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SURVIVAL PROBABILITIES BASED ON PARETO CLAIM DISTRIBUTIONS

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1. INTRODUCTION AND BACKGROUND

It is commonly thought that the characteristic function (Fourier transform) of the Pareto distribution has no known functional form (e.g. SEAL, 1978, pp. 14, 40, 57). This is quite untrue. Nevertheless the characteristic function of the Pareto density is conspicuously absent from standard reference works even when the Pareto distribution itself receives substantial comment (e.g. HAIGHT, 1961; JOHNSON and KOTZ, 1970, Ch. 19; PATEL, KAPADIA and OWEN, 1976, § 1. 5).

The Pareto density may be written

$$(1) \qquad f(x) = \frac{\nu}{b}\left(1 + \frac{x}{b}\right)^{-\nu-1} \qquad 0 < x < \infty; \qquad \nu,\ b > 0$$

with distribution function

$$(2) \qquad F(x) = 1 - \left(1 + \frac{x}{b}\right)^{-\nu}$$

mean $= b/(\nu - 1)$ and variance $= b^2\nu(\nu - 1)^2(\nu - 2)$. These are infinite when $\nu \leq 1$ and $\nu \leq 2$, respectively. Its Laplace transform ($s = c + iu$)

$$\beta(s) = \frac{\nu}{b}\int_0^\infty e^{-sx}\left(1 + \frac{x}{b}\right)^{-\nu-1} dx = \nu\int_0^\infty e^{-sby}(1 + y)^{-\nu-1}\, dy$$

$$(3) \qquad = \nu e^{sb} E_{\nu+1}(sb) \qquad \nu > 0, \qquad Rs \geq 0$$

where E is the generalized exponential integral (PAGUROVA, 1961) and can be written in terms of incomplete gamma or confluent hypergeometric functions (Slater, 1960, Sec. 5.6). When $s = -it$ $\beta(s)$ becomes the characteristic function (see Appendix I).

As BENKTANDER (1970) tells us, the Pareto distribution has been particularly successful at representing the distribution of the larger claim amounts. In earlier years it was employed to represent the distribution of life insurance sums assured but more recently it has been used for the claim distributions of fire and automobile insurance. Table 1 provides the ν-values we have been able to locate. Note that the variance of the distribution is infinite when $\nu \leq 2$ and if it were not for the anomalous ν-values of ANDERSSON (1971) we would have ventured the opinion that modern claim data encourage the assumption that $\nu > 2$. In our numerical work we have used $\nu = 2.7$ and smaller values might change some of the computer rules we have proposed in Appendix II.

TABLE 1

Author	Value(s) of ν	Source of data
Meidell (1912)	Between 1 and 2	Life insurance; no data
Hagstroem (1925)	1.3	Swedish income data
Cramér (1926)	1.5, 1.7	Swedish life insurance companies
Henry (1937)	2.38	French automobile claims exceeding 40,000 fcs.
Meidell (1938)	1.86, 1.85, 1.9, 2.5, 2.67, 2.55, 2.6, 3.1	Norwegian, British, German and Japanese life insurance
	1.73, 3.16, 2.1, 2.3, 2.1	Swedish and German fire insurance
Pellegrin (1948)	1.68	French automobile claims exceeding 75,000 fcs.
Thépaut (1950)	1.67	French fire insurance company
Benckert and Sternberg (1957)	2.45, 2.50, 2.53, 2.53, 2.55, 2.56, 2.56	Swedish fire insurance of dwellings
Hagstroem (1960)	1.40, ... 2.45, 2.54, 2.42, 2.40	Swedish income data for 1912, ... 1950, 1953, 1954, 1955
Benktander (1962)	2.7	Automobile insurance claims
Ammeter (1971)	2.4	American fire insurance portfolio
Andersson (1971)	1.25, 1.26, 1.32, 1.37, 1.38, 1.39, 1.49, 1.76	Scandinavian countries' fire losses in 1950's and 1960's

It is convenient (cp. SEAL, 1978) to make the mean of the claims distribution equal to unity so that $b = \nu - 1$, and write $s = iu$ and $ub = z$. We thus require $\nu e^{iz} E_{\nu+1}(iz)$ for values of z ranging from zero, when $\beta(0) = \nu \int_1^\infty x^{-\nu-1} dx = 1$, up to 500 or more. PAGUROVA (1961) gives two series expansions for small and large arguments of $E_{\nu+1}(x)$, respectively, namely

$$E_{\nu+1}(x) = \Gamma(-\nu) x^\nu - e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{(-\nu)_{k+1}}$$

with $(a)_n = a(a+1) \ldots (a+n-1)$, and

$$E_{\nu+1}(x) \sim \frac{e^{-x}}{x} \left\{ 1 + \sum_{m=1}^{M} (-1)^m \frac{(\nu+m)^{(m)}}{x^m} \right\}$$

with $a^{(n)} = a(a-1) \ldots (a-n+1)$. These were checked from other sources.

