

AN APPLICATION OF GAME THEORY: PROPERTY CATASTROPHE RISK LOAD

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Abstract

Two well known methods for calculating risk load—Marginal Surplus and Marginal Variance—are applied to output from catastrophe modeling software. Risk loads for these marginal methods are calculated for sample new and renewal accounts. Differences between new and renewal pricing are examined. For new situations, both current methods allocate the full marginal impact of the addition of a new account to that new account. For renewal situations, a new concept is introduced which we call “renewal additivity.”

Neither marginal method is renewal additive. A new method is introduced, inspired by game theory, which splits the mutual covariance between any two accounts evenly between those accounts. The new method is extended and generalized to a proportional sharing of mutual covariance between any two accounts. Both new approaches are tested in new and renewal situations.

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1. INTRODUCTION

The calculation of risk load continues to be a topic of interest in the actuarial community—see Bault [1] for a recent survey of well known alternatives. One area of great need, where the

CAS literature is somewhat scarce, is calculation of risk loads for property catastrophe insurance.

Many of the new catastrophe modeling products produce occurrence size-of-loss distributions for a series of simulated events. These output files might contain an event identifier, event probability, and modeled loss amount for that event for the selected portfolio of exposures. Given such output files for a portfolio before and after the addition of a new account, one could calculate the before and after portfolio variance and standard deviation. The difference will be called the *marginal impact* of that new account on the portfolio variance or standard deviation.¹

Two of the more well known risk load methods from the CAS literature—what shall be called “Marginal Surplus” (MS) from Kreps [3] and “Marginal Variance” (MV) from Meyers [6]—use the marginal change in portfolio standard deviation (variance) due to the addition of a new account to calculate the risk load for that new account. However, problems arise when these marginal methods are used to calculate risk loads for the renewal of accounts in a portfolio. These problems can be traced to the *order dependency* of the marginal risk load methods.

Order dependency is a perplexing issue. Many marginal risk load methods—whether based on variance, standard deviation, or even a selected percentile of the loss distribution—suffer from it. It is also not just an actuarial issue; even the financial community struggles with it. “Value at Risk” (VAR) is an attempt by investment firms to capture their risk in a single number. It is a selected percentile of the return distribution (e.g., 95th) for a portfolio of financial instruments over a selected time frame (e.g., 30 days). VAR can be calculated for the entire portfolio or for a desired subset (e.g., asset class). But so-called “marginal”

¹The variance and standard deviation are “between account” and “between event,” and ignore any parameter uncertainty associated with the modeled loss amount for a given event and account.

or “component” VAR has, to this point, eluded satisfactory solution in the finance community precisely because of what will be termed *renewal additivity*. Finance professionals charged with assessing how much VAR a certain financial instrument or asset class contributes to the total VAR are dealing with the same unresolved order dependency issue. As the finance and insurance worlds blend more and more, perhaps actuaries will combine forces with quantitative analysts and Certified Financial Analysts (CFAs) to determine a solution.

The remainder of this paper is organized as follows. Section 2 describes the basic characteristics of a catastrophe occurrence size-of-loss distribution. Sections 3 and 4 describe the application of the MV and MS methods to a simplified occurrence size-of-loss distribution. Sections 5 and 6 calculate risk loads both in assembling or building up a portfolio of risks and in subsequently renewing that portfolio. Section 7 discusses the differences between build-up and renewal results.

Section 8 introduces a new concept to the theory of property catastrophe risk loads—renewal additivity. However, the concept is not new to the field of game theory. Section 9 introduces game theory concepts underlying a new approach. Section 10 extends and generalizes the effect of the new approach to sharing of covariance between accounts. Section 11 concludes by applying the new approach to some examples.

2. THE CATASTROPHE OCCURRENCE SIZE-OF-LOSS DISTRIBUTION

For demonstration purposes throughout the paper, a simplified version of an occurrence size-of-loss distribution will be used. It captures the essence of typical catastrophe modeling software output, while keeping the examples understandable.²

²In particular, only single event or occurrence size-of-loss distributions will be considered. Many models also produce multi-event or aggregate loss distributions. Occurrence

A modeled event denoted by identifier i is considered an independent Poisson process with occurrence rate³ λ_i . To simplify the mathematics, following Meyers [6], the binomial approximation with probability of occurrence p_i [where $\lambda_i = -\ln(1 - p_i)$] will be employed. This is a satisfactory approximation for small⁴ λ_i .

For an individual account or portfolio of accounts, the model produces a modeled loss for each event L_i . A table containing the event identifiers i , the event probabilities p_i and modeled losses L_i will be referred to as an “occurrence size-of-loss distribution.”

From Meyers [6], the formulas for expected loss and variance are

$$E(L) = \sum_i [L_i \times p_i], \quad \text{and} \quad (2.1)$$

$$\text{Var}(L) = \sum_i [L_i^2 \times p_i \times (1 - p_i)], \quad (2.2)$$

where $\sum_i =$ sum over all events.

The formula for covariance of an existing portfolio L (with losses L_i) and a new account n (with losses n_i) is

$$\text{Cov}(L, n) = \sum_i [L_i \times n_i \times p_i \times (1 - p_i)]. \quad (2.3)$$

Note that $\text{Cov}(L, n)$ is always greater than zero when each of L_i , n_i , p_i , and $(1 - p_i)$ are greater than zero.

