TMV-Based Capital Allocation for Multivariate Risks

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ABSTRACT

This paper studies a novel capital allocation framework based on the tail mean-variance (TMV) principle for multivariate risks. The new capital allocation model has many intriguing properties, such as controlling the magnitude and variability of tail risks simultaneously. General formulas for optimal capital allocations are discussed according to the semideviation distance measure. In particular, we discuss the optimal capital allocation for comonotonic risks, and risks from multivariate elliptical distribution and multivariate skew-t distribution. Some numerical examples are given to illustrate the results, and real data from an insurance company is analyzed as well.

KEYWORDS

Capital allocation, Lagrange multiplier, mean-variance, skew-t distribution, tail risks

1. Introduction and motivation

In the actuarial literature, a fundamental question is how to allocate the total amount of risk capital to different subportfolios, divisions, or lines of businesses. The allocation problem is very important since the amount of risk capital allocated to a business consisting of multiple lines of businesses is typically less than the sum of amounts of risk capital that would need to be withheld for each business separately. Heterogeneity and dependence that may exist between the performances of various business units make capital allocation a nontrivial exercise. Therefore, there exists an extensive amount of literature on this subject with a number of proposed capital allocation algorithms. For example, Myers and Read (2001) considered capital allocation principles based on the marginal contribution of each business unit to the company's default option. Denault (2001) discussed capital allocations from the perspective of game theory. The first multivariate top-down model considered in Panjer (2002) studies the particular case of multivariate, normally distributed risks and provides an explicit expression of marginal cost-based allocations using TVaR (tail value-at-risk) risk measure. This work has been extended by Landsman and Valdez (2003) to model risks using multivariate elliptical distributions, which include the multivariate normal as a special case; see also Dhaene et al. (2008). Furman and Landsman (2005) studied the capital allocation for the risks following multivariate gamma distributions. Cossette et al. (2013) discussed the multivariate risks with mixed Erlang marginals and the dependence structure is modeled by the Farlie-Gumbel-Morgenstern copula. One may refer to Dhaene et al. (2012), Xu and Hu (2012), Tsanakas (2004), and Furman and Zitikis (2008) and references therein for the recent developments on this topic.

Assume that a firm has a portfolio of risks $X_1, \dots, X_n$, and wishes to allocate the total capital $K = k_1 + \cdots + k_n$ to the corresponding risks. The total risk is then

$$S = X_1 + \ldots + X_n.$$

Recently, Dhaene et al. (2012) proposed a criterion to set the capital amount k_i to X_i as close as possible to minimize the loss. Specifically, the criterion is to minimize the following loss function

$$L(\boldsymbol{k}) = \sum_{i=1}^{n} D(X_i - k_i), \tag{1.1}$$

where D is some suitable distance measurement function, and $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{R}^n$. A lot of work has been motivated by this criterion; see Xu and Hu (2012), Zaks (2013), Cheung, Rong, and Yam (2014), and others. In fact, the idea of minimizing the loss function has been discussed in the framework of premium calculation. For example, Zaks et al. (2006) used quadratic distance measure $D(x) = x^2$, and Laeven and Goovaerts (2004) used the semi-deviation function $D(x) = \max\{x, 0\}$ as distance measure. This topic was further pursued in Frostig, Zaks and Levikson (2007), where they used the general convex distance measure. However, most of the discussion on capital allocations in the literature has focused only on the magnitude of the loss function L. In practice, the variability also plays an essential role in determining the capital allocations. Indeed, the relevant idea has already appeared in the premium calculation. Furman and Landsman (2006) used the tail variance risk measure to estimate the variability along the tails, and to compute the premium based on the tail variance premium (TVP) model

$$\mathrm{TVP}_q(X) = \mathrm{TCE}_q(X) + \beta \mathrm{TV}_q(X), \qquad \beta \geq 0,$$

where TCE_q and TV_q represent tail conditional expectation, and tail conditional variance, respectively. That is,

$$\mathrm{TCE}_q(X) = \mathbb{E}(X | X > x_q),$$
$$\mathrm{TV}_q(X) = \mathrm{Var}(X | X > x_q),$$

where x_q is qth quantile of risk X. See also Landsman, Pat, and Dhaene (2013) for the discussion on the tail variance related premium calculation. Motivated by this observation, Xu and Mao (2013) proposed a TMV model to discuss the optimal capital allocations, where they defined the loss function as

$$G(\mathbf{X};\boldsymbol{k},q)=\sum_{i=1}^{n}\left[(X_i-k_i)^2\middle|S>\mathrm{VaR}_q(S)\right],$$

where $\mathrm{VaR}_q(S)$ is the qth quantile of S, and considered the following function

$$\pi[G(\mathbf{X};\boldsymbol{k},q)]=\mathbb{E}[G(\mathbf{X};\boldsymbol{k},q)]+\beta\mathrm{Var}(G(\mathbf{X};\boldsymbol{k},q)),$$

where $\pi(\cdot)$ is the mean-variance risk measurement, which has been widely used in practice (Laeven and Goovaerts 2004). The TMV model has many intriguing properties, such as simultaneously controlling the magnitude and variability of tail loss, and providing neat optimal allocation formulas. From the economic perspective, the perfect case is that the company could prepare the capital to match the loss exactly, since too much or less capital would result in the loss of revenue for a company. Therefore, a company should prefer a capital allocation rule which could provide the capital to match the loss as close as possible. It is apparent that controlling the magnitude of deviation of the capital from the loss is important. However, the variability of deviation is also essential in determining the required capital, as the larger variability would lead to more risk for the company. Therefore, the property of controlling the magnitude and variability is appealing in determining the required capital for business lines.

In practice, however, the shortage of capital may often result in much severer consequences than that caused by the excess of capital (Myers and Read 2001; Erel, Myers and Read 2013), which suggests that the semideviation function may be preferred in practice as the distance measure. This issue is also related to the capital allocation of homeland security, an area that has become centrally important since the terrorist attacks of September 11, 2001. Since catastrophes are highly risky and could lead to severe consequences, the Department of Homeland Security (DHS) has endeavored to use risk management to determine the capital allocations on prevention, response, and recovery from such national catastrophes. The budget in DHS is allocated via the program called UASI (Urban Area Security Initiative) each year. For example, DHS allocated a total of \$490.4 million in 2012, \$558.7 million in 2013, and \$587.0 million in 2014 to urban areas to prevent terrorist attacks.[1] The effective allocation of the total capital to urban areas is an important but challenging problem, which has received much attention in the security area (cf. Hu, Homem-de-Mello, and Mehrotra 2011). A popular distance measure used in this area is the semideviation function. In fact, in the literature of actuary science, the semideviation function has been widely used in the stop-loss premium calculation (Dhaene et al. 2012). Based on the above discussion, in this paper, we are motivated to study the capital allocation based on the TMV model with the loss function defined as

$$L(\mathbf{X};\boldsymbol{k},q)=\sum_{i=1}^{n}\left[(X_i-k_i)_+\middle|S>\mathrm{VaR}_q(S)\right].$$

We consider the following general mean-variance model,

$$\begin{cases}\min_{k\in A}\pi[L(\mathbf{X};\boldsymbol{k},q)];\\ s.t.\qquad A=\left\{\boldsymbol{k}\in\mathbb{R}^n:\sum_{i=1}^{n}k_i=K,i=1,\ldots,n\right\}.\end{cases}\tag{1.2}$$

where $\pi(\cdot)$ is the mean-variance risk measurement, and $\beta\geq 0$. It is worth pointing out that Laeven and Goovaerts (2004) considered a special case of the TMV model (1.2). They discussed the case of $n = 2$ but without considering the tail risks. Specifically, they discussed the optimal capital allocation based on minimizing the following loss function,

$$\pi[L(\mathbf{X};\boldsymbol{k})]=\mathbb{E}[(X_1-k_1)_+ +(X_2-k_2)_+]$$
$$+\beta\mathrm{Var}[(X_1-k_1)_+ +(X_2-k_2)_+],$$

over $\boldsymbol{k}\in A$. Therefore, the TMV model (1.2) is a natural extension of their model.

Our main contributions in this paper are summarized as follows. First, we derive the general equations

[1]The data is from the website of Federal Emergency Management Agency, https://www.fema.gov/fy-2014-homeland-security-grant-program-hsgp.

for the TMV model (1.2), based on which the numerical programming could be easily implemented. Second, we discuss the special case of comonotonic risks, and the closed-form solutions are obtained. Third, we compute the key quantities of optimal capital allocation formulas for multivariate elliptical distributions, and Monte Carlo simulation for those quantities of multivariate skew-t distributions is also mentioned. Finally, we conduct a real data analysis and discuss the optimal capital allocation based on the new model.

