TITLE: ADJUSTING SIZE OF LOSS DISTRIBUTIONS FOR TREND

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INTRODUCTION

Size of loss distributions have gained attention in recent years as being the basis upon which many ratemaking and pricing decisions are made. Their usefulness has been enhanced not only by the mathematical relevance that such distributions bring to various pricing problems, but also by the fact that automated equipment and detailed statistical plans have made it possible to gather data and to produce such detailed size of loss distributions. The actuary equipped with historical size of loss distributions is in a position to solve such problems as pricing an excess of loss treaty,[1] producing an increased limits table,[2] and calculating loss elimination ratios in order to calculate deductible discounts.[3]

The solutions to these pricing problems depend however on the ability of the actuary to forecast the shape of these size of loss distributions for the time period during which the determined price will be charged. Historical data are merely the starting point; estimating the adjustments needed to account for changes in the shape of the distributions is the cornerstone of creating rates that are reflective of an ever changing economic environment.

1 The methodology for pricing an excess of loss treaty based on the use of claim size distributions can be found in Patrik, G. and John, R., "Pricing Excess of Loss Casualty Working Cover Reinsurance Treaties," CAS 1980 Discussion Paper Program, Page 399.

2 See Miccolis, R.S., "On the Theory of Increased Limits and Excess of Loss Pricing," PCAS (1977), Page 27, and "Report of the Increased Limits Subcommittee: A Review of Increased Limits Ratemaking," ISO, 1980.

3 See Hurley, R. L., "Commercial Fire Insurance Ratemaking Procedures," PCAS (1973), Page 208.

The aim of this paper is to discuss the methods that can be used to forecast the shape of a size of loss distribution in some future time period based on its shape in the recent past. In other words we will examine the ways one can adjust historical loss distributions for "trend".

Section (1) of this paper provides a working definition and an example of the term "trend" as it is used in the ratemaking sense. Section (2) discusses trend in relation to size of loss distributions and indicates the basic model often used to adjust them for trend. Also included in this section are empirical methods that can be used to test the appropriateness of this model. Section (3) develops an alternate trend model and the paper concludes with Section (4) which exhibits the impact of this alternate model on increased limits pricing.

1. TREND DEFINED

Many definitions are offered for the word trend and they correspond to the various contexts in which the word is used. One may, for example, refer to the view that "opinions are trending towards conservatism", or of "the new fashion trends".

The definitions of trend, offered by Webster's New Collegiate Dictionary, that best explain its use in the ratemaking context however, are those that define trend as, "the general movement in the course of time of a statistically detectable change", and "a statistical curve reflecting such a change". The prior definition relates to the actuary's search for a detectable change in the historical data available to him, while the latter refers to what actuaries generally call "trend factors".

In developing a "statistical curve reflecting such a change", it is first necessary to examine a time series of internal insurance data and external economic data, and reasonable assumptions relating the two, before postulating that there exists a "general movement in the course of time" of some variable.

In examining the data, and plotting values over time, patterns in the data may be revealed and reasonable transformations of the data to fit these patterns may come to mind. Simultaneously, one should be examining reasonable causes for the patterns that appear.

For example, a series often treated in ratemaking is average claim size (severity). A priori, it is not unreasonable to postulate that in times of inflation a severity trend must exist. But what is the nature of this trend? Certainly in times of inflation, frequent upward price changes will affect the average size of a claim.

Masterson[4] illustrates this effect by viewing insurers as huge purchasers of goods and services whose prices are affected by inflation. Thus, for example, an auto accident leading to a property damage claim of $500 when it occurs in time t (goods and services in this example would include auto replacement parts, labor costs and loss adjustment expenses) may lead to a claim of $600 if it occurs in time t + 1. In like manner, the average claim size will also increase over time.

Claim sizes will be affected by phenomena other than just economic (or price) inflation. The growing inclination of juries to grant huge awards in liability cases has concurrently affected claim sizes in general. Also, changes in the scope and definition of tort liability (e.g. enterprise liability) may change the probability of insureds incurring huge claims in the future. These phenomena, often referred to as social inflation, may be difficult to translate into specific trends in average claim size, but certainly must be considered in developing a trend model.

One can either build trend models by trying to relate a series of internal insurance data to external data in an econometric model,[5] or more simply use one's expectations of the impact of

[4] Masterson, N. E., "Economic Factors in Liability and Property Insurance Claim Costs, 1935-1967," PCAS (1968), Page 61.

[5] For examples of econometric modeling applied to insurance ratemaking see Lommele, J.A. and Sturgis, R.W., "An Econometric Model of Workmens' Compensation," PCAS (1974), Page 170; Ferguson, R.E., "Trend Factors-- A Model Approach," CPCU Journal, Volume 31, #3; and Insurance Services Office bulletin #TS-CA-81-2, "Commercial Automobile Bodily Injury Severity Econometric Model."

these variables to help discern a pattern in the internal data. Of course, any pattern discovered over the historical period must be expected to continue in the future and tested and redefined as new data points become available.

In this paper, we shall not discuss how the actuary selects the trend in mean claim size. Rather, we will discuss how overall severity trends affect the shape of a size of loss distribution and how one may project this shape given some overall trend. There will be no further detailed discussion of how to ascertain the overall trend expected in the future or when historically discerned patterns should be judgmentally adjusted for external future influences. Nevertheless, these questions have major impact on what follows.

2. SIZE OF LOSS DISTRIBUTIONS AND TREND

The previous example has illustrated the concept of trend with respect to mean claim size. We now expand this concept to describe how one analyzes trend with respect to a size of loss distribution rather than just the mean of the distribution.

