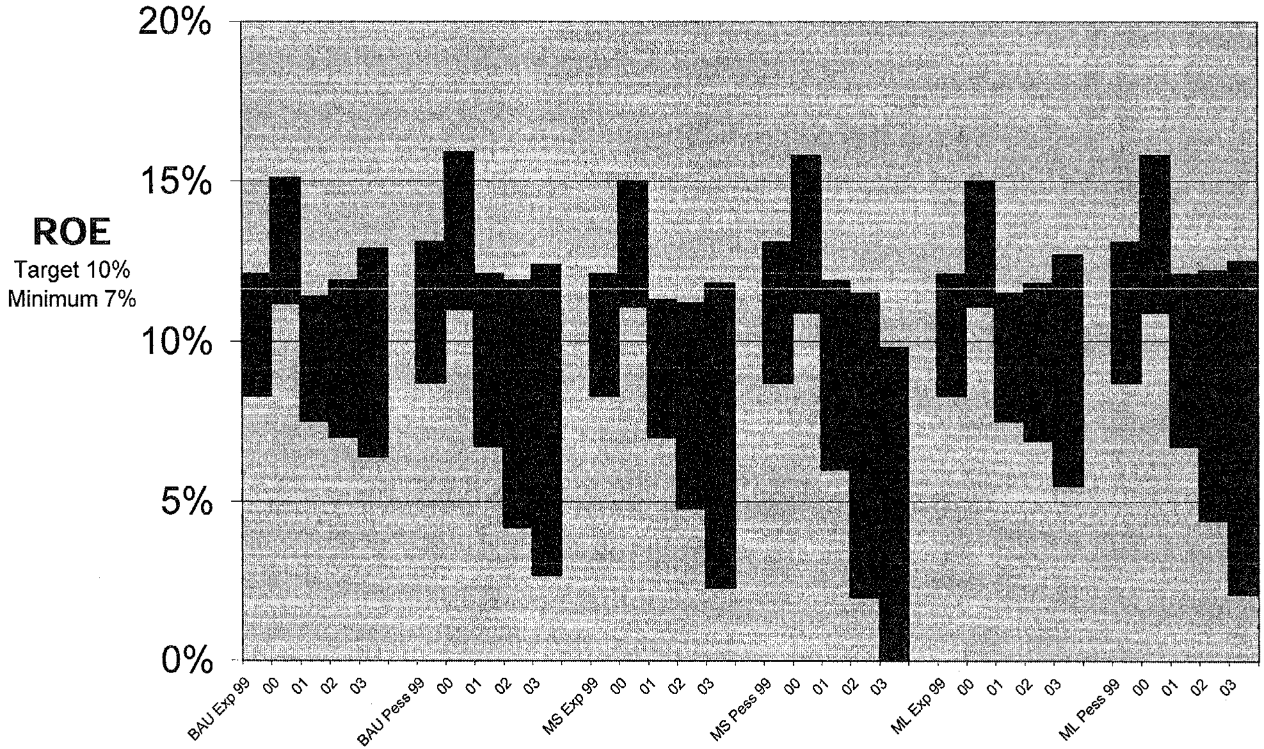
Exhibit 7.2
Sheet 2

Expected Scenario			Pessimistic		
2003 Surplus	2003 ROE	2003 Direct Written Premium	2003 Surplus	2003 ROE	2003 Direct Written Premium
Business as Usual	Multi-Line	Multi-State	Multi-Line	Multi-Line	Multi-State
Option 5	Business as Usual	Multi-Line	Business as Usual	Business as Usual	Multi-Line
Multi-Line	Multi-State	Business as Usual	Mult-State	Multi-State	Option 4
Option 4	Option 4	Option 5	Option 5	Option 4	Option 5
Multi-State	Option 5	Option 4	Option 4	Option 5	Business as Usual

Notes:
1. In practice, efficient frontiers can be done for many combinations of business measures and scenarios

Exhibit 8

ROE Compared to Target



Direct Written Premium Growth

Exhibit 9
Sheet 1

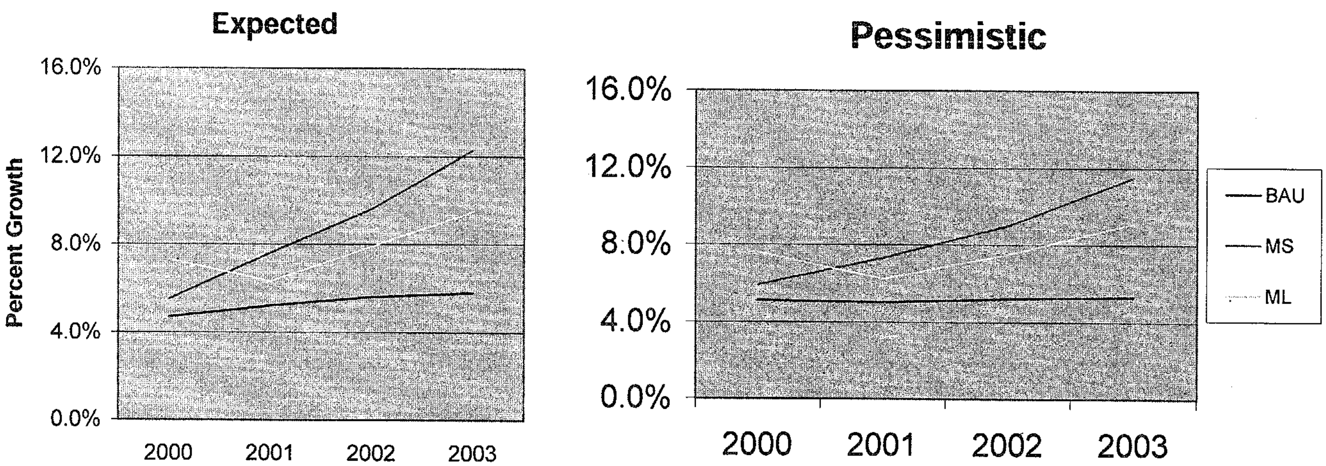
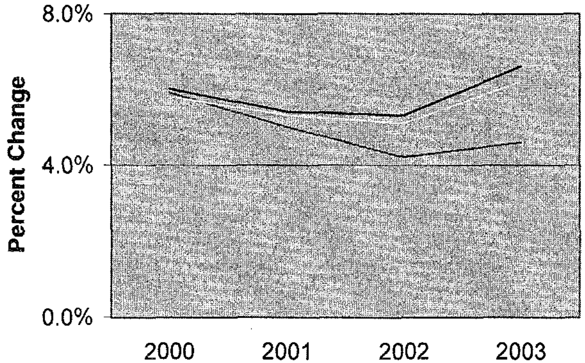


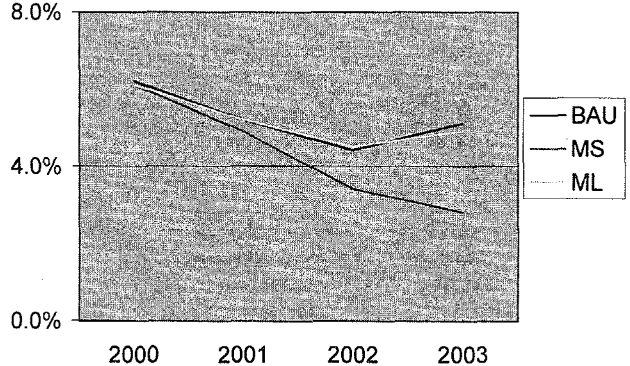
Exhibit 9
Sheet 2

Surplus Growth

Expected



Pessimistic



Portfolio Decomposition: Modeling Aggregate Loss (Ratio) Distributions

Robert K. Bender, FCAS, MAAA

Portfolio Decomposition: Modeling Aggregate Loss (Ratio) Distributions

Abstract

There are two things that may be responsible for differences between the expected loss amount (for a contract or an entire portfolio) and the actual loss amount that is experienced: errors in estimating the long term average (parameter error) and random good or bad luck (process risk). This paper presents a method for using historical data to establish a model for process risk. Because the method does not require individual claim data, it is especially suitable for reinsurance companies for whom individual claim data may not be available. It can also be used when data is obtained from the aggregate policy year and accident year calls that are filed with rating bureaus.

Essentially, the method treats the experience of multiple contract years as if each year were a random sample drawn from a single population consisting of all the outcomes that could have occurred. The techniques of Time Series Decomposition are used to restate the historical data on an "as if current levels" basis. Decomposition is then used to isolate the random fluctuations (process variance). A generalization of the Central Limit Theorem allows a model of these fluctuations to be constructed. While derived from aggregate *portfolio* experience, the model's divisibility property allows it to be scaled down to accurately reflect the aggregate loss distribution of an individual contract or policy.

Portfolio Decomposition: Modeling Aggregate Loss (Ratio) Distributions

Acknowledgements

The author wishes to acknowledge the contributions of several individuals. Professor Mohsen Pourahmadi of the Northern Illinois University Statistical Consulting Laboratory provided me with the generalized Central Limit Theorem estimators. Appendix B is based upon his work and our discussions of the problem. Luen P. Khaw assisted with many of the other statistical applications (e.g., the χ^2 analysis and our investigation of the Cauchy distribution). Melvin S. Silver provided valuable feedback regarding the presentation style of this paper. Finally, I would like to thank Heather L. Chalfant and Christine E. Schindler for their feedback regarding both the style and the substance of this paper. In particular, Ms. Schindler's comments regarding some preliminary work that I had done motivated me to explore using a Gamma distribution rather than simply accepting the limitations of a normal distribution for small portfolios. While all of these individuals contributed to the paper, none of them are responsible for any shortcomings of the paper, all of which are the author's responsibility.

Portfolio Decomposition: Modeling Aggregate Loss (Ratio) Distributions

Process variance enters the analysis of individual insurance (or reinsurance) contracts at two different levels:

1. In the form of the aggregate loss distribution associated with a cohort, or portfolio, of similar contracts (where *similar* may be inclusive enough to encompass the entire company's book of business), and
2. In the form of the aggregate loss distribution associated with individual contracts.

The first application frequently arises in conjunction with the assignment of surplus that is necessary when modeling return on equity (ROE). The second application arises during the analysis of loss sensitive contract provisions. The determination of the necessary supporting surplus that acts as a cushion against ruin requires a knowledge of the aggregate loss distribution for an entire portfolio of contracts or policies, whereas an analysis of loss sensitive contract provisions (e.g., retrospective rating of individual primary company policies, swing rating which is its reinsurance equivalent, sliding scale contingent commissions, profit sharing agreements, contributory dividends, etc.) requires a knowledge of the aggregate loss distribution at the individual policy or contract level.

There are two sources of uncertainty that may be responsible for differences between the estimated loss amount (for a contract or an entire portfolio) and the actual loss amount that is experienced. These are errors in estimating the long term mean (parameter error) and variation from the mean (process risk). This paper presents a method for using historical data to establish a model for process risk. Because the method does not require individual claim data, it is especially suitable for reinsurance companies for which individual claim data may not be available (e.g., proportional reinsurance is often ceded in the form of a bordereau that displays aggregate loss and premium cessions rather than individual claim detail). It can also be used when data is obtained from the aggregate policy year and accident year calls that are filed with rating bureaus.

The first section of the paper presents a very simplified overview of the method. In this section, several very significant assumptions are made without justification or support. Among these assumptions are that homogenous portfolios consisting of identical *exposure units* exist and that the concept of an exposure unit not only has meaning, but that these units can be counted according to some logical rule. The purpose of the first section is to provide a rationale for the more rigorous treatment that follows. Subsequent sections of the paper deal with how to relax the assumptions that were made in the simplified overview.

Throughout the paper, it is assumed that all companies conduct loss reserve adequacy testing and that a byproduct of this type of analysis is the segregation of individual policies or reinsurance contracts into more or less homogenous portfolios. It is further assumed that all of the data that is available to the reserving actuary is also available for modeling aggregate loss distributions. Note that, under many circumstances, the aggregate loss distribution, aggregate pure premium distribution, and aggregate loss ratio distribution differ only by a scale transformation. Where the distinction is not significant, the three terms have been used almost interchangeably.

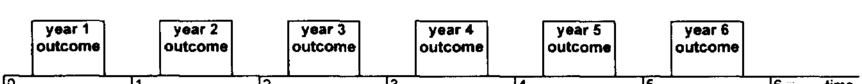
A Very Simplified Overview of the Method

An ideal insurance (or reinsurance) portfolio consists of a large group of identical exposure units. Such a portfolio might consist of identical insurance policies (or reinsurance treaties), all covering the same period of time. If the group of policies all renew coverage upon expiration, the renewal portfolio can be considered as if it were a second year in the life of the original portfolio[1]. In an even more relaxed sense of the definition of a portfolio, the indistinguishable nature of the exposure units (i.e., identical) allows a portfolio to live from year to year, even if the particular constituents (i.e., the particular policyholders or ceding companies) differ from year to year. Later, the definition will be further relaxed to allow the *number* of constituents to vary from year to year[2].

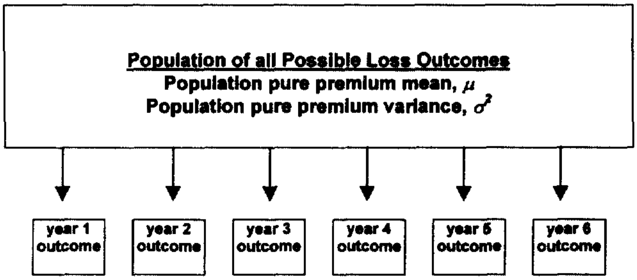
During each year of the portfolio's existence, the N exposure units that make up the portfolio will each experience a loss outcome (where "no loss" may be the most common outcome). The total of the loss outcomes divided by the number of exposure units is the pure premium outcome for the portfolio. In particular, if the loss outcome for the jth exposure unit during year t is given by L_{jt}, then the pure premium outcome for the year, pp_t, is given by

$$pp_t = N^{-1}\sum_{j=1}^{N} L_{jt}.$$

The historical experience of a portfolio is displayed below in timeline form.



If the population does not change over time, the portfolio's historical experience over several years can be considered to be different samples, all drawn from the same population of all possible outcomes. This alternative interpretation is displayed below.



Sample mean pp_1 pp_2 pp_3 pp_4 pp_5 pp_6

If there are N exposure units in the portfolio, then each sample is of size N (one selected outcome for each exposure unit). The mean of each sample is the incurred pure premium for the year.

If N is large, the Central Limit theorem tells us that the sample means will be distributed Normally with a mean equal to the population mean, μ, and with a variance equal to the variance of the population, σ^2, divided by N. The mean of the historical portfolio pure premiums (mean of the sample means), m, can be used as an estimate of the mean of the population. The variance of the portfolio pure premiums from year to year (variance of the sample means), s^2, times the number of exposure units, N, can be used to estimate the population variance.

