QUADRATIC PROGRAMMING IN INSURANCE

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ABSTRACT AND INTRODUCTION

Quadratic programming means *maximizing or minimizing a quadratic function* of one or more variables subject to *linear restrictions* i.e. linear equations and/or inequalities.

Among the numerous insurance problems which can be formulated as quadratic programs we shall only discuss four, namely the *Credibility*, *Retention*, *IBNR* and the *Cost Distribution problems*.

Generally, there is no explicite solution to quadratic optimization problems, only statements about the existence of a solution can be made or some algorithm may be recommended in order to get exact or approximate numerical solutions. Restricting ourselves to typical problems of the above mentioned type, however, enables us to give an explicit solution in terms of general formulae for quite a number of cases, such as the onedimensional credibility problem, the retention problem and—under relatively week assumptions—for the IBNR-problem.

The results given here are by no means new. The only goal of this paper is to describe a few fundamental insurance problems from a common mathematical standpoint, namely that of quadratic programming and at the same time, to draw attention to a few special aspects and open questions in this field.

1. THE CREDIBILITY PROBLEM

We consider a portfolio consisting of N different risk categories j ($j = 1, 2, \ldots\ldots N$) for each of which claims statistics over the last n years ($i = 1, 2, \ldots\ldots n$) are available. With $P_{ij} > 0$, we denote the volume of class no. j for the year no. i (volume = number of risks, total sum insured or underlying premium volume) and with Y_{ij} the corresponding total of claims (or number of claims) so that the yearly loss ratios (or the claims frequency) are given by $X_{ij} = Y_{ij}/P_{ij}$.

For the entire statistical period we therefore have

$$\text{a volume } P_{\cdot j} = \sum_{i=1}^{n} P_{ij}$$

and a loss ratio $X_{\cdot j} = Y_{\cdot j}/P_{\cdot j}$ with $Y_{\cdot j} = \sum_{i=1}^{n} Y_{ij}$.

Now, the so-called credibility problem consists of estimating the expected value of $X_{\cdot k}$ for a fixed risk category k under the condition that the latter depends on a risk parameter θ which characterizes the heterogeneity of the portfolio. Or, expressed a little more mathematically:

$$\begin{aligned} &\text{Estimate } E\,[X_{\cdot k} \mid P_{\cdot k} = P, \theta_k = \theta] \\ &\quad \text{if } E\,[X_{\cdot k} \mid P_{\cdot k} = P, \theta_k = \theta] = \mu(\theta) \text{ independently of } P, \\ &\quad \text{Var}\,[X_{\cdot k} \mid P_{\cdot k} = P, \theta_k = \theta] = \frac{\sigma^2(\theta)}{P} \end{aligned}$$

where $X_{\cdot k}$ and $X_{\cdot j}$ are assumed to be stochastically independent for $j \neq k$ and fixed θ_k and θ_j and furthermore θ_j independent and identically distributed according to a distribution $U(\theta)$ which is called "structure function".

Confining ourselves to linear and unbiased minimum square estimates we may finally write:

For fixed k determine α_{ij} ($i = 1, 2, \ldots n$; $j = 1, 2, \ldots N$) such that

(i) $E\,[\{\mu(\theta_k) - \sum_{j=1}^{N} \sum_{i=1}^{n} \alpha_{ij} X_{ij}\}^2] =$ minimum

(ii) $E\,[\sum_{j=1}^{N} \sum_{i=1}^{n} \alpha_{ij} X_{ij}] = E_\theta\,[\mu(\theta)]$ (unbiasedness)

(iii) $0 \leqslant \alpha_{ij} \leqslant 1$ for $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots N$.

This is a quadratic program for the nN unknowns α_{ij}.

