THE EXPONENTIAL PREMIUM CALCULATION PRINCIPLE REVISITED

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ABSTRACT

In this paper, it is shown how to approximate theoretical premium calculation principles in order to make them useful in practice. The method relies on stochastic extrema in moment spaces and is illustrated with the aid of the exponential principle.

KEYWORDS

Premium Calculation Principles, Moment Spaces, S-Convex Orderings, Stochastic Extrema.

I INTRODUCTION

In actuarial practice, most insurance companies base their tariff on the expected value principle, with a loading coefficient $\theta > 0$, say (for more details about premium calculation principles, the interested reader is referred e.g. to Goovaerts, De Vylder and Haezendonck (1984)); this is to say that they require for the risk X an amount of premium $(1 + \theta)EX$. The reason is, of course, mathematical convenience but this is also often due to a shortage of available statistical data. Against this principle, one can convincingly argue that two risks with the same mean may appear very different whereas the price list will give the same amount of premium for both of them. As an example, think of the family of the normal distributions with fixed mean μ_0 and parameterized by the variance σ^2. The premium will be constant (and equal to $(1 + \theta)\mu_0$) but everybody agrees that the underlying danger will strongly vary. One could therefore opt for the variance principle (with a safety loading proportional to the variance of the risk X) or for the standard deviation principle (with a safety loading proportional to the standard deviation of X). But even risks with identical first two moments may appear very different from the insurer's point of view. The skewness is also an important parameter for the insurance company. This speaks in favour of a premium calculation principle taking into account the whole probability

distribution of the risk X to be covered, instead of only a few numerical characteristics of it. Such principles exist in the actuarial literature (for instance, the Orlicz principle, the Esscher principle or the risk adjusted premium calculation principles introduced by Wang (1996) and recently considered by Wang and Young (1998) and by Silva and Centeno (1998)). In practical business, however, statistical data concerning the risks to be covered are often so scarce that it is hopeless to determine a plausible distribution for X. It is therefore important to construct premium calculation principles which need as less information as possible. The premium principles proposed here are based on the first moments, range and mode. It tries to provide a compromise solution between those two apparently conflicting goals.

The purpose of this paper is to show that theoretical premium calculation principles can also be used in practice, at least if you agree to compute the premium amount, not on the actual risk, but rather on a stochastic upper bound for this risk in its moment space. The premium amount you get then depends upon the mean, the variance, the skewness, the kurtosis, the range and, possibly, the mode of the risk. We illustrate the technique on the zero utility premium calculation principle with an exponential utility function (i.e., on the so-called exponential premium principle). We derive simple analytical lower and upper bounds for exponential premiums. Of course, the ideas developed in the present paper (which theoretically reduce to the derivation of bounds on the moment generating function of a random variable X in a given moment space) can be applied to many other problems in actuarial sciences, as well as in applied probability and economics.

We mention that the approach adopted here is very similar to the ideas contained in Hürlimann (1996). This author obtained simple analytical lower and upper bounds for stop-loss premiums and ruin probabilities of compound Poisson risks in case the mean, the variance, and the range of the claim size distributions are known. The two approaches are complementary since with Hürlimann's method, you get bounds on quantities of the form $E\phi(X)$ when ϕ is non-decreasing and/or convex while the methods proposed here deals with quantities of the form $E\phi(X)$ with smoother functions ϕ (namely, convex functions of degree s).

2 EXPONENTIAL PREMIUM PRINCIPLE

Consider an insurance company with initial wealth w and with a utility function $u(.)$. The company covers a risk X. The amount of premium $\pi(X)$ is determined following the adoption of an economic decision principle. We assume here that the insurance company sets its price for coverage $\pi(X)$ as the solution of the equation

$$Eu(w + \pi(X) - X) = u(w). \tag{2.1}$$

Condition (2.1) expresses that the premium $\pi(X)$ is fair in terms of utility: the right-hand side of (2.1) represents the utility of not issuing the contract; the left-hand side of (2.1) represents the expected utility of the insurer assuming the random financial loss X. Therefore (2.1) means that the expected utility of wealth with the contract is equal to the utility without the contract. Putting $w = 0$ and normalizing the utility function u, we get the so-called zero utility principle: the premium $\pi(X)$ calculated according to this principle is the root of the equation

$$Eu(\pi(X) - X) = 0 \qquad (2.2)$$

which can be interpreted as an equality between the expected utility of the income $\pi(X) - X$ and the utility of not accepting the risk. Despite of its intuitive justification, such a principle is difficult to apply in practical business, since the equation (2.2) determining $\pi(X)$ has no explicit solution in general. However, if we assume that the moment generating function of X exists and that the utility function of the insurance company is of the form

$$u(x) = 1 - e^{-cx}, \quad x \geq 0, \qquad (2.3)$$

for some positive constant c, then (2.2) admits an explicit solution and the premium $\pi(X)$ can be expressed as

$$\pi(X) = \frac{1}{c} \ln Ee^{cX}. \qquad (2.4)$$

The constant c involved in (2.3) is equal to $-\dfrac{d}{dx} \ln \dfrac{d}{dx} u$ and measures the risk aversion of the insurance company; the exponential utility function (2.3) yields a constant risk aversion. But even formula (2.4) will not convince the practitioner of adopting the exponential premium calculation principle and of forsaking his good old expected value principle. Indeed, in order to compute $\pi(X)$, the underlying distribution of X has to be completely known, or at least to be assumed. Therefore, there is a need of approximations for (2.4) if you desire to use such a formula in practice.

