# Munich Chain Ladder: A Reserving Method that Reduces the Gap between IBNR Projections Based on Paid Losses and IBNR Projections Based on Incurred Losses

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*Reserving methods, chain ladder, loss development, reconciliation of paid and incurred projections*

# Munich Chain Ladder

## A reserving method that reduces the gap between IBNR projections based on paid losses and IBNR projections based on incurred losses

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# Preface

The IBNR reserve for a portfolio is usually calculated on the basis of both the run-off triangle of paid losses and the run-off triangle of incurred losses, i.e. the sum of paid losses and case reserves. Often, the problem arises that the projection based on paid losses is far different than the projection based on incurred losses. Even worse, paid losses may yield a higher ultimate loss projection than incurred losses in one accident year, but in the next accident year, the situation may be entirely reversed, with incurred losses yielding the higher projection of ultimate loss.
This paper analyses this problem with regard to the chain ladder method, using examples and generally valid equations, and describes a solution: the Munich chain ladder method. More precisely, the paper shows that between paid losses and incurred losses there are almost always correlations that are ignored in the usual procedure of making a separate chain ladder projection for each of the paid-loss and incurred-loss triangles. The Munich chain ladder method, on the other hand, takes advantage of these correlations, transferring any conjunction of paid and incurred losses that occurred in the past into the projection for the future.
This paper presents in detail the properties, theoretical bases and, not least, capability of the new method, using numerous graphs and a fully elaborated example.

The fundamental insights and ideas presented in this paper were developed by the first of the authors named. The second author contributed the improvement of the paid model vis-à-vis the original version by using $(I/P)$ ratios in place of $(P/I)$ ratios and thus simplifying the formulas.

# Chapter 1

# Introduction to the Munich chain ladder method

One customary method of determining the IBNR reserves for a portfolio is to apply the chain ladder method to both the paid-loss and the incurred-loss triangles independently. For the sake of brevity, this method of performing two separate chain ladder calculations will be referred to in the following as the SCL method.
The new Munich chain ladder (MCL) method, on the other hand, combines the paid-loss $(P)$ and incurred-loss $(I)$ data types by taking $(P/I)$ ratios into account in projections. Here, $(P/I)$ ratio is used to mean the quotient of paid and incurred losses, i.e. the share of the incurred losses that has been paid at the point in time under consideration.

We will start by examining the development of $(P/I)$ ratios using the SCL method.

## 1.1 The $(P/I)$ problem in separate chain ladder calculations

Based on an example, we first will examine how $(P/I)$ ratios influence the degree of concurrence of paid and incurred projections produced using the SCL method. Then, with the aid of an explicit formula, we will show that the problems occurring in the example are not exceptions to the rule, but a systematic weakness in the SCL method.

Please note that we use data from several portfolios in the following examples. This is necessary in order to illustrate effects that do not occur in every data record or that are to be observed in varying degrees of intensity.

### 1.1.1 Example

Let us consider an Asian MTPL (motor third party liability) portfolio. The triangles (paid and incurred) cover fifteen accident years and thus also fifteen development years. The earliest four accident years can for practical purposes be classified as settled.
Figure 1 shows this portfolio's past $(P/I)$ ratios on the $y$-axis plotted against the associated development years on the $x$-axis. The continuous line indicates the average $(P/I)$ ratios from the past. After twelve development years, it has reached (nearly) 100%. Within a given development year, the points are scattered about their average. The range of scattering decreases as the number of development years increases.

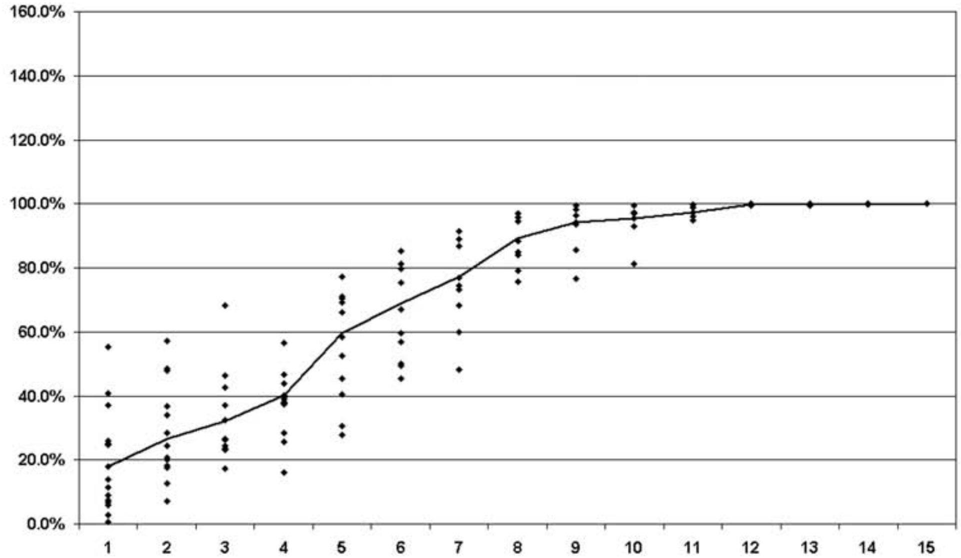


**Figure 1: A $(P/I)$ triangle**

Now we extrapolate both the paid and the incurred triangles using the chain ladder method and from the two resultant quadrangles, create the associated $(P/I)$ quadrangle. If one considers the graph of the entire quadrangle (see Figure 2), i.e. the past (filled diamonds) and projected $(P/I)$ values (unfilled diamonds), this method's lack of plausibility becomes clearly apparent.

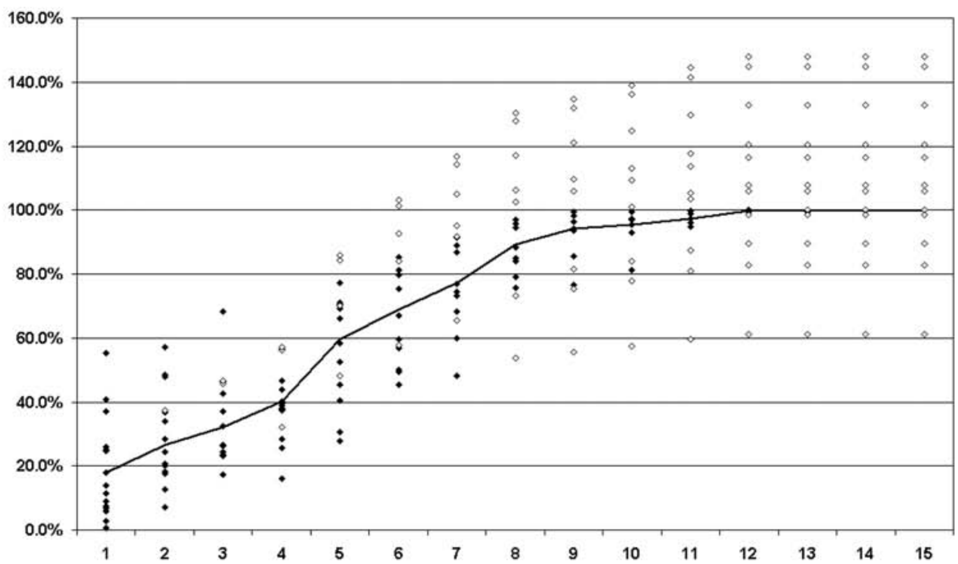


**Figure 2: A $(P/I)$ quadrangle produced using the SCL method**

The ultimate $(P/I)$ ratios, i.e. the $(P/I)$ ratios of the last development year fluctuate between 61% and 148%, which means that the projection based on paid losses remains significantly lower than the projection based on incurred losses in some accident years, but far surpasses it in other accident years.

The new, i.e. the projected points of Figure 2 also show a fundamentally different behaviour than the points in the past, diverging rather than converging. If the current value is above average or below average, this characteristic seems to be intensified in projections produced using the SCL method.

### 1.1.2 Precise formulation

We will now prove that the implausible behaviour of the projected points in Figure 2 is not a feature peculiar to the portfolio used, but a characteristic of the SCL method itself.

To do this, we consider the two quadrangles obtained by applying the SCL method to the paid and the incurred triangles of an arbitrary portfolio. For $i, t = 1, \ldots, n$, we use $P_{i,t}$ and $I_{i,t}$ to denote the cumulative paid-loss and incurred-loss totals, respectively, of the $i$-th accident year after $t$ development years. If $c_i := n + 1 - i$ is the current development time of the $i$-th accident year, the values $P_{i,t}$ and $I_{i,t}$ are given for $1 \leq t \leq c_i$ and projected for $c_i < t \leq n$. The numbers $P_{i,c_i}$ and $I_{i,c_i}$ are thus precisely the entries in the hypotenuses of the corresponding triangles, i.e. the current cumulative totals of the accident years. We also consider on the one hand the $(P/I)$ ratio

$$(P/I)_{i,t} := \frac{P_{i,t}}{I_{i,t}}$$

of the $i$-th accident year at development time $t$ and, on the other, the average $(P/I)$ ratio of all accident years at time $t$

$$(P/I)_t := \frac{\sum_{j=1}^{n} P_{j,t}}{\sum_{j=1}^{n} I_{j,t}} = \frac{1}{\sum_{j=1}^{n} I_{j,t}} \cdot \sum_{j=1}^{n} I_{j,t} \cdot (P/I)_{j,t},$$

which is the average of the $(P/I)$ ratios at development time $t$ weighted with the incurred amounts. Lastly, for $s = 1, \ldots, n-1$, let $f^P_{s \to s+1}$ and $f^I_{s \to s+1}$ denote the average paid and incurred development factors, respectively, from development year $s$ to development year $s+1$ of the chain ladder method:

$$f^P_{s \to s+1} := \frac{\sum_{j=1}^{n-s} P_{j,s+1}}{\sum_{j=1}^{n-s} P_{j,s}} \qquad \text{and} \qquad f^I_{s \to s+1} := \frac{\sum_{j=1}^{n-s} I_{j,s+1}}{\sum_{j=1}^{n-s} I_{j,s}}.$$

For projected amounts $P_{i,s+1}$ and $I_{i,s+1}$, i.e. for $s \geq c_i$, we have by definition

$$P_{i,s+1} = P_{i,s} \cdot f^P_{s \to s+1} \qquad \text{and} \qquad I_{i,s+1} = I_{i,s} \cdot f^I_{s \to s+1}.$$

With these notations, the following formula holds for the $(P/I)$ ratios of the future, i.e. for $t > c_i$:

$$(P/I)_{i,t} = \frac{P_{i,t}}{I_{i,t}} = \frac{P_{i,c_i} \cdot f^P_{c_i \to c_i+1} \cdot \ldots \cdot f^P_{t-1 \to t}}{I_{i,c_i} \cdot f^I_{c_i \to c_i+1} \cdot \ldots \cdot f^I_{t-1 \to t}}. \qquad (*)$$

For the paid development factors, we derive the equation

$$f^P_{s \to s+1} \cdot \sum_{j=1}^{n} P_{j,s} \quad = \quad f^P_{s \to s+1} \cdot \left( \sum_{j=1}^{n-s} P_{j,s} + \sum_{j=n-s+1}^{n} P_{j,s} \right)$$

$$
\begin{aligned}
&= \frac{\sum_{j=1}^{n-s} P_{j,s+1}}{\sum_{j=1}^{n-s} P_{j,s}} \cdot \sum_{j=1}^{n-s} P_{j,s} + \sum_{j=n-s+1}^{n} f^P_{s\to s+1} \cdot P_{j,s} \\
&= \sum_{j=1}^{n-s} P_{j,s+1} + \sum_{j=n-s+1}^{n} P_{j,s+1} \\
&= \sum_{j=1}^{n} P_{j,s+1}.
\end{aligned}
$$

From this and the corresponding expressions for the incurred development factors we get the relations

$$
f^P_{s\to s+1} = \frac{\sum_{j=1}^{n} P_{j,s+1}}{\sum_{j=1}^{n} P_{j,s}} \quad \text{and} \quad f^I_{s\to s+1} = \frac{\sum_{j=1}^{n} I_{j,s+1}}{\sum_{j=1}^{n} I_{j,s}},
$$

in which all summations extend from 1 to $n$. Inserting in formula $(*)$ for the future $(P/I)$ ratios yields after reduction

$$
(P/I)_{i,t} = \frac{P_{i,c_i} \cdot \frac{\sum_{j=1}^{n} P_{j,t}}{\sum_{j=1}^{n} P_{j,c_i}}}{I_{i,c_i} \cdot \frac{\sum_{j=1}^{n} I_{j,t}}{\sum_{j=1}^{n} I_{j,c_i}}}
$$

By rearranging, we obtain the following result for $t > c_i$, which is fundamental for our considerations:

$$
\boxed{\frac{(P/I)_{i,t}}{(P/I)_t} = \frac{(P/I)_{i,c_i}}{(P/I)_{c_i}}}
$$

The essential meaning of this equation can easily be clothed in words:

For each accident year, the ratio of a projected $(P/I)$ value to the corresponding average is the same as the ratio of the current $(P/I)$ value to the corresponding average. Hence, this ratio remains constant in projections based on separate chain ladder calculations.

