A MULTIVARIATE EXTENSION OF EQUILIBRIUM PRICING TRANSFORMS: THE MULTIVARIATE ESSCHER AND WANG TRANSFORMS FOR PRICING FINANCIAL AND INSURANCE RISKS

BY

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ABSTRACT

This paper proposes a multivariate extension of the equilibrium pricing transforms for pricing general financial and insurance risks. The multivariate Esscher and Wang transforms are derived from Bühlmann's equilibrium pricing model (1980) under some assumptions on the aggregate risk. It is shown that the Esscher and Wang transforms coincide with each other when the underlying risks are normally distributed.

KEYWORDS

Equilibrium pricing, Wang transform, Esscher transform, State price density, Gaussian copula.

1. INTRODUCTION

In the actuarial literature, there have been developed many probability transforms for pricing financial and insurance risks. Such methods include the variance loading, the standard deviation loading, and the exponential principle. Among them, the most popular pricing method is the *Esscher transform* given by

$$\pi(Y) = \frac{E\left[Y e^{-\theta Y}\right]}{E\left[e^{-\theta Y}\right]} \tag{1.1}$$

for a random variable Y representing risk, where θ is a positive constant[1] and E is an expectation operator under a probability measure P. The Esscher transform is a simple exponential tilting (or exponential change of measure) and has a sound economic interpretation.

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[1] This paper treats risk as an asset. A liability with loss variable X can be viewed as a negative asset with gain $Y = -X$. See Wang (2002).

Recently, Wang (2000) proposed a universal pricing method based on the following transformation:

$$F^*(x) = \Phi[\Phi^{-1}(F(x)) + \lambda] \tag{1.2}$$

where Φ denotes the standard normal cumulative distribution function (CDF for short), $F(x)$ is the CDF for the risk Y and λ is a positive constant. The transform is now called the *Wang transform* and produces a risk-adjusted CDF $F^*(x)$. The mean value evaluated under $F^*(x)$ will define a risk-adjusted "fair value" of risk Y at some future time, which will then be discounted to the current time using the risk-free interest rate.

The Wang transform not only possesses various desirable properties as a pricing method, but also has a clear economic interpretation which can be justified by a sound economic equilibrium argument for risk exchange. For example, the Wang transform (as well as the Esscher transform) is the only distortion function among the family of distortions that can recover CAPM (the capital asset pricing model) for underlying assets and the Black-Scholes formula for options. See Wang (2002) for details.

On the other hand, in the finance literature, the theory of asset pricing has been developed; the theory is well-developed for the so-called *complete* markets while there are still many blanks for *incomplete* markets. The insurance market is certainly incomplete; new attempts are required to develop a better pricing method.

The pricing methods developed in the actuarial literature based on probability transformations (or distortions) are often criticized by finance researchers, because they are not linear pricing functionals[2], yielding arbitrage opportunities. See, e.g., Harrison and Kreps (1979) for details.

In order to develop a linear pricing method while maintaining probability distortions, we need to develop a multivariate version of such existing methods. To see this, suppose that the underlying risks are described by an n-dimensional random vector, $(X_1, X_2, \dots, X_n)$ say. The probability space is given by $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = \sigma(X_1, X_2, \dots, X_n)$, the smallest σ-algebra generated from $(X_1, X_2, \dots, X_n)$. According to the first fundamental theorem of asset pricing, there is no arbitrage opportunity if and only if there exists a *state price density*[3]. Recall that the state price density is an $\mathcal{F}$-measurable random variable. Hence, supposing that a state price density is given, $\eta = \eta(X_1, X_2, \dots, X_n)$ say, the pricing functional is defined as

$$\pi(Y) = E[\eta Y] = \int_{\Omega} \eta(\omega) Y(\omega) \, dP(\omega) \tag{1.3}$$

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[2] The pricing functional π is said to be linear if $\pi(aX + bY) = a\pi(X) + b\pi(Y)$ for all risks X, Y and constants a, b.

[3] The second fundamental theorem of asset pricing states that the state price density is unique if and only if the market is complete. See, e.g., Pliska (1997) for details.

for risk Y that is an $\mathcal{F}$-measurable random variable. The pricing functional π is linear due to the linearity of the expectation operator.

The classic Esscher transform (1.1) is a special case of (1.3) where $\eta = e^{-\theta Y}/E[e^{-\theta Y}]$. The Wang transform (1.2) can be derived from (1.3) under a set of assumptions on the aggregate risk. See Wang (2003) for details. The aim of this paper is to find an appropriate state price density that covers all the risks under consideration, extending the Esscher and Wang transforms to the multivariate setting[4]. These extensions are derived from Bühlmann's equilibrium pricing model (1980).