3 S-CONVEX APPROXIMATIONS

The idea used here is as follows: Denuit, De Vylder and Lefèvre (1999) (see also Denuit and Lefèvre (1997) and Denuit, Lefèvre and Shaked (1998)) showed that in a given moment space, i.e. among all the risks with common range and first moments, it is possible to determine a minimum and a maximum with respect to some stochastic order relation (for more details about stochastic order relations, the interested reader is referred e.g. to Goovaerts, Kaas, Van Heerwaarden and Bauwelincks (1990), Kaas, Van Heerwaarden and Goovaerts (1994) and Shaked and Shanthikumar (1994)). More precisely, let s be a positive integer and consider the class

$\mathcal{B}_s \equiv \mathcal{B}_s([0,b]; \mu_1, \mu_2, ..., \mu_{s-1})$ of all the risks X with upper bound b and such that $EX^k = \mu_k$ for $k = 1, 2, ..., s - 1$. Let us define the partial order relation $\preceq_{s-cx}$ among elements in $\mathcal{B}_s$ as

$$X \preceq_{s-cx} Y \Leftrightarrow Eu(X) \le Eu(Y) \text{ for all } u \in \bar{\mathcal{U}}_{s-cx} \qquad (3.1)$$

for which the expectations exist, where $\bar{\mathcal{U}}_{s-cx}$ is the closure (in the topology of the pointwise convergence) of the class

$$\mathcal{U}_{s-cx} = \left\{ u : [0,b] \to \mathbb{R} | u^{(s)} \ge 0 \; [0,b] \right\},$$

where $u^{(s)}$ denotes the s-th derivative of the function u. It is worth mentioning that the functions in $\bar{\mathcal{U}}_{s-cx}$ are usually referred to as convex functions of degree s (see, e.g., Roberts and Varberg (1973)). The classes $\cap_{k=1}^{s} \mathcal{U}_{k-cx}$ have been extensively used in economics (see, e.g., Levy (1992)), as well as in actuarial sciences by Kaas and Hesselager (1995) in order to define the so-called stop-loss order of degree $s - 1$.

It is then possible to determine in $\mathcal{B}_s$ two discrete risks $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$, say, with a probability distribution relying only on the upper bound b, and the sequence of moments $(\mu_1, \mu_2, ..., \mu_{s-1})$ defining $\mathcal{B}_s$, such that

$$X_{\min}^{(s)} \preceq_{s-cx} X \preceq_{s-cx} X_{\max}^{(s)} \; \forall \, X \in \mathcal{B}_s. \qquad (3.2)$$

Explicit expressions of these extrema for $s = 1$ to 5 are obtained from the theory given in Denuit, De Vylder and Lefèvre (1999); they are listed in Tables 1 and 3.

Let $\mathcal{B}_s^* \equiv \mathcal{B}_s^*([0,b]; m - \text{unim}; \mu_1, \mu_2, ..., \mu_{s-1})$ be the class of all the risks X with a unique mode m, with upper bound b and such that $EX^k = \mu_k$ for $k = 1, 2, ..., s - 1$. It is possible to find in $\mathcal{B}_s^*$ two risks $X_{\min}^{(s)*}$ and $X_{\max}^{(s)*}$, say, with a probability distribution relying only on the upper bound b, the mode m, and the sequence of moments $(\mu_1, \mu_2, ..., \mu_{s-1})$ defining $\mathcal{B}_s^*$, such that

$$X_{\min}^{(s)*} \preceq_{s-cx} X \preceq_{s-cx} X_{\max}^{(s)*} \; \forall \, X \in \mathcal{B}_s^*. \qquad (3.3)$$

Let us point out that, since $\mathcal{B}_s^* \subset \mathcal{B}_s$, the extrema in (3.3) are more accurate than those in (3.2) in the sense that

$$X_{\min}^{(s)} \preceq_{s-cx} X_{\min}^{(s)*} \quad \text{and} \quad X_{\max}^{(s)*} \preceq_{s-cx} X_{\max}^{(s)}.$$

Explicit expressions of these improved extrema can be obtained from the theory given in Denuit, De Vylder and Lefèvre (1999) and are now discrete mixtures of uniform distributions; they are listed in Tables 2 and 4 for $s = 1$ to 5, where Unif $[\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$, stands for the uniform distribution over the interval $[\min(\alpha, \beta), \max(\alpha, \beta)]$, $\sum_{i=1}^{k} p_i \text{Unif} [\alpha_i, \beta_i]$, $0 \le p_i \le 1$, $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2, ..., k$, represents a mixture of the distributions Unif $[\alpha_i, \beta_i]$, with

respective weights p_i, i.e., a random variable with distribution function $\sum_{i=1}^{k} p_i F_i(x)$ where