The rest of the paper is organized as follows. In Section 2, we derive the general equations for the TMV model, and discuss a special case. Section 3 studies the optimal capital allocations for the comonotonic risks. In Section 4, we present some numerical examples to illustrate the different factors that affect the capital allocations and conduct a real data analysis of capital allocations for an insurance company. In the last section, we summarize the results and present some discussion.

2. Optimal capital allocation: General results

In this section, we provide general capital allocation equations for the TMV model (1.2). To facilitate the discussion, let us denote the conditional survival function of $[X_i | S > \mathrm{VaR}_q(S)]$ by

$$\bar{F}_{i.S}(k_i) = \mathbb{P}(X_i > k_i | S > \mathrm{VaR}_q(S)), \qquad i = 1, \dots, n.$$

The conditional expectation of risk excess $[(X_i - k_i)_+ | S > \mathrm{VaR}_q(S)]$ is denoted by

$$\mathrm{ECT}_S(k_i) = \mathbb{E}\left[(X_i - k_i)_+ | S > \mathrm{VaR}_q(S)\right],$$

and the covariance between $[(X_i - k_i)_+ | S > \mathrm{VaR}_q(S)]$ and $[\mathrm{I}(X_j \geq k_j) | S > \mathrm{VaR}_q(S)]$ is represented by

$$\mathrm{Cov}_{+,S}(k_i, k_j) = \mathrm{Cov}\left[(X_i - k_i)_+, \mathrm{I}(X_j \geq k_j) | S > \mathrm{VaR}_q(S)\right],$$

for $i, j = 1, \dots, n$, where $\mathrm{I}(\cdot)$ is the indicator function.

In the following, by using the methodology of Lagrange multipliers, we present the optimal capital allocation equations based on the TMV model (1.2), and the uniqueness condition is also given. The proof is moved to the Appendix for the sake of readability.

Theorem 2.1. For the TMV model (1.2), assume that $X_1, \dots, X_n$ are continuous risks, then an optimal allocation solution $\boldsymbol{k}^* = (k_1^*, \dots, k_n^*)$ is given by the following equations, for any $l = 1, 2, \dots, n$,

$$\bar{F}_{l.S}(k_l^*) + 2\beta \sum_{j=1}^{n} \mathrm{Cov}_{+,S}(k_j^*, k_l^*) = \bar{F}_{1.S}(k_1^*) + 2\beta \sum_{j=1}^{n} \mathrm{Cov}_{+,S}(k_j^*, k_1^*) \qquad (2.1)$$

and

$$k_1^* + \dots + k_n^* = K.$$

Further, if, for any $l = 1, 2, \dots, n$,

$$1 + 2\beta \sum_{j \neq l}^{n} \mathbb{E}\left[(X_j - k_j^*)_+ | X_l = k_l^*, S > \mathrm{VaR}_q(S)\right] > 2\beta \sum_{j=1}^{n} \mathrm{ECT}_S(k_j^*) \qquad (2.2)$$

then the solution is unique.

From Theorem 2.1, it is seen that the capital allocations based on Model (1.2) depend not only on the magnitude of tail risks but also the covariance among the tail risks. This property would allow the company to control the tail risks from both the magnitude and variability perspectives. In general, there does not exist an analytical solution to Eq. (2.1). The key quantities required to solve the equation are $\bar{F}_{i.S}(\cdot)$, $\mathrm{ECT}_S(\cdot)$, and $\mathrm{Cov}_{+,S}(\cdot, \cdot)$, which, however, could be efficiently computed by using any computer software; see Section 4 for examples. It can be seen from Eq. (2.2) that when β is small, then the uniqueness condition is easily satisfied. In the following, we discuss a special case of $\beta = 0$ for Theorem 2.1, i.e., without considering the penalty on the tail variance. For this case, a closed-form solution could be obtained.

Corollary 2.2. Under the same condition of Theorem 2.1, for $\beta = 0$, a unique optimal allocation solution $\boldsymbol{k}^* = (k_1^*, \ldots, k_n^*)$ is given by

$$k_i^* = F_{i.S}^{-1}(F_{\bar{S}^c}(K)), \qquad i = 1, \ldots, n,$$

where

$$\bar{S}^c = \sum_{i=1}^{n} F_{i.S}^{-1}(U),$$

with $F_{i.S}^{-1}(U) = [X_i | S > \mathrm{VaR}_q(S)]$ almost surely.

Proof: According to Theorem 2.1, the optimal solution should satisfy the following equations:

$$\bar{F}_{l.S}(k_l^*) = \bar{F}_{1.S}(k_1^*) \tag{2.3}$$

for $l = 2, \ldots, n$ and $k_1^* + \ldots + k_n^* = K$. Now define the

$$\bar{S}^c = \sum_{i=1}^{n} F_{i.S}^{-1}(U),$$

where U is the uniform random variable on [0, 1]. It is known from Dhaene et al. (2012) that there exists some $0 \le \alpha \le 1$ such that

$$\sum_{i=1}^{n} F_{i.S}^{-1(\alpha)}(F_{\bar{S}^c}(K)) = K,$$

where $F_{i.S}^{-1(\alpha)}(\cdot)$ is the α-mixed inverse distribution function. Therefore, an optimal solution is given by

$$k_i^* = F_{i.S}^{-1(\alpha)}(F_{\bar{S}^c}(K)), i = 1, \ldots, n,$$

which satisfies Eq. (2.3). Moreover, the uniqueness condition in Eq. (2.2) is fulfilled since $\beta = 0$.

Hence, the required result follows. ■

To conclude this section, we mention that the optimal capital allocation based on the TMV model (1.2) relies on several key quantities of risks from Eq. (2.1). Those quantities are nontrivial to compute since they depend on the tail conditional distribution of multivariate risks. In the following sections, we discuss how to derive the optimal capital allocations for comonotonic risks and some specific multivariate distributions, which are often used in the literature.

3. Comonotonic risks

Comonotonicity, an extremal form of positive dependence, has been widely used in finance and actuarial science over the last two decades. It is well known that the comonotonic random variables are always moving in the same direction simultaneously and hence are considered as extreme dependent risks. Refer to Dhaene et al. (2002a; 2002b) for the properties and applications of this concept in actuarial science and finance. For a company with several business lines, it is particularly important for them to prepare for the worst scenario. It is known in the literature that the aggregate risk of comonotonic risks with finite means may be regarded as the most dangerous case in terms of convex order (Dhaene et al. 2002a). From the perspective of capital allocation allocations, it would be interesting to know whether the comonotonic dependence structure among risks is the most dangerous case in terms of some stochastic measure. Further, if it is the most dangerous scenario, what is the optimal capital allocation strategy? In this section, we first show that the comonotonic risks are the most dangerous risks for the capital allocations in the sense that the expected tail loss is the largest. Then, we discuss the optimal capital allocation based on the TMV model (1.2). We need the following two lemmas.

The first lemma presents an equivalent characterization of a comonotonic random vector (Dhaene et al. 2002a).

Lemma 3.1. A random vector $(X_1, \ldots, X_n)$ is comonotonic if and only if there are increasing real-valued functions $f_1, \ldots, f_n$ and and a random variable W such that

$$(X_1, \ldots, X_n) \stackrel{st}{=} (f_1(W), \ldots, f_n(W)),$$

where $\stackrel{st}{=}$ represents that both sides of equality have the same distribution.

The following lemma, essentially due to Sordo et al. (2013), will also be used in the sequel.

Lemma 3.2. Let X and Y be two continuous risks with strictly increasing distribution functions F and G, respectively. Then, for $q \in (0, 1]$, it holds that

$$[X|Y > G^{-1}(q)] \leq_{st} [X|X > F^{-1}(q)],$$

where $\leq_{st}$ represents the usual stochastic order (Shaked and Shanthikumar 2007). Particularly if X and Y are comonotonic, then

$$[X|Y > G^{-1}(q)] \overset{st}{=} [X|X > F^{-1}(q)].$$

By utilizing the above two lemmas, we show that the comonotonic risks result in the largest tail losses, which may have its own interest.