Now consider next year's experience for a similar portfolio that consists of M identical exposure units (where M is not necessarily equal to N). The portfolio pure premium (i.e., the mean of a sample of size M, drawn from the population) is a random variable. As long as M is large, the Central Limit Theorem can be used to estimate its distribution. The distribution of next year's M exposure unit portfolio pure premium will be Normal, with

$$\text{mean} = \mu = m,$$

and

$$\begin{aligned}\text{variance} &= \sigma^2/M \\ &= [N\,s^2]/M \\ &= [N/M]s^2 \\ &= s^2/\alpha_M,\end{aligned}$$

where

$$\alpha_M = [M/N].$$

Notice that, while the historical experience was used to estimate the population mean and variance, the population parameters are simply abstractions. If all that is desired is to use historical experience to determine the distribution of next year's pure premium, then the population parameters need not ever be explicitly determined. Next year's distribution will be Normally distributed with a mean equal to the mean of the historical means, m, and a variance equal to the variance of the historical pure premiums divided by the ratio of the prospective portfolio size to the size of the historical portfolios.

The remainder of this paper deals with how to relax many of the assumptions, both implicit and explicit, that were made above so that more realistic situations can be addressed. More specifically, the following assumptions must be addressed:

1. Exposure units can defined in such a way that portfolio size can be measured,

2. If the population of possible outcomes is not stationary (i.e., if it changes over time), then there exists a transformation (i.e., a restatement of the historical experience) that makes it possible to treat the population as if it were stationary,

3. If the measure of exposure units is not stationary, then there exists a transformation (i.e., a restatement of the historical portfolio size) that makes it possible to treat the units as if they were stationary[3],

4. The Central Limit Theorem can be generalized so that the parameters of the population can be estimated *even if the samples are not all of the same size*,

5. There is a way in which to extend the method so that individual policy or contract aggregate loss distributions can be modeled even if M is too small to satisfy the requirements of the Central Limit Theorem, and that

6. The method can be applied to situations in which the portfolio is not perfectly homogeneous (i.e., when it reflects a mixture of different outcome spaces).

Assumptions and Issues that Must be Addressed

Portfolio Size: Exposure Units

For the proposed method to work, it is not only necessary to have a well defined, homogeneous portfolio but one must also be able to measure its size. In the overview, the sizes of the portfolios were measured in terms of independent exposure units. Clearly the concept of an **exposure unit** is an abstraction. As such, it will not be readily quantifiable for many (if not all) portfolios. To illustrate why this is true, consider two different homogeneous portfolios.

Portfolio 1 consists of basic limits liability policies covering 200,000 vehicles, each with exactly the same manual classification. Vehicle years is an obvious measure of the exposure. All else being equal, associating twice the number of exposure units to a second portfolio consisting of 400,000 vehicles is consistent with our intuitive notion of what the abstraction, "exposure," means.

As the homogeneity condition is relaxed, vehicles with different manual classifications will enter the portfolio as will drivers with different driving records. In this case, vehicle years becomes a less obvious measure of exposure. To the extent that the manual rate relativities reflect the expected loss amounts, premium might actually be a better proxy for the more abstract concept of exposure for the purpose of determining the size of the "relatively" homogeneous portfolio.

Portfolio 2 consists of 60 excess of loss medical malpractice reinsurance contracts. Each contract covers losses in the layer $750,000 excess of $250,000 per claim arising from any one of the individual policies that the primary company issues to small hospitals. Selecting an appropriate measure of exposure for this portfolio is not as straightforward as it was for Portfolio 1. To the extent that the reinsurance contracts are identical, *number of contracts* might be an acceptable measure of exposure. If the contracts differ in the number of policies issued, *policies issued* or the *number of covered physicians* might be a more appropriate measure of the portfolio exposure. If the individual policyholders are not identical, the number of surgical procedures performed might be used as a measure of exposure.

As with the automobile portfolio, as the homogeneity condition is relaxed, all of the proposed measures of exposure begin to lose their luster. None of the measures seems appropriate if different surgical specialties are covered within a portfolio. Again, when the portfolio is allowed to reflect a mixture of exposures, relying upon actuarially sound rates and using premium as a proxy for the more abstract "number of exposure units" is reasonable.

Three things need to be kept in mind when measuring the size of a portfolio.

1. When the portfolio consists of similar but not identical exposure units, premium may be a better measure of the size of the portfolio than the exposure base that is associated with pricing the underlying coverage.

2. Because there is usually more than one candidate that can be used as a proxy for the size of the portfolio and because they will not always produce the same number (e.g., the number of contracts in Portfolio 2 is not necessarily equal to the number of policies, physicians, or surgical procedures), the size of a portfolio is only a relative number. When determining whether the sample size, N, is large enough for the Central Limit theorem to be used, no absolute standard exists. This issue is addressed more thoroughly in another section of the paper.

3. If the purpose of measuring the size of a portfolio is to compare it to other similarly distributed portfolios, restated premium (at current rate and exposure base levels) can be an appropriate measure of size, even if it is not a good measure of exposure in any absolute sense.

Portfolio Size: Reinsurance Treaty Shares

Reinsurance treaty portfolios introduce an additional complication when size is measured because it is common (especially in the broker market) for several reinsurers to each take a proportional **share of a treaty**. For example, the reinsurer may accept 30% of the total reinsurance premium for the layer $750,000 excess of $250,000 in return for which it pays only 30% of each loss in the layer. Intuitively, it is clear that a portfolio consisting of 25% shares of four identical and independent contracts will not have the same pure premium distribution as a portfolio consisting of 50% shares of two of these contracts. The reason is that the exposure making up the first portfolio composition reflects more *spread*. As shown in Appendix A, the variance of any share of a single contract is the same as the variance of the entire contract. Appendix A also shows that the *effective size* of a reinsurance portfolio that is made up of *m* individual contracts is given by

$$N_{portfolio} = \left(\sum_{j=1}^{m} \left(n_j^2 / N_j \right) \right)^{-1}$$

where

N_j is the size of 100% of the j^{th} contract (in units of exposure, however measured),

n_j is the percentage of the reinsurer's portfolio volume contributed by the j^{th} contract,

$$n_j \equiv \frac{S_j N_j}{\sum_{k=1}^{m} S_k N_k},$$

where S_j is the percentage share taken by the reinsurer.

In our example, all of the contracts were the same size so $N_j = N$ for every j. If four 25% shares are taken, $N_{portfolio} = 4N$. In other words, taking equal *amounts*, *SN*, from four independent contracts of equal size is the same thing as taking the same amount from a single contract that is four times the size. In the case where two 50% shares are taken, $N_{portfolio} = 2N$. Even though both of these situations produce portfolios that have the same premium, the one with twice as many independent contracts acts as if it were twice the size of the other portfolio. The reader is encouraged to consult Appendix A concerning the effective size of a reinsurance portfolio.

Restatement to Produce a Stationary Population

The sample mean is determined by dividing the sum of the loss outcomes by the number of exposure units in the portfolio. When premium is used as a proxy for exposure, the sample means are loss ratios.

As a result of inflationary trends, the loss outcome (dollars of loss) corresponding to a particular event may depend upon when the event occurs and when the loss is settled. To the extent that there is such a time dependence for the portfolio under consideration, the experience of successive years cannot be considered to be the same as multiple loss outcome samples taken from the same population. Clearly, the population of possible outcomes is not stationary over time.

The exposure base used in pricing the primary policy may also be inflation sensitive. For example, *wages* (which are inflation sensitive) are used in Workers Compensation, and *revenues* have been used to rate some liability policies. To the extent that the exposure base inflates at the same rate as the corresponding losses, *pure premium* will be invariant. When such is the case, the portfolio pure premium outcomes from several years can be treated as sample means of multiple samples drawn from a single population ***even if the distribution of the loss outcomes is time dependent.***

If the loss and exposure base inflation trends are not equal, then the pure premium outcomes from different years cannot be taken as the means of samples drawn from a single population. If premium has been selected as the measure of the size of the portfolio (remember, the exposure base that is used to price individual policies need not be adopted as the measure of portfolio exposure units), changes in *rate level adequacy* introduces another factor that can invalidate the assumption that the portfolio incurred loss ratios from successive years can be treated as the means of multiple samples drawn from a single population.

The usual manner in which changes in rate level adequacy is addressed is by restating all historical data on an "as if current levels" basis. If there is sufficient information available, such a restatement is the preferred course of action. If such a procedure could be carried out, individual losses would be trended to a common point in time, the pricing exposure base would be inflated to a common point in time (while not necessarily the same point in time as the losses, using the same common time for both is logical, especially if the common time is the midpoint of the prospective period for which rates are being determined), and historical rates would be replaced by the current rate.

Once the restatement process had been carried out, pure premiums and incurred loss ratios would be equally good measures of the mean of the sample, although loss ratios offer more immunity to changes in the class mix than pure premiums do. The restatement would allow samples drawn from the portfolios of different years to be treated as if they were drawn from a single population. This, of course, is exactly what is assumed when the experience of many years is restated and averaged for the purpose of determining an experience based rate.

When there is insufficient information for restating the historical details (e.g., proportional reinsurance) or when such a process would involve a prohibitive amount of work, an alternative approach, based upon the techniques of time series decomposition, can be used (see, for example, Makridakis [1]). Time series decomposition is based upon the assumption that, at any time t, the portfolio loss ratio (or pure premium) can be expressed as the product of three functions of time, T_t, C_t, and R_t. In symbolic form,

$$ILR_t = T_t x C_t x R_t.$$

T reflects the long-term expected loss ratio trend. In insurance terms, it reflects the degree to which the pricing exposure base trend is exceeded by (for positive trend) or exceeds (for negative trend) the loss trends (both frequency and severity). C reflects cyclic changes in the expected loss ratio. It is in C that the insurance (reinsurance) pricing cycle would be reflected. R reflects the random fluctuations (process risk) that cause the actual incurred loss ratio to differ from its expected value. R has a mean equal to 1.000 (i.e., no long term deviation from the expected loss ratio). For a sufficiently large portfolio, R will be symmetrically distributed (good and bad luck of a given magnitude to be equally likely) with most of its weight near the expected value (large deviations from the expected are much less likely than small deviations in either direction). We note that while none of the insurance interpretations (or the *a priori* notions regarding the shape of R as a function of t) are critical to the method of times series decomposition, they do form the basis of reasonableness tests to which the results must be subjected.

While the usual objective of time series decomposition is to isolate the non-random components, TxC, the objective in the portfolio decomposition application is to isolate the random component and then determine its distribution. An example illustrates how the technique of series decomposition can be used to isolate R_t.

Exhibit 1 demonstrates the method using some rather well behaved (and completely fictitious) data. The column of data titled "ILR" displays ultimate (reinsurance portfolio) incurred loss ratios for each of the 38 contract years, 1960-1997. While the data is fictitious, it could have been the byproduct of a loss reserve adequacy test for the portfolio. In practice, some of these ultimate loss ratios would be estimates as of the most recent valuation date but, for now, we will assume that they are actual ultimate incurred loss ratios, thereby sidestepping the issue of potential bias introduced by the estimation process (both with respect to the loss ratios and their distribution). These issues will be addressed later.

The 38 loss ratios make up a time series. Decomposition of this series begins by observing that the random errors tend to offset each other over time. If a moving average is taken over a suitably large number of years, the random fluctuations should cancel, leaving only the effects of trend and cycle. If a five year period is suitably long, taking a five-year moving average of the *ILR* should result in a series of 34 $T_t x C_t$'s where the time associated with each of the five-year moving averages corresponds to the third year included in the moving average (e.g., the average of the first five years, 1960-1964, was assigned to 1962). Of course, a five year period may not be sufficiently long to remove all of the process variance, but its effect should be greatly diminished by taking a five year moving average. Note that, while a seven year moving average might result in less surviving process variance, it would also decrease the number of TxC points from 34 to 32. There is always an issue of balance regarding the number of points to be included in the moving average and the number of points that remain for analysis (each additional point included in the moving average reduces the number of remaining points). Including more points in the moving average reduces the amount of residual randomness, which is desirable; at the same time, a reduction in the number of moving averages reduces the ability of the analysis to detect rapid, non-random, changes (i.e., there is a reduction in resolution when each moving average reflects a longer period of time).

The trend component can be isolated by fitting a trend curve to the points, $T_t x C_t$. In this example, the trend model,

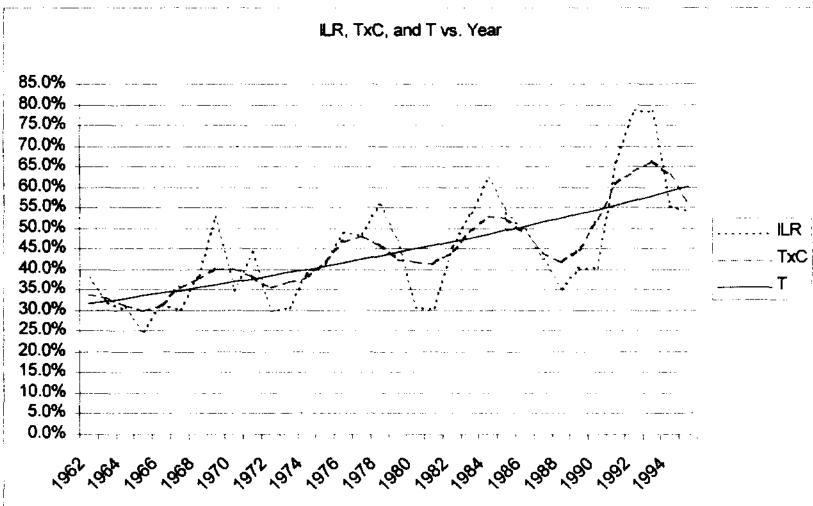
$$T_t = T_{1960}(1+trend)^{(t-1960)},$$

was fit to the TxC series using the method of least square errors. The result was

$$T_{1960} = 30.5\%,$$

$$trend = 1.96\%.$$

The graph, below, displays the original data (ILR), moving average (TxC) and trend component (T) corresponding to the data that is displayed in Exhibit 1.



The degree to which TxC differs from the graph of T can be attributed to the presence of a cycle.

Because the data for this example was generated from known functions, we can compare the model to "reality," something that cannot be done in practice. The fictitious data was generated from a trend curve,

$$T_t=0.30(1.020)^{(t-1960)}.$$

The model overestimated the initial loss ratio and underestimated the annual trend. There are two reasons for these errors. First of all, the underlying cycle was eight years long. As a result, the 38 years did not reflect five full cycles. In this example, the inclusion of a partial cycle introduced a small bias toward understating the long-term trend for the 38 years. Second, while the random component (generated by means of a random number generator) was rescaled so that its 38-year mean was unity, it did not have a mean of unity over every five year period. As a result, there was some residual process variance left in the moving averages. In practice, cycles are not regular (varying in both length and severity) and the random variance factor cannot be expected to average to unity over short periods of time. Because the model of T and C (which is nothing more than the moving average at time t divided by trend component at time t) can be distorted by the mixing of cycle with trend and by the presence of residual process variance, both components should be subjected to a reasonableness test.

