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THE CORE OF A REINSURANCE MARKET*

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In a series of celebrated papers, K Borch characterized the set of the Pareto-optimal risk exchange treaties in a reinsurance market However, the Pareto-optimality and the individual rationality conditions, considered by Borch, do not preclude the possibility that a coalition of companies might be better off by seceding from the whole group In this paper, we introduce this collective rationality condition and characterize the core of this game without transferable utilities in the important special case of exponential utilities The mathematical conditions we obtain can be interpreted in terms of insurance premiums, calculated by means of the zero-utility premium calculation principle We then show that the core is always non-void and conclude by an example

1. UTILITY FUNCTIONS IN INSURANCE

Utility functions were introduced into the actuarial world by BORCH (1961) This notion was since then used mainly in two specific insurance models.

1. *The principle of zero-utility*

Introduced by BUHLMANN (1970), this premium calculation principle requires equality of the company's utility before and after signature of an insurance policy. Denoting by R_j the free reserves, P_j the premium (to be calculated), $F_j(x_j)$ the distribution function of the total claim amount ξ_j, and $u_j(x_j)$ the utility of the amount x_j obtained with certainty, for a given company C_j, the principle demands that

$$u_j(R_j) = \int_0^\infty u_j\,(R_j + P_j - x_j)\,d\,F_j(x_j).$$

Many authors, among which GERBER (1974a, 1974b) and LEEPIN (1975) have shown that the exponential utility functions

$$u_j(x_j) = \frac{1}{c_j}\,(1 - e^{-c_j x_j}), \qquad (c_j \geqslant 0)$$

characterized by a constant risk aversion

$$r_j(x) = \frac{-u_j''(x)}{u_j'(x)} = c_j,$$

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possess very desirable properties. In that case the premium can be explicitly computed, one obtains

$$P_j = \frac{1}{c_j} \text{Log } M_j (c_j),$$

where $M_j(c_j)$ is the moment-generating function of S_j calculated at point c_j. P_j will be referred to in the sequel as the exponential utility premium.

2. *A Model of risk exchange*

Introduced by BORCH (1960a, 1960b, 1962), this model considers a pool of n insurance companies $(C_1, \ldots, C_n)$, willing to improve their secureness by means of an exchange of risks treaty. Let $[R_j, F_j(x_j)]$ be the initial situation of C_j, evaluated by its expected utility

$$U_j(x_j) = U_j[R_j, F_j(x_j)] = \int_0^\infty u_j (R_j - x_j) \, d F_j(x_j)$$

The members of the pool will try to increase their utilities by concluding a treaty

$$\bar{y} = [y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)],$$

where $y_j(x_1, \ldots, x_n) = y_j(\bar{x})$ is the sum C_j has to pay if the claims for the different companies respectively amount to $x_1, \ldots, x_n$

Since all the claims must be indemnified, the treaty has to satisfy the admissibility condition

Condition 1: Admissibility

$$\sum_{j=1}^{n} y_j(\bar{x}) = \sum_{j=1}^{n} x_j = z, \tag{1}$$

the total amount of the claims. After the signature of $\bar{y}$, the utility of C_j becomes

$$U_j(\bar{y}) = \int_{\Theta_N} u_j [R_j - y_j(\bar{x})] \, d F_N(\bar{x}),$$

where Θ_N is the positive orthant of E^n and $F_N(\bar{x})$ the n-dimensional distribution function of the claims $\bar{x} = (x_1, \ldots, x_n)$.

Condition 2. Pareto-optimality

A treaty $\bar{y}$ is efficient or Pareto-optimal if there is no $\bar{y}'$ such that $U_j(\bar{y}') \geq U_j(\bar{y})$ for all j, with at least one strict inequality. DU MOUCHEL (1968) has characterized the Pareto-optimal treaties by means of the following theorem.

Theorem 1

Providing all utility functions are such that $u_j'(x) > 0$ and $u_j''(x) \leqslant 0$, a treaty $\bar{y}$ is Pareto-optimal if and only if there exists n non-negative constants $k_1 = 1$, $k_2, \ldots, k_n$, such that, with probability 1,

$$k_j u_j' [R_j - y_j(\bar{x})] = k_1 u_1' [R_1 - y_1(\bar{x})] \qquad j = 1, \ldots, n \tag{2}$$

Let $K = \{k_1, \ldots, k_n\}$ Using very mild technical conditions, it is not difficult to show [Du Mouchel (1968), Lemaire (1973)] that one and only one Pareto-optimal treaty always exists for given K However, there usually exists an infinity of K that satisfy (1) and (2), even when one takes into consideration the fact that no company will enter the pool if its utility is decreased·

Condition 3· Individual rationality

For all $j = 1, \ldots, n \qquad U_j(\bar{y}) \geqslant U_j(x_j)$.

