PARETO-OPTIMAL CONTRACTS IN AN INSURANCE MARKET

BY

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ABSTRACT

In distinction to the Borch's model of a reinsurance market, this paper treats the problem of optimal risk exchange in an insurance market where treaties are allowed between the insurer and each insured only, not among insureds themselves. A characterization of the Pareto-optimal contract is found. It is shown that the indemnity function in the contract is of a coinsurance kind. We also present a way of finding Pareto-optimal contracts under individual rationality constraints. The obtained results are compared with those of the known model of risk exchange in a reinsurance market.

KEYWORDS

Insurance contract, Premium, Indemnity function, Pareto optimality, Individual rationality.

1. INTRODUCTION

This paper is devoted to a problem of Pareto-optimal risk sharing between an insurance company (insurer) an a group of n potential insurance purchasers (insureds) where an insured stands as a separate decision maker in bargaining with the insurer so that any treaties among insureds themselves, including the pooling of their risks, are not allowed.

Most of previous works on risk sharing in insurance have tended to fall into one of two categories. The first comprises research on optimal risk exchange between an insurer and a single insured. Arrow (1971) found that a policy with a deductible is optimal for the model with a premium containing a fixed percentage loading. Raviv (1979) extended this result to a model with a more general function of operating expenses. Holmstrom (1979), Landsberger and Meilijson (1990) established optimality of a deductible policy respectively in the case of moral hazard and in the case of preferences described by so-called star-shaped utility functions.

In the second category are results on Pareto-optimal risk exchanges in a reinsurance market, stemming from Borch's works (1960a, 1960b, 1990). His theorem characterizing Pareto-optimal risk exchanges in a reinsurance market was extended to a constrained case in Gerber (1978). The Borch's risk exchange model of reinsurance, considered as a special kind of n-person cooperative game, has been developed by Lemaire and Baton (1981), Lemaire (1979, 1990a) where such game theory characteristics as the Pareto-optimal payoffs, core, bargaining set, value of the game were investigated. In an review by Aase (2002), the model was considered in a more general setting, including the competitive equilibrium notion, risk allocation problems in incomplete financial markets. A range of applications of the above-mentioned Borch's theorem were discussed in Lemaire (1990b). Borm et al. (1998) investigated risk exchange in an insurance model, regarding it as a cooperative game with restrictions on forming coalitions and assuming only exponential utility functions of the agents and exponentially distributed losses. Risk exchanges in an insurance market where the premiums are defined by the mean value principle were studied in Golubin (2006) from a view-point of application of the Nash's and Kalai-Smorodinsky's solution concepts to the corresponding bargaining game.

Our setting differs from the classical model of risk exchange in a reinsurance market: In the suggested model, each insured is independent of the other insureds in the sense that he shares his risk with the insurer only, not with the other insureds. Such an isolation of an insured seems natural in the context of insurance market since we consider individual insurance buyers, not (re)insurance companies for which mutual agreements on risk allocation are affordable. We assume premiums paid by the insureds to be chosen jointly with indemnity functions. Thus the Pareto-optimal contract to be found is a pair consisting of an n-dimensional vector of premiums and a set of n indemnity functions. Our main purpose is to derive necessary and sufficient conditions for Pareto optimality, that is, to derive an analogue of the Borch's theorem for the insurance market model. Then we employ the found results for a constrained problem where individual rationality expectations of the agents are met.

The outline of the paper is as follows. In section 2 the notation and assumptions for the insurance market model are given. Section 3 presents a way of characterizing the set of Pareto-optimal contracts in the model. An optimality equation determining the Pareto-optimal policies and premiums (i.e., the contracts) is derived as well as its variants involving the risk aversion functions of the agents. It is shown that the Pareto-optimal indemnity functions are always of the form of coinsurance policies. The optimal contract turns out to be dependent on probability distribution of initial insureds' risks, in distinction to the Borch's result on risk exchanges in reinsurance markets. In section 4 we study a problem of finding Pareto-optimal contracts under the constraints of individual rationality of the agents. We prove the existence and uniqueness theorem and show that in essence the problem reduces to the unconstrained problem with modified Pareto multipliers.

2. The model description

Consider a market consisting of $n + 1$ agents: an insurer and a group of n insureds. The individual insureds' losses $X_j, j = 1,\ldots,n$ are nonnegative stochastic variables defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We assume the losses to be independent, which seems natural in the context of the insurance market under consideration. The distribution function of X_j is denoted by $F_j(x) \overset{def}{=} P\{X_j \le x\}$. The support[1] of $F_j(x)$ is assumed to be a bounded interval $supp\ F_j = [0, T_j]$ or infinite interval $supp\ F_j = [0, \infty)$.

The agents' preferences are represented by their utility functions $u_i(x)$, meaning that $X \succ_i Y$ if and only if $Eu_i(X) > Eu_i(Y)$. We assume smooth increasing and strictly concave utility functions, more exactly, $u_i'(x) > 0$ and $u_i''(x) < 0$ in the relevant domains, for all $i = 0, \ldots, n$.

The insurer and an insured are negotiating to conclude a treaty on risk exchange between them. A peculiarity of the model is that any coalitions within the group of insureds are not allowed, so each insured stands as a separate decision maker in bargaining with the insurer. An *insurance contract* is identified with a pair (P, I), where $P = (P_1, \ldots, P_n)$ is a vector of premiums paid by the n insureds, and $I = (I_1, \ldots, I_n)$ is a set of Borel-measurable functions called indemnity functions or policies defined on $[0, \infty)$ and satisfying $0 \le I_j(x) \le x$ for $j = 1, \ldots, n$. The constraints mean that an indemnity payment $I_j(X_j)$ to the j-th insured is always nonnegative and not greater than the loss size X_j. Thus the summary risk taken by the insurer is $\sum_{j=1}^{n} I_j(X_j)$. Once a contract (P, I) is chosen, the expected utilities of final capitals of the insurer and the j-th insured are $J_0[P, I] \overset{def}{=} Eu_0(w_0 + \sum_{s=1}^{n} P_s - I_s(X_s))$ and $J_j[P, I] \overset{def}{=} Eu_j(w_j - P_j - X_j + I_j(X_j))$, $j = 1, \ldots, n$. Here w_i, $i = 0, \ldots, n$, denote the initial capitals of the agents. Notice that each of the last n functionals representing insureds' utilities depends on the corresponding individual premium and indemnity function only, $J_j[P, I] \equiv J_j[P_j, I_j]$.