$$F_i(x) = \begin{cases} 0 & \text{if } x < \min(\alpha_i, \beta_i), \\[2mm] \dfrac{x - \min(\alpha_i, \beta_i)}{\max(\alpha_i, \beta_i) - \min(\alpha_i, \beta_i)} & \text{if } x \in [\min(\alpha_i, \beta_i),\ \max(\alpha_i, \beta_i)], \\[2mm] 1 & \text{if } x > \max(\alpha_i, \beta_i), \end{cases}$$

and where the following symbols are used:

$$\rho = ((\mu_1 - b)(\mu_4 - b\mu_3) - (\mu_2 - b\mu_1)(\mu_3 - b\mu_2))^2$$
$$- 4\Big((\mu_1 - b)(\mu_3 - b\mu_2) - (\mu_2 - b\mu_1)^2\Big)\Big((\mu_2 - b\mu_1)(\mu_4 - b\mu_3) - (\mu_3 - b\mu_2)^2\Big),$$

and

$$\tilde{\mu}_j = (j+1)\mu_j - jm\mu_{j-1}, \quad j \in \mathbb{N}_0,$$

with the convention that $\mu_0 \equiv 1$.

TABLE 1

PROBABILITY DISTRIBUTION OF $X_{\min}^{(s)} \in \mathcal{B}_s([0,b]; \mu_1, \mu_2, ..., \mu_{s-1})$

s	Support points	Probability masses
1	0	1
2	μ_1	1
3	0	$\dfrac{\mu_2 - \mu_1^2}{\mu_2}$
	$\dfrac{\mu_2}{\mu_1}$	$\dfrac{\mu_1^2}{\mu_2}$
4	$r_+ = \dfrac{\mu_3 - \mu_1\mu_2 + \sqrt{(\mu_3 - \mu_1\mu_2)^2 - 4(\mu_2 - \mu_1^2)(\mu_1\mu_3 - \mu_2^2)}}{2(\mu_2 - \mu_1^2)}$	$\dfrac{\mu_1 - r_-}{r_+ - r_-}$
	$r_- = \dfrac{\mu_3 - \mu_1\mu_2 - \sqrt{(\mu_3 - \mu_1\mu_2)^2 - 4(\mu_2 - \mu_1^2)(\mu_1\mu_3 - \mu_2^2)}}{2(\mu_2 - \mu_1^2)}$	$1 - \dfrac{\mu_1 - r_-}{r_+ - r_-}$
5	0	$1 - p_+ - p_-$
	$t_+ = \dfrac{\mu_1\mu_4 - \mu_2\mu_3 + \sqrt{(\mu_1\mu_4 - \mu_2\mu_3)^2 - 4(\mu_1\mu_3 - \mu_2^2)(\mu_2\mu_4 - \mu_3^2)}}{2(\mu_1\mu_3 - \mu_2^2)}$	$p_+ = \dfrac{\mu_2 - t_-\mu_1}{t_+(t_+ - t_-)}$
	$t_- = \dfrac{\mu_1\mu_4 - \mu_2\mu_3 - \sqrt{(\mu_1\mu_4 - \mu_2\mu_3)^2 - 4(\mu_1\mu_3 - \mu_2^2)(\mu_2\mu_4 - \mu_3^2)}}{2(\mu_1\mu_3 - \mu_2^2)}$	$p_- = \dfrac{\mu_2 - t_+\mu_1}{t_-(t_- - t_+)}$

TABLE 2

PROBABILITY DISTRIBUTION OF $X_{\min}^{(s)*} \in \mathcal{B}_s^*([0,b];\ m-\text{unim};\ \mu_1,\ \mu_2,\ ...,\ \mu_{s-1})$ WHERE $\tilde{r}_\pm,\ \tilde{z}_\pm$ AND $\tilde{p}_\pm$ ARE THOSE FROM TABLE 1, WITH THE $\tilde{\mu}_j$'S SUBSTITUTED FOR THE μ_j'S

s	Distributions
1	Unif $[0,m]$
2	Unif $[m,\tilde{\mu}_1]$
3	$\dfrac{\tilde{\mu}_2-\tilde{\mu}_1^2}{\tilde{\mu}_2}$ Unif $[0,m]+\dfrac{\tilde{\mu}_1^2}{\tilde{\mu}_2}$ Unif $[m,\ \tilde{\mu}_2/\tilde{\mu}_1]$
4	$\dfrac{\tilde{\mu}_1-\tilde{r}_-}{\tilde{r}_+ - \tilde{r}_-}$ Unif $[m,\tilde{r}_+]+\left(1-\dfrac{\tilde{\mu}_1-\tilde{r}_-}{\tilde{r}_+-\tilde{r}_-}\right)$ Unif $[m,\tilde{r}_-]$
5	$(1-\tilde{p}_+-\tilde{p}_-)$Unif $[0,m]+\tilde{p}_+$ Unif $[m,\tilde{t}_+]+\tilde{p}_-$ Unif $[m,\tilde{t}_-]$

