Determination of Optimum Fair Premiums in Property-Liability Insurance: An Optimal Control Theoretic Approach

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ABSTRACT

Dynamic valuation models for the computation of optimum fair premiums are developed using a new framework. The concept of fair premiums which are also "best" is introduced. Optimum fair premiums are defined as the minimum discounted losses for an insurance firm or industry. This notion extends the discrete and continuous discounted cash flow models in many ways. The problem is cast in an optimal control theoretic setting that assumes a world of certainty. It is then extended to incorporate uncertainty in claims occurrence. Using a simple example of the certainty model, a closed-form formula for the rate of return for an insurer or investor is derived. We show that the optimum competitive equilibrium price is the sum of the marginal cost of claims and an economic rent to the insurer.

KEYWORDS

Optimum fair premium, control theoretic, marginal cost, economic rent

1. Introduction

The quest for fair premiums in the property-liability (PL) insurance markets has attracted much attention in recent times. A great deal of attention and controversy in the debate for PL insurance regulation has been focused on the role of investment income in setting prices. There has always been the need for regulatory policies to prevent excessive or inadequate pricing in insurance (Doherty and Garven 1986).

Several models of rate regulation have been developed and used, both in real life and in the academic arena. Other models are the discrete and continuous time discounted cash flow models (DCF). The Myers-Cohn (1987) model is a prime example of the DCF models that have been employed in the rate-setting process. Continuous discounted cash flow models of insurance pricing have been developed by Kraus and Ross (1982). Option pricing models have also been used in insurance. These models are said to provide important insight (Cummins 1988). A recent contribution to the research on the issue of rate regulation is the application of the fuzzy logic approach to insurance pricing (Young 1996).

This paper is an extension of the work by Kraus and Ross (1982) and Cummins (1988), who have developed continuous-time DCF models for the determination of fair premiums in property-liability insurance. They introduced deterministic and stochastic continuous-time models for the analysis of insurance pricing. The stochastic model is based on arbitrage pricing theory. It allows for market-related uncertainty in both frequency and severity.

In this paper, we introduce a new concept in the insurance literature, the notion of optimum (i.e., best) fair premiums (OFP). This leads to the computation of premiums as the optimized total discounted profits. The optimization is done subject to a dynamical constraint that describes the claims process and additional constraints due to, for example, regulation or market structure. Dynamic optimization models are rare in the literature, which may suggest that in the discussion of fair premiums, efficiency has often been ignored. However, insurers are rational economic agents who seek to optimize, for example, their underwriting profits or losses or the utility function (Borch 1990). In contrast to the DCF, this model explicitly models insurers' optimizing behavior. It is a dynamic simulation model for the analysis of long-term insurance markets.

The focus in this paper is on a loss payout pattern rather than incurred loss. The method allows us to derive an equation that tracks the dynamics of claims settlement rates and prices over time. It can be used to analyze different theoretical and practical aspects of insurance pricing. One advantage of this approach is that it can provide a rich variety of dynamic paths that are very useful in analyzing different policy issues. It is also quite straightforward to use this model to study different market structures. Capital investment can also be incorporated in the analysis both under certainty and uncertainty (Dixit and Pindyck 1993).

The paper is structured as follows. We begin by developing general dynamic models of both certain and uncertain worlds. This is then followed by discussions of various concepts and definitions. The concept of optimum fair premiums is introduced and discussed. We then test the model using a simple example of competitive markets.

2. The dynamic model

In this section, a dynamic deterministic model is developed. The dynamics of unsettled claims are described by the differential equation

$$\dot{n} = f(n,u,t) - s \tag{1}$$

where t represents time, $n(t)$ is the number of claims, $u(t)$ is the number of policies in force (also known as the state variable in control theory), and $s(t)$ is the number of claims settled per unit of time or the type of policy written (i.e., the control variable). Note that this analysis could be extended into a multivariable control problem, but for the current paper we present only a single variable control problem (Kamien and Schwartz 1991; Zimbidis and Haberman 2001). The function $f(.)$ is the growth or additions to the claims due to existing policies and current claims. One simplification of $f(.)$ is to assume that it is equal to the accident frequency, as in Kraus and Ross (1982) and Cummins (1988). That is, the frequency process affects the evolution of claims. For example, if we assume that frequency is proportional to claims, that is, that there is a constant percentage growth rate in claims, and if no claims are settled, then the number of claims will grow exponentially. Accordingly, if $f(.)$ is constant, the number of claims will be a linear function of time.