To begin, we define x_t to be a random variable representing the dollar amount of loss incurred by an insured given that an accident has occured in time period t.[6] We also

[6] Ordinarily, t is assumed to be a period of one year. By this definition x_t would represent claim size in calendar accident year t. One could analogously define x_t on a policy year basis.

define $F_t(x_t)$ to be the cumulative distribution of x_t. Throughout, when we refer to $F_t(x_t)$, we mean the cumulative probability distribution of underlying claim sizes; that is the claim amounts incurred by the insured, not the amount paid by the insurer. This latter amount is affected by policy limits as well as deductibles and co-insurance clauses. We will assume that data is available to allow one to estimate a suitable distribution type (i.e. Gamma, Log-Normal, Pareto, etc.) and parameters of $F_t(x_t)$ for several successive values of t.[7] For example, data may be available for t=1972, 1973,...1977. The goal is to develop a trend model that will allow us to project the distribution form and parameters of $F_t(x_t)$ in some future time period, say t=1981.

The general framework in which a trend model is implemented, consists of a transformation of the variable x_t from the historical to the forecast period. This transformation is established either intuitively by making some assumptions about how claim sizes will change over time, or empirically

[7] We will not discuss methods by which one can estimate the distribution type and parameters of $F_t(x_t)$ from the sample data available. This subject is thoroughly discussed in Patrik, G., "Estimating Casualty Insurance Loss Amount Distributions," PCAS (1980). Of particular interest, are the equations developed to estimate the parameters of the underlying claim size distribution where the loss amounts captured in the data base are capped by policy limits.

by analyzing the historical distributions. In the latter case, the transformation is constructed in the following way.

We define the transformation of x_t, from year t to year t+k as follows:

$$g_k(x_t) = F^{-1}_{t+k}(F(x_t))$$

where F^{-1}_{t+k} is the inverse of the cumulative probability distribution F_{t+k}.

In other words the probability of incurring a claim of size less than or equal to x_t at time t is the same as the probability of incurring a claim of size less than or equal to $g(x_t)$ at time t+k. Note that $g(x_t)$ is really dependent on k as well as t and x_t. k represents the length of the projection period. However, since the value of k can be derived from the context of the discussion we will omit it from the functional notation, $g(x_t)$.

With this notation, $g(x_t)$ is said to be the "trended value" of x_t. If a well defined transformation can be established for all values of x_t, then this transformation, combined with knowledge of $F_t(x_t)$ at time t, will lead to the precise definition of $F_{t+k}(x_{t+k})$ at time t+k. That is, given $g(x_t)$ we can solve for $g^{-1}(x_{t+k})$ and then define $F_{t+k}(x_{t+k})$ as:

$$F_{t+k}(x_{t+k})=F_t(g^{-1}(x_{t+k}))$$

Before discussing possible transformations of x_t it would be worthwhile to consider the various reasons why one should expect a trend in x_t to exist. Remember that x_t is a random variable representing claim size in time period t. As we pointed out earlier, trends in individual losses lead to trends in average claim sizes. But, what is the relationship of trend among the various claim sizes? Do inflationary forces (e.g. economic and social) affect all claim sizes equally, or is there some other pattern by size of loss that can be identified and projected? Let us first consider a simple model.

Simple (Uniform) Trend Model

A model that has been shown to be extremely simple to work with, is the transformation:

$$g(x_t) = a.x_t \qquad (1)$$

This model assumes that the economic and social forces creating a claim size of x_t during time t will lead to a claim size of ax_t at time t+k. The implication of this model is that all claim sizes will inflate by the factor "a" regardless of the initial size of the claim. Another way of saying this is that $g(x_t)/ x_t=a$ for all x_t. The factor "a" is the mean severity trend factor.

This model is simple in that it views inflation as affecting only the value of money; it does not anticipate an actual change in the shape of the claim size distribution. That is, given that overall costs as measured by the mean severity trend increase by "a", then it is assumed that each individual claim will also increase by "a". This model is simple to work with because for many distributions (e.g. the Log-Normal and the Pareto), the model implies that the distribution type will not change over time; that is, the trend in claim size can be reflected through a simple algebraic adjustment of the distribution parameters.

Miccolis develops this model to show how one can adjust increased limits factors for trend.[8] Finger employs this model to estimate basic limit trends, once total limits trend is known.[9] Ferguson illustrates the effects of inflation on reinsurers if this model of trend is assumed.[10]

If this model of trend is assumed appropriate, then the determination of the necessary adjustment of the size of loss distribution for trend reduces to estimating the value of "a". One way to do this empirically is to trace the trend in the mean of $f_t(x_t)$, the density function related to $F_t(x_t)$, by fitting an exponential curve to the means of $f_t(x_t)$ for several successive values of t. Since it is assumed that all claim sizes will be trending by the factor "a", the trend in the mean will also be equal to "a". Certainly, this estimate of "a" should be subjectively modified if the inflation rate is expected to be different during the projection period than it was during the historical period.

Testing the validity of $g(x_t)=ax_t$

While the trend model discussed above is convenient and simple to apply, the assumptions underlying the model should be tested and confirmed.

[8] Miccolis, R.S., op. cit.

[9] Finger, R.J., "A Note on Basic Limits Trend Factors," PCAS (1976), Page 106.

[10] Ferguson, R.E., "Non-proportional Reinsurance and the Index Clause," PCAS (1974), Page 141.