The non-uniqueness of the solution is easily explained by the fact that no sharing rule appears in the definition of Pareto-optimality. Cooperation increases global welfare, and nothing is said about the way the companies will divide the benefits of their mutual agreement. The different admissible values of K correspond to all the possible ways of sharing the profits; each company has interest to obtain a k_j as high as possible, in order to pay as less as possible. The interests of the members of the pool are thus partially complementary (as a whole, the group will prefer a Pareto-optimal treaty), and partially conflicting (each company will have to bargain over its constant k_j) This is characteristic of a game-theoretic situation; indeed, it has been shown by Lemaire (1973) that the risk exchange market is in fact a game without transferable utilities

In the case of exponential utilities, the solution of (2), with the constraint (1), is a familiar quota-share treaty

$$y_j(\bar{x}) = q_j z + y_j(0), \quad \text{with} \begin{cases} q_j = \dfrac{1/c_j}{\sum\limits_{i=1}^{n} 1/c_i} \\ \\ y_j(0) = R_j - q_j \sum\limits_{i=1}^{n} \left(R_i + \dfrac{1}{c_i} \operatorname{Log} \dfrac{k_i}{k_j} \right) \end{cases}$$

Each company will pay a share q_j of each claim, inversely proportional to its risk aversion In order to compensate for the fact that the least risk-averse companies will pay greater amounts, side-payments or monetary compensations $y_j(0)$ between the players occur A consequence of the admissibility condition is that $\sum_{j=1}^{n} y_j(0) = 0$. Note that the quotas are determined by the risk aversion parameters only, so that the bargaining process will only involve the monetary compensations: another feature of exponential utilities is that the players will negotiate on amounts of money, not on abstract constants k_j.

2. CHARACTERIZATION OF THE CORE OF THE MARKET

Pareto-optimality has often been called *group rationality*. considered as a group, the members of the pool can do no better than to agree on a Pareto-optimal treaty. However, this condition does not preclude the fact that some of the players might be better off by seceding and forming a sub-coalition. We are going to reduce the set of the Pareto-optimal treaties by computing the core of the game, i.e. by requiring that no sub-coalition has an incentive to quit the pool.

From now on we shall consider only Pareto-optimal treaties. Let N be the set of all the companies, $S \subset N$ any sub-coalition, $v(S)$ the set of the Pareto-optimal treaties for S, i e. the set of all the agreements that S, playing separately from $N \backslash S$, can achieve. $\bar{y}'$ is said to dominate $\bar{y}$ with respect to coalition S if

(i) $U_j(\bar{y}') \geqslant U_j(\bar{y})$ for all $j \in S$ (with at least one strict inequality)

(ii) S can enforce $\bar{y}' : \bar{y}' \in v(S)$.

$\bar{y}'$ is said to dominate $\bar{y}$ if there is a coalition S such that $\bar{y}'$ dominates $\bar{y}$ with respect to S The core is the set of all the non-dominated treaties. In other words, instead of requiring, in addition of (1) and (2), the condition of individual rationality, we shall introduce the much stronger

Condition 4: Collective rationality

No coalition has interest in quitting the pool.

Obviously, this condition implies both conditions 2 and 3 (which are collective rationality applied, respectively, to all the one-player coalitions, and to the grand coalition N).

Assume that coalition $S \subset N$ has decided to form. Let P_j^S be the exponential utility premium C_j would require to take over a share

$$q_{j,S} = \frac{\alpha_j}{\sum\limits_{k \in S} \alpha_k}$$

of the portfolio of all the companies $C_k \in S$, with $\alpha_j = \dfrac{1}{c_j}$.

In particular, $P_j^{\{j\}}$ (or more simply P_j^j) is the premium c_j would demand without any reinsurance.

Let us suppose finally that all the claim amounts ξ_j are independent.

