# Measurement of Risk-Adjusted Profit and Calculation of Fair Premium

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**Abstract**

Two related topics are studied in this paper—profit measurement and fair premium calculation. To measure profits created by a policy during its life time, we compare its ending assets, after all losses, expenses and taxes are paid, with a breakeven value. Fair premium is so calculated as to make the market value of ending assets zero. Both the breakeven ending assets and fair premium are calculated under market equilibrium conditions, and are given in closed-form formulas. The breakeven value of ending assets is a crucial link between profit measurement and pricing, and between loss discount rates and the cost of capital. It also used to measure profit generated by policies separately from that generated by capital investment.

**Keywords**

Fair premium, financial pricing, profitability, EVA, loss discounting, cost of capital, after-tax profit

## 1 Introduction

A typical property-casualty (P&C) insurance policy covers one accident year, but its claims stay open for many years. The ultimate profitability of the policy can only be determined after all claims are paid. In this paper, we develop a method to test a policy's ultimate profitability under a market equilibrium. This essentially measures the profitability of the policy against all other policies and other financial assets. At the end of the policy's life, we trace out its underwriting cash flows, investment incomes, and tax payments, and calculate its net assets. The *ending assets*, which may be positive or negative, are the policy's ultimate contribution to the company. At first glance, if the ending assets are positive, the policy creates value. But since writing the policy adds risk, merely positive ending assets may not be enough. So it is important to find the breakeven value, with risk adjustment, for the ending assets. This will be a main result of the paper.

Schirmacher and Feldblum (2006) discuss emergence of profits over time and calculation of the economic value added (EVA) under two accounting systems. A different approach is taken in this paper. First, only the ultimate profitability, after all claims are paid off, is considered. Valuation of the ultimate profit is independent of accounting system. Second, in computing the EVA, Schirmacher and Feldblum (2006) assume that a cost of capital (COC) is given extraneously. A focus of this paper is to discuss how the COC is related to the claims risk, the investment risk, and the capital level. We emphasize that the amount of profit generated by a risky policy should be sufficient to compensate for its *own* risk.

A *fair premium* may be defined as the amount of premium that exactly covers all costs (losses, expenses, taxes, etc.) related to a policy, or that exactly generates the cost of capital. In this paper, the fair premium is determined by setting the market value of the ending assets to zero. Such an equation can be solved algebraically using the Capital Asset Pricing Model (CAPM). Technically, calculation of the ending assets and their market value is a key element of the paper. This calculation also brings together the two apparently separate issues—pricing and profit measurement, so we can provide a solution to both of them simultaneously.

The paper is organized as follows. Section 2 is a brief review of the economic combined ratio (ECR). Although reflecting the time value of money, the ECR ignores risk, so is not an unambiguous profit measure. In Section 3, the breakeven value of ending assets is derived for the simplest model—one-year, and no tax. An explicit formula is obtained using the CAPM, which shows how the breakeven value varies with the $\beta$ parameter of the policy and the assets. The results are generalized to policies with multi-year payments in the next section. The $\beta$ parameter is well-defined in a single-year model, but for a policy with a multi-year payout pattern, its risk is most conveniently described with the risk-adjusted discount rate.

The main results of the paper are stated in Section 5. The most realistic assumptions are consider here—multi-year loss payments and tax payments. Closed-form formulas are derived for the fair premium and the breakeven ending assets. Our primary concern here is to obtain easily tractable formulas. So tax rules are simplified. The cost of capital, by definition, reflects the combined risk of policies and asset investments. In Section 6, we derive equations to link the breakeven ending assets to the COC. Other things being equal, the COC varies inversely with the capital level. If the investment is assumed risk free, the COC is easily determined from the breakeven ending assets and the held capitals. In Section 7, we show that the EVA can be defined for the policy account only, so profits from policies and from capital investments can be measured separately.

Fair premium calculation is further discussed in Section 8. Premium may be calculated from discounted losses (the direct method), or solved from a given COC (the indirect method). The two methods are equivalent. A clear relationship between them is revealed via the breakeven value of ending assets.

## 2 Review of Economic Combined Ratio

Insurance professionals are more familiar with the nominal combined ratio. It equals the sum of nominal losses and nominal expenses divided by the nominal premium. An obvious shortcoming of this combined ratio is that it does not reflect the time value of money. Two lines of business may have the same combined ratio, but the longer tailed line pays out losses more slowly, generates more investment income along the way, and is more profitable. So the combined ratio is not an unambiguous profit measure. The economic combined ratio (ECR) is introduced to correct this problem. In calculating the ECR all underwriting cash flows are discounted to the time of policy inception.

I will use a numerical example to illustrate various methods in this paper. The example is borrowed from Schirmacher and Feldblum (2006), which will allow us to compare their results with ours. Assume a policy is issued on Dec. 31, 20XX, for accident year 20XX+1. The underwriting cash flows are as follows. On Dec. 31, 20XX (time 0), a premium of \$1000 is collected and acquisition expenses of \$275 paid. General expenses of \$150 are paid six months later (time 0.5). The policyholder has one accident in the year and will receice one payment of \$650 on Dec. 31, 20XX+3 (time 3).

Schirmacher and Feldblum (2006) choose a surplus requirement of 25% of the unearned premium reserve plus 15% of the loss reserve. The risk-free rate is 8% per year compounded semi-annually (4% per half year). The basic policy cash flows are summerized in Table 1.

**Table 1 here**

The nominal losses ($L$) and expenses ($X$) add up to \$1075, and the premium ($p$) is \$1000. The underwriting profit is $\$-75$ and the combined ratio is $1075/1000 = 1.075\%$. But the greater-than-100% combined ratio does not imply that the policy loses money, since the investment income has been ignored. The ECR takes into account both the underwriting profit and the investment income. As defined in Swiss Re (2006), the present value $\mathrm{PV}(L+X)$ is computed at the risk-free rate, and $\mathrm{ECR} = \mathrm{PV}(L+X)/p$. For this policy, $\mathrm{PV}(L+X) = 932.94$, and the ECR equals 93.29%. Swiss Re (2006) states that ECR $< 100\%$ ($> 100\%$) indicates the policy is profitable (unprofitable). By this criterion, this policy is profitable.

The combined ratio and the ECR do not involve the company's capital. These ratios measure profits generated in the *policy account*. Imagin an investment account is set up for the policy. The net premium of \$725 is deposited in the account at time 0. The account earns a 4% interest per half-year; \$150 is withdrawn at time 0.5 to pay general expenses and \$650 withdrawn at time 3 to pay the loss. At time 3, after all payments, the account holds the policy's *ending assets* of \$84.86 (Table 1, column 6). It is easy to check mathematically that $\mathrm{ECR} = 100\% - \mathrm{PV}(\text{Ending Assets})/p$. Therefore, the ending assets' being greater (or less) than zero is equivalent to the ECR's being less (or greater) than 100%.

The significance of the ECR or the ending assets may also be understood from the shareholder's standpoint. Shareholders contribute a certain amount of capital, which the company uses to do two things. First it purchases securities on the capital markets. Second, with the safety margin the capital provides, the company issues insurance policies. The shareholders can do the first thing — investment — themselves.[1] So what they really want from the company is to make additional money on the policies. The shareholders' profit, after all claims are settled, equals the investment income from the invested capital plus the ending assets in the policy account. Therefore, if the policy account returnspositive ending assets, contributing to the insurance company has an advantage. If the ending assets are negative, using the capital to buy securities directly on the markets would be a better choice.