Once a trend rate and cycle function have been selected, each of the historical loss ratios can be restated to reflect a common point in time (i.e., placed at a common point in the cycle and trended to a common point in time). These restated loss ratios should differ from each other only as a result of process variance. The *restated* loss ratios make up 34 (or 38 if C can be extrapolated) samples taken from the same population. For consistency, Exhibit 1 displays both the cyclic component and the trend component in the form of indices. The restated ILR is determined by multiplying each data point, ILR_t, by a restatement factor,

$$(TrendIndex_{1997}/TrendIndex_t)(100/CycleIndex_t).$$

This effectively restates each loss ratio to an "as if 1997 inflation levels" and "as if at the midpoint of the pricing cycle" basis. The fact that 1997 does not appear to be the midpoint of a pricing cycle is not important. The fact that all the loss ratios are restated to a common point in the cycle and common level of inflation is all that is important [4].

Restatement to Produce a Consistent Measure of Exposure (Portfolio Size)

Were it not for the possibility that the portfolio may have changed in actual size (e.g., as a result of true growth or contraction), a restatement of the historical loss ratios to a common level of rate *adequacy* would be all that was necessary. The population loss ratio mean and loss ratio variance could be estimated directly from the restated incurred loss ratios. When there is a possibility that the portfolio size has changed over time (in an absolute sense), it is necessary to remove the impact of changing *rate levels* (not just rate level adequacy) and *pricing exposure base inflation* from the measurement of the historical portfolio size. Once these changes in the measurement "yardstick" have been removed, true size changes can be determined and reflected in the parameter estimates.

Where the sample means (loss ratios) were concerned, only the extent to which premiums and losses changed by different factors was relevant. In other words, only changes in rate level adequacy were relevant. Premiums for a portfolio can change even if relative rate adequacy does not. In some cases, these changes may be indicative of a change in the size of the underlying exposure, and in some cases they may not be. An example of the former would be a change in premium resulting from an increase in the number of identical policies written. This clearly reflects a change in the size of the portfolio. An example of the latter would be a rate change. Whether or not the rate change resulted in a change in rate level adequacy, it clearly does not reflect a change in the size of the portfolio. If premium is selected as the proxy for exposure, then restatements reflecting more than rate level adequacy are necessary.

Clearly, premiums need to be restated to reflect a common *rate level* before they can be used to measure the size of a portfolio. Premiums can also change as a result of exposure base inflation, even in the absence of a rate change. As long as rate level adequacy does not slip, an increase in insured value will have no effect on the aggregate loss ratio distribution. Because of this, historical premiums must be restated to reflect a common level of exposure base inflation as well as a common rate level before they can be used as a measure of true portfolio size.

Ideally, historical premiums should be directly restated Unfortunately, a rate level history for an entire portfolio is rarely available. It is more likely, when full information is lacking, that more will be known concerning losses than rate changes or the pricing exposure base. Frequently, information regarding severity trends may be derived during loss reserve adequacy testing. Econometric data may provide additional information regarding changes in loss severity (e.g., Consumer Price Data or combinations of selected indices such as the Masterson Indices) and frequency. Industry data may be a source of information regarding frequency trends (e.g., the National Council On Compensation Insurance Annual Statistical Bulletin [2]).

Once the impact of changes in rate level adequacy has been eliminated by means of restatement to a common level of rate adequacy, any remaining premium changes (the product of rate and exposure changes) will reflect loss trends (including law and/or benefit changes) exactly. For example, if the ratio of two successive TxC components is 0.90, that means that rate level adequacy has increased by 11.1% (i.e., 0.90 = 1/1.111). If, during the same time, inflated 20%, one can conclude that premiums must have increased by a total of 33.3%. Here, 33.3% not only offsets the inflation but also results in more adequate rates.

In more mathematical detail, let L_X and $Rate_X$, be the Loss and Rate, for year X. Further, let $Exposure_X$ be the exposure base used to determine the premium during year X. Note that it would be very unusual for $Exposure_X$ to be anything more than a proxy for the true exposure.

In terms of these variables, an 11.1% increase in rate level adequacy implies

$$L_{X+1}/(Rate_{X+1}*Exposure_{X+1}) = 0.90*[L_X/(Rate_X*Exposure_X)].$$

Note that, to the extent that the actual number of exposure units change, the loss amount will change proportionately, leaving the rate level adequacy unchanged. In other words, the ratio *L/Exposure* is invariant to changes in the actual number of exposure units. Changes in rate level adequacy reflect factors that can distort the measurement of portfolio size (*i.e.*, it reflects only those factors that do *not* depend upon the portfolio size). The implied change in premium (as far as rate level adequacy is concerned) can be attributed to two factors: changes in the rates themselves and those changes in the proxy used to measure exposure (*i.e.*, the exposure base) that are not related to true changes in size of the population.

In the example, if losses per exposure unit increase by 20%, and Δ represents the unknown fractional change in the combined rate and exposure base product (that portion that is unrelated to true changes in the portfolio size), then

$$0.90*[L_X/(Rate_X*Exposure_X)] = 1.20*L_X/([1+\Delta]Rate_X*Exposure_X).$$

Solving for Δ,

$$[1+\Delta]=1.20/0.90 = 1.333 \text{ or a } 33.3\% \text{ increase.}$$

The conclusion is that the first 33.3% of premium increase must be attributed to changes other than changes in the number of exposure units (i.e., the portfolio size). Any remaining premium change (which might be a decrease) can be attributed to a change in the size of the portfolio.

Another simple example, based upon a different set of well behaved fictitious data, illustrates the required restatement process. The first four columns of Exhibit II display hypothetical data as it might appear together with the results of a series decomposition of the type performed in Exhibit I. As was the case for Exhibit I, the new data is well behaved but no more realistic than the data underlying Exhibit I. It was selected to simplify the illustration of the premium restatement procedure.

For each contract year, there would be a record of the historical premium (restated to reflect reinsurance shares, if necessary), the *TxC* loss ratio component, and a set of loss indices, $\{LossIndex_t\}$, that reflect both severity and frequency. It is assumed that such indices can be obtained. Their derivation is outside the scope of this paper.

Exhibit IIa displays information that could only be known to a privileged, or "all-knowing," observer. The additional information is disclosed only to demonstrate the validity of the restatement procedure. In particular, the privileged observer knows:

- what the true historical exposure (the abstraction) was,
- that the exposure base used to price the primary policies, while related to the true exposure, was inflating relative to the true exposure at a rate of 3.0% per year (e.g., the true exposure might have been products sold but dollars of sales may have been used to determine the premium for individual policies),
- what the historical rates were (where the historical rate times the pricing exposure base equals the historical premium), and
- that losses began as 2.75 times the true exposure and losses per exposure unit increased at a rate of 10% per year.

Of course, none of this detail is directly disclosed by the portfolio data, only those columns that are displayed on Exhibit II would be known.

As was previously discussed, changes in TxC reflect changes in rate level adequacy. Column (8) quantifies these annual changes. Column (10) is the factor necessary to restate historical premiums on an "as if current rate level adequacy" basis. Similarly, Column (11) consists of factors that allow losses to be restated on an "as if current loss levels" basis. The composite factor displayed in Column (12) is simply the loss factor divided by the rate level adequacy factor. Finally, Column (13) displays the restated historical premiums. To see that the year-to-year changes in restated premium mirror the percentage change in the true exposure, Column (13) was rescaled, making the 1960 size equal to 100.00. As can be seen, all of the other entries are then equal to the true exposure. As a result, the relative restated premium is the same as the relative true exposure (i.e., for any pair of years, the ratios are the same). Changes in the true exposure are indicative of changes in portfolio size, N, from year to year.

Central Limit Theorem Estimators when the Samples are of Different Size

It is unlikely that a given portfolio would remain constant in size over time. The number of policies or reinsurance contracts in a portfolio frequently change over time, giving rise to **samples of different sizes**. The Central Limit Theorem makes statements about the distribution of sample means when the samples are all of size N. These statements must be generalized when the samples are of different sizes. In particular, while the distribution of the mean for any single sample of size N continues to be Normal with mean μ and variance σ^2/N, estimators of the population mean and variance are no longer equal to the mean, m, and N times the variance of the sample means, Ns^2, of previously drawn samples when the samples are of different sizes.

Appendix B provides support for the following *generalized estimators* of the population mean and variance.

$$Estimator(pp_{population}) = m' = \sum_{j=1}^{T} \alpha_j pp_j \Big/ \sum_{j=1}^{T} \alpha_j$$

$$Estimator(Var[pp_{population}]) = N_1 (T-1)^{-1} \sum_{j=1}^{T} \alpha_j (pp_j - m')^2 .$$

Where

pp_j is the pure premium experienced by the j^{th} sample (i.e., during the j^{th} year),

α_j is the relative size of the j^{th} sample (i.e., $\alpha_j = N_j/N_1$),

T is the number of samples (i.e., the number of years), and

N_1 is the size of the first sample.

For large sample size, the distribution of the sample mean is Normal with variance equal to $Var[pp_{population}]/N$. It, therefore, follows that the estimate of the variance of the pure premium (i.e., the sample mean) for a sample of size N_1 is given by

$$Var[pp_{population}]/N_1=(T-1)^{-1}\sum_{j=1}^{T}\alpha_j(pp_j-m')^2.$$

The variance of the k^{th} sample is N_1/N_k as large, or

$$Var[pp_{population}]/N_k=(T-1)^{-1}\sum_{j=1}^{T}\alpha_j(pp_j-m')^2/\alpha_k,$$

Since $N_1/N_k=1/\alpha_k$.

A More Realistic Example

Exhibit III illustrates an application of the methodology to a portfolio consisting of similar general casualty excess of loss reinsurance treaties. Over the course of twenty-seven years (contract years 1969-1995) portfolio premiums have increased from approximately \$2,000,000 to almost \$100,000,000. During the same period, incurred loss ratios ranged from a low of less than 40% to a high in excess of 350%. Some of this loss ratio volatility was due to process risk and some (perhaps most) was due to the presence of at least two reinsurance pricing cycles.

Supporting Exhibit IIIa displays the actual loss ratios for the twenty-seven year period together with the series decomposition that allows the loss ratios to be restated on an "as if common rate level adequacy" basis. The first graph displays the actual data, *TxC* component, and isolated trend component, *T*.

The second graph allows for a reasonableness test of the decomposition. Rather than graphing *C vs. Year*, the graph displays the reciprocal of C. When a soft market forces rate level adequacy to slip, incurred loss ratios increase. As a result, C moves in the opposite direction from rate levels. Graphing the reciprocal of C makes the graph more intuitive. The soft reinsurance market of the early to mid 1980's is clearly evident in the graph.

Because this is a reinsurance portfolio and because the reinsurer took less than 100% shares of most of the contracts that it wrote, its premium is not a true indicator of the size of the reinsured entities. Exhibit IIIb displays individual account premium detail for the 1971 contract year. During that year, 35 clients were reinsured. While the reinsurer earned premium equal to $2,120,969, that amount represented over $30,000,000 of premium on a 100% basis. As a result of the manner in which the reinsurer authorized shares, the reinsurer experienced the same variability as if it had reinsured a single $14,193,426 client –regardless of the share taken. While no supporting exhibits were prepared for the other years, the premium for each of the other contract years was similarly adjusted.

Exhibit IIIc begins with the historical premiums (after adjustment to reflect reinsurance shares) and concludes with historical premiums restated on an 'as if current rate levels" premium. Because only relative portfolio size is important, the restated premiums were rescaled to make the 1971 premium equal to 1.000. Column (14) displays the α's of the estimator formulas.

All of the components are brought together in Exhibit III, where the two estimators, m' and Var, were determined. As previously shown, these estimators are the estimated mean and variance of the loss ratio distribution for a portfolio the size of the first one drawn; more precisely, they correspond to the size of the portfolio that is assigned a relative weight equal to 1.000 (i.e., $Var = Var[ILR|\alpha=1.00]$). The expected mean loss ratio is independent of portfolio size, and the expected variance of any portfolio of size α_j (measure relative to the initial sample size) is given by Var/α_j.

The objective of this exercise was to find the distribution of the process variance, R, not the distribution of the incurred loss ratios. If each of the restated *ILR*'s in Column (4) is divided by the mean, m' (which is nothing more than an estimate of the restated *TxC* component), the result is a column of R's. The mean is automatically equal to 1.000 and the variance is equal to the variance of the loss ratios divided by the square of the mean loss ratio. Finding the R's from the *ILR*'s is nothing more than a scale transformation.

Exhibit III concludes with a display of the expected variance, both for the *ILR* distribution and for the corresponding R distribution, for portfolios of various sizes. All of the *ILR* distributions are Normal with mean, m', and the indicated variance, while all of the R distributions are Normal with a mean of unity and the indicated variance.

While the Normal distribution is particularly easy to use and it has an intuitive appeal, wanting it to work isn't evidence that it is an appropriate model. The test displayed in Exhibit IIId begins with the assumption that the random component, R, is Normally distributed with the estimated parameters. While a failure to reject the null hypothesis is not proof that the hypothesis is true, the result of the χ^2 test is evidence that there is no compelling reason to reject the hypothesis.

An Alternative Model: The Gamma Distribution

The parameters of the Normal model, σ in particular, can be adjusted to reflect changes in the size of a portfolio. This adjustment follows directly from the Central limit theorem. If σ_j is known, then σ_k is given by

$$\sigma_j \sqrt{N_j / N_k}$$

where N_j/N_k is the ratio of the size of the j^{th} portfolio to the size of the k^{th} portfolio. One assumption that underlies the Central Limit Theorem is that N is large. That the adjustment is not appropriate when N_k is very small quickly becomes obvious. As N_k decreases, the standard deviation of the model distribution becomes very large. As a result, the model allows for a significant probability of negative loss ratios for small portfolios (e.g., the size of a single account or policy).

The more realistic example (Exhibit III) indicates that σ_I=0.097 when N_I = \$224,626,501 (the restatement of \$14,193,426). A typical client (e.g., #2 on Exhibit IIIb) has approximately \$550,602 of premium on a 100% basis. The

Normal model predicts a random component, R, with a standard deviation equal to 0.492 (=0.097 *(14,193,426/550,602)^{½}). The probability that R will be negative (i.e., fall more than approximately 2.00 standard deviations below the mean) is 0.0212. A significant probability that an account's loss ratio will be negative is not a realistic expectation for most lines of business.

There is another model that is almost Normal for large N, but whose reproductivity/divisibility properties allow it to scale down to the size of an individual account. This is particularly useful if the model is to be derived at the portfolio level and applied at the individual account level. The model is the Gamma distribution, whose pdf is given by

$$\Gamma_{a,r}(x) \equiv \frac{a^{r}}{\Gamma(x)} x^{r-1} e^{-ax}; \begin{bmatrix} x > 0 \\ a > 0 \\ r > 0 \end{bmatrix}.$$

The Gamma distribution has the following properties (Hewett [3]):

- It is divisible. That is to say that if $\Gamma_{a,r}(x)$ is the appropriate model for the aggregate distribution of a portfolio consisting of N independent units of exposure, then $\Gamma_{a,rM/N}(x)$ is the appropriate model for the aggregate distribution of a portfolio consisting of M independent units of exposure.
- The mean of x when x is distributed $\Gamma_{a,r}(x)$ is r/a from which it follows that,
- When the mean of the distribution is known to be unity, $a = r$, and
- When x is distributed $\Gamma_{a,r}(x)$, the variance of x, σ^2, is r/a^2, or $1/r$ when the mean is known to be unity.
- The mode of x, when x is distributed $\Gamma_{a,r}(x)$ is given by $(r-1)/a$ which becomes $(r-1)/r$ when the mean is known to be unity.