Lemma 1

$$P_j^S = \frac{1}{c_j}\sum_{k\in S}\text{Log } M_k(q_{j,S}\, c_j)$$

$$= \frac{1}{c_j}\sum_{k\in S}\text{Log } M_k\left(\frac{1}{\sum\limits_{j\in S}\alpha_j}\right).$$

Proof: we know that

$$P_j^S = \frac{1}{c_j}\sum_{k\in S}\text{Log } M_k^{(*)}(c_j),$$

where $M_k^{*(j)}(x)$ is the moment-generating function of the distribution of the quota $q_{j,S}$ of ξ_k. The fact that

$$M_k^{*(j)}(x) = E\left[e^{x\, q_{j,S}\,\xi_k}\right] = M_k\,(q_{j,S}\, x)$$

completes the proof.

Lemma 2.

Let $\{S_1, \ldots, S_r\}$ be a partition of $S \subset N$. Then

$$\sum_{j\in S}(P_j^S - P_j^N) + \sum_{l=1}^{r}\sum_{j\in \bar{S}_l}(P_j^{\bar{S}_l} - P_j^N) \geqslant 0.$$

Proof. $\sum_{j\in S}(P_j^S - P_j^N) + \sum_{l=1}^{r}\sum_{j\in \bar{S}_l}(P_j^{\bar{S}_l} - P_j^N)$

$$= \sum_{j\in S}\left[\alpha_j\sum_{i\in S}\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in S}\alpha_k}\right) - \alpha_j\sum_{i\in N}\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in N}\alpha_k}\right)\right]$$

$$+ \sum_{l=1}^{r}\sum_{j\in \bar{S}_l}\left[\alpha_j\sum_{i\in \bar{S}_l}\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in \bar{S}_l}\alpha_k}\right) - \alpha_j\sum_{i\in N}\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in N}\alpha_k}\right)\right]$$

$$= \sum_{i\in S}\left(\sum_{k\in S}\alpha_k\right)\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in S}\alpha_k}\right) + \sum_{l=1}^{r}\sum_{i\in \bar{S}_l}\left(\sum_{k\in \bar{S}_l}\alpha_k\right)\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in \bar{S}_l}\alpha_k}\right)$$

$$- \sum_{i\in N}\left[\sum_{k\in S}\alpha_k + \sum_{l=1}^{r}\sum_{j\in \bar{S}_l}\alpha_j\right]\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in N}\alpha_k}\right)$$

$$= \sum_{i\in S}\left(\sum_{k\in S}\alpha_k\right)\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in S}\alpha_k}\right) + \sum_{l=1}^{r}\sum_{i\in \bar{S}_l}\left(\sum_{k\in \bar{S}_l}\alpha_k\right)\text{Log } M_i\left(\frac{1}{\sum\limits_{k\in \bar{S}_l}\alpha_k}\right)$$

$$- r \sum_{i \in N} \left(\sum_{k \in N} \alpha_k \right) \operatorname{Log} M_i \left(\frac{1}{\sum\limits_{k \in S} \alpha_k} \right) \qquad \left(\text{since } \bar{S}_l = \bigcup_{\substack{j=1 \\ j \neq l}}^{r} S_j \cup S; \right.$$

in the sequel we shall note $S_0 = \bar{S}$)

$$= \sum_{i \in S} a \operatorname{Log} M_i \left(\frac{1}{a} \right) + \sum_{l=1}^{r} \sum_{\substack{s=0 \\ s \neq l}}^{r} \sum_{i \in S_s} a_l \operatorname{Log} M_i \left(\frac{1}{a_l} \right) - r \sum_{i \in N} b \operatorname{Log} M_i \left(\frac{1}{b} \right)$$

(where $a = \sum_{k \in S} \alpha_k$, $a_l = \sum_{k \in \bar{S}_l} \alpha_k$, $b = \sum_{k \in N} \alpha_k$)

$$= \sum_{i \in S} \left[a \operatorname{Log} M_i \left(\frac{1}{a} \right) - b \operatorname{Log} M_i \left(\frac{1}{b} \right) + \sum_{l=1}^{r} \sum_{\substack{s=0 \\ s \neq l}}^{r} \sum_{i \in S_s} a_l \operatorname{Log} M_i \left(\frac{1}{a_l} \right) \right.$$

$$- (r-1) \sum_{i \in S} b \operatorname{Log} M_i \left(\frac{1}{b} \right) - r \sum_{i \in \bar{S}} b \operatorname{Log} M_i \left(\frac{1}{b} \right)$$

$$= \sum_{i \in S} \left[a \operatorname{Log} M_i \left(\frac{1}{a} \right) - b \operatorname{Log} M_i \left(\frac{1}{b} \right) \right] + \sum_{l=1}^{r} \sum_{\substack{s=0 \\ s \neq l}}^{r} \sum_{i \in S_s} \left[a_l \operatorname{Log} M_i \left(\frac{1}{a_l} \right) \right.$$