TABLE 3

PROBABILITY DISTRIBUTION OF $X_{\max}^{(s)} \in \mathcal{B}_s([0,b];\mu_1,\ \mu_2,\ ...,\ \mu_{s-1})$

s	Support points	Probability masses
1	b	1
2	0	$\dfrac{b-\mu_1}{b}$
	b	$\dfrac{\mu_1}{b}$
3	$\dfrac{b\mu_1-\mu_2}{b-\mu_1}$	$\dfrac{(b-\mu_1)^2}{(b-\mu_1)^2+\mu_2-\mu_1^2}$
	b	$\dfrac{\mu_2-\mu_1^2}{(b-\mu_1)^2+\mu_2-\mu_1^2}$
4	0	$1-p_1-p_2$
	$\dfrac{\mu_3-b\mu_2}{\mu_2-b\mu_1}$	$p_1=\dfrac{(\mu_2-b\mu_1)^3}{(\mu_3-b\mu_2)(\mu_3-2b\mu_2+b^2\mu_1)}$
	b	$p_2=\dfrac{\mu_1\mu_3-\mu_2^2}{b(\mu_3-2b\mu_2+b^2\mu_1)}$
5	$z_+=\dfrac{(\mu_1-b)(\mu_4-b\mu_3)-(\mu_2-b\mu_1)(\mu_3-b\mu_2)+\sqrt{\rho}}{2\Big((\mu_1-b)(\mu_3-b\mu_2)-(\mu_2-b\mu_1)^2\Big)}$	$p_+=\dfrac{\mu_2-(b+z_-)\mu_1+bz_-}{(z_+-z_-)(z_+-b)}$
	$z_-=\dfrac{(\mu_1-b)(\mu_4-b\mu_3)-(\mu_2-b\mu_1)(\mu_3-b\mu_2)-\sqrt{\rho}}{2\Big((\mu_1-b)(\mu_3-b\mu_2)-(\mu_2-b\mu_1)^2\Big)}$	$p_-=\dfrac{\mu_2-(b+z_+)\mu_1+bz_+}{(z_--z_+)(z_--b)}$
	b	$1-p_+-p_-$

TABLE 4

PROBABILITY DISTRIBUTION OF $X_{max}^{(s)*} \in \mathcal{B}_s^*([0, b]; m - \text{unim}; \mu_1, \mu_2, ..., \mu_{s-1})$ WHERE $\tilde{p}_1, \tilde{p}_2, \tilde{z}_\pm$ AND $\tilde{p}_\pm$ ARE THOSE FROM TABLE 3, WITH THE $\tilde{\mu}_j$'s SUBSTITUTED FOR THE μ_j's

s	Distribution
1	$\text{Unif } [m, b]$
2	$\dfrac{b - \tilde{\mu}_1}{b} \text{Unif } [0, m] + \dfrac{\tilde{\mu}_1}{b} \text{Unif } [m, b]$
3	$\dfrac{(b - \tilde{\mu}_1)^2}{(b - \tilde{\mu}_1)^2 + \tilde{\mu}_2 - \tilde{\mu}_1^2} \text{Unif } \left[m, \dfrac{b\tilde{\mu}_1 - \tilde{\mu}_2}{b - \tilde{\mu}_1} \right] + \dfrac{\tilde{\mu}_2 - \tilde{\mu}_1^2}{(b - \tilde{\mu}_1)^2 + \tilde{\mu}_2 - \tilde{\mu}_1^2} \text{Unif } [m, b]$
4	$(1 - \tilde{p}_1 - \tilde{p}_2)\text{Unif } [0, m] + \tilde{p}_1 \text{Unif } \left[m, \dfrac{\tilde{\mu}_3 - b\tilde{\mu}_2}{\tilde{\mu}_2 - b\tilde{\mu}_1} \right] + \tilde{p}_2 \text{Unif } [m, b]$
5	$\tilde{p}_+ \text{Unif } [m, \tilde{z}_+] + \tilde{p}_- \text{Unif } [m, \tilde{z}_-] + (1 - \tilde{p}_+ - \tilde{p}_-)\text{Unif } [m, b]$

4 PRACTICAL EXPONENTIAL PREMIUM PRINCIPLE

Let $(\mu_1, \mu_2, ..., \mu_{s-1})$ be the $s - 1$ first moments of a risk X to be covered, X assumed to be valued in $[0, b]$, $b \in \mathbb{R}^+$. Since the function $x \mapsto \exp(cx)$ belongs to $\mathcal{U}_{s-cx}$ for all s when $c > 0$, we get from (3.1) together with (3.2) that

$$\pi(X_{min}^{(s)}) \le \pi(X) \le \pi(X_{max}^{(s)}). \tag{4.1}$$

Moreover, if X possesses a unique mode m, we get from (3.1) together with (3.3) that

$$\pi(X_{min}^{(s)*}) \le \pi(X) \le \pi(X_{max}^{(s)*}). \tag{4.2}$$

Explicit expressions of the bounds (4.1) and (4.2) on $\pi(X)$ are listed in Tables 5, 6, 7 and 8. Let us point out that

$$\pi(X_{min}^{(2)}) = \mu_1,$$

so that the expected value principle (with $\theta = 0$) can be seen as an approximation of the exponential principle.