The former series, analogous to that of e^x, converges for all x but involves perhaps hundreds of terms when x is large. The latter is an alternating series with a remainder less in absolute value than $(\nu + M + 1)^{(M+1)} / x^{M+1}$ (JEFFREYS, 1962, Ch. 7).

Coincidentally the generalized exponential integral has appeared before in the actuarial literature. SEAL (1964) showed that for an m-joint-life annuity subject to the m mortality forces

$$\mu_{z_j} = A_j + B_j c^{z_j} \qquad j = 1, 2, \ldots m$$

(Note the uniform c)

$$\bar{a}_{z_1 z_2 \cdots z_m} = \gamma^{-1} e^x E_\nu(x)$$

where

$$\nu = \gamma^{-1} (\delta + \sum_{j=1}^{m} A_j)$$

$$x = \gamma^{-1} \sum_{j=1}^{m} B_j c^{z_j}$$

δ being the force of interest and $\gamma = \ln c$.

Changing x to iz Pagurova's expressions yield

$$\nu e^{iz} E_{\nu+1}(iz) =$$

$$= \nu\Gamma(-\nu) z^\nu e^{iz + i\nu\pi/2} - \nu \left[\sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(-\nu)_{2j+1}} + i \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(-\nu)_{2j+2}} \right.$$

$$= -\Gamma(1-\nu) z^\nu \cos(z + \nu\pi/2) - \nu \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(-\nu)_{2j+1}}$$

$$- i[\Gamma(1-\nu) z^\nu \sin(z + \nu\pi/2) + \nu \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(-\nu)_{2j+2}} \Bigg] \tag{4}$$

and

$$\nu e^{iz} E_{\nu+1}(iz) \sim \frac{\nu}{iz} \left[1 + \sum_{m=1}^{M} (-1)^m \frac{(\nu + m)^{(m)}}{(iz)^m} \right]$$

$$= \frac{\nu}{z} \sum_{j=0} (-1)^j \frac{(\nu + 2j + 1)^{(2j+1)}}{z^{2j+1}}$$

$$- i \frac{\nu}{z} \left[1 + \sum_{j=1} (-1)^j \frac{(\nu + 2j)^{(2j)}}{z^{2j}} \right] \tag{5}$$

For ν in the vicinity of 2.7 numerical work with (4) and (5) needs careful attention. In order to achieve a final term in each of the four series less than 5×10^{-8} the asymptotic relation (5) would not give a sensible result for $z = 23$. For $z = 25$ only nine terms were required in the first series of (5) for the required degree of accuracy. On the other hand 44 terms of the first series in (4) were required when $z = 23$ and the values of the real and imaginary series were then 6114.5 and -10256, respectively, to five significant figures. This indicated that double precision arithmetic (about 30 significant figures) should be used in calculating the series values since the characteristic function has an absolute value not exceeding unity and the first four or five significant figures of the series would be lost in the subtraction from the term involving Γ in the real and imaginary parts, respectively. The value of $\Gamma(1-\nu)$ was obtained as $\Gamma(3-\nu) / (2-\nu)(1-\nu)$, the Γ factor being obtained from Davis's (1964, § 6.134) approximation. Further computational details are supplied in Table 2 and it is noticed how slowly the Laplace transform (with $c=0$) converges towards zero particularly along the imaginary axis. This, of course, poses a problem when we come to invert the Laplace transform of $F(x, t)$, the distribution function of the aggregate claims.

TABLE 2

Real and Imaginary Parts of $2.7e^{iz}E_{3.7}(iz)$

z	Real	Number of terms needed in first series	Imaginary
5	.22663	17	$-.36002$
10	.08124	25	$-.23454$
15	.04002	32	$-.16795$
23	.01801	44	$-.11378$
25	.01535	9	$-.10507$
100	.00100	3	$-$ 02695
200	.00025	3	$-.01349$
250	.00016	3	$-.01080$

2. SURVIVAL PROBABILITIES FOR POISSON/PARETO

The sine and cosine integrals involved in the inversion of a Laplace transform are shown in SEAL (1978, 3.10) and when claims are occurring as a Poisson process the $P(.)$ and $Q(.)$ of that relation are the real and imaginary parts of Laplace transforms implied in SEAL (1978, 3.14 and 4.7). In these relations $\beta(s)$ must, of course, be replaced by its Pareto value developed above.