Theorem 3.3. Let $(X_1, \ldots, X_n)$ be a continuous random vector with strictly increasing distribution functions, and $(X_1^c, \ldots, X_n^c)$ represents its comonotonic counterpart. Then,

$$\mathbb{E}\left[\sum_{i=1}^{n}(X_i - k_i)_+|S > \mathrm{VaR}_q(S)\right] \leq \mathbb{E}\left[\sum_{i=1}^{n}(X_i^c - k_i)_+|S^c > \mathrm{VaR}_q(S^c)\right].$$

Proof: Since (X_i^c, S^c) are comonotonic for $i = 1, \ldots, n$, from Lemma 3.1 it follows that

$$((X_i^c - k_i)_+, S^c)$$

are also comonotonic, since $h(x) = (x - k_i)_+$ is an increasing function of x. Therefore, according to Lemma 3.2, we have

$$\begin{aligned}
&[(X_i^c - k_i)_+|S^c > \mathrm{VaR}_q(S^c)] \\
&\overset{st}{=} [(X_i^c - k_i)_+|(X_i^c - k_i)_+ > \mathrm{VaR}_q((X_i^c - k_i)_+)] \\
&\overset{st}{=} [(X_i - k_i)_+|(X_i - k_i)_+ > \mathrm{VaR}_q((X_i - k_i)_+)] \\
&\geq_{st} [(X_i - k_i)_+|S > \mathrm{VaR}_q(S)].
\end{aligned}$$

Hence, the required result follows immediately. ■

Now, let us discuss the optimal capital allocation based on Model (1.2) for this worst scenario, i.e., $X_1, \ldots, X_n$ are comonotonic risks.

Theorem 3.4. Under Model (1.2), a unique optimal allocation solution $\boldsymbol{k}^* = (k_1^*, \ldots, k_n^*)$ when $(X_1, \ldots, X_n)$ are comonotonic risks with strictly increasing distributions is given by

$$k_i^* = F_{i.S}^{-1}(F_{\bar{S}^c}(K)), \qquad i = 1, \ldots, n, \tag{3.1}$$

where $\bar{S}^c = \sum_{i=1}^{n} F_{i.S}^{-1}(U)$, with $F_{i.S}^{-1}(U) = [X_i|S > \mathrm{VaR}_q(S)]$ almost surely.

Proof: Note that

$$\begin{aligned}
&\sum_{j=1}^{n}\mathrm{Cov}_{+,S}(k_j^*, k_l^*) \\
&= \sum_{j=1}^{n}\mathrm{Cov}[(X_j - k_j^*)_+, \mathrm{I}(X_l \geq k_l^*)|S > \mathrm{VaR}_q(S)] \\
&= \mathrm{Cov}\left[\sum_{j=1}^{n}(X_j - k_j^*)_+, \mathrm{I}(X_l \geq k_l^*)|S > \mathrm{VaR}_q(S)\right].
\end{aligned}$$

Since $(X_1, \ldots, X_n)$ is a comonotonic vector, it holds that

$$[(X_1, \ldots, X_n)|S > \mathrm{VaR}_q(S)]$$

is also comonotonic, and, further,

$$[(X_1 - k_1^*)_+, \ldots, (X_n - k_n^*)_+|S > \mathrm{VaR}_q(S)]$$

is comonotonic. According to Proposition 1 of Cheung (2009), it holds that

$$\left[\sum_{i=1}^{n}(X_i - k_i^*)_+|S > \mathrm{VaR}_q(S)\right] \overset{a.s.}{=} [(S - K)_+|S > \mathrm{VaR}_q(S)],$$

where $\overset{a.s.}{=}$ represents both sides are almost surely equal. Therefore, we have

$$\begin{aligned}
&\mathrm{Cov}\left[\sum_{j=1}^{n}(X_j - k_j^*)_+, \mathrm{I}(X_l \geq k_l^*)|S > \mathrm{VaR}_q(S)\right] \\
&= \mathrm{Cov}[(S - K)_+, \mathrm{I}(X_l \geq k_l^*)|S > \mathrm{VaR}_q(S)] \\
&= \mathrm{Cov}[(S - K)_+, \\
&\qquad \mathrm{I}(F_{l.S}(X_l) \geq F_{\bar{S}^c}(K))|S > \mathrm{VaR}_q(S)] \\
&= \mathrm{Cov}[(S - K)_+, \mathrm{I}(U \geq F_{\bar{S}^c}(K))|S > \mathrm{VaR}_q(S)].
\end{aligned}$$

It is seen that Eq. (2.1) is fulfilled if $\boldsymbol{k}^*$ is a solution. We conclude that $\boldsymbol{k}^*$ is an optimal solution for Model (1.2). Further, the solution $\boldsymbol{k}^*$ is unique, as it does not depend on the parameter β. Hence, the required result follows. ■

Theorem 3.4 presents a closed-form solution of capital allocations for the comonotonic risks. It might be a little surprising to observe that the optimal capital allocation rule based on Model (1.2) for comonotonic risks adopts the same formulas as that in Corollary 2.2, i.e., without the penalty on the tail variance. A careful checking of Theorem 2.1 reveals that, although the formulas are the same, the meanings are quite different for both scenarios. Corollary 2.2 presents the optimal capital allocations for any dependence structure by considering only the magnitude of tail risks. However, Theorem 3.4 presents the optimal capital allocations for the comonotonic risks by considering both the magnitude and variability of tail risks. But, for this particular dependence structure, the magnitude and variability of loss functions are minimized simultaneously, which explains the same optimal capital allocation formulas as in Corollary 2.2. One may wonder whether the magnitude and variability of loss functions could be minimized simultaneously for other general multivariate risks, i.e., β is irrelevant to the optimal capital allocations. The answer is negative from the examples in Section 4. In fact, the penalty parameter β has nonnegligible influence on the capital allocations.

4. Examples and applications

In this section, we present some examples of optimal capital allocations based on Model (1.2) for specific multivariate distributions. We will also apply the new capital allocation rule to real data from one insurance company.

4.1. Elliptical distributions

In the literature of insurance and actuarial science, the elliptical distribution has attracted much attention, mainly due to its mathematical tractability. It includes many well-known distributions, such as multivariate normal distribution, multivariate t distribution, multivariate logistic distribution, and multivariate exponential power distribution, etc. For more discussion of elliptical distribution, one may refer to Fang, Kotz, and Ng (1987) and Landsman and Valdez (2003).

In the following, we first give a brief review of some properties of elliptical distribution, which is pertinent to the discussion of our main results.

Definition 4.1. The random vector $\mathbf{X}$ has a multivariate elliptical distribution, denoted by $\mathbf{X} \sim E_n(\mu, \Sigma, \psi)$, if its characteristic function can be expressed as

$$\phi_{\mathbf{X}}(\mathbf{t}) = \exp\left(i\mathbf{t}^{\mathrm{T}}\mu\right)\psi\left(\mathbf{t}^{\mathrm{T}}\Sigma\mathbf{t}/2\right)$$

for some column-vector μ, $n \times n$ positive definite matrix Σ, and characteristic generator $\psi(\cdot)$.

It should be pointed out that not every multivariate elliptical distribution has a density function. If $\mathbf{X} \sim E_n(\mu, \Sigma, \psi)$, and $\mathbf{X}$ has a density $f_{\mathbf{X}}(\mathbf{x})$, then,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{|\Sigma|^{1/2}} g_n\left(\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1}(\mathbf{x}-\mu)\right), \qquad (4.1)$$

where

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}}\left(\int_0^\infty x^{n/2-1} g_n(x)\,dx\right)^{-1},$$

and

$$\int_0^\infty x^{n/2-1} g_n(x)\,dx < \infty,$$

which guarantees $g_n(x)$ to be the density generator. For this case, one may write $\mathbf{X} \sim E_n(\mu, \Sigma, g_n)$.

If the mean exists, we have $\mathbb{E}(\mathbf{X}) = \mu$. The condition guarantees the existence of the covariance matrix $|\psi'(0)| < \infty$ and hence

$$\mathrm{Cov}(\mathbf{X}) = -\psi'(0)\Sigma.$$

Without loss of generality, in the following discussion, it is assumed that $-\psi'(0) = 1$, and hence $\mathrm{Cov}(\mathbf{X}) = \Sigma$. For the comprehensive discussion of properties of elliptical distributions, please refer to Fang, Kotz, and Ng (1987).

We first recall the well-known property of elliptical distributions.

Proposition 4.2. If $\mathbf{X} \sim E_n(\mu, \Sigma, g_n)$, and A is some $m \times n$ matrix of rank $m \le n$, and $\mathbf{b}$ some m-dimensional column-vector, then

$$A\mathbf{X} + \mathbf{b} \sim E_m\left(A\mu + \mathbf{b}, A\Sigma A^T, g_m\right).$$

Next, we compute the key quantities, including $\bar{F}_{i.S}(\cdot)$, $\mathrm{ECT}_S(\cdot)$, and $\mathrm{Cov}_{+,S}(\cdot, \cdot)$ for the family of elliptical distributions, which would facilitate the computations of Eq. (2.1).

