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An Incremental Approach to Estimating Ultimate Claim Counts and Future Claim Payments

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Abstract

Traditional chain ladder development methods to estimate ultimate claim counts are challenged when the insurer experiences shifts in the speed of claim closure and/or the proportion of claims closed with payment. Similarly, changes in these metrics can impact the estimation of ultimate losses when claims that linger open longer cost more to close. This paper presents an approach to estimate ultimate claim counts by allowing the actuary to explicitly estimate incremental closure rates by age and subsequently apply an inflationary assumption to estimate ultimate losses. The method may also be used as a tool to provide insight into how a change in the speed of claim closures, a shift in the proportion of claims closed with payment and changing inflationary factors may impact the ultimate losses.

Keywords: reserving; internal operational changes; external environmental changes; inflation; claim counts.

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1. INTRODUCTION

Actuaries spend much of their time performing calculations to project future activity. These calculations range from simple to complex, and in some cases may be difficult to explain to stakeholders who are not trained in technical actuarial approaches. When the operating environment changes and the assumptions underlying many traditional techniques are violated, the actuary's job becomes even more challenging.

This paper presents an approach that allows the actuary to infuse judgment into the process of estimating both ultimate claim counts and future claim payments, in a manner that is easily relatable to non-technical audiences. The "incremental method" that we present allows the actuary to break claim closures into component pieces for claims closed both with and without payment. This approach allows the actuary to examine and separately project not only the overall speed of claim closures, but also the proportion of claims that will close with payment in any given time interval. The actuary may then apply an inflationary assumption to future claim payments, with a key underlying premise: Claims that close later tend to cost more to close.

As we write this paper in 2020, the insurance industry is operating in unprecedented times. There are several phenomena that may impact insurance operations, and in turn lead to calls for actuaries to help management and clients understand the impact of these phenomena on ultimate claims payments. The incremental method is useful to project future payments in which the actuary can explicitly make assumptions related to the impact of changes in claim closure rates, shifts in the proportion of claims closed with payment, and variation in future inflationary assumptions. The method may also be used as a diagnostic tool to examine the presence of these metrics in historical data. Scenarios in which this approach may be useful include:

- Social inflation. There has been extensive discussion of this issue, which is credited with leading to increased attorney involvement in claims, more plaintiff-friendly juries, and larger settlements.[1] As social inflation may lead to claims staying open longer, higher severities for indemnity and defense costs, and/or a higher proportion of claims closed with payment, how can the actuary reflect these changes in the unpaid claims analysis?
- Advancements in technology. We have seen the insurance industry embrace technology, particularly as it relates to claims investigation and settlement.[2] Where an auto insurance claim previously required visual inspection, a picture submitted through an app is now common practice. Roof damage that led to an inspector on a ladder has been replaced with inspection by drone. As advances in technology may lead to a faster rate of claims settlement, how can the actuary quantify this impact?
- COVID-19. It remains to be seen what the long-term effects of COVID-19 will be on the insurance industry and the broader economy. But even a few months into the pandemic, we know that there have already been impacts on claims operations as insurers shift to a work-from-home model, inspectors may be limited in their ability to access accident sites, and court closures delay litigation proceedings.[3] To the extent the pandemic drives a slowdown in claim closures, the actuary will need to consider how this impacts the projection of unpaid claims.

The phenomena discussed above are broad, industry-wide events. These phenomena aside, individual insurers may experience internal operational changes, such as a slowdown in claim closures due to turnover in the claims department or a lower proportion of claims closed with payment due to a more aggressive defense strategy. A more active pursuit of claim settlements may also impact the speed with which claims are closed and the ultimate value paid on those claims.

All the aforementioned examples present challenges to one of the key assumptions underlying traditional development methods: that the historical development is predictive of future

[1] https://www.insurance-research.org/sites/default/files/news_releases/IRCSocialInflation2020.pdf
[2] https://www2.deloitte.com/content/dam/Deloitte/us/Documents/human-capital/us-future-of-claims-pov.pdf
[3] https://www2.deloitte.com/content/dam/Deloitte/us/Documents/financial-services/us-COVID-19-impact-property-casualty-insurance.pdf

development. The incremental method presents a framework in which the actuary can examine the impact of these changes on projecting unpaid claims estimates. This framework can be applied in many contexts for property and casualty insurance claims, including an insurer, self-insured, or portfolio of third-party administrator's claims.

The remainder of the paper proceeds as follows:

Section 2 will discuss the mechanics of the incremental method, and then present constructed scenarios in which the incremental method is compared to traditional actuarial methods. An additional scenario will be presented with private passenger auto liability data as an exercise in applying the incremental method in practice. Finally, we discuss considerations in projecting future severities.

Section 3 discusses potential uses of the incremental method, as well as limitations of the method.

2. BACKGROUND AND METHODS

We begin this section by discussing the mechanics of the incremental method. We then present three scenarios in which the incremental method is compared to traditional actuarial methods. The methods included are the reported loss development method, the paid loss development method, and the disposal rate frequency-severity method. Readers unfamiliar with these approaches may refer to "Estimating Unpaid Claims Using Basic Techniques" by Jacqueline Friedland.[4] A final scenario will be presented with private passenger auto liability data as an exercise in applying the incremental method in practice.

2.1 Mechanics of the Incremental Method

The incremental method can be classified as a frequency-severity approach, as claim counts and severities are estimated separately and then multiplied together to arrive at an estimate of ultimate losses. This section lays out the steps to perform the incremental method, applied to a simplified "base case" data set where we have defined development factors and trend rates, as well as the ultimate counts and ultimate losses. The data and diagnostics are displayed on Appendix 1, Exhibits 1-2. We note that calculations may not tie exactly due to additional decimals in the spreadsheet vs.

[4] https://www.casact.org/library/studynotes/Friedland_estimating.pdf

the sample calculations. To provide additional precision in the sample calculations, we show claim counts to the first decimal, acknowledging that in practice, claim counts will be whole numbers.

2.1.1 Definitions

We define the following data elements for use in performing the incremental method to estimate ultimate loss. In this context, we assume that "loss" relates to indemnity payments and that defense and cost containment payments are estimated separately.

- Open counts: claim counts that have been reported to the insurer and are open pending future activity at a given point in time.
- Counts closed with payment: claim counts that have been closed with a loss payment.
- Counts closed without payment: claim counts that have been closed without a loss payment.
- Closed counts: all claim counts that have been closed. Closed counts = closed with payment counts + closed without payment counts.
- Reported counts: all claim counts reported to the insurer. Reported counts = open counts + closed with payment counts + closed without payment counts.
- Non-zero counts: all claim counts reported to the insurer, <u>excluding</u> counts closed without payment. Non-zero counts = open counts + closed with payment counts. Some companies may refer to this as incurred counts.
- Active counts: the claims available to close in a given period
- Ultimate counts: the ultimate number of claim counts closed with payment.
- Paid severity: claim payments divided by counts closed with payment.

All triangles are presented on an accident year basis, with annual evaluations beginning at 12 months maturity.

2.1.2 Projection of Ultimate Claim Counts

In many cases, the actuary can project ultimate claim counts by utilizing a development method applied to non-zero counts. This approach assumes that historical development is predictive of future development. To the extent there have been changes in claim closure rates and/or shifts in the proportion of claims closed with payment, the results of a straightforward development method may be distorted. The approach below requires more intermediate steps than a development method, but it allows the actuary to incorporate the impact of such changes.

The steps for projecting ultimate claim counts are discussed below.

Step 1: Calculate incremental count triangles for reported counts, closed with payment counts, closed without payment counts, and closed counts. The incremental count is the change in the cumulative count triangle from one age to the next. The incremental count triangles are displayed on Appendix 1, Exhibits 3-6.

Step 2: Calculate a triangle of active counts for each incremental period. Active counts for an incremental period are defined as the counts available to close in that period. They are calculated as the counts that were open at the beginning of the period plus newly reported counts during the period. The active count triangle and sample calculations are displayed on Appendix 1, Exhibit 7.

Step 3: Project future reported counts. This step is accomplished through application of the traditional development technique, where the age-to-age factors are utilized to "square the triangle." The underlying assumption is that while there may be changes to the closure rate or proportion of counts closed with payment, the overall reporting pattern of claims to the insurer is relatively stable. The projection of future reported counts is displayed on Appendix 1, Exhibits 8-9.

Step 4: Select the incremental closure rate at each age. The incremental closure rate is defined as the percentage of claims closed within a period of the total claims available to close in that period. It is calculated as the counts closed in a period divided by the active counts for that period. In this simplified dataset, there is no variation in the data by accident year, so our selection is equal to the historical data. We note that the "tail" closure rate should be 100%, as all counts must ultimately close. The incremental closure rate triangle and sample calculations are displayed on Appendix 1, Exhibit 10.

Step 5: Select the incremental closed with payment rate at each age. The incremental closed with payment rate is defined as the percentage of claims closed with payment within a period of the total claims available to close in that period. It is calculated as the counts closed in a period with payment divided by the active counts for that period. Similar to step 4, there is no variation in this data, so our selection is equal to the historical data. The tail selection for this step should equal the actuary's best estimate of the proportion of counts that will close with payment after the last point in the triangle. In this simplified data set where we know by definition what the ultimate counts are, we set the tail equal to our known closed with payment rate for 120 months to ultimate. Section 2.4 will discuss how the tail may be selected in a more realistic data set. The incremental closed with payment rate triangle and sample calculations are displayed on Appendix 1, Exhibit 11.

Step 6: Calculate the implied selected incremental closed without payment rate at each age. The incremental closed without payment rate is defined as the percentage of claims closed without payment in a period of the total claims available to close in that period. It is calculated as the counts closed in a period without payment divided by the active counts for that period. It can also be calculated as the incremental closure rate less the incremental closed with payment rate at each age. In this simplified dataset, the implied selection is equal to the historical data. When working with actual data, the actuary should perform this step to see how the implied pattern aligns to the actual closed without payment rates.[5] The incremental closed without payment rate triangle is displayed on Appendix 1, Exhibit 12.

Step 7: Project future active counts for the next diagonal. To project the next diagonal of active counts, start with open counts at the beginning of the incremental period and add projected newly reported counts (from step 3). The projected active count triangle and sample calculations are displayed on Appendix 1, Exhibit 13.

Step 8: Project future closed with payment counts for the next diagonal. To project the next diagonal of counts closed with payment, multiply the selected incremental closed with payment rate by the active counts for each incremental period. The projected closed with payment triangle and sample calculations are displayed on Appendix 1, Exhibit 14.

Step 9: Project future closed without payment counts for the next diagonal. To project the next diagonal of counts closed without payment, multiply the selected incremental closed without payment rate by the active counts for each incremental period. The projected closed without payment triangle and sample calculations are displayed on Appendix 1, Exhibit 15.

Step 10: Project future open counts for the next diagonal. To project the next diagonal of open counts, start with open counts in the prior diagonal, add newly reported counts (from step 3), subtract incremental counts closed with payment (from step 8), and subtract incremental counts closed without payment (from step 9). The projected open count triangle and sample calculations are displayed on Appendix 1, Exhibit 16.

Steps 7 through 10 are repeated for each subsequent diagonal until the "square" is completed to

[5] This check is conceptually similar to calculating a ground-up trend and a primary limits trend, then examining the implied excess trend for reasonability.

ultimate. The completed triangles are displayed on Appendix 1, Exhibit 17. The sum of the incremental closed with payment triangle represents the projected ultimate claim counts from the incremental method.

2.1.3 Projection of Future Severities

For exposures where claim severity does not vary by age, the actuary could estimate ultimate losses by taking the ultimate claim counts from the results above and multiply by an ultimate severity. However, many property and casualty exposures experience a paid severity that increases over time, where smaller, simpler claims are closed at earlier ages, and larger, more complex claims are closed at later ages. To incorporate this observation in the method, we project future paid severities by age and calendar period.

Step 11: Calculate incremental paid severities. Incremental paid severities are calculated as incremental paid losses divided by incremental closed with payment counts. The incremental paid severity triangle and sample calculations are displayed on Appendix 1, Exhibit 18.

Step 12: Trend incremental paid severities to current calendar year dollars. In this simplified example, we use a single trend rate of 4% per year applied to all historical periods. If desired, the actuary can vary the prior calendar period trend rates. The trended incremental paid severity triangle and sample calculations are displayed on Appendix 1, Exhibit 19.

Step 13: Select an incremental paid severity at each age in current calendar year dollars. There is no variation in this simplified data, so our selection is equal to the historical trended data. The tail selection for this step should equal the actuary's best estimate of the severity for counts that will close with payment after the last point in the triangle. In this simplified data set, we know by definition what the ultimate counts and ultimate losses are; consequently, we set the tail equal to our known paid severity for 120 months to ultimate. Section 2.4 will discuss how the tail may be selected in a more realistic data set. The trended incremental paid severity triangle and selections are displayed on Appendix 1, Exhibit 20.

Step 14: Trend selected incremental severities to future calendar years. Our simplified example again uses a trend rate of 4% per year applied to all future periods. If desired, the actuary can vary the future calendar period trend rates. The trended incremental paid severities and sample calculations are displayed on Appendix 1, Exhibit 21.

2.1.4 Projection of Ultimate Losses

Step 15: To determine ultimate losses, the incremental counts closed with payment are multiplied by the incremental severities, resulting in incremental payments for each period. The sum of the incremental payments across the periods for each accident year result in the estimate of ultimate losses for that accident year. The incremental paid losses, ultimate losses and sample calculations are displayed on Appendix 1, Exhibit 22.

2.2 Scenario 1: Base Case

Now that we have established the mechanics of the incremental method, we will compare the results to traditional methods in a few scenarios. The first scenario is a base case scenario using the simplified data presented in the previous section.

2.2.1 Scenario Description

The simplified data used in this scenario uses the same claim counts in each accident year, with a 4% annual trend in the paid severities. Given the structure of the data, we "know" the value of the ultimate counts and ultimate losses.

2.2.2 Results of Traditional Methods

Appendix 1, Exhibit 23 displays the results of the reported loss development method.

Appendix 1, Exhibit 24 displays the results of the paid loss development method.

Appendix 1, Exhibits 25-27 display the results of the disposal rate frequency-severity method with the same calendar year severities described in section 2.1.3.

We see that with this simplified dataset, the result of all methods is identical to our defined ultimate counts and ultimate losses.

2.2.3 Results of the Incremental Method

Appendix 1, Exhibit 22 displays the results of the incremental method. We see that the result is equal to the other methods for this simplified dataset. We have included this scenario to demonstrate that in a scenario with data structured such that we "know" the value of the ultimate counts and ultimate losses, the mechanics of the incremental method produce a result that is equivalent to traditional approaches.

2.3 Scenario 2: Shift in the proportion of claims closed with payment

In Scenario 2, we take the original dataset and modify the latest diagonal to represent an increase in the proportion of claims closed with payment. While still a simplified dataset, this scenario may be representative of the impacts of a phenomenon such as social inflation, where increased attorney involvement may lead to more claims closing with payment.

2.3.1 Scenario Description

The data and diagnostics for this scenario are displayed on Appendix 2, Exhibits 1-2. The assumptions we make are as follows:

- Total closure rates do not change from the base case. The same number of counts close at each point in time, but 10% more close with payment at each age beginning with calendar year 2020. This is a permanent increase that remains in future calendar years.
- Incremental paid severities do not change from the base case. The revised paid loss triangle is calculated by applying the base case incremental paid severities to the revised closed with payment count triangle. We acknowledge that in practice, an increase in the proportion of claims closed with payment could impact incremental paid severities. The actuary should examine how changes in the proportion of claims closed with payment impact incremental paid severities and consider this in the selection of the incremental paid severity for each age. For example, if the shift in proportion of claims closed with payment has been observed in the most recent three to four calendar years, a severity selection incorporating this data may be most appropriate to reflect the expected severity in the new environment.
- Case reserves do not change from the base case. The number of open counts is the same at each point in the triangle. The revised reported loss triangle is calculated by applying the base case average case reserves per open claim to the revised open claim count triangle and adding this to the revised paid loss triangle. Similar to the above, we acknowledge that in practice, an increase in the proportion of claims closed with payment could impact case reserves. While case reserves are not directly utilized in the incremental method, the actuary should examine how average case reserve levels may impact the results of other methods.
- For simplicity, tail factors are unchanged from the base scenario. In practice, the actuary should consider how changes in operations may impact tails.

2.3.2 Results of Traditional Methods

We examine the results of traditional methods in this scenario. As has been established in actuarial literature, a key underlying assumption of these methods is that the historical development is representative of future development. The shift in the proportion of counts closed with payment violates this assumption.

The reported loss development method is displayed on Appendix 2, Exhibit 3. There is an increase in the latest diagonal where the additional counts are closed with payment. We know that due to the multiplicative nature of this method, utilizing the latest diagonal as the "new normal" will overstate the ultimate losses. Conversely, using the prior development factors may not correctly project the ultimate losses as the prior factors do not account for the increased proportion of counts closing with payment in future development periods.

The paid loss development method is displayed on Appendix 2, Exhibit 4. Similar to the reported loss development method, the increase in the latest diagonal will distort the results of this method, overstating the ultimate losses.

The disposal rate frequency-severity method is displayed on Appendix 2, Exhibits 5-8. Estimating ultimate counts closed with payment using a non-zero count triangle will overstate ultimate counts, similar to the loss development methods. A similar issue would exist if ultimate counts were estimated using closed with payment counts. The severity portion of the method is not impacted; where paid severities are unchanged, the projected severities remain the same as the base case. Multiplying overstated claim counts by "correct" severities results in an overstated ultimate loss.

2.3.3 Results of the Incremental Method

The incremental method is displayed on Appendix 2, Exhibits 9-15. The incremental closed with payment closure rate is 10% higher in the current diagonal, offset by a decrease in the closed without payment closure rates. Selecting this diagonal as the "new normal" projects future closed with payment counts that are 10% higher than the base case. Applying the same approach to project paid severities results in paid severities that remain the same as the base case (shown in Appendix 2, Exhibit 7). A comparison to the base case projected incremental counts is shown on Appendix 2, Exhibit 16. The resulting ultimate losses incorporate the higher proportion of counts closed with payment without overstating the ultimate claim counts.

2.4 Scenario 3: Shift in the rate of claim closures

In Scenario 3, we take the original dataset and modify the latest diagonal to represent a slowdown in closures for counts closed both with and without payment. This scenario may be representative of the impacts of a phenomenon such as COVID-19, where operational limitations may lead to a slowdown in claim closures. Alternatively, this scenario may be representative of an insurer who has experienced a change in staffing in the claims department or has taken a more aggressive approach in defending claims.

2.4.1 Scenario Description

The data and diagnostics for this scenario are displayed on Appendix 3, Exhibits 1-2. The assumptions we make are as follows:

- Total closure rates decline to 85% of the base case in the 2020 diagonal through 60 months. The slowdown impacts both closed with payment counts and closed without payment counts. We assume that the total proportion of claims that will ultimately close with payment is unchanged (i.e., the inherent nature of if a claim has merit is unchanged). However, some of these claims will now close later than they historically would have due to the slowdown in calendar year 2020.
- Incremental paid severities do not change from the base case. The revised paid loss triangle is calculated by applying the base case incremental paid severities to the revised closed with payment count triangle. As in the prior scenario, we acknowledge that in practice, incremental paid severities may change in this scenario, and the actuary should examine such impacts in making selections.
- Average case per open claim does not change from the base case. The revised reported loss triangle is calculated by applying the base case average case reserves to the revised open claim count triangle and adding this to the revised paid loss triangle. Similar to the prior scenario, in practice, the actuary should examine average case reserves to see if there has been any impact from the shift in the rate of claim closures.
- For simplicity, tail factors are unchanged from the base scenario. In practice, the actuary should consider how changes in operations may impact tails.

2.4.2 Results of Traditional Methods

We examine the results of traditional methods in this scenario. As has been established in actuarial literature, a key underlying assumption of these methods is that the historical development is representative of future development. The slowdown in claim closures violates this assumption.

The reported loss development method is displayed on Appendix 3, Exhibit 3. The latest diagonal is distorted compared to the base case, with the 12-month point increasing due to the additional volume of open counts, and subsequent points decreasing due to the lower rate of claim closures. Due to the multiplicative nature of this method, applying historic loss development factors to the latest diagonal will distort the ultimate losses.

The paid loss development method is displayed on Appendix 3, Exhibit 4. The latest diagonal is understated given the slowdown in claim closures, such that applying historic loss development factors to the latest diagonal will understate the ultimate losses.

The disposal rate frequency-severity method is displayed on Appendix 3, Exhibits 5-8. The non-zero count triangle is distorted due to the additional volume of open counts. Estimating ultimate counts using the non-zero count triangle will overstate ultimate counts closed with payment. This occurs because the latest diagonal of the non-zero count triangle contains open counts that would previously have closed without payment and left the non-zero count triangle under claims operations prior to the slowdown; the historic development factors do not account for these claims now leaving the triangle at a later age. Conversely, estimating ultimate counts using the closed with payment count triangle will understate ultimate claim counts due to the slowdown. The distorted claim counts will lead to distorted ultimate losses.

2.4.3 Results of the Incremental Method

The incremental method is displayed on Appendix 3, Exhibits 9-15. The incremental closed with payment closure rate and incremental closed without payment closure rates have both declined from prior diagonals. In this scenario, the actuary must use his or her knowledge of claims operations to determine an appropriate selection. For example, if the actuary believes that claim closure rates will return to normal after a one-time slowdown in calendar year 2020, the actuary may select the historic diagonals, as displayed in this example. If the slowdown is expected to continue, the actuary may select the closure rates in the current diagonal. We recommend that the actuary conduct discussions with the claims department to make these assumptions based on informed judgment. Applying the same approach to project paid severities results in paid severities that remain the same as the base case (shown in Appendix 3, Exhibit 7).

A comparison to the base case projected incremental counts is shown on Appendix 3, Exhibit 16.

In this example, we see that the incremental method produces projected counts that increase over the base case to "make up" for the calendar year 2020 slowdown. Whether the historic closure rates are appropriate to project future closure rates will depend on the driver of the slowdown and how future operations are expected to be impacted; in other words, the actuary must understand the implications of his or her selections on the projected incremental counts.

2.4.4 Comparison to the Berquist-Sherman Approach

Readers may note that there is already a widely accepted actuarial approach to adjust for changes in claim closure rates: the paid Berquist-Sherman method. We note that to apply the paid Berquist-Sherman method, the actuary must first estimate ultimate claim counts closed with payment, prior to adjusting the closed claim count triangle for the change in claim disposal rates. In a scenario where the non-zero count triangle is not distorted, the actuary may estimate the ultimate counts by applying the development method to the non-zero claim count triangle, and then proceed with the paid Berquist-Sherman method. However, in the scenario presented here, the slowdown in claim closures distorts all count triangles other than the reported claim count triangle, such that the actuary may not have a reliable approach to estimate ultimate claim counts closed with payment. The incremental method provides a framework in which the actuary can estimate ultimate claim counts without needing to apply a traditional development approach to the non-zero count triangle.

Additionally, the paid Berquist-Sherman method typically relies upon the selection of a cumulative disposal rate at each age to "square the triangle" of counts closed with payment. It is assumed that the selected cumulative disposal rates are appropriate for all future periods at each age. While it is possible for the actuary to incorporate anticipated changes in future incremental closure rates and/or the proportion of claims closed with payment into the paid Berquist-Sherman approach by calculating the impact of these assumptions on future cumulative disposal rates, the incremental method allows the actuary to approach these assumptions more directly.

Finally, the paid Berquist-Sherman approach utilizes regression between the successive pairs of cumulative paid losses and cumulative closed with payment claims to "restate" the paid loss triangle. While this approach is understood by actuaries who study such methods as part of the credentialing process, it may be difficult to explain to a non-technical audience. The approach to estimating incremental paid severities discussed here may be easier to demonstrate and explain.

2.5 Scenario 4: Private Passenger Auto Data

Scenario 4 presents auto liability data from a private passenger auto insurer. The data and diagnostics are displayed on Appendix 4, Exhibits 1-2.

2.5.1 Scenario Diagnostics

Examining the diagnostics for this data shows:

- There does not appear to be a clear trend in the total incremental closure rates.
- There appears to be a shift to close more claims with payment, particularly at the earlier maturities.
- There appears to be a positive trend in the paid severities.

2.5.2 Selecting Incremental Assumptions

The mechanics of the incremental method operate as presented in section 2.1, and are displayed on Appendix 4, Exhibits 3-10. We select 3-year weighted averages for the incremental selections to reflect recent claims operations.

2.5.3 Selecting the Tail

The data becomes very volatile by 72 months. Rather than continuing to select incremental closure rates, we select a tail.

For the incremental closure rates, we examine the data to estimate approximately how many counts will close at each incremental age. A selection of 50% seems representative of the data observed after 72 months. The tail closure rate for 120 months to ultimate should be 100%, as all counts must ultimately close.

For the incremental closed with payment closure rates, we examine "tail" closure rates at each age by dividing total counts closed with payment by total counts closed for all subsequent maturities. We elect to place the tail at 72 months, where the data becomes thin and volatile. However, the tail selected here includes all future counts closed with payment. To project incremental closed with payment closure rates, we multiply our selected tail by the selected incremental closure rate for each incremental period.

In this example, although we expect 69% of counts that close to close with payment, we need to reflect that at the incremental ages, only about 50% of active counts are closing each period.

Therefore, at the incremental ages, we close 50% of our active counts, 69% of those with payment. 69% * 50% = 35% of active counts close with payment at each incremental age.

The distribution of closures to incremental period matters because we believe that paid severity varies by age. If we were solely trying to project ultimate claim counts, the actuary could simply calculate the tail at 72 months as 69% of remaining active counts. However, because we want to project ultimate losses by applying incremental severities that vary by age, we must project when those counts close.

We may calculate a tail for the severities in a similar manner, electing to place the tail at 72 months.

2.5.4 Results of the Incremental Method

The results of the incremental method are displayed on Appendix 4, Exhibit 10.

2.6 Considerations in Projecting Future Severities

We have discussed that we trend severities to calendar year because claims that linger open longer tend to cost more to close. The flexibility of projecting claim closures by calendar year allows the actuary to easily modify trend rates by calendar year, both historically and prospectively.

However, what should the actuary do if there is a significant impact on the accident year severities, due to a change to reinsurance terms or tort reform with an accident year impact? One option is to run the method on data gross of reinsurance and separately estimate the impact of reinsurance, such as by applying a net-to-gross ratio for each accident period. This may be most appropriate where the impact of reinsurance is relatively small.

If the actuary wishes to examine the impact on net data directly, the actuary can run the severity portion of the method twice. For example, if the actuary expects a significant change in severities beginning in accident year 2019 due to a change in reinsurance terms, the actuary may run the severity triangle twice – once at the pre-2019 reinsurance levels, and once at the 2019 reinsurance levels. The severity triangle at the 2019 reinsurance levels can be estimated by applying the 2019 reinsurance terms to historical data as if those terms had been in place in the prior years. When multiplying projected future severities by the projected future closed with payment counts, the severities for accident years 2018 and prior would be taken from the first triangle, while the severities for accident years 2019-2020 would be taken from the second triangle. This allows the actuary to incorporate accident year impacts on severities while also accounting for calendar year trends.

3. RESULTS AND DISCUSSION

It is important to understand that, like all actuarial methods, the incremental method does not automatically result in the "right" answer for the actuary. It is the actuary's responsibility to understand the insurer's historic operations and anticipated future operations and make selections accordingly. We believe that an advantage of the incremental method is that the nature of relating the incremental closure rates to incremental active claims allows the actuary to explicitly project future closure rates by calendar year based on information from the claims department. For example, in the scenario where there is a slowdown in claim closure rates, the actuary may use his or her knowledge from claims department management to modify the future closure rates by calendar year if it may take multiple years for the closure rates to return to "normal".

3.1 Potential Uses of the Incremental Method

The incremental method may be used as a diagnostic tool along with other methods to estimate ultimate claim counts and ultimate losses. The flexible nature of the assumptions by calendar year allows the actuary to make explicit assumptions about how many claims will close each period, how many of those claims will close with payment, and future inflationary levels. Given this framework, the incremental method may also be used as a sensitivity test to understand the impact of changing these metrics on ultimate loss estimates.

The mechanics of the incremental method also make it easy to describe to a non-technical audience. The major inputs to the method require the actuary to understand and explain assumptions around how fast claims close, what portion of those claims close with payment, and what inflationary impacts will be. The calendar year nature of these assumptions can be more relatable than the traditional age-to-ultimate concept. This may make the method a useful tool for the claims department or a third-party administrator to measure the potential impacts of different claims handling strategies.

Finally, given the incremental nature of the method, it may be helpful in measuring actual versus expected emergence for subsequent calendar periods.

3.2 Limitations of the Incremental Method

As with all approaches for estimating ultimate claim counts and ultimate losses, the actuary must understand the limitations of any method he or she applies. We include limitations of the incremental method below:

- The method relies on a consistent definition of claim counts. If the insurer changes how claims are recorded, historical data may need to be restated to the new definition of claim counts for the method to work.
- If the insurer has made significant changes to the book of business being analyzed that are not yet reflected in the historical data (such as an acquisition or entering a new market), the historic incremental closure rates may not be reflective of future closure rates. The actuary may be able to adjust for this through inquiry with management.
- The assumption that claims that linger open longer cost more to close makes the method very sensitive to the trend assumption. We recommend that the actuary understand the impact of changing the trend assumption on the results of the method.
- The method is sensitive to where the tail is placed and the volume of data underlying the tail selection. We recommend that the actuary understand the impact of changing the tail assumption on the results of the method.
- Given the manner in which paid severities are calculated, the estimation of severities is closely tied to claim closures. Therefore, the severity portion of the method generally will not work well on exposures with significant partial payments, although it can still be used to examine claim closure activity and estimate ultimate claim counts. This may be a consideration for coverages such as workers compensation or liability coverages with significant defense and cost containment payments where indemnity and defense data are not examined separately. When determining if the method may be appropriate for such exposures, the actuary should consider the materiality of the incremental payments in relation to total payments, as well as how closely the incremental payments relate to claim closures.
    - For example, the incremental method may be more useful for a portfolio of workers compensation claims in which many claims are settled with a lump sum payment than a long-tailed workers compensation book with decades of incremental payments.

- Similarly, examining indemnity and defense data on a combined basis with this method may be more appropriate if incremental defense payments occurring prior to claim closure do not make up a significant portion of the ultimate claim payment.

4. CONCLUSIONS

The incremental method allows the actuary to break claim closures into component pieces for claims closed both with and without payment. The flexible nature of the assumptions by calendar year allows the actuary to make explicit assumptions about how many claims will close each period, how many of those claims will close with payment, and future inflationary levels. Given this framework, the incremental method may also be used as a sensitivity test to understand the impact of changing these metrics on ultimate loss estimates. The method may also serve as an actual versus expected tool to measure calendar year activity. Finally, the nature of closure rates by calendar year and inflation rates may be easily related to a non-technical audience.

Acknowledgment

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Scenario 1: Base Case
Data

Appendix 1
Exhibit 1

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	815.0	1,018.0	1,046.0	1,063.0	1,064.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0
2012	822.8	1,027.8	1,056.1	1,073.2	1,074.2	1,075.2	1,075.2	1,075.2	1,075.2		1,075.2
2013	830.7	1,037.7	1,066.2	1,083.5	1,084.6	1,085.6	1,085.6	1,085.6			1,085.6
2014	838.7	1,047.6	1,076.5	1,094.0	1,095.0	1,096.0	1,096.0				1,096.0
2015	846.8	1,057.7	1,086.8	1,104.5	1,105.5	1,106.6					1,106.6
2016	854.9	1,067.9	1,097.3	1,115.1	1,116.1						1,117.2
2017	863.2	1,078.2	1,107.8	1,125.8							1,127.9
2018	871.5	1,088.5	1,118.5								1,138.8
2019	879.8	1,099.0									1,149.7
2020	888.3										1,160.8

Cumulative Closed with Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	375.0	555.0	585.0	600.0	609.0	616.0	622.0	625.0	627.0	628.0	633.0
2012	378.6	560.3	590.6	605.8	614.9	621.9	628.0	631.0	633.0		639.1
2013	382.2	565.7	596.3	611.6	620.8	627.9	634.0	637.1			645.2
2014	385.9	571.2	602.0	617.5	626.7	633.9	640.1				651.4
2015	389.6	576.7	607.8	623.4	632.8	640.0					657.7
2016	393.4	582.2	613.7	629.4	638.8						664.0
2017	397.2	587.8	619.6	635.5							670.4
2018	401.0	593.5	625.5								676.9
2019	404.8	599.2									683.4
2020	408.7										689.9

Cumulative Closed without Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	275.0	375.0	390.0	402.0	411.0	417.0	421.0	424.0	426.0	427.0	432.0
2012	277.6	378.6	393.8	405.9	415.0	421.0	425.0	428.1	430.1		436.2
2013	280.3	382.2	397.5	409.8	418.9	425.1	429.1	432.2			440.3
2014	283.0	385.9	401.4	413.7	423.0	429.1	433.3				444.6
2015	285.7	389.6	405.2	417.7	427.0	433.3					448.9
2016	288.5	393.4	409.1	421.7	431.1						453.2
2017	291.3	397.2	413.0	425.8							457.5
2018	294.1	401.0	417.0								461.9
2019	296.9	404.8									466.4
2020	299.7										470.9

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	165.0	88.0	71.0	61.0	44.0	32.0	22.0	16.0	12.0	10.0	0.0
2012	166.6	88.8	71.7	61.6	44.4	32.3	22.2	16.2	12.1		0.0
2013	168.2	89.7	72.4	62.2	44.9	32.6	22.4	16.3			0.0
2014	169.8	90.6	73.1	62.8	45.3	32.9	22.6				0.0
2015	171.4	91.4	73.8	63.4	45.7	33.2					0.0
2016	173.1	92.3	74.5	64.0	46.2						0.0
2017	174.8	93.2	75.2	64.6							0.0
2018	176.4	94.1	75.9								0.0
2019	178.1	95.0									0.0
2020	179.8										0.0

Cumulative Reported Loss

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	3,750,000	7,500,000	10,500,000	12,600,000	13,860,000	14,970,000	15,720,000	16,030,000	16,190,000	16,270,000	17,500,000
2012	3,937,500	7,875,000	11,025,000	13,230,000	14,553,000	15,718,500	16,506,000	16,831,500	16,999,500		18,375,000
2013	4,134,375	8,268,750	11,576,250	13,891,500	15,280,650	16,504,425	17,331,300	17,673,075			19,293,750
2014	4,341,094	8,682,188	12,155,063	14,586,075	16,044,683	17,329,646	18,197,865				20,258,438
2015	4,558,148	9,116,297	12,762,816	15,315,379	16,846,917	18,196,129					21,271,359
2016	4,786,056	9,572,112	13,400,956	16,081,148	17,689,262						22,334,927
2017	5,025,359	10,050,717	14,071,004	16,885,205							23,451,674
2018	5,276,627	10,553,253	14,774,554								24,624,257
2019	5,540,458	11,080,916									25,855,470
2020	5,817,481										27,148,244

Cumulative Paid Loss

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	1,500,000	3,750,000	6,750,000	9,450,000	11,225,000	12,725,000	14,150,000	14,900,000	15,450,000	15,760,000	17,500,000
2012	1,575,000	3,937,500	7,087,500	9,922,500	11,786,250	13,361,250	14,857,500	15,645,000	16,222,500		18,375,000
2013	1,653,750	4,134,375	7,441,875	10,418,625	12,375,563	14,029,313	15,600,375	16,427,250			19,293,750
2014	1,736,438	4,341,094	7,813,969	10,939,556	12,994,341	14,730,778	16,380,394				20,258,438
2015	1,823,259	4,558,148	8,204,667	11,486,534	13,644,058	15,467,317					21,271,359
2016	1,914,422	4,786,056	8,614,901	12,060,861	14,326,261						22,334,927
2017	2,010,143	5,025,359	9,045,646	12,663,904							23,451,674
2018	2,110,651	5,276,627	9,497,928								24,624,257
2019	2,216,183	5,540,458									25,855,470
2020	2,326,992										27,148,244

Scenario 1: Base Case
Diagnostics

Appendix 1
Exhibit 2

Case Reserves per Open Count

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	13,636	42,614	52,817	51,639	59,886	70,156	71,364	70,625	61,667	51,000
2012	14,182	44,318	54,930	53,705	62,282	72,963	74,218	73,450	64,133	
2013	14,749	46,091	57,127	55,853	64,773	75,881	77,187	76,388		
2014	15,339	47,935	59,412	58,087	67,364	78,916	80,274			
2015	15,953	49,852	61,788	60,411	70,059	82,073				
2016	16,591	51,846	64,260	62,827	72,861					
2017	17,254	53,920	66,830	65,340						
2018	17,945	56,077	69,503							
2019	18,662	58,320								
2020	19,409									

Paid Loss per Closed with Payment Count

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	4,000	6,757	11,538	15,750	18,432	20,657	22,749	23,840	24,641	25,096
2012	4,160	7,027	12,000	16,380	19,169	21,484	23,659	24,794	25,627	
2013	4,326	7,308	12,480	17,035	19,936	22,343	24,606	25,785		
2014	4,499	7,600	12,979	17,717	20,733	23,237	25,590			
2015	4,679	7,904	13,498	18,425	21,563	24,166				
2016	4,867	8,221	14,038	19,162	22,425					
2017	5,061	8,549	14,600	19,929						
2018	5,264	8,891	15,184							
2019	5,474	9,247								
2020	5,693									

Closed Counts / Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%	98.9%	99.1%
2012	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%	98.9%	
2013	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%		
2014	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%			
2015	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%				
2016	79.8%	91.4%	93.2%	94.3%	95.9%					
2017	79.8%	91.4%	93.2%	94.3%						
2018	79.8%	91.4%	93.2%							
2019	79.8%	91.4%								
2020	79.8%									

Closed with Payment Counts / Closed Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%	59.5%	59.5%
2012	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%	59.5%	
2013	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%		
2014	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%			
2015	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%				
2016	57.7%	59.7%	60.0%	59.9%	59.7%					
2017	57.7%	59.7%	60.0%	59.9%						
2018	57.7%	59.7%	60.0%							
2019	57.7%	59.7%								
2020	57.7%									

Scenario 1: Base Case
Step 1

Appendix 1
Exhibit 3

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	815.0	1,018.0	1,046.0	1,063.0	1,064.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0
2012	822.8	1,027.8	1,056.1	1,073.2	1,074.2	1,075.2	1,075.2	1,075.2	1,075.2	
2013	830.7	1,037.7	1,066.2	1,083.5	1,084.6	1,085.6	1,085.6	1,085.6		
2014	838.7	1,047.6	1,076.5	1,094.0	1,095.0	1,096.0	1,096.0			
2015	846.8	1,057.7	1,086.8	1,104.5	1,105.5	1,106.6				
2016	854.9	1,067.9	1,097.3	1,115.1	1,116.1					
2017	863.2	1,078.2	1,107.8	1,125.8						
2018	871.5	1,088.5	1,118.5							
2019	879.8	1,099.0								
2020	888.3									

Incremental Reported Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	815.0	203.0	28.0	17.0	1.0	1.0	0.0	0.0	0.0	0.0
2012	822.8	205.0	28.3	17.2	1.0	1.0	0.0	0.0	0.0	
2013	830.7	206.9	28.5	17.3	1.0	1.0	0.0	0.0		
2014	838.7	208.9	28.8	17.5	1.0	1.0	0.0			
2015	846.8	210.9	29.1	17.7	1.0	1.0				
2016	854.9	212.9	29.4	17.8	1.0					
2017	863.2	215.0	29.7	18.0						
2018	871.5	217.1	29.9							
2019	879.8	219.2								
2020	888.3									

Scenario 1: Base Case
Step 1

Appendix 1
Exhibit 4

Cumulative Closed with Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	375.0	555.0	585.0	600.0	609.0	616.0	622.0	625.0	627.0	628.0
2012	378.6	560.3	590.6	605.8	614.9	621.9	628.0	631.0	633.0	
2013	382.2	565.7	596.3	611.6	620.8	627.9	634.0	637.1		
2014	385.9	571.2	602.0	617.5	626.7	633.9	640.1			
2015	389.6	576.7	607.8	623.4	632.8	640.0				
2016	393.4	582.2	613.7	629.4	638.8					
2017	397.2	587.8	619.6	635.5						
2018	401.0	593.5	625.5							
2019	404.8	599.2								
2020	408.7									

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1		
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2			
2015	389.6	187.0	31.2	15.6	9.4	7.3				
2016	393.4	188.8	31.5	15.7	9.4					
2017	397.2	190.6	31.8	15.9						
2018	401.0	192.5	32.1							
2019	404.8	194.3								
2020	408.7									

Scenario 1: Base Case
Step 1

Appendix 1
Exhibit 5

Cumulative Closed without Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	275.0	375.0	390.0	402.0	411.0	417.0	421.0	424.0	426.0	427.0
2012	277.6	378.6	393.8	405.9	415.0	421.0	425.0	428.1	430.1	
2013	280.3	382.2	397.5	409.8	418.9	425.1	429.1	432.2		
2014	283.0	385.9	401.4	413.7	423.0	429.1	433.3			
2015	285.7	389.6	405.2	417.7	427.0	433.3				
2016	288.5	393.4	409.1	421.7	431.1					
2017	291.3	397.2	413.0	425.8						
2018	294.1	401.0	417.0							
2019	296.9	404.8								
2020	299.7									

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	275.0	100.0	15.0	12.0	9.0	6.0	4.0	3.0	2.0	1.0
2012	277.6	101.0	15.1	12.1	9.1	6.1	4.0	3.0	2.0	
2013	280.3	101.9	15.3	12.2	9.2	6.1	4.1	3.1		
2014	283.0	102.9	15.4	12.3	9.3	6.2	4.1			
2015	285.7	103.9	15.6	12.5	9.4	6.2				
2016	288.5	104.9	15.7	12.6	9.4					
2017	291.3	105.9	15.9	12.7						
2018	294.1	106.9	16.0							
2019	296.9	108.0								
2020	299.7									

Scenario 1: Base Case
Step 1

Appendix 1
Exhibit 6

Cumulative Closed Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	650.0	930.0	975.0	1,002.0	1,020.0	1,033.0	1,043.0	1,049.0	1,053.0	1,055.0
2012	656.3	938.9	984.4	1,011.6	1,029.8	1,042.9	1,053.0	1,059.1	1,063.1	
2013	662.6	948.0	993.8	1,021.4	1,039.7	1,053.0	1,063.2	1,069.3		
2014	668.9	957.1	1,003.4	1,031.2	1,049.7	1,063.1	1,073.4			
2015	675.4	966.3	1,013.0	1,041.1	1,059.8	1,073.3				
2016	681.9	975.6	1,022.8	1,051.1	1,070.0					
2017	688.4	985.0	1,032.6	1,061.2						
2018	695.0	994.4	1,042.5							
2019	701.7	1,004.0								
2020	708.5									

Incremental Closed Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	650.0	280.0	45.0	27.0	18.0	13.0	10.0	6.0	4.0	2.0
2012	656.3	282.7	45.4	27.3	18.2	13.1	10.1	6.1	4.0	
2013	662.6	285.4	45.9	27.5	18.3	13.3	10.2	6.1		
2014	668.9	288.2	46.3	27.8	18.5	13.4	10.3			
2015	675.4	290.9	46.8	28.1	18.7	13.5				
2016	681.9	293.7	47.2	28.3	18.9					
2017	688.4	296.5	47.7	28.6						
2018	695.0	299.4	48.1							
2019	701.7	302.3								
2020	708.5									

Scenario 1: Base Case
Step 2

Appendix 1
Exhibit 7

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	815.0	368.0	116.0	88.0	62.0	45.0	32.0	22.0	16.0	12.0
2012	822.8	371.5	117.1	88.8	62.6	45.4	32.3	22.2	16.2	
2013	830.7	375.1	118.2	89.7	63.2	45.9	32.6	22.4		
2014	838.7	378.7	119.4	90.6	63.8	46.3	32.9			
2015	846.8	382.4	120.5	91.4	64.4	46.8				
2016	854.9	386.0	121.7	92.3	65.0					
2017	863.2	389.7	122.9	93.2						
2018	871.5	393.5	124.0							
2019	879.8	397.3								
2020	888.3									

Accident year 2019 active counts from 0 to 12 months = counts reported from 0 to 12 months = 879.8

Accident year 2019 active counts from 12 to 24 months = open counts at 12 months + counts reported from 12 to 24 months = 178.1 + 219.2 = 397.3

Scenario 1: Base Case **Appendix 1**
Step 3 **Exhibit 8**

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	815.0	1,018.0	1,046.0	1,063.0	1,064.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0
2012	822.8	1,027.8	1,056.1	1,073.2	1,074.2	1,075.2	1,075.2	1,075.2	1,075.2	1,075.2	1,075.2
2013	830.7	1,037.7	1,066.2	1,083.5	1,084.6	1,085.6	1,085.6	1,085.6	1,085.6	1,085.6	1,085.6
2014	838.7	1,047.6	1,076.5	1,094.0	1,095.0	1,096.0	1,096.0	1,096.0	1,096.0	1,096.0	1,096.0
2015	846.8	1,057.7	1,086.8	1,104.5	1,105.5	1,106.6	1,106.6	1,106.6	1,106.6	1,106.6	1,106.6
2016	854.9	1,067.9	1,097.3	1,115.1	1,116.1	1,117.2	1,117.2	1,117.2	1,117.2	1,117.2	1,117.2
2017	863.2	1,078.2	1,107.8	1,125.8	1,126.9	1,127.9	1,127.9	1,127.9	1,127.9	1,127.9	1,127.9
2018	871.5	1,088.5	1,118.5	1,136.6	1,137.7	1,138.8	1,138.8	1,138.8	1,138.8	1,138.8	1,138.8
2019	879.8	1,099.0	1,129.2	1,147.6	1,148.7	1,149.7	1,149.7	1,149.7	1,149.7	1,149.7	1,149.7
2020	888.3	1,109.6	1,140.1	1,158.6	1,159.7	1,160.8	1,160.8	1,160.8	1,160.8	1,160.8	1,160.8