In order, therefore, to calculate successive values of

$$U(w, t) = F(w + \overline{1+\eta}\cdot t, t) - (1+\eta) \int_0^t U(0, t-\tau) f(w + \overline{1+\eta}\cdot\tau, \tau)\, d\tau$$

the probability of survival through time t for $t = 1, 2, 3, \ldots$ we may use the computer program GETUWT in SEAL (1978) once allowance is made for the parameter ν and the Pareto Laplace transform with $c = 0$ has been provided for in the subroutine RANILT of that program. Details of these adjustments together with the new subroutines to calculate $\nu e^{iz} E_{\nu+1}(iz)$ are given in Appendix II.

In illustration of these procedures we have calculated $U(0, t)$ and $U(w, t)$ for $\eta = .1$, $w = 10$ and $t = 1, 2, 3, 4, 5$ in conformity with Table 4.1 of SEAL (1978). The first point to notice is the very large value of T (BIGT) to use instead of infinity in relation (3.10). With $T = 200\pi$ we obtained

	$F(10 + 1.1t, t)$		$f(10 + 1.1t, t)$		$U(0, t)$	
t	$P(T)$	$Q(T)$	$P(T)$	$Q(T)$	$P(T)$	$Q(T)$
1	2×10^{-6}	-9×10^{-4}	-10^{-6}	-6×10^{-4}	2×10^{-15}	7×10^{-13}
5	-2×10^{-7}	-9×10^{-5}	-10^{-7}	-10^{-5}	2×10^{-16}	10^{-14}

It was not considered worth lengthening the range of integration to secure smaller values of $Q(T)$ for $t = 1$.

Using this T-value we ran GETUWT with 1024 panels in the three trapezoidal quadratures at each t-value but convergence appeared to be slow or even non-existent. Further runs with 2048 and 4096 panels (only acceptable because of the 15 significant figure working of the *CDC* computer being used) imply convergence to a four or five decimal result and use of Richardson's "deferred approach to the limit" (BUCKINGHAM, 1957, p. 90) produced final quadratures by Simpson's formula with 4096 panels. Table 3 shows the quadrature results.

TABLE 3

	$F(10 + 1.1t, t)$					$f(10 + 1.1t, t)$					$U(0, t)$				
Number of panels	$t = 1$	2	3	4	5	1	2	3	4	5	1	2	3	4	5
1024	1.7	1.6	1.6	1.6	1 6	.22788	.15859	.12779	.11027	.09883	.57616	.44969	.37631	.32154	.27472
2048	.99380	.98739	.98094	.97456	.96833	.00133	.00265	.00389	.00502	.00604	.58282	.46509	.40284	.36181	.33143
4096	.99421	.98824	.98223	.97631	.97053	.00127	.00256	.00377	.00488	.00587	.58397	.46762	.40701	.36793	.33989
Simpson	.99435	.98852	.98267	.97689	.97127	.00127	.00253	.00373	.00483	.00581	.58436	.46846	.40840	.36998	.34272

These Simpson figures were used in the formula for $U(10, t)$ to obtain the following results comparable with those of Table 4.1 of SEAL (1978).

VALUES OF $U(0, t)$ AND $U(10, t)$ WITH $\eta = 0.1$

$p_n(t)$	$B(y)$	$t = 1$	2	3	4	5
Poisson	Pareto	.5844	.4685	.4084	.3700	.3427
	$(\nu = 2.7)$	.9937	.9865	.9786	.9703	.9618

What we find surprising about these figures is the substantial improvement (increase) in $U(0, t)$ in comparison with Poisson/Exponential in spite of the decrease (which was expected) in $U(10, t)$ for each of $t = 1, 2, 3, 4, 5$. The results for $U(10, t)$ based on Simpson with 4096 panels, are all in excess of the corresponding figures for the trapezoidal, namely by 2, 3, 5, 7 and 8, respectively, in the fourth decimal place.

My thanks go to Peter Nuesch for his helpful suggestions about the evaluation of relation (3) and in connection with Appendix I.

APPENDIX 1

Distributions with Infinite Moments

The characteristic function exists for every probability density. On the other hand the moment generating function, which is obtained from the characteristic function's integral form by replacing the imaginary i by unity, must be tested for existence before it is used (LUKACS, 1970, p. 11). In particular, if any moment of a distribution is infinite the moment generating function does not exist.