Its explicit solution is given by [2]

$$\hat{\mu}_k = \sum_{j=1}^{N} \sum_{i=1}^{n} \alpha_{ij} X_{ij} = \gamma_k X_{\cdot k} + (1 - \gamma_k)\bar{X}$$

where $\gamma_k = \dfrac{wP_{\cdot k}}{v + wP_{\cdot k}}$ — Credibility of risk category k

with $v = E_0[\sigma^2(0)]$, $w = \text{Var}_0[\mu(0)]$

and $\bar{X} = \sum_{j=1}^{N} (\gamma_j/\gamma)\, X_{\cdot j}$ with $\gamma = \sum_{j=1}^{N} \gamma_{\cdot j}$.

We can therefore say that the estimate which is optimal in the above sense is a weighted average of the individual claims experience $X_{\cdot k}$ and the overall claims experience $\bar{X}$. The latter is in general, however, not identical with the "natural" portfolio average $X_{\cdot\cdot}$ where the individual claims experiences are weighted with the relative premium volumes, namely

$$X_{\cdot\cdot} = \sum_{j=1}^{N} (P_{\cdot j}/P_{\cdot\cdot})\, X_{\cdot j} \qquad \text{with } P_{\cdot\cdot} = \sum_{j=1}^{N} P_{\cdot j}$$

but instead, the correct weights are, as we have just seen, the relative credibilities, since

$$X = \sum_{j=1}^{N} (\gamma_{\cdot j}/\gamma)\, X_{\cdot j} \qquad \text{with} \quad \gamma = \sum_{j=1}^{N} \gamma_{\cdot j}$$

From the special form of the credibilities $\gamma_{\cdot j}$ we can also immediately see that

—the larger the risk category j (i.e. the larger $P_{\cdot j}$), the larger is $\gamma_{\cdot j}$

—the larger the variations of the X_{ij}'s *in time* (i.e. the larger v), the smaller is $\gamma_{\cdot j}$

—the larger the variations of the X_{ij}'s *within the portfolio* (i.e. the larger w), the larger is $\gamma_{\cdot j}$

These observations match perfectly with what we may already have intuitively expected and this makes it relatively easy to discuss the general results also with insurance practitioners who are not necessarily mathematically oriented.

However, this very practical advantage seems to be lost as soon as we change to two- or more-dimensional risk parameters, since at least so far, we have not been able to write up a similar kind of explicit solution for $\theta = (\theta_I, \theta_{II})$, i.e. for the case where the portfolio is divided into subgroups by two or more criteria at the same time.

2. Retention Problems

The retention in reinsurance arrangements can be determined by means of quadratic optimization as is demonstrated in the following summary of a chapter from [1].

Let us consider a portfolio consisting of N independent risks, where the i-th risk is characterized by $S^{(i)}$, the accumulated claim in a given time interval. An individual reinsurance arrangement is a function g_i which determines a retained portion $g_i[S^{(i)}]$ of $S^{(i)}$.

Confining ourselves to the proportional case, let $P^{(i)}$ denote the price demanded by the reinsurer for taking over the risk completely, and $P^{(i)}.(1 - a_i)$ for accepting the cession $1 - a_i$. Then the stochastic variable

$$Z = \sum_{i=1}^{N} a_i(P^{(i)} - S^{(i)}) \tag{1}$$

measures the profit earned on the retained portion of the portfolio. Then our problem is that of finding those reinsurance arrangements which guarantee the given expected profit $E(Z)$ in the retention with the smallest possible deviations. In other words, we determine the a_i so that

$$\text{Var}(Z) = \text{min.}$$

under the additional condition that

$$E(Z) = \text{constant.}$$

For this purpose we introduce the Lagrange multiplier λ and differenciate the function

$$\phi = \text{Var}(Z) + \lambda E(Z) \tag{2}$$

partially with respect to a_i. Because of the independence of the $S^{(i)}$ we have

$$\text{Var}(Z) = \sum_{i=1}^{N} a_i^2 \,\text{Var}(S^{(i)})$$

and

$$E(Z) = \sum_{i=1}^{N} a_i(P^{(i)} - E[S^{(i)}]).$$

From

$$\frac{\partial \phi}{\partial a_j} = 2a_j \,\text{Var}[S^{(j)}] + \lambda(P^{(j)} - E[S^{(j)}]) = 0 \tag{3}$$

for all j it follows that

$$a_j = C \frac{P^{(j)} - E[S^{(j)}]}{\text{Var}\,[S^{(j)}]} \tag{4}$$

C is called "absolute Retention" (cf. [1]).

