On the problem of parameter and model uncertainty in risk management

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Introduction

PhD at University of Lausanne, with supervisor Prof. Hansjörg Albrecher

- **Structured reinsurance deals with reference to relative market performance.** Published in *Insurance: Mathematics and Economics*.
  L. Vincent, H. Albrecher et Y. Krvavych (2021)
- **An alternative path to extremes.** Preprint.
  L. Vincent (2022)
- **Risk estimation under parameter and model uncertainty.** Preprint.
  L. Vincent (2022)
- **A note on risk assessment under parameter uncertainty.** Preprint.
  L. Vincent (2022)

Risk modelling 101

Risk estimation under parameter and model uncertainty.

A risk ... e.g. a potential financial loss, natural catastrophe, etc.

- Modelled as a random variable ... $X \sim F$
- $F(x) = \mathbb{P}(X \leq x)$ ... distribution of X
- $F \Rightarrow$ VaR, ES, etc. ... risk measures

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- $F \Rightarrow$ VaR, ES, etc. ... risk measures

In practice ... F is unknown

- Data $\Rightarrow \widehat{F}$ ... estimated/fitted distribution
- $\widehat{F} \Rightarrow \widehat{\text{VaR}}, \widehat{\text{ES}}$, etc. ... estimated risk measures

Aleatoric vs. epistemic uncertainty

Two types of uncertainty

- $Y \sim F$ ... aleatoric uncertainty
- $F \neq \widehat{F}$ ... epistemic uncertainty

Manage risk using a fitted distribution $\widehat{F}$?

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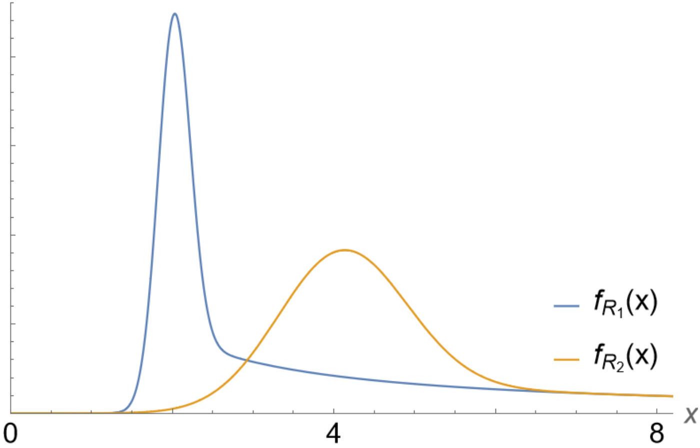
For instance

- X and Y ... two independent risks
- $\widehat{F}_X = \widehat{F}_Y$ ... with identical fitted distributions
- $n_X > n_Y$ ... but unequal sample sizes

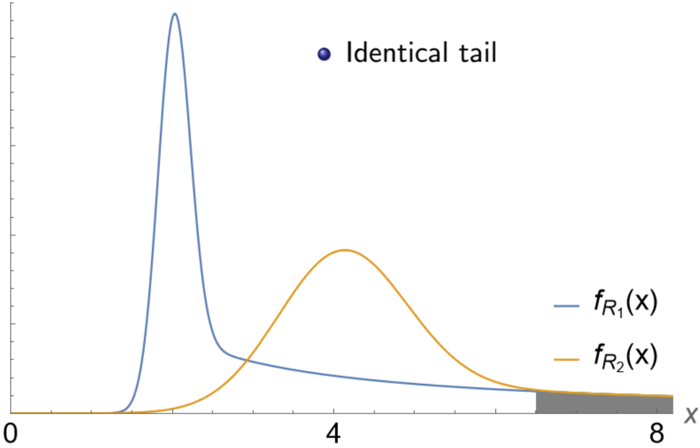
Aleatoric vs. epistemic uncertainty – Another example

- R_1 and R_2 ... two risks



Aleatoric vs. epistemic uncertainty – Another example

- R_1 and R_2 ... two risks
- Identical tail



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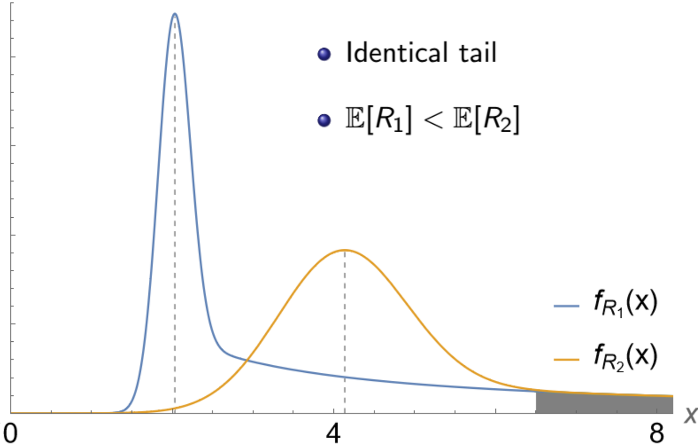