This statement describes precisely the behaviour observed in Figure 2. It is thus not an attribute peculiar to the selected data, but a systematic weakness of the SCL method. An accident year with an above-average or below-average current $(P/I)$ ratio will also have an above-average or below-average, respectively, projected $(P/I)$ ratio at the end of the quadrangle, i.e. at development time $n$.

If there are old, fully settled accident years in a portfolio, these years have $(P/I)$ ratios that grew in the course of their development to approximately 100%. That is why the $(P/I)$ curves of more recent accident years projected using the SCL method, which do not approach 100%, contradict the past. As the title of this section made clear, we refer to this as the SCL method's $(P/I)$ problem.

## 1.2 Correlations between paid and incurred data

The example and the equation presented in the preceding section demonstrate that the usual method of performing two chain ladder calculations, one for the paid losses and one for the incurred losses, has a systematic weakness and therefore frequently produces implausible projections that contradict past experience. It was also normal for old, meanwhile settled accident years to have below-average and above-average $(P/I)$ ratios, but ultimately they still came to a $(P/I)$ ratio of about 100%. This permits only one conclusion:

In the past, a relatively low $(P/I)$ ratio was followed either by relatively high development factors for paid losses or relatively low development factors for incurred losses (or, of course, both). For a relatively high $(P/I)$ ratio, the situation is reversed.

The SCL method's $(P/I)$ problem is that it in fact ignores this fundamental correlation between paid and incurred. The following example demonstrates the dependence of development factors on $(P/I)$ ratios.

### 1.2.1 Example

Figure 3 shows the individual paid development factors $f^P_{i,1\to 2} := P_{i,2}/P_{i,1}$ from 1 to 2 (i.e. from the end of the first development year to the end of the second development year) of accident years $i = 1, \ldots, 24$ for a marine portfolio, plotted against the associated $(P/I)$ ratios after the first development year. For the sake of easier orientation, the average development factor and the average $(P/I)$ ratio are shown by a horizontal and a vertical line, respectively.

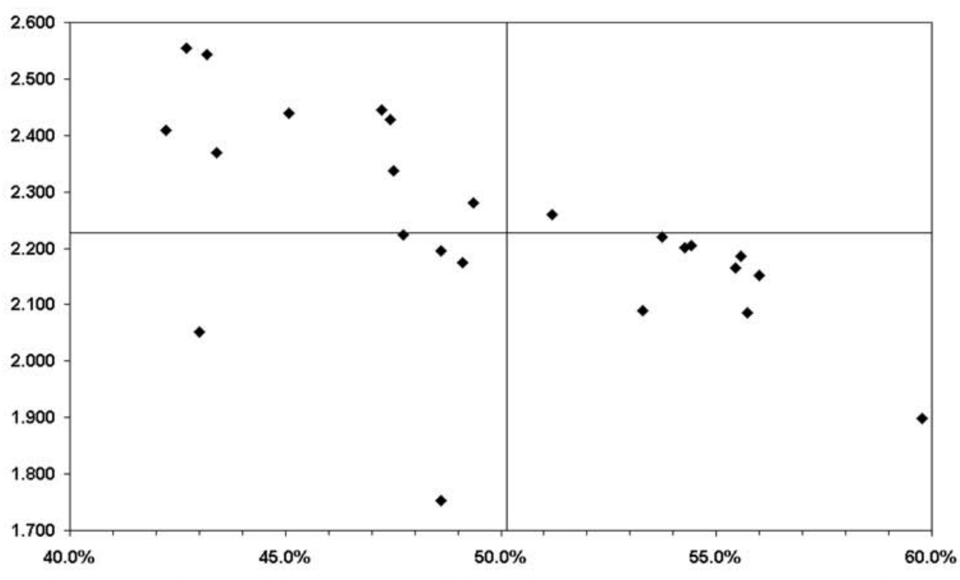


**Figure 3: Dependence of paid development factors on preceding $(P/I)$ ratios**

This obviously confirms the conclusion printed in boldface above. The points show a clearly declining trend with a correlation of $-60\%$. Without accident years 6 and 9, the points of which leap to the eye as outliers, the correlation is even $-89\%$.

Figure 4 shows the corresponding picture with the individual development factors $f^I_{i,1\to 2} := I_{i,2}/I_{i,1}$ of the incurred triangle, again plotted against the $(P/I)$ ratios after the first development year.

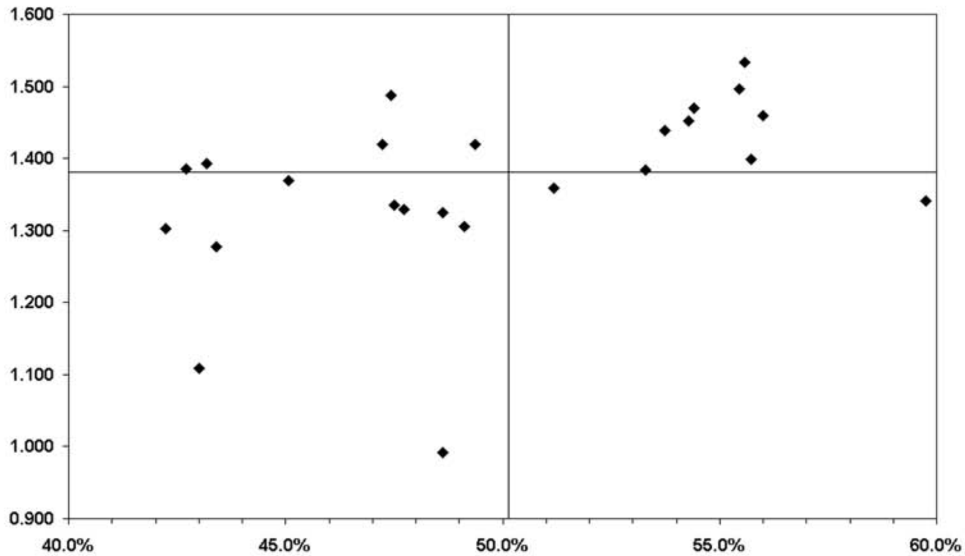


**Figure 4: Dependence of incurred development factors on preceding $(P/I)$ ratios**

This graph, too, confirms the conclusion drawn above. We have a slightly increasing trend with a correlation of 46%, or 51% if we disregard accident years 6 and 9, which are outliers in this chart, as well.

### 1.2.2 The basic idea for solving the $(P/I)$ problem and difficulties in its implementation

The insights gained thus far suggest that, in carrying out an IBNR projection, we should not use the same average chain ladder development factor for all accident years, as is the case with the SCL method, but should adhere to the following rule based on past experience:

Depending on whether the momentary $(P/I)$ ratio is below average or above average, one should use an above-average or below-average paid development factor and/or a below-average or above-average incurred development factor, respectively.

This rule expresses the basic idea for solving the SCL method's $(P/I)$ problem, yet the following important details remain to be clarified. How extensive should the adjustment to the development factors be? And should the adjustment be made more on the paid or on the incurred side? To answer these questions, we must consider data from the past.

If we consider Figures 3 and 4, the following idea suggests itself. In each graph, we draw a regression line that passes through the intersection of the two average lines and, instead of using the average development factor of the horizontal line, use the value determined by the regression line, depending on the $(P/I)$ ratio. We do this not only for the two graphs of the first development year (transition from 1 to 2), but for each development year $s$ (transition from $s$ to $s+1$). In this way, we can complete the paid and the incurred triangles from left to right, creating in each case a quadrangle. In doing so, we use development factors that deviate to a greater or lesser extent, depending on the momentary $(P/I)$ ratio, from those of the SCL method.

However, practice exhibits that this approach is not convincing and leads to great difficulties and implausible results:

1. Often, a linear approach is not suitable for modelling the level of the paid development factors. This is impressively illustrated by Figure 5, which shows the paid development factors $f^P_{i,2\to 3}$ plotted against the $(P/I)$ ratios after two development years for a portfolio with very widely scattered $(P/I)$ ratios.

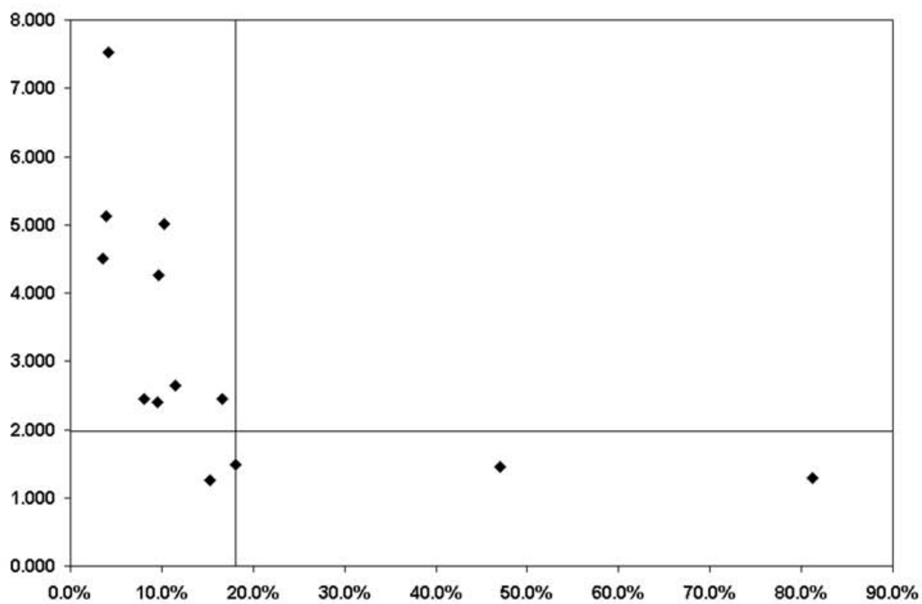


**Figure 5: Paid development factors for widely scattered $(P/I)$ ratios**

Since paid development factors less than 1 are usually untypical and those less than 0 are always impossible, a regression line is not suitable for modelling the level of the development factors in this example. Here, a hyperbolic curve would be appropriate.

2. The estimation of the slopes of the regression lines is frequently very uncertain and volatile, especially if there are only a few accident years, which is always the case in later development years. Sometimes estimated slopes even have the 'wrong' sign. Rather than contradicting our findings thus far, however, this may be expected to occur with a certain probability given the randomness of the

development factors. One customary means of solving this problem, smoothing the parameters over the development years, is hardly possible because it is unclear what pattern the slopes should follow.

3. Sometimes estimates yield relatively steep slopes even though the points of the graph show hardly any correlation. In this case, the correction of the development factors and thus also of the ultimate projection vis-à-vis the SCL method is implausibly strong.