Exhibit IV displays the results of a χ^2 test for the Gamma distribution. It is the Gamma equivalent of Exhibit IIId. Since R_t has a mean equal to unity,

$$a = r = 1/\sigma^2.$$

When the estimated portfolio variance is approximately 0.097, corresponding to an $\alpha = 1.000$ size portfolio (i.e., for the 1971 portfolio),

$$a = r = 10.345.$$

As was the case for the Normal distribution, there is no compelling reason to reject the Gamma distribution. Exhibit Va discloses that for a portfolio this large, the Gamma distribution is almost Normal.

As the size of the portfolio is decreased to 15% of the original size, both models spreadout. The Normal distribution allows a significant probability for negative loss ratios whereas the mode of the Gamma shifts to the left while remaining in the first quadrant (see Exhibit Vb). Exhibit Vc displays the gradual shift of the Gamma distribution as the portfolio size becomes progressively smaller.

The decreasing mode (the mean is always equal to unity) of the Gamma is consistent with reality. That this is so can be seen by allowing the portfolio size to decrease to that of a few exposure units. The most frequent loss outcome for a single exposure unit is often "no claim," yet the expected loss for a large aggregation of such exposure units is rarely zero. A mean equal to unity retains the long term expected average outcome while acknowledging that the most likely outcome is something less than an average loss. It's the possibility of extremely large losses that pushes the mean above the mode.

Applications of the Methodology

Allocation of Surplus

Surplus provides a cushion against unanticipated events. As such, the entire company surplus is available to meet the company's unanticipated obligations, regardless of their source. Strictly speaking, surplus is indivisible and cannot be allocated to lines of business. At the same time, writing one additional unit of exposure increases the amount of surplus that is required (either as a result of Risk Based Capital requirements or to maintain the probability of ruin below some desired amount). In a sense, this additional surplus can be associated with, or allocated to, the particular line of business. To the extent that all lines of business do not have the same marginal surplus requirement, business decisions regarding the mix of business and acceptable profit margins can be influenced by such an allocation of surplus. This paper accepts the premise that, at least for the purpose of assisting in business decisions, surplus can be allocated to lines of business. It also accepts the idea that the role of surplus is to keep the company's probability of ruin below some arbitrarily selected amount. As a result, the establishment of a reserve-to-surplus leverage ratio for a given portfolio requires knowledge of the aggregate loss distribution for the entire portfolio of contracts.

More specifically, the actual loss ratio experience for a given portfolio can differ from the anticipated experience for two reasons:

1. The anticipated result was erroneous. In other words, parameter error was present.

2. The anticipated result was correct, but random bad or good luck resulted in the actual result being different from the anticipated result. In other words, process risk resulted in the unanticipated difference.

It is the process variance, as measured by R_t, that should be reflected in the allocation of surplus when determining the return on equity associated with a particular contract. For a given expected loss ratio, the distribution of R_t would determine how much surplus would be necessary to protect the company against ruin up to some preselected confidence level. The marginal supporting surplus required as a result of introducing an additional contract would depend upon the additional process variance resulting from the introduction of the new contract. The additional process variance would depend upon the portfolio to which the contract was assigned as well as the existing mix of business and any correlations between portfolios.

The resulting return on equity (see Bingham [4] and Bender [5] for a discussion concerning how to measure ROE once surplus has been allocated) will not reflect the presence of potential parameter error. The appropriate reflection of parameter error is in selecting an ROE target. For example, an 8% ROE might be sufficient reward for placing the company surplus at risk if the parameters (e.g., expected loss ratio and payout timing) can be estimated to a high degree of certainty, whereas the target might increase to 15% if less credible estimates are available.

Further discussion regarding how to measure ROE and how to select an appropriate target are beyond the scope of this paper.

Evaluating Loss Sensitive Contract Provisions

Frequently, a portfolio is made up of policies or reinsurance contracts that are subject to loss sensitive elements at the individual contract level. Examples of this would be a portfolio consisting of swing rated reinsurance treaties or retrospectively rated Workers Compensation policies. Because the loss sensitive premium is calculated for each contract separately, substituting the estimated ultimate loss ratio for the entire portfolio into the loss sensitive rating formula will not necessarily produce the best estimate of the ultimate aggregate premium (see Bender [6]).

Charles H. Berry ([7]) proposed a method of estimating the ultimate premium return for such a portfolio. Essentially, his method consists of credibility weighting the reported premium return ratio (to standard or subject premium) with an *a priori* premium return ratio. Over time, more credibility is given to the reported ratio and less is given to the *a priori* ratio. Berry's *a priori* ratio is based on the relationship between historical aggregate portfolio loss ratios and return premium ratios. In his discussion of Berry's method, Roy K. Morell [8] noted a significant limitation to the methodology. Its success depends upon the historical portfolios being subject to similar rating parameters (e.g., swing maximums and minimums) and consisting of similar risks (i.e., exposure units). If there have been material changes in either, the *a priori* estimates will not be appropriate.

As an alternative to using the historical relationship between aggregate loss and aggregate premium, the portfolio aggregate loss ratio distribution could be scaled down to the size of an individual contract using the divisibility property of the Gamma distribution. The individual contract distribution could be used to simulate the possible loss outcomes and to determine the corresponding return premium ratio together with the associated probability of occurrence for each contract in the current portfolio. The aggregate loss ratio and corresponding aggregate expected return premium ratio could serve as the *a priori* estimates for the portfolio. The Gamma Distribution could also be used to determine the sensitivity of the aggregate premium return to changes in the aggregate loss ratio, just as Table M was used by Bender [6].

The results of the sensitivity analysis would be used to develop the reported aggregate return premium ratio to reflect the impact of IBNR loss. At the same time, the sensitivity analysis could be used to revise the *a priori* aggregate return premium ratio to reflect the additional loss ratio information. As of any valuation, the estimated ultimate aggregate return premium ratio would be the credibility weighted average of the developed return premium ratio and the revised *a priori* aggregate return premium ratio.

For example, assume that a portfolio was rated to produce a 60% loss ratio to standard premium and to return 10% of standard premium in the form of retrospective (swing) rated premium adjustments. Further, assume that the expected sensitivity of the formula is 25% (i.e., for every 100 additional points of loss ratio, the return premium ratio decreases by 25 percentage points) for this portfolio. The portfolio's reserve history might look something like Exhibit VI.

While reserving retrospectively rated policies is beyond the scope of this paper, the reader is encouraged to compare the suggestion to the methodology proposed by Berry.

Qualifications and Caveats

Two of the major assumptions regarding the composition of each portfolio are unlikely to be strictly met in reality. These assumptions are that the units of exposure are *identically distributed* and *independent*. While the criteria defining each portfolio can be adjusted to minimize the degree to which either of these assumptions is not met, the cost of doing so will always be a reduction in the size of the portfolio. This is an example of the ubiquitous conflict between obtaining homogeneity while maintaining a credible volume of data. Fortunately, neither of the assumptions has to be met in order for the methodology to be applied. To the extent that the assumptions are not met, parameter error may be introduced into the models.

To see how parameter error arises when the exposure units are not identical requires only that the assumption of identically distributed random variables be removed from the requirements of the Central Limit Theorem. In 1901, Liapounov proved a more general form of the Central Limit Theorem ([9]) that applies when the outcomes for different exposure units are not necessarily identically distributed. In particular, he assumed that a set of random variables, $\{X_j\}$, is distributed with means $\{\mu_j\}$ (not necessarily equal) and variances $\{\sigma^2_j\}$ (again, not the same for all j).

Defining Y_N, the sum of the X_j for a sample of size N, as follows,

$$Y_N \equiv \sum_{j=1}^{N} X_j,$$

Liapounov proved that the sample mean of a finite number of random variables, Y_N/N, will be distributed Normally with mean,

$$\begin{aligned}\mu_N &= N^{-1}\sum_{j=1}^{N}\mu_j \\ &= \langle \mu_j \rangle,\end{aligned}$$

and variance

$$\sigma_N^2 = \left(\sum_{j=1}^{N} \sigma_j^2\right) / N^2$$
$$= \left(N^{-1} \sum_{j=1}^{N} \sigma_j^2\right) / N$$
$$= \left\langle \sigma_j^2 \right\rangle / N$$

where the brackets, <··> indicate taking the average of the indicated sub-population parameters.

It is as if the samples were drawn from a population of exposure units with *identically* distributed losses where the mean pure premium equals the average of the means of the actual distributions and where the variance of the pure premium equals the average of the variances of the actual distributions. For example, consider a portfolio consisting of 50 exposure units whose pure premiums are distributed with a mean of 30.00 and variance 4.00 and 75 exposure units whose pure premiums are distributed with a mean equal to 50.00 and variance equal to 9.00. The 125 exposure unit portfolio will experience Normally distributed pure premiums with a mean equal to 42.00 ([50*30+75*50]/125) and variance equal to 0.056 (the average variance of the population,[50*4+75*9]/125 = 7, divided by the size of the sample, 7/125 =0.056). This is the same distribution of sample means that 125 identically distributed exposure units with mean 42.000 and variance 7.00 would have produced.

When the historical portfolios differ in size and the exposure units are not identically distributed, the generalized estimators will provide the population average mean and variance as long as the proportion of each sub-distribution remains the same as the sample size changes. As long as it is the distribution of a similarly distributed portfolio that is to be modeled, the mixture introduces no parameter error. However, if one type of exposure is to be modeled or if the composition of the portfolio has changed over time, then possibility of parameter error must be considered. Parameter errors could be material if the Gamma distribution for a heterogeneous portfolio is scaled down to the size of an individual contract in the portfolio. *In the case of an extremely heterogeneous portfolio, scaling down the distribution to model an the loss ratio distribution for individual constituent (contract, policy, or treaty) should be performed only as a last resort.*

To the extent that the random loss ratio fluctuations of different exposure units within a single portfolio are correlated, those correlations will be reflected in the volatility of the historical experience. Depending upon the application, correlations within a portfolio may or may not introduce parameter error. There will be no parameter error if the application involves modeling the aggregate loss ratio distribution of a portfolio that is similar to the portfolios that generated the historical data. If the application involves using the historical portfolio experience to model the aggregate loss ratio distribution of a single exposure unit, correlations within the portfolio will result in an inappropriate model; in particular, the model variance may be misstated.

Refining the criteria that define a portfolio can often significantly reduce correlations within a portfolio. For example, one could exclude the reflection of any exposure to catastrophe loss by eliminating catastrophe losses from the data and modeling potential catastrophe losses separately. It is common to treat catastrophe losses (and the corresponding exposure) separately when testing loss reserve adequacy or when pricing a cohort of policies.

Likewise, increasing the homogeneity of the portfolio might reduce correlations that result from the manner in which exposure was quantified. For example, a portfolio consisting of a mixture of $500,000 excess of $500,000 loss reinsurance treaties together with $2,500,000 excess of $500,000 reinsurance treaties may have internal correlations simply by virtue of the proxy in terms of which exposure units are measured. Treaties with a $2,500,000 limit clearly represent more exposure than those with a $500,000 limit. If multiple exposure units are assigned to treaties with the larger limit, correlations will be introduced (between the exposure units assigned to a single treaty). Forming two separate portfolios will allow each type of treaty to be treated as a single exposure unit. Such a separation is not unique to the modeling process. It might also be prudent when attempting to model loss development patterns.

Even if portfolio criteria can be suitably refined without sacrificing predictive credibility, some sources of parameter error will remain. One significant source lies in the method, and another lies in the data itself.

The methodology involved determining the random component, R_t, by restating the actual loss ratios, ILR_t, to an "as if common point in time basis" and then dividing the restated loss ratios by the mean of the historical loss ratios. It was argued that the only reason why the restated historical loss ratios differ is the presence of random fluctuations. Therefore, it followed that the division isolated the random component. Strictly speaking, this is true only if the restated loss ratios were divided by the *population* mean. The method involves dividing by an *estimate* of the population mean, not the *true* population mean. The estimate is, itself, a random variable.

From Appendix B, we see that the estimator, m', is Normally distributed with a mean equal to the population mean and a variance equal to the population variance divided by the size of the super sample (N, the sum of all of the exposure units when all of the samples are combined into a single sample). The result of the division process is not R but rather, it is a stochastic variable, Z. Z is equal to the quotient of two Normally distributed stochastic variables, the *ILR* (which is distributed $N[\mu, \sigma^2]$) and the estimator of μ (which is distributed $N[\mu, \sigma^2/N]$). The distribution of the quotient of two Normally distributed variables is a Cauchy distribution, not a Normal distribution. For a sufficiently large number of years and for sufficiently large portfolios, N will be very large and σ^2/N will be vanishingly small. As a result, the estimator for μ can be treated as if it were not a random variable. Restricting the application of the method to cases where the super sample (all individual samples combined to form a single gigantic sample) size is very large avoids the problem of dealing with a distribution such as the Cauchy distribution, which does not have any moments.

Using loss ratio estimates that arise from loss reserve adequacy tests may introduce an additional source of parameter error. When the method was described, it was assumed that the ultimate loss ratios for each of the historical years were known with certainty. In practice, some of the more recent year's loss ratios will be estimates as of a particular valuation date. To the extent that a Loss

Ratio or Bornhuetter/Ferguson methodology was used to estimate the ultimate loss ratios, random deviations of the more recent loss ratios from the expected will be tempered by the estimates. The reserving methodology automatically reduces the variance of the estimates. As a result, the variance of R_t will be understated. On the other hand, a projection methodology could result in an overstatement of the true variance. Such data induced bias can be detected by applying the methodology to successively shorter historical periods (e.g., 1969-1997, 1969-1996, 1969-1995, etc.) to see if the resulting σ^2 exhibits a constant trend. If σ^2 consistently increases as more recent years are eliminated, then there is evidence that the more recent estimates may be masking the true variance. On the other hand, a decreasing σ^2 would be evidence that the reserving methodology is introducing a misleading indication of actual loss ratio volatility.

<u>*Summary*</u>

The parameters of the aggregate loss ratio distribution corresponding to a large portfolio of identical and independent exposure units can be determined by examining the historical loss experience of portfolios made up of the same type of exposure units. The methodology of time series decomposition allows the historical experience to be restated to a common point in time, making it possible to consider the experience from different years to be equivalent to taking multiple samples from a single year. Even if the *number* of exposure units (i.e., the *size* of the portfolios) varies from historical portfolio to portfolio, a generalized form of the estimators for the population parameters, μ and σ^2, make it possible to model the aggregate loss distribution in terms of a Normal distribution with size dependent parameters.

While a Normal model with size dependent parameters performs satisfactorily for large portfolios, it does not produce a realistic distribution for smaller aggregations of exposure (e.g., small portfolios or individual reinsurance contracts). The Gamma distribution was proposed as an alternative to the Normal distribution for several reasons. It is approximately Normal in the limit as the number of independent exposure units increases without bound. For large portfolios, the two distributions are virtually indistinguishable, making it possible to smoothly make the transition from the Normal distribution produced by application of the Central Limit Theorem to the almost identical Gamma distribution. Unlike the Normal distribution, the Gamma distribution has a divisibility property that allows it to be size transformed in a precise manner. Aside from this attractive, but purely mathematical feature of the Gamma distribution, Hewitt previously reported that the Gamma distribution produces results that are consistent with reality, especially when compared to the Table M tabulation of Workers Compensation loss ratios.