In Kraus and Ross (1982), $s(t)$ is assumed to be a fixed fraction of the number of claims, i.e., $s(t) = \ n(t)$, where θ is the settlement rate. The function can also be interpreted as a Cobb-Douglas production function, i.e., $s(t) = \theta' e(t)^{\alpha} n(t)^{\beta}$, where $e(t)$ represents effort expended at time t. Hence the claims settlement rate is a function of the number of claims and the effort expended, e.g., in claims investigation or processing. This is a reasonable assumption. In this paper, however, we will assume a more general specification.

Note that in control theory, $n(t)$ and $s(t)$ are known as the state and the control variables, respectively. The state variable describes the position of the system, while the control is the variable that is somehow controlled by the insurer. Intuitively, the claims dynamic equation has the interpretation that the number of unsettled claims grows consistent with a function of claims that are not settled at the end of the previous year, the exposure and time. In practice, the function that determines the growth in unsettled claims should be chosen based on historical experience.

3. Fair premium

There are several definitions and methods for calculating fair premiums in the literature. The following definition is taken from Taylor (1994). Taylor defines fair premium as the sum of the pure premium, expense loading, and a margin to service capital. The fair premium methods are those that quantify the profit margin in such a way as to provide shareholders with a fair return but no more.

Let the fair price for an insurance contract be P_0 and $\delta(t)$ be the nominal interest rate; then we have

$$P_0 = \int_0^T e^{-\delta(t)} L(s,t)\,dt, \qquad (2)$$

where L is the losses and T is the time horizon, which can theoretically be finite or infinite. Note that in this formulation, the discount rate is time dependent. To simplify the subsequent analysis, we will assume a constant discount rate, that is, $\delta(t) = tr$. Hence, the fair premium is the total or sum of the discounted losses. The above is the fair premium formula derived in Kraus and Ross. While the notion of fair premium is plausible, it is still vague and arbitrary. The next section introduces practicality to the concept of fair premiums, while at the same time maintaining the goal of optionality. Shareholders may want the best or highest returns while policyholders want to pay the lowest possible premium rates.

4. Optimum fair premium ($\tilde{P}_0$)-deterministic case

The concept of an optimum fair premium is new in that it incorporates the idea that premiums are optimally set to ensure efficiency from the insurer's perspective. Insurers have an incentive to be cost effective in their business operations. Hence, minimizing discounted losses will ensure that not only are premiums fair but that they are efficient as well. This leads to the problem of determining the minimal value of discounted losses, which we shall denote $\tilde{P}_0$:

$$\tilde{P}_0 = \min_s \int_0^\infty e^{-rt} L(s,t)\,dt. \tag{3}$$

Here the discounted losses are minimized subject to the dynamics imposed by the claims equation (1) and additional constraints due to regulations, etc. For example, Kraus and Ross assumed that the settlement rate is bounded from below by regulation or competition, since the notion of competitive premium is meaningless if insurers are free to make the settlement rate as low as desired. Setting rates such that the premiums are not only fair but also optimal will lead to the maximization of both the policyholders' and insurers/investors' surplus.

The above constitutes an optimization problem in an insurance context (Borch 1990). That is, the determination of fair premium is now cast in an optimal control framework, which is a powerful management tool in economics, management, and finance.