## 1.3 Solving the $(P/I)$ problem: Munich chain ladder

We shall now present a method that answers the open questions posed in the preceding section and solves the three problems discussed. We shall begin with the first difficulty, suitable modelling of the paid development factors, and then turn to the other two points.

### 1.3.1 Including the $(I/P)$ ratios

In order to reflect the hyperbolic appearance of the dependency of the paid development factors on the $(P/I)$ ratios in Figure 5, we shift to the reciprocal values, the $(I/P)$ ratios. Figure 6 shows the paid development factors from Figure 5, plotted against the $(I/P)$ ratios after two development years.

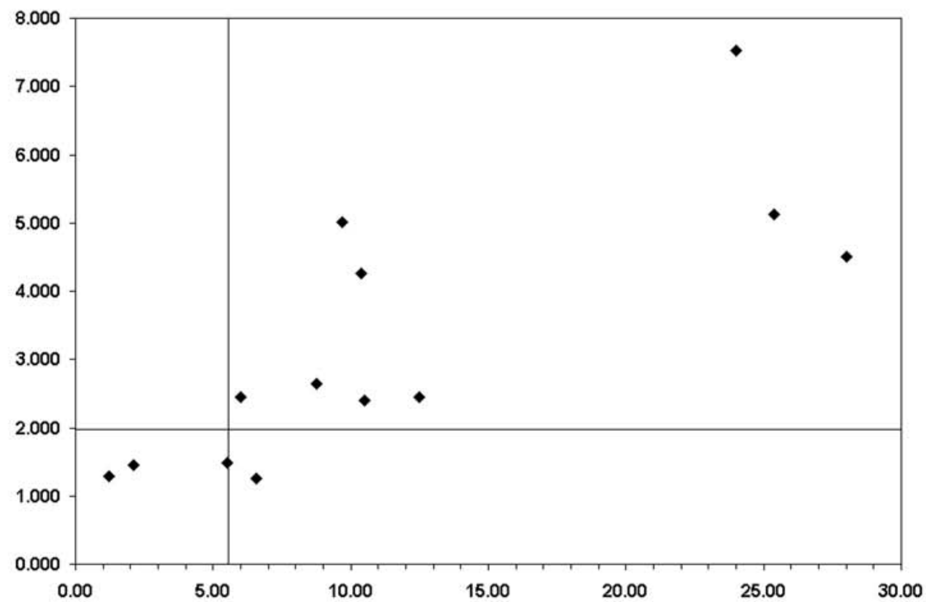


**Figure 6: Dependence of paid development factors from Figure 5 on preceding $(I/P)$ ratios**

It becomes apparent that here, in contrast to Figure 5, a linear approach is quite well-suited for modelling the factors' average. In this example, in which the $(P/I)$ ratios show broad scattering, shifting to the reciprocals has a linearising effect.

In the situation of the marine portfolio from Figure 3, in which the $(P/I)$ ratios show relatively little scattering, the linear impression remains when we shift to the $(I/P)$ ratios as we can see in Figure 7.

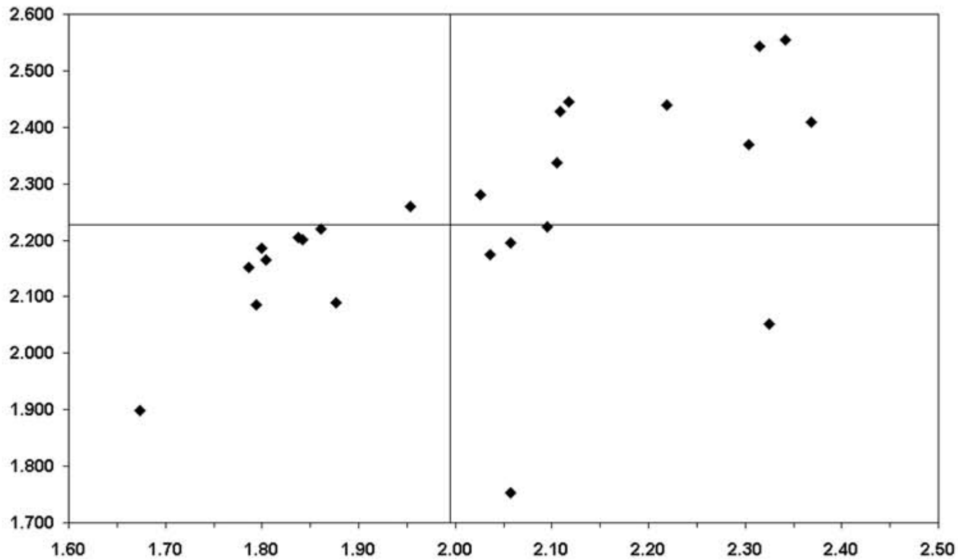


**Figure 7: Dependence of paid development factors from Figure 3 on preceding $(I/P)$ ratios**

These two examples demonstrate that it makes sense to assume that paid development factors have a linear dependency on the preceding $(I/P)$ ratios, rather than on the preceding $(P/I)$ ratios. For the incurred data, on the other hand, the original assumption that the development factors have a linear dependency on the $(P/I)$ ratios can be maintained.

### 1.3.2 Changing over to residuals: the Munich chain ladder method

An important step toward solving the second and third problems described in Section 1.2.2 is to consider the development years all together, rather than individually. This also seems to make sense intuitively because a below-average or above-average $(P/I)$ ratio has to be compensated by suitable development factors not just in the one development year immediately following, but over all subsequent development years.

In order to take the individual development factors and $(P/I)$ and $(I/P)$ ratios of all development years into consideration together, it is necessary to standardise them. This is done by shifting to the residuals of these values. To calculate residuals, we need expectation and variance assumptions. These we will formulate in the next chapter, when we deal with derivations and calculations in detail. At this point in time, we need only the fact that the residual (of a development factor, a $(P/I)$ ratio or an $(I/P)$ ratio) is a standardised measure of the deviation of the value from the average. More precisely, it measures the deviation from the respective expected value in multiples of the standard deviation. Hence, residuals are values of a comparable scale, scattered about zero.

As was already done for a marine portfolio in Figures 7 and 4, we now want to show the dependence of development factors on $(I/P)$ and $(P/I)$ ratios, respectively, for a

European GL (general liability) portfolio, but instead of the values themselves, we now plot the residuals of the paid development factors (paid residuals) against the residuals of the preceding $(I/P)$ ratios ($(I/P)$ residuals) and the residuals of the incurred development factors (incurred residuals) against the residuals of the $(P/I)$ ratios ($(P/I)$ residuals). Due to the standardisation, we can do this for all development years in a single graph. The (residuals of the) paid development factors yield Figure 8, which we call the paid residual plot.

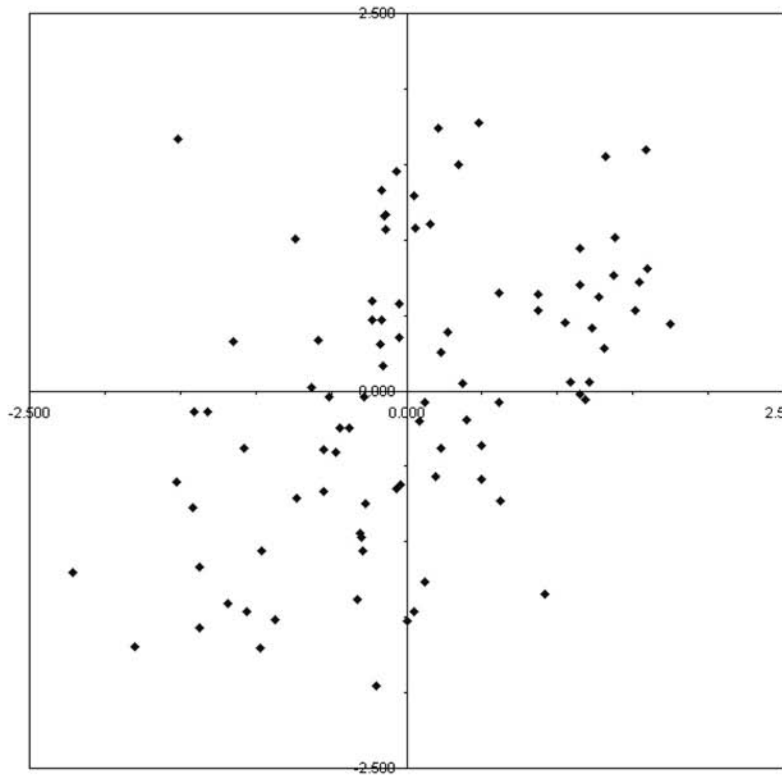


**Figure 8: Paid residual plot**

The graph shows the scatter plot to have a uniformly increasing trend with only a few outliers. This is reflected in a correlation coefficient of 45%. The slope of the regression line through the origin is 0.48.

If, for example, an accident year has a current $(I/P)$ ratio with a residual of $+1.00$, one would forecast for the next development year a paid development factor corresponding to a residual of $+0.48$, according to the slope of the regression line. The common chain ladder calculation uses a development factor with a residual of 0.00.

This portfolio's incurred development factors yield an analogous graph (Figure 9), which we call the incurred residual plot. Here again, the scatter plot shows an increasing trend. The coefficient of correlation is 29%, and the slope of the regression line is 0.31. If, for example, one has a $(P/I)$ ratio with a residual of $+1.00$, one would project an incurred development factor corresponding to a residual of $+0.31$ for the next development year.

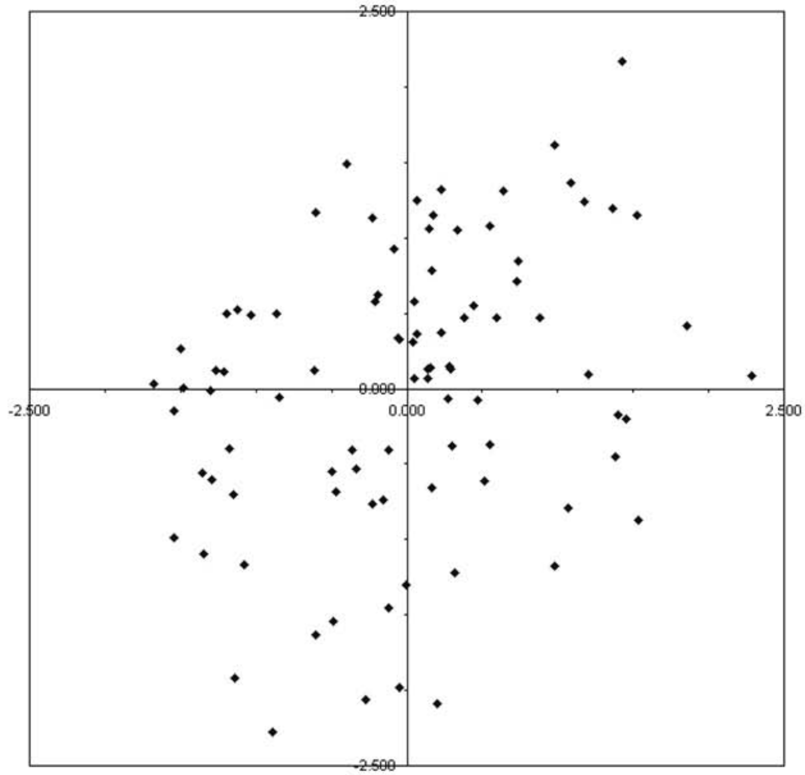


**Figure 9: Incurred residual plot**

This example illustrates the general procedure. First, we produce the two residual plots by plotting, for all development years at once, the residuals of the paid development factors and incurred development factors along the $y$-axis against the residuals of the preceding $(I/P)$ ratios and $(P/I)$ ratios, respectively, along the $x$-axis. We then draw a regression line through the origin in each of the two graphs. For a given $(I/P)$ ratio (or $(P/I)$ ratio), we now calculate the residual, read the associated development factor residuals from the respective regression line and use the development factors calculated from them instead of the average development factors (which have the residual zero). We call this procedure the **Munich chain ladder** method (MCL method).