[1] If each shareholder does his/her own investing, they together may incur higher expenses than doing it through the company. But let us ignore this minor point here.

## 3 Introducing Risk Adjusted Profit Measure

The ECR, however, is also a deficient profit measure. According to Modern Finance, an investment should achieve a return commensurate with its risk. But the risk of policies has been ignored in the calculation of the ECR and the ending assets. If the insurance policies add a great deal of risk, merely positive ending assets may not be enough; if the policies partially offset the investment risk, so the total risk to shareholders is reduced, positive ending assets may not be necessary. Both these scenarios are practically possible. To explain this point quantitatively, I will use the Capital Asset Pricing Model (CAPM). The CAPM can assist us to understand the issue and to find profit measures that encompass risk.

Assume shareholders contribute capital $c$ to start a company. The company issues policies and collects premium $p$ (net of expenses), and invests the total cash $c+p$ in securities. The policy losses $L$ will be paid one year later, and the remaining assets are returned to the shareholders. In the CAPM world, assets are bought and sold at equilibrium prices, and their expected rates of return satisfy the equation $E[R]-r_f=\beta m$, where $r_f$ is the risk-free rate, and $m$ the market risk premium. The $\beta$ coefficient reflects the systematic risk of the asset. Assume the company invests its assets $c+p$ in a portfolio with $\beta_a$. Its rate of return $R_a$ satisfies

$$E[R_a]=r_a, \qquad r_a-r_f=\beta_a m \tag{3.1}$$

To fit insurance liabilities into the CAPM setting, a measure for a liability's systematic risk has to be defined. But let us first stay in the investment world, and substitute shorted assets for liabilities. Suppose the company short-sells a stock that is valued at $p$ and whose $\beta$ is $\beta_l$. The company receives cash $p$—like a policy premium—and invests it, and the investment return is given in (3.1). The liability, $L$, to be paid in one year, is the market value of the shorted stock at that point in time. The "return" of the liability is defined as $R_l=(L-p)/p$, which satisfies

$$E[R_l]=r_l, \quad r_l-r_f=\beta_l m \tag{3.2}$$

From (3.2), the end-of-year (EOY) expected value of the liability is $E[L]=p(1+r_f+\beta_l m)$; from (3.1) the EOY expected value of the premium investment is $p(1+r_f+\beta_a m)$. So the expected ending assets of the "policy"—the combined short and long positions—after the liability is paid, are

$$p(r_a-r_l)=pm(\beta_a-\beta_l) \tag{3.3}$$

This is the risk-adjusted breakeven value under the CAPM assumptions. If the actual ending assets are greater than (3.3), the policy generates a profit; otherwise, a loss. A policy could be profitable for many reasons: a greater (than expected) premium, a higher return on the premium investment, or a lower loss. The breakeven ending assets are determined by the systematic risks of the liability and the premium investment. (It obviously is also in proportion to $p$.) If $\beta_a < \beta_l$, the policy can have negative ending assets and still be profitable (better than breakeven). This means a policy does not have to produce positive ending assets. On the other hand, if $\beta_a > \beta_l$, low positive ending assets are insufficient in regards to risk.

The above results remain valid when $L$ represents a random insured claim. Numerous authors have discussed extensions of the CAPM that cover insurance claims. Early notable papers include Fairley (1979) and Myers and Cohn (1987), and a recent one is Sherris (2003). A widely accepted hypothesis is that the equation (3.2) is satisfied by insurance liabilities, where $r_l$ is the risk adjusted discount rate, i.e., the market value $\mathrm{MV}(L)$ is calculated by $\mathrm{MV}(L) = E[L]/(1 + r_l)$. Intuitively, for a liability $L$, $\mathrm{MV}(L)$ is greater than $\mathrm{PV}(L) = E[L]/(1 + r_f)$, the difference being the liability's risk load. Thus most authors assume that a liability's $\beta$ is less than zero (Myers and Cohn 1987, Bingham 2000, Conger et al. 2004). Alternatively, Feldblum (2006) proposes that most P&C liabilities have no systematic risk (they are uncorrelated with the market return), thus their $\beta$'s equal zero. The disagreement can only be settled with accurate estimation of $\beta$ for some typical liabilities. No reliable approaches have been found so far to empirically verifying the equation (3.2) for liabilities, or to estimating their $\beta_l$. In this paper, we will simply adopt this equation and assume $\beta_l \leq 0$. $\beta_l \leq 0$ implies $r_l \leq r_f$.[2]

In equation (3.3) $p$ is the market value of the shorted assets. In the context of insurance, it is more reasonable to express the breakeven ending assets in terms of the nominal loss. (3.3) is rewritten as

$$\mathrm{MV}(L)(r_a - r_l) = \frac{E[L]}{1 + r_l}(r_a - r_l) = \frac{E[L]}{1 + r_l} m(\beta_a - \beta_l) \tag{3.4}$$

Since $\beta_l \leq 0$, the breakeven value (3.4) is positive. Thus merely greater-than-zero ending assets may not be sufficient to compensate for the claim and investment risks. If expenses $X = 0$, the ECR equals the loss ratio. The breakeven ECR for the CAPM model is $\mathrm{PV}(L)/p = E[L]/(1 + r_f)/p = (1 + r_l)/(1 + r_f)$, which is less than 100% if $r_l < r_f$. Clearly, the ECR of 100% does not "truly indicates the watershed between profit and loss" (cited from Swiss Re 2006, p.24). For a P&C

[2]Some life insurance products can be approximately replicated by an asset portfolio, thus have positive $\beta$ (Day 2004).

insurance policy, the true watershed is typically a value less than 100%, and is determined by risk.

## 4 A Multi-Year Profit Measure

Application of the CAPM gives us a one-period profit measure by comparing a policy's actual ending assets with the risk adjusted breakeven ending assets (3.4). A typical P&C policy has a multi-year payout pattern. A multi-year profit measure will be derived in this section. At the policy inception, claim payment cash flows are estimated, and the market value of claims is established by discounting the payment flows. Afterwards, the market value of unpaid claims at any point in time is computed by discounting the remaining payments. Discount rates for future market value calculation are functions of future interest rates, the market risk premium, and riskiness of remaining payments. It is thus usually different from the initial discount rate. For simplicity, I will assume the discount rates stay unchanged over the life of a policy.[3]

Assume a policy is written at time 0, and all losses incur by time 1. The loss payments are random variables $L_1, \ldots, L_n$, $L_i$ paid at time $i$. The nominal total loss is $L = L_1 + \ldots + L_n$. Assume there is a constant year-to-year loss discount rate $r_l$. Then the market value, at time 0, of the losses is

$$\mathrm{MV}_0(L) = \frac{E[L_1]}{1+r_l} + \ldots + \frac{E[L_n]}{(1+r_l)^n} = \sum_{i=1}^{n} \frac{E[L_i]}{(1+r_l)^i} \tag{4.1}$$

The *fair premium* for the policy, net of all expenses, equals the market value of the losses, $p = \mathrm{MV}_0(L)$. Let the premium and capital be invested risk free, and $r_f$ be the constant risk-free rate.[4] The following formulas give the expected net assets at each time $i$, after loss $L_i$ is paid

$$\begin{array}{rl}
p(1+r_f) - E[L_1] & \text{at time } 1 \\
p(1+r_f)^2 - E[L_1](1+r_f) - E[L_2] & \text{at time } 2 \\
\vdots & \quad\vdots \\
p(1+r_f)^n - E[L_1](1+r_f)^{n-1} - \ldots - E[L_n] & \text{at time } n
\end{array}$$

[3] For a multi-year model, it is more convenient to express assumptions and results with $r_l$ than with $\beta_l$. The latter is even difficult to define.