$$\left. - \text{b} \operatorname{Log} M_i \left(\frac{1}{b} \right) \right].$$

GERBER (1974a) has shown that $\frac{1}{c} \operatorname{Log} M(c)$ is an increasing function of c. It can be deduced that $c \operatorname{Log} M\left(\frac{1}{c}\right)$ is a decreasing function of c. Since $a \leqslant b$ and $a_l \leqslant b$, all the terms between square brackets are non-negative and the lemma is proved.

Corollary 1:
For all $S \subset N$

$$\sum_{j \in S} P_j^S + \sum_{j \in \bar{S}} P_j^{\bar{S}} \geqslant \sum_{j \in N} P_j^N$$

Proof:
Apply lemma 2 to the coalition $S = N$ and the partition $\{S, S\}$. This intuitively obvious corollary enhances the merits of cooperation. It can be extended to all partitions of N.

Lemma 3:

Let $\{S_1, \ldots, S_r\}$ be a partition of N. Then

$$\sum_{i=1}^{r} \sum_{j \in S_i} P_j^{S_i} \geqslant \sum_{j \in N} P_j^N,$$

which amounts to

$$\sum_{i=1}^{r} \sum_{j \in S_i} (P_j^{S_i} - P_j^N) \geqslant 0$$

Proof:

In all respects similar to lemma 2.

Lemma 4:

If $A_1, \ldots, A_r, B_1, \ldots, B_r$ are real numbers such that

$$A_1 + \ldots + A_r < B_1 + \ldots + B_r,$$

there exists real numbers $\alpha_1, \ldots, \alpha_r$ such that $\alpha_1 + \ldots + \alpha_r = 0$ and

$$\begin{gathered} A_1 + \alpha_1 < B_1 \\ \vdots \\ A_r + \alpha_r < B_r \end{gathered}$$

Proof:

The property is true for $r = 2$. In fact, since $A_1 + A_2 < B_1 + B_2$, we have $A_1 + A_2 - B_1 < B_2$, and there exists an $\varepsilon > 0$ such that

$$A_1 + A_2 - B_1 + \varepsilon < B_2.$$

Let $\alpha_1 = B_1 - A_1 - \varepsilon$. Then

$$\begin{aligned} A_1 + \alpha_1 &= A_1 + B_1 - A_1 - \varepsilon = B_1 - \varepsilon < B_1 \\ A_2 + \alpha_2 &= A_2 - \alpha_1 = A_2 - B_1 + A_1 + \varepsilon < B_2. \end{aligned}$$

Suppose the lemma verified for a given r, and let us demonstrate the property for $r + 1$. We have

$$A_1 + \ldots + A_{r+1} < B_1 + \ldots + B_{r+1},$$

or $\quad A_1 + \ldots + A_{r-1} + (A_r + A_{r+1}) < B_1 + \ldots + B_{r-1} + (B_r + B_{r+1}).$

There exists, by induction, $\beta_1, \ldots, \beta_r$ such that $\beta_1 + \ldots + \beta_r = 0$ and

$$\begin{aligned} & A_1 + \beta_1 < B_1 \\ & \vdots \\ & A_{r-1} + \beta_{r-1} < B_{r-1} \\ & (A_r + A_{r+1}) + \beta_r < B_r + B_{r+1}. \end{aligned}$$

The last inequality can be written

$$(A_r + \beta_r) + A_{r+1} < B_r + B_{r+1}$$

There exists a γ such that

$$A_r + \beta_r + \gamma < B_r$$
$$A_{r+1} - \gamma < B_{r+1}.$$

It is then sufficient to put

$$\alpha_1 = \beta_1, \ldots, \alpha_{r-1} = \beta_{r-1}, \alpha_r = \beta + \gamma, \alpha_{r+1} = -\gamma.$$

Theorem 2:

$\bar{y} = (y_1, \ldots, y_n)$ belongs to the core of the pool if and only if

$$y_j(x_1, \ldots, x_n) = q_j z + y_j(0), \text{ with } \begin{cases} \sum_{j=1}^{n} y_j(0) = 0 \\ \sum_{j \in S} y_j(0) \leqslant \sum_{j \in S} (P_j^S - P_j^N), \end{cases}$$

$$\forall S \subset N$$
$$(S \neq \phi).$$

Proof:

(a) Necessity

Suppose $\bar{y}$ belongs to the core. It is then Pareto-optimal, and

$$y_j(x_1, \ldots, x_n) = q_j z + y_j(0), \quad \text{with } \sum_{j=1}^{n} y_j(0) = 0.$$

If the last condition is not verified, there exists a non-void $S \subset N$ such that

$$\sum_{j \in S} y_j(0) > \sum_{j \in S} (P_j^S - P_j^N).$$

Using lemma 1,

$$\sum_{j \in S} y_j(0) > \sum_{j \in S} \left[\frac{1}{c_j} \sum_{i \in S} \text{Log } M_i \left(\frac{1}{\sum_{k \in S} \alpha_k}\right) - \frac{1}{c_j} \sum_{i=1}^{n} \text{Log M}_i \left(\frac{1}{\sum_{k=1}^{n} \alpha_k}\right)\right]$$

Lemma 4 makes sure that there exists $(z_j(0))_{j \in S}$ such that

$$\sum_{j \in S} z_j(0) = 0$$

$$(3) \quad \frac{1}{c_j} \left[\sum_{i \in S} \text{Log} M_i \left(\frac{1}{\sum_{k \in S} \alpha_k}\right) - \sum_{i=1}^{n} \text{Log } M_i \left(\frac{1}{\sum_{k=1}^{n} \alpha_k}\right)\right] + z_j(0) < y_j(0).$$

Consider the sub-treaty $\bar{z} = (z_j)_{j\in S}$, defined by

$$z_j[(x_k)_{k\in S}] = \left(\frac{\alpha_j}{\sum_{k\in S} \alpha_k}\right) \sum_{k\in S} x_k + z_j(0) \qquad (j \in S)$$

For all $j \in S$, we have

$$U_j(\bar{z}) = \frac{1}{c_j} \int_{0_S} \left[1 - e^{-c_j\left[R_j - \frac{\alpha_j}{\sum_{k\in S}\alpha_k} \sum_{k\in S} x_k - z_j(0)\right]}\right] d\,F_S(\bar{x})$$

$$= \frac{1}{c_j}\left[1 - e^{-c_j R_j}\, e^{c_j z_j(0)} \int_{0_S} e^{\frac{1}{\sum_{k\in S}\alpha_k} \sum_{k\in S} x_k}\, d\,F_S(\bar{x})\right]$$

$$= \frac{1}{c_j}\left[1 - e^{-c_j R_j}\, e^{c_j z_j(0)} \prod_{k\in S} M_k\left(\frac{1}{\sum_{i\in S}\alpha_i}\right)\right],$$

where 0_S is the positive orthant of $E^{|S|}$, and $F_S(\bar{x})$ the $|S|$- dimensional distribution function of $[(x_k)_{k\in S}]$.

In the same way, we obtain

$$U_j(\bar{y}) = \frac{1}{c_j}\left[1 - e^{-c_j R_j}\, e^{c_j y_j(0)} \prod_{k=1}^{n} M_k\left(\frac{1}{\sum_{i=1}^{n}\alpha_i}\right)\right].$$

Then $U_j(z) > U_j(\bar{y})$ if and only if

$$e^{c_j z_j(0)} \prod_{k\in S} M_k\left(\frac{1}{\sum_{i\in S}\alpha_i}\right) < e^{c_j y_j(0)} \prod_{k=1}^{n} M_k\left(\frac{1}{\sum_{i=1}^{n}\alpha_i}\right),$$

taking logarithms

$$\frac{1}{c_j}\left[\sum_{k\in S} \operatorname{Log} M_k\left(\frac{1}{\sum_{i\in S}\alpha_i}\right) - \sum_{k=1}^{n} \operatorname{Log} M_k\left(\frac{1}{\sum_{i=1}^{n}\alpha_i}\right)\right] + z_j(0) < y_j(0),$$

which is precisely relation (3) So $U_j(z) > U_j(\bar{y})$, for all $j \in S$, in contradiction with the fact that $\bar{y}$ belongs to the core.

