LARGEST CLAIMS REINSURANCE (LCR).
A QUICK METHOD TO CALCULATE LCR-RISK RATES FROM EXCESS OF LOSS RISK RATES

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Let us denote by $E(x)$ the pure risk premium of an unlimited excess cover with the retention $x$ and by $H(x)$ and $m(x)$ the corresponding expected frequency and severity.

We thus have $E(x) = H(x) \cdot m(x)$.

$H(x)$ is a non-increasing function of $x$ and for practical purposes we can assume that it is decreasing; $H'(x) < 0$. The equation $H(x) = n$ has then only one solution $x_n$, where $n$ is a fixed integer.

Let $E_n$ denote the risk premium for a reinsurance covering the $n$ largest claims from the bottom.

Let us define $E'_n = n x_n + E(x_n) = n(x_n + m(x_n))$. Intuitively we feel that $E'_n$ is a good approximation for $E_n$.

We shall first show that when the claims size distribution is Pareto and the number of claims is Poisson distributed, $E'_n$ is a good approximation for $E_n$, being slightly on the safe side. We further include a proof given by G. Ottaviani that the inequality $E_n < E'_n$ always holds.

In the Pareto case we have

$$H(x) = t(1 - F(x)) = t \cdot x^{-\alpha}$$

where the Poisson parameter $t$ stands for the expected number of claims in excess of 1 (equal to a suitably chosen monetary unit) and

$$m(x) = \frac{x}{\alpha - 1}.$$

The retention $x_n$ over which we expect $n$ claims should satisfy

$$n = H(x_n) = t \cdot x_n^{-\alpha}$$

which gives

$$t = n \cdot x_n^\alpha$$

or

$$x_n = \left(\frac{t}{n}\right)^{1/\alpha}.$$
According to B. Berliner [2] we have, when the number of claims is Poisson distributed

\[ E_n = t^{1/\alpha} \sum_{i=1}^{n} \frac{1}{i!} \cdot \Gamma_t \left( i - \frac{1}{\alpha} \right) \]

where

\[ \Gamma_t(n) = \int_{0}^{t} e^{-u} \cdot u^{n-1} \, du. \]

Replacing the incomplete Gamma function \( \Gamma_t \) by \( \Gamma_n = \Gamma \) we arrive at

\[ E_n = t^{1/\alpha} \cdot \frac{\alpha}{\alpha - 1} \cdot \frac{1}{\Gamma(n)} \cdot \Gamma \left( n + 1 - \frac{1}{\alpha} \right) \]

which formula was given by H. Ammeter already in 1964 [1]. Obviously \( E_n < \hat{E}_n \).

In all cases when \( t \) is large compared to \( n \), we have

\[ \frac{E_n}{\hat{E}_n}(n, \alpha; t) \] very close to 1.

If in a practical situation \( t \) is too small we can always increase \( t \) by decreasing the monetary unit, in other words by enlarging to the left the range of the Pareto distribution.

Inserting \( t = nx_n^\alpha \), as deduced above, in \( E_n \), we obtain

\[ E_n = n^{1/\alpha} \cdot x_n \cdot \frac{\alpha}{\alpha - 1} \cdot \frac{1}{\Gamma(n)} \cdot \Gamma \left( n + 1 - \frac{1}{\alpha} \right). \]

However

\[ E_n = n(x_n + m(x_n)) = n \cdot x_n \cdot \frac{\alpha}{\alpha - 1} = x_n \cdot \frac{\alpha}{\alpha - 1} \frac{\Gamma(n + 1)}{\Gamma(n)}. \]

Thus we have

\[ \frac{E_n}{E'_n} = \frac{n^{1/\alpha} \cdot \Gamma \left( n + 1 - \frac{1}{\alpha} \right)}{\Gamma(n + 1)} . \]
Tabulation of

\[
\begin{array}{ccc}
  n & \alpha = 2 & \alpha = 2.5 & \alpha = 3 \\
  1 & 0.886 & 0.894 & 0.903 \\
  2 & 0.940 & 0.943 & 0.948 \\
  3 & 0.959 & 0.961 & 0.964 \\
  4 & 0.969 & 0.971 & 0.973 \\
  5 & 0.975 & 0.976 & 0.978 \\
  10 & 0.988 & 0.988 & 0.989 \\
\end{array}
\]

The figures illustrate

\textit{that} the approximation is good,
\textit{that} the approximation is on the safe side,
and \textit{that} the approximation is rather invariant to variations of the parameter alpha within the given interval.