Foot Notes

1. Strictly speaking, the life of a portfolio of exposure units consists of the period in which the exposure units are in force together with the time over which losses that arise from the portfolio run off. Under this definition of the life of a portfolio, the renewal of a group of contracts (primary policies, reinsurance treaties, etc.) results in the formation of a new portfolio. In another sense, if all of the contracts renew, then the new portfolio can be considered to be the second year of the original portfolio. In this paper, the context will make clear which sense of the word "portfolio" applies.

2. Portfolios, as defined, cannot actually *change size*. In more precise terms, the historical data consists of the experience of many different portfolios, one for each of the historical contract years. If the same number of identical and independent exposure units make up all of the portfolios, they are all of the same size. If the number varies from year to year, the historical data reflects the historical exposure of portfolios of different size. If the exposure units are indistinguishable, these different portfolios can be treated as if they were a single portfolio with a changing size.

3. The second and third assumptions, while closely related, are not the same. It is possible for the loss outcomes to be inflation sensitive, making the population of possible outcomes time dependent without any corresponding change in the measured size of the portfolio. Conversely, an inflation sensitive exposure base, such as Workers Compensation Payroll, could result in a changing "yardstick" being used to measure portfolio size without any

corresponding change in the possible pure premium outcomes (e.g., Workers Compenstion indemnity loss pure premium for which the losses and exposure would both change at the same rate, leaving the ratio invariant). Because the two inflation sensitivities (outcomes and size) can be different, the restatement of the population of outcomes and measurement of the portfolio size must be addressed separately.

4. To be more precise, once the loss ratios ratios have been restated to reflect a common level of rate adequacy, they may be considered to be multiple samples drawn from the same population. Depending on how the methodology is to be applied, an additional restatement to some particular time might be required. This restatement will involve multiplication of all of the loss ratios by the same restatement factor (taking them all from cycle point 100, and the midpoint of 1997 to the desired time and cycle point). Multiplying all of the loss ratios by the same factor introduces nothing new; it is simply a change in scale.

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Appendix A: Distribution of Portfolio Pure Premium, Mean and Variance

Consider a portfolio consisting of the loss experience of many independent and identical exposure units. These exposure units may arise from reinsuring several different client companies. Let l_{ij} be the loss outcome for the i^{th} exposure unit arising from the j^{th} client. Further, let N_j be the number of independent exposure units arising from the j_{th} client. Since the exposure units are identical and independent, we may assume that

$$E[l_{ij}] = \mu$$

and

$$E[(l_{ij}-\mu)^2] = Var[l_{ij}] = \sigma^2,$$

both of which will be independent of i and j.

The loss outcome for the j^{th} client is given by

$$L_j = \sum_{i=1}^{N_j} l_{ij}$$

and the corresponding account pure premium, pp_j, is given by L_j/N_j.

- It can be demonstrated that the expected account pure premium is equal to the expected pure premium of the individual exposure units, μ.

$$E[pp_j] = E[N_j^{-1} \sum_{i=1}^{N_j} l_{ij}]$$

$$= N_j^{-1} \sum_{i=1}^{N_j} E[l_{ij}]$$

$$= \mu$$

The variance of the account pure premium is equal to the variance of the individual exposure unit pure premium divided by the number of exposure units making up the account.

$$Var(pp_j) = Var[L_j/N_j] =$$

$$= N_j^{-2} Var[L_j]$$

$$= N_j^{-2} \sum_{i=1}^{N_j} Var[l_{ij}]$$

$$= \sigma^2 / N_j$$

These results confirm the notion that all clients have the same pure premium, regardless of size but that the larger clients have less volatility (i.e., the variance of the pure premiums is smaller).

Now, suppose that only a portion, S_j, $(0 < S_j \leq 1)$ of a client's loss is reflected in the portfolio. In other words, for every loss l_{ij} that is incurred, only $S_j l_{ij}$ is reflected in the portfolio. The corresponding exposure contribution is given by $S_j N_j$. An important conclusion that can be drawn is that both the expected pure premium of a share and the variance of the share's pure premium are independent

$$E[pp_j] = (S_j N_j)^{-1} \sum_{i=1}^{N_j} E[S_j l_{ij}]$$

$$= N_j^{-1} \sum_{i=1}^{N_j} E[l_{ij}]$$

$$= \mu,$$

The proof is trivial, since the pure premium is independent of the share,

and, similarly,

$$Var[pp_j] = Var[(S_j N_j)^{-1} \sum_{i=1}^{N_j} S_j l_{ij}]$$

$$= Var[N_j^{-1} \sum_{i=1}^{N_j} l_{ij}]$$

$$= \sigma^2 / N_j.$$

Now, consider a portfolio that is made up of shares of *m* different contracts. The portfolio loss is

$$L = \sum_{j=1}^{m} \sum_{i=1}^{N_j} S_j l_{ij}$$

The corresponding portfolio exposure is given by

$$N = \sum_{j=1}^{m} S_j N_j$$

The portfolio pure premium is the ratio of the two, L/N,

$$pp_{portfolio} = \frac{\sum_{j=1}^{m} S_j \sum_{i=1}^{N_j} l_{ij}}{\sum_{k=1}^{m} S_k N_k}$$

The expected portfolio pure premium is μ,

$$E[pp_{portfolio}] = \frac{\sum_{j=1}^{m} S_j \sum_{i=1}^{N_j} E[l_{ij}]}{\sum_{k=1}^{m} S_k N_k}$$

$$= \frac{\sum_{j=1}^{m} S_j \sum_{i=1}^{N_j} \mu}{\sum_{k=1}^{m} S_k N_k} =$$

$$= \frac{\sum_{j=1}^{m} S_j N_j \mu}{\sum_{k=1}^{m} S_k N_k}$$

$$= \mu.$$

The percentage of the portfolio exposure, n, contributed by the j^{th} contract is given by

$$n_j \equiv \frac{S_j N_j}{\sum_{k=1}^{m} S_k N_k}$$

In terms of n, the variance of the portfolio pure premium can be written as

$$Var[pp_{portfolio}] = Var \sum_{j=1}^{m} \sum_{i=1}^{N_j} n_j \left[l_{ij} / N_j \right]$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{N_j} \left[n_j / N_j \right]^2 \sigma^2$$

$$= \sum_{j=1}^{m} \left(n_j^2 / N_j \right) \sigma^2$$

$$= \sigma^2 / N_{portfolio}$$

where

Consider two examples:

Two identical clients each have an exposure equal to N. A reinsurance portfolio of size ½N is formed in two ways.

1. A 25% share of each contract is written. Since each contract makes up half of the portfolio, $n_1 = n_2 = ½$. $N_{portfolio} = 2N$. In other words, taking equal shares of two identical contracts produces the same variance as taking a share of a single contract for a client with twice the exposure.

$$N_{portfolio} \equiv \left(\sum_{j=1}^{m} \left(n_j^2 / N_j \right) \right)^{-1}.$$

2. A 50% share of a single contract is written. This produces the same amount of premium and expected loss but, $n_1 = 1$ and $n_2 = 0$. Therefore, $N_{portfolio} = N$ which is consistent with the fact that the contract variance is independent of the share of the contract that is taken.

As would be expected, the portfolio variance depends not only upon how large the portfolio is but also the manner in which the portfolio is formed. All else being equal, the portfolio that is made up of small shares of many large contracts will have a smaller pure premium variance than one of equal size that consists of large shares of small contracts. This is nothing more than an application of the concept of *spread*.

Appendix B: Generalized Central Limit Theorem

The Central Limit Theorem is appropriate for situations in which T independent samples, each of size N, are drawn (either with replacement or from a population that is so large that that act of drawing the sample has virtually no effect upon the probabilities affecting subsequently drawn samples). If N is large, then the Central Limit Theorem states that the sample means are distributed Normally with a mean equal to the population mean and a variance equal to the population variance divided by N.

Symbolically,

> let l_{ij} be the i^{th} element in the j^{th} sample,
>
> μ be the population mean, and
>
> σ^2 be the population variance.

In terms of the individual, independent sample elements, the mean of the j^{th} sample, $<l>_j$, is given by

$$\langle l\rangle_j = N^{-1}\sum_{i=1}^{N} l_{ij}$$

and the mean of the sample means is given by

$$\langle l\rangle = T^{-1}\sum_{j=1}^{T}\langle l\rangle_j$$

The variance of the sample means is given by

$$Var(\langle l\rangle_j) = s^2 = (T-1)^{-1}\sum_{j=1}^{T}\left(\langle l\rangle_j - \langle l\rangle\right)^2$$

If N is very large, the Central Limit Theorem states that $\langle l \rangle_j$ is Normally distributed

$$N(\mu, \sigma^2/N),$$

and that $\langle l \rangle$ is an unbiased estimator of μ and Ns^2 is an unbiased estimator of the population variance. Note that there are no restrictions regarding how the l_{ij} are distributed or even that they be independent (see DeGroot [9]).

If all that is desired is to use the T samples to estimate the distribution of the mean of an additional sample of size M (large, but not necessarily equal to N), it is necessary to know neither the population variance, N nor M. Knowledge of the variance of the sample means, s^2 and the *ratio* of N to M is all that is necessary. The distribution of the mean of the $T+1^{st}$ sample will be Normal with mean μ, estimated by $\langle l \rangle$, and variance σ^2/M, estimated by $s^2(N/M)$.

In the Generalized version of the Central Limit Theorem, the condition that the sample sizes all be equal to N is relaxed. T samples of sizes $\{N_j\}$, where Nj is the size of the j^{th} sample, are drawn from a population (with replacement). Alternatively, one can denote the size of the jth sample by $N_1\alpha_j$, where α_j is the *ratio* of the size of the j^{th} sample to that of the first sample drawn.

The conclusion of the Central Limit theorem is unchanged. The distribution of the sample mean of the $T+1^{st}$ sample will be distributed Normally with a mean equal to the population mean and a variance equal to the population variance divided by the size of the $T+1^{st}$ sample, M. This conclusion is still a statement relating the distribution of multiple samples of size M to statistics associated with the entire population, but the estimators of μ and σ^2 need to be modified when the historical samples are not all of the same size.

When the samples differ in size, the **estimator of the population mean** becomes the weighted mean of the sample means,

$$\langle l\rangle = \sum_{j=1}^{T} N_j \langle l\rangle_j \Big/ \sum_{j=1}^{T} N_j$$

$$= \sum_{j=1}^{T} \alpha_j \langle l\rangle_j \Big/ \sum_{j=1}^{T} \alpha_j$$

$$= \sum_{j=1}^{T} \sum_{i=1}^{N_j} l_{ij} \Big/ \sum_{j=1}^{T} N_j$$

When all of the samples are the same size, all of the α's are equal to one and the estimator becomes the same as it was for the Central Limit Theorem.

The estimator, <*l*>, is itself a random variable. If T samples are drawn and the estimator is determined, it will not be exactly the same estimator that would be determined if another set of T samples were drawn. As a result, <*l*> has a distribution.

If the sample sizes, N_j, are large,

- the estimator, <*l*>, will be Normally distributed with mean μ and variance

$$Var(\langle l\rangle) = \sigma^2 / \sum_{j=1}^{T} N_j$$

- the estimator is unbiased, i.e., the expectation of the estimator, $E[<l>] = \mu$, and
- the estimator has minimum variance for all estimators that are linear functions of the sample means.

To prove the first assertion (that the estimator is Normally distributed with the indicated mean and variance), observe that the last form of the definition is simply a sum over all of the observations, regardless of which sample gives rise to the observation. It is identical to the mean of a single, super, sample made up of all T of the individual samples. Thought of in that way, the conventional Central Limit Theorem states that the mean of this sample of size

$$N = \sum_{j=1}^{T} N_j$$

is distributed Normally with mean μ, and variance σ^2/N. This proves the first assertion regarding $<l>$ and the other two follow immediately from the conventional Central Limit Theorem.

The generalized **estimator for the population variance**, σ^2, is given by Ns^2 where

$$s^2 = (T-1)^{-1}\left(\sum_{j=1}^{T} N_j \left(\langle l\rangle_j - \langle l\rangle\right)^2 \Big/ \sum_{j=1}^{T} N_j\right)$$

$$= (T-1)^{-1}\left(N_1 \sum_{j=1}^{T} \alpha_j \left(\langle l\rangle_j - \langle l\rangle\right)^2 \Big/ N_1 \sum_{j=1}^{T} \alpha_j\right)$$

and N is the size of the super sample (i.e., all T samples combined and considered as a single sample).

Ns^2 is an unbiased estimator of σ^2. To show this, begin by considering the expectation of a single term in the numerator,

$$E\left(\langle l\rangle_1-\langle l\rangle\right)^2 = E\left(\frac{\sum_{j=1}^{T} N_j\langle l\rangle_j}{\sum_{j=1}^{T} N_j} - \frac{\sum_{j=1}^{T} N_j}{\sum_{j=1}^{T} N_j}\langle l\rangle_1\right)^2$$

$$=\left(\sum_{j=1}^{T} N_j\right)^{-2} E\left(\sum_{j=2}^{T} N_j\langle l\rangle_j-\left(\sum_{j=2}^{T} N_j\right)\langle l\rangle_1\right)^2$$

$$=\left(\sum_{j=1}^{T} N_j\right)^{-2} E\left(\sum_{j=2}^{T} N_j\left(\langle l\rangle_j-\langle l\rangle_1\right)\right)^2$$

$$=\left(\sum_{j=1}^{T} N_j\right)^{-2} Var\left(\sum_{j=2}^{T} N_j\left(\langle l\rangle_j-\langle l\rangle_1\right)\right)$$

where the last line follows from the fact that the expectation for all of the sample means are equal to μ, hence the mean of the quantity $\langle l\rangle_j-\langle l\rangle_1$ is zero.