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000	1.000	
2012	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000		
2013	1.249	1.028	1.016	1.001	1.001	1.000	1.000			
2014	1.249	1.028	1.016	1.001	1.001	1.000				
2015	1.249	1.028	1.016	1.001	1.001					
2016	1.249	1.028	1.016	1.001						
2017	1.249	1.028	1.016							
2018	1.249	1.028								
2019	1.249									
Age-to-Age	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000	1.000	
Age-to-Ult	1.307	1.046	1.018	1.002	1.001	1.000	1.000	1.000	1.000	1.000

Scenario 1: Base Case
Step 3

Appendix 1
Exhibit 9

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	815.0	1,018.0	1,046.0	1,063.0	1,064.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0
2012	822.8	1,027.8	1,056.1	1,073.2	1,074.2	1,075.2	1,075.2	1,075.2	1,075.2	1,075.2	1,075.2
2013	830.7	1,037.7	1,066.2	1,083.5	1,084.6	1,085.6	1,085.6	1,085.6	1,085.6	1,085.6	1,085.6
2014	838.7	1,047.6	1,076.5	1,094.0	1,095.0	1,096.0	1,096.0	1,096.0	1,096.0	1,096.0	1,096.0
2015	846.8	1,057.7	1,086.8	1,104.5	1,105.5	1,106.6	1,106.6	1,106.6	1,106.6	1,106.6	1,106.6
2016	854.9	1,067.9	1,097.3	1,115.1	1,116.1	1,117.2	1,117.2	1,117.2	1,117.2	1,117.2	1,117.2
2017	863.2	1,078.2	1,107.8	1,125.8	1,126.9	1,127.9	1,127.9	1,127.9	1,127.9	1,127.9	1,127.9
2018	871.5	1,088.5	1,118.5	1,136.6	1,137.7	1,138.8	1,138.8	1,138.8	1,138.8	1,138.8	1,138.8
2019	879.8	1,099.0	1,129.2	1,147.6	1,148.7	1,149.7	1,149.7	1,149.7	1,149.7	1,149.7	1,149.7
2020	888.3	1,109.6	1,140.1	1,158.6	1,159.7	1,160.8	1,160.8	1,160.8	1,160.8	1,160.8	1,160.8

Incremental Reported Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	815.0	203.0	28.0	17.0	1.0	1.0	0.0	0.0	0.0	0.0	0.0
2012	822.8	205.0	28.3	17.2	1.0	1.0	0.0	0.0	0.0	0.0	0.0
2013	830.7	206.9	28.5	17.3	1.0	1.0	0.0	0.0	0.0	0.0	0.0
2014	838.7	208.9	28.8	17.5	1.0	1.0	0.0	0.0	0.0	0.0	0.0
2015	846.8	210.9	29.1	17.7	1.0	1.0	0.0	0.0	0.0	0.0	0.0
2016	854.9	212.9	29.4	17.8	1.0	1.0	0.0	0.0	0.0	0.0	0.0
2017	863.2	215.0	29.7	18.0	1.1	1.1	0.0	0.0	0.0	0.0	0.0
2018	871.5	217.1	29.9	18.2	1.1	1.1	0.0	0.0	0.0	0.0	0.0
2019	879.8	219.2	30.2	18.4	1.1	1.1	0.0	0.0	0.0	0.0	0.0
2020	888.3	221.3	30.5	18.5	1.1	1.1	0.0	0.0	0.0	0.0	0.0

Scenario 1: Base Case
Step 4

Appendix 1
Exhibit 10

Incremental Closure Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	80%	76%	39%	31%	29%	29%	31%	27%	25%	17%	
2012	80%	76%	39%	31%	29%	29%	31%	27%	25%		
2013	80%	76%	39%	31%	29%	29%	31%	27%			
2014	80%	76%	39%	31%	29%	29%	31%				
2015	80%	76%	39%	31%	29%	29%					
2016	80%	76%	39%	31%	29%						
2017	80%	76%	39%	31%							
2018	80%	76%	39%								
2019	80%	76%									
2020	80%										
Selected	80%	76%	39%	31%	29%	29%	31%	27%	25%	17%	100%

Accident year 2019 incremental closure rate from 0 to 12 months = incremental closures / active counts = 701.7 / 879.8 = 80%

Accident year 2019 incremental closure rate from 12 to 24 months = incremental closures / active counts = 302.3 / 397.3 = 76%

Scenario 1: Base Case
Step 5

Appendix 1
Exhibit 11

Incremental Closed with Payment Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	46%	49%	26%	17%	15%	16%	19%	14%	13%	8%	
2012	46%	49%	26%	17%	15%	16%	19%	14%	13%		
2013	46%	49%	26%	17%	15%	16%	19%	14%			
2014	46%	49%	26%	17%	15%	16%	19%				
2015	46%	49%	26%	17%	15%	16%					
2016	46%	49%	26%	17%	15%						
2017	46%	49%	26%	17%							
2018	46%	49%	26%								
2019	46%	49%									
2020	46%										
Selected	46%	49%	26%	17%	15%	16%	19%	14%	13%	8%	50%

Accident year 2019 incremental closed with payment rate from 0 to 12 months = incremental closures with payment / active counts = 404.8 / 879.8 = 46%

Accident year 2019 incremental closed with payment rate from 12 to 24 months = incremental closures with payment / active counts = 194.3 / 397.3 = 49%

120-Ult tail calculated using defined ultimates:
(Ultimate closed with payment counts - closed with payment counts at 120 months) / active counts from 120 months to ultimate = (633.0 - 628.0) / 10.0 = 50%

Scenario 1: Base Case
Step 6

Appendix 1
Exhibit 12

Incremental Closed without Payment Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	34%	27%	13%	14%	15%	13%	13%	14%	13%	8%	
2012	34%	27%	13%	14%	15%	13%	13%	14%	12%		
2013	34%	27%	13%	14%	15%	13%	12%	14%			
2014	34%	27%	13%	14%	15%	13%	12%				
2015	34%	27%	13%	14%	15%	13%					
2016	34%	27%	13%	14%	15%						
2017	34%	27%	13%	14%							
2018	34%	27%	13%								
2019	34%	27%									
2020	34%										
Closure Rate	80%	76%	39%	31%	29%	29%	31%	27%	25%	17%	100%
Closed with Payment Rate	46%	49%	26%	17%	15%	16%	19%	14%	13%	8%	50%
Closed without Payment Rate	34%	27%	13%	14%	15%	13%	13%	14%	13%	8%	50%

Scenario 1: Base Case **Appendix 1**
Step 7 **Exhibit 13**

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	815.0	368.0	116.0	88.0	62.0	45.0	32.0	22.0	16.0	12.0	10.0
2012	822.8	371.5	117.1	88.8	62.6	45.4	32.3	22.2	16.2	12.1	
2013	830.7	375.1	118.2	89.7	63.2	45.9	32.6	22.4	16.3		
2014	838.7	378.7	119.4	90.6	63.8	46.3	32.9	22.6			
2015	846.8	382.4	120.5	91.4	64.4	46.8	33.2				
2016	854.9	386.0	121.7	92.3	65.0	47.2					
2017	863.2	389.7	122.9	93.2	65.7						
2018	871.5	393.5	124.0	94.1							
2019	879.8	397.3	125.2								
2020	888.3	401.1									

Accident year 2019 active counts from 24 to 36 months = open counts at 24 months + counts reported from 24 to 36 months = 95.0 + 30.2 = 125.2

Scenario 1: Base Case **Appendix 1**
Step 8 **Exhibit 14**

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	5.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	1.0	
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1	2.0		
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2	3.1			
2015	389.6	187.0	31.2	15.6	9.4	7.3	6.2				
2016	393.4	188.8	31.5	15.7	9.4	7.3					
2017	397.2	190.6	31.8	15.9	9.5						
2018	401.0	192.5	32.1	16.0							
2019	404.8	194.3	32.4								
2020	408.7	196.2									

Accident year 2019 incremental closed with payment counts from 24 to 36 months = incremental closed with payment rate * active counts = 26% * 125.2 = 32.4

Scenario 1: Base Case **Appendix 1**
Step 9 **Exhibit 15**

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	275.0	100.0	15.0	12.0	9.0	6.0	4.0	3.0	2.0	1.0	5.0
2012	277.6	101.0	15.1	12.1	9.1	6.1	4.0	3.0	2.0	1.0	
2013	280.3	101.9	15.3	12.2	9.2	6.1	4.1	3.1	2.0		
2014	283.0	102.9	15.4	12.3	9.3	6.2	4.1	3.1			
2015	285.7	103.9	15.6	12.5	9.4	6.2	4.2				
2016	288.5	104.9	15.7	12.6	9.4	6.3					
2017	291.3	105.9	15.9	12.7	9.5						
2018	294.1	106.9	16.0	12.8							
2019	296.9	108.0	16.2								
2020	299.7	109.0									

Accident year 2019 incremental closed without payment counts from 24 to 36 months = incremental closed without payment rate * active counts = 13% * 125.2 = 16.2

Scenario 1: Base Case
Step 10

Appendix 1
Exhibit 16

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	165.0	88.0	71.0	61.0	44.0	32.0	22.0	16.0	12.0	10.0	0.0
2012	166.6	88.8	71.7	61.6	44.4	32.3	22.2	16.2	12.1	10.1	
2013	168.2	89.7	72.4	62.2	44.9	32.6	22.4	16.3	12.2		
2014	169.8	90.6	73.1	62.8	45.3	32.9	22.6	16.5			
2015	171.4	91.4	73.8	63.4	45.7	33.2	22.9				
2016	173.1	92.3	74.5	64.0	46.2	33.6					
2017	174.8	93.2	75.2	64.6	46.6						
2018	176.4	94.1	75.9	65.2							
2019	178.1	95.0	76.6								
2020	179.8	95.9									

Accident year 2019 open counts at 36 months:
Open counts at 24 months + counts reported - counts closed with payment - counts closed without payment from 24 to 36 months
= 95.0 + 30.2 - 32.4 - 16.2 = 76.6

Scenario 1: Base Case
Steps 7-10

Appendix 1
Exhibit 17

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	815.0	368.0	116.0	88.0	62.0	45.0	32.0	22.0	16.0	12.0	10.0	0.0
2012	822.8	371.5	117.1	88.8	62.6	45.4	32.3	22.2	16.2	12.1	10.1	0.0
2013	830.7	375.1	118.2	89.7	63.2	45.9	32.6	22.4	16.3	12.2	10.2	0.0
2014	838.7	378.7	119.4	90.6	63.8	46.3	32.9	22.6	16.5	12.3	10.3	0.0
2015	846.8	382.4	120.5	91.4	64.4	46.8	33.2	22.9	16.6	12.5	10.4	0.0
2016	854.9	386.0	121.7	92.3	65.0	47.2	33.6	23.1	16.8	12.6	10.5	0.0
2017	863.2	389.7	122.9	93.2	65.7	47.7	33.9	23.3	16.9	12.7	10.6	0.0
2018	871.5	393.5	124.0	94.1	66.3	48.1	34.2	23.5	17.1	12.8	10.7	0.0
2019	879.8	397.3	125.2	95.0	66.9	48.6	34.5	23.8	17.3	13.0	10.8	0.0
2020	888.3	401.1	126.4	95.9	67.6	49.0	34.9	24.0	17.4	13.1	10.9	0.0

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	5.0	633.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	1.0	5.0	639.1
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1	2.0	1.0	5.1	645.2
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2	3.1	2.1	1.0	5.1	651.4
2015	389.6	187.0	31.2	15.6	9.4	7.3	6.2	3.1	2.1	1.0	5.2	657.7
2016	393.4	188.8	31.5	15.7	9.4	7.3	6.3	3.1	2.1	1.0	5.2	664.0
2017	397.2	190.6	31.8	15.9	9.5	7.4	6.4	3.2	2.1	1.1	5.3	670.4
2018	401.0	192.5	32.1	16.0	9.6	7.5	6.4	3.2	2.1	1.1	5.3	676.9
2019	404.8	194.3	32.4	16.2	9.7	7.6	6.5	3.2	2.2	1.1	5.4	683.4
2020	408.7	196.2	32.7	16.3	9.8	7.6	6.5	3.3	2.2	1.1	5.4	689.9

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	275.0	100.0	15.0	12.0	9.0	6.0	4.0	3.0	2.0	1.0	5.0	432.0
2012	277.6	101.0	15.1	12.1	9.1	6.1	4.0	3.0	2.0	1.0	5.0	436.2
2013	280.3	101.9	15.3	12.2	9.2	6.1	4.1	3.1	2.0	1.0	5.1	440.3
2014	283.0	102.9	15.4	12.3	9.3	6.2	4.1	3.1	2.1	1.0	5.1	444.6
2015	285.7	103.9	15.6	12.5	9.4	6.2	4.2	3.1	2.1	1.0	5.2	448.9
2016	288.5	104.9	15.7	12.6	9.4	6.3	4.2	3.1	2.1	1.0	5.2	453.2
2017	291.3	105.9	15.9	12.7	9.5	6.4	4.2	3.2	2.1	1.1	5.3	457.5
2018	294.1	106.9	16.0	12.8	9.6	6.4	4.3	3.2	2.1	1.1	5.3	461.9
2019	296.9	108.0	16.2	13.0	9.7	6.5	4.3	3.2	2.2	1.1	5.4	466.4
2020	299.7	109.0	16.3	13.1	9.8	6.5	4.4	3.3	2.2	1.1	5.4	470.9

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	165.0	88.0	71.0	61.0	44.0	32.0	22.0	16.0	12.0	10.0	0.0
2012	166.6	88.8	71.7	61.6	44.4	32.3	22.2	16.2	12.1	10.1	0.0
2013	168.2	89.7	72.4	62.2	44.9	32.6	22.4	16.3	12.2	10.2	0.0
2014	169.8	90.6	73.1	62.8	45.3	32.9	22.6	16.5	12.3	10.3	0.0
2015	171.4	91.4	73.8	63.4	45.7	33.2	22.9	16.6	12.5	10.4	0.0
2016	173.1	92.3	74.5	64.0	46.2	33.6	23.1	16.8	12.6	10.5	0.0
2017	174.8	93.2	75.2	64.6	46.6	33.9	23.3	16.9	12.7	10.6	0.0
2018	176.4	94.1	75.9	65.2	47.0	34.2	23.5	17.1	12.8	10.7	0.0
2019	178.1	95.0	76.6	65.9	47.5	34.5	23.8	17.3	13.0	10.8	0.0
2020	179.8	95.9	77.4	66.5	48.0	34.9	24.0	17.4	13.1	10.9	0.0

Scenario 1: Base Case
Step 11

Appendix 1
Exhibit 18

Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	4,000	12,500	100,000	180,000	197,222	214,286	237,500	250,000	275,000	310,000
2012	4,160	13,000	104,000	187,200	205,111	222,857	247,000	260,000	286,000	
2013	4,326	13,520	108,160	194,688	213,316	231,771	256,880	270,400		
2014	4,499	14,061	112,486	202,476	221,848	241,042	267,155			
2015	4,679	14,623	116,986	210,575	230,722	250,684				
2016	4,867	15,208	121,665	218,998	239,951					
2017	5,061	15,816	126,532	227,757						
2018	5,264	16,449	131,593							
2019	5,474	17,107								
2020	5,693									

Accident year 2019 paid severity from 0 to 12 months = paid loss / counts closed with payment = $2,216,183 / 404.8 = $5,474

Accident year 2019 paid severity from 12 to 24 months = paid loss / counts closed with payment = ($5,540,458 - $2,216,183) / 194.3 = $17,107

Scenario 1: Base Case
Step 12

Appendix 1
Exhibit 19

Trended Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000	310,000
2012	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000	
2013	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400		
2014	5,693	17,107	131,593	227,757	239,951	250,684	267,155			
2015	5,693	17,107	131,593	227,757	239,951	250,684				
2016	5,693	17,107	131,593	227,757	239,951					
2017	5,693	17,107	131,593	227,757						
2018	5,693	17,107	131,593							
2019	5,693	17,107								
2020	5,693									

Trend Rate: 4%

Accident year 2019 trended paid severity from 0 to 12 months = nominal severity * (1 + trend rate) ^ trend period = $5,474 * (1.04)^1 = $5,693

Accident year 2019 trended paid severity from 12 to 24 months = nominal severity * (1 + trend rate) ^ trend period = $17,107 * (1.04)^0 = $17,107

Trend Period to Calendar Year 2020

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	9	8	7	6	5	4	3	2	1	0
2012	8	7	6	5	4	3	2	1	0	
2013	7	6	5	4	3	2	1	0		
2014	6	5	4	3	2	1	0			
2015	5	4	3	2	1	0				
2016	4	3	2	1	0					
2017	3	2	1	0						
2018	2	1	0							
2019	1	0								
2020	0									

Scenario 1: Base Case
Step 13

Appendix 1
Exhibit 20

Trended Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000	310,000	
2012	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000		
2013	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400			
2014	5,693	17,107	131,593	227,757	239,951	250,684	267,155				
2015	5,693	17,107	131,593	227,757	239,951	250,684					
2016	5,693	17,107	131,593	227,757	239,951						
2017	5,693	17,107	131,593	227,757							
2018	5,693	17,107	131,593								
2019	5,693	17,107									
2020	5,693										
Selected	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000	310,000	334,615

120-Ult tail in calendar year 2020 dollars calculated using defined ultimates:
(Ultimate paid loss - paid loss at 120 months) / counts closed with payment 120 months to ultimate / (1 + trend factor)
= ($17,500,000 - $15,760,000) / (633.0 - 628.0) / (1.04) = $334,615

Scenario 1: Base Case
Step 14

Appendix 1
Exhibit 21

Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	4,000	12,500	100,000	180,000	197,222	214,286	237,500	250,000	275,000	310,000	348,000
2012	4,160	13,000	104,000	187,200	205,111	222,857	247,000	260,000	286,000	322,400	361,920
2013	4,326	13,520	108,160	194,688	213,316	231,771	256,880	270,400	297,440	335,296	376,397
2014	4,499	14,061	112,486	202,476	221,848	241,042	267,155	281,216	309,338	348,708	391,453
2015	4,679	14,623	116,986	210,575	230,722	250,684	277,841	292,465	321,711	362,656	407,111
2016	4,867	15,208	121,665	218,998	239,951	260,711	288,955	304,163	334,580	377,162	423,395
2017	5,061	15,816	126,532	227,757	249,549	271,140	300,513	316,330	347,963	392,249	440,331
2018	5,264	16,449	131,593	236,868	259,531	281,985	312,534	328,983	361,881	407,939	457,944
2019	5,474	17,107	136,857	246,342	269,912	293,265	325,035	342,142	376,356	424,256	476,262
2020	5,693	17,791	142,331	256,196	280,709	304,995	338,037	355,828	391,411	441,227	495,313

Trend Rate: 4%

Accident year 2019 paid severity from 24 to 36 months = selected severity * (1 + trend rate) ^ trend period = $131,593 * (1.04)^1 = $136,857

Trend Period from Calendar Year 2020 to Future Calendar Years

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011											1
2012										1	2
2013									1	2	3
2014								1	2	3	4
2015							1	2	3	4	5
2016						1	2	3	4	5	6
2017					1	2	3	4	5	6	7
2018				1	2	3	4	5	6	7	8
2019			1	2	3	4	5	6	7	8	9
2020		1	2	3	4	5	6	7	8	9	10

Scenario 1: Base Case
Step 15

Appendix 1
Exhibit 22

Incremental Paid Loss

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	1,500,000	2,250,000	3,000,000	2,700,000	1,775,000	1,500,000	1,425,000	750,000	550,000	310,000	1,740,000	17,500,000
2012	1,575,000	2,362,500	3,150,000	2,835,000	1,863,750	1,575,000	1,496,250	787,500	577,500	325,500	1,827,000	18,375,000
2013	1,653,750	2,480,625	3,307,500	2,976,750	1,956,938	1,653,750	1,571,063	826,875	606,375	341,775	1,918,350	19,293,750
2014	1,736,438	2,604,656	3,472,875	3,125,588	2,054,784	1,736,438	1,649,616	868,219	636,694	358,864	2,014,268	20,258,438
2015	1,823,259	2,734,889	3,646,519	3,281,867	2,157,524	1,823,259	1,732,096	911,630	668,528	376,807	2,114,981	21,271,359
2016	1,914,422	2,871,634	3,828,845	3,445,960	2,265,400	1,914,422	1,818,701	957,211	701,955	395,647	2,220,730	22,334,927
2017	2,010,143	3,015,215	4,020,287	3,618,258	2,378,670	2,010,143	1,909,636	1,005,072	737,053	415,430	2,331,766	23,451,674
2018	2,110,651	3,165,976	4,221,301	3,799,171	2,497,603	2,110,651	2,005,118	1,055,325	773,905	436,201	2,448,355	24,624,257
2019	2,216,183	3,324,275	4,432,366	3,989,130	2,622,483	2,216,183	2,105,374	1,108,092	812,600	458,011	2,570,772	25,855,470
2020	2,326,992	3,490,488	4,653,985	4,188,586	2,753,608	2,326,992	2,210,643	1,163,496	853,231	480,912	2,699,311	27,148,244

Accident year 2019 paid loss from 24 to 36 months = incremental severity * incremental counts closed with payment = $136,857 * 32.4 = $4,432,366

Scenario 1: Base Case **Appendix 1**
Reported Loss Development Method **Exhibit 23**

Cumulative Reported Loss

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	3,750,000	7,500,000	10,500,000	12,600,000	13,860,000	14,970,000	15,720,000	16,030,000	16,190,000	16,270,000	17,500,000
2012	3,937,500	7,875,000	11,025,000	13,230,000	14,553,000	15,718,500	16,506,000	16,831,500	16,999,500	17,083,500	18,375,000
2013	4,134,375	8,268,750	11,576,250	13,891,500	15,280,650	16,504,425	17,331,300	17,673,075	17,849,475	17,937,675	19,293,750
2014	4,341,094	8,682,188	12,155,063	14,586,075	16,044,683	17,329,646	18,197,865	18,556,729	18,741,949	18,834,559	20,258,438
2015	4,558,148	9,116,297	12,762,816	15,315,379	16,846,917	18,196,129	19,107,758	19,484,565	19,679,046	19,776,287	21,271,359
2016	4,786,056	9,572,112	13,400,956	16,081,148	17,689,262	19,105,935	20,063,146	20,458,793	20,662,998	20,765,101	22,334,927
2017	5,025,359	10,050,717	14,071,004	16,885,205	18,573,726	20,061,232	21,066,303	21,481,733	21,696,148	21,803,356	23,451,674
2018	5,276,627	10,553,253	14,774,554	17,729,465	19,502,412	21,064,293	22,119,619	22,555,820	22,780,956	22,893,524	24,624,257
2019	5,540,458	11,080,916	15,513,282	18,615,939	20,477,532	22,117,508	23,225,600	23,683,611	23,920,004	24,038,200	25,855,470
2020	5,817,481	11,634,962	16,288,946	19,546,736	21,501,409	23,223,383	24,386,880	24,867,791	25,116,004	25,240,110	27,148,244

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	2.000	1.400	1.200	1.100	1.080	1.050	1.020	1.010	1.005	
2012	2.000	1.400	1.200	1.100	1.080	1.050	1.020	1.010		
2013	2.000	1.400	1.200	1.100	1.080	1.050	1.020			
2014	2.000	1.400	1.200	1.100	1.080	1.050				
2015	2.000	1.400	1.200	1.100	1.080					
2016	2.000	1.400	1.200	1.100						
2017	2.000	1.400	1.200							
2018	2.000	1.400								
2019	2.000									
Age-to-Age	2.000	1.400	1.200	1.100	1.080	1.050	1.020	1.010	1.005	
Age-to-Ult	4.667	2.333	1.667	1.389	1.263	1.169	1.113	1.092	1.081	1.076

Scenario 1: Base Case **Appendix 1**
Paid Loss Development Method **Exhibit 24**

Cumulative Paid Loss

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	1,500,000	3,750,000	6,750,000	9,450,000	11,225,000	12,725,000	14,150,000	14,900,000	15,450,000	15,760,000	17,500,000
2012	1,575,000	3,937,500	7,087,500	9,922,500	11,786,250	13,361,250	14,857,500	15,645,000	16,222,500	16,548,000	18,375,000
2013	1,653,750	4,134,375	7,441,875	10,418,625	12,375,563	14,029,313	15,600,375	16,427,250	17,033,625	17,375,400	19,293,750
2014	1,736,438	4,341,094	7,813,969	10,939,556	12,994,341	14,730,778	16,380,394	17,248,613	17,885,306	18,244,170	20,258,438
2015	1,823,259	4,558,148	8,204,667	11,486,534	13,644,058	15,467,317	17,199,413	18,111,043	18,779,572	19,156,379	21,271,359
2016	1,914,422	4,786,056	8,614,901	12,060,861	14,326,261	16,240,683	18,059,384	19,016,595	19,718,550	20,114,197	22,334,927
2017	2,010,143	5,025,359	9,045,646	12,663,904	15,042,574	17,052,717	18,962,353	19,967,425	20,704,478	21,119,907	23,451,674
2018	2,110,651	5,276,627	9,497,928	13,297,099	15,794,702	17,905,353	19,910,471	20,965,796	21,739,702	22,175,903	24,624,257
2019	2,216,183	5,540,458	9,972,824	13,961,954	16,584,437	18,800,621	20,905,995	22,014,086	22,826,687	23,284,698	25,855,470
2020	2,326,992	5,817,481	10,471,465	14,660,052	17,413,659	19,740,652	21,951,294	23,114,790	23,968,021	24,448,933	27,148,244

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	2.500	1.800	1.400	1.188	1.134	1.112	1.053	1.037	1.020	
2012	2.500	1.800	1.400	1.188	1.134	1.112	1.053	1.037		
2013	2.500	1.800	1.400	1.188	1.134	1.112	1.053			
2014	2.500	1.800	1.400	1.188	1.134	1.112				
2015	2.500	1.800	1.400	1.188	1.134					
2016	2.500	1.800	1.400	1.188						
2017	2.500	1.800	1.400							
2018	2.500	1.800								
2019	2.500									
Age-to-Age	2.500	1.800	1.400	1.188	1.134	1.112	1.053	1.037	1.020	
Age-to-Ult	11.667	4.667	2.593	1.852	1.559	1.375	1.237	1.174	1.133	1.110

Scenario 1: Base Case **Appendix 1**
Disposal Rate Frequency-Severity Method **Exhibit 25**

Cumulative Non-Zero Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	540.0	643.0	656.0	661.0	653.0	648.0	644.0	641.0	639.0	638.0	633.0
2012	545.2	649.2	662.3	667.4	659.3	654.2	650.2	647.2	645.1	644.1	639.1
2013	550.4	655.4	668.7	673.8	665.6	660.5	656.4	653.4	651.3	650.3	645.2
2014	555.7	661.7	675.1	680.3	672.0	666.9	662.8	659.7	657.6	656.6	651.4
2015	561.1	668.1	681.6	686.8	678.5	673.3	669.1	666.0	663.9	662.9	657.7
2016	566.5	674.5	688.2	693.4	685.0	679.8	675.6	672.4	670.3	669.3	664.0
2017	571.9	681.0	694.8	700.1	691.6	686.3	682.1	678.9	676.8	675.7	670.4
2018	577.4	687.5	701.4	706.8	698.2	692.9	688.6	685.4	683.3	682.2	676.9
2019	583.0	694.2	708.2	713.6	705.0	699.6	695.2	692.0	689.8	688.8	683.4
2020	588.6	700.8	715.0	720.5	711.7	706.3	701.9	698.7	696.5	695.4	689.9

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	1.191	1.020	1.008	0.988	0.992	0.994	0.995	0.997	0.998	
2012	1.191	1.020	1.008	0.988	0.992	0.994	0.995	0.997		
2013	1.191	1.020	1.008	0.988	0.992	0.994	0.995			
2014	1.191	1.020	1.008	0.988	0.992	0.994				
2015	1.191	1.020	1.008	0.988	0.992					
2016	1.191	1.020	1.008	0.988						
2017	1.191	1.020	1.008							
2018	1.191	1.020								
2019	1.191									
Age-to-Age	1.191	1.020	1.008	0.988	0.992	0.994	0.995	0.997	0.998	
Age-to-Ult	1.172	0.984	0.965	0.958	0.969	0.977	0.983	0.988	0.991	0.992

Scenario 1: Base Case
Disposal Rate Frequency-Severity Method

Appendix 1
Exhibit 26

Disposal Rate (Closed with Payment Counts)

Accident Year	12	24	36	48	60	72	84	96	108	120	120-Ult
2011	59%	88%	92%	95%	96%	97%	98%	99%	99%	99%	
2012	59%	88%	92%	95%	96%	97%	98%	99%	99%		
2013	59%	88%	92%	95%	96%	97%	98%	99%			
2014	59%	88%	92%	95%	96%	97%	98%				
2015	59%	88%	92%	95%	96%	97%					
2016	59%	88%	92%	95%	96%						
2017	59%	88%	92%	95%							
2018	59%	88%	92%								
2019	59%	88%									
2020	59%										
Selected	59%	88%	92%	95%	96%	97%	98%	99%	99%	99%	100%

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	5.0	633.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	1.0	5.0	639.1
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1	2.0	1.0	5.1	645.2
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2	3.1	2.1	1.0	5.1	651.4
2015	389.6	187.0	31.2	15.6	9.4	7.3	6.2	3.1	2.1	1.0	5.2	657.7
2016	393.4	188.8	31.5	15.7	9.4	7.3	6.3	3.1	2.1	1.0	5.2	664.0
2017	397.2	190.6	31.8	15.9	9.5	7.4	6.4	3.2	2.1	1.1	5.3	670.4
2018	401.0	192.5	32.1	16.0	9.6	7.5	6.4	3.2	2.1	1.1	5.3	676.9
2019	404.8	194.3	32.4	16.2	9.7	7.6	6.5	3.2	2.2	1.1	5.4	683.4
2020	408.7	196.2	32.7	16.3	9.8	7.6	6.5	3.3	2.2	1.1	5.4	689.9

Scenario 1: Base Case
Disposal Rate Frequency-Severity Method

Appendix 1
Exhibit 27

Incremental Paid Loss

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	1,500,000	2,250,000	3,000,000	2,700,000	1,775,000	1,500,000	1,425,000	750,000	550,000	310,000	1,740,000	17,500,000
2012	1,575,000	2,362,500	3,150,000	2,835,000	1,863,750	1,575,000	1,496,250	787,500	577,500	325,500	1,827,000	18,375,000
2013	1,653,750	2,480,625	3,307,500	2,976,750	1,956,938	1,653,750	1,571,063	826,875	606,375	341,775	1,918,350	19,293,750
2014	1,736,438	2,604,656	3,472,875	3,125,588	2,054,784	1,736,438	1,649,616	868,219	636,694	358,864	2,014,267	20,258,438
2015	1,823,259	2,734,889	3,646,519	3,281,867	2,157,524	1,823,259	1,732,096	911,630	668,528	376,807	2,114,981	21,271,359
2016	1,914,422	2,871,634	3,828,845	3,445,960	2,265,400	1,914,422	1,818,701	957,211	701,955	395,647	2,220,730	22,334,927
2017	2,010,143	3,015,215	4,020,287	3,618,258	2,378,670	2,010,143	1,909,636	1,005,072	737,053	415,430	2,331,766	23,451,674
2018	2,110,651	3,165,976	4,221,301	3,799,171	2,497,603	2,110,651	2,005,118	1,055,325	773,905	436,201	2,448,355	24,624,257
2019	2,216,183	3,324,275	4,432,366	3,989,130	2,622,483	2,216,183	2,105,374	1,108,092	812,600	458,011	2,570,772	25,855,470
2020	2,326,992	3,490,488	4,653,985	4,188,586	2,753,608	2,326,992	2,210,643	1,163,496	853,231	480,912	2,699,311	27,148,244

Accident year 2019 paid loss from 24 to 36 months = incremental severity * incremental counts closed with payment = $136,857 * 32.4 = $4,432,366

Scenario 2: Shift in proportion of claims closed with payment
Data

Appendix 2
Exhibit 1

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	815.0	1,018.0	1,046.0	1,063.0	1,064.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0
2012	822.8	1,027.8	1,056.1	1,073.2	1,074.2	1,075.2	1,075.2	1,075.2	1,075.2	
2013	830.7	1,037.7	1,066.2	1,083.5	1,084.6	1,085.6	1,085.6	1,085.6		
2014	838.7	1,047.6	1,076.5	1,094.0	1,095.0	1,096.0	1,096.0			
2015	846.8	1,057.7	1,086.8	1,104.5	1,105.5	1,106.6				
2016	854.9	1,067.9	1,097.3	1,115.1	1,116.1					
2017	863.2	1,078.2	1,107.8	1,125.8						
2018	871.5	1,088.5	1,118.5							
2019	879.8	1,099.0								
2020	888.3									

Cumulative Closed with Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	375.0	555.0	585.0	600.0	609.0	616.0	622.0	625.0	627.0	628.1
2012	378.6	560.3	590.6	605.8	614.9	621.9	628.0	631.0	633.2	
2013	382.2	565.7	596.3	611.6	620.8	627.9	634.0	637.4		
2014	385.9	571.2	602.0	617.5	626.7	633.9	640.7			
2015	389.6	576.7	607.8	623.4	632.8	640.8				
2016	393.4	582.2	613.7	629.4	639.8					
2017	397.2	587.8	619.6	637.0						
2018	401.0	593.5	628.7							
2019	404.8	618.6								
2020	449.6									

Cumulative Closed without Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	275.0	375.0	390.0	402.0	411.0	417.0	421.0	424.0	426.0	426.9
2012	277.6	378.6	393.8	405.9	415.0	421.0	425.0	428.1	429.9	
2013	280.3	382.2	397.5	409.8	418.9	425.1	429.1	431.9		
2014	283.0	385.9	401.4	413.7	423.0	429.1	432.6			
2015	285.7	389.6	405.2	417.7	427.0	432.5				
2016	288.5	393.4	409.1	421.7	430.2					
2017	291.3	397.2	413.0	424.2						
2018	294.1	401.0	413.8							
2019	296.9	385.4								
2020	258.9									

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	165.0	88.0	71.0	61.0	44.0	32.0	22.0	16.0	12.0	10.0
2012	166.6	88.8	71.7	61.6	44.4	32.3	22.2	16.2	12.1	
2013	168.2	89.7	72.4	62.2	44.9	32.6	22.4	16.3		
2014	169.8	90.6	73.1	62.8	45.3	32.9	22.6			
2015	171.4	91.4	73.8	63.4	45.7	33.2				
2016	173.1	92.3	74.5	64.0	46.2					
2017	174.8	93.2	75.2	64.6						
2018	176.4	94.1	75.9							
2019	178.1	95.0								
2020	179.8									

Cumulative Reported Loss

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	3,750,000	7,500,000	10,500,000	12,600,000	13,860,000	14,970,000	15,720,000	16,030,000	16,190,000	16,301,000
2012	3,937,500	7,875,000	11,025,000	13,230,000	14,553,000	15,718,500	16,506,000	16,831,500	17,057,250	
2013	4,134,375	8,268,750	11,576,250	13,891,500	15,280,650	16,504,425	17,331,300	17,755,763		
2014	4,341,094	8,682,188	12,155,063	14,586,075	16,044,683	17,329,646	18,362,827			
2015	4,558,148	9,116,297	12,762,816	15,315,379	16,846,917	18,378,455				
2016	4,786,056	9,572,112	13,400,956	16,081,148	17,915,802					
2017	5,025,359	10,050,717	14,071,004	17,247,031						
2018	5,276,627	10,553,253	15,196,685							
2019	5,540,458	11,413,343								
2020	6,050,180									

Cumulative Paid Loss

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	1,500,000	3,750,000	6,750,000	9,450,000	11,225,000	12,725,000	14,150,000	14,900,000	15,450,000	15,791,000
2012	1,575,000	3,937,500	7,087,500	9,922,500	11,786,250	13,361,250	14,857,500	15,645,000	16,280,250	
2013	1,653,750	4,134,375	7,441,875	10,418,625	12,375,563	14,029,313	15,600,375	16,509,938		
2014	1,736,438	4,341,094	7,813,969	10,939,556	12,994,341	14,730,778	16,545,355			
2015	1,823,259	4,558,148	8,204,667	11,486,534	13,644,058	15,649,643				
2016	1,914,422	4,786,056	8,614,901	12,060,861	14,552,801					
2017	2,010,143	5,025,359	9,045,646	13,025,730						
2018	2,110,651	5,276,627	9,920,058							
2019	2,216,183	5,872,885								
2020	2,559,692									

Scenario 2: Shift in proportion of claims closed with payment
Diagnostics

Appendix 2
Exhibit 2

Case Reserves per Open Count

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	13,636	42,614	52,817	51,639	59,886	70,156	71,364	70,625	61,667	51,000
2012	14,182	44,318	54,930	53,705	62,282	72,963	74,218	73,450	64,133	
2013	14,749	46,091	57,127	55,853	64,773	75,881	77,187	76,388		
2014	15,339	47,935	59,412	58,087	67,364	78,916	80,274			
2015	15,953	49,852	61,788	60,411	70,059	82,073				
2016	16,591	51,846	64,260	62,827	72,861					
2017	17,254	53,920	66,830	65,340						
2018	17,945	56,077	69,503							
2019	18,662	58,320								
2020	19,409									

Paid Loss per Closed with Payment Count

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	4,000	6,757	11,538	15,750	18,432	20,657	22,749	23,840	24,641	25,141
2012	4,160	7,027	12,000	16,380	19,169	21,484	23,659	24,794	25,710	
2013	4,326	7,308	12,480	17,035	19,936	22,343	24,606	25,903		
2014	4,499	7,600	12,979	17,717	20,733	23,237	25,823			
2015	4,679	7,904	13,498	18,425	21,563	24,423				
2016	4,867	8,221	14,038	19,162	22,746					
2017	5,061	8,549	14,600	20,447						
2018	5,264	8,891	15,778							
2019	5,474	9,494								
2020	5,693									

Closed Counts / Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%	98.9%	99.1%
2012	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%	98.9%	
2013	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%		
2014	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%			
2015	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%				
2016	79.8%	91.4%	93.2%	94.3%	95.9%					
2017	79.8%	91.4%	93.2%	94.3%						
2018	79.8%	91.4%	93.2%							
2019	79.8%	91.4%								
2020	79.8%									

Closed with Payment Counts / Closed Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%	59.5%	59.5%
2012	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%	59.6%	
2013	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%		
2014	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.7%			
2015	57.7%	59.7%	60.0%	59.9%	59.7%	59.7%				
2016	57.7%	59.7%	60.0%	59.9%	59.8%					
2017	57.7%	59.7%	60.0%	60.0%						
2018	57.7%	59.7%	60.3%							
2019	57.7%	61.6%								
2020	63.5%									

Scenario 2: Shift in proportion of claims closed with payment **Appendix 2**
Reported Loss Development Method **Exhibit 3**

Cumulative Reported Loss

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	3,750,000	7,500,000	10,500,000	12,600,000	13,860,000	14,970,000	15,720,000	16,030,000	16,190,000	16,301,000	17,533,344
2012	3,937,500	7,875,000	11,025,000	13,230,000	14,553,000	15,718,500	16,506,000	16,831,500	17,057,250	17,174,196	18,472,552
2013	4,134,375	8,268,750	11,576,250	13,891,500	15,280,650	16,504,425	17,331,300	17,755,763	17,993,909	18,117,277	19,486,930
2014	4,341,094	8,682,188	12,155,063	14,586,075	16,044,683	17,329,646	18,362,827	18,812,552	19,064,873	19,195,583	20,646,755
2015	4,558,148	9,116,297	12,762,816	15,315,379	16,846,917	18,378,455	19,474,164	19,951,108	20,218,699	20,357,320	21,896,318
2016	4,786,056	9,572,112	13,400,956	16,081,148	17,915,802	19,544,512	20,709,741	21,216,945	21,501,514	21,648,930	23,285,573
2017	5,025,359	10,050,717	14,071,004	17,247,031	19,214,698	20,961,489	22,211,197	22,755,173	23,060,374	23,218,478	24,973,777
2018	5,276,627	10,553,253	15,196,685	18,626,793	20,751,874	22,638,408	23,988,093	24,575,587	24,905,204	25,075,956	26,971,680
2019	5,540,458	11,413,343	16,435,214	20,144,877	22,443,152	24,483,438	25,943,122	26,578,498	26,934,978	27,119,646	29,169,871
2020	6,050,180	12,463,371	17,947,254	21,998,206	24,507,922	26,735,915	28,329,889	29,023,719	29,412,996	29,614,654	31,853,500

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	2.000	1.400	1.200	1.100	1.080	1.050	1.020	1.010	1.007	
2012	2.000	1.400	1.200	1.100	1.080	1.050	1.020	1.013		
2013	2.000	1.400	1.200	1.100	1.080	1.050	1.024			
2014	2.000	1.400	1.200	1.100	1.080	1.060				
2015	2.000	1.400	1.200	1.100	1.091					
2016	2.000	1.400	1.200	1.114						
2017	2.000	1.400	1.226							
2018	2.000	1.440								
2019	2.060									
Age-to-Age	2.060	1.440	1.226	1.114	1.091	1.060	1.024	1.013	1.007	
Age-to-Ult	5.265	2.556	1.775	1.448	1.300	1.191	1.124	1.097	1.083	1.076

Scenario 2: Shift in proportion of claims closed with payment **Appendix 2**
Paid Loss Development Method **Exhibit 4**

Cumulative Paid Loss

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	1,500,000	3,750,000	6,750,000	9,450,000	11,225,000	12,725,000	14,150,000	14,900,000	15,450,000	15,791,000	17,534,423
2012	1,575,000	3,937,500	7,087,500	9,922,500	11,786,250	13,361,250	14,857,500	15,645,000	16,280,250	16,639,575	18,476,685
2013	1,653,750	4,134,375	7,441,875	10,418,625	12,375,563	14,029,313	15,600,375	16,509,938	17,180,307	17,559,497	19,498,173
2014	1,736,438	4,341,094	7,813,969	10,939,556	12,994,341	14,730,778	16,545,355	17,510,014	18,220,991	18,623,150	20,679,259
2015	1,823,259	4,558,148	8,204,667	11,486,534	13,644,058	15,649,643	17,577,408	18,602,240	19,357,565	19,784,810	21,969,174
2016	1,914,422	4,786,056	8,614,901	12,060,861	14,552,801	16,691,965	18,748,126	19,841,215	20,646,848	21,102,549	23,432,399
2017	2,010,143	5,025,359	9,045,646	13,025,730	15,717,025	18,027,322	20,247,977	21,428,512	22,298,596	22,790,753	25,306,991
2018	2,110,651	5,276,627	9,920,058	14,284,883	17,236,337	19,769,963	22,205,281	23,499,935	24,454,127	24,993,859	27,753,333
2019	2,216,183	5,872,885	11,041,025	15,899,075	19,184,043	22,003,969	24,714,478	26,155,428	27,217,444	27,818,165	30,889,460
2020	2,559,692	6,783,183	12,752,383	18,363,432	22,157,570	25,414,584	28,545,222	30,209,519	31,436,147	32,129,981	35,677,326

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	2.500	1.800	1.400	1.188	1.134	1.112	1.053	1.037	1.022	
2012	2.500	1.800	1.400	1.188	1.134	1.112	1.053	1.041		
2013	2.500	1.800	1.400	1.188	1.134	1.112	1.058			
2014	2.500	1.800	1.400	1.188	1.134	1.123				
2015	2.500	1.800	1.400	1.188	1.147					
2016	2.500	1.800	1.400	1.207						
2017	2.500	1.800	1.440							
2018	2.500	1.880								
2019	2.650									
Age-to-Age	2.650	1.880	1.440	1.207	1.147	1.123	1.058	1.041	1.022	
Age-to-Ult	13.938	5.260	2.798	1.943	1.610	1.404	1.250	1.181	1.135	1.110

Scenario 2: Shift in proportion of claims closed with payment **Appendix 2**
Disposal Rate Frequency-Severity Method **Exhibit 5**

Cumulative Non-Zero Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	540.0	643.0	656.0	661.0	653.0	648.0	644.0	641.0	639.0	638.1	633.1
2012	545.2	649.2	662.3	667.4	659.3	654.2	650.2	647.2	645.3	644.4	639.4
2013	550.4	655.4	668.7	673.8	665.6	660.5	656.4	653.7	651.9	650.9	645.8
2014	555.7	661.7	675.1	680.3	672.0	666.9	663.4	660.6	658.7	657.8	652.7
2015	561.1	668.1	681.6	686.8	678.5	674.0	670.5	667.7	665.8	664.9	659.6
2016	566.5	674.5	688.2	693.4	685.9	681.4	677.9	675.0	673.1	672.2	666.9
2017	571.9	681.0	694.8	701.7	694.1	689.5	685.9	683.1	681.1	680.2	674.8
2018	577.4	687.5	704.7	711.6	704.0	699.4	695.7	692.8	690.8	689.9	684.4
2019	583.0	713.6	731.3	738.6	730.7	725.8	722.0	719.0	717.0	716.0	710.4
2020	629.4	770.5	789.7	797.5	788.9	783.7	779.6	776.3	774.2	773.1	767.0