[4] Results on multi-year models in this paper are stated for the simpler case of risk-free investment returns. With little additional work, they can be generalized to cover random (risky) investment returns, as long as the returns in different time-periods are independent.

The ending assets of the policy are given by the last formula. Substituting equation (4.1) for $p$, we can rewrite the ending assets as

$$(1+r_f)^n \sum_{i=1}^{n} E[L_i] \left((1+r_l)^{-i} - (1+r_f)^{-i}\right) = (1+r_f)^n \left(\mathrm{MV}_0(L) - \mathrm{PV}_0(L)\right) \quad (4.2)$$

$\mathrm{MV}_0(L) - \mathrm{PV}_0(L)$ is just the market risk load, which is greater than 0 if $r_l < r_f$. A less risky policy—in the sense of systematic risk—has a smaller "spread" $r_f - r_l$, and a smaller value for breakeven ending assets. A very risky policy could have a negative $r_l$,[5] so a much larger breakeven value.

To compute the breakeven value (4.2) for the policy given in Table 1, we need a few more assumptions. Assume $r_l = 3\%$ per half year, and the only loss payment of \$650 at time 3 is both the expected and the actual loss. Since $r_f = 4\%$, the breakeven net assets at time 3 are $(1+0.04)^6 \cdot 650 \cdot \left((1+0.03)^{-6} - (1+0.04)^{-6}\right) = 38.80$. Since the actual ending assets are \$84.86, the policy is profitable under risk-adjustment.

Note that the the breakeven value (4.2) is constructed using the *expected value* of payments, investment rates and discount rates. These expected values are generally forecasted at the beginning of the policy term. They may be revised later as new information about the policy, markets and general economy comes in. But they should not be affected by normal fluctuations in policy losses and investment returns. Practically, the expected values are estimated with long term (multiple-year) averages or class (multiple-policy) averages. Any of the following experience could render a policy profitable: higher (than expected) premium, smaller losses, slower loss payments , or higher investment returns. In the numerical illustration, for simplicity, I assume that the actual loss payments, investment returns and discount rates are equal to their expected counterparts. But these two sets of numbers are usually different.

For the example, the net ending assets \$84.86 exceeds the breakeven value \$38.80 by \$46.06. However, this comparison overstates profits, because the \$1000 premium includes a provision for income taxes, but taxes are omitted in the ending assets calculation. (The combined ratio or the ECR also omits taxes.) Taxes are a significant cost, which I will treat in the following sections.

[5] A catastrophy policy routinely charges a premium far greater than the discounted expected loss, meaning $r_l$ is near $-1$.

# 5 After-Tax Profit Measures

Insurance companies have a greater tax burden than non-financial companies. In addition to taxes on profits from underwriting and premium investment, a company has to pay taxes on investment gains of capital. In the preceeding sections, we calculate the net ending assets without considering capital. But taxes on capital investments have great effects on net operating income. Income taxes are generally a fixed percentage of the pre-tax income. But the precise IRS tax codes are complex. To obtain closed-form, trackable formulas, we have to make simplifying assumptions.

## 5.1 Single-year model

Assume there is one tax rate, denoted by $t$, for both underwriting and investment profits. The pretax operating income is $(p-L)+(p+c)R_a = p(1+R_a)-L+cR_a$. The total income tax, paid at time 1, is $t(p(1+R_a)-L)+tcR_a$.[6] The policy's after-tax net assets are[7]

$$p(1+R_a)-L-t(p(1+R_a)-L)-tcR_a = (1-t)\left(p(1+R_a)-L-\frac{tc}{1-t}R_a\right) \quad (5.1)$$

The fair premium $p$ is such that makes the market value of the net assets zero. In other words, $p$ should equal the market value of $L$ plus that of the tax term $tc/(1-t)\cdot R_a$. Since $\mathrm{MV}(R_a) = r_f/(1+r_f)$,

$$p = \mathrm{MV}(L) + \frac{tcr_f}{(1-t)(1+r_f)} \quad (5.2)$$

The second term is the additional premium needed to cover taxes on investment income of capital. This is the frictional cost due to double taxation—the investment income on capital is taxed twice, at both the corporate level and the personal level. This frictional cost is in direct proportion to $c$. So too much capital hurts the company in price competition. Also note that the fair premium is affected by general economy, via $r_f$ and $r_l$, but it is not affected by how the premium and capital are invested. To derive the expected ending assets for the policy, we substitute (5.2) into the right-hand side of (5.1), and calculate expected values.

[6] In practice, when taxable income is negative, the company may not be able to receive full tax refund in the current year. But we will ignore this complication here.

[7] Note that the assets entirely come from policyholder supplied funds. The capital only produces a tax drag.

Together with (3.1) and (3.2), we get

$$\begin{aligned}
& (1-t)\left(\left(\mathrm{MV}(L)+\frac{tcr_f}{(1-t)(1+r_f)}\right)(1+E[R_a])-E[L]\right)-tcE[R_a] \\
= \;& (1-t)\mathrm{MV}(L)m(\beta_a-\beta_l)-\frac{tcm\beta_a}{1+r_f}=(1-t)\mathrm{MV}(L)(r_a-r_l)-\frac{tc}{1+r_f}(r_a-r_f) \\
= \;& (1-t)\frac{E[L]}{1+r_l}(r_a-r_l)-\frac{tc}{1+r_f}(r_a-r_f) \qquad (5.3)
\end{aligned}$$

This is the risk-adjusted breakeven value for the policy's after-tax ending assets. In (5.3) the first term is essentially the after-tax version of the breakeven value (3.4), and the second term reflects tax on capital. A policy generates a profit if and only if its after-tax ending assets are greater than (5.3).

Consider a special case that the investment is risk free, i.e., $\beta_a = 0$ and $r_a = r_f$. The formula (5.3) reduces to

$$(1-t)\frac{E[L]}{1+r_l}(r_f-r_l) \qquad (5.4)$$

Remember $r_l \leq r_f$, and the riskier the policy, the smaller the $r_l$. So a riskier policy has a greater breakeven value (5.4). Note that capital $c$ does not appear in (5.4). This is because the tax on capital investment is completely predictable, and is exactly covered by the second component of fair premium (5.2). If the investment is risky, the breakeven value (5.3) is affected by $c$.

## 5.2 Multi-year model

In the remainder of the paper, our main results are presented in the most realistic setting — multi-year, with tax. The timing of tax payments is important. Amounts paid out cannot be reinvested for future gain. I assume here taxes are paid at time 1, 2, .... (The length of time between $i-1$ and $i$ need not be one year. In our numerical example, each time period is a half year.) Detailed IRS tax rules can be found in Feldblum and Thandi (2003). In general, tax incurred equal tax rate times taxable income. The calculation of taxable income depends on loss reserve discounting. IRS tax discount rates are different from market value discount rates. In this section, however, I will adopt the following simplified tax rules

$$\begin{aligned}
\text{Tax Paid}_i &= t \times (\text{Underwriting Gain}_i + \text{Investment Gain}_i) \\
\text{Investment Gain}_i &= r_f \times \text{Investible Assets}_{i-1} \\
\text{Underwriting Gain}_1 &= p - L_1 - \text{Loss Reserve}_1 \\
\text{Underwriting Gain}_i &= -L_i - \text{Loss Reserve}_i + \text{Loss Reserve}_{i-1} \\
\text{Loss Reserve}_i &= \mathrm{MV}_i(\text{Unpaid Loss}_i)
\end{aligned}$$

These rules will be used to derive a breakeven value for the after-tax ending assets. The test on profitability is to compare actual ending assets, which follows the IRS tax rules, with their breakeven value. The difference between tax rules will create some inaccuracy—hopefully a small one.