The present paper is organized as follows. In the next section, we review Bühlmann's equilibrium pricing model and derive a multivariate extension of the Esscher transform. Of particular interest is the case of normally distributed risks. It is shown that the multivariate Esscher transform is consistent with CAPM and distorts a normal distribution to another normal distribution with mean vector adjusted by $Cov(Y, \lambda Z)$, the covariance of risk Y and λZ, where Z stands for the aggregate risk, which corresponds to the risk from the market portfolio in CAPM, and λ a risk aversion index. Section 3 develops a multivariate version of the Wang transform. It is shown that, when the underlying risks are normally distributed, the multivariate Wang transform coincides with the Esscher counterpart.

2. BÜHLMANN'S EQUILIBRIUM PRICING MODEL

Consider risk exchanges among a set of agents $j = 1, 2, \ldots, n$. Each agent is characterized by his/her utility function $u_j(x)$, where $u_j'(x) > 0$ and $u_j''(x) \leq 0$, and initial wealth w_j. Suppose that agent j faces a risk of potential loss X_j and is willing to buy/sell a risk exchange Y_j. If agent j is an insurance company, the risk exchange Y_j is thought of the sum of all insurance policies sold by j. While the original risk X_j belongs to agent j, the risk exchange Y_j can be freely bought/sold by the agents in the market.

In this pure risk exchange model, Bühlmann (1980) derived the following equilibrium pricing formula for risk Y, which is an $\mathcal{F}$-measurable random variable, by assuming that each agent j has an *exponential* utility function $u_j(x) = -e^{-\lambda_j x}$, $j = 1, 2, \ldots, n$[5]:

$$\pi(Y) = E[\eta Y], \quad \eta = \frac{e^{-\lambda Z}}{E[e^{-\lambda Z}]};$$

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[4] A similar attempt has been made by Wang (2005) to extend the Wang transform to a multivariate setting, where the Radon-Nikodym derivative is given by the normalized exponential tilting with special structures such as joint and separate distortions. In contrast, this paper assumes no such structures and derives the multivariate extensions directly from Bühlmann's pricing formula.

[5] Because this paper considers the pricing of risky assets, the sign of λ in the state price density η is opposite to that in Bühlmann (1980).

cf. (1.3), where $Z = \sum_{j=1}^{n} X_j$ is the aggregate risk and λ is given by

$$\lambda^{-1} = \sum_{j=1}^{n} \lambda_j^{-1}, \quad \lambda_j > 0.$$

The parameter λ is thought of the *risk aversion index* of the representative agent in the market.

In what follows, we assume that the underlying risks are described by an n-dimensional random vector $X = (X_1, X_2, ..., X_n)$ and the σ-algebra is given by $\mathcal{F} = \sigma(X_1, X_2, ..., X_n)$. Note that a particular risk, Y say, is an $\mathcal{F}$-measurable random variable. That is, there exists an n-variate function $h(x)$, $x = (x_1, x_2, ..., x_n)$, such that $Y = h(X)$ in general. Particular cases include a singleton, i.e. $Y = X_j$ for some j, and a portfolio of $(X_1, X_2, ..., X_n)$, i.e. $Y = \sum_{j=1}^{n} w_j X_j$ for some weights w_j. Bühlmann's equilibrium pricing formula is then given by

$$\pi(Y) = \frac{E\left[h(X)e^{-\lambda Z}\right]}{E\left[e^{-\lambda Z}\right]}, \quad Z = \sum_{j=1}^{n} X_j, \quad Y = h(X). \qquad (2.1)$$

The pricing functional $\pi(Y)$ is linear since

$$\pi(aY + bV) = \frac{E\left[\{ah(X) + bg(X)\}e^{-\lambda Z}\right]}{E\left[e^{-\lambda Z}\right]} = a\pi(Y) + b\pi(V),$$

where $V = g(X)$ for some $g(x)$.

Remark 2.1. Consider a particular risk Y, which is so small that Y and $Z - Y$ are approximately independent. In this case, we have from (2.1) that

$$\pi_\lambda(Y) = \frac{E\left[Ye^{-\lambda Y}e^{-\lambda(Z-Y)}\right]}{E\left[e^{-\lambda Y}e^{-\lambda(Z-Y)}\right]} \approx \frac{E\left[Ye^{-\lambda Y}\right]}{E\left[e^{-\lambda Y}\right]},$$

which is the Esscher transform (1.1). Note however that the resulting functional $\pi_\lambda(Y)$ does not satisfy the linearity, since

$$\frac{E\left[(Y+V)e^{-\lambda(Y+V)}\right]}{E\left[e^{-\lambda(Y+V)}\right]} \neq \frac{E\left[Ye^{-\lambda Y}\right]}{E\left[e^{-\lambda Y}\right]} + \frac{E\left[Ve^{-\lambda V}\right]}{E\left[e^{-\lambda V}\right]},$$

unless Y and V are independent of each other. Also, $\pi_\lambda(aY) = a\pi_{a\lambda}(Y) \neq a\pi_\lambda(Y)$.