Continuing,

$$E\left(\langle l\rangle_1-\langle l\rangle\right)^2=\left(\sum_{j=1}^{T}N_j\right)^{-2}Var\left(\sum_{j=2}^{T}N_j\left(\langle l\rangle_j-\langle l\rangle_1\right)\right)$$

$$=\left(\sum_{j=1}^{T}N_j\right)^{-2}Var\left(\sum_{j=2}^{T}N_j\langle l\rangle_j-\sum_{j=2}^{T}N_j\langle l\rangle_1\right)$$

$$=\left(\sum_{j=1}^{T}N_j\right)^{-2}\left(\sum_{j=2}^{T}N_j^2Var\left(\langle l\rangle_j\right)+\left(\sum_{j=2}^{T}N_j\right)^2Var\left(\langle l\rangle_1\right)\right)$$

From the Generalized Central Limit Theorem, $Var(\langle l\rangle_j) = \sigma^2/N_j$ so,

$$\left(\sum_{j=1}^{T}N_j\right)^{-2}\left(\sum_{j=2}^{T}N_j^2Var\left(\langle l\rangle_j\right)+\sum_{j=2}^{T}N_j^2Var\left(\langle l\rangle_1\right)\right)=\sigma^2\left(\sum_{j=1}^{T}N_j\right)^{-2}\left(\sum_{j=2}^{T}N_j+\left(\sum_{j=2}^{T}N_j\right)^2/N_1\right)$$

$$=\sigma^2\left(\sum_{j=1}^{T}N_j\right)^{-1}\left(\frac{\sum_{j=2}^{T}N_j}{N_1}\right)$$

$$=\sigma^2\left(\sum_{j=1}^{T}N_j\right)^{-1}\left(\frac{\sum_{j=1}^{T}N_j-N_1}{N_1}\right)$$

$$=\sigma^2\left(N_1^{-1}-\left(\sum_{j=1}^{T}N_j\right)^{-1}\right)$$

While the derivation awarded special status to the first sample, any one of the samples could have been given special treatment. In general, then,

$$E\left(\langle l\rangle_k - \langle l\rangle\right)^2 = \sigma^2\left(N_k^{-1} - \left(\sum_{j=1}^{T} N_j\right)^{-1}\right).$$

The expected value of the estimator of the variance, $NE(s^2)$, is given by

$$NE(s^2) = (T-1)^{-1}\sum_{k=1}^{T} N_j E\left(\langle l\rangle_k - \langle l\rangle\right)^2$$

$$= \sigma^2 (T-1)^{-1}\sum_{k=1}^{T} N_k\left(N_k^{-1} - \left(\sum_{j=1}^{T} N_j\right)^{-1}\right)$$

$$= \sigma^2$$

Because the expected value of the population variance estimator is the population variance, the estimator is unbiased.

The conclusion of all this is that the Central Limit Theorem can be used when samples of different sizes are drawn from a single population. The estimators of the population mean and variance are similar to those used when all samples are of the same size except that weighted averages must be taken. The weights are the sample sizes. Note that the estimators when all samples are the same size are special cases of the more general expressions.

In the case when l_{ij} is a loss outcome for the i^{th} exposure unit in the j^{th} sample, <I>$_j$ is the sample pure premium. In terms of portfolio pure premiums, the estimators become:

$$Estimator(pp_{population}) = m' = \sum_{j=1}^{T} \alpha_j pp_j \Big/ \sum_{j=1}^{T} \alpha_j$$

$$Estimator(Var[pp_{population}]) = Ns^2 = N_1 (T-1)^{-1} \sum_{j=1}^{T} \alpha_j (pp_j - m')^2$$

Loss Ratio Series Decomposition

Exhibit I

Year	ILR (1)	Isolated TxC (2)	Isolated T (3)	Isolated C (4)	Trend Index (5)	Cycle Index (6)	Restated ILR (7)
1960	29.9%						
1961	39.9%						
1962	38.0%	34.1%	31.7%	1.07	103.96	107.50	69.85%
1963	32.0%	33.1%	32.3%	1.02	106.01	102.39	60.53%
1964	30.5%	31.4%	32.9%	0.95	108.09	95.44	60.75%
1965	25.0%	29.9%	33.6%	0.89	110.21	89.03	52.21%
1966	31.7%	31.2%	34.2%	0.91	112.37	91.19	63.45%
1967	30.3%	35.7%	34.9%	1.02	114.58	102.24	53.18%
1968	38.6%	37.7%	35.6%	1.06	116.83	105.86	64.15%
1969	52.9%	40.2%	36.3%	1.11	119.12	110.73	82.30%
1970	34.9%	40.1%	37.0%	1.08	121.46	108.44	54.39%
1971	44.2%	38.5%	37.7%	1.02	123.84	102.03	71.83%
1972	30.1%	35.7%	38.5%	0.93	126.27	92.87	52.62%
1973	30.5%	37.0%	39.2%	0.94	128.75	94.38	51.50%
1974	39.0%	38.0%	40.0%	0.95	131.28	95.02	64.24%
1975	41.4%	41.7%	40.8%	1.02	133.86	102.23	62.06%
1976	49.1%	46.7%	41.6%	1.12	136.48	112.32	65.78%
1977	48.5%	48.3%	42.4%	1.14	139.16	113.96	62.77%
1978	55.6%	46.2%	43.2%	1.07	141.89	106.84	75.24%
1979	47.1%	42.5%	44.1%	0.96	144.68	96.30	69.41%
1980	30.7%	41.9%	45.0%	0.93	147.52	93.14	45.88%
1981	30.4%	41.5%	45.8%	0.90	150.42	90.49	45.88%
1982	45.6%	44.5%	46.7%	0.95	153.37	95.17	64.08%
1983	53.6%	49.1%	47.7%	1.03	156.38	103.02	68.30%
1984	62.1%	52.9%	48.6%	1.09	159.45	108.97	73.40%
1985	53.8%	52.4%	49.5%	1.06	162.58	105.69	64.24%
1986	49.7%	48.6%	50.5%	0.96	165.77	96.30	63.90%
1987	42.6%	44.3%	51.5%	0.86	169.02	85.98	60.22%
1988	35.0%	41.6%	52.5%	0.79	172.34	79.24	52.65%
1989	40.3%	44.9%	53.5%	0.84	175.73	83.85	56.17%
1990	40.4%	52.1%	54.6%	0.95	179.18	95.41	48.52%
1991	66.1%	60.8%	55.7%	1.09	182.69	109.17	68.07%
1992	78.6%	63.8%	56.8%	1.12	186.28	112.47	77.01%
1993	78.4%	66.6%	57.9%	1.15	189.94	115.06	73.69%
1994	55.6%	63.1%	59.0%	1.07	193.66	107.00	55.11%
1995	54.2%	55.7%	60.2%	0.93	197.47	92.61	60.83%
1996	48.9%		61.4%		201.34		
1997	41.5%		62.6%		205.29		

(1) Historical Data

(2) Five year moving average of historical data

(3) Model of trend, fit to the isolated TxC of column (2). (3) $=30.5*(1.0196)^{(year-1960)}$

(4) Implied Cycle, TxC/T = (2)/(3)

(5) $=100*(1.0196)^{(year-1960)}$

(6) =100*(4)

(7) =(1)year*[(5)1997/(5)year]*[100/(6)year]

Portfolio Size

Exhibit II

Contract Year (1)	Historical Premium (5)	Isolated TxC (7)	LossIndex (9)	ΔTxC (8)	On Level Factor: adequacy (10)	On Level Factor: Loss (11)	On Level Factor: Composite (12)	Restated Premium (13)	"N" (14)
1960	400.00	68.8%	100.00	1.000	1.088	34.004	31.261	12,504.46	100.00
1961	412.00	73.4%	110.00	1.068	1.019	30.913	30.351	12,504.46	100.00
1962	424.36	78.4%	121.00	1.068	1.019	28.102	27.591	11,708.72	93.64
1963	437.09	83.7%	133.10	1.068	1.019	25.548	25.083	10,963.62	87.68
1964	450.20	89.4%	146.41	1.068	1.019	23.225	22.803	10,265.94	82.10
1965	463.71	95.5%	161.05	1.068	1.019	21.114	20.730	9,612.65	76.87
1966	477.62	102.0%	177.16	1.068	1.019	19.194	18.845	9,000.94	71.98
1967	501.79	108.9%	194.87	1.068	1.019	17.449	17.132	8,596.71	68.75
1968	526.98	116.3%	214.36	1.068	1.019	15.863	15.575	8,207.48	65.64
1969	553.22	124.2%	235.79	1.068	1.019	14.421	14.159	7,832.98	62.64
1970	580.57	132.7%	259.37	1.068	1.019	13.110	12.872	7,472.91	59.76
1971	609.06	141.7%	285.31	1.068	1.019	11.918	11.701	7,126.94	57.00
1972	638.74	151.3%	313.84	1.068	1.019	10.835	10.638	6,794.74	54.34
1973	669.65	161.6%	345.23	1.068	1.019	9.850	9.671	6,475.96	51.79
1974	701.84	172.6%	379.75	1.068	1.019	8.954	8.791	6,170.24	49.34
1975	689.40	196.6%	417.72	1.139	0.955	8.140	8.525	5,877.20	47.00
1976	712.49	212.8%	459.50	1.082	1.005	7.400	7.364	5,246.69	41.96
1977	736.01	230.4%	505.45	1.083	1.005	6.727	6.696	4,928.10	39.41
1978	759.97	249.5%	555.99	1.083	1.005	6.116	6.088	4,626.76	37.00
1979	784.34	270.2%	611.59	1.083	1.004	5.560	5.536	4,341.90	34.72
1980	809.14	292.7%	672.75	1.083	1.004	5.054	5.033	4,072.77	32.57
1981	795.83	317.1%	740.02	1.083	1.004	4.595	4.577	3,642.38	29.13
1982	781.77	343.6%	814.03	1.084	1.004	4.177	4.162	3,253.42	26.02
1983	766.94	372.4%	895.43	1.084	1.004	3.797	3.784	2,902.16	23.21
1984	751.32	403.8%	984.97	1.084	1.003	3.452	3.441	2,585.19	20.67
1985	904.51	355.8%	1,083.47	0.881	1.235	3.138	2.542	2,299.38	18.39
1986	1240.04	304.0%	1,191.82	0.854	1.273	2.853	2.241	2,778.92	22.22
1987	1679.29	270.5%	1,311.00	0.890	1.222	2.594	2.122	3,563.72	28.50
1988	2242.17	247.6%	1,442.10	0.915	1.188	2.358	1.984	4,449.25	35.58
1989	2276.44	264.4%	1,586.31	1.068	1.019	2.144	2.105	4,791.03	38.31
1990	2311.24	286.5%	1,744.94	1.083	1.004	1.949	1.941	4,486.15	35.88
1991	2070.07	351.9%	1,919.43	1.228	0.886	1.772	2.000	4,140.66	33.11
1992	2096.63	382.2%	2,111.38	1.086	1.002	1.611	1.608	3,371.44	26.96
1993	2122.93	415.2%	2,322.52	1.086	1.001	1.464	1.462	3,104.28	24.83
1994	2148.92	451.2%	2,554.77	1.087	1.001	1.331	1.330	2,857.47	22.85
1995	2206.07	490.4%	2,810.24	1.087	1.001	1.210	1.209	2,667.61	21.33
1996	1912.86	533.3%	3,091.27	1.087	1.000	1.100	1.100	2,103.45	16.82
1997	1934.43	580.1%	3,400.39	1.088	1.000	1.000	1.000	1,934.43	15.47

Exhibit IIa displays the assumptions that underlie this exhibit. Column numbers on this exhibit refer to their position in Exhibit IIa.

(8) = 1.000 for 1960 and $(7)_{current}/(7)_{prior}$ for all other years. Note that an increasing TxC denotes rate adequacy slippage.

(10)current= (8)1997/(8)current

(11)current= (9)1997/(9)current

(12)=(11)/(10)

(13)=(5)*(12)

(14)=(13) times a factor that makes the first entry 100. [i.e., (14) is a rescaled version of (13)

Contract Year (1)	Exposure Units (2)	Pricing Exposure Base (3)	Historical Rate (4)	Historical Premium (5)	Historical Loss (6)	Isolated (TxC) (7)	ΔTxC (8)	LossIndex (9)	On Level Factor adequacy (10)	On Level Factor Loss (11)	On Level Factor Composite (12)	Restated Premium (13)	"N" (14)
1960	100	100.00	4.00	400.00	275.00	68.8%	1.000	100.00	1.088	34.004	31.261	12,504.46	100.00
1961	100	103.00	4.00	412.00	302.50	73.4%	1.068	110.00	1.019	30.913	30.351	12,504.46	100.00
1962	100	106.09	4.00	424.36	332.75	78.4%	1.068	121.00	1.019	28.102	27.591	11,708.72	93.64
1963	100	109.27	4.00	437.09	366.03	83.7%	1.068	133.10	1.019	25.548	25.083	10,963.62	87.68
1964	100	112.55	4.00	450.20	402.63	89.4%	1.068	146.41	1.019	23.225	22.803	10,265.94	82.10
1965	100	115.93	4.00	463.71	442.89	95.5%	1.068	161.05	1.019	21.114	20.730	9,612.65	76.87
1966	100	119.41	4.00	477.62	487.18	102.0%	1.068	177.16	1.019	19.194	18.845	9,000.94	71.98
1967	102	125.45	4.00	501.79	546.62	108.9%	1.068	194.87	1.019	17.449	17.132	8,596.71	68.75
1968	104	131.74	4.00	526.98	613.07	116.3%	1.068	214.36	1.019	15.863	15.575	8,207.48	65.64
1969	106	138.31	4.00	553.22	687.34	124.2%	1.068	235.79	1.019	14.421	14.159	7,832.98	62.64
1970	108	145.14	4.00	580.57	770.34	132.7%	1.068	259.37	1.019	13.110	12.872	7,472.91	59.76
1971	110	152.27	4.00	609.06	863.07	141.7%	1.068	285.31	1.019	11.918	11.701	7,126.94	57.00
1972	112	159.69	4.00	638.74	966.64	151.3%	1.068	313.84	1.019	10.835	10.638	6,794.74	54.34
1973	114	167.41	4.00	669.65	1082.29	161.6%	1.068	345.23	1.019	9.850	9.671	6,475.96	51.79
1974	116	175.46	4.00	701.84	1211.40	172.6%	1.068	379.75	1.019	8.954	8.791	6,170.24	49.34
1975	118	183.84	3.75	689.40	1355.52	196.6%	1.139	417.72	0.955	8.140	8.525	5,877.20	47.00
1976	120	192.56	3.70	712.49	1516.34	212.8%	1.082	459.50	1.005	7.400	7.364	5,246.69	41.96
1977	122	201.65	3.65	736.01	1695.77	230.4%	1.083	505.45	1.005	6.727	6.696	4,928.10	39.41
1978	124	211.10	3.60	759.97	1895.93	249.5%	1.083	555.99	1.005	6.116	6.088	4,626.76	37.00
1979	126	220.94	3.55	784.34	2119.16	270.2%	1.083	611.59	1.004	5.560	5.536	4,341.90	34.72
1980	128	231.18	3.50	809.14	2368.08	292.7%	1.083	672.75	1.004	5.054	5.033	4,072.77	32.57
1981	124	230.68	3.45	795.83	2523.49	317.1%	1.083	740.02	1.004	4.595	4.577	3,642.38	29.13
1982	120	229.93	3.40	781.77	2686.29	343.6%	1.084	814.03	1.004	4.177	4.162	3,253.42	26.02
1983	116	228.94	3.35	766.94	2856.42	372.4%	1.084	895.43	1.004	3.797	3.784	2,902.16	23.21
1984	112	227.67	3.30	751.32	3033.72	403.8%	1.084	984.97	1.003	3.452	3.441	2,585.19	20.67
1985	108	226.13	4.00	904.51	3217.91	355.8%	0.881	1,083.47	1.235	3.138	2.542	2,299.38	18.39
1986	115	248.01	5.00	1240.04	3769.12	304.0%	0.854	1,191.82	1.273	2.853	2.241	2,778.92	22.22
1987	126	279.88	6.00	1679.29	4542.61	270.5%	0.890	1,311.00	1.222	2.594	2.122	3,563.72	28.50
1988	140	320.31	7.00	2242.17	5552.08	247.6%	0.915	1,442.10	1.188	2.358	1.984	4,449.25	35.58
1989	138	325.21	7.00	2276.44	6020.04	264.4%	1.068	1,586.31	1.019	2.144	2.105	4,791.03	38.31
1990	138	334.96	6.90	2311.24	6622.05	286.5%	1.083	1,744.94	1.004	1.949	1.941	4,486.15	35.88
1991	138	345.01	6.00	2070.07	7284.25	351.9%	1.228	1,919.43	0.886	1.772	2.000	4,140.66	33.11
1992	138	355.36	5.90	2096.63	8012.68	382.2%	1.086	2,111.38	1.002	1.611	1.608	3,371.44	26.96
1993	138	366.02	5.80	2122.93	8813.95	415.2%	1.086	2,322.52	1.001	1.464	1.462	3,104.28	24.83
1994	138	377.00	5.70	2148.92	9695.34	451.2%	1.087	2,554.77	1.001	1.331	1.330	2,857.47	22.85
1995	140	393.94	5.60	2206.07	10819.44	490.4%	1.087	2,810.24	1.001	1.210	1.209	2,667.61	21.33
1996	120	347.79	5.50	1912.86	10201.18	533.3%	1.087	3,091.27	1.000	1.100	1.100	2,103.45	16.82
1997	120	358.23	5.40	1934.43	11221.30	580.1%	1.088	3,400.39	1.000	1.000	1.000	1,934.43	15.47