To solve the above management problem, we apply the maximum principles of control theory by first constructing the Hamiltonian function. In terms of the Hamiltonian function H, the problem is formulated as

$$H = e^{-rt} L(s,t) + \lambda \dot{n} = e^{-rt} L(s,t) + \lambda (f - s) \tag{4}$$

where λ, often called the shadow price, is the marginal valuation of the number of claims n discounted to time zero. The Hamiltonian is the sum of the current cash flows and the change in the number of claims multiplied by the shadow price. By modifying the settlement rate, two effects are realized. First, the current cash flows are modified and, second, the stock of claims in the future periods are also affected. This can be written in terms of the current value Hamiltonian as

$$\tilde{H} = L(s,t) + m(f - s), \tag{5}$$

where m is the current value multiplier, that is, the marginal valuation of the stock of claims at time t. This has an interesting economic interpretation. Equation (5) is the imputed price of the cost of settling an additional claim at time t to the insurer. Note that this is not the direct sales value but the imputed value from the future productivity of the asset.

The first order conditions satisfy

$$\frac{\partial \tilde{H}}{\partial s} = L_s - m = 0 \tag{6}$$

and

$$\dot{m} = rm - \frac{\partial \tilde{H}}{\partial n} \Rightarrow rm = \frac{\partial \tilde{H}}{\partial n} + \frac{\partial m}{\partial t}. \tag{6a}$$

The above conditions are the optimality conditions for an interior solution. Note that the condition in (6a) can be interpreted as the asset pricing equation for the price of a marginal unit of the claim inventory. The left-hand side is an opportunity cost, i.e., the discount rate times the price of the asset. The first term on the right-hand side is the dividend of the asset measured by the marginal contribution of the number of claims to the Hamiltonian, and the last term is the capital gain or loss when the price of the asset changes over time. Using equations (4)–(6) it is easy to derive an equation for the dynamics of claims settlement and hence the payout pattern. From the dynamics of the number of unsettled claims and the shadow price of the claims' settlement rate, it is in fact possible to perform an equilibrium analysis of the system.

5. Extension to a stochastic model

This model extends the certainty model for optimal fair premium to incorporate uncertainty. The OFP is found by solving the following intertemporal management problem

$$P_0 = \min_s E\left(\int_0^\infty e^{-rt} L(s,n,t)\,dt \right) \tag{7}$$

subject to the dynamical constraint given by the stochastic differential equation

$$dn = f(n,s,t)dt + \sigma(s,n,t)dz, \tag{8}$$

where $E(.)$ denotes expected value and dz is the increment of a stochastic process z that obeys what is called a Brownian motion or Wiener process and $\sigma 2$ is the variance (O'Brien 1986; Kamien and Schwartz 1991; Tapiero 1982). The Wiener process is independent and identically distributed (iid) with mean zero and variance dt. The dynamics of the unsettled claims are described by the dynamic stochastic differential equation (8). Now, the problem is the minimization of the present value of expected losses subject to the constraints.

Following Kraus and Ross (1982), let $V(n_0,t_0)$ be the value at time t_0 of the remaining cash outflows from claims settlement and $n(t_0) = n_0$. This implies

$$V(n_0,t_0) = \min_s E\left(\int_{t_0}^{\infty} e^{-rt}L(n,s)dt\right), \tag{9}$$

which results in the determination of the OFP as an expected value of the discounted net revenues. Then, the basic optimality condition for the stochastic optimal control problem is given by the Hamilton-Jacobi-Bellman equation

$$V_t + \min_s(L(n,s) + V_n g(n,s) + (1/2)\sigma^2(n,s,t)V_{nn}) = 0 \tag{10}$$

where the zero subscript is suppressed since no confusion is likely to arise. This yields a second order partial differential equation in n. The derivation of this equation requires the application of Ito's stochastic calculus (Kamien and Schwartz 1991). Note that the above reduces to the deterministic formulation if the variance term is zero.

6. Example of competitive market pricing under certainty

To elicit the applicability of this model in the policy decision-making process, we study the behavior of insurance markets in a deterministic setting. As in the previous section the dynamics of the unsettled claims are given by an equation of the type

$$\dot{n} = f(n) - s \tag{11}$$

where $f(.)$ is the addition to the stock, which for simplicity is assumed to be a function of the current number of claims and s the number of claims settled per unit of time.