For the n-dimensional random vector $X = (X_1, X_2, ..., X_n)$, suppose that the moment generating function (MGF for short) exists, i.e.

$$m_X(t) = E\left[\exp\left\{\sum_{j=1}^{n} t_j X_j\right\}\right] < \infty, \quad t = (t_1, t_2, ..., t_n),$$

for all t at a neighborhood of the origin. If the MGF exists, the interchange between the partial derivative and the expectation is permissible. Hence, from (2.1), we have

$$\pi(X_j) = \frac{\partial_j\, m_X(-\lambda\mathbf{1})}{m_X(-\lambda\mathbf{1})} = \partial_j\, \log m_X(-\lambda\mathbf{1}), \tag{2.2}$$

where $\mathbf{1}$ denotes the vector with all components being unity, and where ∂_j designates the partial derivative with respect to the jth variable. Hence, if the risky claim Y is a portfolio of X, i.e. $Y = \sum_{j=1}^{n} w_j X_j$, the equilibrium price of Y is given by

$$\pi(Y) = \sum_{j=1}^{n} w_j\, \partial_j\, \log m_X(\lambda\mathbf{1}). \tag{2.3}$$

This means that, in this context, pricing of financial and insurance risks is equivalent to determining the MGF of the underlying multivariate risks.

Remark 2.2. The pricing formula (2.1) rules out log-normal distributions, because the MGF does not exist for any log-normally distributed random variable. On the other hand, according to Iwaki, Kijima and Morimoto (2001), when each agent has a *power* utility function $u_j(x) = \gamma_j^{1-\alpha} x^\alpha / \alpha, \gamma_j > 0$, in Bühlmann's equilibrium setting for the same parameter $\alpha < 1$, the state price density is given by $\eta = Z^{-(1-\alpha)} / E[Z^{-(1-\alpha)}]$. The resulting pricing formula is valid for log-normal distributions.

The pricing formula (2.1) can be described in terms of the change of measures. Suppose for the sake of simplicity that the probability density function (PDF for short) of X exists, which is denoted by $f(x)$, under the original measure P. Let us define

$$f^*(x) = \frac{e^{-\lambda z}}{E[e^{-\lambda Z}]} f(x), \quad z = \sum_{j=1}^{n} x_j. \tag{2.4}$$

Since $f^*(x)$ is nonnegative and satisfies

$$\int_{R^n} f^*(x)\, dx = \frac{E[e^{-\lambda Z}]}{E[e^{-\lambda Z}]} = 1,$$

the function $f^*(x)$ can be thought of as a PDF. Denoting the new probability measure under which the PDF for X is $f^*(x)$ by P^*, the pricing formula (2.1) can be written as

$$\pi(Y) = \int_{R^n} \frac{h(x)\, e^{-\lambda z}}{E[e^{-\lambda Z}]} f(x)\, dx = \int_{R^n} h(x)\, f^*(x)\, dx.$$

Therefore, we obtain Bühlmann's equilibrium price as

$$\pi(Y) = E^*[h(X)], \quad Y = h(X), \tag{2.5}$$

where E^* denotes the expectation operator associated with the new probability measure P^*; cf. Bühlmann (1980). We shall call the transformation (2.4) the *multivariate Esscher transform* by an obvious reason[6].

Remark 2.3. In the case of the univariate Esscher transform, i.e.

$$f^*(x) = \frac{e^{-\lambda x}}{E[e^{-\lambda X}]} f(x), \quad \lambda > 0,$$

the ratio $f^*(x)/f(x)$ is strictly decreasing in x. Hence, denoting the random variable that follows the new PDF $f^*(x)$ by X^*, we conclude that X^* is smaller than X in the sense of likelihood-ratio ordering. This ordering is stronger than the first order stochastic dominance. See Kijima and Ohnishi (1996) for details. On the other hand, from (1.2), we have $\Phi^{-1}(F^*(x)) > \Phi^{-1}(F(x))$ so that $F^*(x) > F(x)$ for all x. Therefore, in the case of the univariate Wang transform, we conclude that X^* is smaller than X in the sense of first order stochastic dominance. In either case, we have $E^*[X] < E[X]$, suggesting a proper (i.e. positive) risk premium.

Of interest is then what will be the new PDF (2.4) under the multivariate Esscher transform. Fortunately, we can determine the transformation explicitly for the normal case, as we shall discuss below.