The safety margin in the approximation—\( E_n' \) replacing \( E_n \)—is roughly of the form constant/\( n \).

This is illustrated below for alpha = 2.5

\[
\begin{array}{ccc}
  n & \frac{E_n}{E_n'} & n \cdot \frac{E_n' - E_n}{E_n'} \\
  1 & 0.894 & 0.11 \\
  2 & 0.943 & 0.11 \\
  3 & 0.961 & 0.12 \\
  4 & 0.971 & 0.12 \\
  5 & 0.976 & 0.12 \\
  10 & 0.988 & 0.12 \\
\end{array}
\]

We have thus shown that in the Pareto case

\[
\frac{E_n}{E_n'} \sim 1
\]

and

\[
E_n < E_n < E_n' = nx_n + E(x_n) = nx_n + n \frac{x_n}{\alpha - 1} = nx_n \frac{\alpha}{\alpha - 1} = \alpha \cdot E(x_n).
\]
Thus
\[
\frac{E_n}{\bar{E}(x_n)} \sim a.
\]

This means that the LCR risk premium is approximately equal to alpha times the risk premium of an XL cover with a retention chosen in such a way that the expected number of claims is equal to the number of LCR-claims protected.

In the Poisson-Pareto case \(E'_n\) gives a handy and fairly good approximation of \(E_n\). The reader is invited to examine other claims size distributions \(F(x)\) which are of importance in the practice.

Most such distributions will for all \(x > x_0\) have \(m''(x) < 0\). We believe that \(m''(x) < 0\) will guarantee that \(E'_n\) will be a good approximation of \(E_n\) with \(E'_n > E_n\).

We now give a proof by G. Ottaviani that the inequality \(E_n < E'_n\) is valid for any \(n\) and for arbitrary distribution functions of the number of claims and of the claim size.

We do not even need the condition of section 2 that the equation \(H(x) = n\) has only one solution since the proof will be valid for any \(X_n\), such that \(H(x) = n\).

Let \(s\) denote the total number of claims which occur and \(N = \min(s, n)\). We thus allow for the possibility that less than \(n\) claims occur.

Let \(X_n\) be the set consisting of the \(N\) largest claims.

Let
\[
\nu(X_n) = E(N)
\]
\[
\nu(X_n) \leq n
\]

(1)

Let \(\mu(X_n) = E_n/\nu(X_n)\) be the expected value of a claim in the set \(X_n\).

Analogously we denote by \(X'_n\) the set consisting of all claims exceeding \(x_n\), the expected number of claims exceeding \(x_n\) by \(\nu(X'_n)\) and the expected value of a claim in the set \(X'_n\) by \(\mu(X'_n)\).

We thus have
\[
\nu(X'_n) = n
\]

(2)

and
\[
\mu(X'_n) = x_n + m(x_n).
\]

Let
\[
Y_n = X_n \cap X'_n
\]
\[
Z_n = (X_n \cap X'_n) - X'_n
\]
\[
Z'_n = (X_n \cap X'_n) - X_n
\]
\( v(Y_n), \mu(Y_n), v(Z_n), \mu(Z_n), v(Z'_n), \mu(Z'_n) \) are defined analogously to \( v(X_n) \) and \( \mu(X_n) \). From the above definition it follows directly that

\[
\begin{align*}
\mu(Z_i) &< x_n \quad \text{and} \quad (3) \\
\mu(Z'_i) &\geq x_n. \quad (4)
\end{align*}
\]

Thus

\[
E_n = v(X_i) \cdot \mu(X_i) = v(Y_i) \mu(Y_i) + v(Z_i) \mu(Z_i)
\]

and

\[
E'_n = v(X'_i) \cdot \mu(X'_i) = v(Y_i) \mu(Y_i) + v(Z'_i) \mu(Z'_i).
\]

From (1) and (2) it follows that

\[
v(Y_i) + v(Z_i) = v(X_i) \leq n = v(X'_i) = v(Y_i) + v(Z'_i).
\]

Thus

\[
v(Z_i) \leq v(Z'_i). \quad (7)
\]

From (3) and (4) it follows that

\[
\mu(Z_i) < \mu(Z'_i) \quad (8)
\]

and from (7) and (8)

\[
v(Z_i) \cdot \mu(Z_i) < v(Z'_i) \mu(Z'_i). \quad (9)
\]

Adding \( v(Y_i) \cdot \mu(Y_i) \) to both sides of (9) and using (5) and (6) leads to

\[
E_n < E'_n \quad \text{q.e.d.}
\]

REFERENCES