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	1.191	1.020	1.008	0.988	0.992	0.994	0.995	0.997	0.999	
2012	1.191	1.020	1.008	0.988	0.992	0.994	0.995	0.997		
2013	1.191	1.020	1.008	0.988	0.992	0.994	0.996			
2014	1.191	1.020	1.008	0.988	0.992	0.995				
2015	1.191	1.020	1.008	0.988	0.993					
2016	1.191	1.020	1.008	0.989						
2017	1.191	1.020	1.010							
2018	1.191	1.025								
2019	1.224									
Age-to-Age	1.224	1.025	1.010	0.989	0.993	0.995	0.996	0.997	0.999	
Age-to-Ult	1.219	0.995	0.971	0.962	0.972	0.979	0.984	0.988	0.991	0.992

Scenario 2: Shift in proportion of claims closed with payment
Disposal Rate Frequency-Severity Method

Appendix 2
Exhibit 6

Disposal Rate (Closed with Payment Counts)

Accident Year	12	24	36	48	60	72	84	96	108	120	120-Ult
2011	59%	88%	92%	95%	96%	97%	98%	99%	99%	99%	
2012	59%	88%	92%	95%	96%	97%	98%	99%	99%		
2013	59%	88%	92%	95%	96%	97%	98%	99%			
2014	59%	88%	92%	95%	96%	97%	98%				
2015	59%	87%	92%	95%	96%	97%					
2016	59%	87%	92%	94%	96%						
2017	59%	87%	92%	94%							
2018	59%	87%	92%								
2019	57%	87%									
2020	59%										
Selected	59%	87%	92%	94%	96%	97%	98%	99%	99%	99%	100%

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.1	5.0	633.1
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.2	1.1	5.0	639.4
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.4	2.2	1.1	5.1	645.8
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.8	3.4	2.3	1.1	5.2	652.7
2015	389.6	187.0	31.2	15.6	9.4	8.0	6.8	3.4	2.3	1.1	5.2	659.6
2016	393.4	188.8	31.5	15.7	10.4	8.0	6.9	3.5	2.3	1.2	5.3	666.9
2017	397.2	190.6	31.8	17.5	10.4	8.1	7.0	3.5	2.3	1.2	5.3	674.8
2018	401.0	192.5	35.3	17.4	10.5	8.2	7.1	3.5	2.4	1.2	5.4	684.4
2019	404.8	213.8	34.0	18.0	10.9	8.5	7.4	3.7	2.5	1.2	5.6	710.4
2020	449.6	218.3	36.7	19.5	11.8	9.2	7.9	4.0	2.7	1.3	6.1	767.0

Scenario 2: Shift in proportion of claims closed with payment **Appendix 2**
Disposal Rate Frequency-Severity Method **Exhibit 7**

Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	4,000	12,500	100,000	180,000	197,222	214,286	237,500	250,000	275,000	310,000	348,000
2012	4,160	13,000	104,000	187,200	205,111	222,857	247,000	260,000	286,000	322,400	361,920
2013	4,326	13,520	108,160	194,688	213,316	231,771	256,880	270,400	297,440	335,296	376,397
2014	4,499	14,061	112,486	202,476	221,848	241,042	267,155	281,216	309,338	348,708	391,453
2015	4,679	14,623	116,986	210,575	230,722	250,684	277,841	292,465	321,711	362,656	407,111
2016	4,867	15,208	121,665	218,998	239,951	260,711	288,955	304,163	334,580	377,162	423,395
2017	5,061	15,816	126,532	227,757	249,549	271,140	300,513	316,330	347,963	392,249	440,331
2018	5,264	16,449	131,593	236,868	259,531	281,985	312,534	328,983	361,881	407,939	457,944
2019	5,474	17,107	136,857	246,342	269,912	293,265	325,035	342,142	376,356	424,256	476,262
2020	5,693	17,791	142,331	256,196	280,709	304,995	338,037	355,828	391,411	441,227	495,313

Trend Rate: 4%

Trended Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000	310,000	
2012	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000		
2013	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400			
2014	5,693	17,107	131,593	227,757	239,951	250,684	267,155				
2015	5,693	17,107	131,593	227,757	239,951	250,684					
2016	5,693	17,107	131,593	227,757	239,951						
2017	5,693	17,107	131,593	227,757							
2018	5,693	17,107	131,593								
2019	5,693	17,107									
2020	5,693										
Selected	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000	310,000	334,615

Scenario 2: Shift in proportion of claims closed with payment
Disposal Rate Frequency-Severity Method

Appendix 2
Exhibit 8

Incremental Paid Loss

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	1,500,000	2,250,000	3,000,000	2,700,000	1,775,000	1,500,000	1,425,000	750,000	550,000	341,000	1,739,727	17,530,727
2012	1,575,000	2,362,500	3,150,000	2,835,000	1,863,750	1,575,000	1,496,250	787,500	635,250	356,940	1,827,285	18,464,475
2013	1,653,750	2,480,625	3,307,500	2,976,750	1,956,938	1,653,750	1,571,063	909,563	665,054	374,962	1,919,548	19,469,501
2014	1,736,438	2,604,656	3,472,875	3,125,588	2,054,784	1,736,438	1,814,577	949,998	698,958	394,077	2,017,403	20,605,791
2015	1,823,259	2,734,889	3,646,519	3,281,867	2,157,524	2,005,585	1,897,385	998,576	734,698	414,228	2,120,561	21,815,091
2016	1,914,422	2,871,634	3,828,845	3,445,960	2,491,940	2,092,182	1,995,000	1,049,950	772,496	435,538	2,229,658	23,127,625
2017	2,010,143	3,015,215	4,020,287	3,980,084	2,585,889	2,201,776	2,099,504	1,104,949	812,962	458,353	2,346,454	24,635,616
2018	2,110,651	3,165,976	4,643,431	4,115,962	2,727,600	2,322,437	2,214,560	1,165,502	857,514	483,472	2,475,044	26,282,149
2019	2,216,183	3,656,702	4,648,125	4,442,742	2,944,154	2,506,824	2,390,382	1,258,036	925,595	521,856	2,671,546	28,182,145
2020	2,559,692	3,883,976	5,219,457	4,988,829	3,306,039	2,814,954	2,684,200	1,412,669	1,039,366	586,001	2,999,923	31,495,108

Scenario 2: Shift in proportion of claims closed with payment **Appendix 2**
Incremental Method **Exhibit 9**

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	815.0	1,018.0	1,046.0	1,063.0	1,064.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0
2012	822.8	1,027.8	1,056.1	1,073.2	1,074.2	1,075.2	1,075.2	1,075.2	1,075.2	1,075.2	1,075.2
2013	830.7	1,037.7	1,066.2	1,083.5	1,084.6	1,085.6	1,085.6	1,085.6	1,085.6	1,085.6	1,085.6
2014	838.7	1,047.6	1,076.5	1,094.0	1,095.0	1,096.0	1,096.0	1,096.0	1,096.0	1,096.0	1,096.0
2015	846.8	1,057.7	1,086.8	1,104.5	1,105.5	1,106.6	1,106.6	1,106.6	1,106.6	1,106.6	1,106.6
2016	854.9	1,067.9	1,097.3	1,115.1	1,116.1	1,117.2	1,117.2	1,117.2	1,117.2	1,117.2	1,117.2
2017	863.2	1,078.2	1,107.8	1,125.8	1,126.9	1,127.9	1,127.9	1,127.9	1,127.9	1,127.9	1,127.9
2018	871.5	1,088.5	1,118.5	1,136.6	1,137.7	1,138.8	1,138.8	1,138.8	1,138.8	1,138.8	1,138.8
2019	879.8	1,099.0	1,129.2	1,147.6	1,148.7	1,149.7	1,149.7	1,149.7	1,149.7	1,149.7	1,149.7
2020	888.3	1,109.6	1,140.1	1,158.6	1,159.7	1,160.8	1,160.8	1,160.8	1,160.8	1,160.8	1,160.8

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000	1.000	
2012	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000		
2013	1.249	1.028	1.016	1.001	1.001	1.000	1.000			
2014	1.249	1.028	1.016	1.001	1.001	1.000				
2015	1.249	1.028	1.016	1.001	1.001					
2016	1.249	1.028	1.016	1.001						
2017	1.249	1.028	1.016							
2018	1.249	1.028								
2019	1.249									
Age-to-Age	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000	1.000	
Age-to-Ult	1.307	1.046	1.018	1.002	1.001	1.000	1.000	1.000	1.000	1.000

Scenario 2: Shift in proportion of claims closed with payment
Incremental Method

Appendix 2
Exhibit 10

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	815.0	368.0	116.0	88.0	62.0	45.0	32.0	22.0	16.0	12.0
2012	822.8	371.5	117.1	88.8	62.6	45.4	32.3	22.2	16.2	
2013	830.7	375.1	118.2	89.7	63.2	45.9	32.6	22.4		
2014	838.7	378.7	119.4	90.6	63.8	46.3	32.9			
2015	846.8	382.4	120.5	91.4	64.4	46.8				
2016	854.9	386.0	121.7	92.3	65.0					
2017	863.2	389.7	122.9	93.2						
2018	871.5	393.5	124.0							
2019	879.8	397.3								
2020	888.3									

Scenario 2: Shift in proportion of claims closed with payment
Incremental Method

Appendix 2
Exhibit 11

Incremental Closed Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	650.0	280.0	45.0	27.0	18.0	13.0	10.0	6.0	4.0	2.0	
2012	656.3	282.7	45.4	27.3	18.2	13.1	10.1	6.1	4.0		
2013	662.6	285.4	45.9	27.5	18.3	13.3	10.2	6.1			
2014	668.9	288.2	46.3	27.8	18.5	13.4	10.3				
2015	675.4	290.9	46.8	28.1	18.7	13.5					
2016	681.9	293.7	47.2	28.3	18.9						
2017	688.4	296.5	47.7	28.6							
2018	695.0	299.4	48.1								
2019	701.7	302.3									
2020	708.5										

Incremental Closure Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	80%	76%	39%	31%	29%	29%	31%	27%	25%	17%	
2012	80%	76%	39%	31%	29%	29%	31%	27%	25%		
2013	80%	76%	39%	31%	29%	29%	31%	27%			
2014	80%	76%	39%	31%	29%	29%	31%				
2015	80%	76%	39%	31%	29%	29%					
2016	80%	76%	39%	31%	29%						
2017	80%	76%	39%	31%							
2018	80%	76%	39%								
2019	80%	76%									
2020	80%										
Selected	80%	76%	39%	31%	29%	29%	31%	27%	25%	17%	100%

Scenario 2: Shift in proportion of claims closed with payment
Incremental Method

Appendix 2
Exhibit 12

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.1	
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.2		
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.4			
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.8				
2015	389.6	187.0	31.2	15.6	9.4	8.0					
2016	393.4	188.8	31.5	15.7	10.4						
2017	397.2	190.6	31.8	17.5							
2018	401.0	192.5	35.3								
2019	404.8	213.8									
2020	449.6										

Incremental Closed with Payment Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	46%	49%	26%	17%	15%	16%	19%	14%	13%	9%	
2012	46%	49%	26%	17%	15%	16%	19%	14%	14%		
2013	46%	49%	26%	17%	15%	16%	19%	15%			
2014	46%	49%	26%	17%	15%	16%	21%				
2015	46%	49%	26%	17%	15%	17%					
2016	46%	49%	26%	17%	16%						
2017	46%	49%	26%	19%							
2018	46%	49%	28%								
2019	46%	54%									
2020	51%										
Selected	51%	54%	28%	19%	16%	17%	21%	15%	14%	9%	50%

Scenario 2: Shift in proportion of claims closed with payment
Incremental Method

Appendix 2
Exhibit 13

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	275.0	100.0	15.0	12.0	9.0	6.0	4.0	3.0	2.0	0.9	
2012	277.6	101.0	15.1	12.1	9.1	6.1	4.0	3.0	1.8		
2013	280.3	101.9	15.3	12.2	9.2	6.1	4.1	2.8			
2014	283.0	102.9	15.4	12.3	9.3	6.2	3.5				
2015	285.7	103.9	15.6	12.5	9.4	5.5					
2016	288.5	104.9	15.7	12.6	8.5						
2017	291.3	105.9	15.9	11.1							
2018	294.1	106.9	12.8								
2019	296.9	88.5									
2020	258.9										

Incremental Closed with Payment Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	34%	27%	13%	14%	15%	13%	13%	14%	13%	7%	
2012	34%	27%	13%	14%	15%	13%	12%	14%	11%		
2013	34%	27%	13%	14%	15%	13%	13%	12%			
2014	34%	27%	13%	14%	15%	13%	11%				
2015	34%	27%	13%	14%	15%	12%					
2016	34%	27%	13%	14%	13%						
2017	34%	27%	13%	12%							
2018	34%	27%	10%								
2019	34%	22%									
2020	29%										
Implied	29%	22%	10%	12%	13%	12%	11%	12%	11%	7%	50%

Scenario 2: Shift in proportion of claims closed with payment
Incremental Method

Appendix 2
Exhibit 14

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	815.0	368.0	116.0	88.0	62.0	45.0	32.0	22.0	16.0	12.0	10.0	0.0
2012	822.8	371.5	117.1	88.8	62.6	45.4	32.3	22.2	16.2	12.1	10.1	0.0
2013	830.7	375.1	118.2	89.7	63.2	45.9	32.6	22.4	16.3	12.2	10.2	0.0
2014	838.7	378.7	119.4	90.6	63.8	46.3	32.9	22.6	16.5	12.3	10.3	0.0
2015	846.8	382.4	120.5	91.4	64.4	46.8	33.2	22.9	16.6	12.5	10.4	0.0
2016	854.9	386.0	121.7	92.3	65.0	47.2	33.6	23.1	16.8	12.6	10.5	0.0
2017	863.2	389.7	122.9	93.2	65.7	47.7	33.9	23.3	16.9	12.7	10.6	0.0
2018	871.5	393.5	124.0	94.1	66.3	48.1	34.2	23.5	17.1	12.8	10.7	0.0
2019	879.8	397.3	125.2	95.0	66.9	48.6	34.5	23.8	17.3	13.0	10.8	0.0
2020	888.3	401.1	126.4	95.9	67.6	49.0	34.9	24.0	17.4	13.1	10.9	0.0

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.1	5.0	633.1
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.2	1.1	5.0	639.4
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.4	2.2	1.1	5.1	645.8
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.8	3.4	2.3	1.1	5.1	652.7
2015	389.6	187.0	31.2	15.6	9.4	8.0	6.9	3.4	2.3	1.1	5.2	659.7
2016	393.4	188.8	31.5	15.7	10.4	8.1	6.9	3.5	2.3	1.2	5.2	667.0
2017	397.2	190.6	31.8	17.5	10.5	8.2	7.0	3.5	2.3	1.2	5.3	675.0
2018	401.0	192.5	35.3	17.6	10.6	8.2	7.1	3.5	2.4	1.2	5.3	684.7
2019	404.8	213.8	35.6	17.8	10.7	8.3	7.1	3.6	2.4	1.2	5.4	710.7
2020	449.6	215.8	36.0	18.0	10.8	8.4	7.2	3.6	2.4	1.2	5.4	758.4

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	275.0	100.0	15.0	12.0	9.0	6.0	4.0	3.0	2.0	0.9	5.0	431.9
2012	277.6	101.0	15.1	12.1	9.1	6.1	4.0	3.0	1.8	0.9	5.0	435.9
2013	280.3	101.9	15.3	12.2	9.2	6.1	4.1	2.8	1.8	0.9	5.1	439.7
2014	283.0	102.9	15.4	12.3	9.3	6.2	3.5	2.8	1.9	0.9	5.1	443.3
2015	285.7	103.9	15.6	12.5	9.4	5.5	3.5	2.8	1.9	0.9	5.2	446.9
2016	288.5	104.9	15.7	12.6	8.5	5.6	3.6	2.8	1.9	0.9	5.2	450.2
2017	291.3	105.9	15.9	11.1	8.6	5.6	3.6	2.9	1.9	1.0	5.3	453.0
2018	294.1	106.9	12.8	11.2	8.7	5.7	3.6	2.9	1.9	1.0	5.3	454.1
2019	296.9	88.5	13.0	11.3	8.7	5.7	3.7	2.9	1.9	1.0	5.4	439.1
2020	258.9	89.4	13.1	11.4	8.8	5.8	3.7	2.9	2.0	1.0	5.4	402.4

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	165.0	88.0	71.0	61.0	44.0	32.0	22.0	16.0	12.0	10.0	0.0
2012	166.6	88.8	71.7	61.6	44.4	32.3	22.2	16.2	12.1	10.1	0.0
2013	168.2	89.7	72.4	62.2	44.9	32.6	22.4	16.3	12.2	10.2	0.0
2014	169.8	90.6	73.1	62.8	45.3	32.9	22.6	16.5	12.3	10.3	0.0
2015	171.4	91.4	73.8	63.4	45.7	33.2	22.9	16.6	12.5	10.4	0.0
2016	173.1	92.3	74.5	64.0	46.2	33.6	23.1	16.8	12.6	10.5	0.0
2017	174.8	93.2	75.2	64.6	46.6	33.9	23.3	16.9	12.7	10.6	0.0
2018	176.4	94.1	75.9	65.2	47.0	34.2	23.5	17.1	12.8	10.7	0.0
2019	178.1	95.0	76.6	65.9	47.5	34.5	23.8	17.3	13.0	10.8	0.0
2020	179.8	95.9	77.4	66.5	48.0	34.9	24.0	17.4	13.1	10.9	0.0

Scenario 2: Shift in proportion of claims closed with payment **Appendix 2**
Incremental Method **Exhibit 15**

Incremental Paid Loss

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	1,500,000	2,250,000	3,000,000	2,700,000	1,775,000	1,500,000	1,425,000	750,000	550,000	341,000	1,740,000	17,531,000
2012	1,575,000	2,362,500	3,150,000	2,835,000	1,863,750	1,575,000	1,496,250	787,500	635,250	358,050	1,827,000	18,465,300
2013	1,653,750	2,480,625	3,307,500	2,976,750	1,956,938	1,653,750	1,571,063	909,563	667,013	375,953	1,918,350	19,471,253
2014	1,736,438	2,604,656	3,472,875	3,125,588	2,054,784	1,736,438	1,814,577	955,041	700,363	394,750	2,014,267	20,609,777
2015	1,823,259	2,734,889	3,646,519	3,281,867	2,157,524	2,005,585	1,905,306	1,002,793	735,381	414,488	2,114,981	21,822,591
2016	1,914,422	2,871,634	3,828,845	3,445,960	2,491,940	2,105,865	2,000,571	1,052,932	772,150	435,212	2,220,730	23,140,261
2017	2,010,143	3,015,215	4,020,287	3,980,084	2,616,537	2,211,158	2,100,600	1,105,579	810,758	456,973	2,331,766	24,659,100
2018	2,110,651	3,165,976	4,643,431	4,179,088	2,747,364	2,321,716	2,205,630	1,160,858	851,296	479,821	2,448,355	26,314,185
2019	2,216,183	3,656,702	4,875,603	4,388,043	2,884,732	2,437,801	2,315,911	1,218,901	893,861	503,812	2,570,772	27,962,322
2020	2,559,692	3,839,537	5,119,383	4,607,445	3,028,968	2,559,692	2,431,707	1,279,846	938,554	529,003	2,699,311	29,593,137

Scenario 2: Shift in proportion of claims closed with payment **Appendix 2**
Incremental Method **Exhibit 16**

Incremental Closed with Payment Counts: Base Case

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	5.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	1.0	5.0
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1	2.0	1.0	5.1
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2	3.1	2.1	1.0	5.1
2015	389.6	187.0	31.2	15.6	9.4	7.3	6.2	3.1	2.1	1.0	5.2
2016	393.4	188.8	31.5	15.7	9.4	7.3	6.3	3.1	2.1	1.0	5.2
2017	397.2	190.6	31.8	15.9	9.5	7.4	6.4	3.2	2.1	1.1	5.3
2018	401.0	192.5	32.1	16.0	9.6	7.5	6.4	3.2	2.1	1.1	5.3
2019	404.8	194.3	32.4	16.2	9.7	7.6	6.5	3.2	2.2	1.1	5.4
2020	408.7	196.2	32.7	16.3	9.8	7.6	6.5	3.3	2.2	1.1	5.4

Incremental Closed with Payment Counts: Scenario 2

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.1	5.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.2	1.1	5.0
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.4	2.2	1.1	5.1
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.8	3.4	2.3	1.1	5.1
2015	389.6	187.0	31.2	15.6	9.4	8.0	6.9	3.4	2.3	1.1	5.2
2016	393.4	188.8	31.5	15.7	10.4	8.1	6.9	3.5	2.3	1.2	5.2
2017	397.2	190.6	31.8	17.5	10.5	8.2	7.0	3.5	2.3	1.2	5.3
2018	401.0	192.5	35.3	17.6	10.6	8.2	7.1	3.5	2.4	1.2	5.3
2019	404.8	213.8	35.6	17.8	10.7	8.3	7.1	3.6	2.4	1.2	5.4
2020	449.6	215.8	36.0	18.0	10.8	8.4	7.2	3.6	2.4	1.2	5.4

Comparison of Scenarios

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	100%	100%	100%	100%	100%	100%	100%	100%	100%	110%	100%
2012	100%	100%	100%	100%	100%	100%	100%	100%	110%	110%	100%
2013	100%	100%	100%	100%	100%	100%	100%	110%	110%	110%	100%
2014	100%	100%	100%	100%	100%	100%	110%	110%	110%	110%	100%
2015	100%	100%	100%	100%	100%	110%	110%	110%	110%	110%	100%
2016	100%	100%	100%	100%	110%	110%	110%	110%	110%	110%	100%
2017	100%	100%	100%	110%	110%	110%	110%	110%	110%	110%	100%
2018	100%	100%	110%	110%	110%	110%	110%	110%	110%	110%	100%
2019	100%	110%	110%	110%	110%	110%	110%	110%	110%	110%	100%
2020	110%	110%	110%	110%	110%	110%	110%	110%	110%	110%	100%

Scenario 3: Shift in rate of claim closure
Data

Appendix 3
Exhibit 1

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	815.0	1,018.0	1,046.0	1,063.0	1,064.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0
2012	822.8	1,027.8	1,056.1	1,073.2	1,074.2	1,075.2	1,075.2	1,075.2	1,075.2	
2013	830.7	1,037.7	1,066.2	1,083.5	1,084.6	1,085.6	1,085.6	1,085.6		
2014	838.7	1,047.6	1,076.5	1,094.0	1,095.0	1,096.0	1,096.0			
2015	846.8	1,057.7	1,086.8	1,104.5	1,105.5	1,106.6				
2016	854.9	1,067.9	1,097.3	1,115.1	1,116.1					
2017	863.2	1,078.2	1,107.8	1,125.8						
2018	871.5	1,088.5	1,118.5							
2019	879.8	1,099.0								
2020	888.3									

Cumulative Closed with Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	375.0	555.0	585.0	600.0	609.0	616.0	622.0	625.0	627.0	628.0
2012	378.6	560.3	590.6	605.8	614.9	621.9	628.0	631.0	633.0	
2013	382.2	565.7	596.3	611.6	620.8	627.9	634.0	637.1		
2014	385.9	571.2	602.0	617.5	626.7	633.9	640.1			
2015	389.6	576.7	607.8	623.4	632.8	640.0				
2016	393.4	582.2	613.7	629.4	637.4					
2017	397.2	587.8	619.6	633.1						
2018	401.0	593.5	620.7							
2019	404.8	570.0								
2020	347.4									

Cumulative Closed without Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	275.0	375.0	390.0	402.0	411.0	417.0	421.0	424.0	426.0	427.0
2012	277.6	378.6	393.8	405.9	415.0	421.0	425.0	428.1	430.1	
2013	280.3	382.2	397.5	409.8	418.9	425.1	429.1	432.2		
2014	283.0	385.9	401.4	413.7	423.0	429.1	433.3			
2015	285.7	389.6	405.2	417.7	427.0	433.3				
2016	288.5	393.4	409.1	421.7	429.7					
2017	291.3	397.2	413.0	423.9						
2018	294.1	401.0	414.6							
2019	296.9	388.6								
2020	254.8									

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	165.0	88.0	71.0	61.0	44.0	32.0	22.0	16.0	12.0	10.0
2012	166.6	88.8	71.7	61.6	44.4	32.3	22.2	16.2	12.1	
2013	168.2	89.7	72.4	62.2	44.9	32.6	22.4	16.3		
2014	169.8	90.6	73.1	62.8	45.3	32.9	22.6			
2015	171.4	91.4	73.8	63.4	45.7	33.2				
2016	173.1	92.3	74.5	64.0	49.0					
2017	174.8	93.2	75.2	68.9						
2018	176.4	94.1	83.1							
2019	178.1	140.3								
2020	286.1									

Cumulative Reported Loss

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	3,750,000	7,500,000	10,500,000	12,600,000	13,860,000	14,970,000	15,720,000	16,030,000	16,190,000	16,270,000
2012	3,937,500	7,875,000	11,025,000	13,230,000	14,553,000	15,718,500	16,506,000	16,831,500	16,999,500	
2013	4,134,375	8,268,750	11,576,250	13,891,500	15,280,650	16,504,425	17,331,300	17,673,075		
2014	4,341,094	8,682,188	12,155,063	14,586,075	16,044,683	17,329,646	18,197,865			
2015	4,558,148	9,116,297	12,762,816	15,315,379	16,846,917	18,196,129				
2016	4,786,056	9,572,112	13,400,956	16,081,148	17,555,819					
2017	5,025,359	10,050,717	14,071,004	16,622,733						
2018	5,276,627	10,553,253	14,643,010							
2019	5,540,458	13,226,584								
2020	7,530,993									

Cumulative Paid Loss

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	1,500,000	3,750,000	6,750,000	9,450,000	11,225,000	12,725,000	14,150,000	14,900,000	15,450,000	15,760,000
2012	1,575,000	3,937,500	7,087,500	9,922,500	11,786,250	13,361,250	14,857,500	15,645,000	16,222,500	
2013	1,653,750	4,134,375	7,441,875	10,418,625	12,375,563	14,029,313	15,600,375	16,427,250		
2014	1,736,438	4,341,094	7,813,969	10,939,556	12,994,341	14,730,778	16,380,394			
2015	1,823,259	4,558,148	8,204,667	11,486,534	13,644,058	15,467,317				
2016	1,914,422	4,786,056	8,614,901	12,060,861	13,986,451					
2017	2,010,143	5,025,359	9,045,646	12,121,165						
2018	2,110,651	5,276,627	8,864,733							
2019	2,216,183	5,041,817								
2020	1,977,943									

Scenario 3: Shift in rate of claim closure
Diagnostics

Appendix 3
Exhibit 2

Case Reserves per Open Count

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	13,636	42,614	52,817	51,639	59,886	70,156	71,364	70,625	61,667	51,000
2012	14,182	44,318	54,930	53,705	62,282	72,963	74,218	73,450	64,133	
2013	14,749	46,091	57,127	55,853	64,773	75,881	77,187	76,388		
2014	15,339	47,935	59,412	58,087	67,364	78,916	80,274			
2015	15,953	49,852	61,788	60,411	70,059	82,073				
2016	16,591	51,846	64,260	62,827	72,861					
2017	17,254	53,920	66,830	65,340						
2018	17,945	56,077	69,503							
2019	18,662	58,320								
2020	19,409									

Paid Loss per Closed with Payment Count

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	4,000	6,757	11,538	15,750	18,432	20,657	22,749	23,840	24,641	25,096
2012	4,160	7,027	12,000	16,380	19,169	21,484	23,659	24,794	25,627	
2013	4,326	7,308	12,480	17,035	19,936	22,343	24,606	25,785		
2014	4,499	7,600	12,979	17,717	20,733	23,237	25,590			
2015	4,679	7,904	13,498	18,425	21,563	24,166				
2016	4,867	8,221	14,038	19,162	21,942					
2017	5,061	8,549	14,600	19,146						
2018	5,264	8,891	14,281							
2019	5,474	8,845								
2020	5,693									

Closed Counts / Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%	98.9%	99.1%
2012	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%	98.9%	
2013	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%	98.5%		
2014	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%	97.9%			
2015	79.8%	91.4%	93.2%	94.3%	95.9%	97.0%				
2016	79.8%	91.4%	93.2%	94.3%	95.6%					
2017	79.8%	91.4%	93.2%	93.9%						
2018	79.8%	91.4%	92.6%							
2019	79.8%	87.2%								
2020	67.8%									

Closed with Payment Counts / Closed Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%	59.5%	59.5%
2012	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%	59.5%	
2013	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%	59.6%		
2014	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%	59.6%			
2015	57.7%	59.7%	60.0%	59.9%	59.7%	59.6%				
2016	57.7%	59.7%	60.0%	59.9%	59.7%					
2017	57.7%	59.7%	60.0%	59.9%						
2018	57.7%	59.7%	60.0%							
2019	57.7%	59.5%								
2020	57.7%									

Scenario 3: Shift in rate of claim closure
Reported Loss Development Method

Appendix 3
Exhibit 3

Cumulative Reported Loss

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	3,750,000	7,500,000	10,500,000	12,600,000	13,860,000	14,970,000	15,720,000	16,030,000	16,190,000	16,270,000	17,500,000
2012	3,937,500	7,875,000	11,025,000	13,230,000	14,553,000	15,718,500	16,506,000	16,831,500	16,999,500	17,083,500	18,375,000
2013	4,134,375	8,268,750	11,576,250	13,891,500	15,280,650	16,504,425	17,331,300	17,673,075	17,849,475	17,937,675	19,293,750
2014	4,341,094	8,682,188	12,155,063	14,586,075	16,044,683	17,329,646	18,197,865	18,556,729	18,741,949	18,834,559	20,258,438
2015	4,558,148	9,116,297	12,762,816	15,315,379	16,846,917	18,196,129	19,107,758	19,484,565	19,679,046	19,776,287	21,271,359
2016	4,786,056	9,572,112	13,400,956	16,081,148	17,555,819	18,961,804	19,911,794	20,304,457	20,507,121	20,608,454	22,166,438
2017	5,025,359	10,050,717	14,071,004	16,622,733	18,285,006	19,749,390	20,738,838	21,147,810	21,358,893	21,464,434	23,087,129
2018	5,276,627	10,553,253	14,643,010	17,571,612	19,328,774	20,876,749	21,922,678	22,354,996	22,578,127	22,689,693	24,405,017
2019	5,540,458	13,226,584	18,517,218	22,220,661	24,442,727	26,400,262	27,722,920	28,269,619	28,551,786	28,692,870	30,862,030
2020	7,530,993	15,061,987	21,086,781	25,304,138	27,834,551	30,063,725	31,569,924	32,192,486	32,513,809	32,674,470	35,144,636

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	2.000	1.400	1.200	1.100	1.080	1.050	1.020	1.010	1.005	
2012	2.000	1.400	1.200	1.100	1.080	1.050	1.020	1.010		
2013	2.000	1.400	1.200	1.100	1.080	1.050	1.020			
2014	2.000	1.400	1.200	1.100	1.080	1.050				
2015	2.000	1.400	1.200	1.100	1.080					
2016	2.000	1.400	1.200	1.092						
2017	2.000	1.400	1.181							
2018	2.000	1.388								
2019	2.387									
Age-to-Age	2.000	1.400	1.200	1.100	1.080	1.050	1.020	1.010	1.005	
Age-to-Ult	4.667	2.333	1.667	1.389	1.263	1.169	1.113	1.092	1.081	1.076

Scenario 3: Shift in rate of claim closure **Appendix 3**
Paid Loss Development Method **Exhibit 4**

Cumulative Paid Loss

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	1,500,000	3,750,000	6,750,000	9,450,000	11,225,000	12,725,000	14,150,000	14,900,000	15,450,000	15,760,000	17,500,000
2012	1,575,000	3,937,500	7,087,500	9,922,500	11,786,250	13,361,250	14,857,500	15,645,000	16,222,500	16,548,000	18,375,000
2013	1,653,750	4,134,375	7,441,875	10,418,625	12,375,563	14,029,313	15,600,375	16,427,250	17,033,625	17,375,400	19,293,750
2014	1,736,438	4,341,094	7,813,969	10,939,556	12,994,341	14,730,778	16,380,394	17,248,613	17,885,306	18,244,170	20,258,438
2015	1,823,259	4,558,148	8,204,667	11,486,534	13,644,058	15,467,317	17,199,413	18,111,043	18,779,572	19,156,379	21,271,359
2016	1,914,422	4,786,056	8,614,901	12,060,861	13,986,451	15,855,464	17,631,027	18,565,534	19,250,838	19,637,101	21,805,157
2017	2,010,143	5,025,359	9,045,646	12,121,165	14,397,892	16,321,886	18,149,681	19,111,678	19,817,143	20,214,768	22,446,602
2018	2,110,651	5,276,627	8,864,733	12,410,626	14,741,722	16,711,663	18,583,106	19,568,077	20,290,388	20,697,509	22,982,640
2019	2,216,183	5,041,817	9,075,270	12,705,378	15,091,838	17,108,565	19,024,455	20,032,818	20,772,285	21,189,075	23,528,478
2020	1,977,943	4,944,859	8,900,746	12,461,044	14,801,610	16,779,554	18,658,600	19,647,572	20,372,818	20,781,593	23,076,007

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	2.500	1.800	1.400	1.188	1.134	1.112	1.053	1.037	1.020	
2012	2.500	1.800	1.400	1.188	1.134	1.112	1.053	1.037		
2013	2.500	1.800	1.400	1.188	1.134	1.112	1.053			
2014	2.500	1.800	1.400	1.188	1.134	1.112				
2015	2.500	1.800	1.400	1.188	1.134					
2016	2.500	1.800	1.400	1.160						
2017	2.500	1.800	1.340							
2018	2.500	1.680								
2019	2.275									
Age-to-Age	2.500	1.800	1.400	1.188	1.134	1.112	1.053	1.037	1.020	
Age-to-Ult	11.667	4.667	2.593	1.852	1.559	1.375	1.237	1.174	1.133	1.110

Scenario 3: Shift in rate of claim closure **Appendix 3**
Disposal Rate Frequency-Severity Method **Exhibit 5**

Cumulative Non-Zero Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	540.0	643.0	656.0	661.0	653.0	648.0	644.0	641.0	639.0	638.0	633.0
2012	545.2	649.2	662.3	667.4	659.3	654.2	650.2	647.2	645.1	644.1	639.1
2013	550.4	655.4	668.7	673.8	665.6	660.5	656.4	653.4	651.3	650.3	645.2
2014	555.7	661.7	675.1	680.3	672.0	666.9	662.8	659.7	657.6	656.6	651.4
2015	561.1	668.1	681.6	686.8	678.5	673.3	669.1	666.0	663.9	662.9	657.7
2016	566.5	674.5	688.2	693.4	686.4	681.2	677.0	673.8	671.7	670.7	665.4
2017	571.9	681.0	694.8	702.0	693.5	688.2	683.9	680.7	678.6	677.5	672.2
2018	577.4	687.5	703.9	709.2	700.6	695.3	691.0	687.8	685.6	684.5	679.2
2019	583.0	710.4	724.7	730.2	721.4	715.9	711.5	708.1	705.9	704.8	699.3
2020	633.5	754.4	769.6	775.5	766.1	760.2	755.5	752.0	749.7	748.5	742.6

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	1.191	1.020	1.008	0.988	0.992	0.994	0.995	0.997	0.998	
2012	1.191	1.020	1.008	0.988	0.992	0.994	0.995	0.997		
2013	1.191	1.020	1.008	0.988	0.992	0.994	0.995			
2014	1.191	1.020	1.008	0.988	0.992	0.994				
2015	1.191	1.020	1.008	0.988	0.992					
2016	1.191	1.020	1.008	0.990						
2017	1.191	1.020	1.010							
2018	1.191	1.024								
2019	1.219									
Age-to-Age	1.191	1.020	1.008	0.988	0.992	0.994	0.995	0.997	0.998	
Age-to-Ult	1.172	0.984	0.965	0.958	0.969	0.977	0.983	0.988	0.991	0.992

Scenario 3: Shift in rate of claim closure
Disposal Rate Frequency-Severity Method

Appendix 3
Exhibit 6

Disposal Rate (Closed with Payment Counts)

Accident Year	12	24	36	48	60	72	84	96	108	120	120-Ult
2011	59%	88%	92%	95%	96%	97%	98%	99%	99%	99%	
2012	59%	88%	92%	95%	96%	97%	98%	99%	99%		
2013	59%	88%	92%	95%	96%	97%	98%	99%			
2014	59%	88%	92%	95%	96%	97%	98%				
2015	59%	88%	92%	95%	96%	97%					
2016	59%	87%	92%	95%	96%						
2017	59%	87%	92%	94%							
2018	59%	87%	91%								
2019	58%	82%									
2020	47%										
Selected	47%	82%	91%	94%	96%	97%	98%	99%	99%	99%	100%

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	5.0	633.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	1.0	5.0	639.1
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1	2.0	1.0	5.1	645.2
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2	3.1	2.1	1.0	5.1	651.4
2015	389.6	187.0	31.2	15.6	9.4	7.3	6.2	3.1	2.1	1.0	5.2	657.7
2016	393.4	188.8	31.5	15.7	8.0	10.1	6.3	3.2	2.1	1.1	5.3	665.4
2017	397.2	190.6	31.8	13.5	10.9	10.2	6.4	3.2	2.1	1.1	5.3	672.2
2018	401.0	192.5	27.3	18.9	11.0	10.3	6.4	3.2	2.1	1.1	5.4	679.2
2019	404.8	165.2	69.1	19.5	11.3	10.6	6.6	3.3	2.2	1.1	5.5	699.3
2020	347.4	257.9	73.4	20.7	12.0	11.3	7.0	3.5	2.3	1.2	5.9	742.6

Scenario 3: Shift in rate of claim closure **Appendix 3**
Disposal Rate Frequency-Severity Method **Exhibit 7**

Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	4,000	12,500	100,000	180,000	197,222	214,286	237,500	250,000	275,000	310,000	348,000
2012	4,160	13,000	104,000	187,200	205,111	222,857	247,000	260,000	286,000	322,400	361,920
2013	4,326	13,520	108,160	194,688	213,316	231,771	256,880	270,400	297,440	335,296	376,397
2014	4,499	14,061	112,486	202,476	221,848	241,042	267,155	281,216	309,338	348,708	391,453
2015	4,679	14,623	116,986	210,575	230,722	250,684	277,841	292,465	321,711	362,656	407,111
2016	4,867	15,208	121,665	218,998	239,951	260,711	288,955	304,163	334,580	377,162	423,395
2017	5,061	15,816	126,532	227,757	249,549	271,140	300,513	316,330	347,963	392,249	440,331
2018	5,264	16,449	131,593	236,868	259,531	281,985	312,534	328,983	361,881	407,939	457,944
2019	5,474	17,107	136,857	246,342	269,912	293,265	325,035	342,142	376,356	424,256	476,262
2020	5,693	17,791	142,331	256,196	280,709	304,995	338,037	355,828	391,411	441,227	495,313

Trend Rate: 4%

Trended Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000	310,000	
2012	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000		
2013	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400			
2014	5,693	17,107	131,593	227,757	239,951	250,684	267,155				
2015	5,693	17,107	131,593	227,757	239,951	250,684					
2016	5,693	17,107	131,593	227,757	239,951						
2017	5,693	17,107	131,593	227,757							
2018	5,693	17,107	131,593								
2019	5,693	17,107									
2020	5,693										
Selected	5,693	17,107	131,593	227,757	239,951	250,684	267,155	270,400	286,000	310,000	334,615

Scenario 3: Shift in rate of claim closure **Appendix 3**
Disposal Rate Frequency-Severity Method **Exhibit 8**

Incremental Paid Loss

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	1,500,000	2,250,000	3,000,000	2,700,000	1,775,000	1,500,000	1,425,000	750,000	550,000	310,000	1,740,000	17,500,000
2012	1,575,000	2,362,500	3,150,000	2,835,000	1,863,750	1,575,000	1,496,250	787,500	577,500	325,500	1,827,000	18,375,000
2013	1,653,750	2,480,625	3,307,500	2,976,750	1,956,938	1,653,750	1,571,063	826,875	606,375	341,775	1,918,350	19,293,750
2014	1,736,438	2,604,656	3,472,875	3,125,588	2,054,784	1,736,438	1,649,616	868,219	636,694	358,864	2,014,268	20,258,438
2015	1,823,259	2,734,889	3,646,519	3,281,867	2,157,524	1,823,259	1,732,096	911,630	668,528	376,807	2,114,981	21,271,359
2016	1,914,422	2,871,634	3,828,845	3,445,960	1,925,590	2,631,922	1,822,461	959,190	703,406	396,465	2,225,321	22,725,217
2017	2,010,143	3,015,215	4,020,287	3,075,519	2,721,635	2,765,327	1,914,837	1,007,809	739,060	416,561	2,338,116	24,024,509
2018	2,110,651	3,165,976	3,588,106	4,475,753	2,859,731	2,905,640	2,011,995	1,058,945	776,560	437,697	2,456,752	25,847,806
2019	2,216,183	2,825,634	9,457,409	4,792,734	3,062,262	3,111,422	2,154,488	1,133,941	831,557	468,696	2,630,744	32,685,070
2020	1,977,943	4,588,585	10,445,175	5,293,305	3,382,096	3,436,391	2,379,511	1,252,374	918,408	517,648	2,905,508	37,096,946

Scenario 3: Shift in rate of claim closure **Appendix 3**
Incremental Method **Exhibit 9**

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	815.0	1,018.0	1,046.0	1,063.0	1,064.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0	1,065.0
2012	822.8	1,027.8	1,056.1	1,073.2	1,074.2	1,075.2	1,075.2	1,075.2	1,075.2	1,075.2	1,075.2
2013	830.7	1,037.7	1,066.2	1,083.5	1,084.6	1,085.6	1,085.6	1,085.6	1,085.6	1,085.6	1,085.6
2014	838.7	1,047.6	1,076.5	1,094.0	1,095.0	1,096.0	1,096.0	1,096.0	1,096.0	1,096.0	1,096.0
2015	846.8	1,057.7	1,086.8	1,104.5	1,105.5	1,106.6	1,106.6	1,106.6	1,106.6	1,106.6	1,106.6
2016	854.9	1,067.9	1,097.3	1,115.1	1,116.1	1,117.2	1,117.2	1,117.2	1,117.2	1,117.2	1,117.2
2017	863.2	1,078.2	1,107.8	1,125.8	1,126.9	1,127.9	1,127.9	1,127.9	1,127.9	1,127.9	1,127.9
2018	871.5	1,088.5	1,118.5	1,136.6	1,137.7	1,138.8	1,138.8	1,138.8	1,138.8	1,138.8	1,138.8
2019	879.8	1,099.0	1,129.2	1,147.6	1,148.7	1,149.7	1,149.7	1,149.7	1,149.7	1,149.7	1,149.7
2020	888.3	1,109.6	1,140.1	1,158.6	1,159.7	1,160.8	1,160.8	1,160.8	1,160.8	1,160.8	1,160.8

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000	1.000	
2012	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000		
2013	1.249	1.028	1.016	1.001	1.001	1.000	1.000			
2014	1.249	1.028	1.016	1.001	1.001	1.000				
2015	1.249	1.028	1.016	1.001	1.001					
2016	1.249	1.028	1.016	1.001						
2017	1.249	1.028	1.016							
2018	1.249	1.028								
2019	1.249									
Age-to-Age	1.249	1.028	1.016	1.001	1.001	1.000	1.000	1.000	1.000	
Age-to-Ult	1.307	1.046	1.018	1.002	1.001	1.000	1.000	1.000	1.000	1.000

Scenario 3: Shift in rate of claim closure **Appendix 3**
Incremental Method **Exhibit 10**

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	815.0	368.0	116.0	88.0	62.0	45.0	32.0	22.0	16.0	12.0
2012	822.8	371.5	117.1	88.8	62.6	45.4	32.3	22.2	16.2	
2013	830.7	375.1	118.2	89.7	63.2	45.9	32.6	22.4		
2014	838.7	378.7	119.4	90.6	63.8	46.3	32.9			
2015	846.8	382.4	120.5	91.4	64.4	46.8				
2016	854.9	386.0	121.7	92.3	65.0					
2017	863.2	389.7	122.9	93.2						
2018	871.5	393.5	124.0							
2019	879.8	397.3								
2020	888.3									

Scenario 3: Shift in rate of claim closure **Appendix 3**
Incremental Method **Exhibit 11**

Incremental Closed Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	650.0	280.0	45.0	27.0	18.0	13.0	10.0	6.0	4.0	2.0	
2012	656.3	282.7	45.4	27.3	18.2	13.1	10.1	6.1	4.0		
2013	662.6	285.4	45.9	27.5	18.3	13.3	10.2	6.1			
2014	668.9	288.2	46.3	27.8	18.5	13.4	10.3				
2015	675.4	290.9	46.8	28.1	18.7	13.5					
2016	681.9	293.7	47.2	28.3	16.0						
2017	688.4	296.5	47.7	24.3							
2018	695.0	299.4	40.9								
2019	701.7	256.9									
2020	602.2										

Incremental Closure Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	80%	76%	39%	31%	29%	29%	31%	27%	25%	17%	
2012	80%	76%	39%	31%	29%	29%	31%	27%	25%		
2013	80%	76%	39%	31%	29%	29%	31%	27%			
2014	80%	76%	39%	31%	29%	29%	31%				
2015	80%	76%	39%	31%	29%	29%					
2016	80%	76%	39%	31%	25%						
2017	80%	76%	39%	26%							
2018	80%	76%	33%								
2019	80%	65%									
2020	68%										
Selected	80%	76%	39%	31%	29%	29%	31%	27%	25%	17%	100%

Scenario 3: Shift in rate of claim closure **Appendix 3**
Incremental Method **Exhibit 12**