Note that if $\mathbf{X} \sim E_n(\mu, \Sigma, g_n)$, then by Proposition 4.2, it holds that

$$S \sim E_1(\mu_S, \sigma_{S,S}, g_1),$$

where $\mu_S = \Sigma_{i=1}^n \mu_i$, and $\sigma_{S,S} = \Sigma_{j=1}^n \Sigma_{i=1}^n \sigma_{ij}$ with $\sigma_{ij} = \mathrm{Cov}(X_i, X_j)$. Further, by Xu and Mao (2013), we have

$$(X_i, X_j | S = s) \sim E_2(\mu_{ij.s}, \Sigma_{ij.S}, g_2), \qquad (4.2)$$

where

$$\mu_{ij.s} = \begin{pmatrix} \mu_{i.s} \\ \mu_{j.s} \end{pmatrix} = \begin{pmatrix} s(\sigma_{i,S}/\sigma_{S,S}) + \mu_i - \mu_S \sigma_{i,S}/\sigma_{S,S} \\ s(\sigma_{j,S}/\sigma_{S,S}) + \mu_j - \mu_S \sigma_{j,S}/\sigma_{S,S} \end{pmatrix},$$

and

$$\Sigma_{ij.S} = \begin{pmatrix} \sigma_{ii.S} & \sigma_{ij.S} \\ \sigma_{ji.S} & \sigma_{jj.S} \end{pmatrix} = \begin{pmatrix} \sigma_{ii} - \sigma_{i,S}^2/\sigma_{S,S} & \sigma_{ij} - \sigma_{i,S}\sigma_{j,S}/\sigma_{S,S} \\ \sigma_{ji} - \sigma_{i,S}\sigma_{j,S}/\sigma_{S,S} & \sigma_{jj} - \sigma_{j,S}^2/\sigma_{S,S} \end{pmatrix}.$$

with $\sigma_{i,S} = \Sigma_{k=1}^n \sigma_{ik}$.

The survival function of $[X_i | S > \mathrm{VaR}_q(S)]$ can be computed as

$$\begin{aligned} \bar{F}_{i.S}(k_i) &= \int_{k_i}^{\infty} f_i(x | S > \mathrm{VaR}_q(S))\,dx \\ &= \int_{k_i}^{\infty} \int_{\mathrm{VaR}_q(S)}^{\infty} f_i(x | S = s)\,dF_S(s | S > \mathrm{VaR}_q(S))\,dx \\ &= \int_{k_i}^{\infty} \int_{\mathrm{VaR}_q(S)}^{\infty} \frac{c_1}{\sqrt{\sigma_{ii.S}}} g_1\left(\frac{(x-\mu_{i.s})^2}{2\sigma_{ii.S}}\right) \\ &\quad \frac{c_1}{\sqrt{\sigma_{S,S}}} g_1\left(\frac{(s-\mu_S)^2}{2\sigma_{S,S}}\right) \frac{1}{1-q} dsdx \\ &= \frac{c_1^2}{\sqrt{\sigma_{ii.S}}\sqrt{\sigma_{S,S}}(1-q)} \\ &\quad \int_{k_i}^{\infty} \int_{\mathrm{VaR}_q(S)}^{\infty} g_1\left(\frac{(x-\mu_{i.s})^2}{2\sigma_{ii.S}}\right) g_1\left(\frac{(s-\mu_S)^2}{2\sigma_{S,S}}\right) dsdx \\ &= \frac{c_1^2}{1-q} \int_{w^*}^{\infty} \left[\int_{z^*}^{\infty} g_1\left(\frac{z^2}{2}\right) dz\right] g_1\left(\frac{w^2}{2}\right) dw, \end{aligned} \qquad (4.3)$$

where $z^* = (k_i - \mu_{i.w'})/\sqrt{\sigma_{ii.S}}$ with $w' = \sqrt{\sigma_{ii.S}}^{\,w} + \mu_S$, and $w^* = (\mathrm{VaR}_q(S) - \mu_S)/\sqrt{\sigma_{SS}}$. The notation $f_i(\cdot | S > \mathrm{VaR}_q(S))$ represents the density function of $[X_i | S > \mathrm{VaR}_q(S)]$. and $F_S(s | S > \mathrm{VaR}_q(S))$ represents the distribution function of $[S | S > \mathrm{VaR}_q(S)]$.

Next, we provide a simple form for computing $\mathrm{ECT}_S(\cdot)$.

$$\begin{aligned} \mathrm{ECT}_S(k_i) &= \mathbb{E}[(X_i - k_i)_+ | S > \mathrm{VaR}_q(S)] \\ &= \int_{k_i}^{\infty} (x - k_i) f_i(x | S > \mathrm{VaR}_q(S))\,dx \\ &= \int_{k_i}^{\infty} (x - k_i) \int_{\mathrm{Var}_q(S)}^{\infty} \frac{c_1}{\sqrt{\sigma_{ii.S}}} g_1\left(\frac{(x-\mu_{i.s})^2}{2\sigma_{ii.S}}\right) \\ &\quad \frac{c_1}{\sqrt{\sigma_{S,S}}} g_1\left(\frac{(s-\mu_S)^2}{2\sigma_{S,S}}\right) \frac{1}{1-q} dsdx \\ &= \frac{c_1^2}{1-q} \int_{w^*}^{\infty} \left[\int_{z^*}^{\infty} \begin{pmatrix} z\sqrt{\sigma_{ii.S}} \\ + \mu_{i.w'} - k_i \end{pmatrix} g_1\left(\frac{z^2}{2}\right) dz\right] \\ &\quad g_1\left(\frac{w^2}{2}\right) dw \end{aligned} \qquad (4.4)$$

The conditional covariance $\mathrm{Cov}_{+,S}(k_i, k_j)$ can be represented as

$$\begin{aligned} \mathrm{Cov}_{+,S}(k_i, k_j) &= \mathbb{E}[(X_i - k_i)_+ \mathrm{I}(X_j \ge k_j) | S > \mathrm{VaR}_q(S)] \\ &\quad - \mathrm{ECT}_S(k_i) \bar{F}_{j.S}(k_j). \end{aligned}$$

Note that

$$\begin{aligned} &\mathbb{E}[(X_i - k_i)_+ \mathrm{I}(X_j \ge k_j) | S > \mathrm{VaR}_q(S)] \\ &= \int_{k_i}^{\infty} \int_{k_j}^{\infty} (x_i - k_i) f_{i,j}(x_i, x_j | S > \mathrm{VaR}_q(S))\,dx_i\,dx_j, \end{aligned} \qquad (4.5)$$

where $f_{i,j}(\cdot, \cdot|S > \mathrm{Var}_q(S))$ is the joint density function of $[(X_i, X_j)|S > \mathrm{VaR}_q(S)]$, which has the following form

$$\begin{aligned} f_{i,j}(x_i, x_j|S > \mathrm{VaR}_q(S)) &= \int_{\mathrm{VaR}_q(S)}^{\infty} f_{i,j}(x_i, x_j|S = s)dF_S(s|S > \mathrm{VaR}_q(S)) \\ &= \frac{1}{1-q}\frac{c_1}{\sqrt{\sigma_{S,S}}}\int_{\mathrm{VaR}_q(S)}^{\infty} f_{i,j}(x_i, x_j|S = s) \\ &\quad g_1\left(\frac{(s-\mu_S)^2}{2\sigma_{S,S}}\right)ds, \end{aligned}$$

where $f_{i,j}(x_i, x_j\,|\,S = s)$ is the density function of $[(X_i, X_j)\,|\,S = s]$, which is the bivariate elliptical distribution by Eq. (4.2). One may easily implement the forms of Eqs. (4.3), (4.4) and (4.5) into Eq. (2.1) to derive the solutions.

In the following, we present a numerical example to study the optimal capital allocations based on Model (1.2).

Example 4.3. An n-dimensional multivariate student-t distribution belongs to an elliptical family if its density generator can be expressed as

$$g_n(x) = \left(1 + \frac{x}{k_p}\right)^{-p}$$

where $p > n/2$, and k_p is some constant depending on p. For simplicity, we assume that $p = n + \nu$ with the degree of freedom ν, and $k_p = \nu/2$. The joint density has the following form:

$$f(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}}\left[1 + \frac{(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)}{\nu}\right]^{-(n+\nu)/2}, \quad (4.6)$$

where

$$c_n = \frac{\Gamma((n+\nu)/2)}{\Gamma(\nu/2)}(\pi\nu)^{-n/2}.$$

Next, by using a specific example, we discuss how the different factors affect the optimal capital allocations based on Model (1.2), which include the dependence, variance penalty parameter β, risk level q, and heavy tail. Assume that an insurance company has three business lines (X_1, X_2, X_3), which follow the multivariate student t distribution with mean vector

$$\mu = (6, 10, 5),$$

and

$$\Sigma = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & 3 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 1 \end{pmatrix}.$$

The total capital is assumed to be $K = 25$. In the following, we examine several scenarios by varying the parameters σ_{12}, σ_{13} and σ_{23}. The results are summarized in Table 1, which are thoroughly discussed as follows.