The total variance of the combined portfolio ($L + n$) is then

$$\text{Var}(L + n) = \text{Var}(L) + \text{Var}(n) + 2\text{Cov}(L, n). \quad (2.4)$$

size-of-loss distributions reflect only the *largest* event that occurs in a given year. Aggregate loss distributions reflect the sum of losses for all events in a given year. Clearly, the aggregate distribution would provide a more complete picture, but for purposes of the exposition here, the occurrence distribution works well and the formulas are substantially less complex.

³This implies that the loss for a given event and account is fixed and known.

⁴An event with a probability of 0.001 (typical of the more severe modeled events) would have $\lambda = 0.0010005$.

3. THE MARGINAL SURPLUS (MS) METHOD

This is a translation of the method described in Rodney Kreps's paper, "Reinsurer Risk Loads from Marginal Surplus Requirements" [3] to property catastrophe calculations.

Consider:

L_0 = losses from a portfolio before a new account is added,

L_1 = losses from a portfolio after a new account is added,

S_0 = Standard deviation of L_0 ,

S_1 = Standard deviation of L_1 ,

R_0 = Return on the portfolio before new account is added, and

R_1 = Return on the portfolio after new account is added.

Borrowing from Mr. Kreps, assume that needed surplus, V , is given by⁵

$$V = z \times \text{standard deviation of loss} - \text{expected return}, \quad (3.1)$$

where z is, to cite Mr. Kreps [3, p. 197], "a distribution percentage point corresponding to the acceptable probability that the actual result will require even more surplus than allocated."

Then

$$\begin{aligned} V_0 &= z \times S_0 - R_0, & \text{and} \\ V_1 &= z \times S_1 - R_1. \end{aligned} \quad (3.2)$$

The difference in returns $R_1 - R_0 = r$, the risk load charged to the new account. The marginal surplus requirement is then

$$V_1 - V_0 = z \times [S_1 - S_0] - r. \quad (3.3)$$

Based on the required return, y , on that marginal surplus (which is based on management goals, market forces and risk appetite),

⁵Mr. Kreps sets needed surplus equal to $z \times \text{standard deviation of return} - \text{expected return}$. If one assumes premiums and expenses are invariant, then $\text{Var}(\text{Return}) = \text{Var}(P - E - L) = \text{Var}(L)$.

the MS risk load would be

$$r = [yz/(1 + y)][S_1 - S_0]. \quad (3.4)$$

4. THE MARGINAL VARIANCE (MV) METHOD

The Marginal Variance Method is based on Glenn Meyers's paper, "The Competitive Market Equilibrium Risk Load Formula for Catastrophe Ratemaking" [6].

For an existing portfolio L and a new account n , the MV risk load r would be

$$\begin{aligned} r &= \lambda \times \text{Marginal Variance of adding } n \text{ to } L \\ &= \lambda \times [\text{Var}(n) + 2\text{Cov}(L, n)], \end{aligned} \quad (4.1)$$

where λ is a multiplier similar to $yz/(1 + y)$ from the MS method although dimensioned to apply to variance rather than standard deviation.⁶

5. BUILDING UP A PORTFOLIO OF TWO ACCOUNTS

Exhibit 1 shows the occurrence size-of-loss distribution and risk load calculations for building up (assembling) a portfolio of two accounts, X and Y . It is assumed X is written first and is the only risk in the portfolio until Y is written.

5.1. MS Method

Pertinent values from Exhibit 1 for the Marginal Surplus method are summarized in Table 1.

Item 1 is the change in portfolio standard deviation from adding each account, or *marginal* standard deviation.

⁶Mr. Meyers develops a variance-based risk load multiplier by converting a standard deviation-based multiplier using the following formula [6, p. 573]: $\lambda = (\text{Rate of Return} \times \text{Std Dev Mult}^2) / (2 \times \text{Avg Capital of Competitors})$.

TABLE 1
BUILDING UP X AND Y : MARGINAL SURPLUS METHOD

	Account X	Account Y	Account X +Account Y	Portfolio ($X + Y$)
(1) Change in Standard Deviation	\$4,429.00	\$356.00	\$4,785.00	\$4,785.00
(2) Risk Load Multiplier	0.33	0.33	—	0.33
(3) Risk Load = (1) \times (2)	\$1,461.71	\$117.43	\$1,579.14	\$1,579.14

Item 2 is the Risk Load multiplier of 0.33. Using Kreps's formula, a return on marginal surplus y of 20% and a standard normal multiplier z of 2.0 (2 standard deviations, corresponding to a cumulative non-exceedance probability of 97.725%) would produce a risk load multiplier of

$$yz/(1 + y) = 0.20 \times 2/1.20 = 0.33 \text{ (rounded)}. \quad (5.1)$$

Item 3 is the Risk Load, the product of Items 1 and 2.

Since X is the first account, the marginal standard deviation from adding X equals the standard deviation of X , $\text{Std Dev}(X) = \$4,429$. This gives a risk load of \$1,461.71.

The marginal standard deviation from writing Y equals $\text{Std Dev}(X + Y) - \text{Std Dev}(X)$ or \$356, implying a risk load of \$117.43.

The sum of these two risk loads $X + Y$ is $\$1,461.71 + \$117.43 = \$1,579.14$. This equals the risk load that this method would calculate for the combined account ($X + Y$).

5.2. *MV Method*

Pertinent values from Exhibit 1 for the Marginal Variance method are summarized in Table 2.

TABLE 2
BUILDING UP X AND Y : MARGINAL VARIANCE METHOD

	Account X	Account Y	Account X +Account Y	Portfolio ($X + Y$)
(1) Change in Variance	19,619,900	3,279,059	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	—	0.000069
(3) Risk Load = (1) \times (2)	\$1,353.02	\$226.13	\$1,579.14	\$1,579.14

Item 1 is the change in portfolio variance from adding each account, or *marginal* variance.