By definition, a contract $(\hat{P}, \hat{I})$ is called *Pareto-optimal* if there is no other contract (P, I) such that $J_i[P, I] \ge J_i[\hat{P}, \hat{I}]$ for $i = 0, \ldots, n$, and at least one of the inequalities is strict. In other words, under such a contract any agent cannot improve his utility without worsening the utility of at least one other agent.

By the same reasonings as that in Borch (1960) and Wilson (1968), or in Gerber (1978), it is easily shown that the set of Pareto-optimal solutions can be obtained by maximization of a weighted sum of the agents' utilities. In our case, this method results in the following: fix a vector $k = (k_0, \ldots, k_n)$ such that $k > 0$ component-wise and $\sum_0^n k_i = 1$, then maximize the functional

$$\sum_{i=0}^{n} k_i J_i[P, I]$$

over the set of insurance contracts (P, I). For convenience we rewrite the problem above as

$$\text{maximize} \quad J[P, I] \equiv J_0[P, I] + \sum_{j=1}^{n} \delta_j J_j[P, I], \tag{1}$$

[1] The support of a probability distribution rigorously defines the notion of the set of "all possible values" of the stochastic variable. By definition, $supp\ F_j$ is the least closed set $S \subseteq R$ such that $P\{X_j \in S\} = 1$. For example, the support of an exponential distribution is $[0, \infty)$.

where $\delta_j > 0$, and maximum is taken over $P \in R^n$ and $I = (I_1,\ldots,I_n) : 0 \le I_j(x) \le x$ for $j = 1,\ldots,n$. As is known (see, e.g., Gerber (1978)), due to concavity of all the functionals J_i in (1) the n-parameter family of maximizers $\{(\hat{P}_\delta, \hat{I}_\delta)\}_{\delta>0}$ consists of all Pareto-optimal contracts (excluding, possibly, corner solutions related to the cases where one or several weights are zero).

Remark 1.
Several approaches to single out a “best” solution from the set of Pareto-optimal ones are known by now. First of all, it seems reasonable to narrow the set of Pareto-optimal contracts by using individual rationality principle. According to it, each of the agents does not accept a contract if it lessens his initial utility. In our notation this means $J_i[P, I] \ge J_i[0, 0]$, $i = 0,\ldots,n$, since every premium and indemnity function are zero before making a contract. Within the game theory approach, the insurance market model may be seen as a bargaining game discussed in Nash (1950), with the pay-offs being the expected utilities $J_i[P, I]$. Following this way, one can use either Nash’s (1950) or Kalai-Smorodinsky’s (1975) concepts for the game solution (see also Borch (1960a, 1960b), Lemaire (1990a), and Golubin (2006) for related insurance applications). Despite a difference in underlying axioms, both the concepts assume that the agents act rationally, which means that an optimal contract should be Pareto-optimal and individually rational for all the parties. The same is true for a notion of competitive equilibrium that is discussed in Aase (2002) in view of a reinsurance market.

Further we focus only on constructing the set of Pareto-optimal contracts $\{(\hat{P}_\delta, \hat{I}_\delta)\}_{\delta>0}$ via treatment of (1) and then employing individual rationality constraints. Application of the game theory concepts as well as equilibrium notion are left beyond the scope.

3. Characterization of Pareto-optimal contracts

The next theorem gives a characterization of the Pareto-optimal contract $(\hat{P}, \hat{I}) = (\hat{P}_\delta, \hat{I}_\delta)$ (below we will omit the subscript δ for convenience) in the form of necessary and sufficient conditions for optimality in (1). To formulate it, we need a notation $E[Y|X]$ that stands for a stochastic value called the conditional expectation of Y with respect to a sigma-algebra $\sigma(X)$ generated by the stochastic value X. Remark that the conditional expectation can also be represented as $E[Y|X] = \phi(X)$ $\mathcal{P}$-a.s., where ϕ is an appropriate Borel-measurable function (see Tucker (1967)). Denote, respectively, by

$$A = w_0 + \sum_{s=1}^{n} \hat{P}_s - \hat{I}_s(X_s) \quad \text{and} \quad B_j = w_j - \hat{P}_j - X_j + \hat{I}_j(X_j) \tag{2}$$

the final capitals of the insurer and j-th insured under a contract $(\hat{P}, \hat{I})$.

Theorem 1. *A contract $(\hat{P}, \hat{I})$ solves* (1) *if and only if*

$$E[u_0'(A)|X_j] = \delta_j u_j'(B_j), \qquad j = 1,\ldots,n \qquad \mathcal{P}-a.s. \tag{3}$$

Moreover, (3) *is equivalent to a pair of conditions* (4)-(5):

$$E[u_0'(A)\,|\,X_j=0] = \delta_j u_j'(w_j-\hat{P}_j), \qquad j=1,\dots,n, \tag{4}$$

$$\text{and} \quad \hat{I}_j'(X_j) = \frac{\delta_j u_j''(B_j)}{E\left[u_0''(A)\,\middle|\,X_j\right]+\delta_j u_j''(B_j)} \qquad \mathcal{P}-a.s. \tag{5}$$

with initial condition $\hat{I}_j(0)=0$, *for* $j=1,\dots,n$.

Proof. An outline of the proof is as follows. Let us ignore, for a while, the constraints $0 \le I_j(x) \le x$ and assume only $I_j(0)=0$, $j=1,\dots,n$. The idea is to show that conditions of optimality in (1) over this wider set of admissible contracts determine a contract $(\hat{P},\hat{I})$ such that $0<\hat{I}(x)<x$ for $x>0$. Thereby, the contract solves problem (1) we start from. First, we show the necessity of (3) for the problem over the wider admissible set. Then we prove that (3) is equivalent to (4)-(5). The latter pair of conditions imply that the contract $(\hat{P},\hat{I})$ is admissible for the initial problem. The second part of the proof establishes the sufficiency of (3) for optimality in (1).