2.1. The case of normal distributions

When X follows an n-variate normal distribution, the pricing formula (2.3) becomes extremely simple. Suppose that X has mean vector $\mu = (\mu_1, \mu_2, ..., \mu_n)$ and covariance matrix $\Sigma = (\sigma_{ij})$. Then, its MGF is given by[7]

$$m_X(t) = \exp\left\{ \mu^{\mathsf{T}} t + \frac{t^{\mathsf{T}} \Sigma t}{2} \right\}, \tag{2.6}$$

where T denotes the transpose. It follows that

$$\partial_j \log m_X(-\lambda \mathbf{1}) = \mu_j - \lambda \sum_{i=1}^{n} \sigma_{ij}.$$

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[6] In Jensen (1995), the measure defined by

$$f^*(x) = \frac{e^{-\theta^{\mathsf{T}} x}}{E[e^{-\theta^{\mathsf{T}} X}]} f(x), \quad \theta = (\theta_1, \theta_2, ..., \theta_n)$$

is called an *exponentially tilted measure*, where T denotes the transpose. The Esscher transform (2.4) is a special case of the exponential tilting with $\theta = \lambda \mathbf{1}$.

[7] See, e.g., Tong (1990) for details of multivariate normal distributions.

Therefore, in the normal case, the multivariate Esscher transform (or Bühlmann's equilibrium price) for risk $Y = \sum_{j=1}^{n} w_j X_j$ is given by

$$\pi(Y) = E[Y] - \lambda Cov(Y,Z), \qquad (2.7)$$

where

$$\kappa \equiv \lambda Cov(Y,Z) = \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \sigma_{ij}$$

is called the *excess loading*. Of course, when Z corresponds to the risk from the *market portfolio* in CAPM, we have

$$\pi(Z) = E[Z] - \lambda V[Z], \qquad (2.8)$$

which is the variance loading. It follows from (2.7) and (2.8) that

$$\frac{E[Y] - \pi(Y)}{E[Z] - \pi(Z)} = \frac{Cov(Y,Z)}{V[Z]},$$

where the right hand side is the beta of Y in the theory of CAPM. In other words, the risk premium for Y, i.e. $E[Y] - \pi(Y)$, is proportional to the market risk premium, i.e. $E[Z] - \pi(Z)$, and the coefficient is given by the beta for Y. Summarizing, we have the following.

Theorem 2.1. *Suppose that the underlying risks* $(X_1, X_2, \ldots, X_n)$ *follow an n-variate normal distribution. If risk Y is a linear combination of the underlying risks, then the multivariate Esscher transform determines the price* $\pi(Y)$ *so that*

$$\pi(Y) = E[Y] - \beta_Y(E[Z] - \pi(Z)); \quad \beta_Y = \frac{Cov(Y,Z)}{V[Z]}, \qquad (2.9)$$

where β_Y *denotes the beta for Y in CAPM.*

Remark 2.4. Note that the pricing formula (2.9) does not depend on the risk aversion index λ of the representative agent. Also, since the beta β_Y is linear by the definition, the formula (2.9) provides a linear pricing functional. That is,

$$\begin{aligned}
\pi(aY + bV) &= E[aY + bV] - \beta_{aY+bV}(E[Z] - \pi(Z)) \\
&= aE[Y] + bE[V] - (a\beta_Y + b\beta_V)(E[Z] - \pi(Z)) \\
&= a\pi(Y) + b\pi(V),
\end{aligned}$$

because $Cov(aY + bV, Z) = aCov(Y,Z) + bCov(V,Z)$.

Now, we consider the new PDF $f^*(x)$ defined in (2.4) for the normal case. Recall that the PDF $f(x)$ under P is given by

$$f(x) = Ke^{-Q/2}; \quad Q = (x - \mu)^{\mathsf{T}} \Sigma^{-1}(x - \mu),$$

for some normalizing constant K, which is determined such that $\int_{R^n} f(x)dx = 1$. Note that

$$(x - \mu - \Sigma t)^\top \Sigma^{-1}(x - \mu - \Sigma t) = Q - 2(x - \mu)^\top t + t^\top \Sigma t.$$

It follows from (2.6) that

$$\exp\left\{-\tfrac{1}{2}Q + x^\top t\right\} = m_X(t) \exp\left\{-\tfrac{1}{2}(x - \mu - \Sigma t)^\top \Sigma^{-1}(x - \mu - \Sigma t)\right\}. \quad (2.10)$$

In particular, when $t = -\lambda \mathbf{1}$, we conclude that

$$f^*(x) = Ke^{-Q^*/2}; \quad Q^* = (x - \mu + \lambda \Sigma \mathbf{1})^\top \Sigma^{-1}(x - \mu + \lambda \Sigma \mathbf{1}),$$

which is the PDF of the n-variate normal distribution with mean vector $\mu - \lambda \Sigma \mathbf{1}$ and the same covariance matrix Σ. The next theorem summarizes[8].