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0		
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1			
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2				
2015	389.6	187.0	31.2	15.6	9.4	7.3					
2016	393.4	188.8	31.5	15.7	8.0						
2017	397.2	190.6	31.8	13.5							
2018	401.0	192.5	27.3								
2019	404.8	165.2									
2020	347.4										

Incremental Closed with Payment Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	46%	49%	26%	17%	15%	16%	19%	14%	13%	8%	
2012	46%	49%	26%	17%	15%	16%	19%	14%	13%		
2013	46%	49%	26%	17%	15%	16%	19%	14%			
2014	46%	49%	26%	17%	15%	16%	19%				
2015	46%	49%	26%	17%	15%	16%					
2016	46%	49%	26%	17%	12%						
2017	46%	49%	26%	14%							
2018	46%	49%	22%								
2019	46%	42%									
2020	39%										
Selected	46%	49%	26%	17%	15%	16%	19%	14%	13%	8%	50%

Scenario 3: Shift in rate of claim closure
Incremental Method

Appendix 3
Exhibit 13

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	275.0	100.0	15.0	12.0	9.0	6.0	4.0	3.0	2.0	1.0	
2012	277.6	101.0	15.1	12.1	9.1	6.1	4.0	3.0	2.0		
2013	280.3	101.9	15.3	12.2	9.2	6.1	4.1	3.1			
2014	283.0	102.9	15.4	12.3	9.3	6.2	4.1				
2015	285.7	103.9	15.6	12.5	9.4	6.2					
2016	288.5	104.9	15.7	12.6	8.0						
2017	291.3	105.9	15.9	10.8							
2018	294.1	106.9	13.6								
2019	296.9	91.8									
2020	254.8										

Incremental Closed with Payment Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	34%	27%	13%	14%	15%	13%	13%	14%	13%	8%	
2012	34%	27%	13%	14%	15%	13%	12%	14%	13%		
2013	34%	27%	13%	14%	15%	13%	13%	14%			
2014	34%	27%	13%	14%	15%	13%	12%				
2015	34%	27%	13%	14%	15%	13%					
2016	34%	27%	13%	14%	12%						
2017	34%	27%	13%	12%							
2018	34%	27%	11%								
2019	34%	23%									
2020	29%										
Implied	34%	27%	13%	14%	15%	13%	13%	14%	13%	8%	50%

Scenario 3: Shift in rate of claim closure
Incremental Method

Appendix 3
Exhibit 14

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	815.0	368.0	116.0	88.0	62.0	45.0	32.0	22.0	16.0	12.0	10.0	0.0
2012	822.8	371.5	117.1	88.8	62.6	45.4	32.3	22.2	16.2	12.1	10.1	0.0
2013	830.7	375.1	118.2	89.7	63.2	45.9	32.6	22.4	16.3	12.2	10.2	0.0
2014	838.7	378.7	119.4	90.6	63.8	46.3	32.9	22.6	16.5	12.3	10.3	0.0
2015	846.8	382.4	120.5	91.4	64.4	46.8	33.2	22.9	16.6	12.5	10.4	0.0
2016	854.9	386.0	121.7	92.3	65.0	50.0	35.6	24.5	17.8	13.3	11.1	0.0
2017	863.2	389.7	122.9	93.2	70.0	50.7	36.1	24.8	18.0	13.5	11.3	0.0
2018	871.5	393.5	124.0	101.3	71.3	51.7	36.7	25.3	18.4	13.8	11.5	0.0
2019	879.8	397.3	170.6	122.8	86.2	62.2	44.3	30.4	22.1	16.6	13.8	0.0
2020	888.3	507.4	151.8	111.5	78.4	56.7	40.3	27.7	20.2	15.1	12.6	0.0

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	5.0	633.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	1.0	5.0	639.1
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1	2.0	1.0	5.1	645.2
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2	3.1	2.1	1.0	5.1	651.4
2015	389.6	187.0	31.2	15.6	9.4	7.3	6.2	3.1	2.1	1.0	5.2	657.7
2016	393.4	188.8	31.5	15.7	8.0	7.8	6.7	3.3	2.2	1.1	5.6	664.1
2017	397.2	190.6	31.8	13.5	10.2	7.9	6.8	3.4	2.3	1.1	5.6	670.3
2018	401.0	192.5	27.3	17.3	10.3	8.0	6.9	3.4	2.3	1.1	5.7	675.9
2019	404.8	165.2	44.1	20.9	12.5	9.7	8.3	4.1	2.8	1.4	6.9	680.7
2020	347.4	248.2	39.3	19.0	11.4	8.8	7.6	3.8	2.5	1.3	6.3	695.5

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	275.0	100.0	15.0	12.0	9.0	6.0	4.0	3.0	2.0	1.0	5.0	432.0
2012	277.6	101.0	15.1	12.1	9.1	6.1	4.0	3.0	2.0	1.0	5.0	436.2
2013	280.3	101.9	15.3	12.2	9.2	6.1	4.1	3.1	2.0	1.0	5.1	440.3
2014	283.0	102.9	15.4	12.3	9.3	6.2	4.1	3.1	2.1	1.0	5.1	444.6
2015	285.7	103.9	15.6	12.5	9.4	6.2	4.2	3.1	2.1	1.0	5.2	448.9
2016	288.5	104.9	15.7	12.6	8.0	6.7	4.4	3.3	2.2	1.1	5.6	453.1
2017	291.3	105.9	15.9	10.8	10.2	6.8	4.5	3.4	2.3	1.1	5.6	457.7
2018	294.1	106.9	13.6	13.8	10.3	6.9	4.6	3.4	2.3	1.1	5.7	462.9
2019	296.9	91.8	22.1	16.7	12.5	8.3	5.5	4.1	2.8	1.4	6.9	469.0
2020	254.8	137.9	19.6	15.2	11.4	7.6	5.0	3.8	2.5	1.3	6.3	465.3

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	165.0	88.0	71.0	61.0	44.0	32.0	22.0	16.0	12.0	10.0	0.0
2012	166.6	88.8	71.7	61.6	44.4	32.3	22.2	16.2	12.1	10.1	0.0
2013	168.2	89.7	72.4	62.2	44.9	32.6	22.4	16.3	12.2	10.2	0.0
2014	169.8	90.6	73.1	62.8	45.3	32.9	22.6	16.5	12.3	10.3	0.0
2015	171.4	91.4	73.8	63.4	45.7	33.2	22.9	16.6	12.5	10.4	0.0
2016	173.1	92.3	74.5	64.0	49.0	35.6	24.5	17.8	13.3	11.1	0.0
2017	174.8	93.2	75.2	68.9	49.6	36.1	24.8	18.0	13.5	11.3	0.0
2018	176.4	94.1	83.1	70.2	50.6	36.7	25.3	18.4	13.8	11.5	0.0
2019	178.1	140.3	104.4	85.1	61.2	44.3	30.4	22.1	16.6	13.8	0.0
2020	286.1	121.3	92.9	77.3	55.6	40.3	27.7	20.2	15.1	12.6	0.0

Scenario 3: Shift in rate of claim closure
Incremental Method

Appendix 3
Exhibit 15

Incremental Paid Loss

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	1,500,000	2,250,000	3,000,000	2,700,000	1,775,000	1,500,000	1,425,000	750,000	550,000	310,000	1,740,000	17,500,000
2012	1,575,000	2,362,500	3,150,000	2,835,000	1,863,750	1,575,000	1,496,250	787,500	577,500	325,500	1,827,000	18,375,000
2013	1,653,750	2,480,625	3,307,500	2,976,750	1,956,938	1,653,750	1,571,063	826,875	606,375	341,775	1,918,350	19,293,750
2014	1,736,438	2,604,656	3,472,875	3,125,588	2,054,784	1,736,438	1,649,616	868,219	636,694	358,864	2,014,268	20,258,438
2015	1,823,259	2,734,889	3,646,519	3,281,867	2,157,524	1,823,259	1,732,096	911,630	668,528	376,807	2,114,981	21,271,359
2016	1,914,422	2,871,634	3,828,845	3,445,960	1,925,590	2,029,288	1,927,823	1,014,644	744,072	419,386	2,353,974	22,475,637
2017	2,010,143	3,015,215	4,020,287	3,075,519	2,534,051	2,138,533	2,031,607	1,069,267	784,129	441,964	2,480,699	23,601,413
2018	2,110,651	3,165,976	3,588,106	4,090,585	2,686,091	2,266,396	2,153,076	1,133,198	831,012	468,389	2,629,020	25,122,499
2019	2,216,183	2,825,634	6,037,189	5,154,450	3,376,217	2,838,987	2,697,037	1,419,493	1,040,962	586,724	3,293,224	31,486,099
2020	1,977,943	4,415,278	5,589,404	4,867,828	3,192,944	2,690,012	2,555,512	1,345,006	986,338	555,936	3,120,414	31,296,616

Scenario 3: Shift in rate of claim closure **Appendix 3**
Incremental Method **Exhibit 16**

Incremental Closed with Payment Counts: Base Case

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	5.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	1.0	5.0
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1	2.0	1.0	5.1
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2	3.1	2.1	1.0	5.1
2015	389.6	187.0	31.2	15.6	9.4	7.3	6.2	3.1	2.1	1.0	5.2
2016	393.4	188.8	31.5	15.7	9.4	7.3	6.3	3.1	2.1	1.0	5.2
2017	397.2	190.6	31.8	15.9	9.5	7.4	6.4	3.2	2.1	1.1	5.3
2018	401.0	192.5	32.1	16.0	9.6	7.5	6.4	3.2	2.1	1.1	5.3
2019	404.8	194.3	32.4	16.2	9.7	7.6	6.5	3.2	2.2	1.1	5.4
2020	408.7	196.2	32.7	16.3	9.8	7.6	6.5	3.3	2.2	1.1	5.4

Incremental Closed with Payment Counts: Scenario 3

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	375.0	180.0	30.0	15.0	9.0	7.0	6.0	3.0	2.0	1.0	5.0
2012	378.6	181.7	30.3	15.1	9.1	7.1	6.1	3.0	2.0	1.0	5.0
2013	382.2	183.5	30.6	15.3	9.2	7.1	6.1	3.1	2.0	1.0	5.1
2014	385.9	185.2	30.9	15.4	9.3	7.2	6.2	3.1	2.1	1.0	5.1
2015	389.6	187.0	31.2	15.6	9.4	7.3	6.2	3.1	2.1	1.0	5.2
2016	393.4	188.8	31.5	15.7	8.0	7.8	6.7	3.3	2.2	1.1	5.6
2017	397.2	190.6	31.8	13.5	10.2	7.9	6.8	3.4	2.3	1.1	5.6
2018	401.0	192.5	27.3	17.3	10.3	8.0	6.9	3.4	2.3	1.1	5.7
2019	404.8	165.2	44.1	20.9	12.5	9.7	8.3	4.1	2.8	1.4	6.9
2020	347.4	248.2	39.3	19.0	11.4	8.8	7.6	3.8	2.5	1.3	6.3

Comparison of Scenarios

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
2012	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
2013	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
2014	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
2015	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
2016	100%	100%	100%	100%	85%	106%	106%	106%	106%	106%	106%
2017	100%	100%	100%	85%	107%	106%	106%	106%	106%	106%	106%
2018	100%	100%	85%	108%	108%	107%	107%	107%	107%	107%	107%
2019	100%	85%	136%	129%	129%	128%	128%	128%	128%	128%	128%
2020	85%	126%	120%	116%	116%	116%	116%	116%	116%	116%	116%

Scenario 4: Private Passenger Auto
Data (Losses in thousands)

Appendix 4
Exhibit 1

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	13,285	14,107	14,278	14,334	14,343	14,345	14,347	14,348	14,348	14,348
2012	12,678	13,480	13,773	13,792	13,796	13,798	13,799	13,799	13,799	
2013	12,202	12,964	13,023	13,035	13,041	13,043	13,044	13,044		
2014	10,667	11,249	11,295	11,317	11,323	11,323	11,323			
2015	10,047	10,703	10,784	10,806	10,809	10,810				
2016	10,220	10,665	10,742	10,762	10,763					
2017	10,710	11,339	11,405	11,429						
2018	11,874	12,501	12,582							
2019	11,451	12,201								
2020	11,163									

Cumulative Closed with Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	6,954	9,392	9,736	9,789	9,833	9,851	9,867	9,872	9,873	9,874
2012	7,045	8,555	8,783	8,866	8,894	8,920	8,922	8,926	8,927	
2013	6,033	7,627	7,841	7,916	7,953	7,963	7,969	7,971		
2014	5,796	7,314	7,541	7,668	7,709	7,727	7,730			
2015	5,535	7,107	7,469	7,564	7,597	7,615				
2016	5,457	7,300	7,579	7,674	7,718					
2017	5,876	7,739	8,029	8,120						
2018	6,173	8,155	8,427							
2019	6,354	8,533								
2020	6,098									

Cumulative Closed without Payment Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	3,644	4,255	4,352	4,431	4,450	4,466	4,471	4,473	4,473	4,473
2012	3,426	4,513	4,801	4,847	4,861	4,865	4,868	4,868	4,870	
2013	3,800	4,954	5,021	5,049	5,057	5,064	5,065	5,068		
2014	3,055	3,476	3,530	3,563	3,582	3,587	3,589			
2015	2,447	2,980	3,130	3,167	3,182	3,184				
2016	2,447	2,913	2,994	3,010	3,021					
2017	2,536	3,174	3,228	3,235						
2018	3,432	3,935	3,999							
2019	2,694	3,239								
2020	2,594									

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	2,687	460	190	114	60	28	9	3	2	1
2012	2,207	412	189	79	41	13	9	5	2	
2013	2,369	383	161	70	31	16	10	5		
2014	1,816	459	224	86	32	9	4			
2015	2,065	616	185	75	30	11				
2016	2,316	452	169	78	24					
2017	2,298	426	148	74						
2018	2,269	411	156							
2019	2,403	429								
2020	2,471									

Cumulative Reported Loss

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	66,864	76,933	82,886	86,662	88,601	89,955	91,425	92,659	93,014	93,652
2012	60,073	67,591	73,451	76,387	77,888	78,630	79,979	81,325	81,549	
2013	58,942	67,331	71,312	73,425	74,761	76,133	77,341	78,456		
2014	57,367	66,125	71,359	74,392	75,449	76,455	77,592			
2015	57,332	66,381	69,709	71,923	73,014	74,372				
2016	60,238	73,707	78,662	82,592	83,763					
2017	68,251	81,617	88,106	90,787						
2018	74,885	90,847	97,546							
2019	85,308	97,740								
2020	90,649									

Cumulative Paid Loss

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	32,674	57,934	72,163	79,523	85,214	88,726	90,877	92,474	92,943	93,558
2012	30,560	53,919	64,638	71,485	75,567	77,868	79,506	80,871	81,390	
2013	28,678	50,761	62,774	69,745	72,641	74,754	76,498	78,015		
2014	27,887	50,158	62,104	69,473	72,672	75,469	77,215			
2015	26,989	49,006	59,694	66,587	70,861	73,706				
2016	29,853	56,665	69,892	76,127	81,361					
2017	33,071	59,463	76,720	84,844						
2018	35,733	70,594	86,043							
2019	39,503	74,123								
2020	41,245									

Scenario 4: Private Passenger Auto
Diagnostics

Appendix 4
Exhibit 2

Case Reserves per Open Count

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	12,724	41,302	56,437	62,623	56,450	43,893	60,889	61,667	35,500	94,000
2012	13,372	33,184	46,630	62,051	56,610	58,615	52,556	90,800	79,500	
2013	12,775	43,264	53,031	52,571	68,387	86,188	84,300	88,200		
2014	16,233	34,786	41,317	57,198	86,781	109,556	94,250			
2015	14,694	28,206	54,135	71,147	71,767	60,545				
2016	13,120	37,704	51,893	82,885	100,083					
2017	15,309	52,005	76,932	80,311						
2018	17,255	49,277	73,737							
2019	19,062	55,051								
2020	19,994									

Paid Loss per Closed with Payment Count

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	4,699	6,168	7,412	8,124	8,666	9,007	9,210	9,367	9,414	9,475
2012	4,338	6,303	7,359	8,063	8,496	8,730	8,911	9,060	9,117	
2013	4,754	6,655	8,006	8,811	9,134	9,388	9,599	9,787		
2014	4,811	6,858	8,236	9,060	9,427	9,767	9,989			
2015	4,876	6,895	7,992	8,803	9,327	9,679				
2016	5,471	7,762	9,222	9,920	10,542					
2017	5,628	7,684	9,555	10,449						
2018	5,789	8,657	10,210							
2019	6,217	8,687								
2020	6,764									

Closed Counts / Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	79.8%	96.7%	98.7%	99.2%	99.6%	99.8%	99.9%	100.0%	100.0%	100.0%
2012	82.6%	96.9%	98.6%	99.4%	99.7%	99.9%	99.9%	100.0%	100.0%	
2013	80.6%	97.0%	98.8%	99.5%	99.8%	99.9%	99.9%	100.0%		
2014	83.0%	95.9%	98.0%	99.2%	99.7%	99.9%	100.0%			
2015	79.4%	94.2%	98.3%	99.3%	99.7%	99.9%				
2016	77.3%	95.8%	98.4%	99.3%	99.8%					
2017	78.5%	96.2%	98.7%	99.4%						
2018	80.9%	96.7%	98.8%							
2019	79.0%	96.5%								
2020	77.9%									

Closed with Payment Counts / Closed Counts

Accident Year	12	24	36	48	60	72	84	96	108	120
2011	65.6%	68.8%	69.1%	68.8%	68.8%	68.8%	68.8%	68.8%	68.8%	68.8%
2012	67.3%	65.5%	64.7%	64.7%	64.7%	64.7%	64.7%	64.7%	64.7%	
2013	61.4%	60.6%	61.0%	61.1%	61.1%	61.1%	61.1%	61.1%		
2014	65.5%	67.8%	68.1%	68.3%	68.3%	68.3%	68.3%			
2015	69.3%	70.5%	70.5%	70.5%	70.5%	70.5%				
2016	69.0%	71.5%	71.7%	71.8%	71.9%					
2017	69.9%	70.9%	71.3%	71.5%						
2018	64.3%	67.5%	67.8%							
2019	70.2%	72.5%								
2020	70.2%									

Scenario 4: Private Passenger Auto **Appendix 4**
Incremental Method **Exhibit 3**

Cumulative Reported Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	13,285	14,107	14,278	14,334	14,343	14,345	14,347	14,348	14,348	14,348	14,348
2012	12,678	13,480	13,773	13,792	13,796	13,798	13,799	13,799	13,799	13,799	13,799
2013	12,202	12,964	13,023	13,035	13,041	13,043	13,044	13,044	13,044	13,044	13,044
2014	10,667	11,249	11,295	11,317	11,323	11,323	11,323	11,323	11,323	11,323	11,323
2015	10,047	10,703	10,784	10,806	10,809	10,810	10,811	10,811	10,811	10,811	10,811
2016	10,220	10,665	10,742	10,762	10,763	10,764	10,764	10,765	10,765	10,765	10,765
2017	10,710	11,339	11,405	11,429	11,432	11,433	11,434	11,434	11,434	11,434	11,434
2018	11,874	12,501	12,582	12,607	12,611	12,612	12,613	12,613	12,613	12,613	12,613
2019	11,451	12,201	12,280	12,305	12,309	12,310	12,310	12,311	12,311	12,311	12,311
2020	11,163	11,821	11,898	11,922	11,925	11,926	11,927	11,927	11,927	11,927	11,927

Age-to-Age Factors

Accident Year	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	1.062	1.012	1.004	1.001	1.000	1.000	1.000	1.000	1.000	
2012	1.063	1.022	1.001	1.000	1.000	1.000	1.000	1.000		
2013	1.062	1.005	1.001	1.000	1.000	1.000	1.000			
2014	1.055	1.004	1.002	1.001	1.000	1.000				
2015	1.065	1.008	1.002	1.000	1.000					
2016	1.044	1.007	1.002	1.000						
2017	1.059	1.006	1.002							
2018	1.053	1.006								
2019	1.065									
Age-to-Age	1.059	1.006	1.002	1.000	1.000	1.000	1.000	1.000	1.000	
Age-to-Ult	1.068	1.009	1.002	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Scenario 4: Private Passenger Auto **Appendix 4**
Incremental Method **Exhibit 4**

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120
2011	13,285	3,509	631	246	123	62	30	10	3	2
2012	12,678	3,009	705	208	83	43	14	9	5	
2013	12,202	3,131	442	173	76	33	17	10		
2014	10,667	2,398	505	246	92	32	9			
2015	10,047	2,721	697	207	78	31				
2016	10,220	2,761	529	189	79					
2017	10,710	2,927	492	172						
2018	11,874	2,896	492							
2019	11,451	3,153								
2020	11,163									

Scenario 4: Private Passenger Auto
Incremental Method

Appendix 4
Exhibit 5

Incremental Closed Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	10,598	3,049	441	132	63	34	21	7	1	1	
2012	10,471	2,597	516	129	42	30	5	4	3		
2013	9,833	2,748	281	103	45	17	7	5			
2014	8,851	1,939	281	160	60	23	5				
2015	7,982	2,105	512	132	48	20					
2016	7,904	2,309	360	111	55						
2017	8,412	2,501	344	98							
2018	9,605	2,485	336								
2019	9,048	2,724									
2020	8,692										

Incremental Closure Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	80%	87%	70%	54%	51%	55%	70%	70%	33%	50%	
2012	83%	86%	73%	62%	51%	70%	36%	44%	60%		
2013	81%	88%	64%	60%	59%	52%	41%	50%			
2014	83%	81%	56%	65%	65%	72%	56%				
2015	79%	77%	73%	64%	62%	65%					
2016	77%	84%	68%	59%	70%						
2017	79%	85%	70%	57%							
2018	81%	86%	68%								
2019	79%	86%									
2020	78%										
Selected	79%	86%	69%	60%	65%	63%	50%	50%	50%	50%	100%

Scenario 4: Private Passenger Auto **Appendix 4**
Incremental Method **Exhibit 6**

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	6,954	2,438	344	53	44	18	16	5	1	1	
2012	7,045	1,510	228	83	28	26	2	4	1		
2013	6,033	1,594	214	75	37	10	6	2			
2014	5,796	1,518	227	127	41	18	3				
2015	5,535	1,572	362	95	33	18					
2016	5,457	1,843	279	95	44						
2017	5,876	1,863	290	91							
2018	6,173	1,982	272								
2019	6,354	2,179									
2020	6,098										

Incremental Closed with Payment Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	52%	69%	55%	22%	36%	29%	53%	50%	33%	50%	
2012	56%	50%	32%	40%	34%	60%	14%	44%	20%		
2013	49%	51%	48%	43%	49%	30%	35%	20%			
2014	54%	63%	45%	52%	45%	56%	33%				
2015	55%	58%	52%	46%	42%	58%					
2016	53%	67%	53%	50%	56%						
2017	55%	64%	59%	53%							
2018	52%	68%	55%								
2019	55%	69%									
2020	55%										
Tail Paid		19,692	3,193	977	358	131	41	14	3	1	
Tail Closed		26,889	4,432	1,361	496	183	59	21	5	1	
Tail		73%	72%	72%	72%	72%	69%	67%	60%	100%	
Selected Tail							69%	69%	69%	69%	69%
Closure Rate	79%	86%	69%	60%	65%	63%	50%	50%	50%	50%	100%
Selected	54%	67%	56%	49%	47%	48%	35%	35%	35%	35%	69%

Scenario 4: Private Passenger Auto **Appendix 4**
Incremental Method **Exhibit 7**

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	3,644	611	97	79	19	16	5	2	0	0	
2012	3,426	1,087	288	46	14	4	3	0	2		
2013	3,800	1,154	67	28	8	7	1	3			
2014	3,055	421	54	33	19	5	2				
2015	2,447	533	150	37	15	2					
2016	2,447	466	81	16	11						
2017	2,536	638	54	7							
2018	3,432	503	64								
2019	2,694	545									
2020	2,594										

Incremental Closed with Payment Rate

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	27%	17%	15%	32%	15%	26%	17%	20%	0%	0%	
2012	27%	36%	41%	22%	17%	9%	21%	0%	40%		
2013	31%	37%	15%	16%	11%	21%	6%	30%			
2014	29%	18%	11%	13%	21%	16%	22%				
2015	24%	20%	22%	18%	19%	6%					
2016	24%	17%	15%	8%	14%						
2017	24%	22%	11%	4%							
2018	29%	17%	13%								
2019	24%	17%									
2020	23%										
Implied	25%	19%	13%	11%	18%	15%	15%	15%	15%	15%	31%

Scenario 4: Private Passenger Auto
Incremental Method

Appendix 4
Exhibit 8

Active Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	13,285	3,509	631	246	123	62	30	10	3	2	1	0
2012	12,678	3,009	705	208	83	43	14	9	5	2	1	0
2013	12,202	3,131	442	173	76	33	17	10	5	3	1	0
2014	10,667	2,398	505	246	92	32	9	4	2	1	1	0
2015	10,047	2,721	697	207	78	31	12	6	3	2	1	0
2016	10,220	2,761	529	189	79	25	10	5	3	1	1	0
2017	10,710	2,927	492	172	77	28	11	6	3	1	1	0
2018	11,874	2,896	492	181	76	27	11	6	3	1	1	0
2019	11,451	3,153	508	183	77	28	11	6	3	1	1	0
2020	11,163	3,129	518	186	78	28	11	6	3	1	1	0

Incremental Closed with Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	6,954	2,438	344	53	44	18	16	5	1	1	1	9,875
2012	7,045	1,510	228	83	28	26	2	4	1	1	1	8,928
2013	6,033	1,594	214	75	37	10	6	2	2	1	1	7,974
2014	5,796	1,518	227	127	41	18	3	1	1	0	0	7,733
2015	5,535	1,572	362	95	33	18	4	2	1	1	1	7,623
2016	5,457	1,843	279	95	44	12	3	2	1	0	0	7,737
2017	5,876	1,863	290	91	37	13	4	2	1	1	1	8,178
2018	6,173	1,982	272	90	36	13	4	2	1	1	1	8,574
2019	6,354	2,179	282	91	37	13	4	2	1	1	1	8,964
2020	6,098	2,100	288	92	37	13	4	2	1	1	1	8,636

Incremental Closed without Payment Counts

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	3,644	611	97	79	19	16	5	2	0	0	0	4,473
2012	3,426	1,087	288	46	14	4	3	0	2	0	0	4,871
2013	3,800	1,154	67	28	8	7	1	3	1	0	0	5,070
2014	3,055	421	54	33	19	5	2	1	0	0	0	3,590
2015	2,447	533	150	37	15	2	2	1	0	0	0	3,188
2016	2,447	466	81	16	11	4	2	1	0	0	0	3,028
2017	2,536	638	54	7	14	4	2	1	0	0	0	3,256
2018	3,432	503	64	19	14	4	2	1	0	0	0	4,039
2019	2,694	545	67	19	14	4	2	1	0	0	0	3,347
2020	2,594	588	68	20	14	4	2	1	0	0	0	3,291

Open Counts

Accident Year	12	24	36	48	60	72	84	96	108	120	Ultimate
2011	2,687	460	190	114	60	28	9	3	2	1	0
2012	2,207	412	189	79	41	13	9	5	2	1	0
2013	2,369	383	161	70	31	16	10	5	3	1	0
2014	1,816	459	224	86	32	9	4	2	1	1	0
2015	2,065	616	185	75	30	11	6	3	2	1	0
2016	2,316	452	169	78	24	9	5	3	1	1	0
2017	2,298	426	148	74	27	10	5	3	1	1	0
2018	2,269	411	156	72	26	10	5	3	1	1	0
2019	2,403	429	159	73	27	10	6	3	1	1	0
2020	2,471	441	162	74	27	10	6	3	1	1	0

Scenario 4: Private Passenger Auto **Appendix 4**
Disposal Rate Frequency-Severity Method **Exhibit 9**

Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	4,699	10,361	41,363	138,868	129,341	195,111	134,438	319,400	469,000	615,000	371,941
2012	4,338	15,470	47,013	82,494	145,786	88,500	819,000	341,250	519,000	371,941	394,258
2013	4,754	13,854	56,136	92,947	78,270	211,300	290,667	758,500	371,941	394,258	417,913
2014	4,811	14,671	52,626	58,024	78,024	155,389	582,000	371,941	394,258	417,913	442,988
2015	4,876	14,006	29,525	72,558	129,515	158,056	371,941	394,258	417,913	442,988	469,567
2016	5,471	14,548	47,409	65,632	118,955	188,588	394,258	417,913	442,988	469,567	497,741
2017	5,628	14,166	59,507	89,275	120,003	199,903	417,913	442,988	469,567	497,741	527,606
2018	5,789	17,589	56,798	84,793	127,203	211,897	442,988	469,567	497,741	527,606	559,262
2019	6,217	15,888	61,260	89,880	134,835	224,611	469,567	497,741	527,606	559,262	592,818
2020	6,764	17,812	64,935	95,273	142,926	238,087	497,741	527,606	559,262	592,818	628,387

Trend Rate: 6%

Trended Incremental Paid Severity

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult
2011	7,938	16,514	62,195	196,987	173,087	246,323	160,117	358,878	497,140	615,000	
2012	6,914	23,260	66,689	110,396	184,051	105,405	920,228	361,725	519,000		
2013	7,148	19,652	75,122	117,343	93,221	237,417	308,107	758,500			
2014	6,825	19,633	66,439	69,107	87,668	164,712	582,000				
2015	6,525	17,682	35,165	81,526	137,286	158,056					
2016	6,906	17,327	53,268	69,569	118,955						
2017	6,703	15,917	63,077	89,275							
2018	6,504	18,644	56,798								
2019	6,590	15,888									
2020	6,764										
Tail Dollars		545,540,196	248,071,328	118,980,556	59,322,027	29,744,735	14,386,401	6,389,429	1,631,140	615,000	
Tail Counts		19,692	3,193	977	358	131	41	14	3	1	
Tail		27,704	77,692	121,782	165,704	227,059	350,888	456,388	543,713	615,000	
Selected	6,618	16,804	57,792	79,993	113,210	177,913	350,888	350,888	350,888	350,888	350,888

Scenario 4: Private Passenger Auto **Appendix 4**
Incremental Method **Exhibit 10**

Incremental Paid Loss

Accident Year	0-12	12-24	24-36	36-48	48-60	60-72	72-84	84-96	96-108	108-120	120-Ult	Ultimate
2011	32,674	25,260	14,229	7,360	5,691	3,512	2,151	1,597	469	615	258	93,816
2012	30,560	23,359	10,719	6,847	4,082	2,301	1,638	1,365	519	258	274	81,922
2013	28,678	22,083	12,013	6,971	2,896	2,113	1,744	1,517	646	342	363	79,367
2014	27,887	22,271	11,946	7,369	3,199	2,797	1,746	552	293	155	164	78,380
2015	26,989	22,017	10,688	6,893	4,274	2,845	1,495	828	439	233	247	76,947
2016	29,853	26,812	13,227	6,235	5,234	2,252	1,357	757	401	213	225	86,567
2017	33,071	26,392	17,257	8,124	4,406	2,657	1,597	889	471	250	265	95,379
2018	35,733	34,861	15,449	7,602	4,597	2,783	1,684	942	500	265	281	104,696
2019	39,503	34,620	17,305	8,159	4,925	2,978	1,798	1,005	532	282	299	111,406
2020	41,245	37,404	18,699	8,757	5,275	3,185	1,919	1,070	567	301	319	118,739

Exceedance Probability in Catastrophe Modeling

July 14, 2020

Abstract

This article explores two of the most important notions in Catastrophic Modeling: the Occurrence Exceedance Probability (OEP) and the Aggregate Exceedance Probability (AEP) curves. Construction of each curve is discussed and comparisons are made. Several numerical and theoretical examples demonstrate introduced metrics and techniques. A separate discussion is dedicated to a connection between the distribution of loss severities and the OEP depending on the distribution of claim counts. The article is concluded with demonstration of OEP and AEP curves for the deadliest, costliest, and most intense US tropical cyclones based on the 2011 National Oceanic and Atmospheric Administration (NOAA) report.

Keywords. Aggregate Exceedance Probability, Average Annual Loss, Catastrophe Modeling, Collective Risk Model, Exceedance Probability, Loss Return Period, Monte Carlo Simulation, Occurrence Exceedance Probability.

1 Introduction

Catastrophe Modeling is a type of estimation technique used in the Property and Casualty (P&C) industry to predict and evaluate damage caused by *natural* catastrophes such as hurricanes, earthquakes, tornados, hail, winter storms, floods and wild fires, as well as *man-made* catastrophes such as terrorism, [1].

Catastrophe models are widely used in ratemaking, portfolio management and optimization, underwriting and risk selection, loss mitigation strategies, allocation of cost of capital, cost of reinsurance, reinsurance and risk transfer analysis, enterprise risk management, as well as financial and capital adequacy analysis utilized by rating agencies, [1].

The Occurrence Exceedance Probability (OEP) and the Aggregate Exceedance Probability (AEP) are two primary metrics used in catastrophe modeling that give an insurer immediate feedback on the financial nature of a disaster.

This paper explores the notions of OEP and AEP and demonstrates their use through several numerical, as well as theoretical examples.

2 Exceedance Probability

Exceedance Probability (EP) is one of the most commonly used metrics in catastrophe modeling. It is the probability that a certain loss value will be exceeded in a predefined future time period. Exceedance probability is used in planning for potential hazards such as river and stream flooding, hurricane storm surges and droughts, reserving for reservoir storage levels and providing homeowners and community members with risk assessment.

To define exceedance probability, let $D_1, D_2, \cdots$ be a set of natural disasters. Let p_i and X_i be an annual probability of occurrence and a corresponding total loss associated with a natural disaster D_i. Thus, D_i is a Bernoulli random variable with

$$\mathbf{P}(D_i \ occurs) = p_i$$
$$\mathbf{P}(D_i \ does\ not\ occur) = 1 - p_i$$

If an event D_i does not occur, the loss is zero. The expected loss for a given event D_i in a given year is $\mathbf{E}[X] = p_i X_i$.

The overall expected loss for the entire set of events is known as the **average annual loss** (AAL) and is defined as the sum of the expected losses of each of the individual events for a given year:

$$\text{AAL} = \sum_{i=1}^{\infty} p_i X_i$$

The Exceedance Probability (EP) is the probability that a loss random variable exceeds a certain amount of loss. This probability is sometimes denoted as $EP(x)$ and is called the **Exceedance Probability Curve**. Let X be a loss random variable. Then

$$\mathbf{EP}(x) = \mathbf{P}(X > x) = 1 - \mathbf{P}(X \leq x)$$

Using probabilistic terminology, $EP(x)$ is the survival function of X.

In particular, if $x = X_i$, which is a loss associated with a disaster D_i, then

$$\mathbf{EP}(X_i) = \mathbf{P}(X > X_i) = 1 - \mathbf{P}(X \leq X_i) = 1 - \prod_{j=1}^{i}(1 - p_j),$$

where $D_1, D_2, \cdots, D_i$ are the events with higher level of losses such that $X_1 \geq X_2 \geq \cdots \geq X_i$.

The probability that all the other events with possible losses above the value X_i have not occurred is

$$\mathbf{P}(X \leq X_i) = \prod_{j=1}^{i}(1 - p_j)$$

and is sometimes called the **Non-Exceedance Probability** (NEP).

A characteristic sometimes associated with the Exceedance Probability is the **Return Period** or the **Loss Return Period** of a natural disaster. It is calculated as a reciprocal of the EP:

$$RP = \frac{1}{EP}.$$

2.1 Example of an Exceedance Probability Curve

Suppose that during a given year no more than one hurricane can occur. The following table shows the probability of each category of hurricane and the associated loss that would incurred.

Event (D_i)	Description	Annual probability of occurrence (p_i)	Loss (X_i)
1	Category 5 Hurricane	0.003	15,000,000
2	Category 4 Hurricane	0.006	8,000,000
3	Category 3 Hurricane	0.011	5,000,000
4	Category 2 Hurricane	0.030	3,000,000
5	Category 1 Hurricane	0.040	1,000,000

Table 1: Event Loss Data

Note that the Saffir/Simpson Hurricane Wind Scale, [6], provides specific wind values for each hurricane category:

Scale Number (Category)	Winds Max 1-min (mph)
1	74 – 95
2	96 – 110
3	111 – 130
4	131 – 155
5	> 155

Table 2: The Saffir/Simpson Hurricane Wind Scale, 1974

Calculating the Exceedance Probability at each level of loss and the Expected Loss for each level of disaster, we obtain

Event (D_i)	Annual probability of occurrence (p_i)	Loss (X_i)	Exceedance Probability $1-(1-p_1)(1-p_2)\cdots$	$\mathbf{E}[X] = p_i X_i$
1	0.003	15,000,000	0.0030	45,000
2	0.006	8,000,000	0.0090	48,000
3	0.011	5,000,000	0.0199	55,000
4	0.030	3,000,000	0.0493	90,000
5	0.040	1,000,000	0.0873	40,000

Table 3: Exceedance Probability and Expected Loss Results

Note that the probability that no hurricane occurs is

$$\mathbf{P}(\text{No Disaster}) = 1 - \sum_{i=1}^{5} p_i = 1 - 0.09 = 0.91.$$

The Average Annual Loss is

$$\text{AAL} = \sum_{i=1}^{\infty} p_i X_i = 45{,}000 + 48{,}000 + 55{,}000 + 90{,}000 + 40{,}000 = 278{,}000.$$

The Exceedance Probability Curve in this example is

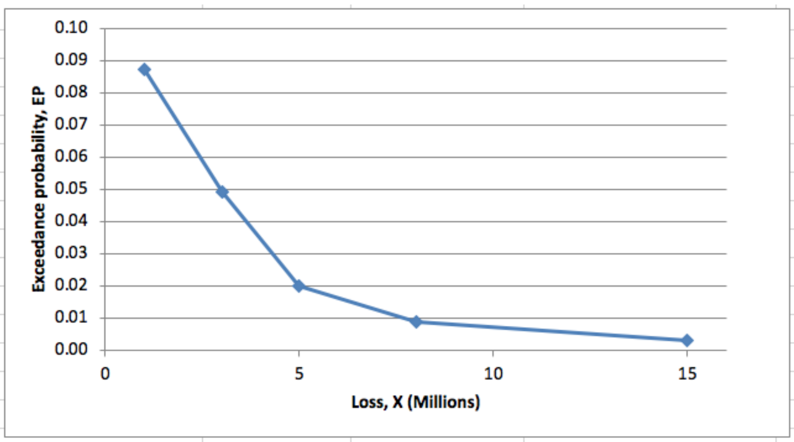

Figure 1: Exceedance Probability Curve in Example 2.1

The probabilities of non-occurrence and non-exceedance are shown in connection with exceedance probability as follows:

Event (D_i)	Annual probability of occurrence p_i	Probability of Non-Occurrence $1-p_i$	Probability of Non-Exceedance $(1-p_1)(1-p_2)\cdots$	Exceedance Probability $1-(1-p_1)(1-p_2)\cdots$
1	0.003	0.997	0.997	0.0030
2	0.006	0.994	0.991	0.0090
3	0.011	0.989	0.980	0.0199
4	0.030	0.970	0.951	0.0493
5	0.040	0.960	0.913	0.0873

Table 4: Non-Occurrence and Non-Exceedance Probabilities

Calculating the Return Period of each event, we have

Event (D_i)	Description	Annual probability of occurrence (p_i)	Exceedance Probability $1-(1-p_1)(1-p_2)\cdots$	Return Period (years) $=1/EP$
1	Category 5 Hurricane	0.003	0.0030	333.33
2	Category 4 Hurricane	0.006	0.0090	111.33
3	Category 3 Hurricane	0.011	0.0199	50.29
4	Category 2 Hurricane	0.030	0.0493	20.29
5	Category 1 Hurricane	0.040	0.0873	11.45

Table 5: Return Period of the Event

The return period is illustrated in the following chart:

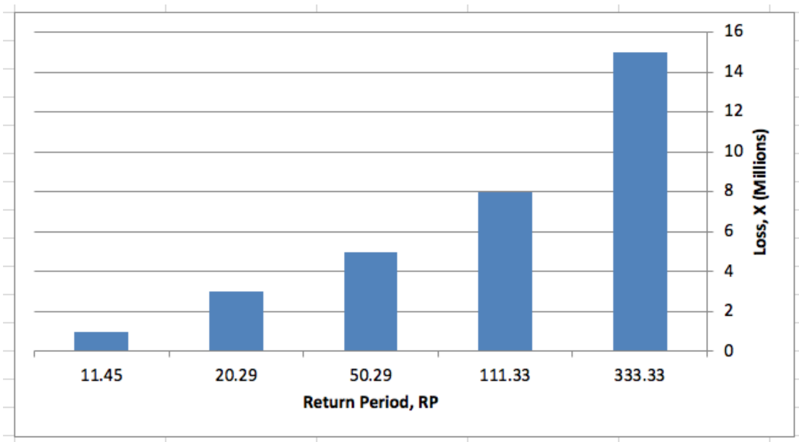

Figure 2: Return Period of the Event in Example 2.1

The exceedance probability can be further broken down into the **occurrence** exceedance probability, OEP, and the **aggregate** exceedance probability, AEP.

3 Occurrence Exceedance Probability

The Occurrence Exceedance Probability (OEP) is the probability that the *largest* loss in a year exceeds a certain amount of loss. This probability is sometimes denoted as $O(x)$ and is called the **Occurrence Exceedance Probability Curve**.

Let $X_1, X_2, \cdots, X_N$ be losses in a given year. Then

$$O(x) = \mathbf{P}(max_{1\leq i\leq N}(X_i) > x) = 1 - \mathbf{P}(max_{1\leq i\leq N}(X_i) \leq x) = 1 - \prod_{i=1}^{N} \mathbf{P}\left(X_i \leq x\right)$$

Using probabilistic terminology, if $X_{(1)}, X_{(2)}, \cdots, X_{(N)}$ is the ordered statistic with $X_{(N)} = \max_{1\leq i\leq N} X_{(i)}$, then $O(x)$ is the survival function of $X_{(N)}$.

Let $F(x)$ be the cumulative distribution function (**CDF**) of X. Then for a fixed N the OEP is

$$O(x) = 1 - \left(F_X(x)\right)^N .$$

If N is the random claim count with the probability mass function (**p.m.f.**) P_N, then by the law of total probability,

$$O(x) = \sum_{n=0}^{\infty} \mathbf{P}(max_{1\le i \le n}(X_i) > x | N = n)\mathbf{P}(N = n) =$$
$$= 1 - \sum_{n=0}^{\infty} \mathbf{P}(max_{1\le i \le n}(X_i) \le x | N = n)\mathbf{P}(N = n) =$$
$$= 1 - \sum_{n=0}^{\infty} \left(\prod_{i=1}^{n} \mathbf{P}\left(X_i \le x\right) \right) \mathbf{P}(N = n) = 1 - \sum_{n=0}^{\infty} \left(F_X(x)\right)^n \mathbf{P}(N = n) =$$
$$= 1 - \mathbf{E}_N \left(\left(F_X(x)\right)^N \right) = 1 - \mathbf{PGF}\left(F_X(x)\right),$$

where $\mathbf{PGF}(x)$ is the probability generating function for N defined as

$$\mathbf{PGF}(t) = \mathbf{E}\left(t^N\right) = \sum_{n=0}^{\infty} t^n \cdot \mathbf{P}\left(N = n\right).$$

Thus,

$$O(x) = 1 - \mathbf{PGF}\left(F_X(x)\right). \tag{3.1}$$

The expected value of $X_{(N)}$ is by definition

$$\mathbf{E}\left[X_{(N)}\right] = \int_0^{\infty} O(x)\, dx.$$

In catastrophe modeling the Occurrence Exceedance Probability is used for occurrence based reinsurance structures such as quota share or working excess.