1. Suppose that $(\hat{P},\hat{I})$ maximizes (1) subject to $I(0)=0$. Let ΔP be an arbitrary vector in R^n and $v(x)=(v_1(x),\dots,v_n(x))$ be an arbitrary Borel-measurable vector function such that $v(0)=0$. Define $P^\lambda=\hat{P}+\lambda\Delta P$ and $I^\lambda=\hat{I}+\lambda v$, where λ is a parameter, and consider a function $J(\lambda)\overset{def}{=}J[P^\lambda,I^\lambda]$ (see (1)). By the assumption, $J(\lambda)$ has a maximum at $\lambda=0$ therefore $J'(0)=0$ or, equivalently,

$$\sum_{s=1}^{n}\Delta P_s E(u_0'(A)-\delta_s u_s'(B_s))-\sum_{s=1}^{n}E\{[u_0'(A)-\delta_s u_s'(B_s)]\,v_s(X_s)\}=0. \tag{6}$$

Set $\Delta P=0$ and, for a fixed j, set $v_s(x)\equiv 0$ for all $s\neq j$. Then from (6) we have

$$0=E\{(u_0'(A)-\delta_j u_j'(B_j))v_j(X_j)\} = E\{(E[u_0'(A)\,|\,X_j]-\delta_j u_j'(B_j))v_j(X_j)\}.$$

As it holds for an arbitrary $v_j(x)$ such that $v_j(0)=0$, we can conclude that $E[u_0'(A)\,|\,X_j]-\delta_j u_j'(B_j)=0$ a.s. with the exception for $X_j=0$. To cope with the latter case, set $v(x)\equiv 0$, $\Delta P_j>0$, and $\Delta P_s=0$ for all $s\neq j$. From (6), $0=E(u_0'(A)-\delta_j u_j'(B_j)) = E\{E[u_0'(A)\,|\,X_j]-\delta_j u_j'(B_j)\}$. Denote the expression in { } by Y. Since $EY=P\{X_j=0\}\,E[Y\,|\,X_j=0]+P\{X_j>0\}E[Y\,|\,X_j>0]$ and, as we have already proved, $E[Y\,|\,X_j>0]=0$, we get $P\{X_j=0\}\,E[Y\,|\,X_j=0]=0$. Thus, (3) must hold for the maximizer $(\hat{P},\hat{I})$ in the considered problem.

Now show the equivalence of (3) and (4)-(5). For given $X_j=x$, (3) yields

$$\mathrm{E}\{u_0'(w_0+\hat{P}_j-\hat{I}_j(x)+\sum_{s\neq j}^{n}\hat{P}_s-\hat{I}_s(X_s))\}-\delta_j u_j'(w_j-\hat{P}_j-x+\hat{I}_j(x))=0, \tag{7}$$

where the expectation in the left-hand side is taken with respect to $X_1,\dots,X_{j-1}$, $X_{j+1},\dots,X_n$ that are assumed to be independent of X_j. Denote the argument of u_0' in (7) by $A_j(x)$ and the argument of u_j' by $B_j(x)$ (note that $A_j(x)$ is a stochastic-valued function). After differentiating (7) with respect to x and combining the terms, we have

$$\hat{I}_j'(x) = \frac{\delta_j u_j''(B_j(x))}{Eu_0''(A_j(x)) + \delta_j u_j''(B_j(x))} \tag{8}$$

for $x \in supp\, F_j$ or, in other words, almost sure with respect to the marginal measure F_j. Note that the initial conditions $\hat{I}_j(0) = 0$, $j = 1,\dots,n$ must hold by assumption. Hence, it is proved that (3) implies (4)-(5). If we suppose (5) to hold (i.e. (8) holds) then equality (7) is determined up to a constant in the right-hand side of (7). If we couple (8) with a condition $Eu_0'(A_j(0)) - \delta_j u_j'(B_j(0)) = 0$, which coincides with (4), we obtain that (7) is equivalent to this pair of conditions, i.e., (3) and (4)-(5) are equivalent.

From (8) it follows that $0 < \hat{I}_j'(x) < 1$, along with $\hat{I}_j(0) = 0$ this gives $0 < \hat{I}_j(x) < x$ for $x > 0$. So $(\hat{P}, \hat{I})$ is admissible in problem (1) and, hence, solves this problem. Thus, we have shown that (3) is necessary for optimality in (1), and (3) is equivalent to (4)-(5).

2. Let (3) hold for some contract $(\hat{P}, \hat{I})$. By the same arguments as above, it is easy to show that $J'(\lambda)|_{\lambda=0} = 0$ for any choice of an (admissible) direction $(\Delta P, v)$. Since the goal functional $J[P, I]$ is concave in (P, I), this implies that $(\hat{P}, \hat{I})$ is a maximizer in the problem with the only constraint $I(0) = 0$ and, hence, $(\hat{P}, \hat{I})$ is a maximizer in (1). □

Remark 2.
One can easily verify (by the same arguments as that in the second part of the Theorem 1's proof) that condition (3) remains sufficient for Pareto optimality even if the individual risks X_j are not independent. In this case, however, equations (5) are not valid any longer and, hence, they cannot be used to prove the necessity of (3), as that was done in the first part of Theorem 1. In the sequel we are going to employ the optimality equations (5), therefore the independence hypothesis is still assumed to hold.

Relation (3) in Theorem 1 resembles the Pareto optimality condition in the Borch's theorem characterizing risk exchanges among n agents in a reinsurance market (see e.g. Borch (1960), Aase (2002)): $\delta_1 u_1'(\tilde{Y}_1) = \dots = \delta_n u_n'(\tilde{Y}_n)$ a.s., where $\tilde{Y}_j = \tilde{y}_j(X_M)$ are functions of total initial loss $X_M = X_1 + \dots + X_n$. The basic differences are the following. First, in our case the indemnity payment $I_j = I_j(X_j)$ is a function of the j-th insured's loss only, not of the total loss X_M — an individual risks pooling is not allowed apriori. Second, the terms $E[u_0'(A)|X_j]$ in (3) depend on distributions of initial risks $X_1,\dots,X_n$, unlike the Borch's characterization where the optimal risk exchange rules are not affected by distribution of initial risk portfolio.

Concerning the form of Pareto-optimal indemnity functions, we see that $0 < \hat{I}_j'(x) < 1$ and $\hat{I}(0) = 0$ by equation (5). Thus, any $\hat{I}_j$ is a coinsurance policy with no deductible. This does not seem surprising in view of Raviv's result (1979) who showed that in the case of risk exchange between the insurer and the only insured, the Pareto-optimal policies are coinsurance policies not involving deductibles if operating expenses are zero. A model of risk exchange based on premium calculation as a function of the mean of indemnity payment gives rise to deductible and so-called upper limit policies as was shown in Golubin (2006). The moral hazard problem was analyzed in Holmstrom (1979) who proved that this results in deductibles. However, all these "sources" of deductible policies are not the case in our model of insurance market.