On an intuitive level, two possible objections to this model are offered in the CAS literature. Ferguson states:

> "It is commonly believed, or at least assumed, that inflation is uniform and does not vary by size of claim. Whether small claims inflate at an annual rate that differs from that affecting large claims has not been explored and remains a matter of conjecture. It is likely, however, that large claims would inflate at a higher rate due to their mix of indemnity and medical/rehabilitation. Large claims may have a higher proportion of medical/rehabilitation costs and thus be more sensitive to inflation."[11]

Thus even if price inflation were the only change influencing claim size, its effect may not necessarily be uniform by size of claim. Another objection to the above model was stated by Fowler in his review of Finger's paper. He argued that:

> "Intuitively I have the notion that where there is a trend in claim costs that trend will make itself felt unevenly according to claim size. I would rather expect that the larger claims would be more heavily affected than the smaller ones. Another way of saying this is that it might be reasonable to assume that superimposed and/or social inflation would exert a greater effect on the claims of greater size."[12]

This argument relates to our discussion earlier regarding the intuitive reasons one expects a trend in claim size to exist. Social inflation may cause the entire shape of the size of loss distribution to change. This seems to be the issue that Fowler is addressing.

[11] Ferguson, R.E., op. cit.

[12] Fowler, T.W., unpublished discussion of Finger, R.J., "A Note on Basic Limits Trend Factors," op. cit.

It is difficult however to decide this issue based on abstract arguments. A good test of the validity of the simple trend model is to check how well it has been confirmed by the loss data available to us.

There are various ways in which to examine size of loss distribution data to determine whether claims do indeed trend uniformly. We will now describe two such methods.

The first method involves estimating the value of "a", the trend factor, under the assumption that $g(x_t)=ax_t$. Earlier we mentioned that this factor can be estimated by fitting an exponential curve to the unlimited means $E(x_1)$, $E(x_2)$, .., $E(x_k)$. We now introduce the following notation. We define $E(x_t,c)$ to be the expected value of x_t given that each claim is limited in size to c. Mathematically:

$$E(x_t,c) = \int_0^c x_t dF_t\,(x_t)+c\int_c^\infty dF_t(x_t)$$

Notice that $E(x_t,\infty)=E(x_t)$. By a phenomenon known as the leveraged effect of inflation,[13] a curve fit to successive values of $E(x_t,c)$, with some fixed finite limit c, will lead to a trend factor that is less than or equal to "a". More specifically, if $g(x_t)=ax_t$ then $E(g(x_t))=aE(x_t)$, but $E(g(x_t),\, c) \leq aE(x_t,\, c)$.

If however, the value of c is not kept fixed but instead is allowed to increase each year by "a" and $g(x_t)=ax_t$, then fitting an exponential curve to $E(x_1,c)$, $E(x_2,\, ac), \ldots, E(x_k,\, a^{k-1}c)$ will lead to a trend

[13] Ferguson, R.E., op. cit.

factor equal to "a" regardless of which value of c is chosen as the limit in the first year.[14] In other words:

$$E(g(x_t),\ ac)=aE(x_t,c) \quad (2)$$

The proof of (2) is included in Appendix A.

Thus, a test of the assumption $g(x_t)=ax_t$ is to try to estimate "a" via an exponential curve fit to $E(x_1,c)$, $E(x_2,\ ac),\ldots,E(x_k,\ a^{k-1}c)$ using several different values of c. This involves an iterative procedure. An initial estimate of "a" must be chosen and the limit c must be trended by this estimate "a". The iteration stops when the trend applied to the limit c and the trend resulting from the exponential curve is the same. If the estimates of "a" so derived from various values of c are about the same (i.e. they do not exhibit an upward or downward trend), then $g(x_t)=ax_t$ would seem an appropriate assumption.

This test was applied to Products BI data of companies reporting to ISO. The data was inclusive of Policy years 1972-1977.[15] The resulting estimated values for "a", the annual trend factor, based on various values of c, the base limit in policy year 1972 are summarized in Table 1.

14 Note the similarity between this statement and the "index clause" discussed in Ferguson, R.E., op. cit.

15 For this test and all the remaining examples in the paper, data included is evaluated at 27 months of maturity. Testing has shown that the results are independent of whether each year is evaluated at 27 months or at ultimate maturity. Also, as mentioned later, the proceeding calculations and tests were based on a smooth Pareto curve fitted to the data, rather than on the reported data itself. The Pareto curve provides an estimate of the distribution of underlying claim sizes whereas the reported data is affected by policy limits purchased. See footnote 7 above.

Table 1

Limit in Policy Year 1972	Estimate of "a"
$ 25,000	+23.0%
300,000	27.0%
500,000	28.4%
1,000,000	30.1%
10,000,000	35.1%

The fact that the estimates of "a" increase as the limits increase leads one to question the assumption that all loss sizes trend at the same rate.

A more direct method to test the assumption that $g(x_t)=ax_t$ is to actually consider the statistic $g(x_t)/x_t$. For Products BI, for Policy Years 1973 and 1977, listed in Table 2 are several values of x_{1973}, with their corresponding trended values, x_{1977}, $(=g(x_{1973}))$, and the statistic x_{1977}/x_{1973}. x_{1977} is the claim size in 1977 whose cumulative probability is the same as the corresponding claim size in 1973, as per our earlier definition of $g(x_t)$.

Table 2

Products BI

(1) x_{1973}	(2) x_{1977}	(3) x_{1977}/x_{1973}	(4) Annual Trend =(column (3))$^{.25}$-1
$ 10,000	$ 21,929	2.193	+21.7%
50,000	116,355	2.327	23.5
100,000	255,310	2.553	26.4
200,000	571,995	2.860	30.0
500,000	1,692,052	3.384	35.6
1,000,000	3,872,216	3.872	40.3

Similarly, Table 3 contains claim sizes in 1972 for OLT BI and Hospital Professional Liability together with their corresponding claim sizes in 1977.