Further on, this example illustrates a pattern that is clearly present in most of the data records studied:

- Both the paid and the incurred development factors contribute to the concurrence of paid and incurred.
- Usually, this tendency is somewhat more marked for paid losses than for incurred, but is significant for both data types.

In practice, however, there are also other patterns. For example, it occurs that the correlation in the residual plot exists only for paid losses or, more rarely, only for incurred losses. In the Munich chain ladder method, these differences are automatically taken into account as they have shown themselves in the data observed so far.

## 1.4 Capability and limits of the Munich chain ladder method

This section compares the results obtained using the MCL method with those yielded by the SCL method on the basis of three examples, focusing particularly on the ultimate $(P/I)$ ratios. As already mentioned, the following chapter is then devoted to the derivation of the necessary formulas.

### 1.4.1 Example: an Asian MTPL portfolio

Figure 10 shows a comparison of the results for the Asian MTPL portfolio from Section 1.1.1. The values shown there (Figure 2) for the last development year, i.e. the ultimate $(P/I)$ ratios of accident years $1, \ldots, 15$ projected using the SCL method are represented here as light-grey bars. The results produced by the MCL method are shown in dark-grey.

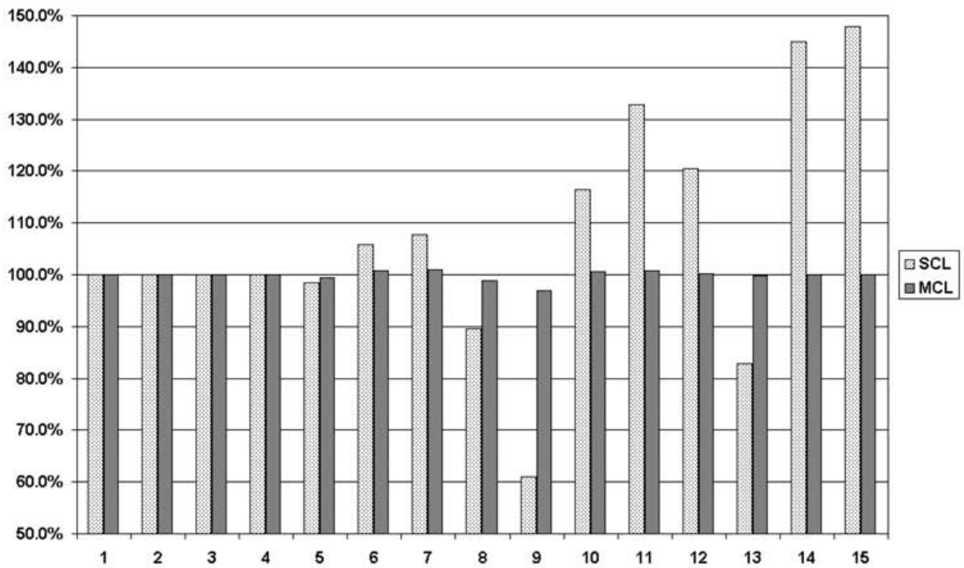


**Figure 10: Comparison of SCL and MCL results (1)**

The ultimate $(P/I)$ ratios of the SCL method fluctuate between 61% and 148%, which means that the paid and incurred projections are markedly different. In all more recent accident years (8 to 15), the paid projection deviates by more than 10% from the incurred projection. The MCL method produces a significantly better picture. Apart from accident year 9, which reaches 'only' 97%, all accident years lie between 99% and 101%. Thus, the MCL projection based on paid losses practically matches that based on incurred losses.

We obtain an even more convincing picture if we depict the entire $(P/I)$ quadrangle calculated with the MCL method as a graph (see Figure 11) and compare it with the corresponding SCL quadrangle (see Figure 2). In Figure 11, in contrast to the SCL chart, the projected points (unfilled diamonds) fit seamlessly into the points actually observed (filled diamonds), yielding a plausible overall picture, with more and more points overlapping from left to right.

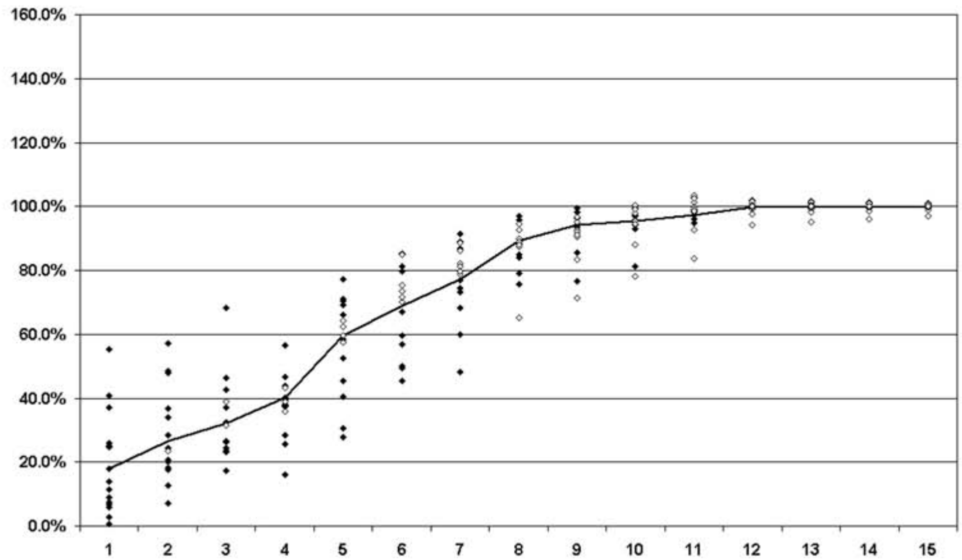


**Figure 11: A $(P/I)$ quadrangle produced using the MCL method (cf. Figure 2)**

### 1.4.2 Example: a European GL portfolio

Figure 12 shows a comparison of the results for the European GL portfolio from Section 1.3.2. While the results obtained using the SCL method vary significantly (between 92% and 102%), those yielded by the MCL method are about 96% for all accident years.

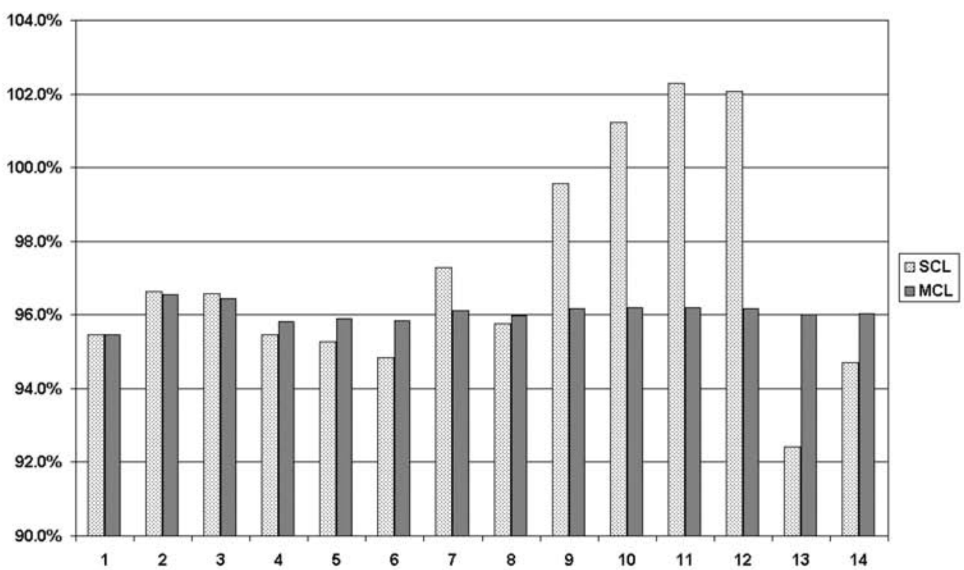


**Figure 12: Comparison of SCL and MCL results (2)**

At first glance, it may seem disappointing that in this case the MCL method failed to close the gap between paid and incurred completely, but that would not have been plausible because the portfolio's run-off is still not settled after the 14 development years available so far. Instead, the $(P/I)$ pattern suggests a value of about 96% after 14 development years, and this is precisely what the MCL method predicts. However, in dealing with the results produced by the MCL method, in contrast to the projection resulting from the SCL method, it makes sense to use a paid and an incurred tail factor (for example by extrapolating the respective development factors) each being the same for all accident years, because in this way the paid and incurred projections can be matched. Thus, we obtain a reliable estimate of the ultimate losses based on both data records.

### 1.4.3 Example: a European MTPL portfolio

In the last example, a European MTPL portfolio, deviations from the average $(P/I)$ ratio are compensated only very slowly. If, in the course of its development, an accident year has a significantly below-average or above-average $(P/I)$ ratio, the deviation usually persists over a number of development years before finally returning to a more or less average value. The correlation in the residual plots is consequently very weak. Since the MCL method makes projections on the basis of the observed data, it is plausible that in this case no uniform $(P/I)$ ratio will result after 15 development years.

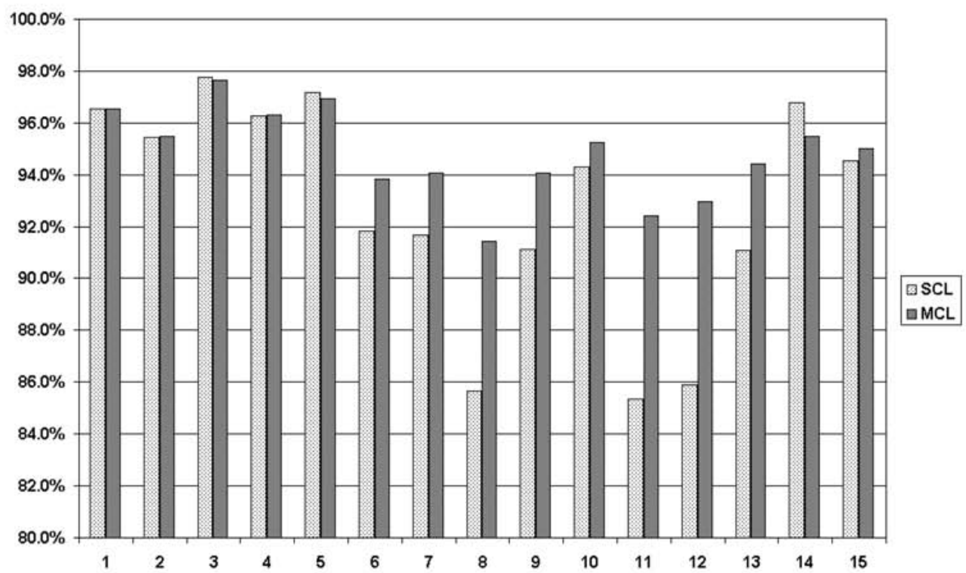


**Figure 13: Comparison of SCL and MCL results (3)**

As can be seen in Figure 13, the diagram of the ultimate $(P/I)$ ratios determined according to the MCL method is not as smooth as in the previous examples, but is still significantly better than that produced by the SCL method. Closer concurrence of the paid and incurred projections can be achieved through extrapolation of all the parameters needed for an MCL calculation (see also Section 3.2.2).

# Chapter 2

# Theoretical bases and formulas

In the preceding chapter, we dealt with the fundamental ideas and properties of the Munich chain ladder method, largely dispensing with a discussion of the mathematical aspects. We will now make up for that by establishing a distribution-free stochastic model, the MCL model, from which we will then be able to derive all the formulas and estimators needed.

The MCL model is based on and generalises the chain ladder model introduced by Mack[1]. More precisely, it is a joint model for paid and incurred that takes into account the dependence of the development factors on the $(P/I)$ ratios (or $(I/P)$ ratios) described in the preceding chapter.