Note also that in the case of a risk-neutral insurer with utility function $u_0(x) = x$, equation (5) yields $I_j(x) = x$ for all j as $u_0''(x) \equiv 0$. That is, the only Pareto-optimal policy is full coverage of the losses, and the range of Pareto-optimal premiums is determined by (4).

The next result gives two modifications of the Pareto optimality conditions in Theorem 1 that employ the notions of risk aversion and risk tolerance functions of the agents. Recall that the absolute risk aversion function of an agent is $r(x) = -\frac{u''(x)}{u'(x)}$ and the reciprocal of it, $\rho(x) = 1/r(x)$, is called the risk tolerance function. Recall also that A and B_j stand for final (stochastic) capitals of the insurer and j-th insured as defined in (2).

Corollary 1. *A contract* $(\hat{P}, \hat{I})$ *solves* (1) *if and only if* (4) *holds and*

$$\hat{I}_j'(X_j) = \frac{r_j(B_j)}{R_{0j}(X_j) + r_j(B_j)} \quad a.s. \tag{9}$$

with initial condition $\hat{I}_j(0) = 0$, *where*

$$R_{0j}(X_j) \stackrel{def}{=} -E[u_0''(A) | X_j] \,/\, E[u_0'(A) | X_j], \;\; j = 1, \ldots, n. \tag{10}$$

Moreover, (9) *is equivalent to*

$$\hat{I}_j'(X_j) = \frac{\rho_{0j}(X_j)}{\rho_j(B_j) + \rho_{0j}(X_j)} \quad a.s. \tag{11}$$

with $\hat{I}_j(0) = 0$, *where* $\rho_{0j}(X_j) \stackrel{def}{=} 1/R_{0j}(X_j)$, $j = 1, \ldots, n$.

Proof. From (3) we have $\delta_j = E[u_0'(A) | X_j] \,/\, u_j'(B_j)$. Inserting this expression for δ_j into (5) yields (9). Division of both the numerator and denominator in the right-hand side of (9) by $R_{0j}(X_j)\, r_j(B_j)$ results in (11). □

Remark 3.
The function $R_{0j}(x)$ introduced by (10) is the ratio of conditional expectations $-E[u_0''(A) | X_j = x]$ and $E[u_0'(A) | X_j = x]$, but not the expectation of the risk aversion $r_0(A) = -u_0''(A) / u_0'(A)$ under $X_j = x$. In this connection, $R_{0j}(x)$ and $\rho_{0j}(x)$ can correspondingly be called a *risk aversion ratio* and a *risk tolerance ratio* of the insurer with respect to j-th insured. The function $\rho_{0j}(x)$ can be rewritten as

$$\rho_{0j}(x) = -Eu_0'(A_j(x))\,/\,Eu_0''(A_j(x)),$$

with $A_j(x) = w_0 + \hat{P}_j - \hat{I}_j(x) + \sum_{s\neq j}^{n} \hat{P}_s - \hat{I}_s(X_s)$ and the expectations being taken with respect to $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n$. Equations (4) and (11) can then be rewritten in the form of an equation

$$E\{u_0'(w_0 + \hat{P}_j + \sum_{s\neq j}^{n} \hat{P}_s - \hat{I}_s(X_s))\} = \delta_j u_j'(w_j - \hat{P}_j) \tag{12}$$

and a differential equation

$$\hat{I}_j'(x) = \frac{\rho_{0j}(x)}{\rho_j\left(w_j - \hat{P}_j - x + \hat{I}_j(x)\right) + \rho_{0j}(x)} \quad \text{on } x \in supp\, F_j. \tag{13}$$

If the group of insureds is *homogeneous* in the sense that the losses are independent and identically distributed, $w_1 = \ldots = w_n$, and $u_1 = \ldots = u_n$ then the Pareto multipliers δ_j can be set identical, and the optimal policies $\hat{I}_j(x) \equiv \hat{I}(x)$ as well as premiums $\hat{P}_j \equiv \hat{P}$ become identical also. System (12) and (13) converts into a pair of equations in which the risk tolerance ratio $\rho_{0j}(x)$ does not depend on the number j of the insured.

Let us compare the Pareto-optimal risk exchange given by Corollary 1 and that given by the Borch's theorem for the model in which treaties among insureds are allowed. In both cases the initial risk portfolio of the $(n+1)$ agents is $(0, X_1, \ldots, X_n)$ – here we consider agents' risks instead of their capitals $(w_0, w_1 - X_1, \ldots, w_n - X_n)$. The Borch's Pareto-optimal risk exchange under a weight vector $(1, \delta_1, \ldots, \delta_n)$ results in a final risk portfolio $\tilde{Y} = (\tilde{Y}_0, \ldots, \tilde{Y}_n)$ where (see, e.g., Wyler (1990) and Aase (2002)) $\tilde{Y}_i = \tilde{y}_i(X_M)$ are functions of the summary initial risk $X_M \overset{def}{=} \sum_1^n X_j$, satisfying

$$\tilde{y}_i'(x) = \rho_i(w_i - \tilde{y}_i(x))\{\sum_{k=0}^{n} \rho_k(w_k - \tilde{y}_k(x))\}^{-1}, \; i = 0, \ldots, n, \tag{14}$$

$$\begin{gathered} u_0'(w_0 - \tilde{y}_0(0)) = \delta_1 u_1'(w_1 - \tilde{y}_1(0)) = \ldots = \delta_n u_n'(w_n - \tilde{y}_n(0)), \\ \text{and } \textstyle\sum_0^n \tilde{y}_i(0) = 0. \end{gathered} \tag{15}$$

In the insurance market model, the final risk portfolio $\hat{Y}$ consists of $\hat{Y}_0 = \sum_1^n \hat{I}_j(X_j) - \sum_1^n \hat{P}_j$ and $\hat{Y}_j = X_j - \hat{I}_j(X_j) + \hat{P}_j, j = 1, \ldots, n$, where $\hat{I}_j$ and $\hat{P}_j$ are determined by (12) and (13). By construction, $\hat{Y}_0 = X_M - \sum_1^n \hat{Y}_j$ and, as $\hat{y}_j'(x) = 1 - I_j'(x)$, from (13) we have

$$\hat{y}_j'(x) = \rho_j(w_j - \hat{y}_j(x))\{\rho_j(w_j - \hat{y}_j(x)) + \rho_{0j}(x)\}^{-1}, \; j = 1, \ldots, n.$$

Note that the argument x in (14) is related to the total initial risk $X_M = x$, while in the equation above the argument is related to the j-th individual risk X_j. The right-hand side of (14) involves all the risk tolerances $\rho_i(\cdot)$ in distinction to the latter equation which in turn differs from (14) by that $\rho_{0j}(x)$ depends on $(n-1)$ distributions of risks X_s, $s \neq j$. In the case of risk exchange between the insurer and the only insured $(n = 1)$, by the definition of $\rho_{0j}(x)$ we, clearly, have

$\rho_{0j}(x) = \rho_0(w_0 + \hat{P} - \hat{I}(x))$ and (12)-(13) become the same as (15) and optimality equation (14) with $i = 0$.