Derivation of results for a multi-year model is inevitably complicated. I will state the results here, and present their proofs in an appendix after the paper. Let $c_0$, $c_1$, $\ldots$, $c_{n-1}$ be the actual capitals held at each time. $c_i$ equals the investable assets less the policy account assets at time $i$. $c_i$ might be an amount required by regulators (as assumed in Schirmacher and Feldblum 2006), or desired by the company management. Again we assume assets are invested risk free, and the risk-free rate $r_f$ is constant for all years.

The fair premium (net of expenses) $p$ makes the market value of the policy account zero at time 0. That is, $p$ equals the market value of losses plus that of all taxes. The following theorem gives a concise, closed-form formula for calculating the fair premium.

**Theorem 1** The fair premium $p$ is given by

$$p = \mathrm{MV}_0(L) + \frac{tr_f}{(1-t)(1+r_f)}\left(c_0 + \frac{c_1}{1+(1-t)r_f} + \ldots + \frac{c_{n-1}}{(1+(1-t)r_f)^{n-1}}\right) \tag{5.5}$$

Premium calculation will be discussed in depth in Section 8. To derive the breakeven value for the after-tax ending assets, we start from time 0 with premium (5.5), compute underwriting, investment, tax cash flows, and the net assets successively at each time $i$. We obtain a simple, closed-form formula for the breakeven ending assets.

**Theorem 2** The breakeven value for the after-tax ending assets in the policy account is given by

$$\begin{aligned} a_n &= \frac{(1-t)(r_f-r_l)(1+(1-t)r_f)^n}{(1-t)r_f-r_l}\sum_{i=1}^{n} E[L_i]\left(\frac{1}{(1+r_l)^i} - \frac{1}{(1+(1-t)r_f)^i}\right) \\ &= \frac{(1-t)(r_f-r_l)(1+(1-t)r_f)^n}{(1-t)r_f-r_l}(\mathrm{MV}_0(L) - \mathrm{PV}_0^{\mathrm{tax}}(L)) \end{aligned} \tag{5.6}$$

where $\mathrm{PV}_0^{\mathrm{tax}}(L)$ stands for the present value discounted with the after-tax interest rate $(1-t)r_f$.

Formula (5.6) does not involve taxes on capital investments ($c_i$'s do not appear in the formula). This is because the tax component in fair premium (5.5) exactly covers all those taxes. (As in the single-year model, if $r_a \neq r_f$, the breakeven ending assets will depend on the $c_i$'s.)

Formulas (5.5) and (5.6) are easy to apply. In Table 2 below, a tax column is added to Table 1, and the policy account assets at each time are recalculated by subtracting taxes. Note that the tax column contains all taxes, including that on capital investments. Given the rates $r_f = 4\%$, $r_l = 3\%$ and $t = 35\%$, we compute $\mathrm{MV}_0(L) = 650/(1+0.03)^6 = 544.36$, and $\mathrm{PV}_0^{\mathrm{tax}}(L) = 650/(1+(1-0.35)\times 0.04)^6 = 557.22$. Substituting the $\mathrm{MV}_0(L)$ into (5.5), yields $p = 569.08$. Loading in the present value of expenses, \$419.23, we get the fair policy premium of \$988.31. This breakeven value is less than the actual charge of \$1000. Thus the policy is a good deal at the beginning. Meanwhile, from (5.6), the breakeven ending assets $a_3 = 24.37$. The policy's after-tax ending assets are \$33.55 (Table 2, column 7), which are better than the breakeven by $33.55 - 24.37 = 9.18$. This is the value added by the policy.

**Table 2 here**

Theorem 2 provides an unambiguous retrospective test on whether a policy has generated profit. Calculation of the breakeven value does not need capital. Calculation of the actual value of the ending assets also mostly involves the policy account cash flows—investment incomes on premium, loss payments, and taxes. But held capital amounts affect the actual value by adding additional taxes. Usually a company would like to measure and compare profitability across lines or business units. For this purpose the capital at each time $i$ has to be allocated.

## 6 Linking Expected Ending Assets to Cost of Capital

The cost of capital is the rate of return on capital required by shareholders. Modern finance tells us the cost of capital is a direct function of the riskiness of the return. For the models in this paper, there are two risk sources that feed into the risk of the return—the investments and the claims. Schirmacher and Feldblum (2006) arbitrarily choose a cost of capital for their example. In practice, when the cost of capital is needed for pricing and performance measurement, it is also mostly judgmentally selected. Using the CAPM, we can link the cost of capital to the beta's of the investments and claims, which provides a theoretical base for its estimation.

## 6.1 Single-year model

In Section 5.1 we found a policy's fair premium and calculated the corresponding after-tax net assets at time 1. Adding the value of capital we have the net assets for the overall operation (insurance transactions, investment of premium, and investment of capital) at time 1

$$(1-t)\mathrm{MV}(L)(R_a - R_l) - \frac{tc}{1+r_f}(R_a - r_f) + c(1+R_a) \tag{6.1}$$

The first two terms of (6.1) are similar to (5.3), but are random instead of expected values. This formula gives us several useful results. First, the expected ending assets of the overall operation equals the expected ending assets of the policy account (5.3) plus the expected value of the capital investment

$$(1-t)\mathrm{MV}(L)(r_a - r_l) - \frac{tc}{1+r_f}(r_a - r_f) + c(1+r_a) \tag{6.2}$$

This is the risk-adjusted breakeven ending assets for the overall operation.

Also, (6.1) implies the following formulas for return on capital

$$\begin{aligned} & \frac{(1-t)\mathrm{MV}(L)}{c}(R_a - R_l) - \frac{t}{1+r_f}(R_a - r_f) + R_a \\ = & \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right) R_a - \frac{(1-t)\mathrm{MV}(L)}{c} R_l + \frac{t}{1+r_f} r_f \end{aligned} \tag{6.3}$$

The latter expression says that the shareholders' investment of $c$ can be replicated in the following way: short-selling a portfolio with return $R_l$ and market value $(1-t)\mathrm{MV}(L)$, lending an amount $tc/(1+r_f)$ at the risk free rate, and buying a portfolio with return $R_a$ with the net cash $c + (1-t)\mathrm{MV}(L) - tc/(1+r_f)$. The cost of capital (COC) is the expected value of this return

$$\begin{aligned} \mathrm{COC} &= \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right) r_a - \frac{(1-t)\mathrm{MV}(L)}{c} r_l + \frac{t}{1+r_f} r_f \\ &= r_a + \frac{(1-t)\mathrm{MV}(L)}{c}(r_a - r_l) - \frac{t}{1+r_f}(r_a - r_f) \end{aligned} \tag{6.4}$$

and the beta of the capital investment is a function of $\beta_a$ and $\beta_l$

$$\beta_c = \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right)\beta_a - \frac{(1-t)\mathrm{MV}(L)}{c}\beta_l \tag{6.5}$$

Although the COC is often called the shareholder required rate of return, it is actually determined by the firm's internal variables, as well as general economic and market conditions. The COC has three components, the investment rate of return $r_a$; the after-tax spread $(1-t)(r_a - r_l)$ times the "leverage ratio" $\mathrm{MV}(L)/c$; and a term related to taxes on capital investment. A greater COC may be caused by a greater investment risk, a greater claims risk, or a higher leverage ratio. Increasing the amount of capital reduces the COC.