It depends on the insurance carrier's stability policy. If e.g. according to the ruin criterion, the probability of ruin should be less than a given P_0 it is shown in [1] that

$$C = \frac{2u}{\rho . |\ln P_0|}$$

under the assumption that $S^{(j)} = \sum_{i=1}^{A_j} Y_i^{(j)}$

where

A_j is a Poisson distributed number of claims variable,

$Y_i^{(j)}$ are independent non-negative variables with identical distribution,

u is the amount of free reserves, and

ρ is the ratio of proportional loading in the retained portion in relation to that in reinsurance.

Replacing those a_j which exceed 1 by 1 yields an optimal solution for a smaller $E(Z)$ than the one given.

3. The IBNR-Problem

In this context IBNR stands for both "Incurred But Not Reported" and inadequate reserving of already reported claims. It is a wellknown fact when dealing with so-called "longtail business" that the final number of claims and final total yearly claims cost are only known after several years. The insurer—and especially the excess of loss reinsurer of Motor Liability e.g.—is therefore forced to estimate final loss ratios either for premium calculation or reserving purposes on the basis of incomplete statistics. These purely statistical observations are usually presented as follows in what is called an "IBNR-triangle":

year of occurrence		Loss ratios as per the end of 1966	1967	1968	1969	1970
1966	i = 5	$X_5^{(1)}$	$X_5^{(2)}$	$X_5^{(3)}$	$X_5^{(4)}$	$X_5^{(5)}$
1967	i = 4		$X_4^{(1)}$	$X_4^{(2)}$	$X_4^{(3)}$	$X_4^{(4)}$
1968	i = 3			$X_3^{(1)}$	$X_3^{(2)}$	$X_3^{(3)}$
1969	i = 2				$X_2^{(1)}$	$X_2^{(2)}$
1970	i = 1					$X_1^{(1)}$

e.g. $X_4^{(3)}$ = loss ratio for 1967 as per December 31st, 1969

If there are m statistical years in total, we have a triangle

$$\nabla X = \{X_i^{(h)} \mid i = 1, 2, \ldots\ldots m;\ h = 1, 2, \ldots\ldots i\}$$

and our problem is to estimate the final loss ratio e.g. for the year $i = q$ or, in other words:

For the conditional expectation $E[X_q^{(\infty)} \mid \nabla X]$ determine an estimator $\hat{\mu}_q^{(\infty)}$ such that

$$E[\{E[X_q^{(\infty)} \mid \nabla X] - \hat{\mu}_q^{(\infty)}\}^2] = \text{minimum}$$

and $$E[\hat{\mu}_q^{(\infty)}] = E[E[X_q^{(\infty)} \mid \nabla X]].$$

We are thus looking for an unbiased minimum square estimator for the—at present unknown—final loss ratio of the year $i = q$.

Under the assumptions

(i) $X_i^{(h)}$ stochastically independent of $X_{i'}^{(h')}$ for $i \neq i'$
(ii) $E[X_i^{(h)}] = e^{(h)}$ independently of i
(iii) $P_i \operatorname{Cov}[X_i^{(h)}, X_i^{(h')}] = c_{hh'}$ independently of i and with P_i = underlying premium volume of year i
(iv) $\hat{\mu}_q^{(\infty)} = \sum_{i=1}^{n} \sum_{h=1}^{i} \alpha_{ih} X_i^{(h)}$ i.e. we confine ourselves to linear estimators
(v) loss ratio already final after m years: $X_i^{(m)} = X_i^{(\infty)}$

we are again confronted with a quadratic program, this time for the $\frac{m(m+1)}{2}$ unknowns α_{ih} and subject to simply one boundary condition, namely