Remark 4.
For the Borch's model of a reinsurance market, Wilson (1968) showed that the Pareto-optimal risk exchanges are affine functions of total loss, i.e., $\hat{y}_j(x) = \theta_j x + \gamma_j$ if the risk tolerances are affine with identical cautiousness, $\rho_j(x) = \alpha x + \beta_j$, $j = 1,\dots,n$. One might expect that a similar result remains valid for the considered model of an insurance market: under appropriate assumptions on $\rho_j(x)$, the Pareto-optimal policies $\hat{I}_j(x)$ are affine or, more exactly, linear as $\hat{I}_j(0) = 0$. However, the situation is more complicated because the risk tolerance ratio $\rho_{0j}(x)$ in equation (11) depends, in general, on distributions of insureds risks. Nevertheless, there are at least two cases where the linearity takes place.

1) Let the insurer and a j-th insured have exponential utility functions, $u_0(x) = c_0^{-1}(1 - \exp(-c_0 x))$ and $u_j(x) = c_j^{-1}(1 - \exp(-c_j x))$. Since $\rho_j(x) \equiv 1/c_j$ and the risk tolerance ratio $\rho_{0j}(x) \equiv 1/c_0$, from (11) (as well as from (9)) we have $\hat{I}_j'(x) = c_j/(c_0 + c_j)$ so that $\hat{I}_j(x) = \theta_j x$ with $\theta_j = c_j/(c_0 + c_j)$.

2) Let the insurer's and j-th insured's utility functions be quadratic, $u_0(x) = -\frac{1}{2}c_0 x^2 + x$ for $x < 1/c_0$, and $u_j(x) = -\frac{1}{2}c_j x^2 + x$ for $x < 1/c_j$. Note that $u_0''(x) \equiv -c_0$ and $u_j''(x) \equiv -c_j$, therefore from (5) it follows that $\hat{I}_j(x) = \theta_j x$ with $\theta_j = \delta_j c_j/(c_0 + \delta_j c_j)$. In distinction to the first case, the Pareto-optimal policy $\hat{I}_j(x)$ depends on the Pareto multiplier δ_j.

Example 1.
Examine a model where all the agents' utility functions are exponential, $u_i(x) = c_i^{-1}(1 - \exp(-c_i x))$, $i = 0,\dots,n$. Without loss of generality we can assume the initial capitals $w_i = 0$, since the i-th utility $J_i[P, I]$ depends on w_i only through a positive multiplier $\exp(-c_i w_i)$, which does not influence the contents of the Pareto-optimal set $\{(\hat{P}_\delta, \hat{I}_\delta)\}_{\delta > 0}$ in spite of a change in its parametrization.

In general, a solution to equation (9) can be parametrized by $\hat{P}$ entering the expressions for final capitals A and B_j. Then $\hat{P}$ is determined by (4) and, thus, depends on the weight vector δ. In the considered case, however, equation (9) is $I_j'(X_j) = c_j/(c_0 + c_j)$ and does not involve $\hat{P}$. Therefore

$$\hat{I}_j(x) = \frac{c_j}{c_0 + c_j} x, \quad j = 1, \dots, n,$$

are the only Pareto-optimal indemnity rule (as was already noted in Remark 4). To obtain the premiums, rewrite (4) as

$$E \exp\left\{- c_0 \left(\sum_{s=1}^{n} P_s - \sum_{s \neq j}^{n} \frac{c_s}{c_0 + c_s} X_s \right) \right\} = \delta_j \exp(c_j P_j), \quad \text{or}$$

$$\sum_{s \neq j}^{n} a_s - c_0 \sum_{s=1}^{n} P_s = \ln \delta_j + c_j P_j, \tag{16}$$

where $a_s \stackrel{def}{=} \ln \left\{ E \exp\left(\frac{c_0\, c_s}{c_0 + c_s} X_s \right) \right\} < \infty$.

Subtraction of (16) from the s-th such equation yields

$$a_j - a_s = \ln\delta_s - \ln\delta_j + c_s P_s - c_j P_j.$$

Expressing P_s from this equation and then inserting it into (16), we obtain

$$\hat{P}_j = \frac{1}{c_j}\left\{-\ln\delta_j - a_j + d\left[\sum_{s=1}^{n} c_s^{-1}(\ln\delta_s + a_s) + a/c_0\right]\right\}, \tag{17}$$

where we introduce d as defined by $d^{-1} = c_0^{-1} + \dots c_n^{-1}$, and $a = \sum_{s=1}^{n} a_s$.

From (16) we can also find the summary premium

$$\sum_{j=1}^{n} \hat{P}_j = \frac{d}{c_0}\sum_{j=1}^{n} c_j^{-1}\left(-\ln\delta_j + \sum_{s \neq j}^{n} a_s\right). \tag{18}$$

The set of Pareto-optimal contracts $\{(\hat{P}_\delta, \hat{I})\}_{\delta>0}$ consists thus of a single indemnity rule $\hat{I}$ and a range of premium vectors $\hat{P} = \hat{P}_\delta$ defined in (17). As is seen from (17), the premiums $\hat{P}$ depend on the distributions of the insured's losses X_j through the quantities $a_j = \ln\{E\exp(c_0\hat{I}_j(X_j))\}$, while the quotas $c_j/(c_0 + c_j)$ in indemnity functions are determined by the risk aversion parameters only. From (17) it follows that $\hat{P}_j$ decreases versus an increase in the insured's weight δ_j: a greater weight allows the j-th insured to reduce his premium $\hat{P}_j$.