Item 2 is the Variance Risk Load multiplier λ of 0.000069. To simplify comparisons between the two methods (recognizing the difficulty of selecting a MV-based multiplier⁷), the MS multiplier was converted to a MV basis by dividing by Std Dev ($X + Y$):

$$\lambda = 0.33/4,785 = 0.000069. \quad (5.2)$$

This means the total risk load calculated for the portfolio by the two methods will be the same, although the individual risk loads for X and Y will differ between the methods.

Item 3 is the Risk Load, the product of Items 1 and 2.

Since X is the first account, the marginal variance from adding X equals the variance of X , $\text{Var}(X) = \$19,619,900$. This gives a risk load of \$1,353.02.

The marginal variance from writing Y equals $\text{Var}(X + Y) - \text{Var}(X)$, or \$3,279,059, implying a risk load of \$226.13.

The sum of these two risk loads is $\$1,353.02 + \$226.13 = \$1,579.14$. This equals the risk load which this method would calculate for the combined account ($X + Y$).

⁷Mr. Meyers [6, p. 572] admits that in practice “it might be difficult for an insurer to obtain the (lambdas) of each of its competitors.” He goes on to suggest an approximate method to arrive at a usable lambda based on required capital being “Z standard deviations of the total loss distribution” [6, p. 574].

TABLE 3
RENEWING X AND Y : MARGINAL SURPLUS METHOD

	Account X	Account Y	Account X +Account Y	Portfolio ($X + Y$)
(1) Change in Standard Deviation	\$4,171.00	\$356.00	\$4,526.00	\$4,785.00
(2) Risk Load Multiplier	0.33	0.33	—	0.33
(3) Risk Load = (1) \times (2)	\$1,376.27	\$117.43	\$1,493.70	\$1,579.14
(4) Build-Up Risk Load	\$1,461.71	\$117.43	\$1,579.14	\$1,579.14
(5) Difference	(\$85.45)	\$0.00	(\$85.45)	\$0.00

6. RENEWING THE PORTFOLIO OF TWO ACCOUNTS

Exhibit 2 shows the natural extension of the build-up scenario—renewal of the two accounts, in what could be termed a “static” or “steady state” portfolio (one with no new entrants).

As for applying these methods in the renewal scenario, renewing policy X is assumed equivalent to adding X to a portfolio of Y ; renewing Y is assumed equivalent to adding Y to a portfolio of X .

6.1. *MS Method*

Pertinent values from Exhibit 2 for the Marginal Surplus method are summarized in Table 3.

The marginal standard deviation for adding Y to X is \$356.00, same as it was during build-up—see Section 5.1. The risk load of \$117.43 is also the same.

However, adding X to Y gives a marginal standard deviation of $\text{Std Dev}(X + Y) - \text{Std Dev}(Y) = \$4,171.00$. This gives a risk load for X of \$1,376.27, which is \$85.45 less than \$1,461.71, the risk load for X calculated in Section 5.1.

TABLE 4
RENEWING X AND Y : MARGINAL VARIANCE METHOD

	Account X	Account Y	Account X +Account Y	Portfolio ($X + Y$)
(1) Change in Variance	22,521,000	3,279,059	25,800,059	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	—	0.000069
(3) Risk Load = (1) \times (2)	\$1,553.08	\$226.13	\$1,779.21	\$1,579.14
(4) Build-Up Risk Load	\$1,353.02	\$226.13	\$1,579.14	\$1,579.14
(5) Difference	\$200.06	\$0.00	\$200.06	\$0.00

The sum of these two risk loads in Table 3 is $\$1,376.27 + \$117.43 = \$1,493.70$. This is also $\$85.45$ less than the total risk load from Section 5.1.

6.2. *MV Method*

Pertinent values from Exhibit 2 for the Marginal Variance method are summarized in Table 4.

The marginal variance for adding Y to X is 3,279,059, same as it was during build-up—see Section 5.2. The risk load of $\$226.13$ is also the same.

However, adding X to Y gives a marginal variance of $\text{Var}(X + Y) - \text{Var}(Y)$, or 22,521,000. The risk load is now $\$1,553.08$, which is $\$200.06$ more than the $\$1,353.02$ calculated in Section 5.2.

The sum of these two risk loads is $\$1,553.08 + \$226.13 = \$1,779.21$. This is also $\$200.06$ more than the total risk load from Section 5.2.

7. EXPLORING THE DIFFERENCES BETWEEN NEW AND RENEWAL

Why are the total Renewal risk loads different from the total Build-Up risk loads?

In Section 5.1 (Build-Up), the marginal standard deviation for X , $\Delta \text{Std Dev}(X)$, was

$$\begin{aligned}\Delta \text{Std Dev}(X) &= \text{Std Dev}(X) \\ &= \sqrt{\sum_i [X_i^2 \times p_i \times (1 - p_i)]},\end{aligned}\quad (7.1)$$

where X_i = modeled losses for X for event i , while in Section 6.1 (Renewal), the marginal standard deviation was

$$\begin{aligned}\Delta \text{Std Dev}(X) &= \text{Std Dev}(X + Y) - \text{Std Dev}(Y) \\ &= \sqrt{\sum_i [(X_i + Y_i)^2 \times p_i \times (1 - p_i)]} \\ &\quad - \sqrt{\sum_i [Y_i^2 \times p_i \times (1 - p_i)]}.\end{aligned}\quad (7.2)$$

For positive Y_i , this value is less than $\text{Std Dev}(X)$. Therefore, one *would* expect the Renewal risk load to be less than the Build-Up.⁸