### 6.2 Multi-year model

In a multi-year model, shareholders contribute an initial capital $c_0$ and establish a *capital account*. The capital account then earns investment income and pays out dividends (releases capital). At any time $i$, the total assets of the company equals the sum of assets in the policy account and those in the capital account.[8] We assume dividends entirely come out of the capital account at intermediate times $i < n$; the policy account only distributes its profit at time $n$.[9] The amount of dividents at time $i$ is so determined as to keep a preset balance $c_i$ in the capital account. Therefore, the capital flow to shareholders at time $i$ is simply $c_{i-1}$ plus the investment income in the year and minus $c_i$. Assume each $c_i$ is invested risk free, with a constant risk-free rate $r_f$. Then the capital flows are $-c_0$, $c_0(1+r_f)-c_1$, $c_1(1+r_f)-c_2$, $\ldots$, $c_{n-1}(1+r_f)$. Obviously, the internal rate of return (IRR) of these flows is $r_f$.

The breakeven ending assets $a_n$ for the policy account is given in (5.6). The breakeven ending assets for the overall operation is thus $c_{n-1}(1+r_f)+a_n$. This amount is returned to the shareholders after time $n$. The breakeven IRR of the total capital flows is given by the following equation

$$c_0 = \frac{c_0(1+r_f)-c_1}{1+\mathrm{IRR}} + \frac{c_1(1+r_f)-c_2}{(1+\mathrm{IRR})^2} + \ldots + \frac{c_{n-1}(1+r_f)+a_n}{(1+\mathrm{IRR})^n} \tag{6.6}$$

This IRR is the average—over $n$ years—cost of capital of the overall operation. After a policy runs its course, we can compute the IRR for the actual capital flows. If the IRR is great than (less than) the average COC given by (6.6), the overall operation is profitable (unprofitable).

In their example, Schirmacher and Feldblum (2006) assume the required capital is 25% of the unearned premium reserve plus 15% of the loss reserve. They then derive the required assets and the capital flows. Table 3 below shows the capital flows from the capital account (column 6), from the overall operation (column 7), and in the breakeven case. These three columns only differ in their last entry. The IRR for column 6 is the investment rate of 4%, as expected. The IRR for column 7 is 6.18%, obtained also in Schirmacher and Feldblum (2006). The IRR for column 8, 5.62%, is the COC. Since the overall IRR is greater than the COC, the insurance operation creates value for the shareholders.

[8] Assets in this paper correspond to the income-producing assets in Schirmacher and Feldblum (2006). Non-income-producing assets, like the DTA, are not considered.

[9] This distinction between the policy and the capital accounts does not affect profit measurement of the company as a whole. But it is important for measuring the policy account profit separately from the capital account.

**Table 3 here**

# 7 Decomposing the EVA

Shareholders invest in a company hoping to earn the cost of capital. If they earn more than (less than) the COC, then the insurance operation adds (destroys) value. The economic value added is defined as (see, e.g., Schirmacher and Feldblum 2006)

$$\text{EVA} = \text{After-tax Net Income} - \text{COC} \times \text{Capital Held}$$

The EVA measures the aggregate gain from all activities of the firm. Using the theories developed in the previous sections, we can separately evaluate the policy account and the capital account.

## 7.1 Single-year model

For the single-year model, the COC is given in formula (6.4), which can be split into two parts. The first term, $r_a$, is the hurdle rate for the capital account. If the actual return on capital is greater than $r_a$, then the investment operation adds value. By Section 5.1, the last two terms of (6.4) can be considered the hurdle rate of the policy account. This leads to the following definition of EVA for the two accounts separately

$$\begin{aligned} \text{EVA}_c &= c \times (\text{Actual Investment Rate} - r_a) && (7.1) \\ \text{EVA}_p &= \text{Actual After-tax Ending Assets} \\ & \qquad -(1-t)\text{MV}(L)(r_a - r_l) - \frac{tc}{1+r_f}(r_a - r_f) && (7.2) \end{aligned}$$

In (7.2) the actual assets in the policy account are after all taxes, including those on capital gains. Readers familiar with investment management may recognize that Actual Investment Rate $- r_a$ in (7.1) is Jensen's alpha for the asset portfolio. Obviously, EVA $=$ EVA$_c$ $+$ EVA$_p$. This decomposition of the EVA allows us to separately assess the policy and the capital accounts. EVA$_p$ provides more accurate information on underwriting than the total EVA.

An analytical method that evaluates various component activities independently is valuable for business planning or for compensation allocation. There have been previous efforts on finding such methods. For example, Bingham (2004) proposes to allocate capital between underwriting and asset investment, find a cost of capital for each function, and calculate value creation for the functions separately. In practice, some companies build models to calculate the underwriting ROE, the investment ROE, and ROEs at various policy group or investment portfolio levels. Our method of decomposing the EVA is rooted in modern finance. Both EVA$_c$ and EVA$_p$ are derived from the CAPM. EVA$_c$ is essentially Jensen's measure, and EVA$_p$, assessing the abnormal return of the policy account, can be considered an extension of Jensen's measure. On the other hand, difficulties exist if we try to further split the policy account into underwriting and investment activities, and measure them independetly. For example, an increase in premium is an achievement of the underwriting department. The resulting increase in profit should be credited entirely to underwriting, not to investment. But the additional premium generates more investment income, which cannot be cleanly attributed to either underwriting or investment. Further, the policy account covers taxes on investment income on capital. It is not clear whether these taxes should be covered by profits from underwriting or from premium investment. Note that in our method, performances of the capital account and the policy account are not completely independent. If capital investment generates a higher return, corresponding taxes increase; this reduces EVA$_p$. But EVA$_c$ is largely independent of policy account activities. (In a multi-year setting, profits in the policy account affect capital releases, which then affect the ultimate investment result.)

### 7.2 Multi-year model

In a multi-year model, the EVA is usually calculated annually. The EVA stream depends on the accounting system used. How loss reserves are set influence the recognition of income and the EVA. Schirmacher and Feldblum (2006) explain the EVA calculation in two accounting systems, the NPV and the IRR. I will not deal with accounting rules here, but only examine the economic gain at the end of the policy life. The main issue is how to separately quantify gains from the policy account and the capital account.

The IRR of the total capital flows, denoted by IRR$_{tot}$, is a standard profit measure of shareholders' investment. In the breakeven case, IRR$_{tot}$ equals the cost of capital. IRR$_{tot}$ of a profitable operation is greater than the COC. As explained in Section 6.2, the total capital flows consist of two component flows: the policy account generates a single flow at time $n$, which is the net ending assets; the capital account distributes dividends at each time $i$. Methods of evaluating the two component flows have essentially been derived in previous sections, which are summarized below.

For the policy account, we compare the actual net assets and their breakeven value at time $n$

$$\text{EVA}_p = \text{Actual After-tax Ending Assets} - a_n \tag{7.3}$$

where $a_n$ is given in (5.6). $\text{EVA}_p$ is the ultimate cash value added by the policy. In our example, $\text{EVA}_p = 33.55 - 24.37 = 9.18$. For the dividend flow out of the capital account, the expected IRR is $r_a$ (or $r_f$, if assets are invested risk free). If the investment and reinvestment operation of the capital account is ultimately profitable, then the IRR of the stream of dividends, denoted by $\text{IRR}_c$, is greater than $r_a$. Both $\text{EVA}_p$ and $\text{IRR}_c$ are independent of the accounting system.