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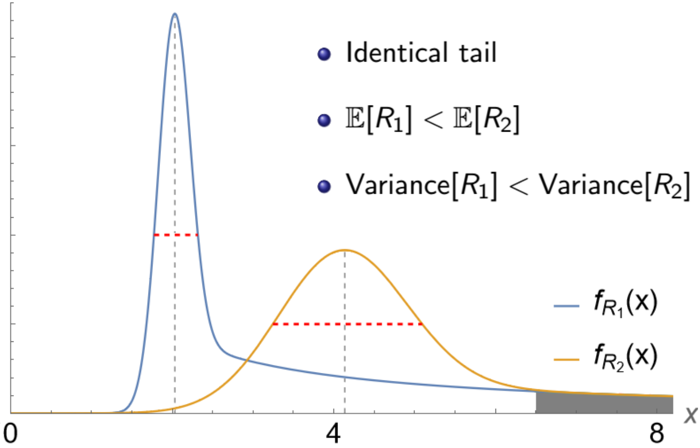

- R_1 and R_2 ... two risks
- Identical tail
- $\mathbb{E}[R_1] < \mathbb{E}[R_2]$
- $\text{Variance}[R_1] < \text{Variance}[R_2]$

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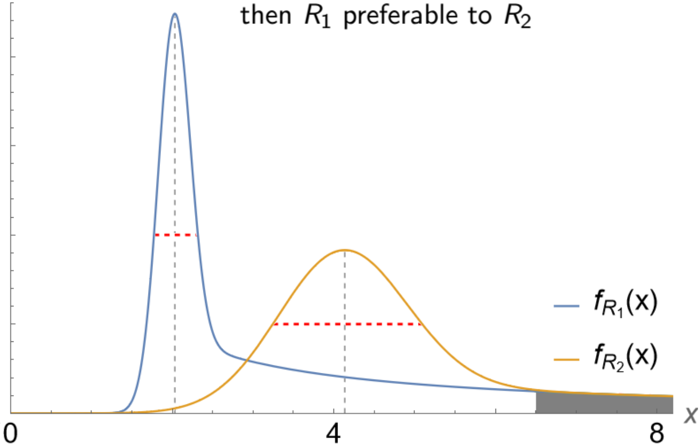
- So if f_{R_1} and f_{R_2} are known, then R_1 preferable to R_2



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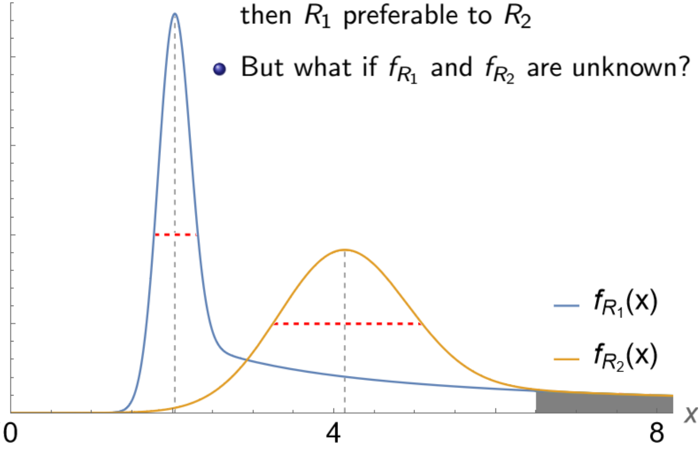

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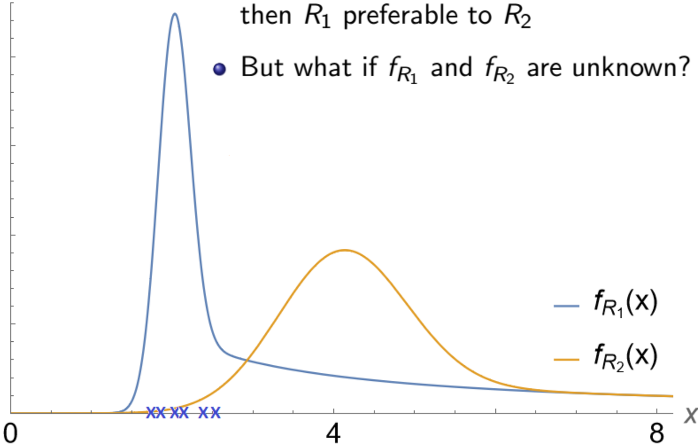