Unfortunately, when the MS method is applied in the renewal of all the accounts in a portfolio, the sum of the individual risk loads will be less than the total portfolio standard deviation times the multiplier. This is because the sum of the marginal standard deviations (found by taking the difference in portfolio standard deviation with and without each account in the portfolio) is less than the total portfolio standard deviation.⁹ Please recall that the square root operator is *sub-additive*: the square root of a sum is less than the sum of the square roots.¹⁰

⁸For example, assume $\text{Var}(X) = 9$, $\text{Var}(Y) = 4$, $\text{Cov}(X, Y) = 1.5$; then

$$\begin{aligned}\Delta \text{Std Dev}(X) &= \sqrt{\text{Var}(X)} = \sqrt{9} = 3, \quad \text{for } X \text{ alone,} \\ \Delta \text{Std Dev}(X) &= \sqrt{9 + 4 + 2 \times 1.5} - \sqrt{4} = 4 - 2 = 2 < 3, \quad \text{for } X \text{ added to } Y.\end{aligned}$$

⁹The same issue is raised in Mr. Gogol's discussion. He suggests correcting for this sub-additivity by using a weighted average of the contract's own standard deviation and its last-in marginal standard deviation. The weight is chosen so the sum of these redefined marginal impacts equals the total portfolio standard deviation [2, p. 363].

¹⁰For example, $\sqrt{9 + 16} < \sqrt{9} + \sqrt{16}$.

What about marginal variance? In Section 5.2 (Build-Up), the marginal variance $\Delta \text{Var}(X)$ was

$$\begin{aligned}\Delta \text{Var}(X) &= \text{Var}(X) \\ &= \sum_i [X_i^2 \times p_i \times (1 - p_i)],\end{aligned}\quad (7.3)$$

while in Section 6.2 (Renewal) the marginal variance was

$$\begin{aligned}\Delta \text{Var}(X) &= \text{Var}(X + Y) - \text{Var}(Y) \\ &= [\text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y)] - \text{Var}(Y) \\ &= \text{Var}(X) + 2\text{Cov}(X, Y) \\ &> \text{Var}(X).\end{aligned}\quad (7.4)$$

Since $2\text{Cov}(X, Y)$ is greater than zero, one would expect the Renewal risk load to be greater than the Build-Up.

However, when the MV method is applied in the renewal of all the accounts in a portfolio, the sum of the individual risk loads will be more than the total portfolio variance times the multiplier. This is because the sum of the marginal variances (found by taking the difference in portfolio variance with and without each account in the portfolio) is greater than the total portfolio variance. The covariance between any two risks in the portfolio is double counted: when each account renews, it is allocated the full amount of its shared covariance with all the other accounts.

8. A NEW CONCEPT: RENEWAL ADDITIVITY

The renewal scenarios point out that these two methods are not what I call *renewal additive*, defined as follows:

For a given portfolio of accounts, a risk load method is *renewal additive* if the sum of the renewal risk loads calculated for each account equals the risk load calculated when the entire portfolio is treated as a single account.

Neither the MS nor the MV method is renewal additive: MS because the square root operator is sub-additive; MV because the covariance is double counted. So why should renewal additivity matter? Consider what happens when either of these non-renewal-additive methods is used to renew the portfolio. The MV method would result in quoted renewal premiums the sum of whose risk loads would be greater than the required total risk load of $(\lambda \times \text{total portfolio variance})$. One would in essence overcharge every account. The opposite is true for the MS case, where one would undercharge every account.

In order for the MS or MV methods to be renewal additive, one must assume an entry order for the accounts. Since the marginal impacts depend on the size of the existing portfolio, the entry order selected for an account could determine whether it is written or declined.

Renewal additivity reduces the renewal risk load calculation to an allocation of the total portfolio amount back to the individual accounts. An objective, systematic allocation methodology for renewals would be desirable. Examples of many such allocation methodologies can be found in the field of game theory.

9. A NEW APPROACH FROM GAME THEORY

Two ASTIN papers by Jean Lemaire—“An Application of Game Theory: Cost Allocation” [4], and “Cooperative Game Theory and Its Insurance Applications” [5]—focus on general insurance applications of game theory. Lemaire also provides an extensive list of real world applications of game theory [4, p. 77], including tax allocation among operating divisions of McDonnell-Douglas, maintenance costs of the Houston medical library, financing of large water resource development projects in Tennessee, construction costs of multi-purpose reservoirs in the U.S., and landing fees at Birmingham airport. Consider this example from [5]:

“The Treasurer of ASTIN (player 1) wishes to invest the amount of 1,800,000 Belgian francs on a short term

(3 months) basis. In Belgium, the annual interest rate is a function of the sum invested.

Deposit (in Belgian Francs)	Annual Interest Rate
0–1,000,000	7.75%
1,000,000–3,000,000	10.25%
3,000,000–5,000,000	12.00%

The ASTIN Treasurer contacts the Treasurers of the International Actuarial Association (I.A.A.–player 2) and of the Brussels Association of Actuaries (A.A.Br.–player 3). I.A.A. agrees to deposit 900,000 francs in the common fund, A.A.Br. 300,000 francs. Hence the 3-million mark is reached and the interest rate will be 12%. How should the interests be split among the three associations?” [5, p. 18]

Games such as this are referred to as “cooperative games with transferable utilities.” They typically feature:

1. participants (players) that have some benefits (or costs) to share (political power, savings, or money),
2. the opportunity to share benefits (costs) results from co-operation of all participants or a sub-group of participants,
3. freedom for players to engage in negotiations, bargaining, and coalition formation, and
4. conflicting player objectives: each wants to secure the largest part of the benefits (smallest share of the costs) for himself. (See [5, p. 20].)