Obviously, if $\text{EVA}_p > 0$ and $\text{IRR}_c > r_a$, then $\text{IRR}_{tot} > \text{COC}$; conversely, if $\text{EVA}_p < 0$ and $\text{IRR}_c < r_a$, then $\text{IRR}_{tot} < \text{COC}$. Using $\text{IRR}_{tot}$ alone we can assess the profitability of the overall, blended operation. But $\text{EVA}_p$ and $\text{IRR}_c$ give us further information on how the two accounts perform independently.

# 8 Comparing Direct and Indirect Pricing Methods

Formula (5.5) can be used for calculating the fair premium, if all inputs are available. Usually in a practical situation, just as in Schirmacher and Feldblum (2006), the capital amounts $c_i$ come from some selected reserve-to-surplus ratios. The risk-adjusted loss discount rate $r_l$ is a key input, but is poorly understood. In theory, the spread $r_f - r_l$ should be proportional to the riskiness of the policy. Unfortunately, little work has been done to quantify the risk of a typical insurance policy, and to find an exact link between the risk and $r_l$. With our selected $r_l = 3\%$, we calculated in Section 5.2 that $\text{MV}_0(L) = 544.36$, $p = 569.08$, and, after loading the present value of expenses of 419.23, the fair policy premium is \$988.31.

Premium formula (5.5) fits in the framework of Myers and Cohn (1987), formula (3.4). But the Myers-Cohn formula (as well as its later improvements in Taylor 1994, Cummins and Phillips 2001) is not in a closed form, since its two terms on taxes are functions of premium. Our closed-form formula is derived under the Section 5.2 assumptions on taxable underwriting gains. Since these assumptions are not identical to the IRS tax rules, formula (5.5) only covers taxes approximately.

Other parameters being fixed, the loss discount rate $r_l$ and the COC uniquely determine each other. The COC is derived from $r_l$ as follows: substitute $r_l$ into (4.1) to get $\text{MV}_0(L)$; compute the fair premium $p$ using (5.5) and the breakeven ending assets $a_n$ using (5.6); then solve the COC from (6.6). (Numerical methods are required to solve the polynomial equation, e.g., Goal Seek or Solver in Excel).

Conversely, the COC determines $r_l$ and $p$ via the following steps: plug the COC into (6.6) and compute $a_n$, then solve (5.6) (numerical methods) to get $r_l$, using which $p$ is calculated from (5.5).

To illustrate the second process, from a COC to $r_l$ and $p$, we assume, as in Schirmacher and Feldblum (2006), that the COC is 5%. Plugging this rate into the denominators of (6.6), and the $c_i$'s in column 3 of Table 3 into the numerators (or directly using column 6 of Table 3 for the numerators), we obtain $a_n = \$14.76$. Since there is only one loss payment of \$650 at time 3, $\mathrm{MV}_0(L) = 650/(1 + r_l)^6$, where $r_l$ is unknown. Plugging all known parameters and $\mathrm{PV}_0^{\mathrm{tax}}(L) = 557.22$ into equation (5.6) and solving for $r_l$, we have $r_l = 3.39\%$. Using this $r_l$ in (5.5) gives $p = 556.98$. Adding in the present value of expenses \$419.23 we obtain the total policy premium of \$976.21.

So we have developed two approaches to pricing a policy, one based on $r_l$ and the other on the COC. If these rates are compatibly selected, the two approaches produce identical premiums. For example, either $r_l = 3\%$ or COC $= 5.62\%$ leads to a premium of \$988.31, and either $r_l = 3.39\%$ or COC $= 5\%$ gives a lower premium of \$976.21. Pricing methods based on loss discounting (with risk adjustment) are generally called *direct* methods, and those using a target COC the *indirect* methods. When applying a direct method in practice, instead of looking for a proper $r_l$, actuaries usually compute a additive risk margin (also called profit margin or risk load).[10] Direct methods have obvious advantages over indirect ones. They are easier to compute; they explicitly calculate the components that cover claims, expenses and taxes, respectively. If a discount rate $r_l$ (or a risk margin) is found, it is a market variable and can be used by other companies. Despite these advantages, the indirect methods are more actively researched recently, and the ROE, IRR and COC measures are gaining popularity. At least part of the reason is that stock return data and asset pricing models are readily available, so that the COC can be estimated and tested (Cummins and Phillips 2005, Schmid and Wolf 2009). But it should be pointed out that the COC is a function of claims risk, asset risk, and the firm's capital level.[11] One firm's COC cannot be used by another firm with different risk profile and financial strength. This is inconvenient for both pricing practice and price regulation. Many pricing models are reviewed and compared in Cummins (1990), Taylor (1994), D'Arcy and Gorvett (1998).

[10]Traditionally a risk margin captures the total risk of the policy, not just its systematic component, thus is inconsistent with modern finance.

[11]In practice, the capital used for pricing a policy is an allocated firm-wide capital. It obviously depends on the other policies the firm carries.

The two methods studied here are expressed in tractable, closed-form formulas, are rigorously derived from assumptions rooted in modern finance, and have clear relations between them.

# 9 Conclusions

The main results of the paper are the two theorems in Section 5.2. They provide a solution to profit measurement and pricing in a multi-year setting. We propose to use the ending assets to test the ultimate profitability of a policy. One benefit of this approach is that profits from the policy account and the capital account can be measured separately. For fair premium calculation, we compare a direct method and an indirect method. The former uses the risk-adjusted loss discount rate and the later the cost of capital. There is a clear relationship between the two parameters via the breakeven ending assets.

To obtain a closed-form formula for the after-tax breakeven ending assets, we adopted the simplified tax rules listed in Section 5.2. Some testing may be done to determine the impact of this deviation from the real tax rules.

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# Appendix

## Proof of Theorem 1

Formula (5.5) can be written as

$$p = \sum_{i=1}^{n} \left( \mathrm{MV}_0(L_i) + \frac{t r_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}} \right) \tag{9.1}$$

We only need to prove the theorem for each $i$, that is, for a policy with a single loss payment $L_i$ at time $i$, and with a single nonzero capital $c_{i-1}$ at time $i-1$ (considered beginning of year $i$), the fair premium is given by

$$p = \mathrm{MV}_0(L_i) + \frac{t r_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}} \tag{9.2}$$

If 9.2 holds for all $i$, then 9.1 is true simply by additivity of premium.

If time $i$ is the only time losses and capital account taxes are paid, then at all other times $j$, $j = 1, \ldots i-1, i+1, \ldots n$, the only payments are taxes on policy account profits. These profits need to be carefully calculated according to the rules stated in Section 5.2.

Similar to $\mathrm{MV}_0(L_i)$, let $\mathrm{MV}_j(L_i)$ be the market value of $L_i$ at time $j < i$. The value of $\mathrm{MV}_j(L_i)$ will not be known until time $j$. Therefore, viewed at time 0, $\mathrm{MV}_j(L_i)$ is a random variable.[12]. To simplify notations, let $\mathrm{MV}_j(L_i)$ be denoted by $V_j$. The loss reserve at time $j < i$ is $V_j$, and loss reserves are zero after time $i$. In the following proof, I start by calculating the net assets at time 1, move forward in time, and end up with the ending net assets at time $n$. I then set the market value of the ending asset to zero, and solve for the fair premium $p$.