Compare the insurer's situation in the presented setting with that in the Borch's model for the case of exponential utilities. For the insurance market model, the components of the final risk portfolio are

$$\hat{Y}_0 = \sum_{j=1}^{n} \frac{c_j}{c_0 + c_j} X_j - \sum_{j=1}^{n} \hat{P}_j, \; \hat{Y}_j = \frac{c_0}{c_0 + c_j} X_j + \hat{P}_j, \quad j = 1,\dots,n.$$

From (14)-(15) we get (see Aase (2002)) for the Borch model

$$\tilde{Y}_i = \frac{d}{c_i} X_M + \gamma_i, \; \gamma_i = -\frac{\ln\delta_i}{c_i} + \frac{d}{c_i}\sum_{s=0}^{n}\frac{\ln\delta_s}{c_s}, \quad i = 0,\dots,n$$

where, recall, $\delta_0 = 1$ so that $\gamma_0 = c_0^{-1} d \sum_1^n c_s^{-1} \ln\delta_s$. Since $c_0^{-1} d = [1 + \sum_1^n c_0/c_s]^{-1} < c_j/(c_0 + c_j) = [1 + c_0/c_j]^{-1}$, the indemnity $c_0^{-1} d \sum_1^n X_j$ taken by the insurer in the Borch model is less (a.s.) than that $\sum_1^n (c_0 + c_j)^{-1} c_j X_j$ in the former model. On the other hand, the payment $-\gamma_0$ is evidently less than the summary premium $\sum_1^n \hat{P}_j$ (see (18)) paid to the insurer, the surplus $c_0^{-1} d \sum_1^n c_j^{-1}\left(\sum_{s \neq j}^n a_s\right)$ depending on individual loss distributions through a_s, $s = 1,\dots,n$. Thus, in the Borch model the insurer is more discrete in the sense that he prefers to take a less summary loss coverage and a less premium.

Such a comparison with respect to the related insureds in the two models, unlike the previous case, does not give a relation with probability one. Although the retained insured's share of loss $(c_0 + c_j)^{-1} c_0 X_j$ in $\hat{Y}_j$ is greater (a.s.) than the share $c_j^{-1}dX_j$ retained by the j-th insured from his own initial risk in the Borch's model, the whole loss $c_j^{-1}dX_M$ of the latter insured depends also on the other risks X_s, $s \neq j$. Difference between premiums $\hat{P}_j$ and γ_j of the insureds may change the sign, based upon concrete values of a_s and c_s.

Example 2.
Examine a case of quadratic utility functions related to a homogeneous group of insureds. Here $u_i(x) = -\frac{1}{2}c_i x^2 + x$ defined on $(-\infty, 1/c_i)$, $i = 0,1$. Theorem 1 gives (see Remark 4) Pareto-optimal policies

$$\hat{I}(x) = \theta x, \text{ where } \theta = \delta c_1/(c_0 + \delta c_1), \ \delta \in (0,\infty).$$

Condition (4) takes the form

$$-Ec_0\left(w_0 + nP - \sum_{s=1}^{n-1}\theta X_s\right) + 1 = \delta(-c_1(w_1 - P) + 1),$$

whence we obtain an equation for determining $\hat{P}$:

$$\hat{P}(\delta c_1 + c_0 n) = c_0(n-1)\theta EX_1 + 1 - c_0 w_0 + \delta(c_1 w_1 - 1). \tag{19}$$

We see that if δ is fixed and the number of insureds $n \to \infty$ (thus, a weight $n\delta$ of the group as a whole increases to infinity) then from (19) it follows $\hat{P} \to \theta EX_1 = E\hat{I}(X_1)$ — the limiting premium is equal to the actuarial value of insurer's risk.

Choose $w_0 = 1$, $w_1 = 0$, $c_0 = c_1 = 0.01$, $n = 10$, and $EX_1 = 5$. For $\delta = 0.5$ we will have $\hat{I}(x) = 0.333x$ and, by (19), $\hat{P} = 6.093$; if the insured's weight δ increases to $\delta = 1$ then the insurer's share of risk becomes greater, $\hat{I}(x) = 0.5x$, while the premium decreases to $\hat{P} = 1.954$.

4. Individual rationality

In this section we impose additional constraints on the set of admissible contracts, namely, an admissible contract (P, I) must now satisfy the individual rationality conditions: $J_i[P,I] \geq J_i[0,0]$, $i = 0,\ldots,n$. Here the agents's initial utilities before making a contract are: $J_0[0,0] = u_0(w_0)$ for the insurer, and $J_j[0,0] = Eu_j(w_j - X_j)$, $j = 1,\ldots,n$, for the insureds. Following the lines in the previous section, our main objective is to find a characterization of the Pareto-optimal individually rational contract, i.e. the solution to the problem

$$\text{maximize } J[P,I] \equiv J_0[P,I] + \sum_{j=1}^{n}\delta_j J_j[P,I] \tag{20}$$

subject to

$$J_0[P,I] \geq u_0(w_0), \ J_j[P,I] \geq Eu_j(w_j - X_j), \ j = 1,\ldots,n. \tag{21}$$

Proposition 1. *For each fixed weight vector* $\delta > 0$, *problem* (20)-(21) *has a unique solution.*

The proof is given in Appendix.

Theorem 2. *Let the Slater condition hold: there is a contract* (P^0, I^0) *such that all the inequalities in* (21) *are strictly satisfied under* $(P, I) = (P^0, I^0)$. *An admissible contract* $(\hat{P}, \hat{I})$ *solves* (20)-(21) *if and only if there exist* $\lambda_i \geq 0$, $i = 0, \ldots, n$ *such that*

$$\lambda_0(J_0[\hat{P}, \hat{I}] - u_0(w_0)) = 0,$$
$$\lambda_j(J_j[\hat{P}, \hat{I}] - Eu_j(w_j - X_j)) = 0, \; j = 1, \ldots, n, \tag{22}$$

and condition (3) *in Theorem* 1 (*or its equivalent* (4)-(5)) *holds with* $\delta'_j = \frac{\delta_j + \lambda_j}{1 + \lambda_0}$, $j = 1, \ldots, n$.