For each value of s, we thus get a margin $\left[\pi(X_{min}^{(s)}), \pi(X_{max}^{(s)}) \right]$ for the premium $\pi(X)$ in terms of the range and the first $s - 1$ moments of X. We even find a more accurate margin $\left[\pi(X_{min}^{(s)*}), \pi(X_{max}^{(s)*}) \right]$ for $\pi(X)$ when X is known to have a unimodal density. In practice, the *modus operandi* should be as follows: the actuary computes the bounds $\pi(X_{min}^{(s)})$ and $\pi(X_{max}^{(s)})$ on the basis of the knowledge of X he has at his disposal. If he charges an amount of premium of $\pi(X_{max}^{(s)})$, the latter includes a positive safety loading. Moreover, the maximal error is given either by $\pi(X_{max}^{(s)}) - \pi(X_{min}^{(s)})$ or by $\pi(X_{max}^{(s)*}) - \pi(X_{min}^{(s)*})$. The main advantage of this method lies in the fact that it only requires the knowledge of the mean, the variance, the skewness, the kurtosis, and possibly, of the mode (for $s = 5$). The amount of premium

charged takes all these characteristics into account. It seems therefore to be an interesting compromise solution between theory and practice. It can also be used as a measure of the underlying danger. The actuary can calculate the aforementioned margins and verify whether the amount of premium he determined (with the help of the expected value principle, for instance) seems reasonable.

TABLE 5

LOWER BOUNDS ON $\pi(X)$

s	$\pi(X_{\min}^{(s)})$
2	μ_1
3	$\dfrac{1}{c}\ln\left\{\dfrac{\mu_2 - \mu_1^2}{\mu_2} + \dfrac{\mu_1^2}{\mu_2}\exp\left(\dfrac{c\mu_2}{\mu_1}\right)\right\}$
4	$\dfrac{1}{c}\ln\left\{\left(1 - \dfrac{\mu_1 - r_-}{r_+ - r_-}\right)\exp(cr_-) + \dfrac{\mu_1 - r_-}{r_+ - r_-}\exp(cr_+)\right\}$
5	$\dfrac{1}{c}\ln\{(1 - p_+ - p_-) + p_+\exp(ct_+) + p_-\exp(ct_-)\}$

TABLE 6

IMPROVED LOWER BOUNDS ON $\pi(X)$

s	$\pi(X_{\min}^{(s)*})$
2	$\dfrac{1}{c}\ln\left\{\dfrac{1}{c(m - \tilde{\mu}_1)}(\exp(cm) - \exp(c\tilde{\mu}_1))\right\}$
3	$\dfrac{1}{c}\ln\left\{\dfrac{\tilde{\mu}_2 - \tilde{\mu}_1^2}{\tilde{\mu}_2 cm}(\exp(cm) - 1) + \dfrac{\tilde{\mu}_1^3}{c\tilde{\mu}_2(m\tilde{\mu}_1 - \tilde{\mu}_2)}\left(\exp(cm) - \exp\left(c\dfrac{\tilde{\mu}_2}{\tilde{\mu}_1}\right)\right)\right\}$
4	$\dfrac{1}{c}\ln\left\{\dfrac{\tilde{\mu}_1 - \tilde{r}_-}{(\tilde{r}_+ - \tilde{r}_-)c(\tilde{r}_+ - m)}(\exp(c\tilde{r}_+) - \exp(cm)) + \dfrac{\tilde{r}_+ - \tilde{\mu}_1}{(\tilde{r}_+ - \tilde{r}_-)c(\tilde{r}_- - m)}(\exp(c\tilde{r}_-) - \exp(cm))\right\}$
5	$\dfrac{1}{c}\ln\left\{\dfrac{1 - \tilde{p}_+ - \tilde{p}_-}{cm}(\exp(cm) - 1) + \dfrac{\tilde{p}_+}{c(\tilde{t}_+ - m)}(\exp(c\tilde{t}_+) - \exp(cm)) + \dfrac{\tilde{p}_-}{c(\tilde{t}_- - m)}(\exp(c\tilde{t}_-) - \exp(cm))\right\}$

TABLE 7

UPPER BOUNDS ON $\pi(X)$

s	$\pi(X_{\max}^{(s)})$
2	$\dfrac{1}{c}\ln\left\{\dfrac{b - \mu_1}{b} + \dfrac{\mu_1}{b}\exp(cb)\right\}$
3	$\dfrac{1}{c}\ln\left\{\dfrac{(b - \mu_1)^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}\exp\left(c\dfrac{b\mu_1 - \mu_2}{b - \mu_1}\right) + \dfrac{\mu_2 - \mu_1^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}\exp(cb)\right\}$
4	$\dfrac{1}{c}\ln\left\{(1 - p_1 - p_2) + p_1\exp\left(c\dfrac{\mu_3 - b\mu_2}{\mu_2 - b\mu_1}\right) + p_2\exp(cb)\right\}$
5	$\dfrac{1}{c}\ln\{p_+\exp(cz_+) + p_-\exp(cz_-) + (1 - p_+ - p_-)\exp(cb)\}$