(b) Sufficiency

Consider $\bar{y}$ such that $y_j(x_1, \ldots, x_n) = q_j\, z + y_j(0)$, with

$$\sum_{j=1}^{n} y_j(0) = 0$$

$$\sum_{j\in S} y_j(0) \leqslant \sum_{j\in S} (P_j^S - P_j^N), \qquad \text{for all } S \subset N \; (S \neq \phi)$$

If $\bar{y}$ does not belong to the core, there exists a coalition $S \subset N$ and a treaty $[(z_j)_{j\in S}]$ such that

$$U_j(\bar{z}) \geqslant U_j(\bar{y}), \qquad \forall\ j \in S,$$

with a least one strict inequality Since we can assume $\bar{z}$ to be Pareto-optimal,

$$z_j[(x_k)_{k\in S}] = \frac{\alpha_j}{\sum\limits_{k\in S} \alpha_k} \sum_{k\in S} x_k + z_j(0) \qquad j \in S$$

with $\sum\limits_{j\in S} z_j(0) = 0$

Since $U_j(\bar{z}) = \dfrac{1}{c_j}\left[1 - e^{-c_j R_j} e^{c_j z_j(0)} \prod\limits_{k\in S} M_k\left(\dfrac{1}{\sum\limits_{i\in S} \alpha_i}\right)\right]$

and $U_j(\bar{y}) = \dfrac{1}{c_j}\left[1 - e^{-c_j R_j} e^{c_j y_j(0)} \prod\limits_{k=1}^{n} M_k\left(\dfrac{1}{\sum\limits_{i=1}^{n} \alpha_i}\right)\right] \qquad \forall\ j \in S.$

We have, taking logarithms,

$$z_j(0) + \frac{1}{c_j}\sum_{k\in S} \operatorname{Log} M_k\left(\frac{1}{\sum\limits_{i\in S} \alpha_i}\right) \leqslant y_j(0) + \frac{1}{c_j}\sum_{k=1}^{n} \operatorname{Log} M_k\left(\frac{1}{\sum\limits_{i=1}^{n} \alpha_i}\right) \qquad \forall\ j \in S.$$

Summing over all $j \in S$, and using $\sum\limits_{j\in S} z_j(0) = 0$, we obtain

$$\sum_{j\in S} y_j(0) > \sum_{j\in S}\left[\frac{1}{c_j}\sum_{k\in S} \operatorname{Log} M_k\left(\frac{1}{\sum\limits_{i\in S} \alpha_i}\right) - \frac{1}{c_j}\sum_{k=1}^{n} \operatorname{Log} M_k\left(\frac{1}{\sum\limits_{i=1}^{n} \alpha_i}\right)\right]$$

or

$$\sum_{j\in S} y_j(0) > \sum_{j\in S} [P_j^S - P_j^N].$$

contradicting the hypothesis.

Corollary

$\bar{y} = (\bar{y}_1, \ldots, y_n)$ belongs to the core of the pool if and only if

$$y_j(x_1, \ldots, x_n) = q_j z + y_j(0), \qquad j = 1, \ldots, n$$

with $\quad \sum\limits_{j\in S} y_j(0) \leqslant \sum\limits_{j\in S} (P_j^S - P_j^N) \qquad\qquad \forall\ S \subset N$

if we define $P_j^{\varnothing} = 0$.

Proof.

Applying condition $\sum_{j\in S} y_j(0) \leqslant \sum_{j\in S} (P_j^S - P_j^N)$ for $S = N$, we obtain

$$\sum_{j\in N} y_j(0) = \sum_{j=1}^{n} y_j(0) \leqslant \sum_{j\in N} (P_j^N - P_j^N) = 0.$$

Since $\sum_{j\in 0} (P_j^0 - P_j^N) = 0$ and $\sum_{j\in 0} y_j(0) + \sum_{j\in N} y_j(0) = 0$, we have successively

$$\sum_{j=1}^{n} y_j(0) = 0$$

$$\sum_{j\in N} y_j(0) \geqslant 0$$

$$\sum_{j\in 0} y_j(0) \leqslant 0 = \sum_{j\in 0} (P_j^0 - P_j^N)$$

In other words, condition $\sum_{j=1}^{n} y_j(0) = 0$ may be replaced by

$$\sum_{j\in N} y_j(0) \leqslant \sum_{j\in N} (P_j^N - P_j^N)$$

$$\sum_{j\in 0} y_j(0) \leqslant \sum_{j\in 0} (P_j^0 - P_j^N).$$

So, not only conditions 2 and 3, but also condition 1 derives from collective rationality.