Now the Pareto distribution is a special case of Fisher's F-distribution. The latter is an example of a beta distribution of the second kind, sometimes called a beta-prime distribution, namely (KENNEY and KEEPING, 1951, p. 96)

$$p(x) = \frac{1}{B(p, q)}\, x^{p-1}\,(1+x)^{-p-q}, \qquad 0 < x < \infty \qquad B(q, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

which has an nth moment about zero equal to

$$\mu_n' = \begin{cases} (p)_n/(q-1)^{(n)} & n < q \\ \infty & n \geq q \end{cases}$$

where $(p)_n$ is the ascending factorial $p(p+1)(p+2)\ldots(p+n-1)$. Putting $p = 1$, $q = \nu$ and introducing the scale factor b the Pareto density is obtained. The F-distribution itself is a scaled beta-prime distribution with $2p$ and $2q$ positive integers.

Both JOHNSON and KOTZ (1970, Ch. 26) and OBERHETTINGER (1973, Table A) state that the characteristic function of the above beta-prime distribution is ${}_1F_1\,(p;\,1-q;\,-it)$—though both books misprint the second argument as $-q$— where the confluent hypergeometric function ${}_1F_1$ is defined by

$${}_1F_1(a;b;x) \equiv \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!} \qquad \text{(Slater, 1960, p. 2)}$$

a series which is absolutely convergent for all values of a, b and x, real or complex, excluding $b = 0, -1, -2, \ldots$, and can thus be differentiated term by term. Hence

$${}_1F_1(p;1-q;-it) = \sum_{n=0}^{\infty} \frac{(p)_n}{(1-q)_n} \frac{(-it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(p)_n}{(q-1)^{(n)}} \frac{(it)^n}{n!}$$

which is in agreement with KENNEY and KEEPING (1951) for the moments of the beta-prime destribution when $n < q$. Now all moments of a distribution function exist if its characteristic function can be differentiated indefinitely (LUKACS, 1970, p. 22) hence the non-existence of moments of the beta-prime distribution for $n \geq q$ invalidates the confluent hypergeometric as its characteristic function. Our first idea of specializing the foregoing confluent hypergeometric function to the Pareto distribution by writing $p = 1$ and introducing the scale factor b had thus to be rejected.

APPENDIX 2

Computer Program for U(w, t)

The following are the *additions* to be made to GETUWT of SEAL (1978):

(I) GNU = 2.7 after XLAM

(II) IBTYPE = 3 after IPTYPE = 1

(III) Extend calling sequence in subroutines GETBGF, RANILT and DOUBLE by GNU after XLAM and accordingly change the CALL instructions:
- (a) two in the main program
- (b) three in GETBGF
- (c) two in DOUBLE

(IV) In subroutine RANILT:
- (a) Change IF(IBTYPE-1) 1, 1, 2 to IF(IBTYPE-2) 1, 2, 8
- (b) Insert after the ninth card (Q = —EXP etc.) the three cards:

```
      GO TO 3
    8 UB = U*(GNU—1.0)
      CALL EXPINT (GNU, UB, P, Q)
```

(V) Insert after subroutine RANILT the two subroutines EXPINT and GAMMA 1 exhibited below.