## 2.1 The chain ladder model introduced by Mack

We shall first introduce some notations and then formulate the assumptions of the chain ladder model.

### 2.1.1 Notations

Let $n \in \mathbb{N}$ be the number of accident years and $T = \{1, \ldots, m\}$ the development time ($m \in \mathbb{N}$; usually, $m = n$). For $i = 1, \ldots, n$, let $P_i = (P_{i,t})_{t \in T}$ denote the paid process of accident year $i$ and $I_i = (I_{i,t})_{t \in T}$ the incurred process. Thus, the random variable $P_{i,t}$ denotes the paid losses for accident year $i$ after $t$ development years, and $I_{i,t}$ denotes the incurred losses for accident year $i$ after $t$ development years. The processes $P_i$ and $I_i$ describe the development of the paid and incurred losses of accident year $i$ throughout the development years.

Furthermore, $\mathcal{P}_i(s) := \{P_{i,1}, \ldots, P_{i,s}\}$ stands for the condition that the paid development of accident year $i$ is given until the end of development year $s$ and $\mathcal{I}_i(s) := \{I_{i,1}, \ldots, I_{i,s}\}$ for the condition that the incurred development of accident year $i$ is given up to and including $s$.

[1] **Mack, Thomas**, Distribution-free Calculation of the Standard Error of Chain Ladder Reserve Estimates, ASTIN Bulletin 23 (1993), 213-225

### 2.1.2 Model assumptions

The chain ladder model assumptions for the paid processes are:

**PE** (expectation assumption) For $s, t \in T$ with $t = s + 1$, there exists a development factor $f^P_{s\to t} > 0$ such that for all $i = 1, \ldots, n$

$$\boxed{\mathbf{E}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right) = f^P_{s\to t}.}$$

**PV** (variance assumption) For $s, t \in T$ with $t = s + 1$, there exists a proportionality constant $\sigma^P_{s\to t} \geq 0$ such that for all $i = 1, \ldots, n$

$$\boxed{\mathbf{Var}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right) = \frac{(\sigma^P_{s\to t})^2}{P_{i,s}}.}$$

**PU** (independence assumption) The various accident years are independent, i.e. the sets $\{P_{1,t} \,|\, t \in T\}, \ldots, \{P_{n,t} \,|\, t \in T\}$ are stochastically independent.

Analogous assumptions apply to the incurred processes:

**IE** (expectation assumption) For $s, t \in T$ with $t = s + 1$, there exists a development factor $f^I_{s\to t} > 0$ such that for all $i = 1, \ldots, n$

$$\boxed{\mathbf{E}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{I}_i(s)\right) = f^I_{s\to t}.}$$

**IV** (variance assumption) For $s, t \in T$ with $t = s + 1$, there exists a proportionality constant $\sigma^I_{s\to t} \geq 0$ such that for all $i = 1, \ldots, n$

$$\boxed{\mathbf{Var}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{I}_i(s)\right) = \frac{(\sigma^I_{s\to t})^2}{I_{i,s}}.}$$

**IU** (independence assumption) The various accident years are independent, i.e. the sets $\{I_{1,t} \,|\, t \in T\}, \ldots, \{I_{n,t} \,|\, t \in T\}$ are stochastically independent.

Thus, the assumptions of the chain ladder model say that the accident years are stochastically independent, but have the same development factors and $\sigma$ parameters in each development year.

The above assumptions are designed for the projection of one triangle and therefore say nothing about relationships between the paid and incurred processes. The conditional expectations describe the best possibility of forecasting $P_{i,t}$ if one only knows the paid process of this accident year up to time $s$. This applies analogously to the incurred processes. In practice, however, one knows both triangles and wants to make projections based on this knowledge, i.e. one is interested in the conditional expectations

$$\mathbf{E}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{B}_i(s)\right) \qquad \text{and} \qquad \mathbf{E}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{B}_i(s)\right),$$

where $\mathcal{B}_i(s) = \{P_{i,1}, \ldots, P_{i,s}, I_{i,1}, \ldots, I_{i,s}\}$ stands for the knowledge of the development of both processes up to the end of development year $s$.

## 2.2 The Munich chain ladder model

In order to obtain reasonable formulas for the above conditional expectations, we must perform some preparatory work.

As the first additional assumption, we want to broaden the independence assumption and, instead of **PU** and **IU**, directly assume **PIU**, i.e. the independence of the accident years across both paid and incurred losses. Formally, this means the stochastic independence of the sets $\{P_{1,t}, I_{1,t} \,|\, t \in T\}, \ldots, \{P_{n,t}, I_{n,t} \,|\, t \in T\}$. Further, for the sake of brevity, let

$$Q_i := \frac{P_i}{I_i} = \left(\frac{P_{i,t}}{I_{i,t}}\right)_{t \in T}$$

denote the $(P/I)$ process.

In order to proceed, we require the concept of the conditional residual: If $X$ is a random variable, $\mathcal{C}$ a condition and

$$\sigma(X \,|\, \mathcal{C}) := \sqrt{\mathbf{Var}(X \,|\, \mathcal{C})}$$

the conditional standard deviation of $X$ given $\mathcal{C}$, then

$$\mathbf{Res}(X \,|\, \mathcal{C}) := \frac{X - \mathbf{E}(X \,|\, \mathcal{C})}{\sigma(X \,|\, \mathcal{C})}$$

is called the conditional residual of $X$ given $\mathcal{C}$. The conditional residual is standardised with regard to its conditional expectation and its conditional variance:

$$\mathbf{E}\Big(\mathbf{Res}(X \,|\, \mathcal{C}) \,|\, \mathcal{C}\Big) = 0 \qquad \text{and} \qquad \mathbf{Var}\Big(\mathbf{Res}(X \,|\, \mathcal{C}) \,|\, \mathcal{C}\Big) = 1.$$

### 2.2.1 Model assumptions

As was suggested at the end of the preceding section (Section 2.1.2), we are interested in the conditional expectations for the paid and incurred development factors and their residuals

$$\mathbf{Res}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right) \qquad \text{and} \qquad \mathbf{Res}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{I}_i(s)\right)$$

provided the previous development of both processes is known. Compared to the SCL model, the Munich chain ladder model's decisive advancement is that it formulates assumptions for these terms, that is to say for:

$$\mathbf{E}\left(\mathbf{Res}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right) \,|\, \mathcal{B}_i(s)\right) \quad \text{and} \quad \mathbf{E}\left(\mathbf{Res}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{I}_i(s)\right) \,|\, \mathcal{B}_i(s)\right).$$

These assumptions, following below, translate the procedure introduced after Figures 8 and 9 of assuming a linear dependence of these conditional expectations on the residuals of the $(I/P)$ or $(P/I)$ ratios

$$\mathbf{Res}\Big(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\Big) \qquad \text{or} \qquad \mathbf{Res}\Big(Q_{i,s} \,|\, \mathcal{I}_i(s)\Big)$$

into a mathematical equation.

**PQ** There exists a constant $\lambda^P$ such that for all $s, t \in T$ with $t = s+1$ and all $i = 1, \ldots, n$

$$\mathbf{E}\left(\mathbf{Res}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right) \,|\, \mathcal{B}_i(s)\right) = \lambda^P \cdot \mathbf{Res}\big(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\big)$$

or equivalently

$$\boxed{\mathbf{E}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{B}_i(s)\right) = f_{s\to t}^P + \lambda^P \cdot \frac{\sigma\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right)}{\sigma\left(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\right)} \cdot \left(Q_{i,s}^{-1} - \mathbf{E}\big(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\big)\right)}$$

**IQ** There exists a constant $\lambda^I$ such that for all $s, t \in T$ with $t = s+1$ and all $i = 1, \ldots, n$

$$\mathbf{E}\left(\mathbf{Res}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{I}_i(s)\right) \,|\, \mathcal{B}_i(s)\right) = \lambda^I \cdot \mathbf{Res}\big(Q_{i,s} \,|\, \mathcal{I}_i(s)\big)$$

or equivalently

$$\boxed{\mathbf{E}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{B}_i(s)\right) = f_{s\to t}^I + \lambda^I \cdot \frac{\sigma\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{I}_i(s)\right)}{\sigma\left(Q_{i,s} \,|\, \mathcal{I}_i(s)\right)} \cdot \left(Q_{i,s} - \mathbf{E}\big(Q_{i,s} \,|\, \mathcal{I}_i(s)\big)\right)}$$

The parameters $\lambda^P$ and $\lambda^I$, which represent the slopes of the regression lines in the respective residual plots, are not dependent on development year $s$. In accordance with the statements made in Sections 1.2 and 1.3, it will usually be the case that $\lambda^P, \lambda^I \geq 0$. The equations in the boxes represent the conditional expectations for the development factors as the sum of the customary chain ladder development factor and a correction term that is a function of both data types. We will analyse this term in greater detail in the next section.

All in all, the MCL model consists of the independence assumption **PIU** for the accident years, the usual chain ladder prerequisites **PE**, **PV**, **IE** and **IV** for paid and incurred losses and the assumptions **PQ** and **IQ**, which describe the dependence of the paid and incurred development factors on the preceding $(I/P)$ and $(P/I)$ ratios, respectively.

### 2.2.2 Analysis of the model assumptions

We shall now examine somewhat more closely the MCL model and especially the two equations from **PQ** and **IQ**, presupposing the normal case $\lambda^P, \lambda^I \geq 0$.

The conditional expectation $\mathbf{E}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{B}_i(s)\right)$, i.e. the incurred development factor to be used to project accident year $i$ from $s$ to $t$, is a monotonously increasing, linear function of the $(P/I)$ ratio $Q_{i,s}$. This shows that the observations from practice (see Section 1.2) have been expressed as theoretical assumptions. More precisely, the equation in the box in **IQ** represents the conditional expectation as the sum of the customary chain ladder development factor $f_{s\to t}^I$ and a correction term that is linear in $Q_{i,s}$. The three factors of the correction term can be explained as follows:

- The factor $\lambda^I$ is the common (i.e. independent from both $i$ and $s$) correlation coefficient of the residuals of the development factors and the residuals of the $(P/I)$ ratios, as we will prove below. We therefore refer to $\lambda^I$ as correlation factor or correlation parameter. The value of $\lambda^I$ is usually between 0 and 1 and measures the dependence of the development factors on the preceding $(P/I)$ ratios. If there is hardly any dependence in the data, then $\lambda^I \approx 0$ and average development factors are projected as in the SCL method.

- The standard deviation factor is the quotient of the conditional standard deviations of the incurred development factor and the momentary $(P/I)$ ratio. It causes deviations of the $(P/I)$ ratios from the average to be rescaled as deviations of the development factors. The greater the standard deviation of the development factor is, the more probable will be a significant deviation from the average, and the greater will be the correction term. The smaller the standard deviation of the $(P/I)$ ratio is, the more untypical will be significant deviations from the average, and again, the greater will be the correction term.

- The linear term $Q_{i,s} - \mathbf{E}(Q_{i,s} \,|\, \mathcal{I}_i(s))$ includes the $(P/I)$ ratio in the projection. Above-average momentary $(P/I)$ ratios have the effect of correcting the development factor upward, while below-average momentary ratios have the effect of correcting it downward. The further removed the momentary $(P/I)$ ratio is from the average, the greater will be the correction term. If the $(P/I)$ ratio is average, the development factor used will be average, as in the SCL method.

The statements made so far in relation to incurred development factors and $(P/I)$ ratios apply of course analogously to paid development factors and $(I/P)$ ratios.

The correlation parameters $\lambda^P$ and $\lambda^I$ represent the link between the paid triangle and the incurred triangle. The magnitude of these parameters indicates the extent to which the development of the paid and incurred losses is affected by the respectively other data type, which is why these parameters are of great significance for the size of the ultimate projection. Since the residual approach makes it possible to consider all development years together, i.e. makes available a sufficient quantity of data points, their estimation is relatively stable. In this way, the MCL method counters the second of the three problems from Section 1.2.2.