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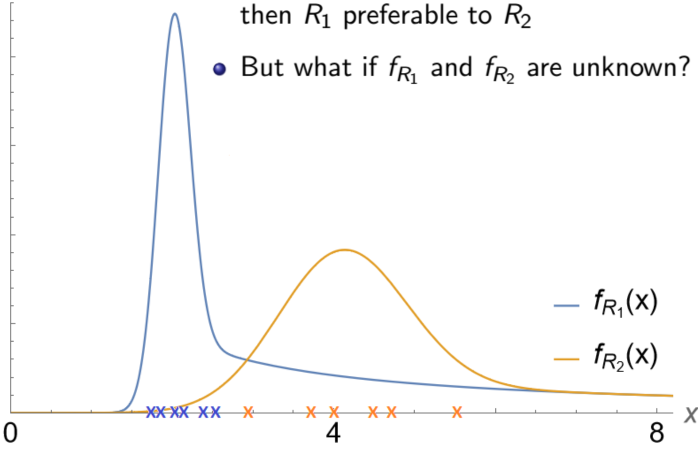

Model and parameter uncertainty

Estimating F through a parametric approach

1) A parametric model is assumed ... $F = F_\theta$

2) Data $\Rightarrow \widehat{\theta}$ ... estimated parameter (e.g. maximum likelihood)

3) $\widehat{F} = F_{\widehat{\theta}}$ ... estimated distribution

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$\Rightarrow$ Two sources for $F \neq F_{\widehat{\theta}}$ (epistemic uncertainty)

- $F \neq F_\theta$ ... wrong model (model uncertainty)
- $\theta \neq \widehat{\theta}$ ... wrong parameter (parameter uncertainty)

Capital calculation

Setting

- Potential future loss ... $X \sim F$
- Backed with some capital ... c
- "Solvency" ... if $X \leq c$

Target solvency probability $\beta \in (0, 1]$

- Target capital ... $c_\beta = F^{-1}(\beta) = \mathrm{VaR}_\beta[X]$
- Solvency probability ... $\mathbb{P}(X \leq c_\beta) = \beta$

Capital calculation under parameter uncertainty

With parameter uncertainty

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- $F_{\widehat{\theta}} \Rightarrow \widehat{c}_\beta$ ... calculated capital

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Resulting solvency probability?

- In general ... $\mathbb{P}(X \leq \widehat{c}_\beta) \neq \beta$

... see e.g. Gerrard and Tsanakas (2011), Fröhlich and Weng (2015)

Expected solvency probability

Performance measure? Before/after data realized

- After ... Ex-post capital $\quad \widehat{c}_\beta \;\Rightarrow\; \mathbb{P}(X \leq \widehat{c}_\beta)$
- Before ... Ex-ante capital $\quad \widehat{C}_\beta \;\Rightarrow\; \mathbb{P}(X \leq \widehat{C}_\beta)$

"Expected solvency probability" $\mathbb{P}(X \leq \widehat{C}_\beta)$ ... $\begin{cases} \neq \beta & \text{in general} \\ < \beta & \text{in relevant cases} \end{cases}$

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Why?

- Remember ... Data $\Rightarrow \widehat{\theta} \Rightarrow F_{\widehat{\theta}} \Rightarrow \widehat{c}_\beta$

$\Rightarrow$ Ignores parameter uncertainty

Capital calculation under parameter uncertainty

Bayesian approach ... Unknown parameter θ treated as random

- Prior $p(\theta)$ $\Rightarrow$ Posterior $p(\theta \mid \text{Data})$
- $\widetilde{F} = \int_\theta F_\theta \cdot p(\theta \mid \text{Data}) \, \mathrm{d}\theta$ ... Predictive distribution for X
- $\widetilde{F} \Rightarrow \widetilde{c}_\beta$ and $\widetilde{C}_\beta$ ... Ex-post and ex-ante capital

Expected solvency probability $\mathbb{P}(X \leq \widetilde{C}_\beta)$?

With parameter uncertainty only

Gerrard and Tsanakas (2011)

- "Transformed location-scale families" ... e.g. $X = g(\theta_1 + \theta_2 Z)$
- "Objective" prior ... $p(\theta_1, \theta_2) \propto \theta_2^{-1}$

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In my paper ... Extension to a more general framework, for instance

- Sample space of data and X ... can depend on θ_1, θ_2
- Data ... not necessarily mutually independent

The Pareto and Pareto-type cases

- Loss $X \ \sim\ $ Pareto(α, t) ... $1 - F(x) = (\frac{t}{x})^{\alpha}$
- Data ... n largest values of a Pareto random sample of size $k \geq n$

$\Rightarrow\ \widetilde{c}_{\beta} \ = \ \exp\left\{u_1(\text{data}) + u_2(\text{data}) \cdot g(k, n, \beta)\right\}$