Cooperative games can be used as models for situations where participants must share or allocate an amount of money. Players may want to maximize or minimize their allocation depending

on the nature of the problem. If the group is deciding who pays what share of the total tax bill, players will want to minimize their share. If on the other hand the group is deciding how to split a pot of bonus money, players will want to maximize their share.

The total amount to be allocated is determined by the *characteristic function*, which associates a real number $v(S)$ to each coalition (group) S of players. It can be either sub-additive or super-additive, defined as follows:

Sub-Additive

$$v(S) + v(T) > v(S \text{ union } T) \quad \text{for every disjoint } S \text{ and } T.$$

Super-Additive

$$v(S) + v(T) < v(S \text{ union } T) \quad \text{for every disjoint } S \text{ and } T.$$

In the actuarial association example above, the characteristic function would be the money earned by each coalition (combination) of associations. It is an example of a super-additive characteristic function where the players seek to maximize their allocation. An example of a sub-additive characteristic function would be the insurance premium for a risk purchasing group: the sum of the individual members' insurance premiums is more than the insurance premium for the risk purchasing group as a whole. These players would seek to minimize their allocations, since they want to be charged the lowest premium. (Equivalently, these players want to maximize their savings as a result of joining the group—savings being the difference between their allocation from the group and their stand-alone premium.)

Allocation Rules

A player's marginal impact depends on its entry order. In the example, the "allocation [to the three associations] of course depends on the order of formation of the grand coalition" [5, p. 27]. In the interests of fairness and stability, a new member should probably receive an allocation amount somewhere between its stand-alone value and its full marginal impact on the coalition

characteristic function—but where in between? How much is fair? These questions must be answered simultaneously for all the players, balancing questions of stability, incentives to split from the group, bargaining power, and marginal impact to the coalition characteristic function value.

To help answer the allocation question, game theory has developed a set of standards or rules for allocations. First, legitimate allocation methods must be additive—the sum of the players' allocations must equal the total amount to be allocated. The MV and MS methods are not (renewal) additive: they either allocate too much (MV) or too little (MS) in the renewal case.

Second, a coalition should be stable, which roughly translates to fair. There must not be incentives for either a single player or a sub-group of players to split from the group and form a faction. These “rules of fairness” are referred to as the conditions of individual and collective rationality (see [4, p. 66–68]). Individual rationality means a player is no worse off for having joined the coalition. Collective rationality means no subgroup would be better off on its own.

These rules can be formalized into a set of acceptable ranges of allocations for each player. This set defines what is known as the *core* of the game. It consists of all allocations satisfying these fairness and stability conditions.

Consider the Brussels Association of Actuaries (A.A.Br.–player 3) from the example. They have 300,000 francs, and on their own could earn 7.75%. If they join as the third player, they will push the coalition rate of return from 10.25% to 12.00%. How much should they earn? Certainly not less than 7.75%—it would not be individually rational for them to join. Conversely, they should not earn so much that players 1 and 2 end up earning less than 10.25%—that would not be collectively rational for them. In that case, players 1 and 2 would be better off forming their own faction. Similar exercises can be performed from the perspective of players 1 and 2. The resulting set of acceptable allocations defines the boundaries of the core (see [5, p. 26]).

TABLE 5
TRANSLATION FROM GAME THEORY TO PROPERTY CAT RISK
LOAD

Game Theory	Property Cat Risk Load
Player	Account
Coalition	Portfolio
Characteristic Function	Portfolio Variance or Standard Deviation

Translating to Property Cat Risk Load

Given this brief introduction, a reasonable first attempt at translating from the game theory context might be as shown in Table 5.

Because of the covariance component, portfolio variance is a super-additive characteristic function: the variance of a portfolio is greater than the sum of the individual account variances. Standard deviation, on the other hand, is a sub-additive characteristic function because of the sub-additivity of the square root operator: the standard deviation of a portfolio is less than the sum of the individual account standard deviations.

This means, from the game theory perspective at least, that the choice between variance and standard deviation is material. It determines whether the characteristic function is sub-additive or super-additive. This is a fundamental paradox of the game theory translation of the risk load problem, and will require further research to resolve.

Setting aside this paradox for the moment, however, the risk load problem fits remarkably well into the game theory framework. The “players” want to minimize their allocations of the portfolio total risk load. The allocation should fairly and objectively assign risk loads to accounts in proportion to their contribution to the total. Using the current definition of marginal im-

pact of a renewal account, however, an entry order would have to be assumed in order to make the allocation additive. The results of that allocation would be heavily dependent on the selected order, however.

How can one choose the entry order of a renewal? A well known allocation method from game theory may provide the answer.

The Shapley Value

The Shapley value (named for Lloyd Shapley, one of the early leaders of the game theory field) is an allocation method that is:

1. additive,
2. at the centroid of the core, and
3. order independent.

It equals the average of the marginal impacts taken over all possible entrance permutations—the different orders in which a new member could have been added to the coalition¹¹ (i.e., a new account could have been added to a portfolio).

For example, consider a portfolio of accounts A and B to which a new account C is added. Shown in Table 6 are the marginal variances for adding C in the 6 possible entrance permutations (“ ABC ” in Column 1 below means A enters first, then B , then C).