At time 1, the underwriting gain is $p - V_1$ and the investment gain is $pr_f$. Then the tax is $t(p(1 + r_f) - V_1)$. The net assets at time 1 are

$$A_1 = p(1 + r_f) - t(p(1 + r_f) - V_1) = p(1 + r_f)(1 - t) + tV_1$$

At time 2, the underwriting gain is $V_1 - V_2$ and the investment gain is $A_1 r_f$. Then the tax is $t(V_1 - V_2 + A_1 r_f)$. The net assets at time 2 are

$$\begin{aligned} A_2 &= A_1(1 + r_f) - t(V_1 - V_2 + A_1 r_f) \\ &= p(1 + r_f)(1 - t)(1 + (1 - t)r_f) + tV_2 + r_f t(1 - t)V_1 \end{aligned}$$

In general, the tax at any time $j < i$ is $t(V_{j-1} - V_j + A_{j-1} r_f)$, and it is not hard to prove by induction that the net assets at $j$ is given by the following formula

$$\begin{aligned} A_j &= p(1 + r_f)(1 - t)(1 + (1 - t)r_f)^{j-1} + tV_j + r_f t(1 - t)\Big(V_{j-1} \\ &\quad + (1 + (1 - t)r_f)V_{j-2} + \ldots + (1 + (1 - t)r_f)^{j-2} V_1\Big) \end{aligned} \tag{9.3}$$

Now this formula holds for $A_{i-1}$. At time $i$, there are two additional payments, loss $L_i$ and tax on capital investment $tr_f c_{i-1}$, and no further loss reserves. So the total tax is $t(V_{i-1} - L_i + A_{i-1} r_f) + tr_f c_{i-1}$, and the net assets are

$$\begin{aligned} A_i &= A_{i-1}(1 + r_f) - t(V_{i-1} - L_i + A_{i-1} r_f) - tr_f c_{i-1} \\ &= p(1 + r_f)(1 - t)(1 + (1 - t)r_f)^{i-1} - (1 - t)L_i + r_f t(1 - t)\Big(V_{i-1} \\ &\quad + (1 + (1 - t)r_f)V_{i-2} + \ldots + (1 + (1 - t)r_f)^{i-2} V_1\Big) - r_f t c_{i-1} \end{aligned} \tag{9.4}$$

After time $i$, every year the assets $A_i$ are reinvested and taxes on the investment gains paid. The after tax investment rate of return is $(1 - t)r_f$. So, the ending assets at time $n$ are

$$A_n = A_i(1 + (1 - t)r_f)^{n-i} \tag{9.5}$$

The fair premium $p$ is defined as such that makes the market value of $A_n$ zero. By (9.5), $\mathrm{MV}_0(A_n) = 0$ if and only if $\mathrm{MV}_0(A_i) = 0$. So we need to calculate the market value of each term in (9.4).

[12] Rigorously, the market values $\mathrm{MV}_0(L_i)$, $\mathrm{MV}_1(L_i)$, $\ldots$, $\mathrm{MV}_{i-1}(L_i)$, $L_i$, are a stochastic process adapted to a filtration indexed by time $j$.

The first and the last term in (9.4) are nonrandom constants. The market value, at time 0, of a constant is the constant divided by $(1+r_f)^i$. The market values of other terms in (9.4) are obtained using the following formula.

$$\mathrm{MV}_0(V_j) = \mathrm{MV}_0(L_i)/(1+r_f)^{i-j} \tag{9.6}$$

I will use this formula now to complete the proof. The formula itself will be proved at the end of the section.

$$\begin{aligned}
& \mathrm{MV}_0(A_i) \\
= \ & \mathrm{MV}_0(p(1+r_f)(1-t)(1+(1-t)r_f)^{i-1}) - (1-t)\mathrm{MV}_0(L_i) + r_f t(1-t)\Big(\mathrm{MV}_0(V_{i-1}) \\
& +(1+(1-t)r_f)\mathrm{MV}_0(V_{i-2}) + \ldots + (1+(1-t)r_f)^{i-2}\mathrm{MV}_0(V_1)\Big) - \mathrm{MV}_0(r_f t c_{i-1}) \\
= \ & \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} - (1-t)\mathrm{MV}_0(L_i) \\
& + \frac{r_f t(1-t)}{1+r_f}\mathrm{MV}_0(L_i)\left(1 + \frac{1+(1-t)r_f}{1+r_f} + \ldots + \frac{(1+(1-t)r_f)^{i-2}}{(1+r_f)^{i-2}}\right) - \frac{r_f t c_{i-1}}{(1+r_f)^i} \\
= \ & \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} - (1-t)\mathrm{MV}_0(L_i) \\
& + \frac{r_f t(1-t)}{1+r_f}\mathrm{MV}_0(L_i)\left(1 - \frac{(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}}\right) \div \left(1 - \frac{1+(1-t)r_f}{1+r_f}\right) - \frac{r_f t c_{i-1}}{(1+r_f)^i} \\
= \ & \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} - \frac{(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}}\mathrm{MV}_0(L_i) - \frac{r_f t c_{i-1}}{(1+r_f)^i}
\end{aligned}$$

Setting $\mathrm{MV}_0(A_i) = 0$ and solving for $p$, we get the formula (9.2). This proves Theorem 1.

Note that in the derivation of (5.5) we do not need the assumption that there is a constant risk-adjusted discount rate $r_l$. Therefore, (5.5) can be used to calculate the fair premium whenever the market value $\mathrm{MV}_0(L)$ can be reasonably estimated.

## Proof of formula (9.6)

In formula (9.4), $A_i$ is a random variable conditioned on all information up to time $i$. This conditioning statement is important when computing the market value of $V_j$. $V_j = \mathrm{MV}_j(L_i)$ is a random variable viewed at any point $j' < j$, but is nonrandom at any $j' > j$. So it is easy to first discount $V_j$ to time $j$,

$$\mathrm{MV}_j(V_j) = V_j/(1+r_f)^{i-j}$$

Then, further discounting the above to time 0, we get

$$\mathrm{MV}_0(V_j) = \mathrm{MV}_0(\mathrm{MV}_j(L_i))/(1+r_f)^{i-j} = \mathrm{MV}_0(L_i)/(1+r_f)^{i-j}$$

which proves (9.6).[13]

## Proof of Theorem 2

Theorem 2 says if a policy charges premium (5.5), then its expected ending assets at time $n$ have the form (5.6). I will again prove the theorem by splitting it into $n$ simpler components. For any $i < n$, assume a sub-policy $i$ has premium (9.2), makes only one loss payment $L_i$ at time $i$, and is supported by one nonzero capital $c_{i-1}$ at time $i-1$. I will prove that the expected ending assets of the sub-policy, at time $n$, are given by the following formula

$$a_{n,i} = \frac{(1-t)(r_f - r_l)(1+(1-t)r_f)^n}{(1-t)r_f - r_l} E[L_i] \left( \frac{1}{(1+r_l)^i} - \frac{1}{(1+(1-t)r_f)^i} \right) \quad (9.7)$$

Obviously, the expected ending assets of the original policy is the sum of these $a_{n,i}$'s. This will prove Theorem 2.

Substituting (9.2) into the righthand side of (9.4), we have

$$\begin{aligned} A_i &= \left( V_0 + \frac{t r_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}} \right) \\ &\quad \times (1+r_f)(1-t)(1+(1-t)r_f)^{i-1} - (1-t)L_i \\ &\quad + r_f t(1-t)\Big( V_{i-1} + (1+(1-t)r_f)V_{i-2} + \ldots + (1+(1-t)r_f)^{i-2}V_1 \Big) - r_f t c_{i-1} \end{aligned}$$

The assumption that there is a constant loss discount rate $r_l$ implies that $E[V_j] =$

[13] A rigorous proof of the formula may be stated with stochastic discount factors. The technique is standard in asset pricing theory.