3.1 Example of an Occurrence Exceedance Probability Curve

Following is a simplified example that demonstrates construction of an Occurrence Exceedance Probability Curve outlined in [3]. Data is simulated over ten years assuming a fixed number of losses per year. Severities are assumed to be Pareto-distributed, with parameters $\alpha = 3$ and $\theta = 1000$. Recall that for a two-parameter Pareto distribution, the cumulative distribution function is of the form

$$F(x) = 1 - \left(\frac{\theta}{x + \theta}\right)^{\alpha}.$$

Using the *inversion method* of the *Monte Carlo Simulation* (MCS) technique, we calculate the inverse function of $F(x)$ as

$$u = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha} \Leftrightarrow 1 - u = \left(\frac{\theta}{x+\theta}\right)^{\alpha} \Leftrightarrow (1-u)^{-1/\alpha} = \frac{x}{\theta} + 1 \Leftrightarrow$$
$$x = \theta\left[(1-u)^{-1/\alpha} - 1\right] \Leftrightarrow F^{-1}(x) = \theta\left[(1-x)^{-1/\alpha} - 1\right].$$

Table 10 of Appendix A contains a 100 simulated losses. Assuming 10 losses per year, the data is simulated over 10 years. Calculating the largest loss within each year, we have

Year	$\max_{1\leq i\leq 10}(X_i)$
1	869.63
2	1,390.24
3	1,713.30
4	3,330.60
5	1,069.76
6	604.58
7	578.61
8	721.97
9	1,644.01
10	1,042.16

Table 6: Maximum Loss by Year

These amounts are highlighted in Table 10. Sorting annual losses from highest to lowest and ranking each year, we obtain

OEP	Rank	Year	$\max_{1\leq i\leq 10}(X_i)$
0.1	1	4	3,330.60
0.2	2	3	1,713.30
0.3	3	9	1,644.01
0.4	4	2	1,390.24
0.5	5	5	1,069.76
0.6	6	10	1,042.16
0.7	7	1	869.63
0.8	8	8	721.97
0.9	9	6	604.58
1.0	10	7	578.61

Table 7: Sorted and Ranked Maximum Losses by Year

The resulting Occurrence Exceedance Probability Curve is

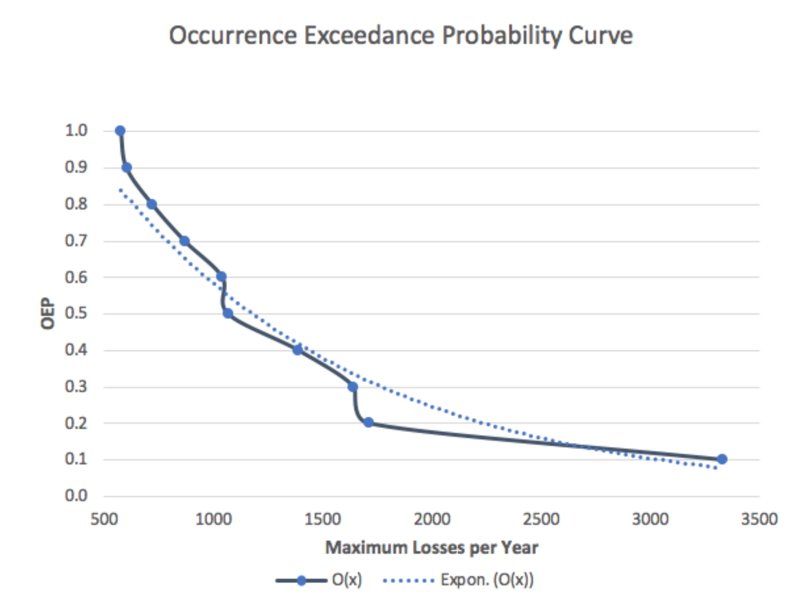

Figure 3: Occurrence Exceedance Probability Curve in Example 3.1

An exponential trend is included to demonstrate the general behavior of the function.

4 Evaluating Severity Distribution Using the OEP

It follows from the equation (3.1) that the cumulative distribution function F_X of losses X can be evaluated using the Occurrence Exceedance Probability $O(x)$ as

$$F_X(x) = \mathbf{PGF}^{-1}\left(1 - O(x)\right), \tag{4.1}$$

where $\mathbf{PGF}^{-1}(x)$ indicates the inverse function of the probability generating function for N.

The loss distribution will be consistent with the starting OEPs and the claim count assumption.

An important property of the probability generating function is outlined in the following Lemma.

Lemma 4.1 *If N and M are independent random variables, then*

$$\boldsymbol{PGF}_{N+M}(t) = \boldsymbol{PGF}_N(t) \cdot \boldsymbol{PGF}_M(t)$$

Proof. By definition,

$$\mathbf{PGF}_{N+M}(t) = \mathbf{E}\left(t^{N+M}\right) = \mathbf{E}\left(t^N \cdot t^M\right) = \mathbf{E}\left(t^N\right) \cdot \mathbf{E}\left(t^M\right) = \mathbf{PGF}_N(t) \cdot \mathbf{PGF}_M(t).$$

Following is the derivation of the cumulative distribution function F_X of losses X for a few standard discrete distributions of claim counts.

4.1 Poisson Distribution of Claim Counts

Suppose claim counts N have a Poisson distribution with mean parameter λ. This is a common assumption when modeling a number of catastrophes. The probability mass function is defined as

$$p_n = \mathbf{P}\left(N = n\right) = e^{-\lambda}\,\frac{\lambda^n}{n!}.$$

Calculating the **PGF**, we obtain

$$PGF(t) = \sum_{n=0}^{\infty} t^n \cdot \mathbf{P}\left(N = n\right) = \sum_{n=0}^{\infty} t^n \cdot e^{-\lambda}\,\frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} t^n\,\frac{\lambda^n}{n!} =$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!} = e^{-\lambda} \cdot e^{t\lambda} = e^{\lambda(t-1)}.$$

Then the inverse function is

$$y = e^{\lambda(t-1)} \Leftrightarrow \lambda(t-1) = \ln y \Leftrightarrow t = \frac{\ln y}{\lambda} + 1 \Leftrightarrow \mathbf{PGF}^{-1}(x) = \frac{\ln x}{\lambda} + 1$$

Using (4.1), cumulative distribution function F_X is

$$F_X(x) = \mathbf{PGF}^{-1}\left(1 - O(x)\right) = \frac{\ln\left(1 - O(x)\right)}{\lambda} + 1$$

4.2 Bernoulli Distribution of Claim Counts

Suppose claim counts N have a Bernoulli distribution with parameter q. The probability mass function is defined as

$$p_0 = \mathbf{P}\left(N = 0\right) = 1 - q, p_1 = \mathbf{P}\left(N = 1\right) = q$$

Calculating the **PGF**, we obtain

$$\mathbf{PGF}(t) = \sum_{n=0}^{1} t^n \cdot \mathbf{P}(N = n) = (1 - q) + qt \tag{4.2}$$

Then the inverse function is

$$y = (1 - q) + qt \Leftrightarrow t = \frac{y - 1 + q}{q} = \frac{y - 1}{q} + 1 \Leftrightarrow$$
$$\mathbf{PGF}^{-1}(x) = \frac{x - 1}{q} + 1$$

Using (4.1), cumulative distribution function F_X is

$$F_X(x) = \mathbf{PGF}^{-1}(1 - O(x)) = \frac{1 - O(x) - 1}{q} + 1 = \frac{O(x)}{q} + 1$$

4.3 Binomial Distribution of Claim Counts

Suppose claim counts N have a binomial distribution with parameters q and m. The probability mass function is defined as

$$p_n = \mathbf{P}(N = n) = \binom{m}{n} q^n (1 - q)^{m-n}.$$

Calculating the **PGF**, we obtain

$$\mathbf{PGF}(t) = \sum_{n=0}^{m} t^n \cdot \mathbf{P}(N = n) = \sum_{n=0}^{m} t^n \cdot \binom{m}{n} q^n (1 - q)^{m-n} = \sum_{n=0}^{m} \binom{m}{n} (qt)^n (1 - q)^{m-n} =$$
$$= ((1 - q) + qt)^m = (1 + q(t - 1))^m$$

Note that the same **PGF** can be obtained using one of the properties of a probability generating function. Since a Binomial (q, m) random variable N can be expressed as a sum of m i.i.d. Bernoulli (q),

$$N = N_1 + N_2 + \cdots + N_m,$$

by Lemma (4.1), using (4.2), its **PGF** is

$$\mathbf{PGF}_N(t) = \prod_{i=1}^{m} \mathbf{PGF}_{N_i}(t) = ((1 - q) + qt)^m$$

The inverse function is

$$y = ((1-q)+qt)^m \Leftrightarrow (1-q)+qt = y^{1/m} \Leftrightarrow t = \frac{y^{1/m}-1+q}{q} = \frac{y^{1/m}-1}{q}+1 \Leftrightarrow$$
$$\mathbf{PGF}^{-1}(x) = \frac{x^{1/m}-1}{q}+1$$

Using (4.1), cumulative distribution function F_X is

$$F_X(x) = \mathbf{PGF}^{-1}\left(1-O(x)\right) = \frac{(1-O(x))^{1/m}-1}{q}+1$$

4.4 Geometric Distribution of Claim Counts

Suppose claim counts N have a geometric distribution with success probability $0 < p < 1$. The probability mass function is defined as

$$p_n = \mathbf{P}\left(N=n\right) = (1-p)^n p.$$

Calculating the **PGF**, we obtain

$$\begin{aligned} \mathbf{PGF}(t) &= \sum_{n=0}^{\infty} t^n \cdot \mathbf{P}\left(N=n\right) = \sum_{n=0}^{\infty} t^n \cdot (1-p)^n p = p\sum_{n=0}^{\infty}(t(1-p))^n = \\ &= \frac{p}{1-t(1-p)} \end{aligned} \tag{4.3}$$

Then the inverse function is

$$y = \frac{p}{1-t(1-p)} \Leftrightarrow y - yt(1-p) = p \Leftrightarrow yt(1-p) = y-p \Leftrightarrow t = \frac{y-p}{y(1-p)} \Leftrightarrow$$
$$\mathbf{PGF}^{-1}(x) = \frac{x-p}{x(1-p)}$$

Using (4.1), cumulative distribution function F_X is

$$F_X(x) = \mathbf{PGF}^{-1}\left(1-O(x)\right) = \frac{1-O(x)-p}{(1-O(x))(1-p)}$$

4.5 Negative Binomial Distribution of Claim Counts

Suppose claim counts N have a negative binomial distribution with parameters p and r. The probability mass function is defined as

$$p_n = \mathbf{P}\,(N = n) = \binom{n+r-1}{n} p^r (1-p)^n.$$

For an integer r, since a Negative Binomial (p, r) random variable N can be expressed as a sum of r i.i.d. geometric (p),

$$N = N_1 + N_2 + \cdots + N_r,$$

by Lemma (4.1), using (4.3), its **PGF** is

$$\mathbf{PGF}_N(t) = \prod_{i=1}^{m} \mathbf{PGF}_{N_i}(t) = \left(\frac{p}{1-t(1-p)}\right)^r$$

Then the inverse function is

$$y = \left(\frac{p}{1-t(1-p)}\right)^r \Leftrightarrow \frac{p}{1-t(1-p)} = y^{1/r} \Leftrightarrow y^{1/r} - y^{1/r}t(1-p) = p \Leftrightarrow$$
$$y^{1/r}t(1-p) = y^{1/r} - p \Leftrightarrow t = \frac{y^{1/r}-p}{y^{1/r}(1-p)} \Leftrightarrow \mathbf{PGF}^{-1}(x) = \frac{x^{1/r}-p}{x^{1/r}(1-p)}$$

Using (4.1), cumulative distribution function F_X is

$$F_X(x) = \mathbf{PGF}^{-1}\,(1 - O(x)) = \frac{(1-O(x))^{1/r} - p}{(1-O(x))^{1/r}(1-p)}$$

5 Aggregate Exceedance Probability

The Aggregate Exceedance Probability (AEP) is the probability that the *sum* of losses in a year exceeds a certain amount of loss. This probability is sometimes denoted as $A(x)$ and is called the **Aggregate Exceedance Probability Curve**.

Let $X_1, X_2, \cdots, X_N$ be losses in a given year. Then

$$A(x) = \mathbf{P}(X_1 + X_2 + \cdots + X_N > x) = 1 - \mathbf{P}(X_1 + X_2 + \cdots + X_N \leq x)$$

Using the terminology of the aggregate loss models, if S is the collective risk model, defined as $S = \sum_{i=1}^{N} X_i$, then $A(x)$ is the survival function of S.

For a fixed N this probability is

$$A(x) = 1 - F_X^{(N)}(x),$$

where $F_X^{(N)}$ is an N-fold convolution of $F_X(x)$, defined as

$$F_X^{(N)}(x) = \int_0^x F_X^{(N-1)}(x-y) f_X(y)\, dy \quad \text{for } N = 2, 3, \cdots .$$

For $N = 1$ this equation reduces to $F_X^{(1)}(x) = F_X(x)$, [5].

If N is the random claim count with the probability mass function (**p.m.f.**) P_N, then by the law of total probability,

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} \mathbf{P}(S > x | N = n)\mathbf{P}(N = n) = \\ &= 1 - \sum_{n=0}^{\infty} \mathbf{P}(S \le x | N = n)\mathbf{P}(N = n) = \\ &= 1 - \sum_{n=0}^{\infty} F_X^{(n)}(x)\mathbf{P}(N = n) = 1 - \mathbf{E}_N \left(F_X^{(N)} \right) \end{aligned}$$

The expected value of S is by definition

$$\mathbf{E}[S] = \int_0^{\infty} A(x)\, dx = \mathbf{E}[X]\,\mathbf{E}[N]\,.$$

In catastrophe modeling the Aggregate Exceedance Probability is used for aggregate based reinsurance structures such as stop loss and reinstatements.

5.1 Example of an Aggregate Exceedance Probability Curve

Following is a simplified example that demonstrates construction of an Aggregate Exceedance Probability Curve outlined in [3]. We use the same data as in Example 3.1.

In that example data was simulated over ten years assuming a fixed number of losses per year. Severities were assumed to be Pareto-distributed, with parameters $\alpha = 3$ and $\theta = 1000$. Losses were simulated using the inversion method of the Monte Carlo Simulation (MCS) technique. Table 10 of Appendix A contains a 100 simulated losses. Assuming 10 losses per year, the data is simulated over 10 years.

Calculating the sum of losses within each year, we have

Year	$\sum_{i=1}^{10} X_i$
1	2,936.52
2	3,867.36
3	4,589.80
4	7,092.26
5	4,125.27
6	2,831.38
7	2,589.09
8	1,832.78
9	5,400.46
10	3,087.66

Table 8: Sum of Losses by Year

These amounts are highlighted in Table 10. Sorting annual losses from highest to lowest and ranking each year, we obtain

AEP	Rank	Year	$\sum_{i=1}^{10} X_i$
0.1	1	4	7,092.26
0.2	2	9	5,400.46
0.3	3	3	4,589.80
0.4	4	5	4,125.27
0.5	5	2	3,867.36
0.6	6	10	3,087.66
0.7	7	1	2,936.52
0.8	8	6	2,831.38
0.9	9	7	2,589.09
1.0	10	8	1,832.78

Table 9: Sorted and Ranked Sum of Losses by Year

The resulting Aggregate Exceedance Probability Curve is

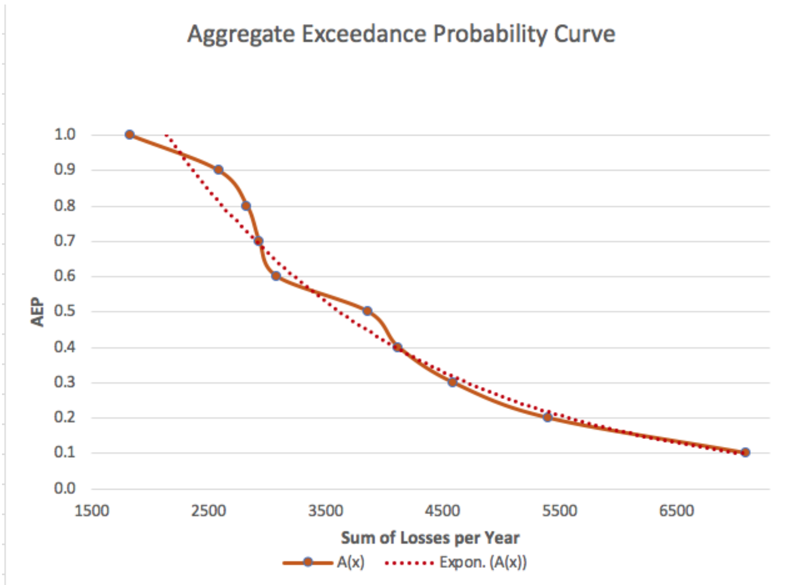

Figure 4: Aggregate Exceedance Probability Curve in Example 5.1

An exponential trend is included to demonstrate the general behavior of the function.

6 Comparison of the OEP and the AEP

In the simplified examples 3.1 and 5.1 we constructed the Occurrence and the Aggregate Exceedance Probability curves using the Monte Carlo simulation technique. These curves are shown in the following Figure.

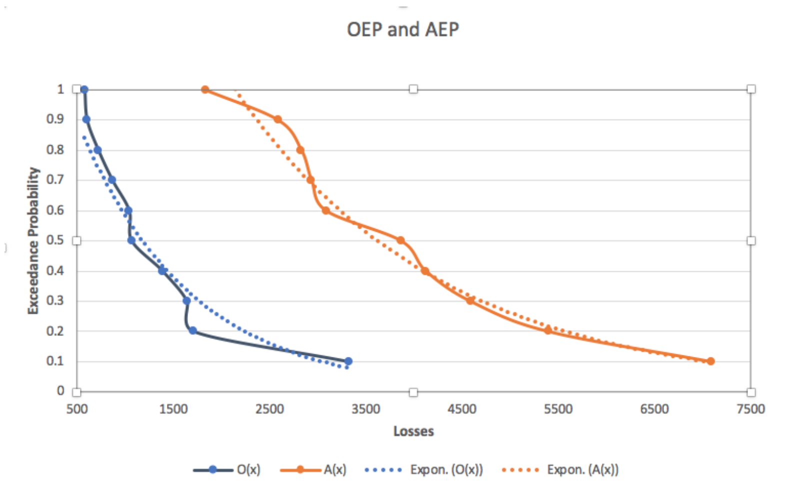

Figure 5: Occurrence and Aggregate EP Curves

In this graph the curves appear to be parallel-shifted due to the nature of the simplified assumption on the fixed number of losses per year.

A more typical visualization of the $O(x)$ and $A(x)$ curves is

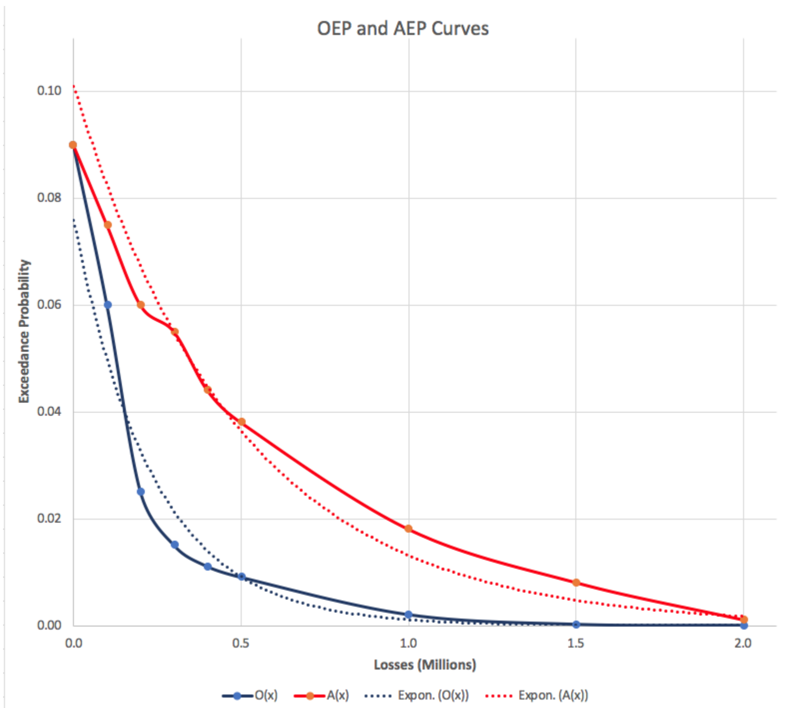

Figure 6: A Standard Visualization of the Occurrence and Aggregate EP Curves

Homer and Li, [2], address a question of when the OEP and the AEP are alike.

Proposition 6.1 *Let X be the severity of loss random variable and N be the number of claims random variable. Suppose that X and N are mutually independent. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\textit{If } \sum_{n=2}^{\infty} \boldsymbol{P}_N(n) < \delta \textit{ then } |A(x) - O(x)| < \epsilon$$

Proof. Let $X_1, X_2, \cdots, X_N$ be losses in a given year. By definition,

$$O(x) = \mathbf{P}\left(\max_{1 \le i \le N} (X_i) > x\right) \text{ and } A(x) = \mathbf{P}\left(\sum_{i=1}^{N} X_i > x\right)$$

We have shown in Sections 3 and 5 that

$$O(x) = 1 - \sum_{n=0}^{\infty} (F_X(x))^n \, \mathbf{P}(N = n) \text{ and}$$
$$A(x) = 1 - \sum_{n=0}^{\infty} F_X^{(n)}(x) \mathbf{P}(N = n),$$

where $F_X^{(n)}$ is an n-fold convolution of $F_X(x)$.

If $P_N(n) = \mathbf{P}(N = n) = 0$ for $n > 1$, then $A(x) = O(x)$. Otherwise, let $\epsilon > 0$. Choose $\delta = \epsilon/2$. Suppose that

$$\sum_{n=2}^{\infty} P_N(n) < \delta$$

Then,

$$|A(x) - O(x)| \le \sum_{n=2}^{\infty} \mathbf{P}(N = n) \left| F_X^{(n)}(x) - (F_X(x))^n \right| \le 2 \sum_{n=2}^{\infty} P_N(n) < 2\delta = \epsilon.$$

The following inequality is always true:

$$\max_{1 \le i \le N} (X_i) \le \sum_{i=1}^{N} X_i.$$

In addition, the following proposition shows connection between the OEP and the AEP with the survival function of the loss severity random variable.

Proposition 6.2 *Let $X_1, X_2, \cdots, X_N$ be losses in a given year, $F_X(x)$ and $S_X(x)$ be the cumulative distribution and survival functions of a loss random variable X. Then*

$$O(x) \geq 1 - F_X(x) = S_X(x) \text{ and}$$
$$A(x) \geq 1 - F_X(x) = S_X(x)$$

Proof. For any N, we have:

$$O(x) = 1 - (F_X(x))^N \geq 1 - F_X(x) = S_X(x)$$
$$A(x) = 1 - F_X^{(N)}(x) = 1 - \int_0^x F_X^{(N-1)}(x-y) f_X(y)\, dy \geq$$
$$\geq 1 - \int_0^x f_X(y)\, dy = 1 - F_X(x) = S_X(x)$$

7 The Deadliest, Costliest, and Most Intense US Tropical Cyclones

In this section we consider the information reported by the National Oceanic and Atmospheric Administration (NOAA) in 2011, [6], on the the deadliest, costliest, and most intense US tropical cyclones from 1851 to 2010 and construct the corresponding OEP and AEP curves for each category.

7.1 Ranking Tropical Cyclones by Deaths

Table 11 of Appendix B lists the tropical cyclones that have caused at least 25 deaths on the U.S. mainland during the period 1851-2010, [6].

Based on this table, the Galveston Hurricane of 1900 was responsible for at least 8000 deaths and remains first on the list. Hurricane Katrina of 2005 remains the third deadliest hurricane to strike the United States. Although these systems are spread out over most of the coast, there is a clustering of tracks on the coasts of Texas, southeastern Louisiana, south Florida, North Carolina and New England.

The following Figure 7, curtesy of [6], shows the paths of these deadly cyclones.

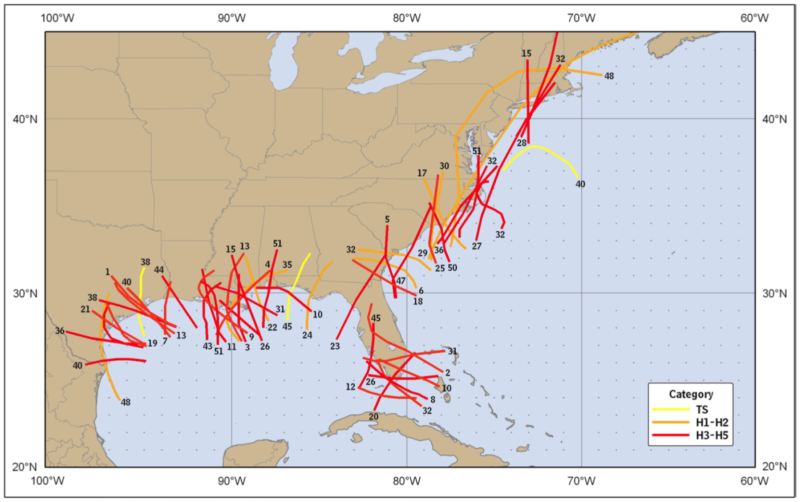

Figure 7: Mainland United States tropical cyclones causing 25 or more deaths, 1851-2010. The black numbers are the ranks of a given storm on Table 11 (e.g. 1 is the deadliest all-time). The colors are the intensity of the tropical cyclone at its maximum impact on the United States.

Table 12 provides maximum deaths and the sum of deaths by year with multiple hurricane years being highlighted. In addition, tables 13 and 14 show maximum number of deaths and the sum of the number of deaths sorted from highest to lowest resulting in the following Occurrence and Aggregate Exceedance Probability Curves

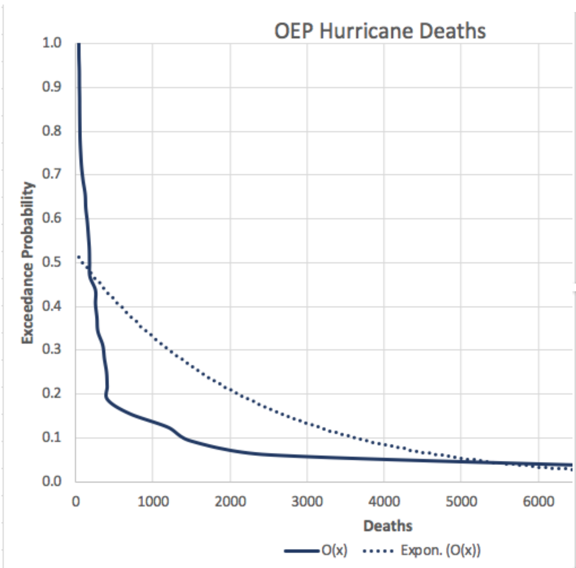

Figure 8: Occurrence EP Curve TC Deaths

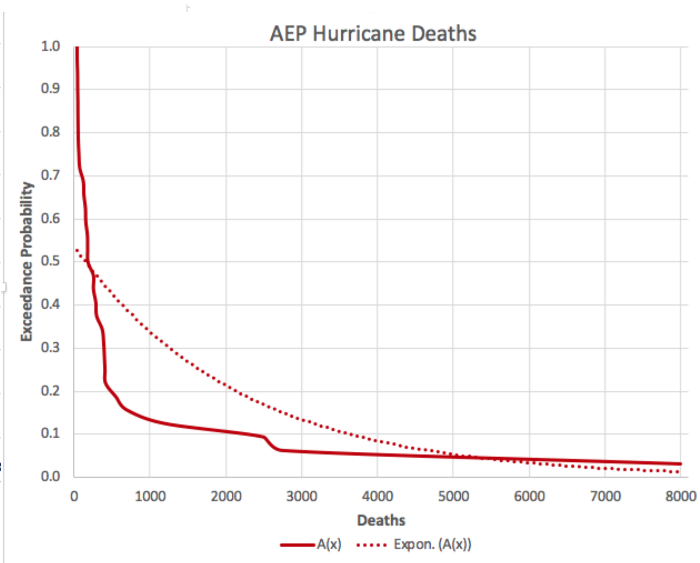

Figure 9: Aggregate EP Curve TC Deaths

An exponential trend is included to demonstrate the general behavior of the functions.

Since there are only a few years with multiple hurricanes, the comparison between the $O(x)$ and the $A(x)$ shown in the following graph is subtle

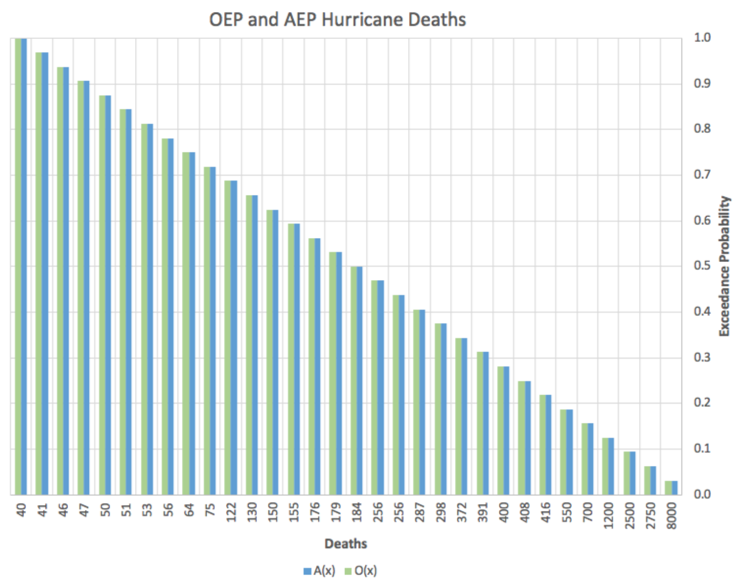

Figure 10: The Deadliest US Tropical Cyclones: Occurrence and Aggregate EP

7.2 Ranking Tropical Cyclones by Costs

Table 15 of Appendix C lists the 30 costliest mainland United States tropical cyclones, 1900-2010, not adjusted for inflation, [6].

Based on this table, hurricane Ike of 2008 was the second-costliest hurricane on record. Hurricane Katrina of 2005 was responsible for at least $108 billion of property damage and is by far the costliest hurricane to ever strike the United States. It is of note that the last ten hurricane seasons have produced 14 out of the 30 costliest systems to affect the United States.

The following Figure 11, curtesy of [6], displays the near-landfall portion of these tropical cyclone tracks and shows concentrations of costly hurricanes along the central Gulf Coast, south Florida and the Carolinas.

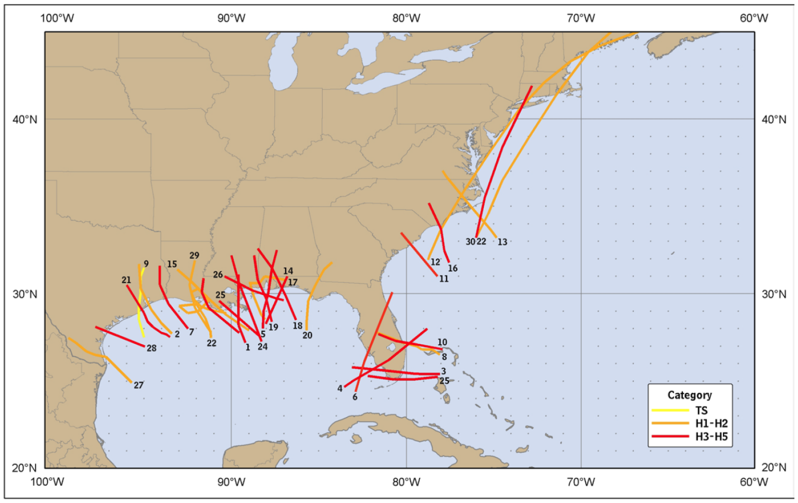

Figure 11: The 30 costliest tropical cyclones to strike the United States, 1900-2010. The black numbers are the ranks of a given storm on Table 15 (e.g. 1 is the costliest all-time). The colors are the intensity of the tropical cyclone at its maximum impact on the United States.

Table 16 re-orders Table 15 and the historical database after adjusting to 2010 dollars, which adds several other hurricanes. After this normalization to todays societal vulnerability, the last decade still accounts for eight of the top 30 tropical cyclones.

The following Figure 12, curtesy of [6], displays the near-landfall portion of these tropical cyclone tracks and shows concentrations of costly hurricanes along the central Gulf Coast, south Florida and the Carolinas.

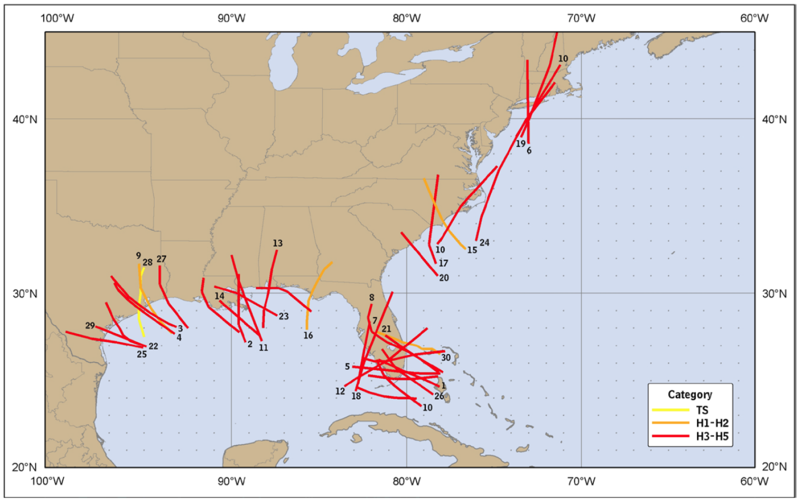

Figure 12: The 30 costliest tropical cyclones to strike the United States, ranked by normalization for inflation, population and wealth, 1900-2010. The black numbers are the ranks of a given storm on Table 16. The colors are the intensity of the tropical cyclone at its maximum impact on the United States.

Table 17 provides maximum costs and the sum of costs by year with multiple hurricane years being highlighted. In addition, tables 18 and 19 show maximum costs and the sum of costs sorted from highest to lowest resulting in the following Occurrence and Aggregate Exceedance Probability Curves

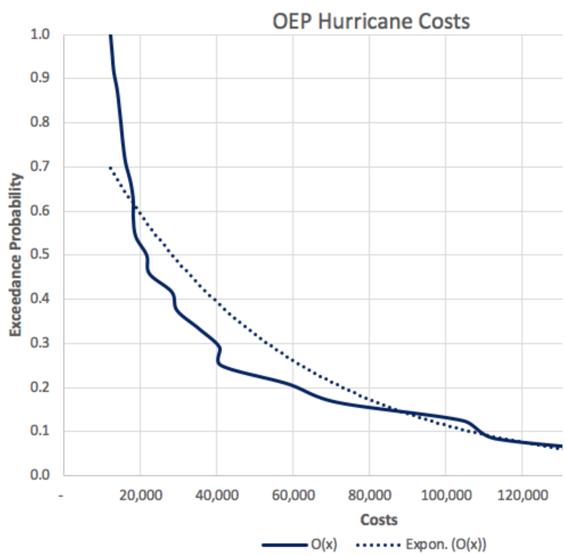

Figure 13: Occurrence EP Curve TC Costs

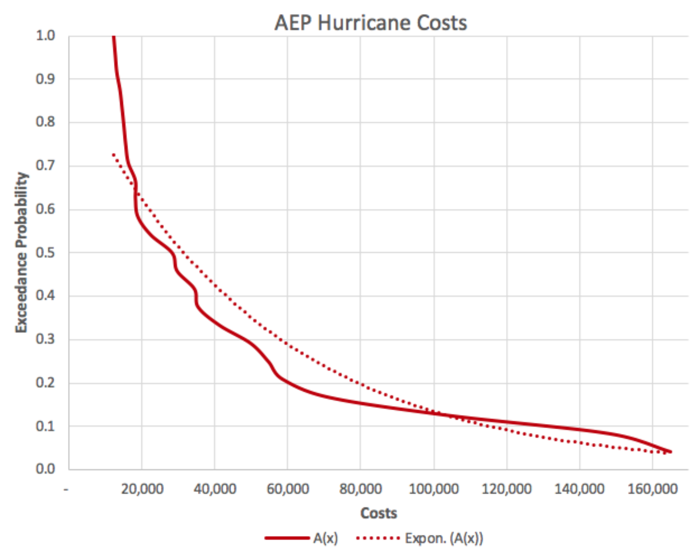

Figure 14: Aggregate EP Curve TC Costs

An exponential trend is included to demonstrate the general behavior of the functions.

Since there are only a few years with multiple hurricanes, the comparison between the $O(x)$ and the $A(x)$ shown in the following graph is subtle

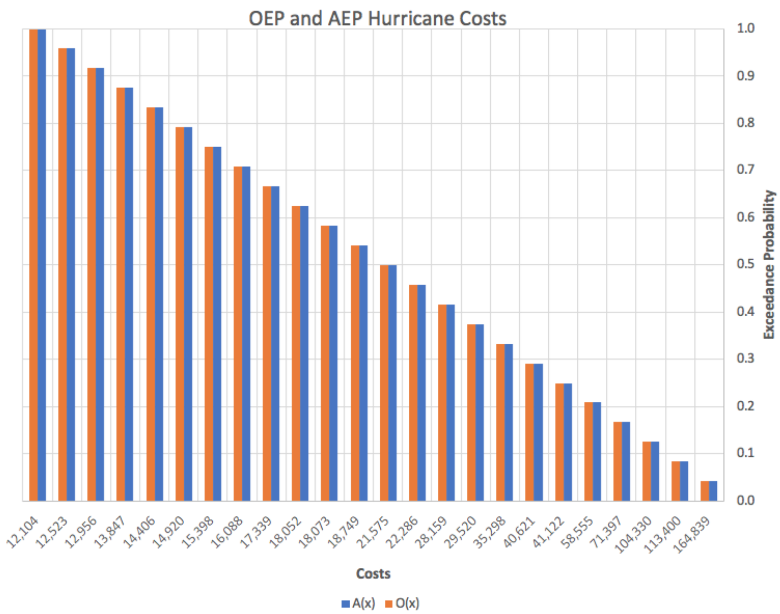

Figure 15: The Costliest US Tropical Cyclones Occurrence and Aggregate EP

7.3 Ranking Tropical Cyclones by Intensity

Table 20 of Appendix D lists the most intense major hurricanes to strike the U.S. mainland during the period 1851– 2010, [6]. In this study, the major hurricanes have been ranked by estimating *central pressure* at time of landfall. Central pressure is used as a proxy for intensity due to the uncertainties in maximum wind speed estimates for many historical hurricanes.

Based on this table, Hurricane Katrina had the third lowest pressure ever noted at landfall, behind the 1935 Florida Keys hurricane and Hurricane Camille in 1969.

The following Figure 16, curtesy of [6], shows where these major hurricanes struck the coast.

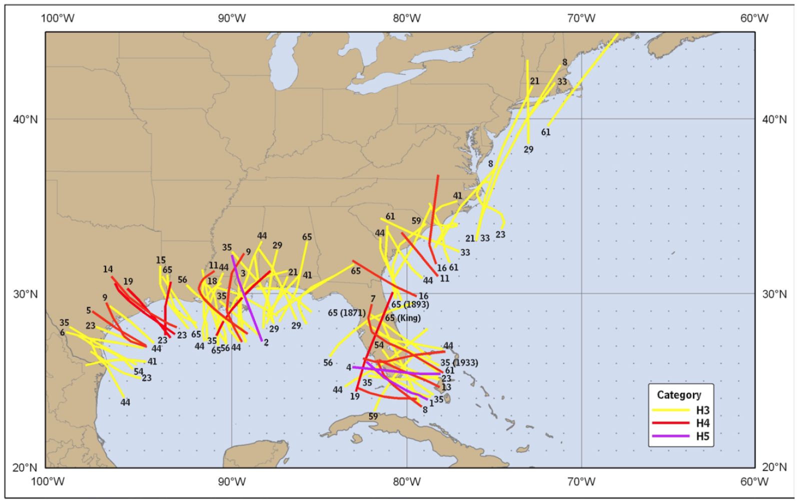

Figure 16: The most intense United States major hurricanes, ranked by pressure at landfall, 1851-2010. The black numbers are the ranks of a given storm on Table 20 (e.g. 1 has the lowest pressure all-time). The colors are the intensity of the tropical cyclone at its maximum impact on the United States.

Table 21 provides minimum and maximum intensities by year with multiple hurricane years being highlighted.

Using the definition of a hurricane intensity, adopted in [6], the most intense tropical storm is the one with the *lowest* central pressure. Thus, the usual definition of exceedance probability must be modified. Let I be an intensity random variable. Then

$$\mathbf{EP}_I(x) = \mathbf{P}(I < x)$$

Using probabilistic terminology, the $EP_I(x)$ is the cumulative distribution function of I.

Tables 22 and 23 show minimum and maximum intensities sorted from lowest to highest resulting in the following Min and Max Exceedance Probability Curves

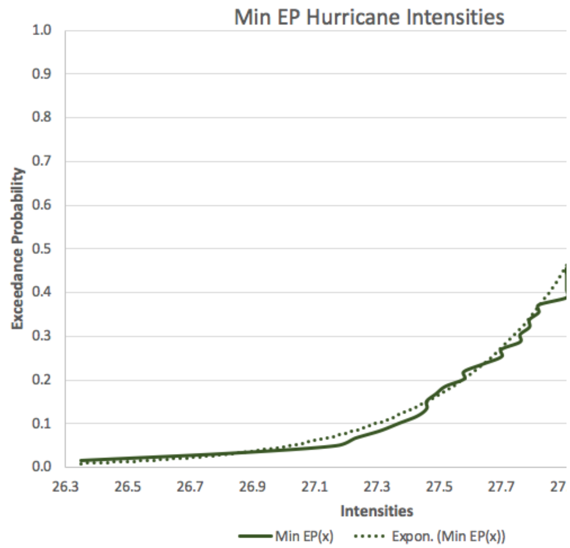

Figure 17: Occurrence EP Curve TC Intensities

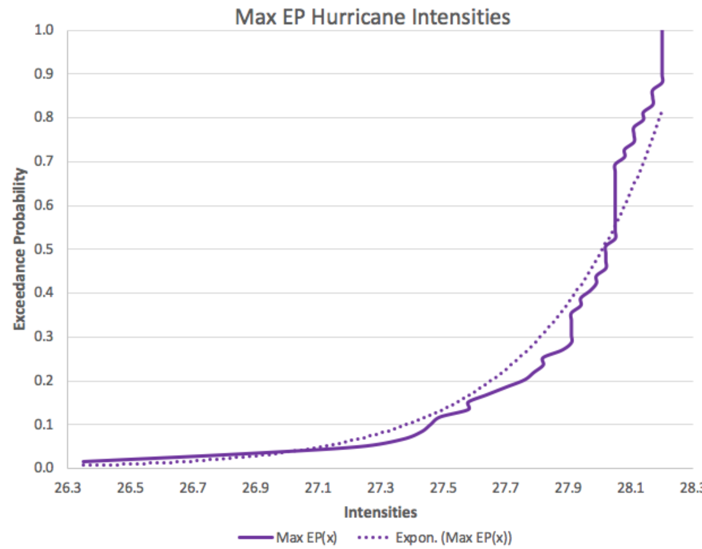

Figure 18: Aggregate EP Curve TC Intensities

An exponential trend is included to demonstrate the general behavior of the functions.

Since there are only a few years with multiple hurricanes, the comparison between the Min EP and the Max EP shown in the following graph is subtle

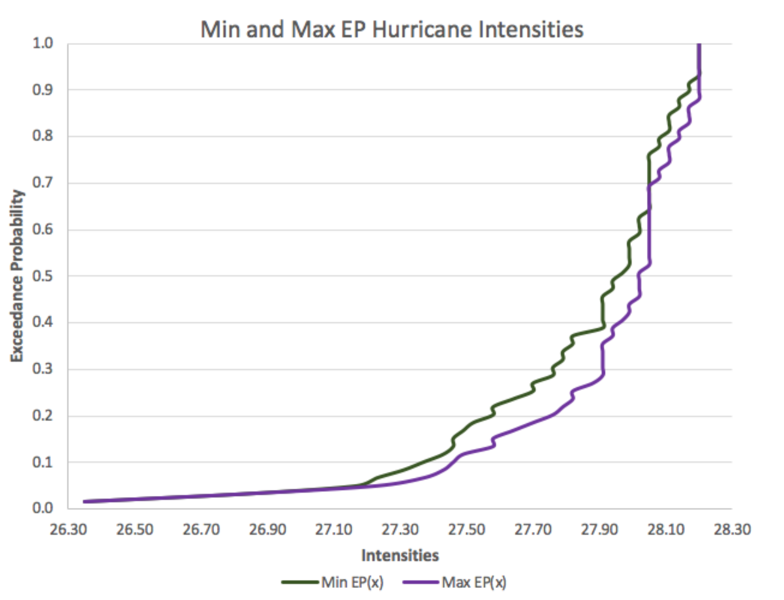

Figure 19: The Most Intense US Tropical Cyclones Occurrence and Aggregate EP Curves

Following [4], the difference between the aggregate and occurrence EP curves would vary depending on:

1. Peril, such as hurricane, earthquake, flood, severe convective storm, etc;
2. Geographic Scope that includes all of the US, by state, by county, by ZIP or by region such as California vs. East Coast vs. Gulf Coast vs. Midwest, etc;
3. Portfolio composition such as construction, occupancy, year built, building height, etc;
4. Insurance structure such as deductibles, endorsements, exclusions, etc.

8 Conclusion

In this paper we explored two of the most important notions in Catastrophic Modeling, the Occurrence Exceedance Probability (OEP) and the Aggregate Exceedance Probability (AEP). We discussed construction of each curve and compared these two metrics in several numeric and theoretical examples. In particular, we discussed a connection between the distribution of loss severities and the OEP depending on the distribution of claim counts. One of the examples involved Monte Carlo Simulation, an important technique that allows to account for risk in quantitative analysis and decision making. Finally, we produced the OEP and AEP curves for the deadliest, costliest, and most intense US tropical cyclones based on the 2011 National Oceanic and Atmospheric Administration (NOAA) report.