¹¹Lemaire provides this more complete definition of the Shapley value [5, p. 29]: “The Shapley value can be *interpreted* as the mathematical expectation of the admission value, when all orders of formation of the grand coalition are equiprobable. In computing the value, one can assume, for convenience, that all players enter the grand coalition one by one, each of them receiving the entire benefits he brings to the coalition formed just before him. All orders of formation of N are considered and intervene with the same weight $1/n!$ in the computation. The combinatorial coefficient results from the fact that there are $(s-1)!(n-s)!$ ways for a player to be the last to enter coalition S : the $(s-1)$ other players of S and the $(n-s)$ players of $N \setminus S$ (those players in N which are not in S) can be permuted without affecting i 's position.”

TABLE 6
ENTRY PERMUTATIONS FOR ACCOUNT *C*

(1) Permutation	(2) <i>C</i> Enters ...	(3) Marginal Variance
<i>ABC</i>	After <i>A</i> & <i>B</i>	$\text{Var}(C) + 2 \times \text{Cov}(C, A) + 2 \times \text{Cov}(C, B)$
<i>ACB</i>	After <i>A</i>	$\text{Var}(C) + 2 \times \text{Cov}(C, A)$
<i>BAC</i>	After <i>B</i> & <i>A</i>	$\text{Var}(C) + 2 \times \text{Cov}(C, A) + 2 \times \text{Cov}(C, B)$
<i>BCA</i>	After <i>B</i>	$\text{Var}(C) + 2 \times \text{Cov}(C, B)$
<i>CAB</i>	First	$\text{Var}(C)$
<i>CBA</i>	First	$\text{Var}(C)$

The Shapley value is the straight average of Column 3, Marginal Variance, over the six permutations:

$$\begin{aligned}
 \text{Shapley Value} &= [\text{Sum}(\text{Column 3})]/6 \\
 &= [6 \text{Var}(C) + 6 \text{Cov}(C, A) + 6 \text{Cov}(C, B)]/6 \\
 &= \text{Var}(C) + \text{Cov}(C, A) + \text{Cov}(C, B). \quad (9.1)
 \end{aligned}$$

Or, to generalize, given

$$\begin{aligned}
 L &= \text{existing portfolio} \quad \text{and} \\
 n &= \text{new account}, \quad (9.2)
 \end{aligned}$$

$$\text{Shapley Value} = \text{Var}(n) + \text{Cov}(L, n).$$

Before seeing this result, there might have been concerns about the practicality of this approach—how much computational time might be required to calculate all the possible entrance permutations for a portfolio of thousands of accounts? This simple reduction formula eliminates those concerns. The Shapley value is as simple to calculate as the marginal variance.

Comparing the Shapley value to the marginal variance formula from Section 4:

$$\text{Marginal Variance} = \text{Var}(n) + 2 \text{Cov}(L, n), \quad (9.3)$$

whereas the Shapley value only takes 1 times the covariance of the new account and the existing portfolio.

One can also calculate the Shapley value under the marginal standard deviation method. However, due to the complex nature of the mathematics—differences of square roots of sums of products—no simplifying reduction formula was immediately apparent.¹²

Therefore, the remainder of the paper will focus on the MV method and the variance-based Shapley value. Life will be much easier (mathematically) working with the variances, and very little is lost by choosing variance. Citing Mr. Bault [1, p. 82], from a risk load perspective, “both (variance and standard deviation) are simply special cases of a unifying covariance framework.” In fact, Bault goes on to suggest “in most cases, the ‘correct’ answer is a marginal risk approach that incorporates covariance.”¹³

10. SHARING THE COVARIANCE

The risk load question, framed in a game-theoretical light, has now become:

How do accounts share their mutual covariance for purposes of calculating risk load?

The Shapley method answers, “Accounts split their mutual covariance equally.” At first glance this appears reasonable, but consider the following example.

Assume two accounts, E and F . F has 100 times the losses of E for each event. Their total shared covariance is

$$\begin{aligned} 2\text{Cov}(E, F) &= 2 \sum_i [E_i \times F_i \times p_i \times (1 - p_i)] \\ &= 2 \sum_i [E_i \times 100E_i \times p_i \times (1 - p_i)]. \end{aligned} \quad (10.1)$$

¹²Those wishing to employ standard deviation can use approximate methods to calculate the Shapley value. Two approaches suggested by John Major are (i) taking the average of marginal value if first in and last in; and (ii) employing Monte Carlo simulation to sample a subset of the possible entrance permutations, presumably large enough to achieve satisfactory convergence while being much more computationally efficient.

¹³Kreps also incorporates covariance in his “Reluctance” R [3, p. 198], which has the formula $R = [yz/(1 + y)]/(2SC + \sigma)/(S' + S)$, where C is the correlation of the contract with the existing book. The Risk Load is then equal to $R\sigma$.

The Shapley value would equally divide this total covariance between E and F , even though their relative contributions to the total are clearly not equal. There is no question that E should be assessed some share of the covariance. The issue is whether there is a more equitable share than simply half.

One could develop a generalized covariance sharing (GCS) method which uses a weight $W_i^X(X, Y)$ to determine an account X 's share of the mutual covariance between itself and another account Y for event i :

$$\text{CovShare}_i^X(X, Y) = W_i^X(X, Y) \times 2 \times X_i \times Y_i \times p_i \times (1 - p_i). \quad (10.2)$$

Then Y 's share of that mutual covariance would simply be

$$\text{CovShare}_i^Y(X, Y) = [1 - W_i^X(X, Y)] \times 2 \times X_i \times Y_i \times p_i \times (1 - p_i). \quad (10.3)$$

The total covariance share allocation for account X over all events would be:

$$\text{CovShare}_{\text{Tot}}^X = \sum_Y \sum_i [\text{CovShare}_i^X(X, Y)], \quad (10.4)$$

where \sum_Y = sum over every other account Y in the portfolio.