Theorem 2.2. *Suppose that the underlying risks $X = (X_1, X_2, \ldots, X_n)$ follow an n-variate normal distribution with mean vector μ and covariance matrix Σ. Then, the multivariate Esscher transform distorts the normal distribution to another normal distribution with mean vector $\mu - \lambda \Sigma \mathbf{1}$ and the same covariance matrix Σ.*

Theorem 2.2 states that, when evaluating a derivative security, $Y = h(X)$ say, each risk X_j should be adjusted by $\lambda Cov(X_j, Z)$ for its mean μ_j. Hence, when the underlying risks are normal, the price of the derivative is given by

$$\pi(Y) = E[h(X - \lambda Cov(X, Z))], \quad (2.11)$$

where $X - \lambda Cov(X, Z)$ denotes the vector whose components are given by $X_j - \lambda Cov(X_j, Z)$. This extends a well-known result in the bivariate case to the multivariate setting[9].

In particular, when the underlying risks $(X_1, X_2, \ldots, X_n)$ follow a *standard* n-variate normal distribution with correlation matrix $\Sigma_\rho = (\rho_{ij})$, the multivariate Esscher transform distorts the standard normal distribution to another normal distribution with mean vector $-\lambda \Sigma_\rho \mathbf{1}$ and the correlation matrix Σ_ρ. In this context, suppose that an asset price in the next period under P is given by

$$S_i = K_i \exp\left\{\left(\mu_i - \frac{\sigma_i^2}{2}\right) + \sigma_i X_i\right\}, \quad i = 1, 2, \ldots, n,$$

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[8]   Alternatively, the theorem can be proved using the MGF given by (2.6).
[9]   For any bivariate normal random vector (X, Y), we have

$$E[h(X)e^{-Y}] = E[e^{-Y}]\,E[h(X - Cov[X, Y])].$$

See, e.g. Kijima and Muromachi (2001) for more general results.

where σ_i denotes the volatility of asset i. According to Theorem 2.2, the current price of asset i is given by

$$s_i(0) = E^*\left[K_i \exp\left\{\left(\mu_i - \frac{\sigma_i^2}{2}\right) + \sigma_i\left(X_i - \lambda\Sigma_p\mathbf{1}\right)\right\}\right]$$

$$= K_i \exp\{\mu_i - \lambda\sigma_i Cov(X_i, Z)\},$$

where $Z = \sum_{i=1}^n X_i$. It follows that

$$S_i = s_i(0)\exp\left\{\left(\lambda\sigma_i Cov(X_i, Z) - \frac{\sigma_i^2}{2}\right) + \sigma_i X_i\right\}, \quad i = 1, 2, \ldots, n, \quad (2.12)$$

under the original probability measure P. Hence, the mean rate of return for asset i must be given by

$$\mu_i = \lambda\sigma_i Cov(X_i, Z), \quad i = 1, 2, \ldots, n,$$

in this equilibrium pricing context. This means that the market price of risk for asset i (or risk X_i) is given by $\lambda Cov(X_i, Z)$, not just λ.

Note that the asset price under the new probability measure P^* is given by

$$S_i = s_i(0)\exp\left\{-\frac{\sigma_i^2}{2} + \sigma_i X_i\right\}, \quad i = 1, 2, \ldots, n. \quad (2.13)$$

Hence, as in Gerber and Shiu (1994), the Black-Scholes formula (1973) is obtained from (2.13) as the current price of a call option written on the asset S_i. More precisely, the pricing formula

$$\pi(Y) = e^{-r}E^*[\max\{S_i - K, 0\}]$$

provides the well-known Black-Scholes formula, where S_i is given by (2.13) under the *risk-neutral measure P^** and r denotes the risk-free interest rate over the period.

Remark 2.5. Even when Y is a portfolio of X, we have

$$\pi(Y) = E[h(Y - \lambda Cov(Y, Z))], \quad Y = \sum_{j=1}^n w_j X_j.$$

Hence, there is no such simple formula as (2.7) for the case of general payoff function $h(x)$. When $Cov(Y, Z)$ is small, however, we have from the first-order approximation to (2.11) that

$$\pi(Y) \approx E[h(Y)] - \lambda Cov(Y, Z)E[h'(Y)].$$

Hence, the beta in Theorem 2.1 needs to be modified by the factor $E[h'(Y)]$ for the non-linear payoff function $h(x)$ when the first-order approximation is applied.

2.2. Archimedean copulas

In general, the dependence structure of n-dimensional random vector $X = (X_1, X_2, \ldots, X_n)$ can be specified by copula. That is, suppose that each marginal CDF of X_j is given by $F_j(x)$. Then, $F_j(X_j)$ follows a standard uniform distribution on [0,1]. The copula specifies the joint CDF of $(F_1(X_1), F_2(X_2), \ldots, F_n(X_n))$. Denoting the copula by $c(x)$, $x = (x_1, x_2, \ldots, x_n)$, we have

$$P\{X_1 \leq x_1, \ X_2 \leq x_2, \ldots, X_n \leq x_n\}$$
$$= P\{F_1(X_1) \leq F_1(x_1), F_2(X_2) \leq F_2(x_2), \ldots, F_n(X_n) \leq F_n(x_n)\}$$
$$= c(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)).$$

Hence, we can recover the joint CDF of X from the copula $c(x)$. See, e.g., Nelson (1999) for details of copula functions.