Table 3

OL&T BI

(1) x_{1972}	(2) x_{1977}	(3) x_{1977}/x_{1972}	(4) Annual Trend =(column (3))$^{.2}$-1
$ 10,000	$ 21,638	2.164	+16.7%
50,000	134,065	2.681	21.8
100,000	317,682	3.177	26.0
200,000	769,278	3.846	30.9
500,000	2,520,619	5.041	38.2
1,000,000	6,228,148	6.228	44.2

Hospital Professional Liability

(1) x_{1972}	(2) x_{1977}	(3) x_{1977}/x_{1972}	(4) Annual Trend =(column (3))$^{.2}$-1
$ 10,000	$ 17,793	1.779	+12.2%
50,000	114,266	2.285	18.0
100,000	269,663	2.697	21.9
200,000	651,383	3.257	26.6
500,000	2,137,138	4.274	33.7
1,000,000	5,300,521	5.301	39.6

The results in Table 2 and Table 3 indicate rather conclusively for the sublines considered that the annual trend is not constant but rather increasing by size of claim.

It should be mentioned that the results in Tables 1, 2 and 3 were attained based on the probabilities yielded by a shifted Pareto curve [16] which was fitted to the reported data for each of the policy years. Use of a

[16] The details of this curve and how it was applied by ISO are contained in "Report of the Increased Limits Subcommittee...", op. cit. Also see Patrik G., "Estimating Casualty Insurance Loss Amount Distributions," op. cit.

continuous probability distribution made it possible to solve for the inverse of any given probability value (i.e. to solve for x_{1977} given that $F_{1977}(x_{1977})$ = a given probability). Similar tests were applied directly to the reported data and the results were quite analogous to those in Tables 1, 2 and 3.

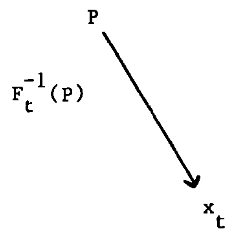
Tables 2 and 3 calculate annual trends corresponding to various claim sizes in 1973 for Products BI and 1972 for OLT BI and Hospital Professional Liability. This was done based only on data for those two policy years as well as policy year 1977. Actually this method can be expanded to incorporate several years of data and to not only test the model $g(x_t)=ax_t$ but to actually develop an alternate model. This is presented in the next section.

3. CHOOSING AN ALTERNATE TREND MODEL

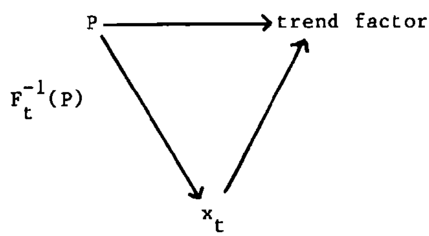
We now make the following assumption. At each level of cumulative probability, P, the claim size values $F_t^{-1}(P)$, $F_{t+1}^{-1}(P)$, .., $F_{t+k}^{-1}(P)$, are such that they are increasing at a rate that is approximately constant. Actually this implies that an exponential curve of the form e^{a+by}, (y=0, 1, 2,...,k), will fit the claim sizes reasonably well at all levels of P. If this assumption is made, then a mapping can be constructed from any level, P, to the annual trend derived by fitting an exponential curve to $F_t^{-1}(P),\ldots, F_{t+k}^{-1}(P)$. Note, this is a mapping from cumulative probabilities to annual trend factors and can be diagramed as follows:

$$P \longrightarrow \text{trend factor}$$

If a point in time is fixed then an additional mapping, namely $F_t^{-1}(P)$, exists. This mapping is from cumulative probabilities to claim sizes; i.e.



We can thus complete this diagram as follows:



We denote the mapping $x_t \longrightarrow$ trend factor as $tr(x_t)$. Note that referring to our earlier notation:

$$tr(x_t) = g(x_t)/x_t.$$

Loosely speaking, $tr(x_t)$, is the trend factor applicable to a claim of size x_t at time t. We refer to $tr(x_t)$ as a "trend function".

Note that by the development above, a trend factor is related to a given cumulative probability independent of time. However, the relationship of trend factors to claim sizes is dependent on time. This is because the claim size related to each cumulative probability will change over time.

Using this notation, the simple trend model becomes a single valued function, $tr(x_t)=a$. By reviewing the data presented in Tables 2 and 3, one is led to believe that $tr(x_t)$ should actually be a monotonically increasing function. We thus proceed by examining suitable alternate functions for $tr(x_t)$.

To do this, we modify and expand the data presented in Tables 2 and 3 in the following way. We pick a value of x_{1972}, say \$10,000, and its associated cumulative probability $F_{1972}(\$10{,}000)$. We then calculate $F_t^{-1}(F_{1972}(\$10{,}000))$ for t=1973, 1974,..., 1977.
For example, for OL&T BI, the distribution of claim sizes for Policy Year 1972 is such that

$$F_{1972}(10{,}000) = .9824$$

That is, the probability that the size of a claim (on a policy written in 1972) will be less than \$10,000, given that an accident has occurred, is .9824. We now search for the claim sizes in Policy Years 1973,...,1977, whose cumulative probabilities are .9824. For OLT BI, these values are given in Table 4.

Table 4

OLT BI

Claim sizes whose cumulative probabilities are .9824

Policy Year	Claim size
1972	10,000
1973	11,751
1974	14,408
1975	16,572
1976	18,036
1977	21,638

An exponential curve is then fit to these six claim sizes to derive an annual trend factor. By repeating this process for several values of x_{1972}, we derive an empirical sample for tr (x_{1972}).

This procedure was applied to data for the sublines mentioned in Tables 2 and 3. Table 5 contains a listing of empirical values for $tr(x_t)$ for several claim sizes.