Interpretation:

In addition to the fact that it characterizes the core, theorem 2 may be interesting in the sense that it links two apparently very different concepts, a collective notion (the core of a game without transferable utilities), and an individual notion (a premium calculation principle)

Applied to a one-player coalition, the core condition becomes

$$y_j(0) \leqslant P_j^j - P_j^N \qquad j = 1, \ldots, n$$

The second member is the difference between the exponential utility premiums before and after reinsurance. Everything happens as if each company evaluates its portfolio by the exponential utility premium: the certainty equivalent of any portfolio is this premium. A positive second member means that C_j finds profitable to participate to the pool It will however only enter the market if its "fee" or side-payment does not exceed the profit.

Applied to the two-player coalition $\{1, 2\}$, the core condition becomes

$$y_1(0) + y_2(0) \leqslant [P_1^{\{1,2\}} - P_1^N] + [P_2^{\{1,2\}} - P_2^N]$$

$P_1^{\{1,2\}}$ is the premium C_1 would ask to assume a share $\frac{\alpha_1}{\alpha_1 + \alpha_2}$ of its own and C_2's portfolios. The term between the first square brackets represents the positive or negative profit C_1 would make by not seceding from the pool with C_2. The condition requires that, globally, coalition $\{1, 2\}$ has no interest to play alone, in the sense that the sum of the side payments required from its members is small enough not to incite them to quit the pool The difference

$$[P_1^{\{1,2\}} - P_1^N] + [P_2^{\{1,2\}} - P_1^N] - y_1(0) - y_2(0)$$

is the benefit coalition $\{1, 2\}$ enjoys from participating to the pool. If this term were negative, $\{1, 2\}$ would have interest to separate and create a 2-company pool Note that nothing is said about the way those 2 companies will share this benefit: the core only introduces global conditions.

Note.

The conditions of theorem 2 not only provide upper limits for the side-payments, but also lower limits. Indeed, since

$$\sum_{j \in S} y_j(0) = - \sum_{j \in \bar{S}} y_j(0) \qquad \forall S,$$

we have

$$(3) \qquad - \sum_{j \in \bar{S}} (P_j^{\bar{S}} - P_j^N) \leqslant \sum_{j \in S} y_j(0) \leqslant \sum_{j \in S} (P_j^S - P_j^N)$$

3. EXISTENCE OF THE CORE

The main disadvantage of the core is that there exists large classes of games for which it is empty. Fortunately, theorem 3 shows that the core of the risk exchange market always exists

Theorem 3:

The core of the market is non-void.

Proof.

The core can be characterized by conditions (3), or, in an equivalent way, by

$$(4) \qquad \begin{cases} \sum_{j \in \bar{S}} (P_j^{\bar{S}} - P_j^N) \leqslant \sum_{j \in S} y_j(0) \leqslant \sum_{j \in S} (P_j^S - P_j^N) & \text{for all } S \subset N \\ & \text{such that } C_n \notin S \\ y_1(0) + \ldots + y_n(0) = 0 & \end{cases}$$

Note that conditions (4) only restrict the values of $y_1(0), \ldots, y_{n-1}(0)$. This is obvious because, if $C_n \in S_0$,

$$\sum_{j \in S_0} y_j(0) = - \sum_{j \in \bar{S}_0} y_j(0) \text{ and } C_n \notin \bar{S}_0,$$

So $$- \sum_{j \in S_0} (P_j^{S_0} - P_j^N) \leqslant \sum_{j \in \bar{S}_0} y_j(0) \leqslant \sum_{j \in \bar{S}_0} (P_j^{\bar{S}_0} - P_j^N)$$

and $$- \sum_{j \in \bar{S}_0} (P_j^{\bar{S}_0} - P_j^N) \leqslant \sum_{j \in S_0} y_j(0) \leqslant \sum_{j \in S_0} (P_j^{S_0} - P_j^N).$$

It only remains to prove that there exists $(n-1)$ constants $y_1(0), \ldots, y_{n-1}(0)$ verifying conditions (4) (then we shall obtain $y_n(0)$ using $y_n(0) = - \sum_{j=1}^{n-1} y_j(0)$). This system has a solution if, for all $S \subset N$ such that $C_n \notin S$ and for all partitions $(S_1, \ldots, S_r)$ of S,

$$- \sum_{j \in \bar{S}} (P_j^{\bar{S}} - P_j^N) \leqslant \sum_{j \in S} (P_j^S - P_j^N)$$

$$\sum_{j \in S} (P_j^S - P_j^N) \geqslant - \sum_{i=1}^{r} \sum_{j \in \bar{S}_i} (P_j^{\bar{S}_i} - P_j^N)$$

$$- \sum_{j \in \bar{S}} (P_j^{\bar{S}} - P_j^N) \leqslant \sum_{i=1} \sum_{j \in S_i} (P_j^{S_i} - P_j^N).$$