So if the loss X is exactly Pareto but with unknown parameters

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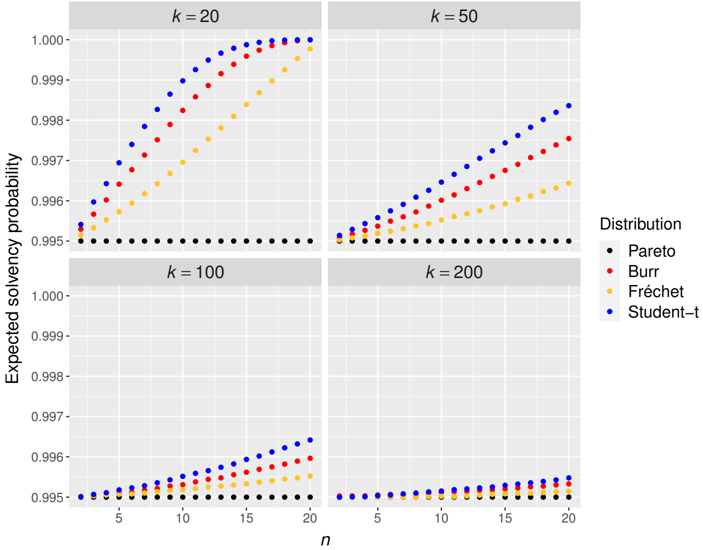
What if not exactly Pareto?

- "Pareto-type" tail ... $1 - F(x) = \ell(x) \cdot (\frac{t}{x})^{\alpha}$
- $\lim_{x \to \infty} \ell(x) = 1$

Illustration



A note on risk assessment under parameter uncertainty.

Predictive distributions under "objective" prior $\Rightarrow$ very conservative ...
For instance

- $X \sim \text{Pareto}(\alpha, 1)$
- Data $\Rightarrow$ $\widehat{\alpha}$ ... maximum likelihood

Then

- Estimated distribution ... $\widehat{F}(x) = 1 - x^{-\widehat{\alpha}}$ (Pareto)
- Predictive distribution ... $\widetilde{F}(x) = 1 - \left(1 + \frac{\ln(x^{\widehat{\alpha}})}{n}\right)^{-n}$ (Log-Pareto)

Too conservative?

Joint solvency probability

Setting ... Risks $X_1 \perp X_2$

- $X_i \sim \text{Pareto}(\alpha_i, 1)$ for both i
- Allocation of a fixed total capital ... $c = c_1 + c_2$
- Joint solvency probability ... $\psi(c_1) = F_1(c_1) \cdot F_2(c - c_1)$

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Possible criterion ... Maximize $\psi(c_1)$

$$\text{Best allocation} \quad \Rightarrow \quad c_1^* = \underset{c_1 \in [0,c]}{\arg\max} \; \psi(c_1)$$

Parameter uncertainty ... What if α_1, α_2 are unknown?

Estimated vs. predictive distributions

Two approaches

- With estimated distributions ... Data $\Rightarrow$ $\widehat{F}_1, \widehat{F}_2$ $\Rightarrow$ $\widehat{c}_1^*$
- With predictive distributions ... Data $\Rightarrow$ $\widetilde{F}_1, \widetilde{F}_2$ $\Rightarrow$ $\widetilde{c}_1^*$

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For one particular dataset

- Resulting joint survival probabilities ... $\psi(\widehat{c}_1^*) \lesseqgtr \psi(\widetilde{c}_1^*)$

Two approaches

- With estimated distributions ... Data $\Rightarrow \widehat{F}_1, \widehat{F}_2 \Rightarrow \widehat{c}_1^*$
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Performance criterion? Before data realized

- Ex-ante joint solvency probabilities ... $\psi(\widehat{C}_1^*)$ and $\psi(\widetilde{C}_1^*)$

Estimated vs. predictive distributions

In short

- $\mathbb{E}[\psi(\widetilde{C}_1^*)] \; > \; \mathbb{E}[\psi(\widehat{C}_1^*)]$
- Variance$[\psi(\widetilde{C}_1^*)] \; < \;$ Variance$[\psi(\widehat{C}_1^*)]$
- Less data $\;\Rightarrow\;$ Larger improvement

So by accounting for parameter uncertainty, predictive distributions here in general lead to better resources (capital) allocations than estimated distributions

Conclusion

Key points

- Epistemic uncertainty shall be taken into account when managing risk
- Parameter and model uncertainty are two types of epistemic uncertainty
- Bayesian approach can help for both capital calculation and resource (capital) allocation problems

... find details in my papers!

Thanks for your attention.

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References I

Fröhlich, A. and Weng, A. (2015). Modelling parameter uncertainty for risk capital calculation. *European Actuarial Journal*, 5(1):79–112.

Gerrard, R. and Tsanakas, A. (2011). Failure probability under parameter uncertainty. *Risk Analysis: An International Journal*, 31(5):727–744.