The Shapley method is in fact an example of the generalized covariance sharing method with $W_i^X(X, Y) = 50\%$ for all X, Y and i .

Returning to the example with E and F , one could develop an example of a weighting scheme that assigns the shared covariance by event to each account in proportion to their loss for that event. $W_i^E(E, F)$, account E 's share of the mutual covariance between itself and account F for event i , equals

$$\begin{aligned} W_i^E(E, F) &= [E_i / (E_i + F_i)] \\ &= [E_i / (E_i + 100E_i)] = (1/101) \\ &= \text{roughly } 1\% \text{ of their mutual covariance for event } i. \end{aligned}$$

This shall be called the ‘‘Covariance Share’’ (CS) method.

TABLE 7
BUILDING UP X AND Y : SHAPLEY VALUE METHOD

	Account X	Account Y	Account X +Account Y	Portfolio ($X + Y$)
(1) Change in Variance	19,619,900	1,828,509	21,448,409	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	—	0.000069
(3) Risk Load = (1) \times (2)	\$1,353.02	\$126.10	\$1,479.11	\$1,579.14

TABLE 8
BUILDING UP X AND Y : COVARIANCE SHARE METHOD

	Account X	Account Y	Account X +Account Y	Portfolio ($X + Y$)
(1) Change in Variance	19,619,000	950,658	20,570,558	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	—	0.000069
(3) Risk Load = (1) \times (2)	\$1,353.02	\$65.56	\$1,418.57	\$1,579.14

11. APPLYING THE SHAPLEY AND CS METHODS TO THE EXAMPLE

Consider the Shapley and CS methods applied to the two Account example for both Build-Up and Renewal.

11.1. Portfolio Build-up

Exhibit 3 shows the Build-Up of accounts X and Y from Section 5, but for the Shapley and CS methods. Pertinent values for the Shapley value are summarized in Table 7.

Pertinent values for the Covariance Share are summarized in Table 8.

Both Shapley and CS produce the same risk load for X as the MV method on Build-Up: \$1,353.02. This is because there is no covariance to share: X is the entire portfolio at this point. How-

TABLE 9
COMPARISON OF BUILD-UP RISK LOADS FOR ACCOUNT Y

Marginal Variance (MV)—Section 5.2	\$226.13
Shapley Value	\$126.10
Difference from MV	\$100.03
Covariance Share (CS)	\$ 65.56
Difference from MV	\$160.57

TABLE 10
RENEWING X AND Y: SHAPLEY VALUE METHOD

	Account X	Account Y	Account X +Account Y	Portfolio (X + Y)
(1) Change in Variance	21,070,450	1,828,509	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	—	0.000069
(3) Risk Load = (1) × (2)	\$1,453.05	\$126.10	\$1,579.14	\$1,579.14
(4) Build-Up Risk Load	\$1,353.02	\$126.10	\$1,479.11	\$1,579.14
(5) Difference	\$100.03	\$0	\$100.03	\$0

ever, compare the results of the three variance-based methods for account Y (see Table 9).

Compared to MV, which charges account Y for the full increase in variance $\text{Var}(Y) + 2\text{Cov}(X, Y)$, the Shapley method only charges Y for $\text{Var}(Y) + \text{Cov}(X, Y)$. The same can be said for the CS method, although the share of the mutual covariance depends on each account's relative contribution by event, weighted and summed over all events. Now consider what happens to that *difference from MV* upon renewal.

11.2. Renewal

Exhibit 4 shows the renewal of X and Y for the Shapley and CS methods. Pertinent values for the Shapley method are summarized in Table 10.

TABLE 11
RENEWING X AND Y : COVARIANCE SHARE METHOD

	Account X	Account Y	Account X +Account Y	Portfolio ($X + Y$)
(1) Change in Variance	21,948,301	950,658	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	—	0.000069
(3) Risk Load = (1) \times (2)	\$1,513.59	\$65.56	\$1,579.14	\$1,579.14
(4) Build-Up Risk Load	\$1,353.02	\$65.56	\$1,418.57	\$1,579.14
(5) Difference	\$160.57	\$0	\$160.57	\$0

TABLE 12
COMPARISON OF BUILD-UP AND RENEWAL RISK LOADS FOR
ACCOUNT X

	Shapley	Cov Share
Renewal	\$1,453.05	\$1,513.59
Build-Up	\$1,353.02	\$1,353.02
Additional Renewal Risk Load over Build-Up	\$100.03	\$160.57
Difference from MV	\$100.03	\$160.57

Pertinent values for the Covariance Share method are summarized in Table 11.

With both the Shapley and CS methods, the sum of the risk loads for Account X and Account Y equals the risk load for Account ($X + Y$), namely \$1,579.14. This means that both new methods are renewal additive.

To see what happened to the *difference from MV*, compare the risk loads calculated at Renewal for X with those at Build-Up (see Table 12).

The difference from MV during Build-Up is simply the portion of X 's risk load attributable to its share of covariance with Y . It was missed during Build-Up because it was unknown—account Y had not been written.

12. CONCLUSION

This paper introduces two new approaches to determination of renewal risk load that address concerns with renewal additivity and point out the issue of covariance sharing between accounts. The ideal solution in practice might involve using a marginal method for the pricing of new accounts, and a renewal additive method for renewals.

This paper also represents a first step in addressing the perplexing question of order dependency. As mentioned in the introduction, order dependency stretches beyond the confines of actuarial pricing to the finance community at large. It will likely take a joint effort between finance professionals and actuaries to reach a satisfactory solution.