$$e^{(m)} = \sum_{i=1}^{m} \sum_{h=1}^{i} \alpha_{ih} e^{(h)}.$$

According to [3] the general solution of this problem is described by the two following equations

$$\hat{\mu}_q^{(m)} = \alpha \sum_{i=1}^{m} P_i(\hat{c}_i, \beta_i^{1} X_i) + (c_{mq}, \beta_q^{-1} X_q) \qquad (1)$$

$$c^{(m)} = \alpha \sum_{i=1}^{m} P_i(c_i, \beta_i^{-1} c_i) + (c_{mq}, \beta_q^{-1} c_q) \qquad (2)$$

where the following vector and matrix notation has been used

$$c_i = \begin{bmatrix} c^{(1)} \\ c^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c^{(i)} \end{bmatrix}, \beta_i = \begin{bmatrix} c_{11} \; c_{12} \; \ldots \; c_{1i} \\ c_{21} \; c_{22} \; \ldots \; c_{2i} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c_{i1} \; c_{i2} \; \ldots \; c_{ii} \end{bmatrix}, X_i = \begin{bmatrix} X^{(1)} \\ X_i^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_i^{(i)} \end{bmatrix}, c_{mi} = \begin{bmatrix} c_{m1} \\ c_{m2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c_{mi} \end{bmatrix}$$

(β_i^{-1} — inverse of the covariance matrix β_i) and where (a, b) denotes the inner product of the two vectors a and b.

If the expectations c_m and covariances β_m are known, we can therefore first calculate the multiplier α from equation (2) and afterwards, by introducing α into (1), directly get the estimator $\hat{\mu}_q^{(m)}$. Equation (2) is, by the way, nothing else than the condition of unbiasedness applied to equation (1).

As shown in [5], these calculations become much more transparent if

either (i) $X^{(h)} = X^{(h-1)} + Y^{(h)}$, $Y^{(h)}$ independent of $X^{(h-1)}$ (*additive* model)

or (ii) $X^{(h)} = \Lambda^{(h)} X^{(h-1)}$, $\Lambda^{(h)}$ independent of $X^{(h-1)}$ (*multiplicative* model)

or (iii) $X^{(h)} = \Lambda^{(h)} X^{(h-1)} + Y^{(h)}$; $X^{(h-1)}$, $Y^{(h)}$, $\Lambda^{(h)}$ independent (the "*mixed*" model)

because in these cases the inverses β^{-1} can all be explicitly written up (their elements are all equal to zero except those in the diagonal and in the two adjacent "diagonals").

When dealing with concrete practical problems, the parameters c_m and β_m are of course unknown, these parameters also have to be estimated from the information contained in the triangle ∇X, a problem which—depending on one's view—may be formulated as yet another quadratic programme. In this context, the estimation of the covariances is of primary interest. Starting e.g. with estimators $\hat{c}_{hk}$ of the form

$$\hat{c}_{hk} = \frac{1}{m-h} \{ \sum_{i=h}^{m} P_i X_i^{(h)} X_i^{(k)} - (\sum_{i=h} P_i)^{1/m} \sum_{i=h}^{m} P_i X_i^{(h)} \sum_{i=h}^{m} P_i X_i^{(k)} \}$$

we may perhaps discover that the corresponding estimators for the correlation coefficient, namely

$$\hat{\rho}_{hk} = \frac{\hat{c}_{hk}}{\sqrt{\hat{c}_{hk}\,\hat{c}_{kh}}}$$

are neither bounded by -1 or $+1$ nor—as it seems reasonable to assume—monotonically increasing in h for $h \leqslant k$. It is then indicated to "isotonize" the above $\hat{\rho}_{hk}$ values which is essentially equivalent to solving a specific quadratic program [4].

4. The Cost Distribution Problem

We consider an insurance portfolio divided with respect to two criteria, e.g. branches i $(i = 1, 2, \ldots, n)$ and countries j $(j = 1, 2, \ldots, m)$, which has to contribute an amount C to a fund, e.g. a catastrophy fund.