For this example, let us assume that the firm's profits are given by the premiums less the undiscounted losses. Then the management problem will be the maximization of the present value of the profits, i.e.,

$$\max_s \int_0^{\infty} [q - c(n)]se^{-rt}dt \tag{12}$$

subject to the dynamics of the claims and a minimum settlement rate that is required by regulation. The discount rate r is assumed to be equal to the market rate of interest. The current or undiscounted shadow price associated with a unit of unsettled claims is given by

$$m\begin{cases} < q - c(n), & s > s^* \\ = q - c(n), & s = s^* \\ > q - c(n), & s < s^* \end{cases} \tag{13}$$

where s^* is the equilibrium rate of claims settlement (the rate at which claims settlement is somewhat steady). The above gives the so-called Bang-Bang solution in control theory. It is obvious that the optimal solution is the corner solution, i.e., to delay payment as long as possible. We will refer to such a solution as the "ideal" solution for the insurer. However, reality will not permit this solution because of competition, regulations, and probably potential lawsuits. It may be assumed that, all things being equal, the equilibrium level of the settlement rate will prevail. Thus, we will assume that $s = s^*$ in the remainder of this section. This yields an equation for the dynamics of price written in terms of the shadow price of coverage. Note that, for a free end point problem and a time horizon T as in equation (2), the boundary condition is given by $m(T) = 0$. This implies that the price of a claim at the end of the time horizon is zero. Using the maximum principles, it can be shown that the imputed price evolves according to the differential equation

$$\dot{m} = rm - f'(n)[q - c(n)] + c'(n)s. \qquad (14)$$

This implies that the rate of capital gain on one unit of unsettled claims must equal the total cost of insuring the unit, i.e., the opportunity cost of foregone interest plus the loss due to increase in the total stock growth rate attributable to that unit plus the change in total settlement cost resulting from that unit. A simple algebraic manipulation of equation (14) results in an expression for the rate of return condition:

$$r = \frac{\dot{m}}{m} + f'(n) - \frac{c'(n)s}{m}. \qquad (15)$$

That is, the total return associated with assuming the risk of a unit of stock of unsettled claims must equal the market rate of interest. Rewriting this in terms of price inflation, i.e., rate of change in price, we have

$$\dot{q} = r(q - c(n)) - f'(n)(p - c(n)) + c'(n)f(n). \qquad (16)$$

This equation describes the price dynamics for competitive insurance. We see that the price of the insurance contract grows at a rate equal to the discount rate.

6.1. The competitive market equilibrium price

To derive the competitive optimum equilibrium market price, a steady-state equilibrium is defined as $\dot{q} = \dot{n} = 0$, which implies that

$$q^* = c(n) + \frac{-c'(n)f(n)}{r - f'(n)}. \qquad (17)$$

That is, the price is equal to the marginal coverage cost plus an economic rent or a profit margin. Since $f(n) = s$ if $\dot{n} = 0$, the competitive equilibrium rent is the capitalized value of the future increases in the cost resulting from a unit increase in the stock of claims. Hence, the capitalization is based on a rate that reflects the opportunity cost of the market interest rate less the contribution of the unit to the stock growth. Note that $r - f'(n)$ is the net interest rate, i.e., the internal rate of return for the insurer or investor. This means that increases in claims will lead to a reduction in total returns.

6.2. Example using the uncertainty case

To make the previous example more realistic, we look at the stochastic version of the certainty case. As in the previous section, the dynamics of unsettled claims are given by an equation of the type

$$dn = (f(n) - s)dt + \sigma(n)dz, \qquad (18)$$

where $\sigma(n)$ is the diffusion term, which is zero in the deterministic case. Again, dz is the increment of a Wiener process. The dynamics of claims are assumed to be in part deterministic and in part random.