- Dependence effect. To study the dependence effect, we vary the values of σ_{12}, σ_{13} and σ_{23}. As seen from Table 1, when σ_{12} ranges from {0, .5, 1.5} and σ_{23} from {–.5, 0, .5}, the more dependence results the more capital requirement. For example, for the case $(\nu, \beta, q) = (5, .01, .95)$, when σ_{12} changes from 0 to .5, and σ_{23} changes from 0 to –.5, it is found that the required capital for risk X_1 increases from 6.618 (26.47%) to 7.280 (29.12%), but risk X_3 reduces from 5.618 (22.47%) to 4.963 (19.85%); when σ_{13} changes from 0 to .1, which indicates increasing the dependence between X_1 and X_3, it is found out that the capital requirement for X_3 increases from 4.793 (19.17%) to 4.849 (19.40%).
- Penalty β. From Table 1, it is observed that when β changes from .01 to .1, the capital requirement on X_2 increases for all the cases. This is very reasonable since X_2 is the riskiest one. For example, for $(\sigma_{12}, \sigma_{13}, \sigma_{23}) = (1.5, .1, .5)$ and $(\nu, q) = (5.99)$, when β changes from .01 to .1, the allocation amount changes from 14.074 (56.30%) to 14.216 (56.86%), which reflects the penalty on the variance of new model as expected.
- Risk level q. Table 1 presents the capital allocations for two risk levels $q = .95$ and $q = .99$. The

Table 1. Optimal capital allocations (amounts and percentages) based on the TMV model (1.2) with a total capital K = 25.

$(\sigma_{12}, \sigma_{13}, \sigma_{23})$	k_1^*	k_2^*	k_3^*	k_1^*	k_2^*	k_3^*
Parameters		$v = 5$, $\beta = .01$, $q = .95$			$v = 5$, $\beta = .01$, $q = .99$	
*(0, 0, 0)	6.618 26.47%	12.763 51.05%	5.618 22.47%	6.296 25.18%	13.408 53.63%	5.296 21.18%
*(1.5, 0, .5)	7.109 28.44%	13.099 52.40%	4.793 19.17%	7.059 28.24%	14.377 57.51%	3.564 14.26%
*(1.5, .1, .5)	7.195 28.78%	12.956 51.83%	4.849 19.40%	7.251 29.00%	14.074 56.30%	3.674 14.70%
*(.5, 0, –.5)	7.280 29.12%	12.758 51.03%	4.963 19.85%	7.393 29.57%	13.379 53.52%	4.229 16.92%
*(.5, .1, –.5)	7.339 29.36%	12.625 50.40%	5.036 20.14%	7.505 30.02%	13.154 52.62%	4.341 17.36%
Parameters		$v = 5$, $\beta = .1$, $q = .95$			$v = 5$, $\beta = .1$, $q = .99$	
*(0, 0, 0)	6.599 26.40%	12.801 51.20%	5.599 22.40%	6.275 25.10%	13.451 53.80%	5.275 21.10%
*(1.5, 0, .5)	7.122 28.49%	13.161 52.64%	4.717 18.87%	7.086 28.34%	14.518 58.07%	3.396 13.58%
*(1.5, .1, .5)	7.213 28.85%	13.013 52.05%	4.773 19.09%	7.285 29.14%	14.216 56.86%	3.499 14.00%
*(.5, 0, –.5)	7.306 29.22%	12.797 51.19%	4.897 19.59%	7.431 29.72%	13.426 53.70%	4.143 16.57%
*(.5,.1, –.5)	7.369 29.48%	12.656 50.62%	4.975 19.90%	7.548 30.19%	13.191 52.76%	4.261 17.04%
Parameters		$v = 50$, $\beta = .01$, $q = .95$			$v = 50$, $\beta = .01$, $q = .99$	
*(0, 0, 0)	6.731 26.92%	12.539 50.16%	5.731 22.92%	6.575 26.30%	12.851 51.40%	5.575 22.30%
*(1.5, 0, .5)	7.109 28.44%	12.752 51.01%	5.138 20.55%	7.078 28.31%	13.350 53.40%	4.572 18.29%
*(1.5, .1, .5)	7.170 28.68%	12.651 50.60%	5.180 20.72%	7.189 28.76%	13.170 52.68%	4.641 18.56%
*(.5, 0, –.5)	7.228 28.91%	12.535 50.14%	5.237 20.95%	7.281 29.12%	12.833 51.33%	4.885 19.54%
*(.5, .1, –.5)	7.274 29.10%	12.435 49.74%	5.291 21.16%	7.352 29.41%	12.689 50.76%	4.959 19.84%

risk level increases, reflecting that the insurance company is more conservative about the risk. Hence the insurance company may be willing to allocate the more capital to the business lines with larger risks. It is seen from Table 1 that the capital allocation to X_2 increases for all cases, which meets the aim of controlling the risk. For example, it is seen that when $(\sigma_{12}, \sigma_{13}, \sigma_{23}) = (1.5, .1, .5)$, for the case of $(\nu, \beta) = (5, .1)$, the capital requirement of X_2 is 14.216 (56.86%) based on $q = .99$ compared to that of 13.013 (52.05%) based on $q = .95$.

- Tail effect. The cases of $\nu = 5$ and $\nu = 50$ are used to calculate the capital allocations in Table 1, which represent different tail thickness of marginal distributions. It is known that when ν is smaller, the tail probability of t distribution is larger. It is clearly seen from Table 1 that when ν is smaller, the capital allocation requirement is larger. For example, it is seen that when $(\sigma_{12}, \sigma_{13}, \sigma_{23}) = (.5, .1, -.5)$, the capital requirement of X_2 is 12.625 (50.40%) based on $(\nu, \beta, q) = (5, .01, .95)$ compared to that of 12.435 (49.74%) based on $(\nu, \beta, q) = (50, .01, .95)$.

From this example, it is observed that Model (1.2) has many intriguing properties, such as reflecting the effects of dependence, penalty, tail, and risk level. The numerical results also possess the intuitive explanations. It should be pointed out that the elliptical distributions discussed here are symmetric. In the following section, we discuss a family of skewed multivariate distributions.

4.2. Multivariate skew-t family

Insurance risks may have skewed distributions, for which the symmetric distributions such as multivariate normal or t distributions are not appropriate models for insurance risks or losses. Therefore, in the literature, the multivariate skewed distributions have been proposed as alternatives to model such risks. Among many multivariate skewed distributions, the multivariate skew-t distribution has been favored since it provides the benefit of flexibility with regard to skewness and thickness of the tails. It allows unlimited range for the indices of skewness and kurtosis for the individual components. For a comprehensive discussion about skewed-distribution family, one may refer to Azzalini (2014).

In the following, we give the definition of a multivariate skew-t distribution.

Definition 4.4. The random vector $\mathbf{X}$ has a multivariate skew-t distribution, denoted by $\mathbf{X} \sim ST(\xi, \Omega, \alpha, \nu)$, if its density function can be expressed as

$$f_{\mathbf{X}}(\mathbf{x}) = 2t_n(\mathbf{x}; \xi, \Omega, \nu) \, T_1\left(\alpha^T w^{-1}(\mathbf{x}-\xi)\left(\frac{\nu+p}{\nu+Q(\mathbf{x})}\right)^{1/2}; \nu+n\right)$$

where $Q(\mathbf{x}) = (\mathbf{x} - \xi)^T\Omega^{-1}(\mathbf{x} - \xi)$, $\alpha \in \mathbb{R}^n$ is the shape parameter, and

$$t_n(\mathbf{x}; \xi, \Omega, \nu) = \frac{\Gamma((\nu+n)/2)}{|\Omega|^{1/2}(\nu\pi)^{n/2}\Gamma(\nu/2)}\left(1+\frac{Q(\mathbf{x})}{\nu}\right)^{-(\nu+n)/2}$$

represents the density function of usual n-dimensional Student's t distribution with location ξ, positive definite $n \times n$ dispersion matrix Ω, and $T_1(.; \nu)$ denotes the univariate standard Student's t cumulative distribution function with degrees of freedom $\nu > 0$.

It should be mentioned that although the multivariate skew-t distribution has many similar properties to the multivariate t distribution, it does not have the preservation property that the conditional distribution is still in the original family of distributions. Therefore, the analytical forms of the key quantities in Eq. (2.1) are infeasible to derive. Instead, we propose to use the Monte Carlo simulation method to compute the key quantities. Specifically, we generate 1,000,000 observations from the multivariate skew-t distribution to compute $\bar{F}_{i,S}(\cdot)$, $\mathrm{ECT}_S(\cdot)$, and $\mathrm{Cov}_{+,S}(\cdot, \cdot)$, which are illustrated by the following specific example.