Table 5

Products BI

1973 Loss Size	Indicated Annual Trend
$ 10,000	1.202
20,000	1.203
50,000	1.233
100,000	1.271
250,000	1.335
500,000	1.392
1,000,000	1.454

OL&T BI

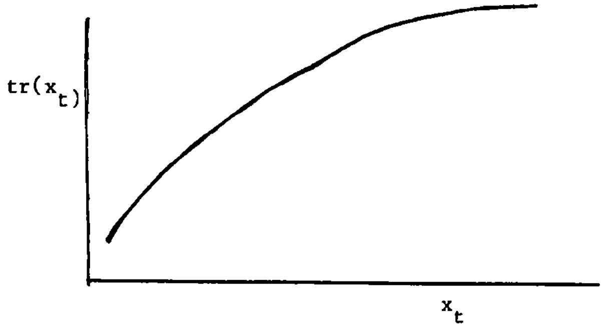
1972 Loss Size	Indicated Annual Trend
$ 10,000	1.163
20,000	1.173
50,000	1.207
100,000	1.243
250,000	1.299
500,000	1.345
1,000,000	1.395

Hospital Professional Liability

1972 Loss Size	Indicated Annual Trend
$ 10,000	1.100
20,000	1.119
50,000	1.160
100,000	1.201
250,000	1.267
500,000	1.324
1,000,000	1.387

For each subline shown above, a graph of $tr(x_t)$ against x_t produced a curve of the form in Figure 1.

Figure 1



From this graph, we observe that any function $tr(x_t)$ for these sublines should possess the following characteristics:

(1) $tr(x_t) \geq 0$ for all x_t

(2) $\frac{d\,tr\,(x_t)}{dx_t} \geq 0$ for all x_t

(3) $\frac{d^2tr(x_t)}{dx_t2} \leq 0$ for all x_t

The next step is to find a function that satisfies these properties and also provides a reasonable fit to the data. Several model forms can be fit to the data via a least squares regression. Among them are:

(1) $tr(x_t) = ax_t^b$

(2) $tr(x_t) = a(\ln(x_t))^b$

(3) $tr(x_t) = a-b/x_t$

Of these three, the first one $tr(x_t)=ax_t^b$, provides the best fit as measured by the coefficient of determination, R^2, when applied to the data for the sublines in Table 5. Table 6 lists several loss size values and the associated empirical values of $tr(x_t)$, as well as the fitted values of $tr(x_t)=ax_t^b$ for Products BI data, OL&T BI data and Hospital Professional Liability data. Note that the parameters "a" and "b" were derived by fitting to more claim sizes than are shown in Table 6.

It should be noted that of the three models listed above, (1) and (2) imply that $tr(x_t)$ increases without bound as $x_t \longrightarrow \infty$. Model (3) implies that $tr(x_t)$ is bounded from above by "a". The data seem to indicate that $tr(x_t)$ is not bounded from above. Note however that $tr(x_t)$ will grow at a very slow pace with the values of parameters "a" and "b" derived via the least squares regression and displayed in Table 6. For example, for Products BI, the fitted annual trend at $1 million is 1.427. At $1 billion the fitted trend increases only to 1.894.

Table 6

Products BI

1973 Loss Size	Indicated Annual Trend	Fitted Annual Trend (ax^b)
$ 10,000	1.202	1.182
20,000	1.203	1.216
50,000	1.233	1.262
100,000	1.271	1.299
250,000	1.335	1.348
500,000	1.392	1.387
1,000,000	1.454	1.427
		a=.810, b=.041

OL&T BI

1972 Loss Size	Indicated Annual Trend	Fitted Annual Trend (ax^b)
$ 10,000	1.163	1.136
20,000	1.173	1.169
50,000	1.207	1.214
100,000	1.243	1.249
250,000	1.299	1.297
500,000	1.345	1.334
1,000,000	1.395	1.373
		a=.779, b=.041

Hospital Professional Liability

1972 Loss Size	Indicated Annual Trend	Fitted Annual Trend (ax^b)
$ 10,000	1.100	1.110
20,000	1.119	1.144
50,000	1.160	1.191
100,000	1.201	1.228
250,000	1.267	1.279
500,000	1.324	1.318
1,000,000	1.387	1.359
		a=.740, b=.044

Applying the model $tr(x_t)=ax_t^b$

The parameter "b" in the trend function ax_t^b provides information regarding how trend varies by claim size. Note by assuming that b=0, one assumes that trend is uniform by claim size. For $b > 0$, trend will increase by claim size. The parameter "a" relates to the overall trend indicated and does not affect the ratio of the trends at various claim sizes. For example, if one were to modify "a" by a multiplicative factor, then the trend factor at each claim size would be multiplied by that same factor. Hence if recent economic activity leads one to believe that historical indications must be modified to reflect higher inflation rates, then this modification should be applied to "a", as is done in the model $tr(x_t)=a$, presented earlier.

It should be noted that the values derived for "a" and "b" were derived by setting t=1972 as the starting point. Had another starting point been chosen, different values would have been calculated for "a" and "b". This is because, as noted earlier, given a level of cumulative probability, the corresponding claim size will change over time. Appendix B describes how a_{t+k} and b_{t+k}, needed to trend claim sizes at time t+k, can be derived from a_t and b_t under the assumptions of our model.

A concern with the trend function ax_t^b is that it tends to underestimate the trend for small x_t. In fact $ax_t^b \longrightarrow 0$ as $x_t \longrightarrow 0$ while small claim size data (e.g. claim sizes of \$250, \$500 and \$1,000) seem to indicate that all claim sizes trend upward, and that the trend factors for small x_t level off to some number such as 1.05. This can be corrected by changing the model to $tr(x_t)=a(x_t+c)^b$, or by using the function $tr(x_t)=ax_t^b$ only for claim sizes greater than a selected value and using empirical data to trend small losses.