TABLE 8

IMPROVED UPPER BOUNDS ON $\pi(X)$

s	$\pi(X_{\max}^{(s)*})$
2	$\frac{1}{c}\ln\left\{\frac{b-\tilde{\mu}_1}{bcm}(\exp(cm)-1)+\frac{\tilde{\mu}_1}{bc(b-m)}(\exp(cb)-\exp(cm))\right\}$
3	$\frac{1}{c}\ln\left\{\frac{(b-\tilde{\mu}_1)^3}{\left((b-\tilde{\mu}_1)^2+\tilde{\mu}_2-\tilde{\mu}_1^2\right)c(mb-m\tilde{\mu}_1-b\tilde{\mu}_1+\tilde{\mu}_2)}\left(\exp(cm)-\exp\left(c\frac{b\tilde{\mu}_1-\tilde{\mu}_2}{b-\tilde{\mu}_1}\right)\right)\right.$ $\left.+\frac{\tilde{\mu}_2-\tilde{\mu}_1^2}{(b-\tilde{\mu}_1)^2+\tilde{\mu}_2-\tilde{\mu}_1^2}\frac{1}{c(b-m)}(\exp(cb)-\exp(cm))\right\}$
4	$\frac{1}{c}\ln\left\{\frac{1-\tilde{p}_1-\tilde{p}_2}{cm}(\exp(cm)-1)+\tilde{p}_1\frac{\tilde{\mu}_2-b\tilde{\mu}_1}{c(m\tilde{\mu}_2-mb\tilde{\mu}_1-\tilde{\mu}_3+b\tilde{\mu}_2)}\left(\exp(cm)-\exp\left(c\frac{\tilde{\mu}_3-b\tilde{\mu}_2}{\tilde{\mu}_2-b\tilde{\mu}_1}\right)\right)\right.$ $\left.+\tilde{p}_2\frac{1}{c(b-m)}(\exp(cb)-\exp(cm))\right\}$
5	$\frac{1}{c}\ln\left\{\frac{\tilde{p}_+}{c(m-\tilde{z}_+)}(\exp(cm)-\exp(c\tilde{z}_+))+\frac{\tilde{p}_-}{c(m-\tilde{z}_-)}(\exp(cm)-\exp(c\tilde{z}_-))\right.$ $\left.+\frac{1-\tilde{p}_+-\tilde{p}_-}{c(b-m)}(\exp(cb)-\exp(cm))\right\}$

5. NUMERICAL ILLUSTRATION

Let $\mathcal{D}_s([0,b])$ be the class of all the possible moment sequences, i.e. the class of all the vectors $(\mu_1, \mu_2, ..., \mu_{s-1}) \in \mathbb{R}^{s-1}$ such that there exists a random variable X valued in $[0, b]$ satisfying

$$EX^k = \mu_k \text{ for } k = 1, 2, ..., s-1.$$

For more details about the sets $\mathcal{D}_s([a, b])$ for $a, b \in \mathbb{R}$, the interested reader is referred e.g. to De Vylder (1996). Moreover, let $\mathcal{D}_s^{\circ}([0,b])$ be the topological interior of $\mathcal{D}_s([0,b])$. Denuit, De Vylder and Lefèvre (1999) provided the following expressions for $\mathcal{D}_s^{\circ}([0,b])$:

$\mathcal{D}_2^{\circ}([0,b])=\{\mu_1 \in \mathbb{R} \mid 0 < \mu_1 < b\}$,

$\mathcal{D}_3^{\circ}([0,b])=\{(\mu_1,\mu_2) \in \mathbb{R}^2 \mid \mu_1 \in \mathcal{D}_2^{\circ}([0,b]) \text{ and } \mu_1^2 < \mu_2 < b\mu_1\}$,

$\mathcal{D}_4^{\circ}([0,b])=\{(\mu_1,\mu_2,\mu_3) \in \mathbb{R}^3 \mid (\mu_1,\mu_2) \in \mathcal{D}_3^{\circ}([0,b]), \text{and}$

$$\frac{\sigma^2}{\mu_1}(\sigma^2-\mu_1^2)-2\mu_1^3+3\mu_1\mu_2 < \mu_3 < (b-\mu_1)\sigma^2-\frac{\sigma^4}{b-\mu_1}-2\mu_1^3+3\mu_1\mu_2\}$$

where $\sigma^2 = \mu_2 - \mu_1^2$.

In order to illustrate the method proposed in this paper, we compute the bounds on the exponential premiums for risks with moments (μ_1,μ_2) in $\mathcal{D}_3^{\circ}([0,b])$. More precisely, we first fix the mean μ_1, $0 < \mu_1 < b$ (so that the

premium computed using the expected value principle remains unchanged) and then, we let μ_2 vary in such a way that $(\mu_1, \mu_2) \in \mathcal{D}_3^{\circ}([0, b])$, i.e.

$$\mu_2 = \mu_1^2 + \frac{j}{n}\mu_1(b - \mu_1), \quad j = 1, 2, ..., n-1.$$

We then assume that the risk X possesses a unique mode m in $[0, b]$. This leads to new conditions on μ_2. Indeed, from the theory developed in Denuit, De Vylder and Lefèvre (1999), we must have that $(\tilde{\mu}_1, \tilde{\mu}_2) \in \mathcal{D}_3^{\circ}([-m, b-m])$, i.e. that

$$\tilde{\mu}_1^2 \leq \tilde{\mu}_2 \leq \tilde{\mu}_1^2 + (b - m - \tilde{\mu}_1)(\tilde{\mu}_1 + m),$$

or, equivalently, in terms of the initial moment μ_2, that

$$\frac{1}{3}(\tilde{\mu}_1^2 + 2m\mu_1) \leq \mu_2 \leq \frac{1}{3}(\tilde{\mu}_1^2 + 2m\mu_1 + (b - m - \tilde{\mu}_1)(\tilde{\mu}_1 + m)).$$