Finally, this paper brings important information from game theory to the *Proceedings*. Game theory is a rich field for actuaries to find new ideas on cost allocation, fairness, and order dependency. Many sticky social issues (taxation, voting rights, utility costs) have been resolved using ideas from game theory. Further research can be done on several questions raised during the review of this paper, including the relative bargaining power of accounts, portfolio departure rules, lack of account information, and the unresolved paradox of the sub-additive MS characteristic function versus the super-additive MV characteristic function.

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EXHIBIT 1

BUILD UP PORTFOLIO OF TWO ACCOUNTS

Event i	Prob $p(i)$	$1-p(i)$	Loss for Account		Portfolio ($X + Y$)
			X	Y	
1	2.0%	98.0%	25,000	200	25,200
2	1.0%	99.0%	15,000	500	15,500
3	3.0%	97.0%	10,000	3,000	13,000
4	3.0%	97.0%	8,000	1,000	9,000
5	1.0%	99.0%	5,000	2,000	7,000
6	2.0%	98.0%	2,500	1,500	4,000

E (L)	1,290	179	1,469
Var (L)	19,619,900	377,959	22,898,959
Std Dev (L)	4,429	615	4,785

Covariances	X	Y
X	19,619,900	1,450,550
Y	1,450,550	377,959

	X	Y	X + Y
Change in Std Deviation	4,429	356	4,785
Risk Load (Std Dev)	1,461.71	117.43	1,579.14
Multiplier :	0.33	Risk Load for X + Y :	1,579.14

Change in Variance	19,619,900	3,279,059	22,898,959
Risk Load (Variance)	1,353.02	226.13	1,579.14
Multiplier :	0.000069	Risk Load for X + Y :	1,579.14

EXHIBIT 2

RENEW THE PORTFOLIO OF TWO ACCOUNTS

Event i	Prob $p(i)$	$1-p(i)$	Loss for Account		
			X	Y	Portfolio (X + Y)
1	2.0%	98.0%	25,000	200	25,200
2	1.0%	99.0%	15,000	500	15,500
3	3.0%	97.0%	10,000	3,000	13,000
4	3.0%	97.0%	8,000	1,000	9,000
5	1.0%	99.0%	5,000	2,000	7,000
6	2.0%	98.0%	2,500	1,500	4,000

E (L)	1,290	179	1,469
Var (L)	19,619,900	377,959	22,898,959
Std Dev (L)	4,429	615	4,785

Covariances	X	Y
X	19,619,900	1,450,550
Y	1,450,550	377,959

	X	Y	X + Y
Change in Std Deviation	4,171	356	4,526
Risk Load (Std Dev)	1,376.27	117.43	1,493.70
0.33 Build Up Risk Load	1,461.71	117.43	1,579.14
Difference	(85.45)		(85.45)

Change in Variance	22,521,000	3,279,059	25,800,059
Risk Load (Variance)	1,553.08	226.13	1,779.21
0.000069 Build Up Risk Load	1,353.02	226.13	1,579.14
Difference	200.06		200.06

EXHIBIT 3

BUILD UP A PORTFOLIO OF TWO ACCOUNTS—ALTERNATIVES

Event <i>i</i>	Prob $p(i)$	1- $p(i)$	Covariance Share \$	
			X	Y
1	2.0%	98.0%	9,920,635	79,365
2	1.0%	99.0%	14,516,129	483,871
3	3.0%	97.0%	46,153,846	13,846,154
4	3.0%	97.0%	14,222,222	1,777,778
5	1.0%	99.0%	14,285,714	5,714,286
6	2.0%	98.0%	4,687,500	2,812,500

		Total
2,328,401	572,699	2,901,100

Chg in Variance		X	Y
If added 1st		19,619,900	377,959
If added 2nd	after 1		3,279,059
	after 2	22,521,000	
Average (Shapley Value)		21,070,450	1,828,509

Shapley Value	19,619,900	1,828,509	21,448,409
Risk Load (Shapley)	1,353.02	126.10	1,479.11
0.000069			1,579.14
	Deferred Risk Load		100.03

Covariance Share	19,619,900	950,658	20,570,558
Risk Load (Cov Share)	1,353.02	65.56	1,418.57
0.000069			1,579.14
	Deferred Risk Load		160.57

EXHIBIT 4

RENEW THE PORTFOLIO OF TWO ACCOUNTS—ALTERNATIVES

Event i	Prob $p(i)$	$1-p(i)$	Covariance Share \$	
			X	Y
1	2.0%	98.0%	9,920,635	79,365
2	1.0%	99.0%	14,516,129	483,871
3	3.0%	97.0%	46,153,846	13,846,154
4	3.0%	97.0%	14,222,222	1,777,778
5	1.0%	99.0%	14,285,714	5,714,286
6	2.0%	98.0%	4,687,500	2,812,500

		Total
2,328,401	572,699	2,901,100

Chg in Variance		X	Y
If added 1st		19,619,900	377,959
If added 2nd	after 1		3,279,059
	after 2	22,521,000	
Average (Shapley Value)		21,070,450	1,828,509

Shapley Value	21,070,450	1,828,509	22,898,959
Risk Load (Shapley)	1,453.05	126.10	1,579.14
0.000069	Risk Load for Portfolio (X + Y)		1,579.14

Covariance Share	21,948,301	950,658	22,898,959
Risk Load (Cov Share)	1,513.59	65.56	1,579.14
0.000069	Risk Load for Portfolio (X + Y)		1,579.14