Example 4.5. Assume that an insurance company has three business lines (X_1, X_2, X_3), which follow the multivariate skew-t distribution with location parameters

$$\xi = (6, 10, 5),$$

and shape parameters

$$\alpha = (10, 30, 20).$$

The dispersion matrix is assumed to be

$$\Omega = \begin{pmatrix} 1 & \omega_{12} & \omega_{13} \\ \omega_{21} & 3 & \omega_{23} \\ \omega_{31} & \omega_{32} & 1 \end{pmatrix}.$$

We note that although the dispersion matrix is not the covariance matrix, it is linearly related to the covariance matrix, which still reflects the dependence between (X_1, X_2, X_3). The specific relation may be found in Eq. (6.26) of Azzalini (2014).

We use the same parameters as that in Table 1 to compute the optimal capital allocations based on Eq. (2.1). The results are summarized in Table 2.

Table 2. Optimal capital allocations (amounts and percentages) based on the TMV model (1.2) with a total capital $K = 25$.

$(\omega_{12}, \omega_{13}, \omega_{23})$	k_1^*	k_2^*	k_3^*	k_1^*	k_2^*	k_3^*
Parameters		$v = 5$, $\beta = .01$, $q = .95$			$v = 5$, $\beta = .01$, $q = .99$	
*(0, 0, 0)	6.448 25.79%	13.094 52.38%	5.458 21.83%	6.032 24.13%	13.902 55.61%	5.066 20.26%
*(1.5, 0, .5)	7.120 28.48%	13.769 55.08%	4.111 16.44%	7.194 28.78%	15.226 60.90%	2.580 10.32%
*(1.5, .1, .5)	7.258 29.03%	13.547 54.19%	4.194 16.78%	7.432 29.73%	14.788 59.15%	2.780 11.12%
*(.5, 0, −.5)	7.349 29.40%	13.073 52.30%	4.578 18.31%	7.499 30.00%	13.828 55.31%	3.673 14.69%
*(.5, .1, −.5)	7.437 29.75%	12.889 51.56%	4.674 18.70%	7.645 30.58%	13.529 54.12%	3.826 15.30%
Parameters		$v = 5$, $\beta = .1$, $q = .95$			$v = 5$, $\beta = .1$, $q = .99$	
*(0, 0, 0)	6.381 25.52%	13.169 52.68%	5.450 21.80%	5.999 24.00%	14.103 56.41%	4.898 19.59%
*(1.5, 0, .5)	7.318 29.27%	14.045 56.18%	3.637 14.55%	7.565 30.26%	15.816 63.26%	1.620 6.48%
*(1.5, .1, .5)	7.448 29.79%	13.838 55.35%	3.713 14.85%	7.750 31.00%	15.347 61.39%	1.903 7.61%
*(.5, 0, −.5)	7.413 29.65%	13.166 52.66%	4.422 17.69%	7.651 30.60%	14.020 56.08%	3.330 13.32%
*(.5, .1, −.5)	7.516 30.06%	12.974 51.90%	4.510 18.04%	7.819 31.28%	13.693 54.77%	3.488 13.95%
Parameters		$v = 50$, $\beta = .01$, $q = .95$			$v = 50$, $\beta = .01$, $q = .99$	
*(0, 0, 0)	6.659 26.64%	12.678 50.71%	5.662 22.65%	6.487 25.95%	13.016 52.06%	5.497 21.99%
*(1.5, 0, .5)	7.100 28.40%	13.030 52.12%	4.870 19.48%	7.088 28.35%	13.621 54.48%	4.291 17.16%
*(1.5, .1, .5)	7.186 28.74%	12.894 51.58%	4.920 19.68%	7.231 28.92%	13.416 53.66%	4.353 17.41%
*(.5, 0, −.5)	7.256 29.02%	12.664 50.66%	5.080 20.32%	7.295 29.18%	12.977 51.91%	4.729 18.92%
*(.5, .1, −.5)	7.311 29.24%	12.544 50.18%	5.145 20.58%	7.374 29.50%	12.820 51.28%	4.806 19.22%

For the multivariate skew-t distributions, we may draw similar conclusions to those in Example 4.3, i.e., the dependence, penalty parameter β, risk level, and tail thickness all have significant effects on the capital allocations based on Model (1.2). It is interesting to observe that the capital requirements on risk X_2 in Table 2 are larger than the corresponding ones in Table 1. This may be intuitively explained by the large skewness of risk X_2. Hence, in practice, one should always seek a suitably skewed distribution if the faced risks are skewed.

4.3. Comparisons to other methods

In this section, we compare the TMV model (1.2) to several models frequently used in the literature. For comprehensive reviews on the methodologies of capital allocations, one may refer to Dhaene et al. (2012), and Bauer and Zanjani (2013). Specifically, the capital allocation rules considered in this section include:

(a) Haircut allocation:

$$k_i = \frac{F_{X_i}^{-1}(q)}{\sum_{j=1}^{n} F_{X_j}^{-1}(q)} K$$

where $F^{-1}_{Xi}(q)$ is the left continuous inverse of the distribution function of X_i at $q > 0$;

(b) Quantile allocation:

$$k_i = \frac{F^{-1}_{X_i}(F_S(K))}{\sum_{j=1}^{n} F^{-1}_{X_j}(F_S(K))} K$$

where $F_S(K) = \mathbb{P}(\Sigma_{i=1}^{n} F^{-1}_{Xi}(U) \leq K)$, and U is a uniform random variable on (0, 1);

(c) Covariance allocation:

$$k_i = \frac{\mathrm{Cov}(X_i, S)}{\sum_{j=1}^{n} \mathrm{Cov}(X_j, S)} K$$

where $S = \Sigma_i^n = 1X_i$;

(d) CTE (conditional tail expectation) allocation

$$k_i = \frac{\mathbb{E}\left[X_i \middle| S > F_S^{-1}(q)\right]}{\sum_{j=1}^{n} \mathbb{E}\left[X_j \middle| S > F_S^{-1}(q)\right]} K$$

where $F_S^{-1}(q)$ is the left continuous inverse of the distribution function of S at $q > 0$.

Example 4.6. For the purpose of comparison, we use the same distribution as Example 4.3. That is, three business lines (X_1, X_2, X_3) follow a multivariate Student t distribution with mean vector

$$\mu = (6, 10, 5),$$

and

$$\Sigma = \begin{pmatrix} 1 & .5 & .1 \\ .5 & 3 & -.5 \\ .1 & -.5 & 1 \end{pmatrix}.$$

The total capital is also assumed to be $K = 25$, and the parameter $\beta = .01$. We calculate the optimal capitals based on different allocation rules. The results are presented in Table 3. It is seen that the quantile allocation rule allocates the smallest amount of capital to risk X_2 (42.43%) compared to the other allocation rules. In particular, the allocation amount based on the quantile rule does not change when the risk level q changes from .95 to .99. The covariance allocation rule allocates the largest amount of capital to risk X_2 (57.91%) compared to the other allocation rules, but it cannot reflect the risk level. Compared to CTE and TMV, the haircut rule allocates a relatively smaller amount of capital to X_2. It is interesting to observe that when the risk level increases from .95 to .99, the allocation amount for the riskiest X_2 decreases from 46.60% to 46.50%. Therefore, the haircut allocation rule does not reflect the risk level very well. The CTE and TMV are similar from the perspectives of allocation amounts and risk levels. Both of them allocate relatively larger capitals to risk X_2, and the allocation amounts increase when the risk level increases from .95 to .99. However, the capital based on TMV model increases from 50.40% to 52.62%, while the

Table 3. Comparisons of optimal capital allocations (amounts and percentages) with a total capital K = 25.

Model	k_1^*	k_2^*	k_3^*	k_1^*	k_2^*	k_3^*
Parameters		$v = 5$, $q = .95$			$v = 5$, $q = .99$	
*TMV	7.339	12.625	5.036	7.505	13.154	4.341
	29.36%	50.40%	20.14%	30.02%	52.62%	17.36%
*Haircut	6.741	11.651	6.607	6.701	11.625	6.674
	26.96%	46.60%	26.43%	26.80%	46.50%	26.70%
*Quantile	7.575	10.608	6.817	7.575	10.608	6.817
	30.30%	42.43%	27.27%	30.30%	42.43%	27.27%
*Covariance	7.667	14.477	2.856	7.667	14.477	2.856
	30.67%	57.91%	11.42%	30.67%	57.91%	11.42%
*CTE	7.293	12.649	5.058	7.344	12.895	4.761
	29.17%	50.60%	20.23%	29.38%	51.58%	19.04%

CTE only increases from 50.60% to 51.58%. This reflects the advantage of TMV model in quickly responding to a large risk level. It is also seen that the allocated capital for risk X_3 based on the TMV model decreases by 2.78% while the allocated capital based on the CTE decreases by 1.19% for X_3. This is because the new TMV model takes into account the negative dependence between X_2 and X_3 for allocations. To conclude, compared to the other models, the new Model (1.2) has many desired properties, such as reflecting the effects of dependence, and risk level.