Furthermore, let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_m$ denote the contributions per branch and country, respectively. We assume

$$\sum_{i=1}^{n} A_i = \sum_{j=1}^{m} B_j = C.$$

The following table represents the distribution of these costs, where $x_{ij} P_{ij}$ stands for the contribution of branch i in country j, P_{ij} being the corresponding volume (e.g. number of risks, sum insured or underlying premium volume).

	country 1	country 2	country m	
branch 1	$x_{11}\ P_{11}$	$x_{12}\ P_{12}$	$x_{1m}\ P_{1m}$	A_1
branch 2	$x_{21}\ P_{21}$	$x_{22}\ P_{22}$	$X_{2m}\ P_{2m}$	A_2
.				
.				
.				
.				
.				
branch n	$X_{n1}\ P_{n1}$	$x_{n2}\ P_{n2}$	$x_{nm}\ P_{nm}$	A_n
	B_1	B_2	B_m	C

We write $A_i = a_i P_{i\cdot}$, $B_j = b_j P_{\cdot j}$, $C = cP_{\cdot\cdot}$

where

$$P_{i\cdot} = \sum_{j=1}^{m} P_{ij},\ P_{\cdot j} = \sum_{i=1}^{n} P_{ij} \text{ and } P_{\cdot\cdot} = \sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}.$$

Now our problem consists in minimizing the function

$$\sum_{i=1}^{n}\sum_{j=1}^{m} (x_{ij} - c)^2 P_{ij} \qquad \text{such that}$$

$$\sum_{j=1}^{m} x_{rj} P_{rj} = a_r P_{r\cdot} \qquad (r = 1, 2, \ldots\ldots, n) \tag{1}$$

$$\sum_{i=1}^{n} x_{is} P_{is} = b_s P_{\cdot s} \qquad (s = 1, 2, \ldots\ldots, m) \tag{2}$$

$$x_{rs} \geqslant 0. \tag{3}$$

As far as we know, this problem cannot be solved explicitly. We are thus forced to confine ourselves to a few remarks about the rare results found until now. First, treating the problem without any sign-restrictions, we form the Lagrangian

$$\phi = \sum_{j=1}^{m}\sum_{i=1}^{n} (x_{ij} - c)^2 P_{ij} + 2 \sum_{i=1}^{m} \theta_i \left(\sum_{j=1}^{n} x_{ij} P_{ij} - A_i\right)$$
$$+ 2 \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} x_{ij} P_{ij} - B_j\right).$$

Putting the partial derivatives with respect to x_{rs} to 0 yields

$$x_{rs} = c - \theta_r - \lambda_s. \tag{4}$$

Summing up over r and s, respectively, we get for the Lagrange multipliers θ_r and λ_s the equations

$$\sum_{i=1}^{n} \theta_i P_{is} + \lambda_s P_{.s} = (c - b_s)\, P_{.s} \qquad (s = 1, 2, \ldots\ldots, m) \qquad (5)$$

$$\theta_r P_{r.} + \sum_{j=1}^{m} \lambda_j P_{rj} = (c - a_r)\, P_{r.} \qquad (r = 1, 2, \ldots\ldots, n)$$

which solved for θ yield

$$\theta_r P_{r.} - \sum_{i=1}^{n} \theta_i\, \alpha_{ri} = \delta_r \qquad (r = 1, 2, \ldots\ldots, n) \qquad (6)$$

$$\text{with } \alpha_{ri} = \sum_{j=1}^{m} \frac{P_{rj} P_{ij}}{P_{.j}} \qquad \text{and } \delta_r = \sum_{j=1}^{m} b_j P_{rj} - A_r$$

or in matrix-notation

$$A\theta = \delta$$

where $$a_{ii} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik},\ a_{ik} = a_{ki} = -\alpha_{ik}$$

and $$\sum_{i=1}^{n} \delta_i = 0.$$

The rank of A is less or equal to $n - 1$. In the sequel we assume rank $A = n - 1$. In this case the one-dimensional subspace $\underline{\theta} = \rho . \underline{1}$ is the solution of the homogenous system $A\underline{\theta} = 0$.