Remember to use N = 4096 and BIGT = 200.0 *PI

```
      SUBROUTINE EXPINT (GNU,X,P,Q)
      DOUBLE PRECISION DGNU,Y,DTERMP,DSUMP,DTERMQ,DSUMQ,DFJ,PP,QQ,GX
      DATA KOUNT/0/
      IF (X.EQ.0.0)   P=1.0
      IF (X.EQ.0.0)   Q=0.0
      IF (X.EQ.0.0) GO TO 6
      PI=3.141592653598
      DGNU=DBLE (GNU)
      Y=DBLE(X)
      IF (X.GE.25.0) GO TO 10
      IF (X.GT. 5.0) GO TO 7
      TERMP=GNU/ (—GNU)
      SUMP=TERMP
      TERMQ=GNU*X/ ((—GNU) * (—GNU+1.0))
      SUMQ=TERMQ
      DO 1 J=1,100
      FJ=FLOAT (J)
      TERMP=TERMP* (—X*X) / ((—GNU+2.*FJ—1.) * (—GNU+2.*FJ))
      SUMP=SUMP+TERMP
      TERMQ=TERMQ* (—X*X) / ((—GNU+2.*FJ    ) * (—GNU+2.*FJ+1.0))
      SUMQ=SUMQ+TERMQ
      IF (ABS(TERMP) .LT.5.0E—8) GO TO 2
    1 CONTINUE
    2 PP=SUMP
      QQ=SUMQ
      GO TO 9
    7 DTERMP=—1.0D0
      DSUMP=DTERMP
      DTERMQ=—Y / (—DGNU+1.0D0)
      DSUMQ=DTERMQ
      DO 20 J=1,100
      DFJ=FLOAT (J)
      DTERMP=DTERMP* (—Y*Y) / ((—DGNU+2.0D0*DFJ—1.0D0) * (—DGNU+2.0D0*DFJ))
      DSUMP=DSUMP+DTERMP
      DTERMQ=DTERMQ* (—Y*Y) / ((—DGNU+2.0D0*DFJ) * (—DGNU+2.0D0*DFJ+1.0D0))
      DSUMQ=DSUMQ+DTERMQ
      IF (DABS (DTERMP).LT.5.0D—8) GO TO 8
   20 CONTINUE
    8 PP=DSUMP
      QQ=DSUMQ
    9 IF (KOUNT.NE.0) GO TO 5
      CALL GAMMA1 (3.0D0—DGNU,GX)
```

```
   GX=GX / ((2.0D0—DGNU) * (1.0D0—DGNU))
 5 PP=—PP—GX*Y**DGNU*DCOS (Y+DGNU*PI/2.)
   QQ=—QQ—GX*Y**DGNU*DSIN (Y+DGNU*PI/2.)
   KOUNT=1
   P=SNGL (PP)
   Q=SNGL (QQ)
   GO TO 6
10 TERMP=GNU* (GNU+1.0) / (X*X)
   SUMP=TERMP
   TERMQ=(GNU*(—1.0) * (GNU+2.0) * (GNU+1.0) / (X*X*X)
   SUMQ=TERMQ
   DO 11 J=1,30
   FJ=FLOAT (J)
   TERMP=TERMP* (—1.) * (GNU+2.*FJ+1.) * (GNU+2.*FJ) / (X*X)
   SUMP=SUMP+TERMP
   TERMQ=TERMQ* (—1.) * (GNU+2.*FJ—1.) * (GNU+2.*FJ) / (X*X)
   SUMQ=SUMQ+TERMQ
   IF (ABS (TERMP).LT.5.E—8) GO TO 12
11 CONTINUE
12  P=SUMP
    Q=—GNU/X — SUMQ
   WRITE (6,3) J
 3 FORMAT (I5)
 6 RETURN
   END


 SUBROUTINE GAMMA1 (X,G)
 DOUBLE PRECISION X,G,C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13,
1   C14,C15,C16,C17,C18,C19,C20,C21,C22,C23,C24,C25
 C1   =   .5772156649015329D0
 C2   =—.6558780715202538D0
 C3 =  —.0420026350340952D0
 C4 =  .1665386113822915D0
 C5  =—.0421977345555443D0
 C6 =  —.0096219715278770D0
 C7 =  .0072189432466630D0
 C8 =—.0011651675918591D0
 C9 =—.0002152416741149D0
 C10=  .0001280502823882D0
 C11=—.0000201348547807D0
 C12=—.0000012504934821D0
```

```
C13=  .0000011330272320D0
C14=-.0000002056338417D0
C15=  .0000000061160950D0
C16=  .0000000050020075D0
C17=-.0000000011812746D0
C18=  .0000000001043427D0
C19=  .0000000000077823D0
C20=-.0000000000036968D0
C21=  .0000000000005100D0
C22=-.0000000000000206D0
C23=-.0000000000000054D0
C24=  .0000000000000014D0
C25=.0000000000000001D0
G=C25*X
G=G*X+C24*X
G = G*X + C23*X
G = G*X + C22*X
G = G*X + C21*X
G = G*X + C20*X
G = G*X + C19*X
G = G*X + C18*X
G = G*X + C17*X
G = G*X + C16*X
G = G*X + C15*X
G = G*X + C14*X
G = G*X + C13*X
G = G*X + C12*X
G = G*X + C11*X
G = G*X + C10*X
G = G*X + C9*X
G = G*X + C8*X
G = G*X + C7*X
G = G*X + C6*X
G = G*X + C5*X
G = G*X + C4*X
G = G*X + C3*X
G = G*X + C2*X
G = G*X + C1*X
G= G*X + X
G=1.0D0/G
RETURN
END
```

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