Application of the Generalized Central Limit Theorem

Exhibit III

Contract Year	Portfolio Premium	Relative Size, α_j	Restated *ILR*	Wt'd ILR	$(ILR-m)^2$	Wt'd square
(1)	(2)	(3)	(4)	(5)=(3)*(4)	(6)	(7)=(3)*(6)
1971	2,120,969	1.000	56.83%	56.83%	0.00346	0.00346
1972	2,911,088	0.956	59.44%	56.81%	0.00108	0.00103
1973	3,743,812	0.917	77.81%	71.35%	0.02278	0.02089
1974	5,013,270	0.910	87.11%	79.25%	0.05947	0.05410
1975	8,152,331	1.130	60.74%	68.64%	0.00039	0.00044
1976	14,309,591	1.438	48.75%	70.10%	0.01952	0.02807
1977	18,575,401	1.483	70.18%	104.08%	0.00557	0.00826
1978	21,062,760	1.758	56.90%	100.03%	0.00338	0.00595
1979	28,171,973	2.260	54.25%	122.59%	0.00717	0.01621
1980	25,932,066	2.205	60.83%	134.11%	0.00036	0.00079
1981	28,123,798	1.845	53.15%	98.07%	0.00916	0.01690
1982	23,631,441	1.295	94.12%	121.92%	0.09859	0.12771
1983	27,239,321	1.104	95.36%	105.25%	0.10654	0.11758
1984	42,815,673	1.506	77.93%	117.39%	0.02312	0.03483
1985	74,217,430	1.892	34.57%	65.40%	0.07922	0.14987
1986	79,620,062	1.580	30.00%	47.40%	0.10705	0.16914
1987	57,935,225	1.010	51.98%	52.51%	0.01154	0.01166
1988	69,423,482	1.925	59.50%	114.56%	0.00104	0.00199
1989	72,130,055	2.118	64.83%	137.31%	0.00044	0.00094
1990	69,647,663	1.910	72.05%	137.64%	0.00870	0.01662
1991	99,383,529	2.014	71.02%	143.03%	0.00690	0.01389
1992	93,804,805	1.745	76.72%	133.90%	0.01961	0.03422
1993	94,513,634	1.570	59.15%	92.86%	0.00127	0.00200
Total	960,479,379	35.57		2231.04%	0.59636	0.83655

Column (2) reflects historical data. The first entry is the total of Column (2) on Exhibit IIIb.
Column (3) is taken from Column (14) of Exhibit IIIc.
Column (4) is the last column on Exhibit IIIa.

Estimators when α = 1.000			
	ILR_t	===>	R_t
m' =	62.7%	===>	1.000
Var =	0.038	===>	0.097

Once the distribution for the incurred loss ratio has been determined, the distribution for the random component follow as a change of scale.

m' = Total (5)/Total(3)

$Var\ [ILR\ |\alpha=1.00]$ = Total(7)/ (number of years -1)

$Var[R\,|\alpha=1.00] = Var[ILR\,|\alpha=1.00]/m'^2$

Portfolio Estimators for *Var* at other α			
Restated $N_{portfolio}$	Size, α_j	$Var[ILR_t]$	$Var[R_t]$
224,626,501	1.000	0.038	0.097
336,939,752	1.500	0.025	0.064
449,253,002	2.000	0.019	0.048
561,566,253	2.500	0.015	0.039
673,879,504	3.000	0.013	0.032
786,192,754	3.500	0.011	0.028

$Var[ILR \text{ or } R\,|\alpha_k] = Var[ILR \text{ or } R\,|\alpha=1.00]/\alpha_k$

Contract Year	Est Ult Loss Ratio ILR (1)	5-year Moving Avg TxC (2)	Isolated T (3)	Isolated C (4)	Trend Index (5)	Cycle Index (6)	Restated ILR (7)
1969	367.46%						
1970	275.03%						
1971	227.32%	270.44%	252.7%	1.070	89.596	107.025	56.83%
1972	219.59%	249.81%	239.2%	1.044	84.807	104.444	59.44%
1973	262.81%	228.36%	226.4%	1.009	80.274	100.868	77.81%
1974	264.32%	205.18%	214.3%	0.957	75.984	95.745	87.11%
1975	167.78%	186.77%	202.8%	0.921	71.923	92.076	60.74%
1976	111.40%	154.51%	192.0%	0.805	68.078	80.472	48.75%
1977	127.55%	122.88%	181.7%	0.676	64.440	67.615	70.18%
1978	101.49%	120.60%	172.0%	0.701	60.995	70.106	56.90%
1979	106.19%	132.36%	162.8%	0.813	57.735	81.283	54.25%
1980	156.37%	173.81%	154.1%	1.128	54.649	112.770	60.83%
1981	170.17%	216.49%	145.9%	1.484	51.729	148.389	53.15%
1982	334.83%	240.54%	138.1%	1.742	48.964	174.188	94.12%
1983	314.87%	223.26%	130.7%	1.708	46.347	170.802	95.36%
1984	226.48%	196.51%	123.7%	1.588	43.869	158.829	77.93%
1985	69.96%	136.82%	117.1%	1.168	41.525	116.823	34.57%
1986	36.44%	82.12%	110.9%	0.741	39.305	74.078	30.00%
1987	36.34%	47.27%	104.9%	0.451	37.204	45.051	51.98%
1988	41.38%	47.03%	99.3%	0.473	35.216	47.348	59.50%
1989	52.24%	54.49%	94.0%	0.580	33.334	57.964	64.83%
1990	68.73%	64.50%	89.0%	0.725	31.552	72.486	72.05%
1991	73.77%	70.23%	84.2%	0.834	29.866	83.377	71.02%
1992	86.39%	76.13%	79.7%	0.955	28.269	95.489	76.72%
1993	70.02%	80.04%	75.5%	1.061	26.758	106.052	59.15%
1994	81.76%		71.4%		25.328		
1995	88.24%		67.6%		23.974		

(1) Historical Data

(2) Five year moving average of historical data

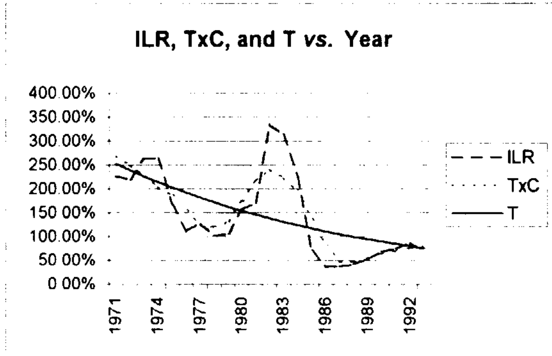
(3) Model of trend, fit to the isolated TxC of column (2). (3) $=30.5*(1.0196)^{(year-1969)}$

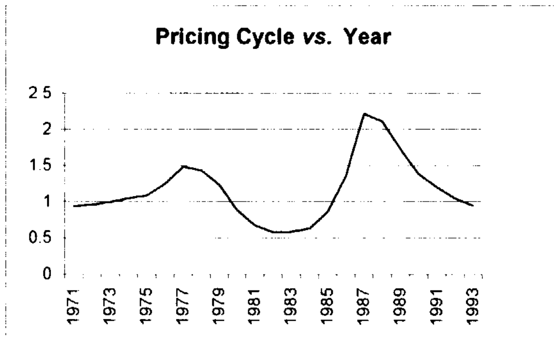
(4) Implied Cycle, TxC/T = (2)/(3)

(5) $=100*(1.0196)^{(year-1969)}$

(6) =100*(4)

(7) =(1)year*[(5)1997/(5)year]*[100/(6)year]





Least square error trend model: $TxC = 266.95*(1-0.0534)^{(yr-1969)}$

Effective Portfolio Size in 1971

Exhibit IIIb

Contract Year: 1971

Client (1)	Portfolio Premium (2)	Percent Cover (3)	Percent Taken (4)	100% Basis (5)	N_j (6)	n_j (7)	n_j^2/N_j (8)
1	57,803	90%	40.0%	160,565	160,565	0.0273	4.626E-09
2	66,072	100%	12.0%	550,602	550,602	0.0312	1.763E-09
3	62,945	100%	18.0%	349,695	349,695	0.0297	2.519E-09
4	55,266	100%	10.0%	552,655	552,655	0.0261	1.229E-09
5	54,678	100%	20.0%	273,389	273,389	0.0258	2.431E-09
6	74,950	100%	10.0%	749,495	749,495	0.0353	1.666E-09
7	73,365	100%	10.0%	733,647	733,647	0.0346	1.631E-09
8	52,155	100%	20.0%	260,777	260,777	0.0246	2.319E-09
9	44,847	100%	5.0%	896,948	896,948	0.0211	4.985E-10
10	72,712	100%	10.0%	727,119	727,119	0.0343	1.616E-09
11	71,820	100%	10.0%	718,201	718,201	0.0339	1.597E-09
12	43,423	100%	10.0%	434,231	434,231	0.0205	9.653E-10
13	53,135	100%	2.0%	2,656,766	2,656,766	0.0251	2.362E-10
14	43,794	100%	10.0%	437,935	437,935	0.0206	9.735E-10
15	71,257	100%	10.0%	712,568	712,568	0.0336	1.584E-09
16	48,097	100%	20.0%	240,487	240,487	0.0227	2.138E-09
17	50,748	100%	20.0%	253,740	253,740	0.0239	2.256E-09
18	62,084	100%	5.0%	1,241,672	1,241,672	0.0293	6.9E-10
19	70,094	85%	15.0%	549,756	549,756	0.0330	1.987E-09
20	82,714	100%	15.0%	551,428	551,428	0.0390	2.758E-09
21	57,830	100%	5.0%	1,156,599	1,156,599	0.0273	6.428E-10
22	71,779	100%	1.0%	7,177,880	7,177,880	0.0338	1.596E-10
23	63,955	100%	5.0%	1,279,091	1,279,091	0.0302	7.108E-10
24	47,139	100%	10.0%	471,388	471,388	0.0222	1.048E-09
25	48,772	100%	10.0%	487,719	487,719	0.0230	1.084E-09
26	74,841	100%	5.0%	1,496,824	1,496,824	0.0353	8.318E-10
27	74,084	100%	25.0%	296,335	296,335	0.0349	4.117E-09
28	44,138	100%	15.0%	294,251	294,251	0.0208	1.472E-09
29	43,995	100%	15.0%	293,300	293,300	0.0207	1.467E-09
30	48,169	100%	30.0%	160,565	160,565	0.0227	3.212E-09
31	76,883	100%	10.0%	768,832	768,832	0.0362	1.709E-09
32	70,557	100%	20.0%	352,785	352,785	0.0333	3.137E-09
33	73,538	50%	5.0%	2,941,520	2,941,520	0.0347	4.087E-10
34	51,082	100%	10.0%	510,818	510,818	0.0241	1.136E-09
35	62,250	100%	100.0%	62,250	62,250	0.0293	1.384E-08
Total	2,120,969		6.9%	30,801,830			7.046E-08

$N_{portfolio}$= 14,193,426

Column (1) is a client contract identifier. It could be the contract number or the name of the client.

Colums (2)-(4) relate to the premium ceded to this particular reinsurance company, the percentage of the total premium ceded to all reinsurers in total, and the percentage of the placement that this particular reinsurer accepted.

Column (5) = (2)/[(3)(4)]. It represents the premium that would have been written had 100% of the business been placed and had a single reinsurer accepted 100% of the placement.

Column (6)=(5)

Column (7) = (2)client/(2)total

Column (8) = $(7)^2/(6)$

$N_{portfolio}$=1/sum(8)

Restated Effective Portfolio Size

Contract Year	$N_{portfolio}$ Premium	Isolated TxC	LossIndex	ΔTxC	On Level Factor: Adequacy	On Level Factor: Loss	On Level Factor: Composite	Restated $N_{portfolio}$ Premium	α
(1)	(5)	(7)	(9)	(8)	(10)	(11)	(12)	(13)	(14)
1971	14,193,426	270.44%	11.412	1.000	1.051	16.637	15.826	224,626,501	1.000
1972	19,091,248	249.81%	14.835	0.924	1.138	12.799	11.246	214,697,754	0.956
1973	24,061,300	228.36%	19.286	0.914	1.150	9.845	8.561	205,979,386	0.917
1974	31,575,645	205.18%	25.072	0.898	1.170	7.573	6.472	204,364,734	0.910
1975	50,319,813	186.77%	32.593	0.910	1.155	5.825	5.044	253,819,452	1.130
1976	86,558,657	154.51%	40.038	0.827	1.271	4.742	3.732	323,011,228	1.438
1977	110,115,274	122.88%	47.483	0.795	1.322	3.999	3.025	333,106,893	1.483
1978	122,363,170	120.60%	54.928	0.981	1.071	3.457	3.227	394,867,336	1.758
1979	160,390,551	132.36%	62.633	1.097	0.958	3.031	3.165	507,575,310	2.260
1980	144,685,401	173.81%	69.296	1.313	0.801	2.740	3.423	495,208,996	2.205
1981	142,840,047	216.49%	77.522	1.246	0.844	2.449	2.902	414,489,368	1.845
1982	126,628,059	240.54%	87.333	1.111	0.946	2.174	2.298	290,970,632	1.295
1983	143,041,514	223.26%	96.721	0.928	1.133	1.963	1.733	247,910,720	1.104
1984	220,340,672	196.51%	103.511	0.880	1.194	1.834	1.536	338,395,876	1.506
1985	374,303,495	136.82%	110.761	0.696	1.510	1.714	1.135	424,927,861	1.892
1986	393,519,772	82.12%	120.196	0.600	1.751	1.580	0.902	354,908,674	1.580
1987	280,616,252	47.27%	128.558	0.576	1.826	1.477	0.809	226,940,464	1.010
1988	329,535,791	47.03%	136.905	0.995	1.057	1.387	1.312	432,472,550	1.925
1989	335,535,545	54.49%	147.594	1.159	0.907	1.286	1.418	475,778,788	2.118
1990	317,508,163	64.50%	158.173	1.184	0.888	1.200	1.352	429,136,831	1.910
1991	386,092,174	70.23%	167.834	1.089	0.966	1.131	1.172	452,362,632	2.014
1992	357,131,160	76.13%	178.354	1.084	0.970	1.065	1.098	392,039,124	1.745
1993	352,633,199	80.04%	189.866	1.051	1.000	1.000	1.000	352,633,199	1.570

Column numbers on this exhibit are consistent with those on Exhibit II

(5) = Historical portfolio premiums, Nportfolio, reflect treaty shares. The supporting detail for 1971 appears on Exhibit IIIb.