Use of the model ax_t^b has advantages in that it can be easily applied for special forms of $F_t(x_t)$. We will now discuss three such cases.

Case 1: $F_t(x_t) = \Phi\left(\frac{\ln x_t - \mu}{\sigma}\right)$ with $\Phi(x)$ being the normal cumulative distribution function, i.e. x_t is lognormally distributed. Under the assumption that $tr(x_t)= ax_t^b$ we know that the trended value of x_t is equal to $(ax_t^b).x_t=ax_t^{b+1}$. We now adjust $F_t(x_t)$ for one year's trend. Thus we attempt to project the distribution type and parameters of $F_{t+1}(x_{t+1})$. We have

$$F_{t+1}(ax_t^{b+1}) = F_t(x_t)$$

$$\therefore \quad F_{t+1}(x_{t+1}) = F_t\left[\left(x_{t+1}/a\right)^{1/(b+1)}\right]$$

$$= \Phi\left(\frac{\ln (x_{t+1}/a)^{1/(b+1)} - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{\ln x_{t+1} - (\ln a + (b+1)\mu)}{\sigma(b+1)}\right)$$

We see that $F_{t+1}(x_{t+1})$ is also lognormally distributed with parameters $\mu' = \ln a + (b+1)\mu$ and $\sigma' = \sigma(b+1)$. Thus to trend a lognormal distribution under the assumption that $tr(x_t)=ax_t^b$, one need only modify the parameters. Note that the coefficient of variation ($\sqrt{Var(x)}/E(x)$) of the lognormal distribution is given by $\sqrt{e^{\sigma^2}-1}$ which increases as σ increases. Similarly, the skewness coefficient[17] increases as σ increases. Under the assumption that trend increases by claim size ($tr(x_t)=ax_t^b$, $b>0$) the skewness of the size of loss distribution will increase over time. Under the assumption of uniform trend ($b=0$), the coefficient of variation and the skewness coefficient will remain unchanged.

Case 2: $F_t(x_t)=1-\exp\left[-(x_t/\beta)^\delta\right]$ i.e. x_t is Weibull distributed

$$F_{t+1}(x_{t+1}) = 1 - \exp\left[-\left(\left(\frac{x_{t+1}}{a}\right)^{\frac{1}{b+1}}\Big/\beta\right)^{\delta}\right]$$

$$= 1 - \exp\left[-\left(\frac{x_{t+1}}{a\beta^{b+1}}\right)^{\delta/(b+1)}\right]$$

So $F_{t+1}(x_{t+1})$ is a Weibull distribution with parameters:

$$\delta' = \delta/(b+1) \qquad \beta' = a\beta^{b+1}$$

[17] In general, skewness = $E[(x-E(x))^3]/Var(x)^{3/2}$. For the lognormal distribution, skewness = $\left[e^{\sigma^2}+2\right]\sqrt{e^{\sigma^2}-1}$

Case 3: $F_t(x_t) = 1 - \left(\frac{\beta}{x_t^{\gamma}+\beta}\right)^{\alpha}$, parameters $\alpha, \beta, \gamma > 0$. We refer to this distribution as the 'Transformed Pareto' distribution.

This distribution is a generalized form of the shifted Pareto distribution[18] $F_t(x_t)=1-\left(\frac{\beta}{x_t+\beta}\right)^{\alpha}$ in which case $\gamma=1$.

For the Transformed Pareto distribution we have:

$$F_{t+1}(x_{t+1}) = 1 - \left(\frac{\beta}{(x_{t+1}/a)^{\gamma/b+1}+\beta}\right)^{\alpha}$$

$$= 1 - \left(\frac{\beta a^{\gamma/b+1}}{x_{t+1}^{\gamma/b+1}+\beta a^{\gamma/b+1}}\right)^{\alpha}$$

We see that $F_{t+1}(x_{t+1})$ is also a Transformed Pareto distribution with parameters

$$\alpha' = \alpha \qquad \beta' = \beta a^{\gamma/b+1} \qquad \gamma' = \gamma/b+1$$

Thus when the distribution of x_t is either lognormal, Weibull, or Transformed Pareto, then under the assumption that $tr(x_t)=ax_t^{\ b}$, the size of loss distribution can be trended by simply modifying the parameters of the distribution.

In the next section we will explore the impact of this trend model in increased limits pricing.

[18] See Footnote 16.

4. COMPARING THE IMPACT OF TWO TREND MODELS ON INCREASED LIMITS PRICING

As an example of an application of $tr(x_t)=ax_t^b$ and a comparison of this model to $tr(x_t)=a$, we will illustrate the difference in increased limits factors derived from each of these models.

In this example we will assume that based on information yielded from the most recent data (e.g. Policy Year 1978) the size of loss distribution for a given line of insurance is lognormally distributed with parameters $\mu=8$ and $\sigma=2$. We wish to trend this distribution to say 1981, a total of three years.

Based on data for several past policy years let us say it has been determined that $tr(x_{1978})= ax_{1978}^b$ is an appropriate trend model with $b\approx .02$ and that overall losses are expected to increase by 15% annually for the next three years. We will choose a value for "a" such that overall trend is 15%. To do this we first develop some formulas.

The mean of a lognormal distribution with parameters μ and σ is $e^{\mu+\sigma^2/2}$. We have shown in Case 1 that when a log-normal distribution is adjusted for one year's trend under the model $tr(x_t)=ax_t^b$, the new parameters become $\mu' = \mu(b+1) + \ln a$ and $\sigma' = \sigma(b+1)$.