We will now justify the description of $\lambda^P$ and $\lambda^I$ as correlation parameters. Using the abbreviation $\mathbf{Cov}^{\mathcal{C}}(X,Y) := \mathbf{Cov}(X,Y \,|\, \mathcal{C})$ for the conditional covariance of two random variables $X$ and $Y$ given condition $\mathcal{C}$, we calculate

$$
\begin{aligned}
& \mathbf{Cov}^{\mathcal{P}_i(s)}\left(Q_{i,s}^{-1}, \frac{P_{i,t}}{P_{i,s}}\right) \\
= \; & \mathbf{Cov}^{\mathcal{P}_i(s)}\left(Q_{i,s}^{-1}, \mathbf{E}\left(\frac{P_{i,t}}{P_{i,s}} \,\Big|\, \mathcal{B}_i(s)\right)\right) \\
= \; & \mathbf{Cov}^{\mathcal{P}_i(s)}\left(Q_{i,s}^{-1}, f_{s\to t}^P + \lambda^P \cdot \frac{\sigma\left(\frac{P_{i,t}}{P_{i,s}} \,\big|\, \mathcal{P}_i(s)\right)}{\sigma\left(Q_{i,s}^{-1} \,\big|\, \mathcal{P}_i(s)\right)} \cdot \left(Q_{i,s}^{-1} - \mathbf{E}\left(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda^P \cdot \frac{\sigma\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right)}{\sigma\left(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\right)} \cdot \mathbf{Var}\left(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\right) \\
&= \lambda^P \cdot \sigma\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right) \cdot \sigma\left(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\right).
\end{aligned}
$$

Together with an analogous calculation for incurred, this leads to the formulas

$$
\mathbf{Corr}\left(Q_{i,s}^{-1}, \frac{P_{i,t}}{P_{i,s}} \mid \mathcal{P}_i(s)\right) = \lambda^P \qquad \text{and} \qquad \mathbf{Corr}\left(Q_{i,s}, \frac{I_{i,t}}{I_{i,s}} \mid \mathcal{I}_i(s)\right) = \lambda^I
$$

for the conditional correlation coefficients and, taking expectations, to

$$
\mathbf{Corr}\left(\mathbf{Res}\big(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\big), \mathbf{Res}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right)\right) = \lambda^P
$$

and

$$
\mathbf{Corr}\left(\mathbf{Res}\big(Q_{i,s} \,|\, \mathcal{I}_i(s)\big), \mathbf{Res}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{I}_i(s)\right)\right) = \lambda^I.
$$

Thus, the $\lambda$ parameters of the MCL model and the correlation coefficients of the corresponding residuals agree. This is an automatic safety mechanism against the third problem of Section 1.2.2: weak correlation in the paid and/or incurred residual plot yields a small estimate of $\lambda^P$ and/or $\lambda^I$, so that the MCL projection deviates only a little from the customary chain ladder calculation for paid and/or incurred losses.

# Chapter 3

# Practical implementation and concrete example

In this chapter, we first provide a detailed description of all the parameter estimators needed for the MCL method, then deal with some special problems arising in practise, before performing a complete MCL calculation for a concrete example.

## 3.1 Estimating the parameters

In order to calculate residuals and expected development factors, we have to estimate every parameter of the MCL model. Here again, let $t = s + 1$.

### 3.1.1 The chain ladder parameters

For the development factors $f^P_{s\to t}$ and $f^I_{s\to t}$ for $s = 1, \ldots, n-1$, we use the usual chain ladder estimators

$$\widehat{f^P_{s\to t}} := \frac{1}{\sum_{i=1}^{n-s} P_{i,s}} \cdot \sum_{i=1}^{n-s} P_{i,s} \cdot \frac{P_{i,t}}{P_{i,s}} = \frac{\sum_{i=1}^{n-s} P_{i,t}}{\sum_{i=1}^{n-s} P_{i,s}}$$

and

$$\widehat{f^I_{s\to t}} := \frac{1}{\sum_{i=1}^{n-s} I_{i,s}} \cdot \sum_{i=1}^{n-s} I_{i,s} \cdot \frac{I_{i,t}}{I_{i,s}} = \frac{\sum_{i=1}^{n-s} I_{i,t}}{\sum_{i=1}^{n-s} I_{i,s}}.$$

For $s = 1, \ldots, n-2$ the $\sigma$ parameters are also estimated as usual through

$$\widehat{(\sigma^P_{s\to t})^2} := \frac{1}{n-s-1} \cdot \sum_{i=1}^{n-s} P_{i,s} \cdot \left(\frac{P_{i,t}}{P_{i,s}} - \widehat{f^P_{s\to t}}\right)^2$$

and

$$\widehat{(\sigma^I_{s\to t})^2} := \frac{1}{n-s-1} \cdot \sum_{i=1}^{n-s} I_{i,s} \cdot \left(\frac{I_{i,t}}{I_{i,s}} - \widehat{f^I_{s\to t}}\right)^2$$

or, to be accurate, through $\widehat{\sigma^P_{s\to t}} := \sqrt{\widehat{(\sigma^P_{s\to t})^2}}$ and $\widehat{\sigma^I_{s\to t}} := \sqrt{\widehat{(\sigma^I_{s\to t})^2}}$.

### 3.1.2 The MCL parameters

In order to calculate the conditional residuals of the $(P/I)$ and $(I/P)$ ratios, we need estimators for the conditional expectations $\mathbf{E}\left(Q_{i,s} \,|\, \mathcal{I}_i(s)\right)$ and $\mathbf{E}(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s))$ and the conditional standard deviations $\sigma(Q_{i,s} \,|\, \mathcal{I}_i(s))$ and $\sigma(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s))$.

At first glance, it seems obvious to assume that $\mathbf{E}\left(Q_{i,s} \,|\, \mathcal{I}_i(s)\right)$ is constant, analogous to condition **IE** of the chain ladder model for incurred. Further, it seems plausible to assume a dependency of the conditional variance of the $(P/I)$ ratio on the incurred volume, analogous to condition **IV**: greater volume means less variance of the $(P/I)$ ratio. For $s = 1, \ldots, n$, these assumptions for the conditional expectations and variances of the $(P/I)$ ratios suggest for $\mathbf{E}\left(Q_{i,s} \,|\, \mathcal{I}_i(s)\right)$ the estimator

$$\widehat{q_s} := \frac{1}{\sum_{j=1}^{n-s+1} I_{j,s}} \cdot \sum_{j=1}^{n-s+1} I_{j,s} \cdot Q_{j,s} = \frac{\sum_{j=1}^{n-s+1} P_{j,s}}{\sum_{j=1}^{n-s+1} I_{j,s}}$$

which is the same for all accident years. For $\sigma(Q_{i,s} \,|\, \mathcal{I}_i(s))$ they suggest the estimator

$$\frac{\widehat{\rho_s^I}}{\sqrt{I_{i,s}}},$$

where $\widehat{\rho_s^I}$ is defined by

$$\widehat{\rho_s^I}^2 = \frac{1}{n-s} \cdot \sum_{j=1}^{n-s+1} I_{j,s} \cdot \left(Q_{j,s} - \widehat{q_s}\right)^2$$

for $s = 1, \ldots, n-1$. Note that $\widehat{\rho_s^I}$ is independent from the accident-year index $i$.

It would then be logical to assume that these considerations for the $(P/I)$ ratios apply analogously to the conditional expectations and variances of the $(I/P)$ ratios, where in compliance with the specified conditions, paid assumes the role of the volume measure. Analogously, one would obtain the estimator

$$\frac{1}{\sum_{j=1}^{n-s+1} P_{j,s}} \cdot \sum_{j=1}^{n-s+1} P_{j,s} \cdot Q_{j,s}^{-1} = \frac{\sum_{j=1}^{n-s+1} I_{j,s}}{\sum_{j=1}^{n-s+1} P_{j,s}} = \widehat{q_s}^{-1}$$

for $\mathbf{E}(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s))$ and

$$\frac{\widehat{\rho_s^P}}{\sqrt{P_{i,s}}}$$

for $\sigma(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s))$, where $\widehat{\rho_s^P}$ is defined by

$$\widehat{\rho_s^P}^2 = \frac{1}{n-s} \cdot \sum_{j=1}^{n-s+1} P_{j,s} \cdot \left(Q_{j,s}^{-1} - \widehat{q_s}^{-1}\right)^2.$$

Here, a problem arises because it follows from the requirement that both conditional expectations $\mathbf{E}\left(Q_{i,s} \,|\, \mathcal{I}_i(s)\right)$ and $\mathbf{E}(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s))$ be constant that $Q_{i,s}$ is already constant, which of course contradicts reality. Hence, this cannot be assumed. There has to

be a more complicated dependency structure of the conditional expectations on $\mathcal{I}_i(s)$ and $\mathcal{P}_i(s)$, respectively.
Given sufficient data, one would therefore estimate $\mathbf{E}(Q_{i,s} \,|\, \mathcal{I}_i(s))$ by averaging over the $(P/I)$ ratios $Q_{j,s}$ of those accident years $j$ for which $\mathcal{I}_j(s)$ is equal or similar to $\mathcal{I}_i(s)$. Being in the chain ladder setting, 'similar' could mean that the level of $I_{j,s}$ is close to $I_{i,s}$ or that the preceding individual development factor $I_{j,s}/I_{j,s-1}$ is close to $I_{i,s}/I_{i,s-1}$. At least, one would omit any accident year $j$ where $\mathcal{I}_j(s)$ is clearly different from $\mathcal{I}_i(s)$. Of course, these concepts apply analogously to the estimation of $\mathbf{E}(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s))$. This approach would yield estimates for the conditional expectations which are not reciprocal by definition and which are individual for each accident year, but it would be very data-dependent and only feasible for the first development years of large triangles.
For the conditional variances the situation is similar. Given sufficient data, a more complicated dependency structure of the conditional variances of $Q_{i,s}$ and $Q_{i,s}^{-1}$ on $\mathcal{I}_i(s)$ and $\mathcal{P}_i(s)$, respectively, could be taken into account.

But, for the clarity and simplicity of the following exposition we keep the above (raw) estimators. This is justified as long as $\mathbf{E}(Q_{i,s} \,|\, \mathcal{I}_i(s))$ and $\mathbf{E}(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s))$ are indeed non-constant functions of $\mathcal{I}_i(s)$ and $\mathcal{P}_i(s)$, respectively, but vary only little in the relevant region of 'normal, non-extreme' incurred and paid values. We leave it to the reader to apply refined estimators if his data permit this.