4.4. Real data analysis

In this section, we analyze a real insurance data set presented in Panjer (2002). The total number of business lines is 10 with

$$\mathbf{X}^T = (X_1, \ldots, X_{10}),$$

which represent a range of insurance and other related financial products. The estimated mean vector (million) is

$$\mu = (25.69, 37.84, 0.85, 12.70, 0.15, 24.05, 14.41, 4.49, 4.39, 9.56).$$

The correlation matrix was reported in Panjer (2002) and Valdez and Chernih (2003) reported the covariance matrix, which is reproduced here for the sake of convenience.

By assuming that the joint distribution of these ten random variables follows a multivariate normal distribution, Panjer (2002) discusses the optimal allocation problems for this data set. We assume that the joint distribution follows a multivariate Student-t distribution with density function defined in Eq. (4.6). Since the original data is not available to us, we use $\nu = 9$ and $\nu = 50$ for the data set, which represent small and large degree of freedoms, separately.

The total capital K is assumed to be 147 million, which is around one standard deviation of estimated means larger than the total sum of estimated means 134.13 million. This value is slightly larger than the $\mathrm{VaR}_{.95}(S) = 145$ based on $\nu = 9$, where $S = X_1 + \ldots + X_{10}$. Table 4 summarizes the optimal capital allocations for various scenarios based on $q = .99$.

From Table 4, it is observed that overall the larger risks are allocated with more capitals. It is seen from the covariance matrix that X_2 has the largest mean and variance, and it has a positive correlation with relatively large risks, say larger than 9 million, (X_4, X_6, X_7), but it is uncorrelated with X_1, and is negatively correlated with X_{10}. When β is increasing, the capital requirement on X_2 is increasing for both of $\nu = 50$ and $\nu = 9$; the capitals for X_2 with $\nu = 9$ are larger than the corresponding ones with $\nu = 50$. This observation reflects that the model penalizes the large variance and heavy tail. For risks X_1 and X_6 with estimated means 25.69 and 24.05 millions, the capital requirements are 26.711 (18.17%) and 27.116 (18.45%) millions for

$$\begin{pmatrix}
7.24 & 0 & 0.07 & -0.07 & 0.28 & -2.71 & -0.51 & 0.28 & 0.23 & -0.21 \\
 & 20.16 & 0.05 & 1.6 & 0.05 & 1.39 & 1.14 & -0.91 & -0.81 & -1.74 \\
 & & 0.04 & 0 & -0.01 & 0.08 & 0.01 & -0.02 & -0.02 & -0.07 \\
 & & & 1.74 & 0.17 & 0.26 & 0.19 & -0.14 & 0.18 & -0.79 \\
 & & & & 0.32 & -0.24 & 0.01 & -0.02 & 0.08 & -0.01 \\
 & & & & & 14.98 & 0.43 & -0.33 & -1.89 & -1.6 \\
 & & & & & & 2.53 & -0.38 & 0.13 & 0.58 \\
 & & & & & & & 0.92 & -0.16 & -0.4 \\
 & & & & & & & & 1.12 & 0.58 \\
 & & & & & & & & & 6.71
\end{pmatrix}$$

Table 4. Optimal capital allocations (amounts and percentages) for various parameters based on TMV model (1.2) with a total capital K = 147 and q = .99.

	$\nu = 50$			$\nu = 9$		
	$\beta = .01$	$\beta = .1$	$\beta = .5$	$\beta = .01$	$\beta = .1$	$\beta = .5$
$^*k_1^*$	26.730 18.18%	26.711 18.17%	26.721 18.18%	26.611 18.10%	26.493 18.02%	26.816 18.24%
$^*k_2^*$	45.358 30.86%	45.476 30.94%	45.655 31.06%	45.732 31.11%	46.257 31.47%	48.162 32.76%
$^*k_3^*$	.840 .57%	.840 .57%	.843 .57%	.832 .57%	.823 .56%	.636 .43%
$^*k_4^*$	13.609 9.26%	13.631 9.27%	13.725 9.34%	13.619 9.26%	13.639 9.28%	13.502 9.19%
$^*k_5^*$	.241 .16%	.244 .17%	.260 .18%	.205 .14%	.194 .13%	.163 .11%
$^*k_6^*$	27.181 18.49%	27.116 18.45%	26.916 18.31%	27.173 18.49%	27.205 18.51%	27.263 18.55%
$^*k_7^*$	15.705 10.68%	15.615 10.62%	15.661 10.65%	15.655 10.65%	15.740 10.71%	16.506 11.23%
k_8^*	3.704 2.52%	3.664 2.49%	3.565 2.43%	3.591 2.44%	3.435 2.34%	2.385 1.62%
k_9^*	4.111 2.80%	3.793 2.58%	2.792 1.90%	3.700 2.52%	3.503 2.38%	2.230 1.52%
k_{10}^*	9.920 6.75%	9.889 6.73%	9.878 6.72%	9.881 6.72%	9.711 6.61%	9.337 6.35%

$\nu = 50$, respectively. The capital requirement on risk X_6 slightly decreases when β changes from .01 to .5, which may be caused by the correlations with the other risks. For the case of $\nu = 9$, it is observed that the capital requirements for risks X_1 and X_6 both increase when β changes from .01 to .5, which may reflect the penalty on the variability again. It is interesting to observe that when β changes from .01 to .1 with $\nu = 9$, the capital requirement on X_1 is slightly less, while on X_6 it is slightly more. It may be explained by noting that the variance of X_1 is less than that of X_6, and further X_6 is positively correlated with X_2. For risk X_{10}, it is seen that it is negatively correlated with (X_1, X_2, X_4, X_6), and therefore, it is not surprising to observe that the capital requirements all decrease when β increases.

5. Conclusion

In this paper, we have suggested a new capital allocation rule which stems from the tail mean-variance premium calculation principle. It is also a variation of the works by Laeven and Goovaerts (2004), Dhaene et al. (2012), and Xu and Mao (2013), which capture both magnitude and variability of tail risks. As seen from the numerical evidence, the TMV model has many intriguing properties, such as penalizing the large risk, variance, positive dependence, and reflecting the tail risk level. It also provides many intuitive explanations on the optimal capital allocations. The penalization parameter β, which is either determined by the historical data or by the experience of the decision maker, provides an additional flexibility for controlling the tail variability. Since the analytical solutions for the TMV model is infeasible, we explore the general equations which could be easily implemented in the software (R code is available upon request). It may be interesting to comprehensively compare the TMV model to those in the literature (Bauer and Zanjani 2013) and use the TMV model for DHS capital allocation. The preliminary study shows that the TMV model provides some promising results, which is currently being pursued, and will be reported when it is completed.

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Appendix

Proof of Theorem 2.1: Define

$$f(\boldsymbol{k}) = \mathbb{E}\left[\sum_{i=1}^{n}(X_i - k_i)_+ | S > \mathrm{VaR}_q(S)\right] + \beta \mathrm{Var}\left[\sum_{i=1}^{n}(X_i - k_i)_+ | S > \mathrm{VaR}_q(S)\right],$$

and

$$h(\boldsymbol{k}) = K - \sum_{i=1}^{n} k_i.$$

Let

$$L(\boldsymbol{k}, \lambda) = f(\boldsymbol{k}) + \lambda h(\boldsymbol{k}).$$

According to Kuhn-Tucker theory (Bertsekas 1999), we need to solve the following equations, for $l = 1, \ldots, n$,

$$\frac{\partial L(\boldsymbol{k},\lambda)}{\partial k_l} = 0, \qquad \frac{\partial L(\boldsymbol{k},\lambda)}{\partial \lambda} = 0.$$

We first observe that

$$\frac{\partial \mathrm{ECT}_S(k_1)}{\partial k_1} = -\bar{F}_{1.S}(k_1),$$

and

$$\mathrm{Var}[(X_1 - k_1)_+ | S > \mathrm{VaR}_q(S)] = \mathbb{E}[(X_1 - k_1)_+^2 | S > \mathrm{VaR}_q(S)] - [\mathrm{ECT}_S(k_1)]^2.$$