A OEP and AEP Curves Simulation

Table 10: Simulated Losses for $O(x)$ and $A(x)$

No	u	X_i	$\max_{1\leq i\leq 10}(X_i)$	$\sum_{i=1}^{10} X_i$
1	0.1244	45.27	**869.63**	**2,936.52**
2	0.2997	126.11	869.63	3,325.47
3	0.8470	869.63	869.63	3,610.52
4	0.4592	227.44	594.64	2,905.16
5	0.3690	165.89	1,390.24	4,067.96
6	0.0547	18.92	1,390.24	4,370.74
7	0.1723	65.05	1,390.24	4,466.30
8	0.4739	238.72	1,390.24	5,122.50
9	0.7534	594.64	1,390.24	4,984.59
10	0.7488	584.86	1,390.24	4,427.47
11	0.6610	434.22	**1,390.24**	**3,867.36**
12	0.6441	411.16	1,390.24	3,434.54
13	0.3664	164.26	1,390.24	3,295.06
14	0.9268	1,390.24	1,713.30	4,844.10
15	0.6843	468.66	1,713.30	4,418.00
16	0.2776	114.49	1,713.30	4,132.96
17	0.8039	721.24	1,713.30	4,811.47
18	0.2503	100.81	1,713.30	4,245.80
19	0.1046	37.52	1,713.30	4,225.72
20	0.0707	24.75	1,713.30	4,334.76
21	0.0042	1.40	**1,713.30**	**4,589.80**
22	0.5137	271.67	1,713.30	5,326.43
23	0.9499	1,713.30	1,713.30	5,126.44
24	0.8680	964.14	964.14	3,548.89
25	0.3969	183.62	793.00	2,796.58
26	0.8265	793.00	793.00	2,959.52
27	0.3519	155.57	1,290.22	3,456.74

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Table 10 – *Continued from previous page*

No	u	X_i	$\max_{1\leq i\leq 10}(X_i)$	$\sum_{i=1}^{10} X_i$
28	0.2078	80.74	3,330.60	6,631.78
29	0.3365	146.55	3,330.60	6,597.18
30	0.5229	279.79	3,330.60	6,533.40
31	0.8095	738.03	**3,330.60**	**7,092.26**
32	0.1875	71.68	3,330.60	7,311.69
33	0.3174	135.76	3,330.60	7,286.00
34	0.4381	211.83	3,330.60	7,624.17
35	0.5904	346.57	3,330.60	8,482.10
36	0.9168	1,290.22	3,330.60	8,340.17
37	0.9877	3,330.60	3,330.60	7,954.98
38	0.1265	46.13	1,069.76	4,786.62
39	0.2123	82.78	1,069.76	4,783.34
40	0.8391	838.65	1,069.76	4,705.71
41	0.8667	957.46	**1,069.76**	**4,125.27**
42	0.1262	46.00	1,069.76	3,484.08
43	0.6877	473.92	1,069.76	3,713.75
44	0.8872	1,069.76	1,069.76	3,482.34
45	0.4279	204.63	905.04	2,696.83
46	0.8554	905.04	905.04	2,802.84
47	0.3631	162.25	316.27	1,945.86
48	0.1183	42.84	316.27	1,855.80
49	0.0153	5.16	568.24	2,381.19
50	0.4980	258.21	604.58	2,980.62
51	0.5615	316.27	**604.58**	**2,831.38**
52	0.5183	275.67	604.58	2,660.73
53	0.4787	242.51	604.58	2,543.30
54	0.5279	284.25	604.58	2,497.96
55	0.5558	310.64	604.58	2,422.68
56	0.1313	48.06	604.58	2,152.96

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Table 10 – *Continued from previous page*

No	u	X_i	$\max_{1\leq i\leq 10}(X_i)$	$\sum_{i=1}^{10} X_i$
57	0.1887	72.18	604.58	2,609.00
58	0.7407	568.24	604.58	2,661.87
59	0.7579	604.58	604.58	2,341.09
60	0.2668	108.98	504.10	2,119.46
61	0.3349	145.61	**578.61**	**2,589.09**
62	0.3564	158.24	578.61	2,473.83
63	0.4172	197.17	578.61	2,465.94
64	0.4341	208.98	721.97	2,990.75
65	0.1133	40.92	721.97	2,826.52
66	0.7061	504.10	721.97	2,988.58
67	0.2978	125.04	721.97	2,689.37
68	0.4849	247.47	721.97	2,607.54
69	0.6219	382.95	721.97	2,590.02
70	0.7458	578.61	721.97	2,312.96
71	0.0858	30.35	**721.97**	**1,832.78**
72	0.3431	150.36	721.97	2,374.31
73	0.8042	721.97	721.97	2,251.83
74	0.1231	44.75	571.89	2,012.57
75	0.4256	202.97	1,254.22	3,222.04
76	0.4283	204.90	1,644.01	4,663.08
77	0.1192	43.22	1,644.01	4,960.95
78	0.4625	229.95	1,644.01	5,008.84
79	0.2606	105.89	1,644.01	4,940.07
80	0.2454	98.42	1,644.01	5,268.69
81	0.7425	571.89	**1,644.01**	**5,400.46**
82	0.0792	27.87	1,644.01	5,155.09
83	0.6932	482.72	1,644.01	5,132.26
84	0.9127	1,254.22	1,644.01	5,691.70
85	0.9459	1,644.01	1,644.01	4,479.08

Continued on next page

Table 10 – *Continued from previous page*

No	u	X_i	$\max_{1\leq i\leq 10}(X_i)$	$\sum_{i=1}^{10} X_i$
86	0.7053	502.77	1,042.16	3,115.89
87	0.2302	91.11	1,042.16	3,180.42
88	0.3613	161.17	1,042.16	3,436.24
89	0.6612	434.51	1,042.16	3,281.85
90	0.4629	230.19	1,042.16	3,311.03
91	0.5716	326.52	**1,042.16**	**3,087.66**
92	0.0150	5.04	1,042.16	2,761.14
93	0.8826	1,042.16	1,042.16	2,756.10
94	0.1151	41.60	567.30	1,713.94
95	0.5241	280.82	567.30	1,672.34
96	0.7403	567.30	567.30	1,391.52
97	0.5908	346.93	463.69	824.23
98	0.0201	6.78	463.69	477.30
99	0.6811	463.69	463.69	470.52
100	0.0202	6.83	6.83	6.83

B The Deadliest US Tropical Cyclones

Table 11: Mainland U.S. Tropical Cyclones Deaths 1851-2010

Rank	Hurricane	Year	Category	Deaths
1	TX (Galveston)	1900	4	8,000
2	FL (SE/Lake Okeechobee)	1928	4	2,500
3	KATRINA (SE LA/MS)	2005	3	1,200
4	LA (Cheniere Caminanda)	1893	4	1,250
5	SC/GA (SeaIs lands)	1893	3	1,500
6	GA/SC	1881	2	700
7	AUDREY (SW LA N TX)	1957	4	416
8	FL (Keys)	1935	5	408
9	LA (Last Island)	1856	4	400
10	FL (Miami) IMS/AUPensacola	1926	4	372
11	LA (Grand Isle)	1909	3	350
12	FL (Keys)/S TX	1919	4	287
13	LA (New Orleans)	1915	3	275
13	TX (Galveston)	1915	4	275
15	New England	1938	3	256
15	CAMILLE (MS/SE LA/VA)	1969	5	256
17	DIANE (NE U.S.)	1955	1	184
18	GA, SC, NC	1898	4	179
19	TX	1875	3	176
20	SE FL	1906	3	164
21	TX (Indianola)	1886	4	150
22	MS/AUPensacola	1906	2	134
23	FL, GA, SC	1896	3	130
24	AGNES (FL/NE U.S.)	1972	1	122
25	HAZEL (SC/NC)	1954	4	95
26	BETSY (SE FL/SE LA)	1965	3	75
27	Northeast U.S.	1944	3	64

Continued on next page

Table 11 – *Continued from previous page*

Rank	Hurricane	Year	Category	Deaths
28	CAROL (NE U.S.)	1954	3	60
29	FLOYD (Mid Atlantic & NE U.S.)	1999	2	56
30	NC	1883	2	53
31	SE FL/SE LA/MS	1947	4	51
32	NC, SC	1899	3	50
32	GA/SCINC	1940	2	50
32	DONNA (FL/Eastem U.S.)	1960	4	50
35	LA	1860	2	47
36	NC, VA	1879	3	46
36	CARLA	1961	4	46
38	TX (Velasco)	1909	3	41
38	ALLISON (SE D9	2001	TS	41
40	Mid-Atlantic	1889	TS	40
40	TX (Freeport)	1932	4	40
40	S TX	1933	3	40

Table 12: Hurricane Max and Sum of Deaths By Year

Year	Max	Sum
1856	400	400
1860	47	47
1875	176	176
1879	46	46
1881	700	700
1883	53	53
1886	150	150
1889	40	40

Continued on next page

Table 12 – *Continued from previous page*

Year	Max	Sum
1893	**1,500**	**2,750**
1896	130	130
1898	179	179
1899	50	50
1900	8,000	8,000
1906	**164**	**298**
1909	**350**	**391**
1915	**275**	**550**
1919	287	287
1926	372	372
1928	2,500	2,500
1932	40	40
1933	40	40
1935	408	408
1938	256	256
1940	50	50
1944	64	64
1947	51	51
1954	95	155
1955	184	184
1957	416	416
1960	50	50
1961	46	46
1965	75	75
1969	256	256
1972	122	122
1999	56	56
2001	41	41
2005	1,200	1,200

Table 13: Hurricane Max Deaths By Year

No	O(x)	Max Deaths Sorted
1	0.031	8,000
2	0.063	2,500
3	0.094	1,500
4	0.125	1,200
5	0.156	700
6	0.188	416
7	0.219	408
8	0.250	400
9	0.281	372
10	0.313	350
11	0.344	287
12	0.375	275
13	0.406	256
14	0.438	256
15	0.469	184
16	0.500	179
17	0.531	176
18	0.563	164
19	0.594	150
20	0.625	130
21	0.656	122
22	0.688	95
23	0.719	75
24	0.750	64
25	0.781	56
26	0.813	53
27	0.844	51
28	0.875	50

Continued on next page

Table 13 – *Continued from previous page*

No	O(x)	Max Deaths Sorted
29	0.906	47
30	0.938	46
31	0.969	41
32	1.000	40

Table 14: Hurricane Sum of Deaths By Year

No	A(x)	Sum Deaths Sorted
1	0.031	8,000
2	0.063	2,750
3	0.094	2,500
4	0.125	1,200
5	0.156	700
6	0.188	550
7	0.219	416
8	0.250	408
9	0.281	400
10	0.313	391
11	0.344	372
12	0.375	298
13	0.406	287
14	0.438	256
15	0.469	256
16	0.500	184
17	0.531	179
18	0.563	176
19	0.594	155

Continued on next page

Table 14 – *Continued from previous page*

No	A(x)	Sum Deaths Sorted
20	0.625	150
21	0.656	130
22	0.688	122
23	0.719	75
24	0.750	64
25	0.781	56
26	0.813	53
27	0.844	51
28	0.875	50
29	0.906	47
30	0.938	46
31	0.969	41
32	1.000	40

C The Costliest US Tropical Cyclones

Table 15: The 30 costliest mainland United States tropical cyclones, 1900-2010, (not adjusted for inflation).

Rank	Hurricane	Year	Category	Damage (Millions)
1	KATRINA (SE FL, LA, MS)	2005	3	108,000
2	IKE (TX, LA)	2008	2	29,520
3	ANDREW (SE FL/LA)	1992	5	26,500
4	WILMA (S FL)	2005	3	21,007
5	IVAN (AL/NW FL)	2004	3	18,820
6	CHARLEY (SW FL)	2004	4	15,113
7	RITA (SW LA, N TX)	2005	3	12,037
8	FRANCES (FL)	2004	2	9,507
9	ALLISON (N TX)	2001	TS	9,000
10	JEANNE (FL)	2004	3	7,660
11	HUGO (SC)	1989	4	7,000
12	FLOYD (Mid-Atlantic & NE U.S.)	1999	2	6,900
13	ISABEL (Mid-Atlantic)	2003	2	5,370
14	OPAL (NW FL/AL)	1995	3	5,142
15	GUSTAV (LA)	2008	2	4,618
16	FRAN (NC)	1996	3	4,160
17	GEORGES (FL Keys, MS,AL)	1998	2	2,765
18	DENNIS (NW FL)	2005	3	2,545
19	FREDERIC (AL/MS)	1979	3	2,300
20	AGNES (FUNE U.S.)	1972	1	2,100
21	ALICIA (N TX)	1983	3	2,000
22	BOB (NC, NE U.S)	1991	2	1,500
22	JUAN (LA)	1985	1	1,500
24	CAMILLE (MS/SE LANA)	1969	5	1,421
25	BETSY (SE FL/SE LA)	1965	3	1,421
26	ELENA (MS/AL/NW FL)	1985	3	1,250

Continued on next page

Table 15 – *Continued from previous page*

Rank	Hurricane	Year	Category	Damage (Millions)
27	DOLLY (S TX)	2008	1	1,050
28	CELIA (S TX)	1970	3	930
29	LILI (SC LA)	2002	1	925
30	GLORIA (Eastern U.S.)	1985	3	900

Table 16: The 30 costliest mainland United States tropical cyclones, 1900-2010, Ranked Using 2010 Inflation, Population and Wealth Normalization.

Rank	Hurricane	Year	Category	Damage (Millions)
1	SE Florida/Alabama	1926	4	164,839
2	KATRINA (SE LA, MS, AL)	2005	3	113,400
3	N Texas (Galveston)	1900	4	104,330
4	N Texas (Galveston)	1915	4	71,397
5	ANDREW (SE FL/LA)	1992	5	58,555
6	New England	1938	3	41,122
7	SW Florida	1944	3	40,621
8	SE Florida/Lake Okeechobee	1928	4	35,298
9	IKE (N TX/SW LA)	2008	2	29,520
10	DONNA (FUEastern U.S.)	1960	4	28,159
11	CAMILLE (MS/LANA)	1969	5	22,286
12	WILMA (S FL)	2005	3	22,057
13	IVAN (NW FL, AL)	2004	3	21,575
14	BETSY (SE FL/LA)	1965	3	18,749
15	DIANE (NE U.S.)	1955	1	18,073
16	AGNES (NW FL, NE U.S.)	1972	1	18,052
17	HAZEL (SC/NC)	1954	4	17,339
18	CHARLEY (SW FL)	2004	4	17,210

Continued on next page

Table 16 – *Continued from previous page*

Rank	Hurricane	Year	Category	Damage (Millions)
19	CAROL (NE U.S.)	1954	3	16,940
20	HUGO (SC)	1989	4	16,088
21	SE Florida	1949	3	15,398
22	CARLA (N & Central TX)	1961	4	14,920
23	SE Florida/Louisiana/Alabama	1947	4	14,406
24	NE U.S.	1944	3	13,881
25	SE FL/S TX	1919	4	13,847
26	SE Florida	1945	3	12,956
27	RITA (SW LA/N TX)	2005	3	12,639
28	ALLISON (N TX)	2001	TS	12,523
29	CELIA (S TX)	1970	3	12,104
30	FRANCES (SE FL)	2004	2	10,899

Table 17: Hurricane Max and Sum of Costs By Year

Year	Max	Sum
1900	104,330	104,330
1915	71,397	71,397
1919	13,847	13,847
1926	164,839	164,839
1928	35,298	35,298
1938	41,122	41,122
1944	**40,621**	**54,502**
1945	12,956	12,956
1947	14,406	14,406
1949	15,398	15,398
1954	**17,339**	**34,279**

Continued on next page

Table 17 – *Continued from previous page*

Year	Max	Sum
1955	18,073	18,073
1960	28,159	28,159
1961	14,920	14,920
1965	18,749	18,749
1969	22,286	22,286
1970	12,104	12,104
1972	18,052	18,052
1989	16,088	16,088
1992	58,555	58,555
2001	12,523	12,523
2004	**21,575**	**49,684**
2005	**113,400**	**148,096**
2008	29,520	29,520

Table 18: Hurricane Max Costs By Year

No	O(x)	Max Costs Sorted
1	0.042	164,839
2	0.083	113,400
3	0.125	104,330
4	0.167	71,397
5	0.208	58,555
6	0.250	41,122
7	0.292	40,621
8	0.333	35,298
9	0.375	29,520
10	0.417	28,159

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Table 18 – *Continued from previous page*

No	O(x)	Max Costs Sorted
11	0.458	22,286
12	0.500	21,575
13	0.542	18,749
14	0.583	18,073
15	0.625	18,052
16	0.667	17,339
17	0.708	16,088
18	0.750	15,398
19	0.792	14,920
20	0.833	14,406
21	0.875	13,847
22	0.917	12,956
23	0.958	12,523
24	1.000	12,104

Table 19: Hurricane Sum of Costs By Year

No	A(x)	Sum Cosths Sorted
1	0.042	164,839
2	0.083	148,096
3	0.125	104,330
4	0.167	71,397
5	0.208	58,555
6	0.250	54,502
7	0.292	49,684
8	0.333	41,122
9	0.375	35,298

Continued on next page

Table 19 – *Continued from previous page*

No	A(x)	Sum Costs Sorted
10	0.417	34,279
11	0.458	29,520
12	0.500	28,159
13	0.542	22,286
14	0.583	18,749
15	0.625	18,073
16	0.667	18,052
17	0.708	16,088
18	0.750	15,398
19	0.792	14,920
20	0.833	14,406
21	0.875	13,847
22	0.917	12,956
23	0.958	12,523
24	1.000	12,104

D The Most Intense US Tropical Cyclones

Table 20: The Most Intense Mainland United States Hurricanes Ranked by Pressure, 1851-2010

Rank	Hurricane	Year	Category (at landfall)	Mimimum Millibars	Pressure (Inches)
1	FL (Keys)	1935	5	892	26.35
2	CAMILLE (MS/SE LA/VA)	1969	5	909	26.84
3	KATRINA (SE LA, MS)	2005	3	920	27.17
4	ANDREW (SE FL/SE LA)	1992	5	922	27.23
5	TX (Indianola)	1886	4	925	27.31
6	FL (Keys)/S TX	1919	4	927	27.37
7	FL (Lake Okeechobee)	1928	4	929	27.43
8	DONNA (FL/Eastern U.S.)	1960	4	930	27.46
8	FL (Miami)/MS/AUPensacola	1926	4	930	27.46
10	CARLA (N & Central TX)	1961	4	931	27.49
11	S TX	1916	4	932	27.52
12	LA (Last Island)	1856	4	934	27.58
12	HUGO (SC)	1989	4	934	27.58
14	TX (Galveston)	1900	4	936	27.64
15	RITA (SW LA/N TX)	2005	3	937	27.67
16	GA/FL (Brunswick)	1898	4	938	27.70
16	HAZEL (SC/NC)	1954	4	938	27.70
18	SE FL/SE LA/MS	1947	4	940	27.76
18	TX (Galveston)	1915	4	940	27.76
20	N TX	1932	4	941	27.79
20	CHARLEY (SW FL)	2004	4	941	27.79
22	GLORIA (Eastern U.S.)	1985	3	942	27.82
22	OPAL (NW FL/AL)	1995	3	942	27.82
24	LA (New Orleans)	1915	3	944	27.88
25	FL (Central)	1888	3	945	27.91
25	E NC	1899	3	945	27.91

Continued on next page

Table 20 – *Continued from previous page*

Rank	Hurricane	Year	Category (at landfall)	Mimimum Millibars	Pressure (Inches)
25	AUDREY (SW LA/N TX)	1957	4	945	27.91
25	CELIA (S TX)	1970	3	945	27.91
25	ALLEN (S TX)	1980	3	945	27.91
30	New England	1938	3	946	27.94
30	FREDERIC (AL/MS)	1979	3	946	27.94
30	/VAN (AL, NW FL)	2004	3	946	27.94
30	DENNIS (NW FL)	2005	3	946	27.94
34	NE U.S.	1944	3	947	27.97
35	LA (Chenier Caminanda)	1893	4	948	27.99
35	BETSY (SE FL/SE LA)	1965	3	948	27.99
35	SE FL/NW FL	1929	3	948	27.99
35	SE FL	1933	3	948	27.99
39	NW FL	1917	3	949	28.02
39	NW FL	1882	3	949	28.02
39	DIANA (NC)	1984	3	949	28.02
39	S TX	1933	3	949	28.02
43	MS/AL	1916	3	950	28.05
43	GA/SC	1854	3	950	28.05
43	LA/MS	1855	3	950	28.05
43	LA/MS/AL	1860	3	950	28.05
43	LA	1879	3	950	28.05
43	BEULAH (S TX)	1967	3	950	28.05
43	HILDA (Central LA)	1964	3	950	28.05
43	GRACIE (SC)	1959	3	950	28.05
43	TX (Central)	1942	3	950	28.05
43	JEANNE (FL)	2004	3	950	28.05
43	WILMA (S FL)	2005	3	950	28.05
54	SE FL	1945	3	951	28.08

Continued on next page

Table 20 – *Continued from previous page*

Rank	Hurricane	Year	Category (at landfall)	Mimimum Millibars	Pressure (Inches)
54	BRET (S TX)	1999	3	951	28.08
56	LA (Grand Isle)	1909	3	952	28.11
56	FL (Tampa Bay)	1921	3	952	28.11
56	CARMEN (Central LA)	1974	3	952	28.11
59	SC/NC	1885	3	953	28.14
59	S FL	1906	3	953	28.14
61	GA/SC	1893	3	954	28.17
61	EDNA (New England)	1954	3	954	28.17
61	SE FL	1949	3	954	28.17
61	FRAN (NC)	1996	3	954	28.17
65	SE FL	1871	3	955	28.20
65	LA/TX	1886	3	955	28.20
65	SC/NC	1893	3	955	28.20
65	NW FL	1894	3	955	28.20
65	ELOISE (NW FL)	1975	3	955	28.20
65	KING (SE FL)	1950	3	955	28.20
65	Central LA	1926	3	955	28.20
65	SW LA	1918	3	955	28.20

Table 21: Hurricane Min and Max of Intensities By Year

Year	Min Pressure	Max Pressure
1854	28.05	28.05
1855	28.05	28.05
1856	27.58	27.58
1860	28.05	28.05

Continued on next page

Table 21 – *Continued from previous page*

Year	Min Pressure	Max Pressure
1871	28.20	28.20
1879	28.05	28.05
1882	28.02	28.02
1885	28.14	28.14
1886	**27.31**	**28.20**
1888	27.91	27.91
1893	**27.99**	**28.20**
1894	28.20	28.20
1898	27.70	27.70
1899	27.91	27.91
1900	27.64	27.64
1906	28.14	28.14
1909	28.11	28.11
1915	**27.76**	**27.88**
1916	**27.52**	**28.05**
1917	28.02	28.02
1918	28.20	28.20
1919	27.37	27.37
1921	28.11	28.11
1926	**27.46**	**28.20**
1928	27.43	27.43
1929	27.99	27.99
1932	27.79	27.79
1933	**27.99**	**28.02**
1935	26.35	26.35
1938	27.94	27.94
1942	28.05	28.05
1944	27.97	27.97
1945	28.08	28.08

Continued on next page

Table 21 – *Continued from previous page*

Year	Min Pressure	Max Pressure
1947	27.76	27.76
1949	28.17	28.17
1950	28.20	28.20
1954	**27.70**	**28.17**
1957	27.91	27.91
1959	28.05	28.05
1960	27.46	27.46
1961	27.49	27.49
1964	28.05	28.05
1965	27.99	27.99
1967	28.05	28.05
1969	26.84	26.84
1970	27.91	27.91
1974	28.11	28.11
1975	28.20	28.20
1979	27.94	27.94
1980	27.91	27.91
1984	28.02	28.02
1985	27.82	27.82
1989	27.58	27.58
1992	27.23	27.23
1995	27.82	27.82
1996	28.17	28.17
1999	28.08	28.08
2004	**27.79**	**28.05**
2005	**27.17**	**28.05**

Table 22: Hurricane Min Intensities By Year

No	$MinEP(x)$	Min Intensities Sorted
1	0.017	26.35
2	0.034	26.84
3	0.051	27.17
4	0.068	27.23
5	0.085	27.31
6	0.102	27.37
7	0.119	27.43
8	0.136	27.46
9	0.153	27.46
10	0.169	27.49
11	0.186	27.52
12	0.203	27.58
13	0.220	27.58
14	0.237	27.64
15	0.254	27.70
16	0.271	27.70
17	0.288	27.76
18	0.305	27.76
19	0.322	27.79
20	0.339	27.79
21	0.356	27.82
22	0.373	27.82
23	0.390	27.91
24	0.407	27.91
25	0.424	27.91
26	0.441	27.91
27	0.458	27.91
28	0.475	27.94
29	0.492	27.94

Continued on next page

Table 22 – *Continued from previous page*

No	$MinEP(x)$	Min Intensities Sorted
30	0.508	27.97
31	0.525	27.99
32	0.542	27.99
33	0.559	27.99
34	0.576	27.99
35	0.593	28.02
36	0.610	28.02
37	0.627	28.02
38	0.644	28.05
39	0.661	28.05
40	0.678	28.05
41	0.695	28.05
42	0.712	28.05
43	0.729	28.05
44	0.746	28.05
45	0.763	28.05
46	0.780	28.08
47	0.797	28.08
48	0.814	28.11
49	0.831	28.11
50	0.847	28.11
51	0.864	28.14
52	0.881	28.14
53	0.898	28.17
54	0.915	28.17
55	0.932	28.20
56	0.949	28.20
57	0.966	28.20
58	0.983	28.20

Continued on next page

Table 22 – *Continued from previous page*

No	$MinEP(x)$	Min Intensities Sorted
59	1.000	28.20

Table 23: Hurricane Max Intensities By Year

No	$MaxEP(x)$	Min Intensities Sorted
1	0.017	26.35
2	0.034	26.84
3	0.051	27.23
4	0.068	27.37
5	0.085	27.43
6	0.102	27.46
7	0.119	27.49
8	0.136	27.58
9	0.153	27.58
10	0.169	27.64
11	0.186	27.70
12	0.203	27.76
13	0.220	27.79
14	0.237	27.82
15	0.254	27.82
16	0.271	27.88
17	0.288	27.91
18	0.305	27.91
19	0.322	27.91
20	0.339	27.91
21	0.356	27.91
22	0.373	27.94

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Table 23 – *Continued from previous page*

No	$MaxEP(x)$	Min Intensities Sorted
23	0.390	27.94
24	0.407	27.97
25	0.424	27.99
26	0.441	27.99
27	0.458	28.02
28	0.475	28.02
29	0.492	28.02
30	0.508	28.02
31	0.525	28.05
32	0.542	28.05
33	0.559	28.05
34	0.576	28.05
35	0.593	28.05
36	0.610	28.05
37	0.627	28.05
38	0.644	28.05
39	0.661	28.05
40	0.678	28.05
41	0.695	28.05
42	0.712	28.08
43	0.729	28.08
44	0.746	28.11
45	0.763	28.11
46	0.780	28.11
47	0.797	28.14
48	0.814	28.14
49	0.831	28.17
50	0.847	28.17
51	0.864	28.17

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Table 23 – *Continued from previous page*

No	$MaxEP(x)$	Min Intensities Sorted
52	0.881	28.20
53	0.898	28.20
54	0.915	28.20
55	0.932	28.20
56	0.949	28.20
57	0.966	28.20
58	0.983	28.20
59	1.000	28.20

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Building and Testing Yield Curve Generators for P&C Insurance

By Gary G. Venter, FCAS, ASA, CERA, and Kailan Shang, FSA, CFA, PRM, SCJP

Abstract Interest-rate risk is a key factor for property-casualty insurer capital. P&C companies tend to be highly leveraged, with bond holdings much greater than capital. For GAAP capital, bonds are marked to market but liabilities are not, so shifts in the yield curve can have a significant impact on capital. Yield-curve scenario generators are one approach to quantifying this risk. They produce many future simulated evolutions of the yield curve, which can be used to quantify the probabilities of bond-value changes that would result from various maturity-mix strategies. Some of these generators are provided as black-box models where the user gets only the projected scenarios. One focus of this paper is to provide methods for testing generated scenarios from such models by comparing to known distributional properties of yield curves.

Typically regulators, security analysts, and customers focus on one to three-year time frames for capital risk. This is much different than risk-management in other financial institutions, where the focus is on how much markets can move from one day's close to the next day's opening. Those institutions trade continuously when the markets are open, and manage risk with derivatives. P&C insurers, on the other hand, hold bonds to maturity and manage cash-flow risk by matching asset and liability flows. Derivative pricing and stochastic volatility are of little concern over the relevant time frames. This requires different models and model testing than what is common in the broader financial markets.

To complicate things further, interest rates for the last decade have not been following the patterns established in the sixty years following WWII. We are now coming out of the period of very low rates, yet are still not returning to what had been thought of as normal before that. Modeling and model testing are in an evolving state while new patterns emerge.

Our analysis starts with a review of the literature on interest-rate model testing, with a P&C focus, and an update of the tests for current market behavior. We then discuss models, and use them to illustrate the fitting and testing methods. The testing discussion does not require the model-building section. We do try to make the modeling more accessible to actuarial modelers, compared to our source papers in the financial literature. Code for MCMC estimation is included at the CAS GitHub site. Model estimation is getting easier as the software advances, and interested actuaries, who often have a better feel for the application

Figure 1: History of Ten Year US Bond Rates

needs than do financial modelers, can use this to fit their own yield-curve generators.

Keywords: Economic scenario generators, affine models, interest rates, inflation, MCMC

1 A historical look

In this Section 1 we look at a long-term history of US interest rates for perspective. Section 2 reviews the literature on properties of yield curves for testing models, and updates the properties in the light of recent data. Section 3 introduces affine models, which often meet most of the yield-curve tests. In section 4 we fit some models. Section 5 then illustrates the tests by applying them to the fitted models. Appendix 1 covers some more general affine models, and Appendix 2 addresses fitting models by Markov-chain Monte Carlo (MCMC).

Figure 1 graphs long Treasury bond yields for 1790–2018. It is hard to identify a long-term equilibrium rate. Before the Civil War, the rate gravitated around 6%, but from 1870 to 1970, it was rarely above 4% – and it has returned to similar levels since about 2010. The postwar period of higher rates – say 1972 to 2007 – was considered the new normal at the

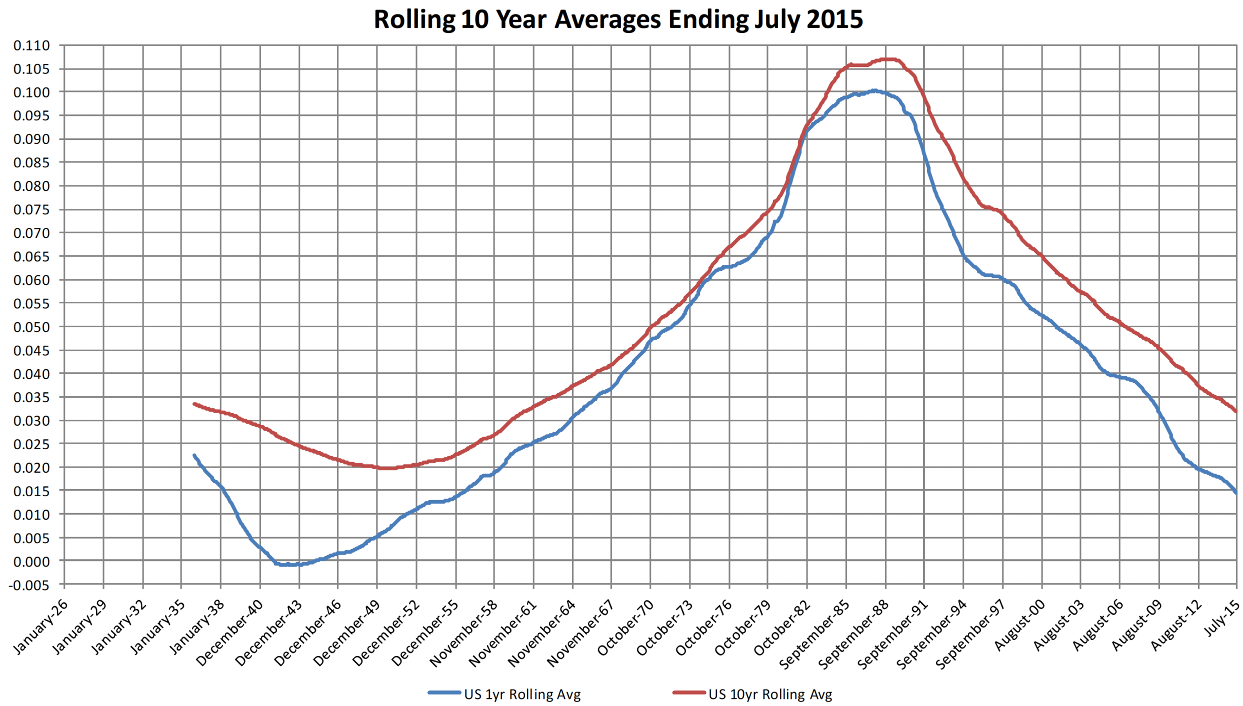

Figure 2: Ten-Year Rolling Average 1 and 10 Year Rates

time, and many economists who spent their careers in those years still consider it normal.

The longer history calls that normality into question. Some economists are now pointing out historical peculiarities of the postwar period. Piketty (2014) reports a few such findings. For instance, that period included a now-completed massive rebuilding of global assets, 50% of which, in monetary terms, were destroyed in the world wars. It was also a time of steadily increasing productivity, since diminished. Both of those elements boosted economic expansion, wages, and interest rates. Absent the postwar period, the current situation looks more like a continuation of the 1870–1970 levels than a return to 1990.

The graph also shows long periods of rising or falling rates. The rate generally declined for 39 years starting in 1861, then rose for 20 years, and then declined for 25 years, to 1.7% in 1945. Then it increased for 36 years, with some fluctuations, and dropped again for 31 years, getting to 1.8% in 2012.

Figures 2 and 3, from Pedersen et al. (2016), the SoA ESG report, show rolling average one and ten year rates and their standard deviations since 1936. The one-year rates are lower, but their standard deviations are higher, compared to the ten-year rates. The standard deviations loosely follow the rates. The Cox-Ingersoll-Ross (CIR) interest rate model assumes that for the shortest rates, the standard deviation is proportional to the rates. The gap between the

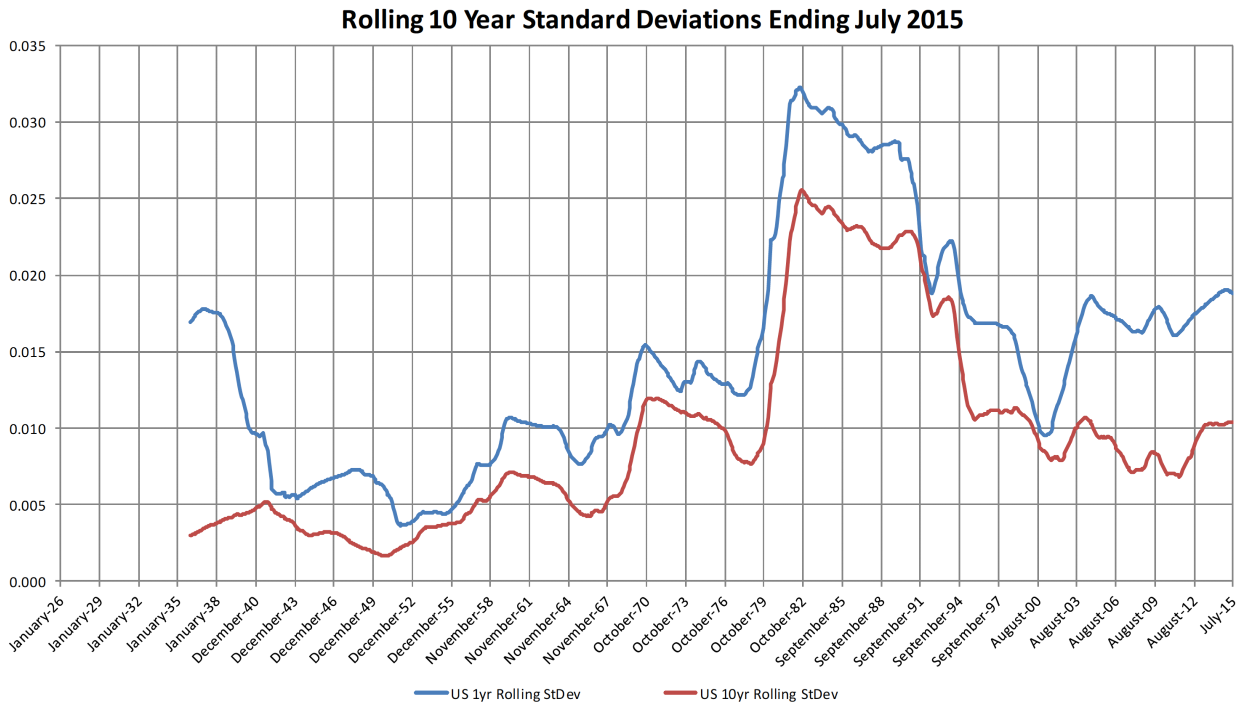

Figure 3: Ten-Year Rolling Average 1 and 10 Year Standard Deviations

one-year and ten-year rates was high in the early 1940s, much like it was recently. In the 1940s the one-year rates were even lower than they were in the 2010s and actually became negative for a few years.

2 Historical model tests and current updates

Several tests have been proposed for yield-curve models. We discuss proposed tests from three papers, and review these with recent data. The period from 2008 to 2018 had unusually low rates – comparable to those from 1938 to 1955. With the recent changes in economic conditions, it seems unlikely that such rates would return over the next several decades. But the historically unusually high rates from 1970 – 2000 had specific causes that also seem unlikely to repeat. This makes it difficult to select a relevant time period for measuring rate properties. What we do below is look at how the properties have been evolving over time in order to come up with reasonable criteria for model behavior. Many of the tests would be performed on scenarios simulated from the fitted models, but a few use fits to the data.

2a Tests from the Feldhűtter paper on fitting affine models

Feldhütter (2016) explores how well affine models fit historical properties of interest rates.

2a.1 Moments by maturity

Table 1 shows his exhibit of moments for US Treasury maturities of 1 – 5 years, using monthly

Table 1: Moments of Treasury Yields

maturity	1	2	3	4	5
mean	5.60	5.81	5.98	6.11	6.19
standard deviation	0.50	0.43	0.40	0.39	0.36
skewness	0.83	0.79	0.78	0.77	0.77
excess kurtosis	0.77	0.57	0.51	0.44	0.35

observations from 1952:6 to 2004:12. This includes the unusually high rates of the 1970s and 80s. What it shows is mean rates that increase by maturity and volatilities that decrease. Higher moments are positive but small and also decline a bit by maturity.

The modestly positive higher moments in Table 1 have been challenging for models to match – most give much higher or virtually zero higher moments. If this holds up for more relevant data, it would be an important feature to capture for risk analysis.

The yield curve is usually upward sloping, and there are good reasons for that. Investors need higher yields to lock up their funds in longer rates, and to take a many-year risk of bond values going down due to rates increasing. The curve gets flat or inverted when the Federal Reserve raises short rates above the level at which the market is trading the longer obligations. This seems reasonably likely to happen over the next several years, so some curve inversions would be a good thing to see in a simulated scenario set.

Although shorter rates are normally more volatile than are longer rates, this pattern was reversed for six years beginning in August 2011. The short rates were too low to have much absolute volatility. As an alternative we looked at the volatility of the log of the rates.

Figure 4 shows 156-week moving standard deviations of rates and the log of rates from 1/2009 to 1/2019. Before August 2011 and after August 2017 the usual pattern held, with volatility decreasing as maturities increased. In between, the pattern was almost exactly reversed. The logs of the rates show more consistent volatilities. Except for a short period where rates longer than two years all had very little volatility, the usual pattern is maintained for those maturities. The one and two-year maturities always have higher log volatility than do the ten and longer year maturities, and drop below the three-to-seven-year log volatilities for only a relatively short period. Going forward it seems that the modeled volatilities of the logs of the rates should decline steadily with maturity. In the near term, it looks like the one-year log volatilities have stabilized around 0.6, while the others show a bit of an upward trend. Numbers a little higher than these would thus seem reasonable targets for model testing.

Figure 5 shows the skewness of the five-year rate over fairly long periods. It displays the rolling skewness of weekly rates over moving 500, 1000, and 1500 week periods, with the

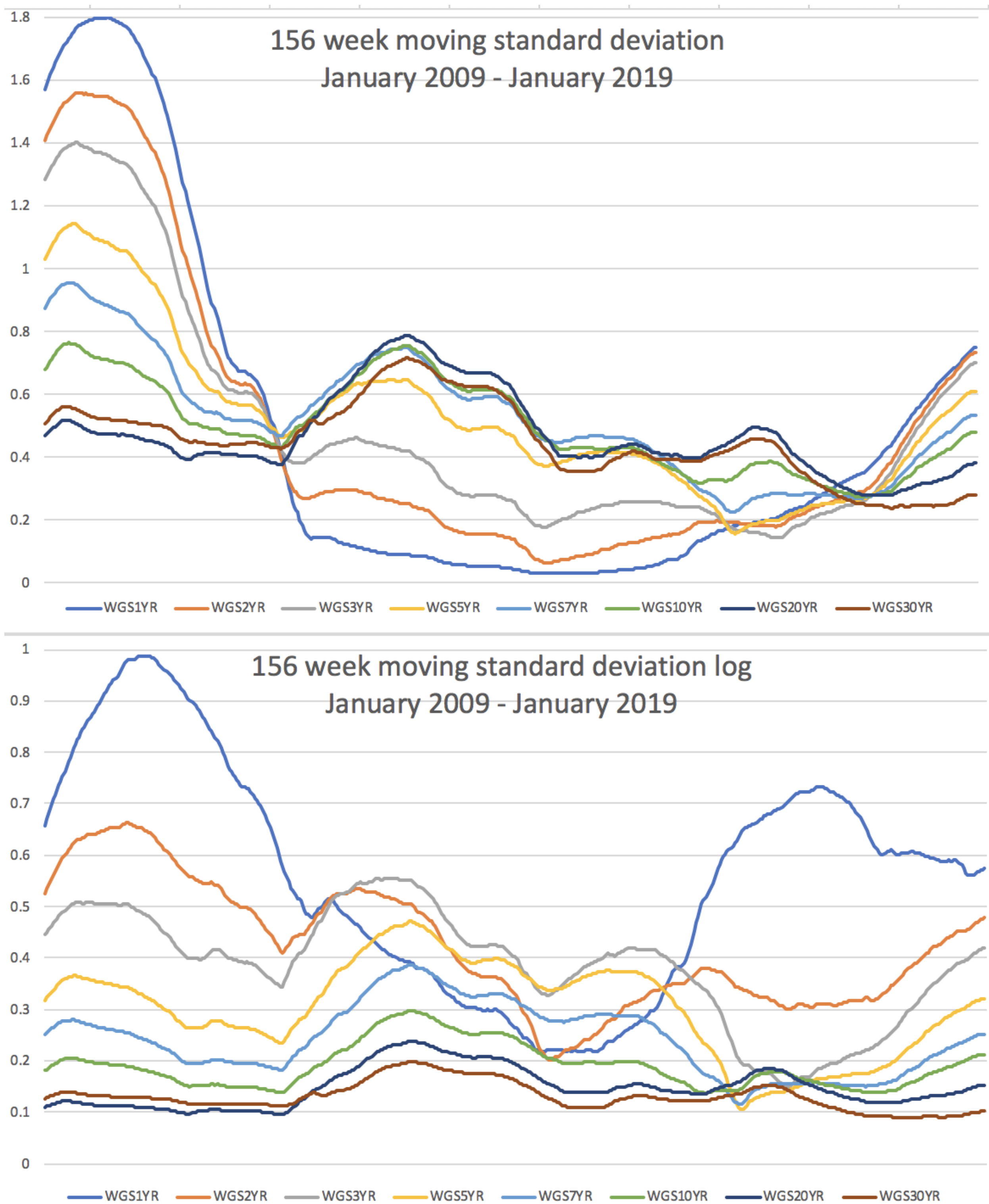

Figure 4: Three-Year Moving Standard Deviation of Rates and Log Rates by Maturity

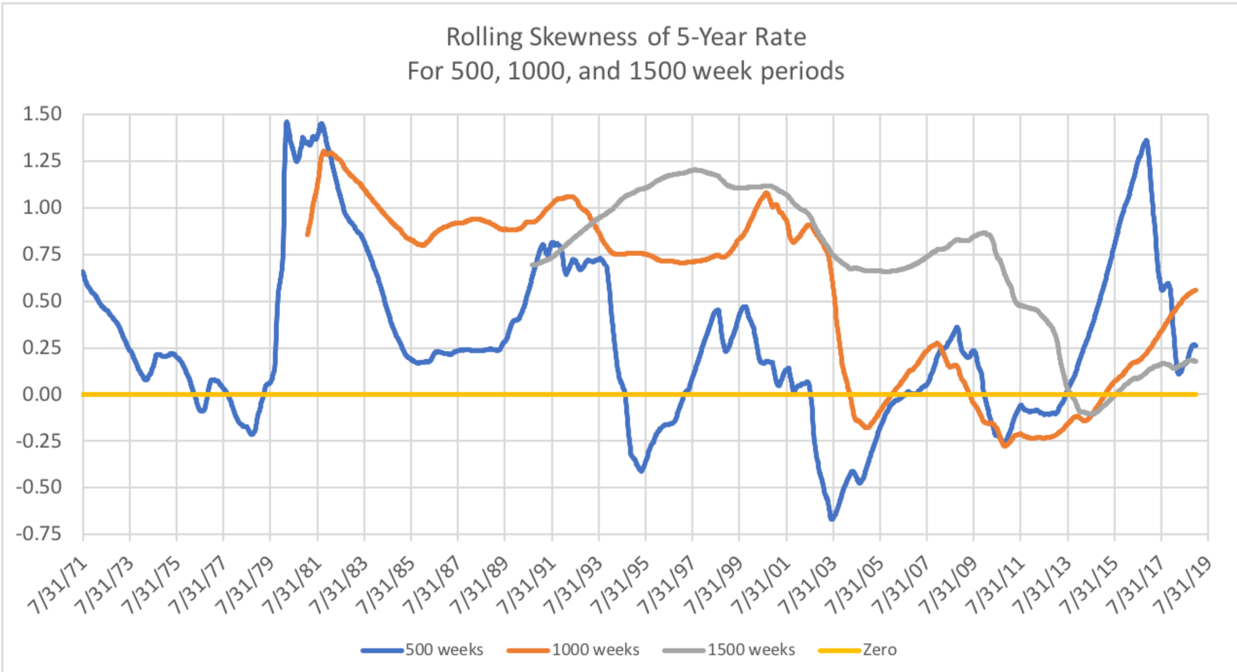

Figure 5: 500, 1000, and 100 Week Moving Skewness of Five-Year Rates

periods' ending points starting in 1971. These correspond roughly to 10, 20, and 30 year periods. The higher values in the earlier years appear to be due to the very high rates in the early 1980s. There is no consistent skewness that holds in general. There is a temporary spike due to the recovery from very low rates (below 1%) starting in 2016. These have since stabilized around 2.5%, so the spike seems to be over. For simulated rates going forward, very little skewness would be a good result – say below 0.25 but not too negative either.