The results are plotted in Figure 1, with $b = 12$, $n = 50$, $c = 5$, $\mu_1 = 5$, $m = 5$. The continuous lines stand for $\pi(X_{\min}^{(3)})$ and $\pi(X_{\max}^{(3)})$, while the dotted lines represent $\pi(X_{\min}^{(3)*})$ and $\pi(X_{\max}^{(3)*})$. These four quantities are functions of μ_2 with μ_2 satisfying both $(\mu_1, \mu_2) \in \mathcal{D}_3^{\circ}([0, b])$ and $(\tilde{\mu}_1, \tilde{\mu}_2) \in \mathcal{D}_3^{\circ}([-m, b-m])$. In our example, μ_2 varies from 25 to 60 in order to satisfy the first condition, and from 25 to 31.6 in order to fulfill the second one. Therefore, Figure 1 is drawn for μ_2 ranging from 25 to 31.6. The margins are moderately accurate and it is clearly seen that a premium of the form $(1 + \theta)\mu_1$ (with $\theta = 20\%$, say) as provided by the expected value principle may be inappropriate for large values of μ_2.

We then fix (μ_1, μ_2) in $\mathcal{D}_3^{\circ}([0, b])$ so that $(\tilde{\mu}_1, \tilde{\mu}_2) \in \mathcal{D}_3^{\circ}([-m, b-m])$ and we let μ_3 vary in such a way that $(\mu_1, \mu_2, \mu_3) \in \mathcal{D}_4^{\circ}(b)$, i.e.

$$\mu_3 = \frac{\sigma^2}{\mu_1}(\sigma^2 - \mu_1^2) - 2\mu_1^3 + 3\mu_1\mu_2$$

$$+ \frac{j}{n}\left((b - \mu_1)\sigma^2 - \frac{\sigma^4}{b - \mu_1} - \frac{\sigma^2}{\mu_1}(\sigma^2 - \mu_1^2)\right),$$

for $j = 1, 2, ..., n-1$. We then assume that X possesses a unimodal density with mode $m \in [0, b]$, so that μ_3 has to satisfy

$$\frac{1}{4}\left(3m\mu_2 + \frac{\bar{\sigma}^2}{\tilde{\mu}_1}(\bar{\sigma}^2 - \tilde{\mu}_1^2) - 2\tilde{\mu}_1^3 + 3\tilde{\mu}_1\tilde{\mu}_2\right) < \mu_3$$

$$< \frac{1}{4}\left(3m\mu_2 + (b - m - \tilde{\mu}_1)\bar{\sigma}^2 - \frac{\bar{\sigma}^4}{b - m - \tilde{\mu}_1} - 2\tilde{\mu}_1^3 + 3\tilde{\mu}_1\tilde{\mu}_2\right).$$

The results are plotted in Figure 2 for $b = 12$, $n = 50$, $c = 5$, $\mu_1 = 5$, $m = 5$, $\mu_2 = 28$. The continuous lines stand for $\pi(X_{\min}^{(4)})$ and $\pi(X_{\max}^{(4)})$, while the dotted lines represent $\pi(X_{\min}^{(4)*})$ and $\pi(X_{\max}^{(4)*})$. These four quantities are functions of μ_3 with μ_3 satisfying both $(\mu_1, \mu_2, \mu_3) \in \mathcal{D}_4^{\circ}([0, b])$ and