The ordinary application of copula to multivariate distributions is to assume a particular copula in an ad hoc manner. Note however that copulas with more than two variables are limited, to the author's best knowledge, to Gaussian copulas, t copulas and Archimedean copulas. While this section considers an Archimedean copula, we employ a Gaussian copula to derive a multivariate Wang transform in the next section[10].

For any positive random variable R, suppose that it has the MGF at a neighbourhood of the origin. Let $\psi(s) = E[e^{-sR}]$, $s \geq 0$. It is readily seen that

$$(-1)^n \psi^{(n)}(s) > 0, \quad s > 0,$$

where $\psi^{(n)}(s)$ denotes the nth derivative of $\psi(s)$. Such a function is called *completely monotone*. We denote the inverse by $\psi^{-1}(s)$. Suppose further that $\psi(0) = 1$ and $\psi(\infty) = 0$. Then, the inverse function is decreasing in s with $\psi^{-1}(0) = \infty$ and $\psi^{-1}(1) = 0$. Now, for any $(x_1, \ldots, x_n) \in (0,1]^n$, let

$$c(x_1, \ldots, x_n) = \psi(\psi^{-1}(x_1) + \cdots + \psi^{-1}(x_n)), \quad 0 < x_j < 1. \qquad (2.14)$$

It is well known that the function $c(x_1, \ldots, x_n)$ defines an *Archimedean copula*.

The Archimedean copula can be generated as follows. Consider a family of independent, exponentially distributed random variables with unit mean, $Y_1, Y_2, \ldots, Y_n$ say. It is also assumed that R and Y_j are independent. Define

$$X_j = F_j^{-1}[\psi(Y_j/R)], \quad j = 1, 2, \ldots, n,$$

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[10] Since there are no MGF's for t-distributions, the multivariate Esscher transform does not apply for t copulas in this framework.

where we assume that each CDF $F_j(x)$ is strictly increasing for the sake of simplicity. According to Kijima (2002), the n-dimensional random vector $X = (X_1, X_2, ..., X_n)$ has the Archimedean copula (2.14). Hence, given the Archimedean copula, we can generate the random samples $x^k = (x_1^k, x_2^k, ..., x_n^k)$ from $X = (X_1, X_2, ..., X_n)$, and evaluate the expectation (2.1) by Monte Carlo simulation as follows.

Algorithm:

1. Generate independently random numbers r^k and y_j^k, $j = 1, 2, ..., n$.
2. Set $x_j^k = F_j^{-1}[\psi(y_j^k/r^k)]$, $j = 1, 2, ..., n$.
3. Repeat this procedure for sufficiently large K, $k = 1, 2, ..., K$, and evaluate (2.1) approximately as

$$\pi(Y) \approx \frac{\sum_{k=1}^{K} h(x^k) e^{-\lambda z^k}}{\sum_{k=1}^{K} e^{-\lambda z^k}}; \quad z^k = \sum_{j=1}^{n} x_j^k.$$

Unfortunately, the Archimedean copula cannot produce a negative dependence. See Marshall and Olkin (1988) for more detailed discussions about the dependence structure of Archimedean copulas. In the next section, we consider a Gaussian copula and show that a multivariate extension of the Wang transform is derived under some conditions on the aggregate risk Z from Bühlmann's equilibrium pricing formula.

3. A MULTIVARIATE EXTENSION OF THE WANG TRANSFORM

In this section, we extend the discussions of Wang (2003) to the multivariate case. For this purpose, we employ a Gaussian copula for the underlying risks $X = (X_1, X_2, ..., X_n)$. That is, define

$$U_j \equiv \Phi^{-1}[F_j(X_j)], \quad j = 1, 2, ..., n,$$

where $\Phi(x)$ is the standard normal CDF and $F_j(x)$ is the marginal CDF for X_j. It is assumed for the sake of simplicity that $F_j(x)$ is strictly increasing in x for all j. A Gaussian copula assumes that $U = (U_1, U_2, ..., U_n)$ follows an n-variate standard normal distribution with correlation matrix $\Sigma_\rho = (\rho_{ij})$.