Proof. The Lagrangian of problem (20)-(21) is

$$\mathcal{L}[P, I; \lambda] = J_0[P, I] + \sum_{j=1}^{n} \delta_j J_j[P, I] + \sum_{i=0}^{n} \lambda_i (J_i[P, I] - J_i[0, 0]). \tag{23}$$

As is known (see, e.g., Bazaraa and Shetty (1979)), in the case of a concave problem (20)-(21) under the Slater condition, a maximizer in it is also a maximizer of $\mathcal{L}[P, I; \lambda]$ over $(P, I) : 0 \leq I(x) \leq x$ for an appropriate vector $\lambda \geq 0$ that satisfies the complementary slackness condition (22). Conversely, a maximizer $(\hat{P}, \hat{I})$ of $\mathcal{L}[P, I; \lambda]$ satisfying (21) is a solution to (20)-(21). In this sense, maximization of the Lagrangian is equivalent to solving the initial problem. After rearranging terms in (23) and division by $1 + \lambda_0$, the goal functional can be written as $J_0[P, I] + \sum_{j=1}^{n} \delta'_j J_j[P, I]$, where $\delta'_j = (\delta_j + \lambda_j)/(1 + \lambda_0)$. This coincides with the goal functional in (1) if the weights δ_j in (1) are changed to δ'_j. □

One may roughly note that individual rationality constraints (21) lead to the following change in the optimal $(\hat{P}, \hat{I}) = (\hat{P}_\delta, \hat{I}_\delta)$. Let first, under some δ, the solution be an interior point for (21). If the weight, say, δ_1 decrease then, as can be seen, $J_1[\hat{P}_\delta, \hat{I}_\delta]$ also decreases up to a point where the constraint for J_1 becomes binding. Further decrease in δ_1 causes a compensating increase in the new weight δ'_1 introduced in Theorem 2 so that the solution $(\hat{P}_{\delta'}, \hat{I}_{\delta'})$ of (1) "glues" to the boundary, making the equality $J_1[\hat{P}_{\delta'}, \hat{I}_{\delta'}] = J_1[0, 0]$ still hold. In the case of a homogeneous group (see Remark 3) with a scalar Pareto multiplier δ, one may expect a range $[\delta_{\min}, \delta_{\max}]$ such that any interior δ defines an interior solution to (20)-(21), and $J_1[\hat{P}_\delta, \hat{I}_\delta] \equiv J_1[0, 0]$ for all $\delta \leq \delta_{\min}$, and $J_0[\hat{P}_\delta, \hat{I}_\delta] = J_0[0, 0]$ for all $\delta \geq \delta_{\max}$. We illustrate it in the example below.

Example 3.
Returning to Example 1, consider a particular case related to the homogeneous group of insureds having identical exponential utility functions $u_j(x) = c_1^{-1}(1 - \exp(-c_1 x))$, $j = 1, \ldots, n$. As we noted above, in this situation a contract (P, I) consists of a scalar P and a scalar indemnity function $I(x)$, the Pareto multiplier $\delta \in (0, \infty)$.

As was shown in Example 1, a unique Pareto-optimal policy is a linear function

$$\hat{I}(x) = \frac{c_1}{c_0 + c_1} x. \tag{24}$$

Let us take this as a testing function I^0 in verification of the Slater condition in Theorem 2. Then the minimum premium $P_{\min}$ admissible for the insurer under policy (24) is determined by $Eu_0(nP_{\min} - \sum_1^n \hat{I}(X_j)) = u_0(0)$ end equal to

$$P_{\min} = a_1 / c_0 (>0), \text{ where } a_1 = \ln\{E \exp(\frac{c_0\, c_1}{c_0 + c_1} X_1)\}. \tag{25}$$

Equation $Eu_1(-P_{\max} - X_1 + \hat{I}(X_1)) = Eu_1(-X_1)$ gives the maximum premium admissible for the insured,

$$P_{\max} = (\ln[E \exp(c_1 X_1)] - a_1)/c_1. \tag{26}$$

Using Jensen's inequality for the case of a strictly convex power function x^d with $d = 1 + c_1/c_0$, we obtain $[E \exp(c_1 X_1/d)]^d < E \exp(c_1 X_1)$ and, hence, $P_{\min} < P_{\max}$. Therefore, choosing any $P^0 \in (P_{\min}, P_{\max})$ suffices for the inequalities $J_0[P, I] > J_0[0,0]$ and $J_1[P, I] > J_1[0,0]$ to be satisfied under $(P, I) = (P^0, \hat{I})$.

Applying Theorem 2, from (17) in Example 1 we have that the Pareto-optimal premium is

$$\hat{P} = \frac{(n-1)\, a_1 - \ln\delta}{c_1 + nc_0}. \tag{27}$$

To determine the admissible range of $\hat{P} = \hat{P}_\delta$, note that the right-hand side of (27) is a decreasing function in δ that takes all values in $(-\infty, \infty)$ when δ runs over $(0, \infty)$. So the range of admissible $\hat{P}_\delta$ is $[P_{\min}, P_{\max}]$ (see (25)-(26)). The set of individually rational Pareto-optimal contracts is thus $[P_{\min}, P_{\max}] \times \hat{I}$. Equating (27) to $P_{\min}$ and then to $P_{\max}$, we get the corresponding boundaries $\delta_{\max}$ and $\delta_{\min}$. If the weight $\delta \le \delta_{\min}$ then $\hat{P}_\delta = P_{\max}$, if $\delta \ge \delta_{\max}$ then $\hat{P}_\delta \equiv P_{\min}$.

Suppose the insureds' losses X_j to be uniformly distributed on $[0,1]$. Choose the insurer's risk aversion coefficient $c_0 = 0.2$, regarding c_1 and n as parameters (recall that $w_0 = \ldots = w_n = 0$). Let c_1 take values in $\{0.1, 0.5, 1.5\}$, then the Pareto-optimal policy $\hat{I}(x)$ becomes $0.333x$, $0.714x$, and $0.882x$ respectively. Values of $P_{\min}$ and $P_{\max}$ are determined from (25) and (26) where now $a_1 = \ln\{d(\exp(1/d) - 1)\}$ with $d = (c_0 + c_1)/(c_0\, c_1)$. The numerical results are summarized in the table below. It is seen that $P_{\min}$ and $P_{\max}$ increase with an increase in the insured's risk aversion c_1, while the number of insureds n affects the left boundary $\delta_{\min}$ only (actually, $\delta_{\min} \to 0$ when $n \to \infty$, as easily follows from (26)-(27)).