2a.2 Volatility as related to levels of the rates

Feldhütter (2016) regresses the squared change in rates from month t to $t+1$ for each maturity against the level, slope, and curvature of the yield curve at month t. The level is the five-year rate, the slope is the difference between the five and one-year rates, and the curvature is the sum of the five and one-year rates less twice the three-year rate, respectively. This measure of curvature is higher when the midpoint of the curve is lower relative to the endpoints, so quantifies upward curvature. Table 2 shows the coefficients for each variable, along with their t-statistics, for each maturity.

The coefficients are small, as the squared change in rates is. The level coefficients all have quite significant t-statistics, showing that the volatility of rates is higher when the level of rates is. This effect is greatest for the shorter maturities, since the volatilities are higher for them. The other coefficients are not individually significant, but the slope coefficients display

Table 2: Volatility Regression Coefficients

maturity	1	2	3	4	5
level	0.11	0.07	0.06	0.05	0.04
t-statistic	-5.4	-5.4	-5.5	-6.9	-7.6
slope	-0.14	-0.08	-0.03	-0.02	0.02
t-statistic	1.5	1.2	0.7	0.5	-0.5
curvature	0.27	0.11	0.17	0.08	0.13
t-statistic	-1.4	-0.8	-1.7	-1.0	-2.1

a clear pattern of lower volatility for the shorter maturities for steeper yield curves. Volatility is also consistently but not significantly higher with less downward curvature.

However, the previous results no longer hold up. Test found that these coefficients were no longer significant, and sometimes had changed sign. The very low rates did not act in accord with the earlier results. This does not appear to be a useful test at this time, but should be reviewed as new data emerges.

2a.3 Annual rate changes by maturity related to curve slope

The value of a bond would generally go up over time as its time to maturity shortens. These changes have been found historically to be stronger (bigger value increase, bigger yield drop) when the current yield spread over the shortest yield is higher. Feldhütter (2016) does a regression for this effect, following the methodology of Campbell and Shiller (1991). With yield $Y(t, n)$ at time t and time to maturity n, he finds the factors for a regression on the change in yield. That is, he estimates:

$$Y(t+1, n-1) - Y(t, n) = const + factor_n \frac{Y(t, n) - Y(t, 1)}{n-1} + res$$

The factors, shown in Table 3, are negative, which means that the drops in yield are greater when the spread over the one-year yield is higher, and are increasingly so as the maturity increases. This is a sort of mean reversion that investors would be looking for. A longer bond that has a particularly high spread relative to the shortest bond would be expected to increase more in value during the next year. Campbell and Shiller (1991) called this the excess-profits effect. Apparently the higher spread led later to a greater decrease in the longer rate. This seems counter-intuitive, as short rates are more volatile, so the wider spread would be expected to narrow due to an increase in the short rate, not a decrease in the long rate. Feldhütter (2016) also finds this effect suspect, but confirms it empirically.

Our Federal Reserve data does not have 4, 6, 8, or 9 year rates, so to update the analysis we looked at the change over two years for bonds that started as 5 or 7 years, and the change

Table 3: Yield Change Regression

maturity	2	3	4	5
factor	-0.78	-1.13	-1.52	-1.49
t-statistic	1.4	1.8	2.2	-2.0

Table 4: Yield Change Slopes by Period

Period	4/53-12/78	1/84-12/08	1/09-1/19	4/53-1/19
2Y	-4.81	-0.36	-1.73	-0.93
3Y	-4.63	-0.34	-2.54	-1.18
5Y	0.91	0.76	-3.46	-0.54
7Y	1.18	0.65	-5.02	-0.66
10Y	1.86	1.39	-4.43	-0.25

over three years for the 10 year bonds. Preliminary analysis suggested that the very high rates around 1981 were distorting the results, but the very low rates of the last ten years were also unique. For this reason, we did regressions over 4 periods: the first 25 years in our data, from mid-1953 to 1978, the 25 years from 1984 through 2008 (thus skipping 1978–1984), 2009 – 2019, and the entire 66 years. Table 4 shows the results.

The entire period shows all negative slopes, but generally getting less negative by maturity. The two 25-year older periods show negative slopes for the 2 and 3-year rates, with increasingly positive slopes for the longer rates. Only the 10 recent years display the pattern found in earlier studies.

The past decade has featured very low short rates, with very low volatility. Thus in this period, a high spread over the 1-year rate would indicate a higher long rate. Then for it to decrease is just a form of mean reversion. The first two columns were 25-year periods of either generally increasing or generally decreasing rates, but their coefficients are similar. It could be that in those periods, higher spreads occurred during expansionary times, where overall rates were increasing more, or decreasing less, than when spreads were lower.

To test simulated rates going forward, negative coefficients would be reasonable for 2 and 3 year rates. For longer rates, right now any result could look plausible – positive, negative, or not significant coefficients. In the future, another look at the regressions would be called for.

2b Test from the Venter (2004) paper on ESGs for P&C companies

2b.1 Yield spreads and the short rate

This paper presents a test of yield curve scenarios based on the way longer yield spreads relate to the short rate. Since longer rates are less volatile, they go up less when the short rate increases. Thus the spread, say between the three and ten-year rates, would be expected

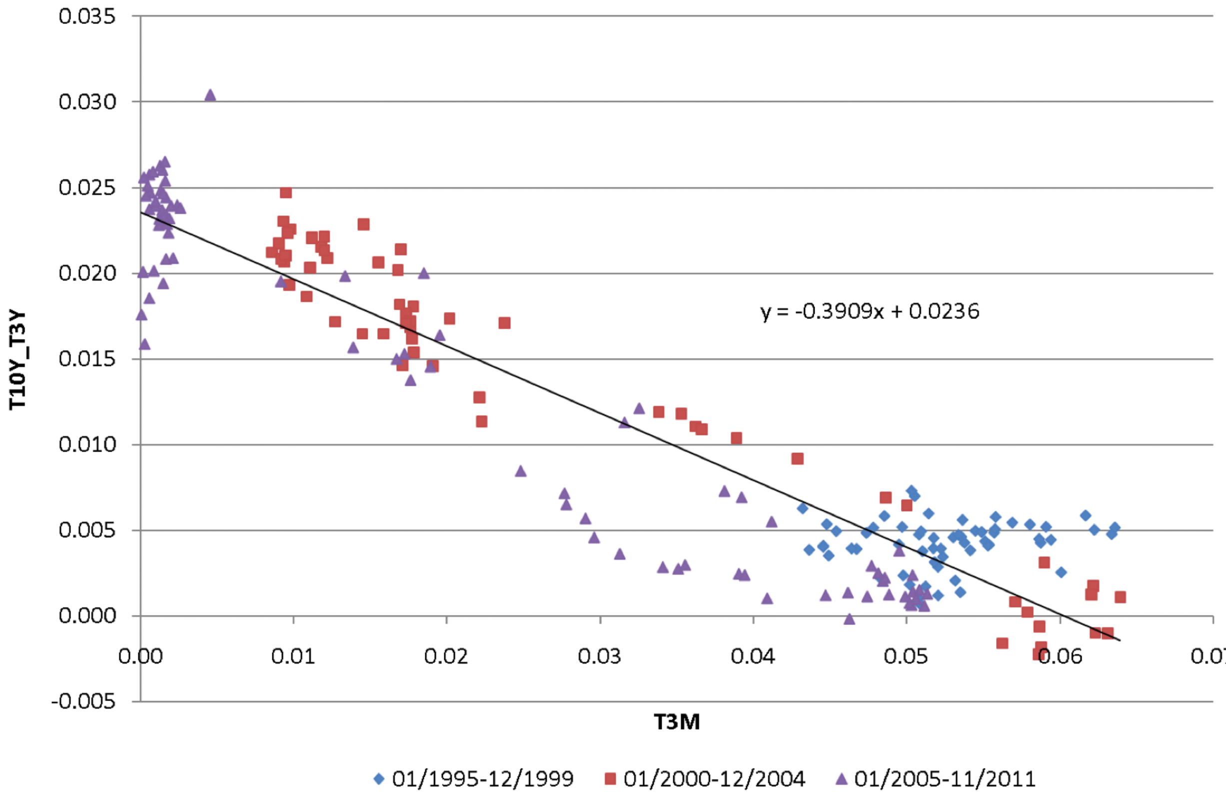

Figure 6: Three-Year to Ten-Year Spreads as a Function of the Three-Month Rate

to decrease when the short rate increases, and widen again when it goes back down. Figure 6 shows data on this from 1995 to 2011 with a regression fit. The scatter around the regression line is fairly consistent for three sub-periods shown.

Figure 7 graphs the 3-year – 30-year spread as a function of the 1-year rate for two periods – 13 years starting in 2006, and about $3^1/_2$ years ending 1/4/19, both with fitted lines. The longer range shows two distinct periods, with similar slopes but different intercepts. The second of these periods is what is fit in the bottom panel. The graphs are similar to Figure 6, which shows the 3-10 spread for 1995-2001, so the general long-term pattern has been continuing. The slope in the upper panel is -0.53, compared to -0.67 in the lower panel. The standard deviation around the slope for the longer period is 0.57, but it is clearly less than this for the two sub-periods. It is 0.12 for the later period.

To use this as a test for simulated scenarios, fitting the trend line would be the starting point, then graphing the data and fitted line. An eyeball test of the overall look of the graph compared to Figures 6 and 7 would be a check of the basic pattern. For the near future, a slope similar to the recent period would be expected. For a longer projection, any slope less than -0.5 would seem reasonable. A flatter line would indicate that the longer spreads do

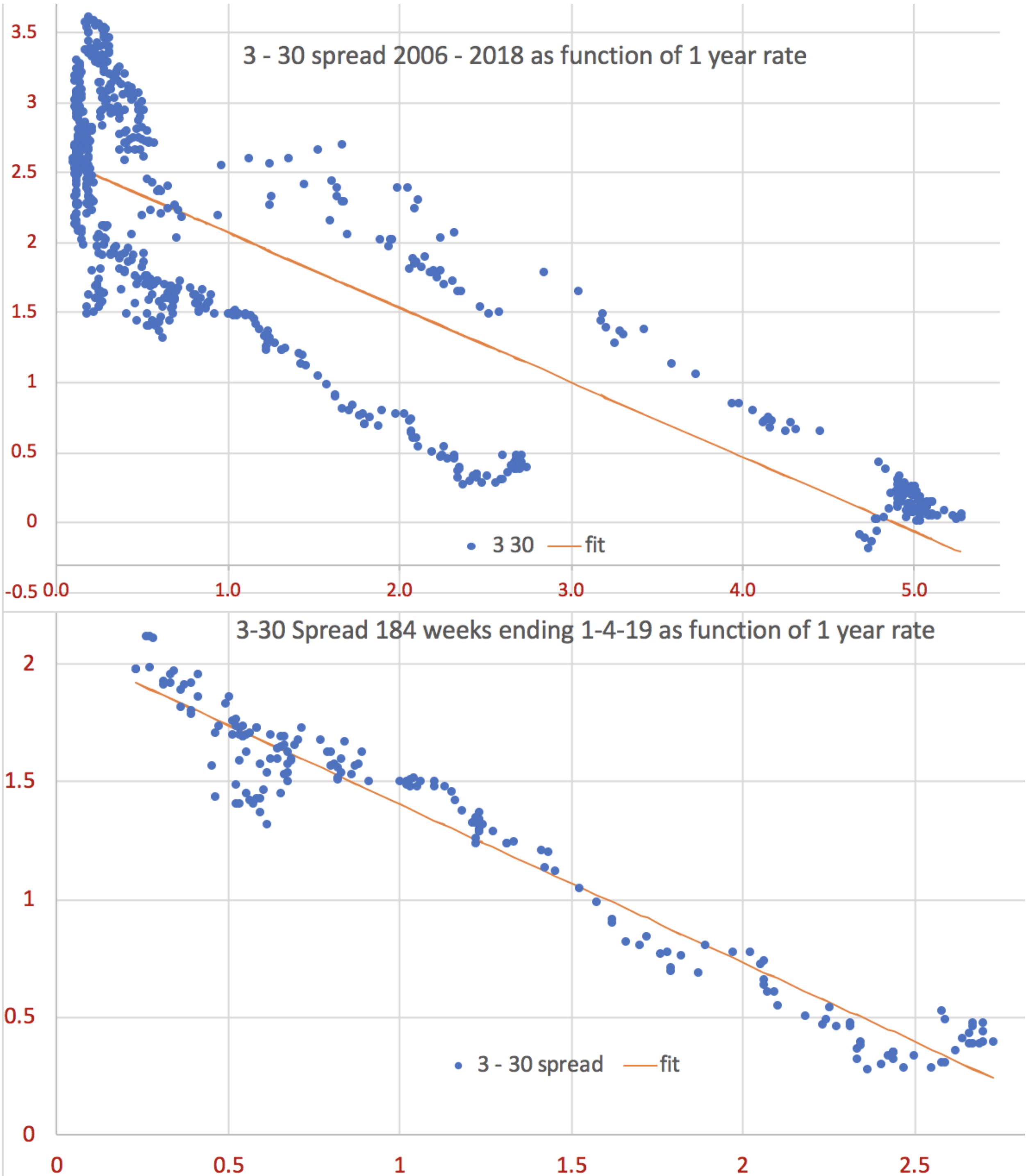

Figure 7: 3 – 30 Year Spread as Function of 1-Year Rates

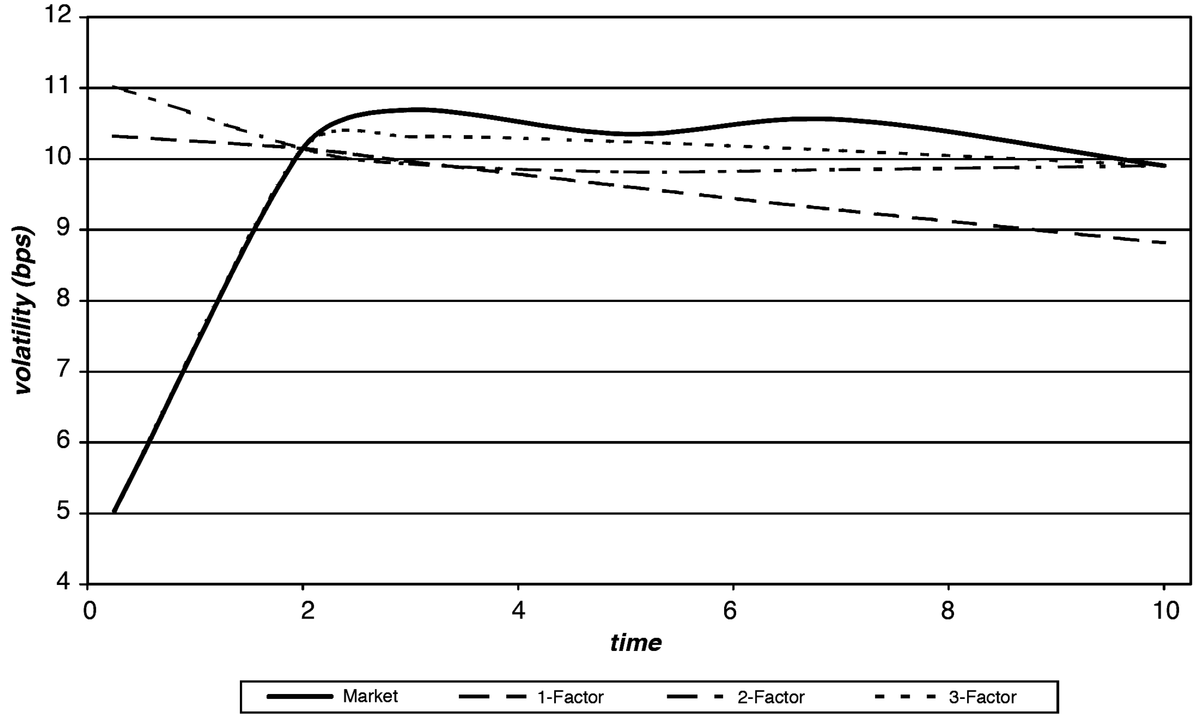

Figure 8: Actual and Implied Volatilites by Maturity for each Model

not compress much with increasing shorter rates, which could arise from problems in the volatilities. For a short-term projection, the spread around the line should be fairly small – below 0.2 perhaps – but a wider spread would be reasonable for a longer projection. Still, any spread above 0.6 would suggest that the historical behavior is not reflected in the model.

2c Tests from the Jagannathan, Kaplin, and Sun (2003) paper about testing CIR models

This paper uses properties of yield curves for expanded model goodness-of-fit tests on the sample data. It compares fits from single-factor, two-factor, and three-factor CIR models. They try several tests, but the two here show ways in which only the three-factor model fits well. They have other tests involving option prices where none of the CIR models work. But since bond option prices depend on stochastic volatility, CIR would not be expected to work. This should not be a problem for the time frames used in P&C models, where the stochastic volatilities average out.

2c.1 Volatility by maturity

They compare volatility by maturity in the sample with that implied by the fits. Figure 8 is their main result for that, and it shows a reasonably good match for the three-factor model. A similar test for simulated data is included in the Feldhütter (2016) discussion above.

Table 5: Weights for the first four principal components of yield changes

Change in:	3M	2Y	3Y	5Y	7Y	10Y	% Explained
Level	0.123	0.430	0.462	0.450	0.454	0.421	93.7
Slope	-0.866	-0.273	-0.112	0.081	0.223	0.326	3.6
Curvature	0.482	-0.601	-0.288	-0.035	0.355	0.4433	2.0
Factor 4	-0.010	-0.606	0.662	0.235	0.016	-0.3738	0.3

2c.2 Yield curve shapes as described by principal component analysis (PCA)

Yield data comes in an array – perhaps rows for each week of observation and columns for each maturity. In their data there are six maturities, so the yield curve at any date can be represented as a vector in six-dimensional space. PCA rotates the axes of this space to put most of the variation into a few dimensions. The first such dimension – i.e., the first principal component – is defined by the line between the two points that are furthest apart in this space. The second axis is the longest line between two points that is perpendicular to the first one, etc. Each axis gives a set of weights for the yields at any date, where the sum product of the weights with the yields is the coordinate on that axis. For interest rates, usually three principal components – often called level, slope, and curvature – are enough to explain most of the variation in the yield curves. That is, there is very little variation in the remaining dimensions. There are standard formulas that produce principal components and software to implement them.

The weights for the first four components for their example are in Table 5, along with the total variance accounted for by each. The first three explain over 99% of the variation in yield curves. Then they look at how well these three principal components can be approximated by the fitted models, using a regression approach. They find that the three-factor model provides a very good approximation to the three principal components, while the other models do not. They conclude that a three-factor CIR can capture the variety of yield curve shapes, but two factors cannot.

With modern fitting, there would be fitted values for each observation, so these could be used to compute fitted principal components for comparison to the actual principal components. That would show how well the fit is able to explain the observed variation in yield curves.

The principal components are essentially two or three new yield curves fit using the observed collection of yield curves. In the example from Jagannathan, Kaplin, and Sun (2003), 94% of the differences among yields across the data set is explained by the first component, and over 97% of the shape differences can be accounted for by regressing each yield curve on the first two components. They stop with three components, which explain 99.3% of the variability.

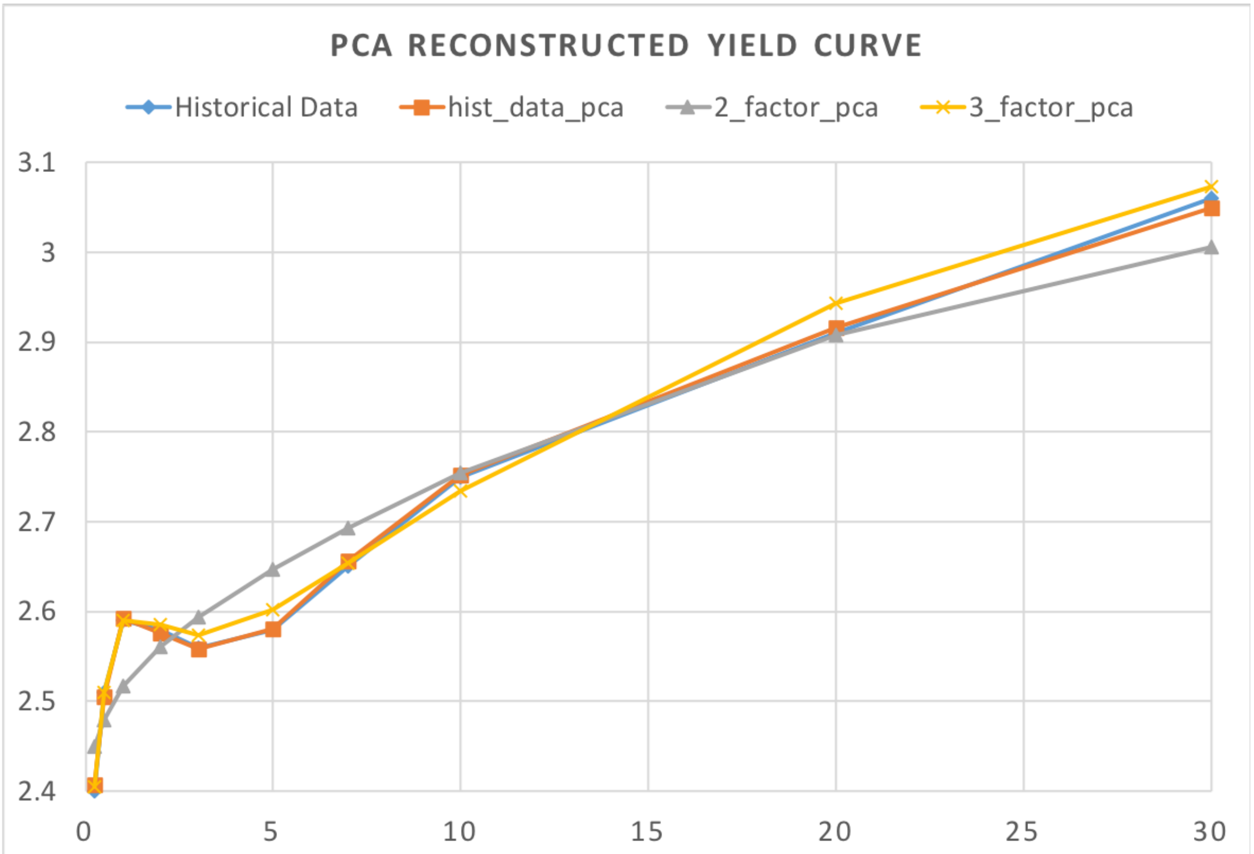

Figure 9: Yield Curve from PCA

For a recent historical dataset, we tried PCA on the actual data, and on two fitted models, which were two and three-factor affine models. These model the yields as linear combinations of partial short rates. Then for each observed time point, we calculated the PCA-fitted yield curves from the three sets of components. Figure 9 graphs the actual and fitted curves at one particular time. This sample has a difficult set of yield curves, due to a reversal in rates from 1 to 5 years. As can be seen, the actual and historical PCA curves are practically identical, and the three-factor model gives a reasonably good approximation. The two-factor model has less flexibility and does not match this particular curve shape.

What we found in doing this exercise, and should have known in advance, is that the two-factor model fits only have two principal components, and the three factor model's have three. That's because the fitted values are linear combinations of two or three sets of short rates. Thus the results of Jagannathan, Kaplin, and Sun (2003) – that the two-factor CIR does not fit the three principal components of actual rates – was more or less pre-ordained. As long as the third actual component is non-trivial, a two-factor model will not have enough variety of curve shapes.

Table 6: R-Squared by Maturity

R^2	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	20Y	30Y
Two-factor fit	95.7%	97.7%	77.6%	94.9%	95.6%	91.6%	96.5%	99.1%	91.7%	81.5%
Three-factor fit	99.4%	99.6%	97.9%	98.1%	99.2%	97.3%	99.6%	99.4%	98.6%	99.2%

This gives another way to do a realism test of simulated yield curves: do a PCA on the set of curves and see if the third component explains enough of the variance – maybe 0.005 or more. An expanded test would be to do a PCA on what is taken to be a relevant historical period, and use that as a guideline on how significant the third component should be.

Another useful indicator of how well the model fits by maturity would be to just compute an R^2 statistic for each maturity. For each maturity, that would be the total variance of the rates at the maturity minus the variance of the model residuals, as a percentage of that total variance. Table 6 illustrates this for the two example models. The problems of the two-factor model show up in some specific maturities.

3 Interest rate models

Generally the academic literature focuses on arbitrage-free models. Some actuarial models do not require that, depending on the applications intended. If the risk of different investment strategies is to be analyzed, having arbitrage possible could easily distort the conclusions. If realistic yield curves are desired, the models should also be arbitrage-free. But if rates for only one or two maturities are to be projected long-term, such constraints might not be necessary. Here we will stick to arbitrage-free models of the whole yield curve.

Lognormal models, like the Libor Market Model, produce more skewed distributions of rates than the historical data supports. They are popular for options' pricing, where the greater skewness may help match market risk pricing of short-term volatility. That is not the main focus of P&C models, though, and so we will use models with normal residuals.

Most popular of these are the affine models. They build up the yield curves from the short rates. In a single-factor model with short-rate $r(t)$ at time t, every maturity τ has a constant term $C(\tau)$ and a factor $D(\tau)$ that do not change over time, with fitted rate $R(t, \tau) = C(\tau) + D(\tau)r(t)$. Thus changes in the yield curve over time depend only on the changes in the short rate $r(t)$. This will make all the yield curves parallel, which is unrealistic.

Multi-factor model postulate short-rate components $r_j(t)$ for each factor j, and constants and factors $C_j(\tau)$ and $D_j(\tau)$ that again do not vary over time. Then $R(t, \tau) = \sum_j [C_j(\tau) + D_j(\tau)r_j(t)]$. Mixtures of processes like that can give a much greater variety of yield-curve shapes, especially mixtures of at least three processes. Still the changes in the yield curves over time come from the short rates components $r_j(t)$, as the $C_j(\tau), D_j(\tau)$ functions still do not vary over time t.

Interest rates are modeled as continuous processes, using Brownian motion, but are fit with discrete data, so can be represented by time-series approximations. (A Brownian motion is a continuous process whose value at any future time is normally distributed with mean equal to the current value and variance equal to the time interval.) Parameters are usually scaled as annual changes, with the rates incremented in fractions of a year dt. For instance, dt might be 0.004, for an increment of 1/250th of a year, which is often used to represent one trading day. The advantage of starting with Brownian motion is that any increment can be taken as an approximation. We will start by assuming an unspecified increment dt.

The basic building blocks of affine models are the CIR and Vasicek processes. These are specified by the distribution of the incremental change $dr(t) = r(t + dt) - r(t)$, which is stochastic. More general affine models are built up as combinations of these two. In both of these processes, the distribution of the increments is normal, which we will write as $\mathcal{N}(\mu, \sigma)$. In the Vasicek process, from Vasicek (1977):

$$dr(t) \sim \mathcal{N}\left([\omega - \kappa r(t)]dt, \sigma\sqrt{dt}\right)$$

where ω, κ, σ are parameters. The square root of dt is taken because the variance is proportional to the time increment dt. The constants $C(\tau), D(\tau)$ for each maturity τ are calculated by closed form but slightly complicated formulas.

The CIR process, from Cox, Ingersoll, and Ross (1985), is similar, with

$$dr(t) \sim \mathcal{N}\left([\omega - \kappa r(t)]dt, \sqrt{\beta r(t)dt}\right)$$

Its incremental variance is proportional to its latest value $r(t)$. The variance parameter here is denoted as β rather than σ^2 because it is not the variance but a factor which when combined with $r(t)$ gives the variance. The processes can be simulated in steps of any desired (but small) dt by sampling from these normal distributions. Also these distributions become the priors for the incremental changes in the values of the processes at each subsequent step when doing Bayesian estimation from the historical processes.

The CIR process cannot go negative: it is a continuous process, so if $r(t) = 0$, the standard deviation also becomes zero, and the process increments by ωdt, which is positive. However when simulating the process, if it gets to a small positive value, the next discrete simulation can become negative. A work-around derives from the fact that the constantly changing volatility results in a gamma-like distribution for the process over time. This is actually the non-central chi-squared distribution, and for step size dt, the next value of the process has mean and variance:

$$\mu = r(t)e^{-\kappa dt} + \omega\left(1 - e^{-\kappa dt}\right)/\kappa$$

$$V = \frac{\beta}{2\kappa}\left(1 - e^{-\kappa dt}\right)\left[2r(t)e^{-\kappa dt} + \omega\left(1 - e^{-\kappa dt}\right)/\kappa\right]$$

The non-central chi-square is close to a gamma distribution, so it can be approximated by a gamma distribution with this mean and variance. This way the process would not become negative in simulations.

The long-term variance is $\beta\omega/2\kappa^2$. For the Vasicek process, this is $\sigma^2/2\kappa$. The Vasicek long-term distribution is normal with mean ω/κ and this variance, while the CIR long-term distribution is gamma. Both the variances follow the algorithm: expected value of the process change variance divided by twice the speed of mean reversion. These could be used in the priors for the first values of each process, in that they have been going on a long time.

As discussed below, multifactor processes can have a few Vasicek and CIR processes, or can even include combined forms. The CIR processes must be independent or positively correlated to be arbitrage-free, but Vasicek processes can be positively or negatively correlated. Apparently negatively correlated processes provide a realistic variety of yield curves.

3a Yield curves and market price of risk

Dai and Singleton (2000) provide a comprehensive characterization of affine models. Like Jagannathan, Kaplin, and Sun (2003) also recommend, they focus on three-factor models. We start with three-factor combinations of the Vasicek and CIR models that have closed form $C_j(\tau)$ and $D_j(\tau)$ functions.

The yield-curve formulas are derived as expectations of related risk-neutral processes. The idea of risk-neutral processes is to add risk load directly into the processes, so the expected value of the bond value at maturity discounted along the risk-neutral rate process is the risk-loaded price of the bond. This allows risk loads to be put into bonds of various maturities in a consistent way. The fundamental theorem of asset pricing says that arbitrage-free prices must be the expected values of admissible transformed processes – i.e., ones that do not change the set of outcomes that have non-zero probabilities. Such transforms here change the ω and κ parameters, but not the σ or β parameters.

The simplest transform is called completely affine. Under it, the mean $[\omega - \kappa r(t)]dt$ of these process changes to $[\widetilde{\omega} - \widetilde{\kappa} r(t)]dt$. This uses a market-price-of-risk parameter λ, which is estimated when fitting the model. For the Vasicek process, it works out that $\widetilde{\kappa} = \kappa$ and

$\widetilde{\omega} = \omega - \sigma\lambda$. For the CIR process, on the other hand, $\widetilde{\kappa} = \kappa + \beta\lambda$ and $\widetilde{\omega} = \omega$. The transformed parameters are called the risk-neutral parameters. The derivation of the $C(\tau)$ and $D(\tau)$ functions assumes that the bond prices are expected values under the transformed process.

There is a more general approach to the market price of risk called essentially affine. It does not affect the previously-transformed CIR process but for each Vasicek process there is another market price parameter ψ. Then

$$\widetilde{\omega} = \omega - \sigma\lambda$$

$$\widetilde{\kappa} = \kappa + \sigma\psi$$

In either case, the values of $C(\tau), D(\tau)$ are derived from the risk-neutral processes. Let $q = \widetilde{\omega}/\widetilde{\kappa}$. Then for the Vasicek process, the constants $C(\tau), D(\tau)$ are:

$$D(\tau) = \frac{1 - e^{-\widetilde{\kappa}\tau}}{\widetilde{\kappa}\tau}$$

$$C(\tau) = q - qD(\tau) + \left[\frac{\sigma}{2\widetilde{\kappa}}\right]^2 \left[\widetilde{\kappa}\tau D(\tau)^2 + 2D(\tau) - 2\right]$$

The CIR formulas require two intermediate values:

$$h = \sqrt{\widetilde{\kappa}^2 + 2\beta}$$

$$Q(\tau) = \left[(\widetilde{\kappa} + h)\left(e^{h\tau} - 1\right) + 2h\right]^{-1}$$

Then:

$$C(\tau) = -\frac{\widetilde{\omega}}{\tau\beta}\left(2log[2hQ(\tau)] + \widetilde{\kappa}\tau + h\tau\right)$$

$$D(\tau) = 2Q(\tau)\left(e^{h\tau} - 1\right)/\tau$$

The D functions start at 1 for $\tau = 0$ and decline from there. That means that the longer maturities change less as $r(t)$ changes, and so have less volatility than do the shorter maturities. This is a pretty standard property of yield curves but did not hold over much of the last decade. These models would probably not do very well when the short rates are so low. Having greater volatility in the shorter rates also builds in the property that longer spreads get lower when the short rate rises.

The $C(\tau), D(\tau)$ functions depend on $\widetilde{\kappa}$ and $\widetilde{\omega}$, so some analysts prefer to make these the primary parameters to estimate, and then calculate κ, ω by reversing the λ and ψ adjustments. In a sense, the $C(\tau)$ and $D(\tau)$ functions basically define the model, as they are constant over time while the short-rates evolve stochastically. Fitting the model then comes down to deriving these functions from the risk-neutral processes. That is why these processes are often taken as the starting point. Doing it this way also seems to help numerically with parameter estimation, as λ and ψ then do not go into estimating the functions. That approach seems to speed up the estimation. However we take it one step further.

For a thee-factor model with one CIR and two Vasiceks, in the completely affine case, this method produces all the risk-neutral parameters, plus three real-world parameters: the CIR κ and the two Vasicek ωs. We actually estimate these directly, and the λs can be backed out later, if desired. In the essentially affine model it is similar, but now the real-world and risk-neutral parameters can all be estimated independently, just with the CIR $\omega = \widetilde{\omega}$. This also holds for the more general models discussed later for the essentially affine risk loads. It is easier for the software to do the estimation this way, as there are less interactions among the parameters that are specified in advance.

In multi-factor models, the CIR and Vasicek processes r_j are taken as unobserved components of the short rate. Any number of independent processes can be combined in this way. Then the yield curves from each model are treated as partial interest rates, and they add up to the interest rates for each maturity. The $C_j(\tau)$ functions add over j, as do the $D_j(\tau)r_j(t)$ terms.

For two correlated Vasicek processes, the $D_j(\tau)r_j(t)$ terms add, but an adjustment to the $C_j(\tau)$ functions is needed. If the correlation is ρ, Brigo and Mercurio (2001), p. 135, as well as Troiani (2017), give the adjustment:

$$C(\tau) = C_1(\tau) + C_2(\tau) + \frac{\rho\sigma_1\sigma_2}{\widetilde{\kappa}_1\widetilde{\kappa}_2}\left[\frac{e^{-\tau(\widetilde{\kappa}_1+\widetilde{\kappa}_2)} - 1}{\tau(\widetilde{\kappa}_1 + \widetilde{\kappa}_2)} + D_1(\tau) + D_2(\tau) - 1\right]$$

If there are additional independent Vasicek or CIR processes, their C functions add in as well, and so do the $D_j(\tau)r_(t)$ terms.

Bolder (2001) shows the adjustment for any number of correlated Vasiceks:

$$C(\tau) = \sum_j C_j(\tau) + \sum_{i,j:j\neq i} \frac{\rho_{ij}\sigma_i\sigma_j}{\widetilde{\kappa}_i\widetilde{\kappa}_j}\left[\frac{e^{-\tau(\widetilde{\kappa}_i+\widetilde{\kappa}_j)} - 1}{\tau(\widetilde{\kappa}_i + \widetilde{\kappa}_j)} + D_i(\tau) + D_j(\tau) - 1\right]$$

This uses the sum of the correlation adjustments across all of the binary correlations.

Multi-factor models allow a slight generalization to the idea that the sum of the vector $r(t)$ of the short-rate components at time t is the actual short rate $r_s(t)$. For instance, three-factor models allow a constant $\delta_0 \geq 0$ and a positive three-vector δ so that $r_s(t) = \delta_0 + \delta' r(t)$. The default assumption, assumed above, is that $\delta_0 = 0$ and $\delta_j = 1, j > 0$. We introduce $\gamma_j = \delta_j - 1 > -1, j > 0$. The result of this is to add δ_0 to $C(\tau)$, and γ_j to $D_j(\tau)$. Dai and Singleton (2000) argue that in a three-factor model, nothing is lost by setting $\gamma_2 = \gamma_3 = 0$. We adopt this approach. Feldhütter (2016) estimates δ_0 and γ_1 as both less than 0.03. He leaves γ_2, γ_3 in the model, but they both come out virtually equal to zero.

All of this provides quite a bit that can be done in closed form. Starting with one CIR and two Vasiceks, you can add correlation to the Vasiceks, which is one more parameter. The Vasiceks can also use essentially affine market prices of risk, adding two more parameters. And two more come from δ_0 and γ_1. This gives quite a lot of flexibility to the model building. You start off with 3 $\tilde{\kappa}$ parameters, 3 $\tilde{\omega}$'s, 2 σ's and β to make the $C(\tau), D(\tau)$ functions. The correlation ρ affects those, as do δ_0 and γ_1. Then the market-price-of-risk adjustments add five more factors to give the real-world yield-curve processes.

The more general affine models, discussed in Appendix 1, require solving systems of ordinary differential equations for the $C(\tau), D(\tau)$ functions. There is convenient software for this, illustrated in the sample code in Appendix 3, but it does add to model fitting time. The fairly complicated formulas for C, D for the CIR and Vasicek models are actually solutions of these differential equations.

In the models discussed so far, the drifts – the incremental mean changes of the processes – depend on the immediately previous values of their processes. The more general models allow the immediately previous values of all the processes to go into the drift of a process. The CIR variance is a multiple of the process itself. In the more general models, every process variance is allowed to be a linear function of the CIR process.

The CIR process is very delicate with respect to any possible generalization and is usually left the same in the more general models. Below is an example of the evolution formula for the first Vasicek process, assumed to be the second process in a model with one CIR and two Vasicek processes:

$$dr_2(t) \sim \mathcal{N}\left(\left[\omega_2 - \kappa_{21} r_1(t) - \kappa_{22} r_2(t) - \kappa_{23} r_3(t)\right] dt, \sigma_2 \sqrt{1 + \beta_2 r_1(t) dt} \right)$$

This lowers subsequent values of this process when any of the processes is high, and also allows the stochastic volatility from the first process to go into each of the process variances. We fit this model below for comparison to the closed-form models. The other Vasicek process is like this one, but has an extra volatility term to produce correlation with this process.

The first process is a plain CIR. With essentially affine market price of risk and the δ_0, γ_1 parameters, this is the maximal model we fit. Below we call this model 7k3b, because it has 7 κ's and 3 β's. In the general form, the κ's are in a 3x3 matrix K, but in this CIR, $\kappa_{12} = \kappa_{13} = 0$.

Appendix 2 goes into the model fitting by MCMC. This requires specifying postulated (prior) distributions for the parameters, but these can be changed after seeing the implied conditional distributions of the parameters given the data. Thus they are not exactly like the Bayesian view of prior beliefs. Historical values of $r_j(t)$ are not parameters of the model, but they are projected as a step in parameter estimation. These are not allowed to fluctuate freely, however. Their prior distributions are defined by the process evolution equations. Thus the next value of a process would be normally distributed according to the evolution assumptions of the process. (In MCMC you do not have to specify the form of the posterior (conditional given the data) parameter distributions – they are sampled numerically based on the priors, the likelihood, and the data.) All of this produces projections for the history of the partial short-rate processes r_j, and so also provides fitted values of the yield curve at every time point in the data. More detail is in Appendix 2.

4 Results of Fitting

A key issue in fitting models to yield-curve data is the choice of period and data to use. The models assume zero-coupon bonds, but there is no raw data on those. Some data series have been constructed, for instance by the Wall Street Journal. Here we instead use US Treasury Constant Maturity Rates, available from the St. Louis Fed FRED database (https://fred.stlouisfed.org/categories/115). It would be rare to have bonds on the market that mature in exactly 5 years, for instance, and the Fed estimates 5-year, etc., rates by interpolating related yields on actual trades. These rates assume semi-annual interest payments at whatever rates the actual bonds carry. The data is considered to be estimates of the yields, each with a distribution around the actual rate. We assume these distributions are all normal, with standard deviations = σ_y, which is estimated in the model fitting using the differences of the fitted and observed rates.

Short-rates have come out of the very-low-yield period following the economic crash of 2008, with the Fed increasing their rates a fair amount. Fitting older data would not be representative of the market we are in now, and even starting at the beginning of the period of rate increases creates a problem of models with built-in upward trend. We chose to use week-ending rates from 1/5/2018 to 6/21/2019 for maturities of 1, 2, 3, 5, 7, 10, 20, and 30 years. Rates shorter than this follow the Fed much more than the market so are not really generated from the same process. This is largely true for the one-year rates as well, but they are too important to leave out. That rate was at 1.82% at the beginning of our data, and ended 1.98%. It got as high as 2.7% along the way. All the other rates also went up for much of this period, but ended up lower than they started. For instance, two-year rates started at 1.95%, got as high as 2.94%, and ended at 1.79%. Thirty-year rates went from 2.8% up to 3.42% then back down to 2.56% The Fed Funds Rate went up for a while then leveled off, not decreasing at the end. The one-year rate seems more influenced by this, and ended higher than the two, three and five-year rates.

We fit four models to that data. VVV is a closed-form model consisting of three correlated Vasicek processes, with essentially affine market prices of risk. This is considered an $A_0(3)$ model as there are three processes and none of them affect the volatility of the rates. The other three models are $A_1(3)$ models, having a single CIR process. CVV is a completely affine model with one CIR and two correlated Vasicek processes. CVV+ is our maximal closed-form model, and is like CVV but is essentially affine and includes parameters for δ_0 and γ_1. 7k3b is the overall maximal ODE model that we fit. It has 7 κ parameters – all but κ_{12}, κ_{13} – and all three β parameters. The CVV model has 14 parameters – 3 for each risk-neutral process, 3 additional real-world parameters, ρ, and σ_y. VVV has 19 parameters, so 5 more: 2 more correlations, and 6 real-world parameters – a κ and a ω for each process. CVV+ has 18 parameters. It only has 1 correlation, but has δ_0 and γ_1 as well as 5 real-world parameters, as $\widetilde{\omega}_1 = \omega_1$. 7k3b actually has 10 more parameters than this, so 28 in total. Compared to CVV+ it has 4 more $\widetilde{\kappa}$s, 2 more βs, and 4 more κs.

Table 7 shows the fitted parameters for each model, along with: σ_y; the implied real-world model means μ_j for each process and their implied total short rate; the loo penalized loglikelihood goodness-of-fit measure for MCMC fits; the loo penalty for parameters (difference from LL); and the implied loglikelihood.

The VVV model does not fit as well as any of the CIR models, according to loo. The Vasicek and CIR yield curves can have somewhat different shapes, which could be contributing to this. Also stochastic volatility may be a feature of the data, and Vasicek models do not capture that. The essentially affine version of the CVV model, including the δ_0, γ_1 adjustments to the $C(\tau), D(\tau)$ functions, is better-fitting than the basic version. The full model, even when considering all the additional parameters, is better yet, both in loo and the residual standard deviation σ_y. The 10 additional parameters make the model much more flexible, and apparently this data can make use of that flexibility.

All of the projected short-rates (here for 3 processes and 77 data points, so 231 in total) are considered parameters for testing goodness of fit, in addition to the model parameters and

Table 7: Fitted Models

Variable	VVV	CVV	CVV+	7k3b
$\widetilde{\omega}_1$	-30.3	1.33	1.19	0.113
$\widetilde{\omega}_2$	-0.627	-62	-19.28	-2.758
$\widetilde{\omega}_3$	30.3	62.2	19.91	1.649
$\widetilde{\kappa}_{11}$	0.07	0.027	0.165	0.007
$\widetilde{\kappa}_{22}$	1.076	0.063	0.07	0.330
$\widetilde{\kappa}_{33}$	0.055	0.052	0.047	0.224
$\widetilde{\kappa}_{21}$	-	-	-	0.047
$\widetilde{\kappa}_{23}$	-	-	-	0.122
$\widetilde{\kappa}_{31}$	-	-	-	-0.708
$\widetilde{\kappa}_{32}$	-	-	-	-0.933
σ_1	1.64	1	1	1
σ_2	0.83	2.58	1.17	0.159
σ_3	1.92	2.9	1.31	0.071
β_1	-	1.85	1.19	9E-05
β_2	-	-	-	1.078
β_3	-	-	-	2.703
ω_1	24.1	1.33	1.19	0.113
ω_2	-0.296	-0.596	7.045	-6.626
ω_3	-19.2	0.43	-22.272	2.908
κ_{11}	0.972	2.432	1.8667	0.012
κ_{22}	1.906	-	0.7369	0.918
κ_{33}	1.0534	-	2.4443	1.094
κ_{21}	-	-	-	0.023
κ_{23}	-	-	-	-0.678
κ_{31}	-	-	-	0.133
κ_{32}	-	-	-	-0.228
ρ_{12}	0.17	-	-	-
ρ_{13}	-0.93	-	-	-
ρ_{23}	-0.47	-0.96	-0.81	-0.65
δ_0	-	-	0.021	0.253
γ_1	-	-	-0.032	0.001
σ_y	0.025	0.023	0.023	0.019
μ_1	9.41	0.55	0.64	9.52
μ_2	-0.46	-13.05	9.36	-5.7
μ_3	-6.35	6.2	-9.09	-0.05
Mean short rate	2.6	-6.3	0.91	4.02
loo	1312.9	1324.7	1333.3	1462.1
penalty	134	175.7	184.2	183.5
LL	1446.9	1500.4	1517.5	1645.6

the market prices of risk. Thus there are between 245 and 259 parameters that go into the parameter penalty. The penalties are much less than this, and are not always higher with more parameters.