In this copula situation, the joint CDF $F(x)$ for X is given by

$$F(x) = P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n\} = P\{U_1 \leq \alpha_1, U_2 \leq \alpha_2, ..., U_n \leq \alpha_n\},$$

where $\alpha_j = \Phi^{-1}[F_j(x_j)]$, $j = 1, 2, ..., n$. Denote by $\phi_n(x)$, $x = (x_1, x_2, ..., x_n)$, the PDF of the n-variate standard normal distribution with correlation matrix Σ_ρ,

$\phi(x)$ the PDF of the univariate standard normal distribution, and $f_j(x)$ the marginal PDF of X_j. Then, the joint PDF $f(x)$ for X exists and is given by

$$f(x) = \phi_n(\alpha) \prod_{j=1}^{n} \frac{d\alpha_j}{dx_j} = \phi_n(\alpha) \prod_{j=1}^{n} \frac{f_j(x_j)}{\phi(\alpha_j)}, \tag{3.1}$$

since $\Phi(\alpha_j) = F_j(x_j)$.

Now, as in Wang (2003), suppose that the aggregate risk Z consists of many individual risks X_j so that it can be approximated by a normal distribution. Let $Z_0 = (Z - \mu_Z)/\sigma_Z$, where $\mu_Z = E[Z]$ and $\sigma_Z^2 = V[Z]$, be the standardized normal random variable. Then, from (2.1), we have

$$\pi(Y) = \frac{E\left[h(X)e^{-\lambda_0 Z_0}\right]}{E\left[e^{-\lambda_0 Z_0}\right]}, \quad \lambda_0 = \lambda\sigma_Z. \tag{3.2}$$

Next, suppose that the standard normal variable Z_0 is expressed as

$$Z_0 \stackrel{d}{=} \xi + \sum_{j=1}^{n} w_j U_j, \quad U_j = \Phi^{-1}\left[F_j(X_j)\right], \tag{3.3}$$

for some constants w_j and a random variable ξ, which is independent of U_j, where $\stackrel{d}{=}$ stands for equality in law. It follows from (3.2) that

$$\pi(Y) = \frac{E\left[h(X)e^{-\sum_{j=1}^{n} \lambda_j U_j}\right]}{E\left[e^{-\sum_{j=1}^{n} \lambda_j U_j}\right]}, \quad \lambda_j = \lambda_0 w_j.$$

Since $U \equiv \sum_{j=1}^{n} \lambda_j U_j$ is normally distributed with mean 0 and variance $\lambda^{\mathsf{T}}\Sigma_\rho\lambda$ with $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, we obtain

$$\pi(Y) = e^{-\sigma_U^2/2} E\left[h(X)e^{-\sum_{j=1}^{n} \lambda_j U_j}\right], \quad \sigma_U^2 \equiv \lambda^{\mathsf{T}}\Sigma_\rho\lambda. \tag{3.4}$$

Remark 3.1. The decomposition (3.3) is always possible for normal random variables due to the linearity. However, when considering the joint distribution of $(h(X), Z_0)$, replacing Z_0 by some Z' with $Z \stackrel{d}{=} Z'$ may not result in the same *joint* distribution. It is therefore of great interest to investigate the accuracy of the approximation (3.3) for the joint distribution of $(h(X), Z_0)$ under the Gaussian copula setting.

Using the approximation (3.3), the change of measure formula (2.4) becomes

$$f^*(x) = e^{-\sigma_U^2/2} e^{-\sum_{j=1}^{n} \lambda_j \alpha_j} f(x), \quad \alpha_j = \Phi^{-1}[F_j(x_j)],$$

in this setting. It follows from (3.1) that

$$f^*(x) = e^{-\sigma_U^2/2} e^{-\Sigma_{j=1}^n \lambda_j \alpha_j} \phi_n(\boldsymbol{\alpha}) \prod_{j=1}^n \frac{f_j(x_j)}{\phi(\alpha_j)}. \tag{3.5}$$

Integrating (3.5) from $-\infty$ to y_j for each component, we obtain the CDF under P^* as

$$F^*(\boldsymbol{y}) = e^{-\sigma_U^2/2} \int^{\boldsymbol{y}} e^{-\Sigma_{j=1}^n \lambda_j \alpha_j} \phi_n(\boldsymbol{\alpha}) \prod_{j=1}^n \frac{f_j(x_j)}{\phi(\alpha_j)} \, d\boldsymbol{x}, \quad \boldsymbol{y} = (y_1, y_2, ..., y_n).$$

We now employ the change of variables; $u_j = \Phi^{-1}[F_j(x_j)]$. Then, since $\alpha_j = u_j$, we obtain

$$F^*(\boldsymbol{y}) = e^{-\sigma_U^2/2} \int^{\boldsymbol{\beta}} e^{-\Sigma_{j=1}^n \lambda_j u_j} \phi_n(\boldsymbol{u}) \, d\boldsymbol{u}, \quad \beta_j = \Phi^{-1}[F_j(y_j)], \tag{3.6}$$

with $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_n)$. Note that, from (2.10), we have

$$e^{-\sigma_U^2/2} e^{-\Sigma_{j=1}^n \lambda_j u_j} \phi_n(\boldsymbol{u}) = \phi_n(\boldsymbol{u} + \Sigma_\rho \boldsymbol{\lambda}).$$