To find a special solution θ^* of the inhomogenous system we add the equation

$$\theta_1^* = 0$$

because of $\sum_{i=1}^{n} \delta_i = 0$ the system

$$\begin{pmatrix} A \\ 1\ 0\ 0\ \ldots\ 0 \end{pmatrix} \theta^* = \underline{\delta, 0}$$

has at least one solution too.

The solution of the system

$$
\begin{bmatrix}
\sum\limits_{\substack{i=1\\ i\neq 2}}^{n} \alpha_{2i} & -\alpha_{23} & -\alpha_{24} & \ldots & -\alpha_{2n} \\
-\alpha_{23} & \sum\limits_{\substack{i=1\\ i\neq 3}}^{n} \alpha_{3i} & -\alpha_{34} & & -\alpha_{3n} \\
-\alpha_{24} & -\alpha_{34} & & & \\
\cdot & \cdot & & & \\
\cdot & & & & \\
\cdot & & & & \\
-\alpha_{2n} & & & & \sum\limits_{i=1}^{n-1} \alpha_{ni}
\end{bmatrix}
\begin{bmatrix} 0_2^* \\ 0_3^* \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0_n^* \end{bmatrix}
=
\begin{bmatrix} \delta_2 \\ \delta_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \delta_n \end{bmatrix}
\qquad (7)
$$

introduced in 5) and 4) yield the x_{rs}.

Explicit expressions become very complicated for $n > 3$.

Yet, there are some rules for the computation of the determinant Δ of the matrix in 7): It is the sum of all products $\alpha_{i_2 k_2}\, \alpha_{i_3 k_3} \ldots . \alpha_{i_n i_n}$ in which every subscript appears at least once and in which no h factors ($h = 2, 3, \ldots, n$) may be ordered in a cycle as $\alpha_{i_1 i_2}\, \alpha_{i_2 i_3} \ldots . . \alpha_{i_h i_h}$.

Similar rules exist for the next lower subdeterminants of Δ.

As an example in the case of $n = 3$, we have

$\Delta = \alpha_{12}\,\alpha_{13} + \alpha_{12}\,\alpha_{23} + \alpha_{13}\,\alpha_{23}$ and the solutions of 5) are

$$x_{1s} = b_s - \frac{\alpha_{23}}{\Delta} \frac{P_{2s} + P_{3s}}{P_{.s}} \delta_1 + \frac{\alpha_{13}}{\Delta} \frac{P_{2s}}{P_{.s}} \delta_2 + \frac{\alpha_{12}}{\Delta} \frac{P_{3s}}{P_{.s}} \delta_3$$

$$x_{2s} = b_s + \frac{\alpha_{23}}{\Delta} \frac{P_{1s}}{P_{.s}} \delta_1 - \frac{\alpha_{13}}{\Delta} \frac{P_{1s} + P_{3s}}{P_{.s}} \delta_2 + \frac{\alpha_{13}}{\Delta} \frac{P_{3s}}{P_{.s}} \delta_3$$

$$x_{3s} = b_s + \frac{\alpha_{23}}{\Delta} \frac{P_{1s}}{P_{.s}} \delta_1 + \frac{\alpha_{13}}{\Delta} \frac{P_{23}}{P_{.s}} \delta_2 - \frac{\alpha_{12}}{\Delta} \frac{P_{1s} + P_{2s}}{P_{.s}} \delta_3.$$

As the problem can be solved explicitly for $n = 2$, presumably even with the sign restrictions, one is encouraged to try to solve the problem "iteratively" in a *first step* by distinguishing the first line of A, combining the second, third a.s.o. to one single line and solving this $2xm$ problem (2 lines, m columns), then in a *second step* distinguishing the originally second line, combining the third to last a.s.o.

It seems, however, that this procedure is not generally applicable, at least the numerical examples we have dealt with so far by this method led, in some cases, to solutions, in others to contradictions.

References

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