By means of the above formulas, we are now able to estimate the conditional residuals. To simplify notation, we denote the estimators obtained that way for

$$\mathbf{Res}\left(\frac{P_{i,t}}{P_{i,s}} \,|\, \mathcal{P}_i(s)\right), \quad \mathbf{Res}\left(\frac{I_{i,t}}{I_{i,s}} \,|\, \mathcal{I}_i(s)\right), \quad \mathbf{Res}\Big(Q_{i,s}^{-1} \,|\, \mathcal{P}_i(s)\Big) \quad \text{and} \quad \mathbf{Res}\Big(Q_{i,s} \,|\, \mathcal{I}_i(s)\Big)$$

by $\widehat{\mathbf{Res}}(P_{i,t})$, $\widehat{\mathbf{Res}}(I_{i,t})$, $\widehat{\mathbf{Res}}(Q_{i,s}^{-1})$ and $\widehat{\mathbf{Res}}(Q_{i,s})$, respectively. Hence,

$$\widehat{\mathbf{Res}}(P_{i,t}) = \frac{\frac{P_{i,t}}{P_{i,s}} - \widehat{f_{s\to t}^P}}{\widehat{\sigma_{s\to t}^P}} \cdot \sqrt{P_{i,s}}, \qquad \widehat{\mathbf{Res}}(I_{i,t}) = \frac{\frac{I_{i,t}}{I_{i,s}} - \widehat{f_{s\to t}^I}}{\widehat{\sigma_{s\to t}^I}} \cdot \sqrt{I_{i,s}}$$

and

$$\widehat{\mathbf{Res}}(Q_{i,s}^{-1}) = \frac{Q_{i,s}^{-1} - \widehat{q}_s^{\,-1}}{\widehat{\rho_s^P}} \cdot \sqrt{P_{i,s}}, \qquad \widehat{\mathbf{Res}}(Q_{i,s}) = \frac{Q_{i,s} - \widehat{q}_s}{\widehat{\rho_s^I}} \cdot \sqrt{I_{i,s}}.$$

For the correlation parameters $\lambda^P$ and $\lambda^I$, we use the estimators that minimise the average quadratic distances of the $y$-coordinates of the points in the residual plot to the regression line through the origin with slope $\lambda^P$ or $\lambda^I$:

$$\widehat{\lambda^P} := \frac{1}{\sum_{i,s} \widehat{\mathbf{Res}}(Q_{i,s}^{-1})^2} \cdot \sum_{i,s} \widehat{\mathbf{Res}}(Q_{i,s}^{-1})^2 \cdot \frac{\widehat{\mathbf{Res}}(P_{i,t})}{\widehat{\mathbf{Res}}(Q_{i,s}^{-1})} = \frac{\sum_{i,s} \widehat{\mathbf{Res}}(Q_{i,s}^{-1}) \cdot \widehat{\mathbf{Res}}(P_{i,t})}{\sum_{i,s} \widehat{\mathbf{Res}}(Q_{i,s}^{-1})^2}$$

and

$$\widehat{\lambda^I} := \frac{1}{\sum_{i,s} \widehat{\mathbf{Res}}(Q_{i,s})^2} \cdot \sum_{i,s} \widehat{\mathbf{Res}}(Q_{i,s})^2 \cdot \frac{\widehat{\mathbf{Res}}(I_{i,t})}{\widehat{\mathbf{Res}}(Q_{i,s})} = \frac{\sum_{i,s} \widehat{\mathbf{Res}}(Q_{i,s}) \cdot \widehat{\mathbf{Res}}(I_{i,t})}{\sum_{i,s} \widehat{\mathbf{Res}}(Q_{i,s})^2}.$$

In all these summations, index $s$ extends from 1 to $n-2$ and index $i$ from 1 to $n-s$. If a triangle's run-off comes to an end in less than $n$ development years, it may be expedient to have index $s$ extend only to the end of the run-off period.

Fixing a development year $s$ in the estimation formulas for $\lambda^P$ and $\lambda^I$ and summing up only over $i$ yields development year estimates for the $\lambda$ parameters. The $\lambda$ parameters for each development year should fluctuate at random and show no clear trends as that would violate the MCL model assumptions. This is usually the case in practice. The often marked fluctuations in these numbers also show the volatility of the estimates for each development year and thus constitute an argument supporting a procedure that considers all development years at once, as is the case with the residual approach of the MCL method.

According to the assumptions **PQ** and **IQ**, we finally obtain the recursion formulas

$$\widehat{P_{i,t}} := \widehat{P_{i,s}} \cdot \left( \widehat{f^P_{s\to t}} + \widehat{\lambda^P} \cdot \frac{\widehat{\sigma^P_{s\to t}}}{\widehat{\rho^P_s}} \cdot \left( \frac{\widehat{I_{i,s}}}{\widehat{P_{i,s}}} - \widehat{q_s}^{-1} \right) \right)$$

and

$$\widehat{I_{i,t}} := \widehat{I_{i,s}} \cdot \left( \widehat{f^I_{s\to t}} + \widehat{\lambda^I} \cdot \frac{\widehat{\sigma^I_{s\to t}}}{\widehat{\rho^I_s}} \cdot \left( \frac{\widehat{P_{i,s}}}{\widehat{I_{i,s}}} - \widehat{q_s} \right) \right)$$

for $s \geq n-i+1$ with the initial values $\widehat{P_{i,s}} := P_{i,s}$ and $\widehat{I_{i,s}} := I_{i,s}$ for $s = n-i+1$.

## 3.2 Handling of special situations

### 3.2.1 Vanishing or miniscule current paid losses

If very low cumulative paid totals in the initial development years are not unusual for a portfolio, a (standard) chain ladder calculation based on the paid triangle frequently yields quite useless ultimate loss prognoses for the most recent accident years. Their levels depend largely on minor fluctuations in current paid values, and IBNR reserves often seem much too low. In cases where no losses have been paid, the chain ladder method yields the usually nonsensical ultimate loss prognosis of zero.

This problem does not exist in the MCL method, provided at least the incurred loss and the paid correlation parameter are of normal magnitude. To see this, let us consider the recursion formula for the paid-loss projection from Section 3.1.2:

$$\begin{aligned} \widehat{P_{i,t}} &= \widehat{P_{i,s}} \cdot \left( \widehat{f^P_{s\to t}} + \widehat{\lambda^P} \cdot \frac{\widehat{\sigma^P_{s\to t}}}{\widehat{\rho^P_s}} \cdot \left( \frac{\widehat{I_{i,s}}}{\widehat{P_{i,s}}} - \widehat{q_s}^{-1} \right) \right) \\ &= \widehat{I_{i,s}} \cdot \widehat{\lambda^P} \cdot \frac{\widehat{\sigma^P_{s\to t}}}{\widehat{\rho^P_s}} + \widehat{P_{i,s}} \cdot \left( \widehat{f^P_{s\to t}} - \widehat{\lambda^P} \cdot \frac{\widehat{\sigma^P_{s\to t}}}{\widehat{\rho^P_s}} \cdot \widehat{q_s}^{-1} \right) \end{aligned}$$

This representation shows that the value prognosticated for $P_{i,t}$ is influenced strongly by the estimation of $I_{i,s}$ and weakly by the estimation of $P_{i,s}$, provided this paid estimation is small and that it does not vanish when the current loss $P_{i,n-i+1} = 0$.

If current cumulative paid losses are extremely low or even zero, the paid values prognosticated with the MCL method depend primarily on the current value of incurred. In practice, the MCL prognoses obtained for paid ultimates are quite realistic, provided that incurred losses are of normal magnitude. Here, in contrast to its implausibly powerful influence in the SCL method, the level of the paid losses plays hardly any role.

### 3.2.2 Smoothing and extrapolation in cases where run-off is unfinished

Whenever the run-off of the earliest accident years in the portfolio under consideration is still unfinished, several problems arise:

- For the latest development years, the little data available is taken as the basis for estimating quite a few parameters, namely development factors, $(P/I)$ pattern and variance parameters. While this is also done in triangles where run-off has come to an end, in this case the parameters have little influence on the final result.
- If run-off has not come to an end by the last development year of the triangle, it is customary to use tail factors. However, this method may not be adequate to achieve extensive concurrence between paid-loss and incurred-loss prognoses, as we saw in Example 1.4.3.

Where necessary, these problems can often be solved by using appropriate regression methods to smooth all the requisite parameters for later development years and extrapolating them beyond the known development years. In doing so, one can take advantage of the fact that the $(P/I)$ pattern generally grows monotonously from 0 to 1 and that the $\sigma$ and $\rho$ parameters generally decrease loglinearly in later development years.

## 3.3 Concrete example

In this section, we perform a complete MCL calculation for a concrete example. For the sake of clarity, the example involves a data record with only 7 accident years that is based on a fire portfolio. The results were calculated with greater precision than is shown here. If you retrace the steps of the calculation, please bear in mind that rounding may lead to minor discrepancies.

### 3.3.1 Initial data

As initial data, a paid and an incurred triangle are given, which we present in the usual way as a table (one line per accident year, one column per development year).

| **Paid** | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|
| **1** | 576 | 1,804 | 1,970 | 2,024 | 2,074 | 2,102 | 2,131 |
| **2** | 866 | 1,948 | 2,162 | 2,232 | 2,284 | 2,348 | |
| **3** | 1,412 | 3,758 | 4,252 | 4,416 | 4,494 | | |
| **4** | 2,286 | 5,292 | 5,724 | 5,850 | | | |
| **5** | 1,868 | 3,778 | 4,648 | | | | |
| **6** | 1,442 | 4,010 | | | | | |
| **7** | 2,044 | | | | | | |

| **Incurred** | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|
| **1** | 978 | 2,104 | 2,134 | 2,144 | 2,174 | 2,182 | 2,174 |
| **2** | 1,844 | 2,552 | 2,466 | 2,480 | 2,508 | 2,454 | |
| **3** | 2,904 | 4,354 | 4,698 | 4,600 | 4,644 | | |
| **4** | 3,502 | 5,958 | 6,070 | 6,142 | | | |
| **5** | 2,812 | 4,882 | 4,852 | | | | |
| **6** | 2,642 | 4,406 | | | | | |
| **7** | 5,022 | | | | | | |

### 3.3.2 Calculating the parameters

The average development factors and $\sigma$ parameters are estimated as described in Section 3.1.1. We obtain:

| | $1 \rightarrow 2$ | $2 \rightarrow 3$ | $3 \rightarrow 4$ | $4 \rightarrow 5$ | $5 \rightarrow 6$ | $6 \rightarrow 7$ |
|---|---|---|---|---|---|---|
| $\widehat{f^P_{s\rightarrow t}}$ | 2.437 | 1.131 | 1.029 | 1.021 | 1.021 | 1.014 |
| $\widehat{f^I_{s\rightarrow t}}$ | 1.652 | 1.019 | 1.000 | 1.011 | 0.990 | 0.996 |
| $\widehat{\sigma^P_{s\rightarrow t}}$ | 13.456 | 3.666 | 0.482 | 0.210 | 0.479 | |
| $\widehat{\sigma^I_{s\rightarrow t}}$ | 9.727 | 2.544 | 1.004 | 0.120 | 0.860 | |

The $(P/I)$ pattern and $\rho$ parameters are determined with the aid of the estimators from Section 3.1.2. We get:

| | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|
| $\widehat{q_s}$ | 53.3% | 84.9% | 92.8% | 94.5% | 94.9% | 96.0% | 98.0% |
| $\widehat{\rho^P_s}$ | 14.943 | 4.990 | 2.167 | 1.619 | 1.791 | 0.236 | |
| $\widehat{\rho^I_s}$ | 5.711 | 3.819 | 1.918 | 1.461 | 1.637 | 0.222 | |

We have now determined all the values that we need to calculate the residuals and can specify the four residual triangles:

| **Paid** | **1 → 2** | **2 → 3** | **3 → 4** | **4 → 5** | **5 → 6** | **6 → 7** |
|---|---|---|---|---|---|---|
| **1** | 1.240 | −0.454 | −0.178 | 0.846 | −0.724 | |
| **2** | −0.410 | −0.258 | 0.293 | 0.572 | 0.690 | |
| **3** | 0.628 | 0.004 | 1.248 | −0.979 | | |
| **4** | −0.433 | −0.985 | −1.151 | | | |
| **5** | −1.330 | 1.661 | | | | |
| **6** | 0.971 | | | | | |
| **7** | | | | | | |

| **Incurred** | **1 → 2** | **2 → 3** | **3 → 4** | **4 → 5** | **5 → 6** | **6 → 7** |
|---|---|---|---|---|---|---|
| **1** | 1.605 | −0.079 | 0.222 | 1.131 | 0.732 | |
| **2** | −1.184 | −1.039 | 0.287 | 0.096 | −0.681 | |
| **3** | −0.846 | 1.565 | −1.415 | −0.843 | | |
| **4** | 0.299 | 0.005 | 0.931 | | | |
| **5** | 0.458 | −0.681 | | | | |
| **6** | 0.082 | | | | | |
| **7** | | | | | | |

| $I/P$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|
| **1** | −0.289 | −0.100 | 0.106 | 0.033 | −0.136 | −0.726 | |
| **2** | 0.496 | 1.168 | 1.343 | 1.547 | 1.188 | 0.687 | |
| **3** | 0.450 | −0.239 | 0.808 | −0.675 | −0.755 | | |
| **4** | −1.106 | −0.761 | −0.615 | −0.388 | | | |
| **5** | −1.077 | 1.406 | −1.075 | | | | |
| **6** | −0.116 | −1.006 | | | | | |
| **7** | 1.753 | | | | | | |

| $P/I$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|
| **1** | 0.309 | 0.103 | −0.107 | −0.033 | 0.137 | 0.728 | |
| **2** | −0.473 | −1.131 | −1.317 | −1.537 | −1.177 | −0.686 | |
| **3** | −0.437 | 0.246 | −0.805 | 0.693 | 0.771 | | |
| **4** | 1.245 | 0.795 | 0.626 | 0.396 | | | |
| **5** | 1.223 | −1.372 | 1.102 | | | | |
| **6** | 0.119 | 1.065 | | | | | |
| **7** | −1.558 | | | | | | |

Using the residuals of the paid and the $(I/P)$ triangles, we can draw the paid residual plot (Figure 14). Note that we do not need the $(I/P)$ residuals of the current calendar year (hypotenuse) and of development year 6.