Therefore, we finally obtain from (3.6) that

$$F^*(\boldsymbol{y}) = \Phi_n(\boldsymbol{\beta} + \Sigma_\rho \boldsymbol{\lambda}).$$

More precisely, we obtain a multivariate extension of the Wang transform as follows:

$$F^*(x) = \Phi_n\left(\Phi^{-1}[F_1(x_1)] + \sum_{j=1}^n \lambda_j \rho_{1j}, ..., \Phi^{-1}[F_n(x_n)] + \sum_{j=1}^n \lambda_j \rho_{nj}\right). \tag{3.7}$$

Note that, when $n = 1$, (3.7) coincides with the Wang transform (1.2) since $\rho_{11} = 1$.

Remark 3.2. Since actual market data exhibit the fat-tailed character, it is wise to replace the normal distribution with a fat-tailed distribution such as t-distributions, as suggested in Wang (2002). However, according to the above derivation, the n-variate standard normal distribution Φ_n used in (3.7) comes from the Gaussian copula to specify the dependence structure of the underlying risks. Because Bühlmann's equilibrium pricing formula does not apply for the case that the underlying risks follow a multivariate t-distribution, it is not possible to use t-copulas directly for $(U_1, U_2, ..., U_n)$ in (3.3). A further investigation is required to develop such pricing models in this framework. See Kijima and Muromachi (2006) for some development in this direction.

3.1. The case of normal distributions

When the underlying risks follow an *n*-variate normal distribution, equation (3.3) holds true, not just an approximation. Hence, in this special case, the multivariate Wang transform (3.7) coincides with the multivariate Esscher transform (2.4), as we shall see below.

Suppose that $X = (X_1, X_2, ..., X_n)$ has mean vector $\mu = (\mu_1, \mu_2, ..., \mu_n)$ and covariance matrix $\Sigma = (\sigma_{ij})$. Then, $U_j = (X_j - \mu_j)/\sigma_j$, where $\sigma_{jj} = \sigma_j^2$, in this special case. Since $Z = \sum_{j=1}^{n} X_j$, we have

$$Z_0 = \frac{Z - \mu_Z}{\sigma_Z} = \sum_{j=1}^{n} \frac{X_j - \mu_j}{\sigma_Z}.$$

Hence, taking $w_j = \sigma_j/\sigma_Z$ and $\xi = 0$, we obtain (3.3) with the ordinary equality, not equality in law.

By the definition of λ_j, we have

$$\lambda_j = \lambda \sigma_Z \frac{\sigma_j}{\sigma_Z} = \lambda \sigma_j, \quad j = 1, 2, ..., n,$$

so that

$$\sum_{j=1}^{n} \lambda_j \rho_{ij} = \lambda Cov(U_i, Z), \quad i = 1, 2, ..., n.$$

Hence, in the normal case, the Wang transform (3.7) is given by

$$F^*(x) = \Phi_n \left(\Phi^{-1}[F_1(x_1)] + \frac{\lambda}{\sigma_1} Cov(X_1, Z), ..., \Phi^{-1}[F_n(x_n)] + \frac{\lambda}{\sigma_n} Cov(X_n, Z) \right).$$
(3.8)

Since $F_j(x_j) = \Phi((x_j - \mu_j)/\sigma_j)$ in this case, we conclude that the multivariate Wang transform (3.8) distorts the normal distribution to another normal distribution with mean vector $\mu - \lambda \Sigma \mathbf{1}$ and the same covariance matrix Σ, which coincides with the multivariate Esscher transform; see Theorem 2.2.

3.2. The case of log-normal distributions

When the underlying risks $X = (X_1, X_2, ..., X_n)$ follow a multivariate log-normal distribution, the Esscher transform cannot define the pricing formula, while does the Wang transform (3.7). This is one of the advantages of the Wang transform over the Esscher counterpart.

Suppose that $(\log X_1, \log X_2, ..., \log X_n)$ has mean vector $\mu = (\mu_1, \mu_2, ..., \mu_n)$ and covariance matrix $\Sigma = (\sigma_{ij})$. In this case, contrary to the normal case, there is no reason to specify $w_j = \sigma_j/\sigma_Z$ and $\xi = 0$. However, if we take the choice, we then obtain the transform (3.8), with X_i being replaced by $\log X_i$, even for this case. Since $F_j(x_j) = \Phi((\log x_j - \mu_j)/\sigma_j)$ for log-normal distributions, we conclude that the multivariate Wang transform (3.8) distorts the log-normal distribution

to another log-normal distribution with mean vector $\boldsymbol{\mu} - \lambda\boldsymbol{\Sigma}\mathbf{1}$ and the same covariance matrix $\boldsymbol{\Sigma}$.

Acknowledgments

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