Therefore, it holds that

$$\frac{\partial \mathrm{Var}[(X_1 - k_1)_+ | S > \mathrm{VaR}_q(S)]}{\partial k_1} = -2F_{1.S}(k_1)\mathrm{ECT}_S(k_1).$$

Further, for any $j = 2, \ldots, n$, we have

$$\begin{aligned}
&\frac{\partial \mathrm{Cov}\{(X_1 - k_1)_+, (X_j - k_j)_+ | S > \mathrm{VaR}_q(S)\}}{\partial k_1} \\
&= -\int_{k_j}^{\infty}\int_{k_1}^{\infty}(x_j - k_j) f_{1,j}(x_1, x_j | S > \mathrm{VaR}_q(S))\, dx_1\, dx_j \\
&\quad + \bar{F}_{1.S}(k_1)\mathrm{ECT}_S(k_j) \\
&= -\mathbb{E}[(X_j - k_j)_+ \mathrm{I}(X_1 \geq k_1) | S > \mathrm{VaR}_q(S)] \\
&\quad + \bar{F}_{1.S}(k_1)\mathrm{ECT}_S(k_j) \\
&= -\mathrm{Cov}\{(X_j - k_j)_+, \mathrm{I}(X_1 \geq k_1) | S > \mathrm{VaR}_q(S)\} \\
&\quad - \bar{F}_{1.S}(k_1)\mathrm{ECT}_S(k_j) + \bar{F}_{1.S}(k_1)\mathrm{ECT}_S(k_j) \\
&= -\mathrm{Cov}_{+,S}(k_j, k_1),
\end{aligned}$$

where $f_{1,j}(\cdot, \cdot | S > \mathrm{VaR}_q(S))$ is the joint density of $[(X_1, X_j) | S > \mathrm{VaR}_q(S)]$. Therefore, we have

$$\begin{aligned}
\frac{\partial f(\boldsymbol{k})}{\partial k_1} &= -\bar{F}_{1.S}(k_1) - 2\beta F_{1.S}(k_1)\mathrm{ECT}_S(k_1) \\
&\quad - 2\beta \sum_{j=2}^{n} \mathrm{Cov}_{+,S}(k_j, k_1) \\
&= -\bar{F}_{1.S}(k_1) - 2\beta \mathrm{ECT}_S(k_1) \\
&\quad + 2\beta \bar{F}_{1.S}(k_1)\mathrm{ECT}_S(k_1) \\
&\quad - 2\beta \sum_{j=2}^{n} \mathrm{Cov}_{+,S}(k_j, k_1) \\
&= -\bar{F}_{1.S}(k_1) - 2\beta \mathrm{Cov}\{(X_1 - k_1)_+, \mathrm{I}(X_1 \geq k_1) | \\
&\quad S > \mathrm{VaR}_q(S)\} - 2\beta \sum_{j=2}^{n} \mathrm{Cov}_{+,S}(k_j, k_1) \\
&= -\bar{F}_{1.S}(k_1) - 2\beta \sum_{j=1}^{n} \mathrm{Cov}_{+,S}(k_j, k_1).
\end{aligned}$$

For $l = 1, 2, \ldots, n$, it follows that

$$\frac{\partial L(\boldsymbol{k},\lambda)}{\partial k_l} = -\bar{F}_{l.S}(k_l) - 2\beta \sum_{j=1}^{n} \mathrm{Cov}_{+,S}(k_j, k_l) - \lambda.$$

Therefore, the optimal solutions should satisfy the following equations:

$$\bar{F}_{l.S}(k_l^*) + 2\beta\sum_{j=1}^{n}\mathrm{Cov}_{+,S}(k_j^*, k_l^*)$$
$$= \bar{F}_{1.S}(k_1^*) + 2\beta\sum_{j=1}^{n}\mathrm{Cov}_{+,S}(k_j^*, k_1^*),$$

and

$$k_1^* + \ldots + k_n^* = K.$$

Next, we discuss the uniqueness condition of solutions. For any $l \neq j$, it holds that

$$\frac{\partial^2 L(\boldsymbol{k},\lambda)}{\partial k_l\,\partial k_j} = -2\beta\frac{\partial\mathrm{Cov}_{+,S}(k_j,k_l)}{\partial k_j}$$
$$= 2\beta\mathrm{Cov}[\mathrm{I}(X_k > k_j), \mathrm{I}(X_l > k_l)|S > \mathrm{VaR}_q(S)].$$

For $l = 1, \ldots, n$, the second derivative of $L(\boldsymbol{k}, \lambda)$ is

$$\frac{\partial^2 L(\boldsymbol{k},\lambda)}{\partial^2 k_l} = f_{l.S}(k_l) - 2\beta\sum_{j=1}^{n}\frac{\partial\mathrm{Cov}_{+,S}(k_j,k_l)}{\partial k_l},$$

where $f_{l.S}(\cdot)$ represents the density function of $[X_l|S > VaR_q(S)]$. Note that

$$\sum_{j=1}^{n}\frac{\partial\mathrm{Cov}_{+,S}(k_j,k_l)}{\partial k_l} = f_{l.S}(k_l)\mathrm{ECT}_S(k_l)$$
$$- F_{l.S}(k_l)\bar{F}_{l.S}(k_l) + \sum_{j\neq l}^{n}\frac{\partial\mathrm{Cov}_{+,S}(k_j,k_l)}{\partial k_j},$$

and

$$\sum_{j\neq l}^{n}\frac{\partial\mathrm{Cov}_{+,S}(k_j,k_l)}{\partial k_l} = -f_{l.S}(k_l)\sum_{j\neq l}^{n}\Big\{\mathbb{E}\big[(X_j-k_j)_+\big|$$
$$X_l = k_l, S > \mathrm{VaR}_q(S)\big] - \mathrm{ECT}_S(k_j)\Big\}.$$

We have

$$\frac{\partial^2 L(\boldsymbol{k},\lambda)}{\partial^2 k_l} = f_{l.S}(k_l) - 2\beta f_{l.S}(k_l)\sum_{j=1}^{n}\mathrm{ECT}_S(k_j)$$
$$+ 2\beta F_{l.S}(k_l)\bar{F}_{l.S}(k_l) + 2\beta f_{l.S}(k_l)$$
$$\sum_{j\neq 1}^{n}\mathbb{E}[(X_j - k_j)_+|X_l = k_l, S > \mathrm{VaR}_q(S)]$$
$$= 2\beta\mathrm{Var}\{\mathrm{I}(X_l > k_l)|S > \mathrm{VaR}_q(S)\}$$
$$+ f_{l.S}(k_l) - 2\beta f_{l.S}(k_l)\sum_{j=1}^{n}\mathrm{ECT}_S(k_j)$$
$$+ 2\beta f_{l.S}(k_l)\sum_{j\neq l}^{n}\mathbb{E}\big[(X_j - k_j)_+\big|$$
$$X_l = k_l, S > \mathrm{VaR}_q(S)\big]$$
$$= 2\beta\mathrm{Var}\{\mathrm{I}(X_l > k_l)|S > \mathrm{VaR}_q(S)\}$$
$$+ f_{l.S}(k_l)\Delta_l,$$

where

$$\Delta_l = 1 - 2\beta\sum_{j=1}^{n}\mathrm{ECT}_S(k_j) + 2\beta\sum_{j\neq l}^{n}\mathbb{E}[(X_j - k_j)_+|X_l = k_l,$$
$$S > \mathrm{VaR}_q(S)].$$

Therefore, the Hessian matrix of the optimal solutions can be represented as

$$H = 2\beta\mathrm{Cov}\Big[(\mathrm{I}(X_1 \geq k_1^*), \ldots, \mathrm{I}(X_n \geq k_n^*))^T\Big|$$
$$S > \mathrm{VaR}_q(S)\Big] + \mathrm{diag}(\Delta_l^*),$$

where $\mathrm{diag}(\Delta_l^*)$ means the diagonal matrix with diagonal elements Δ_l^*, $l = 1, \ldots, n$. Hence, if

$$\Delta_l^* = 1 - 2\beta\sum_{j=1}^{n}\mathrm{ECT}_S(k_j^*) + 2\beta\sum_{j\neq l}^{n}\mathbb{E}\Big[(X_j - k_j^*)_+\Big|$$
$$X_l = k_l^*, S > \mathrm{VaR}_q(S)\Big] > 0,$$

then H is a positive definite matrix, as the covariance matrix is positive semi-definite. Since the set

$$\{\boldsymbol{k}|k_1 + k_2 + \ldots + k_n = K\}$$

is convex, the optimal solution in Eq. (2.1) should also be a globe optimal solution.

The required result follows immediately. ■

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