$(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3) \in \mathcal{D}_4^{\circ}([-m, b-m])$. In our example, μ_3 varies from 156.8 to 189.714 in order to satisfy the first condition, and from 162.8 to 164.375 in order to fulfill the second one. Therefore, Figure 2 is drawn for μ_2 ranging from 162.8 to 164.375. Again, we see that the loading coefficient θ has to be high enough: a value of $\theta < 10\%$ seems clearly unrealistic and the picture argues in favour of a value of at least 30%.
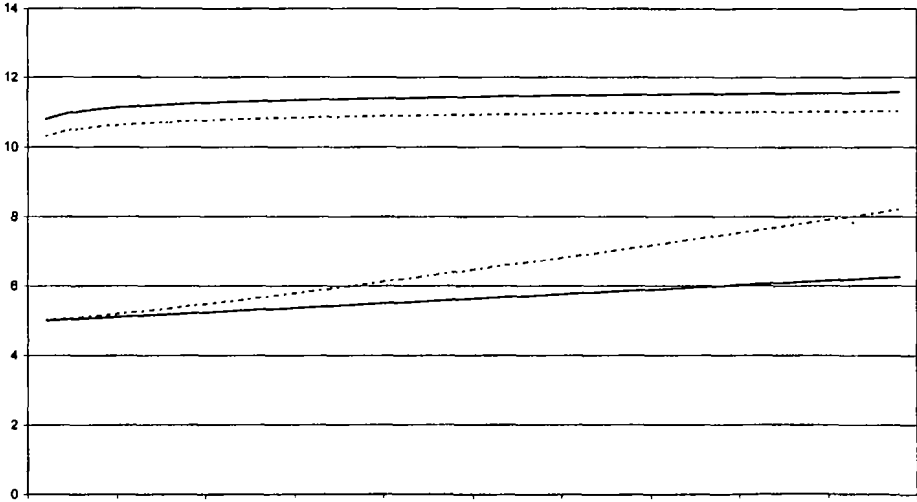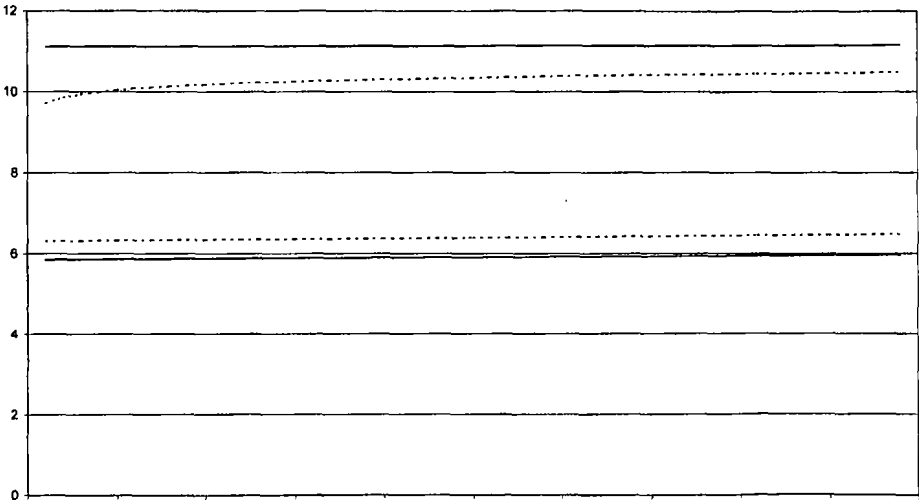

FIGURE 1: Bounds $\pi(X_{\min}^{(3)})$ and $\pi(X_{\max}^{(3)})$ on $\pi(X)$ (continuous lines) as well as $\pi(X_{\min}^{(3)\bullet})$ and $\pi(X_{\max}^{(3)\bullet})$ (dotted lines) as functions of μ_2 ranging from 25 to 31.6.

FIGURE 2: Bounds $\pi(X_{\min}^{(4)})$ and $\pi(X_{\max}^{(4)})$ on $\pi(X)$ (continuous lines) as well as $\pi(X_{\min}^{(4)\bullet})$ and $\pi(X_{\max}^{(4)\bullet})$ (dotted lines) as functions of μ_3 ranging from 162.8 to 164.375.

REFERENCES

DENUIT, M., DE VYLDER, F.E., and LEFÈVRE, Cl. (1999). Extremal generators and extremal distributions for the continuous s-convex stochastic orderings. *Insurance: Mathematics and Economics*, to appear.

DENUIT, M., and LEFÈVRE, Cl. (1997). Some new classes of stochastic order relations among arithmetic random variables, with applications in actuarial sciences. *Insurance: Mathematics and Economics* **20**, 197-214.

DENUIT, M., LEFÈVRE, Cl., and SHAKED, M. (1998). The s-convex orders among real random variables, with applications. *Mathematical Inequalities and Their Applications* **1**, 585-613.

DE VYLDER, F.E. (1996). *Advanced Risk Theory. A Self-Contained Introduction.* Editions de l'Université Libre de Bruxelles - Swiss Association of Actuaries, Bruxelles.

GOOVAERTS, M.J., DE VYLDER, F.E., and HAEZENDONCK, J. (1984). *Insurance Premiums: Theory and Applications.* North Holland. Amsterdam.

GOOVAERTS, M.J., KAAS, R., VAN HEERWAARDEN, A.E., and BAUWELINCKX, T. (1990). *Effective Actuarial Methods.* North Holland. Amsterdam.

HÜRLIMANN, W. (1996). Improval analytical bounds for some risk quantities. *ASTIN Bulletin* .**26**, 185-199.

KAAS, R., and HESSELAGER, O. (1995). Ordering claim size distributions and mixed Poisson probabilities. *Insurance: Mathematics and Economics* **17**, 193-201.

KAAS, R., VAN HEERWAARDEN, A.E., and GOOVAERTS, M.J. (1994). *Ordering of Actuarial Risks.* CAIRE. Brussels.

LEVY, H. (1992). Stochastic dominance and expected utility: survey and analysis. *Management Sciences* **38**, 555-593.

ROBERTS, A.W., and VARBERG, D.E. (1973). *Convex Functions.* Academic Press, New York.

SHAKED, M., and SHANTHIKUMAR, J.G. (1994). *Stochastic Orders and their Applications.* Academic Press. New York.

SILVA, J.M.A., and CENTENO, M.L. (1998). Comparing risk adjusted premiums from the reinsurance point of view. *ASTIN Bulletin* **28**, 221-239.

WANG, S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bulletin* **26**, 71-92.

WANG, S., and YOUNG, V. (1998). Risk-adjusted credibility premiums using distorted probabilities. *Scandinavian Actuarial Journal*, 143-165.

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