Debreu and Scarf (1963) proved for a market model that the core of the corresponding cooperative game shrinks to the point of competitive equilibrium as the number of players infinitely increase. It is not the case in Example 3 because the interval $[P_{\min}, P_{\max}]$ (that can be regarded as a "bid-ask spread" in terms of a financial market) does not change with n. Application of the game theoretical concepts, such as the Nash's solution, will perhaps lead to a

c_1	n	$\delta_{\min}$	$\delta_{\max}$	$P_{\min}$	$P_{\max}$
0.1	10	0.948	0.951	0.167	0.169
0.1	150	0.912	0.951	0.167	0.169
0.5	10	0.748	0.776	0.361	0.376
0.5	150	0.494	0.776	0.361	0.376
1.5	10	0.387	0.467	0.448	0.502
1.5	150	0.085	0.467	0.448	0.502

solution $P_n^* \in [P_{\min}, P_{\max}]$ whose limit as $n \to \infty$ (if exists) bears a relationship to notion of a market equilibrium.

Appendix

The proof of Proposition 1.

Note first that an equality $I_j^1(x) = I_j^2(x)$ is understood as a coincidence $I_j^1(X_j) = I_j^2(X_j)$ with probability one. By assumption, the utility functions $u_j(x), j = 1,\ldots,n$, are strictly concave, therefore the functional $\sum_{j=1}^n \delta_j J_j[P_j, I_j]$ is also strictly concave in (P, I). Since $J_0[P,I] = Eu_0(w_0 + \sum_{s=1}^n P_s - I_s(X_s))$ is concave, the goal functional in (20) is strictly concave in (P, I), which implies that if there exists a maximizer then it is unique.

To prove the existence of a solution to problem (20)-(21), show first that vectors P of admissible premiums satisfying (21) constitute a bounded set in R^n. Since $J_j[P,I] = Eu_j(w_j - P_j - X_j + I_j(X_j)) \geq J_j[0,0] = Eu_j(w_j - X_j)$ and $J_j[P,I] \leq u_j(w_j - P_j)$, we have $w_j - P_j \geq u_j^{-1}(J_j[0,0])$ where u_j^{-1} denotes the inverse of u_j, or $P_j \leq w_j - u_j^{-1}(J_j[0,0])$ for $j = 1,\ldots,n$. On the other hand, $J_0[P,I] = Eu_0(w_0 + \sum_{s=1}^n P_s - I_s(X_s)) \geq J_0[0,0] = u_0(w_0)$ therefore $w_0 + \sum_1^n P_s \geq u_0^{-1}(J_0[0,0]) = w_0$, or $\sum_1^n P_s \geq 0$. Thus the set of admissible premiums is bounded.

Denote by Π the set of admissible in (20)-(21) risk allocations $(P, I(X))$ where $I(X) \stackrel{def}{=} (I_1(X_1),\ldots,I_n(X_n))$ with $0 \leq I_j(x) \leq x$. Fix any sequence $(P^m, Y^m) \subset \Pi$ and prove the existence of a subsequence (P^k, Y^k) such that (a) $P^k \to P'$ and (b) $Y^k \Rightarrow Y'$ (weak convergence) as $k \to \infty$, for some $P' \in R^n$ and $Y' = I'(X)$. In view of the above-proved boundedness of P^m, (a) evidently holds. By Helly's theorem, convergence in (b) takes place but, in general, $P\{Y_1' < \infty,\ldots, Y_n' < \infty\} \leq 1$. Noting that $Y_j^k \leq X_j$ a.s. and, hence, the distribution function $F^k(x_1,\ldots,x_n) \geq F(x_1,\ldots,x_n)$, we have $P\{Y_1' < \infty,\ldots,Y_n' < \infty\} = 1$, i.e., Y' is proved to be a proper stochastic vector. Since each component Y_j' is measurable with respect to sigma-algebra $\sigma(X_j)$ and $0 \leq Y_j' \leq X_j$ a.s., the limit Y' can be represented as $Y' = I'(X)$ for some indemnity rule $I'(x)$, i.e., (b) holds.

Consider a sequence $(P^m, I^m(X)) \subset \Pi$ maximizing (20), that is,

$$\lim_{m \to \infty} J[P^m, I^m] = J^* \stackrel{def}{=} \sup_{(P,I) \in \Pi} J[P, I].$$

By the reasonings above, there exists a subsequence such that

$$P^k \to P^*,\ I^k(X) \Rightarrow I^*(X) \ \text{ as } \ k \to \infty. \tag{28}$$

In order to show

$$J^* = J[P^*, I^*] \ \text{ and } \ (P^*, I^*) \in \Pi, \tag{29}$$

it suffices to prove that $J_0[P^k, I^k] \to J_0[P^*, I^*]$ and $J_j[P_j^k, I_j^k] \to J_j[P_j^*, I_j^*]$ for all $j = 1,\dots,n$. In turn, these convergences hold if the integrands in

$$J_0\left[P^k, I^k\right] = \int_{R_+^n} u_0\left(w_0 + \sum_{s=1}^{n} P_s^k - x_s\right) dF^k(x_1, \dots, x_n), \tag{30}$$

$$J_j\left[P_j^k, I_j^k\right] = \int_{R_+^n} u_j\left(w_j - P_j^k - x\right) dP\left\{X_j - I^k\left(X_j\right) \le x\right\} \tag{31}$$

are uniformly integrable with respect to the corresponding sequence of distributions (Tucker (1967)). (Note that if all *supp* F_j are bounded then (29) is simply implied by (28) along with continuity and boundedness of the integrands in (30) and (31) on the relevant supports, without employing the uniform integrability.) Since u_0 is increasing and concave, $u_0(w_0 + \sum_{s=1}^n P_s^k - x_s) \le 0$ on $\{x_s \ge a,\ s = 1,\dots,n\}$ if a is large enough. Hence

$$0 \ge \int_{\{x \ge a\}} u_0\left(w_0 + \sum_{s=1}^{n} P_s^k - x_s\right) dF^k(x_1, \dots, x_n) \ge$$

$$\int_{\{x \ge a\}} u_0\left(w_0 + \sum_{s=1}^{n} P_s^k - x_s\right) dF(x_1, \dots, x_n) \to 0 \text{ as } a \to \infty.$$

$$\text{Therefore } \lim_{a \to \infty} \sup_k \int_{\{x \ge a\}} u_0\left(w_0 + \sum_{s=1}^{n} P_s^k - x_s\right) dF^k(x_1, \dots, x_n) = 0.$$

The uniform integrability of the integrand in (31) is proved by the same reasonings. Thus we obtain (29), which completes the proof. □

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