The paid residual plot exhibits a correlation of 62%. The estimation of the slope of the regression line through the origin yields $\widehat{\lambda^P} = 0.64$, which confirms the interpretation of $\lambda^P$ as correlation parameter described at the end of Section 2.2.2. The small discrepancy results from the different ways of calculating the correlation coefficient of the scatter plot and the slope of the regression line.

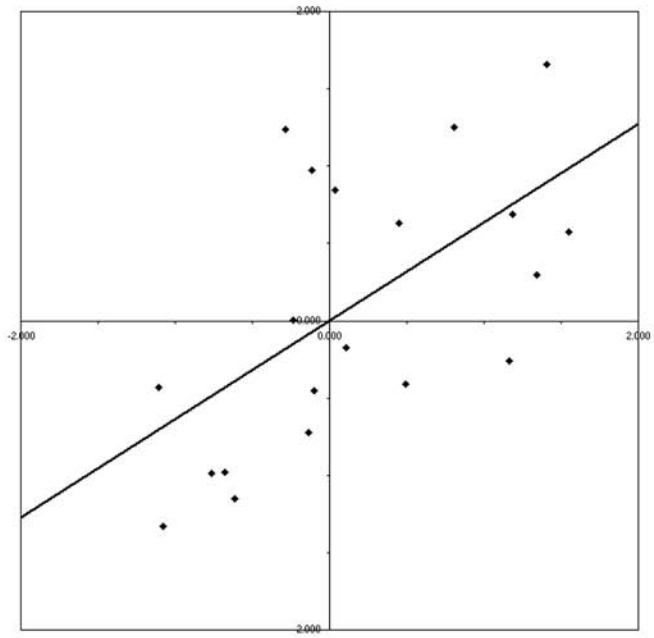


**Figure 14: Paid residual plot**

The incurred residual plot (Figure 15) shows a correlation of 44%. This value matches the estimation $\widehat{\lambda^I} = 0.44$.

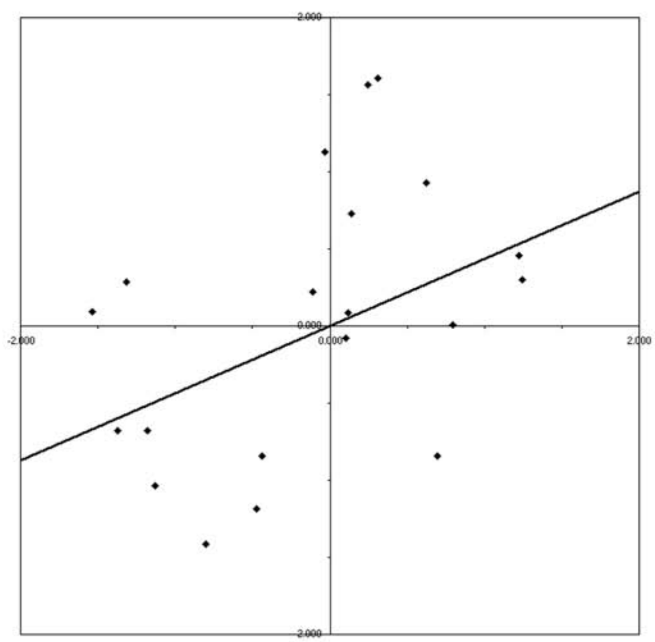


**Figure 15: Incurred residual plot**

Looking at the two residual plots, one could easily get the impression that the regression lines are too flat. This is a widespread phenomenon that can be explained in that, intuitively, one would minimise the (orthogonal) distance of the scatter plot to the regression line, rather than the horizontal distance, and in this way would obtain a steeper regression line.

If we consider the $\lambda$ parameters for each development year, we obtain for paid the values 0.52, 0.71, 0.73, 0.55, 0.64 and for incurred the values 0.66, 0.64, 0.47, −0.27, 0.64. The values are scattered without any clear trends. The negative $\lambda^I$ value from 4 to 5, which is based on only three points of the residual plot, underscores the volatility of a development year-related view.

Now we can use the MCL method to project the paid and the incurred triangles. As an example, we want to take a close look at the most recent accident year 7. As the first paid development factor, we will use not the average value $\widehat{f^P_{1\to 2}} = 2.437$ used in the chain ladder method, but instead

$$\widehat{f^P_{1\to 2}} + \widehat{\lambda^P} \cdot \frac{\widehat{\sigma^P_{1\to 2}}}{\widehat{\rho^P_1}} \cdot (Q^{-1}_{7,1} - \widehat{q_1}^{-1}) = 2.437 + 0.64 \cdot \frac{13.456}{14.943} \cdot (2.457 - 1.878) = 2.768.$$

As the first incurred development factor, we use

$$\widehat{f^I_{1\to 2}} + \widehat{\lambda^I} \cdot \frac{\widehat{\sigma^I_{1\to 2}}}{\widehat{\rho^I_1}} \cdot (Q_{7,1} - \widehat{q_1}) = 1.652 + 0.44 \cdot \frac{9.727}{5.711} \cdot (40.7\% - 53.3\%) = 1.559.$$

This yields as an estimation for $P_{7,2}$ the value 2,044 $\cdot$ 2.768 = 5,659 and for $I_{7,2}$ the value 5,022 $\cdot$ 1.559 = 7,828. In the second step from $s = 2$ to $t = 3$, we proceed analogously. It becomes clear that we must perform the paid and the incurred calculations simultaneously because we need an estimation for the $(P/I)$ ratio $Q_{7,2}$, namely $5,659/7,828 = 72.3\%$, for both the paid and the incurred projections.

### 3.3.3 Results

As the results of the projection, we obtain one paid and one incurred quadrangle.

| **Paid** | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---:|---:|---:|---:|---:|---:|---:|---:|
| **1** | 576 | 1,804 | 1,970 | 2,024 | 2,074 | 2,102 | 2,131 |
| **2** | 866 | 1,948 | 2,162 | 2,232 | 2,284 | 2,348 | 2,383 |
| **3** | 1,412 | 3,758 | 4,252 | 4,416 | 4,494 | 4,573 | 4,597 |
| **4** | 2,286 | 5,292 | 5,724 | 5,850 | 5,967 | 6,081 | 6,119 |
| **5** | 1,868 | 3,778 | 4,648 | 4,762 | 4,848 | 4,923 | 4,937 |
| **6** | 1,442 | 4,010 | 4,388 | 4,493 | 4,574 | 4,643 | 4,656 |
| **7** | 2,044 | 5,659 | 6,944 | 7,177 | 7,330 | 7,485 | 7,549 |

| Incurred | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| **1** | 978 | 2,104 | 2,134 | 2,144 | 2,174 | 2,182 | 2,174 |
| **2** | 1,844 | 2,552 | 2,466 | 2,480 | 2,508 | 2,454 | 2,444 |
| **3** | 2,904 | 4,354 | 4,698 | 4,600 | 4,644 | 4,618 | 4,629 |
| **4** | 3,502 | 5,958 | 6,070 | 6,142 | 6,212 | 6,167 | 6,176 |
| **5** | 2,812 | 4,882 | 4,852 | 4,885 | 4,944 | 4,931 | 4,950 |
| **6** | 2,642 | 4,406 | 4,567 | 4,601 | 4,657 | 4,646 | 4,665 |
| **7** | 5,022 | 7,828 | 7,688 | 7,644 | 7,727 | 7,650 | 7,650 |

Here, we have simply manually set the estimations for $\sigma^P_{6\to 7}$ and $\sigma^I_{6\to 7}$ needed for the calculation to 0.100. In practice, one would use a more sound extrapolation method.

Figure 16 compares the degrees of concurrence between paid and incurred projections achieved using the SCL method and those achieved using the MCL method by comparing the ultimate $(P/I)$ ratios calculated with the two procedures.

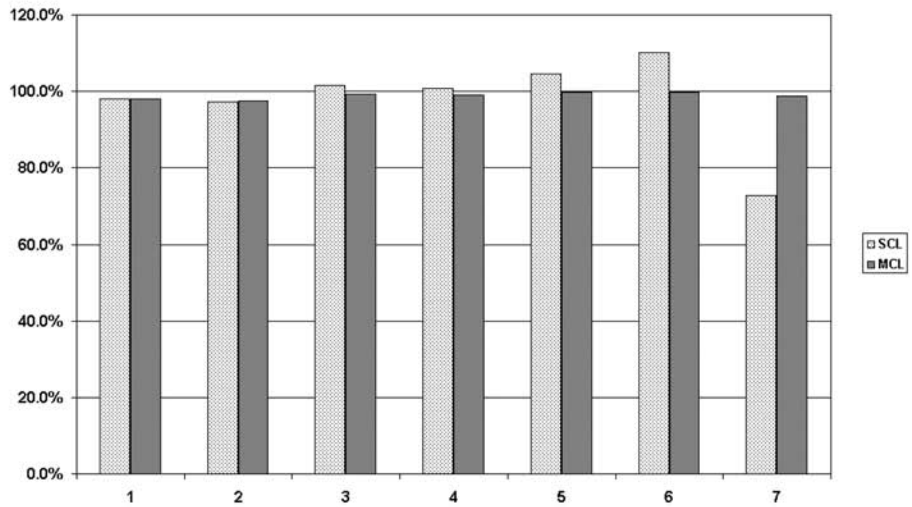


**Figure 16: Comparison of SCL and MCL results**

While separate chain ladder calculations yield a paid projection that exceeds the incurred projection by up to 10% (accident year 6) and falls short of the incurred projection by up to 27% (accident year 7), the results for the two data types obtained with the Munich chain ladder method are practically the same.

# Conclusion

Our analyses have shown that the hitherto customary approach of applying the chain ladder method separately to paid losses and incurred losses generally fails to take into account systematic correlations that exist between the two types of data and consequently produces distorted prognoses.

By contrast, the Munich chain ladder method uses these correlations to precisely the same extent as they have occurred in the past, in calculating prognoses. It yields a paid prognosis and an incurred prognosis that concur as much as it can be expected based on the data observed so far. In those rare cases where the correlations are not significant, the MCL method yields practically the same results as the SCL method.

Ultimate loss prognoses produced using the MCL method are neither systematically higher, nor systematically lower, than those obtained with the SCL method. Whether the MCL prognosis corrects SCL-method results, for example for the incurred triangle, upward or downward, often varies from one accident year to the next. Hence, adopting the Munich chain ladder method will neither have the effect of generally increasing estimates for IBNR reserves, nor generally reducing them, but will produce more realistic results.

The Munich chain ladder method will therefore yield more reliable results for practically all portfolios where chain ladder calculation is appropriate for both the paid and incurred triangles.

The next step is the calculation of the prediction error of the Munich chain ladder reserve estimates. This will be treated in a separate paper.

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