Let us assume that the firm's profits are given by the premiums minus the undiscounted losses. Then the management problem will be the maximization of the present value of the profits, i.e., the problem will be to find $v(n_0 t_0)$, where

$$V(n_0, t_0) = \max_s E\left(\int_{t_0}^{\infty} [q - c(n)]se^{-r(t-t_0)}dt\right) \qquad (19)$$

subject to the dynamics of the claims and a minimum settlement rate that is required by regulation. Here, t_0 is the time at which the policy is issued and $V(n_0, t_0)$ is the OFP. Assume also that the profit is given by $[p(s) - c(n)]s$ in this example. Then the above results in the fundamental equation of optimality (subscripts are again suppressed):

$$rV = \max_s[(q - c(n))s + V_n(n)(f(n) - s) + \tfrac{1}{2}\sigma^2 V_{nn}(n)]. \quad (20)$$

Maximizing the square bracketed term in (20) with respect to s and ignoring the term $[q'(s)s]$ we have

$$V_n = q - c(n). \quad (21)$$

Here, V_n is the rent and can be interpreted as the social and market value of a marginal unit of the stock of claims. It is the profit that can be obtained by insuring the unit. Transforming equation (21) gives $s^*(n) = q^{-1}[V_n + c(n)]$, which implies that

$$rV = \int_0^{s^*(n)} q(s)ds - c(n)s^*(n) + [f(n) - s^*(n)]V_n + \tfrac{1}{2}\sigma^2(n)V_{nn}. \quad (22)$$

Differentiating with respect to n gives

$$rV_n = [p - c(n) - V_n]s_n^* - c'(n)s^* + f'(n)V_n + \sigma'(n)\sigma(n)V_{nn} + [f(n) - s^*]V_{nn} + \tfrac{1}{2}\sigma^2(n)V_{nnn}. \quad (23)$$

The first term in the square brackets in (23) is zero, hence the equation reduces to

$$rV_n = -c'(n)s^* + f'(n)V_n + \sigma'(n)\sigma(n)V_{nn} + (1/dt)E_t d(V_n). \quad (24)$$

Also, equation (21) leads to

$$(1/dt)E(V_n) = (1/dt)E_t(d[p - c(n)]). \quad (25)$$

Using equations (21), (24), and (25) and rearranging yields an explicit condition for the rate of return analogous to the case of certainty:

$$\frac{(1/dt)E_t d(p-c)}{(p-c)} - f'(n) - \frac{c'(n)s^*}{(p-c)} = r + \sigma'(n)\sigma(n)(-V_{nn}/V_n), \quad (26)$$

where $(-V_{nn}/V_n)$ can be referred to us the coefficient (Arrow-Pratt measure) of absolute risk aversion (Silberberg 1990). It reflects in this case premium that policyholders must pay due to uncertainty in the growth of stock of claims. Since $\sigma'(n)$ is positive, the rate of return condition in (26) is the deterministic rate augmented by an "uncertainty premium load" equal to the increase in stock growth variance due to the unit multiplied by the coefficient of implicit risk aversion. The higher the coefficient of absolute risk aversion, the higher the risk premium. That is, more risk-averse insurers will demand higher returns, all else being equal.

7. Summary and conclusion

This paper has used a new technique to analyze rate regulation in an insurance context. A new dimension has been added to the debate on rate regulation and fair pricing in insurance ratemaking. The continuous dynamic discounted cash flow models introduced in this paper are the counterparts of the discrete DCF models such as the Myers-Cohn model used by the Massachusetts Worker Compensation Rating Bureau. They can therefore be used to supplement the discrete models and to shed more light on issues where the DCF models are limited.

The analytical results can be very useful in policymaking and can thus be used in the insurance rate-setting process. The concept of fair premiums is good but it is too broad, subjective, and, at best, vague. If achievable, optimum fair premiums are the ideal prices for insurance contracts. The models presented here can be extended in various ways to include several realistic features of insurance pricing, such as by extension to multivariable models. One important extension would be to model separately the capital dynamics and investment income. Further extension may include the idea of salvage value for a written policy to formulate the problem as dynamic differential games. These are worth pursuing in future research.

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