We can express the overall trend of 1.15 as the ratio of the new mean over the old mean:

$$T=\text{overall trend} = E(x_{t+1})/E(x_t)$$

For the log normal curve we have:

$$T = \exp(\mu' + \sigma'^2/2)/\exp(\mu+\sigma^2/2)$$

$$= \exp(\mu'-\mu+\sigma'^2/2-\sigma^2/2)$$

$$= \exp(\ln a + \mu(b+1) - \mu + ((\sigma(b+1))^2)/2 - \sigma^2/2)$$

$$= \exp(\mu b + \ln a + (\sigma^2/2)(b^2+2b))$$

$$\ln T = \mu b + \ln a + (\sigma^2/2)(b^2+2b) \qquad (3)$$

Thus given values for u, σ, b and T, we can solve for "a". In our example $u=8, \sigma=2$, b=.02, and T=1.15 so that one can calculate a= .904.

Therefore, $tr(x_{1978}) = .904(x_{1978})^{.02}$. The annual trends indicated by this trend function for several claim sizes are listed in Table 7.

Table 7

Loss Size (in 1978)	Annual Trend
$ 10,000	1.087
25,000	1.107
50,000	1.122
100,000	1.138
500,000	1.175
1,000,000	1.192

In order to trend the parameters three years into the future we note that:

$$x_{1981}=(tr(x_{1978}))^3 . x_{1978}$$

$$=(a(x_{1978})^b)^3 . x_{1978}$$

$$=a^3(x_{1978})^{3b+1}$$

In general $g(x_t)$, the trended value of x_t, after n years is given by:

$$a^n x_t^{nb+1}$$

The adjustment of the log-normal parameters then becomes:

$$\mu' = \mu\,(bn+1) + n \ln a$$

$$\sigma' = \sigma\,(bn+1) \qquad (4)$$

In our example:[19]

$$\mu_{1981} = 8(1.06) + 3 \ln (.904) = 8.177$$

$$\sigma_{1981} = 2(1.06) = 2.12$$

For comparison purposes we also calculate μ_{1981} and σ_{1981} under the assumption that trend by claim size is uniform, i.e. that b=0. In this case by using formula (3) above, we calculate a=T or a=1.15. Using (4), we get in the case of uniform trend:

$$\mu_{1981} = 8 + 3 \ln(1.15) = 8.419$$

$$\sigma_{1981} = 2$$

[19] Under the model assumption that at each level of cumulative probability, associated claim sizes are increasing at a rate that is approximately constant, the overall trend cannot demonstrate a constant rate of increase from year to year but rather will show a slightly increasing rate from year to year. This is due to the greater weight assigned to the higher claim sizes. In our example, with a=.904, T=1.15 the first year, 1.152 the second year and 1.154 the third. Given a forecast horizon of only 3 years, the impact on increased limits factors is negligible and has been ignored in our example.

Miccolis[20] develops formulas for calculating increased limits factors given the size of loss distribution. Using our notation, the increased limits factor for limit=c, assuming that the basic limit is $25,000 is given by:

$$E(x, c)/E(x, \$25{,}000)$$

For the log normal curve:

$$E(x, c) = e^{\mu + \sigma^2/2}\,\Phi\left[\frac{\ln x - (\mu + \sigma^2)}{\sigma}\right] + c\left[1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)\right]$$

Listed in Table 8 below are the increased limits factors derived from these formulas under the two assumptions that trend is either uniform by claim size or varying by claim size with parameters as derived above.

Table 8

Increased Limits Factors (1981)

Policy Limit	Uniform Trend	Varying Trend by Claim Size
$ 25,000	1.00	1.00
50,000	1.41	1.41
100,000	1.86	1.87
500,000	2.85	2.95
1,000,000	3.16	3.31
5,000,000	3.55	3.83

Note that at the $5 million limit, the assumption that trend varies by claim size produces an increased limits factor that is almost 8% greater than the corresponding factor developed under the assumption that trend is uniform by claim size.

[20]Miccolis, R. S., *op. cit.*

This comparison is somewhat independent of the overall trend assumed. To illustrate this, we rework the example with overall annual trend assumed to be 20%. The trended parameters then become:

Uniform Trend: $\mu_{1981} = 8.547 \quad \sigma_{1981} = 2$

Varying Trend: $\mu_{1981} = 8.305 \quad \sigma_{1981} = 2.12$

The resulting increased limits factors are included in Table 9 below.

Table 9

Policy Limit	Uniform Trend	Varying Trend by Claim Size
$ 25,000	1.00	1.00
50,000	1.42	1.42
100,000	1.90	1.91
500,000	2.98	3.08
1,000,000	3.34	3.49
5,000,000	3.79	4.08

The relative difference of the factors in Table 9 is almost identical with the differences in Table 8.

The examples above demonstrated the impact on increased limits factors of a varying trend model of the form ax_t^b when such a model is appropriate for all claim sizes. When the trend in losses less than some claim size L does not approach 0 as implied by the model ax_t^b, then the model should be applied to those loss sizes greater than L and the smaller losses should be handled separately in the following manner.