$E[L_i]/(1+r_l)^{i-j}$. Noting that the $c_{i-1}$ terms cancel out, we have

$$\begin{aligned}
E[A_i] &= (1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\frac{E[L_i]}{(1+r_l)^i} - (1-t)E[L_i] \\
&\quad + \frac{r_f t(1-t)}{1+r_l}E[L_i]\left(1+\frac{1+(1-t)r_f}{1+r_l}+\ldots+\frac{(1+(1-t)r_f)^{i-2}}{(1+r_l)^{i-2}}\right) \\
&= (1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\frac{E[L_i]}{(1+r_l)^i} - (1-t)E[L_i] \\
&\quad + \frac{r_f t(1-t)}{1+r_l}E[L_i]\left(1-\frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}}\right) \div \left(1-\frac{1+(1-t)r_f}{1+r_l}\right) \\
&= (1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\frac{E[L_i]}{(1+r_l)^i} - (1-t)E[L_i] \\
&\quad + \frac{r_f t(1-t)}{r_l-(1-t)r_f}E[L_i]\left(1-\frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}}\right) \\
&= \left((1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\frac{E[L_i]}{(1+r_l)^i} - \frac{r_f t(1-t)}{r_l-(1-t)r_f}E[L_i]\frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}}\right) \\
&\quad + \left(-(1-t)E[L_i] + \frac{r_f t(1-t)}{r_l-(1-t)r_f}E[L_i]\right) \\
&= \frac{(1-t)(r_f-r_l)(1+(1-t)r_f)^i}{(1-t)r_f-r_l}\frac{E[L_i]}{(1+r_l)^i} - \frac{(1-t)(r_f-r_l)}{(1-t)r_f-r_l}E[L_i]
\end{aligned}$$

$A_i$ is the net assets at time $i$. After time $i$, the assets grow at the after-tax investment yield $(1-t)r_f$. So the ending net assets at time $n$ is $A_{n,i} = A_i(1+(1-t)r_f)^{n-i}$. Therefore,

$$\begin{aligned}
a_{n,i} &= E[A_{n,i}] = E[A_i](1+(1-t)r_f)^{n-i} \\
&= \frac{(1-t)(r_f-r_l)(1+(1-t)r_f)^n}{(1-t)r_f-r_l}E[L_i]\left(\frac{1}{(1+r_l)^i} - \frac{1}{(1+(1-t)r_f)^i}\right)
\end{aligned}$$

This proves (9.7), thus completes the proof of Theorem 2.

Table 1: Policy Account Assets - No Tax

| Time (1) | Premium (2) | Expense (3) | Loss (4) | Investment Income (5) | Policy Account Assets (6) |
|---|---|---|---|---|---|
| 0.0 | 1000.00 | 275.00 | 0.00 | 0.00 | 725.00 |
| 0.5 | 0.00 | 150.00 | 0.00 | 29.00 | 604.00 |
| 1.0 | 0.00 | 0.00 | 0.00 | 24.16 | 628.16 |
| 1.5 | 0.00 | 0.00 | 0.00 | 25.13 | 653.29 |
| 2.0 | 0.00 | 0.00 | 0.00 | 26.13 | 679.42 |
| 2.5 | 0.00 | 0.00 | 0.00 | 27.18 | 706.59 |
| 3.0 | 0.00 | 0.00 | 650.00 | 28.26 | 84.86 |
| Sum | 1000.00 | 425.00 | 650.00 | 159.86 | |
| PV | 1000.00 | 419.23 | 513.70 | | |

$(6)_{0.0} = (2)_{0.0} - (3)_{0.0} - (4)_{0.0}$
For $i > 0.0$, $(5)_i = (6)_{i-0.5} \times r_f$
For $i > 0.0$, $(6)_i = (6)_{i-0.5} + (2)_i - (3)_i - (4)_i + (5)_i$

Table 2: Policy Account Assets - After Tax

| Time (1) | Premium (2) | Expense (3) | Loss (4) | Tax (5) | Investment Income (6) | Policy Account Assets (7) |
|---|---|---|---|---|---|---|
| 0.0 | 1000.00 | 275.00 | 0.00 | -26.25 | 0.00 | 751.25 |
| 0.5 | 0.00 | 150.00 | 0.00 | 32.45 | 30.05 | 598.86 |
| 1.0 | 0.00 | 0.00 | 0.00 | 29.39 | 23.95 | 593.42 |
| 1.5 | 0.00 | 0.00 | 0.00 | 8.13 | 23.74 | 609.03 |
| 2.0 | 0.00 | 0.00 | 0.00 | 7.97 | 24.36 | 625.43 |
| 2.5 | 0.00 | 0.00 | 0.00 | -3.57 | 25.02 | 654.01 |
| 3.0 | 0.00 | 0.00 | 650.00 | -3.38 | 26.16 | 33.55 |
| Sum | 1000.00 | 425.00 | 650.00 | 44.73 | 153.28 | |
| PV | 1000.00 | 419.23 | 513.70 | 40.55 | | |

Column (5) from Table 2 in Schirmacher and Feldblum (2006).

$(7)_{0.0} = (2)_{0.0} - (3)_{0.0} - (4)_{0.0} - (5)_{0.0}$

For $i > 0.0$, $(6)_i = (7)_{i-0.5} \times r_f$

For $i > 0.0$, $(7)_i = (7)_{i-0.5} + (2)_i - (3)_i - (4)_i - (5)_i + (6)_i$

Table 3: Capital Flows

| Time | Policy Account Assets | Capital Held | Total Assets | Investment Income on Capital | Capital Account Capital Flow | Total Capital Flow | Breakeven Capital Flow |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 0.0 | 751.25 | 428.75 | 1180.00 | 0.00 | -428.75 | -428.75 | -428.75 |
| 0.5 | 598.86 | 362.62 | 961.48 | 17.15 | 83.28 | 83.28 | 83.28 |
| 1.0 | 593.42 | 149.53 | 742.95 | 14.50 | 227.60 | 227.60 | 227.60 |
| 1.5 | 609.03 | 122.54 | 731.58 | 5.98 | 32.97 | 32.97 | 32.97 |
| 2.0 | 625.43 | 94.77 | 720.20 | 4.90 | 32.67 | 32.67 | 32.67 |
| 2.5 | 654.01 | 79.84 | 733.85 | 3.79 | 18.73 | 18.73 | 18.73 |
| 3.0 | 33.55 | 0.00 | 33.55 | 3.19 | 83.03 | 116.58 | 107.40 |
| IRR | | | | | 4.00% | 6.18% | 5.62% |

Column (2) is column (7) in Table 2.

Column (4) from Table 7 in Schirmacher and Feldblum (2006).

For $i > 0.0$, $(5)_i = (3)_{i-0.5} \times r_f$

For $i > 0.0$, $(6)_i = (2)_{i-0.5} + (5)_i - (2)_i$

For $i < 3.0$, $(7)_i = (6)_i$; $(7)_{3.0} = (6)_{3.0} + (7)_{3.0}$ in Table 2

For $i < 3.0$, $(8)_i = (6)_i$; $(8)_{3.0} = (6)_{3.0} + a_{3.0}$ ($a_{3.0}$ calculated in Section 5.2)