(7) Column (2) from Exhibit IIIa

(8) = 1.000 for 1971 and $(7)_{current}/(7)_{prior}$ for all other years. Note that an increasing TxC denotes rate adequacy slippage.

(9) Indices are consistent with Masterson Bodily Injury (other than automobile) indices.

(10)current= (8)1993/(8)current

(11)current= (9)1993/(9)current

(12)=(11)/(10)

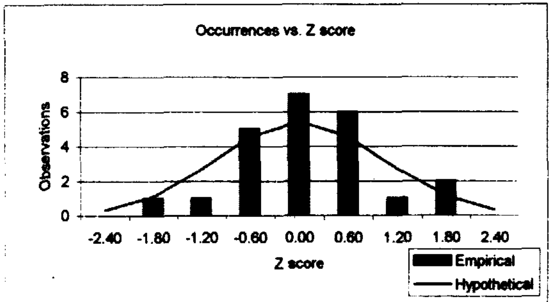
(13)=(5)*(12)

(14)=(13) times a factor that makes the first entry 1.000. [i.e., (14) is a rescaled version of (13)

χ^2 Test for Normal Distribution

Exhibit IIId

Contract Year	Portfolio Size	Adjusted ILR	R_t	Z_R
1971	1.000	56.83%	0.906	-0.302
1972	0.956	59.44%	0.948	-0.168
1973	0.917	77.81%	1.241	0.774
1974	0.910	87.11%	1.389	1.251
1975	1.130	60.74%	0.968	-0.101
1976	1.438	48.75%	0.777	-0.716
1977	1.483	70.18%	1.119	0.383
1978	1.758	56.90%	0.907	-0.298
1979	2.260	54.25%	0.865	-0.434
1980	2.205	60.83%	0.970	-0.097
1981	1.845	53.15%	0.847	-0.491
1982	1.295	94.12%	1.501	1.610
1983	1.104	95.36%	1.520	1.674
1984	1.506	77.93%	1.242	0.780
1985	1.892	34.57%	0.551	-1.443
1986	1.580	30.00%	0.478	-1.678
1987	1.010	51.98%	0.829	-0.551
1988	1.925	59.50%	0.949	-0.165
1989	2.118	64.83%	1.034	0.108
1990	1.910	72.05%	1.149	0.478
1991	2.014	71.02%	1.132	0.426
1992	1.745	76.72%	1.223	0.718
1993	1.570	59.15%	0.943	-0.183
Mean		62.7%	1.000	
Variance			0.097	
Std Dev			0.311	



$\chi 2$ Test

z>	z<=	Midpoint z	Expected Count if N(1.000, 0.311)	Empirical Count	Empirical Weight	Uncorrected Chi-square	Corrected* Chi-square
-2.70	-2.10	-2.40	0	0	0.000		
-2.10	-1.50	-1.80	1	1	1.580	<=	<=
-1.50	-0.90	-1.20	3	1	1.892	1.117	1.368
-0.90	-0.30	-0.60	5	5	7.553	0.044	0.001
-0.30	0.30	0.00	5	7	11.661	0.458	0.214
0.30	0.90	0.60	5	6	9.576	0.459	0.197
0.90	1.50	1.20	3	1	0.910	0.320	0.103
1.50	2.10	1.80	1	2	2.399	>=	>=
2.10	2.70	2.40	0	0	0.000		
		Total	23	23	35.571	0.422	0.266

Empirical normal

μ = 1.000

σ = 0.311

Degrees of freedom

23 = k = # of obversations

1 = *m = # of estimated parameters*

22 = degrees of freedom

Chi-square

Uncorrected $\chi 2$ = 0.422

Corrected $\chi 2$ = 0.266

χ 2 at .95 = 32.67

χ 2 at .99 = 41.40

$\chi 2$ at .05 = 11.59

$\chi 2$ at .01 = 8.90

Conclusion: at 95% - Fit is good ==> CANNOT REJECT null hypothesis

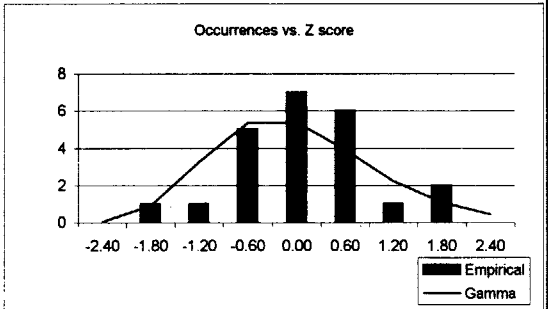
at 99% - Fit is good ==> CANNOT REJECT null hypothesis

* *Corrected to reflect the application of a continuous distribution to discrete data*

χ^2 Test for a Gamma Distribution

Exhibit IV

Contract Year	Portfolio Size	Adjusted ILR	R_t	z_R
1971	1.000	56.83%	0.906	-0.302
1972	0.956	59.44%	0.948	-0.168
1973	0.917	77.81%	1.241	0.774
1974	0.910	87.11%	1.389	1.251
1975	1.130	60.74%	0.968	-0.101
1976	1.438	48.75%	0.777	-0.716
1977	1.483	70.18%	1.119	0.383
1978	1.758	56.90%	0.907	-0.298
1979	2.260	54.25%	0.865	-0.434
1980	2.205	60.83%	0.970	-0.097
1981	1.845	53.15%	0.847	-0.491
1982	1.295	94.12%	1.501	1.610
1983	1.104	95.36%	1.520	1.674
1984	1.506	77.93%	1.242	0.780
1985	1.892	34.57%	0.551	-1.443
1986	1.580	30.00%	0.478	-1.678
1987	1.010	51.98%	0.829	-0.551
1988	1.925	59.50%	0.949	-0.165
1989	2.118	64.83%	1.034	0.108
1990	1.910	72.05%	1.149	0.478
1991	2.014	71.02%	1.132	0.426
1992	1.745	76.72%	1.223	0.718
1993	1.570	59.15%	0.943	-0.183
Mean		62.7%	1.000	
Variance			0.097	
Std Dev			0.311	



χ2 Test

z>	z<=	Midpoint z	Expected Count $\Gamma_{r,r}(x)$ (r= 10.345)	Empirical Count	Empirical Weight	Uncorrected Chi-square	Corrected* Chi-square
-2.70	-2.10	-2.40	0	0	0.000		
-2.10	-1.50	-1.80	1	1	1.580	<=	<=
-1.50	-0.90	-1.20	3	1	1.892	1.204	1.561
-0.90	-0.30	-0.60	5	5	7.553	0.025	0.003
-0.30	0.30	0.00	5	7	11.661	0.477	0.226
0.30	0.90	0.60	4	6	9.576	1.107	0.640
0.90	1.50	1.20	2	1	0.910	0.167	0.023
1.50	2.10	1.80	1	2	2.399	>=	>=
2.10	2.70	2.40	0	0	0.000		
		Total	23	23	35.571	0.593	0.402

Empirical Gamma
mean= 1.000
std dev = 0.311
r= 10.345

Degrees of freedom
23 = k = # of obversations
1 = *m = # of estimated parameters*
22 = degrees of freedom

Chi-square

Uncorrected χ2 = 0.593 | χ 2 at .95 = 32.67
Corrected χ2 = 0.402 | χ 2 at .99 = 41.40
χ2 at .05 = 11.59
χ2 at .01 = 8.90

Conclusion: at 95% - Fit is good ==> CANNOT REJECT null hypothesis

at 99% - Fit is good ==> CANNOT REJECT null hypothesis

* *Corrected to reflect the application of a continuous distribution to discrete data*

Exhibit Va

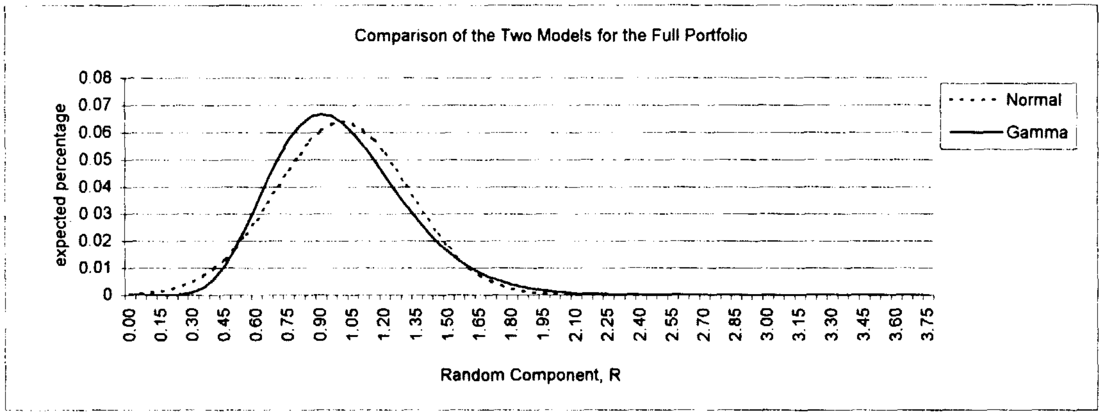
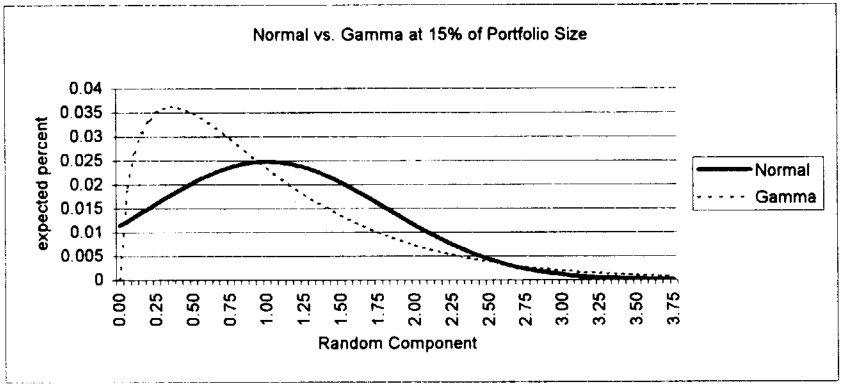

Comparison of the Two Models for the Full Portfolio
expected percentage
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.00
0.15
0.30
0.45
0.60
0.75
0.90
1.05
1.20
1.35
1.50
1.65
1.80
1.95
2.10
2.25
2.40
2.55
2.70
2.85
3.00
3.15
3.30
3.45
3.60
3.75
Random Component, R
Normal
Gamma

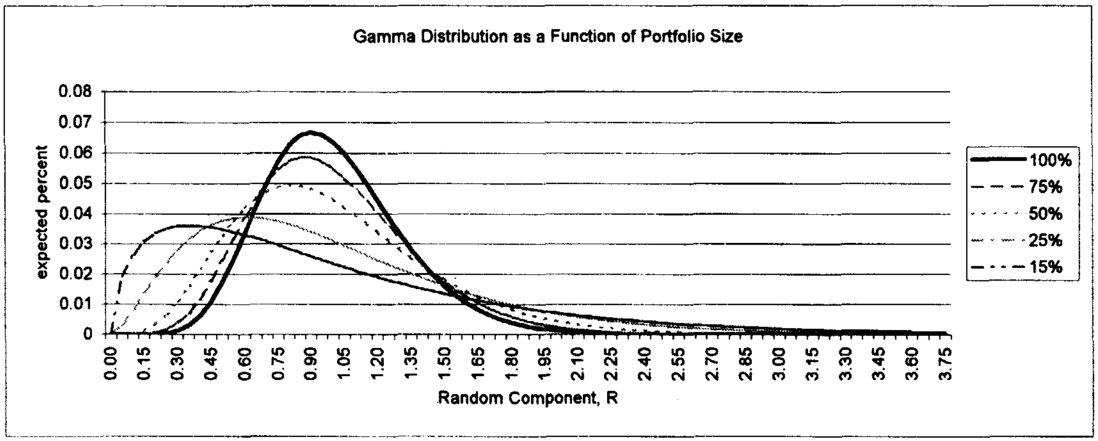
Exhibit Vb



Normal Parameters: μ =1.000, σ^2 = 0.097/0.15

Gamma parameter: r = 10.345 * 0.15

Exhibit Vc


Gamma Distribution as a Function of Portfolio Size
expected percent
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.00 0.15 0.30 0.45 0.60 0.75 0.90 1.05 1.20 1.35 1.50 1.65 1.80 1.95 2.10 2.25 2.40 2.55 2.70 2.85 3.00 3.15 3.30 3.45 3.60 3.75
Random Component, R
100%
75%
50%
25%
15%

Reserving Loss Sensitive Elements

Valuation	Loss Ratio			Return Premium Ratio				
Months	Reported	IBNR	Est Ult	Reported	Developed	*a priori*	Weight	Est Ult
	(1)	(2)	(3)	(4)	(5)	*(6)*	(7)	(8)
0	0.00%	60.00%	60.00%	0.00%	-15.00%	10.00%	0.00	10.00%
12	25.00%	55.00%	80.00%	30.00%	16.25%	5.00%	0.05	5.56%
24	40.00%	32.00%	72.00%	25.00%	17.00%	7.00%	0.20	9.00%
36	55.00%	18.00%	73.00%	10.00%	5.50%	6.75%	0.50	6.13%
48	70.00%	4.00%	74.00%	9.00%	8.00%	6.50%	0.70	7.55%
60	70.00%	5.00%	75.00%	8.00%	6.75%	6.25%	0.80	6.65%
72	75.00%	0.00%	75.00%	8.40%	8.40%	6.25%	0.90	8.19%
84	75.00%	0.00%	75.00%	8.40%	8.40%	6.25%	1.00	8.40%

(1) From company data
(2) Determined by means of standard loss reserve development techniques
(3) =(1)+(2)
(4) From company data
(5) =(4)-.25(2), reflects expected 25% sensitivity to future loss development.*
(6) =10%-.25[(3)-60%], reflects expected 25% sensitivity to changes in expected aggregate loss ratio*
(7) Illustrative weights increase over time from 0% initially to 100% by 84 months
(8) =(7)(5)+[1-(7)]*(6)*