This is a consequence of corollary 1 and lemmas 2 and 3

4. EXAMPLE

Let us consider the following example, introduced by LEMAIRE (1979). Suppose that the pool consists of 3 companies, whose risk aversion coefficients are respectively $c_1 = .3$, $c_2 = .6$, $c_3 = .1$. All the companies have the same distribution of claim amounts, namely a Γ-distribution

$$\frac{d\,F_j(x)}{dx} = \begin{cases} \dfrac{\tau^a}{\Gamma(a)} e^{-\tau x} x^{a-1} & x > 0 \\ 0 & \text{elsewhere} \end{cases} \qquad j = 1, 2, 3,$$

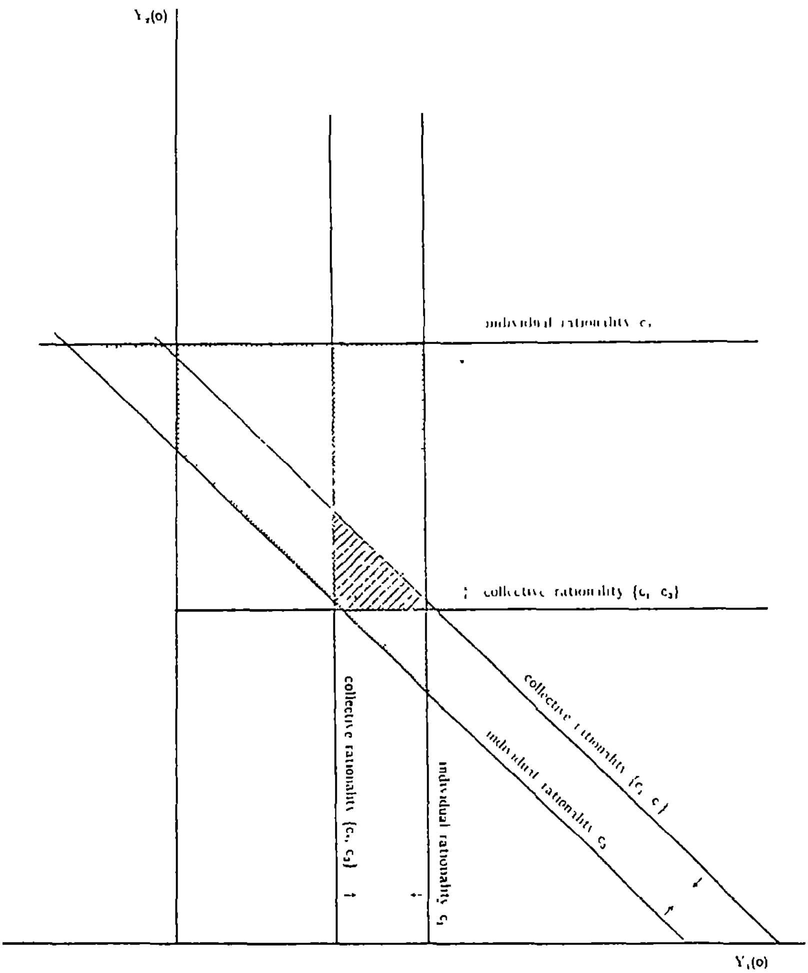
with parameters $a = 1.152$ and $\tau = .96$ The mean is equal to $m = \dfrac{a}{\tau} = 1.2$, the variance $\sigma^2 = \dfrac{a}{\tau^2} = 1.25$ Using the moment-generating function of this distribution

$$M(t) = \left(1 - \frac{t}{\tau}\right)^{-a},$$

we obtain the following conditions for the core

$$.388 \leqslant y_1(0) \leqslant .610$$
$$.818 \leqslant y_2(0) \leqslant 1.469$$
$$1.22 \leqslant y_1(0) + y_2(0) \leqslant 1.448.$$

The core is shown in Figure 1 Note that the core (hachured area) is substantially smaller than the set of the Pareto-optimal treaties (dotted area).



Note:

For $n > 3$, the core is more difficult to represent, since it forms a convex compact polyhedron in the n-1-dimensional Euclidian space with axis $y_1(0)$, ..., $y_{n-1}(0)$. It is characterized by a set of $2^{n-1} - 1$ double inequalities. This number of constraints increases tremendously with n.

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