The loo penalty is not calculated as a multiple of the number of parameters but rather comes from a cross-validation approach. The penalized loglikelihood is an estimate of the loglikelihood (LL) of the fitted model for a new independent sample. The penalty is how much lower the penalized likelihood is from the actual LL. It is an estimate of the sample bias, which tends to increase for more parameters, but not in a readily-predictable way for non-linear models like these. The penalties here are less than the actual number of parameters partly because the estimated historical short-rates are highly constrained and do not act like independent parameters.

Ye (1998) defines the generalized degrees of freedom used by a data point in a nonlinear model as the derivative of the fitted point wrt the actual data point. These derivatives constitute the diagonal of the hat matrix in linear models. The sum of the resulting dofs is the effective number of parameters, and these can then be penalized by AIC, etc. Constrained parameters do not give the data points much power to move the fitted values towards them, so the derivatives and the effective number of parameters are reduced. Something similar happens in loo. The loglikelihood at a point is penalized by how much it would be reduced if that point were left out of the sample. Again if the parameters are highly constrained, the loglikelihood at a point is not affected much by leaving it out of the estimation.

Figures 10 and 11 show the fitted short rates, the $C(\tau), D(\tau)$ functions, and the resulting rate fits for the 7k3b model.

5 Applying Tests

We now apply several of the tests from Section 2 to the VVV and 7k3b models for illustration. We first extracted the sampled parameter distributions from the Stan output for each model. Then we simulated two years of future rates for the two models, and these simulated yield curves were the used for the tests. This is similar to the common situation of the insurer only having the generated scenario sets and not the model fitting comparisons.

The simulations started from the sampled parameter distributions, which is a way of including parameter uncertainty. Each simulation starts by drawing a parameter sample at random. Then the ending values of the three processes in that sample are the starting previous values for their first simulations. The simulations are done by computing the drift and the variance using the real-world processes, then doing a random draw for the next value of each process. The C and D functions (already in the parameter samples) are based on the risk-neutral

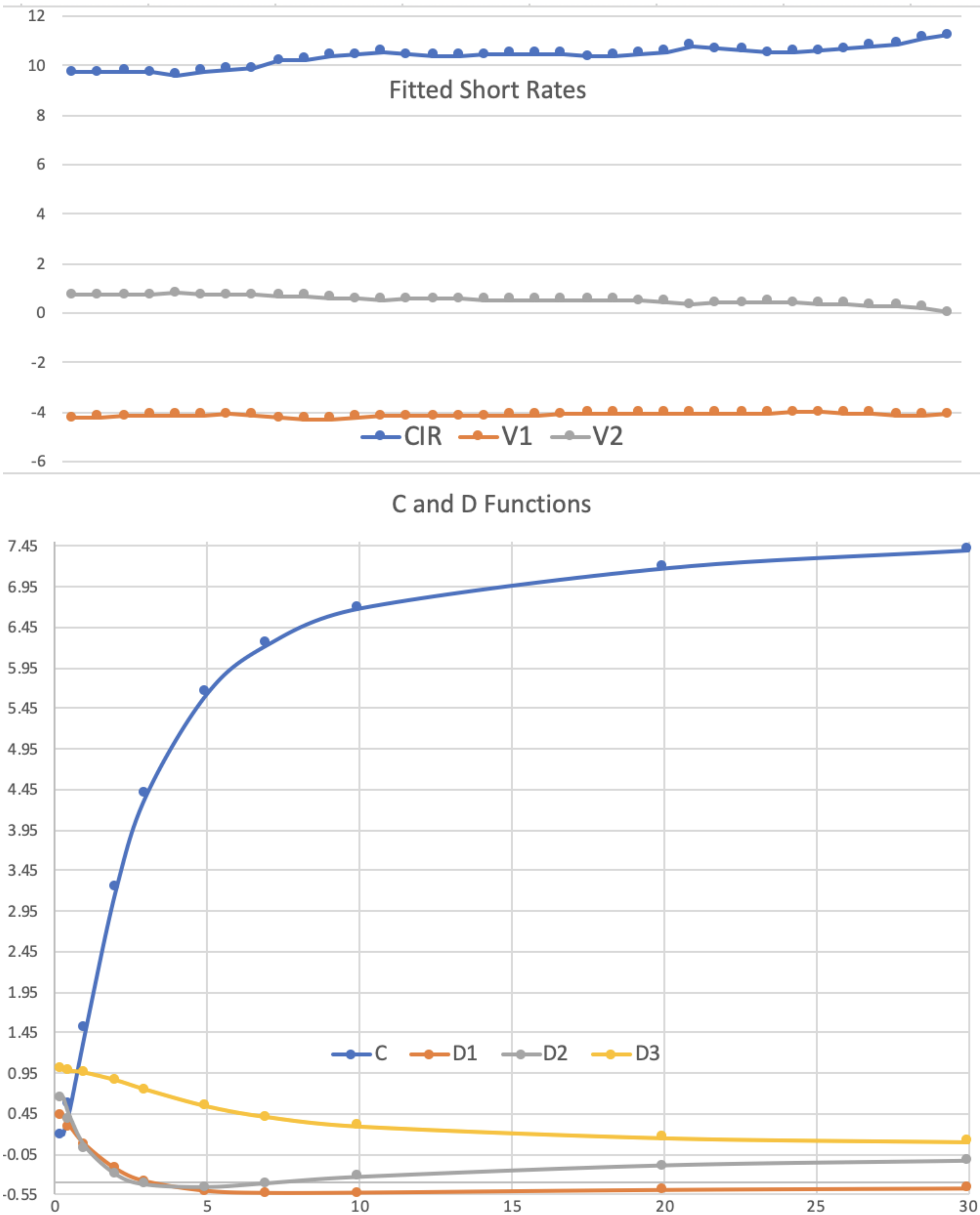

Figure 10: Projected Historical Short Rates and C, D Functions

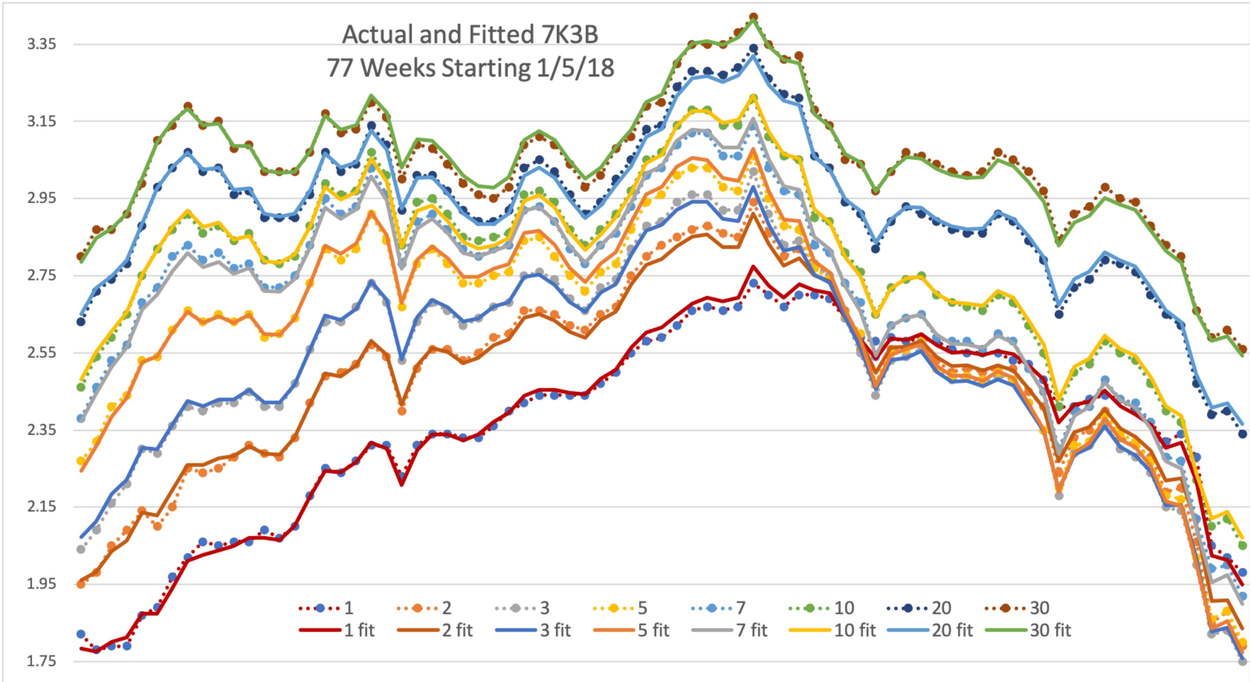

Figure 11: Actual vs. Fitted Yield Curves

processes and are not simulated except as implied by the draw of the sample. We simulate at monthly intervals for two years for each process, and the two year-ending yield curves are used in the tests.

Table 8 shows the moments of the simulated rates for the two models by year. The mean yield curves are basically upward for the VVV model, while the 7k3b model projects the current curve shape, which is high on both ends and low in the middle, which seems better. The standard deviations generally decline with maturity, which is probably all you can look for there at this point, as the right shape of the volatilities by maturity is still emerging. The skewness is moderately negative for the VVV model, indicating more downward risk in yields. It is also negative for the longer rates in the 7k3b model, but is positive for the shorter rates. That could be coming from the recent experience, which had movements like that.

Table 9 has the spread regressions for the 3-year – 30-year spreads as a function of the 1-year rates and the Campbell-Schiller regressions for 2 and 3-year rates. The spread regressions' slopes and standard errors are compatible with recent history for 7k3g but are pretty small for VVV. The expected slopes for the Campbell-Schiller regressions are negative. That is all consistent with the targets for these tests, and what we would want for the exact magnitudes of the slopes is not clear beyond those signs.

We did principal component analysis on the actual data, the fitted means for all four models,

Table 8: Scenario Moments

Maturity	1	2	3	5	7	10	20	30
VVV Year 1								
mean	1.90	1.87	1.89	1.97	2.06	2.18	2.48	2.70
std	0.40	0.44	0.46	0.47	0.47	0.45	0.38	0.32
skw	-0.55	-0.50	-0.45	-0.41	-0.39	-0.37	-0.35	-0.34
VVV Year 2								
mean	1.94	1.92	1.95	2.03	2.12	2.24	2.52	2.73
std	0.53	0.56	0.58	0.58	0.57	0.54	0.45	0.38
skw	-0.86	-0.77	-0.71	-0.64	-0.61	-0.59	-0.55	-0.53
7k3b Year 1								
mean	2.74	2.56	2.45	2.38	2.42	2.51	2.71	2.86
std	0.82	0.76	0.76	0.73	0.68	0.60	0.45	0.40
skw	0.18	0.04	-0.08	-0.23	-0.28	-0.30	-0.30	-0.30
7k3b Year 2								
mean	3.29	2.85	2.57	2.32	2.30	2.39	2.65	2.82
std	1.47	1.01	0.90	0.93	0.92	0.83	0.59	0.49
skw	0.94	0.68	0.24	-0.85	-1.20	-1.27	-1.02	-0.78

Table 9: Scenario Regressions

	Spread Slope	Spread SE	Campbell-Schiller Slope
VVV Year 1 (2 Year τ for C-S)	-0.42	0.01	-0.99
VVV Year 2 (3 Year τ for C-S)	-0.29	0.01	-1.63
7k3b Year 1 (2 Year τ for C-S)	-0.61	0.06	-2.63
7k3b Year 2 (3 Year τ for C-S)	-0.45	0.09	-1.99

and the two years of simulated scenarios for the VVV and 7k3b models. For this we used the R function prcomp, which takes as input an array, with the variables in the columns and the observation times in rows. The first three components explain the vast majority of the variation in the data. An indicator of the degree of complexity of the yield curves is provided by the percentage of variance explained by the third principal component. This is small for the short time period we have, but was greatest in the data itself, and was higher for the fitted values than for the simulated scenarios. The VVV model looks too weak in the third PC's proportion of variance in the simulations. Table 10 shows the results.

Conclusion

The last 30 or 40 years of yield curve history, instead of being the new normal, are starting to look more like a one-time aberration, driven by the post-WWII financing needs. We are now seeing interest rates more like the 1870 – 1970 period, where the long rates rarely got above 4%. This makes constructing tests based on yield-curve standard behavior challenging.

We started by reviewing the tests that had been used historically in the related literature. Most of them are still useful with updated targets, although these are fairly broad given the uncertainty about the properties of interest rates in the emerging era. We applied the tests to a few models fit to recent data, and the models did reasonably well in the context of these broad guidelines, with the better-fitting model also doing better on the tests.

The models used were affine models, which build up the yield curves as weighted sums of partial short-rate processes. The weights for this are time invariant but are a bit complicated to calculate, and for the most general models require numerical solutions of ordinary differential equations. Several closed form models where the differential equations have been solved in closed form provide reasonable fits, but not as good as the more general models.

MCMC estimation provides a direct way to fit the models, and produces fitted values for the historical processes.

Table 10: PCA Comparisons

data	PC1	PC2	PC3
Standard deviation	0.657	0.254	0.083
Proportion of Variance	0.857	0.128	0.014
Cumulative Proportion	0.857	0.985	0.999
fit vvv	PC1	PC2	PC3
Standard deviation	0.655	0.251	0.070
Proportion of Variance	0.863	0.127	0.010
Cumulative Proportion	0.863	0.990	1.000
fit cvv	PC1	PC2	PC3
Standard deviation	0.656	0.252	0.074
Proportion of Variance	0.862	0.127	0.011
Cumulative Proportion	0.862	0.989	1.000
fit cvv+	PC1	PC2	PC3
Standard deviation	0.655	0.253	0.074
Proportion of Variance	0.861	0.128	0.011
Cumulative Proportion	0.861	0.989	1.000
fit 7k	PC1	PC2	PC3
Standard deviation	0.657	0.254	0.079
Proportion of Variance	0.859	0.128	0.013
Cumulative Proportion	0.859	0.987	1.000
vvv yr1	PC1	PC2	PC3
Standard deviation	1.189	0.187	0.062
Proportion of Variance	0.973	0.024	0.003
Cumulative Proportion	0.973	0.997	1.000
vvv yr2	PC1	PC2	PC3
Standard deviation	1.481	0.211	0.072
Proportion of Variance	0.978	0.020	0.002
Cumulative Proportion	0.978	0.998	1.000
7k yr1	PC1	PC2	PC3
Standard deviation	3.067	0.749	0.165
Proportion of Variance	0.789	0.203	0.009
Cumulative Proportion	0.789	0.991	1.000
7k yr2	PC1	PC2	PC3
Standard deviation	1.975	1.734	0.222
Proportion of Variance	0.561	0.432	0.007
Cumulative Proportion	0.561	0.993	1.000

Appendix 1 – General Affine Models

A1a Matrix Notation

We follow the notation of Dai and Singleton (2000) for affine models. The model combining two Vasiceks and one CIR is classified as an $A_1(3)$ model. That means that there are three processes, with one of them affecting the variance – in this case the CIR process. For consistency with more general models, we replace κ_j with κ_{jj} and σ_j with σ_{jj}. Then our model, with the CIR model first, can be written:

$$dr_1(t) \sim \mathcal{N}\left([\omega_1 - \kappa_{11} r_1(t)]dt, \sqrt{\beta_1 r_1(t) dt}\right)$$

$$dr_2(t) \sim \mathcal{N}\left([\omega_2 - \kappa_{22} r_2(t)]dt, \sigma_{22}\sqrt{dt}\right)$$

$$dr_3(t) \sim \mathcal{N}\left([\omega_3 - \kappa_{33} r_3(t)]dt, \sigma_{33}\sqrt{dt}\right)$$

$$corr\,(dr_2(t), dr_3(t)) = \rho$$

While this is a convenient way to show the correlation, more calculation detail is needed for use as a prior or for simulation. The bivariate normal prior can be used for the two Vasicek processes, specifying their covariance matrix as:

$$Cov(dr_2(t), dr_3(t)) = \begin{pmatrix} \sigma_{22}^2 & \rho\sigma_{22}\sigma_{33} \\ \rho\sigma_{22}\sigma_{33} & \sigma_{33}^2 \end{pmatrix} dt$$

We do that in the code. For simulation, it is useful to be able to show these evolution equations in terms of independent standard normal draws. For this we write $dr_j(t) = \mu_j(t)dt + z_j(t)\sqrt{dt}$, where $z_j(t)dt$ is a mean zero normal variable. Also we define $\epsilon_j(t)$ to be a standard normal random variable observed at time t. Then the vector $z(t)dt$ that combines the three processes can be expressed as:

$$z(t)\sqrt{dt} = \begin{pmatrix} \sqrt{\beta_1 r_1(t)} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & \rho\sigma_{22} & \sqrt{1-\rho^2}\sigma_{33} \end{pmatrix} \begin{pmatrix} \epsilon_1(t) \\ \epsilon_2(t) \\ \epsilon_3(t) \end{pmatrix} \sqrt{dt}$$

This now can be set up in matrix notation as the evolution of a vector of three processes. Let $r(t)$ be a column vector of the three processes, with $dr(t) \sim \mu(t)dt + z(t)\sqrt{dt}$, and vector of standard normals $\epsilon(t)$. Then the general $A_1(3)$ model diffusion can be expressed by:

$$\mu(t)dt = [\Omega - Kr(t)]dt$$

$$z(t)\sqrt{dt} = \Sigma D(t)\epsilon(t)\sqrt{dt}$$

where: K is the 3x3 matrix of $\kappa_{i,j}$, with $\kappa_{12} = \kappa_{13} = 0$; Ω is a 3-vector; Σ is the matrix of $\sigma_{i,j}$ values, with $\sigma_{11} = 1, \sigma_{12} = \sigma_{13} = 0$ and ρ combined into the σ coefficients; $D(t)$ is a diagonal matrix with the three values $\sqrt{\beta_1 r_1(t)}, \sqrt{\alpha_2 + \beta_2 r_1(t)}, \sqrt{\alpha_3 + \beta_3 r_1(t)}$, with all β_j non-negative and β_1 positive. This allows the CIR process to also contribute to the variances of the other processes. D^2 is a variance factor, scaled by Σ^2. We are going to assume that $\alpha_2 = \alpha_3 = 1$ in the fitting, as the σ's can pick up a lot of the α effects.

Such general $A_1(3)$ models provide for interaction among the processes. For the two Vasicek processes, σ_{32} takes the place of $\rho\sigma_{22}$. We set $\sigma_{23} = 0$ for identifiability, but this is not always done. The off-diagonal κ terms make a Vasicek process's mean movements responsive to the current levels of both other processes, as well as its own. That cannot be done for the CIR process, as it can easily lose its arbitrage-free property with any adjustments.

Some constraints on the parameters are needed for identifiability. Dai and Singleton (2000) set up a canonical form of the model that builds in numerous constraints, but it is not clear how our original three-factor model would be expressed in it, so we rely instead on the use of priors on the parameters to get unique parameters. The yield curves are built up from the short rates using market prices of risk. The CIR and Vasicek market prices of risk discussed previously are special cases. Here we follow Christensen (2015) who clarifies the notation and provides examples.

A1b Market Price of Risk and Bond Yields

The basic form of market price of risk is the completely affine risk premium. It uses a 3-vector $\Lambda(t) = D(t)\lambda$, where λ is a 3-vector of constants. The risk-neutral process is produced by a change in the drift term. $\Sigma D(t)\Lambda(t)dt$ is subtracted from the drift $[\Omega - Kr(t)]dt$. Some algebra on this can then be used to express the risk-neutral drift term as $[\widetilde{\Omega} - \widetilde{K}r(t)]dt$, and this is used to compute the yield curve, as it was in the three-factor model above.

The term subtracted is similar to the stochastic term of the process. We need to compute:

$$\widetilde{\Omega}dt - \widetilde{K}r(t)dt = \Omega dt - Kr(t)dt - \Sigma D(t)\Lambda(t)dt$$

The right side of this is already the risk-neutral drift, but the formulas for the yield curve use the notation of the left side, so we have to put it in that form. For the completely affine case, $\Sigma D(t)\Lambda(t)dt = \Sigma D(t)^2\lambda$, where $D(t)^2$ is diagonal with elements $\alpha_j + \beta_j r_1(t)$. Then $D(t)^2\lambda$ is the vector with elements $\lambda_j[\alpha_j + \beta_j r_1(t)]$, which means that $\Sigma D(t)\Lambda(t)$ is a 3-vector, but with $\alpha_1 = 0$. Each element is a sum of multiples of α_j and $\beta_j r_1(t)$ terms. The terms with

α_j's are subtracted from the ω_j's terms, and this gives $\widetilde{\Omega}$. All the factors with $\beta_j r_1(t)$'s are subtracted from the drift, so the coefficients of $r_j(t)$ are added to K to produce $\widetilde{K}$. Also note that the stochastic part of the process does not change in the risk-neutral process.

Dai and Singleton (2000) solve some of this for any $A_1(3)$ model. Let L be the diagonal matrix with elements $\lambda_j\beta_j$ and H be the vector of $\lambda_j\alpha_j$. Then $\widetilde{K} = K + \Sigma L$, and $\widetilde{\Omega} = \Omega - \Sigma H$. In practice we reverse this for estimation efficiency, so we start with the risk-neutral process and get the real-world process by $K = \widetilde{K} - \Sigma L$, and $\Omega = \widetilde{\Omega} + \Sigma H$.

Above we used formulas from Brigo and Mercurio (2001) to compute the $C(\tau)$ and $D_j(\tau)$ functions in the correlated Vasicek case. Troiani (2017) illustrate these formulas for the essentially affine case. For more general models, C and D_j functions are not closed form and need to be computed numerically, from systems of ordinary differential equations (ODEs). Fast software for solving ODEs is widely available. The closed-form calculation is considerably faster, but solving the system numerically is feasible on personal computers. That's why these functions are considered to be "almost closed form."

The system for the $A_1(3)$ models is expressed in terms of functions $A(\tau)$ and $B_j(\tau)$, with $C(\tau) = -A(\tau)/\tau$ and $D_j(\tau) = B_j(\tau)/\tau$. B is the vector of the B_j's. Let $Q(\tau) = \Sigma' B(\tau)B(\tau)'\Sigma$, which is 3x3 and can be considered to be the square of the three-vector $\Sigma' B(\tau)$. The jth element on the diagonal of Q is the square of the jth element of $\Sigma' B(\tau)$, so $Q_{jj} = ([\Sigma' B(\tau]_j)^2$. Also let $\beta 0_j$ be the vector consisting of β_j followed by two zeros. The equations are then:

$$\frac{dA(\tau)}{d\tau} = -\widetilde{\Omega}' B(\tau) + \frac{1}{2}\sum_{j=1}^{3}([\Sigma' B(\tau)]_j)^2\alpha_j$$

$$\frac{dB(\tau)}{d\tau} = \mathbf{1} - \widetilde{K}' B(\tau) - \frac{1}{2}\sum_{j=1}^{3}([\Sigma' B(\tau)]_j)^2\beta 0_j$$

Here $\mathbf{1}$ is a vector of 1's. The starting values are $A(0) = B_j(0) = 0$.

We have been assuming that the sum of the vector $r(t)$ of the three short-rate components is the actual short rate $r_s(t)$. But affine models allow a slight generalization of that with a constant δ_0 and a three-vector δ so that $r_s(t) = \delta_0 + \delta' r(t)$. The differential equations above are based on the default assumptions that $\delta_0 = 0$ and $\delta = \mathbf{1}$. We set $\gamma = \delta - \mathbf{1}$. Then the differential equations become:

$$\frac{dA(\tau)}{d\tau} = -\widetilde{\Omega}' B(\tau) + \frac{1}{2}\sum_{j=1}^{3}([\Sigma' B(\tau)]_j)^2\alpha_j - \delta_0$$

$$\frac{dB(\tau)}{d\tau} = \gamma + \mathbf{1} - \widetilde{K}'B(\tau) - \frac{1}{2}\sum_{j=1}^{3}([\Sigma' B(\tau)]_j)^2 \beta 0_j$$

This ends up subtracting $\delta_0\tau$ from $A(\tau)$ and adding $\gamma\tau$ to $B(\tau)$. Thus $C(\tau)$ is increased by δ_0 and $D_j(\tau)$ is increased by γ_j. Dai and Singleton (2000) assume $\delta_0 \geq 0$ and $\gamma_2 = \gamma_3 = 0$. Feldhütter (2016) estimates δ_0 and γ_1 as both less than 0.03. These same adjustments to C and D can be done for the closed form cases as well, as discussed before.

A1c Essentially Affine Market Price of Risk

Analysts have generally concluded that completely affine risk premiums are overly restrictive on yield-curve dynamics. Recall that these subtract $\Sigma D(t)\Lambda(t)dt$ from the drift $[\Omega - Kr(t)]dt$, where $D(t)\Lambda(t)$ is set to $D(t)^2\lambda$, with λ a 3-vector. Greater flexibility is provided by the essentially affine risk premium, with $D(t)\Lambda(t) = D(t)^2\lambda + J\Psi r(t)$. Here J is a diagonal matrix with elements $0, \alpha_2^{-1/2}, \alpha_3^{-1/2}$, and Ψ is a 3x3 matrix with the first row all zeros. This makes $D(t)\Lambda(t) =$

$$\begin{pmatrix} \beta_1 r_1(t) & 0 & 0 \\ 0 & \alpha_2 + \beta_2 r_1(t) & 0 \\ 0 & 0 & \alpha_3 + \beta_3 r_1(t) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2^{-1/2} & 0 \\ 0 & 0 & \alpha_3^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{pmatrix} \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix}$$

Multiplying by Σ shows that $(0, \sigma_{22}\lambda_2\alpha_2, \sigma_{33}\lambda_3\alpha_3)$' is subtracted from Ω, all of the other pieces are added to K. Alternatively, starting with the risk-neutral coefficients can produce the real-world coefficients.

A1d Our maximal model

The maximal model that we fit, which we call 7k3b, is the full general model with the following restrictions:

- Σ is diagonal except for $\sigma_{32} = \rho\sigma_{22}$, with $\sigma_{33} = \sqrt{1-\rho^2}\sigma_3$, where σ_3 is the standard deviation of the original correlated second Vasicek process.
- $\sigma_{11} = 1$
- $\kappa_{12} = \kappa_{13} = 0$
- $\alpha_2 = \alpha_3 = 1$

This means that in the model $dr(t) \sim \mu(t)dt + z(t)\sqrt{dt}$

$$\mu(t) = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} - \begin{pmatrix} \kappa_{11} & 0 & 0 \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix} r(t)$$

$$z(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \sqrt{\beta_1 r_1(t)}\epsilon_1(t) \\ \sqrt{1+\beta_2 r_1(t)}\epsilon_2(t) \\ \sqrt{1+\beta_3 r_1(t)}\epsilon_3(t) \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_1 r_1(t)}\epsilon_1(t) \\ \sigma_{22}\sqrt{1+\beta_2 r_1(t)}\epsilon_2(t) \\ \sigma_{32}\sqrt{1+\beta_2 r_1(t)}\epsilon_2(t) + \sigma_{33}\sqrt{1+\beta_3 r_1(t)}\epsilon_3(t) \end{pmatrix}$$

There are thus 7 κs and 3 βs. The variances of these processes over a short time dt are: $\beta_1 r_1(t)$, $\sigma_{22}^2(\beta_2 r_1(t)+1)dt$, $[\sigma_{32}^2(\beta_2 r_1(t)+1)+\sigma_{33}^2(\beta_3 r_1(t)+1)]dt$. The last is from the sum of two normal distributions. The covariance of the two Vasicek processes is $\sigma_{22}\sigma_{32}(1+\beta_2 r_1(t))dt$.

The risk-adjustment for the essentially affine version for this model comes from

$$D(t)\Lambda(t) = \begin{pmatrix} \lambda_1\beta_1 r_1(t) \\ \lambda_2 + \lambda_2\beta_2 r_1(t) \\ \lambda_3 + \lambda_3\beta_3 r_1(t) \end{pmatrix} + \begin{pmatrix} 0 \\ {}_{21}r_1(t) + \psi_{22}r_2(t) + \psi_{23}r_3(t) \\ {}_{31}r_1(t) + \psi_{32}r_2(t) + \psi_{33}r_3(t) \end{pmatrix}$$

Then

$$\Sigma D(t)\Lambda(t) = \begin{pmatrix} \lambda_1\beta_1 r_1(t) \\ \sigma_{22}\lambda_2 + \sigma_{22}\lambda_2\beta_2 r_1(t) \\ \sigma_{32}\lambda_2 + \sigma_{32}\lambda_2\beta_2 r_1(t) + \sigma_{33}\lambda_3 + \sigma_{33}\lambda_3\beta_3 r_1(t) \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \sigma_{22}\psi_{21}r_1(t) + \sigma_{22}\psi_{22}r_2(t) + \sigma_{22}\psi_{23}r_3(t) \\ \sigma_{32}\psi_{21}r_1(t) + \sigma_{32}\psi_{22}r_2(t) + \sigma_{32}\psi_{23}r_3(t) + \sigma_{33}\psi_{31}r_1(t) + \sigma_{33}\psi_{32}r_2(t) + \sigma_{33}\psi_{33}r_3(t) \end{pmatrix}$$

From above, the r_2 mean for this model is:

$$\mu_2(t) = \omega_2 - \kappa_{21}r_1(t) - \kappa_{22}r_2(t) - \kappa_{23}r_3(t)$$

Starting with the risk-neutral version, we can see that $\omega_2 = \tilde{\omega}_2 + \sigma_{22}\lambda_2$. Also, $\kappa_{21} = \tilde{\kappa}_{21} - \sigma_{22}(\lambda_2\beta_2 + \psi_{21})$, from the middle row of $z(t)$. The other coefficients are calculated similarly.

As the real-world parameters are linear combinations of the risk-neutral parameters and the market-price-of-risk parameters, we can fit the risk-neutral and real-world parameters separately and then back out the market-price-of-risk parameters if we want them. It is just necessary to keep $\omega_1 = \widetilde{\omega}_1$. This seems to make it easier and faster for the Stan software. Perhaps Stan uses numerical derivatives of the posterior probabilities with respect to the parameters as a fitting step, and having the real-world and risk-neutral parameters related algebraically complicates this.

The reverting mean of the CIR process is $\mu_1 = \omega_1/\kappa_{11}$, as it is for any CIR process. For the Vasicek processes, the changes to the next period will have mean zero if all the processes are at their means. A little algebra can give these means. First, let $w_2 = \omega_2 - \kappa_{21}\mu_1$ and $w_3 = \omega_3 - \kappa_{31}\mu_1$. Also define $\kappa_\Delta = \kappa_{22}\kappa_{33} - \kappa_{23}\kappa_{32}$. Then $\mu_2 = (\kappa_{33}w_2 - \kappa_{23}w_3)/\kappa_\Delta$ and $\mu_3 = (\kappa_{22}w_3 - \kappa_{32}w_2)/\kappa_\Delta$.

The long-term variances of the Vasicek processes, calculated as expected incremental variance for one year divided by twice the speed of mean reversion, come out: $0.5\sigma_{22}^2\left(\beta_2\omega_1/\kappa_{11}+1\right)/\kappa_{22}$ and $0.5\left[\sigma_{32}^2(\beta_2\omega_1/\kappa_{11}+1)+\sigma_{33}^2(\beta_3\omega_1/\kappa_{11}+1)\right]/\kappa_{33}$.

A1e Unspanned Stochastic Volatility (USV)

In an $A_1(3)$ model, or any model that includes a single CIR component, the current variance of the CIR piece is a constant multiple of the latest value of that process, and the other variances are linear functions of the CIR process. That means that the variance of a rate can be expressed as a linear function of all the rates. This is also the case when there are multiple CIR processes. The value of the processes at any point in time can be estimated from the $C(\tau), D_j(\tau)$ values at each maturity and the current rates. Thus the variance at any time can be well estimated by a regression on the rates.

This turns out not to be true of actual rates. Typically a regression like that has an R^2 of about 20%. Collin-Dufresne, Goldstein, and Jones (2003) call this situation "unspanned stochastic volatility." Knowing the yield curve at a given time is not enough to know the variances of the rates. Since it would be enough in $A_1(3)$ generated rates, these models have too close a relationship among the rates and their volatilities.

Collin-Dufresne, Goldstein, and Jones (2003) then look for affine models that have stochastic volatility but for which the variance cannot be computed as a linear function of the rates. They come up with a closed-form $A_1(3)$ model with a lot of related parameters which interact to produce $D(\tau) = 0$ for the CIR process. Then the rates are not a linear function of the CIR process, even though its variance does affect the other processes. Apparently their model did not fit very well, however. Because only two factors enter into the rate calculation, it acts more like a two-factor model, for PCA for example.

Joslin (2018) comes up with general constraints for an affine model to display USV, and gives an $A_1(4)$ example. Filipovic, Larsson, and Statti (2018) work out conditions for an $A_3(3)$

model – 3 CIRs – to have USV. USV models do not necessarily fit better – this is more of a constraint, like being arbitrage-free. We found in our own fitting that mistakes in our code that gave models that were not arbitrage-free often produced better fits. This is typical for constraints. Similarly, adding more data to the fit usually makes the fit a bit worse for the original smaller dataset.

Appendix 2 – Fitting models by MCMC

MCMC (Markov Chain Monte Carlo) estimation simulates sample sets of parameter estimates. It requires a postulated distribution for each parameter and samples from the conditional distribution of the parameters given the data, using efficient numerical techniques. Sometimes it is presented in a Bayesian context, with the postulated distributions labeled as the priors and the conditional distributions called posteriors. However these priors often have no connection to any prior beliefs, or any subjective view of probability, and Bayes Theorem is not needed to sample from the conditional distributions. The postulated distributions can be revised in response to the conditional distributions they generate. MCMC can be done in a frequentist context if the parameters are instead treated as being random effects with the postulated distributions. We will use the prior/posterior terminology, but with the understanding that they are not the same as prescribed by traditional Bayesian methods, and also have a frequentist interpretation.

MCMC has goodness of fit measures analogous to the AIC, BIC, etc., the best one being the leave-one-out (loo) loglikelihood. From the sample of estimates it is possible to numerically approximate what the likelihood would be for a point from a fit done with the data excluding that point – basically by giving more weight to the parameter sets that fit worst at this point. Loo is a good estimate of what the loglikelihood would be for an entirely new sample using the parameters fit to this sample, which is the goal of the AIC measure as well. All of this is in line with the idea that model estimation should optimize the fit to the entire population instead of optimizing the fit to the given sample.

MCMC fitting of yield-curve models includes an intermediate calculation of the $r_j(t)$ values for each process j at each point in time t. These are not parameters of the model per se, but are still given priors and produce posteriors. The estimation assumes that there is noise in the observation process. The interest rates by maturity produced by a model are the estimated mean values, and the data is (typically) assumed to be normally distributed with those means and variance σ_y. The likelihood function is calculated from those probabilities.

In the CVV model, we start with four parameters for each process: $\widetilde{\kappa}, \widetilde{\omega}, \omega$ and σ or β. Those are all given wide-enough priors so that the priors do not restrict the conditional distributions.

Sometimes the priors have to be narrowed to exclude poor-fitting local maxima, and possibly to speed the calculations. We use normal priors, but for parameters that must be positive, we use gamma priors or give their logs normal or uniform priors. This eliminates a problem with wide priors over-estimating positive parameters. A similar prior is used for σ_y. The prior for ρ is initially uniform$(-1,1)$. These parameters are then used to calculate the $C(\tau)$ and $D_j(\tau)$ functions according to the formulas above.

The first $r_j(t)$ for each process is given the prior of the long-term distribution of the process defined by the parameters, so is gamma with mean ω/κ and variance $\beta\omega/2$ for the CIR, and is normal$(\omega_j/\kappa_j, \sigma_j^2/2\kappa_j)$ for the Vasicek processes. This is like assuming that the process has been going a long time up to that point.

Then the sample value at $r_j(t)$ is used to produce the prior for the process at the next period $r_j(t+dt)$, using the evolution equations for the processes. Each Vasicek prior for $r_j(t+dt)$ is normal with mean $r_j(t)+[\omega_j-\kappa_j r_j(t)]dt$ and variance $\sigma_j^2 dt$. The CIR process is approximated by a gamma with mean and variance $\mu = r(t)e^{-\kappa dt}+\kappa\left(1-e^{-\kappa dt}\right)/\omega$ and $V = \beta^2\left(1-e^{-\kappa dt}\right)\left[2r(t)e^{-\kappa dt}+c\left(1-e^{-\kappa dt}\right)\right]/2\kappa$. This has the same mean and variance as the CIR evolution equation, but for non-instantaneous jumps, the gamma is a better approximation. The Vasicek priors are bivariate normals with correlation $=\rho$.

MCMC simulates the conditional distribution of the parameters given the data. For each simulated set of parameters, it simulates a value of each $r_j(t+dt)$ from the parameters and all the $r_j(t)$'s for the processes. Again this gives the conditional parameter distribution given the data. The different processes can end up with correlated parameters. Finally the fitted parameters and short rates are taken to be the means of the conditional distributions.

There are some model diagnostics. Each parameter has a convergence measure Rhat that will be close to 1.0 if the estimates have converged. We also check to see if the posterior distributions are pushing against the boundaries of the priors. If so, the priors are adjusted to accommodate. There are further diagnostics when parameters do not converge. Finally, loo can be used to compare alternative models.

A2a Estimation Using Ordinary Differential Equations - ODEs

For MCMC model estimation, the prior for $dr_2(t), r_3(t)$ given $r(t)$ is bivariate normal. To get the moments for this, we expand the evolution matrices. This gives:

$$\mu_2(t)dt = [\kappa_{21}(\theta_1 - r_1(t)) + \kappa_{22}(\theta_2 - r_2(t)) + \kappa_{23}(\theta_3 - r_3(t))]dt$$

$$z_2(t)\sqrt{dt} = \sigma_{21}\sqrt{\beta_1 r_1(t)dt}\epsilon_1(t) + \sigma_{22}\sqrt{\alpha_2 dt + \beta_2 r_1(t)dt}\epsilon_2(t) + \sigma_{23}\sqrt{\alpha_3 dt + \beta_3 r_1(t)dt}\epsilon_3(t)$$

and:

$$\mu_3(t)dt = [\kappa_{31}(\theta_1 - r_1(t)) + \kappa_{32}(\theta_2 - r_2(t)) + \kappa_{33}(\theta_3 - r_3(t))]dt$$
$$z_3(t)\sqrt{dt} = \sigma_{31}\sqrt{\beta_1 r_1(t)dt}\epsilon_1(t) + \sigma_{32}\sqrt{\alpha_2 dt + \beta_2 r_1(t)dt}\epsilon_2(t) + \sigma_{33}\sqrt{\alpha_3 dt + \beta_3 r_1(t)dt}\epsilon_3(t)$$

Using $E[(X - EX)^2]$ for the variance shows that it is the expected value of the stochastic part squared. All the terms of that that are mixtures of different ϵ_j's have mean zero. The expected squared of a mean-zero normal is its variance, so, for a short time period dt, we have:

$$Variance(dr_2(t)) = \sigma_{21}^2\beta_1 r_1(t)dt + \sigma_{22}^2(\alpha_2 + \beta_2 r_1(t))dt + \sigma_{23}^2(\alpha_3 + \beta_3 r_1(t))dt$$

$$Variance(dr_3(t)) = \sigma_{31}^2\beta_1 r_1(t)dt + \sigma_{32}^2(\alpha_2 + \beta_2 r_1(t))dt + \sigma_{33}^2(\alpha_3 + \beta_3 r_1(t))dt$$

These depend on $r_(t)$, the CIR process, but for an $A_1(3)$ model, the variances do not depend on the Vasicek processes.

We calculate the covariance using $E[(X - EX)(Y - EY)]$. This is the product of the two stochastic terms, and again any mixed products have mean zero. This gives the incremental covariance:

$$Cov(dr_2(t), dr_3(t)) = \sigma_{21}\sigma_{31}\beta_1 r_1(t)dt + \sigma_{22}\sigma_{32}(\alpha_2 + \beta_2 r_1(t))dt + \sigma_{23}\sigma_{33}(\alpha_3 + \beta_3 r_1(t))dt$$

For the prior for the starting point of each process, we again assume that each one has the long-term distribution for the process. The mean for each is its θ_j, the reverting mean. The variances are the one-year variances (i.e., for $dt = 1$) divided by twice the speed of mean reversion κ_{jj}, where for this purpose, r_1 takes its mean value θ_1. By convention, mean reversion is expressed in annual terms.

Stan has a solver for systems of differential equations. You write the system as a single function at the top of the code, in the functions section. It gives as output the vector of left-hand sides of the system – here the $d/d\tau$ terms. We need $C(\tau)$ and $D_j(\tau)$ functions for each maturity τ, but for a given set of parameters, these are fixed across the observation times. Thus the function takes as arguments the current values of: $\tau, \widetilde{K}, \tilde{\Omega}, \Sigma, \beta 0_j, j = 1, 2, 3$. This does not require $r(t)$. Then to solve the system, the differential-equation solver function is applied to the output of the differential equation function. For some reason, Stan's name for this solver is "integrate_ode_rk45."

Appendix 3 – Code

Stan and R code is up on the CAS GitHub site. A few comments are below.

Stan is run in R and other platforms. Some R code is needed for that, but most of the code is for the models in Stan. It is easy to have the R code open in RStudio, then run a few lines as needed. What is up is typical R code for our models. We are not Stan experts and just reuse and modify code from online examples. This is undoubtedly inelegant and probably inefficient. We are probably using vectors and matrices when arrays would be better, looping more than we need to, etc. But the code can give some idea about how to approach MCMC fitting for these models. There is some R code up, but most of the code is for rstan, the R implementation of Stan. The R code includes some variations for different models.

The introductory Stan file is for a single CIR process. The structure of a Stan file is to first read in variables already populated from the R space, then define all the variables to be used. That takes a fair amount of real estate in a Stan file, and is needed because it will be translated to C++ then compiled. We make the risk-neutral parameters the first ones to estimate, and the model begins with calculating the C and D functions. Then the real-world parameters are defined, and from these the processes at each point are estimated. In the CIR file that just uses the CIR evolution equation to define the prior for the CIR process at each time point, conditional on the previous point. The gamma distribution approximation is used for that. From all this, the fitted values are calculated, then the model for the data is just normal in these values. Finally the likelihoods are computed to pass to loo.

We also put up this model done by solving the differential equations, as an example of how to do that, although no one ever would as the equations have already been solved in closed form. To do it, you start with a new functions section above the data section, and define the system of differential equations there. We call that system AB_eq, as it solves for the A and B functions. These produce the C and D functions after the system has been solved. It is solved by the function integrate_ode_rk45. This takes as arguments the name of the system to be solved, starting values at $\tau = 0$ – here a vector of zeros, the values of the risk-neutral parameters, the precision wanted for the iteration, and some other arguments. The latter are not well documented and are required even though not used, so we put in some (obscure) values that we found in Stan examples, and it all seems to work well. The values of the parameters are strung out in a linear array to pass to the function, which then puts them back into vectors, etc. There is probably a simpler way to do that, but we got it working by doing it, so kept doing it that way. The intermediate W and h calculations for CIR are not needed here, as those are for solving the system in closed form. All the other parts of the code after getting the C and D functions are the same as in the closed-form case.

For the three-factor models, the Vasicek formulas and the correlation adjustment are needed. There is code provided for the CVV+ essentially affine model with the δ and γ terms included, and for the 7k3b model. The correlation adjustment is not needed for the 7k3b model, since it solves for C and D numerically. The priors for it are what gave the fit above, but alternative priors are shown as comments. They gave an even better fit, but the priors are very narrow, and with much deviation from these ranges the model deteriorates rapidly. That makes them suspect as a fluke set of priors that works for the current data but is not really representative of a longer-term population, but both sets are worth trying. Probably for new data they would both have to be modified after seeing how they perform. The posterior histograms can help show if the parameters are trying to move in one direction or another from the priors used.

Some miscellaneous R code is shown below for defining the paras variable used in the print and plot statements, and for extracting the simulated samples for use in simulating future scenarios.

```
#for 7k3b model
paras <- c("kaprn","kaprn21","kaprn23","kaprn31","kaprn32","omrn","lb1","lb2",
"lb3","ls2","ls3", "corr","kap", "kap21","kap23","kap31","kap32","om","ldel","gam1")

#for CVV_plus model
paras <-
c("kaprn", "omrn", "lb", "ls2", "ls3", "sigma_y", "corr", "kap", "om","ldel","gam1")

us1_ss = extract(us_1, permuted = FALSE) # this gets all the samples
#Need permuted = FALSE to get it in array form
dim(us1_ss) # shows dimensions, like 1000 x 4 x 2000 for 1000 sampling draws,
#4 chains and 2000 things computed
us1_ss = us1_ss[,,c(1:28,105,182,259:277,286:336)]
# keeps only variables that are needed for 7k3b;
#for simulation. Here 105, 182, 259 were the last (90th) period's r values
#these numbers would change with more periods fit; also taus (278-285) left out
#careful though as arrays show in different order than in print output
#just keeping 1 chain will keep variables names to check: us1_ss = us_1_ss[,1,]
dim(us1_ss) = c(4000,100) # collapses first two dimensions (samples and chains)
write.csv(us1_ss, file="samples_out.csv")
```

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