Suppose that trend at claim size x_t is equal to ax_t^b, but for claim sizes less than some value, L, trend is observed to be uniform and equal to aL^b. When a log-normal distribution with parameters, μ and σ^2, is trended (n years into the future) under this assumption then the trended distribution becomes:

$$F_{t+n}(x_{t+n}) = \begin{cases} \Phi\left(\dfrac{\ln x_{t+n} - \mu'}{\sigma'}\right) & \text{for } x_{t+n} < a^n L^{nb+1} \\ \mu' = \mu + n \ln a + nb \ln L & \\ \sigma' = \sigma & \\ \Phi\left(\dfrac{\ln x - \mu''}{\sigma''}\right) & \\ \mu'' = n \ln a + (nb+1)\mu & \text{for } x_{t+n} \geq a^n L^{nb+1} \\ \sigma'' = \sigma(nb+1) & \end{cases}$$

If we denote the distribution for $x_{t+n} < a^n L^{nb+1}$ as $G_{t+n}(x_{t+n})$ and the distribution for $x_{t+n} \geq a^n L^{nb+1}$ as $H_{t+n}(x_{t+n})$ then $E(x_{t+n}, c)$ becomes (assuming $c > a^n L^{nb+1}$):

$$\int_0^{a^n L^{nb+1}} x_{t+n}\, dG_{t+n}(x_{t+n}) + \int_{a^n L^{nb+1}}^{c} x_{t+n}\, dH_{t+n}(x_{t+n}) + c(1 - H_{t+n}(c))$$

As an example consider the trend function displayed in our first example:

$$tr(x_{1978}) = .904(x_{1978})^{.02}$$

In table 7, it was shown that this trend function implies that a \$10,000 claim will be trending at 8.7% annually. Now suppose that all claims less than or equal to \$10,000 are trending at 8.7% annually. The trend function then becomes:.

$$tr(x_{1978}) = \begin{cases} .904(10{,}000)^{.02} = 1.087 & \text{for } x_{1978} \leq 10{,}000 \\ .904(x_{1978})^{.02} & \text{for } x_{1978} > 10{,}000 \end{cases}$$

The corresponding log-normal parameters for claim sizes in 1981 then become, (assuming as we did earlier that in 1978, μ=8 and σ=2).

$$\mu_{1981} = 8.250, \ \sigma_{1981} = 2 \qquad \text{for } x_{1981} < 10{,}000(1.087)^3 = 12{,}844$$

$$\mu_{1981} = 8.177, \ \sigma_{1981} = 2.12 \qquad \text{for } x_{1981} \geq 12{,}844$$

Table 10 now adds a third column to the two columns listed in Table 8. This column represents the 1981 increased limits factors calculated when the assumed trend function is bounded from below by 1.087.

Table 10

Increased Limits Factors (1981)

Policy Limit	Uniform Trend	Varying Trend by Claim Size	Varying Trend by Claim Size With Minimum Trend = 1.087
$ 25,000	1.00	1.00	1.00
50,000	1.41	1.41	1.40
100,000	1.86	1.87	1.86
500,000	2.85	2.95	2.92
1,000,000	3.16	3.31	3.28
5,000,000	3.55	3.83	3.79

Note that there is still a significant difference between the $5 million increased limits factors calculated under the uniform trend assumption and the assumption that trend varies by claim size even though a minimum trend has been imposed.

CONCLUSION

This paper has examined various methods that can be used to adjust size of loss distributions for trend. Data has been presented that indicates that trend tends to increase by claim size. We have examined the implications of a model assuming an increasing trend by claim size for increased limits pricing. Presumably this trend assumption would affect other pricing areas that depend on size of loss distributions. Such implications remain to be explored.

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APPENDIX A

"TRENDED LIMITS TREND"

We prove in this appendix that under the assumption that all claims are trending at the same rate, the leveraged effect of inflation can be negated by allowing the policy limit or retention level to trend at the same rate.

Using the notation developed in the paper:

Let: $x_{t+k} = g(x_t) = ax_t$

Then: $E(x_{t+k}, ac) = aE(x_t, c)$

Proof: Without any loss of generality, and for ease of notation we let t=k=1.

Now $E(x_2, ac) = \int_0^{ac} x_2 f_2(x_2)dx_2 + ac\ (1-F_2(ac))$

Since $x_2 = ax_1$

We have $F_2(x_2) = F_2(ax_1) = F_1(x_1)$

and $af_2(ax_1) = f_1(x_1)$

So $E(x_2, ac) = \int_0^{c} ax_1 f_2(ax_1)adx_1 + ac\ (1-F_1(c))$

$= a\ (\int_0^{c} x_1 f_1(x_1)dx_1 + c(1-F_1(c)))$

$= aE(x_1, c)$

Q.E.D.

APPENDIX B

SOLVING FOR a_{t+k} AND b_{t+k} IN TERMS OF a_t AND b_t

The function $tr_t(x_t) = a_t x_t^{b_t}$ is appropriate for trending loss sizes at time t one year into the future. To trend loss sizes at time t+k one year into the future we must solve for a_{t+k} and b_{t+k} in terms of a_t and b_t.

To keep the subscripts simple we will deal with the case t=1 and k=2.

Now by our assumption of exponential trend at each level of cumulative probability, we have:

$$x_3 = (a_1 x_1^{b_1})^2 x_1$$

$$= a_1^2 x_1^{2b_1+1}$$

This implies that $x_1 = \left(\frac{x_3}{a_1^2}\right)^{\frac{1}{2b_1+1}}$

Also $tr_3(x_3) = tr_1(x_1)$

So $a_3 x_3^{b_3} = a_1 x_1^{b_1}$

$$= a_1 \left(\frac{x_3}{a_1^2}\right)^{\frac{b_1}{2b_1+1}}$$

$$= \left(a_1\right)^{\left(1-\frac{2b_1}{2b_1+1}\right)} \cdot x_3^{\left(\frac{b_1}{2b_1+1}\right)}$$

So we have

$$a_3 = \left(a_1\right)^{\left(\frac{1}{2b_1+1}\right)} \quad \text{and } b_3 = \frac{b_1}{2b_1+1}$$

In general

$$a_{t+k} = \left(a_t\right)^{\left(\frac{1}{kb_t+1}\right)} \quad \text{and } b_{t+k} = \frac{b_t}{kb_t+1}$$